The Principle of Linearised Stability for Non-Linear Parabolic Volterra Equations

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aus Halle (Saale) im Jahr 2018

Dissertation zur Erlangung des Doktorgrades Dr. rer. nat. der Fakultät für Mathematik und Wirtschaftswissenschaften der Universität Ulm
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Tag der Promotion:
18. Mai 2018
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Introduction

This thesis is concerned with the long-time behaviour of solutions to non-linear abstract parabolic Volterra equations. The focus is both on a wide class of semilinear parabolic Volterra equations and on quasilinear fractional evolution equations. We aim at results which generalise the well-known principle of linearised stability for classical evolution equations to these classes of non-local in time equations. On the basis of stability properties of the linearised problem we derive conclusions about the stability behaviour of the non-linear problem. The key idea is the same for both problem classes and consists in a perturbation argument that allows to control small (understood in an appropriate sense) non-linear terms. However, the required analytical tools and methods to make this work differ significantly for the considered class of problems. In particular, in the quasilinear setting we use the method of maximal $L_p$-regularity.

The Classical Principle of Linearised Stability for ODEs. The principle of linearised stability is a well-known result in the theory of ordinary differential equations (ODEs). To describe it, let $g : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map, where $U \subset \mathbb{R}^n$ is an open set. We consider the differential equation
\[
\partial_t v = g(v).
\]
Assuming that $v_* \in U$ is an equilibrium of this equation, i.e. $v = v_*$ is a stationary solution with $g(v_*) = 0$, we can rewrite the differential equation as follows. Setting $u = v - v_*$, $A = g'(v_*) \in \mathbb{R}^{n \times n}$, and defining $f : U - v_* \subset \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = g(x + v_*) - g'(v_*)x$, for $x \in V = U - v_*$, the differential equation $\partial_t v = g(v)$ is equivalent to the transformed equation
\[
\partial_t u + Au = f(u),
\]
with $f(0) = 0$ and $f'(0) = 0$, for which $u_* = 0$ is an equilibrium. The linear system $\partial_t w + Aw = 0$ is the corresponding homogeneous linearised problem.

The stability behaviour of linear autonomous systems of ODEs is well understood. By the principle of linearised stability it is possible to draw conclusions for the non-linear equation by referring to the linearised problem. Only the eigenvalues of the matrix $A$ and their position in the complex plane are necessary to decide whether the equilibrium $u_* = 0$ is stable or not. Denoting as usual by $\sigma(A)$ the spectrum of the matrix $A$ the principle of linearised stability reads as follows.

**Theorem.** Let $A \in \mathbb{R}^{n \times n}$, $f \in C^1(V; \mathbb{R}^n)$ with $f(0) = 0$, $f'(0) = 0$.

If $\sigma(A) \subset \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \}$, then the equilibrium $u_* = 0$ is asymptotically stable for the equation (1). If there is an eigenvalue $\lambda_0 \in \sigma(A)$ with $\Re \lambda_0 < 0$, then the equilibrium $u_* = 0$ is unstable.
A proof of this result can be found in many textbooks on the qualitative theory of ordinary differential equations, for example see Prüss and Wilke [PW10], Amann [Ama90b] as well as Chicone [Chi06]. We point out that in the case of asymptotic stability one gets an exponential convergence rate for the decay to the equilibrium.

**Semilinear Evolution Equations.** The equation $\partial_t u + Au = f(u)$ can be generalised to abstract semilinear evolution equations in an arbitrary Banach space $X$ where $A$ is an unbounded linear operator. It can be treated in the framework of linear semigroup theory. We refer to the monographs of Pazy [Paz92], Lunardi [Lun95], Cazenave and Haraux [CH98], as well as Engel and Nagel [EN00] for more details about the semigroup approach. There are several works which transfer the stability result for ODEs to the abstract semilinear setting. A principle of linearised stability for abstract evolution equations can be found in Kielhöfer [Kie75], he considered the Hilbert space case. For results in Banach spaces we refer for example to Henry [Hen81], N. Kato [Kat95b] and Ruess [Rue03]. It is an important fact, that also in these generalisations we have the exponential decay estimate in the case of asymptotic stability.

**Fractional Differential Equations.** In the last decades fractional differential equations have attracted a lot of interest for the modelling of processes in science and engineering. For example they can be used to model transmission processes for media with memory, see [Prü12], and diffusion of fluids in porous media with memory, see [Cap99]. Moreover, they play an important role in describing anomalous diffusion processes, for more details we refer to the survey articles [MK00] and [MK04], as well as [KLM12] with further applications of fractional dynamics. Furthermore, we mention the monographs about fractional differential equations of Podlubny [Pod99], Kilbas, Srivastava and Trujillo [KST06] and Diethelm [Die10], which provide a good account of the theory at least in the finite dimensional case.

Before turning to non-linear fractional differential equations we first consider the linear scalar fractional differential equation

$$\partial_t^\alpha (u - u_0) + \mu u = 0, \quad t \in \mathbb{R}_+, \quad u(0) = u_0, \quad (2)$$

where $\alpha \in (0,1)$ and $\mu \in \mathbb{C}$.

Here, $\partial_t^\alpha (u - u_0)$ denotes the Riemann-Liouville derivative of $u - u_0$ given by $\partial_t^\alpha (u - u_0) = \partial_t [g_1 - \alpha *(u - u_0)]$, with $g_1 - \alpha (t) = t^{-\alpha}/\Gamma(1 - \alpha)$, $t \in (0,\infty)$, and $f * g$ denotes the convolution on the positive half-line of the functions $f$ and $g$.\(^1\)

\(^1\)Note that we have to distinguish between the Riemann-Liouville derivative of $u$, given by $\partial_t^\alpha u = \partial_t [g_1 - \alpha *u]$, and the so-called Caputo derivative of $u$ which given by $\mathcal{C} \partial_t^\alpha u = \partial_t^\alpha (u - u_0)$. 

The solution of the equation (2) is given by \( u(t; u_0) = E_\alpha(-\mu t^\alpha)u_0, t \in \mathbb{R}_+ \); where \( E_\alpha \) denotes the so-called Mittag-Leffler function – a generalisation of the exponential function. The decay behaviour of the Mittag-Leffler function \( E_\alpha(-\mu t^\alpha) \) depends on the parameter \( \mu \). Indeed, for all \( \mu \in \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi - \alpha \frac{\pi}{2} \} \) there is some constant \( C > 0 \) such that \( |E_\alpha(-\mu t^\alpha)| \leq Ct^{-\alpha}, t \in (0, \infty) \). This means the Mittag-Leffler function decays to zero algebraically like \( t^{-\alpha} \) and it can be shown that this rate is optimal. For \( \mu \in \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| > \pi - \alpha \frac{\pi}{2} \} \) the Mittag-Leffler function is unbounded. Note that for a fixed \( \alpha \in (0, 1) \) the behaviour of the Mittag-Leffler function is substantially different from the exponential function, in particular the decay as \( t \to \infty \) is much slower.

A similar behaviour is also known for linear fractional differential equations \( \partial_t^\alpha (u - u_0) + Au = 0, t \in \mathbb{R}_+, u(0) = u_0, \) with \( A \in \mathbb{R}^{n \times n} \). Depending on the location of the eigenvalues of the matrix \( A \) in the complex plane one gets asymptotic stability if \( \sigma(A) \subset \{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \pi - \alpha \frac{\pi}{2} \} \) or instability if there is one eigenvalue \( \lambda_0 \in \sigma(A) \) with \( |\arg \lambda_0| > \pi - \alpha \frac{\pi}{2} \). For this result we refer to the work of Matignon [Mat96], which also provides the decay rate \( t^{-\alpha} \) in case of asymptotic stability.

Now, we come to non-linear fractional differential equations. There is an analogous version of the principle of linearised stability.

**Theorem.** Let \( \alpha \in (0, 1), A \in \mathbb{R}^{n \times n}, \) and \( f \in C^1(V; \mathbb{R}^n) \) with \( f(0) = 0, f'(0) = 0, \) where \( V \subset \mathbb{R}^n \) is open with \( 0 \in V \). Then the following statements concerning the stability of the equilibrium \( u_\star = 0 \) of the equation

\[
\partial_t^\alpha (u - u(0)) + Au = f(u),
\]  

hold true.

If \( \sigma(A) \subset \{ \lambda \in \mathbb{C} : |\arg \lambda| < \pi - \alpha \frac{\pi}{2} \} \), then the equilibrium \( u_\star = 0 \) is asymptotically stable for (3). If there is an eigenvalue \( \lambda_0 \in \sigma(A) \) with \( |\arg \lambda_0| > \pi - \alpha \frac{\pi}{2} \), then the equilibrium \( u_\star = 0 \) is unstable.

This result of linearised stability had been an open problem for a long time. Only in 2016, Cong, Doan, Siegmund and Tuan [CDST16] gave a correct proof of the stability part of this result. The instability result has been established in [CDST17]. Their proofs rely essentially on properties of the matrix-valued Mittag-Leffler function and the Jordan normal form of the linearisation. There is no explicit statement about the decay rate in case of asymptotic stability.

**Volterra Integral Equations.** For the so-called standard kernel \( g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha), t \in (0, \infty), \) we have the property that \( g_\alpha * g_{1-\alpha} \equiv 1 \) on \( (0, \infty) \). Convoluting the fractional differential equation (3) with \( g_\alpha \) (and assuming that \( u \)
is sufficiently smooth with $u(0) = u_0$ leads to the Volterra integral equation

$$u + g_\alpha * Au = u_0 + g_\alpha * f(u).$$

In general, Volterra equations can be considered with an arbitrary locally integrable kernel instead of the standard kernel $g_\alpha$. There is a vast literature about such Volterra equations. Moreover, these equations can be also treated in the abstract setting of Banach spaces with unbounded linear operators $A$. Important monographs in this context are the monograph of Gripenberg, Londen and Staffans [GLS90] for the finite-dimensional case and the monograph of Prüss [Prü12] for arbitrary Banach spaces. Further important contributions are the works of Londen [Lon78], Clément et al. [CLN78], [CN79] and [CN81], Gripenberg [Gri79] and [Gri85], as well as Arendt and Prüss [AP92], about the existence and uniqueness, regularity and long-time behaviour of solutions of abstract linear and non-linear Volterra equations.

For the class of semilinear Volterra equations a principle of linearised stability is obtained by N. Kato [Kat95a] and [Kat97] as well as by Londen and Ruess [LR17]. Here $A$ is an accretive operator and the kernel of the Volterra equation is completely positive, but bounded at zero. In particular, these results do not include the important case of the (singular) standard kernel.

**Abstract Semilinear Volterra Equations.** One goal of this thesis is to establish a principle of linearised stability for a wide class of abstract semilinear Volterra equations which can be also reformulated as abstract integro-differential equations and include in particular the case of fractional evolution equations. Let us describe the class of problems to be studied. We suppose that $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ is an unbounded, completely monotonic kernel and denote by $b \in L_{1,\text{loc}}(\mathbb{R}_+)$ the corresponding completely monotonic kernel with $a * b \equiv 1$ on $(0, \infty)$; it is known that there is always a unique kernel $b$ with this property. Furthermore, let $A$ be a linear, closed and densely defined operator on the Banach space $X$, $f \in C^1(X;X)$ with $f(0) = 0$ and $f'(0) = 0$, as well as $u_0 \in X$. For the unknown function $u : J \to X$ we consider the semilinear Voltera equation

$$u + a * Au = u_0 + a * f(u), \quad t \in J; \quad (4)$$

where $J$ is either a compact interval $[0, T]$, $T > 0$, or $\mathbb{R}_+$. Due to the kernel property $a * b \equiv 1$ we are able to rewrite the equation (4) as integro-differential equation given by

$$\partial_t [b * (u - u_0)] + Au = f(u), \quad t \in J, \quad u(0) = u_0. \quad (5)$$

An equilibrium of the Volterra equation (4) and the integro-differential equation (5), respectively, is given by $u_* = 0$, which is a steady solution for the
problem. Note that the stationary solutions of the Volterra equation (4) and integro-differential equation (5), respectively, are the same as in the case of the corresponding classical semilinear evolution equation \( \partial_t u + Au = f(u) \), \( t \in J \), \( u(0) = u_0 \), but the dynamical behaviour is completely different.

In contrast to classical evolution equations there are no semigroups for Volterra equations. In fact one has to generalise the concept of semigroups, leading to the notions of so-called resolvent families \( \{S(t)\}_{t \in \mathbb{R}^+} \subset B(X) \) and integral resolvent families \( \{R(t)\}_{t \in \mathbb{R}^+} \subset B(X) \), see [Prü12]. With these two objects one can represent the solution of the Volterra equation in a way similar to the well-known variation of constants formula in the classical case, provided that these objects exist. By means of these operator families one can introduce a mild solution \( u \in C(J;X) \) of the semilinear problem (4) as solution of the fixed point equation

\[
 u(t) = S(t)u_0 + (R \ast f(u))(t), \quad t \in J.
\]

We prove the existence of a unique local mild solution in case of a locally Lipschitz continuous function \( f \). Subsequently, we define the continuation of such a solution. Unlike the classical case one cannot restart the solution at a later time with a new initial value, since the memory effect of the convolution term must be taken into account. Note that all these considerations assume that there are a resolvent family and an integral resolvent family for the problem (4), see Chapter 3 for definitions.

The main result about the stability of semilinear Volterra equations reads as follows.

**Theorem.** Let \( X \) be a complex Banach space and \( A \) be a linear closed and densely defined operator in \( X \). Furthermore, let \( a \in L^1_{\text{loc}}(\mathbb{R}_+) \) be an unbounded, completely monotonic kernel and \( b \in L^1_{\text{loc}}(\mathbb{R}_+) \) be the corresponding kernel such that \( a \ast b \equiv 1 \) on \((0, \infty)\). Moreover, let \( f \in C^1(X;X) \) with \( f(0) = 0 \) and \( f'(0) = 0 \). We assume, that the Volterra equation (4) admits a bounded resolvent \( \{S(t)\}_{t \in \mathbb{R}_+} \) and an integrable integral resolvent \( \{R(t)\}_{t \in \mathbb{R}_+} \).

Then the equilibrium \( u^* = 0 \) is stable for the Volterra equation (4). Moreover, if \( a \in L^1(\mathbb{R}_+) \) the equilibrium \( u^* = 0 \) is asymptotically stable with rate \( s_\epsilon(t) \) for some sufficiently small \( \epsilon > 0 \), where \( s_\epsilon \) is the unique solution of the scalar resolvent equation

\[
 s_\epsilon + \epsilon (a \ast s_\epsilon) = 1 \quad \text{on} \quad \mathbb{R}_+.
\]

Note that in case of the standard kernel \( a = g_\alpha \) we have \( s_\epsilon(t) = E_\alpha(-\epsilon t^\alpha) \) and so the decay rate is algebraic. It is important to note that exactly the same decay rate as for the scalar resolvent is obtained for the abstract semilinear Volterra equation.

The proof of the exact decay rate is one of the hardest and for our considerations most important parts of this thesis. In order to succeed we use appropriate time-weights and prove the boundedness of the time-weighted solution, which in turn solves a more complicated Volterra equation in the
mild sense. The difficulty comes from an additional commutator term, which has to be studied in detail. Actually, this is the same approach like in the case of classical evolution equations. However, in the classical case the situation is much simpler since an exponential time-weight merely leads to a shift of the operator $A$, which is no longer the case in our situation.

A large part of the theory about Volterra equations heavily relies on the study of the behaviour of the Laplace transformed equation. It is necessary to develop further the known tools relying on the Laplace transform to get more precise results on the asymptotic behaviour. For this reason we prove an extended version of the known Analytic Representation Theorem from Arendt et al. [ABHN01, Theorem 2.6.1] and the Tauberian Theorem [ABHN01, Theorem 2.6.4] for the Laplace transform of holomorphic functions. In contrast to the known result we allow for a more general decay behaviour such as $t^{-\alpha}$, $\alpha \in (0, 1]$. Note that these generalisations also are interesting in its own.

In case of parabolic Volterra equations the existence of a bounded resolvent $\{S(t)\}_{t \in \mathbb{R}^+}$ is well-understood, see the monograph of Prüss [Prü12]. The existence of an integral resolvent family has not been finally clarified. Although there is a generation theorem for the integral resolvent $\{R(t)\}_{t \in \mathbb{R}^+}$, sufficient and easily verifiable conditions for the existence of the integral resolvent and their integrability seem to be not known. Some comments concerning this question can be found in [Prü12, Comment 2.7 d)] and [Prü96b, p. 297], but the proofs of the sufficiency of the given conditions are missing. In this thesis, we prove criteria for the existence and integrability of an integral resolvent in case of parabolic Volterra equations which are easy to check. These criteria cover a large class of kernels, in particular the standard kernel, and lead to the following version of the stability result.

**Theorem.** Let $a \in L^1_{1, \text{loc}}(\mathbb{R}_+)$ be an unbounded, completely monotonic kernel which is $\theta_a$-sectorial with some $\theta_a \in (0, \frac{\pi}{2}]$. We denote by $b \in L^1_{1, \text{loc}}(\mathbb{R}_+)$ the corresponding kernel with $a * b \equiv 1$ on $(0, \infty)$. Assume that $\int_0^1 \hat{a}(1/t) \frac{dt}{t} < \infty$, $\int_0^1 \hat{b}(1/t) \frac{dt}{t} < \infty$ and $[t \mapsto \min\{\hat{a}(1/t)/t, \hat{b}(1/t)/t^2\}] \in L^1(\mathbb{R}_+)$. Moreover, let $A$ be an invertible and sectorial operator with spectral angle $\varphi_A < \pi - \theta_a$ and let $f \in C^1(X; X)$ with $f(0) = 0$ and $f'(0) = 0$.

Then the equilibrium $u_* = 0$ is stable for the Volterra equation (4). If $a \notin L^1(\mathbb{R}_+)$ then $u_* = 0$ is asymptotically stable with rate $s_\epsilon(t)$ for some $\epsilon > 0$.

Observe that, apart from the decay behaviour of the kernels, we only require that the invertible sectorial operator $A$ is compatible to the kernel $a$ in the sense of parabolicity.

Moreover, we are also able to give sufficient conditions for the instability of the solution $u_* = 0$. With the spectral projection along a compact spectral set we show that our Volterra equation as well as the resolvent and integral
resolvent family decomposes in the expected sense. In case of the standard kernel we get instability if the spectrum of the operator $A$ has a compact part in the complement of the stability sector. In the general case the instability condition depends on the mapping properties of the Laplace transform of the kernel $a$. The used estimates for the proof require a deep insight into the growth behaviour of the scalar resolvent and integral resolvent.

In case of the standard kernel the instability result reads as follows.

**Theorem.** Let $X$ be a complex Banach space, $\alpha \in (0,1)$ and $A$ be a closed linear operator in $X$ with dense domain such that $\sigma(-A) \cap \partial \Sigma_{\alpha \pi/2} = \emptyset$ and $-\sigma_+ = \sigma(-A) \cap \Sigma_{\alpha \pi/2}$ is a non-empty compact set and $\sigma_- = \sigma(A) \setminus \sigma_+$ is a closed subset of $\mathbb{C}$.

We denote by $P_+$ the spectral projection of the operator $A$ associated with the compact spectral set $\sigma_+$. Let $A_-$ be the part of $A$ in $\text{Rg}(\text{Id} - P_+)$ with $\sigma(A_-) = \sigma_-$. We assume that $A_-$ is an invertible sectorial operator with spectral angle $\varphi_{A_-} < \pi - \alpha \pi/2$.

Moreover, suppose that $f \in C^1(X;X)$ with $f(0) = 0$ and $f'(0) = 0$.

Then, the equilibrium $u^* = 0$ is unstable for the Volterra equation (4) with the standard kernel $a = g_\alpha$.

It should be noted that the admissible pairs of kernels and operators in our theorems cover a large class of Volterra equations. Observe that in particular the finite dimensional case for fractional differential equations from [CDST16] and [CDST17] is included.

**Quasilinear Equations.** A more general and particularly with regard to applications interesting class of equations are quasilinear evolution equations. A good example is provided by diffusion equations where the diffusion coefficients depend on the concentration and/or its gradient. Important contributions to the theory of abstract quasilinear parabolic evolution equations are the monographs of Lunardi [Lun95] and the very recent monograph of Prüss and Simonett [PS16] which give a very good overview about the current state of research in the field of abstract evolution equations. Furthermore, we refer to the publications of Amann [Ama86, Ama88, Ama90b, Ama90a, Ama93].

For abstract evolution equations it is known that linearisation techniques and maximal regularity, together with a fixed point argument, yield an elegant approach to quasilinear problems, in particular local well-posedness can be shown. Here, we refer to the papers of Clément and Li [CL94], Clément and Prüss [CP92], Prüss [Prü02], Prüss and Simonett [PS04] and Amann [Ama05]. Especially with regard to the application of the abstract theory to partial differential equations the work [DHP03] and [DHP07] of Denk, Hieber and Prüss have to be mentioned here.

Maximal $L^p$-regularity and the linearisation method can be also used for the stability analysis of equilibria of quasilinear parabolic evolution equations,
see the work of Prüss et al. [PSZ09], [PWS13] and [PS16]. Their considerations even go beyond the case of discrete equilibria. Here, smooth manifolds of equilibria are studied, for this reason the concept is called *generalised principle of linearised stability*.

Beside the approach based on maximal $L_p$-regularity one can also use the framework of continuous maximal regularity. As to evolution equations we refer to the work of Simonett [Sim94] and Clément and Simonett [CS01]. Fractional quasilinear parabolic evolution equations are considered in this setting by Clément, Londen and Simonett in [CLS04]. Via continuous maximal regularity results for linear equations they obtain result on existence, uniqueness and continuation of solutions for the quasilinear equation. Note that in this context there is no linearised stability result known.

The second focus of this thesis is on the stability of equilibria of quasilinear fractional evolution equations in the setting of maximal $L_p$-regularity.

Firstly, we want to mention the work of Prüss [Prü91], where quasilinear Volterra equations in spaces of integrable functions are investigated, and his monograph [Prü12, Chapter 8] about parabolic Volterra equations in $L_p$-spaces. Next, we refer to Zacher, who considered abstract Volterra equations in the setting of maximal $L_p$-regularity in [Zac03] and [Zac05], as well as [Zac06]. Moreover, in this framework it is important to refer to the work [Zac09], [PVZ10], [Zac10] and [Zac12], as well as to the work about long-time behaviour of a special class of Volterra equations [VZ15], [KSVZ16], [VZ17] and [KSZ17]. Additionally, we mention the thesis from Bajlekova [Baj01], here also abstract fractional evolution equation are considered, but in particular the well-posedness result for quasilinear equation is formulated for the Riemann-Liouville derivative of the unknown function and not the Caputo one.

Let us now describe the maximal $L_p$-regularity setting for the quasilinear problem, for details see Part III. For Banach spaces $X_0,X_1$ of class $\mathcal{HT}$ with dense embedding $X_1 \hookrightarrow X_0$, $p \in (1,\infty)$ and $\alpha \in (1/p,1)$ we consider the quasilinear fractional evolution equation

$$\partial_t^\alpha (u-u_0) + A(u)u = F(u), \quad u(0) = u_0 \in V,$$

where $V$ is an open subset of the real interpolation space $(X_0,X_1)_{1-\frac{\alpha}{p},p}$, and $(A,F) : V \to \mathcal{B}(X_1;X_0) \times X_0$. We are interested in solutions of maximal $L_p$-regularity, i.e. $u \in H^\alpha_p([0,T];X_0) \cap L_p([0,T];X_1)$.

We want to use the maximal $L_p$-regularity results for the linear equation to get results on existence, uniqueness and continuation for the quasilinear equation. The local well-posedness of the quasilinear problem for initial values in the neighbourhood of $u_0$ is proved under the assumption that the maps $A$ and $F$ are locally Lipschitz continuous and the operator $A(u_0)$ has
the property of maximal $L_p$-regularity on $[0,T]$. The idea of the proof is exactly the same as in the classical case of quasilinear evolution equations. Subsequently, we show that any local solution can be continued to a maximal interval of existence. Observe that in contrast to the classical case, it is not possible to restart straightforwardly a local solution of the quasilinear fractional evolution at a later time with a new initial value, due to the non-local nature of the equation.

Our main result about the stability of equilibria of quasilinear fractional evolution equations reads as follows. Here, $\mathcal{R}\mathcal{S}(X_0)$ denotes the class of $\mathcal{R}$-sectorial operators in $X_0$, see Chapter 6 for the definition.

**Theorem.** Let $p \in (1, \infty)$, $\alpha \in (1/p, 1)$ and $X_0, X_1$ as described above. Suppose that $(A,F) \in C^1(V;B(X_1;X_0) \times X_0)$ and assume that $u_* \in V \cap X_1$ is an equilibrium of the fractional evolution equation (6), i.e. $A(u_*)u_* = F(u_*)$.

For the linearisation of (6) given by $A_0v = A(u_*)v + (A'(u_*)v)u_* - F'(u_*)v$, $v \in X_1$, we assume that the operator $A_0 \in \mathcal{R}\mathcal{S}(X_0)$ is invertible with $q_{A_0}^R < \pi - \alpha \pi$, i.e. the operator $A_0$ has the property of maximal $L_p$-regularity on $\mathbb{R}_+$. Then the equilibrium $u_*$ is asymptotically stable in $X_\gamma$ for the quasilinear equation (6) with an algebraic decay rate with exponent between 0 and $\alpha - 1/p$.

Under the additional assumption $(A,F) \in C^2(V;B(X_1;X_0) \times X_0)$ the decay rate in $X_\gamma$ is algebraic with exponent $\alpha$.

Also in the quasilinear case the main idea for the proof rests on time-weights and the study of a suitably reformulated equation for the weighted solution. As before this involves a commutator term for which we now need $L_p$-estimates. The main difference to the classical result is the fact that for the optimal decay rate a higher regularity of the maps $A$ and $F$ is required. This additional regularity allows us to get better estimates for the non-linearity and to guarantee the known decay rate from the linear problem. It is subject to further research whether this additional regularity assumption is really necessary or if it can be weakened. We point out that for the proof of asymptotic stability with a weaker algebraic rate this additional regularity assumption is not needed.

Overall, this result is the only known generalisation to quasilinear fractional evolution equations of the known principle of linearised stability for quasilinear evolution equations. Up to the additional technical regularity assumption for the optimal decay rate it generalises the classical result in the expected manner.
Organization of the Thesis. The thesis is organized as follows.

Chapter 1. The Laplace transform is the main tool for the treatment of abstract Volterra equations. In the first chapter we introduce the Laplace transform for vector-valued functions, summarise their most important properties and link to the convolution on $\mathbb{R}_+$. In order to examine the asymptotic behaviour of the Laplace transform of holomorphic functions we prove an Analytic Representation Theorem and a Tauberian Theorem. These theorems are fundamental for the following considerations.

Chapter 2. In this chapter basic definitions and properties concerning scalar Volterra equations and their kernels are summarised. We start with the definition of the scalar resolvent and integral resolvent and give a representation formula for the solutions of Volterra equations. We introduce the class of completely positive kernels and the resulting properties for the solutions of Volterra equations with these kernels. Next, we restrict our interest to completely monotonic kernels since their Laplace transform is very well-understood, which is a convenient circumstance with regard to our used tools. In addition, for completely monotonic kernels we prove that properties like sectoriality and regularity can be extended to a slightly larger sector then the right complex half-plan. We close this chapter with permanence properties for completely monotonic kernels and give examples of such kernels.

Chapter 3. For Volterra equations the concept of resolvent and integral resolvent families is paramount, the definitions of these objects are given in this chapter and we quote generation theorems for these objects. In case of parabolic Volterra equations the existence of resolvent families is well-understood. For these resolvent families we prove decay estimates and examine their stability properties. Furthermore, we give sufficient conditions which ensure the existence and integrability of integral resolvents for parabolic Volterra equations. We close this chapter with a time-weighting argument for linear Volterra equations and the calculation of the raising commutator term for a special choice of time-weights. These arguments are essential for the proofs of the decay rates for asymptotic stability in case of semilinear Volterra equations in Part II and quasilinear fractional evolution equations in Part III.

Chapter 4. In this chapter we introduce semilinear Volterra equations. On the basis of the resolvent and integral resolvent family we define mild solutions and prove for locally Lipschitz continuous non-linearities the existence of unique mild solutions. Moreover, we explain how to continue these
solutions and define maximal solutions. The main result of this chapter is the stability result for semilinear Volterra equations. Assuming the existence of a bounded resolvent and an integrable integral resolvent for the Volterra equation we can prove the stability of the equilibrium. Moreover, in case of asymptotic stability we can show that the decay rate is similar to the asymptotic behaviour of the corresponding scalar resolvent. Using the sufficient conditions for the existence of a bounded resolvent and integrable integral resolvent from Chapter 3 we formulate a further version of the stability result. Finally, we pay special attention to the situation of parabolic Volterra equations with standard kernels.

**Chapter 5.** The aim of this chapter is to prove an instability result for semilinear parabolic problems. In order to do this we consider the spectral projection of an operator associated with a compact spectral set. We show the compatibility of this projection with the resolvent and integral resolvent family and their resulting decomposition. Since the part of the operator which is associated with the compact spectral set is a bounded operator, the corresponding resolvent and integral resolvent family can be represented using Dunford’s functional calculus. To understand these objects, it is necessary to study the scalar resolvents and integral resolvents, as well as the mapping behaviour of the Laplace transform of the kernel in detail. We consider appropriate exponential weights for the scalar resolvents and integral resolvents and prove the convergence and the decay rate of these weighted objects. An important issue during these considerations is the uniformity of the estimates along compact sets. Finally, we can prove our instability result for semilinear parabolic Volterra equations. This chapter closes with two corollaries which consider the situation of parabolic Volterra equations with standard kernels.

**Chapter 6.** To treat quasilinear fractional evolution equations in the framework of maximal $L^p$-regularity it is necessary to introduce the most important definitions and underlying objects. Next, we give sufficient conditions for maximal $L^p$-regularity of linear fractional evolution equations and quote the necessary trace space theorem. This chapter is for informational purposes only since we use the framework of maximal $L^p$-regularity merely as a concept.

**Chapter 7.** In this chapter we prove the local well-posedness of quasilinear fractional evolution equations under the assumption that an appropriated linear problem has the property of maximal $L^p$-regularity on a compact interval. Together with the contraction mapping principle we are able to draw conclusions about the solution of the quasilinear problem. This approach
is exactly the same as in the case of classical quasilinear evolution equation. On the basis of the existence of a unique local solution we explain how to continue such a local solution and define a maximal solution.

Chapter 8. In the last chapter of this thesis we turn to the stability study of quasilinear fractional evolution equations. Similar to our approach for the semilinear equations we use time-weights and study a suitable reformulated equation for the weighted solution. Here an additional commutator term arises, which has to be studied in detail. For two special choices of time-weighting functions we show $L_p$-estimates for the commutator term. Under the same assumptions as in the classical case we can prove the asymptotic stability of an equilibrium of the quasilinear fractional evolution equation. Assuming an additional regularity of the non-linearities the optimal algebraic decay rate with exponent $\alpha$ can be shown.

We close this chapter with the application of our results to a fractional diffusion equation on bounded domains.

Basic Notations. We fix some notations used throughout this thesis, which are fairly standard in the modern mathematical literature. By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ we denote the set of natural numbers, integers, real and complex numbers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{C}_+ = \{z \in \mathbb{C}: \Re z > 0\}$. We denote by $\Sigma_\varphi = \{z \in \mathbb{C} \setminus \{0\}: |\arg z| < \varphi\}$ the open sector in $\mathbb{C}$ symmetric to $\mathbb{R}_+$ with opening angle $\varphi \in (0, \pi]$. The Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$. For a metric space $(M, d)$ and $N \subset M$ we designate by $N^c$, $\overline{N}$, $\partial N$ the complement, closure and boundary of $N$, respectively.

$X$, $Y$, $Z$ will always be Banach spaces endowed with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$. $B(X, Y)$ denotes the space of all bounded linear operators from $X$ to $Y$, we write $B(X) = B(X, X)$ for short. $B^X_\epsilon(x_0)$ and $\overline{B}^X_\epsilon(x_0)$ are the open and closed balls, respectively, with centre $x_0$ and radius $\epsilon$ in the Banach space $X$; the superscript is dropped in case no confusion is likely.

If $A$ is a linear operator in $X$, $D(A)$, $\text{Rg}(A)$, $\text{Ker}(A)$ denote the domain, range and kernel of the operator $A$, respectively, while $\sigma(A)$ and $\rho(A)$ designate the spectrum and resolvent set of the operator $A$. For a closed operator $A$ we denote by $X_A$ the domain of $A$ equipped with the graph norm, $\|x\|_A = \|x\|_X + \|Ax\|_X$, $x \in D(A)$.

If $(\Omega, A, \mu)$ is a measure space and $X$ a Banach space, then $L_p((\Omega, A, \mu); X)$, $p \in [1, \infty)$, denotes the space of all Bochner-measurable functions $f: \Omega \to X$ such that $\|f(\cdot)\|_X^p$ is integrable. This space is a Banach space, when it is endowed with the norm $\|f\|_{L_p((\Omega, A, \mu); X)} = \left(\int_{\Omega} \|f(t)\|_X^p \, d\mu(t)\right)^{1/p}$, and functions equal almost everywhere are identified.
For $\Omega \subset \mathbb{R}^n$ open, $\mathcal{A}$ the Lebesgue $\sigma$-algebra, and $\mu$ the Lebesgue measure, we abbreviate $L^p(\Omega; \mathcal{A}, \mu; X)$ to $L^p(\Omega; X)$, $p \in [1, \infty)$. In this case $H^m_p(\Omega; X)$ is the space of all functions $f : \Omega \to X$ having distributional derivatives $\partial^\alpha f \in L^p(\Omega; X)$ of order $|\alpha| \leq m$. The norm in $H^m_p(\Omega; X)$ is given by

$$
\|f\|_{H^m_p(\Omega; X)} = \left( \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p(\Omega; X)}^p \right)^{1/p}.
$$

For $\Omega \subset \mathbb{R}^n$ open, $C^m(\Omega; X)$ denotes the space of all functions $f : \Omega \to X$ which admit continuous partial derivatives $\partial^\alpha f$ in $\Omega$, for each multi-index $\alpha$ with $|\alpha| \leq m$.

For $f \in C(\Omega; X)$ the support of $f$ is defined by $\text{supp } f = \{x \in \Omega : f(x) \neq 0\}$.

We call a map $F : X \to Y$ locally Lipschitz continuous, if for all $x \in X$ there is some $r = r(x) > 0$ and $L = L(x, r) > 0$ such that for all $y, z \in B_X^r(x)$ we have the estimate $\|f(x) - f(y)\|_Y \leq L \|x - y\|_X$. We define $C^{1, \text{loc}}(X; Y) = \{f : X \to Y \text{ is locally Lipschitz continuous}\}$, note that $C^1(X; Y) \subset C^{1, \text{loc}}(X; Y)$.

The subscript 'loc' assigned to any of the above function spaces means membership to the corresponding space when restricted to compact subsets of its domain. In the scalar case $X = \mathbb{R}$ or $X = \mathbb{C}$ we usually omit the image space in the function space notation.

Acknowledgement. First of all I want to thank my supervisor Prof. Dr. Rico Zacher for his support and guidance over the past years. Moreover, I want to thank Prof. Dr. Wolfgang Arendt for accepting to be the second referee of this thesis. I also want to thank all the former and current members of the Institute of Applied Analysis at the University of Ulm for creating such a pleasant working atmosphere.

In particular, I want to thank Jochen Glück for many interesting discussions and his valuable hints. A special thanks goes to Marcel Kreuter and Manfred Sauter for their moral support, particularly in the last phase of this doctorate.

Above and all, I want to thank my parents, my brother and Lars for their enduring support over all the years.
Part I

Preliminaries
The Laplace Transform

In the first part of this chapter we introduce the Laplace transform for vector-valued functions and summarise important general properties of the Laplace transform. Moreover, we respond to the connection between the Laplace transform and the convolution of \( \mathbb{R}_+ \), which is introduced as well.

In the further course of this chapter we deal with the Laplace transform of holomorphic functions and their asymptotic behaviour. We prove important results which are fundamental for the following considerations.

Here, \((X, \|\cdot\|_X)\) always denotes a complex Banach space.

1.1 Definition and General Properties

We start with the definition of the Laplace integral of locally integrable vector-valued functions.

Definition 1.1.1 (Laplace Transform and Abscissa of Convergence). Let \( f \in L^1_{1, \text{loc}}(\mathbb{R}_+; X) \) and \( \lambda \in \mathbb{C} \). If the improper integral

\[
\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) \, dt,
\]

exists in the sense of Bochner, we call it the Laplace integral or the Laplace transform of the function \( f \) at the point \( \lambda \).

The abscissa of convergence of \( \hat{f} \) is given by \( \text{abs}(f) = \inf \{ \text{Re} \lambda : \hat{f}(\lambda) \text{ exists} \} \).

Remark 1.1.2. a) We know from Arendt, Batty, Hieber and Neubrander [ABHN01, Proposition 1.4.1] that the Laplace integral \( \hat{f}(\lambda) \) converges if \( \text{Re} \lambda > \text{abs}(f) \) and diverges if \( \text{Re} \lambda < \text{abs}(f) \). Moreover, according to Section 1.4 from Arendt et al. [ABHN01] we have: If \( \hat{f}(\lambda) \) converges for all \( \lambda \in \mathbb{C} \), then \( \text{abs}(f) = -\infty \), if the domain of convergence is empty, then \( \text{abs}(f) = \infty \). In the case of \( \text{abs}(f) < \infty \), the function \( f \) is called Laplace transformable.

b) If the function \( f \in L^1_{1, \text{loc}}(\mathbb{R}_+; X) \) is of exponential growth, i.e. there is some \( \omega \in \mathbb{R} \) such that \( \int_0^\infty e^{-\omega t} \|f(t)\|_X \, dt < \infty \), then \( \hat{f}(\lambda) \) is well-defined for all \( \text{Re} \lambda \geq \omega \). In particular, we have \( \text{abs}(f) < \omega \).

c) For more detailed information about the vector-valued Laplace transform we refer to Arendt, Batty, Hieber and Neubrander [ABHN01] as well as Hille and Phillips [HP57, Chapter VI]. The standard references for the classical Laplace transform are Widder [Wid41] and Doetsch [Doe50].
1. The Laplace Transform

For a proof of the following theorem about the analytic behaviour of the vector-valued Laplace transform we refer Arendt, Batty, Hieber and Neubrander, see [ABHN01, Theorem 1.5.1].

**Theorem 1.1.3 (Analytic Behaviour of the Laplace Transform).** Let \( f \in L_{1,\text{loc}}(\mathbb{R}_+; X) \) with \( \text{abs}(f) < \infty \). Then \( \lambda \mapsto \hat{f}(\lambda) \) is holomorphic for \( \text{Re} \lambda > \text{abs}(f) \) and, for all \( n \in \mathbb{N}_0 \) and \( \text{Re} \lambda > \text{abs}(f) \) we have

\[
\hat{f}^{(n)}(\lambda) = \int_0^\infty e^{-\lambda t}(-t)^n f(t) \, dt,
\]

as an improper Bochner integral.

We have the following uniqueness theorem for the vector-valued Laplace transform, cf. [ABHN01, Theorem 1.7.3].

**Theorem 1.1.4 (Uniqueness of the Laplace Transform).** Assume that \( f, g \in L_{1,\text{loc}}(\mathbb{R}_+; X) \) with \( \text{abs}(f) < \infty \) and \( \text{abs}(g) < \infty \), and let \( \lambda_0 > \max\{\text{abs}(f), \text{abs}(g)\} \). Suppose that \( \hat{f}(\lambda) = \hat{g}(\lambda) \) whenever \( \lambda > \lambda_0 \). Then \( f(t) = g(t) \) a.e. on \( \mathbb{R}_+ \).

Now, we introduce the convolution on \( \mathbb{R}_+ \) of a scalar and a locally integrable vector-valued function.

**Definition 1.1.5 (Convolution).** For \( k \in L_{1,\text{loc}}(\mathbb{R}_+) \) and \( f \in L_{1,\text{loc}}(\mathbb{R}_+; X) \) we define the convolution for \( t \in \mathbb{R}_+ \) by

\[
(k \ast f)(t) = \int_0^t k(t-s)f(s) \, ds,
\]

and understand the integral in the sense of Bochner.

**Remark 1.1.6.** One can be shown that \((k \ast f)(t)\) exists for almost all \( t \in \mathbb{R}_+ \) and \((k \ast f) \in L_{1,\text{loc}}(\mathbb{R}_+)\). Moreover, we have \((k \ast f)(t) = \int_0^t k(s)f(t-s) \, ds\), hence one can write \((f \ast k)\) instead of \((k \ast f)\). For further information, we refer to Arendt, Batty, Hieber and Neubrander, Section 1.3 about the convolution of vector-and operator-valued functions, cf. [ABHN01, Section 1.3].

Similarly the classical case, the Laplace integral of the convolution of a scalar-valued function \( k \) with a vector-valued function \( f \) is given by the the product of the Laplace transform of the single functions, if they exist, see [ABHN01, Proposition 1.6.4].

**Proposition 1.1.7.** Let \( k \in L_{1,\text{loc}}(\mathbb{R}_+) \) and \( f \in L_{1,\text{loc}}(\mathbb{R}_+; X) \) and suppose that \( \text{Re} \lambda > \max\{\text{abs}(|k|), \text{abs}(f)\} \). Then \((k \ast f)(\lambda)\) exists and \((k \ast f)(\lambda) = \hat{k}(\lambda)\hat{f}(\lambda)\).
1.2 About the Laplace Transform of Holomorphic Functions

In this section we give a generalisation of the Analytic Representation Theorem from [ABHN01, Theorem 2.6.1]. In particular, the decay behaviour of the considered functions is weakened. While the Analytic Representation Theorem from Arendt, Batty, Hieber and Neubrander requires a behaviour of the Laplace transform as $s^{-1}$, we allow a weaker decay as $s^{-\alpha}$, with $\alpha \in (0, 1]$.

On the basis of this result we are able to prove a Tauberian Theorem in terms of the Tauberian Theorem of [ABHN01, Theorem 2.6.4]. All arguments which are used in the proofs of our results are quite similar to the arguments in the corresponding proofs of the theorems in [ABHN01].

1.2.1 An Analytic Representation Theorem

The following theorem extends the Analytic Representation Theorem 2.6.1 from [ABHN01], the proof uses similar arguments as the proof of the mentioned theorem from Arendt, Batty, Hieber and Neubrander.

**Theorem 1.2.1 (Analytic Representation Theorem).** Let $g: (0, \infty) \to \mathbb{R}_+$ be a non-negative, non-increasing function with $\lim_{t \to \infty} g(t) = 0$, which satisfies $\int_1^\infty \frac{g(t)}{t} \, dt < \infty$ and $\sup_{t \in (0, 1]} tg(t) < \infty$. Furthermore, let $0 < \alpha \leq \frac{\pi}{2}$ and $\omega \in \mathbb{C}$.

We suppose that the function $q: \omega + \Sigma_{\alpha + \pi/2} \to X$ is holomorphic and that for all $\gamma \in (0, \alpha)$ there is a constant $M = M(\gamma) > 0$ satisfying for all $\lambda \in \omega + \Sigma_{\gamma + \pi/2}$

$$
\|q(\lambda)\|_X \leq Mg(|\lambda - \omega|).
$$

Then there exists a holomorphic function $f: \Sigma_{\alpha} \to X$ such that for each $\gamma \in (0, \alpha)$ and all $z \in \Sigma_{\gamma}$ there holds

$$
\|e^{-\omega z} f(z)\|_X \leq \frac{C}{|z|} g(1/|z|)
$$

with some constant $C = C(\gamma) > 0$ and $\hat{f}(\lambda) = q(\lambda)$ for all $\lambda > \text{Re} \omega$.

**Proof.** Let $\gamma \in (0, \alpha)$ and $\delta > 0$ arbitrary, but fixed. There exists $M = M((\alpha + \gamma)/2) > 0$ such that $\|q(\lambda)\|_X \leq Mg(|\lambda - \omega|)$ for all $\lambda \in \omega + \Sigma_{\gamma/2} \setminus \{\omega\}$. We consider an oriented path $\Gamma = \Gamma_0 \cup \Gamma_+ \cup \Gamma_-$ (depending on $\delta$ and $\gamma$) consisting of

$$
\Gamma_0 = \left\{ \lambda \in \mathbb{C}: \lambda = \omega + \delta e^{i\varphi}, -\gamma - \frac{\pi}{2} \leq \varphi \leq \gamma + \frac{\pi}{2} \right\},
$$

$$
\Gamma_+ = \left\{ \lambda \in \mathbb{C}: \lambda = \omega + r e^{i(\gamma + \frac{\pi}{2})}, \delta \leq r \right\},
$$

$$
\Gamma_- = \left\{ \lambda \in \mathbb{C}: \lambda = \omega - \delta e^{i\varphi}, -\gamma - \frac{\pi}{2} \leq \varphi \leq \gamma + \frac{\pi}{2} \right\}.
$$
1. The Laplace Transform

Let $\varepsilon \in (0, \gamma)$, as well as $z \in \Sigma_{\gamma-\varepsilon}$ and $\lambda \in \Gamma_\pm$. Then $\lambda z = \omega z + rze^{\pm i(y+\frac{\pi}{2})}$ and thus

$$\text{Re}(\lambda z) = \text{Re}(\omega z) + r|z|\cos(\arg z \pm (\gamma + \pi/2)) \leq \text{Re}(\omega z) - r|z|\sin \varepsilon.$$  

For $\lambda \in \Gamma_\pm$ we have:

$$\left\| e^{\lambda z} q(\lambda) \right\|_X = \left\| e^{(\omega + re^{i(\gamma+\pi)} \frac{\pi}{2})} z q(\omega + re^{\pm i(y+\frac{\pi}{2})}) \right\|_X \leq e^{\text{Re}(\omega z) - |z|\sin \varepsilon} Mg(r); \quad (1.1)$$

and for $\lambda \in \Gamma_0$ we get the estimate

$$\left\| e^{\lambda z} q(\lambda) \right\|_X = \left\| e^{\omega z + z\delta e^{i\theta}} q(\omega + \delta e^{i\theta}) \right\|_X \leq e^{\text{Re}(\omega z) + |z|\delta \cos(\arg z + \theta) Mg(\delta)}. \quad (1.2)$$

We consider for $z \in \Sigma_{\gamma-\varepsilon}$ the expression

$$\int_{\Gamma} e^{\lambda z} q(\lambda) d\lambda = \int_{\Gamma_0 \cup \Gamma_+ \cup \Gamma_-} e^{\lambda z} q(\lambda) d\lambda$$

$$= \int_{\gamma - \frac{\pi}{2}}^{\gamma + \frac{\pi}{2}} e^{\omega z + z\delta e^{i\theta}} q(\omega + \delta e^{i\theta}) i\delta e^{i\theta} d\theta$$

$$+ \int_{-\delta}^{\infty} e^{\omega z + z\delta e^{i\theta}} q(\omega + re^{i(y+\frac{\pi}{2})}) e^{(y+\frac{\pi}{2})} dr$$

$$+ \int_{-\infty}^{\delta} e^{\omega z + z\delta e^{i\theta}} q(\omega + re^{-i(y+\frac{\pi}{2})}) e^{-i(y+\frac{\pi}{2})} dr.$$

We have

$$\left\| \int_{\Gamma_\pm} e^{\lambda z} q(\lambda) d\lambda \right\|_X \leq \int_{-\delta}^{\infty} \left\| e^{\omega z + z\delta e^{i\theta}} q(\omega + re^{\pm i(y+\frac{\pi}{2})}) e^{\pm i(y+\frac{\pi}{2})} \right\|_X dr,$$

$$\left\| \int_{\Gamma_0} e^{\lambda z} q(\lambda) d\lambda \right\|_X \leq \int_{-\gamma - \frac{\pi}{2}}^{\gamma + \frac{\pi}{2}} \left\| e^{\omega z + z\delta e^{i\theta}} q(\omega + \delta e^{i\theta}) i\delta e^{i\theta} \right\|_X d\theta.$$

With the above estimate (1.1) along $\Gamma_\pm$ and the property that the functions $g$ is non-increasing on $(0, \infty)$ we obtain that

$$\left\| \int_{\Gamma_\pm} e^{\lambda z} q(\lambda) d\lambda \right\|_X \leq \int_{-\delta}^{\infty} e^{\text{Re}(\omega z) - |z|\sin \varepsilon} Mg(r) dr$$

$$\leq M e^{\text{Re}(\omega z)} \overline{g(\delta)} e^{-|z|\delta \sin \varepsilon} \left| \frac{1}{\sin \varepsilon} \right|. \quad (1.3)$$

The estimate (1.2) along $\Gamma_0$ yields
1.2. About the Laplace Transform of Holomorphic Functions

\[
\begin{aligned}
\left\| \int_{\Gamma} e^{iz} q(\lambda) d\lambda \right\|_X &\leq \int_{-\gamma - \frac{i\pi}{2}}^{\gamma + \frac{i\pi}{2}} e^{\text{Re}(\omega z) + |z| \delta \cos(\arg z + \theta)} M \mathcal{g}(\delta) \delta d\theta \\
&\leq 2\pi M e^{\text{Re}(\omega z)} e^{\frac{|z|\delta}{2}} g(\delta) \delta.
\end{aligned}
\]

All in all we have the estimate

\[
\left\| \int_{\Gamma} e^{iz} q(\lambda) d\lambda \right\|_X \leq 2\pi M e^{\text{Re}(\omega z)} e^{\frac{|z|\delta}{2}} g(\delta) \delta + 2 M e^{\text{Re}(\omega z)} g(\delta) e^{-|\delta\sin\epsilon\sin\epsilon}.
\]

Since the exponential function and the function \( q \) are holomorphic functions on \( \omega + \Sigma_{a+\frac{\pi}{2}} \) these estimates show that \( f(z) = \frac{1}{2\pi i} \int e^{iz} q(\lambda) d\lambda \) is absolutely convergent, uniformly for \( z \) in compact subsets of \( \Sigma_r \), and therefore defines a holomorphic function in \( \Sigma_r \).

We claim that by Cauchy’s theorem the function \( f \) (given by the above integral) is independent of \( \delta > 0 \), and also independent of \( \gamma \in (0, \alpha) \) as long as \( |\arg z| < \gamma \) (use the assumptions on \( q \) for \( \lambda \in \omega + \Sigma_{a+\frac{\pi}{2}} \) with large norm). In fact, let \( \delta_1, \delta_2 > 0 \) and \( \gamma_1, \gamma_2 \in (0, \alpha) \). W. l. o. g. we assume that \( \gamma_1 \leq \gamma_2 \). We define for \( k \in \{1, 2\} \) an oriented path \( \Gamma_k = \Gamma_k^0 \cup \Gamma_k^+ \cup \Gamma_k^- \) consisting of

\[
\Gamma_k^0 = \{ \lambda \in \mathbb{C} : \lambda = \omega + \delta_k e^{i\theta}, -\gamma_k - \frac{\pi}{2} \leq \theta \leq \gamma_k + \frac{\pi}{2} \},
\]

\[
\Gamma_k^+ = \{ \lambda \in \mathbb{C} : \lambda = \omega + re^{i(\gamma_1 + \frac{\pi}{2})}, \delta_k \leq r \leq R \},
\]

as well as an oriented path \( \Gamma_k^R = \Gamma_k^0 \cup \Gamma_k^+ \cup \Gamma_k^- \) consisting of \( \Gamma_0^k \) as above and

\[
\Gamma_k^R = \{ \lambda \in \mathbb{C} : \lambda = \omega + re^{i(\gamma_1 + \frac{\pi}{2})}, \delta_k \leq r \leq R \}.
\]

Moreover, we set

\[
\Gamma_k^+ = \{ \lambda \in \mathbb{C} : \lambda = \omega + Re^{i\theta}, \gamma_1 + \frac{\pi}{2} \leq \theta \leq \gamma_2 + \frac{\pi}{2} \},
\]

\[
\Gamma_k^- = \{ \lambda \in \mathbb{C} : \lambda = \omega + Re^{-i\theta}, \gamma_1 + \frac{\pi}{2} \leq \theta \leq \gamma_2 + \frac{\pi}{2} \}.
\]

The function \( e^{iz} q(\lambda) \) is holomorphic on \( \omega + \Sigma_{a+\frac{\pi}{2}} \), hence Cauchy’s integral theorem yields

\[
\int (\Gamma_k^+ + \Gamma_k^-) e^{iz} q(\lambda) d\lambda = 0.
\]

Now, let \( z \in \Sigma_{\gamma_1-\epsilon} \) for some \( \epsilon \in (0, \gamma_1) \). We have that

\[
\int_{\Gamma_k^+} e^{iz} q(\lambda) d\lambda = \int_{\gamma_1 + \frac{\pi}{2}}^{\gamma_1 + \frac{\pi}{2} + \pi \epsilon} e^{\omega z + zRe^{i\theta}} q(\omega + Re^{i\theta}) iRe^{i\theta} d\theta,
\]

\[
\int_{\Gamma_k^-} e^{iz} q(\lambda) d\lambda = \int_{\gamma_1 + \frac{\pi}{2}}^{\gamma_1 + \frac{\pi}{2} - \pi \epsilon} e^{\omega z + zRe^{-i\theta}} q(\omega + Re^{-i\theta}) (-i)Re^{-i\theta} d\theta,
\]

\[7\]
and hence
\[
\left\| \int_{\Gamma_1} e^{\lambda z} q(\lambda) d\lambda \right\|_X \leq \int_{\gamma_1 + \frac{\pi}{2}}^{\gamma_2 + \frac{\pi}{2}} \left\| e^{(\omega + Re^{i\theta}) t} q \left( \omega + Re^{i\theta} \right) (\pm i) Re^{i\theta} \right\|_X d\theta
\]
\[
\leq M g(R) Re^{-|z| R \sin \epsilon} e^{\omega z} (\gamma_2 - \gamma_1) \to 0,
\]
for \( R \to \infty \). This implies
\[
\int_{\Gamma_1} e^{\lambda z} q(\lambda) d\lambda = \lim_{R \to \infty} \int_{\Gamma_1} e^{\lambda z} q(\lambda) d\lambda = \int_{\Gamma_2} e^{\lambda z} q(\lambda) d\lambda,
\]
and hence the function \( f(z) \) is independent of the parameters \( \delta \) and \( \gamma \) of the integral along the oriented path \( \Gamma \). Hence \( \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda z} q(\lambda) d\lambda \) defines a holomorphic function \( f \) on \( \Sigma_{\alpha} \).

Let \( z \in \Sigma_\alpha \) be arbitrary, but fixed. We want to estimate \( f(z) \). Since the sector \( \Sigma_\alpha \) is open there is some \( \gamma < \alpha \) and some sufficiently small \( 0 < \varepsilon < \gamma \) such that \( z \in \Sigma_{\gamma - \varepsilon} \). Let \( M = M(\gamma) > 0 \) the constant from the requirements. Choosing \( \delta = |z| - 1 \) we obtain with the aid of (1.4) for all \( \lambda \in \Gamma_0 \) the estimate
\[
\left\| \int_{\Gamma_0} e^{\lambda z} q(\lambda) d\lambda \right\|_X \leq 2\pi e^{Re(\omega z) + 1} \frac{M}{|z|} g(1/|z|),
\]
and for \( \lambda \in \Gamma_{\pm} \) we have together with estimate (1.3) that
\[
\left\| \int_{\Gamma_1} e^{\lambda z} q(\lambda) d\lambda \right\|_X \leq e^{Re(\omega z)} e^{-\sin \varepsilon} \frac{M}{\sin \varepsilon} \frac{1}{|z|} g(1/|z|).
\]
These estimates establish that for all \( z \in \Sigma_{\gamma - \varepsilon} \) we have
\[
\| e^{-\omega z} f(z) \|_X \leq \frac{C}{|z|} g(1/|z|), \quad (1.5)
\]
with \( C = M \max \{ e^{-\sin \varepsilon}/\sin \varepsilon, 2\pi e \} \).

Next, we will show that \( \tilde{f}(\lambda) = q(\lambda) \) whenever \( \lambda > Re \omega \). Using relation (1.5) we have for all \( \lambda > Re \omega \) that
\[
\int_0^\infty e^{-\lambda t} \| f(t) \|_X dt \leq C \int_0^\infty e^{-\lambda t} e^{Re \omega t} \frac{g(1/t)}{t} dt
\]
\[
\leq C \int_0^1 e^{-\lambda t (1-Re \omega)} \frac{g(1/t)}{t} dt + C \int_1^\infty e^{-\lambda t (1-Re \omega)} \frac{g(1/t)}{t} dt
\]
\[
\leq C \int_0^1 \frac{g(1/t)}{t} dt + C \sup_{s \in (0,1]} s g(s) \int_1^\infty e^{-\lambda t (1-Re \omega)} dt < \infty.
\]
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This means that $f$ is Laplace transformable for all $\Re \lambda > \Re \omega$. Given $\lambda > \Re \omega$, choose $0 < \delta < \lambda - \Re \omega$, and $\gamma \in (0, \alpha)$. Since $\lambda$ is on the right of the path $\Gamma$ it follows that

$$
\int_{\Gamma} \int_{0}^{\infty} |e^{-(\lambda - \mu)t}| \|q(\mu)\|_X \, dt \, d\mu = \int_{\Gamma} \|q(\mu)\|_X |d\mu| \\
\leq M \int_{\Gamma} \frac{g(|\mu - \omega|)}{\lambda - \Re \mu} \, d\mu < \infty,
$$

because

$$
\int_{\Gamma} \frac{g(|\mu - \omega|)}{\lambda - \Re \mu} \, d\mu = \int_{\delta}^{\infty} \frac{g(r)}{\lambda - [\Re \omega + r \cos(\gamma + \frac{\pi}{2})]} \, dr \\
\leq \frac{1}{\sin(\gamma)} \int_{\delta}^{\infty} \frac{g(r)}{r} \, dr < \infty,
$$

since we have for all $\delta \in (0, 1)$

$$
\int_{\delta}^{\infty} \frac{g(r)}{r} \, dr = \int_{\delta}^{1} \frac{g(r)}{r} \, dr + \int_{1}^{\infty} \frac{g(r)}{r} \, dr \\
\leq \sup_{s \in [\delta, 1]} sg(s) \int_{\delta}^{1} \frac{ds}{s^2} + \int_{0}^{1} \frac{g(1/t)}{t} \, dt < \infty,
$$

by assumptions on the map $g$, as well as

$$
\int_{\Gamma} \frac{g(|\mu - \omega|)}{\lambda - \Re \mu} \, d\mu = \int_{-\gamma - \frac{\pi}{2}}^{\gamma + \frac{\pi}{2}} \frac{g(\delta)}{\lambda - (\Re \omega + \delta \cos \theta)} \delta \, d\theta \\
\leq \delta g(\delta) \frac{2\pi}{\lambda - (\Re \omega + \delta)} < \infty.
$$

Fubini’s theorem yields

$$
\int_{\Gamma} \int_{0}^{\infty} e^{-\lambda t} e^{\mu t} q(\mu) \, dt \, d\mu = \int_{0}^{\infty} \int_{\Gamma} e^{-\lambda t} e^{\mu t} q(\mu) \, d\mu \, dt.
$$

Due to the choice of $\delta \in (0, \lambda - \Re \omega)$, we know $\lambda$ is to the right of the path $\Gamma$, and Fubini’s theorem and Cauchy’s residue theorem imply that

$$
f'(\lambda) = \int_{0}^{\infty} e^{-\lambda t} \frac{1}{2\pi i} \int_{\Gamma} e^{\mu t} q(\mu) \, d\mu \, dt \\
= \frac{1}{2\pi i} \int_{\Gamma} \frac{q(\mu)}{\lambda - \mu} \, d\mu \\
= q(\lambda) + \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\Gamma} \frac{q(\mu)}{\lambda - \mu} \, d\mu,
where \( \widetilde{\Gamma}_R = \{ \lambda \in \mathbb{C} : \lambda = \omega + \text{Re}i\theta, -\gamma - \frac{\pi}{2} \leq \theta \leq \gamma + \frac{\pi}{2} \} \). Then

\[
\left\| \int_{\widetilde{\Gamma}_R} \frac{q(\mu)}{\lambda - \mu} \, d\mu \right\|_X \leq \int_{-\gamma - \frac{\pi}{2}}^{\gamma + \frac{\pi}{2}} \left\| q(\omega + \text{Re}i\theta) i\text{Re}i\theta \right\|_X \, d\theta,
\]

for sufficient large \( R \) we have \( R > 2|\lambda - \omega| \) and hence

\[
\left\| \int_{\widetilde{\Gamma}_R} \frac{q(\mu)}{\lambda - \mu} \, d\mu \right\|_X \leq 2\pi M \frac{g(R)R}{1 - |\lambda - \omega|} \leq 4\pi M g(R),
\]

and this expression tends to zero as \( R \to \infty \) by assumption on the function \( g \). This proves that \( \hat{f}(\lambda) = q(\lambda) \) for \( \lambda > \text{Re}\omega \). \( \square \)

**Remark 1.2.2.** For the proof of the above theorem it would be enough to require that \( \int_1^\infty e^{-\varepsilon t}g(1/t)\frac{d}{t} < \infty \) for all \( \varepsilon > 0 \), instead of our requirement \( \sup_{t \in (0,1]} t g(t) < \infty \). But for our application in the following the slightly stronger assumption is always satisfied and from our point of view easier to check.

The next converse statement is for our following considerations non-essential, but we insert it into this treatise for the sake of completeness.

**Theorem 1.2.3.** Let \( g : (0,\infty) \to \mathbb{R}_+ \) be a non-negative, non-increasing function with \( \lim_{t \to \infty} g(t) = 0 \) and \( \lim_{t \to \infty} g(1/t) e^{-\eta t} = 0 \) for all \( \eta > 0 \). Furthermore, let \( tg(t) \) be non-decreasing and \( g(t) \leq -ctg'(t) \) for a.e. \( t > 0 \) and some \( c > 0 \). In addition, let \( \alpha \in (0, \frac{\pi}{2}) \), \( \omega \in \mathbb{C} \) and \( q : (\text{Re}\omega + |\text{Im}\omega| \tan \alpha, \infty) \to X \). Suppose there is a holomorphic function \( f : \Sigma_{\alpha} \to X \) such that for all \( \gamma \in (0, \alpha) \) and all \( z \in \Sigma_{\gamma} \) there holds

\[
\| e^{-\omega z} f(z) \|_X \leq C \frac{g(1/|z|)}{|z|}
\]

with some constant \( C = C(\gamma) > 0 \) and \( f(\lambda) = q(\lambda) \) for all \( \lambda > \text{Re}\omega + |\text{Im}\omega| \tan \alpha \). Then the function \( q \) has a holomorphic extension \( q : \omega + \Sigma_{\alpha+\frac{\pi}{2}} \to X \) such that for all \( \gamma \in (0, \alpha) \) and all \( \lambda \in \omega + \Sigma_{\gamma+\frac{\pi}{2}} \) we have

\[
\| q(\lambda) \|_X \leq Mg(|\lambda - \omega|)
\]

with some constant \( M = M(\gamma) > 0 \).
Moreover, we get

\[ \int_{-\gamma}^{\gamma} e^{-\lambda Re^{i\theta}} f(Re^{i\theta})(\pm i)Re^{i\theta} d\theta = \int_{R}^{0} e^{-\lambda s e^{i\gamma}} f(se^{i\gamma})e^{i\gamma} ds. \]

Moreover, we get

\[
\| \int_{0}^{\gamma} e^{-\lambda Re^{i\theta}} f(Re^{i\theta})(\pm i)Re^{i\theta} d\theta \|_{\infty}
\leq \int_{0}^{\gamma} \left| e^{-\lambda Re^{i\theta}} e^{\lambda Re^{i\theta}} \right| \frac{C}{|Re^{i\theta}|} g \left( |Re^{i\theta}| \right) |Re^{i\theta}| d\theta
\]

\[ = Cg(1/R) \int_{0}^{\gamma} e^{-Re[(\lambda - \omega)Re^{i\theta}]} d\theta \]

\[ \leq Cg(1/R) \frac{\pi}{2} e^{-(|\lambda - \omega| - |\omega| |\tan \gamma| |R\cos \gamma|)} \to 0, \]

as \( R \to \infty \). We deduce

\[ \int_{0}^{\gamma} e^{-\lambda f(t)} dt = \int_{0}^{\gamma} e^{-\lambda \omega e^{i\gamma}} f(se^{i\gamma})e^{i\gamma} ds = \int_{\Gamma_\pm} e^{-\lambda z} f(z) dz, \]

for all \( \lambda > Re \omega + |Im \omega| \tan \alpha > Re \omega + |Im \omega| \tan \gamma \), whenever \( \gamma \in (0, \alpha) \).

Let \( \epsilon \in (0, \frac{\pi}{2} - \gamma) \) and \( \lambda \in \mathbb{C} \) with \( -\frac{\pi}{2} + \gamma + \epsilon < \arg(\lambda - \omega) < \frac{\pi}{2} - \gamma - \epsilon \). Then \( -\frac{\pi}{2} + \epsilon < \arg((\lambda - \omega)e^{i\gamma}) < \frac{\pi}{2} - \epsilon \), thus

\[ Re(\lambda - \omega)e^{i\gamma} = |\lambda - \omega|cos(\arg(\lambda - \omega) + i\gamma) \]

\[ \geq |\lambda - \omega|cos \left( \frac{\pi}{2} - \epsilon \right) \]

\[ = |\lambda - \omega|\sin \epsilon. \]

For all \( s > 0 \) we get the estimate

\[
\left\| e^{\lambda se^{i\gamma}} f(se^{i\gamma}) \right\|_{L} \leq e^{-\operatorname{Re}(\lambda - \omega)e^{i\gamma}} \frac{C}{s} g(1/s) \leq e^{-\epsilon |\lambda - \omega|\sin \epsilon} \frac{C}{s} g(1/s),
\]

and hence we have for

\[
\int_{\Gamma_\pm} e^{-\lambda z} f(z) dz = e^{i\gamma} \int_{0}^{\infty} e^{-\lambda se^{i\gamma}} f(se^{i\gamma}) ds,
\]

the estimate
1. The Laplace Transform

With integration by parts it follows that

\[ \left\| \int_{\Gamma_a} e^{-\lambda z} f(z) \, dz \right\|_X \leq C \int_0^\infty e^{-|\lambda - \omega| \sin \varepsilon} \frac{g(1/s)}{s} \, ds \]

\[ = C \int_0^{1/|\lambda - \omega|} e^{-|\lambda - \omega| \sin \varepsilon} \frac{g(1/s)}{s} \, ds \]

\[ + C \int_{1/|\lambda - \omega|}^\infty e^{-|\lambda - \omega| \sin \varepsilon} \frac{g(1/s)}{s} \, ds. \]

For the first integral we use the estimate \( g(s) \leq -c s g'(s) \) and the resulting inequality \( \frac{1}{s} g(1/s) \leq c \frac{d}{ds} g(1/s) \). Furthermore, the function \( s g(s) \) is non-decreasing, hence \( \frac{1}{s} g(1/s) \) in non-increasing, this property is used for the second integral. We obtain that

\[ \left\| \int_{\Gamma_a} e^{-\lambda z} f(z) \, dz \right\|_X \leq C c \left[ e^{-|\lambda - \omega| \sin \varepsilon} \frac{g(1/s)}{s} \right]_0^{1/|\lambda - \omega|} \]

\[ - \int_0^{1/|\lambda - \omega|} (-|\lambda - \omega| \sin \varepsilon) e^{-|\lambda - \omega| \sin \varepsilon} \frac{d}{ds} g(1/s) \, ds \]

\[ + C g(|\lambda - \omega|) |\lambda - \omega| \int_{1/|\lambda - \omega|}^\infty e^{-|\lambda - \omega| \sin \varepsilon} \, ds. \]

Since the function \( g \) is non-increasing, we have that \( s \mapsto g(1/s) \) is non-decreasing. With integration by parts it follows that

\[ \left\| \int_{\Gamma_a} e^{-\lambda z} f(z) \, dz \right\|_X \leq C c \left[ e^{-|\lambda - \omega| \sin \varepsilon} g(1/s) \right]_0^{1/|\lambda - \omega|} \]

\[ - \int_0^{1/|\lambda - \omega|} (-|\lambda - \omega| \sin \varepsilon) e^{-|\lambda - \omega| \sin \varepsilon} g(1/s) \, ds \]

\[ + C g(|\lambda - \omega|) |\lambda - \omega| \int_{1/|\lambda - \omega|}^\infty e^{-|\lambda - \omega| \sin \varepsilon} \, ds. \]

Consequently, the integrals \( \int_{\Gamma_a} e^{-\lambda z} f(z) \, dz \) are absolutely convergent and define a holomorphic function in the region \( -\frac{\pi}{2} + \gamma + \varepsilon < \arg(\lambda - \omega) < \frac{\pi}{2} + \gamma - \varepsilon \), with some constant \( M > 0 \) such that

\[ \left\| \int_{\Gamma_a} e^{-\lambda z} f(z) \, dz \right\|_X \leq M g(|\lambda - \omega|). \]

For \( \lambda \in \mathbb{C} \) with \( -\frac{\pi}{2} + \gamma + \varepsilon < \arg(\lambda - \omega) < \frac{\pi}{2} + \gamma - \varepsilon \) we set

\[ q_+(\lambda) = \int_{\Gamma_a} e^{-\lambda z} f(z) \, dz, \]
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both \( q_+ \) and \( q_- \) are extensions of \( q \), and together they define a holomorphic extension (again denoted by \( q \)) to \( \omega + \Sigma \frac{\gamma' - \varepsilon}{\varepsilon} \), satisfying

\[ \|q(\lambda)\|_X \leq Mg(\lambda - \omega), \]

in the sector \( \omega + \Sigma \frac{\gamma' - \varepsilon}{\varepsilon} \). Since \( \gamma \in (0, \alpha) \) and \( \varepsilon \in \left(0, \frac{\pi}{2} - \gamma\right) \) are arbitrary, this proves the claim. \( \square \)

For \( \omega \in \mathbb{R} \) we have a quite similar result.

