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On second order elliptic stochastic partial differential equations and almost periodically stationary processes

Dissertation

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Abstract

This thesis covers various topics in the field of probability theory and stochastic processes and consists of three parts. In the first part, we are interested in second order elliptic stochastic partial differential equations driven by Lévy white noise and derive existence results for generalized and mild solutions. In the second part, we investigate stochastic processes with almost periodic finite dimensional distributions and derive a characterization for infinitely divisible processes in terms of their characteristic triplets. Moreover, we obtain a central limit theorem for such processes, where m-dependence and uniform integrability is assumed. In this regard, we investigate in the third part the generalized uniform integrability of infinitely divisible processes and especially of certain stochastic integrals.

In **Chapter 2** we obtain generalized solutions s in the space of distributions of the stochastic partial differential equation $p(x, D)s = \dot{L}$, where $\dot{L}$ is a Lévy white noise and p is a second order elliptic partial differential operator. Furthermore, we discuss moment properties of the generalized random processes s and show that if $\dot{L}$ has finite $\beta > 0$ moment, then also s has finite β-moment under further conditions. Moreover, we give sufficient conditions for the existence and stochastic continuity of mild solutions. As an application, we achieve generalized and stochastically continuous mild solutions for the generalized electric Schrödinger operator driven by Lévy white noise.

In the context of stochastic processes, there are miscellaneous concepts of almost periodicity, such as almost periodic in distribution, in probability, almost sure, etc. In **Chapter 3** we derive a sufficient and necessary condition for stochastic processes to have almost periodic finite dimensional distributions and especially obtain a characterization for infinitely divisible processes to be almost periodic in terms of the characteristic triplets. We apply these results to stochastic processes which are defined as stochastic integrals driven by a Lévy basis and also state a sufficient condition for such processes to be almost periodic in probability. Afterwards, we show the existence of an Ornstein-Uhlenbeck-type process with almost periodic finite dimensional distributions and also state a central limit theorem for m-dependent and L^2-uniformly integrable processes with almost periodic finite dimensional distributions.

Motivated by the assumption of uniform integrability in our central limit theorem, we discuss in **Chapter 4** the generalized uniform integrability of infinitely divisible processes. We establish sufficient conditions under which stochastic integrals driven by a Lévy basis are generalized uniformly integrable and apply our results to the mild solution of the electric Schrödinger equation and to Ornstein-Uhlenbeck-type processes.

Zusammenfassung

Diese Arbeit behandelt verschiedene Themen im Bereich der Wahrscheinlichkeitstheorie und stochastischen Prozesse und besteht aus drei Teilen. Im ersten Teil interessieren wir uns für elliptische stochastische partielle Differentialgleichungen zweiter Ordnung, die durch Lévy weißes Rauschen angetrieben werden und leiten Existenzresultate für verallgemeinerte und milde Lösungen her. Im zweiten Teil untersuchen wir stochastische Prozesse mit fastperiodischen endlichdimensionalen Verteilungen und leiten eine Charakterisierung für unendlich teilbare Prozesse bezüglich ihrer charakteristischen Triplets ab. Außerdem erhalten wir für solche Prozesse einen zentralen Grenzwertsatz, bei dem m-Abhängigkeit und gleichgradige Integrierbarkeit vorausgesetzt werden. In diesem Zusammenhang untersuchen wir im dritten Teil die verallgemeinerte gleichgradige Integrierbarkeit unendlich teilbarer Prozesse und insbesondere gewisser stochastischer Integrale.

In **Kapitel 2** erhalten wir generalisierte Lösungen s im Raum der Distributionen der stochastischen partiellen Differentialgleichung $p(x, D)s = \dot{L}$, wobei $\dot{L}$ ein Lévy weißes Rauschen und p ein elliptischer partieller Differentialoperator zweiter Ordnung ist. Des Weiteren diskutieren wir Momenteneigenschaften des generalisierten stochastischen Prozess s und zeigen, dass wenn $\dot{L}$ endliches $\beta > 0$ Moment hat, dann hat auch s endliches β-Moment unter weiteren Bedingungen. Außerdem geben wir hinreichende Bedingungen für die Existenz stochastisch stetiger milder Lösungen an. Als Anwendung zeigen wir die Existenz stochastisch stetiger milder Lösungen für den verallgemeinerten elektrischen Schrödinger-Operator welcher von Lévy weißem Rauschen angetrieben wird.

Im Zusammenhang mit stochastischen Prozessen gibt es verschiedene Konzepte von Fastperiodizität, wie etwa fast periodisch in Verteilung, in Wahrscheinlichkeit, fast sicher usw. In **Kapitel 3** erhalten wir eine hinreichende und notwendige Bedingung wann stochastische Prozesse fastperiodische endlichdimensionale Verteilungen haben und erhalten insbesondere eine Charakterisierung für unendlich teilbare Prozesse abhängig von deren charakteristischen Triplets. Wir wenden diese Resultate auf stochastische Prozesse an, die als stochastische Integrale definiert sind, welche von einer Lévy Basis angetrieben werden. Zudem geben wir eine hinreichende Bedingung dafür an, wann solche Prozesse in Wahrscheinlichkeit fastperiodisch sind. Anschließend zeigen wir die Existenz eines Ornstein-Uhlenbeck-typ Prozesses mit fastperiodischen endlichdimensionalen Verteilungen und geben auch einen zentralen Grenzwertsatz für m-abhängige und L^2-gleichgradig integrierbare Prozesse mit fastperiodischen endlichdimensionalen Verteilungen an.

Motiviert durch die Annahme gleichgradiger Integrierbarkeit in unserem zentralen Grenzwertsatz diskutieren wir in **Kapitel 4** die verallgemeinerte gleichgradige Integrierbarkeit unendlich teilbarer Prozesse. Wir erhalten hinreichende Bedingungen unter denen stochastische Integrale, die von einer Lévy Basis angetrieben werden, gleichgradig integrierbar sind und wenden unsere Resultate auf die milde Lösung der elektrischen Schrödinger-Gleichung und auf Ornstein-Uhlenbeck-typ Prozesse an.

Parts of this dissertation have already been published in:

1. D. Berger and F. Mohamed, *Second order elliptic partial differential equations driven by Lévy white noise*, Modern Stoch. Theory Appl. 8 (2021), no. 2, 179-207.
2. D. Berger and F. Mohamed, *Almost periodic stationary processes*, arXiv:2208.08240, (2022).

