ulm university universität uulm

Fakultät für Mathematik und Wirtschaftswissenschaften

Periodic ARMA, Ornstein-Uhlenbeck and CARMA Processes as Periodic Time Series

Dissertation

zur Erlangung des Doktorgrades Dr. rer. nat.
der Fakultät für Mathematik und Wirtschaftswissenschaften
der Universität Ulm

vorgelegt von

Abdulkahar Mohamed Alkadour

aus Aleppo/ Syrien

2018

Amtierender Dekan: Prof. Dr. Alexander Lindner
Erstgutachter: Prof. Dr. Alexander Lindner
Zweitgutachter: Prof. Dr. Jens-Peter Kreiß

Abgabe der Doktorarbeit: 22. März 2018
Mündliche Prüfung: 05. Juli 2018

Abstract

This thesis consists of two parts, the first one deals with time series with discrete time index and the second deals with time series with continuous time index.

In Chapter 2 we consider the periodic autoregressive moving average model with period $d \in \mathbb{N}$ (short $PARMA_d$) of the form

$$X_{nd+s} - \sum_{i=1}^{p} \phi_i(s) X_{nd+s-i} = Z_{nd+s} + \sum_{i=1}^{q} \theta_i(s) Z_{nd+s-i}\,, \quad n \in \mathbb{Z}\,,\ s = 1, \dots, d\,,$$

where $p, q \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\phi_i(s)$ and $\theta_i(s)$ are d-periodic complex-valued polynomials and $(Z_{nd+s})_{n\in\mathbb{Z}}$ is a d-periodic or i.i.d. noise sequence. We focus on the periodic autoregressive model i.e. when $q = 0$. Necessary and sufficient conditions for the existence and uniqueness of strictly and weakly periodic stationary solutions of an autoregressive model will be specified.

In Chapter 3 we study the periodic Ornstein-Uhlenbeck process $(V_t)_{t\in\mathbb{R}}$ driven by a real-valued two-sided Lévy process $(L_t)_{t\in\mathbb{R}}$ fulfilling the stochastic differential equation

$$dV_t = \lambda(t) V_t dt + dL_t\,, \qquad t \in \mathbb{R}\,,$$

where $\lambda(\cdot)$ is a d-periodic, real-valued bounded and measurable function. We give necessary and sufficient conditions for the existence of strictly and weakly periodic stationary solutions of the periodic Ornstein-Uhlenbeck process. We also give the autocovariance function and consider the 1-periodic Ornstein-Uhlenbeck process when sampled equidistantly.

In Chapter 4 we consider the multivariate periodic Ornstein-Uhlenbeck equation

$$dV_t = A(t) V_t dt + dL_t\,, \qquad t \in \mathbb{R}\,,$$

where $A(\cdot) \in \mathbb{R}^{d\times d}$ a 1-periodic, continuous, non identically zero and known function and $(L_t)_{t\in\mathbb{R}}$ is a $\mathbb{R}^d$-valued two-sided Lévy process. Necessary and sufficient conditions for the existence of a strictly and weakly periodic stationary solutions of this stochastic differential equation are obtained.

In Chapter 5 we study the sufficient conditions for the existence of a periodic strictly and weakly stationary solution of the periodic CARMA-process $(Y_t)_{t\in\mathbb{R}}$ of the form $Y_t = \mathbf{b}_t^T \mathbf{X}_t$, where $\mathbf{b}_t \in \mathbb{R}^p$ is a d-periodic vector and $(\mathbf{X}_t)_{t\in\mathbb{R}} \in \mathbb{R}^p$ fulfils a multivariate periodic Ornstein-Uhlenbeck equation. We also calculate the autocovariance function of periodic weakly stationary PCARMA processes.

Zusammenfassung

Diese Dissertation besteht aus zwei Teilen, der erste Teil handelt von Zeitreihen in diskreter Zeit und der andere Teil handelt von Zeitreihen in stetiger Zeit.

In Kapitel 1 betrachten wir das periodische autoregressive moving average Modell mit Periode $d \in \mathbb{N}$ ($PARMA_d$) von der Form

$$X_{nd+s} - \sum_{i=1}^{p} \phi_i(s) X_{nd+s-i} = Z_{nd+s} + \sum_{i=1}^{q} \theta_i(s) Z_{nd+s-i}\,, \quad n \in \mathbb{Z}\,,\ s = 1, \dots, d\,,$$

wobei $p, q \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\phi_i(s)$ und $\theta_i(s)$ sind d-periodische komplex-wertige Polynome und $(Z_{nd+s})_{n\in\mathbb{Z}}$ ist eine d-periodische oder i.i.d. Noise Sequenz. Wir konzentrieren uns auf das periodische autoregressive Modell, d.h. auf $q = 0$. Wir geben notwendige und hinreichende Bedingungen für die Existenz und Eindeutigkeit der strikt und schwach stationären periodischen Lösung des periodischen autoregressiven Modells.

In Kapitel 3 studieren wir den periodischen Ornstein-Uhlenbeck Prozess $(V_t)_{t\in\mathbb{R}}$, der die folgende stochastische Differentialgleichung

$$dV_t = \lambda(t) V_t dt + dL_t\,, \qquad t \in \mathbb{R}\,,$$

erfüllt, wobei $(L_t)_{t\in\mathbb{R}}$ ein reellwertiger zwei-seitiger Lévy Prozess $(L_t)_{t\in\mathbb{R}}$ ist und $\lambda(\cdot)$ eine d-periodische, reellwertige beschränkte und messbare Funktion. Die notwendigen und hinreichenden Bedingungen für die Existenz und Eindeutigkeit der strikt und schwach periodischen stationären Lösung des periodischen Ornstein-Uhlenbeck Prozesses werden gegeben. Außerdem geben wir die Autokovarianzfunktion des 1-periodischen Ornstein-Uhlenbeck Prozesses an, und betrachten den äquidistant gesampelten Prozess.

In Kapitel 4 betrachten wir die multivariate periodische Ornstein-Uhlenbeck Gleichung der Form

$$dV_t = A(t) V_t dt + dL_t\,, \qquad t \in \mathbb{R}\,,$$

wobei $A(\cdot) \in \mathbb{R}^{d\times d}$ eine 1-periodische, stetige und bekannte Funktion (nicht identisch null) und $(L_t)_{t\in\mathbb{R}}$ ein $\mathbb{R}^d$-wertiger zwei-seitiger Lévy Prozess ist. Die notwendigen und hinreichenden Bedingungen für die Existenz und Eindeutigkeit der strikt und schwach periodischen stationären Lösung des multivariaten periodischen Ornstein-Uhlenbeck Prozesses werden gegeben.

In Kapitel 5 geben wir die hinreichenden Bedingungen für die Existenz und Eindeutigkeit der strikt und schwach periodischen stationären Lösung des periodischen CARMA Prozesses $(Y_t)_{t\in\mathbb{R}}$, der die folgende Form $Y_t = \mathbf{b}_t^T \mathbf{X}_t$ hat, wobei $\mathbf{b}_t \in \mathbb{R}^p$ d-periodisch ist und

$(\mathbf{X}_t)_{t\in\mathbb{R}} \in \mathbb{R}^p$ erfüllt eine multivariate periodische Ornstein-Uhlenbeck Gleichung. Außerdem berechenen wir die Autocovarianzfunktion des periodisch schwach stationären PCARMA Prozesses.

Contents

CHAPTER 1

Introduction

Stationary stochastic processes have been extensively studied and their prediction theory is fairly complete. A wide class of non-stationary stochastic processes is that of periodically correlated sequences.
Periodically correlated processes have recently been studied by several authors, including Gladyshe [18], Hurd [21], [22], Gudzenko [20], Pagano [31], etc.

We are going to introduce the class of periodic Ornstein-Uhlenbeck processes analogous to the class of the usual Ornstein-Uhlenbeck processes but the coefficients will pe periodic functions.

In the following section we give some basic definitions and some preliminary results that are important and useful for this work.

1.1 ARMA, Ornstein-Uhlenbeck and CARMA Processes

In this chapter we will consider that $T = \mathbb{Z}$ for the discrete-time processes and $T = \mathbb{R}$ for the continuous-time processes. Firstly, let us recall the definition of a stationary stochastic process.

Definition 1.1. *(a) An $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\in T}$ is said to be* weakly stationary *if*

(i) $\mathbb{E}|X_t|^2 < \infty$ *for all* $t \in T$,

(ii) $\mathbb{E}X_t$ *is independent of* t *for all* $t \in T$ *and*

(iii) $\boldsymbol{Cov}(X_{t+h}, X_t)$ *is independent of* t *for all* $h \in T$.

(Hence, $\boldsymbol{Cov}(X_{t+h}, X_t) = \mathbb{E}X_{t+h}X_t^T - \mathbb{E}X_{t+h}\mathbb{E}X_t^T$.*)*

(b) A stochastic process $(X_t)_{t\in T}$ is said to be strictly stationary *if the joint distributions of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ and $(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h})$ are the same for all positive integers n and for all $t_1, \ldots, t_n, h \in T$.*

1.1.1 ARMA Processes

In the statistical analysis of time series, autoregressive-moving average (ARMA) models provide a parsimonious description of a (weakly) stationary stochastic process in terms

of two polynomials, one for the autoregression and the second for the moving average. The general ARMA model was described in the 1951 thesis of Peter Whittle, Hypothesis testing in time series analysis, and it was popularized in the 1971 book by George E. P. Box and Gwilym Jenkins.
Given a time series of data X_t, the ARMA model is a tool for understanding and, perhaps, predicting future values in this series. The model consists of two parts, an autoregressive (AR) part and a moving average (MA) part. The AR part involves regressing the variable on its own lagged (i.e., past) values. The MA part involves modeling the error term as a linear combination of error terms occurring contemporaneously and at various times in the past.
This model is usually referred to as the ARMA(p,q) model where p is the order of the autoregressive part and q is the order of the moving average part.

Univariate ARMA Processes

Let us here give the mathematical definition of the ARMA(p,q) model:

Definition 1.2. *Let* $(Z_t)_{t\in\mathbb{Z}}$ *be an* $\mathbb{R}$*-valued weak white noise sequence,* $p, q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\phi_1, \phi_2 \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q \in \mathbb{R}$, $\phi_p \neq 0, \theta_q \neq 0$. *Then any weakly stationary stochastic process* $(X_t)_{t\in\mathbb{Z}}$ *which satisfies*

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \qquad t \in \mathbb{Z}$$

is called a weak autoregressive moving average process of the autoregressive order p and moving average order q, *short* weak ARMA(p,q) process.
Defining the polynomials

$$\Phi(z) := 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p, \qquad \Theta(z) := 1 + \theta_1 z + \cdots + \theta_q z^q, \qquad z \in \mathbb{C},$$

and letting B *be the backwards shift operator defined by* $B^j X_t = X_{t-j}$, $j \in \mathbb{Z}$, *then we can write the ARMA(p,q) equation in the form*

$$\Phi(B)X_t = \Theta(B)Z_t, \qquad t \in \mathbb{Z}. \tag{1.1}$$

An $ARMA(p,0)$ *process is called an* autoregressive process of order p, *short* $AR(p)$ *process. An* $ARMA(0,q)$ *process is called a* moving average process of order q, *short* $MA(q)$ *process.*

The following Theorem gives necessary and sufficient conditions for a weak ARMA process to exist. It is a reformulation of Theorem 7.4 in Kreiß and Neuhaus [26] and its proof relies heavily on the spectral representation of X_t .

Theorem 1.3. *[Weak ARMA(p,q) processes]*
Let $(Z_t)_{t\in\mathbb{Z}}$ *be an* $\mathbb{R}$*-valued weak white noise sequence with mean* μ *and variance* $\sigma^2 > 0$. *Then the ARMA(p,q) equation (1.1) admits a weakly stationary solution if and only if all singularities of* $\Theta(z)/\Phi(z)$ *on the unit circle are removable. In this case, a weakly stationary solution of (1.1) is given by*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \qquad t \in \mathbb{Z}, \tag{1.2}$$

where

$$\sum_{j=-\infty}^{\infty} \psi_j z^j = \frac{\Theta(z)}{\Phi(z)}, \quad 1-\delta < |z| < 1+\delta, \quad \textit{for some} \;\; \delta \in (0,1)\,,$$

is the Laurent expansion of $\Theta(z)/\Phi(z)$. The sum in (1.2) converges absolutely almost surely. If $\Phi(z)$ does not have a zero on the unit circle, the solution is unique.

The following Theorem gives necessary and sufficient conditions for a strict ARMA process to exist. This Theorem is the Theorem 1 in Brockwell and Lindner [10].

Theorem 1.4. *[Strict ARMA(p,q) processes]*
Suppose that $(Z_t)_{t\in\mathbb{Z}}$ is a nondeterministic i.i.d. noise sequence. Then the ARMA equation (1.1) admits a strictly stationary solution $(X_t)_{t\in\mathbb{Z}}$ if and only if

(i) all singularities of $\Theta(z)/\Phi(z)$ on the unit circle are removable and $\mathbb{E}\log^+|Z_1| < \infty$, or

(ii) all singularities of $\Theta(z)/\Phi(z)$ in $\mathbb{C}$ are removable.

If (i) or (ii) above holds, then a strictly stationary solution of (1.1) is given by

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \qquad t \in \mathbb{Z}, \tag{1.3}$$

where

$$\sum_{j=-\infty}^{\infty} \psi_j z^j = \frac{\Theta(z)}{\Phi(z)}, \quad 1-\delta < |z| < 1+\delta, \quad \textit{for some} \;\; \delta \in (0,1)\,,$$

is the Laurent expansion of $\Theta(z)/\Phi(z)$. The sum in (1.3) converges absolutely almost surely. If $\Phi(z)$ does not have a zero on the unit circle, then (1.3) is the unique strictly stationary solution of (1.1).

Multivariate ARMA processes

In this definition we give the definition of multidimensional ARMA processes.

Definition 1.5. *Let $d,m \in \mathbb{N}$, $p,q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Let $(Z_t)_{t\in\mathbb{Z}}$ be a d-variate weak white noise sequence of random vectors and $\Psi_1,\ldots,\Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0,\Theta_1,\ldots,\Theta_q \in \mathbb{C}^{m\times d}$ be deterministic complex-valued matrices. Then any m-variate weakly stationary stochastic process $(X_t) = (X_{t,1},X_{t,2},\ldots,X_{t,m})^T$ which satisfies almost surely the equation*

$$X_t - \Psi_1 X_{t-1} - \cdots - \Psi_p X_{t-p} = \Theta_0 Z_t + \Theta_1 Z_{t-1} + \cdots + \Theta_q Z_{t-q}, \quad t \in \mathbb{Z} \tag{1.4}$$

is called a weak (multivariate) ARMA(p,q) process*. Such a process is often also called a weak* vector ARMA *(VARMA) process to distinguish it from the scalar case.*
Denoting the identity matrix in $\mathbb{C}^{m\times m}$ by Id_m, the characteristic polynomials $P(z)$ and $Q(z)$ of the ARMA(p,q) equation (1.4) are defined as

$$P(z) := Id_m - \sum_{j=1}^{p} \Psi_j z^j \quad \textit{and} \quad Q(z) := \sum_{j=0}^{q} \Theta_j z^j. \tag{1.5}$$

With the aid of the backwards shift operator B, Equation (1.4) can be written more compactly in the form

$$P(B)X_t = Q(B)Z_t\,, \qquad t \in \mathbb{Z}. \tag{1.6}$$

The following Theorem gives necessary and sufficient conditions for a weak vector ARMA process to exist. This Theorem is the Theorem 3 in Brockwell, Lindner and Vollenbröker [13].

Theorem 1.6. *[Weak VARMA(p,q) processes]*
Let $m, d, p \in \mathbb{N}$, $q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be a weak white noise sequence in $\mathbb{C}^d$ with expectation $\mathbb{E}Z_0$ and covariance matrix Σ. Let $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \Theta_1, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$, and define the matrix polynomials $P(z)$ and $Q(z)$ by (1.5). Let $U \in \mathbb{C}^{d\times d}$ be unitary such that $U\Sigma U^\star = \begin{pmatrix} D & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s}\end{pmatrix}$, where D is a real $(s \times s)$-diagonal matrix with the strictly positive eigenvalues of Σ on its diagonal for some $s \in \{0, \ldots, d\}$. (The matrix U exists since Σ is positive semi-definite). Then the ARMA(p, q) equation (1.4) admits a weakly stationary solution $(X_t)_{t\in\mathbb{Z}}$ if and only if the $\mathbb{C}^{m\times d}$-valued rational function

$$z \mapsto M(z) := P^{-1}(z)Q(z)U^\star \begin{pmatrix} Id_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s}\end{pmatrix}$$

has only removable singularities on the unit circle and if there is some $g \in \mathbb{C}^m$ such that

$$P(1)g = Q(1)\mathbb{E}Z_0\,.$$

In that case, a weakly stationary solution of (1.4) is given by

$$X_t = g + \sum_{j=-\infty}^{\infty} M_j U(Z_{t-j} - \mathbb{E}Z_0), \qquad t \in \mathbb{Z} \tag{1.7}$$

where $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ is the Laurent expansion of $M(z)$ in a neighbourhood of the unit circle, which converges absolutely there.

The necessary and sufficient condition for a strictly stationary ARMA(1,q) processes will depend on the value of the eigenvalue of Ψ_1. The following Theorem, which is the Theorem 1 in Brockwell, Lindner and Vollenbröker [13], gives necessary and sufficient conditions for a strict vector ARMA(1,q) process to exist.

Theorem 1.7. *[Strict VARMA(1, q) processes]*
Let $m, d \in \mathbb{N}$, $q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{C}^d$-valued random vectors. Let $\Psi_1 \in \mathbb{C}^{m\times m}$ and $\Theta_0, \Theta_1, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$ be complex-valued matrices. Let $S \in \mathbb{C}^{m\times m}$ be an invertible matrix such that $S^{-1}\Psi_1 S$ is in Jordan block form, with H Jordan blocks $\Phi_1, \Phi_2, \ldots, \Phi_H$, the h^{th} block beginning in row r_h, where $r_1 := 1 < r_2 < \cdots < r_H < m + 1 := r_{H+1}$. Let λ_h be the associated eigenvalues of the blocks Φ_h for , $h \in \{1, \ldots, H\}$, and

let I_h be the $(r_{h+1}-r_h)\times m$ matrix with (i,j) components $I_h(i,j) = \begin{cases} 1, & \text{if } j = i + r_h - 1 \\ 0, & \text{otherwise} \end{cases}$ *for all $h \in \{1,\ldots,H\}$. Then the ARMA(1, q) equation*

$$X_t - \Psi_1 X_{t-1} = \sum_{j=0}^{q} \Theta_j Z_{t-j}, \qquad t \in \mathbb{Z} \tag{1.8}$$

has a strictly stationary solution Y if and only if the following statements (i) - (iii) hold:

(i) For every $h \in \{1,\ldots,H\}$ such that $|\lambda_h| \neq 0, 1$,

$$\mathbb{E}\log^+ \parallel \left(\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1}\Theta_k\right) Z_0 \parallel < \infty$$

(ii) For every $h \in \{1,\ldots,H\}$ such that $|\lambda_h| = 1$, but $\lambda_h \neq 1$, there exists a constant $\alpha_h \in \mathbb{C}^{r_{h+1}-r_h}$ such that

$$\left(\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1}\Theta_k\right) Z_0 = \alpha_h \quad a.s.$$

(iii) For every $h \in \{1,\ldots,H\}$ such that $\lambda_h = 1$, there exists a constant $\alpha_h = (\alpha_{h,1},\ldots,\alpha_{h,r_{h+1}-r_h})^T \in \mathbb{C}^{r_{h+1}-r_h}$ such that $\alpha_{h,1} = 0$ and

$$\left(\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1}\Theta_k\right) Z_0 = \alpha_h \quad a.s.$$

If these conditions are satisfied, then a strictly stationary solution to (1.8) is given by

$$X_t := S(X_t^{(1)^T},\ldots,X_t^{(H)^T})^T, \qquad t \in \mathbb{Z} \tag{1.9}$$

with

$$X_t^{(h)} := \begin{cases} \sum_{j=0}^{\infty} \Phi_h^{j-q}\left(\sum_{k=0}^{j\wedge q} \Phi_h^{q-k} I_h S^{-1}\Theta_k\right) Z_{t-j}, & |\lambda_h| \in (0,1)\,, \\ -\sum_{j=1-q}^{\infty} \Phi_h^{-j-q}\left(\sum_{k=(1-j)\vee 0}^{j\wedge q} \Phi_h^{q-k} I_h S^{-1}\Theta_k\right) Z_{t+j} & |\lambda_h| > 1\,, \\ \sum_{j=0}^{m+q-1}\left(\sum_{k=0}^{j\wedge q} \Phi_h^{j-k} I_h S^{-1}\Theta_k\right) Z_{t-j}, & \lambda = 0\,, \\ f_h + \sum_{j=0}^{q-1}\left(\sum_{k=0}^{j} \Phi_h^{j-k} I_h S^{-1}\Theta_k\right) Z_{t-j}, & |\lambda_h| = 1, \end{cases} \tag{1.10}$$

where $f_h \in \mathbb{C}^{r_{h+1}-r_h}$ is a solution to

$$(Id_h - \Phi_h) f_h = \alpha_h\,,$$

which exists for $\lambda_h = 1$ by (iii) and, for $|\lambda_h| = 1, \lambda_h \neq 1$, by the invertibility of $(Id_h - \Phi_h)$. The series in (1.10) converge a.s. absolutely.
If the necessary and sufficient conditions stated above are satisfied, then, provided the underlying probability space is rich enough to support a random variable which is uniformly distributed on $[0,1)$ and independent of $(Z_t)_{t\in\mathbb{Z}}$, the solution given by (1.9) and (1.10) is the unique strictly stationary solution of (1.8) if and only if $|\lambda_h| \neq 1$ for all $h \in \{1,\ldots,H\}$.

The characterization for the existence of strictly stationary ARMA(p,q) processes was given by Brockwell, Lindner and Vollenbröker [13]. More precisely, in [13], Theorem 2, they state the following.

Theorem 1.8. *[Strict ARMA(p,q) processes]*
Let $m, d, p \in \mathbb{N}, q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{C}^d$-valued random vectors. Let $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \Theta_1, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$ be complex-valued matrices, and define the characteristic polynomials as in (1.5). Define the linear subspace

$$K := \{a \in \mathbb{C}^d : \text{ the distribution of } a^\star Z_0 \text{ is degenerate to a Dirac measure}\}$$

of $\mathbb{C}^d$, denote by $K^\perp$ its orthogonal complement in $\mathbb{C}^d$, and let $s := \dim K^\perp$ the vector space dimension of $K^\perp$. Let $U \in \mathbb{C}^{d\times d}$ be unitary such that $UK^\perp = \mathbb{C}^s \times \{0_{d-s}\}$ and $UK = \{0_s\} \times \mathbb{C}^{d-s}$, and define the $\mathbb{C}^{m\times d}$-valued rational function $M(z)$ by

$$z \mapsto M(z) := P^{-1}(z)Q(z)U^\star \begin{pmatrix} Id_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}.$$

Then there is a constant $u \in \mathbb{C}^{d-s}$ and a $\mathbb{C}^s$-valued i.i.d. sequence $(w_t)_{t\in\mathbb{Z}}$ such that $UZ_t = \begin{pmatrix} w_t \\ u \end{pmatrix}$ a.s. $\forall t \in \mathbb{Z}$, and the distribution of $b^\star w_0$ is not degenerate to a Dirac measure for any $b \in \mathbb{C}^s\backslash\{0\}$. Further, a strictly stationary solution to the ARMA(p,q) equation (1.4) exists if and only if the following statements (i)-(iii) hold:

(i) All singularities on the unit circle of the meromorphic function $M(z)$ are removable.

(ii) If $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ denotes the Laurent expansion of M in a neighbourhood of the unit circle, then

$$\mathbb{E}\log^+ \| M_j U Z_0 \| < \infty \quad \forall j \in \{mp+q-p+1, \ldots, mp+q\} \cap \{-p, \ldots, -1\}.$$

(iii) There exist $\nu \in \mathbb{C}^s$ and $g \in \mathbb{C}^m$ such that g is a solution to the linear equation

$$P(1)g = Q(1)U^\star(\nu^T, u^T)^T$$

.

Further, if (i) above holds, then condition (ii) can be replaced by
(ii') If $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ denotes the Laurent expansion of M in a neighbourhood of the unit circle, then $\sum_{j=-\infty}^{\infty} M_j U Z_{t-j}$ converges almost surely absolutely for every $t \in \mathbb{Z}$, and condition (iii) can be replaced by
(iii') For all $\nu \in \mathbb{C}^s$ there exists a solution $g = g(v)$ to the linear equation $P(1)g = Q(1)U^\star(\nu^T, u^T)^T$.
If the conditions (i)-(iii) given above are satisfied, then a strictly stationary solution X_t of the ARMA(p,q) equation (1.4) is given by

$$X_t = g + \sum_{j=-\infty}^{\infty} M_j(UZ_{t-j} - (\nu^T, u^T)^T), \quad t \in \mathbb{Z}, \tag{1.11}$$

the series converging almost surely absolutely. Further, provided that the underlying probability space is rich enough to support a random variable which is uniformly distributed on $[0, 1)$ and independent of $(Z_t)_{t\in\mathbb{Z}}$, the solution given by (1.11) is the unique strictly stationary solution of (1.4) if and only if $\det P(z) \neq 0$ for all z on the unit circle.

1.1.2 Ornstein Uhlenbeck Processes

The Ornstein-Uhlenbeck process originally has been developed to describe the motion of a free particle in a fluid. In 1905 Albert Einstein modelled this movement by a Brownian motion. Twenty five years later the two physicists Leonard Ornstein and George Uhlenbeck added the concept of friction to Einstein's model. This led to the following differential equation for the velocity v_t, $t \geq 0$ of a free particle in a fluid

$$m dv_t = -\lambda v_t dt + dB_t$$

where $(B_t)_{t \geq 0}$ is a Brownian motion (times a constant), m is the mass of the given particle and $\lambda > 0$ is a friction coefficient.
The previous equation is also known as *Langevin equation* and its solution with a starting value v_0 is given by

$$v_t = e^{-\lambda t/m} v_0 + \frac{1}{m} \int_{(0,t]} e^{\lambda(s-t)/m} dB_s, \quad t \geq 0 \, .$$

This solution is called an *Ornstein-Uhlenbeck process* (abbreviated as *OU-process.*)

In this section we are going to give some definitions and results about the OU-processes.

Definition 1.9. *A (two-sided)* Lévy process *with values in $\mathbb{R}^d$, $d \in \mathbb{N}$, is a stochastic process $(L_t)_{t \in \mathbb{R}}$ such that the following properties hold:*

(i) $(L_t)_{t \in T}$ starts almost surely at 0, i.e. $L_0 = 0$ a.s.

(ii) It has independent increments, i.e. for all $n \in \mathbb{N}$, and $t_1, t_2, \ldots, t_n \in \mathbb{R}$ with $t_1 \leq t_2 \leq \cdots \leq t_n$ the random variables $L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent.

(iii) It has stationary increments, i.e. for all $s, t \in \mathbb{R}$ it holds $L_{t+s} - L_s \stackrel{d}{=} L_t - L_0$.

(iv) $(L_t)_{t \in \mathbb{R}}$ has almost surely cádlág paths.

When restricted to $t \in [0, \infty)$, we also speak of (one-sided) Lévy process. It can be shown that every Lévy process $(L_t)_{t \in \mathbb{R}}$ is continuous in probability, i.e. it holds $\lim_{s \to t} P(|L_t - L_s| > \varepsilon) = 0$ for all $t \in \mathbb{R}$ and $\varepsilon > 0$.

Lévy processes are related to infinitely divisible distributions. Denote the $n-th$ convolution of a probability distribution μ by $\mu^{n*} = \underbrace{\mu * \cdots * \mu}_{n \ times}$

Definition 1.10. *A probability distribution μ of a random variable Z in $\mathbb{R}^d$ is called* infinitely divisible *if for any $n \in \mathbb{N}$ there exists another probability distribution μ_n (there exists a sequence of i.i.d. random variables $Z_{1,n}, \ldots, Z_{n,n}$ having law μ_n) such that*

$$\mu = \mu_n^{n*} \qquad (Z \stackrel{d}{=} Z_{1,n} + \cdots + Z_{n,n}) \, .$$

Examples for infinitely divisible distributions are the normal distribution and the compound Poisson distributions. It follows directly from the definition of Lévy processes that the distribution of a Lévy process $(L_t)_{t\in\mathbb{R}}$ for fixed time $t \in \mathbb{R}$ is infinitely divisible. Conversely, given any infinitely divisible law μ, we can define a Lévy process $(L_t)_{t\in\mathbb{R}}$ such that $\mathcal{L}(L_1) = \mu$.

Definition 1.11. *A stochastic process $X = (X_t)_{t\geq 0}$ is called a* semimartingale *with respect to a filtration $\mathbb{F}$, where $\mathbb{F}$ satisfies the usual hypotheses, if it can be written as a sum $X_t = X_0 + M_t + A_t$ where*

(*i*) *M is a* local martingale*, i.e. it is adapted, cádlág and there exists a sequence of increasing stopping times $(T_n)_{n\in\mathbb{N}}$ with $T_n \to \infty$ a.s., $n \to \infty$, such that the stopped process $(M_{t\wedge T_n}\mathbb{1}_{t_n>0})_{t\geq 0}$ is a uniformly integrable martingale for each n.*

(*ii*) *A is a cádlág, adapted process with paths of finite variation on compacts, starting in 0.*

It can be shown that all one-sided Lévy processes $(L_t)_{t\geq 0}$ are semimartingales with respect to the augmented natural filtration.