**Corollary 1.2.4.** Let \( g: (0, \infty) \to \mathbb{R}_+ \) be a non-negative, non-increasing function with \( \lim_{t \to \infty} g(t) = 0 \) and \( \lim_{t \to 0} g(1/t) = 0 \) for all \( \eta > 0 \). Furthermore, let \( t g(t) \) be non-decreasing and \( g(t) \leq -ctg'(t) \) for a.e. \( t > 0 \) with some \( c > 0 \). In addition, let \( \alpha \in (0, \frac{\pi}{2}) \), \( \omega \in \mathbb{R} \) and \( q: (\omega, \infty) \to X \). Suppose there is a holomorphic function \( f: \Sigma_\alpha \to X \) such that for all \( \gamma \in (0, \alpha) \) and all \( z \in \Sigma_{\gamma'} \), we have

\[ \|e^{-\omega z}f(z)\|_X \leq \frac{C}{|z|^\gamma}g(1/|z|) \]

with some constant \( C = C(\gamma) > 0 \) and \( f(\lambda) = q(\lambda) \) for all \( \lambda > \omega \). Then the function \( q \) has a holomorphic extension \( q: \omega + \Sigma_{\alpha + \frac{\gamma'}{2}} \to X \) such that for all \( \gamma \in (0, \alpha) \) and all \( \lambda \in \omega + \Sigma_{\gamma' + \frac{\gamma'}{2}} \) holds

\[ \|q(\lambda)\|_X \leq Mg(\lambda - \omega), \]

with some constant \( M = M(\gamma) > 0 \).

**Proof.** The proof is quite similar to the proof of the Theorem 1.2.3. The only difference is the following estimate for \( \lambda > \omega \):

\[ \left|e^{-\lambda \omega}Re^{-s^z \theta}\right| = e^{-\lambda \omega}Re^{Re^{-s^z \theta}} = e^{-\lambda \omega}R \cos \theta, \]

and hence we have for all \( \theta \in [0, \gamma] \) the estimate

\[ \left\|\int_0^\gamma e^{-\lambda Re^{i\theta}}f(Re^{i\theta})(\pm i)Re^{\pm i\theta}d\theta\right\|_X \leq Cg(1/R)\frac{\pi}{2}e^{-(\lambda - \omega)R \cos \gamma} \to 0, \]

as \( R \to \infty \) by assumptions on the function \( g \). Thus, we have that

\[ \int_0^\infty e^{-\lambda t}f(t)dt = \int_0^\infty e^{-\lambda se^{i\gamma}}f(s e^{i\gamma}) e^{i\gamma} ds = \int_{i\gamma} e^{-\lambda z}f(z)dz, \]

for all \( \lambda > \omega \). The subsequent arguments are all the same as in the proof of Theorem 1.2.3. \( \square \)

**Remark 1.2.5.** The assumptions on the function \( g \) in Theorem 1.2.3 and Corollary 1.2.4 are technical. Nevertheless, these assumptions are satisfied for the function \( g(s) = s^{-\alpha}, \alpha \in (0, 1] \). Thus, we can understand these both results as an extension of the corresponding part of the Analytic Representation Theorem from Arendt et al., cf. [ABHN91, Theorem 2.6.1].
1.2.2 Asymptotic Behaviour of Vector-Valued Holomorphic Functions

For the proof of our Tauberian Theorem we use the following two auxiliary results, taken from [ABHN01, Proposition A. 2 and Theorem A. 5].

Proposition 1.2.6 (Identity Theorem for Holomorphic Functions). Let $Y$ be a closed subspace of a Banach space $X$ and $f : \Omega \to X$ be holomorphic. Assume that there exists a convergent sequence $(z_n)_{n \in \mathbb{N}} \subset \Omega$ such that $\lim_{n \to \infty} z_n \in \Omega$ and $f(z_n) \in Y$ for all $n \in \mathbb{N}$. Then $f(z) \in Y$ for all $z \in \Omega$.

Theorem 1.2.7 (Vitali's Convergence Theorem). Let $\Omega \subset \mathbb{C}$ be open and connected. For every $n \in \mathbb{N}$ let $f_n : \Omega \to X$ be holomorphic such that
\[
\sup_{n \in \mathbb{N}, z \in B_{r}(z_0)} \|f_n(z)\|_X < \infty
\]
whenever $B_r(z_0) \subset \Omega$. Assume that the set
\[
\Omega_0 = \{z \in \Omega : \lim_{n \to \infty} f_n(z) \text{ exists} \}
\]
has a limit point in $\Omega$. Then there exists a holomorphic function $f : \Omega \to X$ such that
\[
f^{(k)}(z) = \lim_{n \to \infty} f_n^{(k)}(z)
\]
uniformly on all compact subsets of $\Omega$ for all $k \in \mathbb{N}_0$.

We start with the following adapted version of Proposition 2.6.3 from [ABHN01]. The difference to the known result is, that only the boundedness in the neighbourhood of the origin is required and not on the whole sector. The proof uses the same arguments as in the book from Arendt, Batty, Hieber and Neubrander [ABHN01].

Proposition 1.2.8. Let $0 < \alpha \leq \pi$ and let $f : \Sigma_{\alpha} \to X$ be holomorphic such that for some $r > 0$ and all $0 < \beta < \alpha$ we have
\[
\sup_{z \in \Sigma_{\beta} \cap B_r(0)} \|f(z)\|_X < \infty.
\]
Let $x \in X$. If $\lim_{t \to 0} f(t) = x$, then $\lim_{|z| \to 0, z \in \Sigma_{\beta} \cap B_r(0)} f(z) = x$ for all $0 < \beta < \alpha$.

Proof. We set $\Omega = \Sigma_{\alpha} \cap B_r(0)$ and for $n \in \mathbb{N}$ we define $f_n(z) = f\left(\frac{z}{n}\right)$, $z \in \Omega$. We have for all $n \in \mathbb{N}$, $0 < \beta < \alpha$ and $z \in \Sigma_{\beta} \cap B_r(0)$ that $z/n \in \Sigma_{\beta} \cap B_{r/n}(0)$, this implies
We set \( z \) weakened in the same sense as for the Analytic Representation Theorem 1.2.1. The idea of the proof is analogue to the book [ABHN01]. This implies that \( \| \Omega \) The set \( \Omega \) cf. [ABHN01, Theorem 2.6.4]. Note, that the required decay behaviour is ilar to the Tauberian Theorem from Arendt, Batty, Hieber and Neubrander, Now, let \( z \) get lim identity theorem for holomorphic functions Proposition 1.2.6. All in all, we ˜hence \( \Omega \)

\[
\sup_{n \in \mathbb{N}} \sup_{z \in \Sigma_\beta \cap B_r(0)} \| f_n(z) \|_X = \sup_{n \in \mathbb{N}} \sup_{z \in \Sigma_\beta \cap B_r(0)} \left\| f \left( \frac{z}{n} \right) \right\|_X
\]

\[
= \sup_{n \in \mathbb{N}} \sup_{z \in \Sigma_\beta \cap B_r(0)} \| f(z) \|_X
\]

\[
\leq \sup_{n \in \mathbb{N}} \sup_{z \in \Sigma_\beta \cap B_r(0)} \| f(z) \|_X
\]

\[
= \sup_{z \in \Sigma_\beta \cap B_r(0)} \| f(z) \|_X < \infty,
\]

by assumption.

In particular, let \( \overline{B}_r(z_0) \subset \Omega \), then there is \( \beta' \in (0, \alpha) \) such that \( \overline{B}_r(z_0) \subset \Sigma_{\beta'} \cap B_r(0) \) and hence

\[
\sup_{n \in \mathbb{N}, z \in \overline{B}_r(z_0)} \| f_n(z) \|_X \leq \sup_{n \in \mathbb{N}, z \in \Sigma_\beta \cap B_r(0)} \| f_n(z) \|_X < \infty.
\]

The set \( \Omega_0 = \{ z \in \Omega : \lim_{n \to \infty} f_n(z) \text{ exists} \} \supset (0, r) \) has a limit point in the open and connected set \( \Omega \). Now, Vitali’s Theorem 1.2.7 implies that there is a holomorphic function \( \tilde{f} : \Omega \to X \) such that \( \tilde{f}(z) = \lim_{n \to \infty} f_n(z) \) uniformly on compact subsets of \( \Omega \).

Let \( K \subset \Omega \) be a connected and open set with \( [a, b] \subset K \) for some \( 0 < a < b < r \). We set \( z_n = a + 1/n \) for \( n \geq k_0 \in \mathbb{N} \) with \( a + 1/k_0 < b \). Then \( z_n \to a \) as \( n \to \infty \) and

\[
\tilde{f}(z_k) = \lim_{n \to \infty} f_n(z_k) = \lim_{n \to \infty} f(z_k/n) = \lim_{t \to 0} f(t) = x,
\]

hence \( \tilde{f}(z_k) = x \) for all \( k \geq k_0 \); and thus we have \( \tilde{f}(z) = x \) for all \( z \in \Omega \) by the identity theorem for holomorphic functions Proposition 1.2.6. All in all, we get \( \lim_{n \to \infty} f_n(z) = x \) uniformly on compact subsets of \( \Omega \).

Let \( 0 < \beta < \alpha \), \( 0 < \eta \leq r \) and \( \varepsilon > 0 \) arbitrary, but fixed. There is some \( k_0 = k_0(\eta) \in \mathbb{N} \) such that \( \| f_k(z) - x \|_X \leq \varepsilon \) for all \( z \in \overline{\Sigma}_\beta \) with \( \frac{\eta}{2} \leq |z| \leq \eta, k \geq k_0 \geq 2 \). Now, let \( z \in \overline{\Sigma}_\beta \) with \( |z| \leq \frac{k_0}{2} \leq \frac{\eta}{2} \), choose \( k \in \mathbb{N} \) (\( k \geq k_0 \)) such that \( \frac{k}{k+1} \eta < |z| \leq \frac{\eta}{k} \). This implies that \( \| f(z) - x \|_X = \| f_k(kz) - x \|_X \leq \varepsilon \). Hence, the claim follows. \( \square \)

Finally, we are in the situation to formulate our Tauberian Theorem similar to the Tauberian Theorem from Arendt, Batty, Hieber and Neubrander, cf. [ABHN01, Theorem 2.6.4]. Note, that the required decay behaviour is weakened in the same sense as for the Analytic Representation Theorem 1.2.4. The idea of the proof is analogue to the book [ABHN01].
1. The Laplace Transform

Theorem 1.2.9 (Tauberian Theorem). Let \( g : (0, \infty) \to \mathbb{R}_+ \) be non-negative, non-increasing such that \( \sup_{s \in [0,1]} s g(s) < \infty \) and \( \int_0^1 g(1/t) \frac{dt}{t} < \infty \). Suppose \( q : \Sigma_{\alpha + \frac{\pi}{2}} \to X \) is holomorphic for some \( \alpha \in (0, \frac{\pi}{2}] \) and satisfies for all \( \gamma \in (0, \alpha) \) and all \( \lambda \in \Sigma_{\gamma + \frac{\pi}{2}} \) the estimate

\[
\|q(\lambda)\|_X \leq Mg(|\lambda|),
\]

with some constant \( M = M(\gamma) > 0 \). Let \( f \) be such as in Theorem 1.2.1 with \( \hat{f}(\lambda) = q(\lambda) \) for all \( \lambda > 0 \).

Then \( \lim_{t \to \infty} f(t) = x \) if and only if \( \lim_{\lambda \to 0} \lambda q(\lambda) = x \).

Proof. Assume that \( \lim_{t \to \infty} f(t) = x \). The claim follows by [ABHN01, Theorem 4.1.2], since \( f \in L^1_{1,\text{loc}}(\mathbb{R}_+; X) \). Indeed, let \( T > 0 \) be arbitrary but fixed, Theorem 1.2.1 yields the estimate

\[
\int_0^T \|f(t)\|_X dt \leq C \int_0^T \frac{g(1/t)}{t} dt \leq C \int_0^1 \frac{g(1/t)}{t} dt + C \int_{1/T}^{1} \frac{g(s)}{s} ds \leq C \int_0^1 \frac{g(1/t)}{t} dt + C \sup_{s \in [1/T,1]} s g(s) \int_{1/T}^{1} \frac{ds}{s^2} < \infty,
\]

by the assumptions on the function \( g \).

For the converse implication we assume w.l.o.g. that \( x = 0 \), otherwise we replace \( f(t) \) by \( f(t) - x \). Assume that \( \lim_{\lambda \to 0} \lambda q(\lambda) = x \). Let \( \gamma \in (0, \alpha) \) be arbitrary, but fixed. By assumption we have for all \( \lambda \in \Sigma_{\gamma + \frac{\pi}{2}} \) the estimate \( \|\lambda q(\lambda)\|_X \leq M(\gamma) g(|\lambda|) \), with some constant \( M = M((\alpha + \gamma)/2) > 0 \). Since \( \sup_{(0,1]} s g(s) < \infty \) we have

\[
\sup_{\lambda \in \Sigma_{\gamma + \frac{\pi}{2}} \cap B_1(0)} \|\lambda q(\lambda)\|_X \leq M \sup_{s \in [0,1]} s g(s) < \infty.
\]

In particular, we have by Proposition 1.2.8 that

\[
\lim_{|\lambda| \to 0, \lambda \in \Sigma_{\gamma + \frac{\pi}{2}} \cap B_1(0)} \lambda q(\lambda) = x.
\]

Let \( \varepsilon > 0 \) be arbitrary, but fixed. There exists \( \delta_0 \in (0,1] \) with \( \|\lambda q(\lambda)\|_X \leq \varepsilon \) whenever \( \lambda \in \Sigma_{\gamma + \frac{\pi}{2}} \) and \( |\lambda| \leq \delta_0 \). Let \( 0 < \frac{1}{T} \leq \delta_0 \). Now, we choose the contour \( \Gamma \) as in the proof of Theorem 1.2.1 with \( \delta = \frac{1}{T} \) and \( f(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} q(\lambda) d\lambda \). We have
Thus, we have a domination by an integrable function. The dominated convergence theorem implies the convergence of the integral to zero. It follows since
$$
\epsilon = \frac{e}{2\pi} \int_{-\gamma - \frac{\pi}{2}}^{\gamma + \frac{\pi}{2}} e^{i\theta} d\theta
$$
$$
\leq \epsilon e,
$$
since \( \|q(\lambda)\|_X \leq \epsilon \) for \( |\lambda| = \|\frac{1}{\lambda}\| = \frac{1}{\gamma} \leq \delta_0 \). Furthermore, we have
$$
\frac{1}{2\pi i} \int_{\Gamma_0} e^{\lambda t} q(\lambda) d\lambda = \frac{1}{2\pi i} \int_1^\infty e^{re^{i\pi}(y + \frac{\pi}{2})} q(re^{i\pi}(y + \frac{\pi}{2})) e^{r} dr
$$
$$
= \frac{1}{2\pi i} \int_1^\infty e^{se^{i\pi}(y + \frac{\pi}{2})} q\left( \frac{s}{t} e^{i\pi}(y + \frac{\pi}{2}) \right) \frac{s}{t} e^{i\pi}(y + \frac{\pi}{2}) ds \rightarrow 0,
$$
as \( t \rightarrow \infty \) by the dominated convergence theorem. Indeed, for \( s \geq 1 \) we set \( t_0 = t_0(s) = \max\{1/\delta_0, 1/s\} \). For all \( t > t_0 \) we have \( \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \in \Sigma_{y + \frac{\pi}{2}} \cap \overline{B}_1(0) \) and hence \( \lim_{t \rightarrow \infty} \left\| q\left( \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right) \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right\|_X = 0 \). This implies
$$
\left\| e^{se^{i\pi}(y + \frac{\pi}{2})} q\left( \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right) \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right\|_X \leq \frac{e^{-s\sin \gamma}}{s} \left\| q\left( \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right) \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right\|_X \rightarrow 0,
$$
as \( t \rightarrow \infty \); the integrand converges pointwise to zero.

Moreover, the property that the function \( g(s) \) is non-increasing implies for \( s \geq 1 \) that
$$
\left\| e^{se^{i\pi}(y + \frac{\pi}{2})} q\left( \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right) \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right\|_X \leq Me^{-s\sin \gamma} \frac{g(s/t)}{t} \leq Me^{-s\sin \gamma} \frac{g(1/t)}{t},
$$
Using the fact that \( \sup_{s \in [0,1]} s g(s) = M_0 < \infty \) and that \( t > 1 \) we obtain for all \( s \geq 1 \) the estimate
$$
\left\| e^{se^{i\pi}(y + \frac{\pi}{2})} q\left( \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right) \frac{s}{t} e^{i\pi(y + \frac{\pi}{2})} \right\|_X \leq Me^{-s\sin \gamma} M_0.
$$
Thus, we have a domination by an integrable function. The dominated convergence theorem implies the convergence of the integral to zero. It follows from the representation of \( f \) given above that \( \lim_{t \rightarrow \infty} \|f(t)\|_X \leq \epsilon e \). Since \( \epsilon > 0 \) was chosen arbitrary it follows the claim. \( \square \)
Scalar Volterra Equations and Kernels

In this chapter we summarise basic definitions and properties concerning scalar Volterra equations and the associated kernels. At first we introduce the scalar resolvent and integral resolvent for linear scalar Volterra equations and discuss a representation of solutions. On the basis of these considerations we give a motivation why we restrict ourselves in the following to the class of completely positive kernels. We define this class of kernels, give a characterisation for unbounded, completely positive kernels and collect general properties of these kernels and the corresponding scalar resolvent and integral resolvent.

Next, we turn to the more special class of completely monotonic kernels. Kernels of this class are of special interest, since their Laplace transform is very well understood, and they are characterised by the famous theorem of Bernstein, as Laplace transform of positive measures with support on the non-negative real axis.

An important part of this section is the paragraph about the properties of the Laplace transform of completely monotonic kernels. Beside some known facts, we are able to extend the properties of sectoriality and regularity to a sector which is larger than the right complex half-plane.

Finally, we summarise some permanence properties of completely monotonic kernels. The so-called standard kernel is an important example for such kernels. We also give an explicit expression for the corresponding scalar resolvent and integral resolvent in terms of the Mittag-Leffler function.

2.1 Scalar Volterra Equations

We are interested in linear scalar Volterra equations of the type

\[ u(t) + \mu(a * u)(t) = f(t), \quad t \in \mathbb{R}_+, \quad (2.1) \]

where \( a : \mathbb{R}_+ \rightarrow \mathbb{R} \) is a given kernel, \( \mu \in \mathbb{C} \) is a complex number and \( f : \mathbb{R}_+ \rightarrow \mathbb{C} \) is a given function. A more general setting can be found in Gripenberg, Londen and Staffans [GLS90] as well as Clément and Nohel [CN81]. We introduce the scalar resolvent and integral resolvent as solutions of linear Volterra equations with special right-hand side. These functions are essential for the treatment of linear Volterra equations.
### Definition 2.1.1 (Scalar Resolvent and Integral Resolvent)

Let $\mu \in \mathbb{C}$ and $a \in L_{1,\text{loc}}(\mathbb{R}_+)$. The solution $s_\mu : \mathbb{R}_+ \to \mathbb{C}$ of the scalar Volterra equation

$$s_\mu(t) + \mu(a \ast s_\mu)(t) = 1, \quad t \in \mathbb{R}_+,$$

is called **scalar resolvent** of the Volterra equation (2.1). The solution $r_\mu : \mathbb{R}_+ \to \mathbb{C}$ of the scalar Volterra equation

$$r_\mu(t) + \mu(a \ast r_\mu)(t) = a(t), \quad t \in \mathbb{R}_+,$$

is called **scalar integral resolvent** of the Volterra equation (2.1).

We know from Gripenberg, Londen and Staffans [GLS90, Theorem 2.3.1] that for each locally integrable kernel $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ there is a unique and locally integrable scalar integral resolvent $r_\mu \in L_{1,\text{loc}}(\mathbb{R}_+)$. With the aid of the integral resolvent we can express the solution of our Volterra equation via the **variation of constants formula**. We have the following existence and uniqueness theorem. For the proof we refer to [GLS90, Theorem 2.3.5].

**Theorem 2.1.2.** Let $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ and $\mu \in \mathbb{C}$. Then, for each $f \in L_{1,\text{loc}}(\mathbb{R}_+)$, there is a unique solution $u \in L_{1,\text{loc}}(\mathbb{R}_+)$ of the Volterra equation (2.1). This solution is given by

$$u(t) = f(t) - \mu(r_\mu \ast f)(t), \quad t \in \mathbb{R}_+.$$

In particular, if $f \in L_{p,\text{loc}}(\mathbb{R}_+)$, with $1 \leq p \leq \infty$, or $f \in C(\mathbb{R}_+)$, then $u \in L_{p,\text{loc}}(\mathbb{R}_+)$ or $u \in C(\mathbb{R}_+)$, respectively.

This theorem implies the existence and uniqueness of the scalar resolvent $s_\mu \in C(\mathbb{R}_+)$ whenever $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ and the following relation between the scalar resolvent and integral resolvent:

$$s_\mu(t) = 1 - \mu(r_\mu \ast 1)(t), \quad t \in \mathbb{R}_+.$$  \hspace{1cm} (2.2)

In particular, we know that the scalar resolvent $s_\mu$ is absolutely continuous on $\mathbb{R}_+$ and differentiable almost everywhere on $\mathbb{R}_+$ with $s_\mu'(t) = -\mu r_\mu(t)$ for almost all $t \in \mathbb{R}_+$.

In the following we consider a special case of Volterra equations

$$u(t) + \mu(a \ast u)(t) = u_0 + (a \ast g)(t), \quad t \in \mathbb{R}_+,$$  \hspace{1cm} (2.3)

with some $g \in L_{1,\text{loc}}(\mathbb{R}_+)$. The unique solution can be expressed in terms of the scalar resolvent and integral resolvent and is given by

$$u(t) = s_\mu(t)u_0 + (r_\mu \ast g)(t), \quad t \in \mathbb{R}_+.$$  \hspace{1cm} (2.4)
2.2. Completely Positive Kernels

Hence, if one understands the behaviour of the scalar resolvent and integral resolvent very well one understands in particular the solution of the Volterra equation (2.3). We want to motivate in the following, why we restrict ourselves to Volterra equations with a special class of kernels. J. Levin proved in the 1970s the following properties of solutions of the Volterra equation (2.1), cf. [Lev77, Lemma 1.3].

**Lemma 2.1.3.** Let \( \mu \in \mathbb{R}_+ \), \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be non-negative and non-increasing on \( \mathbb{R}_+ \) and \( f \in L_{1,\text{loc}}(\mathbb{R}_+) \) be non-negative and non-decreasing. Then, the solution \( u \) of the Volterra equation (2.1) is non-negative on \( \mathbb{R}_+ \) and we have \( 0 \leq u(t) \leq f(t) \) a.e. on \( \mathbb{R}_+ \).

The representation of the solution of (2.3) via the scalar resolvent and integral resolvent suggests that their properties are primarily responsible for the properties of the solution. In particular, positivity of the solution depends on the positivity of the scalar resolvent and integral resolvent. For this reason, we restrict our considerations onto a class of kernels which guarantees the positivity for the scalar resolvent and integral resolvent at least for non-negative parameters \( \mu \in \mathbb{R}_+ \).

### 2.2 Completely Positive Kernels

We start this paragraph with the definition of completely positive kernels, cf. [CN81, Definition 1.1]

**Definition 2.2.1 (Completely Positive Kernel).** A kernel \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) is called **completely positive** if for all \( \mu \in \mathbb{R}_+ \) the functions \( r_\mu \) and \( s_\mu \) are non-negative on \( \mathbb{R}_+ \).

It is known from Clement and Nohel [CN81] that the kernel \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) is completely positive if \( a \) is non-negative, non-increasing and \( \log a \) is convex on \( \mathbb{R}_+ \). Furthermore, Clement and Nohel give in [CN81, Theorem 2.2] the following characterisation of unbounded, completely positive kernels:

**Theorem 2.2.2.** Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \setminus L_{\infty,\text{loc}}(\mathbb{R}_+) \) and \( a \not\equiv 0 \). Then \( a \) is a completely positive kernel if and only if there exists \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) non-negative and non-increasing satisfying: \( (a \ast b)(t) = 1 \) for all \( t \in (0,\infty) \).

**Remark 2.2.3.** N. Sonine was the first in the 1880s who considered so-called Abel integral equations with kernels \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) satisfying the condition that there is another kernel \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) with \( (a \ast b)(t) = 1 \) for all \( t \in (0,\infty) \), see [SKM93, p. 85]. For this reason we speak in this situation of the **Sonine condition** and call the kernel \( b \) the corresponding **Sonine kernel** (to the kernel \( a \)).
On the basis of the Sonine condition it is possible to deduce some general properties of the kernels, cf. [SC03, Lemma 4.3]. Note that these properties are satisfied for any pair of completely positive kernels.

Lemma 2.2.4. Let \( a, b \in L_{1,\text{loc}}(\mathbb{R}_+) \) non-negative and non-increasing kernels satisfying the Sonine condition \( (a \ast b)(t) = 1 \) for all \( t \in (0, \infty) \). Then the following relations hold for all \( t > 0 \):

(i) \( ta(t)b(t) \leq 1 \),
(ii) \( a(t)\int_0^t b(\tau)d\tau \leq 1 \), \( b(t)\int_0^t a(\tau)d\tau \leq 1 \),
(iii) \( a(t)\int_0^t b(\tau)d\tau + b(t)\int_0^t a(\tau)d\tau \geq 1 \),
(iv) \( \int_0^t a(\tau)d\tau \int_0^t b(\tau)d\tau \geq t \),
(v) \( \lim_{\tau \to 0^+} a(\tau) = \lim_{\tau \to 0^+} b(\tau) = +\infty \),
(vi) \( \lim_{\tau \to 0^+} \tau a(\tau) = \lim_{\tau \to 0^+} \tau b(\tau) = 0 \).

In the following proposition we summarise some important facts about the scalar resolvent and the scalar integral resolvent of Volterra equations with completely monotonic kernels. For the proof we refer to Clement and Nohel [CN81, Section 2].

Proposition 2.2.5. Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be a completely positive kernel. Then:

(i) The kernel \( a \) is non-negative on \( \mathbb{R}_+ \) and, for all \( \mu \in \mathbb{R}_+ \) the scalar resolvent \( s_\mu \) is non-negative and non-increasing on \( \mathbb{R}_+ \) and we have the estimate \( s_\mu(t)[1 + \mu(1 \ast a)(t)] \leq 1 \) for all \( t \in \mathbb{R}_+ \).

(ii) For each \( \mu > 0 \), the scalar integral resolvent \( r_\mu \) is itself completely positive on \( \mathbb{R}_+ \).

(iii) If \( a \notin L_1(\mathbb{R}_+) \), then for all \( \mu > 0 \) we have \( \lim_{t \to \infty} s_\mu(t) = 0 \) and the equation \( \mu \| r_\mu \|_{L_1(\mathbb{R}_+)} = 1 \).

(iv) If \( a \notin L_{\infty,\text{loc}}(\mathbb{R}_+) \), then for all \( \mu \in \mathbb{C} \) we have \( s_\mu(t) = (b \ast r_\mu)(t) \), for all \( t \in \mathbb{R}_+ \), where \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) denotes the corresponding Sonine kernel to the kernel \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \).

The following lemma gives a lower pointwise estimate for the scalar resolvent in the case \( \mu \in \mathbb{R}_+ \).

---

1 We are mainly interested in unbounded kernels which are not integrable on \( \mathbb{R}_+ \) but have an integrable singularity at zero. Therefore we focus on results in this situation. For further conclusions we refer to [CN81].
2.3. Completely Monotonic Functions and Kernels

**Lemma 2.2.6.** Let \(a \in L^1_{\text{loc}}(\mathbb{R}_+)\) be unbounded and completely positive. For each \(\mu \in \mathbb{R}_+\) we have the estimate \([\mu + b(t)] s_\mu(t) \geq b(t)\), for all \(t \in \mathbb{R}_+\), where \(b \in L^1_{\text{loc}}(\mathbb{R}_+)\) denotes the corresponding Sonine kernel.

**Proof.** Let \(\mu \in \mathbb{R}_+\) be arbitrary, but fixed. Since the kernel \(a \in L^1_{\text{loc}}(\mathbb{R}_+)\) is completely positive, we know from Proposition 2.2.5 that we have for all \(t \in \mathbb{R}_+\) the relation \(\mu s_\mu(t) = \mu(b \ast r_\mu)(t)\). The corresponding Sonine kernel \(b \in L^1_{\text{loc}}(\mathbb{R}_+)\) is non-increasing, so we can estimate \(\mu s_\mu(t) \geq \mu b(t)(1 \ast r_\mu)(t)\). The relation between the scalar resolvent and integral resolvent (2.2) yields \(\mu s_\mu(t) \geq b(t)[1 - s_\mu(t)]\) and the claim follows immediately. \(\square\)

This estimate in combination with Proposition 2.2.5 (i) yields a very precise description of the behaviour of the scalar resolvent in case of completely positive kernels and \(\mu \in \mathbb{R}_+\):

\[
\frac{b(t)}{\mu + b(t)} \leq s_\mu(t) \leq \frac{1}{1 + \mu(1 \ast a)(t)}, \quad t \in \mathbb{R}_+.
\] (2.5)

### 2.3 Completely Monotonic Functions and Kernels

A large class of completely positive kernels is the class of completely monotonic kernels. These kernels are of special interest, since their Laplace transform is very well understood. We start with the definition of completely monotonic kernels and subsequently we discuss their properties.

**Definition 2.3.1 (Completely Monotonic Function).** A function \(a: (0, \infty) \to \mathbb{R}\) is completely monotonic if \(a \in C^\infty((0, \infty))\) and for all \(n \in \mathbb{N}_0\) and all \(t > 0\) we have \((-1)^n a^{(n)}(t) \geq 0\).

By [Mil68, Lemma 1] we know that a completely monotonic function either vanishes or is positive on \((0, \infty)\). Furthermore, a non-vanishing completely monotonic function is log-convex, see [Mil68, Lemma 2].\(^2\) In particular, a completely monotonic kernel \(a \in L^1_{\text{loc}}(\mathbb{R}_+)\) is completely positive. Moreover, if \(a \in L^\infty_{\text{loc}}(\mathbb{R}_+)\), then the corresponding Sonine kernel \(b \in L^1_{\text{loc}}(\mathbb{R}_+)\) is completely monotonic, too, cf. [GLS90, Theorem 5.5.4].

Gripenberg, Staffans and Londen give us the following characterisation for the scalar integral resolvent of completely monotonic kernels, see [GLS90, Theorem 5.3.1].

**Theorem 2.3.2.** Let \(a \in L^1_{\text{loc}}(\mathbb{R}_+)\) and for \(\mu > 0\) we consider the corresponding scalar integral resolvent \(r_\mu: \mathbb{R}_+ \to \mathbb{R}\), the solution of the scalar Volterra equation

\[
r_\mu(t) + \mu(a \ast r_\mu)(t) = a(t), \quad t \in \mathbb{R}_+.
\]

\(^2\)In the following we always exclude the case of vanishing functions whenever we speak about complete monotonicity.
2. Scalar Volterra Equations and Kernels

Then the kernel $a$ is completely monotonic if and only if the scalar integral resolvent $r_\mu$ is completely monotonic, $r_\mu \in L_1(\mathbb{R}_+)$, and $\|r_\mu\|_{L_1(\mathbb{R}_+)} \leq 1/\mu$.

**Remark 2.3.3.** In particular, the corresponding scalar resolvent $s_\mu$, the solution of the scalar Volterra equation

$$s_\mu(t) + \mu(a * s_\mu)(t) = 1, \quad t \in \mathbb{R}_+,$$

is completely monotonic, too. Since we have the relation $s_\mu(t) = 1 - \mu(r_\mu * 1)(t)$, $t \in \mathbb{R}_+$, the differentiation of this expression yields the claim. Moreover, the relation $s'_\mu(t) = -\mu r_\mu(t)$ is valid for all $t > 0$.

A characterisation of completely monotonic functions is given by Bernstein’s Theorem. For a proof of this result we refer to [Wid41, Chapter IV.12].

**Theorem 2.3.4 (Bernstein’s Theorem).** A function $a$ is completely monotonic if and only if there is a non-decreasing function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$a(x) = \int_0^\infty e^{-xt} d\beta(t), \quad x > 0.$$

In other words: Completely monotonic functions are Laplace transforms of positive measures supported on $\mathbb{R}_+$; the characterisation is unique if one normalises the function $\beta$ by the assumptions $\beta(0) = 0$ and left-continuity, cf. [Prü12, p. 90].

2.3.1 Properties of the Laplace Transform of Completely Monotonic Kernels

Many aspects in the studies of Volterra equations base on the Laplace transform of the Volterra equation. The Laplace transform of completely monotonic kernels has special properties, which will summarised in this section.

Firstly, we introduce the term of kernels with subexponential growth.

**Definition 2.3.5 (Subexponential Growth).** A kernel $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ is of subexponential growth if we have

$$\int_0^\infty e^{-\epsilon t} |a(t)| dt < \infty$$

for all $\epsilon > 0$.

This property ensures that the Laplace transform of the kernel $a$ exists the open right half-plane $\mathbb{C}_+$, a important fact since we want to use the Laplace transform to study our Volterra equations.

**Remark 2.3.6.** a) For a kernel $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ of subexponential growth we have $\text{abs} |a| \leq 0$. In particular, the map $\lambda \to \hat{a}(\lambda)$ is holomorphic for all $\lambda \in \mathbb{C}_+$, as well as for all $n \in \mathbb{N}_0$ and all $\lambda \in \mathbb{C}_+$ we have

$$\hat{a}^{(n)}(\lambda) = \int_0^\infty e^{-\lambda t} (-t)^n a(t) dt,$$

see Theorem 1.1.3.
Moreover, if the kernel \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \) is non-negative and of subexponential growth, then its Laplace transform \( \hat{a} : (0, \infty) \to \mathbb{R} \) is completely monotonic.

b) For a kernel \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \) of subexponential growth, we have for all \( \lambda \in \mathbb{C}_+ \) that \( \tilde{a}(\lambda) = \hat{a}(\lambda) \). Hence, for the examination of the Laplace transform it is enough to study \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda > 0 \).

We refer to Gripenberg, Londen and Staffans [GLS90, Theorem 5.2.6] for the following theorem which summarises properties of the Laplace transform of completely monotonic and locally integrable kernels.

**Theorem 2.3.7.** Let \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \) be completely monotonic. The Laplace transform \( \hat{a} \) has the following properties:

(i) The Laplace transform \( \hat{a} \) has a holomorphic extension to \( \Sigma \pi \) via

\[
\hat{a}(\lambda) = \int_0^\infty \frac{d\beta(t)}{\lambda + t},
\]

where \( \beta \) is the uniquely determined function from Bernstein’s Theorem.

(ii) The Laplace transform \( \hat{a}(x) \) is real and non-negative for \( x > 0 \).

(iii) We have \( \lim_{x \to \infty} \hat{a}(x) = 0 \).

(iv) \( \text{Im} \hat{a}(z) \leq 0 \) for all \( \text{Im} z > 0 \).

Moreover, \( a \in L_1(\mathbb{R}_+) \) if and only if \( \limsup_{x \to 0} |\hat{a}(x)| < \infty \).

The representation as Laplace transform of a positive measure shows that \( \hat{a}(\lambda) \neq 0 \) for all \( \lambda \in \Sigma \pi \), see [Prü12, p. 56].

Now, we study the mapping behaviour of the Laplace transform \( \hat{a} \) of a completely monotonic kernel \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \), where we use the representation of \( \hat{a} \) from Theorem 2.3.7 (i). At first we prove that \( \hat{a} \) is injective.

**Lemma 2.3.8.** Let \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \) be a completely monotonic kernel.

Then \( \hat{a} : \mathbb{C}_+ \to \mathbb{C} \) is injective.

**Proof.** Suppose, to the contrary, that there are \( \lambda_1, \lambda_2 \in \mathbb{C}_+ \) with \( \lambda_1 \neq \lambda_2 \) such that \( \hat{a}(\lambda_1) = \hat{a}(\lambda_2) \).

In the first instance we know that \( \text{Im} \lambda_1 \) and \( \text{Im} \lambda_2 \) have the same sign.\(^3\)

Indeed, if not, that is \( \text{sgn} \text{Im} \lambda_1 \neq \text{sgn} \text{Im} \lambda_2 \), then \( \text{sgn} \text{Im}(\lambda_1 + \sigma) \neq \text{sgn} \text{Im}(\lambda_2 + \sigma) \) for all \( \sigma \in \mathbb{R}_+ \). Thus, we have

\[
\text{sgn} \text{Im} \int_0^\infty \frac{d\beta(\sigma)}{\lambda_1 + \sigma} \neq \text{sgn} \text{Im} \int_0^\infty \frac{d\beta(\sigma)}{\lambda_2 + \sigma},
\]

\(^3\)We set for convenience \( \text{sgn} 0 = 0 \).
where \( \beta \) is the positive measure supported on \( \mathbb{R}_+ \) from the representation of \( \hat{a} \) in Theorem 2.3.7 (i). But relation (2.6) is a contradiction to the assumption that \( \hat{a}(\lambda_1) = \hat{a}(\lambda_2) \).

In case of \( \text{Im} \lambda_1 = \text{Im} \lambda_2 = 0 \), the assumption \( \hat{a}(\lambda_1) = \hat{a}(\lambda_2) \) implies directly that \( \lambda_1 = \lambda_2 \), since \( \hat{a} \) is itself completely monotonic on \((0, \infty)\), cf. Remark 2.3.6 a), and thus injective on \((0, \infty)\).

Now, we assume w.l.o.g. that \( \text{Im} \lambda_1 > 0 \). By the representation of \( \hat{a} \) in Theorem 2.3.7 (i) we obtain

\[
0 = \hat{a}(\lambda_1) - \hat{a}(\lambda_2) = \int_0^\infty \frac{\lambda_2 - \lambda_1}{(\lambda_1 + \sigma)(\lambda_2 + \sigma)} \, d\beta(\sigma).
\]

Due to the assumption that \( \lambda_1 \neq \lambda_2 \) we deduce that

\[
\int_0^\infty \frac{d\beta(\sigma)}{(\lambda_1 + \sigma)(\lambda_2 + \sigma)} = 0. \tag{2.7}
\]

Since \( \text{sgn} \text{Im} \lambda_1 = \text{sgn} \text{Im} \lambda_2 > 0 \), we have for all \( \sigma \in \mathbb{R}_+ \) that \( \text{Im}(\lambda_i + \sigma) > 0 \) and \( \lambda_i + \sigma \in \mathbb{C}_+, i \in \{1, 2\} \). Moreover, we obtain \( (\lambda_1 + \sigma)(\lambda_2 + \sigma) \in \Sigma_\pi \) and \( \text{Im}[(\lambda_1 + \sigma)(\lambda_2 + \sigma)] > 0 \) for all \( \sigma \in \mathbb{R}_+ \) and hence

\[
\text{Im} \int_0^\infty \frac{d\beta(\sigma)}{(\lambda_1 + \sigma)(\lambda_2 + \sigma)} \neq 0,
\]

in contradiction to relation (2.7). Hence, we have \( \hat{a}(\lambda_1) \neq \hat{a}(\lambda_2) \) for all \( \lambda_1, \lambda_2 \in \mathbb{C}_+ \) with \( \lambda_1 \neq \lambda_2 \).

In a certain manner we can extend the property of injectivity of the Laplace transform of completely monotonic kernel to some slightly larger sector.

**Lemma 2.3.9.** Let \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \) be a completely monotonic kernel and choose some \( \delta \in (0, \frac{\pi}{2}) \).

Then for all \( \rho \in \Sigma_{\frac{\pi}{2} - \delta} \) and all \( \lambda \in \left\{ \Sigma_{\frac{\pi}{2} + \delta} - \rho \right\} \setminus \{0\} \) we have \( \hat{a}(\lambda + \rho) \neq \hat{a}(\rho) \).

**Proof.** We set \( \varepsilon = \delta/2 \) and suppose, to the contrary, that there is some \( \rho_0 \in \Sigma_{\frac{\pi}{2} - \delta} \) and some \( \lambda_0 \in \left\{ \Sigma_{\frac{\pi}{2} + \varepsilon} - \rho_0 \right\} \setminus \{0\} \) such that \( \hat{a}(\lambda_0 + \rho_0) = \hat{a}(\rho_0) \). Analogously to the proof of the previous Lemma 2.3.8 it follows that \( \text{Im}(\lambda_0 + \rho_0) \) and \( \text{Im} \rho_0 \) have the same sign. It is not possible that \( \lambda_0 + \rho_0 \) and \( \rho_0 \) are real, since \( \hat{a} \) is itself completely monotonic on \((0, \infty)\), cf. Remark 2.3.6 a), and thus injective on \((0, \infty)\).

W.l.o.g. we assume that \( \text{Im} \rho_0 > 0 \). By the representation of \( \hat{a} \) in Theorem 2.3.7 (i) we obtain

\[
0 = \hat{a}(\rho_0) - \hat{a}(\lambda_0 + \rho_0) = \int_0^\infty \frac{\lambda_0}{(\rho_0 + \sigma)(\lambda_0 + \rho_0 + \sigma)} \, d\beta(\sigma).
\]
Due to the assumption $\lambda_0 \neq 0$ we deduce that
\[ \int_0^\infty \frac{d\beta(\sigma)}{(\rho_0 + \sigma)(\lambda_0 + \rho_0 + \sigma)} = 0. \tag{2.8} \]
We have $\text{sgn Im}(\rho_0) = \text{sgn Im}(\lambda_0 + \rho_0) > 0$, thus we obtain for all $\sigma \in \mathbb{R}_+$ that $\text{Im}(\rho_0 + \sigma) > 0$ and $\rho_0 + \sigma \in \Sigma_{\frac{\pi}{2} - \delta}$, as well as $\text{Im}(\lambda_0 + \rho_0 + \sigma) > 0$ and that $\lambda_0 + \rho_0 + \sigma \in \Sigma_{\frac{\pi}{2} + \epsilon}$. Moreover, we deduce that $(\rho_0 + \sigma)(\lambda_0 + \rho_0 + \sigma) \in \Sigma_{\pi - \epsilon}$ and $\text{Im}[(\rho_0 + \sigma)(\rho_0 + \lambda_0 + \sigma)] > 0$ for all $\sigma \in \mathbb{R}_+$. It follows that
\[ \text{Im} \int_0^\infty \frac{d\beta(\sigma)}{(\rho_0 + \sigma)(\lambda_0 + \rho_0 + \sigma)} 
eq 0, \]
in contradiction to relation (2.8). Hence, we have for all $\rho \in \Sigma_{\frac{\pi}{2} - \delta}$ and all $\lambda \in \{\Sigma_{\frac{\pi}{2} + \epsilon} - \rho\} \backslash \{0\}$ that $d(\lambda + \rho) \neq d(\rho)$.

To prove the following theorems we need a technical result for estimates in the complex plane.

**Lemma 2.3.10.** Let $\eta \in (0, \pi]$. Then there exists a constant $c_\eta > 0$ such that we have for all $\lambda \in \Sigma_{\pi - \eta}$ the estimate
\[ 1 + |\lambda| \leq c_\eta |1 + \lambda|. \]

**Proof.** Let $\lambda \in \Sigma_{\pi - \eta}$. Then there are $r \in (0, \infty)$ and $\varphi \in (-\pi + \eta, \pi - \eta)$ such that $\lambda = re^{i\varphi}$.
\[
|1 + \lambda|^2 = (1 + r \cos \varphi)^2 + r^2 \sin^2 \varphi \\
\geq 1 + r^2 - 2r \cos \varphi \\
\geq 1 - \cos \eta \left(1 + r^2\right) = \frac{1 - \cos \eta}{2}(1 + |\lambda|^2).
\]
The claim follows with $c_\eta^2 = (1 - \cos \eta)/2$. For a illustration of the proof we refer to [HJ11, p. 4].

An easy consequence is the next corollary.

**Corollary 2.3.11.** Let $\theta_1, \theta_2 \in [0, \pi)$ be such that $\theta_1 + \theta_2 \in [0, \pi)$. Then, for all $\lambda \in \Sigma_{\theta_1}$ and all $\mu \in \Sigma_{\theta_2}$ there is a constant $c_\eta > 0$ such that
\[ |\lambda + \mu| \geq c_\eta (|\lambda| + |\mu|), \]
with $\eta = \pi - (\theta_1 + \theta_2) \in (0, \pi]$.

The following theorem gives a upper and lower estimate for the Laplace transform of a completely monotonic function along a sector in the complex plane by the Laplace transform along the positive axis.
Theorem 2.3.12. Let $a \in L^1_{1, \text{loc}}(\mathbb{R}_+)$ be completely monotonic. For each $\varepsilon \in (0, \pi/4]$ there are constants $C_1 = C_1(\varepsilon), C_2 = C_2(\varepsilon) > 0$ such that for all $\lambda \in \Sigma_{3\pi/4-\varepsilon}$ we have

$$C_1 \lambda(\lambda) \leq |\lambda(\lambda)| \leq C_2 \lambda(\lambda).$$

Proof. The statement of this theorem is true for $\lambda \in \overline{C}_+ \setminus \{0\}$, cf. [Prü12, Lemma 8.1], since completely monotonic kernels are in particular 1-regular. Let $\varepsilon \in (0, \pi/4)$ be arbitrary, but fixed and let $\lambda \in \Sigma_{3\pi/4-\varepsilon} \setminus \overline{C}_+$ be arbitrary, i.e. $\text{Re} \lambda < 0$. We have

$$\lambda(\lambda) = \int_0^{\infty} \frac{\lambda + \sigma}{|\lambda + \sigma|^2} \, d\beta(\sigma),$$

and hence

$$\lambda(\lambda) = \int_0^{\infty} \frac{\lambda + \sigma}{|\lambda + \sigma|^2} \, d\beta(\sigma), \quad \text{and} \quad \lambda(\lambda) = -\int_0^{\infty} \frac{\lambda + \sigma}{|\lambda + \sigma|^2} \, d\beta(\sigma).$$

Because of $\text{Re} \lambda < 0$ we can write $\lambda = -|\lambda|$ and $\lambda = \mp |\lambda|$. We have

$$|\lambda(\lambda)|^2 = (\lambda(\lambda))^2 + (|\lambda(\lambda)|)^2 \geq \frac{1}{2} (\lambda(\lambda) + |\lambda(\lambda)|)^2$$

$$= \frac{1}{2} \left( \int_0^{\infty} -|\lambda| + \sigma + |\lambda| \frac{d\beta(\sigma)}{|\lambda + \sigma|^2} \right)^2.$$

Since $\lambda \in \Sigma_{3\pi/4-\varepsilon} \setminus \overline{C}_+$ there is some $\lambda \in (0, \pi/4-\varepsilon)$ such that $\tan \gamma |\lambda| = |\lambda|$. With this relation it follows that

$$-|\lambda| + |\lambda| + \sigma = (1 - \tan \gamma) |\lambda| + \sigma \geq 0$$

for all $\sigma \geq 0$, since $\gamma \in (0, \pi/4)$. Thus we get for all $\gamma \in (0, \pi/4 - \varepsilon)$ that

$$-|\lambda| + |\lambda| + \sigma \geq \frac{(-\tan \gamma + 1) \lambda}{\sqrt{\tan^2 \gamma + 1 \lambda}} \geq \frac{1 - \tan(\pi/4 - \varepsilon)}{\sqrt{1 + \tan^2(\pi/4 - \varepsilon)}} = C(\varepsilon) > 0.$$
2.3. Completely Monotonic Functions and Kernels

For the other estimate we use Lemma 2.3.10. There is a constant $C_\ast > 0$ such that for all $\lambda \in \Sigma_{3\pi/4}$ and all $\sigma \in \mathbb{R}_+$ we have $|\lambda| + \sigma \leq C_\ast |\lambda + \sigma|$. Thus, we obtain for all $\lambda \in \Sigma_{3\pi/4}$

$$|\hat{a}(\lambda)| \leq \int_{0}^{\infty} \frac{d\beta(\sigma)}{|\lambda + \sigma|} \leq C_\ast \int_{0}^{\infty} \frac{d\beta(\sigma)}{|\lambda| + \sigma} = C_\ast |\hat{a}(\lambda)|.$$  

The claim follows with $C_1 = \min\{C(\varepsilon)/\sqrt{2}, c^{-1}\}$ and $C_2 = \max\{C_\ast, c\}$, where $c > 0$ is the constant from [Prü12, Lemma 8.1].

**Further Properties of Kernels.** We introduce two special properties of kernels and their Laplace transform. These properties are important for our subsequent work with parabolic Volterra equations, cf. [Prü12, Chapter 3, Definition 3.1 and Definition 3.2].

The first property describes a special mapping property of the Laplace transform of kernels with subexponential growth.

**Definition 2.3.13 (Sectorial Kernel).** Let $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ be of subexponential growth and suppose $\hat{a}(\lambda) \neq 0$ for all $\text{Re} \lambda > 0$. The kernel $a$ is called $\theta$-sectorial if we have for all $\lambda \in \mathbb{C}_+$ that $|\arg \hat{a}(\lambda)| \leq \theta$.  

Here $\arg \hat{a}(\lambda)$ is defined as the imaginary part of a fixed branch of $\log \hat{a}(\lambda)$, and $\theta$ in the definition of sectorial kernels can be greater $\pi$. If the kernel $a$ is sectorial, the branch of $\log \hat{a}$ which gives the smallest angle $\theta$ is chosen. If $\hat{a}(\lambda)$ is real for real $\lambda$ we choose the principal branch.

The second definition we introduce is a notation of regularity of kernels.

**Definition 2.3.14 (Regular Kernel).** Let $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. The kernel $a$ is called $k$-regular if there is a constant $c > 0$ such that we have for $0 \leq n \leq k$ and all $\lambda \in \mathbb{C}_+$ that

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \leq c |\hat{a}(\lambda)|.$$  

We know from [Prü12, p. 69] that for a $k$-regular kernel $a$, $k \in \mathbb{N}$ arbitrary, its Laplace transform $\hat{a}$ has no zeros in the open right half-plane. Furthermore, each completely monotonic kernel is $k$-regular for all $k \in \mathbb{N}$, cf. [Prü12, p. 72].

In our next step we extend for completely monotonic kernels the properties of sectoriality and the 1-regularity onto a larger sector $\Sigma_{\frac{\pi}{4} + \varepsilon}$, with $\varepsilon > 0$.

**Extended Sectoriality for Completely Monotonic Kernels.** We show that the property of $\theta$-sectoriality can be continuously extended for completely monotonic kernels.
Lemma 2.3.15. Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be completely monotonic and \( \theta \)-sectorial with \( \theta \in (0, \frac{\pi}{2}) \). For each \( \varepsilon \in (0, \frac{\pi}{2}) \) there is some \( \delta = \delta(\varepsilon) \in (0, \frac{\pi}{2}) \) such that we have for all \( \lambda \in \Sigma_{\frac{\pi}{2} + \delta} \) the property \( \hat{a}(\lambda) \in \Sigma_{\theta + \varepsilon} \).

Proof. Since the kernel \( a \) is \( \theta \)-sectorial, we have for all \( \lambda \in \Sigma_{\frac{\pi}{2}} \) that \( \hat{a}(\lambda) \in \Sigma_{\theta} \). We are done if we can prove that \( \hat{a}(\lambda) \) depends continuously on rotations of the argument \( \lambda \). That is, for all \( \lambda \in \Sigma_{\frac{\pi}{2}} \) and all \( \varepsilon > 0 \) there is a \( \delta = \delta(\varepsilon) > 0 \) such that for all \( |\varphi| \leq \delta \) we have \( \left| \hat{a}(\lambda) - \hat{a}(\lambda e^{\pm i\varphi}) \right| \leq \varepsilon \).

By continuity of \( \hat{a} \) on \( \Sigma_{\pi} \), the claim is clear for all \( \lambda \in \Sigma_{\frac{\pi}{2}} \cap \overline{B}(0) \). Indeed, for each \( \varepsilon > 0 \) there is some \( \delta_0 = \delta_0(\varepsilon) \in (0, \frac{\pi}{2}) \) such that for \( |\varphi| \leq \delta_0 \) and all \( \lambda \in \Sigma_{\frac{\pi}{2}} \cap \overline{B}(0) \) we have \( \left| \hat{a}(\lambda) - \hat{a}(\lambda e^{\pm i\varphi}) \right| \leq \varepsilon \).

Now, let \( \lambda \in \Sigma_{\frac{\pi}{2}} \) with \( |\lambda| \geq 1 \). Then we have by the representation of \( \hat{a} \) as Laplace transform of a positive measure that

\[
\left| \hat{a}(\lambda) - \hat{a}(\lambda e^{\pm i\varphi}) \right| \leq \left| e^{\pm i\varphi} - 1 \right| \int_0^\infty \frac{|\lambda|}{|\lambda + t||\lambda e^{\pm i\varphi} + t|} \, d\beta(t),
\]

For all \( |\varphi| \leq \delta_0 \) there is a constant \( C_{\delta_0} > 0 \) such that \( t + |\lambda| \leq C_{\delta_0} |t + \lambda| \) for all \( \lambda \in \Sigma_{\frac{\pi}{2} + \delta_0} \) and all \( t \in \mathbb{R}_+ \), cf. Lemma 2.3.10. This implies the estimate

\[
\left| \hat{a}(\lambda) - \hat{a}(\lambda e^{\pm i\varphi}) \right| \leq C_{\delta_0}^2 \left| e^{\pm i\varphi} - 1 \right| \int_0^\infty \frac{|\lambda|}{(|\lambda| + t)^2} \, d\beta(t).
\]

Since \( |\lambda|/(|\lambda| + t)^2 \leq 1/(|\lambda| + 2t) \) and \( \hat{a} \) is decreasing on \((0, \infty)\) we get

\[
\left| \hat{a}(\lambda) - \hat{a}(\lambda e^{\pm i\varphi}) \right| \leq C_{\delta_0}^2 \sin(|\varphi|/2)\hat{a}(1/2).
\]

(2.9)

Obviously, there is some \( \delta \leq \delta_0 \) such that this expression is smaller \( \varepsilon \) for all \( \lambda \in \Sigma_{\frac{\pi}{2}} \) with \( |\lambda| \geq 1 \), if \( |\varphi| \leq \delta \). The claim follows with the continuity of the complex logarithm.

Extended 1-Regularity for Completely Monotonic Kernels. Finally, we extend the property of 1-regularity for completely monotonic kernels onto a larger sector.

Lemma 2.3.16. Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be completely monotonic. For each \( \eta \in (0, \pi] \) there is a constant \( C_{\eta} > 0 \) such that for all \( \lambda \in \Sigma_{\pi - \eta} \) we have the estimate

\[
\left| \lambda \hat{a}'(\lambda) \right| \leq C_{\eta} |\hat{a}(\lambda)|.
\]

Proof. For all \( \lambda \in \mathbb{C} \setminus \mathbb{R}_- \) we have

\[
\hat{a}(\lambda) = \int_0^\infty \frac{d\beta(\sigma)}{\lambda + \sigma},
\]
as well as
\[ \hat{a}'(\lambda) = -\int_0^\infty \frac{d\beta(\sigma)}{(\lambda + \sigma)^2}. \]

For each \( \eta \in (0, \pi] \) there is a constant \( c_\eta > 0 \) such that \( c_\eta |\lambda + \sigma| \geq |\lambda| + \sigma \) for all \( \sigma \in \mathbb{R}_+ \) and all \( \lambda \in \Sigma_{\pi - \eta} \), cf. Lemma 2.3.10. Hence, we have
\[
|\lambda \hat{a}'(\lambda)| \leq \int_0^\infty \frac{|\lambda|}{|\lambda + \sigma|^2} d\beta(\sigma) \leq c_\eta^2 \int_0^\infty \frac{|\lambda|}{(|\lambda| + \sigma)^2} d\beta(\sigma)
\]
\[
\leq c_\eta^2 \int_0^\infty \frac{d\beta(\sigma)}{|\lambda| + \sigma} = c_\eta^2 \hat{a}(|\lambda|).\]

\[ \square \]

**Remark 2.3.17.** a) As already mentioned, the Laplace transform of a completely monotonic kernel \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) has no zeros in \( \Sigma_\pi \). On the basis of the representation as the Laplace transform of a positive measure it follows that for every \( n \in \mathbb{N} \) and all \( \lambda \in \Sigma_\pi \) we have
\[ \hat{a}^{(n)}(\lambda) = (-1)^n n! \int_0^\infty \frac{d\beta(\sigma)}{(\lambda + \sigma)^{n+1}}. \]

Hence, the \( n \)-th derivative \( \hat{a}^{(n)} \) has no zeros in \( \Sigma_\pi \).

b) We know from Remark 2.3.6 a) that \( \hat{a} : (0, \infty) \to \mathbb{R} \) is completely monotonic. Together with the property that \( \hat{a}' \) has no zero in \( \Sigma_\pi \) it is obvious that \( \hat{a} \) is strictly decreasing on \( (0, \infty) \), in particular \( \hat{a} \) is injective on \( (0, \infty) \).

If additionally \( a \notin L_1(\mathbb{R}_+) \) we have \( \lim_{t \to 0} \hat{a}(t) = \infty \), and \( \lim_{t \to \infty} \hat{a}(t) = 0 \) by Theorem 2.3.7 (iii). Thus, the Laplace transform \( \hat{a} : (0, \infty) \to (0, \infty) \) is bijective.

### 2.3.2 Permanence Properties of Completely Monotonic Functions

It is a useful result to know that the sum and the product of completely monotonic functions is completely monotonic, too. For a proof of this result we refer to Schilling, Song and Vondraček [SSV12, Corollary 1.6].

**Corollary 2.3.18.** The set of all completely monotonic functions is a convex cone which is closed under multiplication and under pointwise convergence.
Remark 2.3.19. Examples for completely monotonic functions are
\[
\begin{align*}
  f(t) &= t^\alpha, \quad \alpha \leq 0, \\
  g(t) &= e^{-ct}, \quad c \in \mathbb{R}_+, \\
  h(t) &= \ln(1 + 1/t), \\
  k(t) &= e^{1/t},
\end{align*}
\]
for \( t > 0 \), see [SSV12].

Observe that the composition of two completely monotonic functions is in general not completely monotonic. For this we introduce another class of functions.

Definition 2.3.20 (Bernstein Function). A function \( f : (0, \infty) \to \mathbb{R} \) is a Bernstein function if \( f \in C^\infty((0, \infty)) \), \( f(t) \geq 0 \) for all \( t > 0 \) and \( f' \) is completely monotonic.

The class of Bernstein functions has similar properties like the class of completely monotonic functions, cf. [SSV12, Corollary 3.8].

Corollary 2.3.21. The set of all Bernstein functions is a convex cone, which is closed under pointwise limits and composition. Furthermore, for Bernstein functions \( f, g \in C^\infty((0, \infty)) \) and \( \alpha, \beta \in (0, 1) \) with \( \alpha + \beta \leq 1 \) is \( t \mapsto f(t^\alpha)g(t^\beta) \) again a Bernstein function.

The following structural characterisation for Bernstein functions originates from Bochner [Boc55, Chapter 4.1]. For a proof we refer to [SSV12, Theorem 3.7].

Theorem 2.3.22. Let \( f \in C^\infty((0, \infty)) \) be a positive function. Then the following statements are equivalent:

(i) The function \( f \) is a Bernstein function.

(ii) For each completely monotonic function \( g \in C^\infty((0, \infty)) \) the composition \( g \circ f \in C^\infty((0, \infty)) \) is completely monotonic.

(iii) For all \( \alpha > 0 \) the function \( e^{-\alpha f} \in C^\infty((0, \infty)) \) is completely monotonic.

In particular, the composition of a completely monotonic function with a Bernstein function always yields a completely monotonic function. Furthermore, Corollary 3.8 (iv) from Schilling, Song and Vondraček [SSV12] tells us, that for all Bernstein functions \( f \in C^\infty((0, \infty)) \) the map \( t \mapsto f(t)/t \) is completely monotonic. Thus, we can construct other completely monotonic functions with the aid of Bernstein functions.
Remark 2.3.23. Examples for Bernstein functions are

\[ f(t) = t^\alpha, \quad \alpha \in (0, 1), \]
\[ g(t) = \frac{t}{1+t}, \]
\[ h(t) = \ln(1+t), \]
for \( t > 0 \), see [SSV12, Remark 3.13].

2.3.3 Examples for Completely Monotonic Kernels

The possibly most important example for an unbounded, locally integrable and completely monotonic kernel is the so-called standard kernel, which is introduced as first example in this paragraph. The following examples base on this kernel.

Standard Kernel. For \( \alpha > 0 \) we define \( g_\alpha : (0, \infty) \to \mathbb{R}_+ \) for \( t > 0 \) by

\[ g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha), \]
and call \( g_\alpha \) the standard kernel (with exponent \( \alpha \)); here \( \Gamma \) denotes the Gamma function. Then \( g_\alpha \) is locally integrable on \( \mathbb{R}_+ \) and of sub-exponential growth, and its Laplace transform is given for all \( \lambda \in \mathbb{C}_+ \) by \( \hat{g}_\alpha(\lambda) = \lambda^{-\alpha} \). For all \( \alpha, \beta > 0 \) we have the property \( g_\alpha * g_\beta = g_{\alpha+\beta} \) on \( (0, \infty) \).

Remark 2.3.24. For \( \alpha \in (0, 1) \) the standard kernel \( a = g_\alpha \) is unbounded, locally integrable and completely monotonic, and the corresponding Sonine kernel is given by \( b = g_{1-\alpha} \). Their Laplace transforms are given by

\[ \hat{a}(\lambda) = \lambda^{-\alpha}, \quad \hat{b}(\lambda) = \lambda^{\alpha-1}, \quad \lambda \in \mathbb{C}_+. \]

In the situation of the standard kernel we can express the scalar resolvent and integral resolvent in terms of the (generalized) Mittag-Leffler function.

Definition 2.3.25. Let \( \alpha \in \mathbb{C}_+ \) and \( \beta \in \mathbb{C} \) be arbitrary.

(i) The entire function \( E_\alpha : \mathbb{C} \to \mathbb{C} \) given by

\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \]
for all \( z \in \mathbb{C} \), is called Mittag-Leffler function.

(ii) The entire function \( E_{\alpha,\beta} : \mathbb{C} \to \mathbb{C} \) given by

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \]
for all \( z \in \mathbb{C} \), is called generalised Mittag-Leffler function.
Remark 2.3.26.  

a) The functions $E_\alpha$ were introduced by G. M. Mittag-Leffler in the 1900s as generalisation of the exponential function. For example, we have $E_1(z) = e^z$ and $E_2(z^2) = \cosh(z)$, cf. [BE55, Chapter XVII], as well as $E_2(-z^2) = \cos(z)$, see [GKMR14, Proposition 3.2]. Special cases of the generalised Mittag-Leffler function are for instance $E_2,1(z) = \cosh(\sqrt{z})$ and $E_2,2(z) = \sinh(\sqrt{z})/\sqrt{z}$, see [GKMR14, Chapter 4.2]. For further information about the order and type of the entire functions $E_\alpha$ and $E_{\alpha,\beta}$, we refer to Gorenflo et al. [GKMR14].

b) For $\alpha \in (0,1]$ the completely monotonicity of the Mittag-Leffler function $E_\alpha(-x), x > 0$, was proved by Pollard [Pol48] without using methods of probability theory. A short proof for the completely monotonicity of the generalised Mittag-Leffler function $E_{\alpha,\beta}(-x), x > 0$, for all $\alpha \in (0,1]$ and $\beta \geq \alpha$, can be found in the article of Miller and Samko [MS97].

We consider the scalar Volterra equation with the standard kernel $g_\alpha$

$$u(t) + \mu(g_\alpha * u)(t) = u_0 + (g_\alpha * f)(t), \quad t \in \mathbb{R}_+, \quad (2.10)$$

where $f \in L_{1,loc}(\mathbb{R}_+)$. The scalar resolvent $s_\mu$ and integral resolvent $r_\mu$ are given for all $t \in \mathbb{R}_+$ by

$$s_\mu(t) = E_\alpha(-\mu t^\alpha),$$

$$r_\mu(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\mu t^\alpha).$$

The solution of the scalar Volterra equation (2.10) is given by $u(t) = s_\mu(t)u_0 + (r_\mu * f)(t), t \in \mathbb{R}_+$. In particular, for $\alpha \in (0,1]$ this solution is non-negative for all $\mu, u_0 \in \mathbb{R}_+$ and non-negative $f \in L_{1,loc}(\mathbb{R}_+)$. Note that for the standard kernel the scalar resolvent $s_\mu, \mu \in \mathbb{R}_+$, decays to zero as $t \to \infty$ like $t^{-\alpha}$. This rate is optimal, see estimate (2.5).

**Standard Kernel with Exponential Weight.** The standard kernel with exponential weight is a further example for an unbounded, locally integrable and completely monotonic kernel. Let $\gamma \geq 0$ and $\alpha \in (0,1)$, then we have the pair of Sonine kernels

$$a(t) = g_\alpha(t)e^{-\gamma t} + \gamma(1 * [g_\alpha e^{-\gamma}])t(t),$$

$$b(t) = g_{1-\alpha}(t)e^{-\gamma t},$$

for $t > 0$. Their Laplace transforms are given by

$$\hat{a}(\lambda) = \frac{1 + \gamma/\lambda}{(\lambda + \gamma)^\alpha}, \quad \hat{b}(\lambda) = \frac{1}{(\lambda + \gamma)^{1-\alpha}}, \quad \lambda \in \mathbb{C}_+,$$

cf. [Zac10, p. 3].
Sum of Standard Kernels. We can consider arbitrary sums of standard kernels. Let \( n \in \mathbb{N} \) and \( \alpha_i \in (0,1) \) for \( i \in \{1,\ldots,n\} \). For \( t > 0 \) we define the kernel

\[
b(t) = \sum_{i=1}^{n} g_{1-\alpha_i}(t),
\]

which is unbounded, locally integrable and completely monotonic. Hence, there is a completely monotonic corresponding Sonine kernel \( a \in L_{1,\text{loc}}(\mathbb{R}^+) \). Their Laplace transforms are given by

\[
\hat{b}(\lambda) = \sum_{i=1}^{n} \lambda^{\alpha_i-1}, \quad \hat{a}(\lambda) = \left( \sum_{i=1}^{n} \lambda^{\alpha_i} \right)^{-1}, \quad \lambda \in \mathbb{C}^+.
\]

Distributed Order Kernel. It is also possible to consider integrals over standard kernels. Let \( 0 < c < d < 1 \), for \( t > 0 \) we define the kernel

\[
b(t) = \int_{c}^{d} g_{1-\alpha}(t) d\alpha,
\]

which is unbounded, locally integrable and completely monotonic. Thus, there is a completely monotonic corresponding Sonine kernel \( a \in L_{1,\text{loc}}(\mathbb{R}^+) \). Their Laplace transforms are given by

\[
\hat{b}(\lambda) = \begin{cases} \frac{\lambda^{d-c}}{\lambda \log \lambda} & \text{for } \lambda \in \mathbb{C}^+ \setminus \{1\}, \\ d-c & \text{for } \lambda = 1, \end{cases} \quad \hat{a}(\lambda) = \begin{cases} \frac{\log \lambda}{\lambda^{d-c}} & \text{for } \lambda \in \mathbb{C}^+ \setminus \{1\}, \\ \frac{1}{\pi^{2}} & \text{for } \lambda = 1. \end{cases}
\]

Ultraslow Diffusion Kernel. If one consider distributed order kernels in the limiting case of \( c \to 0 \) and \( d \to 1 \) we get the so-called ultraslow diffusion kernel, which is an unbounded, locally integrable and completely monotonic kernel. We have

\[
b(t) = \int_{0}^{1} g_{a}(t) d\alpha, \quad a(t) = \int_{0}^{\infty} \frac{e^{-st}}{1+s} ds, \quad t > 0,
\]

and their Laplace transforms are given by

\[
\hat{b}(\lambda) = \begin{cases} \frac{1}{\lambda \log \lambda} & \text{for } \lambda \in \mathbb{C}^+ \setminus \{1\}, \\ 1 & \text{for } \lambda = 1, \end{cases} \quad \hat{a}(\lambda) = \begin{cases} \frac{\log \lambda}{\lambda-1} & \text{for } \lambda \in \mathbb{C}^+ \setminus \{1\}, \\ 1 & \text{for } \lambda = 1. \end{cases}
\]

see [VZ15, Example 6.5].
Linear Volterra Equations

In this section we summarise the most important definitions and results about the concepts of resolvents and integral resolvents for Banach space valued linear Volterra equations of scalar type introduced in the first chapter of [Prü12].