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1 Introduction

Gaussian white noise is a well-known subject in white noise analysis and is widely used in areas such as mathematical physics or stochastic partial differential equations. As a natural extension the Lévy white noise theory was established, which is closely connected to Lévy processes: a Lévy process $(L_t)_{t\in\mathbb{R}}$ is a stochastically continuous process with independent and stationary increments satisfying $L_0 = 0$. Its characteristic function is given by $\mathbb{E}e^{izL_t} = \exp(t\psi(z))$ for every $z \in \mathbb{R}$ and $t \geq 0$. The function ψ is called the Lévy exponent of $(L_t)_{t\in\mathbb{R}}$ and is characterized by an $a \geq 0$, $\gamma \in \mathbb{R}$ and a Lévy measure ν, i.e. a measure such that

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty.$$

For all $z \in \mathbb{R}$ it holds

$$\psi(z) = i\gamma z - \frac{1}{2}az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbb{1}_{|x|\leq 1})\nu(dx).$$

The function ψ is uniquely characterized by the triplet (a, γ, ν) known as the characteristic triplet of $(L_t)_{t\in\mathbb{R}}$. There exist several ways to construct Lévy white noise and one possible construction is to define Lévy white noise as a generalized random process in the sense of Gelfand and Vilenkin (see [20]). Given a probability space $(\Omega, \mathcal{F}, P)$, a generalized random process s is a linear and continuous mapping from the space $\mathcal{D}(\mathbb{R}^d)$ of compactly supported and smooth functions (equipped with the usual topology) into $L^0(\Omega)$, the space of almost surely finite random variables. The continuity means that if $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$, then $s(\varphi_n)$ converges to $s(\varphi)$ in probability. A generalized random process has a version which is a measurable function from $(\Omega, \mathcal{F})$ to $(\mathcal{D}'(\mathbb{R}^d), \mathcal{C})$, where $\mathcal{D}'(\mathbb{R}^d)$ denotes the topological dual space of $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{C}$ the cylindrical σ-field generated by the sets

$$\{u \in \mathcal{D}'(\mathbb{R}^d) | \, (\langle u, \varphi_1\rangle, \dots, \langle u, \varphi_N\rangle) \in B\}$$

with $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in \mathcal{D}(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^N)$. Accordingly, we can define the probability law of a generalized random process s as the probability measure on $\mathcal{D}'(\mathbb{R}^d)$ given by

$$\mathcal{P}_s(B) := \mathcal{P}(s \in B) = \mathcal{P}(\{\omega \in \Omega : s(\omega) \in B\})$$

for $B \in \mathcal{C}$, and its characteristic functional $\widehat{\mathcal{P}}_s : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$ as

$$\widehat{\mathcal{P}}_s(\varphi) = \int_{\mathcal{D}'(\mathbb{R}^d)} \exp(i\langle u, \varphi\rangle) d\mathcal{P}_s(u).$$

The characteristic functional characterizes the law of s in the sense that two random processes are equal in law if and only if they have the same characteristic functional. In this context a Lévy white noise $\dot{L}$ on $\mathbb{R}^d$ is defined as a generalized random process with characteristic functional of the form $\widehat{\mathcal{P}}_{\dot{L}}(\varphi) = \exp\big(\int_{\mathbb{R}^d} \psi(\varphi(x))\, dx\big)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, where $\psi : \mathbb{R} \to \mathbb{C}$ is a Lévy exponent. The domain of a Lévy white noise can be extended to indicator functions $\mathbb{1}_A$ for bounded Borel sets A. Moreover, in [19] the domain of $\dot{L}$ was characterized allowing also more general functions.
Another approach in constructing a Lévy white noise is to define it as a Lévy basis in the sense of Rajput and Rosinski (see [38]). A Lévy basis L is a family of random variables whose test functions are indicator functions of bounded Borel sets and satisfy that two indicator functions with disjoint supports define independent random variables. Moreover, the characteristic function of the Lévy basis L is given by $\mathbb{E}e^{izL(A)} = \exp(\psi(z)\lambda^d(A))$ for every Borel set A with finite Lebesgue measure λ^d and ψ as the Lévy exponent of L. By setting $L(A) := \langle \dot{L}, \mathbb{1}_A\rangle$ for bounded Borel sets A, the extension of a Lévy white noise $\dot{L}$ can be identified with a Lévy basis L. Conversely, a Lévy basis L can be identified with a Lévy white noise $\dot{L}$ by setting $\langle \dot{L}, \varphi\rangle := \int_{\mathbb{R}^d} \varphi(x) dL(x)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and a function $f : \mathbb{R}^d \to \mathbb{R}$ is in the domain of $\dot{L}$ if and only if f is integrable with respect to the Lévy basis L (see [19]). These two constructions define the same mathematical object seen from different perspectives. In the context of linear stochastic partial differential equations, we use generalized random processes since the (weak) derivative or the integration of a generalized random process is again a generalized random process.
In the general theory of deterministic linear partial differential equations, the notion of weak solutions or solutions in the sense of distributions is widely used. In the context of stochastic partial differential equations, these concepts lead to the theory of mild solutions and generalized solutions. Given a partial differential operator $p(x, D)$, $x \in \mathbb{R}^d$, we call $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ a weak fundamental solution of the operator $p(x, D)$ if

$$E(\varphi) := \int_{\mathbb{R}^d} E(x, y)\varphi(y)\, dy$$

solves $p(x, D)E(\varphi) = \varphi$ in the weak sense for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. For stochastic partial differential equations of the form $p(x, D)u = \dot{L}$, where $\dot{L}$ is a Lévy white noise with characteristic triplet (a, γ, ν), we call the random field $u(x) := \langle \dot{L}, E(x, \cdot)\rangle$ the mild solution of above equation, if $u(x)$ exists for all $x \in \mathbb{R}^d$. A sufficient condition for u to exist would be that $E(x, \cdot) \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ and $\dot{L}$ having finite first moment, i.e. $\int_{|r|>1} |r|\nu(dr) < \infty$. Inspired by distributional solutions, a generalized random process $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ is called a generalized solution of the equation $p(x, D)s = \dot{L}$ if it holds

$$\langle s, p(x, D)^*\varphi\rangle = \langle \dot{L}, \varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where $p(x,D)^*$ is the adjoint of $p(x,D)$.