Definition 1.12. *For a real-valued semimartingale X satisfying $X_0 = 0$ the* stochastic exponential *of X, written $\mathcal{E}(X)$, is the unique semimartingale Z, such that $Z_t = 1 + \int_{(0,t]} Z_{s-}dX_s$ holds for all $t \geq 0$.*

Univariate Ornstein-Uhlenbeck Processes

Firstly, we are going to give the definition of a Lévy-driven Ornstein–Uhlenbeck process.

Definition 1.13. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a real valued non-zero Lévy process and $\lambda \in \mathbb{R}$ a fixed constant. Then a* Lévy-driven Ornstein-Uhlenbeck process *$V = (V_t)_{t\in\mathbb{R}}$ is defined by the stochastic differential equation*

$$dV_t = \lambda V_t dt + dL_t\,, \quad t \in \mathbb{R}\,, \tag{1.12}$$

or in integral form as

$$V_t = V_0 + \lambda \int_0^t V_u du + L_t\,, \quad t \in \mathbb{R}\,. \tag{1.13}$$

The solution to this equation is given by

$$V_t = e^{\lambda t}\left(V_0 + \int_0^t e^{-\lambda u} dL_u\right)\,, \quad t \in \mathbb{R}\,, \tag{1.14}$$

where V_0 is a starting random variable, which is often assumed to be independent of the driving Lévy process $(L_t)_{t\geq 0}$.

Equations (1.13) and (1.14) extend to

$$V_t = V_s + \lambda \int_s^t V_u du + L_t - L_s\,, \quad s \leq t \in \mathbb{R}\,, \tag{1.15}$$

and
$$V_t = e^{\lambda t - \lambda s} V_s + \int_s^t e^{\lambda t - \lambda u} dL_u \,, \quad s \le t \in \mathbb{R}. \tag{1.16}$$
Obviously, if additionally L_t is a Brownian motion, we get the classical Ornstein-Uhlenbeck process.
Brockwell and Lindner gave in [12] necessary and sufficient conditions for (1.12) to have a causal stationary solution as in the following proposition. By a *causal* solution we mean a solution $(V_t)_{t\in\mathbb{R}}$ such that each V_t is measurable with respect to the σ-algebra $\sigma(L_u - L_v : u, v \le t)$.

Proposition 1.14. *Let $(V_t)_{t\in\mathbb{R}}$ be a one-dimensional Ornstein–Uhlenbeck process driven by the real valued Lévy process $(L_t)_{t\in\mathbb{R}}$ with parameter $\lambda \in \mathbb{R}$.*
There exists a causal strict stationary solution of the equation (1.12) if and only if $\lambda < 0$ and $\mathbb{E}\log^+|L_1| < \infty$. If such a solution exists then it is unique and given by
$$V_t = \int_{-\infty}^t e^{\lambda(t-u)} dL_u \,, \qquad t \in \mathbb{R}\,. \tag{1.17}$$
If $\mathbb{E}\,L_1^2 < \infty$, then the causal OU-process given by (1.17) is weakly stationary.

More details are given in the following Theorem, which also treats non-causal solutions:

Theorem 1.15. *[Stationary OU-Processes]*
Let $V = (V_t)_{t\in\mathbb{R}}$ be a one-dimensional Ornstein–Uhlenbeck process driven by the real valued Lévy process $L = (L_t)_{t\in\mathbb{R}}$ with parameter $\lambda \in \mathbb{R}$, where the Lévy process is defined on the probability space $(\Omega, \mathcal{F}, P)$ and assumed to be non-deterministic. Then the following are true:

(i) A random variable $V_0 : \Omega \to \mathbb{C}$ can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is strictly stationary if and only if $\lambda \neq 0$ and $\mathbb{E}\log^+|L_1| < \infty$. The stationary solution V as well as V_0 are then unique (almost surely) and given by
$$V_t = \begin{cases} e^{\lambda t} \int_{-\infty}^t e^{-\lambda s} dL_s, & \lambda < 0 \\ -e^{\lambda t} \int_{t+}^\infty e^{-\lambda s} dL_s, & \lambda > 0, \end{cases} \tag{1.18}$$
for $t \in \mathbb{R}$, where the integrals converge almost surely as limits $\int_{(-u)+}^t$ and $\int_{t+}^u$ as $u \to \infty$, respectively. A cádlág modification of this solution is given by
$$V_t = \begin{cases} L_t + \int_{-\infty}^t \lambda e^{\lambda(t-s)} L_s ds, & \lambda < 0 \\ L_t - \int_t^\infty \lambda e^{\lambda(t-s)} L_s ds, & \lambda > 0\,, \end{cases} \tag{1.19}$$
for $t \in \mathbb{R}$, where almost surely the integrals exist as Lebesgue-integrals.

(ii) A random variable $V_0 : \Omega \to \mathbb{R}$ can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is weakly stationary if and only if $\lambda \neq 0$ and $\mathbb{E}L_1^2 < \infty$. In that case, the weakly stationary solution is unique and given by (1.18), or equivalently by (1.19), with the integrals in (1.18) converging almost surely and in mean-square as $u \to \infty$.

Proof. The proof of this theorem can be found in the book of Brockwell and Lindner [14] (in preparation). Alternatively, it can be deduced from the more general results on CARMA-processes, see Theorem 3.3 in Brockwell and Lindner [9]. □

Multivariate Ornstein-Uhlenbeck Processes

Definition 1.16. *Let $A \in \mathbb{R}^{d\times d}$ and $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided $\mathbb{R}^d$-valued Lévy process defined on a probability space $(\Omega, \mathcal{F}, P)$. The stochastic differential equation*

$$dV_t = AV_t dt + dL_t\,, \quad t \in \mathbb{R}\,, \tag{1.20}$$

is called a multivariate Ornstein-Uhlenbeck equation*, and for every starting variable $V_0 : \Omega \to \mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, P)$, its cádlág solution $V = (V_t)_{t\in\mathbb{R}}$ is called a* multivariate Ornstein-Uhlenbeck process*. We then also speak of V as being driven by L with parameter A and starting random variable V_0. It hence satisfies the stochastic integral equation*

$$V_t = V_0 + \int_0^t AV_u du + L_t\,, \quad t \in \mathbb{R}. \tag{1.21}$$

For each starting random variable V_0, Equation (1.20) has an almost surely unique cádlág solution V, given by

$$V_t = e^{At}\left(V_0 + \int_0^t e^{-Au} AL_u du\right) + L_t\,, \quad t \in \mathbb{R}, \tag{1.22}$$

equivalent to

$$V_t = e^{At}\left(V_0 + \int_0^t e^{-Au} dL_u\right)\,, \quad t \in \mathbb{R}. \tag{1.23}$$

In particular, the multivariate Ornstein–Uhlenbeck process is uniquely determined by A, L and V_0.

Sato and Yamazato [[37] , Theorem 5.1] gave sufficient conditions for an MOU-equation to have a causal strictly stationary solution as in the following proposition.

Proposition 1.17. *Let $(V_t)_{t\in\mathbb{R}}$ be a multivariate Ornstein–Uhlenbeck process driven by the $\mathbb{R}^d$-valued Lévy process $(L_t)_{t\in\mathbb{R}}$ and let A be a squared matrix with values in $\mathbb{R}^{d\times d}$. If the eigenvalues of A are negative and the Lévy process L satisfies $\mathbb{E}L_1 = 0$ and $\mathbb{E}\log^+ \| L_1 \|^2 < \infty$, then the stochastic differential equation of Ornstein-Uhlenbeck type*

$$dV_t = AV_t dt + dL_t\,, \quad t \in \mathbb{R}$$

has a unique causal strictly stationary solution given by

$$V_t = \int_{-\infty}^t e^{A(t-u)} dL_u\,, \qquad t \in \mathbb{R}\,, \tag{1.24}$$

which has the same distribution as $\int_0^\infty e^{Au} dL_u$.

Actually, when all the eigenvalues of A are strictly negative, then the condition $\mathbb{E}\log^+ \| L_1 \| < \infty$ is also necessary for a strictly stationary solution to exists (see [37]).

1.1.3 CARMA Processes

Continuous-time autoregressive (CAR) processes have been of interest to physicists and engineers for many years. Early papers deal with the properties and statistical analysis of such processes and of the more general continuous-time autoregressive moving average (CARMA) processes. In recent years there has been a resurgence of interest in these processes and in continuous-time processes more generally, partly as a result of the very successful application of stochastic differential equation models to problems in finance, particularly to the pricing of options. The proliferation of high-frequency data, especially in fields such as finance and turbulence, has stimulated interest also in the connections between CARMA processes and the discrete-time processes obtained by sampling them at high frequencies. and the possible use of continuous-time models to suggest inferential methods for such data. CARMA models have also been utilized very successfully for the modelling of irregularly-spaced data. Recent applications include the CARMA interest rate model, the application of stable CARMA processes to futures pricing in electricity markets and applications to signal extraction. An asset price model with CAR(1) spot volatility was introduced and CARMA spot volatility models have been studied. The potential for further applications, with the proliferation of high-frequency data in so many fields, and further theoretical developments, particularly with respect to multivariate models, nonlinear models, non-causal modelling, sampling and embedding remains broad and challenging (see [7].)

Definition 1.18. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a (real-valued) Lévy process and p, q integers such that $0 \leq q < p$. We define a* (real-valued) continuous-time ARMA process *with autoregressive order p and moving average order q* (CARMA(p,q)-process) $Y = (Y_t)_{t\in\mathbb{R}}$ *driven by L, by the equation*

$$Y_t = \boldsymbol{b}^T \boldsymbol{X}_t \quad t \in \mathbb{R} \tag{1.25}$$

where $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\in\mathbb{R}}$ is a $\mathbb{R}^p$-valued process satisfying the stochastic differential equation,

$$d\boldsymbol{X}_t = A\boldsymbol{X}_t dt + \boldsymbol{e} dL_t\,, \tag{1.26}$$

or equivalently

$$\boldsymbol{X}_t = e^{A(t-s)}\boldsymbol{X}_s + \int_s^t e^{A(t-u)}\boldsymbol{e} dL_u, \quad \forall s \leq t \in \mathbb{R}, \tag{1.27}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 0 \\ -a_p & -a_{p-1} & -a_{p-2} & \vdots & -a_1 \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix} \tag{1.28}$$

where $a_1, \dots, a_p, b_0, \dots, b_{p-1}$ are real-valued coefficients such that $b_q \neq 0$ and $b_j = 0$ for $j > q$. For $p = 1$ the matrix A is to be understood as $-a_1$. The equations (1.25) and (1.26) constitute the state-space representation of the formal pth-order stochastic differential equation,

$$a(D)Y_t = b(D)DL_t, \quad t \in \mathbb{R}\,, \tag{1.29}$$

where D denotes differentiation with respect to t, $a(\cdot)$ and $b(\cdot)$ are polynomials,

$$a(z) = z^p + a_1 z^{p-1} + \cdots + a_p\,, \tag{1.30}$$

and

$$b(z) = b_0 + b_1 z + \cdots + b_q z^q\,. \tag{1.31}$$

Equation (1.29) is the natural continuous-time analogue of the p'th-order linear difference equations used to define a discrete-time ARMA process (1.1). However, since the derivatives on the right-hand side of (1.27) do not exist as random functions, we base the definition on the state-space formulation (1.25) and (1.26).

Remark 1.19. *If the zeroes of $a(z)$ all have strictly negative real parts then the state-vector process and the CARMA process are both said to be* causal *since the increments of L_u ; $u > t$ do not enter the definitions of $\boldsymbol{X}_t$ and Y_t as follows from Theorem 1.20 below.*

Strictly stationary CARMA:

Brockwell and Lindner [9] establish necessary and sufficient conditions for the existence and uniqueness of a strictly stationary solution of the Lévy-driven CARMA process. More precisely they prove the following theorems, see [9], Theorem 3.3, Theorem 4.2 and Proposition 5.1.

Theorem 1.20. *Let L be a (real-valued) Lévy process which is not deterministic and suppose that $a(\cdot)$ and $b(\cdot)$ (given in (1.30) and (1.31)) have no common zeroes. Then the CARMA equations (1.25) and (1.26) have a strictly stationary solution Y on $\mathbb{R}$ if and only if $\mathbb{E}\log^+|L_1| < \infty$ and $a(\cdot)$ is non-zero on the imaginary axis. In this case the solution Y is unique and is given by*

$$Y_t = \int_{-\infty}^{\infty} g(t-u)dL_u, \qquad t \in \mathbb{R}\,, \tag{1.32}$$

where

$$g(t) = \left(\sum_{\lambda:\Re\lambda<0} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} \boldsymbol{1}_{(0,\infty)}(t) - \sum_{\lambda:\Re\lambda>0} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} \boldsymbol{1}_{(-\infty,0)}(t) \right), \quad t \in \mathbb{R} \tag{1.33}$$

with $\mu(\lambda)$ being the multiplicity of the zero λ and $\sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} = \frac{1}{(\mu(\lambda)-1)!}[D_z^{\mu(\lambda)-1}((z-\lambda)^{\mu(\lambda)} e^{zt} b(z)/a(z))]_{z=\lambda}$ and D_z denotes differentiation with respect to z. The corresponding state vector $(\boldsymbol{X}_t)_{t\in\mathbb{R}}$ can be chosen to be strictly stationary as

$$\boldsymbol{X}_t := e^{At} \left(\int_{-\infty}^{t} e^{-Au} \boldsymbol{l}(0) dL_u - \int_{t}^{\infty} e^{-Au} \boldsymbol{r}(0) dL_u \right), \quad t \in \mathbb{R}$$

where $\boldsymbol{l}(t), \boldsymbol{r}(t)$ are the sums of the residues of the column vector $e^{zt}a^{-1}(z)[1\ z\ \ldots\ z^{p-1}]^T$ at the zeroes of $a(\cdot)$ with strictly negative and strictly positive real parts, respectively.

Theorem 1.21. *Suppose that $p \geq 1$, that $\boldsymbol{b} \neq 0$ and that the Lévy process L is not deterministic. Then the CARMA equations (1.25) and (1.26) have a strictly stationary solution Y on $\mathbb{R}$ if and only if $\mathbb{E}\log^+|L_1| < \infty$ and all singularities of the meromorphic function $z \mapsto \frac{b(\cdot)}{a(\cdot)}$ on the imaginary axis are removable, i.e. if $a(\cdot)$ has a zero λ_1 of multiplicity $\mu(\lambda_1)$ on the imaginary axis, then $b(\cdot)$ has also a zero at λ_1 of multiplicity greater than or equal to $\mu(\lambda_1)$. In this case, the solution is unique and is given by (1.32) and (1.33).*

Proposition 1.22. *Let L be a deterministic Lévy process, i.e. suppose there is $\sigma \in \mathbb{R}$ such that $L_t = \sigma t$ for all $t \in \mathbb{R}$. Suppose further that $\boldsymbol{b} \neq 0$. Denote by $\mu_a(\lambda)$ and $\mu_b(\lambda)$ the multiplicity of λ as a zero of $a(\cdot)$ and of $b(\cdot)$, respectively. Then the following results hold:*
(a) If $a_p \neq 0$, then the CARMA equations (1.25) and (1.26) have a strictly stationary solution Y, one of which is $Y_t = \sigma b_0/a_p$ for all $t \in \mathbb{R}$. This solution is unique if and only if $\mu_b(\lambda) \geq \mu_a(\lambda)$ for every zero λ of $a(\cdot)$ such that $\boldsymbol{Re}(\lambda) = 0$.
(b) If $a_p = 0$ and $\sigma \neq 0$, then the CARMA equations (1.25) and (1.26) have a strictly stationary solution Y if and only if $\mu_b(0) \geq \mu_a(0)$. If this condition is satisfied, one solution is $Y_t = \sigma b_{\mu_a(0)}/a_{p-\mu_a(0)}$, and this solution is unique if and only if $\mu_b(\lambda) \geq \mu_a(\lambda)$ for all zeroes λ of $a(\cdot)$ such that $\boldsymbol{Re}(\lambda) = 0$.
(c) If $a_p = \sigma = 0$, then $Y_t = 0$, $t \in \mathbb{R}$ is a strictly stationary solution of the CARMA equations (1.25) and (1.26), and this solution is unique if and only if $\mu_b(\lambda) \geq \mu_a(\lambda)$ for all zeroes λ of $a(\cdot)$ such that $\boldsymbol{Re}(\lambda) = 0$.

Weakly stationary CARMA:
Similar results for the existence of weakly stationary CARMA process exist as well.

Proposition 1.23. *Suppose that $p \geq 1$, that $\boldsymbol{b} \neq 0$ and that the Lévy process L is not deterministic. If $\mathbb{E}L_1^2 < \infty$ and all singularities of the meromorphic function $z \mapsto \frac{b(\cdot)}{a(\cdot)}$ on the imaginary axis are removable, then there exists a weakly stationary solution to the CARMA equations (1.25) and (1.26). The solution is unique and is given by (1.32) and (1.33).*

The proof is similar to the proof of Theorem 1.20.

Proposition 1.24. *Let L be a Lévy process which is not deterministic and suppose that $a(\cdot)$ and $b(\cdot)$ (given in (1.30)and (1.31)) have no common zeroes. Let $\mathbb{E}L_1^2 < \infty$ and suppose that X_0 is independent of $(L_t)_{t\geq 0}$. Then the CARMA process defined by equation (1.25) and (1.26) is weakly stationary solution if and only if the zeroes of the polynomial $a(\cdot)$ (which are also the eigenvalues of the matrix A) have strictly negative real parts and X_0 has the same mean and covariance as $\int_0^\infty e^{Au}\boldsymbol{e}dL_u$. (See [6], Proposition 1.)*

Theorem 1.25. *Suppose that $p \geq 1$, that $\boldsymbol{b} \neq 0$ and that the Lévy process L is not deterministic. Then the CARMA equations (1.25) and (1.26) have a weakly stationary solution Y on $\mathbb{R}$ if and only if $\mathbb{E}L_1^2 < \infty$ and all singularities of the meromorphic function $z \mapsto \frac{b(\cdot)}{a(\cdot)}$ on the imaginary axis are removable, i.e. if $a(\cdot)$ has a zero λ_1 of multiplicity $\mu(\lambda_1)$ on the imaginary axis, then $b(\cdot)$ has also a zero at λ_1 of multiplicity greater than or equal to $\mu(\lambda_1)$. In this case, the solution is unique and is given by (1.32) and (1.33).*

The proof of this theorem can be found in the book of Brockwell and Lindner [14], and can be proved analogously to the corresponding result (Theorem 1.21) for strictly stationary solutions.

1.2 PARMA, PCARMA and POU Processes

Seasonal phenomena are frequently observed in many fields such as hydrology, climatology, air pollution, radio astronomy, econometrics, communications, signal processing, among others. A standard approach in the literature is to fit a stationary seasonal model after removing any trend. This strategy can be suggested by standard time series tools even if the true covariance structure has a periodic (or cyclic) nonstationary behaviour. In this case, adjusting a seasonal model is inappropriate and deteriorates the forecast performance and this model mispecification is not revealed by the usual residual diagnostic checking. The simplest way to build models for periodically stationary processes is to allow the parameters of stationary models to vary periodically with time.

Recall that $T = \mathbb{Z}$ or $T = \mathbb{R}$.

Definition 1.26. *Let $f : T \to T$ be a function and $d \in T$, $d > 0$. f is said to be* periodic *with period d if we have*

$$f(x + d) = f(x), \quad \forall x \in T\,.$$

Definition 1.27. *(a) An $\mathbb{R}^m$-valued stochastic process $(X_t)_{t\in T} \in L^2(\Omega, \mathcal{F}, P)$ is called* periodically stationary *or* periodic weakly stationary *with period $d \in T$ ($d > 0$), if $\mathbb{E}|X_t|^2 < \infty$ for all $t \in T$, and for every $s, t \in T$ the mean $m(t) := \mathbb{E}X_t$ and autocovariance function $\gamma(s,t) := \boldsymbol{Cov}(X_s, X_t)$ are periodic with period d, i.e.*

$$m(t) = m(t + d) \;\; \text{and} \;\; \gamma(s,t) = \gamma(s + d, t + d)\,, \qquad \forall s, t \in T. \tag{1.34}$$

(b) An $\mathbb{R}^m$-valued stochastic process $(X_t)_{t\in T}$ is called periodic strictly stationary *with period $d \in T$ ($d > 0$) if, for every $n \in \mathbb{N}$, any collection of times $t_1, \dots, t_n \in T$, the joint distributions of $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ and of $(X_{t_1+md}, X_{t_2+md}, \dots, X_{t_n+md})$ are the same for all $m \in \mathbb{Z}$.*

1.2.1 Periodic ARMA processes

An important class of stochastic models for describing the periodically stationary processes is constituted by the periodic Autoregressive Moving Average (ARMA) models, which allows the model parameters in the classical ARMA model to vary with the season. The PARMA models can be used to model a large class of periodic process. The relationship between the periodic processes and the PARMA models is akin to that between stationary processes and ARMA models.

In the following, we are going to give the definition of an PARMA process.

Definition 1.28. *A stochastic process $(X_t)_{t\in\mathbb{Z}}$ is said to be a* periodic autoregressive moving average (PARMA) process with period $d \in \mathbb{N}$ *if it is a solution to the periodic linear difference equation*

$$X_{nd+s} - \sum_{i=1}^{p(s)} \phi_i(s) X_{nd+s-i} = Z_{nd+s} + \sum_{i=1}^{q(s)} \theta_i(s) Z_{nd+s-i} \tag{1.35}$$

where Z_{nd+s} is i.i.d. or periodic white noise (i.e. Z_{nd+s} is uncorrelated with $\mathbb{E}Z_{nd+s} = 0$ and $\boldsymbol{Var}\, Z_{nd+s} = \sigma_s^2 > 0$ for all seasons s). The periodic notation X_{nd+s} refers to the process X_t during the s'th season, $1 \le s \le d$, and cycle n. The orders of the autoregressive and moving-average during season s are respectively $p(s)$ and $q(s)$, and the model coefficients of the autoregressive and moving-average are respectively $\phi_i(s) \in \mathbb{R}$ for $i = 1, \dots, p(s)$ and $\theta_i(s) \in \mathbb{R}$ for $i = 1, \dots, q(s)$.

A process $(X_t)_{t\in\mathbb{Z}}$ satisfying the equation (1.35) will be referred to as a $PARMA_d\big(p(s), q(s)\big)$ model. For mathematical purposes $p(s)$ and $q(s)$ can be taken as constant in s-merely set

$$p = \max_{1\le s\le d} p(s) \qquad \text{and} \qquad q = \max_{1\le s\le d} q(s).$$

and take $\phi_i(s) = 0$ for $i > p(s)$ and $\theta_i(s) = 0$ for $i > q(s)$ (see [4]).

Hence the equation (1.35) can be written in the form:

$$X_{nd+s} - \sum_{i=1}^{p} \phi_i(s) X_{nd+s-i} = Z_{nd+s} + \sum_{i=1}^{q} \theta_i(s) Z_{nd+s-i}\,. \tag{1.36}$$

A process X_t satisfying the equation (1.36) will also be referred to as a $PARMA_d(p, q)$ model. We can rewrite (1.36) in operator form by using the backward shift operator B such that

$$B^k X_{nd+s} = X_{nd+s-k} \text{ and } B^k Z_{nd+s} = Z_{nd+s-k}$$

as follows

$$\Phi_s(B) X_{nd+s} = \Theta_s(B) Z_{nd+s}, \tag{1.37}$$

where

$$\Phi_s(z) = 1 - \phi_1(s)z - \cdots - \phi_p(s)z^p$$

is the AR operator of order p for season s and

$$\Theta_s(z) = 1 + \theta_1(s)z + \cdots + \theta_q(s)z^q$$

is the MA operator of order q for season s.

Applications of PARMA Models

Generation of synthetic river flow data is important in planning, design and operation of water resources systems. PARMA models provide a powerful tool for the modeling of periodic hydrologic series in general and river flow series in particular.

Accurate forecasting of river flows is one of the most important applications in hydrology, especially for the management of reservoir systems.

In the following we represent some previous studies of the applications of PARMA models:

1. Application of PAR model to the modeling of the Garonne river flows:

Ursa et. al. in [44] developed an approach to identify and to estimate periodic autoregressive (PAR) models. They applied the PAR model to average monthly and quarter-monthly flow data for the period from 1959 to 2010 for the Garonne river in the southwest of France.

2. Application of PARMA model to particulate matter concentrations:

Sarnaglia et. al. in [41] considered the series of the Particulate Matter with an aerodynamic diameter smaller than or equal to 10 μm (PM10), that is observed between January 1, 2005 and December 31, 2009 at the monitoring station of Environment and Water Resources State Institute located in Cariacica, ES, Brazil. They took the first 1603 observations for fitting the model and the remaining 223 observations are used for the out-of-sample forecast study. Since the data are collected daily, they suggested a PARMA model with period $d = 7$ to fit the series. The sample periodic autocorrelation and partial autocorrelation functions indicate a PARMA models.

3. Application of PARMA models to monthly Fraser river flows:

Anderson et al. in [1] have developed and implemented a practical methodology for forecasting periodic ARMA models. They applied this methodology to forecast future values for monthly average flow for the Fraser River at Hope, British Columbia between October 1912 to September 1984. A $PARMA_{12}(1,1)$ model was found adequate to capture the seasonal covariance structure in the mean centered series.
In another article, Tesfaye et al. [42] developed model identification and simulation techniques based on a periodic autoregressive moving average (PARMA) model to capture the seasonal variations in river flow statistics. They applied this techniques to monthly flow data for the Fraser River in British Columbia. They considered the data, that are obtained from daily discharge measurements, in cubic meter per second, averaged over each of the respective months to obtain the monthly series. The series contains 72 years of data from October 1912 to September 1984. They found that a $PARMA_{12}(1,1)$ model is sufficient in adequately capturing the series autocorrelation structure.

1.2.2 Periodic Ornstein-Uhlenbeck processes

Univariate Periodic OU-process:

In the Definition 1.13 we defined the Lévy-driven Ornstein-Uhlenbeck process for a constant $\lambda \in \mathbb{R}$. In the following, we are going to give the definition of a periodic Lévy-driven Ornstein-Uhlenbeck process.

Definition 1.29. *Let $(L_t)_{t\in\mathbb{R}}$ be a real valued Lévy process, and let*

$$\lambda : \mathbb{R} \to \mathbb{R}$$

be a periodic, bounded and measurable function. Then a periodic Lévy-driven Ornstein-Uhlenbeck (POU) process $(V_t)_{t\in\mathbb{R}}$ *is defined as a solution to the stochastic differential equation*

$$dV_t = \lambda(t)V_t dt + dL_t\,; \quad t \in \mathbb{R}\,, \tag{1.38}$$

or in integral form as

$$V_t = V_0 + \int_0^t \lambda(s)V_s ds + L_t\,; \quad t \in \mathbb{R}\,, \tag{1.39}$$

where V_0 is a a starting random variable.

We can set $\lambda(t)dt =: dU_t$, then the equation (1.38) becomes

$$dV_t = V_t dU_t + dL_t\,; \quad t \in \mathbb{R}\,. \tag{1.40}$$

And the integral form of (1.40) is

$$V_t = V_0 + \int_0^t V_s dU_s + L_t\,; \quad t \in \mathbb{R}\,, \tag{1.41}$$

where V_0 is a finite random variable (the starting value of V_t).

Multivariate Periodic OU-process:

In Definition (1.16) we defined the multivariate Lévy-driven Ornstein-Uhlenbeck process for a real-valued matrix A. In the following, we are going to give the same definition but instead of the matrix A we have a periodic, continuous matrix function $A(t)$. We will call this new process a Lévy-driven multivariate periodic Ornstein-Uhlenbeck process.

Definition 1.30. *Let $L = (L_1, \ldots, L_d)^T : \mathbb{R} \to \mathbb{R}^d$ be a real-valued Levy process, and let $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ be a periodic (with period $d > 0$), continuous, non-zero and known function. Then a* Lévy-driven multivariate periodic Ornstein Uhlenbeck process $V = (V_1, \ldots, V_d)^T : \mathbb{R} \to \mathbb{R}^d$ *is the solution of the stochastic differential equation*

$$dV_t = A(t)V_t dt + dL_t\,, \qquad t \in \mathbb{R}, \tag{1.42}$$

it hence satisfies the stochastic integral equation

$$V_t = V_0 + \int_0^t A(s)V_s ds + L_t, \tag{1.43}$$

or more generally,

$$V_t = V_u + \int_u^t A(s)V_s ds + L_t - L_u, \qquad \textit{for } u \leq t\,. \tag{1.44}$$

We shall abbreviate it as MPOU-process.

If we set $A(t)dt =: dU_t$, then $U_t = \int_0^t A(s)ds$, with $U_t \in \mathbb{R}^{d\times d}$. Then the equation (1.42) becomes

$$dV_t = dU_t V_t + dL_t\ , \qquad t \in \mathbb{R}, \tag{1.45}$$

and the integral form of (1.45) is

$$V_t = V_0 + \int_0^t dU_k V_k + L_t. \tag{1.46}$$

The solution of the equation (1.45) or (1.46) can be given by

$$V_t = \mathcal{E}(U)_t \left(V_0 + \int_0^t \mathcal{E}(U)_k^{-1} dL_k \right); \qquad t \in \mathbb{R}\,, \tag{1.47}$$

where $\mathcal{E}(U)$ is the matrix stochastic exponential of U. See Chapter 4 for more details on the definition of the matrix stochastic exponential $\mathcal{E}(U)$.

1.2.3 Periodic CARMA processes

A natural continuous-time analogue of the periodic difference equation (1.37) is the formal periodic differential equation,

$$a_t(D)Y_t = b_t(D)DL_t\,, \tag{1.48}$$

where L is an $\mathbb{R}$-valued Lévy process, D denotes differentiation with respect to t, and $a_t(z)$ and $b_t(z)$ are periodic and continuous real-valued polynomials of the form,

$$a_t(z) = z^p + a_1(t)z^{p-1} + \cdots + a_p(t)\,,$$

$$b_t(z) = b_0(t) + b_1(t)z + \cdots + b_q(t)z^q\,,$$

with $b_q(t) = 1$ and $q < p$. We shall refer to $a_t(z)$ as the *autoregressive polynomial* and to $b_t(z)$ as the *moving-average polynomial.* Since $a_t(z)$ and $b_t(z)$ are periodic with period $d \in \mathbb{R}$, it is $a_t(z) = a_{t+d}(z)$ and $b_t(z) = b_{t+d}(z)$ for all $t, s \in \mathbb{R}$.