Afterwards we focus on parabolic Volterra equations. For this setting we quote a fundamental theorem about the existence and regularity of resolvents. With regards to Part III of this work we analyse the asymptotic behaviour of the resolvent and introduce terms of stability of a resolvent. In particular, we pay attention to the situation of the standard kernel and the resulting estimates.

In the following we give sufficient conditions for the existence and integrability of the integral resolvent for parabolic Volterra equations. Here, the standard kernel situations is discussed, too.

This chapter closes with a time-weighting argument for linear Volterra equations and the calculation of the commutator term for special choice of time-weights. These arguments are essential components in the proofs of our stability results for semilinear equations in Part II and for quasilinear equations in Part III.

Throughout this chapter we assume:

**Assumptions.** Let $X$ be a complex Banach space equipped with the norm $\| \cdot \|_X$, and let $A$ be a closed linear unbounded operator in $X$ with dense domain $\mathcal{D}(A)$. We denote by $X_A = (\mathcal{D}(A), \| \cdot \|_A)$ the domain of the operator $A$ equipped with the operator norm defined for all $x \in \mathcal{D}(A)$ by $\|x\|_A = \|x\|_X + \|Ax\|_X$. Since the operator $A$ is closed the space $X_A$ is a Banach space which is continuously and densely embedded into $X$. Furthermore, let $a \in L_{1,loc}(\mathbb{R}_+)$ be a scalar non-vanishing kernel, and $J = [0, T]$ with some $T > 0$ or $J = \mathbb{R}_+$.

### 3.1 Resolvents and Integral Resolvents

For $f \in C(J; X)$ we consider the linear Volterra equation

$$u(t) + (a \ast Au)(t) = f(t), \quad t \in J.$$  \hspace{1cm} (3.1)

Following Prüss, we introduce the concepts of strong solutions and wellposedness of the linear Volterra equation (3.1), see [Prü12, Definition 1.1 and 1.2].

**Definition 3.1.1 (Strong Solutions of Volterra Equations).** A function $u \in C(J; X)$ is called strong solution of (3.1) on $J$ if $u \in C(J; X_A)$ and (3.1) holds on $J$. 


3. Linear Volterra Equations

Definition 3.1.2 (Well-Posedness of Volterra Equations). The Volterra equation (3.1) is called well-posed if for each \( x \in \mathcal{D}(A) \) there is a unique strong solution \( u(t;x) \) on \( \mathbb{R}_+ \) of

\[
    u(t) + (a \ast Au)(t) = x, \quad t \in \mathbb{R}_+,
\]

and \( (x_n) \subset \mathcal{D}(A), \ x_n \to 0 \) imply \( u(t;x_n) \to 0 \) in \( X \), uniformly on compact intervals of \( \mathbb{R}_+ \).

We will only consider well-posed Volterra equations. It can be shown that the well-posedness of the linear Volterra equation is equivalent to the existence of a unique resolvent family, defined below, cf. [Prü12, pp. 31 and Definition 1.3].

Definition 3.1.3 (Resolvent). A family \( \{S(t)\}_{t \in \mathbb{R}_+} \subset B(X) \) of bounded linear operators in \( X \) is called resolvent for the Volterra equation (3.1) if the following conditions are satisfied:

\begin{enumerate}[(S1)]
    \item \( S(\cdot) \) is strongly continuous on \( \mathbb{R}_+ \) and \( S(0) = \text{Id} \);
    \item \( S(t)\mathcal{D}(A) \subset \mathcal{D}(A) \) and \( AS(t)x = S(t)Ax \) for all \( x \in \mathcal{D}(A) \) and \( t \in \mathbb{R}_+ \);
    \item \( S(t)x + (a \ast [ASx])(t) = x, \) for all \( x \in \mathcal{D}(A) \) and \( t \in \mathbb{R}_+ \);
\end{enumerate}

the resolvent equation is satisfied.

Remark 3.1.4. \( \text{a) If the Volterra equation (3.1) is well-posed, there is a unique resolvent family} \{S(t)\}_{t \in \mathbb{R}_+} \text{ and we have for all} \ x \in X \text{ and all} \ t \in \mathbb{R}_+ \text{ that} \ (a \ast [Sx])(t) \in \mathcal{D}(A) \text{ as well as} \ S(t)x + A(a \ast [Sx])(t) = x, \text{ see [Prü12, Proposition 1.1 and Corollary 1.1].}

\( \text{b) If there exists a resolvent} \{S(t)\}_{t \in \mathbb{R}_+} \text{ for the linear Volterra equation (3.1), we call} \ u(\cdot;u_0) = S(\cdot)u_0 \in C(\mathbb{R}_+;X), \ u_0 \in X, \ \text{a mild solution of the linear Volterra equation} \ u(t) + (a \ast Au)(t) = u_0, \ t \in \mathbb{R}_+ .

We want to discuss the stability of the equilibrium \( u_\ast = 0 \) for a well-posed linear Volterra equation

\[
    u(t) + (a \ast Au)(t) = u_0, \quad t \in \mathbb{R}_+, \quad (3.2)
\]

with \( u_0 \in X \). According to the classical theory of dynamical systems we introduce the terms of stable and asymptotically stable equilibria.

Definition 3.1.5 (Stability). (i) We call the equilibrium \( u_\ast = 0 \) stable for the linear Volterra equation (3.2) if for each \( \epsilon > 0 \) there is some \( \delta = \delta(\epsilon) > 0 \) such that for all \( u_0 \in B_\delta(0) \) the corresponding mild solution \( u = u(\cdot;u_0) \) exists on \( \mathbb{R}_+ \) and for all \( t \in \mathbb{R}_+ \) we have \( \|u(t)\|_X < \epsilon \).
3.1. Resolvents and Integral Resolvents

(ii) The equilibrium \( u_* = 0 \) is called \textit{asymptotically stable} for the linear Volterra equation (3.2) if \( u_* = 0 \) is stable and there exists some \( \delta_0 > 0 \) such that for all \( u_0 \in B_{\delta_0}(0) \) the corresponding mild solution \( u = u(\cdot; u_0) \) of the Volterra equation (3.2) satisfies \( \|u(t)\|_X \to 0 \) as \( t \to \infty \).

(iii) The equilibrium \( u_* = 0 \) is called \textit{unstable} for the linear Volterra equation (3.2) if \( u_* \) is not stable.

Analogous to classical linear evolution equations it is possible to characterise the stability of the equilibrium \( u_* = 0 \) of the linear equations in terms of the resolvent family.

\textbf{Theorem 3.1.6 (Stability of Linear Volterra Equations).} Let \( \{S(t)\}_{t \in \mathbb{R}_+} \) be a resolvent family for the linear Volterra equation (3.1). Then,

(i) \( u_* = 0 \) is stable for the linear Volterra equation (3.2) if and only if we have \( \sup_{t \in \mathbb{R}_+} \|S(t)\|_{B(X)} < \infty \);

(ii) \( u_* = 0 \) is asymptotically stable for the linear Volterra equation (3.2) if and only if \( \|S(t)\|_{B(X)} \to 0 \) as \( t \to \infty \).

\textit{Proof.} The proof is completely analogous to the known result for the classical case, cf. [PW10, p. 93]. Since the linear Volterra equation (3.2) admits a resolvent family, for all \( u_0 \in X \) the corresponding mild solution \( u(\cdot; u_0) \) is given by \( S(\cdot)u_0 \).

(i) We assume that \( u_* = 0 \) is stable for the linear Volterra equation (3.2), i.e. for each \( \epsilon > 0 \) there is some \( \delta = \delta(\epsilon) > 0 \) such that for all \( u_0 \in B_\delta(0) \) the corresponding solutions \( u(\cdot; u_0) \) satisfies for all \( t \in \mathbb{R}_+ \) the estimate \( \|u(t; u_0)\|_X < \epsilon \). We fix \( \epsilon = 1 \) and the corresponding parameter \( \delta = \delta(1) > 0 \). Then, for all \( u_0 \in B_\delta(0) \) and all \( t \in \mathbb{R}_+ \), we have \( \|u(t; u_0)\|_X < 1 \), and hence, \( \|S(t)u_0\|_X < 1 \). Moreover, let \( v \in X \setminus \{0\} \) be arbitrary, we set \( v_0 = \delta v/\|v\|_X \in B_\delta(0) \). This implies for all \( t \in \mathbb{R}_+ \) that

\[
\|S(t)v\|_X = \frac{\|v\|_X}{\delta} \|S(t)v_0\|_X < \frac{\|v\|_X}{\delta}.
\]

In particular, we have \( \|S(t)\|_{B(X)} < 1/\delta \), for all \( t \in \mathbb{R}_+ \). Thus, the resolvent family is bounded on \( \mathbb{R}_+ \).

Now, we suppose that \( \|S(t)\|_{B(X)} \leq M \) for all \( t \in \mathbb{R}_+ \). For each \( \epsilon > 0 \) we set \( \delta = \epsilon/M \) such that for all \( u_0 \in B_\delta(0) \) and all \( t \in \mathbb{R}_+ \), we have

\[
\|u(t, u_0)\|_X = \|S(t)u_0\|_X \leq M \|u_0\|_X < \epsilon,
\]

i.e. \( u_* = 0 \) is stable for the Volterra equation (3.2).

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(ii) Initially, we assume that \( u_* = 0 \) is asymptotically stable for the linear Volterra equation (3.2). That is, \( u_* = 0 \) is stable for (3.2) and there is some \( \delta_0 > 0 \) such that for all \( u_0 \in B_{\delta_0} \) the corresponding solution \( u(\cdot; u_0) \) exists on \( \mathbb{R}_+ \) and \( \|u(t; u_0)\|_X = \|S(t)u_0\|_X \to 0 \) for \( t \to \infty \). Analogous to the proof of part (i), we deduce for all \( v \in X \) that \( \|S(t)v\|_X \to 0 \). This implies \( \|S(t)\|_{\mathcal{B}(X)} \to 0 \) as \( t \to \infty \).

Conversely, suppose that \( \|S(t)\|_{\mathcal{B}(X)} \to 0 \). Obviously, we have for all \( \delta_0 > 0 \) and \( u_0 \in B_{\delta_0}(0) \) that \( \|u(t; u_0)\|_X = \|S(t)u_0\|_X \leq \|S(t)\|_{\mathcal{B}(X)} \|u_0\|_X \to 0 \) as \( t \to \infty \). Clearly, in this situation the resolvent is bounded. Hence, \( u_* = 0 \) is asymptotically stable for the linear Volterra equation (3.2).

\[ \square \]

**Remark 3.1.7.** The above theorem shows that the stability properties of the equilibrium \( u_* = 0 \) for the linear Volterra equation (3.2) only depend on the behaviour of the resolvent family. For this reason, we speak in the situation of a bounded resolvent family on \( X \) also of a stable resolvent family on \( X \). In the situation that the \( \mathcal{B}(X) \)-norm of the resolvent family tends to zero as \( t \to \infty \), we call the resolvent family asymptotically stable on the Banach space \( X \).

We refer to Section 3.3 where we examine the existence and asymptotic behaviour of the resolvent family in detail.

In order to treat linear Volterra equations with arbitrary right hand sides, we define the integral resolvent family, cf. [Prü12, Definition 1.6], in a similar way to the resolvent family.

**Definition 3.1.8 (Integral Resolvent).** A family \( \{R(t)\}_{t \in \mathbb{R}_+} \subset \mathcal{B}(X) \) of bounded linear operators in \( X \) is called integral resolvent for the Volterra equation (3.1) if the following conditions are satisfied:

1. \( R(\cdot)x \in L_{1,\text{loc}}(\mathbb{R}_+; X) \) for each \( x \in X \), and \( \|R(t)\|_{\mathcal{B}(X)} \leq \varphi(t) \) a.e. on \( \mathbb{R}_+ \), for some \( \varphi \in L_{1,\text{loc}}(\mathbb{R}_+) \);

2. \( R(t)D(A) \subset D(A) \) and \( AR(t)x = R(t)Ax \) for all \( x \in D(A) \) and a. a. \( t \in \mathbb{R}_+ \);

3. \( R(t)x + (a \ast [ARx])(t) = a(t)x \), for all \( x \in D(A) \) and a. a. \( t \in \mathbb{R}_+ \);

the integral resolvent equation is satisfied.

**Remark 3.1.9.**

a) If there exists an integral resolvent \( \{R(t)\}_{t \in \mathbb{R}_+} \) then we have for all \( x \in X \) and almost all \( t \in \mathbb{R}_+ \) that \( (a \ast [Rx])(t) \in D(A) \) and \( R(t)x + A(a \ast [Rx])(t) = a(t)x \). For a proof, we refer to Lizama’s work about \( k \)-regularized resolvents, see [Liz00, Lemma 2.2].

b) For \( u \in C(J; X) \) the convolution \( (R \ast u) \) is well-defined on \( J \) and we have the estimate \( \|(R \ast u)(t)\|_X \leq (\varphi \ast \|u\|_X)(t) \), \( t \in J \).
c) Assuming the existence of an integral resolvent family \( \{R(t)\}_{t \in \mathbb{R}_+} \), one can prove the uniqueness of this object. In case of \( f = g * a \) with \( g \in C(J; X_A) \) a representation of the strong solution of the linear Volterra equation (3.1) is given by \( u(t) = f(t) - A(R * f)(t), \ t \in J, \) cf. [Prü12, p. 46].

d) If both resolvent and integral resolvent exist for the Volterra equation (3.1), they are related by the following identities:

\[
\begin{align*}
R(t)Ax &= -\partial_t S(t)x, \quad \text{for all } x \in \mathcal{D}(A), \text{ and a. a. } t \in \mathbb{R}_+, \\
R(t)x &= \partial_t (a * S)(t)x, \quad \text{for all } x \in X, \text{ and a. a. } t \in \mathbb{R}_+
\end{align*}
\]

We will mostly confine ourselves to a special case of the linear Volterra equation (3.1)

\[
u(t) + (a * Au)(t) = u_0 + (a * f)(t), \quad t \in J, \tag{3.3}
\]

with \( u_0 \in X \) and \( f \in C(J; X) \).

**Definition 3.1.10 (Mild Solution).** Let \( u_0 \in X \) and \( f \in C(J; X) \). Assume that the equation Volterra equation (3.3) admits both a resolvent \( \{S(t)\}_{t \in \mathbb{R}_+} \) and an integral resolvent \( \{R(t)\}_{t \in \mathbb{R}_+} \). Then we call

\[
u(t) = S(t)u_0 + (R * f)(t), \quad t \in J, \tag{3.4}
\]

the (unique) **mild solution** of the linear Volterra equation (3.3).

**Remark 3.1.11.**

a) Using the properties of the resolvent and integral resolvent one assures oneself that the mild solution is even a strong solution if \( u_0 \in \mathcal{D}(A) \) and \( f \in C(J; X_A) \).

b) In the case \( a \equiv 1 \) both resolvent and integral resolvent coincides with the \( C_0 \)-semigroup \( e^{-At} \). In this situation the formula (3.4) is nothing else than the well-known variation of parameters formula.

Analogous to the classical case there are generation theorems which characterise the existence of a resolvent and an integral resolvent. For the proof we refer to Prüss [Prü12, Theorem 1.3 and 1.4].
Theorem 3.1.12 (Generation Theorem for Resolvents and Integral Resolvents). Let $A$ be a closed linear unbounded operator in $X$ with dense domain $\mathcal{D}(A)$ and let $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ satisfy $\int_0^\infty e^{-\omega t} |a(t)| \, dt < \infty$ for some $\omega \in \mathbb{R}$.

(i) Then the Volterra equation (3.1) admits a resolvent family $\{S(t)\}_{t \in \mathbb{R}_+}$ such that there is a constant $M \geq 1$ with $\|S(t)\|_{\mathcal{B}(X)} \leq Me^{\omega t}$, for all $t \in \mathbb{R}_+$ if and only if the following conditions hold.

(H1) $\hat{a}(\lambda) \neq 0$ and $-1/\hat{a}(\lambda) \in \rho(A)$ for all $\lambda > \omega$;

(H2) $H(\lambda) = (\text{Id} + \hat{a}(\lambda)A)^{-1}/\lambda$ satisfies the estimate

$$\|H^{(n)}(\lambda)\|_{\mathcal{B}(X)} \leq M n! (\lambda - \omega)^{-(n+1)},$$

for all $\lambda > \omega$ and all $n \in \mathbb{N}_0$, with some constant $M \geq 1$.

(ii) The Volterra equation (3.1) admits an integral resolvent family $\{R(t)\}_{t \in \mathbb{R}_+}$ such that $\|R(t)\|_{\mathcal{B}(X)} \leq e^{\omega t} \varphi(t)$, for a.a. $t \in \mathbb{R}_+$, holds for some $\varphi \in L_1(\mathbb{R}_+)$ if and only if the following conditions are satisfied.

(K1) $\hat{a}(\lambda) \neq 0$ and $-1/\hat{a}(\lambda) \in \rho(A)$ for all $\lambda > \omega$;

(K2) $K(\lambda) = \hat{a}(\lambda)(\text{Id} + \hat{a}(\lambda)A)^{-1}$ satisfies the estimate

$$\sum_{n=0}^\infty (\lambda - \omega)^n \|K^{(n)}(\lambda)\|_{\mathcal{B}(X)} \leq M,$$

for all $\lambda > \omega$, with some constant $M \geq 1$.

In general the conditions in the generation theorem are hard to verify. Thus, we restrict our considerations to the import class of parabolic Volterra equation.

3.2 Parabolic Equations

In this section we summarise the most important definitions and results about parabolic Volterra equations taken from chapter 3 in [Prü12].

We assume additionally to the section above:

Assumption. The kernel $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ is of subexponential growth, cf. Definition 2.3.5. If this is the case it is obvious that the Laplace transform $\hat{a}(\lambda)$ of the kernel exists for all $\lambda \in \mathbb{C}_+$.

Again, we consider the linear Volterra equation

$$u(t) + (a * Au)(t) = f(t), \quad t \in \mathbb{R}_+.$$  \hfill (3.5)

Following Prüss, we introduce the notion of parabolicity, cf. [Prü12, Definition 3.1].
Definition 3.2.1 (Parabolic Volterra Equation). The Volterra equation (3.5) is called parabolic, if the following conditions hold.

(P1) \( \hat{a}(\lambda) \neq 0 \), and \(-1/\hat{a}(\lambda) \in \rho(A)\) for all \( \lambda \in \mathbb{C}_+ \).

(P2) There is a constant \( M > 0 \) such that \( \| (\text{Id} + \hat{a}(\lambda)A)^{-1} \|_{B(X)} \leq M \) for all \( \lambda \in \mathbb{C}_+ \).

The next proposition provides sufficient conditions for the parabolicity of the Volterra equation (3.5), see [Prü12, Proposition 3.1].

Proposition 3.2.2. Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be \( \theta \)-sectorial for some \( \theta < \pi \), suppose \( A \) is a closed linear densely defined operator, such that \( \rho(-A) \supset \Sigma_{\theta} \), and for all \( \mu \in \Sigma_{\theta} \) we have \( \| (\mu + A)^{-1} \|_{B(X)} \leq M/|\mu| \). Then (3.5) is a parabolic Volterra equation.

Additionally, we introduce the notation of sectorial operators, see [DHP03, Definition 1.1].

Definition 3.2.3 (Sectorial Operator). Let \( X \) be a complex Banach space, and \( A \) be a closed linear operator in \( X \). The operator \( A \) is called sectorial if the following conditions are satisfied.

(i) \( \overline{D(A)} = X, \text{Rg}(A) = X, (\infty, 0) \subset \rho(A) \).

(ii) \( \| t(t + A)^{-1} \|_{B(X)} \leq M \) for all \( t > 0 \) and some \( M < \infty \).

Remark 3.2.4. It is not difficult to see that for a sectorial operator \( A \) there is some \( \theta > 0 \) such that \( \rho(-A) \supset \Sigma_{\theta} \) and \( \sup \{ \| \lambda(\lambda + A)^{-1} \|_{B(X)} : |\arg \lambda| < \theta \} < \infty \).

We define the spectral angle \( \varphi_A \) of the sectorial operator \( A \) by
\[
\varphi_A = \inf \left\{ \varphi : \rho(-A) \supset \Sigma_{\pi - \varphi}, \sup_{\lambda \in \Sigma_{\pi - \varphi}} \| \lambda(\lambda + A)^{-1} \|_{B(X)} < \infty \right\},
\]
cf. [DHP03, p. 9].

Remark 3.2.5. In particular, if \( A \) is a sectorial operator with spectral angle \( \varphi_A \), and the kernel \( a \) is \( \theta \)-sectorial, then (3.5) is a parabolic Volterra equation provided \( \varphi_A + \theta < \pi \).

The subsequent technical corollary shows that the additional assumption of invertibility of the operator \( A \) yields an uniform bound for the resolvent operator \( (\mu + A)^{-1} \) on a closed sector with smaller opening angle.

Corollary 3.2.6. We assume additionally to the requirements of Proposition 3.2.2 that \( 0 \in \rho(A) \). Then for each \( \psi < \theta \) there is some constant \( M_0 = M_0(\psi) > 0 \) such that for all \( \mu \in \Sigma_{\psi} \) we have
\[
\| (\mu + A)^{-1} \|_{B(X)} \leq M_0.
\]
Proof. Let $\psi < \theta$ be arbitrary, but fixed. For all $\mu \in \overline{\Sigma}_\psi$ with $|\mu| > 1$ the estimate for the resolvent $(\mu + A)^{-1}$ implies $\| (\mu + A)^{-1} \|_{\mathcal{B}(X)} \leq M/|\mu| \leq M$. Moreover, the set $\overline{\Sigma}_\psi \cap B_1(0)$ is compact and $\mu \mapsto \| (\mu + A)^{-1} \|_{\mathcal{B}(X)}$ is continuous, hence $\| (\mu + A)^{-1} \|_{\mathcal{B}(X)} \leq \sup_{\lambda \in \overline{\Sigma}_\psi \cap B_1(0)} \| (\lambda + A)^{-1} \|_{\mathcal{B}(X)} \leq M_1$. The claim follows with $M_0 = \max\{M, M_1\}$. \hfill \Box

3.3 Existence and Estimates for Resolvents and Integral Resolvents of Parabolic Volterra Equations

3.3.1 Existence and Boundedness of Resolvents

We have the following theorem about the existence and boundedness of resolvents for parabolic Volterra equations, see [Prü12, Theorem 3.1]

**Theorem 3.3.1 (Resolvents for Parabolic Equations).** Let $X$ be a Banach space, $A$ a closed linear operator in $X$ with dense domain $D(A)$ and $a \in L_{1, \text{loc}}(\mathbb{R}_+)$. Assume the Volterra equation (3.5) is parabolic, and the kernel $a$ is 1-regular.

Then there is a uniformly bounded resolvent $S \in C(\mathbb{R}_+; \mathcal{B}(X))$ for (3.5).

**Remark 3.3.2.** A similar existence result for the integral resolvent is not known. From Comment 2.7.d) in [Prü12] we know that the existence of an integral resolvent is not clear even in the case of analytic resolvents\(^1\). Although the resolvent is differentiable on $(0, \infty)$ the local integrability on $\mathbb{R}_+$ can only be ensured under additional assumptions onto the kernel $a \in L_{1, \text{loc}}(\mathbb{R}_+)$.  

For Part II of this work it is enough to have knowledge about the existence of a bounded resolvent. With regard to Part III it is necessary to know the asymptotic behaviour of the resolvent family. The result reads as follows:

**Theorem 3.3.3 (Asymptotic Behaviour of Resolvents).** Let $a \in L_{1, \text{loc}}(\mathbb{R}_+)$ be an unbounded, completely monotonic kernel which is $\theta_a$-sectorial with $\theta_a \in (0, \frac{\pi}{2})$. We assume that the corresponding completely monotonic Sonine kernel $b \in L_{1, \text{loc}}(\mathbb{R}_+)$ satisfies the estimate $\int_0^1 \hat{b}(1/t)\frac{dt}{t} < \infty$. Furthermore, let $A$ be a closed, linear and densely defined operator on the Banach space $X$ with $\rho(-A) \supset \Sigma_\phi \cup \{0\}$ and $\| (\mu + A)^{-1} \|_{\mathcal{B}(X)} \leq M/|\mu|$ for all $\mu \in \Sigma_\phi$, with some $\phi > \theta_a$.

Then, there is a constant $C > 0$ such that the resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$ for the Volterra equation (3.5) satisfies for each $t > 0$ the estimate

$$\| S(t) \|_{\mathcal{B}(X)} \leq C \hat{b}(1/t)/t.$$  

\(^1\)Analogous to the concept of analytic $C_0$-semigroups, it is possible to extend the concept of resolvents to some sector $\Sigma_\psi$ where we have an analytic extension of the resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$. For further information see chapter 2 in [Prü12].
Remark 3.3.4. The proof of Theorem 3.3.3 as well as the proof of the subsequent Theorem 3.3.8 heavily relies on the Analytic Representation Theorem 1.2.1. For this reason, we explain why the function \( \hat{a} \) and \( \hat{b} \), respectively, satisfies the assumptions for this theorem.

The kernel \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) is unbounded and completely monotonic. Hence, \( \hat{a} \) is a completely monotonic function, see Remark 2.3.6; in particular \( \hat{a} \) is non-negative and non-increasing on \((0,\infty)\). By the properties of the Laplace transform of completely monotonic functions it follows that \( \lim_{t \to \infty} \hat{a}(t) = 0 \), see Theorem 2.3.7. The same properties hold for the corresponding completely monotonic Sonine kernel \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \).

The Laplace transform of the Sonine condition \( \hat{a}(\lambda)\hat{b}(\lambda) = 1/\lambda \), for all \( \lambda \in \mathbb{C}_+ \), implies that \( \sup_{t \in (0,1]} t\hat{a}(t) = \sup_{t \in (0,1]} 1/\hat{b}(t) \). Note that \( \lim_{t \to 0} t\hat{a}(t) \) either vanishes in the case of \( b \in L_1(\mathbb{R}_+) \) or is finite with \( \lim_{t \to 0} t\hat{a}(t) = 1/\hat{b}(0) \) in the case of \( b \in L_1(\mathbb{R}_+) \). Since \( \hat{b} \) is non-increasing, it follows that \( 1/\hat{b} \) is increasing. In particular, we have \( \sup_{t \in (0,1]} t\hat{a}(t) \leq 1/\hat{b}(1) < \infty \). The same argument works if one changes the role of the kernels \( a \) and \( b \).

Proof of Theorem 3.3.3. The assumptions on the kernel \( a \) imply that the kernel \( a \) is \( \varphi \)-sectorial. Together with the assumptions on the operator \( A \) we know that the Volterra equation (3.5) is a parabolic, see Proposition 3.2.2.

The Laplace transform \( \hat{a} \) of the kernel \( a \) can be extended to the sector \( \Sigma_\varphi \), cf. Theorem 2.3.7. Furthermore, the mapping property of the Laplace transform \( \hat{a} \) onto the sector \( \Sigma_{\theta_\varphi} \) is continuously extendable, see Lemma 2.3.15. In particular, we may choose \( \epsilon_0 < \min\{\varphi - \theta_\varphi, \pi/2\} \), and fix \( \delta_0 \in (0,\pi/4) \) such that \( \hat{a}(\lambda) \in \Sigma_{\theta_\varphi + \epsilon_0} \) for all \( \lambda \in \Sigma_{\varphi + \delta_0} \).

Moreover, the Laplace transform of the corresponding completely monotonic kernel \( \hat{b}(\lambda) \) has a holomorphic extension to \( \Sigma_\varphi \) and we have for all \( \lambda \in \Sigma_{\varphi + \delta_0} \) the lower and upper estimate of \( |\hat{b}(\lambda)| \) by \( \hat{b}(|\lambda|) \), see Theorem 2.3.12.

Due to the choice of \( \epsilon_0 \) it is obvious that for all \( \lambda \in \Sigma_{\varphi + \delta_0} \) we have \( 1/\hat{a}(\lambda) \in \Sigma_{\theta_\varphi + \epsilon_0} \subseteq \Sigma_\varphi \). We set

\[
H(\lambda) = (1/\hat{a}(\lambda) + A)^{-1}/(\lambda \hat{a}(\lambda)) = \hat{b}(\lambda)(1/\hat{a}(\lambda) + A)^{-1},
\]

for all \( \lambda \in \Sigma_{\varphi + \delta_0} \). Obviously, \( H \) is holomorphic on \( \Sigma_{\varphi + \delta_0} \) with values in \( B(X) \). Since \( \theta_\varphi + \epsilon_0 < \varphi \) and because of the invertibility of the operator \( A \) it follows that the expression \( \|(1/\hat{a}(\lambda) + A)^{-1}\|_{B(X)} \) is uniformly bounded on \( \Sigma_{\theta_\varphi + \epsilon_0} \), cf. Corollary 3.2.6. Together with the upper estimate of \( |\hat{b}(\lambda)| \) by \( \hat{b}(|\lambda|) \), see Theorem 2.3.12, we deduce for all \( \lambda \in \Sigma_{\varphi + \delta_0} \) the estimate

\[
\|H(\lambda)\|_{B(X)} \leq M \hat{b}(|\lambda|),
\]

with some constant \( M > 0 \). The map \( \hat{b} : (0,\infty) \to \mathbb{R}_+ \) satisfies the condition in Theorem 1.2.1. It follows that there is a holomorphic function \( h : \Sigma_{\delta_0} \to B(X) \).
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satisfying the estimate \( \|h(t)\|_{B(X)} \leq Cb(1/t)/t \) on \((0, \infty)\) and \( \hat{h}(\lambda) = H(\lambda) \) for all \( \lambda > 0 \). The uniqueness of the Laplace transform, Theorem 1.1.4, together with the uniqueness of the resolvent, implies the desired estimate for the resolvent family. For each \( t > 0 \) we have \( \|S(t)\|_{B(X)} \leq Cb(1/t)/t \).

In the following we give sufficient conditions for the stability and asymptotic stability of the resolvent family \( \{S(t)\}_{t \in \mathbb{R}_+} \), cf. Remark 3.1.7 and Theorem 3.1.6.

**Lemma 3.3.5.** Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be an unbounded, completely monotonic kernel which is \( \theta_a \)-sectorial with \( \theta_a \in (0, \frac{\pi}{2}] \). Furthermore, let \( A \) be a closed, linear and densely defined operator on the Banach space \( X \) with \( \rho(-A) \supset \Sigma_{\varphi} \cup \{0\} \) and

\[
\|\rho(A)\|_{B(X)} \leq M|\mu| \quad \text{for all } \mu \in \Sigma_{\varphi}, \text{ with some } \varphi > \theta_a.
\]

Then, the resolvent family \( \{S(t)\}_{t \in \mathbb{R}_+} \) is stable on \( X \).

Moreover, if \( a \in L_1(\mathbb{R}_+) \) and the corresponding completely monotonic Sonine kernel satisfies the estimate \( \int_0^1 \hat{b}(1/t)dt < \infty \), then the resolvent family \( \{S(t)\}_{t \in \mathbb{R}_+} \) is asymptotically stable on the Banach space \( X \).

**Proof.** The conclusion about the stability follows immediately from the existence of a bounded resolvent, cf. Theorem 3.1.1.

Using the estimate from Theorem 3.3.3, we have in case of \( a \notin L_1(\mathbb{R}_+) \) that \( \lim_{t \to \infty} \hat{b}(1/t)/t = \lim_{t \to \infty} 1/\hat{a}(1/t) = 1/\hat{a}(0) = 0 \). The claim follows.

We conclude with an estimate of the resolvent family \( \{S(t)\} \) on \( X_A = (D(A), \|\cdot\|_A) \), which is densely embedded in \( X \), and on various real interpolation spaces between \( X_A \) and \( X \), the spaces \( (X, X_A)^{\gamma,q} \) for \( \gamma \in (0, 1) \) and \( q \in [1, \infty] \). For convenience we set \( (X, X_A)^{0,q} = X \) and \( (X, X_A)^{1,q} = X_A \), for all values of \( q \). We refer to Bergh and Löfström [BL76], Lunardi [Lun09] as well as Triebel [Tri78] and [Tri83] for further information about real interpolation spaces and general interpolation theory.

**Corollary 3.3.6.** Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be an unbounded, completely monotonic kernel which is \( \theta_a \)-sectorial with \( \theta_a \in (0, \frac{\pi}{2}] \). We assume that the corresponding completely monotonic Sonine kernel \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) satisfies the estimate \( \int_0^1 \hat{b}(1/t)dt < \infty \).

Furthermore, let \( A \) be a closed, linear and densely defined operator on the Banach space \( X \) with \( \rho(-A) \supset \Sigma_{\varphi} \cup \{0\} \) and

\[
\|\rho(A)\|_{B(X)} \leq M|\mu| \quad \text{for all } \mu \in \Sigma_{\varphi},
\]

with some \( \varphi > \theta_a \).

Then, for each \( \gamma \in [0, 1] \) and all \( q \in [1, \infty] \) there is some constant \( M = M(\gamma, q) \) > 0 such that for all \( t \in \mathbb{R}_+ \) we have

\[
\|S(t)\|_{B((X, X_A)^{\gamma,q})} \leq M \min\{1, \hat{b}(1/t)/t\}.
\]

In particular, if \( a \notin L_1(\mathbb{R}_+) \), then the resolvent family is asymptotically stable on \( (X, X_A)^{\gamma,q} \), for all \( \gamma \in [0, 1] \) and all \( q \in [1, \infty] \).
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Proof. Since equation (3.5) is parabolic, the claim for the Banach space X follows by Theorem 3.3.1 and Theorem 3.3.3. For the Banach space $X_A$, we have for $x \in D(A)$ and all $t \in \mathbb{R}_+$ that $\|S(t)x\|_{X_A} = \|S(t)x\|_X + \|AS(t)x\|_X$. On $D(A)$ the resolvent commutes with the operator $A$, thus $\|S(t)x\|_{X_A} \leq \|S(t)\|_{B(X)} \|x\|_X + \|S(t)\|_{B(X)} \|Ax\|_X$. This implies $\|S(t)\|_{B(X_A)} \leq \|S(t)\|_{B(X)}$ and thus the desired pointwise estimate.

The claim for the real interpolation spaces $(X, X_A)_{\gamma,q}$ between $X$ and $X_A$ follows by the fact that we have an exact interpolation pair of exponent $\gamma$. \qed

Finally, we consider the case of the standard kernel.

Corollary 3.3.7. Let $X$ be a Banach space, $p \in (1, \infty)$ and $\alpha \in (0, 1)$. Assume, that the operator $A$ is invertible and sectorial with spectral angle $\varphi_A < \pi - \frac{\alpha \pi}{2}$.

We denote by $\{S(t)\}_{t \in \mathbb{R}_+}$ the resolvent family of the linear Volterra equation with standard kernel

$$u(t) + (g_a * Au)(t) = f(t), \quad t \in [0, T],$$

where $f \in C([0, T]; X)$. The following statements hold true.

(i) For each $\gamma \in (0, 1]$ and all $q \in [1, \infty]$ the resolvent family $\{S(t)\}_{t \in \mathbb{R}_+}$ is asymptotically stable on $(X, X_A)_{\gamma,q}$.

(ii) For all $\alpha \in (1/p, 1)$ we have $S \in L_p\left(\mathbb{R}_+; B\left((X, X_A)_{\gamma,q}\right)\right)$.

(iii) There is some constant $M > 0$ such that for all $t > 0$ we have the estimate $\|S(t)\|_{B(X; X_A)} \leq Mt^{-\alpha}$.

Proof. (i) We know from Remark 3.2.5 and Proposition 3.2.2 that the operator $A$ together with the standard kernels $a = g_a$ and $b = g_{1-\alpha}$ satisfy the conditions of Corollary 3.3.6. Since $g_a \notin L_1(\mathbb{R}_+)$ we obtain the asymptotic stability of the resolvent family on $(X, X_A)_{\gamma,q}$ for all $\gamma \in (0, 1]$ and all $q \in [1, \infty]$.

(ii) In view of $b(1/t)/t = t^{-\alpha}$ we know that $t \mapsto \min\{1, b(1/t)/t\} \in L_p(\mathbb{R}_+)$ for all $\alpha \in (1/p, 1)$.

(iii) For the estimate of the resolvent family as bounded operator from $X$ to $X_A$ we refer to Corollary 3 in [Prü96a]. \qed

3.3.2 Existence and Integrability of Integral Resolvents

In this section we give sufficient conditions for the existence of an integral resolvent and their integrability. Following Prüss [Prü12, Definition 10.2], we call an integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$ integrable if there is a $\varphi \in L_1(\mathbb{R}_+)$ such that $\|R(t)\|_{B(X)} \leq \varphi(t)$ for a.a. $t \in \mathbb{R}_+$.  47
Theorem 3.3.8 (Existence and Integrability of Integral Resolvents). Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be an unbounded, completely monotonic kernel which is \( \theta_a \)-sectorial with \( \theta_a \in (0, \frac{\pi}{2}) \) and satisfies the estimate \( \int_0^1 \hat{a}(1/t) \frac{dt}{t^2} \leq \infty \). Furthermore, let \( A \) be a closed, linear and densely defined operator on the Banach space \( X \) with \( \rho(-A) \supset \Sigma_{\varphi} \) and \( \|(\mu + A)^{-1}\|_{B(X)} \leq M/|\mu| \) for all \( \mu \in \Sigma_{\varphi} \), with some \( \varphi > \theta_a \).

(i) Then there exists an integral resolvent \( \{R(t)\}_{t \in \mathbb{R}_+} \) for the Volterra equation (3.5), and we have for all \( t > 0 \) the estimate

\[
\|R(t)\|_{B(X)} \leq C_a \hat{a}(1/t)/t,
\]

with some constant \( C_a > 0 \).

Assume additionally that the corresponding Sonine kernel \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) satisfies the estimate \( \int_0^1 \hat{b}(1/t) \frac{dt}{t^2} \leq \infty \) and that \( 0 \in \rho(A) \).

(ii) Then the integral resolvent for the Volterra equation (3.5) satisfies for all \( t > 0 \) the estimate

\[
\|R(t)\|_{B(X)} \leq C_b \hat{b}(1/t)/t^2,
\]

with some constant \( C_b > 0 \).

(iii) If the map \( t \mapsto \min\{\hat{a}(1/t)/t, \hat{b}(1/t)/t^2\} \) is integrable on \( \mathbb{R}_+ \), then the integral resolvent is an integrable integral resolvent.

In particular, if \( \int_1^\infty \hat{b}(1/t) \frac{dt}{t^2} \leq \infty \), then the integral resolvent is integrable.

Proof of Theorem 3.3.8. Analogous to the proof of the above Theorem 3.3.3 we know that the assumptions on the kernel \( a \) imply that the kernel \( \hat{a} \) is \( \varphi \)-sectorial. Together with the assumptions on the operator \( A \) we know that the Volterra equation (3.5) is a parabolic, see Proposition 3.2.2.

The Laplace transform \( \hat{a} \) of the kernel \( a \) can be extended to the sector \( \Sigma_\varphi \), cf. Theorem 2.3.7. Furthermore, the mapping property of the Laplace transform \( \hat{a} \) onto the sector \( \Sigma_{\theta_a} \) is continuously extendable, see Lemma 2.3.15.

In particular, we may choose \( \epsilon_0 < \min\{\varphi - \theta_a, \frac{\pi}{2}\} \), and fix \( \delta_0 \in (0, \pi/4) \) such that \( \hat{a}(\lambda) \in \Sigma_{\theta_a + \epsilon_0} \) for all \( \lambda \in \Sigma_{\frac{\pi}{2} + \delta_0} \). Moreover, we have for all \( \lambda \in \Sigma_{\frac{\pi}{2} + \delta_0} \) the lower and upper estimate of \( |\hat{a}(\lambda)| \) by \( \hat{a}(|\lambda|) \), see Theorem 2.3.12, as well as the extended 1-regularity on \( \Sigma_{\frac{\pi}{2} + \delta_0} \), cf. Lemma 2.3.16.

(i) Due to the choice of \( \epsilon_0 \) it is obvious, that for all \( \lambda \in \Sigma_{\frac{\pi}{2} + \delta_0} \) we have \( 1/\hat{a}(\lambda) \in \Sigma_{\theta_a + \epsilon_0} \subset \Sigma_\varphi \). We set \( K(\lambda) = (1/\hat{a}(\lambda) + A)^{-1} \) for all \( \lambda \in \Sigma_{\frac{\pi}{2} + \delta_0} \). Obviously, \( K \) is holomorphic on \( \Sigma_{\frac{\pi}{2} + \delta_0} \) with values in \( B(X) \). The assumed estimate for the resolvent operator \( (\mu + A)^{-1} \) implies for all \( \lambda \in \Sigma_{\frac{\pi}{2} + \delta_0} \) that \( \|K(\lambda)\|_{B(X)} = \|(1/\hat{a}(\lambda) + A)^{-1}\|_{B(X)} \leq M|\hat{a}(\lambda)| \). Since \( |\hat{a}(\lambda)| \) behaves as
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\( \hat{a}(\lambda) \) on \( \Sigma^{\pi}_{\frac{\pi}{2}} \), we have for all \( \lambda \in \Sigma^{\pi}_{\frac{\pi}{2}} \) that there is some constant \( M_a > 0 \) such that

\[ \| K(\lambda) \|_{\mathcal{B}(X)} \leq M_a \hat{a}(\lambda). \]

The map \( \hat{a} : (0, \infty) \to \mathbb{R}_+ \) satisfies the conditions for Theorem 1.2.1. Therefore, there is a holomorphic function \( f : \Sigma_{\delta_0} \to \mathcal{B}(X) \) which satisfies the estimate \( \| f(t) \|_{\mathcal{B}(X)} \leq C_a \hat{a}(1/t)/t \) on \((0, \infty)\) and \( \hat{f}(\lambda) = K(\lambda) \) for all \( \lambda > 0 \). In particular, \( f(\cdot)x \in L_{1,\text{loc}}(\mathbb{R}_+;X) \) and \( f \in L_{1,\text{loc}}(\mathbb{R}_+;\mathcal{B}(X)) \), hence \( f \) satisfies condition (R1) from Definition 3.1.8. The commutation property (R2) is obviously satisfied, since the operator \( A \) commutes with its resolvent operator on \( \mathcal{D}(A) \). Due to the uniqueness of the Laplace transform, Theorem 1.1.4, this implies that \( f \) satisfies the integral resolvent equation (R3). Since there is only one integral resolvent for the Volterra equation (3.5) we know that on \((0, \infty)\) the map \( f \) is the integral resolvent \( \{ R(t) \}_{t \in \mathbb{R}} \) we searched for and we have for all \( t > 0 \) the estimate

\[ \| R(t) \|_{\mathcal{B}(X)} \leq C_a \hat{a}(1/t)/t; \]

obviously, the map \( t \mapsto \hat{a}(1/t)/t \) is locally integrable on \( \mathbb{R}_+ \).

(ii) Assuming that the operator \( A \) is invertible, there is some constant \( M_0 > 0 \) such that for all \( \lambda \in \Sigma^{\pi}_{\frac{\pi}{2}} \), we have the uniform estimate \( \| K(\lambda) \|_{\mathcal{B}(X)} \leq M_0 \), cf. Corollary 3.2.6.

The map \( K \) is holomorphic on \( \Sigma^{\pi}_{\frac{\pi}{2}} \), so we can consider the holomorphic function \( K' \) on \( \Sigma^{\pi}_{\frac{\pi}{2}} \) with values in \( \mathcal{B}(X) \). We have \( K'(\lambda) = K^2(\lambda)\hat{a}'(\lambda)/\hat{a}^2(\lambda) \), and with the corresponding Sonine kernel \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) we can rewrite this expression as \( K'(\lambda) = K^2(\lambda)\hat{a}'(\lambda)\hat{b}(\lambda)/\hat{a}(\lambda) \) on \( \Sigma^{\pi}_{\frac{\pi}{2}} \).

Using the 1-regularity of the kernel \( a \) as well as the uniform boundedness of \( \| K(\lambda) \|_{\mathcal{B}(X)} \) on the sector \( \Sigma^{\pi}_{\frac{\pi}{2}} \) we obtain for all \( \lambda \in \Sigma^{\pi}_{\frac{\pi}{2}} \) the estimate \( \| K'(\lambda) \|_{\mathcal{B}(X)} \leq M_0^2 C |\hat{b}(\lambda)| \). The corresponding Sonine kernel \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) is itself completely monotonic. We can estimate \( |\hat{b}(\lambda)| \) on \( \Sigma^{\pi}_{\frac{\pi}{2}} \) by \( \hat{b}(|\lambda|) \), see Theorem 2.3.12. So, we see that for all \( \lambda \in \Sigma^{\pi}_{\frac{\pi}{2}} \) that there is some constant \( M_b > 0 \) such that

\[ \| K'(\lambda) \|_{\mathcal{B}(X)} \leq M_b |\hat{b}(\lambda)|. \]

The map \( \hat{b} : (0, \infty) \to \mathbb{R}_+ \) satisfies the properties of Theorem 1.2.1. Hence, there is a holomorphic function \( g : \Sigma_{\delta_0} \to X \) satisfying the estimate \( \| g(t) \|_{\mathcal{B}(X)} \leq C_b |\hat{b}(1/t)/t \) for all \( t > 0 \), with \( \hat{g}(\lambda) = K'(\lambda) \) for all \( \lambda > 0 \). Again, via the uniqueness of the Laplace transform, Theorem 1.1.4, this implies \( g(t) = -tR(t) \) for all \( t > 0 \). In particular, we have for all \( t > 0 \) the estimate

\[ \| R(t) \|_{\mathcal{B}(X)} \leq C_b |\hat{b}(1/t)/t|^2. \]
(iii) Combining the estimates from part (i) and (ii) implies for each $t > 0$ that

$$\|R(t)\|_{B(X)} \leq C \min\{\hat{a}(1/t)/t, \hat{b}(1/t)/t^2\},$$

with some constant $C > 0$. The map $t \mapsto \min\{\hat{a}(1/t)/t, \hat{b}(1/t)/t^2\}$ is integrable on $\mathbb{R}_+$, by assumption. This implies directly the integrability of the integral resolvent on $\mathbb{R}_+$.

Moreover, if $\int_1^\infty \hat{b}(1/t)\frac{dt}{t^2} < \infty$, then we have $\int_0^\infty \min\{\hat{a}(1/t)/t, \hat{b}(1/t)/t^2\}dt \leq \int_0^1 \hat{a}(1/t)\frac{dt}{t} + \int_1^\infty \hat{b}(1/t)\frac{dt}{t^2} < \infty$, and the integrability of the integral resolvent on $\mathbb{R}_+$ follows.

$\square$

In the situation of Volterra equations with standard kernel and a compatible sectorial operator there always exists an integrable integral resolvent.

**Corollary 3.3.9.** Let $\alpha \in (0, 1)$ and $A$ be a sectorial operator on the Banach space $X$ with spectral angle $\varphi_A < \pi - \alpha \frac{\pi}{2}$.

Then the Volterra equation with standard kernel

$$u(t) + (g_{\alpha} * Au)(t) = f(t), \quad t \in [0, T], \quad (3.6)$$

where $f \in C([0, T]; X)$, admits an integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$. Moreover, if the operator $A$ is invertible, then the integral resolvent is integrable on $\mathbb{R}_+$.

**Proof.** We consider the situation of the standard kernel with $a = g_{\alpha}$ and $b = g_{1-\alpha}$, $\alpha \in (0, 1)$. Their Laplace transforms are given by $\hat{a}(\lambda) = \lambda^{-\alpha}$ and $\hat{b}(\lambda) = \lambda^{-(1-\alpha)}$, for $\lambda \in \mathbb{C}_+$. In particular, we have $\int_0^1 \hat{a}(1/t)\frac{dt}{t} = \int_0^1 t^{-\alpha}\frac{dt}{t} = \frac{1}{1-\alpha} < \infty$, as well as $\int_0^1 \hat{b}(1/t)\frac{dt}{t^2} = \int_0^1 t^{1-\alpha}\frac{dt}{t^2} = \frac{1}{1-\alpha} < \infty$ and $\int_0^\infty \hat{b}(1/t)\frac{dt}{t^2} = \int_0^1 t^{1-\alpha}\frac{dt}{t^2} = \frac{1}{1-\alpha} < \infty$. Together with the assumptions on the operator $A$ it follows by Theorem 3.3.8 that there is an integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$ for the Volterra equation (3.6). Assuming the invertibility of the operator $A$, the integral resolvent is integrable on $\mathbb{R}_+$.

$\square$

The following corollary gives sufficient conditions for the integrability requirements in the above Theorem 3.3.8. These conditions are integrability conditions for the given kernel and not for its Laplace transform.

**Corollary 3.3.10.** Let $c \in L_{1,loc}(\mathbb{R}_+)$ be a non-negative kernel of subexponential growth.

(i) If $-\int_0^1 c(t)\ln t\,dt < \infty$, then $\int_0^1 \hat{c}(1/t)\frac{dt}{t} < \infty$.

(ii) The condition $\int_1^\infty c(t)\frac{dt}{t} < \infty$ implies $\int_1^\infty \hat{c}(1/t)\frac{dt}{t^2} < \infty$. 

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Proof. (i) By substitution we have \( \int_0^1 \hat{c}(t) \frac{dt}{t} = \int_1^\infty \hat{c}(r) \frac{dr}{r} \). The definition of the Laplace transform yields

\[
\int_0^1 \hat{c}(t) \frac{dt}{t} = \int_1^\infty \int_0^\infty e^{-rt} c(t) dt \frac{dr}{r}.
\]

Note that the integrand is non-negative for all \((r, t) \in [1, \infty) \times (0, \infty)\). We change the order of integration and show that the integral exists. Thus,

\[
\int_0^\infty c(t) \int_1^\infty e^{-rt} \frac{dr}{r} dt \leq \int_0^\infty c(t) e^{-t} \ln \left(1 + \frac{1}{t}\right) dt,
\]

here we applied the estimate for the exponential integral \( \int_1^\infty e^{-rt} \frac{dr}{r} < e^{-t} \ln \left(1 + \frac{1}{t}\right) \) for \( t > 0 \), see [AS64, p. 229, 5.1.10]. On the interval \([0, 1]\) we have the estimate \( \ln \left(1 + \frac{1}{t}\right) = \ln(t + 1) - \ln t \leq \ln 2 - \ln t \), and on the interval \([1, \infty)\) that \( \ln \left(1 + \frac{1}{t}\right) \leq \ln 2 \). This yields

\[
\int_0^\infty c(t) \int_1^\infty e^{-rt} \frac{dr}{r} dt \leq \ln 2 \int_0^\infty c(t) e^{-t} dt - \int_0^1 c(t) \ln t dt < \infty,
\]

by the assumption that \(-\int_0^1 c(t) \ln t dt < \infty\). By Tonelli’s Theorem we know that both repeated integrals coincide and thus \( \int_0^1 \hat{c}(t) \frac{dt}{t} < \infty \).

(ii) An easy substitution yields that \( \int_1^\infty \hat{c}(1/t) \frac{dt}{t} = \int_0^1 \hat{c}(r) dr \). With the definition of the Laplace transform it follows that

\[
\int_1^\infty \hat{c}(1/t) \frac{dt}{t^2} = \int_0^1 \int_0^\infty e^{-rt} c(t) dt dr.
\]

Observe that the integrand is non-negative for all \((r, t) \in [0, 1] \times (0, \infty)\). Once more we change the order of integration and prove that the integral exists, thus

\[
\int_0^\infty c(t) \int_0^1 e^{-rt} dr dt = \int_0^\infty c(t) \left[1 - e^{-t}\right] \frac{dt}{t}.
\]

We can estimate the term \( \left[1 - e^{-t}\right]/t \) on the interval \([0, 1]\) uniformly by one. Together with the triangle inequality we deduce

\[
\int_0^\infty c(t) \int_0^1 e^{-rt} dr dt \leq \int_0^1 c(t) dt + \int_1^\infty c(t) \frac{dt}{t} + \int_1^\infty e^{-t} c(t) dt < \infty,
\]

since \( c \in L_{1,loc}(\mathbb{R}_+) \) is of subexponential growth and \( \int_1^\infty c(t) \frac{dt}{t} < \infty \), by assumption.

\[\square\]
Remark 3.3.11. Observe that the idea for the condition in Corollary 3.3.10 (i) originates from Comment 2.7 c) in [Prü12]. Here, Prüss gives similar sufficient conditions for the integrability of the integral resolvent without any reasoning. Our sufficient conditions on the kernel \( a \) for the existence of an integrable integral resolvent correspond to the conditions in the monograph of Prüss.

### 3.4 Time-Weights for Linear Volterra Equations

An essential step in the proof of our stability result in Part II is the time-weighting of the Volterra equation and the rewriting of the corresponding mild solution. For this reason we introduce the used time-weighting argument for linear Volterra equations. At first, we have the following auxiliary result.

**Lemma 3.4.1.** Let \( b \in L^1_{\text{loc}}(\mathbb{R}_+) \cap C^1((0,\infty)) \) such that \( b \geq 0 \) with \( \lim_{t \to 0} tb(t) = 0 \) and \( -\dot{b} \geq 0 \) is non-increasing on \((0,\infty)\). For some \( T > 0 \) we consider \( \phi \in C([0,T]) \cap C^1((0,T)) \) with \( \phi > 0 \) on \([0,T]\), and \( u \in C([0,T];X) \).

Then the map \( t \mapsto \phi(t)(b \ast u)(t) - (b \ast [\phi u])(t) \) is continuous on \([0,T]\) and differentiable on \((0,T)\) with

\[
\partial_t [\phi(b \ast u) - (b \ast [\phi u])](t) = \phi(t)(b \ast u)(t) + \int_0^t \dot{b}(t-\tau)[\phi(t) - \phi(\tau)]u(\tau) d\tau,
\]

for all \( t \in (0,T) \).

**Remark 3.4.2.**

a) The assumptions of Lemma 3.4.1 imply that \([t \mapsto \dot{b}(t)] \in L^1_{\text{loc}}(\mathbb{R}_+)\). Indeed, for arbitrary \( T > 0 \) we have by integration by parts that

\[
\left| \int_0^T \dot{t}b(t) dt \right| \leq T b(T) + \lim_{t \to 0} t b(t) + \int_0^T b(t) dt < \infty,
\]

since \( \lim_{t \to 0} t b(t) = 0 \) and \( b \in L^1_{\text{loc}}(\mathbb{R}_+) \) is non-negative.

b) Note, that for all completely monotonic kernels \( b \in L^1_{\text{loc}}(\mathbb{R}_+) \), which admit a corresponding Sonine kernel, the assumptions of Lemma 3.4.1 are satisfied.

**Proof.** The continuity of the map \( t \mapsto \phi(t)(b \ast u)(t) - (b \ast [\phi u])(t) \) is obvious, since the convolution of a locally integrable kernel \( b \) with the continuous function \( u \) and \( \phi u \), respectively, is continuous on \([0,T]\). We prove the differentiability by showing the limit of the differential quotient exists. Let \( t \in (0,T) \) be arbitrary, but fixed, and \( |h| \in (0,\min\{t/4,T-t\}) \) be sufficiently small. We have
The function \( \varphi \) is differentiable on \((0, T)\) and the convolution \((b * u)\) is continuous on \([0, T]\). This implies

\[
\frac{[\varphi(t+h) - \varphi(t)](b * u)(t+h)}{h} \to \varphi(t)(b * u)(t),
\]
as \( |h| \to 0 \).

**Right-Hand Differential Quotient.** We consider the second summand in (3.7) with \( h \in (0, \min\{t/4, T-t\}) \) and have

\[
\frac{1}{h} \left\{ \varphi(t)(b * u)(t+h) - (b * [\varphi u])(t+h) - [\varphi(t)(b * u)(t) - (b * [\varphi u])(t)] \right\}
\]
\[
= \left[ \int_0^t \frac{b(t+h-\tau) - b(t-\tau)}{h} [\varphi(t) - \varphi(\tau)] u(\tau) d\tau \right.
\]
\[
+ \left. \int_t^{t+h} b(t+h-\tau) \frac{\varphi(t) - \varphi(\tau)}{h} u(\tau) d\tau \right].
\]

Since \( \varphi \) is continuously differentiable on \((0, T)\) we have for all \( \tau \in (t, t+h) \) that there is some \( \xi \in (t, \tau) \subset (t, t+h) \) such that \(|\varphi(t) - \varphi(\tau)| \leq |\varphi(\xi)| |t-\tau| \leq \|\varphi\|_{C([t, \min\{5t/4, T]\}]} h\). This yields

\[
\int_t^{t+h} \frac{|b(t+h-\tau)|}{h} \frac{|\varphi(t) - \varphi(\tau)|}{h} \|u(\tau)\|_X d\tau 
\]
\[
\leq \int_t^{t+h} |b(t+h-\tau)| d\tau \|\varphi\|_{C([t, \min\{5t/4, T]\}]} \|u\|_{C([0, T]; X)} ;
\]

since \( b \in L_1 ((0, T)) \), this expression tends to zero as \( h \to 0 \).

Now, we want to apply the vector-valued version of the dominated convergence theorem from Arendt et al. [ABHN01, Theorem 1.1.8] to deduce that

\[
\int_0^t \frac{b(t+h-\tau) - b(t-\tau)}{h} [\varphi(t) - \varphi(\tau)] u(\tau) d\tau \to \int_0^t b(t-\tau) [\varphi(t) - \varphi(\tau)] u(\tau) d\tau,
\]
as \( h \to 0 \). We define \( f_h : [0, t] \to X \) by

\[
f_h(\tau) = \begin{cases} 
\frac{b(t+\tau-h) - b(t-\tau)}{h} [\varphi(t) - \varphi(\tau)] u(\tau), & \tau \in [0, t), \\
0, & \tau = t.
\end{cases}
\]
3. Linear Volterra Equations

Obviously, for all \( \tau \in [0, t) \) we have \( f_h(\tau) \rightarrow b(t-\tau)[\varphi(t) - \varphi(\tau)]u(\tau) \) as \( h \rightarrow 0 \).

We will show that the map \( g: [0, t] \rightarrow \mathbb{R}_+ \), given by

\[
g(\tau) = \begin{cases} 
|\dot{b}(t-\tau)|2\|\varphi\|_{C([0,T])}\|u\|_{C([0,T],X)}, & \tau \in [0, t/2], \\
|\dot{b}(t-\tau)(t-\tau)|\|\dot{\varphi}\|_{C([t/2,T])}\|u\|_{C([0,T],X)}, & \tau \in (t/2, t), \\
0, & \tau = t,
\end{cases}
\]

dominates \( f_h(\tau) \) for almost all \( \tau \in [0, t] \) and all \( h > 0 \). The assumptions on \( b \) and \( \dot{b} \) ensure the integrability of \( s \mapsto \dot{b}(s) \) on \( (0, t) \). Hence, we have \( g \in L_1((0,t)) \).

Using the continuous differentiability of the kernel \( b \) we deduce that for each \( \tau \in [0, t) \) there is some \( \xi \in (0, h) \) with \( |b(t-\tau+h) - b(t-\tau)| \leq |\dot{b}(t-\tau+\xi)|h \). Since \( -\dot{b} = |b| \) is non-increasing, we have \( |b(t-\tau+h) - b(t-\tau)| \leq h|\dot{b}(t-\tau)| \).

For \( \tau \in [0, t/2] \) we have the uniform estimate

\[
\|f_h(\tau)\|_{X} \leq |\dot{b}(t-\tau)|2\|\varphi\|_{C([0,T])}\|u\|_{C([0,T],X)}.
\]

For \( \tau \in (t/2, t) \) we use the continuous differentiability of the map \( \varphi \). There is some \( \xi \in (\tau, t) \subset (t/2, t) \) such that \( |\varphi(t) - \varphi(\tau)| \leq |\dot{\varphi}(\xi)||t-\tau| \). This yields for \( \tau \in [t/2, t) \) that

\[
\|f_h(\tau)\|_{X} \leq |\dot{b}(t-\tau)(t-\tau)|\|\dot{\varphi}\|_{C([t/2,T])}\|u\|_{C([0,T],X)}.
\]

Finally, the vector-valued dominated convergence theorem implies that

\[
\int_{0}^{t} f_h(\tau) d\tau \rightarrow \int_{0}^{t} \dot{b}(t-\tau)[\varphi(t) - \varphi(\tau)]u(\tau) d\tau,
\]
as \( h \rightarrow 0 \). Thus, \( \varphi(t)(b * u)(t) - (b * [\varphi u])(t) \) is differentiable from the right.

**Left-Hand Differential Quotient.** Now, we consider the second summand in (3.7) with \( h \in (0, \min(t/4, T - t)) \) and have

\[
\frac{1}{h} \{ \varphi(t)(b * u)(t) - (b * [\varphi u])(t) - [\varphi(t)(b * u)(t-h) - (b * [\varphi u])(t-h)] \}
\]

\[
= \int_{0}^{t-h} \frac{b(t-\tau) - b(t-h-\tau)}{h}[\varphi(t) - \varphi(\tau)]u(\tau) d\tau \\
+ \int_{t-h}^{t} \frac{b(t-\tau)\varphi(t) - \varphi(\tau)}{h}u(\tau) d\tau.
\]

Analogous to the situation above, we use the continuous differentiability of \( \varphi \) on \( (0, T) \). For all \( \tau \in [t-h, t] \) there is some \( \xi \in (\tau, t) \subset (t-h, t) \subset (3t/4, t) \) such that \( |\varphi(t) - \varphi(\tau)| \leq |\varphi(\xi)||t-\tau| \leq \|\dot{\varphi}\|_{C([3t/4,T])}h \). This implies

\[
\int_{t-h}^{t} |b(t-\tau)||d\tau|\|\dot{\varphi}\|_{C([3t/4,T])}\|u\|_{C([0,T],X)};
\]
3.4. Time-Weights for Linear Volterra Equations

since $b \in L_1((0, T))$ this expression tends to zero as $h \to 0$.

Once more we want to apply the vector-valued version of the dominated
convergence theorem from Arendt et al. [ABHN01, Theorem 1.1.8] to prove
that

$$
\int_0^{t-h} b(t - \tau) - b(t - h - \tau) \frac{\|\varphi(t) - \varphi(\tau)\|}{h} u(\tau) \, d\tau
= \int_h^t b(t - \tau + h) - b(t - \tau) \frac{\|\varphi(t) - \varphi(\tau - h)\|}{h} u(\tau - h) \, d\tau
\to \int_0^t b(t - \tau) [\varphi(t) - \varphi(\tau)] u(\tau) \, d\tau,
$$

as $h \to 0$. We define $k_h: [0, t] \to X$ by

$$
k_h(\tau) = \begin{cases} 
\frac{b(t - \tau + h) - b(t - \tau)}{h} [\varphi(t) - \varphi(\tau - h)] u(\tau - h), & \tau \in [h, t) \\
0, & \tau \in [0, h) \cup \{t\}.
\end{cases}
$$

Obviously, for all $\tau \in [0, t)$ we have $k_h(\tau) \to \dot{b}(t - \tau) [\varphi(t) - \varphi(\tau)] u(\tau)$ as $h \to 0$.

We will show that the map $l: [0, t] \to \mathbb{R}_+$, given by

$$
l(\tau) = \begin{cases} 
[\dot{b}(t - \tau)]2 \|\varphi\|_{C([0, t])} \|u\|_{C([0, t]; X)}, & \tau \in [0, t/2],

2 |\dot{b}(t - s)| + |\dot{b}(t - \tau)(t - \tau)| \|\dot{\varphi}\|_{C([t/2, t])} \|u\|_{C([0, t]; X)}), & \tau \in (t/2, t),

0, & \tau = t,
\end{cases}
$$

dominates $k_h(\tau)$ for almost all $\tau \in [0, t]$ and all $h > 0$. By the assumptions on $b$
and $\dot{b}$ we obtain $l \in L_1((0, t))$.

The continuous differentiability of the kernel $b$ implies that for each
$\tau \in [0, t)$ there is some $\xi \in (0, h)$ such that $|b(t - \tau + h) - b(t - \tau)| \leq |b(t - \tau + \xi) - h|.$
Since $-\dot{b} = |\dot{b}|$ is non-increasing, we have $|b(t - \tau + h) - b(t - \tau)| \leq h|\dot{b}(t - \tau)|$.

For $\tau \in [0, t/2]$ it follows the uniform estimate

$$
\|k_h(\tau)\|_X \leq [\dot{b}(t - \tau)]2 \|\varphi\|_{C([0, t])} \|u\|_{C([0, t]; X)}.
$$

For $\tau \in (t/2, t)$ we employ the continuous differentiability of the map $\varphi$.
There is some $\xi \in (t - h, t) \subset (3t/4, t)$ such that $|\varphi(t) - \varphi(t - h)| \leq |\varphi(\xi)|h$. The
same statement remains true for all $\tau \in (t/2, t)$, there is some $\eta \in (t - h, t - h) \subset (t/4, t)$ such that $|\varphi(t - h) - \varphi(\tau - h)| \leq |\varphi(\eta)||t - \tau|$. Combining these estimates with the known estimates for $\dot{b}$ from above, we have for $\tau \in (t/2, t)$ that

$$
\|k_h(\tau)\|_X \leq [\dot{b}(t + h - \tau) - b(t - \tau)] \frac{|\varphi(t) - \varphi(t - h)|}{h} \|u\|_{C([0, t]; X)}
+ \frac{|\dot{b}(t + h - \tau) - b(t - \tau)|}{h} |\varphi(t - h) - \varphi(\tau - h)| \|u\|_{C([0, t]; X)}
\leq \left[2 |\dot{b}(t - \tau)| \|\dot{\varphi}\|_{C([3t/4, t])} + |\dot{b}(t - \tau)(t - \tau)| \|\dot{\varphi}\|_{C([t/4, t])} \right] \|u\|_{C([0, t]; X)}.
$$
Finally, the vector-valued dominated convergence theorem yields that
\[ \int_0^t k_h(\tau) \, d\tau \to \int_0^t b(t-\tau)[\varphi(\tau) - \varphi(t)]u(\tau) \, d\tau, \]
as \( h \to 0 \). Hence, \( \varphi(t)(b \ast u)(t) - (b \ast [\varphi u])(t) \) is differentiable from the left.

Since the right-hand limit and the left-hand limit of the differential quotient coincide, the map \( t \mapsto \varphi(t)(b \ast u)(t) - (b \ast [\varphi u])(t) \) is differentiable at \( t \in (0,T) \). In particular, \( \int_0^t b(t-\tau)[\varphi(t) - \varphi(\tau)]u(\tau) \, d\tau \) exists for all \( t \in (0,T) \) as a Bochner integral. \( \square \)

**Corollary 3.4.3.** Let the assumptions of Lemma 3.4.1 be satisfied.

If for some \( t \in (0,T) \) it is known that \((b \ast u)(t)\) or \((b \ast [\varphi u])(t)\) is differentiable at the point \( t \), then we have
\[ \varphi(t)\partial_t(b \ast u)(t) - \partial_t(b \ast [\varphi u])(t) = \int_0^t b(t-\tau)[\varphi(t) - \varphi(\tau)]u(\tau) \, d\tau. \] (3.8)

**Remark 3.4.4.** In the situation of Corollary 3.4.3 we call the expression
\[ \int_0^t [-b(t-\tau)]\varphi(t)(b \ast u)(t) \] commutator term.

**Proof of Corollary 3.4.3.** Firstly, we consider the case that \((b \ast u)(t)\) is differentiable at the point \( t \in (0,T) \). The differentiability of \( \varphi(t)(b \ast u)(t) - (b \ast [\varphi u])(t) \) and \((b \ast u)(t)\) at the point \( t \in (0,T) \) yields the differentiability of \((b \ast [\varphi u])(t)\) at \( t \). Now, we consider the case that \((b \ast [\varphi u])(t)\) is differentiable at the point \( t \in (0,T) \). The differentiability of \( \varphi(t)(b \ast u)(t) - (b \ast [\varphi u])(t) \) and \((b \ast [\varphi u])(t)\) implies the differentiability of \( \varphi(t)(b \ast u)(t) \) at \( t \). Since \( \varphi(t) > 0 \) and \( \varphi(t) \) is differentiable it follows that \((b \ast u)(t)\) is differentiable at \( t \).

The relation (3.8) follows by Lemma 3.4.1 and the product rule for differentiable functions. \( \square \)

Suppose that \( u \in C([0,T];X_A) \) is a strong solution of the linear Volterra equation
\[ u(t) + (a \ast Au)(t) = u_0 + (a \ast f)(t), \quad t \in [0,T], \] (3.9)
where \( A \) is a closed, linear and densely defined operator on the Banach space \( X \), \( u_0 \in D(A) \) and \( f \in C([0,T];X_A) \). Applying the above Corollary 3.4.3 we deduce a Volterra equation for the product \( \varphi u \), with \( \varphi \in C([0,T]) \cap C^1((0,T)) \).
Lemma 3.4.5. Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) be a unbounded, completely monotonic kernel and \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) denote the corresponding completely monotonic Sonine kernel. Furthermore, let \( A \) be a closed, linear and densely defined operator in the Banach space \( X \). We assume that the Volterra equation (3.9) admits a resolvent and an integral resolvent. Moreover, let \( u \in C([0, T]; X_A) \) be a strong solution of the linear Volterra equation (3.9) for some \( T > 0 \), \( f \in C([0, T]; X_A) \) and \( u_0 \in D(A) \), as well as \( \varphi \in C([0, T]) \cap C^1((0, T)) \) with \( \varphi > 0 \) on \([0, T]\).