In **Chapter 2** we discuss second order elliptic stochastic partial differential equations driven by Levy white noise. In this framework our partial differential operator $p(x,D)$ is of the form

$$p(x,D)u = -\mathrm{div}(A(x)\nabla u) + b(x)\cdot\nabla u + V(x)u, \quad u \in C^\infty(\mathbb{R}^d),$$

where A is a uniformly elliptic $\mathbb{R}^d$-valued matrix function and given functions $b : \mathbb{R}^d \to \mathbb{R}^d$, $V : \mathbb{R}^d \to \mathbb{R}$. We obtain generalized solutions for such stochastic partial differential equations and especially achieve generalized and mild solutions for the generalized electric Schrödinger operator driven by a Lévy white noise, i.e. solutions u of the equation

$$-\mathrm{div}(A(x)\nabla u) + V(x)u = \dot{L},$$

where A is a uniformly elliptic $d\times d$ matrix and $V : \mathbb{R}^d \to (0,\infty)$ a function.
As a first result, we derive in Theorem 2.4 conditions for a large class of stochastic partial differential equations to have a generalized solution, meaning that we obtain sufficient conditions for the existence of the generalized random process s defined by $s(\varphi) = \langle \dot{L}, E(\varphi)\rangle$, where $E : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a suitable kernel. This extends the work of Berger in [6], where generalized processes of the form $s(\varphi) = \langle \dot{L}, E * \varphi\rangle$ were considered. Moreover, we see in Theorem 2.8 that if $\dot{L}$ has finite $\beta > 0$ moment, then so has the generalized process $s(\varphi) = \langle \dot{L}, E(\varphi)\rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$ under further conditions on the kernel E and conversely if s has finite β-moment, then also $\dot{L}$ has finite β-moment. In Theorem 2.9 we state our general result for generalized solutions of elliptic stochastic partial differential operators of second order with variable coefficients in divergence form, i.e. we derive generalized solutions s of the equation $p(x,D)s = \dot{L}$, where

$$p(x,D)u = -\mathrm{div}(A(x)\nabla u),$$

where A is a uniformly elliptic matrix. This general result holds only in dimension $d \geq 5$ and we have to assume that $\dot{L}$ satisfies a polynomial moment condition. However, we see in Theorem 2.17 that logarithmic moments are sufficient for generalized solutions of the generalized electric Schrödinger operator driven by Lévy white noise in dimensions $d \geq 3$. Additionally, we also obtain in dimension $d = 3$ a stochastically continuous mild solution u of the generalized electric Schrödinger operator driven by Lévy white noise. Moreover, if we assume that $\dot{L}$ has finite first moment, our mild solution u gives rise to a generalized solution s via

$$\langle s, \varphi\rangle := \int_{\mathbb{R}^d} u(x)\varphi(x)\lambda^d(dx).$$

These results follow from Proposition 2.19, where we state sufficient conditions for the existence of stochastically continuous random fields $u(x) = \langle \dot{L}, E(x,\cdot)\rangle$, and from Theorem 2.20, where we analyze under which conditions a mild solution of second order elliptic stochastic partial differential equations driven by Lévy white noise gives rise to a generalized solution.

As mentioned before, the existence of generalized random processes s of the form $s(\varphi) = \langle \dot{L}, E * \varphi\rangle$ was shown in [6]. These kind of processes are stationary and our extension to processes of the form $s(\varphi) = \langle \dot{L}, E(\varphi)\rangle$ allows us now to also model different kinds of stationarity assumptions (see e.g. Proposition 2.14). Stationary processes have been studied extensively and play a key role in time series analysis, since these are processes for which some properties do not vary with time. To be more precise, a stochastic process $(X_t)_{t\in\mathbb{R}}$ is called (strictly) stationary if its finite dimensional distributions are shift-invariant, i.e. for all $n \in \mathbb{N}$ and for all $t_1, \dots, t_n \in \mathbb{R}$ it holds

$$\mathcal{L}(X_{t_1+h}, \dots X_{t_n+h}) = \mathcal{L}(X_{t_1}, \dots X_{t_n}) \quad \text{for every } h \in \mathbb{R},$$

where $\mathcal{L}(X_{t_1}, \dots X_{t_n})$ denotes the n-dimensional distribution of X in $(t_1, \dots, t_n)$. As an extension of stationary processes, the concepts of periodic strictly stationary processes were introduced, which were discussed in [1], among others. A stochastic process $(X_t)_{t\in\mathbb{R}}$ is called periodic strictly stationary with period $l \in \mathbb{R}$ if for all $n \in \mathbb{N}$ and for all $t_1, \dots, t_n \in \mathbb{R}$ it holds

$$\mathcal{L}(X_{t_1+l}, \dots X_{t_n+l}) = \mathcal{L}(X_{t_1}, \dots X_{t_n}).$$

In **Chapter 3** we consider processes for which the shift operator functions on the finite dimensional distributions are almost periodic. Morozan and Tudor defined such processes in [34] as processes with almost periodic finite dimensional distributions, whereas we call such processes almost periodically stationary instead. Given a complete metric space (M, d), a continuous function $f : \mathbb{R} \to M$ is called almost periodic if for every $\varepsilon > 0$ there exists an $L_\varepsilon > 0$ and a $\tau = \tau(a, \varepsilon) \in [a, a + L_\varepsilon]$ for all $a \in \mathbb{R}$ such that

$$d(f(x + \tau), f(x)) < \varepsilon \quad \text{for all } x \in \mathbb{R}.$$

Clearly, every periodic function is almost periodic. Moreover, the sum and product of almost periodic functions is again almost periodic, and Bohr also showed in [10] a Fourier series representation for almost periodic functions which differs from the periodic case in so far that the summation is over periodic functions with different periods. The definition above seems to imply that the metric is important for the definition of almost periodicity, but it turns out that only the generated topology is relevant as there exist at least two equivalent definitions, which do not include the notion of a metric, but a topology. Therefore, we can exchange the metric as long as the different metrics generate the same topology.

For a stochastic process $(X_t)_{t\in\mathbb{R}}$ we define for $n \in \mathbb{N}$ and $(t_1, \dots, t_n) \in \mathbb{R}^n$ the function $\mu_{t_1,\dots,t_n} : \mathbb{R} \to \mathcal{P}(\mathbb{R}^n)$, $\mu_{t_1,\dots,t_n}(x) := \mathcal{L}(X_{t_1+x,\dots,t_n+x})$, where $\mathcal{P}(\mathbb{R}^n)$ denotes the collection of probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. We call $(X_t)_{t\in\mathbb{R}}$ almost periodically stationary if for any $n \in \mathbb{N}$ and $(t_1, \dots, t_n) \in \mathbb{R}^n$ the function $x \mapsto \mu_{t_1,\dots,t_n}(x)$ is almost periodic in $x \in \mathbb{R}$. We assume that the metric used is complete on the space of probability measures and induces the topology of weak convergence. Such a metric could be, for example, the well-known Prokhorov metric $\delta_n : \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \to [0, \infty)$ defined as

$$\delta_n(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(A) \le \nu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(\mathbb{R}^n)\},$$