Since the derivatives on the right-hand side of (1.48) do not exist in the usual sense, we give the equation a meaningful interpretation by rewriting it in the state-space form,

$$Y_t = \mathbf{b}_t^T \mathbf{X}_t \quad t \in \mathbb{R} \tag{1.49}$$

where $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ is an $\mathbb{R}^p$-valued process satisfying the stochastic differential equation,

$$d\mathbf{X}_t = A_t\mathbf{X}_t dt + \mathbf{e}dL_t\,, \tag{1.50}$$

or equivalently

$$\mathbf{X}_t - \mathbf{X}_s = \int_s^t A_u\mathbf{X}_u du + \mathbf{e}(L_t - L_s), \quad \forall s \le t \in \mathbb{R}, \tag{1.51}$$

with the periodic matrices A_t, periodic vector $\mathbf{b}_t$ and vector $\mathbf{e} \in \mathbb{R}^p$ given by

$$A_t = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 0 \\ -a_p(t) & -a_{p-1}(t) & -a_{p-2}(t) & \vdots & -a_1(t) \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_t = \begin{bmatrix} b_0(t) \\ b_1(t) \\ \vdots \\ b_{p-2}(t) \\ b_{p-1}(t) \end{bmatrix} \tag{1.52}$$

where $a_1(t), \dots, a_p(t), b_0(t), \dots, b_{p-1}(t)$ are the real-valued coefficients of the periodic polynomials $a_t(z),\ b_t(z)$, satisfying $b_q(t) = 1$ and $b_j = 0$ for $j > q$. For $p = 1$ the matrix A_t is to be understood as $-a_1(t)$.

Remark 1.31. *Since* $\det(zId - A_t) = a_t(z)$, *the eigenvalues of the matrix* A_t *are the same as the zeros of the autoregressive polynomial* $a_t(z)$ *for all* t. *We shall denote these zeros by* $\lambda_1(t), \dots, \lambda_r(t)$ *and their multiplicities by* $m_1(t), \dots, m_r(t)$ *respectively. Thus* $\sum_{i=1}^r m_i(t) = p$.

Definition 1.32. *[Lévy-driven periodic CARMA process]*
A Periodic CARMA process (PCARMA-process) *(not necessarily strictly stationary) driven by the Lévy process* $(L_t)_{t\in\mathbb{R}}$ *with autoregressive polynomial* $a_t(\cdot)$ *and moving-average polynomial* $b_t(.)$ *is a solution* Y *of equations (1.49) and (1.50). We also write* $PCARMA_d(p,q)$ *to indicate the orders* p *and* q *as well the period* d.

1.3 Main results of the thesis

Periodic ARMA model
In **Chapter 2** we consider the periodic ARMA (PARMA) model defined by the Equation (1.36). We recall the vector ARMA representation of the periodic ARMA model and give an equivalence between the periodic stationary solutions of periodic AR processes and the stationary solutions of the vector AR processes. Then we represent the Markovian dual process of the periodic AR process and recall in Section 2.2 some well-known results of Boshnakov about the periodic weakly stationary solutions of the PAR(p) equation. In Section 2.3 we deal with the periodic strictly stationary PAR(p) process. We represent the PAR(p) process as a vector AR process, then we apply the well-known results of Brockwell, Lindner and Vollenbröker [13] to get the necessary and sufficient conditions for the existence of periodic strictly stationary solutions of (2.6) in the case that $p \le d$.

In Theorem 2.18 we then give new necessary and sufficient conditions for the existence and uniqueness of periodic strictly stationary solutions to the PAR(p) process for general orders p and d using the Markovian dual representation, i.e. without the assumption that $p \leq d$.

Univariate periodic OU process
We study in **Chapter 3** the one-dimensional periodic Ornstein-Uhlenbeck (POU) model of the type

$$dV_t = \lambda(t)V_t dt + dL_t\,; \quad t \in \mathbb{R}\,.$$

We show that this model has a solution of the form

$$V_t = \mathcal{E}(U)_t \left(V_0 + \int_0^t \mathcal{E}(U)_u^{-1} dL_u \right)\,; \quad t \in \mathbb{R}\,,$$

where $\mathcal{E}(U_t)$ is the stochastic exponential of the process U_t defined as $U_t := \int_0^t \lambda(s)ds$.

In Lemma 3.1 and Lemma 3.2 we give two equivalences, the first between the log moments of the Lévy process L_t and the log moments of the integral of the form $\int_0^1 \lambda(s)dL_s$, and the second between the p-th moments L_t and the p-th moments of $\int_0^1 \lambda(s)dL_s$. In Theorem 3.3 we give the necessary and sufficient conditions for the existence of the periodic strictly stationary solution of (1.40). We consider the constant $\alpha := \int_0^1 \lambda(s)ds$ (where the period $d = 1$) and show that (1.40) has a periodic strictly stationary solution if and only if $\alpha \neq 0$ and $\mathbb{E}\log^+|L_1| < \infty$. In this case, the solution is unique and given by $V_t = e^{U_t}\int_{-\infty}^t e^{-U_u}dL_u$ if $\alpha < 0$ and by $V_t = -e^{U_t}\int_{t+}^{\infty} e^{-U_u}dL_u$ if $\alpha > 0$.

The necessary and sufficient conditions for the existence of the periodic weakly stationary solution of (1.40) will be given in Theorem 3.4, namely the equation (1.40) has a unique periodic weakly stationary solution if and only if $\alpha \neq 0$ and $\mathbb{E}|L_1|^2 < \infty$. The solution has the same form as the periodic strictly stationary one above. We also give the autocovariance function in this case. In section 3.3 we sample the d-periodic OU process at equidistant times $h := \frac{d}{m}$ for $m \in \mathbb{N}$ and show that the sampled process $(V_{nh})_{n\in\mathbb{Z}}$ is a periodic AR(1) process with period m.

Multivariate Periodic OU process
In **Chapter 4** we consider the multi-dimensional periodic Ornstein-Uhlenbeck (MPOU) model defined as

$$dV_t = A(t)V_t dt + dL_t\,, \qquad t \in \mathbb{R},$$

where the coefficient $A(t)$ is a periodic matrix. We also show that the solution of this model has the form (with $U_t = \int_0^t A(s)ds$)

$$V_t = \mathcal{E}(U)_t \left(V_0 + \int_0^t \mathcal{E}(U)_k^{-1} dL_k \right)\,; \qquad t \in \mathbb{R}\,.$$

Before studying the existence of the periodic stationary solution for this model, we recall some results in the Floquet Theory, that will be needed in this chapter.

We then give necessary and sufficient conditions for the existence of strictly periodic and weakly periodic solutions in Section 4.2. In particular we show that if $\mathcal{E}(U)_1$ has no eigenvalues of absolute size 1, then the MPOU model has a unique periodic strictly (weakly) stationary solution if and only if $\mathbb{E}\log^+|L_1| < \infty$ ($\mathbb{E}|L_1|^2 < \infty$, respectively). In this case the solution has the form

$$V_t = \mathcal{E}(U)_t \int_{-\infty}^{t} \mathcal{E}(U)_k^{-1} dL_k \,,$$

if all eigenvalues of $\mathcal{E}(U)_1$ have absolute value in (0,1), and the form

$$V_t = -\mathcal{E}(U)_t \int_{t}^{\infty} \mathcal{E}(U)_k^{-1} dL_k,$$

if all eigenvalues of $\mathcal{E}(U)_1$ have absolute value greater than one.

The case of general eigenvalues of $\mathcal{E}(U)_1$ is also considered and the form of the solution given.

Periodic CARMA process
In **Chapter 5** we study the stationary periodic continuous-time ARMA (PCARMA) model driven by a Lévy process that is given by

$$Y_t = \mathbf{b}_t^T \mathbf{X}_t \,, \quad t \in \mathbb{R}$$

where $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ is a $\mathbb{R}^p$-valued process satisfies

$$d\mathbf{X}_t = A_t \mathbf{X}_t dt + \mathbf{e} dL_t \,,$$

with A_t as in Section 1.2.3. In Theorem 5.3 we give sufficient conditions for the existence of a periodic strictly (weakly) stationary solution of the PCARMA model, namely $\mathbb{E}\log^+|L_1| < \infty$ ($\mathbb{E}|L_1|^2 < \infty$, respectively), and some further conditions if $\mathcal{E}(U)_1$ has eigenvalues of absolute size 1. The solution is again given explicitly. In Section 5.2 we give the autocovariance function of the PCARMA model in the case that all the eigenvalues of $\mathcal{E}(U)_1$ have absolute values in the interval $(0, 1)$.

CHAPTER 2

Stationary Periodic ARMA Processes

Time series with periodically varying parameters are natural modelling vehicles for series with cyclic autocovariances. Such series arise in climatology, economics, hydrology, electrical engineering and many other disciplines.
Analogous to autoregressive moving-average (ARMA) models and short memory stationary series, periodic autoregressive moving-average (PARMA) models are fundamental periodic time series models.
In this chapter, we are going to give a periodic stationary solution for a periodic ARMA process. There is a close link between the periodically stationary processes and the multivariate stationary processes. The univariate series X_t can be turned into a multivariate one by forming d-dimensional vectors of d consecutive X's. In the following Sections 2.1 and 2.2 we recall some well-known results of Boshnakov [3] and others, that are useful in this work. In section 2.3 we present new results regarding periodically strictly stationary solutions.

Definition 2.1. *The* dual *or more precisely d-dual process* $(Y_n)_{n\in\mathbb{Z}}$ *of a stochastic process* $(X_n)_{n\in\mathbb{Z}}$ *is defined as*

$$Y_n = (X_{nd+1}, X_{nd+2}, \dots, X_{nd+d})^T .$$

For example $Y_0 = (X_1, X_2, \dots, X_d)^T$, $Y_1 = (X_{d+1}, X_{d+2}, \dots, X_{2d})^T$, etc.

The following Theorem holds and is due to Gladishev [18]:

Theorem 2.2. *The process* $(X_t)_{t\in\mathbb{Z}}$ *is periodically weakly stationary with period* $d \in \mathbb{N}$ *if and only if its dual process* $(Y_t)_{t\in\mathbb{Z}}$ *is weakly stationary.*

Analogous to the result of Gladishev we can get the same result for the periodic strict stationarity:

Proposition 2.3. *The process* $(X_t)_{t\in\mathbb{Z}}$ *is periodically strictly stationary with period* $d \in \mathbb{N}$ *if and only if its dual process* $(Y_t)_{t\in\mathbb{Z}}$ *is strictly stationary.*

The proof of this Proposition follows directly from the definitions of dual process, strict stationary and periodic strict stationary.

As we saw in Definition 1.28 of Chapter 1, a stochastic process $(X_t)_{t\in\mathbb{Z}}$ is said to be a *periodic autoregressive moving average (PARMA) process* with period d if it is a solution to the periodic linear difference equation

$$X_{nd+s} - \sum_{i=1}^{p} \phi_i(s) X_{nd+s-i} = Z_{nd+s} + \sum_{i=1}^{q} \theta_i(s) Z_{nd+s-i}\,, \tag{2.1}$$

where Z_{nd+s} is a periodic white noise (or Z_{nd+s} is i.i.d.), i.e. $Z_{nd+s} \sim PWN(0,\sigma_s^2)$ (or $Z_{nd+s} \sim I.I.D$) for $1 \le s \le d$ and the coefficients $\phi_i(s), \theta_i(s)$ are periodic.
It is common to write equation (2.1) in operator form by using the backward shift operator B $(BX_t = X_{t-1})$:

$$\Phi_s(B) X_{nd+s} = \Theta_s(B) Z_{nd+s}, \tag{2.2}$$

where

$$\Phi_s(z) = 1 - \phi_1(s)z - \cdots - \phi_p(s)z^p \quad \textit{and} \quad \Theta_s(z) = 1 + \theta_1(s)z + \cdots + \theta_q(s)z^q\,.$$

2.1 Represent PARMA-process as vector ARMA-process

As we saw in Theorem 2.2, Gladeshev's result states that X_{nd+s} is periodically weakly stationary with period d if and only if its dual process Y_t is a second-order stationary vector process. Hence, we will express (2.1) in terms of a vector process on Y_t for which the corresponding parameter restrictions will be easily determined.
Define $P := \lceil p/d \rceil$ and $Q := \lceil q/d \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Then (2.1) is equivalent to

$$\Phi_0 Y_t - \sum_{i=1}^{P} \Phi_i Y_{t-i} = \Theta_0 \eta_t + \sum_{i=1}^{Q} \Theta_i \eta_{t-i} \tag{2.3}$$

where Y_t is the dual process of X_t and $\eta_t = (Z_{td+1}, \ldots, Z_{td+d})^T$. The coefficient Φ_i, Θ_i are matrices of the order $d \times d$. The entries of the coefficients matrices Φ_i and Θ_i are given by

$$(\Phi_0)_{i,j} = \begin{cases} 1 & \text{if i=j} \\ 0 & \text{if } i < j \\ -\phi_{i-j}(i) & \text{if } i > j \end{cases}$$

$$(\Phi_k)_{i,j} = \phi_{kd+i-j}(i) \quad \text{for } 1 \le k \le P\,,$$

and

$$(\Theta_0)_{i,j} = \begin{cases} 1 & \text{if i=j} \\ 0 & \text{if } i < j \\ -\theta_{i-j}(i) & \text{if } i > j \end{cases}$$

$$(\Theta_k)_{i,j} = \theta_{kd+i-j}(i) \quad \text{for } 1 \le k \le Q\,,$$

with the conventions $\phi_k(i) = 0$ for $k > p$ and $\theta_k(i) = 0$ for $k > q$. (e.g. Basawa and Lund [2]).

Example 2.4. *Let X_{nd+s} be a PAR(2) process with period $d = 3$, given by*

$$X_{nd+s} - \phi_1(s)X_{nd+s-1} - \phi_2(s)X_{nd+s-2} = Z_{nd+s}\,,$$

for $s = 1, 2, 3$. Then the dual process Y_n of X_{nd+s} is a VAR(1) process of the form

$$\Phi_0 Y_n - \Phi_1 Y_{n-1} = \Theta_0 \eta_t\,.$$

This is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 \\ -\phi_1(2) & 1 & 0 \\ -\phi_2(3) & -\phi_1(3) & 1 \end{pmatrix} \begin{pmatrix} X_{nd+1} \\ X_{nd+2} \\ X_{nd+3} \end{pmatrix} - \begin{pmatrix} 0 & \phi_2(1) & \phi_1(1) \\ 0 & 0 & \phi_2(2) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{(n-1)d+1} \\ X_{(n-1)d+2} \\ X_{(n-1)d+3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_{nd+1} \\ Z_{nd+2} \\ Z_{nd+3} \end{pmatrix}$$

$$\begin{aligned} \Leftrightarrow & X_{nd+1} - \phi_1(1)X_{nd} - \phi_2(1)X_{nd-1} = Z_{nd+1}\,, \\ & X_{nd+2} - \phi_1(2)X_{nd+1} - \phi_2(2)X_{nd} = Z_{nd+2}\,, \\ & X_{nd+3} - \phi_1(3)X_{nd+2} - \phi_2(3)X_{nd+1} = Z_{nd+3}\,. \end{aligned}$$

and these are the PAR(2) equations for $s = 1, 2, 3$.

Many results for PARMA model can be extracted from its vector ARMA representation. For example Lund and Basawa [28] provide a sufficient condition for PARMA causality. Specifically, let

$$\Phi(z) = \Phi_0 - \sum_{i=1}^{P} \Phi_i z^i$$

be the d-variant vector AR polynomial. If $\Phi(z)$ has no roots within or on the complex unit circle in sense that $\det \Phi(z) \neq 0$ for all $z \in \mathbb{C} : |z| \leq 1$ and if $Z_{nd+s} \sim PWN(0, \sigma_s^2)$, the solution to (2.3) exists and can be uniquely (in mean square) expressed in the infinite order moving average form as

$$X_{nd+s} = \sum_{k=0}^{\infty} \psi_k(s) Z_{nd+s-k}, \tag{2.4}$$

where the sequence (ψ_k) is absolutely summable, with $\psi_0(s) = 1$. The solution is periodically weakly stationary.

We mention a generalization of the above: when the vector AR polynomial has no roots on the unit circle, solutions to PARMA difference equation exist and are unique and periodically weakly stationary, but may not be causal [28].

The standard form of the multivariate ARMA model can be obtained by left multiplication of (2.3) by Φ_0^{-1},

$$Y_t - \sum_{i=1}^{P} \underbrace{\Phi_0^{-1}\Phi_i}_{:=\Phi_i^\star} Y_{t-i} = \sum_{i=0}^{Q} \underbrace{\Phi_0^{-1}\Theta_i}_{:=\Theta_i^\star} \eta_{t-i}\,.$$

Hence

$$Y_t - \sum_{i=1}^{P} \Phi_i^\star Y_{t-i} = \sum_{i=0}^{Q} \Theta_i^\star \eta_{t-i} \,. \tag{2.5}$$

Most applications of periodic ARMA models exclude the MA-part for easyness. Therefore we are going to study here the PAR(p) process, given by

$$X_{nd+s} - \sum_{i=1}^{p} \phi_i(s) X_{nd+s-i} = Z_{nd+s} \,, \tag{2.6}$$

where the operator form of (2.6) can be given by

$$\Phi_s(B) X_{nd+s} = Z_{nd+s} \,. \tag{2.7}$$

From (2.5) we get the dual multivariate equation of (2.7)

$$Y_t - \sum_{i=1}^{P} \Phi_i^\star Y_{t-i} = \xi_t \,, \tag{2.8}$$

where $\xi_t := \Theta_0^\star \eta_t$.

For the PARMA process we replace Z_{nd+s} in (2.6) by $\sum_{i=0}^{q} \theta_i(s) \varepsilon_{nd+s-i}$.
We can formulate now the following result:

Theorem 2.5. *The innovation variances σ_s^2 of Z_{nd+s}, $s = 1, 2, \ldots, d$ are all positive if and only if the covariance matrix of the dual innovations is nonsingular.*

The proof can be found in [31].

Now we are going to give the conditions for the existence and uniqueness of a periodically weakly stationary solution of the equation (2.7). They can be found in [3], Theorem 15, Corollary 16.

Theorem 2.6. *Let $(Z_{nd+s}) \sim PWN(0, \sigma_s^2)$ with $0 < \sigma_s^2 < \infty$ for all $s \in \{1, 2, \ldots, d\}$. Then the equation (2.7) has a periodically weakly stationary solution if and only if the dual multivariate equation (2.8) has a weakly stationary solution. If such solution exists it is unique.*

Corollary 2.7. *Under the conditions of Theorem 2.6, the equation (2.7) has a unique periodically weakly stationary solution if and only if the determinant of the dual matrix polynomial*

$$Id - \Phi_1^* z - \cdots - \Phi_p^* z^p$$

has no roots on the unit circle.

2.2 The Markovian dual processes of PAR(p) processes

There is another multivariate representation of the PAR model that also turns out to be useful, namely the Markovian dual model. Firstly, we give the following useful definition:

Definition 2.8. *Let $p, d \in \mathbb{N}$, and let $\phi_1(t), \dots, \phi_p(t)$; $t = 1, .., d$ be constants. Then the matrices of the form*

$$A_t = \begin{pmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_{p-1}(t) & \phi_p(t) \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{2.9}$$

are called companion matrices*. The determinant of a companion matrix is nonzero if and only if $\phi_p(t) \neq 0$ for $t = 1, 2, \dots, d$.*

In the following we give a definition of the Markovian dual process of a PAR process.

Definition 2.9. *Let $(X_t)_{t\in\mathbb{Z}}$ be a periodic autoregression process of order p. Then the Markovian dual process $(\boldsymbol{Z}_t)_{t\in\mathbb{Z}}$ of $(X_t)_{t\in\mathbb{Z}}$ is defined to be a vector of p consecutive $X'es$*

$$\boldsymbol{Z}_t = (X_t, X_{t-1}, \dots, X_{t-p+1})^T. \tag{2.10}$$

Clearly $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$ is not stationary. In the following we give some properties of $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$.

Proposition 2.10. *The Markovian dual process $(\boldsymbol{Z}_t)_{t\in\mathbb{Z}}$ of a periodic autoregression process $(X_t)_{t\in\mathbb{Z}}$ with a companion matrices A_t has the following properties:*
(i) $\boldsymbol{Z}_t = A_t\boldsymbol{Z}_{t-1} + E_t$, where $E_t := (Z_t, 0, \dots, 0)^T$.
(ii) $\boldsymbol{Z}_t = \alpha_t\boldsymbol{Z}_{t-d} + V_t$, where

$$\alpha_t := A_t \cdot A_{t-1} \cdot \dots \cdot A_{t-d+1} \tag{2.11}$$

and

$$V_t := (A_tA_{t-1} \dots A_{t-d+2}E_{t-d+1} + \dots + A_tE_{t-1} + E_t).$$

(iii) Assume that $Z_{nd+s} \sim PWN(0, \sigma_s^2)$ with $0 < \sigma_s^2 < \infty$ for every $s \in \{1, \dots, d\}$ and that $(X_t)_{t\in\mathbb{Z}}$ is periodically weakly stationary. For every $i \in \{1, \dots, d\}$ the process $(z_t^{(i)})$ obtained by taking every d'th $\boldsymbol{Z}_t$, $z_t^{(i)} = \boldsymbol{Z}_{i+td}$ is a weakly stationary multivariate AR(1) process.

The eigenvalues of the matrix α_t, given in (2.11), play a main role in the solution to the PAR process. Recall this proposition([3], Proposition 22).

Proposition 2.11. *The eigenvalues of the matrices α_t are the same for all $t = 1, \dots, d$.*

The following theorem gives the conditions for existence and uniqueness of the solution of the PAR process (2.7).

Theorem 2.12 (Boshnakov, [4]). *The PAR process (2.7) has a periodically weakly stationary solution if and only if the modulus of all eigenvalues of α_t are different from one for $t = 1, \dots, d$. When such a solution exists, it is unique.*

Remark 2.13. *Karanasos, Paraskevopoulos and Dafnos in [27] gave a general solution of the Periodic ARMA model, they expressed the PARMA model as an infinite linear system. This solution is derived from a general method for solving infinite linear systems in row-finite form.*

2.3 Strictly stationary periodic AR(p)-processes

As we saw above, we can represent a PAR process as a vector AR process, hence the periodic strictly stationary solution for a PAR process is equivalent to the strictly stationary solution for its dual multivariate AR process. Hence we can use this equivalence to get a periodic strictly stationary solution to an PAR(p) process using results of Brockwell, Lindner and Vollenbröker [13] for vector autoregressive processes.

2.3.1 Periodic strictly stationary AR(1) process

Firstly we give necessary and sufficient conditions for the existence of a periodic strictly stationary solution of a PAR(1) process, then we can generalize this conditions to PAR(p) processes with $p \leq d$. From (2.6), a PAR(1) process has the form

$$X_{nd+s} - \phi(s) X_{nd+s-1} = Z_{nd+s} \,. \tag{2.12}$$

We now have:

Theorem 2.14. *Let Z_{nd+s} be an i.i.d. non-deterministic random sequence and let X_{nd+s} be a PAR(1) process with period $d \in \mathbb{N}$. Suppose that $\prod_{s=1}^{d} \phi(s) \neq 0$. Then (2.12) has a periodic strictly stationary solution X_{nd+s} if and only if*

$$|\prod_{s=1}^{d} \phi(s)| \neq 1 \quad \text{and} \quad \mathbb{E} \log^+ |Z_{nd+s}| < +\infty \quad \forall s = 1, \dots, d \,.$$

If the solution exists, it is unique and given by

$$X_{nd+s} = \sum_{k=0}^{\infty} \left(\prod_{j=1}^{k} \phi(s+1-j) \right) Z_{nd+s-k}, \qquad \text{if} \quad |\prod_{s=1}^{d} \phi(s)| < 1 \tag{2.13}$$

and

$$X_{nd+s} = -\sum_{k=1}^{\infty} \left(\prod_{j=1}^{k} \phi(s+j) \right)^{-1} Z_{nd+s+k}, \qquad \text{if} \quad |\prod_{s=1}^{d} \phi(s)| > 1, \tag{2.14}$$

where we denote $\phi(s+dk) := \phi(s)$ for $s \in \{1, \dots, d\}$ and $k \in \mathbb{Z} \setminus \{0\}$, i.e. we continue $\phi(\cdot)$ periodically. All series in (2.13) and (2.14) converge almost surely absolutely.

Proof. (i) Necessity of the conditions and uniqueness:

For showing that the stated conditions are necessary and that the solution is then unique, we give two different proofs, one which is based on the VAR(1) representation of the dual process, and one which is based on the Markovian dual representation. So assume that X_{nd+s} is a periodically strictly stationary solution of (2.12).

*P*roof using the VAR(1) representation of the dual process

Define the $d \times d$-matrices

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -\phi(2) & 1 & 0 & \dots & \vdots \\ 0 & -\phi(3) & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & -\phi(d) & 1 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} 0 & \dots & 0 & \phi(1) \\ 0 & \dots & 0 & \phi(2) \\ \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & \phi(d) \end{pmatrix}$$

and

$$\Phi^\star := \Phi_0^{-1}\Phi_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & \phi(1) \\ 0 & 0 & \dots & 0 & \phi(1)\phi(2) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \prod_{s=1}^d \phi(s) \end{pmatrix}.$$

Further, let $Y_t = (X_{td+1}, \dots, X_{td+d})^T$ and $\eta_t = (Z_{td+1}, \dots, Z_{td+d})^T$, $t \in \mathbb{Z}$, be the dual processes. Then we know from (2.3) that $\Phi_0 Y_t - \Phi_1 Y_{t-1} = \eta_t$, equivalently $Y_t - \Phi^\star Y_{t-1} = \Phi_0^{-1}\eta_t$, and that (Y_t) is strictly stationary. Since the eigenvalues of $\Phi^\star$ are 0 (with multiplicity $d-1$) and $\prod_{s=1}^d \phi(s)$ (with multiplictiy 1), and since $(\Phi_0)^{-1}\eta_t$, $t \in \mathbb{Z}$, is i.i.d., it follows from Theorem 1 in [13] that necessarily $|\prod_{s=1}^d \phi(s)| \neq 1$. Denote by $J = S^{-1}\Phi^\star S$ the (complex) Jordan decomposition of $\Phi^\star$ with invertible $S \in \mathbb{C}^{d\times d}$, such that the first Jordan block corresponds to the eigenvalue $\prod_{s=1}^d \phi(s)$; it is a 1×1-Jordan block. Denote by I_1 the projection of $\mathbb{C}^d$ to the first component. Again from Theorem 1 in [13], we obtain that

$$\mathbb{E} \log^+ |I_1 S^{-1} \Phi_0^{-1} \eta_0| < \infty.$$

But $S^{-1}\Phi_0^{-1}\eta_0 = a_1 Z_1 + \ldots + a_d Z_d$ for some coefficients $a_1, \dots, a_d$ that are not all zero. By the independence of $Z_1, \dots, Z_d$ follows that $\mathbb{E}\log^+ |a_s Z_s| < \infty$ for all $s \in \{1, \dots, d\}$, hence that $\mathbb{E}\log^+ |Z_0| < \infty$ since the Z_s are identically distributed. This shows the necessity of the conditions. Uniqueness follows also from Theorem 1 in [13], namely that if $|\prod_{s=1}^d \phi(s)| \neq 1$, then there is at most one strictly stationary solution Y of the VAR(1) equation, and hence at most one periodically strictly stationary solution of (2.12).

*P*roof using the Markovian dual representation of the dual process

Since we have a PAR(1) process, the Markovian dual of X_{nd+s} is X_t, $t \in \mathbb{Z}$, itself. For each $s \in \{1, \dots, d\}$ and $n \in \mathbb{Z}$ we then have by (2.11) that

$$X_{nd+s} = \alpha_{nd+s} X_{(n-1)d+s} + V_{nd+s},$$

where $\alpha_{nd+s} = \prod_{j=1}^{d} \phi(j)$ and

$$V_{nd+s} = Z_{nd+s} + \phi(s)Z_{nd+s-1} + \phi(s)\phi(s-1)Z_{nd+s-2} + \ldots + \left(\prod_{j=0}^{s-2} \phi(j)\right) Z_{nd+s-d+1}.$$

For each fixed $s \in \{1, \ldots, d\}$, this is a 1-dimensional AR(1) equation with i.i.d. noise $(V_{nd+s})_{n\in\mathbb{Z}}$. By Theorem 1 in [10] (alternatively by Theorem 1 in [13]), it then follows that $|\prod_{j=1}^{s} \phi(j)| \neq 1$, that $\mathbb{E}\log^+ |V_{nd+s}| < \infty$ and that $(X_{nd+s})_{n\in\mathbb{Z}}$ is unique. Since $\mathbb{E}\log^+ |V_{nd+s}| < \infty$ implies $\mathbb{E}\log^+ |Z_{nd+s}| < \infty$, this finishes the proof of the necessity and uniqueness.