Then we have for the product \( \varphi u \) the following Volterra equation

\[
[q u](t) + (a * A[q u])(t) = u_0 + (a * [b[\varphi - 1]u_0])(t) + (a * [\varphi f])(t) + \left( a * \int_0^t [-\hat{b}(\cdot - \tau)] [\varphi(\cdot) - \varphi(\tau)] u(\tau) d\tau \right)(t), \quad t \in [0, T].
\]

(3.10)

Remark 3.4.6. The strong solution \( u \in C([0, T]; X_A) \) is given by the mild formulation \( u(t) = S(t)u_0 + (R * f)(t), \quad t \in [0, T]. \) According to this representation we have \( [q u](t) = \varphi(t)S(t)u_0 + \varphi(t)(R * f)(t). \) The equation (3.10) from the above Lemma 3.4.5 yields for the product \( \varphi u \) another representation

\[
[q u](t) = S(t)u_0 + (R * [b[\varphi - 1]u_0])(t) + (R * [\varphi f])(t) + \left( R * \int_0^t [-\hat{b}(\cdot - \tau)] [\varphi(\cdot) - \varphi(\tau)] u(\tau) d\tau \right)(t),
\]

for all \( t \in [0, T] \).

Proof of Lemma 3.4.5. Let \( u \in C([0, T]; X_A) \) be a strong solution of the linear Volterra equations (3.9). If we convolve the equation (3.9) with the corresponding Sonine kernel \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) and use the Sonine condition \( a * b = 1 \) on \([0, \infty)\), we obtain the equation

\[
(b * u)(t) + (1 * Au)(t) = (b * 1)(t)u_0 + (1 * f)(t), \quad t \in [0, T].
\]

Since \((1 * Au), (1 * f) \in C^1([0, T]; X)\) and \((1 * b) \in C^1((0, T]; X)\), we know that every term of the equation is differentiable on \([0, T]\). In particular, \( b * u \) differentiable on \([0, T]\) with \( \partial_t(b * u)(t) = b(t)u_0 + f(t) - Au(t) \). Obviously \( \partial_t(b * u)(t) - b(t)u_0 \) is continuous on \([0, T]\). Hence, we get for \( t \in (0, T] \) the equation

\[
\partial_t(b * u)(t) + Au(t) = b(t)u_0 + f(t).
\]

Now, we multiply this equation by the scalar function \( \varphi \) and apply Corollary 3.4.3 to replace the expression \( \varphi(t) \partial_t(b * u)(t) \). This yields for all \( t \in (0, T] \) that

\[
\partial_t(b * [q u])(t) + A[q u](t) = (\varphi b)(t)u_0 + (\varphi f)(t) + \int_0^t (-\hat{b}(t - \tau)) [\varphi(t) - \varphi(\tau)] u(\tau) d\tau.
\]
We convolve both sides of this equation with the kernel \( a \in \mathbb{L}_{1,\text{loc}}(\mathbb{R}_+) \) and obtain for all \( t \in [0, T] \) the equation
\[
(a \ast \partial_t (b \ast [\varphi u]))(t) + (a \ast A[\varphi u])(t) = (a \ast [\varphi b])(t)u_0 + (a \ast [\varphi f])(t) \\
+ \bigg( a \ast \int_0^t [-b(\cdot - \tau)] \left[ \varphi(\cdot) - \varphi(\tau) \right] u(\tau) \, d\tau \bigg)(t).
\]
Finally, we can replace the term \( a \ast \partial_t (b \ast [\varphi u]) \) by \( \varphi u \) on \( [0, T] \), since we have \( (b \ast [\varphi u])(0) = 0 \) and \( a \ast b \equiv 1 \) on \( (0, \infty) \). Thus,
\[
[\varphi u](t) + (a \ast A[\varphi u])(t) = (a \ast [b\varphi])(t)u_0 + (a \ast [\varphi f])(t) \\
+ \bigg( a \ast \int_0^t [-b(\cdot - \tau)] \left[ \varphi(\cdot) - \varphi(\tau) \right] u(\tau) \, d\tau \bigg)(t), \quad t \in [0, T].
\]
Obviously, we can add a zero \( u_0 - (a \ast b)\cdot u_0 \) on the right hand side of the equation. The linearity of the integral yields the claim. \( \square \)

**Lemma 3.4.7.** Let \( a \in \mathbb{L}_{1,\text{loc}}(\mathbb{R}_+) \) be an unbounded, completely monotonic kernel and \( b \in \mathbb{L}_{1,\text{loc}}(\mathbb{R}_+) \) denote the corresponding Sonine kernel. Furthermore, let \( A \) be a closed, linear and densely defined operator in the Banach space \( X \). We assume that the Volterra equation (3.9) admits a resolvent and an integral resolvent. Moreover, let \( u_0 \in X \) and \( f \in C([0, T]; X) \) for some \( T > 0 \), as well as \( \varphi \in C([0, T]) \cap C^1((0, T)) \) with \( \varphi > 0 \) on \([0, T]\). We denote by \( u \in C([0, T]; X) \) the mild solution of the Volterra equation (3.9). Then,
\[
[\varphi u](t) = S(t)u_0 + (R \ast [b[\varphi - 1]u_0])(t) + (R \ast [\varphi f])(t) \\
+ \bigg( R \ast \int_0^t [-b(\cdot - \tau)] \left[ \varphi(\cdot) - \varphi(\tau) \right] u(\tau) \, d\tau \bigg)(t),
\]
for all \( t \in [0, T] \).

**Proof.** Lemma 3.4.5 and Remark 3.4.6 imply this representation for the product only for \( u_0 \in D(A) \) and \( f \in C([0, T]; X_A) \). Let \( \left\{ u_0^{(n)} \right\}_{n \in \mathbb{N}} \subset D(A) \) be such that \( u_0^{(n)} \to u_0 \) in \( X \) and \( \left\{ f^{(n)} \right\}_{n \in \mathbb{N}} \subset C([0, T]; X_A) \) be such that \( \left\| f^{(n)} - f \right\|_{C([0,T];X)} \to 0 \) as \( n \to \infty \). We denote by \( u^{(n)} \in C([0, T]; X_A) \) the strong solution of the Volterra equation
\[
u^{(n)}(t) + (a \ast Au^{(n)})(t) = u_0^{(n)} + (a \ast f^{(n)})(t), \quad t \in [0, T].
\]
It is given by the mild formulation \( u^{(n)}(t) = S(t)u_0^{(n)} + (R \ast f^{(n)})(t), \ t \in [0, T] \).
On one hand, we have
\[
\left\| u - u^{(n)} \right\|_{C([0,T];X)} \leq \sup_{t \in [0,T]} \left\| S(t) \cdot G(X) \right\| \left\| u_0 - u_0^{(n)} \right\|_X + (\psi * 1)(T) \left\| f - f^{(n)} \right\|_{C([0,T];X)} \to 0,
\]
as \( n \to \infty \). Here \( \psi \in L_{1, \text{loc}}(\mathbb{R}_+) \) denotes the locally integrable function from the Definition 3.1.8 of the integral resolvent. Note that the resolvent is uniformly bounded on compact intervals. On the other hand, we have by Lemma 3.4.5 that
\[
[\varphi u]^{(n)}(t) = S(t)u_0^{(n)} + \left( R * \left[ b[\varphi - 1]u_0^{(n)} \right] \right)(t) + \left( R * \left[ \varphi f^{(n)} \right] \right)(t) + \left( R * \int_0^t [-b(\cdot - \tau)] [\varphi(\cdot) - \varphi(\tau)] (u_0^{(n)}(\tau) - u^{(n)}(\tau)) d\tau \right)(t).
\]
In particular, we obtain for each \( t \in [0, T] \) that
\[
\begin{align*}
\left\| S(t) \left[ u_0 - u_0^{(n)} \right] \right\| + \left( R * \left[ b[\varphi - 1] \left[ u_0 - u_0^{(n)} \right] \right] \right)(t) + \left( R * \left[ \varphi \left[ f - f^{(n)} \right] \right] \right)(t) & \\
+ \left( R * \int_0^t [-b(\cdot - \tau)] [\varphi(\cdot) - \varphi(\tau)] (u_0^{(n)}(\tau) - u^{(n)}(\tau)) d\tau \right)(t) & \\
\leq \sup_{t \in [0,T]} \left\| S(t) \cdot G(X) \right\| \left\| u_0 - u_0^{(n)} \right\|_X + (\psi * [b[\varphi - 1]])(T) \left\| u_0 - u_0^{(n)} \right\|_X + (\psi * \varphi)(T) \left\| f - f^{(n)} \right\|_{C([0,T];X)} + (\psi * \int_0^T [b(\cdot - \tau)] [\varphi(\cdot) - \varphi(\tau)] d\tau \right)(T) \left\| u - u^{(n)} \right\|_{C([0,T];X)} & \\
\to 0,
\end{align*}
\]
as \( n \to \infty \). The claim follows.

\[
\square
\]

### 3.5 Commutator Term

For a special choice of the maps \( \varphi \) and \( u \) we are able to calculate the commutator term.

**Lemma 3.5.1.** Let \( \varepsilon > 0 \) and \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \) be an unbounded, completely monotonic kernel and \( b \in L_{1, \text{loc}}(\mathbb{R}_+) \) be the corresponding completely monotonic Sotn2ine kernel. Let \( s_\varepsilon \) denote the scalar resolvent, i.e. the solution of the equation \( s_\varepsilon + \varepsilon (s_\varepsilon * a) = 1 \) on \( \mathbb{R}_+ \). We set \( \varphi_\varepsilon = 1/s_\varepsilon \) on \( \mathbb{R}_+ \).

Then we have for all \( t > 0 \) the equation
\[
\int_0^t [-b(t - \tau)] [\varphi_\varepsilon(t) - \varphi_\varepsilon(\tau)] \frac{1}{\varphi_\varepsilon(\tau)} d\tau = \varepsilon + b(t) [1 - \varphi_\varepsilon(t)] \geq 0. \tag{3.11}
\]
We recall the relations

Applying again the fundamental theorem of calculus yields

The preceding calculations imply

Now, we use the relation

By the properties of the scalar resolvent \( s_{\varepsilon} \), we know for the map \( q_{\varepsilon} \) that \( q_{\varepsilon}(0) = 1 \), \( q_{\varepsilon} \geq 1 \) is increasing and absolutely continuous on \( \mathbb{R}_+ \), and \( q_{\varepsilon} \in L_1,\text{loc}(\mathbb{R}_+) \), with \( \dot{q}_{\varepsilon}(t) = \varepsilon r_{\varepsilon}(t)/s^2_{\varepsilon}(t) \) for all \( t \in (0, \infty) \). The relation (3.11) implies on \((0, \infty)\) the estimate

Note that this estimate is also known from Lemma 2.2.6.

**Proof of Lemma 3.5.1.** We bring the expression in the bracket down to a common denominator and obtain

By the fundamental theorem of calculus it follows that

Changing the order of integration yields

Applying again the fundamental theorem of calculus yields

Now, we use the relation

where \( r_{\varepsilon} \) denotes the scalar integral resolvent, cf. Remark 2.3.3, and get

We recall the relations \( r_{\varepsilon} \ast b = s_{\varepsilon} \) on \((0, \infty)\) and \( s_{\varepsilon}(0) = 1 \), see Proposition 2.2.5, and obtain the relation

The preceding calculations imply

The non-negativity of the integral relies on the non-negativity of \(-b\) and the fact that \( s_{\varepsilon} \) is non-increasing on \( \mathbb{R}_+ \). Therefore the integrand is non-negative, too. It is also possible to see the non-negativity of the expression by using the estimate from Lemma 2.2.6.\( \Box \)
Part II

Semilinear Parabolic Volterra Equations
Stability for Semilinear Parabolic Volterra Equations

In this chapter we consider semilinear Volterra equations. We define mild solutions for these equations, prove the local existence and uniqueness of mild solutions for semilinear problems and show how to continue local solutions. We close this chapter with the principle of linearised stability for semilinear parabolic Volterra equations, Theorem 4.2.2. The simple subsequent corollaries pay attention to the special case of Volterra equations with standard kernels.

Throughout this chapter we assume the following:

Assumptions. Let $X$ be a complex Banach space equipped with the norm $\|\cdot\|_X$, and let $A$ be a closed, unbounded linear operator in $X$ with dense domain $D(A)$ and $X_A = (D(A),\|\cdot\|_A)$. Furthermore, let $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ be an unbounded, non-vanishing scalar kernel and $J = [0,T]$ with some $T > 0$, as well as $u_0 \in X$. For $T > 0$ and some $r > 0$ the map $F: [0,T] \times B_r(u_0) \to X$ is continuous and satisfies a Lipschitz condition in the second component, i.e. there is some $L > 0$ such that for all $u,v \in B_r(u_0)$ and all $t \in [0,T]$ we have the estimate

$$\|F(t,u) - F(t,v)\|_X \leq L\|u - v\|_X.$$

4.1 Local Existence and Uniqueness of Mild Solutions and Maximal Solutions

We study the following semilinear Volterra equation

$$u(t) + (a*Au)(t) = u_0 + (a*F(\cdot,u))(t), \quad t \in J, \quad (4.1)$$

under the additional assumption, that this equation admits both a resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$ and an integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$ in the sense of Section 3.1.

Definition 4.1.1 (Mild Solution). We call a function $u \in C(J;X)$ mild solution of the Volterra equation (4.1) if $u$ solves the equation

$$u(t) = S(t)u_0 + (R*F(\cdot,u))(t), \quad t \in J.$$

We start with a result on existence and uniqueness of mild solutions of the semilinear equation (4.1) under the above assumption on the map $F$. 63
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4.1.1 Local Existence and Uniqueness

We prove that the semilinear Volterra equation admits a unique local mild solution.

Theorem 4.1.2 (Local Existence and Uniqueness). Let \( u_0 \in X \) and \( F: [0, T] \times B_r(u_0) \to X \) be continuous and Lipschitz continuous in the second component. We assume that there is a resolvent \( \{S(t)\}_{t \in \mathbb{R}_+} \) and an integral resolvent \( \{R(t)\}_{t \in \mathbb{R}_+} \) for the Volterra equation (4.1).

Then there is some \( \delta = \delta(u_0) > 0 \) such that there is a unique local mild solution \( u \in C([0, \delta]; X) \) for (4.1).

Proof. For a given \( u_0 \in X \) and some sufficiently small \( \delta > 0 \), which will be chosen later, we define the map \( K: C([0, \delta]; X) \to C([0, \delta]; X) \) via

\[
(Ku)(t) = S(t)u_0 + (R* F(\cdot, u))(t),
\]

for all \( t \in [0, \delta] \subset [0, T] \). A fixed point of \( K \) yields the desired mild solution. We define the set

\[
\mathcal{B}(u_0, \delta, r) = \{ u \in C([0, \delta]; X): \|u - u_0\|_{C([0, \delta], X)} \leq r \},
\]

and we will prove that \( K \) maps the set \( \mathcal{B}(u_0, \delta, r) \) into itself and \( K \) is a contraction.

Self-mapping. Let \( u \in \mathcal{B}(u_0, \delta, r) \) be arbitrary. We show that for sufficiently small \( \delta > 0 \) the function \( Ku \) belongs to \( \mathcal{B}(u_0, \delta, r) \), too. Obviously, \( Ku \) is continuous on \([0, \delta]\) and we have for all \( t \in [0, \delta] \) that

\[
\| (R* F(\cdot, u))(t) \|_X \leq \int_0^t \| R(t-\tau)F(\tau, u(\tau)) \|_X \, d\tau \\
\leq \| F(\cdot, u) \|_{C([0, \delta], X)} \| \varphi \|_{L_1([0, \delta])}.
\]

Via the triangle inequality and the continuity of the map \( F \) it is obvious that

\[
\| F(\cdot, u) \|_{C([0, \delta], X)} \leq L \| u - u_0 \|_{C([0, \delta], X)} + M,
\]

with \( M = \| F(\cdot, u_0) \|_{C([0, T], X)} \). Thus, we have

\[
\| Ku - u_0 \|_{C([0, \delta], X)} \leq \| S(t)u_0 - u_0 \|_{C([0, \delta], X)} + \| R* F(\cdot, u) \|_{C([0, \delta], X)} \\
\leq \| S(t)u_0 - u_0 \|_{C([0, \delta], X)} + (Lr + M) \| \varphi \|_{L_1([0, \delta])}.
\]

We choose \( \delta_1 > 0 \) such that for all \( \delta \leq \delta_1 \) we have \( (Lr + M) \| \varphi \|_{L_1([0, \delta])} \leq r/2 \). Then, we choose \( \delta_2 > 0 \) such small that we have for all \( \delta \leq \delta_2 \) the estimate

\[
\| S(t)u_0 - u_0 \|_{C([0, \delta], X)} \leq r/2.
\]

Hence, for all \( \delta \leq \min\{\delta_1, \delta_2\} \) we obtain the self-mapping property of the map \( K \) onto the set \( \mathcal{B}(u_0, \delta, r) \).
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**Contraction.** Let \( u, \tilde{u} \in B(u_0, \delta, r) \) be arbitrary. We show that for sufficiently small \( \delta > 0 \) the map \( K \) is a contraction. We have

\[
\|Ku - K\tilde{u}\|_{C([0,\delta];X)} \leq \sup_{t\in[0,\delta]} \int_0^t \|R(t-\tau)[F(\tau, u(\tau)) - F(\tau, \tilde{u}(\tau))]\|_X \, d\tau,
\]

\[
\leq L\|u - \tilde{u}\|_{C([0,\delta];X)} \|\varphi\|_{L_1([0,\delta])}.
\]

We choose \( \delta_3 > 0 \) such small that for all \( \delta \leq \delta_3 \) we have \( L\|\varphi\|_{L_1([0,\delta])} \leq 1/2 \) and the contraction property of the map \( K \) is obtained.

Altogether, we have for \( \delta \leq \min\{\delta_1, \delta_2, \delta_3\} \) that \( K \) is a contractive self-mapping of the set \( B(u_0, \delta, r) \). This set is not empty and a complete metric space when endowed with the sup-norm. Hence, the contraction mapping principle yields a unique fixed point of \( K \) in the set \( B(u_0, \delta, r) \). It follows the claim.

\( \Box \)

4.1.2 Maximal Solutions

In this section we show that a local unique mild solution \( u \in C([0, \tau]; X) \) of (4.1) can be continued to a mild solution \( \tilde{u} \in C([0, \tau + \delta]; X) \) for some sufficiently small \( \delta > 0 \), where \( \tilde{u} \) restricted to the interval \([0, \tau]\) is equal to \( u \). On the basis of this continuation we are able to define a maximal (mild) solution for the semilinear problem (4.1).

**Definition 4.1.3.** We call a function \( F: \mathbb{R}_+ \times X \to X \) locally Lipschitz continuous in the second component, uniformly on bounded intervals of \( \mathbb{R}_+ \), if for all \( T > 0 \) and all \( u_0 \in X \) there is some \( r = r(u_0) > 0 \) and a constant \( L = L(u_0, r, T) > 0 \) such that for all \( t \in [0, T] \) and all \( x, y \in B_r(u_0) \) we have the Lipschitz estimate

\[
\|F(t, x) - F(t, y)\|_X \leq L\|x - y\|_X.
\]

**Lemma 4.1.4.** Let \( F: \mathbb{R}_+ \times X \to X \) be continuous and locally Lipschitz continuous in the second component, uniformly on bounded intervals of \( \mathbb{R}_+ \). We assume that there is a resolvent \( \{S(t)\}_{t\in\mathbb{R}_+} \) and an integral resolvent \( \{R(t)\}_{t\in\mathbb{R}_+} \) for the Volterra equation (4.1). Let \( u \in C([0, \tau]; X) \) be the local mild solution of (4.1) with \( u_0 \in X \).

Then there is some \( \delta > 0 \) such that there is a mild solution \( \tilde{u} \in C([0, \tau + \delta]; X) \) of (4.1) and for all \( t \in [0, \tau] \) we have \( u(t) = \tilde{u}(t) \).

**Proof.** The local mild solution \( u \) of (4.1) with \( u_0 \in X \) is continuous on \([0, \tau]\), hence we set \( u_\tau = u(\tau) \). Due to the local Lipschitz continuity in the second component of \( F \) there is some \( r = r(\tau) > 0 \) and some \( L = L(\tau, r, \tau) > 0 \) such that for all \( x, y \in B_r(u_\tau) \) and all \( t \in [\tau + 1] \) we have the Lipschitz estimate

\[
\|F(t, x) - F(t, y)\|_X \leq L\|x - y\|_X.
\]
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Once more, using a fixed point argument we will prove that the there is a continuation on some larger interval. For some sufficiently small $\delta \in (0, 1]$, we define the map $K: C([0, \tau + \delta]; X) \to C([0, \tau + \delta]; X)$ via

$$(Kv)(t) = S(t)u_0 + (R \ast F(\cdot, v))(t),$$

for all $t \in [0, \tau + \delta]$. A fixed point of $K$ yields the wanted continuation of our mild solution. Therefore, we define the set

$$\mathcal{B}(\tau, \delta, r) = \{ v \in C([0, \tau + \delta]; X): v = u$ on $[0, \tau]$ and $\|u_\tau - v\|_{C([0, \tau+\delta]; X)} \leq r/2\},$$

and we will show that $K$ maps $\mathcal{B}(\tau, \delta, r)$ into itself and it is a contraction, provided that $\delta \in (0, 1]$ is small enough. We set $\tilde{u}(t) = (Kv)(t)$ for $t \in [0, \tau + \delta]$.

**Self-mapping.** Let $v \in \mathcal{B}(\delta, \tau, r)$ be arbitrary. We will prove that for sufficiently small $\delta > 0$ the function $Kv$ belongs to $\mathcal{B}(\tau, \delta, r)$, too. It is obvious that the map $\tilde{u}$ is continuous on $[0, \tau + \delta]$. We consider $\tilde{u} - u_\tau$ on $[0, \tau + \delta]$. Since $u_\tau = u(\tau) = S(\tau)u_0 + (R \ast F(\cdot, u))(\tau)$ we have

$$\tilde{u}(t) - u_\tau = S(t)u_0 - S(\tau)u_0 + (R \ast F(\cdot, v)x|_{[\tau, \tau+\delta]})(t).$$

For all $t \in [\tau, \tau + \delta]$ we obtain

$$\|\tilde{u}(t) - u_\tau\|_X \leq \|S(t)u_0 - S(\tau)u_0\|_X + \int_\tau^t \|R(s - t)F(s, v(s))\|_X ds.$$

We know that $S(\cdot)u_0$ is continuous on $\mathbb{R}_+$, in particular $S(\cdot)u_0$ uniformly continuous on $[0, \tau + 1]$, thus we may choose $\delta_1$ such small that for all $\delta \leq \delta_1$ and all $|t - \tau| \leq \delta$ we have $\|S(t)u_0 - S(\tau)u_0\|_X \leq r/4$. Moreover, we have for all $t \in [\tau, \tau + \delta]$ that $v(t) \in B(\delta, u_\tau)$ and hence the estimate

$$\|F(t, v(t))\|_X \leq L\|v(t) - u_\tau\|_X + \|F(t, u_\tau)\|_X,$$

Since $v \in \mathcal{B}(\tau, \delta, r)$ we deduce $\|F(\cdot, v)\|_{C([\tau, \tau+\delta]; X)} \leq Lr/2 + N(\tau)$, where $N(\tau) = \|F(\cdot, u_\tau)|_{C([\tau, \tau+\delta]; X)}$. Thus,

$$\|\tilde{u}(t) - u_\tau\|_X \leq \|S(t)u_0 - S(\tau)u_0\|_X + \|F(\cdot, v)\|_{C([\tau, \tau+\delta]; X)} \|\varphi\|_{L_1([0,\tau-\tau])} \|\varphi\|_{L_1([0,\delta])} \leq r/4 + (Lr/2 + N(\tau)) \|\varphi\|_{L_1([0,\delta])}.$$

We choose $\delta_2 \in (0, \delta_1]$ such that for all $\delta \leq \delta_2$ we have the estimate $(Lr/2 + N(\tau)) \|\varphi\|_{L_1([0,\delta])} \leq r/4$, and hence we conclude the self-mapping property of the map $K$ onto the set $\mathcal{B}(\tau, \delta, r)$.
Theorem 4.1.6. Let \( v, \tilde{v} \in \mathcal{B}(\tau, \delta, r) \) be arbitrary. We show that for sufficiently small \( \delta \in (0, 1) \) the map \( K \) is a contraction. Since \( v(s), \tilde{v}(s) \in B_{C}(u_{\tau}) \) for all \( s \in [\tau, \tau + \delta] \) we obtain that

\[
\|Kv - K\tilde{v}\|_{C([0, \tau+\delta];X)} \leq \sup_{t\in[0,\tau+\delta]} \int_{\tau}^{t} \|R(t-s)[F(s,v(s)) - F(s,\tilde{v}(s))}\|_{X} \, ds,
\]

\[
\leq L\|v - \tilde{v}\|_{C([0,\tau+\delta];X)}\|\phi\|_{L_{1}([0,\delta])}.
\]

The choice of \( \delta_{2} \) implies in particular that for all \( \delta \leq \delta_{2} \) that \( L\|\phi\|_{L_{1}([0,\delta])} \leq 1/2 \). Hence, we obtain for all \( \delta \leq \delta_{2} \) the contraction property of the map \( K \).

Altogether, we have for \( \delta \leq \min\{\delta_{2}, 1\} \) that \( K \) is a contractive self-mapping of the set \( \mathcal{B}(\tau, \delta, r) \); of course \( \delta \) might depend on \( \tau \). This set is not empty and it becomes a complete metric space when endowed with the sup-norm. Hence, the contraction mapping principle yields a unique fixed point of \( K \) in the set \( \mathcal{B}(\tau, \delta, r) \). This fixed point \( \tilde{u} \in C([0, \tau + \delta]; X) \) has the property that \( \tilde{u} = u \) on \( [0, \tau] \), and it is a mild solution of our semilinear Volterra equation (4.1) on \( [0, \tau + \delta] \).

Additionally, we introduce the term of Lipschitz continuity on bounded sets.

Definition 4.1.5. A map \( F: \mathbb{R}_{+} \times X \to X \) is called Lipschitz continuous on bounded sets in the second component, uniformly on bounded intervals of \( \mathbb{R}_{+} \), if for each \( T > 0 \) and \( C > 0 \) there is some \( L = L(C, T) \) such that for all \( x, y \in B_{C}(0) \) and all \( t \in [0, T] \) we have the Lipschitz estimate

\[
\|F(t,x) - F(t,y)\|_{X} \leq L\|x - y\|_{X}.
\]

Finally, we can define a maximal mild solution for the semilinear Volterra equation (4.1).

Theorem 4.1.6. Let \( F: \mathbb{R}_{+} \times X \to X \) be continuous and locally Lipschitz continuous in the second component, uniformly on bounded intervals of \( \mathbb{R}_{+} \). Suppose that there is a resolvent \( \{S(t)\}_{t \in \mathbb{R}_{+}} \) and an integral resolvent \( \{R(t)\}_{t \in \mathbb{R}_{+}} \) for (4.1).

(i) Then for each \( u_{0} \in X \) there is a \( \tau(u_{0}) \leq \infty \), such that the Volterra equation (4.1) has a unique mild solution on \([0, \tau(u_{0})]\) which cannot be extended further. Moreover, if \( \tau(u_{0}) < \infty \) then \( \lim_{t \to \tau(u_{0})^{-}} u(t) \) does not exists in \( X \).

Assume additionally that \( F \) is Lipschitz continuous on bounded sets in the second component, uniformly on bounded intervals of \( \mathbb{R}_{+} \).

(ii) If \( \tau(u_{0}) < \infty \) then \( \lim_{t \to \tau(u_{0})^{-}} \|u(t)\|_{X} = \infty \).
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Remark 4.1.7. We call the interval \([0, \tau(u_0))\) maximal interval of existence, and the corresponding mild solution \(u\) on \([0, \tau(u_0))\) the maximal mild solution of (4.1) with \(u_0 \in X\). Moreover, we have

\[
\tau(u_0) = \sup\{T > 0: u \in C([0, T]; X) \text{ is solution of (4.1)}\}.
\]

Proof. (i) Let \(u_0 \in X\) be arbitrary, but fixed. The existence of a maximal mild solution on the maximal interval of existence \([0, \tau(u_0))\) follows immediately from Lemma 4.1.4 about the continuation of local solutions. In case of \(\tau(u_0) < \infty\) suppose, to the contrary, that \(\lim_{t \to \tau(u_0)} u(t) = u_1\) exists in \(X\). Thus, \(u\) can continuously extended to \(\tilde{u} \in C([0, \tau(u_0)]; X)\) with \(\tilde{u} = u\) on \([0, \tau(u_0))\) and \(\tilde{u}(\tau(u_0)) = u_1\). Using Lemma 4.1.4 there is some sufficiently small \(\delta > 0\) such that we can extend \(\tilde{u}\) to the interval \([0, \tau(u_0) + \delta]\). Then \(\tilde{u} \in C([0, \tau(u_0) + \delta]; X)\) is an extension of \(u\). This contradicts the definition of \(\tau(u_0)\). Hence, the limit \(\lim_{t \to \tau(u_0)} u(t)\) cannot exists in \(X\).

(ii) This proof is completely analogous to the classical case of \(C_0\)-semigroups presented by Pazy, see [Paz92, Theorem 6.1.4].

Let \([0, \tau(u_0))\) be the maximal interval of the mild solution of (4.1). If \(\tau(u_0) < \infty\) then \(\lim_{t \to \tau(u_0)} \|u(t)\|_X = \infty\). Since otherwise there is a sequence \((t_n)_{n \in \mathbb{N}}\) with \(t_n \to \tau(u_0)\) such that \(\|u(t_n)\|_X \leq r/2\), for some \(r > 0\) and for all \(n \in \mathbb{N}\).

Let \(n \in \mathbb{N}\) and consider the proof of Lemma 4.1.4 where we replace \(u_t\) by \(u(t_n)\). Since \(S(t)u_0\) is uniformly continuous on \([0, \tau(u_0) + 1]\) we can chose \(\delta_1\) independent of \(t_n\). The Lipschitz continuity on the bounded set \(B_r(0)\) with \(L = L(r, \tau(u_0)) > 0\) implies for all \(t \in [0, \tau(u_0) + 1]\) that

\[
\|F(t, u(t_n))\|_X \leq \|F(t, u(t_n)) - F(t, 0)\|_X + \|F(t, 0)\|_X \\
\leq L\|u(t_n)\|_X + \|F(t, 0)\|_X,
\]

Thus, we have with the uniform constant \(N_0 = \|F(\cdot, 0)\|_{C([0, \tau(u_0) + 1]; X)}\) that

\[
\|F(\cdot, u(t_n))\|_{C([0, \tau(u_0) + 1]; X)} \leq N,
\]

where \(N = Lr/2 + N_0\) is independent of \(t_n\). Thus, analogous to the proof of Lemma 4.1.4, we can choose \(\delta_2 \in (0, \delta_1]\) such that \((Lr/2 + N)\|Q\|_{L_1([0, \delta])} \leq r/4\) independent of \(t_n\). Hence, we can extend the solution \(u\) to \([t_n, t_n + \delta]\), with \(\delta > 0\) independent of \(t_n\). (By uniqueness, on \([t_n, \tau(u_0))\) this is of course the solution we already have.)

This would imply by the previous considerations that for each \(t_n\), close enough to \(\tau(u_0)\), \(u\) defined on \([0, t_n]\) can be extended to \([0, t_n + \delta]\) where \(\delta > 0\) is independent of \(t_n\) and hence \(u\) can be extended beyond \(\tau(u_0)\) contradicting the definition of \(\tau(u_0)\). 

\[\square\]
4.2 Stability Theorem

In this section we study semilinear Volterra equations

\[ u(t) + (a \ast Au)(t) = u_0 + (a \ast f(u))(t), \quad t \in [0, T], \]  

(4.2)

where \( u_0 \in X \) and \( f \in C^1(X; X) \) with \( f(0) = 0 \) and \( f'(0) = 0. \) \(^1\) At first, we introduce the notion of stability of the equilibrium \( u_* = 0. \) This concept is similar to the well-known definitions for classical evolution equations.

**Definition 4.2.1 (Stability).**

(i) We call the equilibrium \( u_* = 0 \) stable for the Volterra equation (4.2) if for each \( \varepsilon > 0 \) there is some \( \delta = \delta(\varepsilon) > 0 \) such that for all \( u_0 \in B_\delta(0) \) a mild solution \( u = u(\cdot; u_0) \) of the Volterra equation (4.2) exists on \( \mathbb{R}_+ \) and \( \|u\|_{C(\mathbb{R}_+; X)} < \varepsilon. \)

(ii) We call the equilibrium \( u_* = 0 \) asymptotically stable for the Volterra equation (4.2) if \( u_* = 0 \) is stable and there is some \( \delta_0 > 0 \) such that for all \( u_0 \in B_{\delta_0}(0) \) the mild solution \( u = u(\cdot; u_0) \) of the Volterra equation (4.2) satisfies \( \|u(t)\|_X \to 0 \) as \( t \to \infty. \)

(iii) We call the equilibrium \( u_* = 0 \) asymptotically stable with rate \( \omega \) if \( u_* \) is asymptotically stable and there is a constant \( C > 0 \) such that for each \( u_0 \in B_{\delta_0}(0) \) we have for the mild solution \( u(\cdot; u_0) \) the estimate

\[ \|u(t; u_0)\|_X \leq C \|u_0\|_X \omega(t), \quad t \in (0, \infty), \]

where \( \omega: (0, \infty) \to \mathbb{R}_+ \) is non-negative with \( \omega(t) \to 0 \) as \( t \to \infty. \)

(iv) We call the equilibrium \( u_* = 0 \) unstable for the Volterra equation (4.2) if \( u_* \) is not stable.

**Theorem 4.2.2 (Stability Theorem).** Let \( X \) be a complex Banach space and \( A \) be a linear, closed and densely defined operator in \( X. \) Furthermore, let \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \) be an unbounded, completely monotonic kernel and \( b \in L_{1, \text{loc}}(\mathbb{R}_+) \) be the corresponding Sonine kernel. We assume, that the linear Volterra equation

\[ u(t) + (a \ast Au)(t) = g(t), \quad t \in [0, T], \]

with \( g \in C([0, T]; X), \) admits a bounded resolvent \( \{S(t)\}_{t \in \mathbb{R}_+} \) and an integrable integral resolvent \( \{R(t)\}_{t \in \mathbb{R}_+}. \) Suppose that \( f \in C^1(X; X) \) with \( f(0) = 0, \ f'(0) = 0. \)

Then the equilibrium \( u_* = 0 \) is stable for the semilinear Volterra equation

\[ u(t) + (a \ast Au)(t) = u_0 + (a \ast f(u))(t), \quad t \in [0, T], \]

with \( u_0 \in X. \) Moreover, if \( a \not\in L_1(\mathbb{R}_+) \) the equilibrium \( u_* = 0 \) is asymptotically stable with rate \( s_\varepsilon(t) \) for some sufficiently small \( \varepsilon > 0. \)

\(^1\)Note that unlike the previous section we drop the explicit time-dependence of the function \( f, \) the results about local solutions etc. remain true.
Here, $s_{\varepsilon}$ denotes as usual the scalar resolvent, i.e. the solution of the scalar Volterra equation

$$s_{\varepsilon}(t) + \varepsilon(s_{\varepsilon} \ast a)(t) = 1, \quad t \in \mathbb{R}_+.$$  

**Proof.** Since the resolvent family is bounded, there is a constant $M_0 \geq 1$ such that $\|S(t)\|_{B(X)} \leq M_0$. The assumption of an integrable integral resolvent yields a $\psi \in L_1(\mathbb{R}_+)$ such that $\|R(t)\|_{B(X)} \leq \psi(t)$ for almost all $t \in \mathbb{R}_+$ and $\|\psi\|_{L_1(\mathbb{R}_+)} = N_0 < \infty$.

Since $f$ is continuously differentiable on $X$ we know that $f$ is locally Lipschitz continuous. Thus, there is some $B_r(0)$ such that $f|_{B_r(0)}$ is Lipschitz continuous with constant $L > 0$. Moreover, the continuous differentiability of the map $f$, together with the assumption $f(0) = 0$ and $f'(0) = 0$, implies that for each $\rho > 0$ there is some $\eta = \eta(\rho) \in (0, r/2]$ such that for all $x \in X$ with $\|x\|_X \leq \eta$ it follows that $\|f(x)\|_X \leq \rho \|x\|_X$. Fix $\rho \in (0, 1/(4N_0)]$ and $\eta = \eta(\rho) \in (0, r/2]$ according to the previously described property. We know by the Local Existence and Uniqueness Theorem 4.1.2 that for all $u_0 \in B_r(0)$ there is some $T = T(u_0) > 0$ such there is a unique mild solution $u = u(\cdot; u_0) \in C([0, T]; X)$. This solution with initial value $u_0 \in B_r(0)$ can be extended to a maximal interval of existence $[0, t_\ast(u_0))$. If $t_\ast(u_0)$ is finite, then the limit $\lim_{t \rightarrow t_\ast(u_0)^-} u(t)$ does not exist in $X$, see Theorem 4.1.6.

Now, let $u_0 \in B_r(0)$ be arbitrary, but fixed, and let $u = u(\cdot; u_0)$ denote the corresponding mild solution with maximal interval of existence $[0, t_\ast)$. We define the exit time for the ball $B_r(0)$, that is

$$t_0 = \sup \{t \in (0, t_\ast): \|u(\tau)\|_X \leq \eta \text{ for all } \tau \in [0, t]\},$$

and suppose $t_0 < t_\ast$. We know that $u \in C([0, t_0]; X)$ is the unique local mild solution of (4.2), hence $u(t) = S(t)u_0 + (R \ast f(u))(t)$ for $t \in [0, t_0]$, and $\|u\|_{C([0, t_0]; X)} \leq \eta$ by definition of the exit time.

On $[0, t_0]$ we consider for $\varepsilon \in (0, 1/(4N_0)]$ the function $v = \varphi_\varepsilon u$, with $\varphi_\varepsilon = 1/s_{\varepsilon}$, where $s_{\varepsilon}$ denotes the scalar resolvent. Our aim is to prove that $v$ is uniformly bounded.

Lemma 3.4.7 gives us the representation for $v(t) = \varphi_\varepsilon(t)u(t)$ on $[0, t_0]$ by

$$v(t) = S(t)u_0 + (R \ast [b[\varphi_\varepsilon - 1]])(t)u_0 + (R \ast [\varphi_\varepsilon f(v/\varphi_\varepsilon)])(t) + \left(R \ast \int_0^t \left[-b(\cdot - \tau)[\varphi_\varepsilon(\cdot) - \varphi_\varepsilon(t)]\frac{v(\tau)}{\varphi_\varepsilon(\tau)}\right]d\tau\right)(t).$$

In addition, we know that $v \in C([0, t_0]; X)$ and $\|v\|_{C([0, t_0]; X)} \leq \varphi_\varepsilon(t_0)\eta$, since $\varphi_\varepsilon$ is continuous and increasing on $\mathbb{R}_+$.

Now, let $t \in [0, t_0]$ be arbitrary. Recall that $\varphi_\varepsilon \geq 1$, thus we have the estimate

$$\|v(t)\|_X \leq \|S(t)\|_{B(X)}\|u_0\|_X + (\psi \ast b[\varphi_\varepsilon - 1])(t)\|u_0\|_X$$

with $\psi = \psi(\cdot; u_0)$ as in (4.2) and $\psi \in L_1(\mathbb{R}_+)$ such that $\|\psi\|_{L_1(\mathbb{R}_+)} = N_0 < \infty$. Thus, we get

$$\|v(t)\|_X \leq M_0\|u_0\|_X + N_0\|u_0\|_X \leq (M_0 + N_0)\|u_0\|_X.$$
The growth property of the map $f$ yields on $[0,t_0]$ the uniform estimate
\[ \|v(t)/\varphi_\varepsilon(t)\|_X \leq \varphi_\varepsilon(t) \rho \|v(t)/\varphi_\varepsilon(t)\|_X = \rho \|v(t)\|_X, \]

since $\|v(t)/\varphi_\varepsilon(t)\|_X = \|u(t)\|_X \leq \eta$ for all $t \in [0,t_0]$. By the boundedness of the resolvent and the integrability of the integral resolvent it follows that
\[
\|v(t)\|_X \leq M_0 \|u_0\|_X + \varepsilon (\psi \ast 1)(t) \|u_0\|_X + \rho (\psi \ast \|v\|_X)(t) + \rho (\psi \ast \|v\|_X)(t)
\]
\[ + C \int_0^t \int_0^t \int_0^t [-\dot{b}(\cdot - \tau)] [\varphi_\varepsilon(\cdot) - \varphi_\varepsilon(\tau)] \frac{d\tau}{\varphi_\varepsilon(\tau)} \frac{d\tau}{\varphi_\varepsilon(\tau)} \frac{d\tau}{\varphi_\varepsilon(\tau)} (t). \]

We are able to calculate the commutator on $(0,\infty)$, see Lemma 3.5.1. We have
\[
\int_0^t [-\dot{b}(t - \tau)] [\varphi_\varepsilon(t) - \varphi_\varepsilon(\tau)] \frac{d\tau}{\varphi_\varepsilon(\tau)} = \varepsilon + b(t) [1 - \varphi_\varepsilon(t)] \in [0,\varepsilon],
\]
and hence we have for all $t \in [0,t_0]$ that
\[
\left( \psi \ast \int_0^t [-\dot{b}(\cdot - \tau)] [\varphi_\varepsilon(\cdot) - \varphi_\varepsilon(\tau)] \frac{d\tau}{\varphi_\varepsilon(\tau)} \right)(t) \leq \varepsilon \|\varphi_\varepsilon\|_{L^1([0,t_0])}.
\]
By the integrability of $\psi$ on $\mathbb{R}_+$ we obtain the uniform estimate
\[
\|v(t)\|_X \leq (M_0 + \varepsilon N_0) \|u_0\|_X + (\rho + \varepsilon) N_0 \|v\|_{C([0,t_0];X)},
\]
and thus
\[
\|v\|_{C([0,t_0];X)} \leq (M_0 + \varepsilon N_0) \|u_0\|_X + (\rho + \varepsilon) N_0 \|v\|_{C([0,t_0];X)}.
\]
With the restriction $\rho \in (0,1/(4N_0)]$ and $\varepsilon \in (0,1/(4N_0)]$ it follows that
\[
\|v\|_{C([0,t_0];X)} \leq C_0 \|u_0\|_X,
\]
with $C_0 = 2M_0 + 1/2$. We have for all initial values $u_0 \in B_\delta(0)$ with $\delta \leq \delta_1 = \eta/(2C_0)$ that $\|v\|_{C([0,t_0];X)} \leq \eta/2$. In particular, we have $\|v(t_0)\|_X \leq \eta/2$. This
contradicts the definition of the exit time $t_0$ for the ball $B_{\eta}(0)$. Thus, we conclude $t_0 = t$. The above arguments yield for each $\delta \leq \delta_1$ and all $u_0 \in B_\delta(0)$ the uniform estimate $\|v\|_{C((0,T),X)} \leq \eta/2$ for all $T < t$. This implies $t_* = \infty$, here we use that $\|v(t)\|_X < r$ for all $t \in [0,t_*)$ and that we have a uniform Lipschitz estimate for the map $f$ on $B_r(0)$. Thus, we can extend the mild solution to $\mathbb{R}_+$, see the proof of Theorem 4.1.6 (ii), which in turn, together with the previous estimates for $v$ gives $\|v\|_{C(\mathbb{R}_+,X)} \leq C_0 \|u_0\|_X$. Rephrasing this inequality in term of $u$, we obtain

$$\|u(t)\|_X \leq C_0 s_* \|u_0\|_X \leq C_0 \|u_0\|_X,$$

for all $t \in \mathbb{R}_+$. In particular, the equilibrium $u_* = 0$ is stable. Indeed, let $\epsilon_0 > 0$ be arbitrary, but fixed. With $\delta < \min\{\eta/(2C_0), \epsilon_0/C_0\}$, we have $\|u\|_{C(\mathbb{R}_+,X)} < \epsilon_0$ for all $u_0 \in B_\delta(0)$. Moreover, for the completely monotonic kernel $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ we have the estimate $0 \leq s_*(t) \leq (1 + \epsilon(1 + a)(t))^{-1}$, see Lemma 2.2.5. Hence, $s_*(t)$ tends to zero as $t \to \infty$, if $a \notin L_1(\mathbb{R}_+)$. This implies the asymptotic stability of the equilibrium $u_* = 0$ with the rate $s_*(t)$ in case of $a \notin L_1(\mathbb{R}_+)$. \hfill \Box

**Remark 4.2.3.** One can weaken the assumption on the map $f$, it suffices to know that $f$ is continuously differentiable in a small neighbourhood of zero with $f(0) = 0$ and $f'(0) = 0$.

Using the sufficient conditions for the existence of a bounded resolvent and an integrable integral resolvent, see Section 3.3, the stability result reads as follows.

**Theorem 4.2.4.** Let $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ be an unbounded, completely monotonic kernel which is $\Theta_a$-sectorial with some $\Theta_a \in (0,\frac{\pi}{2}]$. We denote by $b \in L_{1,\text{loc}}(\mathbb{R}_+)$ the corresponding Sonine kernel. Assume that $\int_0^1 a(t) \frac{dt}{t} < \infty$, $\int_0^1 b(t) \frac{dt}{t^2} < \infty$ and $[t \mapsto \min\{a(1/t)/t, b(1/t)/t^2\}] \in L_1(\mathbb{R}_+)$. Moreover, let $A$ be an invertible and sectorial operator with spectral angle $\varphi_A < \pi - \Theta_a$ and let $f \in C^1(X;X)$ such that $f(0) = 0$ and $f'(0) = 0$. Then the equilibrium $u_* = 0$ is stable for the Volterra equation (4.2). If in addition $a \notin L_1(\mathbb{R}_+)$ then $u_* = 0$ is asymptotically stable with rate $s_*(t)$ for some sufficiently small $\epsilon > 0$.

**Proof.** The assumptions on the operator $A$ together with the $\Theta_a$-sectoriality of the kernel imply that the Volterra equation (4.2) is parabolic. In particular, completely monotonic kernels are $1$-regular, hence, we know by Theorem 3.3.1 that there is a bounded resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$ for the Volterra equation (4.2). The integrability requirements on the kernels $a$ and $b$ as well as the assumptions on the operator $A$ ensure the existence of an integrable integral resolvent, see Theorem 3.3.8. Hence, all requirements for Theorem 4.2.2 are satisfied. It follows the claim. \hfill \Box
It is easy to see that for a Volterra equation with standard kernel and a proper sectorial operator $A$ the equilibrium $u_e = 0$ is asymptotically stable.

**Corollary 4.2.5.** Let $\alpha \in (0, 1)$ and suppose that the invertible operator $A$ is sectorial on the Banach space $X$ with spectral angle $\theta_A < \pi - \alpha \frac{\pi}{2}$. Moreover, let $f \in C^1(X; X)$ with $f(0) = 0$, $f'(0) = 0$.

Then, the equilibrium $u_e = 0$ is asymptotically stable for the Volterra equation

$$u(t) + (g_\alpha * Au)(t) = u_0 + (g_\alpha * f(u))(t), \quad t \in [0, T],$$

where $u_0 \in X$; the decay rate is given by $t^{-\alpha}$.

**Proof.** We know that the operator $A$ together with the standard kernel $g_\alpha \in L_1(\mathbb{R}_+)$ satisfy all assumptions from Theorem 4.2.4. In case of the standard kernel we know that $s_\varepsilon$ decays to zero with optimal rate $t^{-\alpha}$. The claim follows.

The following corollary is a further easy consequence of Theorem 4.2.2 for the finite dimensional case with the standard kernel.

**Corollary 4.2.6.** Let $\alpha \in (0, 1)$ and assume that the matrix $A \in \mathbb{R}^{n \times n}$ is invertible and that every eigenvalue $\lambda \in \sigma(A)$ satisfies $|\text{arg} \lambda| < \pi - \alpha \frac{\pi}{2}$. Moreover, let $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ with $f(0) = 0$, $f'(0) = 0$.

Then, the equilibrium $u_e = 0$ is asymptotically stable for the Volterra equation

$$u(t) + (g_\alpha * Au)(t) = u_0 + (g_\alpha * f(u))(t), \quad t \in [0, T],$$

where $u_0 \in \mathbb{R}^n$, and the decay rate is given by $t^{-\alpha}$.

Finally, we use Theorem 4.2.2 to deduce the following further result about the decay behaviour of the resolvent family.

**Corollary 4.2.7.** Assume the situation of Theorem 4.2.2 with $f \equiv 0$.

Then there is a constant $C > 0$ such that we have for the resolvent family $\{S(t)\}_{t \in \mathbb{R}_+}$ the estimate

$$\|S(t)\|_{B(X)} \leq Cs_\varepsilon(t), \quad t \in \mathbb{R}_+,$$

where $s_\varepsilon$ denotes the scalar resolvent for some sufficiently small $\varepsilon > 0$.

Consequently, the decay behaviour of the resolvent family $\{S(t)\}_{t \in \mathbb{R}_+}$ is the same as the decay behaviour of the scalar resolvent $s_\varepsilon$.

**Remark 4.2.8.** We consider the situation of exponentially weighted standard kernels with

$$\hat{a}(\lambda) = \frac{1 + \gamma/\lambda}{(\lambda + \gamma)\alpha}, \quad \hat{b}(\lambda) = \frac{1}{(\lambda + \gamma)^{1-\alpha}}.$$
where $\gamma \geq 0$ and $\alpha \in (0,1)$. Easy calculations show that these completely monotonic kernels satisfy the assumptions $\int_0^1 \hat{a}(1/t) \frac{dt}{t} < \infty$, $\int_0^1 \hat{b}(1/t) \frac{dt}{t} < \infty$ and $\int_1^\infty \hat{b}(1/t) \frac{dt}{t} < \infty$. Thus, we know that the Volterra equation with exponentially weighted standard kernel and an invertible sectorial operator $A$ with $\varphi_A < \frac{\pi}{2}$ admits a bounded resolvent family $\{S(t)\}_{t \in \mathbb{R}_+}$ and an integrable integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$. Corollary 4.2.7 implies that we have for the resolvent family $\{S(t)\}_{t \in \mathbb{R}_+}$ the estimate

$$
\|S(t)\|_{\mathcal{B}(X)} \leq C_s \varepsilon(t), \quad t \in \mathbb{R}_+,
$$

where $C > 0$ is a constant and $s_\varepsilon$ denotes the scalar resolvent for a sufficiently small $\varepsilon > 0$. Vergara and Zacher proved in [VZ17] that in the case of exponentially weighted standard kernels the scalar resolvent decays to zero as $t \to \infty$ with an appropriate exponential rate. Hence, $\|S(\cdot)\|_{\mathcal{B}(X)}$ decays to zero with the same exponential rate as the scalar resolvent.
Instability for Semilinear Parabolic Volterra Equations

The aim of this chapter is to prove the instability result, Theorem 5.3.1, for semilinear parabolic Volterra equations. In order to do this, firstly we introduce the spectral projection of an operator associated with a compact spectral set and show its compatibility with the resolvent and integral resolvent, see Section 5.1. Afterwards, we consider exponential weights for the scalar resolvents and integral resolvents and determine their long-time behaviour including estimates for the convergence rate, cf. Section 5.2. In doing so one of the major challenges is the proof of the uniformity of the estimates along compact sets. These results are interesting in its own and crucial for the proof of our instability result Theorem 5.3.1. Finally, we close this chapter with an easy consequence for Volterra equations with standard kernel.

5.1 Spectral Projections associated with Spectral Sets and Volterra Equations

5.1.1 Spectral Sets and Projections

In this section we introduce the spectral projection of a linear operator $A$ associated with a compact spectral set. The subsequent result can be found in Arendt et al. [ABHN01, Proposition B.9].

Proposition 5.1.1. Let $A$ be closed, linear operator on the Banach space $X$ with $\rho(A) \neq \emptyset$. We suppose that $\sigma(A) = \sigma_+ \cup \sigma_-$ with $\sigma_+ \cap \sigma_- = \emptyset$, where $\sigma_+$ is a compact subset and $\sigma_-$ a closed subset of $\mathbb{C}$.

Then there is a bounded projection $P_+ \in \mathcal{B}(X)$ on $X$ such that $(\lambda - A)^{-1} P_+ = P_+(\lambda - A)^{-1}$, for all $\lambda \in \rho(A)$, $\text{Rg} P_+ \subset D(A)$, $\sigma(A_+) = \sigma_+$ and $\sigma(A_-) = \sigma_-$, where $A_+$ denotes the part of $A$ in $\text{Rg} P_+$ and $A_-$ the part of $A$ in $\text{Rg}(\text{Id} - P_+)$. Moreover, the projection $P_+$ is unique and $A|_{\text{Rg} P_+} \in \mathcal{B}(\text{Rg} P_+)$. 

Remark 5.1.2.  

a) The projection $P_+$ is called the spectral projection of the operator $A$ associated with the compact spectral set $\sigma_+$.

b) We set $P_- = \text{Id} - P_+$. Obviously, $P_- \in \mathcal{B}(X)$ is also a projection and commutes with the resolvent operator $(\lambda - A)^{-1}$, $\lambda \in \rho(A)$, too.

Moreover, $A$ commutes with $P_\pm$ on $D(A)$, cf. [ABHN01, Proposition B.7].

c) Introducing $X_+ = \text{Rg} P_+$ and $X_- = \text{Rg}(\text{Id} - P_+)$, we know from Kato [Kat66, p. 155] that $X_\pm$ are closed linear manifolds of $X$ and $X = X_+ \oplus X_-$. 

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We have $A_+: X_+ \to X_+$ and $A_+x = Ax$ for all $x \in X_+ = D(A_+)$ and $A_-: D(A_-) \to X_-$ and $A_-x = Ax$ for all $x \in D(A_-) = X_- \cap D(A)$.

The operators $A_\pm$ are closed operators on $X_\pm$, cf. [Kat66, p. 172].

For each $\lambda \in \rho(A)$ we have the decomposition of the resolvent operator $(\lambda - A)^{-1} = (\lambda - A_+)^{-1} + (\lambda - A_-)^{-1}$, where $(\lambda - A_+)^{-1} = P_+(\lambda - A)^{-1} = (\lambda - A)^{-1}P_+$, the part of $(\lambda - A)^{-1}$ in $X_+$, and $(\lambda - A_-)^{-1} = P_-(\lambda - A)^{-1} = (\lambda - A)^{-1}P_-$, the part of $(\lambda - A)^{-1}$ in $X_-$, see [Kat66, p. 179].

d) According to Lunardi [Lun95, Appendix A.1] we know that there exists a bounded open set $\Omega \subset \mathbb{C}$ containing $\sigma_+$ and such that its closure is disjoint from $\sigma_-$ (since the distance between $\sigma_-$ and $\sigma_+$ is positive) and we may assume that the boundary $\gamma$ of $\Omega$ consists of a finite number of rectifiable closed Jordan curves, oriented counterclockwise. Then the spectral projection $P_+ \in \mathcal{B}(X)$ of the operator $A$ associated to the compact spectral set $\sigma_+$ is given by

$$P_+ = \frac{1}{2\pi i} \int_\gamma (\xi - A)^{-1} d\xi.$$  \hspace{1cm} (5.1)

### 5.1.2 Projections of Resolvents and Integral Resolvents

In this section we use the above introduced projections to decompose the resolvent and integral resolvent of a linear Volterra equation into the respective objects on the subspaces $X_\pm$. Throughout this section we assume:

**Assumptions.** The operator $A$ is a linear, closed and densely defined operator on the Banach space $X$ with domain $D(A)$. Let the resolvent set of the operator $A$ be non-empty and assume that $\sigma(A) = \sigma_+ \cup \sigma_-$ with $\sigma_+ \cap \sigma_- = \emptyset$ and $\sigma_+$ is a compact and $\sigma_-$ is a closed subset of $\mathbb{C}$. The spectral projection of $A$ associated with $\sigma_+$ is denoted by $P_+$ and $P_- = \text{Id} - P_+$. We have $X_\pm = \text{Rg} P_\pm$ and $A_\pm$ denotes the part of $A$ in $X_\pm$. Furthermore, let $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ be a locally integrable kernel. For $T > 0$ we set $J = [0, T]$.

We consider the linear Volterra equation

$$u(t) + (a * Au)(t) = f(t), \quad t \in J,$$  \hspace{1cm} (5.2)

where $f \in C(J; X)$.

**Lemma 5.1.3.** Assume that the Volterra equation (5.2) admits a resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$.

Then the resolvent operator $(\lambda - A)^{-1} \in \mathcal{B}(X)$, $\lambda \in \rho(A)$, as well as the spectral projections $P_\pm$ commute with the resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$.

If the Volterra equation (5.2) admits an integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$ then the same statement holds true for the integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$.
5.1. Spectral Projections associated with Spectral Sets and Volterra Equations

Proof. Due to condition (S2)/(R3) we know that the resolvent $\{S(t)\}_{t \in \mathbb{R}_+} \subset B(X)$ and the integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+} \subset B(X)$ commute with the operator $A$ on $D(A)$. Using [ABHN01, Proposition B.7] it follows directly that the resolvent and integral resolvent family, respectively, commute with the resolvent operator $(\lambda - A)^{-1}$, $\lambda \in \rho(A)$. Let $t \in \mathbb{R}_+$ and $x \in X$ be arbitrary. We set $T = S(t)$ and $T = R(t)$, respectively. We have with equation (5.1) that

$$P_+ Tx = \frac{1}{2\pi i} \int_\gamma (\lambda - A)^{-1} Tx d\lambda = \frac{1}{2\pi i} \int_\gamma T(\lambda - A)^{-1} x d\lambda,$$

here we used the fact that both the resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$ and the integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$ commute with the resolvent operator. Since $\lambda \mapsto (\lambda - A)^{-1} x$ is Bochner integrable along $\gamma$, we have by [ABHN01, Proposition 1.1.6] that

$$P_+ Tx = \frac{1}{2\pi i} T \int_\gamma (\lambda - A)^{-1} x d\lambda = TP_+ x.$$

In addition, we get $P_- x = Tx - P_+ Tx = Tx - TP_+ x = TP_- x$. 

Lemma 5.1.4. Assume that the Volterra equation (5.2) admit a resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$ and an integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$.

Then $\{S_\pm(t)\}_{t \in \mathbb{R}_+} = \{P_\pm S(t)\}_{t \in \mathbb{R}_+}$ is a resolvent and $\{R_\pm(t)\}_{t \in \mathbb{R}_+} = \{P_\pm R(t)\}_{t \in \mathbb{R}_+}$ is an integral resolvent for the Volterra equation

$$u(t) + (a \ast A u)(t) = f_\pm(t), \quad t \in J,$$

with $f_\pm \in C(f; X_\pm)$.

Furthermore, if both Volterra equations (5.3) admits a resolvent $\{S_\pm(t)\}_{t \in \mathbb{R}_+}$ and an integral resolvent $\{R_\pm(t)\}_{t \in \mathbb{R}_+}$, then for all $t \in \mathbb{R}_+$ we have $S_\pm(t) = S_\pm(t)$ and $R_\pm(t) = R_\pm(t)$ on $X_\pm$. Moreover, $\{S(t)\}_{t \in \mathbb{R}_+} = \{S_+ (t) + S_- (t)\}_{t \in \mathbb{R}_+}$ is a resolvent and $\{R(t)\}_{t \in \mathbb{R}_+} = \{R_+ (t) + R_- (t)\}_{t \in \mathbb{R}_+}$ is an integral resolvent for the Volterra equation (5.2).

Proof. Let $t \in \mathbb{R}_+$ be arbitrary, but fixed. Lemma 5.1.3 implies that $S(t)X_\pm \subset X_\pm$ and $R(t)X_\pm \subset X_\pm$. Furthermore, we have $\|P_\pm S(t)\|_{B(X_\pm)} \leq \|P_\pm\|_{B(X)} \|S(t)\|_{B(X)} < \infty$, as well as $\|P_\pm R(t)\|_{B(X_\pm)} \leq \|P_\pm\|_{B(X)} \|R(t)\|_{B(X)} < \infty$. Hence, $\{S_\pm(t)\}_{t \in \mathbb{R}_+} \subset B(X_\pm)$ and $\{R_\pm(t)\}_{t \in \mathbb{R}_+} \subset B(X_\pm)$ are families of bounded operators on $X_\pm$.

In our next steps, we prove that $\{S_\pm(t)\}_{t \in \mathbb{R}_+}$ satisfies the properties (S1) – (S3) from Definition 3.1.3 of the resolvent and that $\{R_\pm(t)\}_{t \in \mathbb{R}_+}$ satisfies the properties (R1) – (R3) from Definition 3.1.8 of the integral resolvent.
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Condition (S1). We show that $P_xS(t)$ is strongly continuous on $\mathbb{R}_+$ and $P_xS(0) = \text{Id}$ on $X_x$.

Let $t \in \mathbb{R}_+$ and $x_x \subset X_x \subset X$ be arbitrary, but fixed. For each $\varepsilon > 0$ exists $\delta = \delta(\varepsilon, t) > 0$ such that for all $t, s \in \mathbb{R}_+$ with $|t - s| \leq \delta$ we have

$$
\|P_xS(s)x_x - P_xS(t)x_x\|_X \leq \|P_x\|_{\mathcal{B}(X)}\|S(s)x_x - S(t)x_x\|_X \leq \varepsilon,
$$

because $S(\cdot)x$ is strongly continuous on $\mathbb{R}_+$ for all $x \in X$. Moreover, we have $P_xS(0)x_x = S(0)x_x = x_x$, since $S(0) = \text{Id}$ on $X$. This implies $P_xS(0) = \text{Id}$ on $X_x$. Thus, $\{S_x(t)\}_{t \in \mathbb{R}_+}$ satisfies (S1).

Condition (R1). We prove that $P_xR(\cdot)x_x \in L_{1,\text{loc}}(\mathbb{R}_+; X_x)$ for each $x_x \in X_x$ and $\|P_xR(\cdot)t\|_{\mathcal{B}(X)} \leq \varphi_x(t)$ a.e. $t \in \mathbb{R}_+$ for some $\varphi_x \in L_{1,\text{loc}}(\mathbb{R}_+)$. Let $x_x \in X_x \subset X$ be arbitrary, but fixed. The integral resolvent commutes with $P_x$, see Lemma 5.1.3, we have $P_xR(\cdot)x_x = R(\cdot)x_x : \mathbb{R}_+ \to X_x$ and thus $P_xR(\cdot)x_x \in L_{1,\text{loc}}(\mathbb{R}_+; X_x)$, since $R(\cdot)x$ is locally Bochner integrable for all $x \in X$ and $X_x$ are closed subspaces of $X$. Moreover, for a.e. $t \in \mathbb{R}_+$ we have

$$
\|P_xR(\cdot)t\|_{\mathcal{B}(X)} \leq \|P_x\|_{\mathcal{B}(X)}\|R(\cdot)t\|_{\mathcal{B}(X)} \leq \varphi_x(t),
$$

with $\varphi_x = \|P_x\|_{\mathcal{B}(X)}\varphi \in L_{1,\text{loc}}(\mathbb{R}_+)$, since $\varphi \in L_{1,\text{loc}}(\mathbb{R}_+)$ from the property (R1) of the definition of the integral resolvent $\{R(\cdot)t\}_{t \in \mathbb{R}_+}$. Hence, $\{R_x(t)\}_{t \in \mathbb{R}_+}$ satisfies condition (R1).

Condition (S2)/(R2). We show that for each $t \in \mathbb{R}_+$ commutes $P_xS(t)$ and $P_xR(t)$ with $A_x \subset D(A_x)$.

Let $t \in \mathbb{R}_+$ and $x_x \in D(A_x)$ be arbitrary, but fixed. We set $T = S(t)$ or $T = R(t)$, respectively. Since $D(A_x) \subset D(A)$, we know that $T$ commutes with $A_x$ on $D(A_x)$. We have $x_x \subset X$ be invariant for $T$, i.e. $Tx_x \subset X$. Moreover, the domain $D(A)$ is invariant for $T$, i.e. $Tx_x \in D(A)$. In particular, we have $Tx_x \in D(A) \cap X = D(A_x)$. We know that the operators $A$ and $A_x$ coincide on $D(A_x)$, by definition. Hence, we have $P_xTA_xT_x = P_xT_xA_x = P_xAT_x$. The operators $P_x$ commute with the operator $A$ on $D(A)$, so we get $P_xT_xA_x = A_xP_xT_x$. Since we have $Tx_x \in D(A_x)$ and we know that $A$ and $A_x$ coincide on $D(A_x)$, we obtain $P_xTA_xT_x = A_xP_xT_x$. Thus, $\{S_x(t)\}_{t \in \mathbb{R}_+}$ and $\{R_x(t)\}_{t \in \mathbb{R}_+}$, respectively, satisfy condition (S2) and (R2), respectively.

Condition (S3). We prove that for each $t \in \mathbb{R}_+$ and all $x_x \in D(A_x)$ the resolvent equation

$$
S_x(t)x_x + (a \ast [A_xS_xT_x])(t) = x_x,
$$

is satisfied.
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Let \( t \in \mathbb{R}_+ \) and \( x_\pm \in \mathcal{D}(A_\pm) \subset \mathcal{D}(A) \) be arbitrary, but fixed. We know that \( S(t) \) satisfies the resolvent equation, so \( S(t)x_\pm + (a \ast [ASx_\pm])(t) = x_\pm \). On one hand, we have \( P_\pm S(\tau)x_\pm = S(\tau)x_\pm \in \mathcal{D}(A_\pm) \), \( \tau \in \mathbb{R}_+ \), see proof of condition (S2). On the other hand, we know that \( A \) coincide with \( A_\pm \) on \( \mathcal{D}(A_\pm) \). Combining these facts, we get for all \( \tau \in \mathbb{R}_+ \) that \( AS(\tau)x_\pm = A_\pm P_\pm S(\tau)x_\pm \). In particular, the resolvent equation is equivalent to \( P_\pm S(t)x_\pm + (a \ast [A_\pm P_\pm Sx_\pm])(t) = x_\pm \), for \( x_\pm \in \mathcal{D}(A_\pm) \). This implies, that \( \{S_\pm(t)\}_{t \in \mathbb{R}_+} \) satisfies condition (S3).

**Condition (R3).** We show that for almost all \( t \in \mathbb{R}_+ \) and all \( x_\pm \in \mathcal{D}(A_\pm) \) the integral resolvent equation

\[
\mathcal{R}_\pm(t)x_\pm + (a \ast [A_\pm \mathcal{R}_\pm x_\pm])(t) = a(t)x_\pm,
\]

is satisfied.

Let \( x_\pm \in \mathcal{D}(A_\pm) \) be arbitrary, but fixed. Let \( t \in \mathbb{R}_+ \) be such that \( R(t) \) satisfies the integral resolvent equation, i.e. \( R(t)x_\pm + (a \ast [ARx_\pm])(t) = a(t)x_\pm \). Analogous to the argumentation for condition (S2), we have for all \( \tau \in \mathbb{R}_+ \) that \( P_\pm R(\tau)x_\pm = R(\tau)x_\pm \in \mathcal{D}(A_\pm) \) as well as \( AR(\tau)x_\pm = A_\pm P_\pm R(\tau)x_\pm \). Thus, the integral resolvent equation is equivalent to \( P_\pm R(t)x_\pm + (a \ast [A_\pm P_\pm Rx_\pm])(t) = a(t)x_\pm \), for \( x_\pm \in \mathcal{D}(A_\pm) \). This proves that \( \{\mathcal{R}_\pm(t)\}_{t \in \mathbb{R}_+} \) satisfies condition (R3).

We can conclude, that \( \{S_\pm(t)\}_{t \in \mathbb{R}_+} \) and \( \{\mathcal{R}_\pm(t)\}_{t \in \mathbb{R}_+} \) are resolvent and integral resolvent families for the Volterra equations (5.3). The resolvent family as well as the integral resolvent family are unique provided they exist. Thus, for all \( t \in \mathbb{R}_+ \) we know that \( S_\pm(t) \) and \( \mathcal{R}_\pm(t) \) coincide on \( \mathcal{D}(A_\pm) \) and by the density they coincide on \( X_\pm \).

Using the decomposition of the space \( X \) and of the operator \( A \), which is introduced by the spectral projection \( P_\pm \), together with the uniqueness of the resolvent and integral resolvent it follows directly that \( S(t) = S_+(t) + S_-(t) \) and \( R(t) = R_+(t) + R_-(t) \) are the desired objects.

As a direct consequence of Lemma 5.1.4 we obtain the following corollary.