where $A^\varepsilon = \bigcup_{x\in A}\{y \in \mathbb{R}^n : |y-x| < \varepsilon\}$. There exist further concepts of almost periodicity in the context of stochastic processes, like almost periodic in probability, almost sure, in quadratic mean, etc. In [4] and [45] those different types of almost periodicity were compared. In Proposition 3.8 we state general results for almost periodically stationary processes $(X_t)_{t\in\mathbb{R}}$ such as the relative compactness of the set of probability measures $(\mathcal{L}_{X_t})_{t\in\mathbb{R}}$ or the fact that, if we assume additionally uniform integrability of $(X_t^2)_{t\in\mathbb{R}}$, then $(X_t)_{t\in\mathbb{R}}$ is almost periodically correlated, which means that the mean and the covariance functions of $(X_t)_{t\in\mathbb{R}}$ are almost periodic functions in $t \in \mathbb{R}$. In Theorem 3.9 we characterize the almost periodically stationarity of stochastic processes in terms of their characteristic function. Thereby we get in Theorem 3.10 a characterization for infinitely divisible processes to be almost periodically stationary in terms of their characteristic triplets. As an application, we derive in Corollary 3.17 sufficient conditions for the almost periodically stationarity of $(X_t)_{t\in\mathbb{R}}$ defined as $X_t = \int_{\mathbb{R}^d} f(t,s)dL(s)$, where $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is a suitable deterministic function and L is a Lévy basis. Moreover, we study the concept of almost periodicity in probability: a stochastically continuous process $(X_t)_{t\in\mathbb{R}}$ is almost periodic in probability if the function $\mathbb{R} \to L^0(\Omega,\mathbb{R})$, $t \mapsto X_t$ is almost periodic with respect to a metric on $L^0(\Omega,\mathbb{R})$ which induces the convergence in probability. In this context we choose the Ky-Fan metric $\alpha(X,Y) := \inf\{\varepsilon \geq 0 : P(|X-Y| > \varepsilon) \leq \varepsilon\}$ and obtain in Lemma 3.19 a sufficient condition for stochastic processes to be almost periodic in probability. Analogous to before, we apply this to get in Theorem 3.21 a sufficient condition for the almost periodicity in probability of stochastic integrals. Further, we study Ornstein-Uhlenbeck-type processes for which we obtain an almost periodically stationary solution. An Ornstein-Uhlenbeck process $X^{(\mu)} = (X_t^{(\mu)})_{t\in\mathbb{R}}$ driven by a Lévy process L is a solution of the stochastic differential equation $dX_t = \mu X_t dt + dL_t$, where $\mu \in \mathbb{R}$. In our framework, we understand an Ornstein-Uhlenbeck-type process as a stochastic process $(X_t)_{t\in\mathbb{R}}$ which solves the stochastic differential equation $dX_t = \mu(t)X_t dt + dL_t$, $X_0 = X(0)$, where $\mu : \mathbb{R} \to \mathbb{R}$ is almost periodic and $X(0)$ a starting random variable. The case where μ is a periodic function was discussed in [1]. In Theorem 3.24 we prove the existence and uniqueness of the almost periodically stationary Ornstein-Uhlenbeck-type process $(X_t)_{t\in\mathbb{R}}$, which is given by $X_t = \int_{-\infty}^t \exp(\int_s^t \mu(u)du)\, dL(s)$. Another result related to almost periodically stationary is our central limit theorem for almost periodically stationary processes $(X_t)_{t\in\mathbb{R}}$, which are m-dependent and L^2-uniformly integrable, i.e. $(X_t)_{t\leq u}$ and $(X_t)_{t>u+m}$ are independent for all $u \in \mathbb{R}$ and $(X_t^2)_{t\in\mathbb{R}}$ is uniformly integrable (see Theorem 3.27).

In **Chapter 4** we discuss the generalized uniform integrability of infinitely divisible random processes and especially of stochastic integrals of the form $\int_{\mathbb{R}^d} f(t,s)dL(s)$. By generalized uniform integrability we mean that $(g(X_t))_{t\in\mathbb{R}}$ is uniformly integrable for suitable submultiplicative functions g and we call such processes g-uniformly integrable. A non-negative function $g : \mathbb{R}^d \to (0,\infty)$ is called submultiplicative if there is a constant $C > 0$ such that $g(x+y) \leq Cg(x)g(y)$ for all $x,y \in \mathbb{R}^d$. For example, for $p \geq 1$, the functions $\max(|x|^p, 1)$, $\log(\max(|x|, e))$, $\exp(|x|^{1/p})$ are submultiplicative. The reason for choosing submultiplicative and locally bounded functions g is the well-known result that a Lévy process $(L_t)_{t\geq 0}$ has finite g-moment for some $t > 0$ (hence for all $t > 0$) if and only if its Lévy measure has finite g-moment. This also carries on when considering g-uniform integrability of infinitely divisible random variables. Given an infinitely divisible random variable $(X_t)_{t\in\mathbb{R}}$ such that $(P_{X_t})_{t\in\mathbb{R}}$ is relatively compact, it was shown in [28] for continuous submultiplicative functions g that $(X_t)_{t\in\mathbb{R}}$ is g-uniformly integrable if and only if the Lévy measure of $(X_t)_{t\in\mathbb{R}}$ is g-uniformly integrable. In Corollary 4.3 we reformulate this result for locally bounded submultiplicative functions g and drop the assumption of relative compactness of $(P_{X_t})_{t\in\mathbb{R}}$. This leads us to Theorem 4.5, where we derive conditions under which $(X_t)_{t\in\mathbb{R}}$ defined by $X_t = \int_{\mathbb{R}^d} f(t,s)dL(s)$ is g-uniformly integrable for polynomially bounded submultiplicative functions g. Finally, we show the L^2-uniform integrability of the mild solution of the generalized electric Schrödinger equation, considered in Chapter 2, and for $p \geq 2$ the L^p-uniform integrability of Ornstein-Uhlenbeck-type processes discussed in Chapter 3.

Chapter 2 is based on the published article [7], which is a joint work with David Berger and Chapter 3 is based on Berger, Mohamed [8] (preprint). While also Chapter 4 contains new results, at the moment there does not exist a corresponding preprint.

2 Second order elliptic partial differential equations driven by Lévy white noise

This chapter is based on the published article by Berger, Mohamed [7] "Second order elliptic partial differential equations driven by Lévy white noise" and deals with linear stochastic partial differential equations with variable coefficients driven by Lévy white noise. We first derive an existence theorem for integral transforms of Lévy white noise and prove the existence of generalized and mild solutions of second order elliptic partial differential equations. Furthermore, we discuss the generalized electric Schrödinger operator for different potential functions V.