*(*ii) Sufficiency of the conditions

Now suppose that $|\prod_{s=1}^{d} \phi(s)| \neq 1$ and define X_{nd+s} by (2.13) and (2.14), respectively. Since by periodicity, $\prod_{j=1}^{k} \phi(s+1-j)$ decays exponentially as $k \to \infty$ when $|\prod_{j=1}^{s} \phi(s)| < 1$, and since $\mathbb{E}\log^+ |Z_0| < \infty$, the sum in (2.13) converges almost surely absolutely. That X_{nd+s} satisfies indeed (2.12) and that it is periodically strictly stationary is easily checked directly. The argument for (2.14) is similar. □

Remark 2.15. *The periodically strictly solution given in (2.13) and (2.14) is of the form*

$$X_{nd+s} = \sum_{k=-\infty}^{\infty} a_{k,s} Z_{nd+s-k}\,, \tag{2.15}$$

where for each fixed $s \in \{1, \ldots, d\}$, $(a_{k,s})_{k\in\mathbb{Z}}$ are constants such that $\sum_{k=-\infty}^{\infty} |a_{k,s}| < \infty$.

2.3.2 Periodic strictly stationary AR(p) process

In this section we shall generalise the results of Section 2.3.1 to AR(p) processes, i.e. we consider the PAR(p) process (2.6) given by

$$X_{nd+s} - \sum_{i=1}^{p} \phi_i(s) X_{nd+s-i} = Z_{nd+s},$$

where Z_{nd+s} is non-deterministic i.i.d. When using the VAR-representation, we will have to restrict to $p \leq d$, but when we use the Markovian dual representation, we can show necessary and sufficient conditions for general $p, d \in \mathbb{N}$ for a periodic strictly stationary solution to exist. Let us start with conditions using the VAR representation and assume that $p \leq d$.

Theorem 2.16. *Let Z_{nd+s} be a non-deterministic i.i.d. sequence and let X_{nd+s} be a PAR(p) process satisfying (2.6) with period $d \in \mathbb{N}$, such that $p \leq d$. Define the matrices Φ_0, Φ_1 and $\Phi^\star \in \mathbb{R}^{d\times d}$ by*

$$\begin{aligned}
(\Phi_0)_{i,j} &:= \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i < j, \\ -\phi_{i-j}(i), & \text{if} \quad i > j, \end{cases} \\
(\Phi_1)_{i,j} &:= \begin{cases} \phi_{d+i-j}(i), & \text{if} \quad 1 \leq d + i - j \leq p, \\ 0, & \text{else}, \end{cases} \\
\Phi^\star &:= \Phi_0^{-1}\Phi_1.
\end{aligned}$$

(a) Then (2.6) has a periodic strictly stationary solution X_{nd+s} if and only if one of the following two conditions (i) or (ii) is satisfied:

(i) The only eigenvalue of $\Phi^\star$ is 0;

(ii) $\Phi^\star$ has an eigenvalue that is different from 0, all eigenvalues of $\Phi^\star$ have absolute size different from 1, and $\mathbb{E}\log^+|Z_0| < \infty$.

If the periodic strictly stationary solution exists, it is unique.

(b) Define the matrix

$$A(z) := \Phi_0 - z\Phi_1, \quad z \in \mathbb{C}.$$

Then the condition that all eigenvalues of $\Phi^\star$ have absolute size different from 1 is equivalent to the fact that

$$\det A(z) \neq 0 \quad \text{for all } z \in \mathbb{C} : |z| = 1.$$

If this condition is satisfied and $\mathbb{E}\log^+|Z_0| < \infty$, then the unique periodic strictly stationary solution of (2.6) is given by the almost surely absolutely convergent sum

$$X_{nd+s} = \sum_{i=1}^{d} \sum_{k\in\mathbb{Z}} e_s^T B_k e_i Z_{(n-k)d+i}, \quad n \in \mathbb{Z}, \; s \in \{1, 2, \ldots, d\}, \tag{2.16}$$

where $(B_k)_{k\in\mathbb{Z}}$ is given as coefficients of the Laurent expansion

$$\sum_{k=-\infty}^{\infty} B_k z^k = (\det(A(z))^{-1} \operatorname{adj}(A(z)),$$

which converges absolutely in $\{z \in \mathbb{C} : 1 - \delta < |z| < 1 + \delta\}$ for some $\delta > 0$, and $e_i \in \mathbb{R}^d$ is the i'th unit vector in $\mathbb{C}^d$ and $\operatorname{adj}(A(z))$ denotes the adjugate matrix of $A(z)$.

Proof. (a) Denote the dual process of X_{nd+s} by $Y_t = (X_{td+1}, \ldots, X_{td+d})^T$, $t \in \mathbb{R}$, and the dual process of Z_{nd+s} by $\eta_t = (Z_{td+1}, \ldots, Z_{td+d})^T$, $t \in \mathbb{R}$. Then from (2.5) and Proposition 2.3, X_{nd+s} is a periodic strictly stationary solution of (2.6) if and only if $(Y_t)_{t\in\mathbb{R}}$ is a strictly stationary solution of the VAR(1) equation $Y_t - \Phi^\star Y_{t-1} = \Phi_0^{-1}\eta_t$, $t \in \mathbb{R}$. But by Theorem 1 of [13] (observe that no non-trival linear combination of the components of $\Phi_0^{-1}\eta_0$ can be deterministic), this VAR(1) equation admits a strictly stationary solution if and only if either all eigenvalues of $\Phi^\star$ are zero, or there exists a non-zero eigenvalue, all eigenvalues have absolute size different from 1, and

$$\mathbb{E}\log^+ \|I_h S^{-1}\Phi_0^{-1}\eta_0\| < \infty \tag{2.17}$$

for all h that correspond to Jordan blocks with eigenvalues different from 0. Here, we take a Jordan decomposition $S^{-1}\Phi^\star S$ of $\Phi^\star$ and denote by I_h a projection matrix which corresponds to the h'th Jordan block as in Equation (7) of [13]. When we assume the existence of at least one Jordan block with an eigenvalue different from 0, the first component of $I_h S^{-1}\Phi_0^{-1}\eta_0$ is a non-trivial linear combination of $Z_1, \ldots, Z_d$, and hence by the i.i.d. property the condition (2.17) is equivalent to $\mathbb{E}\log^+|Z_0| < \infty$. Hence, the derived conditions are exactly conditions (i) and (ii) above. From Theorem 1 in [13] follows also that the strictly stationary solution Y and hence the periodic strictly stationary solution X_{nd+s} are unique under the given conditions.

(b) Define $M(z) := \mathrm{Id}_{d\times d} - \Phi^\star z$ for $z \in \mathbb{C}$. The fact that all eigenvalues of $\Phi^\star$ have absolute size different from 1 can be restated as

$$\det(\Phi^\star - z\mathrm{Id}_{d\times d}) \neq 0 \quad \forall\, z \in \mathbb{C} : |z| = 1.$$

Replacing z by $1/z$ this is equivalent to $\det M(z) \neq 0$ for all $z \in \mathbb{C} : |z| = 1$. But since

$$M(z) = \Phi_0^{-1}(\Phi_0 - \Phi_1 z) = \Phi_0^{-1} A(z),$$

we have $\det M(z) = \det \Phi_0^{-1} \cdot \det A(z)$, giving the equivalence with $\det A(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = 1$.

Now assume that $\det(A(z)) \neq 0$ for all $z \in \mathbb{C} : |z| = 1$ and that $\mathbb{E} \log^+ |Z_0| < \infty$. Then

$$A(z)^{-1} = (\det A(z))^{-1} \mathrm{adj}(A(z))$$

by Cramer's rule, and since $\det(A(z))$ is a polynomial in z which is non-zero on the unit circle, there is a neighbourhood of the form $\{z \in \mathbb{C} : 1-\delta < |z| < 1+\delta\}$ of the unit circle (for some $\delta \in (0,1)$) such that $\det(A(z))$ is non-zero there, hence that $(A(z))^{-1}$ is holomorphic there and hence admits a Laurent expansion of the form $\sum_{k\in\mathbb{Z}} B_k z^k$. Furthermore, since $M(z) = \Phi_0^{-1} A(z)$, we have

$$(M(z))^{-1} = (A(z))^{-1}\Phi_0$$

on this ring. Define

$$K := \{a \in \mathbb{C}^d : \text{the distribution of } a^* \Phi_0^{-1} \eta_0 \text{ is degenerate to a Dirac measure}\},$$

where a^* denotes the complex conjugate transpose. Since the components of $\eta_0 = (Z_1, \dots, Z_d)^T$ are independent and identically non-trivial distributed, a linear combination of the components of η_0 can be degenerate to a constant only if it is the trivial linear combination. This implies that $K = \{0\}$. Let $U = \mathrm{Id}_{d\times d}$, $w_t = \Phi_0^{-1}\eta_t$, $v = g = 0 \in \mathbb{R}^d$ (these are the notations of Theorem 2 in [13]). Let X_{nd+s} be the unique periodic strictly stationary solution of (2.6). As seen in (a), this implies that the dual process $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of the VAR(1) equation $Y_t - \Phi^\star Y_{t-1} = \Phi_0^{-1}\eta_t$. Then it follows from Theorem 2 in [13] that

$$Y_t = \sum_{k\in\mathbb{Z}} M_k \Phi_0^{-1} \eta_{t-k}, \quad t \in \mathbb{Z},$$

with an almost surely absolutely converging sum, where $\sum_{k\in\mathbb{Z}} M_k z^k$ is the Laurent expansion of $(M(z))^{-1}$ on $\{z \in \mathbb{C} : 1-\delta < |z| < 1+\delta\}$. Since $(M(z))^{-1} = (A(z))^{-1}\Phi_0$, this shows that $M_k = B_k \Phi_0$ and hence that

$$Y_n = \sum_{k\in\mathbb{Z}} B_k \eta_{n-k} = \sum_{k\in\mathbb{Z}} B_k \sum_{i=1}^{d} e_i Z_{(n-k)d+i}.$$

Since $X_{nd+s} = e_s^T Y_n$, this gives (2.16), finishing the proof. □

Remark 2.17. *Observe that the solution given in (2.16) is also of the form*

$$X_{nd+s} = \sum_{k=-\infty}^{\infty} a_{k,s} Z_{nd+s-k}, \quad n \in \mathbb{Z},\ s \in \{1,\dots,d\}$$

with absolute summable coefficient sequences $(a_{k,s})_{k\in\mathbb{Z}}$ for each $s \in \{1,\dots,d\}$.

It is also possible to get a characterisation of periodic strictly stationary solutions using the Markovian dual process of X_{nd+s} without the restriction $p \leq d$. This gives a complete characterisation of the existence of periodic strictly stationary solutions of PAR(p) equations and is our main theorem of this section.

Theorem 2.18. *Let Z_{nd+s} be a non-deterministic i.i.d. sequence. Consider the PAR(p) equation (2.6) with $p, d \in \mathbb{N}$. Recall the matrices $A_t \in \mathbb{R}^{p\times p}$ from (2.9) (with $A_{s+md} := A_s$ for $s \in \{1,\dots,d\}$ and $m \in \mathbb{Z}\setminus\{0\}$) and the matrices $\alpha_t = A_t A_{t-1}\cdots A_{t-d+1}$, $t \in \mathbb{Z}$. Then a periodic strictly stationary solution X_{nd+s} to Equation (2.6) exists if and only if (i) and (ii) below are satisfied:*

(i) all eigenvalues of the matrix α_d have absolute size different from 1,

(ii) $E\log^+\|Z_0\| < \infty$ or all eigenvalues of α_d are 0.

When a periodic strictly stationary solution exists, it is unique.

Proof. Denote

$$\begin{aligned}
E_t &:= (Z_t, 0, \dots, 0)^T \in \mathbb{R}^p, \\
\mathbf{Z}_t &:= (X_t, X_{t-1}, \dots, X_{t-p+1})^T \in \mathbb{R}^p, \quad \text{and} \\
V_t &:= A_t A_{t-1}\cdots A_{t-d+2}E_{t-d+1} + \ldots + A_t E_{t-1} + E_t \in \mathbb{R}^{p\times p}.
\end{aligned}$$

We have seen that X_{nd+s} satisfies (2.6) if and only if $\mathbf{Z}_t$ satisfies $\mathbf{Z}_t = A_t\mathbf{Z}_{t-1} + E_t$. For the process $(\mathbf{Z}_{td+s})_{t\in\mathbb{Z}}$ for fixed $s \in \{1,\dots,d\}$, this results in the VAR(1) equation

$$\mathbf{Z}_{td+s} = \alpha_{td+s}\mathbf{Z}_{(t-1)d+s} + V_{td+s} \tag{2.18}$$

with i.i.d. noise $(V_{td+s})_{t\in\mathbb{Z}}$. Observe that $\alpha_{td} = \alpha_d$.

Proof of the necessity of the conditions and uniqueness:

Now assume that X_{nd+s} is a periodic strictly stationary of (2.6). Then $(\mathbf{Z}_{td})_{t\in\mathbb{Z}}$ is strictly stationary. If we can show that $a^T V_d$ cannot degenerate to a constant for all $a \in \mathbb{C}^d\setminus\{0\}$, then it will follow from Theorem 1 in [13] that all eigenvalues of α_d must have absolute size different from 1. So let $a = (a_1,\dots,a_d)^T \in \mathbb{C}^d$ with $a^T V_d$ being constant almost surely. Since $E_1,\dots,E_d$ are independent, this implies that $a^T E_j$ is constant for all $j = 1,\dots,d$. In particular, we obtain $a_1 E_d = a^T Z_d$ is constant almost surely, which results in $a_1 = 0$ since Z_d is non-deterministic. Next, observe that $A_{d-1}E_{d-1} = (\phi_1(d-1), 1, 0, \dots, 0)^T Z_{d-1}$, hence $a^T A_{d-1}E_{d-1} = a_2 Z_{d-1}$ is constant almost surely, which results in $a_2 = 0$. Similarly, we see that $A_d A_{d-1} E_{d-2}$ is of the form $(*, *, Z_{t-2}, 0, \dots, 0)$ and we obtain $a_3 = 0$. Continuing inductively in this way, we obtain $a = 0$ and hence that all eigenvalues of α_d must have absolute size different from 1, i.e. condition (i) is satisfied.

Now assume additionally that α_d has an eigenvalue which is different from 0. We have to show that $\mathbb{E}\log^+|Z_0| < \infty$. We do this in a similar way as above, and assume by way of contradiction that $\mathbb{E}\log^+|Z_0| = \infty$. Choose a Jordan decomposition $J = S^{-1}\alpha_d S$ of α_d such that the first Jordan block of J corresponds to an eigenvalue $\lambda \neq 0$; call this Jordan block

$$J_1 = \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix} \in \mathbb{C}^{r_1 \times r_1}.$$

By what we have already shown, $|\lambda| \neq 1$. Denote by I_1 the projection matrix that maps $(x_1, \ldots, x_p)^T \in \mathbb{C}^p$ to $(x_1, \ldots, x_{r_1})^T \in \mathbb{C}^{r_1}$ and by e_i the i'th unit vector in $\mathbb{C}^p$. Since $(\mathbf{Z}_{td})_{t\in\mathbb{Z}}$ is a strictly stationary VAR(1) process with i.i.d. noise $(V_{dt})_{t\in\mathbb{Z}}$, it follows from Theorem 1 in [13] that

$$\mathbb{E}\log^+\|I_1 S^{-1} V_d\| < \infty.$$

From the form of V_d and observing that $E_d, E_{d-1}, \ldots, E_1$ are all independent, this implies

$$\mathbb{E}\log^+\|I_1 S^{-1} A_d A_{d-1} \cdots A_{d-k+1} E_{d-k}\| < \infty \quad \forall\, k = 0, 1, \ldots, d-1. \tag{2.19}$$

Applying this for $k = 0$ we obtain

$$\mathbb{E}\log^+\|I_1 S^{-1} E_d\| = \mathbb{E}\log^+\|I_1 S^{-1} e_1 Z_d\| < \infty.$$

Since we assumed $\mathbb{E}\log^+|Z_d| = \infty$, this shows $I_1 S^{-1} e_1 = 0$. Now apply (2.19) for $k = 1$. Then

$$\infty > \mathbb{E}\log^+\|I_1 S^{-1} A_d E_{d-1}\| = \mathbb{E}\log^+\|I_1 S^{-1}(\phi_1(d) e_1 + e_2) Z_{d-1}\| = \mathbb{E}\log^+\|I_1 S^{-1} e_2 Z_{d-1}\|.$$

Again, since $\mathbb{E}\log^+|Z_{d-2}| = \infty$, this implies $I_1 S^{-1} e_2 = 0$. Next, apply (2.19) for $k = 2$. Then, by what we have already shown for $k = 0$ and $k = 1$,

$$\infty > \mathbb{E}\log^+\|I_1 S^{-1} A_d A_{d-1} E_{d-2}\| = \mathbb{E}\log^+\|I_1 S^{-1} e_3 Z_{d-2}\|,$$

so that again $I_1 S^{-1} e_3 = 0$. Continuing inductively in this way, we conclude that $I_1 S^{-1} e_i = 0$ for all $i \in \{1, \ldots, p\}$, resulting in $I_1 S^{-1} = 0$. This contradicts $I_1 S^{-1} \alpha_d S I_1^T = I_1 J I_1^T = J_1 \neq 0$. We conclude that necessarily $\mathbb{E}\log^+|Z_0| < \infty$ if α_d has an eigenvalue different from 0, so that we have obtained condition (ii).

If condition (i) is satisfied, then the eigenvalues of all α_t have absolute size different from 1, since the eigenvalues of these matrices are the same. Then by Theorem 1 in [13], for each $s \in \{1, \ldots, d\}$, there is at most one strictly stationary solution $(\mathbf{Z}_{td+s})_{t\in\mathbb{Z}}$ of (2.18), showing that there exists at most one periodically strictly stationary solution X_{nd+s}.

*P*roof of the sufficiency of the conditions:

Assume that (i) and (ii) are satisfied. By Theorem 1 in [13], there exists a strictly stationary $\mathbb{R}^p$-valued solution $(\mathbf{W}_{nd})_{n\in\mathbb{Z}}$ to the VAR(1) equation

$$\mathbf{W}_{nd} = \alpha_d \mathbf{W}_{(n-1)d} + V_{nd}, \quad n \in \mathbb{Z}. \tag{2.20}$$

More precisely, choose a matrix $S \in \mathbb{C}^{p\times p}$ such that $B := S^{-1}\alpha_d S$ is in block form $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ with $B_1 \in \mathbb{C}^{p_1\times p_1}$ having only eigenvalues of absolute size less than 1, and $B_2 \in \mathbb{C}^{p_2\times p_2}$ having only eigenvalues of absolute size greater than 1, where $p_1 + p_2 = p$. Then the unique strictly stationary to (2.20) is given by

$$S^{-1}\mathbf{W}_{nd} = \sum_{k\in\mathbb{Z}} \begin{pmatrix} B_1^k \mathbf{1}_{\{k\geq 0\}} & 0 \\ 0 & -B_2^k \mathbf{1}_{\{k<0\}} \end{pmatrix} S^{-1} V_{(n-k)d},$$

i.e.

$$\mathbf{W}_{nd} = \sum_{k\in\mathbb{Z}} S \begin{pmatrix} B_1^k \mathbf{1}_{\{k\geq 0\}} & 0 \\ 0 & -B_2^k \mathbf{1}_{\{k>0\}} \end{pmatrix} S^{-1} V_{(n-k)d};$$

this is an immediate consequence of Theorem 1 in [13]. From the specific form of $\mathbf{W}_{nd}$ it follows that not only $(\mathbf{W}_{nd})_{n\in\mathbb{Z}}$, but also the $\mathbb{R}^{p+d}$-valued stochastic process $(\mathbf{W}_{nd}^T, E_{nd}, E_{nd+1}, \ldots, E_{nd+d-1})_{n\in\mathbb{Z}}$ is strictly stationary.

Given this solution, define $\mathbf{W}_{nd+s}$ for $n \in \mathbb{Z}$ and $s \in \{1, \ldots, d-1\}$ recursively by

$$\begin{aligned} \mathbf{W}_{nd+1} &:= A_{nd+1}\mathbf{W}_{nd} + E_{nd+1}, \\ \mathbf{W}_{nd+2} &:= A_{nd+2}\mathbf{W}_{nd+1} + E_{nd+2}, \\ \vdots \ &\ \vdots \ \ \vdots \\ \mathbf{W}_{nd+d-1} &:= A_{nd+d-1}\mathbf{W}_{nd+d-2} + E_{nd+d-1}. \end{aligned}$$

Since $(\mathbf{W}_{nd}, E_{nd}, E_{nd+1}, \ldots, E_{nd+d-1})_{n\in\mathbb{Z}}$ is strictly stationary and $A_{nd+s} = A_s$ for $n \in \mathbb{Z}$, it follows that $(\mathbf{W}_t)_{t\in\mathbb{Z}}$ is periodically strictly stationary. By construction, we have

$$\mathbf{W}_{nd+s} = A_s\mathbf{W}_{nd+s-1} + E_{nd+s} \quad \forall\, s \in \{1, \ldots, d-1\}.$$

To see that this also holds for $s = d$, observe that

$$\begin{aligned} A_d\mathbf{W}_{nd+d-1} + E_{nd+d} &= A_d(A_{d-1}\mathbf{W}_{nd+d-2} + E_{nd+d-1}) + E_{nd+d} \\ &= A_dA_{d-1}\mathbf{W}_{nd+d-2} + A_dE_{nd+d-1} + E_{nd+d} \\ &= A_dA_{d-1}(A_{d-2}\mathbf{W}_{nd+d-3} + E_{nd+d-2}) + A_dE_{nd+d-1} + E_{nd+d} \\ &= \ldots \\ &= \alpha_d\mathbf{W}_{nd} + V_{nd+d} \\ &= \mathbf{W}_{nd+d}. \end{aligned}$$

Hence we have shown that $\mathbf{W}_t = A_t\mathbf{W}_{n-1} + E_t$ for all $t \in \mathbb{Z}$, and that $(\mathbf{W}_t)_{t\in\mathbb{Z}}$ is periodically strictly stationary. Denoting by X_t the first component of $\mathbf{W}_t$, it follows that $(X_t)_{t\in\mathbb{Z}}$ satisfies (2.6) and is periodically strictly stationary, showing the existence of the periodically strictly stationary solution. $\square$

CHAPTER 3

Stationary Periodic Ornstein-Uhlenbeck Processes

Kwakernaak (1975) considered Gaussian periodic Ornstein–Uhlenbeck processes. These processes have been applied in statistical shape analysis. They were generalized by Pederson to the Lévy driven case. More precisely, he calls a stochastic process $X = (X_t)_{t\geq 0}$ a periodic Ornstein-Uhlenbeck process, if its paths are almost surely periodic and it satisfies $dX_t = -\lambda X_t dt + dL_t$, where L_t is a Lévy process. In particular, since $X_0 = X_1$, this implies $X_0 = X_1 = \frac{1}{e^{\lambda}-1}\int_0^1 e^{\lambda s} dL_s$, see Pederson [32]. In this thesis, by a periodic Ornstein-Uhlenbeck process we refer to a different process, namely one for which the coefficient function $\lambda(\cdot)$ is periodic as defined in Definition 1.29. In this chapter we shall give necessary and sufficient conditions for the existence of periodic stationary solutions. Let us recall the Definition 1.29.

Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process and let $\lambda : \mathbb{R} \to \mathbb{R}$ be a periodic, bounded and measurable function with period $d > 0$. Consider the periodic OU-equation

$$dV_t = \lambda(t)V_t dt + dL_t\,; \quad t \in \mathbb{R}\,. \tag{3.1}$$

Set $U_t = \int_0^t \lambda(s)ds$, then the equation (3.1) is equivalent to the equation

$$dV_t = V_t dU_t + dL_t\,; \quad t \in \mathbb{R}\,. \tag{3.2}$$

Then the integral form of (3.2) is given by

$$V_t = V_s + \int_s^t V_u dU_u + L_t - L_s\,; \quad s \leq t \in \mathbb{R}\,, \tag{3.3}$$

where V_s is a finite random variable. The solution to (3.2) or (3.3) can be given by the equation (see Jaschke [23], Theorem 1).

$$V_t = e^{U_t - U_s}\left(V_s + \int_s^t e^{U_s - U_u} dL_u\right)\,; \quad s \leq t \in \mathbb{R}\,, \tag{3.4}$$

or equivalently

$$V_t = e^{U_t}\left(V_0 + \int_0^t e^{-U_u} dL_u\right)\,; \quad t \in \mathbb{R}\,, \tag{3.5}$$

with $\int_0^t = -\int_t^0$ for $t < 0$.

Here in this case we have $\mathcal{E}(U)_t = e^{U_t} = e^{\int_0^t \lambda(s)ds}$, where $\mathcal{E}(U)_t$ is the stochastic exponential of U_t. Hence we can rewrite (3.5) as

$$V_t = \mathcal{E}(U)_t\left(V_0 + \int_0^t \mathcal{E}(U)_u^{-1} dL_u\right)\,; \quad t \in \mathbb{R}. \tag{3.6}$$

In the following we are going to give a short proof that V_t given in (3.4) is indeed a solution to the equation (3.2) or (3.3):
We have from the equation (3.4)

$$V_t = \underbrace{e^{U_t - U_s}}_{=:A_{s,t}} \underbrace{\Big(V_s + \int_s^t e^{U_s - U_u} dL_u \Big)}_{=:B_{s,t}} ,$$

hence

$$\begin{aligned}
V_t &= A_{s,t} B_{s,t} \overset{\text{partial integration}}{=} \\
&= V_s + \int_{s+}^t A_{s,u-} dB_{s,u} + \int_{s+}^t B_{s,u-} dA_{s,u} \\
&= V_s + \int_{s+}^t e^{U_u - U_s} d(V_s + \int_{s+}^u e^{U_s - U_v} dL_v) + \int_{s+}^t (V_s + \int_{s+}^{u-} e^{U_s - U_v} dL_v) d(e^{U_u - U_s}) \\
&= V_s + \int_{s+}^t e^{U_u - U_s} \cdot e^{U_s - U_u} dL_u + \int_{s+}^t \underbrace{(V_s + \int_{s+}^{u-} e^{U_s - U_v} dL_v)(e^{U_u - U_s})}_{=V_{u-}} dU_u \\
&= V_s + L_t - L_s + \int_{s+}^t V_{u-} dU_u \,.
\end{aligned}$$

Hence (3.4) holds and V_t is indeed a solution.

Now we are going to give the following useful lemma:

Lemma 3.1. *Let $\lambda : [0,1) \to \mathbb{R}$ be measurable, bounded and $\lambda \not\equiv 0$, such that $\int_0^1 |\lambda(s)| ds > 0$, and L be a Lévy process. Then*

$$\mathbb{E} \log^+ | \int_0^1 \lambda(s) dL_s | < +\infty \Leftrightarrow \mathbb{E} \log^+ |L_1| < +\infty \,. \tag{3.7}$$

Proof. Set $f(s) := \lambda(s)$ for $s \in [0,1]$, i.e f is bounded and $\not\equiv 0$. Let $|f| \leq c$ for $c \geq 1$. Let $(L_t)_{t \geq 0}$ be a Levy process with the triplet (A, ν, γ), then

$$\int_0^1 f^2(s) A ds < +\infty \,.$$

Hence

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}} \Big((f(s)x)^2 \wedge 1 \Big) \nu(dx) ds \leq \\
\leq &\int_0^1 \int_{\mathbb{R}} \Big(|cx|^2 \wedge 1 \Big) \nu(dx) ds \leq \\
\leq &c^2 \int_0^1 \underbrace{\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx)}_{<+\infty} ds < +\infty \,.
\end{aligned}$$

Set $X := \int_0^1 f(s)dL_s$, this exists since f is bounded. X is infinitely divisible and has the triplet (A_X, ν_X, γ_X) given by

$$\begin{aligned}
A_X &= \int_0^1 f^2(s)Ads \\
\nu_X(B) &= \int_0^1 ds \int_{\mathbb{R}} \mathbb{1}_B(f(s)x)\nu(dx) \ ; \quad B \in \mathcal{B}(\mathbb{R})\backslash\{0\} \\
\gamma_X &= \int_0^1 \Big(f(s)\gamma + \int_{\mathbb{R}} f(s)x(\mathbb{1}_{[-1,1]}((f(s)x)) - c(x))\nu(dx)\Big)\, ds\,.
\end{aligned}$$

(See Sato [36], Proposition 57. 10).
Also

$$\nu_X(B) = \int_0^1 \int_{\mathbb{R}} \mathbb{1}_B(f(s)x)\nu(dx)ds\,,$$

hence

$$\int_{\mathbb{R}} \mathbb{1}_B(x)\nu_X(dx) = \int_0^1 \int_{\mathbb{R}} \mathbb{1}_B(f(s)x)\nu(dx)ds\,,$$

and by measure theoretic inductionn follows

$$\int_{\mathbb{R}} g(x)\nu_X(dx) = \int_0^1 \int_{\mathbb{R}} g(f(s)x)\nu(dx)ds\,, \tag{3.8}$$

for all $g : \mathbb{R} \to [0, \infty)$ measurable.
Set $g(x) := \log^+(x)$, then

$$\begin{aligned}
&\mathbb{E}\log^+|X| < +\infty \\
\Leftrightarrow &\int_{x\in\mathbb{R}\backslash[-1,1)} \log|x|\nu_X(dx) < +\infty \qquad\qquad \text{(Sato [36], Theorem 25.3)} \\
\Leftrightarrow &\int_{\mathbb{R}} \log^+|x|\nu_X(dx) < +\infty \\
\overset{(3.8)}{\Leftrightarrow} &\int_0^1 \int_{\mathbb{R}} \log^+(|f(s)x|)\nu(dx)ds < +\infty \\
\overset{(**)}{\Leftrightarrow} &\mathbb{E}\log^+|L_1| < \infty \qquad\qquad \text{(Sato [36], Theorem 25.3).}
\end{aligned}$$

Proof of the equivalence (**):
$" \Rightarrow "$

$$\int_0^1 \int_{\mathbb{R}} \log^+(|f(s)||x|)\nu(dx)ds < \infty$$

$$\Rightarrow \int_{\mathbb{R}} \log^+(|f(s)||x|)\nu(dx) < \infty$$

for Lebesgue almost all $s \in [0, 1]$. Let $b := |f(s)| \neq 0$ for $s \in [0, 1]$ and $\int_{\mathbb{R}} \log^+(|f(s)||x|)\nu(dx) < \infty$.