**Corollary 5.1.5.** Assume that the Volterra equation (5.2) admits a resolvent \( \{S(t)\}_{t \in \mathbb{R}_+} \) and an integral resolvent \( \{R(t)\}_{t \in \mathbb{R}_+} \). Suppose further that both Volterra equations (5.3) admits a resolvent \( \{S_\pm(t)\}_{t \in \mathbb{R}_+} \) and an integral resolvent \( \{R_\pm(t)\}_{t \in \mathbb{R}_+} \).

Then, the mild solution \( u(t) = S(t)u_0 + (R \ast f)(t), t \in J \), of the Volterra equation

\[
u(t) + (a \ast Au)(t) = u_0 + (a \ast f)(t), \quad t \in J,
\]

where \( u_0 \in X \) and \( f \in C(J;X) \), decomposes into its components on \( X_\pm \) given by the mild solutions of the Volterra equations

\[
u_\pm(t) + (a \ast A_\pm u_\pm)(t) = P_\pm u_0 + (a \ast [P_\pm f])(t), \quad t \in J,
\]
that is
\[ u_\pm(t) = S_\pm(t)u_0 + (R_\pm \ast f)(t), \quad t \in J, \]
and \( u(t) = P_+ u(t) + P_- u(t) = u_+(t) + u_-(t), \quad t \in J. \)

The next result shows the holomorphic dependence of the scalar resolvent \( s_\mu \) and integral resolvent \( r_\mu \) on the parameter \( \mu \in \mathbb{C} \).

**Lemma 5.1.6.** Let \( a \in L_{1, \text{loc}}(\mathbb{R}_+) \) be scalar kernel. Assume that \( \Omega \subset \mathbb{C} \) is a non-empty, open, connected and bounded set in \( \mathbb{C} \).

Then, for each \( T > 0 \) the maps \( r: \Omega \to L_1([0, T]; \mathbb{C}), \mu \mapsto r_\mu \), as well as \( s: \Omega \to L_1([0, T]; \mathbb{C}), \mu \mapsto s_\mu \), are holomorphic on \( \Omega \).

**Proof.** Let \( T > 0 \) and \( \mu \in \Omega \) be arbitrary, but fixed. We consider on \([0, T]\) the scalar integral resolvent equation
\[ r_\mu(t) + \mu(a \ast r_\mu)(t) = a(t), \quad t \in [0, T]. \]
We introduce for \( \sigma \in \mathbb{C} \) the notation \( g^\sigma(t) = g(t)e^{-\sigma t} \), the multiplication with an exponential weight. We choose \( \sigma > 0 \) sufficiently large, such that \( \sup_{\mu \in \Omega} \|\mu a^\sigma\|_{L_1([0, T]; \mathbb{C})} < 1 \). We multiply the Volterra equation for \( r_\mu \) with \( e^{-\sigma t} \), this yields the equation
\[ r_\mu^\sigma(t) + (r_\mu^\sigma \ast [\mu a^\sigma])(t) = a^\sigma(t), \quad t \in [0, T]. \tag{5.4} \]
We define for every \( n \in \mathbb{N} \) and all \( t \in [0, T] \) the expression
\[ r_\mu^{\sigma, n}(t) = a^\sigma(t) + \sum_{k=1}^n (-1)^k \left[ (\mu a^\sigma)^k \ast a^\sigma \right](t). \]
Here, for every \( k \in \mathbb{N} \) we denote by \((\mu a^\sigma)^k \) the \((k-1)\)-fold convolution of \( \mu a^\sigma \) by itself and \((\mu a^\sigma)^1 = \mu a^\sigma \). Obviously, the expressions \( r_\mu^{\sigma, n} \in L_1([0, T]; \mathbb{C}) \) are holomorphic in \( \mu \in \Omega \), for all \( n \in \mathbb{N} \). Furthermore, for each fixed \( \mu \in \Omega \) the sequence \( (r_\mu^{\sigma, n})_{n \in \mathbb{N}} \) is a Cauchy sequence in \( L^1([0, T]; \mathbb{C}) \), since \( \|(\mu a^\sigma)^k \ast a^\sigma\|_{L_1([0, T]; \mathbb{C})} \leq \|\mu a^\sigma\|_{L_1([0, T]; \mathbb{C})}\|a^\sigma\|_{L_1([0, T]; \mathbb{C})}, \ k \in \mathbb{N}, \) and \( \|\mu a^\sigma\|_{L_1([0, T]; \mathbb{C})} < 1 \) by the choice of \( \sigma > 0 \). The space \( L_1([0, T]; \mathbb{C}) \) is complete, so we know that there is a \( r_\mu^\sigma \in L_1([0, T]; \mathbb{C}) \) such that \( r_\mu^{\sigma, n} \to r_\mu^\sigma \) in \( L_1([0, T]; \mathbb{C}) \) as \( n \to \infty \), and \( r_\mu^\sigma \) satisfies our equation (5.4). These arguments are completely analogous to the arguments which Gripenberg, Londen and Staffans give in the proof of [GLS90, Theorem 2.3.1].

Now, we want to apply Vitali’s Theorem 1.2.7 to show that \( r_\mu^\sigma \) is holomorphic on \( \Omega \). Obviously, we have that \( \Omega = \{ \mu \in \Omega : \lim_{n \to \infty} r_\mu^{\sigma, n} \text{ exists} \} \neq \emptyset \). Furthermore, we have for each ball \( \overline{B}_r(z_0) \subset \Omega \) that...
5.1. Spectral Projections associated with Spectral Sets and Volterra Equations

\[ \sup_{n \in \mathbb{N}, \mu \in B_r(z_0)} \| r^{\sigma,n}_\mu \|_{L_1([0,T];\mathbb{C})} \leq \sup_{n \in \mathbb{N}, \mu \in B_r(z_0)} \left[ \| a^{\sigma} \|_{L_1([0,T];\mathbb{C})} \sum_{j=0}^{N} \left\| \mu a^{\sigma} \|_{L_1([0,T];\mathbb{C})} \right\| \right] \]

\[ \leq \frac{\| a^{\sigma} \|_{L_1([0,T];\mathbb{C})}}{1 - \| \mu a^{\sigma} \|_{L_1([0,T];\mathbb{C})}} < \infty. \]

Via Vitali’s Theorem 1.2.7 we know that there is a holomorphic function \( r^{\sigma}_\mu : \Omega \rightarrow L_1([0,T];\mathbb{C}) \) such that \( r^{\sigma}_\mu = \lim_{n \rightarrow \infty} r^{\sigma,n}_\mu \) in \( L_1([0,T];\mathbb{C}) \) uniformly on all compact subsets of \( \Omega \).

Hence, \( r : \Omega \rightarrow L_1([0,T];\mathbb{C}), \mu \mapsto r^{\sigma}_\mu \), is holomorphic on \( \Omega \). Since the resolvent \( s^{\sigma}_\mu \) is given by \( s^{\sigma}_\mu(t) = 1 - \mu(r^{\sigma}_\mu * 1)(t), t \in \mathbb{R}_+ \), see (2.2), it is obvious that \( s : \Omega \rightarrow L_1([0,T];\mathbb{C}), \mu \mapsto s^{\sigma}_\mu \), is holomorphic on \( \Omega \), too.

**Corollary 5.1.7.** We consider on \( X_+ \) the Volterra equation

\[ u(t) + (a * A_+ u)(t) = f(t), \quad t \in J, \tag{5.5} \]

where \( A_+ \in B(X_+) \) with \( \sigma(A_+) = \sigma_+ \).

Then, the resolvent \( \{ S_+(t) \}_{t \in \mathbb{R}_+} \) and the integral resolvent \( \{ R_+(t) \}_{t \in \mathbb{R}_+} \) of the Volterra equation (5.5) are given by Dunford’s integral representation

\[ S_+(t) = \frac{1}{2\pi i} \int_{\gamma} s_\lambda(t)(\lambda - A_+)^{-1} d\lambda, \quad t \in \mathbb{R}_+, \]

\[ R_+(t) = \frac{1}{2\pi i} \int_{\gamma} r_\lambda(t)(\lambda - A_+)^{-1} d\lambda, \quad t \in \mathbb{R}_+, \]

where \( \gamma \) is an appropriately chosen path in the complex plane \( \mathbb{C} \) around \( \sigma_+ \), cf. Remark 5.1.2 d).

**Proof.** Since \( \sigma_+ \) is compact in \( \mathbb{C} \), there is some non-empty, open, connected and bounded set \( \Omega \subset \mathbb{C} \) which contains \( \sigma_+ \subset \Omega \) and \( \gamma \) denotes the boundary of \( \Omega \), cf. Remark 5.1.2 d). Note that in this situation it is possible to choose \( \gamma \) sufficiently smooth. Moreover, the operator \( A_+ \in B(X_+) \) is a bounded operator on \( X_+ \), so Dunford’s integral representation, cf. [Yos95, Section VIII.7], yields the claim.

\[ \square \]
5. Exponential Weighted Scalar Resolvents and Integral Resolvents

Throughout this chapter we assume:

Assumptions. Let \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \setminus L_1(\mathbb{R}_+) \) be an unbounded, completely monotonic kernel and by \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) we denote the corresponding completely monotonic kernel with \( a \ast b \equiv 1 \) on \((0,\infty)\). Furthermore, let the kernel \( a \) be \( \theta_a \)-sectorial with \( \theta_a \in (0,\pi/2) \).

First, we recall that for arbitrary \( \mu \in \mathbb{C} \) the scalar resolvent \( s_{\mu} \in L_{1,\text{loc}}(\mathbb{R}_+;\mathbb{C}) \) and integral resolvent \( r_{\mu} \in L_{1,\text{loc}}(\mathbb{R}_+;\mathbb{C}) \) are the unique solutions of the following linear Volterra equations on \( \mathbb{R}_+ \):

\[
\begin{align*}
    s_{\mu}(t) + \mu(s_{\mu} \ast a)(t) &= 1, \\
    r_{\mu}(t) + \mu(r_{\mu} \ast a)(t) &= a(t).
\end{align*}
\]

In the following we use for \( \omega \in \mathbb{C} \) the notation \( k^\omega(t) = k(t)e^{-\omega t} \), the multiplication with an exponential weight.

From Lemma 2.3.8 we know that the Laplace transform \( \hat{a} \) of the completely monotonic kernel \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \) is injective. Hence, the Laplace transform \( \hat{a} \) maps the right half-plane \( \mathbb{C}_+ \) bijectively to its image set \( \hat{a}(\mathbb{C}_+) \subset \Sigma_{\theta_a}. \) This implies that for all \(-1/\mu \in \hat{a}(\mathbb{C}_+) \subset \Sigma_{\theta_a} \) there is a unique \( \rho \in \mathbb{C}_+ \) such that \( \hat{a}(\rho) = -1/\mu \), i.e. \( \rho = \hat{a}^{-1}(-1/\mu) \). Note that the condition \(-1/\mu \in \hat{a}(\mathbb{C}_+) \) is equivalent to the condition \( \mu \in -1/\hat{a}^{-1}(\mathbb{C}_+) \), since \( \hat{a} \) has no zeros in \( \Sigma_{\theta_a} \).

For each \(-1/\mu \in \hat{a}(\mathbb{C}_+) \subset \Sigma_{\theta_a} \) we multiply the Volterra equation for \( s_{\mu} \) and \( r_{\mu} \) with the exponential weight \( e^{-\rho t} \), where \( \rho \) is chosen such that \( \hat{a}(\rho) = -1/\mu \). We deduce for all \( t \in \mathbb{R}_+ \) that

\[
\begin{align*}
    s_{\mu}^\rho(t) + \mu(s_{\mu}^\rho \ast a^\rho)(t) &= 1^\rho(t), \\
    r_{\mu}^\rho(t) + \mu(r_{\mu}^\rho \ast a^\rho)(t) &= a^\rho(t).
\end{align*}
\]

Since \( \mu = -1/\hat{a}(\rho) \), we may simplify the notation by writing \( s^\rho \equiv s_{\mu}^\rho \) and \( r^\rho \equiv r_{\mu}^\rho \). Thus, we have on \( \mathbb{R}_+ \) the equations

\[
    s^\rho(t) - \frac{(s^\rho \ast a^\rho)(t)}{\hat{a}(\rho)} = 1^\rho(t),
\]

1We denote by \( \hat{a}(\mathbb{C}_+) \) the image set of the mapping \( \hat{a} \) of the complex right half-plane \( \mathbb{C}_+ \), i.e. \( \hat{a}(\mathbb{C}_+) = \{ \lambda \in \mathbb{C} : \text{ there is some } z \in \mathbb{C}_+ \text{ such that } \lambda = \hat{a}(z) \} \). Since \( \hat{a} \) is a non-trivial holomorphic function \( \hat{a}(\mathbb{C}_+) \) is an open, non-empty, connected set in \( \mathbb{C} \). The assumption, that the kernel \( a \) is \( \theta_a \)-sectorial, implies that \( \hat{a}(\mathbb{C}_+) \subset \Sigma_{\theta_a} \). Since \( \hat{a} \) has no zeros in \( \Sigma_{\theta_a} \), cf. Remark 2.3.17 a), the set \(-1/\hat{a}(\mathbb{C}_+) = \{ \lambda \in \mathbb{C} : \text{ there is some } z \in \mathbb{C}_+ \text{ such that } \mu = -1/\hat{a}(z) \} \subset -\Sigma_{\theta_a} \) is also well-defined.
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\[ r^\rho(t) - \frac{(r^\rho \ast a^\rho)(t)}{\hat{a}(\rho)} = a^\rho(t). \]

Evidently, we have \( a^\rho, 1^\rho \in L_1(\mathbb{R}_+; \mathbb{C}) \), therefore it is a direct consequence of the Paley-Wiener Theorem on the Half-Line [GLS90, Theorem 4.4.1], that \( s^\rho, r^\rho \in L_{1,loc}(\mathbb{R}_+; \mathbb{C}) \) are of subexponential growth. Thus, the Laplace transform of \( s^\rho \) and \( r^\rho \) exists for all \( \lambda \in \mathbb{C}_+ \) given by

\[
\tilde{s}^\rho(\lambda) = \frac{1}{\lambda + \rho} \cdot \frac{1}{1 - \frac{\hat{a}^\rho(\lambda)}{\hat{a}(\rho)}} = \frac{1}{\lambda + \rho} \cdot \frac{1}{1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)}},
\]

\[
\tilde{r}^\rho(\lambda) = \tilde{s}^\rho(\lambda) \cdot \frac{1}{1 - \frac{\hat{a}^\rho(\lambda)}{\hat{a}(\rho)}} = \hat{a}(\lambda + \rho) \cdot \frac{1}{1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)}}.
\]

For each \( \rho \in \mathbb{C}_+ \) there is some \( \delta_0 = \delta_0(\rho) \in (0, \frac{\pi}{2}) \) such that \( \rho \in \Sigma_{\frac{\pi}{2} - \delta_0} \). Using Lemma 2.3.9 we know that for all \( \lambda \in \left\{ \Sigma_{\frac{\pi}{2} + \delta_0} - \rho \right\} \setminus \{0\} \) we have \( \hat{a}(\lambda + \rho) \neq \hat{a}(\rho) \).

Moreover, the Laplace transform \( \hat{a} \) possesses a holomorphic extension to \( \Sigma_{\pi} \), since \( a \) is completely monotonic, cf. Theorem 2.3.7. Due to the structure of \( \tilde{s}^\rho(\lambda) \) and \( \tilde{r}^\rho(\lambda) \) we can conclude that both expressions have a holomorphic extension at least to the set \( \{ \Sigma_{\frac{\pi}{2} + \delta_0} - \rho \} \setminus \{0\} \).

5.2.1 Long-Time Behaviour

We are interested in the long-time behaviour of \( s^\rho \) and \( r^\rho \). We want to examine their behaviour via the Tauberian Theorem 1.2.9. By L’Hospital’s rule we have

\[
\lim_{\lambda \to 0^+} \frac{\lambda}{1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)}} = \lim_{\lambda \to 0^+} \frac{1}{\frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)}} = -\frac{\hat{a}(\rho)}{\hat{a}'(\rho)},
\]

and hence

\[
\lim_{\lambda \to 0^+} \lambda \tilde{s}^\rho(\lambda) = -\frac{\hat{a}(\rho)}{\hat{a}'(\rho)} =: s^*_\rho(\rho),
\]

\[
\lim_{\lambda \to 0^+} \lambda \tilde{r}^\rho(\lambda) = -\frac{\hat{a}'(\rho)}{\hat{a}''(\rho)} =: r^*_\rho(\rho).
\]

We point out that the kernel \( a \in L_{1,loc}(\mathbb{R}_+) \) is completely monotonic, so neither \( \hat{a} \) nor \( \hat{a}' \) has zeros in \( \mathbb{C}_+ \), see Remark 2.3.17. Thus, we have for all \( \rho \in \mathbb{C}_+ \) that

\[ |s^*_\rho(\rho)|, |r^*_\rho(\rho)| < \infty, \]

as well as \( s^*_\rho(\rho) \neq 0 \) and \( r^*_\rho(\rho) \neq 0 \). In other words, \( \lambda = 0 \) is a pole of first order of \( \tilde{s}^\rho \) and \( \tilde{r}^\rho \). This implies, together with the holomorphic extension properties of \( \hat{a} \) on \( \Sigma_{\pi} \), that for fixed \( \rho \in \Sigma_{\frac{\pi}{2} - \delta_0} \) the both expressions \( \tilde{s}^\rho(\lambda) - s^*_\rho(\rho)/\lambda \) and \( \tilde{r}^\rho(\lambda) - r^*_\rho(\rho)/\lambda \) have a holomorphic extension at least to the shifted sector \( \Sigma_{\frac{\pi}{2} + \delta_0} - \rho \).

To prove the convergence \( s^\rho(t) \to s^*_\rho(\rho) \) and \( r^\rho(t) \to r^*_\rho(\rho) \) as \( t \to \infty \), we use the following auxiliary result.
Proposition 5.2.1. For \( \delta_0 \in (0, \frac{\pi}{2}) \) we consider some \( \rho \in \Sigma_{\pi/2-\delta_0} \). We set \( \alpha_0 = \delta_0/2 \). Then we have for all \( \lambda \in \Sigma_{\pi/2+\alpha_0} \) the following estimates.

(i) There are constants \( C_{a_0}, c_{a_0} > 0 \) such that \( c_{a_0} \hat{\alpha}(|\lambda|) \leq |\hat{\alpha}(\lambda)| \leq C_{a_0} \hat{\alpha}(|\lambda|) \).

(ii) There is some constant \( C = C(\delta_0) > 0 \) such that \( |\lambda + \rho| \geq C |\lambda| \).

(iii) We have the estimate \( |\hat{\alpha}(\lambda + \rho)| \leq C_{a_0} \hat{\alpha}(C |\lambda|) \), where \( C > 0 \) is the constant from part (ii).

(iv) There is some \( \xi_0 = \xi_0(\delta_0,|\rho|) > 0 \) such that for all \( |\lambda| \geq \xi_0 \) we have the estimate

\[
\left| 1 - \frac{\hat{\alpha}(\lambda + \rho)}{\hat{\alpha}(\rho)} \right|^{-1} \leq 2.
\]

(v) For each \( C > 0 \) the function \( t \mapsto t \hat{\alpha}(Ct) \) is non-decreasing on \( (0, \infty) \).

Proof. (i) The angle \( \alpha_0 \) is chosen smaller that \( \pi/4 \), hence the claim follows by Theorem 2.3.12.

(ii) The choice of the angle \( \alpha_0 \) ensures that we have \( \pi/2 + \alpha_0 + (\pi/2 - \delta_0) = \pi - \delta_0/2 < \pi \). Corollary 2.3.11 implies that there is some constant \( C = C(\delta_0) > 0 \) such that for \( \rho \in \Sigma_{\pi/2-\delta_0} \) and all \( \lambda \in \Sigma_{\pi/2+\alpha_0} \) we have \( |\lambda + \rho| \geq C (|\lambda| + |\rho|) \). The claim follows immediately.

(iii) We know that \( \hat{\alpha} \) is decreasing on \( (0, \infty) \), see Remark 2.3.6. Together with the estimate in part (ii) we obtain for all \( \lambda \in \Sigma_{\pi/2+\alpha_0} \) that \( |\hat{\alpha}(\lambda + \rho)| \leq C_{a_0} \hat{\alpha}(C |\lambda|) \), where \( C > 0 \) is the constant from part (ii).

(iv) On the whole we have the estimate

\[
\left| \frac{\hat{\alpha}(\lambda + \rho)}{\hat{\alpha}(\rho)} \right| \leq \frac{C_{a_0} \hat{\alpha}(|\lambda + \rho|)}{c_{a_0} \hat{\alpha}(|\rho|)} \leq \frac{C_{a_0} \hat{\alpha}(C |\lambda|)}{c_{a_0} \hat{\alpha}(|\rho|)}.
\]

Due to the assumption \( a \in L_{1,\text{loc}}(\mathbb{R}_+) \setminus L_1(\mathbb{R}_+) \) we deduce for the continuous and decreasing function \( \hat{\alpha}: (0, \infty) \to \mathbb{R}_+ \) that \( \hat{\alpha} \) is bijective on \( (0, \infty) \), cf. Remark 2.3.17 b).

Now, we choose \( \xi_0 = \xi_0(\delta_0,|\rho|) > 0 \) such that

\[
\xi_0 = \frac{1}{C} \hat{\alpha}^{-1} \left( \frac{1}{2} \frac{c_{a_0} \hat{\alpha}(|\rho|)}{C_{a_0}} \right).
\]

Hence, we have for all \( \lambda \in \Sigma_{\pi/2+\alpha_0} \) with \( |\lambda| \geq \xi_0 \) the estimate

\[
\left| \frac{\hat{\alpha}(\lambda + \rho)}{\hat{\alpha}(\rho)} \right| \leq \frac{1}{2}.
\]
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and consequently

\[ \left| 1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)} \right|^{-1} \leq 2. \]

(v) The Sonine condition implies for all \( t \in (0, \infty) \) that \( t\hat{a}(t) = 1/b(t) \). Since the corresponding Sonine kernel \( b \in L_{1,\text{loc}}(\mathbb{R}_+) \) is completely monotonic, its Laplace transform \( \hat{b} \) is decreasing on \((0, \infty)\), cf. Remark 2.3.6. Thus, it is obvious that \( t \mapsto t\hat{a}(Ct) \) is non-decreasing on \((0, \infty)\), for each constant \( C > 0 \).

Now, we can prove the following convergence theorem.

**Lemma 5.2.2.** Assume additionally that \( \int_0^1 \hat{a}(1/t) dt < \infty \).

Then we have for each \( \rho \in \mathbb{C}_+ \) that

(i) \( s^\rho(t) \to s_\star(\rho) = -\frac{1}{\rho} \cdot \frac{\hat{a}(\rho)}{\hat{a}(\rho)}, \) as \( t \to \infty \),

(ii) \( r^\rho(t) \to r_\star(\rho) = -\frac{\hat{a}(\rho)}{\hat{a}(\rho)}, \) as \( t \to \infty \).

**Proof.** Let \( \rho \in \mathbb{C}_+ \) be arbitrary, but fixed. Then there is some \( \delta_0 = \delta_0(\rho) \in (0, \pi/2) \) such that \( \rho \in \Sigma_{\pi/2-\delta_0} \). We set \( \alpha_0 = \delta_0/2 < \pi/4 \). Obviously, we have \( \Sigma_{\pi/2+\alpha_0} \subset \Sigma_{\pi/2-\alpha_0} \setminus \{0\} \). Thus, \( s^\rho \) and \( r^\rho \) are holomorphic on the sector \( \Sigma_{\pi/2+\alpha_0} \). We want to apply the Tauberian Theorem 1.2.9. In particular, we show that for all \( \lambda \in \Sigma_{\pi/2+\alpha_0} \) there are constants \( M_s, M_r, C > 0 \) with

\[
|\hat{s}(\lambda)| \leq M_s/|\lambda|,
\]

\[
|\hat{r}(\lambda)| \leq M_r [1/|\lambda| + \hat{a}(C|\lambda|)].
\]

(i) Firstly, we consider for \( \lambda \in \Sigma_{\pi/2+\alpha_0} \) the expression

\[
|\hat{s}(\lambda)| = \left| \frac{1}{\lambda + \rho} \cdot \frac{1}{1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)}} \right|.
\]

By Proposition 5.2.1 (ii) there is some constant \( C = C(\delta_0) > 0 \) such that for all \( \lambda \in \Sigma_{\pi/2+\alpha_0} \) we have \( |\lambda/(\lambda + \rho)| \leq 1/C(\delta_0) \). Together with the estimate from Proposition 5.2.1 (iv), we obtain for all \( \lambda \in \Sigma_{\pi/2+\alpha_0} \) with \( |\lambda| \geq \xi_0 \) that

\[
|\lambda s(\lambda)| = \left| \frac{\lambda}{\lambda + \rho} \cdot \frac{1}{1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)}} \right| \leq \frac{2}{C(\delta_0)}.
\]
Due to the previous considerations we know \( \lambda \tilde{s}(\lambda) \to s_\ast(\rho) \) for \( \lambda \to 0 \) and \( \tilde{s}(\lambda) \) is holomorphic on \( \{ \Sigma_{n/2+a_0} - \rho \} \setminus \{ 0 \} \). Thus, we conclude \( \lambda \tilde{s}(\lambda) \) is holomorphic on \( \Sigma_{n/2+a_0} - \rho \). The set \( \Sigma_{n/2+a_0} \cap B_{\delta_0}(0) \) is a compact set in \( \Sigma_{n/2+a_0} - \rho \). Hence

\[
\sup_{\lambda \in \Sigma_{n/2+a_0} \cap B_{\delta_0}(0)} \left| \lambda \tilde{s}(\lambda) \right| \leq M_1,
\]

with some constant \( M_1 = M_1(a_0, \rho, \xi_0) = M_1(\delta_0, |\rho|) \). Thus, we obtain for all \( \lambda \in \Sigma_{n/2+a_0} \) that

\[
\left| \lambda \tilde{s}(\lambda) \right| \leq \max\{M_1, 2/C(\delta_0)\} = M_2(\delta_0, |\rho|) < \infty.
\]

It is easy to see that \( g(s) = 1/s, \ s \in (0, \infty) \) satisfies the condition of the Tauberian Theorem 1.2.9. The claim follows.

(ii) Now, we estimate for \( \lambda \in \Sigma_{n/2+a_0} \) the expression:

\[
\left| \tilde{r}(\lambda) \right| = \left| \hat{a}(\lambda + \rho) \cdot \frac{1}{1 - \hat{a}(\lambda + \rho) \hat{a}(\rho)} \right|
\]

Just like above, we have with Proposition 5.2.1 for all \( \lambda \in \Sigma_{n/2+a_0} \) with \( |\lambda| \geq \xi_0 \) that

\[
\left| \tilde{r}(\lambda) \right| \leq 2C_{a_0} \hat{a}(C |\lambda|),
\]

where \( C = C(\delta_0) > 0 \) is the constant from Proposition 5.2.1 (ii). We know from the previous considerations that \( \lim_{\lambda \to 0} \lambda \tilde{r}(\lambda) = r_\ast(\rho) \) and \( \tilde{r}(\lambda) \) is holomorphic on \( \{ \Sigma_{n/2+a_0} - \rho \} \setminus \{ 0 \} \). Thus, we conclude that \( \lambda \tilde{r}(\lambda) \) is holomorphic on \( \Sigma_{n/2+a_0} - \rho \). The set \( \Sigma_{n/2+a_0} \cap B_{\delta_0}(0) \) is a compact subset of \( \Sigma_{n/2+a_0} - \rho \). Hence,

\[
\sup_{\lambda \in \Sigma_{n/2+a_0} \cap B_{\delta_0}(0)} \left| \lambda \tilde{r}(\lambda) \right| \leq M_2,
\]

with some constant \( M_2 = M_2(a_0, \rho, \xi_0) = M_2(\delta_0, |\rho|) \). All together, we have for each \( \lambda \in \Sigma_{n/2+a_0} \) that

\[
\left| \tilde{r}(\lambda) \right| \leq \frac{M_2}{|\lambda|} + 2C_{a_0} \hat{a}(C |\lambda|) \leq \max\{M_3, 2C_{a_0}\left( \frac{1}{|\lambda|} + \hat{a}(C |\lambda|) \right) \}.
\]

This means that we have \( \left| \tilde{r}(\lambda) \right| \leq M_3 g(|\lambda|) \) with \( g(s) = 1/s + \hat{a}(Cs) \), for \( s \in (0, \infty) \). The function \( g \) is a positive, non-increasing function on \( (0, \infty) \) and \( sg(s) = 1 + s\hat{a}(Cs) \) is non-decreasing on \( (0, \infty) \), cf. Proposition 5.2.1 (iv). Thus, we have \( \sup_{t \in (0,1]} tg(t) \leq 1 + \hat{a}(C), \) and \( \int_0^1 g(1/t) \frac{dt}{t} \leq 1 + \int_0^1 \hat{a}(C/t) \frac{dt}{t} < \infty \). The Tauberian Theorem 1.2.9 yields the claim.
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5.2.2 Convergence Rate

To prove the convergence rate of $s^\rho$ and $r^\rho$ to $s^\star_\rho(\rho)$ and $r^\star_\rho(\rho)$, respectively, we need the following auxiliary estimates which are uniform in $\rho$ on compact sets of $C_+$. 

**Proposition 5.2.3.** Let $K \subset C_+$ be a compact set such that for suitable constants $\delta_0 \in (0, \pi/2)$ and $\rho_{\text{max}} \geq \rho_{\text{min}} > 0$, depending on $K$, we have $K \subset \Sigma_{\pi/2-\delta_0} \cap \left( \overline{B}_{\rho_{\text{max}}}(0) \setminus B_{\rho_{\text{min}}}(0) \right)$. Furthermore, we set $\alpha_0 = \delta_0/2$.

Then we have for each $\rho \in K$ and all $\lambda \in \Sigma_{\pi/2} + \alpha_0 - \rho$ the following statements.

(i) We have $\hat{a}(\rho_{\text{max}}) \leq \hat{a}(|\rho|) \leq \hat{a}(\rho_{\text{min}})$.

(ii) There are constants $C_{\alpha_0}, c_{\alpha_0} > 0$ such that $c_{\alpha_0} \hat{a}(|\lambda + \rho|) \leq |\hat{a}(\lambda + \rho)| \leq C_{\alpha_0} \hat{a}(|\lambda + \rho|)$.

(iii) For $|\lambda + \rho| \geq 2\rho_{\text{max}}$ we have $|\lambda| \geq \rho_{\text{max}}$ and $|\lambda + \rho|/|\lambda| \leq 2$.

(iv) For $|\lambda + \rho| \leq \rho_{\text{min}}/2$ we have $|\lambda| \geq \rho_{\text{min}}/2$ and $|\lambda + \rho|/|\lambda| \leq 1 + 2\rho_{\text{max}}/\rho_{\text{min}}$.

(v) There is some $\xi_1 = \xi_1(K) > 0$ such that for all $|\lambda + \rho| \geq \xi_1$ we have

\[ \left| 1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)} \right|^{-1} \leq 2. \]

(vi) There is some $\xi_2 = \xi_2(K) > 0$ such that for all $|\lambda + \rho| \geq \xi_2$ we have

\[ |\lambda \hat{a}(\lambda + \rho)| \geq \frac{1}{2}. \]

(vii) There is some $\xi_\star = \xi_\star(K) > 0$ such that for all $|\lambda + \rho| \geq \xi_\star$ we have

\[ \left| \frac{1}{\hat{a}(\lambda + \rho) - \hat{a}(\rho)} \right|^{-1} \leq 2c_{\alpha_0} \hat{a}(\rho_{\text{max}}). \]

(viii) There is some $\nu_1 = \nu_1(K) > 0$ such that for all $|\lambda + \rho| \leq \nu_1$ we have

\[ \left| 1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)} \right|^{-1} \leq \frac{1}{2}. \]

(ix) There is some $\nu_2 = \nu_2(K) > 0$ such that for all $|\lambda + \rho| \leq \nu_2$ we have

\[ |\lambda \hat{a}(\lambda + \rho)| \geq \frac{1}{2}. \]
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(x) There is some \( \nu_\bullet = \nu_\bullet(K) > 0 \) such that for all \( |\lambda + \rho| \leq \nu_\bullet \) we have

\[
\left| \frac{1}{\hat{a}(\lambda + \rho)} - \frac{1}{\hat{a}(\rho)} \right|^{-1} \leq 2C_{a_0}\hat{a}(\rho_{\min}).
\]

Proof. (i) The Laplace transform \( \hat{a} \) is decreasing on \((0, \infty)\), see Remark 2.3.6, the claim follows directly.

(ii) Since \( \lambda + \rho \in \Sigma_{\pi/2 + \alpha_0} \) and \( \alpha_0 < \pi/4 \) the assertion follows by Theorem 2.3.12.

(iii) We have \( |\lambda| \geq |\lambda + \rho| - |\rho| \geq 2\rho_{\max} - \rho_{\max} = \rho_{\max}, \) as well as \( |\lambda + \rho|/|\lambda| \leq 1 + |\rho|/|\lambda| \leq 1 + \rho_{\max}/\rho_{\max} = 2. \)

(iv) We have \( |\lambda| \geq |\rho| - |\lambda + \rho| \geq \rho_{\min} - \rho_{\min}/2 = \rho_{\min}/2, \) as well as \( |\lambda + \rho|/|\lambda| \leq 1 + |\rho|/|\lambda| \leq 1 + 2\rho_{\max}/\rho_{\min}. \)

(v) From part (i) and (ii) it follows the estimate

\[
\left| \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)} \right| \leq \frac{C_{a_0}\hat{a}(\lambda + \rho)}{c_{a_0}\hat{a}(\rho_{\min})} \leq \frac{C_{a_0}\hat{a}(\lambda + \rho)}{c_{a_0}\hat{a}(\rho_{\min})}.
\]

Due to the bijectivity of \( \hat{a} \) on \((0, \infty)\), we choose \( \xi_1 = \xi_1(\alpha_0, \rho_{\min}) = \xi_1(K) > 0 \) such that

\[
\xi_1 = \hat{a}^{-1}\left( \frac{1}{2} \frac{c_{a_0}\hat{a}(\rho_{\min})}{C_{a_0}} \right).
\]

Thus, for each \( \rho \in K \) and all \( \lambda \in \Sigma_{\pi/2 + \alpha_0} - \rho \) with \( |\lambda + \rho| \geq \xi_1 \) we obtain that

\[
\left| \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)} \right| \leq \frac{1}{2},
\]

and consequently

\[
\left| 1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)} \right|^{-1} \leq 2.
\]

(vi) The corresponding Sonine kernel \( b \in L_{1,loc}(\mathbb{R}_+) \) is completely monotonic, hence \( \hat{b}: (0, \infty) \rightarrow \mathbb{R}_+ \) is strictly decreasing. In case of \( b \in L_1(\mathbb{R}_+) \) and \( \lim_{t \to 0^+} \hat{b}(t) = \beta < \infty \) we assume w. l. o. g. that \( \beta > c_{a_0} \) (if this is not true, we replace in part (ii) \( c_{a_0} \) by \( \beta/2 \)), then \( \hat{b}: (0, \infty) \rightarrow (0, \beta) \) is bijective and \( \hat{b}^{-1}(c_{a_0}) > 0 \) is well-defined. We set \( \xi_2 = \max\{\hat{b}^{-1}(c_{a_0}), 2\rho_{\max}\} = \xi_2(K) > 0. \) For each \( \rho \in K \) and \( \lambda \in \Sigma_{\pi/2 + \alpha_0} - \rho \) with \( |\lambda + \rho| \geq \xi_2 \) we obtain, together with the estimate from part (iii), that
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\[ |\lambda \hat{a}(\lambda + \rho)| \geq \frac{|\lambda|}{|\lambda + \rho|} c_0 a |\lambda + \rho| \hat{a}(\lambda + \rho) \]
\[ \geq \frac{c_0}{2b(|\lambda + \rho|)} \]
\[ \geq \frac{c_0}{2b(\epsilon_2)} \geq 1/2. \]

(vii) Now, we choose \( \xi_* = \xi_*(a_0, \rho_{\text{max}}) = \xi_*(K) > 0 \) such that

\[ \xi_* = \hat{a}^{-1}\left(\frac{2c_0 \hat{a}(\rho_{\text{max}})}{3 c_0}\right). \]

For each \( \rho \in K \) and all \( \lambda \in \Sigma_{\pi/2 + a_0} \) with \( |\lambda + \rho| \geq \xi_* \) we have on one hand \( |\hat{a}(\rho)| \geq c_0 \hat{a}(|\rho|) \geq c_0 \hat{a}(\rho_{\text{max}}) \) and on the other hand that \( |\hat{a}(\lambda + \rho)| \leq C_{a_0} \hat{a}(|\lambda + \rho|) \leq C_{a_0} \hat{a}(\xi_*) = 2c_0 \hat{a}(\rho_{\text{max}})/3 \). Thus it follows that

\[ \left| \frac{1}{\hat{a}(\lambda + \rho)} - \frac{1}{\hat{a}(\rho)} \right| \geq \left| \hat{a}(\lambda + \rho) \right|^{-1} - \left| \hat{a}(\rho) \right|^{-1} \]
\[ \geq \left[ C_{a_0} \hat{a}(\xi_*) \right]^{-1} - \left[ c_0 \hat{a}(\rho_{\text{max}}) \right]^{-1} \]
\[ = \left[ 2c_0 \hat{a}(\rho_{\text{max}}) \right]^{-1}. \]

(viii) For each \( \rho \in K \) and all \( \lambda \in \Sigma_{\pi/2 + a_0} \) we obtain by the previous estimates from part (i) and (ii) that

\[ \left| \hat{a}(\lambda + \rho) \right| \geq \frac{c_0 \hat{a}(|\lambda + \rho|)}{C_{a_0} \hat{a}(|\rho|)} \geq \frac{c_0 \hat{a}(|\lambda + \rho|)}{C_{a_0} \hat{a}(\rho_{\text{min}})}. \]

Using the bijectivity of \( \hat{a} \) on \((0, \infty)\), we choose \( \nu_1 = \nu_1(a_0, \rho_{\text{max}}) = \nu_1(K) > 0 \) such that

\[ \nu_1 = \hat{a}^{-1}\left(\frac{3 C_{a_0} \hat{a}(\rho_{\text{max}})}{c_0 a_0}\right). \]

For each \( \rho \in K \) and all \( \lambda \in \Sigma_{\pi/2 + a_0} \) with \( |\lambda + \rho| \leq \nu_1 \) we obtain the estimate

\[ \left| \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)} \right| \geq 3, \]

and consequently

\[ \left| 1 - \frac{\hat{a}(\lambda + \rho)}{\hat{a}(\rho)} \right|^{-1} \leq \frac{1}{2}. \]
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(ix) We set \( \nu_2 = \min \{ \hat{a}^{-1}(1/(\rho_{\min}c_0)), \rho_{\min}/2 \} = \nu_2(K) > 0 \). For each \( \rho \in K \) and all \( \lambda \in \Sigma_{\pi/2+a_0} - \rho \) with \(|\lambda + \rho| \leq \nu_2 \) we deduce, together with the previous estimates from part (iv), that

\[
|\lambda \hat{a}(|\lambda + \rho|)| \geq \rho_{\min}c_0 \hat{a}(|\lambda + \rho|)/2 \\
\geq \rho_{\min}c_0 \hat{a}(\nu_2)/2 \geq 1/2.
\]

(x) Using the bijectivity of \( \hat{a} \) on \((0,\infty)\), we choose \( \nu_* = \nu_*(\alpha_0, \rho_{\min}) = \nu_*(K) > 0 \) such that

\[
\nu_* = \hat{a}^{-1} \left( 2 \frac{C_{\alpha_0} \hat{a}(\rho_{\min})}{c_0} \right).
\]

For each \( \rho \in K \) and all \( \lambda \in \Sigma_{\pi/2+a_0} - \rho \) with \(|\lambda + \rho| \leq \nu_* \) we have the estimates \(|\hat{a}(\rho)| \leq C_{\alpha_0} \hat{a}(\rho_{\min}) \) and \(|\hat{a}(\lambda + \rho)| \geq c_0 \hat{a}(\rho_{\min}) \). Thus it follows that

\[
\left| \frac{1}{\hat{a}(|\lambda + \rho|)} - \frac{1}{\hat{a}(\rho)} \right| \geq |\hat{a}(\rho)|^{-1} - |\hat{a}(\lambda + \rho)|^{-1} \\
\geq \left[ C_{\alpha_0} \hat{a}(\rho_{\min}) \right]^{-1} - \left[ c_0 \hat{a}(\nu_*) \right]^{-1} \\
= \left[ 2C_{\alpha_0} \hat{a}(\rho_{\min}) \right]^{-1}.
\]

\[\square\]

After the previous preparations we are able to determine the convergence rate of the exponentially weighted scalar resolvent \( s^\rho(t) \) and integral resolvent \( r^\rho(t) \) to \( s_*(\rho) \) and \( r_*(\rho) \), respectively, as \( t \to \infty \).

**Lemma 5.2.4.** We assume additionally that \( \int_0^1 \hat{a}(1/t) \frac{dt}{t}, \int_0^1 \hat{b}(1/t) \frac{dt}{t} < \infty \).

For each \( \rho \in \mathbb{C}_+ \) there are some constants \( C_s, C_r, C_r' > 0 \) such that for all \( t \in \mathbb{R}_+ \) we have the estimates

(i) \( \left| s_{-1/\hat{a}(\rho)}(t) - e^{\rho t} s_*(\rho) \right| \leq C_s \),

(ii) \( \left| r_{-1/\hat{a}(\rho)}(t) - e^{\rho t} r_*(\rho) \right| \leq C_r \hat{a}(1/t)/t \),

(iii) \( \left| r_{-1/\hat{a}(\rho)}(t) - e^{\rho t} r_*(\rho) \right| \leq C'_r \hat{b}(1/t)/t^2 \).

These estimates are uniform in \( \rho \) on compact sets of \( \mathbb{C}_+ \).

**Remark 5.2.5.**

a) The estimates from Lemma 5.2.4 remain true even on \( \Sigma_{\epsilon_0} \) with \( 0 < \epsilon_0 < \alpha_0, \alpha_0 > 0 \) sufficiently small.

b) Together with the fact that for each \( \rho \in \mathbb{C}_+ \) we have \( s^\rho(t) \to s_*(\rho) \neq 0 \), as \( t \to \infty \), see Lemma 5.2.2, it follows immediately that \( s_{-1/\hat{a}(\rho)} \) is unbounded on \( \mathbb{R}_+ \) for each \( \rho \in \mathbb{C}_+ \).
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Proof. Let $K \subset \mathbb{C}_+$ be a compact set. Then there are suitable constants $\delta_0 \in (0, \pi/2)$ and $\rho_{\text{max}} \geq \rho_{\text{min}} > 0$ such that $K \subset \Sigma_{\pi/2 - \delta_0} \cap \left( \overline{\mathbb{B}_{\rho_{\text{max}}} (0)} \setminus B_{\rho_{\text{min}}} (0) \right)$. We set $\alpha_0 = \delta_0 / 2 < \pi / 4$. We use the estimates and notations from Proposition 5.2.3.

(i) For each $\rho \in K$ we consider the holomorphic expression

$$\widehat{s}^\rho (\lambda) - \frac{s_*(\rho)}{\lambda},$$

on $\Sigma_{\pi/2 + \alpha_0} - \rho$. We want to apply the Analytic Representation Theorem 1.2.1 to this holomorphic expression on the shifted sector $\Sigma_{\pi/2 + \alpha_0} - \rho$.

We set $\xi_3 = \max \{ \xi_1, 2 \rho_{\text{max}} \} = \xi_3 (K) > 0$. This implies for all $\rho \in K$ and all $\lambda \in \Sigma_{\pi/2 + \alpha_0} - \rho$ with $|\lambda + \rho| \geq \xi_3$ that

$$\left| \widehat{s}^\rho (\lambda) - \frac{s_*(\rho)}{\lambda} \right| \leq \frac{1}{|\lambda + \rho|} \left| 1 - \frac{\hat{a} (\lambda + \rho)}{\hat{a} (\rho)} \right|^{-1} + \frac{|\lambda + \rho|}{|\lambda|} |s_*(\rho)| \leq M_1 \frac{1}{|\lambda + \rho|},$$

with a constant $M_1 = M_1 (K) = 2 + 2 \max_{\rho \in K} |s_*(\rho)|$.

Now, we choose $\nu_2 = \min \{ \nu_1, \rho_{\text{min}} / 2 \} = \nu_2 (K) > 0$. Thus, it follows for all $\rho \in K$ and all $\lambda \in \Sigma_{\pi/2 + \alpha_0} - \rho$ with $|\lambda + \rho| \leq \nu_2$ that

$$\left| \widehat{s}^\rho (\lambda) - \frac{s_*(\rho)}{\lambda} \right| \leq \frac{1}{|\lambda + \rho|} \left| 1 - \frac{\hat{a} (\lambda + \rho)}{\hat{a} (\rho)} \right|^{-1} + \frac{|\lambda + \rho|}{|\lambda|} |s_*(\rho)| \leq M_2 \frac{1}{|\lambda + \rho|},$$

with a constant $M_2 = M_2 (K) = 1 / 2 + (1 + 2 \rho_{\text{max}} / \rho_{\text{min}}) \max_{\rho \in K} |s_*(\rho)|$.

Let $\epsilon_0 \in (0, \alpha_0)$, $\rho \in K$ and $\lambda \in \Sigma_{\pi/2 + \epsilon_0} - \rho$ such that we have $\nu_2 \leq |\lambda + \rho| \leq \xi_3$. We set $G = \Sigma_{\pi/2 + \epsilon_0} \cap \left( \overline{\mathbb{B}_{\xi_3} (0)} \setminus B_{\nu_2} (0) \right)$. For fixed $\rho \in \mathbb{C}_+$, the function $\widehat{s}^\rho (z - \rho) - s_*(\rho) / (z - \rho)$ is holomorphic on $\Sigma_{\pi/2 + \alpha_0}$. Furthermore, there is a constant $M_0 = M_0 (K, \epsilon_0, \delta_0, \nu_2, \xi_3, M_0 (K, \epsilon_0) > 0$ such that

$$\sup_{\rho \in K} \sup_{z \in G} \left| \widehat{s}^\rho (z - \rho) - s_*(\rho) / (z - \rho) \right| \leq \sup_{(\rho, z) \in K \times G} \left| \widehat{s}^\rho (z - \rho) - s_*(\rho) / (z - \rho) \right|,$$

since $\widehat{s}^\rho (z - \rho) - s_*(\rho) / (z - \rho)$ is continuous on the compact set $\partial G \times K$, where $K \cap \partial G = \emptyset$. Observe here that $\nu_2 \leq \rho_{\text{min}} / 2$, $\xi_3 \geq 2 \rho_{\text{max}}$ as well as $K \subset \Sigma_{\pi/2 - \delta_0}$. Thus, we obtain for all $\lambda \in \Sigma_{\pi/2 + \epsilon_0} \cap \left( \overline{\mathbb{B}_{\xi_3} (0)} \setminus B_{\nu_2} (0) \right) - \rho$ that

$$\left| \widehat{s}^\rho (\lambda) - \frac{s_*(\rho)}{\lambda} \right| \frac{|\lambda + \rho|}{|\lambda + \rho|} \leq M_0 \frac{\epsilon_2}{|\lambda + \rho|} = \frac{M_3}{|\lambda + \rho|}. $$
with $M_3 = M_3(K, \varepsilon_0) > 0$. All together, we have for each $\rho \in K$ and all $\lambda \in \Sigma_{\pi/2+\varepsilon_0} - \rho$ that

$$|s^\rho(\lambda) - s^\rho(\rho)| \leq \max[M_1, M_2, M_3] \frac{|\lambda + \rho|}{|\lambda + \rho|},$$

where $M_4 = M_4(K, \varepsilon_0) > 0$ a uniform constant for all $\rho \in K$. The Analytic Representation Theorem 1.2.1 yields with $g(s) = 1/s$ for all $\varepsilon_0 \in (0, \alpha_0)$ and all $z \in \Sigma_\varepsilon$ the estimate

$$|e^{\rho z} [s^\rho(z) - s^\rho(\rho)]| \leq C_\varepsilon,$$

and consequently

$$|s_{-1/\rho}(z) - e^{\rho z} s^\rho(\rho)| \leq C_\varepsilon,$$

where $C_\varepsilon = C_\varepsilon(K, \varepsilon_0, \alpha_0) > 0$ is a uniform constant for all $\rho \in K$.

(ii) For each $\rho \in K$ we consider the holomorphic expression

$$\widehat{r}^\rho(\lambda) - \frac{r^\rho(\rho)}{\lambda},$$

on $\Sigma_{\pi/2+\alpha_0} - \rho$. We want to apply the Analytic Representation Theorem 1.2.1 to this holomorphic expression on the shifted sector $\Sigma_{\pi/2+\alpha_0} - \rho$.

We set $\xi_3 = \max\{\xi_1, \xi_2, 2\rho_{\max}\} = \xi_3(K) > 0$. This implies for all $\rho \in K$ and all $\lambda \in \Sigma_{\pi/2+\alpha_0} - \rho$ with $|\lambda + \rho| \geq \xi_3$ that

$$|\rho^\rho(\lambda) - \frac{r^\rho(\rho)}{\lambda}| \leq |\widehat{a}(\lambda + \rho)| \left[1 - \frac{\widehat{a}(\lambda + \rho)}{\widehat{a}(\rho)}\right] + \frac{|r^\rho(\rho)|}{|\lambda \widehat{a}(\lambda + \rho)|} \leq M_4 |\widehat{a}(\lambda + \rho)|.
$$

with $M_4 = M_4(K) = 2 + 2 \max_{\rho \in K} |r^\rho(\rho)|$. Now, we choose the parameter $\nu_3 = \min\{\nu_1, \nu_2, \rho_{\min}/2\} = \nu_3(K) > 0$. Thus, it follows for all $\rho \in K$ and all $\lambda \in \Sigma_{\pi/2+\alpha_0} - \rho$ with $|\lambda + \rho| \leq \nu_3$ that

$$|\rho^\rho(\lambda) - \frac{r^\rho(\rho)}{\lambda}| \leq |\widehat{a}(\lambda + \rho)| \left[1 - \frac{\widehat{a}(\lambda + \rho)}{\widehat{a}(\rho)}\right] + \frac{|r^\rho(\rho)|}{|\lambda \widehat{a}(\lambda + \rho)|} \leq M_5 |\widehat{a}(\lambda + \rho)|,
$$

with a constant $M_5 = M_5(K) = 1/2 + 2 \max_{\rho \in K} |r^\rho(\rho)|$.

Let $\varepsilon_0 \in (0, \alpha_0)$, $\rho \in K$ and $\lambda \in \Sigma_{\pi/2+\varepsilon_0} - \rho$ such that we have $\nu_3 \leq |\lambda + \rho| \leq \xi_4$. We set $G = \Sigma_{\pi/2+\varepsilon_0} \cap \left\{B_{\xi_3}(0) \setminus B_{\nu_3}(0)\right\}$. For fixed $\rho \in C_\varepsilon$ the function
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\( \tilde{r}^\beta(z - \rho) - r_*(\rho)/(z - \rho) \) is holomorphic on \( \Sigma_{n/2+n_0} \). Furthermore there is some constant \( M_6 = M_6(K, \varepsilon_0, \delta_0, \nu_3, \xi_4) = M_0(K, \varepsilon_0) > 0 \) such that

\[
\sup_{\rho \in K} \sup_{z \in G} \left| \tilde{r}^\beta(z - \rho) - r_*(\rho)/(z - \rho) \right| \leq \sup_{(\rho, z) \in K \times G} \left| \tilde{r}^\beta(z - \rho) - r_*(\rho)/(z - \rho) \right| = \sup_{(\rho, z) \in K \times G} \left| \tilde{r}^\beta(z - \rho) - r_*(\rho)/(z - \rho) \right| = M_6 < \infty,
\]

since \( \tilde{r}^\beta(z - \rho) - r_*(\rho)/(z - \rho) \) is continuous on the compact set \( \partial G \times K \), because \( \partial G \cap K = \emptyset \). Thus, we obtain that for all \( \lambda \in \Sigma_{n/2+\varepsilon_0} \cap \{ \overline{B}_{\xi_4}(0) \setminus B_{\nu_3}(0) \} - \rho \) the estimate

\[
\left| \tilde{r}^\beta(\lambda) - \frac{r_*(\rho)}{\lambda} \right| \frac{|\hat{a}(\lambda + \rho)|}{|\hat{a}(\lambda + \rho)|} \leq \frac{M_6|\hat{a}(\lambda + \rho)|}{c_{a_0}\hat{a}(|\lambda + \rho|)} = M_7|\hat{a}(\lambda + \rho)|,
\]

with \( M_7 = M_7(K, \varepsilon_0) = M_6/[c_{a_0}\hat{a}(\xi_3)] > 0 \). All together, we have for all \( \rho \in K \) and all \( \lambda \in \Sigma_{n/2+\varepsilon_0} - \rho \) that

\[
\left| \tilde{r}^\beta(\lambda) - \frac{r_*(\rho)}{\lambda} \right| \leq \max\{M_4, M_5, M_7\}|\hat{a}(\lambda + \rho)| \leq M_7|\hat{a}(|\lambda + \rho|)|,
\]

where \( M_7 = M_7(K, \varepsilon_0) = \max\{M_4, M_5, M_7\}c_{a_0} > 0 \) is a uniform constant for all \( \rho \in K \). Note that \( \hat{a} : (0, \infty) \to \mathbb{R}_+ \) is non-negative, non-increasing with \( \lim_{\lambda \to 0} \hat{a}(t) = 0 \), since it is the Laplace transform of a completely monotonic kernel, and \( \sup_{t \in [0, 1]} |\tilde{t}\hat{a}(t)| = \sup_{t \in [0, 1]} 1/\tilde{b}(t) = 1/\tilde{b}(1) < \infty \) as well as \( \int_0^1 \tilde{a}(1/t) \frac{dt}{t} < \infty \) by assumption. The Analytic Representation Theorem 1.2.1 yields with \( g = \hat{a} \) for all \( \varepsilon_0 \in (0, \alpha_0) \) and all \( z \in \Sigma_{\varepsilon_0} \) that

\[
|e^{\rho z} [\tilde{r}^\beta(z) - r_*(\rho)]| \leq C_r \hat{a}(1/|z|)/|z|,
\]

and consequently

\[
|r_{-1/\hat{a}(\rho)}(z) - e^{\rho z}r_*(\rho)| \leq C_r \hat{a}(1/|z|)/|z|,
\]

where \( C_r = C_r(K, \varepsilon_0, \alpha_0) > 0 \) is a uniform constant for all \( \rho \in K \).

(iii) For each \( \rho \in K \) we consider the derivative of \( \tilde{r}^\beta(\lambda) - r_*(\rho)/\lambda \), the holomorphic expression

\[
\tilde{r}^\beta(\lambda) + \frac{r_*(\rho)}{\lambda^2}.
\]

---

3The constant \( C_r \) is given by \( C_r = M_7(K, \varepsilon_0, \alpha_0) \max\left\{ \frac{e^{-\sin(\xi_0)\lambda}}{\sin(\xi_0)\cdot 2\pi\nu}, 2\pi e \right\} \).
on $\Sigma_{n/2+a_0} - \rho$. We want to apply the Analytic Representation Theorem 1.2.1 to this holomorphic expression on the shifted sector $\Sigma_{n/2+a_0} - \rho$.

Easy calculations yield the identity

$$\bar{r}^\prime(\lambda) = \left(\frac{1}{\bar{a}(\lambda + \rho)} - \frac{1}{\bar{a}(\rho)}\right)^{-2} \frac{\bar{a}'(\lambda + \rho)}{\bar{a}(\lambda + \rho)} = \bar{r}^\prime(\lambda) \frac{\bar{a}'(\lambda + \rho)}{\bar{a}(\lambda + \rho)}.$$  

With the relation $\hat{\alpha}(\lambda + \rho) = 1/[(\lambda + \rho)\hat{b}(\lambda + \rho)]$ it follows that

$$\bar{r}^\prime(\lambda) + \frac{r_*(\rho)}{\lambda^2} = \hat{b}(\lambda + \rho) \left[ \bar{r}^\prime(\lambda)^2 \frac{\hat{a}'(\lambda + \rho)}{\hat{a}(\lambda + \rho)^2} + \frac{r_*(\rho)}{\hat{b}(\lambda + \rho)\lambda^2} \right] = \hat{b}(\lambda + \rho) \left[ \frac{\bar{r}^\prime(\lambda)^2 \hat{a}'(\lambda + \rho)(\lambda + \rho)}{\hat{a}(\lambda + \rho)} + r_*(\rho) \frac{\hat{a}(\lambda + \rho)}{\lambda^2} \right].$$

By Lemma 2.3.16 we know there is constant $k = k(\delta_0) > 0$ such that $|\lambda \hat{a}'(\lambda)| \leq k |\hat{a}(\lambda)|$ for all $\lambda \in \Sigma_{n/2+a_0}$.

Now, we choose $\xi_5 = \max\{\xi_*; 2\rho_{\max}\} = \xi_5(K) > 0$ and obtain for each $\rho \in K$ and all $\lambda \in \Sigma_{n/2+a_0} - \rho$ with $|\lambda + \rho| \geq \xi_5$ that

$$\left| \bar{r}^\prime(\lambda) + \frac{r_*(\rho)}{\lambda^2} \right| \leq |\hat{b}(\lambda + \rho)| \left[ |\bar{r}^\prime(\lambda)|^2 \frac{\hat{a}'(\lambda + \rho)(\lambda + \rho)}{\hat{a}(\lambda + \rho)} \right] + |r_*(\rho)| \frac{\hat{a}(\lambda + \rho)}{\lambda^2} \leq M_8 |\hat{b}(\lambda + \rho)|,$$

with a constant $M_8 = [2C_{a_0} \hat{a}(\rho_{\min})]^2 k + \max_{\rho \in K} |r_*(\rho)| C_{a_0} \hat{a}(\xi_5) 2/\rho_{\min} = M_8(K) > 0$. We set $\nu_4 = \min\{\nu_*; \rho_{\min}/2\} = \nu_4(K) > 0$. Thus, we have for each $\rho \in K$ and all $\lambda \in \Sigma_{n/2+a_0} - \rho$ and with $|\lambda + \rho| \leq \nu_4$ that

$$\left| \bar{r}^\prime(\lambda) + \frac{r_*(\rho)}{\lambda^2} \right| \leq |\hat{b}(\lambda + \rho)| \left[ |\bar{r}^\prime(\lambda)|^2 \frac{\hat{a}'(\lambda + \rho)(\lambda + \rho)}{\hat{a}(\lambda + \rho)} \right] + |r_*(\rho)| \frac{\hat{a}(\lambda + \rho)}{\lambda^2} \leq M_9 |\hat{b}(\lambda + \rho)|,$$

where the constant $M_9 = M_9(K) > 0$ is given by the expression $M_9 = [2C_{a_0} \hat{a}(\rho_{\max})]^2 k + \max_{\rho \in K} |r_*(\rho)| C_{a_0} \hat{a}(\xi_5) 2/\rho_{\min}]^2$. Keep in mind, that $\hat{a}(t) = 1/\hat{b}(t)$ is non-decreasing on $(0, \infty)$.

Let $\epsilon_0 \in (0, a_0), \rho \in K \subset \Sigma_{n/2+\epsilon_0}$ and $\lambda \in \Sigma_{n/2+\epsilon_0} - \rho$ with $\nu_4 \leq |\lambda + \rho| \leq \xi_5$. We set $G = \Sigma_{n/2+\epsilon_0} \cap \{B_{\xi_5}(0) \setminus B_{\nu_4}(0)\}$. For fixed $\rho \in C$, the function $\bar{r}^\prime(z - \rho) + r_*(\rho)/(z - \rho)^2$ is holomorphic on $\Sigma_{n/2+a_0}$. Furthermore, there is some constant $M_{10} = M_{10}(K, \epsilon_0, \delta_0, \nu_4, \xi_5) = M_{10}(K, \epsilon_0) > 0$ such that
4. The constant $C$ is given by $C' = C'_r(M, \varepsilon_0, \alpha_0) \max \left\{ e^{-\sin(\varepsilon_0)/2\pi}, z \right\}$.
5.3 Instability Theorem

**Theorem 5.3.1 (Instability Theorem).** Let $X$ be a complex Banach space and $a \in L_{1, \text{loc}}(\mathbb{R}_+) \setminus L_1(\mathbb{R}_+)$ be a unbounded, completely monotonic kernel. We denote by $b \in L_{1, \text{loc}}(\mathbb{R}_+)$ the corresponding completely monotonic Sonine kernel. Moreover, let the kernel $a$ be $\theta_\sigma$-sectorial with $\theta_\sigma \in (0, \pi/2]$. Furthermore, we suppose that

$$\int_0^1 \dot{a}(1/t) \frac{dt}{t} < \infty, \quad \int_0^1 \dot{b}(1/t) \frac{dt}{t} < \infty \quad \text{and} \quad \left[t \mapsto \min\{\dot{a}(1/t)/t, \dot{b}(1/t)/t^2\}\right] \in L_1(\mathbb{R}_+).$$

Moreover, let $A$ be a closed linear operator in $X$ with dense resolvent $\rho(A)$ and $\sigma(A) \cap \{1/\dot{a}(\lambda + 1/\dot{b})\} \subset \{1/\dot{a}(\lambda + 1/\dot{b})\} \subset \{1/\dot{a}(\lambda + 1/\dot{b})\}$ is a non-empty compact set and $\sigma_- = \sigma(A) \setminus \sigma_+$ is a closed subset of $\mathbb{C}$.

We denote by $P_+$ the spectral projection of the operator $A$ associated with the compact spectral set $\sigma_+$. Let $A_+ = \rho(A)$ be the part of $A$ in $\text{Rg}(\text{Id} - P_+)$ with $\sigma(A) = \sigma_-$. We assume for some $q > \theta_\sigma$ that $\rho(-A_+) \supset \Sigma_q \cup \{0\}$ and $\|\mu + A_+\|_{\text{B}(X_+)} \leq M/|\mu|$ for all $\mu \in \Sigma_q$. Suppose that $h \in C([-T, T])$ with $h(0) = 0$, $h'(0) = 0$.

Then, the equilibrium $w_0 = 0$ is unstable for the Volterra equation

$$w(t) + (a \ast Aw)(t) = w_0 + (a \ast h(w))(t), \quad t \in [0, T], \quad (5.6)$$

where $w_0 \in X$.

**Proof.** We use the same notation as introduced in Section 5.1. By the spectral projection $P_+$ of the operator $A$ associated with the compact spectral set $\sigma_+$ we get a decomposition of the space $X = X_+ \oplus X_-$, see Remark 5.1.2. The operator $A_+ \in \text{B}(X_+)$ denotes the part of $A$ in $\text{Rg}P_+$.

We consider for $J = [0, T]$ and $f_\pm \in C(J; X_\pm)$ the Volterra equations on $X_\pm$

$$w_\pm(t) + (a \ast A_\pm w_\pm)(t) = f_\pm(t), \quad t \in J.$$

The boundedness of the operator $A_+ \in \text{B}(X_+)$ yields the existence of the resolvent $\{S_+(t)\}_{t \in \mathbb{R}_+}$ and the integral resolvent $\{R_+(t)\}_{t \in \mathbb{R}_+}$ on $X_+$ given by Dunford's integral representation, see Corollary 5.1.7.

The assumptions on the operator $A_-$ yield the parabolicity of the corresponding Volterra equation on $X_-$. Together with the completely monotonicity of the kernel $a \in L_{1, \text{loc}}(\mathbb{R}_+)$ this implies by Theorem 3.3.1 that there exists a bounded resolvent $\{S_-(t)\}_{t \in \mathbb{R}_+} \in \text{B}(X_-)$ on $X_-$, thus $\|S_-(t)\|_{\text{B}(X_-)} \leq M_-$, $t \in \mathbb{R}_+$, with $M_- \geq 1$. Moreover, the integrability conditions $\dot{a}$ and $\dot{b}$ yield the existence of an integrable integral resolvent $\{R_-(t)\}_{t \in \mathbb{R}_-}$ on $X_-$, so $\|R_-(t)\|_{L_1(\mathbb{R}_-, \text{B}(X_-))} = N_-$ with some $N_- > 0$.

In particular, the resolvent $\{S(t)\}_{t \in \mathbb{R}_+}$ and integral resolvent $\{R(t)\}_{t \in \mathbb{R}_+}$ for the linear Volterra equation

$$w(t) \ast (a \ast Aw)(t) = f(t), \quad t \in J,$$
5.3. Instability Theorem

where \( f \in C(J; X) \), exist and they are given by

\[
S(t) = S_+(t) + S_-(t), \quad t \in \mathbb{R}_+;
\]
\[
R(t) = R_+(t) + R_-(t), \quad t \in \mathbb{R}_+.
\]

We assume, that the equilibrium \( w_0 = 0 \) is stable of the Volterra equation (5.6), i.e. for each \( \varepsilon > 0 \) there is some \( \delta = \delta(\varepsilon) > 0 \) such that for all \( w_0 \in B_\delta(0) \) there exists a mild solution \( w = w(\cdot; w_0) \) of the Volterra equation (5.6) on \( \mathbb{R}_+ \) and \( \|w\|_{C(\mathbb{R}_+ ; X)} \leq \varepsilon \).

Let \( \varepsilon > 0 \) be arbitrary, but fixed. We consider for the corresponding parameter \( \delta = \delta(\varepsilon) > 0 \) some \( w_0 \in B_\delta(0) \). The mild solution \( w = w(\cdot; w_0) \) of the Volterra equation (5.6) is given as solution of

\[
w(t) = S(t)w_0 + (R \ast h(w))(t), \quad t \in \mathbb{R}_+.
\]

The spectral projection \( P_+ \) yields the decomposition \( w(t) = P_+w(t) + P_-w(t) = v(t) + u(t) \). Whereupon \( u \) and \( v \) solve the equations

\[
v(t) = S_+(t)v_0 + (R_+ \ast h(u + v))(t), \quad t \in \mathbb{R}_+ \text{ on } X_+;
\]
\[
u(t) = S_-(t)u_0 + (R_- \ast h(u + v))(t), \quad t \in \mathbb{R}_+ \text{ on } X_-.
\]

with \( w_0 = P_+w_0 + P_-w_0 = v_0 + u_0 \).

We know there is some bounded set \( \Omega \subset \{-1/\hat{a}(\mathbb{C}_+)\} \subset -\Sigma_{\partial_+} \) such that \( \sigma_+ \subset \Omega \). We denote by \( \gamma \) the boundary of \( \Omega \) and assume that it is sufficiently smooth, cf. Remark 5.1.2, as well as \( \text{dist}(0, \gamma) > 0 \).

Firstly, we consider the expression

\[
x = \frac{1}{2\pi i} \int_\gamma (\mu - A_+)^{-1} \left[ \int_0^\infty \left( -\frac{1}{\mu} \right) \hat{a}^{-1}(1+1/\mu) e^{-\hat{a}^{-1}(1+1/\mu) t} h(w(t)) \, dt \right] \, d\mu
\]

\[
= \frac{1}{2\pi i} \int_\gamma (\mu - A_+)^{-1} g(\mu) \, d\mu;
\]

where

\[
g(\mu) = \int_0^\infty \left( -\frac{1}{\mu} \right) \hat{a}^{-1}(1+1/\mu) e^{-\hat{a}^{-1}(1+1/\mu) t} h(w(t)) \, dt.
\]

By definition of \( x \) it follows that \( x \in X_+ \). Note that we can change the order of integration, since the path \( \gamma \) is compact and thus the continuous map \( h(w) \) is bounded along the path \( \gamma \) with \( \text{dist}(0, \gamma) > 0 \). One convince oneself easily, that \( g: -1/\hat{a}(\mathbb{C}_+) \to X \) is a holomorphic map, because \( \hat{a}^{-1}: \hat{a}(\mathbb{C}_+) \to \mathbb{C}_+ \) is holomorphic as inverse function of the bijective holomorphic map \( \hat{a}: \mathbb{C}_+ \to \hat{a}(\mathbb{C}_+) \) with \( \hat{a}'(\lambda) \neq 0 \) on \( \mathbb{C}_+ \). In particular, we have some bound

\[
\|x\|_X \leq \frac{1}{2\pi} \sup_{\mu \in \gamma} \left\| (\mu - A_+)^{-1} \right\|_{B(X)} \|g(\mu)\|_X \mathcal{L}(\gamma) < \infty,
\]

where \( \mathcal{L}(\gamma) < \infty \) denotes the length of the closed path \( \gamma \).
5. Instability for Semilinear Parabolic Volterra Equations

Now, we take another bounded set \( \Omega' \subset -1/\partial(C_+) \subset -\Sigma \Theta \) such that \( \Omega' \supset \Omega \). We denote by \( \gamma' \) the boundary of \( \Omega' \) and assume that it is sufficiently smooth with \( \text{dist}(0, \gamma') > 0 \), too. Using Dunford’s integral representation, for each \( t \in \mathbb{R}_+ \) the resolvent \( S_+ (t) \) is given by

\[
S_+ (t) = \frac{1}{2\pi i} \int_{\gamma'} s_{\mu'} (t) (\mu' - A_+) \mu' d\mu,
\]

see Corollary 5.1.7. Due to the choice of the paths \( \gamma \) and \( \gamma' \) as well as the resolvent identity and Cauchy’s integral formula it follows that

\[
S_+ (t)x = \frac{1}{2\pi i} \int_{\gamma'} s_{\mu'} (t) (\mu' - A_+) \mu' g(\mu) d\mu \quad (5.7)
\]

Using Dunford’s integral representation for the integral resolvent \( R_+ (t), t \in \mathbb{R}_+ \), yields

\[
(R_+ \ast h(w))(t) = \int_0^t \left[ \int_{\gamma'} r_\mu (t - \tau) (\mu - A_+) \mu g(\mu) d\mu \right] h(w(\tau)) d\tau
\]

\[
= \int_{\gamma'} (\mu - A_+) g(\mu) \left[ \int_0^t r_\mu (t - \tau) h(w(\tau)) d\tau \right] d\mu;
\]

note that we can change the order of integration, since \( \gamma \) is a compact path and therewith \( h(w) \) is bounded along this path, and \( r_\mu \in L_{1, \text{loc}}(\mathbb{R}_+) \).