2.1 Introduction

Since the beginning of studying partial differential equations the Laplacian operator $\Delta := \sum_{j=1}^{d} \partial_j^2$ was of great interest in different mathematical theories and applications. For example, the solution of the Poisson equation

$$-\Delta u = f$$

for some function f can be interpreted as a stationary solution of the heat equation and is therefore important in thermodynamics. In order to study different heterogeneity assumptions in the space, the divergence operator

$$\operatorname{div}(A(x)\nabla u) := \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j u)$$

was introduced, where the matrix function A satisfies some ellipticity condition. This kind of operator is for example used in the Maxwell equations in general media (see 48). The fundamental solution of the Laplace equation is well-known, but there is no explicit form for a fundamental solution of a general divergence form operator, but there exist upper and lower bounds, see for example [32].
The goal of this chapter is to obtain generalized solutions of the equation

$$p(x, D)s = \dot{L},$$

where $\dot{L}$ is a so-called generalized Lévy white noise and p is a partial differential operator of the form

$$-\operatorname{div}(A(x)\nabla u) + b(x) \cdot \nabla u + V(x)u, \quad u \in C^\infty(\mathbb{R}^d), \tag{2.1}$$

for a uniformly elliptic $\mathbb{R}^d$-valued matrix function A and functions $b : \mathbb{R}^d \to \mathbb{R}^d$, $V : \mathbb{R}^d \to \mathbb{R}$. We especially achieve generalized and mild solutions for the generalized electric Schrödinger operator driven by a Lévy white noise, i.e. we are looking for a solution u of the stochastic partial differential equation

$$-\operatorname{div}(A(x)\nabla u) + V(x)u = \dot{L}, \tag{2.2}$$

where A is a uniformly elliptic $d \times d$ matrix, the potential $V > 0$ belongs to the reverse Hölder class and $\dot{L}$ is a Lévy white noise. Since the fundamental solution of the Schrödinger operator has exponential decay, we will derive weaker assumptions on the Lévy white noise in comparison to the general case (2.1) to show the existence of generalized and mild solutions. This can be seen as an extension of the theory founded in [6] by D. Berger, but the results are not directly applicable. In order to overcome this shortcoming we derive existence results for generalized random processes constructed by integral transforms of the underlying Lévy white noise. Furthermore, we study different distributional properties of these solutions and show that we can construct periodically stationary generalized random processes.
We are solving the stochastic partial differential equations in distributional sense, i.e. a solution s is a distribution valued random variable such that $\langle s, p(x,D)^*\varphi\rangle = \langle \dot{L}, \varphi\rangle$ for every φ in our function space. For a good introduction to distributional solutions of partial differential equations see for example [24]. Until now there does not exist a good understanding of Lévy white noise driven stochastic partial differential equations under general moment conditions, but there exists literature for the case of Gaussian white noise and Lévy white noise with stricter moment conditions. In [49] SPDEs driven by Gaussian white noise where studied. In the case of stochastic partial differential equations with constant coefficients see also [15] and [6]. Our method is inspired by the papers of [19] and the results of [38]. We also mention the monograph [36] by S. Peszat and J. Zabczyk, which gives a good overview about SPDEs driven by Lévy noise, where another approach motivated by the semigroup theory is used to consider parabolic and hyperbolic SPDEs driven by Lévy noise in Banach spaces.

In Section 2.2 we provide the general framework needed to discuss stochastic partial differential equations driven by Lévy white noise, whose solutions are defined as generalized random process. We introduce Lévy white noise as a generalized random process in the sense of I.M. Gelfand and N.Y. Vilenkin (see [20]). Theorem 2.4 implies that a large class of linear stochastic partial differential equations driven by a Lévy white noise has a generalized solution, where we used a more general kernel $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ compared to Theorem 3.4 of D. Berger in [6]. Furthermore, we study the moment properties of generalized random processes s driven by Lévy white noise $\dot{L}$. For a well-defined random process $s(\varphi) = \langle \dot{L}, G(\varphi)\rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we show in Theorem 2.8 that if $\dot{L}$ has finite

$\beta > 0$ moment, then s has also finite β-moment under further conditions on the kernel G. Moreover, we show that if s has finite β-moment, then also $\dot{L}$ has finite β-moment. In Section 2.3 we discuss our first example, the partial differential operators of the form (2.1) and give existence results for generalized solutions. Furthermore, we discuss periodically stationary solutions s for this example. Afterwards we consider the generalized electric Schrödinger operator driven by Lévy white noise and show under weaker conditions, as in the example above, the existence of generalized solutions. We also study the concept of mild solutions of (2.2), i.e. a solution u which is a random field and given by the convolution of the Lévy white noise with the fundamental solution of (2.2). In Proposition 2.19 we mention when such a solution u exists and is stochastically continuous. Most of the notation used later on is standard or self explanatory. We mention only that λ^d denotes the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{D}(\mathbb{R}^d)$ the space of test functions on $\mathbb{R}^d$, i.e. the space of infinitely differentiable real valued functions on $\mathbb{R}^d$ with compact support and $\mathcal{D}'$ its dual space, i.e. the space of distributions.

2.2 Integral transforms and generalized stochastic processes driven by Lévy white noise

We provide the general framework needed to discuss stochastic partial differential equations driven by Lévy white noise and introduce Lévy white noise as generalized random processes in the sense of I.M. Gelfand and N.Y. Vilenkin (see [20]). In [6] it was shown that a convolution operator, with certain properties regarding his integrability, defines a generalized random process, assuming low moment conditions on the Lévy white noise. Similar to [6], we will use the characterization of the extended domain (see [19, Proposition 3.4]) and achieve new results for a more general kernel $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$, which allows us in Section 2.3 to model different kinds of stationarity assumptions and also to obtain generalized solutions of Lévy driven stochastic partial differential equations.
Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space.

Definition 2.1. (see [19], Definition 2.1.) A *generalized random process* is a linear and continuous function $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$. The linearity means that, for every $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d)$ and $\mu \in \mathbb{R}$,

$$s(\varphi_1 + \mu\varphi_2) = s(\varphi_1) + \mu s(\varphi_2) \text{ almost surely.}$$

The continuity means that if $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$, then $s(\varphi_n)$ converges to $s(\varphi)$ in probability.

Due to the nuclear structure on $\mathcal{D}(\mathbb{R}^d)$ it follows with [49], Corollary 4.2 that a generalized random process has a version which is a measurable function from $(\Omega, \mathcal{F})$ to $(\mathcal{D}'(\mathbb{R}^d), \mathcal{C})$ with respect to the cylindrical σ-field $\mathcal{C}$ generated by the sets

$$\{u \in \mathcal{D}'(\mathbb{R}^d) | \, (\langle u, \varphi_1 \rangle, \dots, \langle u, \varphi_N \rangle) \in B\}$$

with $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in \mathcal{D}(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^N)$. From now on it is always meant such a version.
The probability law of a generalized random process s is the probability measure on $\mathcal{D}'(\mathbb{R}^d)$ given by

$$\mathcal{P}_s(B) := \mathcal{P}(s \in B) = \mathcal{P}(\{\omega \in \Omega : s(\omega) \in B\})$$

for $B \in \mathcal{C}$, where $\mathcal{C}$ is the cylindrical σ-field on $\mathcal{D}'(\mathbb{R}^d)$.
The characteristic functional of a generalized random process s is the functional $\widehat{\mathcal{P}}_s : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$ defined by

$$\widehat{\mathcal{P}}_s(\varphi) = \int\limits_{\mathcal{D}'(\mathbb{R}^d)} \exp(i\langle u, \varphi\rangle) d\mathcal{P}_s(u).$$