If $b > 1$, then $\log^+(b|x|) \geq \log^+(|x|)$. If $b \leq 1$, then

$$\begin{aligned}
\log^+(b|x|) &= \begin{cases} 0, & |x| < \frac{1}{b} \\ \log(b|x|), & |x| \geq \frac{1}{b} \end{cases} \\
&= \begin{cases} 0, & |x| < \frac{1}{b} \\ \log(b) + \log(|x|), & |x| \geq \frac{1}{b} \end{cases} \\
&\geq \frac{\log|x|}{2}, \text{ for } |x| \geq b^{-2}
\end{aligned}$$

Hence

$$\begin{aligned}
&\Rightarrow \int_{|x|\geq b^{-2}} \log^+|x|\nu(dx) \leq 2\int_{|x|\geq b^{-2}} \log^+(b|x|)\nu(dx) < \infty \\
&\Rightarrow \mathbb{E}\log^+|L_1| < \infty \qquad \text{(Sato [36], Theorem 25.3).}
\end{aligned}$$

" $\Leftarrow$ "
Let $\mathbb{E}\log^+|L_1| < \infty$, then

$$\int_{|x|\geq 1} \log^+|x|\nu(dx) < \infty \qquad \text{(Sato [36], Theorem 25.3).}$$

Let $|f(s)| \leq c$ with $c \geq 1$, then

$$\begin{aligned}
&\Rightarrow \log^+|f(s)x| \leq \log^+|cx| \leq \log(c) + \log^+|x| \\
&\Rightarrow \int_0^1\int_{\mathbb{R}} \log^+|f(s)x|\nu(dx)ds \leq \int_0^1\int_{\mathbb{R}} \log^+|cx|\nu(dx)ds \\
&= \int_{\mathbb{R}} \log^+|cx|\nu(dx) = \int_{|x|\geq\frac{1}{c}} \left(\log|x| + \log|c|\right)\nu(dx) + \int_{|x|<\frac{1}{c}} 0\nu(dx) < \infty\,.
\end{aligned}$$

□

Analogous to the previous Lemma we can give the following lemma

Lemma 3.2. *Let $\lambda : [0,1) \to \mathbb{R}$ be measurable, bounded and $\lambda \not\equiv 0$, such that $\int_0^1 |\lambda(s)|ds > 0$ and L a Lévy process. Then*

$$\mathbb{E}|\int_0^1 \lambda(s)dL_s|^p < +\infty \Leftrightarrow \mathbb{E}|L_1|^p < +\infty\,, \quad p \in (0,\infty)\,. \tag{3.9}$$

In the following we are going to give a helpful representation of V_t. In the equation (3.4), we set

$$V_t = \underbrace{e^{U_t-U_s}}_{=:A_{s,t}} V_s + \underbrace{\int_s^t e^{U_t-U_u}dL_u}_{=:B_{s,t}}\,, \qquad s,t \in \mathbb{R}\ ;\ s \leq t \tag{3.10}$$

then

$$V_t = A_{s,t}V_s + B_{s,t} \qquad s,t \in \mathbb{R}\ ;\ s \leq t. \tag{3.11}$$

3.1 Periodic strictly stationary periodic OU processes

Necessary and sufficient conditions for the existence of periodic strictly stationary solutions of (3.3) are given in the following theorem. For simplicity, we restrict to the period $d = 1$.

Theorem 3.3. *Let $\lambda : \mathbb{R} \to \mathbb{R}$ be 1-periodic, bounded, non identically zero and measurable and let $\alpha := \int_0^1 \lambda(s)ds$. Let $L = (L_t)_{t\in\mathbb{R}}$ be a non-zero two-sided real-valued Levy process. Consider the periodic Ornstein-Uhlenbeck equation (3.3) and the process V_t satisfying (3.3). Then the following are true*

(*i*) *If $\alpha < 0$, then a random variable V_0 can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary if and only if $\mathbb{E}\log^+|L_1| < \infty$.*
In this case the solution is unique and given by

$$V_t = e^{U_t}\int_{-\infty}^{t} e^{-U_u}dL_u\,. \tag{3.12}$$

The integral in the equation (3.12) converges almost surely.

(*ii*) *If $\alpha > 0$, then a random variable V_0 can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary, if and only if $\mathbb{E}\log^+|L_1| < \infty$.*
In this case, the solution is unique and given by

$$V_t = -e^{U_t}\int_{t+}^{\infty} e^{-U_u}dL_u\,. \tag{3.13}$$

The integral in the equation (3.13) converges almost surely.

(*iii*) *If $\alpha = 0$, then there is no choice of V_0 making $(V_t)_{t\in\mathbb{R}}$ 1-periodic strictly stationary.*

Proof. **Proof of (i)**: *The necessary condition and uniqueness:*
Firstly we are going to show that necessarily $\mathbb{E}\log^+|L_1| < \infty$ and that the solution must then have the form $V_t = e^{U_t}\int_{-\infty}^{t} e^{-U_u}dL_u$.
We have from the equation (3.11)

$$V_{n+1} = A_{n,n+1}V_n + B_{n,n+1}$$

where $A_{n,n+1} = e^{U_{n+1}-U_n}$ and $B_{n,n+1} = \int_{n+}^{n+1} e^{U_{n+1}-U_u}dL_u$.
It easy to show that $A_{n,n+1} = e^{\alpha}$. We set $B_{n,n+1} =: Z_{n+1}$, and note that $(Z_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence. Hence

$$V_{n+1} = e^{\alpha}V_n + Z_{n+1}\,. \tag{3.14}$$

This is an AR(1) process with coefficient $\phi = e^{\alpha} < 1$.

Since $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary by assumption, $(V_n)_{n\in\mathbb{Z}}$ is strictly stationary. By Brockwell and Lindner ([10], Theorem 1), it follows that $\mathbb{E}\log^+|Z_1| < \infty$, and that

$$V_n = \sum_{j=0}^{\infty}(e^{\alpha})^j Z_{n-j} \tag{3.15}$$

and that this sum converges almost surely absolutely. Since $e^{U_1-U_u}$ is non-zero and bounded on $[0,1]$, Lemma 3.1 implies $\mathbb{E}\log^+|L_1| < \infty$.

Recall that the equation (3.14) has the solution given in the equation (3.15) namely

$$\begin{aligned}
V_{n+1} &= \sum_{j=0}^{\infty} e^{\alpha j} Z_{n+1-j} \\
&= \sum_{j=0}^{\infty} e^{\alpha j} \int_{(n-j)+}^{n-j+1} e^{U_{n-j+1}-U_u} dL_u \\
&= \sum_{j=0}^{\infty} e^{\alpha j} \int_{(n-j)+}^{n-j+1} e^{\alpha(n-j+1)-U_u} dL_u \\
&= e^{\alpha(n+1)} \sum_{j=0}^{\infty} \int_{(n-j)+}^{n-j+1} e^{-U_u} dL_u \\
&= e^{\alpha(n+1)} \int_{-\infty}^{n+1} e^{-U_u} dL_u \,.
\end{aligned}$$

Hence

$$V_n = e^{U_n} \int_{-\infty}^{n} e^{-U_u} dL_u,$$

with the integral converging almost surely.

Let $t \in [n, n+1)$, then from equation (3.4)

$$\begin{aligned}
V_t &= e^{U_t-U_n} V_n + e^{U_t} \int_{n+}^{t} e^{-U_u} dL_u \\
&= e^{U_t-U_n} e^{U_n} \int_{-\infty}^{n} e^{-U_u} dL_u + e^{U_t} \int_{n+}^{t} e^{-U_u} dL_u \\
&= e^{U_t} \int_{-\infty}^{t} e^{-U_u} dL_u \,.
\end{aligned}$$

Hence $V_0 = \int_{-\infty}^{0} e^{-U_u} dL_u$.

The sufficient condition:
Suppose that $\mathbb{E}\log^+|L_1| < \infty$ and define V_t by the equation (3.12). Since e^{U_t} decreases exponentially as $t \to \infty$, it is easy to show that $\int_{-\infty}^{t} e^{-U_u} dL_u$ converges almost surely absolutely (see Sato [35], Theorem 1.2 and Proposition 4.3.)

We are going to show that V_t given in (3.12) is a solution to the equation (3.4).

From the equation (3.12) we have

$$V_t = \underbrace{e^{U_t}}_{=:A_t}\underbrace{\Big(\int_{-\infty}^{t} e^{-U_u} dL_u\Big)}_{=:B_t}$$

$$\Rightarrow V_t = A_t B_t \overset{\text{partial Integration}}{=}$$

$$= V_s + \int_{s+}^{t} A_{v-} dB_v + \int_{s+}^{t} B_{v-} dA_v + [A,B]_{(s,t]}$$

$$= V_s + \int_{s+}^{t} e^{U_v} d\Big(\int_{-\infty}^{v} e^{-U_u} dL_u\Big) + \int_{s+}^{t} \Big(\int_{-\infty}^{v-} e^{-U_u} dL_u\Big) d(e^{U_v}) + \underbrace{[A,B]_{(s,t]}}_{=0 \text{ since U cont. and of finite var.}}$$

$$= V_s + \int_{s+}^{t} e^{U_v} \cdot e^{-U_v} dL_v + \int_{s+}^{t} \underbrace{\Big(\int_{-\infty}^{v-} e^{-U_v} dL_u\Big)(e^{U_v})}_{=V_{v-}} dU_u$$

$$= V_s + L_t - L_s + \int_{s}^{t} V_{u-} dU_u \,.$$

Hence (3.3) holds and V_t indeed a solution.

Now we are going to show that V_t given in (3.12) is 1-periodic strictly stationary. Let $t_1 \le t_2 \le \cdots \le t_n$ for all $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}$, then

$$\begin{aligned}
&(V_{t_1+1}, \ldots, V_{t_n+1}) \overset{(3.12)}{=} \\
=&\Big(e^{U_{t_1+1}} \int_{-\infty}^{t_1+1} e^{-U_u} dL_u, \ldots, e^{U_{t_n+1}} \int_{-\infty}^{t_n+1} e^{-U_u} dL_u\Big) \\
=&\Big(e^{U_{t_1}+U_1} \int_{-\infty}^{t_1+1} e^{-U_u} dL_u, \ldots, e^{U_{t_n}+U_1} \int_{-\infty}^{t_n+1} e^{-U_u} dL_u\Big) \\
\overset{v=u-1}{=}&\Big(e^{U_{t_1}+U_1} \int_{-\infty}^{t_1} e^{-U_{v+1}} dL_{v+1}, \ldots, e^{U_{t_n}+U_1} \int_{-\infty}^{t_n} e^{-U_{v+1}} dL_{v+1}\Big) \\
=&\Big(e^{U_{t_1}+U_1} \int_{-\infty}^{t_1} e^{-U_v-U_1} dL_{v+1}, \ldots, e^{U_{t_n}+U_1} \int_{-\infty}^{t_n} e^{-U_v-U_1} dL_{v+1}\Big) \\
\overset{d}{=}&\Big(e^{U_{t_1}} \int_{-\infty}^{t_1} e^{-U_v} dL_v, \ldots, e^{U_{t_n}} \int_{-\infty}^{t_n} e^{-U_v} dL_v\Big) \\
=&(V_{t_1}, \ldots, V_{t_n}) \,.
\end{aligned}$$

Proof of (ii): *The necessary condition and uniqueness:*

Let $(V_t)_{t\in\mathbb{R}}$ be 1-periodic strictly stationary solution and $\alpha > 0$. Hence $(V_n)_{n\in\mathbb{Z}}$ is an AR(1)-process with i.i.d. noise and coefficient $e^{\alpha} > 1$. We are going to show that $V_t = -e^{U_t} \int_{t+}^{\infty} e^{-U_u} dL_u$.

Since $(V_n)_{n\in\mathbb{Z}}$ is strictly stationary it follows from Theorem 1 in Brockwell and Lindner [10] that $\mathbb{E}\log^+ |Z_0| < \infty$ and that

$$V_n = -\sum_{j=1}^{\infty} (e^{-\alpha})^j Z_{n+j} \,, \tag{3.16}$$

the sum being absolutely convergent almost surely. Hence $\mathbb{E}\log^+|L_1| < \infty$ by Lemma 3.1. To show the equation (3.13), observe that $(V_t)_{t\in\mathbb{R}}$ satisfies for $n \in \mathbb{Z}$

$$\begin{aligned}
V_n &= -\sum_{j=1}^{\infty} e^{-\alpha j} Z_{n+j} \\
&= -\sum_{j=0}^{\infty} e^{-\alpha j} e^{-U_1} \int_{(n+j)+}^{n+j+1} e^{U_{n+j+1}-U_u} dL_u \\
&= -\sum_{j=0}^{\infty} e^{-\alpha j} e^{-\alpha} e^{\alpha(n+j+1)} \int_{(n+j)+}^{n+j+1} e^{-U_u} dL_u \\
&= -e^{\alpha n} \sum_{j=0}^{\infty} \int_{(n+j)+}^{n+j+1} e^{-U_u} dL_u \\
&= -e^{\alpha n} \int_{n+}^{\infty} e^{-U_u} dL_u \,.
\end{aligned}$$

Hence

$$V_n = -e^{U_n} \int_{n+}^{\infty} e^{-U_u} dL_u. \tag{3.17}$$

Let $t \in [n, n+1)$, then from equation (3.4)

$$\begin{aligned}
V_t &= e^{U_t - U_n} V_n + e^{U_t} \int_{n+}^{t} e^{-U_u} dL_u \\
&\overset{(3.13)}{=} e^{U_t-U_n}(-e^{U_n} \int_{n+}^{\infty} e^{-U_u} dL_u) + e^{U_t} \int_{n+}^{t} e^{-U_u} dL_u \\
&= -e^{U_t}(\int_{n+}^{\infty} e^{-U_u} dL_u - \int_{n+}^{t} e^{-U_u} dL_u) \\
&= -e^{U_t} \int_{t+}^{\infty} e^{-U_u} dL_u \,.
\end{aligned}$$

In particular, $V_0 = -\int_{0+}^{\infty} e^{-U_u} dL_u$.

The sufficient condition:
Define V_t by the equation (3.13). We are going to show that V_t is a solution to the equation (3.4).

From the equation (3.13) we have

$$V_t = \underbrace{-e^{U_t}}_{=:A_t} \underbrace{\Big(\int_{t+}^{\infty} e^{-U_u} dL_u\Big)}_{=:B_t} \quad \Rightarrow$$

$$\begin{aligned}
V_t = A_t B_t &\overset{\text{part. integ.}}{=} \\
&= V_s + \int_{s+}^{t} A_{u-} dB_u + \int_{s+}^{t} B_{u-} dA_u + [A,B]_{(s,t]} \\
&= V_s + \int_{s+}^{t} \Big(-e^{U_v}\Big) d\Big(\int_{v}^{\infty} e^{-U_u} dL_u\Big) + \int_{s+}^{t} \Big(\int_{v}^{\infty} e^{-U_u} dL_u\Big) d(-e^{U_v}) + \underbrace{[A,B]_{(s,t]}}_{=0 \text{ since U cont. and of finite var.}} \\
&= V_s + \int_{s+}^{t} \Big(-e^{U_u}\Big) \cdot \Big(-e^{-U_u}\Big) dL_u + \int_{s+}^{t} \underbrace{\Big(\int_{v}^{\infty} e^{-U_u} dL_u\Big)(-e^{U_v})}_{=V_{v-}} dU_v \\
&= V_s + L_t - L_s + \int_{s}^{t} V_{u-} dU_u \,.
\end{aligned}$$

Hence V_t satisfies the equation (3.4).

Now we are going to show that V_t given in (3.13) is 1-periodic strictly stationary.
Let $t_1 \leq t_2 \leq \cdots \leq t_n$ for all $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}$, then

$$\begin{aligned}
&(V_{t_1+1}, \ldots, V_{t_n+1}) \\
\overset{(3.13)}{=}&\Big(-e^{U_{t_1+1}} \int_{(t_1+1)+}^{\infty} e^{-U_u} dL_u, \ldots, -e^{U_{t_n+1}} \int_{(t_n+1)+}^{\infty} e^{-U_u} dL_u\Big) \\
=&\Big(-e^{U_{t_1}+U_1} \int_{(t_1+1)+}^{\infty} e^{-U_u} dL_u, \ldots, -e^{U_{t_n}+U_1} \int_{(t_n+1)+}^{\infty} e^{-U_u} dL_u\Big) \\
\overset{v=u-1}{=}&\Big(-e^{U_{t_1}+U_1} \int_{t_1+}^{\infty} e^{-U_{v+1}} dL_{v+1}, \ldots, -e^{U_{t_n}+U_1} \int_{t_n+}^{\infty} e^{-U_{v+1}} dL_{v+1}\Big) \\
=&\Big(-e^{U_{t_1}+U_1} \int_{t_1+}^{\infty} e^{-U_v-U_1} dL_{v+1}, \ldots, -e^{U_{t_n}+U_1} \int_{t_n+}^{\infty} e^{-U_v-U_1} dL_{v+1}\Big) \\
\overset{d}{=}&\Big(-e^{U_{t_1}} \int_{t_1+}^{\infty} e^{-U_v} dL_v, \ldots, -e^{U_{t_n}} \int_{t_n+}^{\infty} e^{-U_v} dL_v\Big) \\
=&(V_{t_1}, \ldots, V_{t_n}) \,.
\end{aligned}$$

Proof of (iii): Suppose that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary. With the same arguments like in the proof of (i) we can show that

$$V_{n+1} = A_{n,n+1} V_n + Z_{n+1} \tag{3.18}$$

with i.i.d. noise $(Z_n)_{n\in\mathbb{Z}}$.

This is an AR(1) process with coefficient $\phi = 1$. Hence $Z_n = 0$ (cf. Brockwell and Lindner [10], Theorem 1). This is equivalent to $L_1 = 0$ almost surely, hence a contradiction to the assumption that L_t is a non-zero Levy process. Hence there is no choice of V_0 making $(V_t)_{t\in\mathbb{R}}$ 1-periodic strictly stationary. $\square$

3.2 Weakly stationary periodic OU processes

In this section we are going to give the necessary and sufficient condition for the existence of the periodic weakly stationary solution to the one dimensional periodic Ornstein-Uhlenbeck process.

Theorem 3.4. *Let $\lambda : \mathbb{R} \to \mathbb{R}$ be 1-periodic, bounded, non identically zero and measurable and let $U_t := \int_0^t \lambda(s)ds$ and $\alpha := \int_0^1 \lambda(s)ds = U_1$. Let $L = (L_t)_{t\in\mathbb{R}}$ be a non-zero two-sided Levy process. Consider the Ornstein-Uhlenbeck equation (3.3) and the process V_t satisfying (3.3). Then the following are true:*

(i) If $\alpha < 0$, then a random variable V_0 can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic weakly stationary if and only if $\mathbb{E}|L_1|^2 < \infty$.
In this case the solution is unique and given by

$$V_t = e^{U_t} \int_{-\infty}^{t} e^{-U_u} dL_u \,. \tag{3.19}$$

The integral in the equation (3.19) converges almost surely.

(ii) If $\alpha > 0$, then a random variable V_0 can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic weakly stationary, if and only if $\mathbb{E}|L_1|^2 < \infty$.
In this case, the solution is unique and given by

$$V_t = -e^{U_t} \int_{t+}^{\infty} e^{-U_u} dL_u \,. \tag{3.20}$$

The integral in the equation (3.20) converges almost surely.

(iii) If $\alpha = 0$, then there is no choice of V_0 making $(V_t)_{t\in\mathbb{R}}$ 1-periodic weakly stationary.

Proof. The proof of this theorem is similar to the proof of Theorem 3.3, therefore we are going to sketch the proof of the first case and omit the rest.

The necessary condition and uniqueness:
Firstly we are going to show that necessarily $\mathbb{E}|L_1|^2 < \infty$ and that the solution must then have the form $V_t = e^{U_t} \int_{-\infty}^{t} e^{-U_u} dL_u$. So suppose a 1-periodic weakly stationary solution exists.
As above we can write V_n as AR(1)-equation with coefficient $\phi = e^{\alpha} < 1$. Iterating we get

$$V_n = \sum_{j=0}^{m} (e^{\alpha})^j Z_{n-j} + e^{\alpha(m+1)} V_{n-m-1} \,.$$

Since $\mathbb{E}V_n^2$ is constant, we obtain that $e^{\alpha(m+1)}V_{n-m-1}$ converges in L^2 and hence in probability to zero as $m \to \infty$. Hence

$$V_n = \sum_{j=0}^{\infty} (e^{\alpha})^j Z_{n-j} \,, \tag{3.21}$$

where $Z_n = e^{U_n} \int_{n-1}^{n} e^{-U_s} dL_s$. Since an independent sum of two variables has finite variance if and only if both summands have finite variance, this shows $\mathbb{E}|Z_0|^2 < \infty$. This implies $\mathbb{E}|L_1|^2 < \infty$ by Lemma 3.2.
Now let V_t be a 1-periodic weakly stationary solution to (3.4), then V_n is unique and given in (3.21) for $n \in \mathbb{Z}$.
Similar to the proof of Theorem 3.3, it can be shown that

$$V_t = e^{U_t} \int_{-\infty}^{t} e^{-U_u} dL_u \,, \qquad \forall t \in \mathbb{R}.$$

The sufficient condition:
Suppose that $\mathbb{E}|L_1|^2 < \infty$ and define V_t by the equation (3.19).
Like above it can be shown that V_t given by (3.19) is a solution to the equation (3.3).
It is sufficient to show that $\mathbb{E}|V_t|^2 < \infty$ for all $t \in \mathbb{R}$. Then V_t, given by (3.19) is 1-periodic weakly stationary by the same argument as in Theorem 3.3.

We have from (3.19)

$$\begin{aligned} \mathbb{E}|V_t|^2 &= \mathbb{E}\left|e^{U_t} \int_{-\infty}^{t} e^{-U_k} dL_k\right|^2 \\ &= \mathbb{E}\left|e^{U_t} \int_{-\infty}^{t} e^{-U_k} d(L_k - k\mathbb{E}L_1) + e^{U_t} \int_{-\infty}^{t} e^{-U_k} dk \cdot \mathbb{E}L_1\right|^2 \\ &\leq |e^{U_t}|^2 \left(2\int_{-\infty}^{t} |e^{-U_k}|^2 \mathbf{Var}\,(L_1) dk + 2\Big(\int_{-\infty}^{t} e^{-U_k} dk \cdot \mathbb{E}L_1\Big)^2\right) \\ &< +\infty \qquad \forall t \in \mathbb{R}\,, \end{aligned}$$

since $\mathbb{E}|L_1|^2 < \infty$ and $\alpha = U_1 < 0$.

□

Now let us give the autocovariance function of the 1-periodic Ornstein-Uhlenbeck process. We have

Theorem 3.5. *Let $\lambda : \mathbb{R} \to \mathbb{R}$ be 1-periodic, bounded, non identically zero and measurable and let $U_t := \int_0^t \lambda(s)ds$ and $\alpha := \int_0^1 \lambda(s)ds = U_1$. Let $L = (L_t)_{t\in\mathbb{R}}$ be a non-zero two-sided Lévy process with finite variance. Assume that $\alpha \neq 0$ and let $(V_t)_{t\in\mathbb{R}}$ be the unique 1-periodic weakly stationary Ornstein-Uhlenbeck process given by (3.19) and (3.20), respectively. Define*

$$J_t := \int_0^t e^{-2U_u} du \qquad \textit{for } t \in \mathbb{R}.$$

Then if $\alpha < 0$,

$$\boldsymbol{Cov}\,(V_t, V_{t+h}) = \boldsymbol{Var}\,(L_1) e^{U_{t-\lfloor t\rfloor}} e^{U_{t-\lfloor t\rfloor+h}} \left(\frac{1}{e^{-2\alpha} - 1} J_1 + J_{t-\lfloor t\rfloor}\right), \tag{3.22}$$

for $h \geq 0$ and $t \in \mathbb{R}$.
If $\alpha > 0$, then

$$\boldsymbol{Cov}(V_t, V_{t+h}) = \boldsymbol{Var}(L_1) e^{U_{t-\lfloor t \rfloor}} e^{U_{t-\lfloor t \rfloor + h}} e^{-2\alpha(\lfloor t+h \rfloor - \lfloor t \rfloor)} \left(\frac{1}{1-e^{-2\alpha}} J_1 + J_{t+h-\lfloor t+h \rfloor} \right), \tag{3.23}$$

for $h \geq 0$ and $t \in \mathbb{R}$.

Proof. Since $(V_t)_{t\in\mathbb{R}}$ is 1-periodic weakly stationary, for the calculation of $\mathbf{Cov}(V_t, V_{t+h})$ we can and shall assume w.l.o.g. that $t \in [0,1)$.
(i) Let $\alpha < 0$, so that

$$V_t = e^{U_t} \int_{-\infty}^{t} e^{-U_u} dL_u \qquad \textit{and} \qquad V_{t+h} = e^{U_{t+h}} \int_{-\infty}^{t+h} e^{-U_u} dL_u \,.$$

Since

$$\mathbb{E}(V_t) = e^{U_t} \int_{-\infty}^{t} e^{-U_u} du \mathbb{E} L_1$$

and similarly for V_{t+h}, we have

$$\mathbf{Cov}(V_t, V_{t+h}) = \mathbb{E}\Big((V_t - \mathbb{E}(V_t))(V_{t+h} - \mathbb{E}(V_{t+h})) \Big)$$

$$= \mathbb{E}\left(e^{U_t} \int_{-\infty}^{t} e^{-U_u} d(L_u - u\mathbb{E}L_1) \cdot e^{U_{t+h}} \int_{-\infty}^{t+h} e^{-U_u} d(L_u - u\mathbb{E}L_1) \right).$$

By considering $\tilde{L}_u := L_u - u\mathbb{E}L_1$ and noting that $\mathbf{Var}(\tilde{L}_1) = \mathbf{Var}(L_1) = \mathbb{E}\tilde{L}_1^2$, we can assume w.l.o.g. that $\mathbb{E}L_1 = 0$. Then since $h \geq 0$,

$$\mathbf{Cov}(V_t, V_{t+h}) = \mathbb{E}\left(e^{U_t} \int_{-\infty}^{t} e^{-U_u} dL_u \cdot \left(e^{U_{t+h}} \int_{-\infty}^{t} e^{-U_u} dL_u + \underbrace{e^{U_{t+h}} \int_{t}^{t+h} e^{-U_u} dL_u}_{\text{indp. of } V_t\text{, expect } 0} \right) \right)$$

$$= \mathbb{E}\left(e^{U_t} \int_{-\infty}^{t} e^{-U_u} dL_u \cdot e^{U_{t+h}} \int_{-\infty}^{t} e^{-U_u} dL_u \right)$$

$$= e^{U_t} e^{U_{t+h}} \int_{-\infty}^{t} e^{-2U_u} du \mathbf{Var}(L_1),$$

where we used the Ito-isometry.
It remains to calculate $\int_{-\infty}^{t} e^{-2U_u} du$. Observe that $U_1 = \alpha$ and that $U_{s+k} = U_s + kU_1 = U_s + k\alpha$ for all $s \in \mathbb{R}$ and $k \in \mathbb{Z}$.
Hence

$$\int_{-\infty}^{t} e^{-2U_u} du = \sum_{k=0}^{\infty} \int_{-k-1}^{-k} e^{-2U_u} du + \int_{0}^{t} e^{-2U_u} du$$

$$\overset{v=u+k+1}{=} \sum_{k=0}^{\infty} \int_{0}^{1} e^{-2U_{v-k-1}} dv + \int_{0}^{t} e^{-2U_u} du$$

$$= \sum_{k=0}^{\infty} \int_{0}^{1} e^{-2U_v} dv \cdot e^{2\alpha(k+1)} + \int_{0}^{t} e^{-2U_u} du$$

$$e^{2\alpha} \cdot \frac{1}{1-e^{2\alpha}} \int_0^1 e^{-2U_v} dv + \int_0^t e^{-2U_u} du$$
$$e^{2\alpha} \cdot \frac{1}{1-e^{2\alpha}} J_1 + J_t \,.$$

Altogether we obtain

$$\mathbf{Cov}\,(V_t, V_{t+h}) = \mathbf{Var}\,(L_1) e^{U_t} e^{U_{t+h}} \bigg(e^{2\alpha} \cdot \frac{1}{1-e^{2\alpha}} J_1 + J_t \bigg) \,,$$

for $t \in [0,1)$ and $h \geq 0$, from which the result follows.