Now, we consider the component \( v \) of the mild solution on \( X_+ \). For each \( t \in \mathbb{R}_+ \) we insert a zero in terms of \( S_+ (t)x \), given by (5.7), and \( -S_+ (t)x \), given by (5.8), where we split the integral in the definition of \( g \) at the point \( t \). Hence, we obtain for all \( t \in \mathbb{R}_+ \) that

\[
(2\pi i) v(t)
\]

\[
= \int_{\gamma'} s_{\mu'} (t) (\mu' - A_+) \mu' \left[ v_0 + \frac{1}{2\pi i} \int_{\gamma'} (\mu - A_+) \mu g(\mu) d\mu \right] d\mu'
\]

\[
+ \int_{\gamma'} (\mu - A_+) \mu g(\mu) \left[ \int_0^t \left\{ r_\mu (t - \tau) - s_{\mu} (t) \left\{ -\frac{1}{\mu} \hat{\alpha}^{-1} (-1/\mu) e^{-\hat{\alpha}^{-1} (-1/\mu) \tau} \right\} h(w(\tau)) d\tau \right. \right. d\mu
\]

\[
\left. \left. - \int_t^\infty s_{\mu} (t) \left\{ -\frac{1}{\mu} \hat{\alpha}^{-1} (-1/\mu) e^{-\hat{\alpha}^{-1} (-1/\mu) \tau} h(w(\tau)) d\tau \right\} d\mu. \right] \right]
\]

Now, we substitute \( \rho = \hat{\alpha}^{-1} (-1/\mu) \) in the integral along the contour \( \gamma \). This yields a unique path \( \Gamma = \hat{\alpha}^{-1} (-1/\gamma) \subset C_+ \) with \( \text{dist}(0, \Gamma) > 0 \). Thus, we have for all \( t \in \mathbb{R}_+ \) that
\[(2\pi i)\nu(t)\]
\[= \int_Y s_{\mu}'(t)(\mu - A_+)^{-1} \left[ v_0 + \frac{1}{2\pi i} \int_Y (\mu - A_+)^{-1} g(\mu) d\mu \right] d\mu' \]
\[- \int_{\Gamma} (1/\hat{\alpha}(\rho) + A_+)^{-1} \left\{ \int_0^t \left[ r_{-1/\hat{\alpha}(\rho)}(t - \tau) - s_{-1/\hat{\alpha}(\rho)}(t)\right] \rho \hat{\alpha}(\rho)e^{-\rho t} \right\} h(w(\tau)) d\tau \]
\[- \int_t^\infty s_{-1/\hat{\alpha}(\rho)}(t)\rho \hat{\alpha}(\rho)e^{-\rho t} h(w(\tau)) d\tau \]
\[\hat{\alpha}'(\rho) d\rho.\]

In our next step we will show that the integral over \(\Gamma\) is uniformly bounded for all \(t \in \mathbb{R}_+\). Since the map \(h\) is continuously differentiable with \(h(0) = 0\) and \(h'(0) = 0\), we have for all \(\sigma > 0\) that there exists \(\eta = \eta(\sigma) > 0\) such that for all \(z \in B_\eta(0)\) we have \(\|h(z)\|_X \leq \sigma \|z\|_X\). For sufficiently small \(\sigma > 0\) we choose \(\epsilon = \eta(\sigma)\) and \(\delta = \delta(\epsilon) > 0\) according to the stability property of the equilibrium \(w_* = 0\). Hence, we have for all \(w_0 \in B_\delta(0)\) that \(\|w\|_{C(\mathbb{R}_+,X)} \leq \epsilon\), thus \(\|h(w)\|_{C(\mathbb{R}_+,X)} \leq \sigma\|w\|_{C(\mathbb{R}_+,X)} \leq \sigma\eta\). Consequently, we deduce for all \(\rho \in \Gamma\) and all \(t \in \mathbb{R}_+\) that

\[\left\| \int_0^t \left[ r_{-1/\hat{\alpha}(\rho)}(t - \tau) - \rho \hat{\alpha}(\rho)e^{-\rho t} s_{-1/\hat{\alpha}(\rho)}(t) \right] h(w(\tau)) d\tau \right\|_X \]
\[\leq \sigma\eta \int_0^t \left| r_{-1/\hat{\alpha}(\rho)}(t - \tau) - \rho \hat{\alpha}(\rho)e^{-\rho t} s_{-1/\hat{\alpha}(\rho)}(t) \right| d\tau.\]

Now, we insert for all \(\tau \in [0,t]\) the zero \(e^{\rho(t-\tau)} [r_*(\rho) - \rho \hat{\alpha}(\rho)s_*(\rho)] = 0\), where \(s_*(\rho)\) and \(r_*(\rho)\) the limits from Lemma 5.2.2. We obtain

\[\left| r_{-1/\hat{\alpha}(\rho)}(t - \tau) - \rho \hat{\alpha}(\rho)e^{-\rho t} s_{-1/\hat{\alpha}(\rho)}(t) \right| \]
\[\leq \left| r_{-1/\hat{\alpha}(\rho)}(t - \tau) - e^{\rho(t-\tau)} r_*(\rho) \right| + \left| \rho \hat{\alpha}(\rho) \right| \left| s_*(\rho)e^{\rho t} - s_{-1/\hat{\alpha}(\rho)}(t) \right| e^{-\rho t}.\]

In combination with an easy substitution we get the estimate

\[\left\| \int_0^t \left[ r_{-1/\hat{\alpha}(\rho)}(t - \tau) - \rho \hat{\alpha}(\rho)e^{-\rho t} s_{-1/\hat{\alpha}(\rho)}(t) \right] h(w(\tau)) d\tau \right\|_X \]
\[\leq \sigma\eta \int_0^t \left| r_{-1/\hat{\alpha}(\rho)}(t - \tau) - e^{\rho t} r_*(\rho) \right| d\tau + \left| \rho \hat{\alpha}(\rho) \right| \int_0^t \left| s_*(\rho)e^{\rho t} - s_{-1/\hat{\alpha}(\rho)}(t) \right| e^{-\rho t} d\tau.\]

Since \(\Gamma\) is a compact set in \(\mathbb{C}_+\), by Lemma 5.2.4, there are uniform constants \(C_+, M_+ > 0\) such that for all \(\rho \in \Gamma\) and all \(t \in \mathbb{R}_+\) we have the estimates

\[\left| s_{-1/\hat{\alpha}(\rho)}(t) - e^{\rho t} s_*(\rho) \right| \leq M_+,\]
\[\left| r_{-1/\hat{\alpha}(\rho)}(t) - e^{\rho t} r_*(\rho) \right| \leq C_+ \min \left\{ \frac{\hat{\alpha}(1/t)}{t}, \frac{\hat{b}(1/t)}{t^2} \right\},\]

5.3. Instability Theorem
we define \( N_+ = \| t \mapsto C_+ \min \{ \hat{a}(1/t), \hat{b}(1/t)/t^2 \} \|_{L^1(\mathbb{R}_+)} \). Thus, we have for all \( t \in \mathbb{R}_+ \) that

\[
\left\| \int_0^t \left[ r_{-1/\hat{a}(\rho)}(t - \tau) - \rho \hat{a}(\rho)e^{-\rho \tau} s_{-1/\hat{a}(\rho)}(t) \right] h(w(\tau)) \, d\tau \right\|_X
\leq \sigma \eta \left[ N_+ + \left| \rho \hat{a}(\rho) \right| \frac{M_+}{\text{Re} \rho} \right] \leq \sigma \eta M_0,
\]

with some uniform constant \( M_0 = N_+ + \sup_{\rho \in \Gamma} \left| \rho \hat{a}(\rho) \right| \frac{M_+}{\text{Re} \rho} \), for all \( \rho \in \Gamma \) and all \( t \in \mathbb{R}_+ \); note that \( \text{dist}(\Gamma, 0) > 0 \).

Furthermore, we have by the differentiability of the map \( h \) the following estimate for each \( t \in \mathbb{R}_+ \)

\[
\left\| \int_t^\infty s_{-1/\hat{a}(\rho)}(t) \rho \hat{a}(\rho)e^{-\rho \tau} h(w(\tau)) \, d\tau \right\|_X
\leq \sigma \eta \left| \rho \hat{a}(\rho) \right| \int_t^\infty \left| e^{-\rho(\tau-t)} \left[ e^{-\rho t} s_{-1/\hat{a}(\rho)}(t) \right] \right| \, d\tau.
\]

Using the relation

\[
\left| e^{-\rho t} s_{-1/\hat{a}(\rho)}(t) \right| \leq \left| e^{-\rho t} \left[ s_{-1/\hat{a}(\rho)}(t) - e^{\rho t} s_*(\rho) \right] \right| + \left| s_*(\rho) \right| \leq M_+ + \sup_{\rho \in \Gamma} \left| s_*(\rho) \right| = M_1,
\]

which is uniform in \( \rho \in \Gamma \subset C_+ \) and valid for all \( t \in \mathbb{R}_+ \). Hence we conclude for each \( t \in \mathbb{R}_+ \) that

\[
\left| \int_t^\infty s_{-1/\hat{a}(\rho)}(t) \rho \hat{a}(\rho)e^{-\rho \tau} h(w(\tau)) \, d\tau \right|
\leq \sigma \eta M_1 \left| \rho \hat{a}(\rho) \right| \int_t^\infty \left| e^{-\rho(\tau-t)} \right| \, d\tau = \sigma \eta \left| \rho \hat{a}(\rho) \right| \frac{M_1}{\text{Re} \rho} \leq \sigma \eta M_2,
\]

with some uniform constant \( M_2 = \sup_{\rho \in \Gamma} \left| \rho \hat{a}(\rho) \right| \frac{M_1}{\text{Re} \rho} \), since \( \text{dist}(\Gamma, 0) > 0 \).

This shows for all \( t \in \mathbb{R}_+ \) that the expression

\[
\int_\Gamma (1/\hat{a}(\rho) + A_+)^{-1} \left[ \int_0^t \left( r_{-1/\hat{a}(\rho)}(t - \tau) - s_{-1/\hat{a}(\rho)}(t) \rho \hat{a}(\rho)e^{-\rho \tau} \right) h(w(\tau)) \, d\tau 
- \int_t^\infty s_{-1/\hat{a}(\rho)}(t) \rho \hat{a}(\rho)e^{-\rho \tau} h((u + v)(\tau)) \, d\tau \right] \frac{\hat{a}(\rho)}{\hat{a}(\rho)^2} \, d\rho
\]

is uniformly bounded by
where as well as the resolvent identity and Cauchy's integral formula it follows that

\[
\left\| \int_{\Gamma} \left( \frac{1}{\dot{a}(\rho)} + A_+ \right)^{-1} \left[ \int_0^t \left\{ r_{-1/\dot{a}(\rho)}(t - \tau) - s_{-1/\dot{a}(\rho)}(t) \rho \dot{a}(\rho) e^{-\rho \tau} \right\} h(w(\tau)) d\tau \right.ight.
\]

\[
- \int_t^\infty s_{-1/\dot{a}(\rho)}(t) \rho \dot{a}(\rho) e^{-\rho \tau} h((u + v)(\tau)) d\tau \left. \right\|_{\mathcal{X}} \right.
\]

\[
\leq \sigma \eta [M_0 + M_2] K < \infty,
\]

with \( K = \sup_{\rho \in \Gamma} \left\| (1/\dot{a}(\rho) + A_+)^{-1} \right\|_{\mathcal{B}(\mathcal{X})} \left\| \frac{\dot{a}(\rho)}{\mu} \right\|_{\mathcal{L}(\Gamma)} \), where \( \mathcal{L}(\Gamma) < \infty \) denotes the length of the closed contour \( \Gamma \).

Since the mild solution \( w \) is assumed to be bounded, its components \( u \) and \( v \) have to be bounded for all \( t \in \mathbb{R}^+ \) as well. Thus the expression

\[
S_+(t) \left[ v_0 + x \right] = \frac{1}{2\pi i} \int_{\gamma} s_\mu(t)(\mu' - A_+)^{-1} \left[ v_0 + \frac{1}{2\pi i} \int_{\gamma} (\mu - A_+)^{-1} g(\mu) d\mu \right] d\mu'
\]

has to be bounded in \( X_+ \). Indeed, assuming that there is some constant \( M > 0 \) such that for all \( t \in \mathbb{R}^+ \) we have \( \|S_+(t) [v_0 + x]\|_{X_+} \leq M \). We take a bounded set \( \Omega'' \subset -1/\dot{a}(C_+) \subset -\Sigma_{\dot{a}} \) such that \( \Omega'' \supset \Omega' \) and denote by \( \gamma'' \) the boundary of \( \Omega'' \). For each \( \mu \in \Omega'' \) there is some \( \rho \in K \subset \Sigma_{n/2+\delta_0} \cap (\overline{B}_{\rho_{\max}}(0) \setminus B_{\rho_{\min}}(0)) \), where \( \delta_0 \in (0, \pi/2) \) and \( 0 < \rho_{\min} \leq \rho_{\max} \) are suitable parameters, such that \( \mu = -1/\dot{a}(\rho) \). By Lemma 5.2.4 there is some uniform constant \( C > 0 \) such that for each \( \mu \in \Omega'' \) and all \( t \geq t_0 = \ln(2C/M)/\rho_{\min} \) we have the estimate

\[
\left| s_\mu(t) \right| \geq e^{\rho_{\min} t} \frac{M}{2} > 0,
\]

where \( M = \min_{\rho \in K} |s_\mu(\rho)| > 0 \).

Hence, for \( t \geq t_0 \) the expression

\[
F(t) = \frac{1}{2\pi i} \int_{\gamma''} \frac{1}{s_\mu(t)(\mu'' - A_+)^{-1} d\mu''} \in \mathcal{B}(X_+),
\]

is well-defined and we have the estimate

\[
\|F(t)\|_{\mathcal{B}(X_+)} \leq \frac{e^{\rho_{\min} t}}{M\pi} \sup_{\mu'' \in \gamma''} \| (\mu'' - A_+)^{-1} \|_{\mathcal{B}(X_+)} \mathcal{L}(\gamma'') < \infty,
\]

where \( \mathcal{L}(\gamma'') < \infty \) denotes the length of the closes path \( \gamma'' \). In particular, we have \( \|F(t)\|_{\mathcal{B}(X_+)} \to 0 \) as \( t \to \infty \). Due to the choice of the paths \( \gamma' \) and \( \gamma'' \) as well as the resolvent identity and Cauchy's integral formula it follows that

\[
F(t)S_+(t) = \frac{1}{2\pi i} \int_{\gamma'} \frac{1}{s_\mu(t)(\mu'' - A_+)^{-1} d\mu''} \left[ \frac{1}{2\pi i} \int_{\gamma'} s_\mu(t)(\mu' - A_+)^{-1} d\mu' \right] d\mu''
\]

\[
= \frac{1}{2\pi i} \int_{\gamma'} (\mu' - A_+)^{-1} d\mu' = P_+.
\]
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This implies on one hand that
\[ \|F(t)S_+(t)[v_0 + x]\|_{X_+} \leq \|F(t)\|_{B(L^2)} \|S_+(t)[v_0 + x]\|_{X_+} \]
\[ \leq \|F(t)\|_{B(L^2)} M \to 0 \]
as \( t \to \infty \), and on the other hand that
\[ F(t)S_+(t)[v_0 + x] = P_+[v_0 + x] = v_0 + x, \]
since \( v_0 + x \in X_+ \). So, we conclude \( v_0 + x = 0 \).

Consequently, we have
\[ -(2\pi i)v(t) = \int_\Gamma (1/\hat{\alpha}(\rho) + A_+)^{-1} \left[ \int_0^t \{ r_{-1/\hat{\alpha}(\rho)}(t - \tau) - \rho \hat{\alpha}(\rho)e^{-\rho t} s_{-1/\hat{\alpha}(\rho)}(\tau) \} h((u + v)(\tau))d\tau \right. \]
\[ \left. - \int_0^\infty \rho \hat{\alpha}(\rho)e^{-\rho t} s_{-1/\hat{\alpha}(\rho)}(\tau) h((u + v)(\tau))d\tau \right] \frac{\hat{\alpha}'(\rho)}{\hat{\alpha}(\rho)^2} d\rho. \]

This implies for all \( t \in \mathbb{R}_+ \) the estimate
\[ \|v(t)\|_X \leq \frac{\sigma K}{2\pi} \sup_{\rho \in \Gamma} \left\{ \int_0^t \left| r_{-1/\hat{\alpha}(\rho)}(t - \tau) - \rho \hat{\alpha}(\rho)e^{-\rho t} s_{-1/\hat{\alpha}(\rho)}(\tau) \right| \|h((u + v)(\tau))\|_X d\tau \right. \]
\[ + \left| \rho \hat{\alpha}(\rho) \right| \int_0^\infty |e^{\rho t - \tau}| e^{-\rho t} s_{-1/\hat{\alpha}(\rho)}(\tau) \|h((u + v)(\tau))\|_X d\tau \right] \]
\[ \leq \frac{\sigma K}{2\pi} \left[ M_0 + M_2 \right] \sup_{\tau \in \mathbb{R}_+} \left\{ \|u(\tau)\|_X + \|v(\tau)\|_X \right\}. \quad (5.9) \]

Now, we consider the component \( u \) of the mild solution on \( X_- \). The boundedness of the resolvent \( \{\mathcal{S}_-(\tau)\}_{\tau \in \mathbb{R}_-} \), the integrability of the integral resolvent \( \{\mathcal{R}_-(\tau)\}_{\tau \in \mathbb{R}_-} \) on \( X_- \) and the differentiability of the map \( h \) yield for all \( t \in \mathbb{R}_+ \) that
\[ \|u(t)\|_X \leq \|\mathcal{S}_-(\cdot)\|_{B(L^2)} \|u_0\|_X + \int_0^t \|\mathcal{R}_-(\tau)\|_{B(L^2)} \|h((u + v)(\tau))\|_X d\tau \]
\[ \leq M_- \|u_0\|_X + \sigma N_- \sup_{\tau \in \mathbb{R}_-} \left\{ \|u(\tau)\|_X + \|v(\tau)\|_X \right\}. \quad (5.10) \]

Combining the estimates for the component \( v \) from (5.9) and for the component \( u \) from (5.10), we get for each \( t \in \mathbb{R}_+ \) the uniform estimate
\[ \|u(t)\|_X + \|v(t)\|_X \leq M_- \|u_0\|_X + \sigma \left[ N_- + \frac{K}{2\pi} \left[ M_0 + M_2 \right] \right] \sup_{\tau \in \mathbb{R}_-} \left\{ \|u(\tau)\|_X + \|v(\tau)\|_X \right\}. \]

We set \( C = N_- + \frac{K}{2\pi} [M_0 + M_2] \) and we choose \( \sigma > 0 \) sufficiently small, such that \( \sigma C < 1/2 \), hence we have for all \( t \in \mathbb{R}_+ \) the estimate
\[ \sup_{\tau \in \mathbb{R}_-} \left\{ \|u(\tau)\|_X + \|v(\tau)\|_X \right\} \leq 2M_- \|u_0\|_X , \]
5.3. Instability Theorem

in particular we have \( \|u(0)\|_X + \|v(0)\|_X \leq 2M_-\|u_0\| \) and therewith \( \|v(0)\|_X \leq (2M_- - 1)\|u_0\|_X \). This is a contradiction to the stability of the equilibrium \( u_* = 0 \), since we have a restriction to the initial values inside the ball \( B_\delta(0) \). It the claim follows.

In case of the standard kernel the previous theorem reads as follows.

**Corollary 5.3.2.** Let \( X \) be a complex Banach space, \( \alpha \in (0, 1) \) and \( A \) be a closed linear operator in \( X \) with dense domain such that \( \sigma(-A) \cap \partial \Sigma_{\alpha \frac{\pi}{2}} = \emptyset \) and \( -\sigma_+ = \sigma(-A) \cap \Sigma_{\alpha \frac{\pi}{2}} \) is a non-empty compact set and \( \sigma_- = \sigma(A) \setminus \sigma_+ \) is a closed set in \( \mathbb{C} \).

We denote by \( P_\pm \) the spectral projection of the operator \( A \) associated with the compact spectral set \( \sigma_\pm \). Let \( \Lambda_- \) be the part of \( \sigma(A) \) in \( \text{Rg}(\text{Id} - P_-) \) with \( \sigma(\Lambda_-) = \sigma_- \). We assume that \( \Lambda_- \) is an invertible sectorial operator with spectral angle \( \varphi_{\Lambda_-} < \pi - \alpha \frac{\pi}{2} \).

Moreover, suppose that \( f \in C^1(X; X) \) with \( f(0) = 0, f'(0) = 0 \).

Then, the equilibrium \( u_* = 0 \) is unstable of the Volterra equation with standard kernel

\[
\begin{align*}
    u(t) + (g_\alpha * Au)(t) &= u_0 + (g_\alpha * f(u))(t), \quad t \in [0, T],
\end{align*}
\]

where \( u_0 \in X \).

The following corollary is an easy consequence for Volterra equations with standard kernel in the finite dimensional case.

**Corollary 5.3.3.** Let \( \alpha \in (0, 1) \). Suppose that the spectrum of the matrix \( A \in \mathbb{R}^{n \times n} \) be such that \( \sigma(-A) \cap \partial \Sigma_{\alpha \frac{\pi}{2}} = \emptyset \) and there is at least one eigenvalue \( \lambda_0 \in \sigma(A) \) with \( |\arg \lambda_0| > \pi - \alpha \frac{\pi}{2} \). Furthermore, let \( f \in C^1(\mathbb{R}^n; \mathbb{R}^n) \) with \( f(0) = 0 \) and \( f'(0) = 0 \).

Then, the equilibrium \( u_* = 0 \) is unstable for the Volterra equation

\[
\begin{align*}
    u(t) + (g_\alpha * Au)(t) &= u_0 + (g_\alpha * f(u))(t), \quad t \in [0, T],
\end{align*}
\]

where \( u_0 \in \mathbb{R}^n \).
Part III

Quasilinear Parabolic
Fractional Evolution Equations
Maximal $L_p$-Regularity

The goal of the last part of this thesis are stability studies of quasilinear fractional evolution equations. A possible approach for quasilinear problems is the use of the property of maximal $L_p$-regularity for the linearised problem. For this reason we explain in this chapter the necessary underlying objects, see Section 6.1, and introduce the concept of maximal $L_p$-regularity for linear fractional evolution equations, see Section 6.2. We refer to Zacher’s work [Zac03] that all these considerations keep true if one replace the time-fractional evolution equation with the standard kernel by a wider class of kernels for which also the concept of maximal $L_p$-regularity is known.

6.1 Preliminaries

As in the classical case certain geometrical properties of the underlying Banach spaces and operators are relevant for the considerations about the maximal $L_p$-regularity of the equation (6.1). We introduce these notions.

Definition 6.1.1 (Banach space of class $\mathcal{H}T$). A Banach space $X$ is said to belong to the class $\mathcal{H}T$, if the Hilbert transform is bounded on $L^p(\mathbb{R};X)$ for some (and then all) $p \in (1, \infty)$. Here the Hilbert transform $Hf$ of a function $f \in S(\mathbb{R};X)$, the Schwartz space of rapidly decreasing $X$-valued functions, is defined by

$$(Hf)(t) = \lim_{R \to \infty} \int_{R^{-1} \leq |s| \leq R} f(t-s) \frac{ds}{\pi s}, \quad t \in \mathbb{R}.$$ 

Remark 6.1.2. a) We refer to Burkholder [Bur83] and Bourgain [Bou83] for the proof that the class of Banach spaces of class $\mathcal{H}T$ coincides with the class of UMD spaces, where UMD is the abbreviation for unconditional martingale difference property.

b) Each Banach space of class $\mathcal{H}T$ is super-reflexive and all Hilbert spaces belong to the class of $\mathcal{H}T$. If $(\Omega, \mathcal{A}, \mu)$ is a $\sigma$-finite measure space, $p \in (1, \infty)$, then $L^p(\Omega;X)$ is of class $\mathcal{H}T$ if $X$ has this property. Moreover, closed subspaces, quotients, duals and finite products are preserved, and complex interpolation spaces $(X,Y)_\theta$ as well as real interpolation spaces $(X,Y)_{\theta,p}$ are of class $\mathcal{H}T$, provided $X$ and $Y$ have this property and $\theta \in (0,1)$, $p \in (1,\infty)$. For proofs of these results, we refer to the survey article by Burkholder [Bur86] and Amann [Ama95].
Moreover, let \( \Omega \neq \emptyset \) be an open subset of \( \mathbb{R}^n \). Then for all \( m \in \mathbb{N}_0 \) the Sobolev spaces \( H^m_p(\Omega) \) are of class \( \mathcal{H}T \) if and only if \( p \in (1, \infty) \), see [Are04, Section 6.1.3 Example d)].

But of course not each Banach space is class \( \mathcal{H}T \). A counterexample is given by the well-known space of continuous functions \( C(\Omega) \), where \( \Omega \subset \mathbb{R}^n \) is a non-empty open set, cf. [Are04, Section 6.1.3 Example e)].

Furthermore, we are interested in the class of operators which are \( \mathcal{R} \)-sectorial. We start with the definition of \( \mathcal{R} \)-bounded families of bounded linear operators.

\textbf{Definition 6.1.3 (\( \mathcal{R} \)-Boundedness).} Let \( X \) and \( Y \) be Banach spaces. A family of operators \( T \subset B(X,Y) \) is called \( \mathcal{R} \)-bounded, if there is a constant \( C > 0 \) and \( p \in [1, \infty) \) such that for each \( N \in \mathbb{N} \), \( T_j \in T \), \( x_j \in X \) and for all independent, symmetric, \( \{-1,1\} \)-valued random variables \( \varepsilon_j \) on a probability space \((\Omega,A,\mu)\) the inequality

\[
\left\| \sum_{j=1}^N \varepsilon_j T_j x_j \right\|_{L_p(\Omega;Y)} \leq C \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|_{L_p(\Omega;X)}
\]

is valid. The smallest such \( C \) is called \( \mathcal{R} \)-bound of \( T \), we denote it by \( \mathcal{R}(T) \).

\textbf{Remark 6.1.4.}  
1. Consider \( \Omega = [0,1] \), \( A \) the Borel sets in \([0,1]\) and \( \mu \) the Lebesgue measure. Then the \textit{Rademacher functions} \( r_k(t) = \text{sgn}(\sin(2^k \pi t)) \), \( k \in \mathbb{N} \), are the prototype for the random variables \( \varepsilon_k \), cf. [PS16, Example 4.1.2].

   By means of the notion of \( \mathcal{R} \)-boundedness stochastic analysis is introduced in operator theory.

2. The definition of \( \mathcal{R} \)-boundedness is independent of \( p \in [1, \infty) \). If the operator family \( T \subset B(X,Y) \) is \( \mathcal{R} \)-bounded, then it is uniformly bounded, \( \sup\|T\|_{B(X,Y)} : T \in T \leq \mathcal{R}(T) \). If \( X \) and \( Y \) are Hilbert spaces, then the \( \mathcal{R} \)-boundedness of the operator family \( T \subset B(X,Y) \) is equivalent to its uniform boundedness. For a proof of these results, we refer to Prüss and Simonett [PS16, Remark 4.1.3].

With the aid of the definition of \( \mathcal{R} \)-boundedness we are able to define \( \mathcal{R} \)-sectorial operators by replacing \textit{bounded} with \( \mathcal{R} \)-bounded in the definition of sectorial operators, cf. Definition 3.2.3.

\textbf{Definition 6.1.5 (\( \mathcal{R} \)-Sectorial Operator).} Let \( X \) be a complex Banach space, and assume that \( A \) is a sectorial operator in \( X \). Then the operator \( A \) is called \( \mathcal{R} \)-sectorial if

\[
\mathcal{R}_A(0) = \mathcal{R}\{t(t+A)^{-1} : t > 0\} < \infty.
\]
The $\mathcal{R}$-angle $\varphi^R_A$ of the operator $A$ is defined by means of

$$\varphi^R_A = \inf\{\theta \in (0, \pi): \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) = \mathcal{R}[\lambda(\lambda + A)^{-1}: |\arg \lambda| < \theta].$$

The class of $\mathcal{R}$-sectorial operators in $X$ will be denoted by $\mathcal{RS}(X)$.

**Remark 6.1.6.**

a) The $\mathcal{R}$-angle of an $\mathcal{R}$-sectorial operator $A$ is well-defined and it is always not smaller than the spectral angle of $A$, see Denk, Hieber and Prüss [DHP03, p. 43].

b) A sufficient condition for $\mathcal{R}$-sectoriality is proved by Clément and Prüss in [CP01]. Let $X$ be a Banach space of class $\mathcal{H}T$ and let $A \in B\text{IP}(X)$ with power angle $\theta_A$. Then $A$ is $\mathcal{R}$-sectorial with $\varphi^R_A \leq \theta_A$.

A sectorial operator $A$ is said to admit bounded imaginary powers (abbreviation: $B\text{IP}$) if $A^{i\lambda}$ is in $B(X)$ for each $\lambda \in \mathbb{R}$, and there is a constant $C > 0$ such that $\|A^{i\lambda}\|_{B(X)} \leq C$ for all $|\lambda| \leq 1$. The class of such operators is denoted by $B\text{IP}(X)$. The power angle $\theta_A$ of $A$ is given by $\theta_A = \limsup_{|\lambda| \to \infty} \log \|A^{i\lambda}\|_{B(X)}/|\lambda|$.

c) Denk, Hieber and Prüss proved in [DHP03, Chapter II] that differential operators which are elliptic in a special sense and satisfy the Lopatinski-Shapiro condition on domains $G \subset \mathbb{R}^n$ with sufficiently smooth boundary are $\mathcal{R}$-sectorial operators on $L_p(G)$.

**The Vector-Valued Bessel-Potential Space $H^\alpha_p$.** For an arbitrary Banach space $X$ of class $\mathcal{H}T$ we define the derivation operator of second order $B_0$ in $L_p(\mathbb{R}; X)$, $p \in (1, \infty)$, with domain $D(B_0) = H^2_p(\mathbb{R}; X)$ by means of $(B_0u)(t) = -\partial_t^2 u(t), t \in \mathbb{R}, u \in D(B_0)$. We refer to Prüss [Prü12, p. 226] for the fact that $B_0 \in \mathcal{BIP}(L_p(\mathbb{R}; X))$ with power angle $\theta_{B_0} = 0$.

For $\alpha \in \mathbb{R}_+$ we define $H^\alpha_p(\mathbb{R}; X) = D(B_{0}^{\alpha/2})$. Then it is known, that for $m \in \mathbb{N}$ and $\alpha \in (0, 1)$ we have $H^{2m}_p(\mathbb{R}; X) = [L_p(\mathbb{R}; X); H^{2m}_p(\mathbb{R}; X)],$ the complex interpolation space of order $\alpha$, cf. [Prü12, p. 226]. For $I = [0, T]$ or $J = \mathbb{R}$, we follow Zacher [Zac05, p. 85] and put $H^\alpha_p(I; X) = \{f_I: f \in H^\alpha_p(\mathbb{R}; X)\}$, endowed with the norm $\|f\|_{H^\alpha_p(I; X)} = \inf\{\|g\|_{H^\alpha_p(\mathbb{R}; X)}: g_{|I} = f\}$. Furthermore, we set $\partial H^\alpha_p(I; X) = \{f_I: f \in H^\alpha_p(\mathbb{R}; X)\}$ and $supp f \subset \mathbb{R}_+$. These spaces are vector-valued Bessel-potential spaces.

For $\alpha \in (1/p, 1)$ one can prove that these spaces coincide with the definition of the vector-valued Bessel-potential spaces $H^\alpha_p(I; X) = [L_p(I; X); H^p(I; X)]_\alpha$ and $\partial H^\alpha_p(I; X) = [L_p(I; X); \partial H^p(I; X)]_\alpha$, respectively; the domains of the $\alpha$-th power of the time-derivative $\partial_t$ on $L_p(I; X)$ (with trace zero).
Moreover, for \( \alpha \in (0,1) \) the operator \( \partial_t^\alpha \) with domain \( \mathcal{D}(\partial_t^\alpha) = \mathcal{D}^\alpha \) given by
\[
\partial_t^\alpha u = \partial_t (g_{1-\alpha} \ast u), \quad u \in \mathcal{D}^\alpha,
\]
coincides with \( (\partial_t)^\alpha \), the derivation operator of (fractional) order \( \alpha \); \( \partial_t^\alpha u \) is called the fractional derivative of \( u \) of order \( \alpha \), cf. [Zac05, Example 2.1].

## 6.2 Maximal \( L_p \)-Regularity for Fractional Evolution Equations

Now, we are in the situation to introduce the concept of maximal \( L_p \)-regularity for linear fractional evolution equation.

Let \( X \) be a Banach space of class \( \mathcal{H}^T, \ p \in (1,\infty), \ \alpha \in (1/p,1) \) and let \( f : \mathbb{R}_+ \to X \). We assume that the operator \( A \in \mathcal{R}(X) \) with \( \mathcal{R} \)-angle \( \varphi_A^{R} < \pi - \alpha \frac{\pi}{2} \). We consider the fractional evolution equation
\[
\partial_t^\alpha u + Au = f, \quad t \in \mathbb{R}_+, \quad u(0) = 0,
\]
in the space \( L_p(\mathbb{R}_+;X) \) for \( p \in (1,\infty) \). For this purpose, we extend the operator \( A \) in the canonical way to \( L_p(\mathbb{R}_+;X) \) with natural domain \( L_p(\mathbb{R}_+;X_A) \).

We denote by \( J \) either a compact interval \([0,T], T > 0, \) or \( \mathbb{R}_+ \). The definition of maximal \( L_p \)-regularity for the equation (6.1) is as follows.

**Definition 6.2.1 (Maximal \( L_p \)-Regularity).** Suppose the operator \( A : \mathcal{D}(A) \subset X \to X \) is linear, closed and densely defined. Then the operator \( A \) has maximal \( L_p \)-regularity on \( J \) if for each \( f \in L_p(J;X) \) there exists a unique solution \( u \in \mathcal{D}^\alpha \cap L_p(J;X_A) \) satisfying the fractional evolution equation (6.1) almost everywhere in \( J \).

**Remark 6.2.2.**

(a) By the closed graph theorem one can show that there exists some constant \( C_{MR} > 0 \) such that for each \( f \in L_p(J;X_0) \) and the unique solution \( u \) of (6.1) we have
\[
\|u\|_{\mathcal{D}^\alpha \cap L_p(J;X_A)} \leq C_{MR} \|f\|_{L_p(J;X)}.
\]
Note that the constant \( C_{MR} \) can be chosen independently of \( J \).

(b) Via Sobolev embedding one can prove that the space \( \mathcal{D}^\alpha \cap L_p(J;X_A) \) is continuously embedded in the space \( C(J;X_\gamma) \) with the real interpolation space \( X_\gamma = (X,X_A)^{1-\frac{1}{p}} \). In particular, there is some constant \( C_{emb} > 0 \) such that for \( u \in \mathcal{D}^\alpha \cap L_p(J;X_A) \) we have
\[
\|u\|_{C(J;X_\gamma)} \leq C_{emb} \|u\|_{\mathcal{D}^\alpha \cap L_p(J;X_A)},
\]
where the embedding constant \( C_{emb} \) does not depend of \( J \) (here it is crucial that \( u \) is vanishing at \( t = 0 \)).
We have the following theorem about maximal $L_p$-regularity for fractional evolution equations. This theorem is a special cases of Theorem 3.1.1 considering Remark 3.1.1 in [Zac10].

**Theorem 6.2.3.** Let $X$ be a Banach space of class $\mathcal{H}(T, p \in (1, \infty)$ and $\alpha \in (1/p, 1)$. Let the operator $A$ be an invertible and $\mathcal{R}$-sectorial operator on $X$ with $\mathcal{R}$-angle $\varphi_A^R < \pi - \alpha \frac{\pi}{2}$.

Then the fractional evolution equation (6.1) has a unique solution in the space $\mathcal{R}_p(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; X_A)$ if and only if $f \in L_p(\mathbb{R}_+; X)$.

**Remark 6.2.4.** The theorem reads in the same way, if one replaces $\mathbb{R}_+$ by some compact interval $J = [0, T]$, with $T > 0$. In this situation the assumptions on the operator $A$ can be weakened. It suffices that the operator $\mu + A$ is $\mathcal{R}$-sectorial on $X$ with $\mathcal{R}$-angle $\varphi_{\mu + A}^R < \pi - \alpha \frac{\pi}{2}$, for some $\mu \geq 0$.

Now, we consider nontrivial initial values

$$\partial_t^\alpha (u - u_0) + Au = f, \quad t \in \mathbb{R}_+, \quad u(0) = u_0. \tag{6.2}$$

Using the notation $g_\alpha(t) = t^{\alpha - 1}/\Gamma(\alpha)$, $t \in \mathbb{R}_+$, we are able to rewrite (6.2) as an evolutionary integral equation

$$u + g_\alpha * Au = g_\alpha * f + u_0, \quad t \in \mathbb{R}_+. \tag{6.3}$$

We denote by $\{S(t)\}_{t \in \mathbb{R}_+}$ the resolvent of the Volterra equation (6.3). For the definition of the resolvent we refer to Section 3.1.

For a proof of the following result about the trace space, we refer to Prüss and Simonett [PS16, Proposition 4.5.14, p.190].

**Proposition 6.2.5 (Trace Space).** Let $X$ be a Banach space of class $\mathcal{H}(T, p \in (1, \infty), \alpha \in (1/p, 1)$, and suppose $A \in \mathcal{RS}(X)$ is invertible with $\varphi_A^R < \pi - \alpha \frac{\pi}{2}$. Let $\{S(t)\}_{t \in \mathbb{R}_+}$ denote the resolvent of (6.3).

Then we have $S(\cdot)x \in \mathcal{H}_p^\alpha(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; X_A)$ if and only if $x \in (X, X_A)_{1 - \frac{1}{mp}}$. The corresponding map $x \mapsto S(\cdot)x$ is continuous between the relevant spaces. In particular, we have a constant $C_1 > 0$ such that for all $x \in (X, X_A)_{1 - \frac{1}{mp}}$ we have

$$\|S(\cdot)x\|_{\mathcal{H}_p^\alpha(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; X_A)} \leq C_1 \|x\|_{(X, X_A)_{1 - \frac{1}{mp}}}.$$

From the preceding results we obtain the theorem about maximal $L_p$-regularity of fractional evolution equations with non-trivial initial values, cf. Theorem 4.5.15 in [PS16].

**Theorem 6.2.6 (Maximal $L_p$-Regularity).** Let $p \in (1, \infty)$, $\alpha \in (1/p, 1)$ and $X$ be a Banach space of class $\mathcal{H}(T, p \in (1, \infty)$ and $\alpha \in (1/p, 1)$ and $X$ be a Banach space of class $\mathcal{H}(T$. Suppose that $A \in \mathcal{RS}(X)$ is invertible, $\varphi_A^R + \alpha \frac{\pi}{2} < \pi$. Then (6.2) admits a unique solution $u \in \mathcal{H}_p^\alpha(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; X_A)$ if and only if $f \in L_p(\mathbb{R}_+; X)$ and $u_0 \in (X, X_A)_{1 - \frac{1}{mp}}$.
Remark 6.2.7. a) The maximal $L_p$-regularity theorem takes the same form if one replaces $\mathbb{R}_+$ by an interval $[0, T]$ with $T > 0$. In this case one can weaken the assumptions on the operator $A$; it suffices to assume that for some $\mu \geq 0$ we have $\mu + A \in \mathcal{R}\mathcal{S}(X)$ with $\Phi_{\mu + A}^2 < \pi - \alpha \frac{2}{T}$, see [Zac03, Remark 3.1.3].

b) We introduce the following notation

$$E_1(\mathbb{R}_+) = H^0_p(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; X),$$
$$0E_1(\mathbb{R}_+) = 0H^0_p(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; X),$$
$$E_0(\mathbb{R}_+) = L_p(\mathbb{R}_+; X).$$

In case of a finite interval $[0, T]$, we write

$$E_1(T) = H^0_p([0, T]; X) \cap L_p([0, T]; X),$$
$$0E_1(T) = 0H^0_p([0, T]; X) \cap L_p([0, T]; X),$$
$$E_0(T) = L_p([0, T]; X).$$

c) Let $f \in L_p(\mathbb{R}_+; X)$ be arbitrary, but fixed. We can decompose the solution $u$ of (6.2) as $u = v + w$, where $v$ solves the problem with zero initial value

$$\partial_t^\alpha v + Av = f, \quad t \in \mathbb{R}_+, \quad v(0) = 0,$$

and $w$ solves (6.2) with zero right hand side

$$\partial_t^\alpha (w - u_0) + Aw = 0, \quad t \in \mathbb{R}_+, \quad w(0) = u_0.$$

In particular, the solution $w$ is given as $w = S(.)u_0$, where, as usual, $\{S(t)\}_{t \in \mathbb{R}_+}$ denotes the resolvent of (6.3). Consequently, $\|w\|_{E_1(\mathbb{R}_+)}$ can be estimated with the aid of Proposition 6.2.5. Due to $v(0) = 0$ we have $\|v\|_{E_1(\mathbb{R}_+)} = \|v\|_{0E_1(\mathbb{R}_+)}$ and we can use the maximal $L_p$-regularity estimate. Hence,

$$\|u\|_{E_1(\mathbb{R}_+)} \leq \|v\|_{0E_1(\mathbb{R}_+)} + \|w\|_{E_1(\mathbb{R}_+)} \leq CMR \|f\|_{E_0(\mathbb{R}_+)} + C_1 \|H_0\|_{X, X_A, 1-p^\alpha}.$$

These considerations are the same in the case where one replaces $\mathbb{R}_+$ by a finite interval $f = [0, T], T > 0$. Here, all constants can be chosen independently of $f$.

For $\alpha = 1$ we have the known results for classical evolution equations.
Remark 6.2.8. Note that it is possible to consider the property of maximal $L^p$-regularity for a wider class of equations, cf. [Zac03]. But in contrast to the classical case there are still some basic questions about maximal $L^p$-regularity open. For example it is not known what properties of the operator $A$, such as sectoriality, are implied by maximal $L^p$-regularity. Moreover, the relation between maximal $L^p$-regularity on a compact interval $[0,T]$ and on $\mathbb{R}_+$ is not known. For this thesis these questions are not immediately relevant, since here the property that a certain operator has maximal $L^p$-regularity is paramount. But independently of these considerations it is certainly worthwhile to think about the above raised questions.
Local Well-Posedness and Maximal Solutions of Quasilinear Problems

This section concerns with the strong solvability of quasilinear fractional evolution equations. In order to establish local well-posedness, the basic idea is to use a fixed point argument which relies on the maximal $L^p$-regularity for an associated linear problem, see Theorem 7.1.1.

Afterwards, we explain how to continue a local solution and thus we are able to define the maximal solution of the quasilinear problem. Moreover, we give a characterisation of the maximal interval of existence, see Section 7.2.

Throughout this chapter we assume:

Assumptions. The spaces $X_0$ and $X_1$ are Banach spaces of class HT such that $X_1 \hookrightarrow X_0$ with dense embedding. For $p \in (1, \infty)$ and $\alpha \in (1/p, 1)$ let the $V$ be an open subset of the real interpolation space $X_\gamma = ([X_0, X_1]_{1-\alpha/p}, p)$, and $u_0 \in V$. Moreover, we have maps $(A, F) : V \to B(X_1, X_0) \times X_0$. We denote by $J$ either a compact interval $[0, T^*]$, $T^* > 0$, or $\mathbb{R}_*$.

We consider abstract quasilinear parabolic problems of the form

$$\partial_t^\alpha (u - u_0) + A(u)u = F(u), \quad t \in J, \quad u(0) = u_0. \tag{7.1}$$

We are interested in (local) solutions $u$ of (7.1) with maximal $L^p$-regularity, i.e.

$$u \in H^\alpha_p([0, T]; X_0) \cap L^p([0, T]; X_1) \cap C([0, T]; V).$$

The trace space of this class of functions is given by $X_\gamma$.

7.1 Local Well-Posedness

First of all we prove the local well-posedness of the quasilinear problem (7.1). The following theorem including its proof is analogous to Prüss and Simonett, Theorem 5.1.1, [PS16, Theorem 5.1.1, pp. 195-200].

**Theorem 7.1.1 (Local Well-Posedness).** Let $p \in (1, \infty)$, $\alpha \in (1/p, 1)$ and $u_0 \in V$ be given and suppose that $(A, F) \in C^1(V; B(X_1, X_0) \times X_0)$. Assume in addition that for some $\mu \geq 0$ the operator $\mu + A(u_0) \in \mathcal{R}(\mathcal{S}(X_0))$ with $\varphi^R_{\mu + A(u_0)} < \pi - \alpha \frac{\pi}{2}$, i.e. the operator $A(u_0)$ has maximal $L^p$-regularity on a compact interval $[0, T^*]$.

Then there exist $T = T(u_0) \in (0, T_*]$ and $\varepsilon = \varepsilon(u_0) > 0$ with $B_{\varepsilon}^{X_\gamma}(u_0) \subset V$ such that (7.1) has a unique solution

$$u(\cdot, u_1) \in H^\alpha_p([0, T]; X_0) \cap L^p([0, T]; X_1) \cap C([0, T]; V),$$

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on \([0, T]\), for any initial value \(u_1 \in \overline{B}_{X^p}^X(u_0)\).

Moreover, there exists a constant \(c = c(u_0) > 0\) such that for all \(u_1, u_2 \in \overline{B}_{X^p}^X(u_0)\) the estimate

\[
\|u(\cdot; u_1) - u(\cdot; u_2)\|_{H^p_0([0, T]; X) \cap L_p([0, T]; X)} \leq c \|u_1 - u_2\|_{X^p}
\]

is valid.

**Proof.** This proof is analogous to the proof of [PS16, Thm. 5.1.1, pp. 195-199].

Since \(u_0 \in V\) and by Lipschitz continuity of \((A, F)\), there exists some \(\varepsilon_0 > 0\) and a constant \(L > 0\) such that \(\overline{B}_{X^0}^X(u_0) \subset V\) and

\[
\|A(w_1)v - A(w_2)v\|_{X^0} \leq L \|w_1 - w_2\|_{X^p}, \quad (7.2)
\]

as well as

\[
\|F(w_1) - F(w_2)\|_{X^0} \leq L \|w_1 - w_2\|_{X^p}, \quad (7.3)
\]

is valid for all \(w_1, w_2 \in \overline{B}_{X^0}^X(u_0), v \in X_1\).

For \(T \in (0, T]\) we introduce a reference function \(u_0^* \in \mathcal{E}_1(T) = H^p_0([0, T]; X) \cap L_p([0, T]; X)\) as the solution of the linear problem

\[
\partial_t^p (w - u_0) + A(u_0)w = 0, \quad t \in [0, T], \quad w(0) = u_0. \quad (7.4)
\]

Observe that by the assumption, the operator \(\mu + A(u_0) \in \mathcal{R}(X_0)\) with \(\mathcal{R}\)-angle \(\phi_{\mu + A(u_0)} < \pi - \frac{\alpha}{2}\), the corresponding Volterra equation with standard kernel

\[
u + (g_\alpha \ast [\mu + A(u_0)]u) = u_0, \quad t \in [0, T],
\]

is parabolic and therefore admits a bounded resolvent family \(\{S_{\mu + A(u_0)}(t)\}_{t \in \mathcal{R}_+}\) on \(X_0\), cf. Theorem 3.3.1. Using a perturbation result for parabolic Volterra equation, see [Prü12, Theorem 3.2], with the term \(-\mu (g_\alpha \ast u)\), we know that the linear Volterra equation

\[
u + g_\alpha \ast A(u_0)u = u_0, \quad t \in [0, T],
\]

also admits a resolvent family \(\{S(t)\}_{t \in \mathcal{R}_+} = \{S_{A(u_0)}(t)\}_{t \in \mathcal{R}_+} \subset \mathcal{B}(X_0)\). Thus, the solution of the linear reference problem (7.4) is given by \(u_0^* = S(t)u_0\).

The strong continuity on \(\mathcal{R}_+\), together with the uniform boundedness principle, implies that there is a constant \(M_0 > 0\) such that \(\sup_{t \in [0, T]} \|S(t)\|_{\mathcal{B}(X_0)} \leq M_0 < \infty\). Moreover, the resolvent family \(\{S(t)\}_{t \in \mathcal{R}_+}\) considered as operator family on \(X_1\) is also strongly continuous on \(\mathcal{R}_+\), and hence there is some constant \(M_1 > 0\) such that \(\sup_{t \in [0, T]} \|S(t)\|_{\mathcal{B}(X_1)} \leq M_1 < \infty\). Via the properties of the real interpolation space \(X^p\), we deduce that there is some constant \(M_\gamma > 0\) such that \(\sup_{t \in [0, T]} \|S(t)\|_{\mathcal{B}(X^\gamma)} \leq M_\gamma < \infty\).
Let \( u_1 \in B_{X_p}^{X_p} (u_0) \) with \( \varepsilon \in (0, \varepsilon_0] \) be given. For \( r \in (0, 1] \) we define the set \( B_{r,T,u_1} \subset \mathcal{E}_1(T) \) by

\[
B_{r,T,u_1} = \{ v \in \mathcal{E}_1(T) : v(0) = u_1 \text{ and } \| v - u_0 \|_{\mathcal{E}_1(T)} \leq r \}.
\]

Note that \( B_{r,T,u_1} \) is a non-empty complete metric space if it is endowed with the \( \mathcal{E}_1(T) \)-norm. We will show that for all \( v \in B_{r,T,u_1} \) we have \( v(t) \in B_{X_p}^{X_p} (u_0) \) for all \( t \in [0, T] \), provided that \( r, T, \varepsilon > 0 \) are sufficiently small. For this purpose we define \( u_1^* \in \mathcal{E}_1(T) \) as the unique solution of

\[
\partial_t^\varepsilon (w - u_1) + A(u_0)w = 0, \quad t \in [0, T], \quad w(0) = u_1.
\]

Note that \( u_1^* = S(\cdot)u_1 \). Given \( v \in B_{r,T,u_1} \) we estimate as follows

\[
\| v - u_0 \|_{C([0,T];X_p)} \leq \| v - u_1^* \|_{C([0,T];X_p)} + \| u_1^* - u_0^* \|_{C([0,T];X_p)} + \| u_0^* - u_0 \|_{C([0,T];X_p)}.
\]

Since \( u_0 \) is fixed, there exists a number \( T_0 = T_0(u_0) \in (0, T] \) such that

\[
\| u_0^* - u_0 \|_{C([0,T_0];X_p)} = \sup_{t \in [0,T_0]} \| u_0^*(t) - u_0 \|_{X_p} \leq \varepsilon_0/3,
\]

since the resolvent family is strongly continuous on \( \mathbb{R}_+ \), even considered as operator family on the space \( X_p \). Observe that \( v(0) - u_1^*(0) = u_1 - u_1 = 0 \), and via the continuous embedding of \( _0\mathcal{E}_1(T) = _0H_p^\alpha ([0,T];X_0) \cap L_p([0,T];X_1) \hookrightarrow C([0,T];X_p) \) we have

\[
\| v - u_1^* \|_{C([0,T];X_p)} \leq C_{em}\| v - u_1^* \|_{\mathcal{E}_1(T)}.
\]

with the constant \( C_{em} > 0 \) being independent of \( T > 0 \). Therefore, we obtain via the triangle inequality

\[
\| v - u_1^* \|_{C([0,T];X_p)} \leq C_{em} \left( \| v - u_0 \|_{\mathcal{E}_1(T)} + \| u_0^* - u_1^* \|_{\mathcal{E}_1(T)} \right),
\]

and with the definition of \( v \in B_{r,T,u_1} \) it follows that

\[
\| v - u_1^* \|_{C([0,T];X_p)} \leq C_{em} \left( r + \| u_0^* - u_1^* \|_{\mathcal{E}_1(T)} \right).
\]

As mentioned above, the assumptions on the operator \( \mu + A(u_0) \) ensure that \( \sup_{t \in [0,T]} \| S(t) \|_{B(X_p)} \leq M_y \), with some constant \( M_y > 0 \). So, we get with \( u_0^* = S(\cdot)u_0 \) and \( u_1^* = S(\cdot)u_1 \) that

\[
\| u_0^* - u_1^* \|_{C([0,T];X_p)} = \sup_{t \in [0,T]} \| S(t)(u_0 - u_1) \|_{X_p} \leq M_y \| u_0 - u_1 \|_{X_p},
\]

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since \( T \in (0, T_*] \). Via the trace space result from Proposition 6.2.5 we know that there is some constant \( C_1 > 0 \) which is independent of \( T > 0 \) such that

\[
\|u_0 - u_1\|_{E_1(T)} \leq C_1 \|u_0 - u_1\|_{X_T}.
\]

Hence, we have with \( C_y = \max\{M_y, C_{emb} C_1\} \) that

\[
\|u_0^* - u_1^*\|_{C([0, T_*]; X_T)} + C_{emb} \|u_0 - u_1\|_{E_1(T)} \leq C_y \|u_0 - u_1\|_{X_T}. \tag{7.5}
\]

For \( \varepsilon \leq \varepsilon_0/(3C_y) \), \( r \leq \varepsilon_0/(3C_{emb}) \) and \( T \in [0, T_0] \), we obtain

\[
\|v - u_0\|_{C([0, T_*]; X_T)} \leq C_{emb} r + C_y \varepsilon + \|u_0^* - u_0\|_{C([0, T_*]; X_T)} \leq \varepsilon_0. \tag{7.6}
\]

Throughout the remaining part of this proof we will assume that \( u_1 \in \mathcal{B}_\varepsilon^{X_y}(u_0) \) with \( \varepsilon \leq \varepsilon_0/(3C_y) \), \( T \in (0, T_0] \), and \( r \leq \varepsilon_0/(3C_{emb}) \). Under these assumptions, we may define a mapping

\[
T_{u_1} : \mathcal{B}_{r, T, u_1} \to \mathbb{E}_1(T), \quad v \mapsto T_{u_1} v = u,
\]

where \( u \) is the unique solution of the linear problem

\[
\partial_t^\beta (u - u_1) + A(u_0) u = F(v) + (A(u_0) - A(v)) v, \quad t \in [0, T], \quad u(0) = u_1.
\]

In order to apply the contraction mapping principle, firstly, we show that \( T_{u_1} \) defines a strict contraction on \( \mathcal{B}_{r, T, u_1} \), i.e. there is some \( \kappa \in (0, 1) \) such that for all \( v, \bar{v} \in \mathcal{B}_{r, T, u_1} \) we have

\[
\|T_{u_1} v - T_{u_1} \bar{v}\|_{E_1(T)} \leq \kappa \|v - \bar{v}\|_{E_1(T)}.
\]

**Self-mapping.** At first we take care of the self-mapping property. For each \( v \in \mathcal{B}_{r, T, u_1} \), we know that \( T_{u_1} v - u_1^* \) solves the equation

\[
\partial_t^\beta w + A(u_0)w = F(v) + (A(u_0) - A(v))v, \quad t \in \mathbb{R}_+, \quad w(0) = 0.
\]

The property of maximal \( L_p \)-regularity on the compact interval \([0, T_*]\) of the operator \( A(u_0) \) implies the estimate

\[
\|T_{u_1} v - u_1^*\|_{E_1(T)} \leq C_{MR} \|F(v) + (A(u_0) - A(v))v\|_{E_0(T)},
\]

and the constant \( C_{MR} > 0 \) is independent of \( T \in (0, T_0] \). Now, we estimate \((A(u_0) - A(v))v\) in \( \mathbb{E}_0(T) \). Using the Lipschitz continuity (7.2) we obtain that
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\[\| (A(u_0) - A(v))v \|_{E_0(T)} \leq L \left( \int_0^T \| u_0 - v(t) \|_{X_\gamma}^p \| v(t) \|_{X_\gamma}^{p'} \ dt \right)^{1/p} \]
\[\leq L \| u_0 - v \|_{C([0,T];X_\gamma)} \| v \|_{E_1(T)}. \]

In view of the definition of \( v \in \mathcal{B}_{r,T,u_1} \) we have \( \| v \|_{E_1(T)} \leq r + \| u_0^* \|_{E_1(T)} \) and hence

\[\| (A(u_0) - A(v))v \|_{E_0(T)} \leq L \| u_0 - v \|_{C([0,T];X_\gamma)} \left( r + \| u_0^* \|_{E_1(T)} \right). \]

Next we use the assumed Lipschitz continuity of the map \( F \) from (7.3) to estimate

\[\| F(v) \|_{E_0(T)} \leq \| F(v) - F(u_0) \|_{E_0(T)} + \| F(u_0) \|_{E_0(T)} \leq T^{1/p} \left( L \| v - u_0 \|_{C([0,T];X_\gamma)} + \| F(u_0) \|_{X_0} \right). \]

Combining these estimates we have

\[\| T_{u_1} v - u_0^* \|_{E_1(T)} \leq \| u_1^* - u_0^* \|_{E_1(T)} + \| T_{u_1} v - u_1^* \|_{E_1(T)} \]
\[\leq \| u_1^* - u_0^* \|_{E_1(T)} + C_{MR} \left( \| F(v) \|_{E_0(T)} + \| (A(u_0) - A(v))v \|_{E_0(T)} \right) \]
\[\leq \| u_1^* - u_0^* \|_{E_1(T)} + C_{MR} T^{1/p} \left( L \| v - u_0 \|_{C([0,T];X_\gamma)} + \| F(u_0) \|_{X_0} \right) \]
\[+ C_{MR} L \| u_0 - v \|_{C([0,T];X_\gamma)} \left( r + \| u_0^* \|_{E_1(T)} \right). \]

One easily verifies that \( \| u_0^* - u_0 \|_{C([0,T];X_\gamma)} \) and \( \| u_0^* \|_{E_1(T)} \) tend to zero as \( T \to 0 \). Therefore we get for sufficiently small \( r > 0, T > 0 \) and \( \varepsilon > 0 \) that

\[\| T_{u_1} v - u_0^* \|_{E_1(T)} \leq \| u_1^* - u_0^* \|_{E_1(T)} + r/2, \]

here we employ the estimate \( \| v - u_0 \|_{C([0,T];X_\gamma)} \leq C_{emb} r + C_{\gamma} \varepsilon + \| u_0^* - u_0 \|_{C([0,T];X_\gamma)} \) from (7.6). With the estimate (7.5) we get

\[\| u_0^* - u_1^* \|_{E_1(T)} \leq (C_{\gamma}/C_{emb}) \| u_0 - u_1 \|_{X_\gamma}. \]

The preceding estimates imply that

\[\| T_{u_1} v - u_0^* \|_{E_1(T)} \leq (C_{\gamma}/C_{emb}) \| u_1 - u_0 \|_{X_\gamma} + r/2 \leq r/2 + r/2 = r, \]

with a possible smaller \( \varepsilon > 0 \). This proves the self-mapping property of \( T_{u_1} \) on \( \mathcal{B}_{r,T,u_1} \).
Contraction. Now, we turn to the contraction property. Let \( u_1, u_2 \in B_{e_0}^{X_T}(u_0) \) be arbitrary but fixed as well as \( v_1 \in B_{r,T,u_1} \) and \( v_2 \in B_{r,T,u_2} \). This means, that \( T_{u_i} v_i \) is the solution of the equation

\[
\partial_t^2 (u - u_i) + A(u_0)u = F(v_i) + (A(u_0) - A(v_i))v_i, \quad t \in [0, T], \quad u(0) = u_i,
\]

for \( i \in \{1, 2\} \). One convines oneself that \( T_{u_i} v_i - S(\cdot)u_i \) solves the equation

\[
\partial_t^2 u + A(u_0)u = F(v_i) + (A(u_0) - A(v_i))v_i, \quad t \in [0, T], \quad u(0) = 0,
\]

for \( i \in \{1, 2\} \). Due to the property of maximal \( L_p \)-regularity of the operator \( A(u_0) \) on the compact interval \([0, T]\) we have

\[
\| T_{u_1} v_1 - T_{u_2} v_2 \|_{E_i(T)} \leq \| S(\cdot)(u_1 - u_2) \|_{E_i(T)} + C_{MR} \| F(v_1) - F(v_2) \|_{E_i(T)}
\]

\[
+ C_{MR} \| (A(u_0) - A(v_1))v_1 - (A(u_0) - A(v_2))v_2 \|_{E_i(T)}.
\]

The first term on the right-hand side of equation (7.7) can be estimated by Proposition 6.2.5, thereby we obtain that

\[
\| S(\cdot)(u_1 - u_2) \|_{E_i(T)} \leq C_1 \| u_1 - u_2 \|_{X_T},
\]

with some constant \( C_1 > 0 \) independent of \( T > 0 \). For the second term on the right-hand side of (7.7) we use the Lipschitz continuity of the map \( F \) from (7.3) and get

\[
\| F(v_1) - F(v_2) \|_{E_i(T)} \leq L \left( \int_0^T \| v_1(t) - v_1(t) \|_{X_T}^p \, dt \right)^{1/p}
\]

\[
\leq LT^{1/p} \| v_1 - v_2 \|_{C([0, T]; X_T)},
\]

Moreover, since \( v_1(0) - u_1 = 0 \) as well as \( v_2(0) - u_2 = 0 \) and by the boundedness of the resolvent family \( \{S(t)\}_{t \in \mathbb{R}} \) on \( X_T \), we deduce that

\[
\| v_1 - v_2 \|_{C([0, T]; X_T)} \leq \| v_1 - v_2 - S(\cdot)(u_1 - u_2) \|_{C([0, T]; X_T)} + \| S(\cdot)(u_1 - u_2) \|_{C([0, T]; X_T)}
\]

\[
\leq C_{emb} \| v_1 - v_2 - S(\cdot)(u_1 - u_2) \|_{E_i(T)} + M_T \| u_1 - u_2 \|_{X_T},
\]

with \( T \)-independent constants \( M_T \) and \( C_{emb} \). By the triangle inequality and the estimate (7.8) it follows that

\[
\| v_1 - v_2 \|_{C([0, T]; X_T)} \leq C_{emb} \| v_1 - v_2 \|_{E_i(T)} + C_1 \| u_1 - u_2 \|_{X_T} + M_T \| u_1 - u_2 \|_{X_T}
\]

\[
= C_{emb} \| v_1 - v_2 \|_{E_i(T)} + 2 C_T \| u_1 - u_2 \|_{X_T},
\]

(7.10)

with \( C_T = \max\{M_T, C_{emb} C_1\} \) like above. Together with (7.9) we have

\[
\| F(v_1) - F(v_2) \|_{E_i(T)} \leq LT^{1/p} \left( C_{emb} \| v_1 - v_2 \|_{E_i(T)} + 2 C_T \| u_1 - u_2 \|_{X_T} \right).
\]
For the remaining term in (7.7) we use the Lipschitz continuity of the map $A$ from (7.2):

$$
\| (A(u_0) - A(v_1)) (v_1 - (u_0) - A(v_2)) v_2 \|_{E_0(T)} \\
\leq \| (A(v_1) - A(u_0)) (v_1 - v_2) \|_{E_0(T)} + \| (A(v_1) - A(v_2)) v_2 \|_{E_0(T)} \\
\leq L \left( \| v_1 - u_0 \|_{C([0,T];X_p)} \| v_1 - v_2 \|_{E_1(T)} + \| v_1 - v_2 \|_{C([0,T];X_p)} \| v_2 \|_{E_1(T)} \right).
$$

As in (7.6), we can made the term $\| v_1 - u_0 \|_{C([0,T];X_p)}$ as small as we wish by choosing $r > 0$, $T \in (0, T_*]$ and $\varepsilon > 0$ sufficiently small. By the definition of $v_2 \in \mathbb{B}_{r,T,u_2}$ we have

$$
\| v_2 \|_{E_1(T)} \leq \| v_2 - u_0^* \|_{E_1(T)} + \| u_0^* \|_{E_1(T)} \leq r + \| u_0^* \|_{E_1(T)},
$$

hence $\| v_2 \|_{E_1(T)}$ is as small as we want, provided $r > 0$ and $T \in (0, T_*]$ are small enough. Lastly, the term $\| v_1 - v_2 \|_{C([0,T];X_p)}$ can be estimated like in (7.10). In summary, if we choose $r > 0$, $T > 0$ and $\varepsilon > 0$ sufficiently small, we obtain

$$
\| T_{u_1} v_1 - T_{u_2} v_2 \|_{E_1(T)} \leq C_1 \| u_1 - u_2 \|_{X_p} \\
+ C_{MR} L^{1/p} \left( C_1 \| v_1 - v_2 \|_{E_1(T)} + 2 C_p \| u_1 - u_2 \|_{X_p} \right) \\
+ C_{MR} L \left( \| v_1 - u_0 \|_{C([0,T];X_p)} \| v_1 - v_2 \|_{E_1(T)} \\
+ C_{emb} \| v_1 - v_2 \|_{E_1(T)} + 2 C_p \| u_1 - u_2 \|_{X_p} \| v_2 \|_{E_1(T)} \right) \\
\leq \frac{1}{2} \| v_1 - v_2 \|_{E_1(T)} + c \| u_1 - u_2 \|_{X_p},
$$

with some constant $c = c(u_0) > 0$, for all $u_1, u_2 \in \mathbb{B}_{r,T,u_1}^X$ and all $v_1 \in \mathbb{B}_{r,T,u_1}$ and all $v_2 \in \mathbb{B}_{r,T,u_2}$. In the very special case $u_1 = u_2$ we get the contraction mapping property of $T_{u_1}$ on $\mathbb{B}_{r,T,u_1}$.

Now we are in position to apply Banach’s fixed point theorem to obtain a unique fixed point $\tilde{u} \in \mathbb{B}_{r,T,u_1}$ of $T_{u_1}$, i.e. $T_{u_1} \tilde{u} = \tilde{u}$. Therefore $\tilde{u} \in \mathbb{B}_{r,T,u_1}$ is the unique local solution of (7.1). Furthermore, if $u(\cdot;u_1)$ and $u(\cdot;u_2)$ denote the solutions of (7.1) with initial values $u_1, u_2 \in \mathbb{B}_{r,T,u_1}^X$ respectively, the last assertion of the theorem follows from the last estimate above. This completes the proof. \[\Box\]
7. Local Well-Posedness and Maximal Solutions of Quasilinear Problems

7.2 Maximal Solutions

In this section we show that a local solution \( u \in \mathcal{E}_1(\tau) \cap C([0, \tau]; X_\rho) \) of the quasilinear parabolic fractional evolution equation (7.1) can be continued to a solution \( \bar{u} \in \mathcal{E}_1(\tau + \delta) \cap C([0, \tau + \delta]; X_\rho) \) with some sufficiently small \( \delta > 0 \), where \( \bar{u} \) restricted to the interval \([0, \tau]\) is equal to \( u \). On the basis of this continuation we are able to define a maximal solution for the quasilinear problem (7.1).

**Lemma 7.2.1.** Let \( p \in (1, \infty) \) and \( \alpha \in (1/p, 1) \) be given and suppose that \( (A, F) \in C^1(V; \mathcal{B}(X_1, X_0) \times X_0) \). Assume that for all \( v \in V \) there is some \( \mu = \mu(v) \geq 0 \) such that \( \mu + A(v) \in \mathcal{R}(X_0) \) with \( \varphi_{\mu+}^R < \pi - \alpha \frac{T}{2} \), i.e. the operator \( A(v) \) has maximal \( L_p \)-regularity on each compact interval \([0, T_*]\). Let \( u \in \mathcal{E}_1(\tau) \cap C([0, \tau]; V) \) be a local unique solution of the quasilinear problem (7.1) with \( u_0 \in V \).

Then there exists some \( \delta > 0 \) such that there is a unique solution of maximal \( L_p \)-regularity \( \bar{u} \in \mathcal{E}_1(\tau + \delta) \cap C([0, \tau + \delta]; V) \) of the quasilinear problem (7.1) and \( u(t) = \bar{u}(t) \) for all \( t \in [0, \tau] \).