The characteristic functional characterizes the law of s in the sense that two random processes are equal in law if and only if they have the same characteristic functional. Now we define the Lévy white noise, which is closely connected to a Lévy process. In general, a Lévy process is a stochastically continuous process with independent and stationary increments starting in 0. A Lévy process $(L_t)_{t \geq 0}$ is characterized by its characteristic function, it holds that

$$\mathbb{E}e^{izL_t} = \exp(t\psi(z)),$$

for every $z \in \mathbb{R}$ and $t \geq 0$. We call ψ the Lévy exponent which can be characterized by an $a \geq 0$, $\gamma \in \mathbb{R}$ and a Lévy measure ν, i.e. a measure such that

$$\nu(\{0\}) = 0 \text{ and } \int\limits_{\mathbb{R}\setminus\{0\}} \min\{1, x^2\}\nu(dx) < \infty.$$

For all $z \in \mathbb{R}$ it holds that

$$\psi(z) = i\gamma z - \frac{1}{2}az^2 + \int\limits_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbb{1}_{|x|\leq 1})\nu(dx).$$

Definition 2.2. (see [7], Definition 2.) A *Lévy white noise* $\dot{L}$ on $\mathbb{R}^d$ is a generalized random process with characteristic functional of the form

$$\widehat{\mathcal{P}}_{\dot{L}}(\varphi) = \exp\left(\int\limits_{\mathbb{R}^d} \psi(\varphi(x))\lambda^d(dx) \right)$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, where $\psi : \mathbb{R} \to \mathbb{C}$ is a Lévy exponent, i.e. there exist $a \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$ and ν a Lévy-measure, such that

$$\psi(z) = i\gamma z - \frac{1}{2}az^2 + \int\limits_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbb{1}_{|x|\leq 1})\nu(dx).$$

The function ψ is uniquely characterized by the triplet (a, γ, ν) known as the *characteristic triplet*.

The existence of the Lévy white noise was shown in [20]. Another possible way to construct Lévy white noise would be as an independently scattered random measures, i.e. a random process whose test functions are indicator functions and are independently scattered when two indicator functions with disjoint supports define independent random variables (see B.S. Rajput and J. Rosinski [38]). In [19] J. Fageot and T. Humeau unified these two approaches by extending the Lévy white noise, defined as generalized random processes, to independently scattered random measures. This connection led to results in [19], which made it possible to extend the domain of definition of Lévy white noise to some Borel-measurable functions $f : \mathbb{R}^d \to \mathbb{R}$. We say that the function f is in the domain of $\dot{L}$ if there exists a sequence of elementary functions f_n converging almost everywhere to f such that $\langle \dot{L}, f_n \mathbb{1}_A \rangle$ converges in probability for $n \to \infty$ for every Borel set A and set $\langle \dot{L}, f \rangle$ as the limit in probability of $\langle \dot{L}, f_n \rangle$ for $n \to \infty$, where $\langle \dot{L}, f_n \rangle$ is defined by $\sum_{j=1}^m a_j \langle \dot{L}, \mathbb{1}_{A_j} \rangle$ for a elementary function $f_n := \sum_{j=1}^m a_j \mathbb{1}_{A_j}$, see also [19], Definition 3.3. For the maximal domain of the Lévy white noise $\dot{L}$ we write $D(\dot{L})$. By setting $L(A) := \langle \dot{L}, \mathbb{1}_A \rangle$ for bounded Borel sets A, the extension of a Lévy white noise $\dot{L}$ can be identified with a Lévy basis L in the sense of Rajput and Rosinski [38], see [19], Theorem 3.2 and Proposition 3.4. As a Lévy basis can be identified with a Lévy white noise in a canonical way, i.e. $\langle \dot{L}, \varphi \rangle := \int_{\mathbb{R}^d} \varphi(x) dL(x)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we make no difference between a Lévy white noise and a Lévy basis. In particular, a Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is in $D(\dot{L})$ if and only if f is integrable with respect to the Lévy basis L in the sense of Rajput and Rosinski [38], see [19], Definition 3.3.

Definition 2.3. (see [21], Definition 1.1.1.) For a measurable function $f \in L^0(\mathbb{R}^d)$ we define the *distribution function* of f as

$$d_f(\alpha) = \lambda^d(\{x \in \mathbb{R}^d : |f(x)| > \alpha\}),\ \alpha > 0.$$

With the aid of the distribution function we can now obtain a sufficient condition for the existence of the generalized random process s defined by $s(\varphi) = \langle \dot{L}, G(\varphi) \rangle$, where $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a suitable kernel. In order to do so, we use the results from [38] and [19] regarding integrability conditions for Lévy white noises. In contrast to [6], where the existence of the stationary generalized random process $s(\varphi) = \langle \dot{L}, G * \varphi \rangle$ was obtained, this more general kernel $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ allows us to model different kinds of stationarity assumptions. Furthermore, this will be crucial in Section 2.3 for proving the existence of generalized processes as solutions to stochastic partial differential equations as in (2.1).

Theorem 2.4. (see [7], Theorem 1.) *Let $\dot{L}$ be a Lévy white noise on $\mathbb{R}^m$ with characteristic triplet (a, γ, ν) and $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Define for every $x \in \mathbb{R}^m$ and $R > 0$*

$$G_R(x) := \int_{B_R(0)} |G(x,y)| \lambda^d(dy) \in [0, \infty]$$

and

$$h_R(x) := x \int_0^{1/x} d_{G_R}(\alpha)\lambda^1(d\alpha) \text{ for } x > 0.$$

Assume that $G_R \in L^1(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$ *and*

$$\int_{\mathbb{R}} \mathbb{1}_{|r|>1} h_R(|r|)\nu(dr) < \infty \tag{2.3}$$

for every $R > 0$*. Then for* $\big(G(\varphi)\big)(x) := \int_{\mathbb{R}^d} G(x,y)\varphi(y)\lambda^d(dy)$ *we have that*

$$s(\varphi) := \langle \dot{L}, G(\varphi)\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d)$$

defines a generalized random process.