(ii) Let $\alpha > 0$, so that

$$V_t = -e^{U_t} \int_{t+}^{\infty} e^{-U_u} dL_u \qquad \textit{and} \qquad V_{t+h} = -e^{U_{t+h}} \int_{(t+h)+}^{\infty} e^{-U_u} dL_u \,.$$

As in (i) we can assume w.l.o.g. that $\mathbb{E}L_1 = 0$, and by using the independence of $\int_t^{t+h} e^{-U_u} dL_u$ and $\int_{(t+h)+}^{\infty} e^{-U_u} dL_u$ and the Ito-isometry, we obtain (for $t \in [0,1)$)

$$\begin{aligned}
\mathbf{Cov}\,(V_t, V_{t+h}) &= \mathbb{E}\bigg(e^{U_t} \int_{t+}^{\infty} e^{-U_u} dL_u \cdot e^{U_{t+h}} \int_{(t+h)+}^{\infty} e^{-U_u} dL_u \bigg) \\
&= e^{U_t} e^{U_{t+h}} \mathbb{E}\bigg(\Big(\int_{t+}^{t+h} e^{-U_u} dL_u + \int_{(t+h)+}^{\infty} e^{-U_u} dL_u \Big) \int_{(t+h)+}^{\infty} e^{-U_u} dL_u \bigg) \\
&= e^{U_t} e^{U_{t+h}} \mathbb{E}\bigg(\int_{(t+h)+}^{\infty} e^{-U_u} dL_u \int_{(t+h)+}^{\infty} e^{-U_u} dL_u \bigg) \\
&= e^{U_t} e^{U_{t+h}} \mathbf{Var}\,(L_1) \int_{(t+h)+}^{\infty} e^{-2U_u} du \,.
\end{aligned}$$

For the calculation of $\int_{(t+h)+}^{\infty} e^{-2U_u} du$, we have

$$\begin{aligned}
\int_{(t+h)+}^{\infty} e^{-2U_u} du &= -\int_{\lfloor t+h \rfloor}^{t+h} e^{-2U_u} du + \int_{\lfloor t+h \rfloor}^{\infty} e^{-2U_u} du \\
&= -\int_0^{t+h-\lfloor t+h \rfloor} e^{-2U_{v+\lfloor t+h \rfloor}} dv + \sum_{k=0}^{\infty} \int_{\lfloor t+h \rfloor + k}^{\lfloor t+h \rfloor + k+1} e^{-2U_u} du \\
&\overset{v=u-k-\lfloor t+h \rfloor}{=} -e^{-2\alpha \lfloor t+h \rfloor} J_{t+h-\lfloor t+h \rfloor} + \sum_{k=0}^{\infty} \int_0^1 e^{-2U_{v+k+\lfloor t+h \rfloor}} dv \\
&= -e^{-2\alpha \lfloor t+h \rfloor} J_{t+h-\lfloor t+h \rfloor} + \sum_{k=0}^{\infty} e^{-2\alpha \lfloor t+h \rfloor} e^{-2\alpha k} \int_0^1 e^{-2U_v} dv \\
&= e^{-2\alpha \lfloor t+h \rfloor} \Big(\frac{1}{1-e^{-2\alpha}} J_1 - J_{t+h-\lfloor t+h \rfloor} \Big) \,.
\end{aligned}$$

This gives the claim.

□

Remark 3.6. *Observe that* $\boldsymbol{Cov}(V_t, V_{t-h})$ *is in general different from* $\boldsymbol{Cov}(V_t, V_{t+h})$, *unless* $h \in \mathbb{Z}$. *However,* $\boldsymbol{Cov}(V_t, V_{t-h})$ *for* $h \geq 0$ *and* $t \in \mathbb{R}$ *can be calculated from the results of Theorem 3.5 by observing*

$$\boldsymbol{Cov}(V_t, V_{t-h}) = \boldsymbol{Cov}(V_{t-h}, V_t) = \boldsymbol{Cov}(V_{t-h}, V_{(t-h)+h}),$$

hence by replacing t *by* $t-h$ *in Theorem 3.5 we get the claim.*

3.3 Sampling POU Processes

We have seen at the begin of this chapter that the periodic OU process (3.2) has a solution of the form

$$V_t = e^{U_t - U_s} V_s + \int_s^t e^{U_t - U_u} dL_u\,,\ s \leq t \in \mathbb{R}$$

equivalently

$$V_t = \exp\left(\int_s^t \lambda(k)dk\right) V_s + \int_s^t \exp\left(\int_u^t \lambda(k)dk\right) dL_u\,, \tag{3.24}$$

where $\lambda(\cdot)$ is a periodic function with period d.

If we sample the process V_t at equidistant times, then we can write by (3.24)

$$V_{nh} = \exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right) V_{(n-1)h} + \int_{(n-1)h}^{nh} \exp\left(\int_u^{nh} \lambda(k)dk\right) dL_u\,, \tag{3.25}$$

where $h > 0$ and $n \in \mathbb{Z}$.

3.3.1 V_t has a period equals one

Let us consider first that the periodic d of the process V_t is one and discuss the following cases:
Case I: Consider that $h = \frac{1}{m}$ for $m \in \mathbb{N}$ then it is easy to see that $\exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right)$ is periodic with period m, since

$$\begin{aligned}\exp\left(\int_{(n-1+m)h}^{(n+m)h} \lambda(k)dk\right) &= \exp\left(\int_{(n-1)h+1}^{nh+1} \lambda(k)dk\right) \\ = \exp\left(\int_{(n-1)h}^{nh} \lambda(k+1)dk\right) &= \exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right)\end{aligned}$$

and $\int_{(n-1)h}^{nh} \exp\left(\int_u^{nh} \lambda(k)dk\right) dL_u$ is an independent m-periodic stationary noise, since the kernel is periodic with period m. Hence

$$V_{nh} = \exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right) V_{(n-1)h} + \int_{(n-1)h}^{nh} \exp\left(\int_u^{nh} \lambda(k)dk\right) dL_u\,, \qquad n \in \mathbb{Z}$$

is an m-periodic AR(1) process with independent m-periodic stationary noise for all $h = \frac{1}{m}$ with $m \in \mathbb{N}$.

Case II: Consider that $h = m$ for $m \in \mathbb{N}$, then we can see that $\exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right)$ is a constant independent of n (periodic with period one) and $\int_{(n-1)h}^{nh} \exp\left(\int_u^{nh} \lambda(k)dk\right)dL_u$ is an i.i.d. noise. Hence

$$V_{nh} = \exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right) V_{(n-1)h} + \int_{(n-1)h}^{nh} \exp\left(\int_u^{nh} \lambda(k)dk\right) dL_u \,,$$

is an AR(1) process with i.i.d. noise for all $n \in \mathbb{Z}$ and $h = m$ with $m \in \mathbb{N}$.

Case III: Consider that $h = \frac{s}{m}$ for $s, m \in \mathbb{N}$ distinct, then we can see that $\exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right)$ is a periodic with period m, since

$$\begin{aligned}
&\exp\left(\int_{(n-1+m)h}^{(n+m)h} \lambda(k)dk\right) = \exp\left(\int_{(n-1+m)\frac{s}{m}}^{(n+m)\frac{s}{m}} \lambda(k)dk\right) = \\
&\exp\left(\int_{(n-1)h+s}^{nh+s} \lambda(k)dk\right) = \exp\left(\int_{(n-1)h}^{nh} \lambda(k+s)dk\right) = \\
&\exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right)
\end{aligned}$$

and $\int_{(n-1)\frac{s}{m}}^{n\frac{s}{m}} \exp\left(\int_u^{n\frac{s}{m}} \lambda(k)dk\right)dL_u$ is an independent m-stationary noise, since the kernel is periodic with period m. Hence

$$V_{nh} = \exp\left(\int_{(n-1)h}^{nh} \lambda(k)dk\right) V_{(n-1)h} + \int_{(n-1)h}^{nh} \exp\left(\int_u^{nh} \lambda(k)dk\right) dL_u \,,$$

is an m-periodic AR(1) process with independent m-periodic stationary noise for all $n \in \mathbb{Z}$ and $h = \frac{s}{m}$ with $s, m \in \mathbb{N}$.

3.3.2 V_t has a period $d \in \mathbb{N}$

Let us consider that V_t has period equal to $d > 0$ and let us define $h := d \cdot \frac{s}{m}$. Let us sample the process V_t at the times hn, i.e.

$$V_{nd\cdot\frac{s}{m}} = \exp\left(\int_{(n-1)d\cdot\frac{s}{m}}^{nd\cdot\frac{s}{m}} \lambda(k)dk\right)V_{(n-1)d\cdot\frac{s}{m}} + \int_{(n-1)d\cdot\frac{s}{m}}^{nd\cdot\frac{s}{m}} \exp\left(\int_u^{nd\cdot\frac{s}{m}} \lambda(k)dk\right)dL_u \,.$$

Note that $\exp\left(\int_{(n-1)d\cdot\frac{s}{m}}^{nd\cdot\frac{s}{m}} \lambda(k)dk\right)$ is periodic with period m, since

$$\exp\left(\int_{(n-1+m)d\cdot\frac{s}{m}}^{(n+m)d\cdot\frac{s}{m}} \lambda(k)dk\right) = \exp\left(\int_{(n-1)d\cdot\frac{s}{m}+sd}^{nd\cdot\frac{s}{m}+sd} \lambda(k)dk\right)$$

$$= \exp\left(\int_{(n-1)d\cdot\frac{s}{m}}^{nd\cdot\frac{s}{m}} \lambda(k+sd)dk\right) = \exp\left(\int_{(n-1)d\cdot\frac{s}{m}}^{nd\cdot\frac{s}{m}} \lambda(k)dk\right)$$

and $\int_{(n-1)d\cdot\frac{s}{m}}^{nd\cdot\frac{s}{m}} \exp\left(\int_u^{nd\cdot\frac{s}{m}} \lambda(k)dk\right)dL_u$ is an independent m-periodic stationary noise. Hence $(V_{nd\cdot\frac{s}{m}})_{n\in\mathbb{Z}}$ is an m-periodic AR(1) process with independent m-periodic stationary noise.

CHAPTER 4

Stationary Multivariate Periodic Ornstein Uhlenbeck Processes

In this chapter we are going to determine the necessary and sufficient conditions for the existence and uniqueness of the periodic stationary solution to the multivariate periodic Ornstein-Uhlenbeck processes.

Before we start with the stationary multivariate periodic Ornstein-Uhlenbeck process, we need some facts about solutions of homogeneous linear differential equations when the matrix function is periodic.

4.1 Floquet Theory and the matrix exponential

Let $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ be a continuous function. Then it is well-known (e.g. Forster, [16], §12) that for each $b^i \in \mathbb{R}^d$ there exists a unique solution $y^i : \mathbb{R} \to \mathbb{R}^d$ (which is continuously differentiable) such that

$$\frac{d}{dt} y^i_t = A(t) y^i_t \,, \qquad t \in \mathbb{R} \,,$$

with $y^i_0 = b^i$. For $b^1, b^2, \ldots, b^d \in \mathbb{R}^d$, the $\mathbb{R}^{d\times d}$-valued function $(y_t)_{t\in\mathbb{R}}$ with $y_t = (y^1_t, \ldots, y^d_t)$ is the unique solution of the differential system

$$\frac{d}{dt} y_t = A(t) y_t \,, \qquad t \in \mathbb{R} \,,$$

with $y_0 = (b^1, \ldots, b^d) \in \mathbb{R}^{d\times d}$. Using the theory of Wronskian determinants, it is well-known that y_t is invertible for fixed $t \in \mathbb{R}$ if and only if $y_0 = (b^1, \ldots, b^d)$ is invertible. Denoted by $(X_t)_{t\in\mathbb{R}}$, $X_t : \mathbb{R} \to \mathbb{R}^{d\times d}$ the unique solution of the differential equation

$$\frac{d}{dt} X_t = A(t) X_t \;\; ; \qquad with \qquad X_0 = Id_d \,, \tag{4.1}$$

i.e specialised to the initial condition $X_0 = Id_d$. Then X_t is invertible for all $t \in \mathbb{R}$ by the previous observation.

Define $U_t := \int_0^t A(s)ds$. Then (4.1) can be rewritten as

$$dX_t = A(t) X_t dt \,, \qquad X_0 = Id_d \,,$$

or

$$dX_t = dU_t X_t \,, \qquad X_0 = Id_d \,.$$

Hence the solution $(X_t)_{t\in\mathbb{R}}$ is also called the *(right) matrix exponential* of U_t and we also write

$$\mathcal{E}(U)_t = X_t\,.$$

Now assume that $A:\mathbb{R}\to\mathbb{R}^{d\times d}$ is additionally periodic, and for simplicity assume that it has period 1. Let $n\in\mathbb{Z}$. Then

$$\frac{d}{dt}(X_tX_n^{-1}) = (\frac{d}{dt}X_t)X_n^{-1} = A(t)X_tX_n^{-1}\,,$$

hence also

$$\frac{d}{dt}(X_{t+n}X_n^{-1}) = A(t+n)X_{t+n}X_n^{-1} = A(t)X_{t+n}X_n^{-1}\,,$$

and since the starting value of $t\mapsto X_{t+n}X_n^{-1}$ is Id_d, we obtain by the uniqueness of the solution to (4.1)

$$X_{t+n}X_n^{-1} = X_t\,,$$

i.e.

$$X_{t+n} = X_t\cdot X_n\,. \tag{4.2}$$

In particular, $X_n = X_1^n$ for all $n\in\mathbb{Z}$ and if $t\in[n,n+1)$ for some $n\in\mathbb{Z}$, this implies

$$X_t = X_{t-n}\cdot X_1^n\,,\qquad t\in(n,n+1)\,.$$

It is clear from this that for the long time behavior of X_t, the eigenvalues of $X_1=\mathcal{E}(U)_1$ will be of special importance. The eigenvalues $\rho_1,\dots,\rho_d$ of X_1 counted with multiplicity, are called *characteristic multipliers* of (4.1). Any set $\mu_1,\dots,\mu_d$ of complex numbers such that

$$\rho_1 = e^{\mu_1},\dots,\rho_n = e^{\mu_n}\,,$$

are called *Floquet exponents* of (4.1). Let $B\in\mathbb{R}^{d\times d}$ be a matrix such that $\exp(B)=X_1$ (this exists since X_1 is invertible.) Then the eigenvalues of B are Floquet exponents of (4.1).

Define

$$P(t) := X_te^{-Bt}\,,\qquad t\in\mathbb{R}\,.$$

Then P is continuously differentiable and

$$P(t+1) = X_{t+1}e^{-B(t+1)} \overset{(4.2)}{=} X_t\underbrace{X_1e^{-B}}_{=Id_d}e^{-Bt} = P(t)\,,$$

so that P is 1-periodic. This is the Floquet's Theorem 2.1 in [43] which we state again for convenience:

Theorem 4.1. *Let $A:\mathbb{R}\to\mathbb{R}^{d\times d}$ be continuous and 1-periodic and $(X_t)_{t\in\mathbb{R}}$ be the unique solution of (4.1), i.e. the matrix exponential of $U_t=\int_0^t A(s)ds$, $t\in\mathbb{R}$. Then there exists a continuously differentiable 1-periodic function $P:\mathbb{R}\to\mathbb{R}^{d\times d}$ and a matrix $B\in\mathbb{R}^{d\times d}$ such that*

$$X_t = P(t)e^{Bt}\,,\qquad t\in\mathbb{R}\,.$$

The calculation of the explicit solution X_t is often difficult. In some special cases it is however possible to give the Floquet exponents:

Theorem 4.2. *Given the homogeneous linear periodic system (4.1) as above, with the coefficient matrix $A(t)$, the following are true:*

(i) If $A(t) \cdot A(s) = A(s) \cdot A(t)$ for all s, t, then $X_1 = \exp(\int_0^1 A(t)dt)$, and the Floquet exponents can be selected as the eigenvalues of the matrix $B = \int_0^1 A(t)dt$.

(ii) If $A(t)$ is a periodic lower triangular matrix function

$$\begin{pmatrix} a_{11}(t) & 0 & \dots & 0 \\ a_{21}(t) & a_{22}(t) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{d1}(t) & a_{d2}(t) & \dots & a_{dd}(t) \end{pmatrix},$$

then the eigenvalues of X_1 are given by

$$\exp(\int_0^1 a_{11}(t)dt), \exp(\int_0^1 a_{22}(t)dt) \dots, \exp(\int_0^1 a_{dd}(t)dt),$$

and a set of its Floquet exponents are given by

$$\int_0^1 a_{11}(t)dt, \int_0^1 a_{22}(t)dt \dots, \int_0^1 a_{dd}(t)dt.$$

([43], Theorem 2.5).

4.2 Strictly stationary multivariate periodic OU processes

Let $L = (L_1, \dots, L_d)^T : \mathbb{R} \to \mathbb{R}^d$ be a Lévy process, and let $A : \mathbb{R} \to \mathbb{R}^{d \times d}$ be a periodic (with period one), continuous, non-zero function. The multivariate periodic Ornstein-Uhlenbeck (MPOU) process $V = (V_t^1, \dots, V_t^d)^T : \mathbb{R} \to \mathbb{R}^d$ is the solution of the stochastic differential equation

$$dV_t = A(t)V_t dt + dL_t, \qquad t \in \mathbb{R}, \tag{4.3}$$

or in the integral equation

$$V_t = V_0 + \int_0^t A(s)V_s ds + L_t, \qquad t \in \mathbb{R}. \tag{4.4}$$

(cf. Definition 1.30).

We set $A(t)dt =: dU_t$, hence $U_t = \int_0^t A(s)ds$, with $U \in \mathbb{R}^{d \times d}$. Then the equation (4.3) becomes

$$dV_t = dU_t V_t + dL_t, \qquad t \in \mathbb{R}, \tag{4.5}$$

and the integral form of (4.5) is

$$V_t = V_0 + \int_0^t dU_k V_k + L_t, \qquad t \in \mathbb{R}. \tag{4.6}$$

Recall the matrix exponential $\mathcal{E}(U)_t = X_t$ of Section 4.1, with X_t the solution of (4.1). By the theory of stochastic differential equations, (4.6) has a unique solution (e.g. Protter [33], Theorem V.7). This solution is given by

$$V_t = \mathcal{E}(U)_t \left(V_0 + \int_{0+}^{t} \mathcal{E}(U)_k^{-1} dL_k \right) = X_t \left(V_0 + \int_{0+}^{t} X_k^{-1} dL_k \right); \qquad t \in \mathbb{R}\,, \tag{4.7}$$

as we now show. For that we set $B_t := V_0 + \int_0^t \mathcal{E}(U)_k^{-1} dL_k$, then we have from (4.7)

$$\begin{aligned}
V_t = \mathcal{E}(U)_t B_t &\overset{Multi.\ par.\ Integration}{=} V_0 + \int_{0+}^{t} \mathcal{E}(U)_k dB_k + \int_{0+}^{t} d\mathcal{E}(U)_k B_{k-} + \underbrace{[\mathcal{E}(U), B]_{0+}^{t}}_{=0 \text{ since } \mathcal{E}(U) \text{ continous}} \\
&= V_0 + \int_{0+}^{t} \mathcal{E}(U)_k d(V_0 + \int_{0+}^{k} \mathcal{E}(U)_u^{-1} dL_u) + \int_{0+}^{t} dU_k \mathcal{E}(U)_k (V_0 + \int_{0+}^{k-} \mathcal{E}(U)_u^{-1} dL_u) \\
&= V_0 + \int_{0+}^{t} \mathcal{E}(U)_k \mathcal{E}(U)_k^{-1} dL_k + \int_{0+}^{t} A(k) dk \underbrace{\mathcal{E}(U)_k \left(V_0 + \int_{0+}^{k-} \mathcal{E}(U)_u^{-1} dL_u \right)}_{=V_{k-}} \\
&= V_0 + L_t + \int_0^t A(k) V_{k-} dk = V_0 + L_t + \int_0^t A(k) V_k dk = V_0 + \int_0^t dU_k V_k + L_t\,,
\end{aligned}$$

hence (4.6) holds.

The used multivariate integration rules can be found e.g. in Karandikar [25].

4.2.1 Necessary and sufficient conditions for the existence of periodic strictly stationary solution

It is well-known, that the Ornstein-Uhlenbeck process sampled at equidistant times is a vector AR(1) process with i.i.d. noise. The same fact holds for periodic multivariate Ornstein-Uhlembeck processes. More precisely, we have from the equation (4.7)

$$V_t = X_t V_0 + X_t \int_0^t X_k^{-1} dL_k\ .$$

Hence we obtain for $n \in \mathbb{Z}$

$$\begin{aligned}
V_n &= X_n (V_0 + \int_0^n X_s^{-1} dL_s) \\
&= X_n X_{n-1}^{-1} X_{n-1} (V_0 + \int_0^{n-1} X_s^{-1} dL_s + \int_{n-1}^{n} X_s^{-1} dL_s) \\
&= X_n X_{n-1}^{-1} \underbrace{X_{n-1} (V_0 + \int_0^{n-1} X_s^{-1} dL_s)}_{=V_{n-1}} + X_n X_{n-1}^{-1} X_{n-1} \int_{n-1}^{n} X_s^{-1} dL_s \\
&= X_1^n X_1^{-(n-1)} V_{n-1} + \underbrace{X_n \int_{n-1}^{n} X_s^{-1} dL_s}_{=:Z_n}\,,
\end{aligned}$$

hence

$$V_n = X_1 V_{n-1} + Z_n\,, \qquad n \in \mathbb{Z} \tag{4.8}$$

where

$$Z_n = X_n \int_{n-1}^{n} X_s^{-1} dL_s \,, \tag{4.9}$$

is i.i.d. noise, by (4.2) and since L is Lévy process.

Note that if V_t given in (4.7) is 1-periodic strictly stationary, then V_n given in (4.8) is strictly stationary.

Recall that for a given eigenvalue $\lambda \in \mathbb{C}$ of a squared matrix $A \in \mathbb{C}^{d\times d}$, a vector $v \in \mathbb{C}^d$ is called a *generalised eigenvector* of A (to λ) if there exists $p \in \mathbb{N}$ such that $(A - \lambda Id_d)^p v = 0 \neq (A-\lambda Id_d)^{p-1} v$. The unique number p is called the *rank* of v, and generalised eigenvectors of rank 1 are the eigenvectors. Further, if v is a generalised eigenvector of rank $p \geq 2$ to λ, then $(A - \lambda Id_d)v$ is a generalised eigenvector to λ of rank $p-1$.

We will need the following lemma to prove the next Theorem:

Lemma 4.3. *Let $(Z_n)_{n\in\mathbb{Z}}$ be an $\mathbb{R}^d$-valued i.i.d. noise and let $C \in \mathbb{R}^{d\times d}$. Consider the vector AR(1) equation*

$$V_n = CV_{n-1} + Z_n \,, \qquad n \in \mathbb{Z} \,, \tag{4.10}$$

and suppose that a strictly or weakly stationary solution $(V_n)_{n\in\mathbb{Z}}$ of (4.10) exists. Then the following are true:

(i) For every generalised eigenvector ν of C^T to an eigenvalue λ of C^T with $|\lambda| = 1$ the random variable $\nu^T Z_0$ is almost surely constant.

(ii) If $\lambda = 1$ is an eigenvalue of C, then for every eigenvector ν of C^T to the eigenvalue λ the random variable $\nu^T Z_0$ is almost surely 0.

(iii) If a weakly stationary solution exists, then necessarily $\mathbb{E}|Z_0|^2 < \infty$.

The proof of this theorem can be found in the book of Brockwell and Lindner [14], currently Lemma 11.12 there.

We also need another lemma from Brockwell and Lindner [14], which is analogous to Lemma 3.1 and 3.2. Namely

Lemma 4.4. *Let L be a Lévy process in $\mathbb{R}^d$ and $f : [0,1] \to \mathbb{R}^{d\times d}$ be measurable, bounded and such that $f(t)$ is invertible for every $t \in [0,1]$. Let $p \in (0,\infty)$. Then*

$$\mathbb{E} \log^+ |L_1| < \infty \quad \Leftrightarrow \mathbb{E} \log^+ | \int_0^1 f(t) dL_t| < \infty \,,$$

and

$$\mathbb{E}|L_1|^p < \infty \quad \Leftrightarrow \mathbb{E}| \int_0^1 f(t) dL_t|^p < \infty \,,$$

(Brockwell and Lindner [14], Corollary 10.8).

In the following theorem we are going to give the necessary and sufficient conditions for the existence and uniqueness of a periodic strictly stationary solution to the MPOU process in the case that all the eigenvalues of X_1 lie in special regions.

Theorem 4.5. *Let $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ be a 1-periodic, continuous, non identically zero function and let $(X_t)_{t\in\mathbb{R}}$ as above. Let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued, two-sided Lévy process. Consider the multivariate Ornstein-Uhlenbeck equation (4.5) and its solution given by (4.7). Then the following are true*

(*i*) *If all eigenvalues of X_1 have absolute value in (0,1), then a random vector V_0 can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary if and only if $\mathbb{E}\log^+|L_1| < \infty$.*
In this case the solution is unique and given by

$$V_t = X_t \int_{-\infty}^{t} X_k^{-1} dL_k \,. \tag{4.11}$$

The integral in the equation (4.11) converges almost surely absolutely, since all eigenvalues of X_1 have absolute value in (0,1) and $\mathbb{E}\log^+|L_1| < \infty$.

(*ii*) *If all eigenvalues of X_1 have absolute value greater than one, then a random vector V_0 can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary if and only if $\mathbb{E}\log^+|L_1| < \infty$.*
In this case the solution is unique and given by

$$V_t = -X_t \int_{t}^{\infty} X_k^{-1} dL_k \,. \tag{4.12}$$

The integral in the equation (4.12) converges almost surely absolutely, since all eigenvalues of X_1 have absolute value grater than one and $\mathbb{E}\log^+|L_1| < \infty$.

(*iii*) *Assume all eigenvalues of X_1 have absolute value one and that L_t is symmetric. Then a random vector V_0 can be chosen such that V_t is a 1-periodic strictly stationary solution of (4.5) if and only if $L_t = 0$ almost surely. In this case, $V_t = 0$ defines a 1-periodic strictly stationary solution.*

(*iv*) *Assume all eigenvalues of X_1 have absolute value one. Then a random vector V_0 can be chosen such that V_t is 1-periodic strictly stationary if and only if L is deterministic and $X_1 \int_0^1 X_s^{-1} dL_s$ is equal to a constant α, and $\nu^T\alpha = 0$ for every eigenvector ν of X_1^T to the eigenvalue 1. In this case, a vector $f \in \mathbb{R}^d$ exists such that $(Id - X_1)f = \alpha$, and for every such vector f a 1-periodic strictly stationary solution is given by*

$$V_t = X_{t-\lfloor t\rfloor}\left(f + \int_0^{t-\lfloor t\rfloor} X_s^{-1} dL_s\right), \qquad t \in \mathbb{R}. \tag{4.13}$$

Proof. Recall from (4.8) and (4.9) that if $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary, then $(V_n)_{n\in\mathbb{Z}}$ is strictly stationary solution of the AR(1) process (4.8) with i.i.d. noise $(Z_n)_{n\in\mathbb{Z}}$ given by (4.9).

(i) All eigenvalues of X_1 have absolute value in (0,1):
To see the necessity of the conditions and the specific form, assume that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary. Hence $(V_n)_{n\in\mathbb{Z}}$ satisfies a vector AR(1) equation with all eigenvalues of X_1 having absolute values in $(0,1)$. Hence $\mathbb{E}\log^+|Z_0| < \infty$ by Corollary 1 in Brockwell et al. [13]. Theorem 1 in [13] shows that

$$V_n = \sum_{j=0}^{\infty} X_1^j Z_{n-j}\,,$$

with the sum converging almost surely absolutely. Hence, from (4.8) and (4.9) we have

$$\begin{aligned}
V_n &= \sum_{k=0}^{\infty} X_1^k Z_{n-k} = \sum_{k=0}^{\infty} X_1^k X_{n-k} \int_{n-k-1}^{n-k} X_s^{-1} dL_s \\
&= \sum_{k=0}^{\infty} X_n \int_{n-k-1}^{n-k} X_s^{-1} dL_s = \\
&= X_n \int_{-\infty}^{n} X_s^{-1} dL_s\,.
\end{aligned}$$

Let $t \in \mathbb{R}$ then from (4.7) and the last equality follows

$$\begin{aligned}
V_t &= X_t(V_0 + \int_0^t X_s^{-1} dL_s) = \\
&= X_t(X_0 \int_{-\infty}^{0} X_s^{-1} dL_s + \int_0^t X_s^{-1} dL_s) = \\
&= X_t(\int_{-\infty}^{t} X_s^{-1} dL_s)\,.
\end{aligned}$$

Hence the solution to (4.7) has the form given by (4.11).
To see that $\mathbb{E}\log^+|L_1| < \infty$, observe that $Z_1 = \int_0^1 X_1 X_s^{-1} dL_s$, and that $\mathbb{E}\log^+|Z_1| < \infty$. Since $X_1 X_s^{-1}$ is invertible, this implies $\mathbb{E}\log^+|L_1| < \infty$ by Lemma 4.4.

The sufficient condition:
Suppose that $\mathbb{E}\log^+|L_1| < \infty$ and define V_t by the Equation (4.11), which converges almost surely. We are going to show that V_t given in (4.11) is a solution to the equation (4.6).
From (4.11) we set $B_t := \int_{-\infty}^t X_s^{-1} dL_s$, then (By using the multivariate partition integration for $t \geq 0$)

$$V_t = X_t B_t = V_0 + \int_0^t X_s dB_s + \int_0^t dX_s B_s + [X,B]_{(0,t]}$$

$$\begin{aligned}
&= V_0 + \int_0^t X_s d(\int_{-\infty}^{s} X_k^{-1} dL_k) + \int_0^t dU_s X_s(\int_{-\infty}^{s} X_k^{-1} dL_k) \\
&= V_0 + \int_0^t X_s X_s^{-1} dL_s + \int_0^t A(s) ds X_s \int_{-\infty}^{s} X_k^{-1} dL_k \\
&= V_0 + L_t + \int_0^t A(s) V_s ds = V_0 + \int_0^t dU_s V_s + L_t\,.
\end{aligned}$$

Hence V_t given by (4.11) satisfies (4.6) for $t \geq 0$. That it satisfies (4.6) also for general $t \in \mathbb{R}$ can be seen by showing similarly that

$$V_t - V_s = \int_{s+}^{t} dU_k V_k + L_t - L_s\,, \quad \text{for } s \leq t \in \mathbb{R}\,.$$

Now we are going to show that V_t given by (4.11) is 1-periodic strictly stationary. Let $t_1 \leq t_2 \leq \cdots \leq t_n$ for all $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}$, then

$$\begin{aligned}
(V_{t_1+1}, \ldots, V_{t_n+1}) &\overset{(4.11)}{=} (X_{t_1+1} \int_{-\infty}^{t_1+1} X_k^{-1} dL_k, \ldots, X_{t_n+1} \int_{-\infty}^{t_n+1} X_k^{-1} dL_k) \\
&= (X_{t_1} X_1 \int_{-\infty}^{t_1} X_1^{-1} X_k^{-1} dL_{k+1}, \ldots, X_{t_n} X_1 \int_{-\infty}^{t_n} X_1^{-1} X_k^{-1} dL_{k+1}) \\
&\overset{d}{=} (X_{t_1} \int_{-\infty}^{t_1} X_k^{-1} dL_k \ldots, X_{t_n} \int_{-\infty}^{t_n} X_k^{-1} dL_k) \\
&= (V_{t_1}, \ldots, V_{t_n})\,.
\end{aligned}$$

Hence V_t is 1-periodic strictly stationary solution to (4.6).