**Proof.** Let \( u_0 \in V \) be arbitrary, but fixed. By the Local Well-Posedness Theorem 7.1.1 there is a unique solution \( u \in \mathcal{E}_1(\tau) \cap C([0, \tau], V) \) on the interval \([0, \tau]\), for some \( \tau > 0 \). This solution can be continued to some larger interval \([0, \tau + \delta] \subset [0, \tau + 1] \). Indeed, for \( u_\tau = u(\tau) \in V \) we introduce a reference function \( w \in \mathcal{E}_1(\tau + 1) \cap C([0, \tau + 1]; X_\rho) \) as the solution of the linear problem

\[
\frac{\partial}{\partial t} (w - u_0) + A(u_\tau)w = [(A(u_\tau) - A(u))u + F(u)] \chi_{[0, 1]}, \quad t \in [0, \tau + 1],
\]

with \( w(0) = u_0 \). Obviously, we have \( w = u \) on \([0, \tau]\) by uniqueness. In particular, \( w(\tau) = u_\tau \). By the continuity of \( w \) in \( X_\rho \) it follows that for all \( \eta > 0 \) there is some \( r = r(\eta) \in (0, 1] \) such that for all \( \delta \in (0, r) \) we have

\[
\|w - u_\tau\|_{C([\tau, \tau + \delta]; X_\rho)} = \sup_{t \in [\tau, \tau + \delta]} \|w(t) - u_\tau\|_{X_\rho} \leq \eta. \tag{7.11}
\]

Since \( V \) is an open set with \( u_\tau \in V \) there is some \( \varepsilon_0 > 0 \) such \( \mathcal{B}_{\varepsilon_0}^X(u_\tau) \subset V \). We choose the parameter \( \eta = \varepsilon_0/2 \) according to the relation (7.11) and fix the corresponding parameter \( r = r(\varepsilon_0/2) \in (0, 1] \). We denote by \( C_{emb} \) the uniform embedding constant for \( \varepsilon_0 \mathcal{E}_1(T) \hookrightarrow C([0, T]; X_\rho) \), with \( T > 0 \). We consider for \( \delta \in (0, r) \) and \( \rho \in (0, \varepsilon_0/(2C_{emb})] \) the set

\[
\mathcal{B}(\tau, \delta, \rho) = \{v \in \mathcal{E}_1(\tau + \delta): v(t) = u(t) \text{ for all } t \in [0, \tau] \text{ and } \|v - u\|_{\varepsilon_0 \mathcal{E}_1(\tau + \delta)} \leq \rho\}.
\]

Since \( u = u \) on \([0, \tau]\) by uniqueness, the set \( \mathcal{B}(\tau, \delta, \rho) \) is not empty and it becomes a complete metric space when endowed with the metric induced by the norm of \( \varepsilon_0 \mathcal{E}_1(\tau + \delta) \).

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Firstly, we will show that for all $\delta \in (0, r]$ and $\rho \in (0, \epsilon_0/(2C_{emb})]$ we have for $v \in B(\tau, \delta, \rho)$ that $\|v - u_\tau\|_{C([\tau, \tau+\delta]; X_\gamma)} \leq \epsilon_0$. In fact,

$$\|v - u_\tau\|_{C([\tau, \tau+\delta]; X_\gamma)} \leq \|v - w\|_{C([\tau, \tau+\delta]; X_\gamma)} + \|w - u_\tau\|_{C([\tau, \tau+\delta]; X_\gamma)} \leq C_{emb} \|v - w\|_{E_1(\tau+\delta)} + \epsilon_0/2 \leq C_{emb} \rho + \epsilon_0/2 \leq \epsilon_0.$$

In particular, $v(t) \in \overline{B}_{\epsilon_0}(u_\tau) \subset V$ for all $t \in [\tau, \tau + \delta]$ and hence, $A(v(t))$ and $F(v(t))$ are well-defined for all $t \in [0, \tau + \delta]$.

Next, we define the mapping $G : B(\tau, \delta, \rho) \to E_1(\tau + \delta)$, which assigns to $v \in B(\tau, \delta, \rho)$ the solution $\tilde{u} = G(v)$ of the linear problem

$$\partial_t \tilde{u} - (\tilde{u} - u_\tau) + A(u_\tau)\tilde{u} = (A(u_\tau) - A(v))v + F(v), \quad t \in [0, \tau + \delta], \quad \text{(7.12)}$$

with $\tilde{u}(0) = u_\tau$. Since $v = u_\tau$ on $[0, \tau]$, we have also $\tilde{u} = u_\tau$ on $[0, \tau]$, by uniqueness. We will prove that $G$ is a self-mapping of the set $B(\tau, \delta, \rho)$ and a contraction for sufficiently small parameters.

**Self-mapping.** To prove the self-mapping property we have to estimate $\|\tilde{u} - w\|_{E_1(\tau + \delta)}$. The difference $\tilde{u} - w$ satisfies on $[0, \tau + \delta]$ the equation

$$\partial_t^2 (\tilde{u} - w) + A(u_\tau)(\tilde{u} - w) = [(A(u_\tau) - A(v))v + F(v)] \chi_{[\tau, \tau + \delta]},$$

with $(\tilde{u} - w)(0) = 0$. Via the maximal $L_p$-regularity property of the operator $A(u_\tau)$ on the interval $[0, \tau + 1]$ it follows that

$$\|\tilde{u} - w\|_{E_1(\tau + \delta)} \leq C_{MR}^{A(u_\tau)} \left\{ \|A(u_\tau) - A(v)v\|_{L_p([\tau, \tau + \delta]; X_\gamma)} + \|F(v)\|_{L_p([\tau, \tau + \delta]; X_\gamma)} \right\}.$$

Using the Lipschitz continuity of the maps $A$ and $F$ on the ball $\overline{B}_{\epsilon_0}(u_\tau)$ we obtain that

$$\|A(u_\tau) - A(v)v\|_{L_p([\tau, \tau + \delta]; X_\gamma)} \leq L \left( \int_{\tau}^{\tau+\delta} \|u_\tau - v(t)\|_{X_\gamma} \|v(t)\|_{X_1} dt \right)^{1/p} \leq L \|u_\tau - v\|_{C([\tau, \tau+\delta]; X_\gamma)} \left( \int_{\tau}^{\tau+\delta} \|v(t)\|_{X_1}^{p} dt \right)^{1/p} \leq L \left[ C_{emb} \rho + \|w - u_\tau\|_{C([\tau, \tau+\delta]; X_\gamma)} \right] \left( \rho + \|w\|_{L_p([\tau, \tau+\delta]; X_1)} \right),$$

as well as

$$\|F(v)\|_{L_p([\tau, \tau + \delta]; X_\gamma)} \leq \delta^{1/p} \left( L \|u_\tau - v\|_{C([\tau, \tau+\delta]; X_\gamma)} + \|F(u_\tau)\|_{X_\gamma} \right) \leq \delta^{1/p} \left( L C_{emb} \rho + \|w - u_\tau\|_{C([\tau, \tau+\delta]; X_\gamma)} + \|F(u_\tau)\|_{X_\gamma} \right).$$

Since $\|u_\tau - w\|_{C([\tau, \tau+\delta]; X_\gamma)}$, as well as $\|w\|_{L_p([\tau, \tau+\delta]; X_1)}$ tends to zero as $\delta \to 0$ it
follows that for sufficiently small $\rho > 0$ we have
\[ \| \tilde{u} - w \|_{0, E_1(\tau + \delta)} \leq \rho. \]
This proves the self-mapping property of the map $G$.

**Contraction.** To prove the contraction property of the map $G$, we define for $v_1, v_2 \in B(\tau, \delta, \rho)$ the corresponding solutions of the equation (7.12) denoted by $\tilde{u}_1 = G(v_1), \tilde{u}_2 = G(v_2)$, like above. This implies for the difference $\tilde{u}_1 - \tilde{u}_2$ on $[0, \tau + \delta]$ the equation
\[
\partial_t (\tilde{u}_1 - \tilde{u}_2) + A(u_\tau)(\tilde{u}_1 - \tilde{u}_2) = \{(A(u_\tau) - A(v_1))v_1 - (A(u_\tau) - A(v_2))v_2
\]
\[+ F(v_1) - F(v_2)\} X_{\tau, \tau + \delta}, \]
with $(\tilde{u}_1 - \tilde{u}_2)(0) = 0$. By means of the maximal $L_p$-regularity property on the interval $[0, \tau + 1]$ of the operator $A(u_\tau)$ we get the estimate
\[
\| \tilde{u}_1 - \tilde{u}_2\|_{0, E_1(\delta + \tau)} \leq C_{MR} A(u_\tau) \left\{ \| (A(u_\tau) - A(v_1))v_1 + (A(u_\tau) - A(v_2))v_2 \|_{L_p([\tau, \tau + \delta]; X_0)} + \| F(v_1) - F(v_2) \|_{L_p([\tau, \tau + \delta]; X_0)} \right\}.
\]
Using the Lipschitz estimate from (7.2) we obtain
\[
\| (A(u_\tau) - A(v_1))v_1 + (A(u_\tau) - A(v_2))v_2 \|_{L_p([\tau, \tau + \delta]; X_0)}
\]
\[\leq \| (A(u_\tau) - A(v_1))v_1 - v_2 \|_{L_p([\tau, \tau + \delta]; X_0)} + \| (A(v_1) - A(v_2))v_2 \|_{L_p([\tau, \tau + \delta]; X_0)}
\]
\[\leq L \left( \| v_1 - u_\tau \|_{C([\tau, \tau + \delta]; X_p)} \| v_1 - v_2 \|_{L_p([\tau, \tau + \delta]; X_1)} + \| v_1 - v_2 \|_{L_p([\tau, \tau + \delta]; X_1)} \right)
\]
\[\leq L \left( C_{emb} \rho + \| w - u_\tau \|_{C([\tau, \tau + \delta]; X_p)} \right) \| v_1 - v_2 \|_{L_p([\tau, \tau + \delta]; X_1)}
\]
\[+ \| v_1 - v_2 \|_{C([\tau, \tau + \delta]; X_p)} \left( \rho + \| w \|_{L_p([\tau, \tau + \delta]; X_1)} + \delta^{1/p} \right). \]
Furthermore, with the Lipschitz estimate from (7.3) we get
\[
\| F(v_1) - F(v_2) \|_{L_p([\tau, \tau + \delta]; X_0)} \leq L \delta^{1/p} \| v_1 - v_2 \|_{C([\tau, \tau + \delta]; X_p)}.
\]
We use the continuous embedding $\| \cdot \|_{0, E_1(\tau + \delta)} \hookrightarrow C([0, \tau + \delta]; X_p)$ to estimate the term $\| v_1 - v_2 \|_{C([\tau, \tau + \delta]; X_p)} \leq C_{emb} \| v_1 - v_2 \|_{0, E_1(\tau + \delta)}$. This yields the estimate
\[
\| \tilde{u}_1 - \tilde{u}_2\|_{0, E_1(\tau + \delta)}
\leq C_{MR} A(u_\tau) \left[ \| w - u_\tau \|_{C([\tau, \tau + \delta]; X_p)} + C_{emb} \left( \rho + \| w \|_{L_p([\tau, \tau + \delta]; X_1)} + \delta^{1/p} \right) \right] \| v_1 - v_2 \|_{0, E_1(\tau + \delta)}.
\]
Since \( \|w - u_\tau\|_{C([\tau, \tau + \delta]; X_\gamma)} \) and \( \|w\|_{L_p([\tau, \tau + \delta]; X_1)} \) tend to zero as \( \delta \to 0 \), the contraction property follows.

Thus, we can use the contraction mapping principle to show that there is a unique fixed point \( \tilde{u} \in \mathcal{E}_1(\tau + \delta) \cap C([0, \tau + \delta]; V) \) on \([0, \tau + \delta]\). This solution coincides with \( u \) on \([0, \tau]\). Thus, \( \tilde{u} \) is the desired continuation of the local solution \( u \).

We call \( u \) defined on an interval \( J \) maximal solution if there does not exist a solution \( v \) on an interval \( J' \) strictly containing \( J \) such that \( v \) restricted to \( J \) equals \( u \). If \( u \) is a maximal solution on \( J \), then \( J \) is called the maximal interval of existence, and \( J = [0, \tau(u_0)) \) where

\[
\tau(u_0) = \sup\{T > 0 : u \in \mathcal{E}_1(T) \cap C([0, T]; V) \text{ is solution of (7.1)}\}.
\]

**Lemma 7.2.2.** Let \( p \in (1, \infty) \) and \( \alpha \in (1/p, 1) \) be given and suppose that \( (A, F) \in C^{1-}(V; B(X_1, X_0) \times X_0) \). Assume that for all \( v \in V \) there is some \( \mu = \mu(v) \geq 0 \) such that \( \mu + A(v) \in \mathcal{R}(X_0) \) with \( q_{p, \alpha}^R < \pi - \alpha \frac{\pi}{2} \), i.e. the operator \( A(v) \) has maximal \( L_p \)-regularity on any compact interval \([0, T_*]\).

Then the quasilinear problem (7.1) with initial value \( u_0 \in X_\gamma \) has a unique solution \( u \) on a maximal interval of existence \([0, \tau(u_0))\), which is characterised by one of the following options.

(i) \( \tau(u_0) = \infty \), i.e. global existence;

(ii) \( \tau(u_0) < \infty \) and \( \liminf_{t \to \tau(u_0)} \text{dist}_{X_\gamma}(u(t), \partial V) = 0 \);

(iii) \( \tau(u_0) < \infty \) and \( \lim_{t \to \tau(u_0)^-} u(t) \) does not exists in \( X_\gamma \).

**Proof.** Suppose \( \tau(u_0) < \infty \), \( \text{dist}_{X_\gamma}(u(t), \partial V) \geq \eta \), for some \( \eta > 0 \) and all \( t \in [0, \tau(u_0)) \) and assume that the solution \( u \) converges in \( X_\gamma \) to some \( u_1 \in V \), \( u_1 = \lim_{t \to \tau(u_0)} u(t) \). Thus, there is a continuous extension \( \tilde{u} \) on \([0, \tau(u_0)]\) with \( \tilde{u} = u \) on \([0, \tau(u_0))\) and \( \tilde{u}(\tau(u_0)) = u_1 \).

Let \( \bar{\tau} \in [0, \tau(u_0)) \). Analogous to the preceding proof there is some \( \delta = \delta(\tilde{u}(\bar{\tau})) > 0 \) such that we can continue the solution \( u \) from \([0, \bar{\tau}]\) to \([0, \bar{\tau} + \delta]\). By uniqueness, on \([\bar{\tau}, \tau(u_0))\) this is of course the solution we already have. The parameter \( \delta \) can be chosen independently of \( u(\bar{\tau}) \). In fact, the set \( K = \tilde{u}(\tau(u_0)) \) is a compact set in \( X_\gamma \). Thus, for sufficiently small \( \varepsilon > 0 \) there is some \( N = N(\varepsilon) \in \mathbb{N} \) and \( x_1, \ldots, x_N \in \mathcal{E}_\gamma \) such that \( K \subset \bigcup_{i=1}^{N} B_{x_i}^{X_\gamma}(x_i) \subset V \). We fix \( i \in \{1, \ldots, N(\varepsilon)\} \) with \( \|u(\bar{\tau}) - x_i\|_{X_\gamma} \leq \varepsilon \). We follow the arguments in the proof of the above Lemma 7.2.1 and replace \( u_\tau \) by \( x_i \). By a perturbation argument, all estimates are valid for sufficiently small parameters \( \varepsilon, \rho \) and \( \delta \), which can be chosen independently of \( u(\bar{\tau}) \). The solution may be continued past \( \tau(u_0) \) (take \( \bar{\tau} \) sufficiently close to \( \tau(u_0) \)) and it follows a contradiction. \( \square \)
8

Stability for Quasilinear Fractional Evolution Equations

An essential step in our proof of the stability result for quasilinear fractional evolution equations, Theorem 8.2.3, is the time-weighting and rewriting of the equation in a way similar to the procedure for the semilinear Volterra equation in Part II. To achieve this we will first introduce a product rule for fractional derivatives. Here special attention will be paid to the occurring commutator term. Subsequently, for two different time-weighting functions we prove $L_p$-estimates on $\mathbb{R}_+$ for the commutator term, see Lemma 8.1.2 and Lemma 8.1.6.

On the basis of these considerations we are able to prove with the aid of maximal $L_p$-regularity of the linearised equation a stability result for quasilinear fractional evolution equations, Theorem 8.2.3. We can prove asymptotic stability with the same regularity assumptions as in the classical case. But to prove the optimal asymptotic decay behaviour we need an additional regularity assumption on the non-linearities.

Finally, we apply our stability result to an example, see Section 8.3.

8.1 Commutator Term Estimates

We start with a product rule for fractional derivatives. Afterwards, we introduce the so-called commutator term which is motivated by the second term in the product rule. For two special choices of the time-weighting function we prove $L_p$-estimates on $\mathbb{R}_+$, see Lemma 8.1.2 and Lemma 8.1.6.

8.1.1 Product Rule

Like our approach for the semilinear problem in Part II, we want to rewrite our equation after multiplication with some function. The following helpful lemma concerning a product rule for fractional derivatives is taken from [KZ15, Lemma 2.3].

Lemma 8.1.1 (Product Rule). Let $J = [0, T]$ with $T > 0$ be a compact interval and $p \in (1, \infty)$, $\alpha \in (1/p, 1)$ and $\varepsilon \in (0, 1 - \alpha)$ be given. Furthermore, let $X$ be a Banach space of class $\mathcal{H}T$, $v \in \dot{H}_p^\alpha (J; X)$ and $\varphi \in C^{\alpha + \varepsilon} (J; \mathcal{B}(X))$.

Then, we have $\varphi v \in \dot{H}_p^\alpha (J; X)$ and for all $t \in J$ that

$$
\partial_t^\alpha (\varphi v)(t) = \varphi(t) \partial_t^\alpha v(t) + \int_0^t [-\mathring{g}_{1-\alpha}(t-\tau)](\varphi(t) - \varphi(\tau)) v(\tau) d\tau.
$$
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8.1.2 \( L_p \)-Estimates for the Commutator Term

Let \( p \in (0, \infty) \) and \( \alpha \in (1/p, 1) \) be arbitrary, but fixed. We are interested in \( L_p \)-estimates for the so-called \textit{commutator term} for \( t \in \mathbb{R}_+ \) given by

\[
\int_0^t [-\dot{g}_{1-\alpha}(t - \tau)] \frac{q(p(t) - q(p(\tau))}{q(p(\tau))} \, d\tau,
\]

with two special choices for \( \varphi \).

**Lemma 8.1.2.** Let \( \xi \in (0, \alpha - 1/p) \) and \( \varepsilon_0 > 0 \). We define the map \( \varphi_{\varepsilon_0} : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
\varphi_{\varepsilon_0}(t) = 1 + \varepsilon_0 \frac{t}{1 + t^{1-\xi}}, \quad t \in \mathbb{R}_+.
\]

Then, there are constants \( C(\alpha, p, \xi) > 0 \) and \( M(\alpha, p, \xi) > 0 \) which are independent of \( \varepsilon_0 \) such that

\[
\| t \mapsto g_{1-\alpha}(t)[\varphi_{\varepsilon_0}(t) - 1] \|_{L_p(\mathbb{R}_+)} \leq \varepsilon_0 M(\alpha, p, \xi),
\]

as well as

\[
\left\| t \mapsto \int_0^t [-\dot{g}_{1-\alpha}(t - \tau)] \frac{\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau)}{\varphi_{\varepsilon_0}(\tau)} \, d\tau \right\|_{L_p(\mathbb{R}_+)} \leq \varepsilon_0 C(\alpha, p, \xi).
\]

**Proof.** Obviously, the map \( \varphi_{\varepsilon_0} \) is continuously differentiable on \( \mathbb{R}_+ \). Considering the derivative of \( \varphi_{\varepsilon_0} \) it is easy to see that \( \varphi_{\varepsilon_0} \) is strictly increasing on \( \mathbb{R}_+ \), since

\[
\dot{\varphi}_{\varepsilon_0}(t) = \varepsilon_0 \frac{1 + \xi t^{1-\xi}}{(1 + t^{1-\xi})^2} > 0.
\]

We have by the definition of the standard kernel \( g_{1-\alpha} \) that

\[
\left( \int_0^\infty [g_{1-\alpha}(t)[\varphi_{\varepsilon_0}(t) - 1]]^p \, dt \right)^{1/p}
\]

\[
= \frac{\varepsilon_0}{\Gamma(1-\alpha)} \left( \int_0^\infty \frac{t^{1-\alpha}}{1 + t^{1-\xi}} \, dt \right)^{1/p}
\]

\[
\leq \frac{\varepsilon_0}{\Gamma(1-\alpha)} \left[ \left( \int_1^\infty t^{1-\alpha} \, dt \right)^{1/p} + \left( \int_1^\infty t^{-\alpha+\xi} \, dt \right)^{1/p} \right]
\]

\[
= \frac{\varepsilon_0}{\Gamma(1-\alpha)} \left[ ((1-\alpha)p + 1)^{-1/p} + ((\alpha - \xi)p - 1)^{-1/p} \right]
\]

\[
= \varepsilon_0 M(\alpha, p, \xi),
\]
8.1. Commutator Term Estimates

and hence \( t \mapsto [g_{1-a}(t)[\varphi_{e_0}(t) - 1]] \in L_2(\mathbb{R}_+) \). Next, we consider the map

\[
t \mapsto \int_0^t [-\dot{g}_{1-a}(t-\tau)] \frac{\varphi_{e_0}(t)-\varphi_{e_0}(\tau)}{\varphi_{e_0}(\tau)} \, d\tau.
\]

At first, we estimate this integral for \( t \leq 1 \). We have \( \varphi_{e_0}(t) \geq 1, t \in \mathbb{R}_+ \), hence

\[
\int_0^t [-\dot{g}_{1-a}(t-\tau)] \frac{\varphi_{e_0}(t)-\varphi_{e_0}(\tau)}{\varphi_{e_0}(\tau)} \, d\tau \leq \int_0^t [-\dot{g}_{1-a}(t-\tau)] \varphi_{e_0}(t) \, d\tau.
\]

For each \( t \in \mathbb{R}_+ \) we have the trivial estimate \( \dot{\varphi}_{e_0}(t) \leq \varepsilon_0 (1 + \xi^{1-\xi}) \), and hence for \( t \in [0,1] \) we obtain \( \dot{\varphi}_{e_0}(t) \leq 2\varepsilon_0 \). By the mean value theorem there is some \( \nu \in (t, \tau) \subset [0,1] \) such that \( \varphi_{e_0}(t) - \varphi_{e_0}(\tau) = \dot{\varphi}_{e_0}(\nu)(t-\tau) \), and so we have the uniform estimate \( \varphi_{e_0}(t) - \varphi_{e_0}(\tau) \leq 2\varepsilon_0(t-\tau) \). Hence,

\[
\int_0^t [-\dot{g}_{1-a}(t-\tau)] \frac{\varphi_{e_0}(t)-\varphi_{e_0}(\tau)}{\varphi_{e_0}(\tau)} \, d\tau \leq 2\varepsilon_0 \int_0^t [-\dot{g}_{1-a}(t-\tau)](t-\tau) \, d\tau
\]

\[
= 2\varepsilon_0 \int_0^t \frac{\alpha s^{-\alpha}}{\Gamma(1-\alpha)} \, ds
\]

\[
= \varepsilon_0 C_1(\alpha) t^{1-\alpha}.
\] (8.1)

To estimate for \( t \geq 1 \) the integral

\[
\int_0^t [-\dot{g}_{1-a}(t-\tau)] \frac{\varphi_{e_0}(t)-\varphi_{e_0}(\tau)}{\varphi_{e_0}(\tau)} \, d\tau,
\] (8.2)

we split the integral at \( t/2 \) and estimate the summands separately.

For the integral (8.2) from 0 to \( t/2 \) we use on one hand the completely monotonicity of the standard kernel and get for all \( \tau \in (0, t/2) \) that \(-\dot{g}_{1-a}(t-\tau) \leq -\dot{g}_{1-a}(t/2) = (t/2)^{-1-a}/\Gamma(1-a) \). On the other hand we use monotonicity of \( \varphi_{e_0}(t) \geq 1, \ t \in \mathbb{R}_+ \), to obtain that

\[
\frac{\varphi_{e_0}(t)-\varphi_{e_0}(\tau)}{\varphi_{e_0}(\tau)} \leq \varphi_{e_0}(t) - 1 = \varepsilon_0 \frac{t}{1+t^{-\varepsilon}} \leq \varepsilon_0 t^{1-\varepsilon}.
\]

So, we get

\[
\int_0^{t/2} [-\dot{g}_{1-a}(t-\tau)] \frac{\varphi_{e_0}(t)-\varphi_{e_0}(\tau)}{\varphi_{e_0}(\tau)} \, d\tau \leq \int_0^{t/2} \left( \frac{t}{2} \right)^{-1-a} \frac{\alpha}{\Gamma(1-a)} \varepsilon_0 t^{1-\varepsilon} \, d\tau
\]

\[
= \varepsilon_0 C_2(\alpha) t^{-(a-\varepsilon)}.
\]

To estimate the integral (8.2) from \( t/2 \) to \( t \) we use again the mean value theorem. Thus, there is some \( \nu \in (t, \tau) \subset (t/2, t) \) such that \( \varphi_{e_0}(t) - \varphi_{e_0}(\tau) \leq \varepsilon_0 \frac{t}{1+t^{-\varepsilon}} \leq \varepsilon_0 t^{1-\varepsilon} \).
Due to the monotonicity of the enumerator and denominator we have for $\nu \in (t/2, t)$ that

$$
\dot{q}_\nu(t) = \varepsilon_0 \frac{1 + \xi \nu^{1-\xi}}{(1 + \nu^{1-\xi})^2} \leq \varepsilon_0 \frac{1 + \xi t^{1-\xi}}{(1 + (t/2)^{1-\xi})^2} \leq \varepsilon_0 2^{3-2\xi} t^{-1-\xi}.
$$

Using this estimate we deduce that

$$
\int_{t/2}^t \left[ -\dot{g}_{1-a}(t-\tau) \right] \frac{q_{\nu} (t) - q_{\nu} (\tau)}{q_{\nu} (\tau)} \, d\tau \leq \varepsilon_0 2^{3-2\xi} t^{-1-\xi} \int_{t/2}^t \left[ -\dot{g}_{1-a}(t-\tau) \right] (t-\tau) \, d\tau
$$

$$
= \varepsilon_0 2^{3-2\xi} t^{-1-\xi} \int_{0}^t \frac{as^{-\alpha}}{\Gamma(1-a)} \, ds
$$

$$
= \varepsilon_0 C_3(\alpha, \xi) t^{-(\alpha-\xi)}.
$$

Combining these both estimates we obtain for $t \geq 1$ that

$$
\int_{0}^t \left[ -\dot{g}_{1-a}(t-\tau) \right] \frac{q_{\nu} (t) - q_{\nu} (\tau)}{q_{\nu} (\tau)} \, d\tau \leq \varepsilon_0 C_4(\alpha, \xi) t^{-(\alpha-\xi)},
$$

with $C_4(\alpha, \xi) = \max\{C_2(\alpha), C_3(\alpha, \xi)\}$. Together with the estimate (8.1) for small $t \leq 1$ we thus have

$$
\int_{0}^t \left[ -\dot{g}_{1-a}(t-\tau) \right] \frac{q_{\nu} (t) - q_{\nu} (\tau)}{q_{\nu} (\tau)} \, d\tau \leq \varepsilon_0 C_0(\alpha, \xi) \min\{t^{1-\alpha}, t^{-(\alpha-\xi)}\},
$$

with $C_0(\alpha, \xi) = \max\{C_4(\alpha, \xi), C_1(\alpha)\}$. In particular, we have

$$
\left\| t \mapsto \int_{0}^t \left[ -\dot{g}_{1-a}(t-\tau) \right] \frac{q_{\nu} (t) - q_{\nu} (\tau)}{q_{\nu} (\tau)} \, d\tau \right\|_{L_p(\mathbb{R}_+)} \leq \varepsilon_0 C(\alpha, p, \xi),
$$

with $C(\alpha, p, \xi) = C_0(\alpha, \xi) \left[(p(1-\alpha) + 1)^{-1/p} + (p(\alpha - \xi) - 1)^{-1/p}\right]$. \hfill $\Box$

Now, we are interested in $L_p$-estimates for the commutator term in the case $q_\xi = 1/s_\xi$, $\varepsilon > 0$, where $s_\xi(t) = E_\alpha(-\varepsilon t^\alpha)$, $t \in \mathbb{R}_+$ denotes the scalar resolvent for the scalar Volterra equation with standard kernel, see Section 2.3.3. In this situation Lemma 3.5.1 yields for all $t \in (0, \infty)$ that

$$
\int_{0}^t \left[ -\dot{g}_{1-a}(t-\tau) \right] \frac{q_\xi (t) - q_\xi (\tau)}{q_\xi (\tau)} \, d\tau = \int_{0}^t \left[ -\dot{g}_{1-a}(t-\tau) \right] \left[ \frac{1}{s_\xi(t)} - \frac{1}{s_\xi(\tau)} \right] s_\xi(\tau) \, d\tau
$$

$$
= \varepsilon + \dot{g}_{1-a}(t) \left[ 1 - \frac{1}{s_\xi(t)} \right] \geq 0.
$$

We need the following auxiliary lemma about the scaling-invariance of the scalar resolvent $s_\mu(t) = E_\alpha(-\mu t^\alpha)$.
Lemma 8.1.3. The function \( s_\mu(t) \) is scaling-invariant in the sense that \( s_\mu(\lambda t) = s_{\lambda \mu}(t) \), for all \( \lambda \in \Sigma_\pi \), \( \mu \in \mathbb{C} \) and \( t \in \mathbb{R}_+ \).

Proof. It is obvious, that \( s_\mu(\lambda t) = E_\alpha E_\alpha(-\mu(\lambda t)^\alpha) = E_\alpha(-\mu(\lambda t)^\alpha) = s_{\lambda \mu}(t) \), for all \( \lambda \in \Sigma_\pi \), \( \mu \in \mathbb{C} \) and \( t \in \mathbb{R}_+ \). \( \square \)

Lemma 8.1.4. There is a constant \( C = C(\alpha) > 0 \) such that for \( \varepsilon > 0 \)

\[
\varepsilon - \frac{g_{1-\alpha}(t)}{s_\varepsilon(t)} + g_{1-\alpha}(t) \leq Cg_{1-\alpha}(t),
\]

is valid for all \( t > 0 \).

Proof. Due to the scaling-invariance of the standard kernel \( g_{1-\alpha} \) and the scalar resolvent \( s_\varepsilon \), it is enough to show this relation for \( \varepsilon = 1 \) and all \( t > 0 \). In fact, suppose that we have for all \( t > 0 \) the inequality

\[
1 - \frac{g_{1-\alpha}(t)}{s_1(t)} + g_{1-\alpha}(t) \leq Cg_{1-\alpha}(t).
\]

With the scaling \( t = \lambda s \ (\lambda > 0) \) and the scaling behaviour of the standard kernel \( g_{1-\alpha}(\lambda t) = \lambda^{-\alpha}g_{1-\alpha}(t) \) this inequality is equivalent to the relation

\[
1 - \frac{\lambda^{-\alpha}g_{1-\alpha}(s)}{s_1(\lambda s)} + \lambda^{-\alpha}g_{1-\alpha}(s) \leq C\lambda^{-\alpha}g_{1-\alpha}(s).
\]

From Lemma 8.1.3 we know that \( s_1(\lambda s) = s_{\lambda^\alpha}(s) \). Consequently, we obtain for all \( \lambda > 0 \) and \( s > 0 \) that

\[
\lambda^\alpha - \frac{g_{1-\alpha}(s)}{s_{\lambda^\alpha}(s)} + g_{1-\alpha}(s) \leq Cg_{1-\alpha}(s),
\]

the claim follows with \( \varepsilon = \lambda^\alpha \).

Now, we will prove for \( t > 0 \) that

\[
|s_1(t) - g_{1-\alpha}(t)| \leq C_0g_{1-\alpha}(t)s_1(t).
\]

For all \( t \in (0, 1] \) we have via the triangle inequality that

\[
|s_1(t) - g_{1-\alpha}(t)| \leq s_1(t)\frac{g_{1-\alpha}(t)}{g_{1-\alpha}(1)} + g_{1-\alpha}(t)\frac{s_1(t)}{s_1(t)}.
\]

Since \( s_1(t) \) and \( g_{1-\alpha}(t) \) are monotonically decreasing, we can estimate as follows

\[
|s_1(t) - g_{1-\alpha}(t)| \leq \left[ \frac{1}{g_{1-\alpha}(1)} + \frac{1}{s_1(1)} \right] g_{1-\alpha}(t)s_1(t).
\]
We know from Lemma 2.2.6 that for each $t \in \mathbb{R}_+$ we have the pointwise estimate $s_1(t) \geq (1 + 1/g_{1-a}(t))^{-1}$, in particular $s_1(1) \geq (1 + 1/g_{1-a}(1))^{-1} = (1 + \Gamma(1-a))^{-1}$. Together with $g_{1-a}(1) = 1/\Gamma(1-a)$ we obtain for all $t \in (0,1]$

$$|s_1(t) - g_{1-a}(t)| \leq C_1g_{1-a}(t)s_1(t),$$

where $C_1 = C_1(\alpha) = [1 + 2\Gamma(1-a)] > 0$.

To prove the corresponding inequality for $t \geq 1$ we use Lemma 8.1.5. There is some constant $\tilde{C} = \tilde{C}(\alpha) > 0$ such that for all $t \in (0,\infty)$ we have the estimate $|s_1(t) - g_{1-a}(t)| \leq \tilde{C}t^{-2\alpha}$. Moreover, we have for each $t \in (0,\infty)$ the lower estimate $s_1(t) \geq (1 + 1/g_{1-a}(t))^{-1}$. Since $g_{1-a}$ is monotonically decreasing on $(0,\infty)$ we obtain for $t \geq 1$ the lower estimate $s_1(t) \geq g_{1-a}(t)(1 + g_{1-a}(1))^{-1} = g_{1-a}(t)(1 + 1/\Gamma(1-a))^{-1}$. Together with the definition of the standard kernel we then have that

$$|s_1(t) - g_{1-a}(t)| \leq \tilde{C}\Gamma(1-a)^2g_{1-a}(t)^2 \leq C_2g_{1-a}(t)s_1(t),$$

where $C_2 = C_2(\alpha) = \tilde{C}\Gamma(1-a)^2[1 + 1/\Gamma(1-a)] > 0$. All in all, we have for all $t \in (0,\infty)$ the estimate

$$|s_1(t) - g_{1-a}(t)| \leq C_0g_{1-a}(t)s_1(t),$$

with the constant $C_0 = C_0(\alpha) = \max\{C_1, C_2\}$.

This implies for all $t \in (0,\infty)$ that

$$\left|1 - \frac{g_{1-a}(t)}{s_1(t)}\right| \leq C_0g_{1-a}(t),$$

as well as

$$0 \leq 1 - \frac{g_{1-a}(t)}{s_1(t)} + g_{1-a}(t) \leq \left|1 - \frac{g_{1-a}(t)}{s_1(t)}\right| + g_{1-a}(t) \leq (C_0 + 1)g_{1-a}(t).$$

Hence, the claim follows with $C = C(\alpha) = C_0 + 1$. \hfill $\square$

In the above proof we used the following lemma about the decay behaviour of the difference $s_1 - g_{1-a}$.

**Lemma 8.1.5.** There is some constant $\bar{C} > 0$ such that for all $t \in (0,\infty)$ we have the estimate

$$|s_1(t) - g_{1-a}(t)| \leq \bar{C}t^{-2\alpha}.$$

**Proof.** For each $\lambda \in \mathbb{C}_+$ we know that the Laplace transforms of the functions $s_1$ and $g_{1-a}$ are given by

$$\mathcal{L}s_1(\lambda) = \frac{1}{\lambda + 1 + \lambda^{-\alpha}}, \quad \mathcal{L}g_{1-a}(\lambda) = \lambda^{a-1}.$$
For this reason we have for each \( \lambda \in \mathbb{C}_+ \) that
\[
\widetilde{s}_1(\lambda) - \widetilde{g}_{1-a}(\lambda) = -\frac{\lambda^{2\alpha-1}}{\lambda^a + 1},
\]
as well as
\[
\frac{d}{d\lambda} [\widetilde{s}_1(\lambda) - \widetilde{g}_{1-a}(\lambda)] = -\lambda^{2\alpha-2} \frac{(\alpha - 1)\lambda^a + 2\alpha - 1}{(\lambda^a + 1)^2} := \lambda^{2\alpha-2} h(\lambda).
\]

We want to apply the Analytic Representation Theorem 1.2.1 to the expressions \( \widetilde{s}_1(\lambda) - \widetilde{g}_{1-a}(\lambda) \), for \( \alpha \in (0,1/2) \), and \( d/d\lambda [\widetilde{s}_1(\lambda) - \widetilde{g}_{1-a}(\lambda)] \), for \( \alpha \in [1/2,1) \).

**Case \( \alpha \in (0,1/2) \).** The term \( \lambda^{2\alpha-1}/(\lambda^a + 1) \) possesses an holomorphic extension to \( \Sigma_{\eta} \). By Lemma 2.3.10 we have for each \( \eta \in (0,\pi/2) \) some constant \( c = c(\eta) > 0 \) such that for all \( z \in \Sigma_{\pi/2+\eta} \) we have \( |1 + z| \geq c(1 + |z|) \). We fix some \( \eta \in (0,\pi/2) \) and consider the expression \( \widetilde{s}_1(\lambda) - \widetilde{g}_{1-a}(\lambda) \) on \( \lambda \in \Sigma_{\pi/2+\gamma} \) with \( \gamma = \eta/2 \). We have \( \lambda^a \in \Sigma_{\pi(\frac{\pi}{2} + \gamma)} \subset \Sigma_{\pi/2+\gamma} \) and obtain that
\[
\frac{\lambda^{2\alpha-1}}{\lambda^a + 1} \leq \frac{|\lambda|^{2\alpha-1}}{c(\lambda^a + 1)} \leq \frac{|\lambda|^{2\alpha-1}}{c}.
\]
The map \( g : (0,\infty) \to \mathbb{R}_+ \) with \( g(s) = s^{2\alpha-1}, s \in (0,\infty) \), satisfies all conditions for the Analytic Representation Theorem 1.2.1, thus we have that there is a constant \( C_0 > 0 \) such that for all \( t \in (0,\infty) \) we have \( |s_1(t) - g_{1-a}(t)| \leq C_0 t^{-2\alpha} \).

**Case \( \alpha \in [1/2,1) \).** Obviously, the term \( \lambda^{2\alpha-2}(\lambda^a h(\lambda)) \) possesses a holomorphic extension to \( \Sigma_{\eta} \). We fix some \( \eta \in (0,\pi/2) \) and consider the expression \( \lambda^{2\alpha-2} h(\lambda) \) for all \( \lambda \in \Sigma_{\pi/2+\eta} \), with \( \gamma = \eta/2 \), as above. Since \( \lambda^a \in \Sigma_{\pi/2+\eta} \) we have
\[
\left| \frac{\lambda^{2\alpha-2}(\alpha - 1)\lambda^a + 2\alpha - 1}{(\lambda^a + 1)^2} \right| \leq |\lambda|^{2\alpha-2} \frac{(1 - \alpha)|\lambda|^a + 2|\alpha - 1|}{c^2(|\lambda|^a + 1)^2}.
\]
In our next step we will show that \( h(\lambda) \) is uniformly bounded on \( \Sigma_{\pi/2+\gamma} \). For all \( \lambda \in \Sigma_{\pi/2+\gamma} \) with \( |\lambda| \leq 1 \) we know that
\[
|h(\lambda)| \leq \frac{(1 - \alpha) + |2\alpha - 1|}{c^2} = M_1.
\]
For all \( \lambda \in \Sigma_{\pi/2+\gamma} \) with \( |\lambda| \geq 1 \) we have
\[
|h(\lambda)| \leq \frac{(1 - \alpha) + |2\alpha - 1||\lambda|^{-a}}{c^2(|\lambda|^a + 2 + |\lambda|^{-a})} \leq \frac{(1 - \alpha) + |2\alpha - 1|}{2c^2} = M_2.
\]
All together we have for all \( \lambda \in \Sigma_{\pi/2+\gamma} \) the estimate \( |h(\lambda)| \leq \max\{M_1, M_2\} =: M \). In particular, we obtain for all \( \lambda \in \Sigma_{\pi/2+\gamma} \) the estimate
\[
\left| \frac{d}{d\lambda} [\widetilde{s}_1(\lambda) - \widetilde{g}_{1-a}(\lambda)] \right| \leq M |\lambda|^{2\alpha-2}.
\]
The Analytic Representation Theorem 1.2.1 with \(g(s) = s^{2\alpha - 2}, s \in (0, \infty)\) together with the uniqueness of the Laplace transform, Theorem 1.1.4, yield that there is a constant \(C > 0\) such that for all \(t \in (0, \infty)\) we have the estimate
\[
|t(s_1(t) - g_{1-\alpha}(t))| \leq Ct^{-2\alpha + 1}.
\]
The claim follows with \(\widetilde{C} = \max\{C_0, C\}\).

Finally, we are able to prove the following.

**Lemma 8.1.6.** For each \(\eta > 0\) there is some \(\varepsilon = \varepsilon(\eta) > 0\) such that
\[
\left\| t \mapsto \int_0^t \left[ -\dot{g}_{1-\alpha}(t - \tau) \right] \frac{\varphi_{\varepsilon}(t) - \varphi_{\varepsilon}(\tau)}{\varphi_{\varepsilon}(\tau)} \, d\tau \right\|_{L_p(\mathbb{R}_+)} = \left( \int_0^{\infty} \left| \varepsilon - \frac{g_{1-\alpha}(t)}{s_{\varepsilon}(t)} \right|^p \, dt \right)^{1/p} \leq \eta;
\]
where \(\varphi_{\varepsilon} = 1/s_{\varepsilon}\) and \(s_{\varepsilon}\) denotes the scalar resolvent for the scalar Volterra equation with standard kernel.

**Proof.** Let \(\varepsilon > 0\). We define \(\psi_{\varepsilon} : (0, \infty) \rightarrow \mathbb{R}_+\) via
\[
\psi_{\varepsilon}(t) = \varepsilon + g_{1-\alpha}(t) \left[ 1 - \frac{1}{s_{\varepsilon}(t)} \right].
\]
For each \(t \in (0, \infty)\) we have on one hand that \(0 \leq \psi_{\varepsilon}(t) \leq \varepsilon\), and on the other hand there is some constant \(C = C(\alpha) > 0\) (independent of \(\varepsilon\)) such that we have the estimate \(\psi_{\varepsilon}(t) \leq Cg_{1-\alpha}(t)\), cf. Lemma 8.1.4. Now, let \(\eta > 0\) be arbitrary, but fixed. Then there is some \(t_* = t_*(\eta) > 0\) such that
\[
\left( \int_{t_*}^{\infty} C^p \varepsilon^p g_{1-\alpha}^p(t) \, dt \right)^{1/p} = \frac{Ct_*^{1-p}}{\Gamma(1-\alpha)(1-\alpha p)^{1/p}} \leq \eta/2.
\]
Now, we choose \(\varepsilon = \varepsilon(\eta) > 0\) so small that \(\varepsilon^p t_* \leq (\eta/2)^p\). Then we have the following estimate for the \(L^p\)-norm of \(\psi_{\varepsilon}\):
\[
\left\| \psi_{\varepsilon} \right\|_{L_p(\mathbb{R}_+)} = \left( \int_0^{\infty} |\psi_{\varepsilon}(t)|^p \, dt \right)^{1/p} \leq \left( \int_0^{t_*} |\psi_{\varepsilon}(t)|^p \, dt \right)^{1/p} + \left( \int_{t_*}^{\infty} |\psi_{\varepsilon}(t)|^p \, dt \right)^{1/p} \leq \left( \int_0^{t_*} \varepsilon^p \, dt \right)^{1/p} + \left( \int_{t_*}^{\infty} C^p g_{1-\alpha}^p(t) \, dt \right)^{1/p} \leq \eta/2 + \eta/2 = \eta.
\]
8.2 Stability Theorem

After these preparations we are in the situation to examine quasilinear fractional evolution equations in regards to stability of equilibria. On the basis of time-weighting and rewriting of the equation, together with the maximal $L_p$-regularity assumption on the linearised problem, we can prove asymptotic stability. The optimal decay rate of $t^{-\alpha}$, like the scalar resolvent, we can only prove under an additional regularity assumption on the non-linearities.

We consider again the setting from Chapter 7.

**Assumptions.** The spaces $X_1$ and $X_0$ are Banach spaces for class $\mathcal{HT}$ such that $X_1 \hookrightarrow X_0$ with dense embedding. For $p \in (1, \infty)$ and $\alpha \in (1/p, 1)$ let $V$ be an open subset of the real interpolation space $X_\gamma \equiv (X_0, X_1)_{\frac{1}{\frac{1}{p} + \frac{1}{\alpha}}}$, and $u_0 \in V$. For simplicity we choose the norms in the involved spaces in such a way that $\|v\|_{X_\gamma} \leq \|v\|_{X_1} \leq \|v\|_{X_0}$ for all $v \in X_1$. Moreover, we have maps $(A, F) : V \to \mathcal{B}(X_1, X_0) \times X_0$.

We consider abstract quasilinear parabolic problems of the form

$$\partial^\alpha_t (u - u_0) + A(u)u = F(u), \quad t \in \mathbb{R}_+, \quad u(0) = u_0. \quad (8.3)$$

The trace space of this class of functions is given by $X_\gamma$. We assume for the open set $V \subset X_\gamma$ that

$$(A, F) \in C^1(V; \mathcal{B}(X_1, X_0) \times X_0).$$

Let $u_* \in V \cap X_1$ be an equilibrium of equation (8.3), i.e. $A(u_*)u_* = F(u_*)$.

Introducing the deviation $v = u - u_*$ from the equilibrium $u_*$ and linearising around $u_*$, the equation for $v$ then reads as follows

$$\partial^\alpha_t (v - v_0) + A_0v = G(v), \quad t \in \mathbb{R}_+, \quad v(0) = v_0. \quad (8.4)$$

where $v_0 = u_0 - u_*$ and for all $v \in X_1$ we have

$$A_0v = A(u_*)v + (A'(u_*)v)u_* - F'(u_*)v.$$  

We set $V_* = V - u_*$, then the map $G : V_* \cap X_1 \to X_0$ can be written as $G(v) = G_1(v) + G_2(v, v)$, where for all $w \in X_1$ and all $v \in V_* := V - u_* \subset X_\gamma$ we have

$$G_1(v) = (F(u_* + v) - F(u_*) - F'(u_*)v) - (A(u_* + v) - A(u_*) - A'(u_*)v)u_*, \quad (8.5)$$

$$G_2(v, w) = -(A(u_* + v) - A(u_*))w. \quad (8.6)$$

Before we turn our attention to the asymptotic stability of the equilibrium, we prove the next lemma concerning the growth behaviour of the map $G$ in a neighbourhood of zero on the basis of different regularity assumptions on the maps $A$ and $F$. 

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Lemma 8.2.1. Let $V \subset X_y$ be an open subset, $(A,F) \in C^1(V;B(X_1;X_0) \times X_0)$ be such that for some $u_* \in V \cap X_1$ we have $A(u_*)u_* = F(u_*)$. We set $V_* = V - u_*$. Furthermore, let the map $G: V_* \cap X_1 \to X_0$, given by $G(v) = G_1(v) + G_2(v,v)$, where $G_1$ and $G_2$ are like in (8.5) and (8.6).

Then, for some sufficiently small $r_0 > 0$ we have $B_{r_0}^X(0) \subset V_*$ and the following is true.

(i) There is some constant $C_0 = C_0(r_0) > 0$ such that for all $\eta > 0$ there is some $r = r(\eta) \in (0,r_0]$ such that for all $v_1, v_2 \in B_{r}(0) \cap X_1$ we have
\[
\|G(v_1) - G(v_2)\|_{X_0} \leq C_0 \bigl( \eta + r \|v_2\|_{X_1} \bigr) \|v_1 - v_2\|_{X_1}.
\]

In particular, we have for all $v \in B_{r}(0) \cap X_1$ that
\[
\|G(v)\|_{X_0} \leq C_0 \eta r \|v\|_{X_1}.
\]

(ii) For $(A,F) \in C^2(V;B(X_1;X_0) \times X_0)$ we have that there is some constant $M_0 = M_0(r_0) > 0$ such that for all $\delta \in (0,\eta] \cap \mathbb{R}$ and $v \in B_{\delta}^X(0) \cap X_1$ we have the estimate
\[
\|G(v)\|_{X_0} \leq M_0 \|v\|_{X_1} \|v\|_{X_1}.
\]

Proof. From the regularity assumption we obtain that $G_1 \in C^1(V;X_0)$ and $G_2 \in C^1(V \times X_1;X_0)$. Moreover, we have
\[
G_1(0) = G_2(0,0) = 0 \in X_0,
\]
\[
G'_1(0) = 0 \in B(X_1;X_0),
\]
\[
G'_2(0,0) = 0 \in B(X_1 \times X_1;X_0),
\]
where $G'_1$ and $G'_2$ denote the Fréchet derivatives of $G_1$ and $G_2$, respectively.

The set $V$ is open in $X_y$, with $u_* \in V$, this implies that there is some $r_0 > 0$ such that $B_{r_0}^X(0) \subset V_* = V - u_*$. 

(i) The assumptions imply that $A: V \to B(X_1;X_0)$ and $F: V \to X_0$ are continuously Fréchet-differentiable at $u_* \in V$. Hence, there is a bounded operator $A'(u_*) \in B(X_1;B(X_1;X_0))$ such that for all $\eta > 0$ there is an $r = r(\eta) \in (0,r_0]$ such that for all $v \in B_r^X(0)$ we have the estimate
\[
\|A(u_* + v) - A(u_*) - A'(u_*)v\|_{B(X_1;X_0)} \leq \eta \|v\|_{X_1}.
\]

The analogous statement holds for the map $F$. There is an operator $F'(u_*) \in B(X_1;X_0)$ such that for all $\eta > 0$ there is an $r = r(\eta) \in (0,r_0]$ such that for all $v \in B_r^X(0)$ we have
\[
\|F(u_* + v) - F(u_*) - F'(u_*)v\|_{X_0} \leq \eta \|v\|_{X_1}.
\]
A continuously Fréchet-differentiable map is locally Lipschitz continuous, for this reason there is a constant $L > 0$ (depending only on a appropriately small $r_0$) such that for all $r \in (0, r_0]$ and all $v \in B_r^{X_r}(0)$ it follows that

$$
\|A(u_\ast + v) - A(u_\ast)\|_{g(X_1; X_0)} \leq L \|v\|_{X_r}.
$$

With these properties it is clear, that we have the following estimates for the map $G = G_1 + G_2 : V_\ast \rightarrow X_0$. For each $\eta > 0$ there is an $r = r(\eta) \in (0, r_0]$ such that for all $v_1, v_2 \in B_r^{X_r}(0)$ we obtain

$$
\|G_1(v_1) - G_1(v_2)\|_{X_0} \leq \eta \|v_1 - v_2\|_{X_r},
$$

for all $w \in X_1$ and $v_1, v_2 \in B_r^{X_r}(0)$ we have

$$
\|G_2(v_1, w) - G_2(v_2, w)\|_{X_0} \leq L \|w\|_{X_1} \|v_1 - v_2\|_{X_r},
$$

and for $w_1, w_2 \in X_1$ and $v \in B_r^{X_r}(0)$ we get

$$
\|G_2(v, w_1) - G_2(v, w_2)\|_{X_0} \leq L \|w_1 - w_2\|_{X_1}.
$$

Note that the constant $L$ does not depend on $r \in (0, r_0]$ with $r_0 > 0$ appropriately chosen. Combining these estimates, we have for all $v_1, v_2 \in B_r^{X_r}(0) \cap X_1$ the estimate

$$
\|G(v_1) - G(v_2)\|_{X_0} \leq (\eta + L \|v_2\|_{X_1}) \|v_1 - v_2\|_{X_r} + L \|v_1 - v_2\|_{X_1}.
$$

where the constant $C_0 = \max\{1, L\}$ is independent of $r \in (0, r_0]$. Now, we choose $v_2 = 0$ and remember that $G(0) = 0 \in X_0$. Hence, for all $v \in B_r^{X_r}(0) \cap X_1$ it follows that

$$
\|G(v)\|_{X_0} \leq C_0(\eta + r) \|v\|_{X_1}.
$$

(ii) By the assumption on the maps $A$ and $F$ we deduce $G_1 \in C^2(V_\ast; X_0)$ and $G_2 \in C^2(V \times X_1; X_0)$. Now, we consider the Taylor expansion of $G$ at the point 0. For all $v \in V_\ast \cap X_1$ we have

$$
G(v) = G(0) + G'(0)v + \int_0^1 (1 - t)G''(tv)(v,v) dt.
$$

Due to the linearity of the Fréchet derivative we have to consider the Taylor expansion of $G_1$ and $G_2$ at the point 0, respectively.
Taylor expansion of $G_1$ at the point $0$. For all $v \in V$, we have

$$G_1(v) = G_1(0) + G'_1(0)v + \int_0^1 (1-t)G''_1(tv)(v,v)\,dt$$

since $G_1(0) = 0$ and $G'_1(0)v = 0$. The first derivative $G'_1: V \to B(X_Y;X_0)$ and second derivative $G''_1: V \to B(X_Y;B(X_Y;X_0))$ are given by

$$G'_1(v)h = [F'(u_0 + v) - F'(u_0)] h - [A'(u_0 + v) - A'(u_0)] h_0,$$

$$G''_1(v)(h,\bar{h}) = F''(u_0 + v) (h,\bar{h}) - (A''(u_0 + v) (h,\bar{h})) u_0,$$

for all $v \in V$ and $h, \bar{h} \in X_Y$. For each $v \in V$, we have

$$\|G_1(v)\|_{X_0} \leq \sup_{t \in [0,1]} \|G''_1(tv)(v,v)\|_{X_0}$$

$$\leq \sup_{t \in [0,1]} \left( \|F''(u_0 + tv)(v,v)\|_{X_0} + \|(A''(u_0 + tv)(v,v))u_0\|_{X_0} \right)$$

$$\leq \sup_{t \in [0,1]} \|F''(u_0 + tv)\|_{B^2(X_Y;X_0)} \|v\|_{X_Y}^2$$

$$+ \sup_{t \in [0,1]} \|(A''(u_0 + tv))u_0\|_{X_0} \|v\|_{X_Y} \|u_0\|_{X_1}.$$

Taylor expansion of $G_2$ at the point $(0,0)$. We have for all $v \in V$ and $w \in X_1$ that

$$G_2(v, w) = G_2(0,0) + G'_2(0,0)(v, w) + \int_0^1 (1-t)G''_2(tv,tw)((v, w), (v, w))\,dt$$

since $G_2(0,0) = 0$ and $G'_2(0,0) = 0$. Furthermore, the first derivative $G'_2: V \times X_1 \to B(X_Y \times X_1;X_0)$ as well as second derivative $G''_2: V \times X_1 \to B(X_Y \times X_1;B(X_Y \times X_1;X_0))$ are given by

$$G'_2(v, w)(h_1, h_2) = -(A'(u_0 + v)h_1) w - (A(u_0 + v) - A(u_0)) h_2,$$

$$G''_2(v, w)((h_1, h_2), (\bar{h}_1, \bar{h}_2)) = -\left( A''(u_0 + v)(h_1, \bar{h}_1) \right) w$$

$$- (A'(u_0 + v) h_1) \bar{h}_2 - \left( A'(u_0 + v) \bar{h}_1 \right) h_2,$$

for all $v \in V$ and $h_1, \bar{h}_1 \in X_Y$, and all $w \in X_1$ and $h_2, \bar{h}_2 \in X_1$. For all $v \in V$ and $w \in X_1$ we have

$$\|G_2(v, w)\|_{X_0} \leq \sup_{t \in [0,1]} \left\| G''_2(tv,tw)((v, w), (v, w)) \right\|_{X_0}.$$
This yields for all $\delta \in (0, r_0]$ and all $v \in B_{X_0}^\delta$ that

$$
\|G(v)\|_{X_0} \leq \| G_1(v) \|_{X_0} + \| G_2(v, v) \|_{X_0}
\leq M_F \|v\|_{X_0} \|v\|_{X_i} + M_{A_1} (\|u_1\|_{X_i} + \|v\|_{X_0}) \|v\|_{X_i} + 2 M_{A_1} \|v\|_{X_0} \|v\|_{X_i}.
$$

and hence

$$
\|G(v)\|_{X_0} \leq M_0 \|v\|_{X_0} \|v\|_{X_1} + 2 M_{A_1}.
$$

with $M_0(r_0) = M_F + M_{A_2} (\|u_1\|_{X_1} + r_0) + 2 M_{A_1}$. 

\[\square\]
We have the following terms of stability for the equilibrium $u_*$ of the quasilinear equation (8.3). Basically, these notions are similar to the semilinear case, cf. Definition 4.2.1, but we recall them for the sake of completeness.

**Definition 8.2.2 (Stability).**

(i) We call the equilibrium $u_*$ **stable in** $X_\gamma$ for the quasilinear equation (8.3) if for each $\varepsilon > 0$ there is some $\delta = \delta(\varepsilon) > 0$ such that for all $u_0 \in B_X^{X_\gamma}(u_*)$ a solution $u = u(\cdot; u_0) \in \mathcal{E}_1(\mathbb{R}_+) \cap C(\mathbb{R}_+; X_\gamma)$ of the equation (8.3) exists and $\|u - u_*\|_{C(\mathbb{R}_+; X_\gamma)} < \varepsilon$.

(ii) We call the equilibrium $u_*$ **asymptotically stable in** $X_\gamma$ with an algebraic decay rate with exponent $\alpha$, if $u_*$ is stable in $X_\gamma$ for the quasilinear equation (8.3) and there is some $\delta_0 > 0$ and a constant $C > 0$ such that for all $u_0 \in B_X^{X_\gamma}(u_*)$ the solution $u = u(\cdot; u_0)$ satisfies the estimate $\|u(t) - u_*\|_{X_\gamma} \leq C\|u_0\|_{X_\gamma} t^{-\alpha}, t \in (0, \infty)$.

Now, we formulate our stability result for the abstract quasilinear parabolic problem (8.3). The peculiarity of this result is the different decay rates according to the regularity assumptions on the maps $A$ and $F$.

**Theorem 8.2.3 (Stability Theorem).** Let $X_0$ and $X_1$ be Banach spaces of class $\mathcal{HT}$ such that $X_1 \hookrightarrow X_0$ is a dense embedding. Let $p \in (1, \infty)$ and $\alpha \in (1/p, 1)$. Furthermore, let $V$ be an open subset of the real interpolation space $X_\gamma = (X_0, X_1)_{1-\frac{1}{\alpha p}, p}$.

Suppose that $(A, F) \in C^1(V; B(X_1; X_0) \times X_0)$ and assume that $u_* \in V \cap X_1$ is an equilibrium of (8.3), i.e. $A(u_*)u_* = F(u_*)$.

For the linearisation of (8.3) given by $A_0v = A(u_*)v + (A'(u_*)v)u_* - F'(u_*)v$, $v \in X_1$, we assume that the operator $A_0 \in \mathcal{R}(X_0)$ is invertible with $\varphi_{A_0}^\mathbb{R} > \pi - \frac{\pi}{\alpha}$, i.e. the operator $A_0$ has the property of maximal $L_p$-regularity on $\mathbb{R}_+$.

(i) Then the equilibrium $u_*$ is stable in $X_\gamma$ for the quasilinear equations (8.3).

(ii) Moreover, the equilibrium $u_*$ is asymptotically stable in $X_\gamma$ with an algebraic decay rate with exponent between 0 and $\alpha - 1/p$.

Under the additional assumption $(A, F) \in C^2(V; B(X_1; X_0) \times X_0)$ we have that

(iii) The decay rate in $X_\gamma$ is algebraic with exponent $\alpha$.

**Remark 8.2.4.** Obviously, the asymptotic stability property of the equilibrium $u_*$ in part (ii) is stronger than the stability statement in part (i) of Theorem 8.2.3. But in regard to the following proof and the used arguments it seems to be useful to split these both statements.

**Proof of Theorem 8.2.3.** Firstly, we show that the assumptions on the operator $A_0$ imply that there is some $\mu \geq 0$ such that $\mu + A(u_*) \in \mathcal{R}(X_0)$ with $\varphi_{\mu + A(u_*)}^\mathbb{R} < \pi - \frac{\pi}{\alpha}$, i.e. the operator $A(u_*)$ has maximal $L_p$-regularity on each compact
interval $[0,T]$. Indeed, we have $A(u_t) = A_0 - S$, where $Sv = (A'(u_t)v)u_t + F'(u_t)v$ for all $v \in X_y$. The assumption on the maps $A$ and $F$ imply that $S \in B(X_y; X_0)$. Moreover, observe that due to the invertibility of the operator $A_0$ and the definition of the real interpolation space $X_y$, we have for all $v \in X_1$ that $\|Sv\|_{X_0} \leq C \|v\|_{X_0}^{1-\theta} \|A_0v\|_{X_0}^\theta$, where $\theta = 1 - \frac{1}{\alpha p} \in (0,1)$ is the interpolation parameter. Hence, by means for Young’s inequality we deduce for all $\varepsilon > 0$ and all $v \in X_1$ that

$$\|Sv\|_{X_0} \leq C \left( \varepsilon \theta \|A_0v\|_{X_0} + (1-\theta)e^{-\theta/(1-\theta)}\|v\|_{X_0} \right),$$

cf. [DHP03, p. 12]. Thus, $S$ is a relatively bounded perturbation of $A_0$ with small relative bound. Now, we choose $\varepsilon > 0$ sufficiently small, such that $\varepsilon < C_{A_0}/(1 + a)$, where $a = R^\lambda (\lambda + A_0)^{-1} : \lambda \in \Sigma_0$, $\theta > \varphi_{A_0}^R$, and $C_{A_0} = \sup_{\lambda > 0} \|A_0(\lambda + A_0)^{-1}\|_{B(X_0)}$. Then, we choose a sufficiently large parameter $\mu > \beta M_{A_0}(1 + a)/(1 - \varepsilon C_{A_0}(1 + a))$, where $M_{A_0} = \sup_{\lambda > 0} \|\lambda (\lambda + A_0)^{-1}\|_{B(X_0)}$. The perturbation result for relatively bounded perturbations [DHP03, Proposition 4.3] implies that $\mu + A(u_t) \in \mathcal{R}(X_0)$ and $\varphi_{\mu + A_0}(u_t) = \varphi_{A_0}^R < \pi - \alpha \frac{\pi}{2}$. Hence $A(u_t)$ has maximal $L_p$-regularity on each compact interval $[0,T]$.

Thus, the quasilinear problem (8.3) is locally well-posed for all initial values $u_0 \in X_y$ in a small ball around $u_0$, see Theorem 7.1.1.

(i) We first choose $r_0 > 0$ sufficiently small and $C_0 > 0$ as in Lemma 8.2.1 (i); next, we choose $\eta \in (0,1/(8C_{MR} C_0)]$. By Lemma 8.2.1 we may choose $r = r(\eta) \in (0, \min\{r_0, 1/(8C_{MR} C_0)\}]$ such that the estimates in Lemma 8.2.1 (i) hold true. Here, $C_{MR}$ denotes the constant of maximal $L_p$-regularity, cf. Remark 6.2.2 a).

If $r_0 > 0$ is sufficiently small, we know by the local well-posedness of the equation that there is some $T > 0$ such that for all $v_0 \in \overline{B}_{r_0}^{X_y}(0) \subset V$, there is a unique local strong solution $v$ of the equation (8.4) in the class $v \in \mathcal{E}_1(T) \cap C([0,T];X_y)$.

The solution $v$ with initial value $v_0 \in B_{r_0}^{X_y}(0)$ can be extended to a maximal interval of existence $[0,t_*]$. If $t_*$ is finite, then either $v$ leaves the ball $B_{r}^{X_y}(0)$ at time $t_*$, or the limit $\lim_{t \to t_*} v(t)$ does not exist in $X_y$. We show that this cannot happen for initial values $v_0 \in B_{\delta}^{X_y}(0)$, with sufficiently small $\delta \leq r$ to be chosen later. Now, we define the exit time for the ball $B_{\delta}^{X_y}(0)$, that is

$$t_0 := \sup\{t \in (0,t_*): \|v(t)\|_{X_y} \leq r \text{ for all } t \in [0,t]\},$$

1For the proof of part (i) and (ii) it would be enough to claim $C_{MR} C_0(\eta + r) \leq 1/2$, but in part (iii) it is a technical necessity to have $C_{MR} C_0(\eta + r) \leq 1/4$. For uniform estimates in all three parts of the proof we require the parameters with the stronger property.
and suppose that \( t_0 < t_* \). We know that \( v \in \mathcal{E}_1(t_0) \cap C([0, t_0]; X_\gamma) \) is the unique local strong solution of (8.4), hence we have for almost all \( t \in [0, t_0] \) that \( v(t) \in X_1 \cap B_{r^\gamma}(0) \). The properties of the map \( G \) yield for almost all \( t \in [0, t_0] \) that \( \|G(v(t))\|_{X_0} \leq C_0(\eta + r)\|v(t)\|_{X_1} \), and hence

\[
\|G(v)\|_{\mathcal{E}_0(t_0)} \leq C_0(\eta + r)\|v\|_{\mathcal{E}_1(t_0)}. \tag{8.7}
\]

To prove the stability of the equilibrium \( v_* = 0 \) of the equation (8.4), we decompose our solution into the part which comes from the initial value \( v_0 \) and the remainder \( \tilde{v}(t) = v(t) - S(t)v_0 \). The remainder \( \tilde{v} \) solves the problem

\[
\partial_t^\gamma \tilde{v} + A_0\tilde{v}(t) = G(v), \quad t \in [0, t_*), \quad \tilde{v}(0) = 0.
\]

As usual \( S(\cdot)v_0 \) solves the corresponding resolvent equation for the operator \( A_0 \). We estimate the \( \mathcal{E}_1(t_0) \)-norm of \( v \) via \( \|v\|_{\mathcal{E}_1(t_0)} \leq \|S(\cdot)v_0\|_{\mathcal{E}_1(t_0)} + \|\tilde{v}\|_{\mathcal{E}_1(t_0)} \). By Proposition 6.2.5 there is some constant \( C_1 > 0 \) independent of \( t_0 \) such that \( \|S(\cdot)v_0\|_{\mathcal{E}_1(t_0)} \leq \|S(\cdot)v_0\|_{\mathcal{E}_1(R)} \leq C_1\|v_0\|_{X_\gamma} \). On the other hand we have by maximal \( L_p \)-regularity of the operator \( A_0 \) for the remainder \( \tilde{v} \) the estimate

\[
\|\tilde{v}\|_{\mathcal{E}_1(t_0)} \leq C_{MR}\|G(v)\|_{\mathcal{E}_0(t_0)}.
\]

observe the independence of the constant \( C_{MR} > 0 \) of the time \( t_0 \). By the estimate (8.7) we obtain with the above choice of \( \eta \) and \( r \) that

\[
\|v\|_{\mathcal{E}_1(t_0)} \leq C_0C_{MR}(\eta + r)\|v\|_{\mathcal{E}_1(t_0)} \leq \frac{1}{2}\|v\|_{\mathcal{E}_1(t_0)}.
\]

The decomposition of the solution \( v \) yields

\[
\|\tilde{v}\|_{\mathcal{E}_1(t_0)} \leq \frac{1}{2}\left(\|S(\cdot)v_0\|_{\mathcal{E}_1(t_0)} + \|\tilde{v}\|_{\mathcal{E}_1(t_0)}\right),
\]

and hence \( \|\tilde{v}\|_{\mathcal{E}_1(t_0)} \leq C_1\|v_0\|_{X_\gamma} \). On the whole, we have with the uniform constant \( C_1 > 0 \) the estimate \( \|v\|_{\mathcal{E}_1(t_0)} \leq 2C_1\|v_0\|_{X_\gamma} \).

Additionally, we consider the \( X_\gamma \)-norm of \( v(t) \) for \( t \in [0, t_0] \): \( \|v(t)\|_{X_\gamma} \leq \|S(t)v_0\|_{X_\gamma} + \|\tilde{v}(t)\|_{X_\gamma} \). By Corollary 3.3.6, we have for all \( t \in \mathbb{R}_+ \) that \( \|S(t)v_0\|_{X_\gamma} \leq C_\gamma\|v_0\|_{X_\gamma} \). The remainder \( \tilde{v} \) has trace zero, so we can use the embedding \( \mathcal{E}_1(t_0) \hookrightarrow C([0, t_0], X_\gamma) \) to obtain with a uniform constant \( C_{emb} > 0 \) that \( \|\tilde{v}\|_{C([0, t_0], X_\gamma)} \leq C_{emb}\|\tilde{v}\|_{\mathcal{E}_1(t_0)} \). Together with the above estimate on the \( \mathcal{E}_1(t_0) \)-norm of \( \tilde{v} \) we have

\[
\|\tilde{v}\|_{C([0, t_0], X_\gamma)} \leq C_{emb}C_1\|v_0\|_{X_\gamma}.
\]
Combining the estimates for the initial-value part and the remainder yields
\[ \|v\|_{C([-t_0,0])} \leq \|S(t)v_0\|_{C([-t_0,0])} + \|v\|_{C([-t_0,0])} \leq C_{\gamma} \|v_0\|_{X_{\gamma}} + C_{emb} \|v_0\|_{X_{\gamma}}, \]

i.e. we have the uniform estimate \( \|v(t)\|_{X_{\gamma}} \leq K \|v_0\|_{X_{\gamma}} \) for all \( t \in [0,t_0] \), where \( K = C_{\gamma} + C_{emb} > 0 \). In particular, for all \( v_0 \in B_{\delta}^{X_{\gamma}}(0) \) with \( 0 < \delta \leq \delta_1 = \min\{\varepsilon/(2K), \varepsilon\} \) we have for \( t = t_0 \) that
\[ \|v(t_0)\|_{X_{\gamma}} \leq r/2. \]

This is a contradiction to the definition of \( t_0 \), and we conclude that \( t_0 = t^* \).