Proof. The proof is quite similar to that of [6], Theorem 3.4. For the sake of completeness we give a detailed proof. We need to show that $G(\varphi) \in D(\dot{L})$ and $\langle \dot{L}, G(\varphi_n)\rangle \to \langle \dot{L}, G(\varphi)\rangle$ as $n \to \infty$ in probability for a sequence $(\varphi_n)_{n\in\mathbb{N}}$ converging to φ in $\mathcal{D}(\mathbb{R}^d)$. As $\langle \dot{L}, G(\cdot)\rangle$ is linear, this is equivalent to check that $\langle \dot{L}, G(\varphi_n - \varphi)\rangle \to 0$ as $n \to \infty$ in probability (see [19], Theorem 3.6). Now given [38], Theorem 2.7, we have to show

$$\int_{\mathbb{R}^m} \Big|\gamma\Big(G(\varphi_n)\Big)(x) + \int_{\mathbb{R}} r\Big(G(\varphi_n)\Big)(x)\left(\mathbb{1}_{|r(G(\varphi_n))(x)|\le 1} - \mathbb{1}_{|r|\le 1}\right)\nu(dr)\Big|\lambda^m(dx) \to 0, \tag{2.4}$$

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}} \min\Big(1, \Big|r\Big(G(\varphi_n)\Big)(x)\Big|^2\Big)\nu(dr)\lambda^m(dx) \to 0 \quad \text{and} \tag{2.5}$$

$$a^2 \int_{\mathbb{R}^m} \Big|\Big(G(\varphi_n)\Big)(x)\Big|^2 \lambda^m(dx) \to 0 \tag{2.6}$$

as $n \to \infty$ if $\varphi_n \to 0$ for $n \to \infty$ in $\mathcal{D}(\mathbb{R}^d)$.
In the following we give a pointwise upper bound for $G(\varphi)$. Let therefore be $R > 0$ such that $\operatorname{supp}(\varphi_n) \subset B_r(0)$ for some $r < R$. Then it holds for every $x \in \mathbb{R}^m$

$$\begin{aligned} \Big|\Big(G(\varphi_n)\Big)(x)| &\le \int_{\mathbb{R}^d} \Big|G(x,y)\varphi_n(y)|\lambda^d(dy) \\ &= \int_{B_R(0)} |G(x,y)\varphi_n(y)|\lambda^d(dy) \le G_R(x)\|\varphi_n\|_\infty \end{aligned} \tag{2.7}$$

and we obtain for $\alpha > 0$

$$\begin{aligned} d_{G(\varphi_n)}(\alpha) =& \lambda^m\left(\{x \in \mathbb{R}^m : |(G(\varphi_n))(x)| > \alpha\}\right) \\ \le& \lambda^m\left(\Big\{x \in \mathbb{R}^m : |G_R(x)| > \frac{\alpha}{\|\varphi_n\|_\infty}\Big\}\right) = d_{G_R}\left(\frac{\alpha}{\|\varphi_n\|_\infty}\right). \end{aligned} \tag{2.8}$$

Since $G_R \in L^2(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} |\Big(G(\varphi_n)\Big)(x)| \mathbb{1}_{|(G(\varphi_n))(x)| > \frac{1}{|r|}} \lambda^m(dx) \le \int_{\mathbb{R}^m} |G\Big(\varphi_n\Big)(x)|^2 |r| \lambda^m(dx)$$
$$\le \|\varphi_n\|_\infty^2 \|G_R\|_{L^2(\mathbb{R}^m)}^2 |r|. \tag{2.9}$$

Now we show (2.4). Since $G_R \in L^1(\mathbb{R}^m)$, we have

$$\int_{\mathbb{R}^m} \Big|\gamma\Big(G(\varphi_n)\Big)(x)\Big| \lambda^m(dx) \le |\gamma| \, \|\varphi_n\|_\infty \|G_R\|_{L^1(\mathbb{R}^m)} \to 0 \tag{2.10}$$

for $n \to \infty$. We rewrite the second term in (2.4) in the following way

$$\begin{aligned}
&\int_{\mathbb{R}^m} \int_{\mathbb{R}} \Big| r\Big(G(\varphi_n)\Big)(x)\Big| \left(\mathbb{1}_{|r(G(\varphi_n))(x)| \le 1} - \mathbb{1}_{|r| \le 1}\right) \nu(dr) \lambda^m(dx) \\
&= \int_{\mathbb{R}} |r| \mathbb{1}_{|r|>1} \int_{\mathbb{R}^m} \Big|\Big(G(\varphi_n)\Big)(x)\Big| \mathbb{1}_{|r(G(\varphi_n))(x)| \le 1} \lambda^m(dx) \nu(dr) \\
&\quad - \int_{\mathbb{R}} |r| \mathbb{1}_{|r| \le 1} \int_{\mathbb{R}^m} \Big|\Big(G(\varphi_n)\Big)(x)\Big| \mathbb{1}_{|r(G(\varphi_n))(x)| > 1} \lambda^m(dx) \nu(dr)
\end{aligned}$$

and by [21], Exercise 1.1.10, p. 14 and (2.8) we observe

$$\begin{aligned}
\int_{\mathbb{R}^m} \Big|\Big(G(\varphi_n)\Big)(x)\Big| \mathbb{1}_{|r(G(\varphi_n))(x)| \le 1} \lambda^m(dx) &\le \int_0^{\frac{1}{|r|}} d_{G(\varphi_n)}(\alpha) \lambda^1(d\alpha) \\
&\le \int_0^{\frac{1}{|r|}} d_{G_R} \left(\frac{\alpha}{\|\varphi_n\|_\infty}\right) \lambda^1(d\alpha).
\end{aligned}$$

We see that the right hand side converges to 0 for $n \to \infty$ and for n large enough we have

$$\int_0^{\frac{1}{|r|}} d_{G_R} \left(\frac{\alpha}{\|\varphi_n\|_\infty}\right) \lambda^1(d\alpha) \le \int_0^{\frac{1}{|r|}} d_{G_R}(\alpha) \, \lambda^1(d\alpha) = \frac{1}{|r|} h_R(|r|).$$

Lebesgue's dominated convergence theorem using (2.3) implies

$$\int_{\mathbb{R}} |r| \mathbb{1}_{|r|>1} \int_{\mathbb{R}^m} \Big|\Big(G(\varphi_n)\Big)(x)\Big| \mathbb{1}_{|r(G(\varphi_n))(x)| \le 1} \lambda^m(dx) \nu(dr) \to 0$$

for $n \to \infty$. We observe with (2.9) for the remaining term that

$$\int_{\mathbb{R}} |r| \mathbb{1}_{|r| \le 1} \int_{\mathbb{R}^m} \Big|\Big(G(\varphi_n)\Big)(x)\Big| \mathbb{1}_{|r(G(\varphi_n))(x)| > 1} \lambda^m(dx) \nu(dr)$$

$$\leq \|\varphi_n\|_\infty^2 \|G_R\|_{L^2(\mathbb{R}^m)}^2 \int_{\mathbb{R}} |r|^2 \mathbb{1}_{|r|\leq 1} \nu(dr)$$