(ii) All eigenvalues of X_1 have absolute value grater than one:
Since all eigenvalues of X_1 have absolute value greater than one, all eigenvalues of X_1^{-1} have absolute value in $(0, 1)$. The proof then follows in complete analogous to (i) above.

(iii) All eigenvalues of X_1 have absolute value one and L_1 is symmetric:
First, we are going to show the necessity of the conditions. Assume that V_t is a 1-periodic strictly stationary solution to (4.5), then V_n is a strictly stationary solution to (4.8). Since L_1 is symmetric, hence $\int_0^1 X_s^{-1} dL_s$ is symmetric, hence Z_0 is symmetric. Choose a basis $\nu_1, \ldots, \nu_d$ of generalized eigenvectors of X_1^T. By Lemma 4.3, $\nu_i^T Z_0$ is constant almost surely for all $i = 1, \ldots, d$. Since $\nu_1, \ldots, \nu_d$ is basis of $\mathbb{R}^d$ also $w^T Z_0$ is constant almost surely for all $w \in \mathbb{R}^d$. In particular, taking for w the i'th unit vector in $\mathbb{R}^d$, the i'th component of Z_0 is constant almost surely. Since Z_0 is symmetric, Z_0 must be zero, hence $\int_0^1 X_s^{-1} dL_s = 0$, hence $L_t = 0$ almost surely.
To see sufficiency of the conditions, observe that $L_t = 0$, then $\int_0^1 X_s^{-1} dL_s = 0$ i.e. $Z_0 = 0$ then $Z_n = 0$ i.e. $V_t = 0$ defines a 1-periodic strictly stationary solution.

(iv) All eigenvalues of X_1 have absolute value one:
The necessity of the conditions: Assume that V_t is a 1-periodic strictly stationary solution to (4.5), then we have $V_n = X_1 V_{n-1} + Z_s$ by (4.8), hence $Z_1 = X_1 \int_0^1 X_s^{-1} dL_s$ is a constant $\alpha \in \mathbb{R}^d$ by the same reasoning as under (iii). By Lemma 4.3, $wZ_1 = 0$ for every eigenvector w of X_1^T to the eigenvalue one.

Since $Z_1 = \alpha$ is constant, also $\int_0^1 X_s^{-1} dL_s =: w$ is constant. Denote the characteristic triplet of L by $(\Sigma_L, \nu_L, \gamma_L)$. Then w has characteristic triplet $(\Sigma_w, \nu_w, \gamma_w)$ with

$$\Sigma_w = \int_0^1 X_s^{-1} \Sigma_L (X_s^{-1})^T ds\,,$$

and
$$\nu_w(C) = \int_0^1 \int_{\mathbb{R}^d} \mathbf{1}_{C\setminus\{0\}}(X_t^{-1}x)\nu_L(dx)dt\,, \qquad C \in \mathcal{B}_d,$$
see Brockwell and Lindner [14], Theorem 10.7; alternatively, this is an easy multivariate extension of Proposition 57.10 in Sato [36].

Since $\Sigma_w = 0$ and $\nu_w = 0$, this gives $\Sigma_L = 0$ and $\nu_L = 0$, hence L is constant almost surely.

To see the *sufficiency of the conditions*, put $\alpha := X_1 \int_0^1 X_s^{-1} dL_s \in \mathbb{R}^d$ and $L_t = bt$ for some $b \in \mathbb{R}^d$, and suppose $w^T\alpha = 0$ for all $w \in \mathbb{R}^d$ eigenvector to the eigenvalue one to X_1^T. By Lemma 11.13 of Brockwell and Lindner [14], there exists a solution $f \in \mathbb{C}^d$ and hence, by taking $\mathbf{Re}$(f) (the real part of f), a solution $f \in \mathbb{R}^d$ to the equation $X_1 f = f - \alpha$. Now define X_t by (4.13). Then obviously $V_t = V_{t+1}$ (since $L_t = bt$), hence V_t is 1-periodic strictly stationary.

Now define
$$W_t := X_t(f + \int_0^t X_s^{-1} b ds)\,, \qquad t \in \mathbb{R}\,.$$
Then W obviously solves (4.6) and $W_t = V_t$ for $t \in [0,1]$. To see that $W_t = V_t$ for all $t \in \mathbb{R}$, it suffices to show that $W_t = W_{t+1}$. To see that, note that
$$\begin{aligned} W_{t+1} &= X_{t+1}\Big(f + \int_0^{t+1} X_s^{-1} b ds\Big) \\ &= X_t X_1\Big(f + \int_0^1 X_s^{-1} b ds + \int_1^{t+1} X_s^{-1} b ds\Big) \\ &= X_t\Big(\underbrace{X_1 f}_{=f-\alpha} + \underbrace{X_1 \int_0^1 X_s^{-1} b ds}_{=\alpha} + \int_1^{t+1} \underbrace{X_1 X_s^{-1}}_{=X_{s-1}^{-1}} b ds\Big) \\ &= X_t\Big(f - \alpha + \alpha + \int_1^{t+1} X_{s-1}^{-1} b ds\Big) \\ &= X_t\Big(f + \int_0^t X_s^{-1} b ds\Big) \\ &= W_t. \end{aligned}$$

□

Now we are going to give the necessary and sufficient conditions in the general case when X_1 has arbitrary eigenvalues:

Theorem 4.6. *[The General case of X_1]*
Let $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ be a 1-periodic, continuous, non identically zero function. Let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued, non-zero two-sided Lévy process and let $X_t = \mathcal{E}(U)_t$, where $U_t = \int_0^t A(s)ds$. Define $Z_1 := X_1 \int_0^1 X_s^{-1} dL_s$.
Let $S \in \mathbb{R}^{d\times d}$ be an invertible matrix such that
$$B = S X_1 S^{-1} = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{pmatrix}$$

where $B_1 \in \mathbb{R}^{d_1\times d_1}$ has only eigenvalues of absolute size in (0,1), $B_2 \in \mathbb{R}^{d_2\times d_2}$ has only eigenvalues of absolute size greater than one, and $B_3 \in \mathbb{R}^{d_3\times d_3}$ has only eigenvalues of absolute size one; here $d_1 + d_2 + d_3 = d$ (such matrices S and B exist; e.g. B could be the real Jordan canonical form of X_1). Denote by $I_i \in \mathbb{R}^{d_i\times d}$, $i = 1, 2, 3$ the projection matrix which maps $(x_1, \dots, x_d)^T \in \mathbb{R}^d$ to $(x_{1+\sum_{j=1}^{i-1} d_j}, \dots, x_{\sum_{j=1}^{i} d_j})^T \in \mathbb{R}^{d_i}$. Consider the MOU-equation (4.5) and the MOU-process V_t satisfying (4.6).
Then equation (4.5) admits a 1-periodic strictly stationary solution if and only if

$$\mathbb{E}\log^+|L_1| < \infty, \qquad I_3SZ_1 = \alpha = constant.$$

and

$$w^T I_3 S Z_1 = 0 \quad \text{for all eigenvectors } w \in \mathbb{R}^{d_3} \text{ of } B_3^T \text{ to the eigenvalue one.}$$

If the corresponding conditions are satisfied, then a 1-periodic strictly stationary solution is given by

$$V_t = X_t \left(S^{-1} \int_{-\infty}^{0} \xi_1 S X_s^{-1} dL_s - S^{-1} \int_{0}^{\infty} \xi_2 S X_s^{-1} dL_s + S^{-1}\xi_3 g + \int_0^t X_s^{-1} dL_s \right) \quad \forall t \in \mathbb{R}, \tag{4.14}$$

equivalently

$$V_t = X_t \left(S^{-1} \int_{-\infty}^{t} \xi_1 S X_s^{-1} dL_s - S^{-1} \int_{t}^{\infty} \xi_2 S X_s^{-1} dL_s + S^{-1}\xi_3 g + S^{-1}\xi_3 S \int_0^t X_s^{-1} dL_s \right) \quad \forall t \in \mathbb{R}\,, \tag{4.15}$$

where

$$\xi_1 := \begin{pmatrix} Id_{d_1\times d_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{d\times d}, \qquad \xi_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & Id_{d_2\times d_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{d\times d}, \qquad \xi_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Id_{d_3\times d_3} \end{pmatrix}_{d\times d},$$

and g is an $\mathbb{R}^d$-valued vector equal to $(0, 0, f)^T$ where $f \in \mathbb{R}^{d_3}$ is a solution to $(Id_{d_3\times d_3} - B_3)f = \alpha$ (which exists.)
The 1-periodic strictly stationary solution is unique if $d_3 = 0$, i.e. if X_1 has no eigenvalue of absolute size one.

Proof.
Necessity of the conditions and uniqueness:
Let $(V_t)_{t\in\mathbb{R}}$ be a 1-periodic strictly stationary solution of (4.5). Then $(V_n)_{n\in\mathbb{Z}}$ is strictly stationary and satisfies the vector AR(1)-equation (4.8), i.e. $V_n = X_1 V_{n-1} + Z_n$, with the i.i.d. noise $Z_n = X_n \int_{n-1}^{n} X_s^{-1} dL_s$.

Define the process $(Y_n)_{n\in\mathbb{Z}}$ by $Y_n = SV_n$ and the process $(W_n)_{n\in\mathbb{Z}}$ by $W_n = SZ_n$. From (4.8) we can write

$$SV_n = SX_1S^{-1}SV_{n-1} + SZ_n\,,$$

equivalently

$$Y_n = BY_{n-1} + W_n\,, \qquad n \in \mathbb{Z}\,. \tag{4.16}$$

Since $(V_n)_{n\in\mathbb{Z}}$ is strictly stationary, so is $(Y_n)_{n\in\mathbb{Z}}$ and since $(Z_n)_{n\in\mathbb{Z}}$ is i.i.d. noise, so is $(W_n)_{n\in\mathbb{Z}}$. We can write

$$Y_n = \left((Y_n^1)^T, (Y_n^2)^T, (Y_n^3)^T\right), \qquad n \in \mathbb{Z}\,,$$

and

$$W_n = \left((W_n^1)^T, (W_n^2)^T, (W_n^3)^T\right), \qquad n \in \mathbb{Z}\,,$$

where Y_n^i and W_n^i are $\mathbb{R}^{d_i}$-valued for $i = 1, 2, 3$, and given by $Y_n^i = I_i Y_n$ and $W_n^i = I_i W_n$.

Since B is in block matrix form, it is easy to see that each Y_n^i solves

$$Y_n^i = B_i Y_{n-1}^i + W_n^i\,, \qquad n \in \mathbb{Z}\,, i = 1, 2, 3\,, \tag{4.17}$$

and $(Y_n^i)_{n\in\mathbb{Z}}$ is strictly stationary and $(W_n^i)_{n\in\mathbb{Z}}$ is i.i.d.

From Corollary 1 in Brockwell and Lindner [13] we obtain $\mathbb{E}\log^+|W_n^1| < \infty$, $\mathbb{E}\log^+|W_n^2| < \infty$ and from Lemma 4.4 we obtain (as in proof of Theorem 4.5 (iii), (iv)) that W_n^3 is constant almost surely and that $w^T W_n^3 = 0$ for all eigenvectors $w \in \mathbb{R}^{d_3}$ of B_3^T to the eigenvalue one. Observe that $W_1^3 = I_3 S Z_1$.

Finally, observe that $\mathbb{E}\log^+|W_1| < \infty$ since $\mathbb{E}\log^+|W_1^1| < \infty$, $\mathbb{E}\log^+|W_1^2| < \infty$ and W_1^3 is constant.

Hence also $\mathbb{E}\log^+|Z_1| = \mathbb{E}\log^+|S^{-1}W_1| < \infty$. Since $Z_1 = X_1 \int_0^1 X_s^{-1} dL_s$, this gives

$\mathbb{E}\log^+|\int_0^1 X_s^{-1} dL_s| < \infty$, and Lemma 4.4 implies $\mathbb{E}\log^+|L_1| < \infty$. This finishes the proof of the necessity part.

For the uniqueness, observe that $(Y_n^1)_{n\in\mathbb{Z}}$ and $(Y_n^2)_{n\in\mathbb{Z}}$ are unique by Theorem 1 in Brockwell and Lindner [13]. Hence, if $d_3 = 0$, then $(Y_n)_{n\in\mathbb{Z}}$ and hence $(V_n)_{n\in\mathbb{Z}}$ are unique.

Since $(V_t)_{t\in\mathbb{R}}$ is completely determined by V_0 (and L_0), this gives uniqueness if $d_3 = 0$.

Sufficiency of the conditions:

Observe first that, by (4.2)

$$\begin{aligned} SX_s^{-1} &= S(X_{s-\lfloor s\rfloor}X_{\lfloor s\rfloor})^{-1} = SX_{\lfloor s\rfloor}^{-1}X_{s-\lfloor s\rfloor}^{-1} \\ &= SX_1^{-\lfloor s\rfloor}S^{-1}SX_{s-\lfloor s\rfloor}^{-1} \\ &= \begin{pmatrix} B_1^{-\lfloor s\rfloor} & 0 & 0 \\ 0 & B_2^{-\lfloor s\rfloor} & 0 \\ 0 & 0 & B_3^{-\lfloor s\rfloor} \end{pmatrix} SX_{s-\lfloor s\rfloor}^{-1}\,. \end{aligned}$$

Hence

$$\xi_1 S X_s^{-1} = \begin{pmatrix} B_1^{-\lfloor s \rfloor} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S X_{s-\lfloor s \rfloor}^{-1} .$$

This shows that $\xi_1 S X_s^{-1}$ decays exponentially as $s \to -\infty$ (since all eigenvalues of B_1 have absolute size less than one), hence $\int_{-\infty}^t \xi_1 S X_s^{-1} dL_s$ converges for each $t \in \mathbb{R}$. Similarly, $\int_t^\infty \xi_2 S X_s^{-1} dL_s$ converges for each $t \in \mathbb{R}$. This shows that the right-hand sides of (4.14) and (4.15)are indeed well-defined. Observe also that the vector $f \in \mathbb{R}^{d_3}$ as a solution to the equation $(Id_{d_3 \times d_3} - B_3) f = \alpha$ exists by Lemma 11.13 in Brockwell and Lindner [14].

Now define $(V_t)_{t \in \mathbb{R}}$ by the right-hand side of (4.14). Then

$$V_0 = S^{-1} \int_{-\infty}^0 \xi_1 S X_s^{-1} dL_s - S^{-1} \int_0^\infty \xi_2 S X_s^{-1} dL_s + S^{-1} \xi_3 g ,$$

and (4.14) reads

$$V_t = X_t (V_0 + \int_0^t X_s^{-1} dL_s) , \qquad t \in \mathbb{R} .$$

Hence $(V_t)_{t \in \mathbb{R}}$ is a solution of (4.5).

Next, let us show that $(V_t)_{t \in \mathbb{R}}$ satisfies (4.15). To see that observe that

$$\begin{aligned} V_t &= X_t \Big(S^{-1} \int_{-\infty}^0 \xi_1 S X_s^{-1} dL_s - S^{-1} \int_0^\infty \xi_2 S X_s^{-1} dL_s + S^{-1} \xi_3 g + \underbrace{S^{-1}(\xi_1 + \xi_2 + \xi_3) S}_{= Id_{d \times d}} \int_0^t X_s^{-1} dL_s \Big) \\ &= X_t \Big(S^{-1} \int_{-\infty}^t \xi_1 S X_s^{-1} dL_s - S^{-1} \int_t^\infty \xi_2 S X_s^{-1} dL_s + S^{-1} \xi_3 (g + S \int_0^t X_s^{-1} dL_s) \Big) , \end{aligned}$$

which is (4.15).

It remains to show that $(V_t)_{t \in \mathbb{R}}$ is 1-periodic strictly stationary.

For that observe that

$$\begin{aligned} & X_{t+1} \Big(S^{-1} \xi_3 g + S^{-1} \xi_3 S \int_0^1 X_s^{-1} dL_s \Big) = \\ & X_t X_1 \Big(S^{-1} \xi_3 g + S^{-1} \xi_3 \underbrace{S X_1^{-1} S^{-1}}_{= B^{-1}} S \underbrace{X_1 \int_0^1 X_s^{-1} dL_s}_{= Z_1} \Big) = \\ & X_t S^{-1} B S \Big(S^{-1} \xi_3 g + S^{-1} \xi_3 B^{-1} S Z_1 \Big) , \end{aligned}$$

Noting that

$$\xi_3 B^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_3^{-1} \end{pmatrix} = B^{-1} \xi_3 ,$$

and that

$$\xi_3 S Z_1 = \begin{pmatrix} 0 \\ 0 \\ I_3 S Z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} = constant\,,$$

we continue and write

$$\begin{aligned} & X_t S^{-1} B\Big(\xi_3 g + B^{-1}\xi_3 S Z_1\Big) \\ &= X_t S^{-1}\Big(B\xi_3 g + \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}\Big) \\ &= X_t S^{-1}\Big(B\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}\Big)\,. \end{aligned}$$

Since $(Id_{d_3\times d_3} - B_3)f = \alpha$, this can be further simplified to

$$X_t S^{-1}\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} = X_t S^{-1}\xi_3 g\,.$$

So we have

$$X_{t+1}\Big(S^{-1}\xi_3 g + S^{-1}\xi_3 S\int_0^1 X_s^{-1} dL_s\Big) = X_t S^{-1}\xi_3 g\,. \tag{4.18}$$

Next, observe that for $i \in \{1,2,3\}$,

$$\begin{aligned} & X_1 S^{-1}\xi_i S X_1^{-1} = \\ & S^{-1} B S S^{-1}\xi_i S S^{-1} B^{-1} S = \\ & S^{-1}\underbrace{B\xi_i}_{=\xi_i B} B^{-1} S = S^{-1}\xi_i S\,. \end{aligned}$$

Hence

$$V_{t+1} \overset{(4.15)}{=} X_{t+1}\Bigg(S^{-1}\int_{-\infty}^{t+1}\xi_1 S X_s^{-1} dL_s - S^{-1}\int_{t+1}^{\infty}\xi_2 S X_s^{-1} dL_s$$

$$
\begin{aligned}
&+ S^{-1}\xi_3 S \int_1^{t+1} X_s^{-1} dL_s\Big) + X_{t+1}\Big(S^{-1}\xi_3 g + S^{-1}\xi_3 S \int_0^1 X_s^{-1} dL_s\Big) \\
&\overset{(4.18)}{=} X_t\Big(\int_{-\infty}^{t+1} X_1 S^{-1}\xi_1 S X_s^{-1} dL_s - \int_{t+1}^{\infty} X_1 S^{-1}\xi_2 S X_s^{-1} dL_s \\
&+ \int_1^{t+1} X_1 S^{-1}\xi_3 S X_s^{-1} dL_s\Big) + X_t S^{-1}\xi_3 g \\
&= X_t\Big(\int_{-\infty}^{t} X_1 S^{-1}\xi_1 S X_{s+1}^{-1} dL_{s+1} - \int_{t}^{\infty} X_1 S^{-1}\xi_2 S X_{s+1}^{-1} dL_{s+1} \\
&+ \int_0^{t} X_1 S^{-1}\xi_3 S X_{s+1}^{-1} dL_{s+1} + S^{-1}\xi_3 g\Big) \\
&= X_t\Big(\int_{-\infty}^{t} \underbrace{X_1 S^{-1}\xi_1 S X_1^{-1}}_{=S^{-1}\xi_1 S} X_s^{-1} dL_{s+1} - \int_{t}^{\infty} \underbrace{X_1 S^{-1}\xi_2 S X_1^{-1}}_{S^{-1}\xi_2 S} X_s^{-1} dL_{s+1} \\
&+ \int_0^{t} \underbrace{X_1 S^{-1}\xi_3 S X_1^{-1}}_{S^{-1}\xi_3 S} X_s^{-1} dL_{s+1} + S^{-1}\xi_3 g\Big) \\
&\overset{d}{=} V_t
\end{aligned}
$$

by (4.15), since L is a Lévy process. With the same arguments, it can be shown that for any $n \in \mathbb{N}$ and $t_1, \dots, t_n \in \mathbb{R}$ it follows

$$
(V_{t_1+1,\dots,V_{t_n+1}}) \overset{d}{=} (V_{t_1,\dots,V_{t_n}}),
$$

showing that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic strictly stationary.

$\square$

Corollary 4.7. *Let $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ be 1-periodic, continuous, non-identically zero and bounded function. Let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued, non-zero two-sided Lévy process and let $X_t = \mathcal{E}(U)_t$, where $U_t = \int_0^t A(s)ds$. Assume that X_1 has no eigenvalues of absolute size one. Then the equation (4.5) admits a 1-periodic strictly stationary solution if and only if $\mathbb{E}\log^+|L_1| < \infty$, and in this case this solution is unique.*

This is immediate from Theorem 4.6.

4.3 Weakly stationary multivariate periodic OU processes

In the following theorem we are going to give the necessary and sufficient conditions for the existence and uniqueness of a periodic weakly stationary solution to multivariate periodic OU processes in the case that all the eigenvalues of X_1 lie in a special region.

Theorem 4.8. *Let $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ be a 1-periodic, continuous, non identically zero and known function, and let $X_t = \mathcal{E}(U)_t$, where $U_t = \int_0^t A(s)ds$. Let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued, two-sided Lévy process. Consider the multivariate Ornstein-Uhlenbeck equation (4.5) and its solution given by (4.7). Then the following are true* *(i) If all eigenvalues of X_1 have absolute value in (0,1), then a random vector V_0 can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic weakly stationary if and only if $\mathbb{E}|L_1|^2 < \infty$. In this case the solution is unique and given by*

$$V_t = X_t \int_{-\infty}^{t} X_k^{-1} dL_k \,. \tag{4.19}$$

The integral in the equation (4.19) converges almost surely.

(ii) If all eigenvalues of X_1 have absolute value greater than one, then a random vector V_0 can be chosen such that $(V_t)_{t\in\mathbb{R}}$ is 1-periodic weakly stationary if and only if $\mathbb{E}|L_1|^2 < \infty$.
In this case the solution is unique and given by

$$V_t = -X_t \int_{t}^{\infty} X_k^{-1} dL_k \,. \tag{4.20}$$

The integral in the equation (4.20) converges almost surely.

(iii) If all eigenvalues of X_1 have absolute value one and L_1 is symmetric, then a random vector V_0 can be chosen such that V_t is a 1-periodic weakly stationary solution of (4.5) if and only if $L_1 = 0$ almost surely. In this case, $V_t = 0$ defines a 1-periodic weakly stationary solution.

(iv) Assume that all eigenvalues of X_1 have absolute value one. Then a random vector V_0 can be chosen such that V_t is 1-periodic weakly stationary if and only if L_1 is deterministic and $X_1 \int_0^1 X_s^{-1} dL_s$ is almost surely equal to a constant α, and $\nu^T\alpha = 0$ for every eigenvector ν of X_1^T to the eigenvalue 1. In this case, a vector $f \in \mathbb{R}^d$ exists such that $(Id - X_1)f = \alpha$, and for every such vector f a 1-periodic weakly stationary solution is given by

$$V_t = X_{t-\lfloor t\rfloor} \left(f + \int_0^{t-\lfloor t\rfloor} X_s^{-1} dL_s \right) \,. \tag{4.21}$$

Proof. The proof of this theorem is analogous to the proof of Theorem 4.5, therefore we are going to sketch the proof of the first case and omit the rest.
(i) All eigenvalues of X_1 have absolute value in (0,1):
The necessary conditions: We have from (4.8) and (4.9)

$$V_n = \sum_{k=0}^{m} X_1^k Z_{n-k} + X_1^{m+1} V_{n-k-1} \,; \qquad n \in \mathbb{Z} \,, m \in \mathbb{N} \,. \tag{4.22}$$

Now if $(V_t)_{t\in\mathbb{R}}$ is a 1-periodic weakly stationary, then $(V_n)_{n\in\mathbb{Z}}$ is weakly stationary, then $\mathbb{E}(\nu^T X_1^{m+1} V_{n-k-1}) \to 0$ and $\mathbf{Var}\,(\nu^T X_1^{m+1} V_{n-k-1}) \to 0$ as $m \to \infty$ for every $\nu \in \mathbb{R}^d$, so that $\sum_{k=0}^{\infty} X_1^k Z_{n-k}$ converges in probability and hence almost surely to V_n as $m \to \infty$. Hence $V_n = \sum_{k=0}^{\infty} X_1^k Z_{n-k}$ and since $\mathbb{E}|V_n|^2 < \infty$ and Z_n and $\sum_{k=1}^{\infty} X_1^k Z_{n-k}$ are independent, also $\mathbb{E}|Z_0|^2 < \infty$. By Lemma 4.4 this is equivalent to $\mathbb{E}|L_1|^2 < \infty$. That the solution is unique and given by (4.19) follows as in the proof of Theorem 4.5.

The sufficient condition: Suppose that $\mathbb{E}|L_1|^2 < \infty$ and define V_t by the equation (4.19). It is sufficient to show that $\mathbb{E}|V_t|^2 < \infty$ for all $t \in \mathbb{R}$ since V_t, as given above, is 1-periodic strictly stationary by Theorem 4.5.

We have from (4.19)

$$\mathbb{E}|V_t|^2 = \mathbb{E}\left|X_t \int_{-\infty}^{t} X_k^{-1} dL_k\right|^2$$

$$= \mathbb{E}\left|X_t \int_{-\infty}^{t} X_k^{-1} d(L_k - k\mathbb{E}L_1) + X_t \int_{-\infty}^{t} X_k^{-1} \cdot dk\mathbb{E}L_1\right|^2$$

$$\leq 2|X_t|^2 \mathbb{E}\left|\int_{-\infty}^{t} X_k^{-1} d(L_k - k\mathbb{E}L_1)\right|^2 + 2|X_t|^2 \mathbb{E}\left|\int_{-\infty}^{t} X_k^{-1} dk\right|^2 |\mathbb{E}L_1|^2$$

and the latter is finite since $\mathbb{E}|L_1|^2 < \infty$ and all the eigenvalues of X_1 have absolute value in the interval $(0,1)$. This shows that $(V_t)_{t\in\mathbb{R}}$ is a 1-periodic weakly (and strictly) stationary solution.

□

Now we are going to give the necessary and sufficient conditions in the general case when X_1 has arbitrary eigenvalues:

Theorem 4.9. ***[The General case of** X_1**]***
Let $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ be a 1-periodic, continuous, non identically zero and known function. Let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued, non-zero two-sided Lévy process and let $X_t = \mathcal{E}(U)_t$, where $U_t = \int_0^t A(s)ds$. Define $Z_1 := X_1 \int_0^1 X_s^{-1} dL_s$.
Let $S \in \mathbb{R}^{d\times d}$ be an invertible matrix such that

$$B = SX_1S^{-1} = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{pmatrix}$$

where $B_1 \in \mathbb{R}^{d_1\times d_1}$ has only eigenvalues of absolute size in (0,1), $B_2 \in \mathbb{R}^{d_2\times d_2}$ has only eigenvalues of absolute size greater than one, and $B_3 \in \mathbb{R}^{d_3\times d_3}$ has only eigenvalues of absolute size one; here $d_1 + d_2 + d_3 = d$ (a possible choice for B is the real Jordan canonical form of X_1). Denote by $I_i \in \mathbb{R}^{d_i\times d}$, $i = 1,2,3$ the projection matrix which maps $(x_1, \dots, x_d)^T \in \mathbb{R}^d$ to $(x_{1+\sum_{j=1}^{i-1} d_j}, \dots, x_{\sum_{j=1}^{i} d_j})^T \in \mathbb{R}^{d_i}$. Consider the MOU-equation (4.5) and the MOU-process V_t satisfying (4.6).
Then the equation (4.5) admits a 1-periodic weakly stationary solution if and only if

$$\mathbb{E}|L_1|^2 < \infty, \qquad I_3SZ_1 = \alpha = constant$$

and

$$w^T I_3 S Z_1 = 0 \quad \text{for all eigenvectors } w \in \mathbb{R}^{d_3} \text{ of } B_3^T \text{ to the eigenvalue one.}$$

If the corresponding conditions are satisfied, then a 1-periodic weakly stationary solution is given by

$$V_t = X_t \left(S^{-1} \int_{-\infty}^{0} \xi_1 S X_s^{-1} dL_s - S^{-1} \int_{0}^{\infty} \xi_2 S X_s^{-1} dL_s + S^{-1}\xi_3 g + \int_0^t X_s^{-1} dL_s\right) \quad \forall t \in \mathbb{R}, \tag{4.23}$$

equivalently

$$V_t = X_t\left(S^{-1}\int_{-\infty}^{t}\xi_1 S X_s^{-1}dL_s - S^{-1}\int_{t}^{\infty}\xi_2 S X_s^{-1}dL_s + S^{-1}\xi_3 g + S^{-1}\xi_3 S\int_0^t X_s^{-1}dL_s\right) \quad \forall t\in\mathbb{R}\,, \tag{4.24}$$

where

$$\xi_1 := \begin{pmatrix} Id_{d_1\times d_1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_{d\times d}, \qquad \xi_2 := \begin{pmatrix} 0 & 0 & 0\\ 0 & Id_{d_2\times d_2} & 0\\ 0 & 0 & 0\end{pmatrix}_{d\times d}, \qquad \xi_3 := \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & Id_{d_3\times d_3}\end{pmatrix}_{d\times d},$$

and g is an $\mathbb{R}^d$-valued vector equal to $(0,0,f)^T$, where $f\in\mathbb{R}^{d_3}$ is a solution to $(Id_{d_3\times d_3} - B_3)f = \alpha$ (which exists.)
The 1-periodic weakly stationary solution is unique if $d_3 = 0$, i.e. if X_1 has no eigenvalue of absolute size one.