The above arguments yield for each \( \delta \in (0,\delta_1] \) and all \( v_0 \in B_{\delta}^{X_{\gamma}}(0) \) the uniform estimates \( \|v\|_{E_1(T)} \leq 2C_{1} \|v_0\|_{X_{\gamma}} \), as well as \( \|v(t)\|_{C([-t_0,0])} \leq r/2 \) for all \( T < t^* \). This implies \( t^* = \infty \), the global existence of the solution \( v \in E_1(\mathbb{R}_+) \) with \( \|v\|_{E_1(\mathbb{R}_+)} \leq 2C_{1} \|v_0\|_{X_{\gamma}} \) and \( v(t) \in B_{\delta}^{X_{\gamma}}(0) \subset V_{\ast} \) for all \( t \in \mathbb{R}_+ \), and the stability of the equilibrium \( v_\ast = 0 \) in \( X_{\gamma} \). Indeed, given \( \varepsilon > 0 \) we choose \( \eta \) and \( r \) as before and \( \delta < \min\{\delta_1, \varepsilon/K\} \), thus we have \( \|v\|_{C(\mathbb{R}_+,X_{\gamma})} \leq \varepsilon \) for all \( v_0 \in B_{\delta}^{X_{\gamma}}(0) \).

(ii) We choose \( r_0 > 0 \), \( C_0 > 0 \) as well as \( \eta \in (0,1/(8C_{MR}C_0)] \) and \( r = r(\eta) \in (0,\min\{r_0, 1/(8C_{MR}C_0)\}) \) as in the first part (i). We use the stability property of the equilibrium \( v_\ast = 0 \) and fix for \( \varepsilon > r \) the corresponding parameter \( \delta_0 = \delta_0(\varepsilon) > 0 \) such that for all \( v_0 \in B_{\delta_0}^{X_{\gamma}}(0) \) there is a solution \( v \in E_1(\mathbb{R}_+) \cap C(\mathbb{R}_+,X_{\gamma}) \) and \( v(t) \in B_{\delta_0}^{X_{\gamma}}(0) \) for all \( t \in \mathbb{R}_+ \). We use again the decomposition \( v = S(t)v_0 + \tilde{v} \) where \( \tilde{v} \) solves the problem
\[ \partial_t^\alpha \tilde{v} + A_0 \tilde{v} = G(v), \quad t \in \mathbb{R}_+, \quad \tilde{v}(0) = 0. \]

To study the decay behaviour of the solution \( v \) we examine every part of the decomposition by its own.

a) Decay behaviour of the initial-value part \( S(\cdot)v_0 \) in \( X_{\gamma} \). The decay behaviour of the linear part is given by Corollary 3.3.6, saying that for all \( t \in \mathbb{R}_+ \) we have a constant \( C_{\gamma} > 0 \) such that \( \|S(t)v_0\|_{X_{\gamma}} \leq C_{\gamma} \min\{1,t^{-\alpha}\} \|v_0\|_{X_{\gamma}} \).

b) Decay behaviour of the remainder term \( \tilde{v} \) in \( X_{\gamma} \). To examine the decay behaviour of the remainder term \( \tilde{v} \), we consider for some
appropriate function \( \varphi_{\varepsilon_0} \) the product \( w = \varphi_{\varepsilon_0} \tilde{v} \) and show that this expression is uniformly bounded in \( X_\gamma \). This yields the decay behaviour of \( 1/\varphi_{\varepsilon_0} \) for the remainder \( \tilde{v} \).

b) (i) Introducing the time-weighted remainder \( w \). We fix some \( \xi \in (0, \alpha - 1/p) \) and set for all \( t \in \mathbb{R}_+ \)

\[
\varphi_{\varepsilon_0}(t) = 1 + \varepsilon_0 \frac{t}{1 + t^{1-\xi}},
\]

with some arbitrary \( \varepsilon_0 > 0 \) which will be chosen later. Now, we multiply the equation (8.8) for the remainder term \( \tilde{v} \) by \( \varphi_{\varepsilon_0} \). For \( w = \varphi_{\varepsilon_0} \tilde{v} \), this yields the equation

\[
\partial_t^\alpha w(t) + A_0 w(t) = \varphi_{\varepsilon_0}(t) G\big( w(t)/\varphi_{\varepsilon_0}(t) + S(t)v_0 \big) + \int_0^t \left[-\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau) \right] \frac{\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau)}{\varphi_{\varepsilon_0}(\tau)} w(\tau) d\tau, \quad t \in \mathbb{R}_+,
\]

with \( w(0) = 0 \), using Lemma 8.1.1. Our aim is to show that \( \|w(t)\|_{X_\gamma} \) is uniformly bounded for all \( t \in \mathbb{R}_+ \). To achieve this, we estimate the solution \( w \) by using the property of maximal \( L_p \)-regularity of the operator \( A_0 \). We will show that every term on the right hand side of (8.9) is an element of \( L_p(\mathbb{R}_+;X_\gamma) = \mathcal{E}_0(\mathbb{R}_+) \).

b) (ii) Estimates for the commutator term of the time-weighted remainder \( w \). Firstly, we consider the map

\[
\begin{align*}
\mathbb{R}_+ & \ni t \mapsto \int_0^t \left[-\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau) \right] \frac{\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau)}{\varphi_{\varepsilon_0}(\tau)} w(\tau) d\tau.
\end{align*}
\]

Since \( \tilde{v} = v - S(\cdot)v_0 \in 0 \mathcal{E}_1(\mathbb{R}_+) \cap C(\mathbb{R}_+;X_\gamma) \) and \( \varphi_{\varepsilon_0} \) is continuous on \( \mathbb{R}_+ \) it is obvious that \( w \) is continuous on \( \mathbb{R}_+ \) with values in \( X_\gamma \) and for each \( T > 0 \) we have \( w \in 0 \mathcal{E}_1([0,T]) \cap C([0,T];X_\gamma) \).

Let w.l.o.g. \( T \geq 1 \) be arbitrary, but fixed. We have by Lemma 8.1.2 that

\[
\begin{align*}
\int_0^t \left[-\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau) \right] \frac{\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau)}{\varphi_{\varepsilon_0}(\tau)} w(\tau) d\tau & \leq \|w\|_{C([0,T];X_\gamma)} \int_0^t \left[-\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau) \right] \frac{\varphi_{\varepsilon_0}(t) - \varphi_{\varepsilon_0}(\tau)}{\varphi_{\varepsilon_0}(\tau)} d\tau \\
& \leq \varepsilon_0 C \|w\|_{C([0,T];X_\gamma)} \\
& \leq \varepsilon_0 C C_{emb} \|w\|_{\mathcal{E}_1([0,T])},
\end{align*}
\]
where the constant $C = C(\alpha, p, \varepsilon)$ is independent of $T$ and $\varepsilon_0$, the embedding constant $C_{emb} > 0$ enjoys the same property since $w$ has trace zero. Thus, we know that

$$
t \mapsto \int_0^t [-\dot{g}_{1-a}(t-\tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} w(\tau) d\tau \in L_p([0,T];X_0).
$$

**b) (iii) Estimates for the map $t \mapsto q_{\varepsilon_0}(t)G(v(t))$.** Now, we turn to the expression

$$
t \mapsto q_{\varepsilon_0}(t)G(v(t)) = q_{\varepsilon_0}(t)G\left(\frac{w(t)}{q_{\varepsilon_0}(t)} + S(t)v_0\right)
$$

and the question, whether this map belongs to $L_p([0,T];X_0)$. Due to the choice of $\varepsilon = r \in (0,r_0]$ and $\delta_0 = \delta_0(\varepsilon)$, we have for all initial values $v_0 \in B_\varepsilon^{X_0}(0)$ that $v(t) \in B_\varepsilon^{X_0}(0)$, $t \in \mathbb{R}_+$. Moreover, the previous proof of the stability part (i) implies $v \in \mathcal{E}_1(\mathbb{R}_+)$ and the estimate $\|v\|_{\mathcal{E}_1(\mathbb{R}_+)} \leq 2C_1 \|v_0\|_{X_0}$. In particular, we have $v(t) \in X_1$ for almost all $t \in \mathbb{R}_+$. Together with the properties of the map $G$ from Lemma 8.2.1 (i) we obtain the estimate

$$
\|G(v)\|_{L_p([0,1];X_0)} = \|G(v)\|_{\mathcal{E}_0(1)} \\
\leq C_0(\eta + \varepsilon) \|v\|_{\mathcal{E}_1(1)} \\
\leq C_0(\eta + \varepsilon) 2C_1 \|v_0\|_{X_0} \\
\leq C_1 \|v_0\|_{X_0},
$$

since $C_0(\eta + \varepsilon) \leq 1/2$. Moreover, for all $t \in [0,1]$ and $\varepsilon_0 \in [0,1]$ we have $1 \leq q_{\varepsilon_0}(t) \leq 2$, it follows that $\|q_{\varepsilon_0}G(v)\|_{L_p([0,1];X_0)} \leq 2C_1 \|v_0\|_{X_0}$. Using the properties of the map $G$ from Lemma 8.2.1 (i) as well as the fact that for almost all $t \in \mathbb{R}_+$ we have that $v(t) \in B_\varepsilon^{X_0} \cap X_1$, the same considerations as mentioned above imply

$$
\|q_{\varepsilon_0}G(v)\|_{L_p([1,T];X_0)} \\
\leq C_0(\eta + \varepsilon) \left(\int_1^T \left\|q_{\varepsilon_0}(t)v(t)\right\|_{X_1}^p dt\right)^{1/p} \\
\leq C_0(\eta + \varepsilon) \left[\left(\int_1^T \|w\|_{X_1}^p dt\right)^{1/p} + \left(\int_1^\infty \left\|q_{\varepsilon_0}(t)v_0\right\|_{X_1}^p dt\right)^{1/p}\right].
$$

Via Corollary 3.3.7 (iii) there is a constant $M > 0$ such that we have the estimate $\|S(t)\|_{\mathcal{B}(X_0;X_1)} \leq Mt^{-\alpha}$ for all $t > 0$, and hence
8. Stability for Quasilinear Fractional Evolution Equations

Using again the embedding theorem yields the uniform estimate

\[ \| q_{\varepsilon_0} G(v) \|_{L_p([1,T];X_0)} \leq C_0(\eta + \varepsilon) \left( \| w \|_{\mathcal{E}_1(T)} + M \| \nu_0 \|_{X_0} \left( \int_1^T q_{\varepsilon_0}(t) t^{-\alpha} \, dt \right)^{1/p} \right). \]

On can easily show that for \( \varepsilon_0 \in [0,1] \) we have the uniform estimate

\[ \| t \mapsto q_{\varepsilon_0}(t) t^{-\alpha} \|_{L_p((1,\infty))} \leq (\alpha p - 1)^{-1/p} + (p(\alpha - \xi) - 1)^{-1/p} = M_1(\alpha, p, \xi) = M_1; \]

observe the independence of \( T \) and \( \varepsilon_0 \). Consequently, we have

\[ \| q_{\varepsilon_0} G(v) \|_{L_p([0,T];X_0)} \leq 2C_1 \| \nu_0 \|_{X_p} + C_0(\eta + \varepsilon) \left( \| w \|_{\mathcal{E}_1(T)} + M \| \nu_0 \|_{X_0} M_1 \right) < \infty, \]

thus \( t \mapsto q_{\varepsilon_0}(t) G(v(t)) \in L_p([0,T];X_0). \)

b) (iv) Estimates for the time-weighted remainder \( w \). By the maximal \( L_p \)-regularity property of the operator \( A_0 \) in equation (8.9) we have the estimate

\[ \| w \|_{\mathcal{E}_1(T)} \leq C_{MR} \left\{ \| q_{\varepsilon_0} G(v) \|_{\mathcal{E}_0(T)} + \left\| \int_0^t \left[ -\dot{g}_{1-\alpha}(t-\tau) \right] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} w(\tau) \, d\tau \right\|_{\mathcal{E}_0(T)} \right\} \]

\[ \leq C_{MR} \left\{ 2C_1 \| \nu_0 \|_{X_p} + C_0(\eta + \varepsilon) \| w \|_{\mathcal{E}_1(T)} + C_0(\eta + \varepsilon) M \| \nu_0 \|_{X_0} M_1 + \varepsilon_0 C_{emb} \| w \|_{\mathcal{E}_1(T)} \right\}. \]

Since \( C_{MR} C_0(\eta + \varepsilon) \leq 1/2 \) by the choice of the parameters above, we have

\[ \| w \|_{\mathcal{E}_1(T)} \leq C_{MR} \left\{ 4C_1 + MM_1 \right\} \| \nu_0 \|_{X_p} + 2 \varepsilon_0 C_{emb} \| w \|_{\mathcal{E}_1(T)}. \]

Now, we fix \( \varepsilon_0 = \min\{(4C_{emb} C_{MR} C)^{-1}, 1\} \), and it follows

\[ \| w \|_{\mathcal{E}_1(T)} \leq 2C_{MR} \left[ 4C_1 + MM_1 \right] \| \nu_0 \|_{X_p}. \]

Using again the embedding theorem yields the uniform estimate

\[ \| \dot{w} \|_{C([0,T];X_0)} \leq C_{emb} \| w \|_{\mathcal{E}_1(T)} \leq 2C_{emb} C_{MR} \left[ 4C_1 + MM_1 \right] \| \nu_0 \|_{X_0}. \]

Hence, we have for each \( t \in [0,T] \) and all \( \nu_0 \in B_{\delta_0}^{X_p}(0) \) that

\[ \| w(t) \|_{X_p} \leq \| \dot{w} \|_{C([0,T];X_0)} \leq M_2 \| \nu_0 \|_{X_p} < \infty, \]
with $M_2 = 2C_{emb}C_{MR}[4C_1 + MM_1]$ independent of $T > 0$. This uniform estimate implies for each $t \in \mathbb{R}_+$ that $\|w(t)\|_{X_r} \leq M_2\|v_0\|_{X_r}$ for all $v_0 \in B_{\delta_0}^{X_r}(0)$. Recalling the definition $w = \varphi_{\varepsilon_0}\tilde{v}$, together with the lower estimate $\varphi_{\varepsilon_0}(t) \geq (\varepsilon_0/2)t^\xi$, $t \in \mathbb{R}_+$, it follows that

$$\|\tilde{v}(t)\|_{X_r} \leq \frac{M_2\|v_0\|_{X_r}}{\varphi_{\varepsilon_0}(t)} \leq \frac{2M_2\|v_0\|_{X_r}}{\varepsilon_0} t^{-\xi} \to 0,$$

as $t \to \infty$, with an algebraic decay rate of exponent $\xi$.

c) Conclusion. Together with the decay rate of the initial-value part $S(\cdot)v_0$ we have for $t > 0$ that

$$\|v(t)\|_{X_r} \leq \|\tilde{v}(t)\|_{X_r} + \|S(t)v_0\|_{X_r} \leq \bar{M}\|v_0\|_{X_r} t^{-\xi},$$

for all $v_0 \in B_{\delta_0}^{X_r}(0)$, with $\bar{M} = \max\{2M_2/\varepsilon_0, C_r\}$.

(iii) We choose $r_0 > 0$, $C_0 > 0$ as well as $\eta \in (0, 1/(8C_{MR}C_0)]$ and $r = r(\eta) \in (0, \min\{r_0, 1/(8C_{MR}C_0)\}]$ as above in part (i). We use the stability property of the equilibrium $v_\ast = 0$ and fix for $\varepsilon = r$ the corresponding parameter $\delta_0 = \delta_0(\varepsilon) > 0$ such that for all $v_0 \in B_{\delta_0}^{X_r}(0)$ with $\delta \leq \delta_0$ there is a solution $v \in \mathcal{E}_1(\mathbb{R}_+) \cap C(\mathbb{R}_+, X_r)$ and $v(t) \in B_{\delta}^{X_r}(0)$ for all $t \in \mathbb{R}_+$. We decompose the global solution into the initial-value part $S(\cdot)v_0$ and two remainder terms $\tilde{v} = S(\cdot)v_0 + \bar{v} + \tilde{v}$, where $\bar{v}$ and $\tilde{v}$ solve the subsequent equations

$$\begin{align*}
\partial_t^a \bar{v} + A_0\bar{v} &= G(S(\cdot)v_0), \quad t \in \mathbb{R}_+, \quad \bar{v}(0) = 0, \quad (8.10)\\
\partial_t^a \tilde{v} + A_0\tilde{v} &= G(v) - G(S(\cdot)v_0), \quad t \in \mathbb{R}_+, \quad \tilde{v}(0) = 0, \quad (8.11)
\end{align*}$$

We study the decay behaviour of each part of the decomposition.

a) Decay behaviour of the initial-value part $S(\cdot)v_0$ in $X_r$. The decay behaviour of the initial-value part is given by Corollary 3.3.6, we have for all $t \in \mathbb{R}_+$ that $\|S(t)v_0\|_{X_r} \leq C_r\min\{1, t^{-a}\}\|v_0\|_{X_r}$.

b) Introducing the time-weighted remainders $\tilde{v}$ and $\bar{v}$. To prove the decay behaviour of the terms $\tilde{v}$ and $\bar{v}$ we consider the function $\varphi_{\varepsilon_0} = 1/s_{\varepsilon_0}$, where $s_{\varepsilon_0}$ denotes the scalar resolvent, the parameter $\varepsilon_0 > 0$ will be fixed later. The aim is to show that $\bar{w} = \varphi_{\varepsilon_0}\bar{v}$ and $\tilde{w} = \varphi_{\varepsilon_0}\tilde{v}$ are uniformly bounded in $X_r$. This shows that $\bar{v}$ and $\tilde{v}$ decay like $s_{\varepsilon_0}$ as $t \to \infty$. 

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We multiply the equations (8.10) and (8.11) by \( q_{\varepsilon_0} \) and obtain with the notation \( \tilde{w} = q_{\varepsilon_0} \tilde{v} \) and \( \tilde{w} = q_{\varepsilon_0} \tilde{v} \), respectively, the equations
\[
\partial_t^\alpha \tilde{w}(t) + A_0 \tilde{w}(t) = q_{\varepsilon_0}(t)G(S(t)v_0) + \int_0^t [-\partial_{\varepsilon_0}(t - \tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} \tilde{w}(\tau) d\tau, \quad t \in \mathbb{R}_+,
\]
with \( \tilde{w}(0) = 0 \), and
\[
\partial_t^\alpha \tilde{w}(t) + A_0 \tilde{w}(t) = q_{\varepsilon_0}(t) [G(v(t)) - G(S(t)v_0)] + \int_0^t [-\partial_{\varepsilon_0}(t - \tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} \tilde{w}(\tau) d\tau, \quad t \in \mathbb{R}_+,
\]
with \( \tilde{w}(0) = 0 \), using Lemma 8.1.1. To reach our goal we make use of the maximal \( L_p \)-regularity property of the operator \( A_0 \). Thus, we have to show that every term on the right-hand side of the equation (8.12) and (8.13), respectively, are elements of \( L_p([0, T]; X_0) = E_0(T) \) for each \( T > 0 \).

Trivially, we have for each \( t \in [0, 1] \) and all \( \varepsilon_0 \in [0, 1] \) that there is some constant \( M_p > 1 \) such that \( q_{\varepsilon_0}(t) \in [1, M_p] \). It is well-known that for all \( t \geq 1 \) there is some constant \( C_p > 0 \) (independent of \( \varepsilon_0 \in [0, 1] \)) such that \( q_{\varepsilon_0}(t) \leq C_q t^\alpha \).

c) **Decay behaviour of the term \( \tilde{v} \) in \( X_p \).** At first, we show that \( \tilde{v} \in \bigcap_{0} \mathbb{E}_1(R_+) \cap \mathbb{C}(R_+; X_p) \). For this end, we show that \( t \mapsto G(S(t)v_0) \in L_p(R_+; X_0) \) for each \( v_0 \in B^{X_0}_0(0) \) with sufficiently small \( \delta > 0 \).

c) **(i) Estimates for the map \( t \mapsto q_{\varepsilon_0}(t)G(S(t)v_0) \).** By Corollary 3.3.6 we have for all \( t \in \mathbb{R}_+ \) that \( \|S(t)v_0\|_{X_p} \leq M_p \min\{1, t^{-\alpha}\} \|v_0\|_{X_0} \).

Hence, for \( \delta \leq \delta_2 = \min\{\delta_0, \varepsilon/M_p, 1\} \) we have for all \( v_0 \in B^{X_0}_{0}(0) \) that \( S(t)v_0 \in B^{X_0}_{0}(0) \) as well as for almost all \( t \in \mathbb{R}_+ \) that \( S(t)v_0 \in B^{X_0}_{\varepsilon_0}(0) \cap X_1 \).

Using the properties of the map \( G \) it follows for almost all \( t \in \mathbb{R}_+ \) that \( \|G(S(t)v_0)\|_{X_0} \leq C_0(\eta + \varepsilon)\|S(t)v_0\|_{X_1} \) and therefore
\[
\|G(S(t)v_0)\|_{L_p([0, 1]; X_0)} \leq C_0(\eta + \varepsilon)\|S(t)v_0\|_{L_p([0, 1]; X_1)},
\]

and considering Proposition 6.2.5 we have \( \|S(t)v_0\|_{\mathbb{E}_1(R_+)} \leq C_1 \|v_0\|_{X_0} \) and hence
\[
\|G(S(t)v_0)\|_{L_p([0, 1]; X_0)} \leq C_0(\eta + \varepsilon)C_1 \|v_0\|_{X_0}.
\]

Next, using the estimate \( \|S(t)v_0\|_{X_1} \leq M t^{-\alpha} \|v_0\|_{X_0} \), cf. Corollary 3.3.7 (iii), we deduce with the same considerations as above that
with $M_3 = M_3(\alpha, p) = (\alpha p - 1)^{-1}$. Altogether, we have
\[
\|G(S(\cdot)v_0)\|_{L_p([0,1];X_0)} \leq C_0(\eta + \varepsilon)\|S(\cdot)v_0\|_{L_p([1,\infty);X_1)}
\leq C_0(\eta + \varepsilon) M \left( \int_1^\infty r^{-\alpha p} \, dt \right)^{1/p} \|v_0\|_{X_0}
\leq C_0(\eta + \varepsilon) M M_3 \|v_0\|_{X_0},
\]
and thus $t \mapsto G(S(t)v_0) \in L_p(\mathbb{R}_+;X_0)$. The maximal $L_p$-regularity property of the operator $A_0$ applied to equation (8.10) yields $\tilde{\psi} \in \mathbb{E}_1(\mathbb{R}_+) \cap C(\mathbb{R}_+;X_y)$. Since $\varphi(t)$ is continuous it follows directly that $\tilde{\psi}$ is continuous on $\mathbb{R}_+$, a function taking values in $X_y$, and for each $T > 0$ we have $\tilde{\psi} \in \mathbb{E}_1(T) \cap C([0,T];X_y)$. Let w.l.o.g. $T \geq 1$ be arbitrary, but fixed. In our second step, we will show that the terms on the right-hand side of equation (8.12) are elements of $L_p([0,T];X_0)$. We begin by looking at the map $t \mapsto \varphi(t)G(S(t)v_0)$. With the aid of the the considerations above we have
\[
\left\| \varphi(t)G(S(\cdot)v_0) \right\|_{L_p([0,1];X_0)} \leq C_0(\eta + \varepsilon) M_\psi \|S(\cdot)v_0\|_{L_p([0,1];X_1)}
\leq C_0(\eta + \varepsilon) M_\psi C_1 \|v_0\|_{X_y}.
\]
As mentioned before, for almost all $t \in \mathbb{R}_+$ we have $S(t)v_0 \in B_{X_y} \cap X_1$. Using Lemma 8.2.1 (ii) as well as Corollary 3.3.7 there are constants $M_0, M_\psi, M_\gamma, M_\psi > 0$ such that for almost all $t \geq 1$ we deduce
\[
\|G(S(t)v_0)\|_{X_0} \leq M_0 \|S(t)v_0\|_{X_y} \|S(t)v_0\|_{X_1}
\leq M_0 M_\gamma \|\varphi(t)\|_{X_y} M t^{-\alpha} \|v_0\|_{X_0}.
\]
Consequently, we have
\[
\left\| \varphi(t)G(S(\cdot)v_0) \right\|_{L_p([1,T];X_0)} = \left( \int_1^T \varphi(t) \|G(S(t)v_0)\|_{X_y}^p \, dt \right)^{1/p}
\leq C_\psi M_0 M_\gamma M \left( \int_1^{\infty} t^{-\alpha p} \, dt \right)^{1/p} \|v_0\|_{X_y} \|v_0\|_{X_0}
\leq C_\psi M_0 M_\gamma M_\psi M \|v_0\|_{X_y}^2,
\]
with $M_3$ independent of $T > 0$, see above. Combining the previous estimates we obtain
\[
\left\| \varphi(t)G(S(\cdot)v_0) \right\|_{L_p([0,T];X_0)} \leq C_0(\eta + \varepsilon) M_\psi C_1 \|v_0\|_{X_y}
+ C_\psi M_0 M_\gamma M \|v_0\|_{X_y}^2,
\]
and hence we have $t \mapsto \varphi(t)G(S(t)v_0) \in L_p([0,T];X_0)$.
c) (ii) Estimates for the commutator term of the time-weighted remainder $\tilde{w}$. Next, we consider the map

$$t \mapsto \int_0^t [-\mathcal{g}_{1-\alpha}(t-\tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} \tilde{w}(\tau) d\tau.$$ 

Lemma 8.1.6 implies for each $\rho > 0$ that there is an $\varepsilon_0 = \varepsilon_0(\rho) > 0$ such that we have the estimates

$$\left\| t \mapsto \int_0^t [-\mathcal{g}_{1-\alpha}(t-\tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} \tilde{w}(\tau) d\tau \right\|_{L_p([0,T];X_0)} \leq \rho \left\| \tilde{w} \right\|_{C([0,T];X_0)} + \rho C_{emb} \left\| \tilde{w} \right\|_{\mathbb{E}_1(T)},$$

using the embedding property with embedding constant $C_{emb} > 0$ independent of $T > 0$. Hence, this map is an element of $L_p([0,T];X_0)$.

c) (iii) Estimates for the time-weighted remainder $\tilde{w}$. Now, we are in the situation to apply the maximal $L_p$-regularity property of the operator $A_0$, thereby getting that

$$\left\| \tilde{w} \right\|_{\mathbb{E}_1(T)} \leq C_{MR} \left[ \left\| q_{\varepsilon_0} G(S(\cdot)v_0) \right\|_{E_0(T)} + \left\| t \mapsto \int_0^t [-\mathcal{g}_{1-\alpha}(t-\tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} \tilde{w}(\tau) d\tau \right\|_{E_0(T)} \right] \leq C_{MR} \left[ C_0(\eta + \varepsilon) M_p C_1 \left\| v_0 \right\|_{X_\gamma} + C_q M_0 M_\gamma M M_3 \left\| v_0 \right\|_{X_\gamma}^2 + \rho C_{emb} \left\| \tilde{w} \right\|_{\mathbb{E}_1(T)} \right].$$

Now, we choose $\rho = (2 C_{emb} C_{MR})^{-1}$ and fix the corresponding $\varepsilon_0 = \varepsilon_0(\rho) > 0$. Note that $\rho$ and $\varepsilon_0$ are independent of $T > 0$. This implies with $C_{MR} C_0(\eta + \varepsilon) \leq 1/2$ that

$$\left\| \tilde{w} \right\|_{\mathbb{E}_1(T)} \leq M_p C_1 \left\| v_0 \right\|_{X_\gamma} + 2 C_{MR} C_q M_0 M_\gamma M M_3 \left\| v_0 \right\|_{X_\gamma}^2.$$

In particular, we have for all $v_0 \in B_0^{X_\gamma}(0)$ with $\delta \leq \delta_2$ the uniform estimate

$$\left\| \tilde{w} \right\|_{\mathbb{E}_1(T)} \leq M_4 \left\| v_0 \right\|_{X_\gamma},$$
with the $T$-independent constant $M_4 = M_p C_1 + 2 C_{MR} C_p M_0 M_y M M_3$. Finally, we use the continuous embedding $\bar{\mathfrak{B}}_1(T) \hookrightarrow C([0, T]; X_Y)$ with $T$-independent embedding constant $C_{emb} > 0$ to see that

$$
\| \bar{\omega} \|_{C([0, T]; X_Y)} \leq C_{emb} \| \bar{\omega} \|_{\bar{\mathfrak{B}}_1(T)} \leq C_{emb} M_4 \| v_0 \|_{X_Y}.
$$

Since $T \geq 1$ was arbitrary, this uniform estimate yields for all $t \in \mathbb{R}_+$ that

$$
\| w(t) \|_{X_Y} \leq M_5 \| v_0 \|_{X_Y},
$$

with $M_5 = C_{emb} M_4$ which is $T$-independent. With the definition $\bar{\omega} = \tilde{v}/\delta_0$, it follows for all $v_0 \in B^X_{\delta_0}(0)$ that

$$
\| \bar{\omega}(t) \|_{X_Y} \leq M_5 \| v_0 \|_{X_Y} s_{\delta_0}(t) \to 0,
$$

as $t \to \infty$, with the algebraic rate $t^{-\alpha}$.

d) Decay behaviour of the term $\bar{\omega}$ in $X_Y$. From the considerations above we know that $v, S(\cdot) v_0, \tilde{v} \in \mathfrak{B}_1(\mathbb{R}_+) \cap C(\mathbb{R}_+; X_Y)$, for all $v_0 \in B^X_{\delta_0}(0)$, and $\delta \leq \delta_2$. This implies in particular $\bar{\omega} = v - S(\cdot) v_0 - \tilde{v} \in \mathfrak{B}_1(\mathbb{R}_+) \cap C(\mathbb{R}_+; X_Y)$. We know that $\varphi_{s_0}$ is continuous on $\mathbb{R}_+$, hence $\bar{\omega}$ is continuous on $\mathbb{R}_+$ with values in $X_Y$ and for all $T > 0$ we have $\bar{\omega} \in \mathfrak{B}_1(T) \cap C([0, T]; X_Y)$.

Let w.l.o.g. $T \geq 1$ be arbitrary, but fixed. In our next step, we will show that every term on the right-hand side of equation (8.13) is an element of $L_p([0, T]; X_0)$.

d) (i) Estimates for the map $t \mapsto \varphi_{s_0}(t)[G(v(t)) - G(S(t) v_0)]$. At first we consider the map $t \mapsto \varphi_{s_0}(t)[G(v(t)) - G(S(t) v_0)]$. Analogous to the examination above we have with a suitable choice of the parameters for almost all $t \in \mathbb{R}_+$ that $v(t), S(t) v_0 \in B^X_{\delta_0}(0) \cap X_1$ for all $v_0 \in B^X_{\delta_0}(0)$ with $\delta \leq \delta_2$. Thus, Lemma 8.2.1 (i) implies

$$
\| \varphi_{s_0} G(v) - G(S(\cdot) v_0) \|_{L_p([0, 1]; X_0)} 
\leq \| \varphi_{s_0} G(v) \|_{L_p([0, 1]; X_0)} + \| \varphi_{s_0} G(S(\cdot) v_0) \|_{L_p([0, 1]; X_0)} 
\leq M_p C_0(\eta + \epsilon) \| \varphi \|_{C(\mathbb{R}_+)} + \| S(\cdot) v_0 \|_{C(\mathbb{R}_+)}.
$$

The proof of part (i) yields, that $\| v \|_{C(\mathbb{R}_+)} \leq 2 C_1 \| v_0 \|_{X_Y}$ and with Lemma 6.2.5 it follows that

$$
\| \varphi_{s_0} G(v) - G(S(\cdot) v_0) \|_{L_p([0, 1]; X_0)} \leq M_p C_0(\eta + \epsilon) 3 C_1 \| v_0 \|_{X_Y};
$$

with $M_p = \sup_{t \in [0, 1], \epsilon_0 \in [0, 1]} | \varphi_{s_0}(t) |$. Using again Lemma 8.2.1 (i) we have for almost all $t \geq 1$ that
\[ \|G(v(t)) - G(S(t)v_0)\|_{X_0} \leq C_0(\eta + \varepsilon + \|S(t)v_0\|_{X_1}) \|v(t) - S(t)v_0\|_{X_1} \]
\[ \leq C_0(\eta + \varepsilon + M\|v_0\|_{X_0}) \|v(t) - S(t)v_0\|_{X_1}. \]

Here we apply Corollary 3.3.7 (iii) which gives us a constant \( M > 0 \) such that for all \( t \geq 1 \) we have the estimate \( \|S(t)v_0\|_{X_1} \leq M\|v_0\|_{X_0} \). The decomposition of our solution \( v - S(\cdot)v_0 = \bar{v} + \tilde{v} \) leads us to the estimate

\[
\|q_{\varepsilon_0}[G(v) - G(S(\cdot)v_0)]\|_{L_p([0,T];X_0)} \\
\leq C_0(\eta + \varepsilon + M\|v_0\|_{X_0}) \left( \left( \int_1^T \|q_{\varepsilon_0}(t)\bar{v}\|^p_{X_1} \, dt \right)^{1/p} + \left( \int_1^T \|q_{\varepsilon_0}(t)\tilde{v}\|^p_{X_1} \, dt \right)^{1/p} \right) \\
\leq C_0(\eta + \varepsilon + M\|v_0\|_{X_0}) \left[ \|\bar{v}\|_{E_1(T)} + \|\tilde{v}\|_{E_1(T)} \right].
\]

On the whole we have

\[
\|q_{\varepsilon_0}[G(v) - G(S(\cdot)v_0)]\|_{L_p([0,T];X_0)} \\
\leq M_p C_0(\eta + \varepsilon) 3C_1\|v_0\|_{X_0} + C_0(\eta + \varepsilon + M\|v_0\|_{X_0}) \left[ \|\bar{v}\|_{E_1(0,T)} + \|\tilde{v}\|_{E_1(0,T)} \right],
\]

and thus we have \( t \mapsto q_{\varepsilon_0}(t)[G(v(t)) - G(S(t)v_0)] \in L_p([0,T];X_0) \).

d) (ii) Estimates for the commutator term for the time-weighted remainder \( \bar{w} \). Next, we consider the map

\[
t \mapsto \int_0^t [-\hat{g}_{1-\alpha}(t - \tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} \bar{w}(\tau) \, d\tau.
\]

Lemma 8.1.6 yields that for all \( \rho > 0 \) there is an \( \varepsilon_0 = \varepsilon_0(\rho) > 0 \) such that

\[
\left\| t \mapsto \int_0^t [-\hat{g}_{1-\alpha}(t - \tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} \bar{w}(\tau) \, d\tau \right\|_{L_p([0,T];X_0)} \\
\leq \|\bar{w}\|_{C([0,T];X_0)} \left\| t \mapsto \int_0^t [-\hat{g}_{1-\alpha}(t - \tau)] \frac{q_{\varepsilon_0}(t) - q_{\varepsilon_0}(\tau)}{q_{\varepsilon_0}(\tau)} \, d\tau \right\|_{L_p(\mathbb{R}_+)} \\
\leq \rho \|\bar{w}\|_{C([0,T];X_0)} \|\bar{w}\|_{E_1(T)} \]

using the embedding property with embedding constant \( C_{emb} > 0 \) independent of \( T > 0 \). Hence, this map is an element of \( L_p([0,T];X_0) \).

d) (iii) Estimates for the time-weighted remainder \( \tilde{w} \). Now, we are in the situation to apply the maximal \( L_p \)-regularity property of the operator \( A_0 \) to the equation (8.13). This gives
\[ \|\tilde{w}\|_{E^1(T)} \leq C_{MR} \left[ \|Q_{e_0}[G(v) - G(S(\cdot)v_0)]\|_{E^0(T)} \right. \\
+ \left. \left\| t \mapsto \int_0^t [-g_{1-a}(t - \tau)] \frac{q_{e_0}(\tau) - q_{e_0}(t)}{q_{e_0}(\tau)} \tilde{w}(\tau) \, d\tau \right\|_{E^0(T)} \right) \\
\leq C_{MR} \left[ C_0(\eta + \varepsilon)M_\gamma 3 C_1 \|v_0\|_{X'} \right. \\
+ C_0(\eta + \varepsilon + M \|v_0\|_{X'}) \left\{ \|\tilde{w}\|_{E^1(T)} + \|\tilde{w}\|_{E^1(T)} \right\} \\
+ \rho C_{emb} \|\tilde{w}\|_{E^1(T)} \right]. \\
\]

Now, we use the same choice as above for \( \rho = (2 C_{emb} C_{MR})^{-1} \) and the corresponding \( \varepsilon_0 = \varepsilon_0(\rho) > 0 \). This implies

\[ \|\tilde{w}\|_{E^1(T)} \leq 2 C_{MR} \left[ C_0(\eta + \varepsilon)M_\gamma 3 C_1 \|v_0\|_{X'} \right. \\
+ C_0(\eta + \varepsilon + M \|v_0\|_{X'}) \left\{ \|\tilde{w}\|_{E^1(T)} + \|\tilde{w}\|_{E^1(T)} \right\} \]. \\
\]

Now, we choose \( \delta \leq \delta_3 = \min\{\delta_2, (8 C_{MR} C_0 M)^{-1}\} \) and consider \( v_0 \in B_{\delta}^{X'}(0) \). This, together with \( C_{MR} C_0(\eta + \varepsilon) \leq 1/4 \), implies the estimate

\[ 2 C_{MR} C_0(\eta + \varepsilon + M \|v_0\|_{X'}) \leq 3/4 \] \\
and consequently

\[ \|\tilde{w}\|_{E^1(T)} \leq 12 M_\gamma C_1 \|v_0\|_{X'} + 3 \|\tilde{w}\|_{E^1(T)} \leq \left[ 12 M_\gamma C_1 + 3 M_4 \right] \|v_0\|_{X'}. \]

The map \( \tilde{w} \) has trace zero, so we can use the continuous embedding with \( T \)-independent embedding constant \( C_{emb} > 0 \) and get

\[ \|\tilde{w}\|_{C([0,T];X')} \leq C_{emb} \|\tilde{w}\|_{E^1(T)} \leq C_{emb} \left[ 12 M_\gamma C_1 + 3 M_4 \right] \|v_0\|_{X'}. \]

In particular, we have for all \( v_0 \in B_{\delta}^{X'}(0) \) with \( \delta \leq \delta_3 \) the uniform estimate

\[ \|\tilde{w}\|_{C([0,T];X')} \leq M_6 \|v_0\|_{X'}, \quad (8.14) \]

with \( T \)-independent constant \( M_6 = C_{emb} \left[ 12 M_\gamma C_1 + 3 M_4 \right] \). This uniform estimate yields for all \( t \in \mathbb{R}_+ \) that \( \|\tilde{w}(t)\|_{X'} \leq M_6 \|v_0\|_{X'}. \) With the definition \( \tilde{w} = \tilde{v}/s_{e_0} \) it follows for all \( v_0 \in B_{\delta_3}^{X'}(0) \) that

\[ \|\tilde{v}(t)\|_{X'} \leq M_6 \|v_0\|_{X'} s_{e_0}(t) \rightarrow 0, \]

for \( t \rightarrow \infty. \)
8. Stability for Quasilinear Fractional Evolution Equations

d) Conclusion. With the well-known estimate \( s_\alpha(t) \leq K_0 t^{-\alpha}, \) cf. Section 2.3.3 Standard Kernel, it follows, together with the decay behaviour of \( S(t)v_0, \tilde{v}, \) and \( \bar{v}, \) that

\[
\|v(t)\|_{X_\gamma} \leq \|S(t)v_0\|_{X_\gamma} + \|	ilde{v}(t)\|_{X_\gamma} + \|\bar{v}(t)\|_{X_\gamma} \\
\leq \left[ C_\gamma t^{-\alpha} + M_5 s_\alpha(t) + M_6 s_\alpha(t) \right] \|v_0\|_{X_\gamma} \\
\leq \bar{M} t^{-\alpha} \|v_0\|_{X_\gamma},
\]

for all \( v_0 \in B_{X_\delta}(0), \) where \( \delta \leq \delta_3 \) and \( \bar{M} = C_\gamma + K_0(M_5 + M_6) > 0. \) The claim follows.

\[\square\]

Remark 8.2.5. At this point should be mentioned that the above considerations can be adapted to the more general class of quasilinear equations with the property of maximal \( L^p \)-regularity treated by Zacher [Zac03].

8.3 Example – Time-Fractional Diffusion Equation

We want to apply our theory to a concrete equation. This equation has already been studied in the classical case by Prüss [Prü02, Example 7.2]. Initially, we collect important facts about the Neumann Laplacian on bounded domains in \( L^p \)-setting for each \( p \in (1, \infty) \). Afterwards, we turn to the question of local well-posedness and then to the stability of certain equilibria.

The Neumann Laplacian on Bounded Domains

On \( L^2(\Omega) \). Firstly, we consider the Laplacian with homogeneous Neumann boundary conditions on a bounded domain \( \Omega \subset \mathbb{R}^n \) with \( C^2 \)-boundary \( \Gamma = \partial \Omega \) on the space \( L^2(\Omega) \). We collect important properties of its spectrum and the semigroup generated by this operator. As usual, we denote by \( H^1_0(\Omega) \) the first Sobolev space.

We define \( a: H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow \mathbb{C} \) by \( a(u, v) = \int_\Omega \nabla u \nabla v \, dx \), for all \( u, v \in H^1_0(\Omega) \). The form \( a \) is closed and symmetric. We denote by \( -\Delta_N \) the operator associated with the form \( a \). Then \( \Delta_N \) is a self-adjoint operator, called Neumann Laplacian on \( \Omega, \) with \( D(\Delta_N) = \{ u \in H^1_0(\Omega): \Delta u \in L^2(\Omega) \text{ and } \partial_\nu u = 0 \text{ weakly} \}, \) cf. [Are04, Example 5.3.3 c)]. Here, \( \partial_\nu u = 0 \) in the weak sense can be defined by the condition \( \int_\Omega [Vu v + \Delta uv] \, dx = 0 \) for all \( v \in C^1(\Omega) \).

Using regularity theory of elliptic operators, one can show that \( D(\Delta_N) = \{ u \in H^2_0(\Omega): \partial_\nu u = 0 \text{ on } \Gamma \}, \) cf. [Yag10, Theorem 2.6].
8.3. Example – Time-Fractional Diffusion Equation

Properties of the $C_0$-Semigroup. Since the Neumann Laplacian is induced by a form, it is the generator of a $C_0$-semigroup $\left( e^{t\Delta_N} \right)_{t \in \mathbb{R}_+}$ which admits a holomorphic continuation to some sector $\Sigma_\theta$ with angle $\theta \in \left(0, \frac{\pi}{2}\right)$. Additionally, we know that the semigroup $\left( e^{t\Delta_N} \right)_{t \in \mathbb{R}_+}$ generated by the Neumann Laplacian on $L^2(\Omega)$ is contractive. A proof of this property is given in [Vra03, Theorem 4.2.1].

Thus, the Neumann Laplacian is self-adjoint and generates a bounded holomorphic, positive and contractive $C_0$-semigroup $\left( e^{i\lambda_N} \right)_{t \in \mathbb{R}_+}$ on $L^2(\Omega)$, cf. [Are04, Section 2.6].

The Spectrum. It is known that on $L^2(\Omega)$ the Neumann Laplacian operator has compact resolvent, its spectrum is discrete consisting only of eigenvalues with finite multiplicity and zero is an isolated eigenvalue, cf. [BM16, Appendix A]. Thus, $\sigma(-\Delta_N) = \{ \lambda_k \in [0, \infty) : k \in \mathbb{N}_0 \}$ with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$, $k \in \mathbb{N}$.

On $L^p(\Omega)$. Now, we consider the Neumann Laplacian on $L^p(\Omega)$, $p \in [1, \infty]$. The semigroup $\left( e^{t\Delta_N} \right)_{t \in \mathbb{R}_+}$ generated by the Neumann Laplacian on $L^2(\Omega)$ is symmetric submarkovian, see [Are04, Example 7.1.4], i.e. for each $t \in \mathbb{R}_+$, $e^{t\Delta_N}$ is positivity preserving and $\left\| e^{t\Delta_N} u \right\|_\infty \leq \left\| u \right\|_\infty$ for all $0 \leq u \in L^2(\Omega) \cap L^\infty(\Omega)$. Hence, there exist consistent extrapolation $C_0$-semigroups $\left( e^{i\lambda_N} \right)_{t \in \mathbb{R}_+}$ on $L^p(\Omega)$, $p \in [1, \infty]$; the corresponding generators are denoted by $-\Delta_N^p$.

The $L^p$-theory of elliptic operators implies that $D(\Delta_N^p) = \{ u \in H^2_p(\Omega) : \partial\nu u = 0 \text{ on } \Gamma \}, p \in (1, \infty)$, see [Yag10, Theorem 2.15].

Properties of the $C_0$-Semigroup. It is well-known that for $p \in (1, \infty)$ the generated semigroup $\left( e^{t\Delta_N} \right)_{t \in \mathbb{R}_+}$ on $L^p(\Omega)$ is bounded, holomorphic, positive and contractive, see [Are04, The Heritage List 7.2.2].

Indeed, the symmetric submarkovian property of $\left( e^{t\Delta_N} \right)_{t \in \mathbb{R}_+}$ implies, together with the fact that the Neumann Laplacian is self-adjoint, that its adjoint operator is submarkovian, too. Thus, for $p \in (1, \infty)$ we have $\left\| e^{t\Delta_N^p} f \right\|_{L^p(\Omega)} \leq \left\| f \right\|_{L^p(\Omega)}$, for all $f \in L^p(\Omega) \cap L^2(\Omega)$ and all $t \in \mathbb{R}_+$; and by interpolation also for each $p \in (1, \infty)$. In other words, the extrapolation semigroup on $L^p(\Omega)$ is contractive, for all $p \in (1, \infty)$.

Furthermore, due to the holomorphy of the semigroup generated by the Neumann Laplacian on $L^2(\Omega)$ and together with Gaussian estimates, see [Are04, Example 7.4.1 (b)], it follows that the extrapolation semigroups on
All in all, we have that the Neumann Laplacian generates a bounded holomorphic, positive and contractive $C_0$-semigroup on $L_p(\Omega)$ for all $p \in (1, \infty)$. Now, it follows from Remark 4.9 c), together with Theorem 4.2 and Corollary 4.4, from Weis [Wei01], that for each $\mu > 0$ the operator $-\Delta_N^p + \mu$ on $L_p(\Omega)$ is $R$-sectorial with $R$-angle $< \frac{\pi}{2}$.

**The Spectrum.** The Neumann Laplacian has compact resolvent, see above. Hence, the spectrum is $p$-independent, $p \in (1, \infty)$, cf. [Are04, The Heritage List 7.2.2], i.e. $\sigma(-\Delta_N^p) = \sigma(-\Delta_N)$.

### Time-Fractional Diffusion Equation on Bounded Domains

We want to illustrate our theory on a quasilinear time-fractional diffusion equation on a bounded domain.

**Assumptions.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^2$-boundary $\Gamma = \partial \Omega$. Suppose that $f \in C^1(\mathbb{R}^n; \mathbb{R})$ and $a \in C^1(\mathbb{R}^n; \mathbb{R})$ such that $a(u, v) > 0$ for all $u \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Moreover, let $p \in (1, \infty)$ and $\alpha \in (1/p, 1)$.

We consider the quasilinear time-fractional diffusion equation

\[\begin{align*}
\partial^\alpha_t (u - u_0) - a(u, \nabla u)\Delta u &= f(u, \nabla u), & t > 0, & x \in \Omega, \\
\partial_\nu u &= 0, & t > 0, & x \in \Gamma, \\
u = u_0, & t = 0, & x \in \Omega.
\end{align*}\]

(8.15)

Here, we denote by $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)^T$ the gradient of the map $u$ with respect to the spatial variable $x$. By $\Delta u$ we mean the Laplacian of the map $u$ with respect to the spatial variable $x$, that is $\Delta u = \sum_{i=1}^n \partial^2_{x_i} u$. Since we have a $C^2$-boundary $\Gamma$ the outward pointing unit normal $\nu : \Gamma \to \mathbb{R}^n$ is well-defined and the normal derivative of $u$ along $\Gamma$ is given by $\partial_\nu u(x) = \sum_{i=1}^n \nu_i(x) \partial_{x_i} u(x), \ x \in \Gamma$.

Firstly, we are looking for a unique local solution of the problem (8.15) in the space of maximal $L_p$-regularity

\[H^\alpha_p([0, T]; L_p(\Omega)) \cap L_p([0, T]; H^2_p(\Omega)),\]

for some $T > 0$.

In the following we always consider the case $p > n + \frac{2}{\alpha}$. We set $X_0 = L_p(\Omega)$ and

\[X_1 = \{v \in H^2_p(\Omega) : \partial_\nu v = 0 \text{ on } \Gamma\}.

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8.3. Example – Time-Fractional Diffusion Equation

Note that $X_1$ coincides with the domain of the $L_p$-realisation of the Neumann Laplacian. The space $X_1$ is densely embedded in $X_0$ and both spaces are Banach spaces of class $\mathcal{H}T$. By the Besov space $B_{pp}^{2-\frac{2}{n}}(\Omega) = \left(L_p(\Omega), H^2_p(\Omega)\right)_{1-\frac{2}{np}, p}$ the trace space is given by

$$X_\gamma = (X_0, X_1)_{1-\frac{2}{np}, p} = \{v \in B_{pp}^{2-\frac{2}{n}}(\Omega): \partial_\gamma v = 0 \text{ on } \Gamma\},$$

see [Ama09, Section 4.9]. Since we consider $p > n + \frac{2}{\alpha}$ we have the embeddings $B_{pp}^{2-\frac{2}{n}}(\Omega) \hookrightarrow C(\overline{\Omega})$ and also $B_{pp}^{1-\frac{2}{np}}(\Omega) \hookrightarrow C(\overline{\Omega}),$ see [AF03, Chapter 7].

Together with the assumptions on $a$ and $f$ we have for

$$A: X_\gamma \to \mathcal{B}(X_1; X_0), \quad u \mapsto A(u) = -a(u, \nabla u)\Delta,$$

that $A \in C^1\left(X_\gamma; \mathcal{B}(X_1; X_0)\right)$ and for

$$F: X_\gamma \to X_0, \quad u \mapsto F(u) = f(u, \nabla u),$$

that $F \in C^1\left(X_\gamma; X_0\right)$.

For $u \in H_p^a\left([0, T]; L_p(\Omega)\right) \cap L_p\left([0, T]; H^2_p(\Omega)\right)$ we have the embedding

$$H_p^a\left([0, T]; L_p(\Omega)\right) \cap L_p\left([0, T]; H^2_p(\Omega)\right) \hookrightarrow C\left([0, T]; B_{pp}^{2-\frac{2}{n}}(\Omega)\right) \hookrightarrow C\left([0, T] \times \overline{\Omega}\right),$$

since we consider $\alpha > 1/p$. Moreover, for

$$\partial_i x, u \in H_p^a\left([0, T]; L_p(\Omega)\right) \cap L_p\left([0, T]; H^1_p(\Omega)\right),$$

$i \in \{1, \ldots, n\}$, we have the embedding

$$H_p^{\frac{a}{2}}\left([0, T]; L_p(\Omega)\right) \cap L_p\left([0, T]; H^1_p(\Omega)\right) \hookrightarrow C\left([0, T]; B_{pp}^{1-\frac{2}{np}}(\Omega)\right) \hookrightarrow C\left([0, T] \times \overline{\Omega}\right),$$

due to the condition $p > n + \frac{2}{\alpha}$, cf. [Zac03, p. 96].

Let $u_0 \in X_\gamma$ be arbitrary, but fixed. We consider the operator $A(u_0) = -a(u_0, \nabla u_0)\Delta$; a differential operator of second order with variable coefficients. By the above embeddings it is obvious that the coefficient $a(u_0, \nabla u_0)$ is continuous on $\overline{\Omega}$. Moreover, the principal symbol of $A(u_0)$ is given by $a(u_0(x), \nabla u_0(x))|\xi|^2$, $\xi \in \mathbb{R}^n$, it is parameter elliptic with angle zero for each $x \in \overline{\Omega}$, and the Lopatinskii-Shapiro Condition is satisfied (essentially, one has to realise this condition for the Neumann Laplacian). Thus, we conclude with Theorem 8.2 from Denk, Hieber and Prüss [DHP03] that up to some shift the operator $A(u_0)$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $< \frac{\pi}{4}$. Thus, the problem is locally
8. Stability for Quasilinear Fractional Evolution Equations

well-posed for \( p > n + \frac{2}{\alpha} \), see Theorem 7.1.1.

Now, we want to study the stability of equilibria for equation (8.15). The Neumann boundary condition guarantees that each constant function \( u_\ast \) with \( f(u_\ast, 0) = 0 \) defines an equilibrium, i.e. \( A(u_\ast)u_\ast = F(u_\ast) \).

The linearisation of \( A(u)u - F(u) \) around the constant solution \( u_\ast \) is given for all \( w \in X_1 \) by

\[
A_0 w = A(u_\ast)w + (A'(u_\ast)w)u_\ast - F'(u_\ast)w \\
= -a(u_\ast, 0)\Delta w - [a(u_\ast, 0)w + a_\nu(u_\ast, 0)\nabla w] \Delta u_\ast - f'(u_\ast, 0)w \\
= -a(u_\ast, 0)\Delta w - f'(u_\ast, 0)w.
\]

The eigenvalues of the operator \( A_0 \) are given by \( (\mu_k)_{k \in \mathbb{N}_0} \)

\[
\mu_k = a(u_\ast, 0)\lambda_k - f'(u_\ast, 0), \quad k \in \mathbb{N}_0,
\]

where \( (\lambda_k)_{k \in \mathbb{N}_0} \) are the eigenvalues of the Neumann Laplacian, see above. Thus, for \( f'(u_\ast, 0) < 0 \) we have \( (\mu_k)_{k \in \mathbb{N}_0} \subset (0, \infty) \). Hence, the operator \( A_0 \) is invertible. Furthermore, this operator is \( \mathcal{R} \)-sectorial with \( \mathcal{R} \)-sectoriality, by the permanence properties of \( \mathcal{R} \)-sectorial operators which are similar to those for sectorial operators, cf. [DHP03, p. 43 and Proposition 1.3]. In particular, the equilibrium is asymptotically stable, see Theorem 8.2.3.

Moreover, if we consider \( f \in C^2(\mathbb{R}^{n+1}; \mathbb{R}) \) and \( a \in C^2(\mathbb{R}^{n+1}; \mathbb{R}) \) then we have \( A \in C^2(\mathcal{X}_\gamma; \mathcal{B}(X_1; X_0)) \) as well as \( F \in C^2(\mathcal{X}_\gamma; X_0) \). By Theorem 8.2.3 we know that there is a sufficiently small \( \delta_0 > 0 \) and a constant \( C > 0 \) such that for all \( u_0 \in B_{\delta_0}^{X_\gamma}(u_\ast) \) we have for the corresponding solution \( u(\cdot; u_0) \) of (8.15) for each \( t \in (0, \infty) \) the estimate

\[
\|u(t; u_0) - u_\ast\|_{X_\gamma} \leq Ct^{-\alpha} \|u_0\|_{X_\gamma}.
\]

We summarise our results concerning the stability of the equilibrium \( u_\ast \).

**Theorem 8.3.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^2 \)-boundary \( \Gamma = \partial \Omega \). Suppose that \( f \in C^1(\mathbb{R}^{n+1}; \mathbb{R}) \) and \( a \in C^1(\mathbb{R}^{n+1}; \mathbb{R}) \) such that \( a(u, v) > 0 \) for all \( u \in \mathbb{R} \) and \( v \in \mathbb{R}^n \). Moreover, let \( p \in (1, \infty) \) and \( \alpha \in (1/p, 1) \) with \( p > n + \frac{2}{\alpha} \).

Suppose that \( u_\ast \) is a constant function with \( f(u_\ast, 0) = 0 \) satisfies the stability condition

\[
f'(u_\ast, 0) < 0.
\]

(i) Then the equilibrium \( u_\ast \) is asymptotically stable in \( X_\gamma \) for the quasilinear time-fractional diffusion equation (8.15).

(ii) If we assume additionally that \( f \in C^2(\mathbb{R}^{n+1}; \mathbb{R}) \) and \( a \in C^2(\mathbb{R}^{n+1}; \mathbb{R}) \), then \( u_\ast \) is asymptotically stable in \( X_\gamma \) with an algebraic decay rate of exponent \( \alpha \).
Appendices
Operators and their Spectrum

In the following we always denote by $X$ a complex Banach space with norm $\|\cdot\|_X$. We refer to Arendt, Batty, Hieber and Neubrander [ABHN01, Appendix B] and Lunardi [Lun95, Appendix A.0] for the following definitions and results as well as their proofs.

A.1 Closed Operators

Definition A.1.1 (Linear, Densely Defined and Closed Operator). Let $D(A)$ be a linear subspace of the complex Banach space $X$.

(i) A linear map $A : D(A) \to X$ is called linear operator on $X$ with domain $D(A)$. We denote by $\text{Rg} A = \{Ax : x \in D(A)\}$ the range of the operator $A$ and by $\text{Ker} A = \{x \in D(A) : Ax = 0\}$ the kernel of the operator $A$.

(ii) The linear operator $A$ is densely defined, if $D(A)$ is dense in $X$.

(iii) The linear operator $A$ is closed if its graph $G(A) = \{(x,Ax) : x \in D(A)\}$ is closed in $X \times X$.

(iv) The linear operator $A$ is invertible if there is bounded operator $A^{-1}$ on $X$ such that $A^{-1}Ax = x$ for all $x \in D(A)$ as well as $A^{-1}y \in D(A)$ and $AA^{-1}y = y$ for all $y \in X$.

Remark A.1.2. The definition of closed linear operators is equivalent to the condition that for all $(x_n)_{n \in \mathbb{N}} \subset D(A)$ such that $\lim_{n \to \infty} x_n = x$ in $X$ we have $x \in D(A)$ and $Ax = y$.

Proposition A.1.3. Let $A$ be a linear operator on $X$ with domain $D(A)$. We denote by $X_A = (D(A), \|\cdot\|_A)$ the domain of the operator equipped with the graph norm defined for all $x \in D(A)$ by $\|x\|_A = \|x\|_X + \|Ax\|_X$.

Then the operator $A : D(A) \to X$ is always bounded with respect to the graph norm. Furthermore, the linear operator $A$ is closed if and only if $X_A$ is a Banach space.

Definition A.1.4 (Part of an Operator in an Subspace). Let $A$ be a linear operator on $X$, and let $Y$ be a closed subspace of $X$. The part of the operator $A$ in the subspace $Y$ is the operator $A_Y : D(A_Y) \to Y$ on $Y$ with domain $D(A_Y) = \{y \in D(A) : y \in Y\}$ given by $A_Yy = Ay$. 

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A.2 Spectrum and Resolvent Operator

Definition A.2.1. Let \( A : D(A) \subset X \to X \) be a linear operator. The resolvent set \( \rho(A) \) and the spectrum \( \sigma(A) \) of \( A \) are defined by

\[
\rho(A) = \{ \lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is invertible} \}, \quad \sigma(A) = \mathbb{C} \setminus \rho(A).
\]

For \( \lambda \in \rho(A) \) the operator \((\lambda - A)^{-1} : X \to X\) is called the resolvent operator (of \( A \) at \( \lambda \)).

Remark A.2.2. If \( \rho(A) \neq \emptyset \), then the linear operator \( A \) is closed. Let \( A \) be a closed linear operator and \( \lambda \in \rho(A) \). The resolvent operator \((\lambda - A)^{-1}\) has the range \( D(A) \), it is closed and belongs to \( B(X) \).

Proposition A.2.3. Let \( A \) be a linear operator on \( X \). Then the resolvent set \( \rho(A) \) is open and the spectrum \( \sigma(A) \) is closed in \( \mathbb{C} \).

Let \( \mu \in \rho(A) \) and \( \lambda \in \mathbb{C} \) with \( |\lambda - \mu| \leq \| (\mu - A)^{-1} \|^{-1}_{B(X)} \), then \( \lambda \in \rho(A) \), and

\[
(\lambda - A)^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu - A)^{-(n+1)},
\]

where the series in norm-convergent. Hence,

\[
\| (\lambda - A)^{-1} \|_{B(X)} \leq \frac{\| (\mu - A)^{-1} \|_{B(X)}}{1 - |\lambda - \mu| \| (\mu - A)^{-1} \|_{B(X)}}.
\]

Moreover, the map \( \lambda \mapsto (\lambda - A)^{-1} \) is holomorphic on \( \rho(A) \) with values in \( B(X) \) and for all \( \lambda \in \rho(A) \) and all \( n \in \mathbb{N} \) we have

\[
\left( \frac{d}{d\lambda} \right)^n (\lambda - A)^{-1} = (-1)^n n! (\lambda - A)^{-(n+1)}.
\]

The resolvent set \( \rho(A) \) is the biggest domain of holomorphy of the map \( \lambda \mapsto (\lambda - A)^{-1} \).

Corollary A.2.4. Let \( A \) be a linear operator on \( X \) with domain \( D(A) \). For each \( \mu, \lambda \in \rho(A) \) we have

(i) \((\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - \lambda)(\mu - A)^{-1}(\lambda - A)^{-1}, \)

i.e. the so-called resolvent identity is satisfied and the resolvent operators commute;

(ii) \( A(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - \text{Id} \);

(iii) for all \( x \in D(A) \) we have \( A(\lambda - A)^{-1} x = (\lambda - A)^{-1} A x \), i.e. the resolvent operator commutes with the linear operator \( A \) on its domain \( D(A) \).
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Zusammenfassung in deutscher Sprache


Abstrakte Semilineare Volterra Gleichungen. Wir formulieren das Prinzip der linearisierten Stabilität für eine große Klasse von abstrakten semilinearen Volterra Gleichungen, welche auch als abstrakte Integro-Differentialgleichungen formuliert werden können und insbesondere den Fall der fraktionalen Evolutionsgleichungen einschließt. Zunächst beschreiben wir diese Klasse von Volterra Gleichungen. Es sei $a \in L_{1,\text{loc}}(\mathbb{R}_+)$ ein unbeschränkter, vollständig monotoner Kern und $b \in L_{1,\text{loc}}(\mathbb{R}_+)$ bezeichne den zugehörigen vollständig monotonen Kern mit der Eigenschaft $a \ast b \equiv 1$ auf $(0,\infty)$; es ist bekannt, dass es stets genau einen Kern $b$ mit dieser Eigenschaft gibt. Ferner sei $A$ ein linearer, abgeschlossener und dicht definierter Operator im Banachraum $X$, $f \in C^1(X;X)$ mit $f(0) = 0$ und $f'(0) = 0$, sowie $u_0 \in X$. Für die unbekannte Funktion $u: J \to X$ betrachten wir die semilineare Volterra Gleichung

$$u + a \ast Au = u_0 + a \ast f (u), \quad t \in J; \quad (A.1)$$

wobei $J$ entweder ein kompaktes Intervall $[0,T]$, $T > 0$, oder $\mathbb{R}_+$ ist. Auf Grund der Kerneigenschaft $a \ast b \equiv 1$ ist es möglich die Gleichung (A.1) zu einer Integro-Differentialgleichung umzuschreiben:

$$\partial_t [b \ast (u - u_0)] + Au = f (u), \quad t \in J, \quad u(0) = u_0. \quad (A.2)$$
Ein Equilibrium der Volterra Gleichung (A.1) bzw. der Integro-Differentialgleichung (A.2) ist durch die stationäre Lösung \( u^* = 0 \) gegeben.

Im Gegensatz zu klassischen Evolutionsgleichungen gibt es keine Halbgruppen für Volterra Gleichungen. Daher wird ein verallgemeinertes Konzept der sogenannten Resolventen und Integral-Resolventen benötigt. Diese beiden Objekte ermöglichen eine Lösungsdarstellung für die Volterra Gleichungen in ähnliche Weise zu der aus dem klassischen Fall bekannten Variation der Konstanten Formel.

Das hier formulierte Prinzip der linearisierten Stabilität für semilineare Volterra Gleichungen fordert für die Stabilität des Equilibriums \( u^* = 0 \) die Existenz einer beschränkten Resolvente und einer integrierbaren Integral-Resolvente. Darüber hinaus erhält man im Fall \( a \not\in L_1(\mathbb{R}_+) \) die asymptotische Stabilität mit exakt derselben Abklingrate wie für die skalare Resolvente.

Für parabolische Volterra Gleichungen kennt man einfache hinreichende Bedingungen für die Existenz einer beschränkten Resolvente, jedoch keine für die Existenz und Integrierbarkeit der Integral-Resolvente. Es werden daher einfach überprüfbare Kriterien für die Existenz und Integrierbarkeit der Integral-Resolvente im Fall von parabolischen Volterra Gleichungen bewiesen.

Unter Verwendung der hinreichenden Bedingungen für die Existenz einer beschränkten Resolvente und integrierbaren Integral-Resolvente erkennt man, dass die wesentliche Bedingung für die Stabilität des Equilibriums \( u^* = 0 \), neben einem gewissen Abklingverhalten der Kerne, letztlich die Kompatibilität des Operators \( A \) mit dem Kern \( a \), im Sinne der Parabolizität, ist.

Darüber hinaus können wir auch hinreichende Bedingungen für die Instabilität des Equilibriums \( u^* = 0 \) geben. Im Falle des Standardkerns erhält man die Instabilität sobald das Spektrum des Operators \( A \) einen kompakten Teil im Komplement des Stabilitätssektors hat. Im allgemeinen Fall hängt die Instabilitätsbedingung von den Abbildungseigenschaften der Laplace-Transformation des Kerns \( a \) ab.

Die zulässigen Paare von Kernen und Operatoren in dem hier formulierten Prinzip der linearisierten Stabilität decken eine große Klasse von parabolischen Volterra Gleichungen ab.

### Quasilineare Fraktionelle Evolutionsgleichungen.

Der zweite Schwerpunkt dieser Arbeit liegt auf der Verallgemeinerung des Prinzips der linearisierten Stabilität für quasilineare fraktionelle Evolutionsgleichungen. Zur Behandlung dieses Gleichungstypes wird die Methode der maximalen \( L_p \)-Regularität verwendet. Dabei seien \( X_0 \) und \( X_1 \) Banachräume der Klasse \( \mathcal{HT} \) mit dichter Einbettung \( X_1 \hookrightarrow X_0 \), \( p \in (1, \infty) \) und \( \alpha \in (1/p, 1) \). Wir betrachten die quasilineare fraktionale Evolutionsgleichung

\[
\partial_t^\alpha (u - u_0) + A(u)u = F(u), \quad u(0) = u_0 \in V, \quad (A.3)
\]
wobei $V$ eine offene Teilmenge des reellen Interpolationsraumes $(X_0, X_1)_{\frac{1}{p}, p}$ ist, und $(A, F): V \to \mathcal{B}(X_1; X_0) \times X_0$. Wir untersuchen Lösungen mit maximaler $L_p$-Regularität, d. h. $u \in H_0^\alpha([0, T]; X_0) \cap L_p([0, T]; X_1)$.

Zunächst nutzen wir Resultate zur maximal $L_p$-Regularität von linearen Gleichungen, um die Existenz und Eindeutigkeit von Lösungen der quasilinearen Gleichung zu beweisen.

Im Hauptresultat über die Stabilität von Equilibria für quasilineare fraktionielle Evolutionsgleichungen betrachten wir $(A, F) \in C^1(V; \mathcal{B}(X_1; X_0) \times X_0)$ und das Equilibrium $u_* \in V \cap X_1$, d. h. $A(u_*)u_* = F(u_*)$. Unter der Voraussetzung, dass die Linearisierung von $A(u)u - F(u)$ um $u_*$ maximale $L_p$-Regularität auf $\mathbb{R}_+$ hat, können wir die asymptotische Stabilität des Equilibriums $u_*$ zeigen. Unter der zusätzlichen Voraussetzung $(A, F) \in C^2(V; \mathcal{B}(X_1, X_0) \times X_0)$ können wir auch die optimale algebraische Abklingrate mit Exponent $\alpha$ beweisen.

Das in dieser Arbeit formulierte Theorem ist die einzige bekannte Verallgemeinerung des Prinzips der linearisierten Stabilität auf quasilineare fraktionielle Evolutionsgleichungen. Bis auf die zusätzliche technische Regularitätsannahme für die optimale Abklingrate verallgemeinert es das klassische Resultat in zu erwartender Art und Weise.
Ehrenwörtliche Erklärung

Ich versichere hiermit, dass ich die Arbeit selbstständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe und die Satzung der Universität Ulm zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

Ulm, den 13. März 2018

Marie-Luise Hein
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