Proof. This follows in complete analogy to the proof of Theorem 4.6. For the sufficiency, observe that the given solution is also 1-periodic strictly stationary by Theorem 4.6, hence it suffices to show that $\mathbb{E}|V_t|^2 < \infty$, which follows as in the proof of Theorem 4.8. $\square$

Corollary 4.10. *Let $A:\mathbb{R}\to\mathbb{R}^{d\times d}$ be a 1-periodic, continuous, non-identically zero and known function. Let $L=(L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued, non-zero two-sided Lévy process and let $X_t = \mathcal{E}(U)_t$, where $U_t = \int_0^t A(s)ds$. Assume that X_1 has no eigenvalues of absolute size one. Then the equation (4.5) admits a 1-periodic weakly stationary solution if and only if $\mathbb{E}|L_1|^2<\infty$, and in this case this solution is unique.*

This is immediate from Theorem 4.9.

CHAPTER 5

Stationary Periodic CARMA Processes

In this chapter we are going to study the periodic continuous-time ARMA process driven by a two-sided Lévy process. First of all let us recall from Definition 1.32 the definition of the periodic CARMA process.

Definition 5.1. *[Lévy-driven periodic CARMA process]*
Let $d > 0$. A Periodic CARMA process (PCARMA-process or $PCARMA_d$-process) $(Y_t)_{t\in\mathbb{R}}$ of period d driven by the $\mathbb{R}$-valued Lévy process $(L_t)_{t\in\mathbb{R}}$ with auroregressive polynomial $a_t(\cdot)$

$$a_t(z) = z^p + a_1(t)z^{p-1} + \cdots + a_p(t)\,,$$

and moving-average polynomial $b_t(.)$

$$b_t(z) = b_0(t) + b_1(t)z + \cdots + b_q(t)z^q\,,$$

with $b_q(t) = 1$ and $q < p$, is a solution Y of the equation

$$Y_t = \boldsymbol{b}_t^T \boldsymbol{X}_t\,, \quad t \in \mathbb{R} \tag{5.1}$$

where $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\in\mathbb{R}}$ is an $\mathbb{R}^p$-valued process satisfying the stochastic differential equation,

$$d\boldsymbol{X}_t = A_t\boldsymbol{X}_t dt + \boldsymbol{e} dL_t\,, \tag{5.2}$$

or equivalently

$$\boldsymbol{X}_t - \boldsymbol{X}_s = \int_s^t A_u \boldsymbol{X}_u du + \boldsymbol{e}(L_t - L_s), \quad \forall s \le t \in \mathbb{R}, \tag{5.3}$$

with the d-periodic matrices $A_t \in \mathbb{R}^{p\times p}$, the d-periodic vectors $\boldsymbol{b}_t \in \mathbb{R}^p$ and the vector $\boldsymbol{e} \in \mathbb{R}^p$ given by

$$A_t = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 0 \\ -a_p(t) & -a_{p-1}(t) & -a_{p-2}(t) & \vdots & -a_1(t) \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_t = \begin{bmatrix} b_0(t) \\ b_1(t) \\ \vdots \\ b_{p-2}(t) \\ b_{p-1}(t) \end{bmatrix}. \tag{5.4}$$

Here $a_1(t), \dots, a_p(t), b_0(t), \dots, b_{p-1}(t)$ are the d-periodic real-valued coefficients of the periodic polynomials $a_t(z)$, $b_t(z)$, satisfying $b_q(t) = 1$ and $b_j = 0$ for $j > q$. For $p = 1$ the matrix A_t is to be understood as $-a_1(t)$.

We set $A_t dt =: dU_t$, hence $U_t = \int_0^t A_s ds$, with $U_t \in \mathbb{R}^{p\times p}$. Then the equation (5.2) is equivalent to

$$d\mathbf{X}_t = dU_t\mathbf{X}_t + \mathbf{e}dL_t, \qquad t \in \mathbb{R}, \tag{5.5}$$

or equivalently

$$\mathbf{X}_t - \mathbf{X}_0 = \int_0^t dU_s\mathbf{X}_s + \mathbf{e}L_t, \qquad t \in \mathbb{R}. \tag{5.6}$$

The solution of Equation (5.2) (equivalently (5.5)) is unique for any given $\mathbf{X}_0 \in \mathbb{R}^{p\times p}$ and satisfies

$$\mathbf{X}_t = \mathcal{E}(U)_t \left(\mathbf{X}_0 + \int_{(0,t]} \mathcal{E}(U)_s^{-1}\mathbf{e}dL_s\right) \quad \forall t \in \mathbb{R}\,, \tag{5.7}$$

(cf Equation (4.7)).

In the Equation (5.7) we set $\Psi_t := \mathcal{E}(U)_t$, hence $d\Psi_t = dU_t\Psi_t = A_t dt\Psi_t$, with Ψ_t being invertible, $\mathbb{R}^{p\times p}$-valued and $\Psi_0 = Id_d$. The equation (5.7) becomes

$$\mathbf{X}_t = \Psi_t \left(\mathbf{X}_0 + \int_0^t \Psi_s^{-1}\mathbf{e}dL_s\right) \quad \forall t \in \mathbb{R}. \tag{5.8}$$

Remark 5.2. *(i) Since $a_t(z)$ and $b_t(z)$ are periodic with period $d > 0$, it is $a_t(z) = a_{t+d}(z)$ and $b_t(z) = b_{t+d}(z)$ for all $t, s \in \mathbb{R}$.*
(ii) Since $\det(zI - A_t) = a_t(z)$, the eigenvalues of the matrix A_t are the same as the zeros of the autoregressive polynomial $a_t(z)$ for all t. We shall denote these zeros by $\lambda_1(t), \dots, \lambda_r(t)$ and their multiplicities by $m_1(t), \dots, m_r(t)$, respectively. Thus $\sum_{i=1}^r m_i(t) = p$.

5.1 The sufficient conditions for the existence of stationary solution to the periodic CARMA process

In the following section we are going to give sufficient conditions for the existence of a periodic strictly stationary solution of the periodic CARMA-equation. For simplicity we restrict to $d = 1$.

Theorem 5.3. *Let $A_t, \boldsymbol{e}, \boldsymbol{b}_t$ be as in (5.4) with period $d = 1$, and let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}$-valued, non-zero two-sided Lévy process. Let $\Psi_1 = \mathcal{E}(U)_1$ as above, $Z_1 = \Psi_1 \int_0^1 \Psi_s^{-1}\boldsymbol{e}dL_s$. Let $S \in \mathbb{R}p \times p$ be an invertible matrix such that*

$$B = S\Psi_1 S^{-1} = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{pmatrix},$$

where $B_1 \in \mathbb{R}^{p_1\times p_1}$ has only eigenvalues of absolute size in (0,1), $B_2 \in \mathbb{R}^{p_2\times p_2}$ has only eigenvalues of absolute size greater than one, and $B_3 \in \mathbb{R}^{p_3\times p_3}$ has only eigenvalues of absolute size one; here $p_1 + p_2 + p_3 = p$.

Let

$$\xi_1 := \begin{pmatrix} Id_{p_1\times p_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{p\times p}, \qquad \xi_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & Id_{p_2\times p_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{p\times p}, \qquad \xi_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Id_{p_3\times p_3} \end{pmatrix}_{p\times p},$$

and denote by $I_3 \in \mathbb{R}^{p_3 \times p_3}$ *the projection matrix which maps* $(x_1, \dots, x_p)^T \in \mathbb{R}^p$ *to* $(x_{1+p_1+p_2}, \dots, x_p)^T \in \mathbb{R}^{p_3}$.

Assume that $I_3 S Z_1 = \alpha =$ *constant and* $w^T I_3 S Z_1 = 0$ *for all eigenvectors* $w \in \mathbb{R}^{p_3}$ *of* B_3^T *to the eigenvalue one. Let* $f \in \mathbb{R}^{p_3}$ *be a solution to the equation* $(Id_{p_3 \times p_3} - B_3) f = \alpha$ *(which exists) and define* $g = (0, 0, f)^T \in \mathbb{R}^p$.

(*i*) *If*

$$\mathbb{E} \log^+ |L_1| < \infty \,,$$

then there exists a 1-periodic strictly stationary solution of (5.2) (equivalently (5.5)), which is given by

$$\boldsymbol{X}_t = \Psi_t \left(S^{-1} \int_{-\infty}^{0} \xi_1 S \Psi_s^{-1} \boldsymbol{e} dL_s - S^{-1} \int_0^{\infty} \xi_2 S \Psi_s^{-1} \boldsymbol{e} dL_s + S^{-1} \xi_3 g + \int_0^t \Psi_s^{-1} \boldsymbol{e} dL_s \right) \quad \forall t \in \mathbb{R}, \tag{5.9}$$

equivalently

$$\boldsymbol{X}_t = \Psi_t \left(S^{-1} \int_{-\infty}^{t} \xi_1 S \Psi_s^{-1} \boldsymbol{e} dL_s - S^{-1} \int_t^{\infty} \xi_2 S \Psi_s^{-1} \boldsymbol{e} dL_s + S^{-1} \xi_3 g + S^{-1} \int_0^t \xi_3 S \Psi_s^{-1} \boldsymbol{e} dL_s \right) \quad \forall t \in \mathbb{R} \,. \tag{5.10}$$

(*ii*) *If*

$$\mathbb{E} |L_1|^2 < \infty,$$

then there exists a 1-periodic weakly stationary solution of (5.2) (equivalently (5.5)), which is given by (5.9) and (5.10).

Hence, under these conditions, a 1-periodic strictly (weakly) stationary solution of (5.1) can be given by

$$Y_t = \boldsymbol{b}_t^T \left[\Psi_t \left(S^{-1} \int_{-\infty}^{0} \xi_1 S \Psi_s^{-1} \boldsymbol{e} dL_s - S^{-1} \int_0^{\infty} \xi_2 S \Psi_s^{-1} \boldsymbol{e} dL_s + S^{-1} \xi_3 g + \int_0^t \Psi_s^{-1} \boldsymbol{e} dL_s \right) \right] \quad \forall t \in \mathbb{R}, \tag{5.11}$$

equivalently

$$Y_t = \boldsymbol{b}_t^T \left[\Psi_t \left(S^{-1} \int_{-\infty}^{t} \xi_1 S \Psi_s^{-1} \boldsymbol{e} dL_s - S^{-1} \int_t^{\infty} \xi_2 S \Psi_s^{-1} \boldsymbol{e} dL_s + S^{-1} \xi_3 g + S^{-1} \int_0^t \xi_3 S \Psi_s^{-1} \boldsymbol{e} dL_s \right) \right], \quad \forall t \in \mathbb{R} \,. \tag{5.12}$$

Proof. Suppose that the conditions of this theorem are satisfied, then by Theorem 4.6 (Theorem 4.9), $\mathbf{X}_t$ given by (5.9) is a 1-periodic strictly (weakly) stationary solution to (5.5) (equivalently to (5.2)). Hence Y_t, given by (5.11) is a 1-periodic strictly (weakly) stationary solution to (5.1), since also $\mathbf{b}_t$ is 1-periodic.

□

5.2 Autocovariance function of PCARMA processes

In this section we calculate the autocovariance function of periodic weakly stationary PCARMA processes.

Again we restrict to the case when the period $d = 1$. For simplicity, we also restrict the case when all the eigenvalues of Ψ_1 have absolute size in (0,1).

As usual, we denote by $A \otimes B$ the Kronecker product of two matrices, by $\mathbf{vec}(A)$ the vector which arises from A by stacking the columns of A in a vector (starting with the first column) and by **unvec** the inverse operator to **vec**.

For the properties of the Kronecker product, we refer to Lütkepohl [29].

The result now reads as follows:

Theorem 5.4. *Let $A_t, \boldsymbol{e}, \boldsymbol{b}_t$ as in (5.4) with period $d = 1$, let $L = (L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}$-valued two-sided Lévy process, let $U_t = \int_0^t A(s)ds$ and $\Psi_t = \mathcal{E}(U)_t$. Assume that all eigenvalues of Ψ_1 have absolute size less than one, and that $\mathbb{E}|L_1|^2 < \infty$.*

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}}$ and $(Y_t)_{t\in\mathbb{R}}$ be the 1-periodic weakly stationary solutions given by (5.10) and (5.12) (where $S = Id_{p\times p}$), respectively, i.e.

$$\boldsymbol{X}_t = \Psi_t \int_{-\infty}^t \Psi_s^{-1} \boldsymbol{e} dL_s \,, \qquad Y_t = \boldsymbol{b}_t^T \boldsymbol{X}_t \,.$$

Define

$$M_t := \int_0^t \Psi_s^{-1} \otimes \Psi_s^{-1} ds \,, \qquad t \in \mathbb{R} \,.$$

Then

$$\boldsymbol{Cov}(Y_t, Y_{t+h}) = \boldsymbol{b}_{t-\lfloor t\rfloor} \Psi_{t-\lfloor t\rfloor} \cdot \boldsymbol{unvec}\Bigg(\Bigg[\Big(Id_{p^2\times p^2} - (\Psi_1 \otimes \Psi_1)\Big)^{-1} (\Psi_1 \otimes \Psi_1) M_1 + M_{t-\lfloor t\rfloor}\Bigg]$$

$$\cdot \quad \boldsymbol{vec}(\boldsymbol{e}\boldsymbol{e}^T)\Bigg) \cdot \Psi_{t-\lfloor t\rfloor+h}^T \boldsymbol{b}_{t-\lfloor t\rfloor+h} \cdot \boldsymbol{Var}(L_1) \,, \qquad t \in \mathbb{R} \,, h \geq 0. \tag{5.13}$$

Proof. W.l.o.g. assume that $t \in [0, 1)$ (since Y_t and $\mathbf{X}_t$ are 1-periodic weakly stationary). We have

$$\mathbf{Cov}(Y_t, Y_{t+h}) = \mathbb{E}\left((Y_t - \mathbb{E}(Y_t))(Y_{t+h} - \mathbb{E}(Y_{t+h}))^T\right) \,.$$

Observe that

$$\mathbb{E}Y_t = \mathbf{b}_t^T \mathbb{E}\mathbf{X}_t = \mathbf{b}_t^T \Psi_t \int_{-\infty}^t \Psi_k^{-1} \mathbf{e} dk \cdot \mathbb{E}L_1 \,,$$

hence

$$\mathbf{Cov}(Y_t, Y_{t+h}) = \mathbb{E}\Bigg[\mathbf{b}_t^T \Psi_t \int_{-\infty}^t \Psi_k^{-1} \mathbf{e} d(L_k - k\mathbb{E}L_1) \cdot$$

$$\left.\left(\mathbf{b}_{t+h}^T \Psi_{t+h} \int_{-\infty}^{t+h} \Psi_k^{-1} \mathbf{e} d(L_k - k\mathbb{E}L_1)\right)^T\right]$$

$$= \mathbf{b}_t^T \Psi_t \mathbb{E}\left[\int_{-\infty}^{t} \Psi_k^{-1} \mathbf{e} d(L_k - k\mathbb{E}L_1) \cdot \int_{-\infty}^{t+h} \mathbf{e}^T (\Psi_k^{-1})^T d(L_k - k\mathbb{E}L_1)\right] \cdot \Psi_{t+h}^T \mathbf{b}_{t+h}^T .$$

Since $\int_t^{t+h} \mathbf{e}^T (\Psi_k^{-1})^T d(L_k - k\mathbb{E}L_1)$ is independent from $\int_{-\infty}^t \Psi_k^{-1} \mathbf{e} d(L_k - k\mathbb{E}L_1)$ for $h \geq 0$, we obtain with the aid of the Ito-isometry for $h \geq 0$

$$\mathbf{Cov}\,(Y_t, Y_{t+h}) = \mathbf{b}_t^T \Psi_t \mathbb{E}\left[\int_{-\infty}^{t} \Psi_k^{-1} \mathbf{e} d(L_k - k\mathbb{E}L_1) \cdot \int_{-\infty}^{t} \mathbf{e}^T (\Psi_k^{-1})^T d(L_k - k\mathbb{E}L_1)\right] \cdot \Psi_{t+h}^T \mathbf{b}_{t+h}^T$$

$$= \mathbf{b}_t^T \Psi_t \int_{-\infty}^{t} \Psi_k^{-1} \mathbf{e}\mathbf{e}^T (\Psi_k^{-1})^T dk \cdot \mathbf{Var}\,(L_1) \cdot \Psi_{t+h}^T \mathbf{b}_{t+h}^T . \tag{5.14}$$

Let us now calculate $\int_{-\infty}^t \Psi_k^{-1} \mathbf{e}\mathbf{e}^T (\Psi_k^{-1})^T dk$. We have (recall $t \in [0,1)$)

$$\begin{aligned}
&\mathbf{vec}\left(\int_{-\infty}^{t} \Psi_s^{-1} \mathbf{e}\mathbf{e}^T (\Psi_s^{-1})^T ds\right) = \\
&= \int_{-\infty}^{0} \Psi_s^{-1} \otimes \Psi_s^{-1} ds \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) + \int_{0}^{t} \Psi_s^{-1} \otimes \Psi_s^{-1} ds \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) \\
&= \sum_{j=0}^{\infty} \int_{-j-1}^{-j} \Psi_s^{-1} \otimes \Psi_s^{-1} ds \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) + M_t \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) \\
&\overset{v=s+j+1}{=} \sum_{j=0}^{\infty} \int_{0}^{1} \Psi_{v-j-1}^{-1} \otimes \Psi_{v-j-1}^{-1} dv \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) + M_t \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) \\
&= \sum_{j=0}^{\infty} \underbrace{\Psi_{-j-1}^{-1} \otimes \Psi_{-j-1}^{-1}}_{=(\Psi_1 \otimes \Psi_1)^{j+1}} \underbrace{\int_{0}^{1} \Psi_v^{-1} \otimes \Psi_v^{-1} dv}_{=M_1} \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) + M_t \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) \\
&= \Big(Id_{p^2 \times p^2} - (\Psi_1 \otimes \Psi_1)\Big)^{-1} (\Psi_1 \otimes \Psi_1) \cdot M_1 \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) + M_t \quad \mathbf{vec}(\mathbf{e}\mathbf{e}^T) .
\end{aligned}$$

Together with (5.14) this gives (5.13).

□

Remark 5.5. *The only expressions in (5.13) that depend on h are $\boldsymbol{b}_{t-\lfloor t \rfloor+h}$ and $\Psi_{t-\lfloor t \rfloor+h}$. The first remains bounded, and $\Psi_{t-\lfloor t \rfloor+h}$ decays by periodicity exponentially as $h \to \infty$. All other terms can be bounded uniformly in t. Hence if $\rho = \max\{|\lambda_i| : \lambda_i$ eigenvalue of $\Psi_1\}$ with $\rho < 1$, then for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\boldsymbol{Cov}\,(Y_t, Y_{t+h}) \leq C_\varepsilon . e^{-(\rho-\varepsilon)h} , \qquad \forall t \in \mathbb{R},\ h \geq 0 .$$

Since $\boldsymbol{Cov}(Y_t, Y_{t+h}) = \boldsymbol{Cov}(Y_{t+h}, Y_t)$*, a similar estimate also holds when* $h < 0$*, leading to*

$$\boldsymbol{Cov}(Y_t, Y_{t+h}) \leq C_\varepsilon . e^{-(\rho-\varepsilon)|h|}, \qquad \forall t \in \mathbb{R},\ h \in \mathbb{R}.$$

Bibliography

[1] Anderson, P.L., Meerschaert, M.M. and Zhang, K. (2013). *Forecasting with prediction intervals for periodic arma models.* Journal of Time Series Analysis, 34(2), 187-193.

[2] Basawa, I.V. and Lund, R. (2001). *Large sample properties of parameter estimates for periodic ARMA models.* Journal of Time Series Analysis Vol. 22, No. 6, 651-663.

[3] Boshnakov, G. N. (1994) *Periodically correlated sequences: Some properties and recursions.* Research Report 1, Division of Quality Technology and Statistics, Luleo University, Sweden, Mar 1994.

[4] Boshnakov, G. N. (1997). *Periodically correlated solution to a class of stochastic difference equations.* Stochastic Differential and Difference Equations,Progress in Systems and Control Theory Volume 23, 1-9 .

[5] Brockwell, P. J. (2001). *Continuous-time ARMA processes.* Handbook of Statistics, volume 19, 249-276.

[6] Brockwell, P. J. (2009). *Lévy–Driven Continuous–Time ARMA Processes.* Handbook of Financial Time Series, pp 457-480.

[7] Brockwell, P. J. (2014). *Recent results in the theory and applications of CARMA processes.* Ann Inst Stat Math 66, 647-685.

[8] Brockwell, P. J. and Davis, R.A. (1991) *Time Series: Theory and Methods, Second Edition.* Springer, New York.

[9] Brockwell, P.J. and Lindner, A. (2009). *Existence and uniqueness of stationary Lévy-driven CARMA processes.* Stoch. Proc. Appl. 119, 2660-2681.

[10] Brockwell, P. J. and Lindner, A. (2010). *Strictly Stationary Solutions of Autoregressive Moving Average Equations.* Biometrika 97, 765-772.

[11] Brockell, P.J. and Lindner, A. (2012). *Ornstein-Uhlenbeck related models driven by Lévy processes.* In Statistical Methods for Stochastic Differential Equations. Chapman Hall / CRC Press, 383-427.

[12] Brockwell, P.J. and Lindner, A. (2012). *Lévy-driven time series models for financial data.* In: T. Subba Rao and C.R. Rao (Eds), Handbook of Statistics, Volume 30, Time Series Analysis: Methods and Applications, pp. 543-563. Elsevier, Amsterdam.

[13] Brockwell, P. J., Lindner, A. and Vollenbröker, B. (2012). *Strictly stationary solutions of multivariate ARMAequations with i.i.d. noise.* Ann. Inst. Statist. Math. 64, 1089-1119.

[14] Brockwell, P.J. and Lindner, A. M.(in preparation). *Analysis of Time Series with Continuous Parameter.* Springer.

[15] Chow, Y. S. and Teicher, H. (1997) *Probability Theory. Independence, Interchangeability, Martingales.* 3rd ed., Springer, New-York.

[16] Forster, O. (1977). *Analysis 2. Differentialrechnung im $\mathbb{R}^n$, Ggewöhnliche Differentialgleichungen.* Wiesbaden, Verlag Vieweg, 1977.

[17] Granger, C.W.J. and Morris, M.J. (1976). *Time Series Modelling and Interpretation.* Journal of the Royal Statistical Society. Series A (General), Vol. 139, No. 2, 246-257.

[18] Gladishev, E.G. (1961). *Periodically correlated random sequences.* Soviet Math., Vol. 2, 385-388.

[19] Golub, G. H. and van Loan, C. F. (1996) *Matrix Computations. Third Edition*, Johns Hopkins, Baltimore and London.

[20] Gudzenko, L. I. (1959). *On periodically nonstationary processes*, Radiotekhn. i Electron, 6. pp. 1062-1064.

[21] Hurd, H. L. (1969). *An investigation of periodically correlated stochastic processes*, Ph.D. dissertation, Duke University, Durham NC, 1969.

[22] Hurd, H. L. (2007). *Periodically correlated random sequenses: spectral, theory and practice.* Published by John Wiley and Sons, Inc., Hoboken, New Jersey.

[23] Jaschke, S. (2003) *A note on the inhomogeneous linear stochastic differential equation.* Insurance: Math. and Econ., vol.32, issue 3, p. 461-464.

[24] Kallenberg, O. (2002). *Foundations of Modern Probability* . Springer, Berlin, 2nd edition.

[25] Karandikar, R. L. (1982). *Multiplicative Decomposition of Non-Singular matrix Valued Continuous Semimartingales.* The Annals of Probability, Vol. 10, No. 6, 1088-1091.

[26] Kreiß, J.-P. and Neuhaus, G. (2006). *Einführung in die Zeitreihenanalyse.* Springer, Berlin.

[27] Karanasos, M., Paraskevopoulos, A. G. and Dafnos, S. (2014) *A univariate time varying analysis of periodic ARMA processes.* arXiv:1403.4803v1 [stat.ME].

[28] Lund, R. and Basawa, I. V. (1999). *Modeling and inference for periodically correlated time series.* Asymptotics, Nonparametrics, and Time Series (ed. S.Ghosh) 37-62. Marcel Dekker, New York.

[29] Lütkepohl, H. (1996). *Handbook of Matrices.* John Wiley and Sons Ltd (1. August 1996).

[30] Miamee, A. G. (1990). *Periodically correlated processes and their stationary dilations.* SIAM Journal on Applied Mathematics, Vol. 50, No. 4 (Aug., 1990), pp. 1194-1199.

[31] Pagano, M. (1978). *On periodic and multiple autoregression.* Ann. Statist. 6, 1310-1317.

[32] Pedersen, J. (2002). *Periodic Ornstein-Uhlenbeck processes driven by Lévy processes.* J. Appl. Prob. 39, 748–763.

[33] Protter, P. (2005). *Stochastic Integration and Differential equations.* Second Edition. Springer, Berlin.

[34] Sato, K. (2004). *Stochastic integrals in additive processes and application to semi-Levy processes.* Oska J. Math. 41 (2004), 211-236.

[35] Sato, K. (2006) *Monotonicity and non-monotonicity of domains of stochastic integral operators.* Probab. Math. Statist. 26, 23-39.

[36] Sato, K. (2013). *Levy Processes and Infinitely Divisible Distributions.* Revised Edition. Cambridge University Press, Cambridge.

[37] K. Sato, M. Yamazato (1983) *Stationary processes of Ornstein-Uhlenbeck type*. Probability Theory and Mathematical Statistics, Fourth USSR-Japan Symp., Proc. 1982 (ed. K. Ito and J. V. Prokhorov, Lect. Notes in Math. No. 1021, Springer, Berlin), 541-551.

[38] Sato, K. and Yamazato, M. (1984). *Operator-self-decomposable distributions as limit distributions of processes of Ornstein–Uhlenbeck type*. Stochastic Process. Appl. 17, 73-100.

[39] Schlemm, E. and Stelzer, R. (2012). *Multivariate CARMA Processes, Continuous-Time State Space Models and Complete Regularity of the Innovations of the Sampled Processes*. Bernoulli, 18, No. 1, 46-63.

[40] Shao, Q. (2002). *Inference for a Class of Periodic Time Series Models and their Applications*. Doctoral thesis, Athens, Georgia.

[41] Sarnaglia, A.J.Q., Reisen, V.A. and Bondon, P. (2015) *Periodic ARMA models: Application to particulate matter concentrations*. Signal Processing Conference (EUSIPCO), 23rd European, 2181-2185.

[42] Tesfaye, Y.G., Anderson, P.L. and Meerschaert, M.M. (2006). *Identification of periodic ARMA models and thier application to the modeling of river flows*. American Grophysical Union.

[43] Tiana, J. P. and Wangb, J. (2014). *Some results in Floquet theory, with application to periodic epidemic models*. Applicable Analysis, 1128-1152.

[44] Ursu, E. and Pereau, J.C. (2015). *Application of periodic autoregressive process to the modeling of the Garonne river flows*. Stoch Environ Res Risk Assess. DOI 10.1007/s00477-015-1193-3.

[45] Vecchia, A. V. (1985). *Periodic autoregressive-moving average modeling with applications to water resources*. Water Resources Bulletin. Volume 21, Issue 5, Pages 721-730.

[46] Vollenbröker, Bernd (2011). *Strictly stationary solutions of multivariate ARMA and univariate ARIMA equations*. Doctoral thesis, Technische Universität Braunschweig.

Acknowledgements

First and foremost, I want to express a deep gratitude to my supervisor Prof. Dr. Alexander Lindner for giving me the opportunity to write this thesis and offering me a very pleasant work environment. He always had time for me and to give constructive advices since I have started my master's degree at the Technische Universität Braunschweig till I finished this thesis.

Secondly, I would like to thank Prof. Dr. Jens-Peter Kreiß, who agreed to act as the second referee for my thesis which I highly appreciate.

I would like also to thank Prof. Dr. Anna Dall'Acqua, Prof. Dr. Helmut Maier and Prof. Dr. Rico Zacher who agreed to participate in the examination committee for my defense.

Further thanks go to my colleagues of the Institute of Mathematical Finance at University Ulm for the pleasant atmosphere.

Last but not least I want to thank my parents for praying for me all the time. Also I want to thank my wife Yasmin, who encouraged and supported me during my study.

Erklärung

Hiermit versichere ich, Abdulkahar Mohamed Alkadour, dass ich die vorliegende Arbeit selbständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wortliche oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe. Ich erkläre außerdem, dass diese Arbeit weder im In- noch im Ausland in dieser oder änlicher Form in einem anderen Promotionsverfahren vorgelegt wurde.

Ulm, der 22. März 2018

(Abdulkahar Mohamed Alkadour)

Curriculum Vitae

Personal data

Name	Abdulkahar Mohamed Alkadour
Born	10/12/1986
in	Aleppo/Syria

Professional Experience

04/2015 - current	Research and teaching assistant at the Institute of financial mathematics at Ulm University.
10/2008 - 09/2011	Research and teaching assistant at the department of mathematical statistics, Faculty of sciences at Aleppo University, Syria.

Education

14/08/2014	Mster-Mathematik (master in mathematics)
04/2012 - 08/2014	Studies master in mathematics at TU-Braunschweig major subject: mathematics minor subject: informatics
07/2008	Bachelor in mathematical statistics
09/2004 - 08-2008	Studies bachelor in mathematical statistics at Aleppo university, Siria
08/2004	Abitur
09/1992 - 08/2004	School

Ulm, March 22, 2018

(Abdulkahar Mohamed Alkadour)