and by Lebesgue's dominated convergence theorem (since $\int_{|r|\leq 1} r^2 \nu(dr) < \infty$)

$$\int_{\mathbb{R}} |r| \mathbb{1}_{|r|\leq 1} \int_{\mathbb{R}^m} \left| \Big(G(\varphi_n) \Big)(x) \right| \mathbb{1}_{|r(G(\varphi_n))(x)|>1} \lambda^m(dx) \nu(dr) \to 0$$

for $n \to \infty$. This gives (2.4). In order to show (2.5) we observe

$$\begin{aligned} \min\left(1, \left| r\Big(G(\varphi_n)\Big)(x)\right|^2\right) \leq & \mathbb{1}_{|rG(\varphi_n)(x)|>1} \mathbb{1}_{|r|>1} + |rG(\varphi_n)(x)| \mathbb{1}_{|rG(\varphi_n)(x)|>1} \mathbb{1}_{|r|\leq 1} \\ & + \left| rG(\varphi_n)(x)\right|^2 \mathbb{1}_{|rG(\varphi_n)(x)|\leq 1} \mathbb{1}_{|r|\leq 1} \\ & + |rG(\varphi_n)(x)| \mathbb{1}_{|rG(\varphi_n)(x)|\leq 1} \mathbb{1}_{|r|>1}. \end{aligned}$$

From the previous calculations we conclude that the second and fourth term (when integrated with respect to $\nu(dr)\lambda^m(dx)$) converge to 0 for $n \to \infty$ and for the first term we note that

$$\int_{\mathbb{R}^m} \mathbb{1}_{|rG(\varphi_n)(x)|>1} \lambda^m(dx) = d_{G(\varphi_n)}\left(\frac{1}{|r|}\right) \leq d_{G_R}\left(\frac{1}{|r|\, \|\varphi_n\|_\infty}\right)$$

and by Lebesgue's dominated convergence theorem we conclude that

$$\int_{\mathbb{R}} \mathbb{1}_{|r|>1} d_{G_R}\left(\frac{1}{|r|\, \|\varphi_n\|_\infty}\right) \nu(dr) \to 0$$

for $n \to \infty$, as $h_R(|r|) \geq d_{G_R}(1/|r|)$. For the third term we directly see

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}^m} \left| rG(\varphi_n)(x)\right|^2 \mathbb{1}_{|rG(\varphi_n)(x)|\leq 1} \mathbb{1}_{|r|\leq 1} \lambda^m(dx)\nu(dr) \\ & \quad \leq \|G(\varphi_n)(\cdot)\|_{L^2(\mathbb{R}^m)}^2 \int_{\mathbb{R}} \mathbb{1}_{|r|\leq 1} |r|^2 \nu(dr) \\ & \quad \leq \|\varphi_n\|_\infty^2 \|G_R\|_{L^2(\mathbb{R}^m)}^2 \int_{\mathbb{R}} \mathbb{1}_{|r|\leq 1} |r|^2 \nu(dr) \to 0 \end{aligned}$$

for $n \to \infty$. This gives (2.5) and (2.6) follows with (2.9). Hence $G(\varphi_n) \to G(\varphi)$ in $D(\dot{L})$ as $n \to \infty$. □

In Theorem 2.4 we assumed that $G_R \in L^1(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$. In the following Proposition we will show that, if the Lévy white noise has no Gaussian part and it holds $\int_{\mathbb{R}} |r|^\beta \mathbb{1}_{|r|\leq 1} \nu(dr) < \infty$, for $\beta \in (1, 2)$, then we can assume $G_R \in L^1(\mathbb{R}^m) \cap L^\beta(\mathbb{R}^m)$ instead.

Proposition 2.5. (see [7], Proposition 1.) *Let $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function and for $R > 0$ let G_R and h_R be defined as in Theorem 2.4. Furthermore, let $\dot{L}$ be a Lévy white noise on $\mathbb{R}^m$ with characteristic triplet $(0, \gamma, \nu)$ such that (2.3) holds. If further $G_R \in L^1(\mathbb{R}^m) \cap L^\beta(\mathbb{R}^m)$ for some $\beta \in (1, 2)$ and*

$$\int_{\mathbb{R}} |r|^\beta \mathbb{1}_{|r|\leq 1} \nu(dr) < \infty,$$

then

$$s(\varphi) := \langle \dot{L}, G(\varphi) \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d)$$

defines a generalized random process, where $G(\varphi)$ is defined as in Theorem 2.4.

Proof. The proof follows with similar steps as in the proof of Theorem 2.4 and hence we only mention the needed modifications. As $G_R \in L^1(\mathbb{R}^m)$ we only have to consider the terms which were estimated with $\|G_R\|_{L^2(\mathbb{R}^m)}$ in the proof of Theorem 2.4. These are

$$\int_{\mathbb{R}} |r| \mathbb{1}_{|r|\leq 1} \int_{\mathbb{R}^m} |\big(G(\varphi_n)\big)(x)| \mathbb{1}_{|(G(\varphi_n))(x)| > \frac{1}{|r|}} \lambda^m(dx) \nu(dr) \tag{2.11}$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^m} |r|^2 |\big(G(\varphi_n)\big)(x)|^2 \mathbb{1}_{|r(G(\varphi_n))(x)|\leq 1} \mathbb{1}_{|r|\leq 1} \lambda^m(dx) \nu(dr). \tag{2.12}$$

and we have to show that they converge to 0 as $\varphi_n \to 0$ in $\mathcal{D}(\mathbb{R}^d)$. We have

$$\begin{aligned}\int_{\mathbb{R}^m} |\big(G(\varphi_n)\big)(x)| \mathbb{1}_{|(G(\varphi_n))(x)| > \frac{1}{|r|}} \lambda^m(ds) &\leq \|(G(\varphi_n))\|^\beta_{L^\beta(\mathbb{R}^m)} |r|^{\beta-1} \\ &\leq \|\varphi_n\|^\beta_\infty \|G_R\|^\beta_{L^\beta(\mathbb{R}^m)} |r|^{\beta-1}.\end{aligned}$$

So it follows that the term (2.11) converges to 0 as $\varphi_n \to 0$ in $\mathcal{D}(\mathbb{R}^d)$. Furthermore, it holds

$$\begin{aligned}&\int_{\mathbb{R}} \int_{\mathbb{R}^m} |r|^2 |\big(G(\varphi_n)\big)(x)|^2 \mathbb{1}_{|r(G(\varphi_n))(x)|\leq 1} \mathbb{1}_{|r|\leq 1} \lambda^m(dx) \nu(dr) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^m} |r|^\beta |\big(G(\varphi_n)\big)(x)|^\beta |r|^{2-\beta} |\big(G(\varphi_n)\big)(x)|^{2-\beta} \mathbb{1}_{|r(G(\varphi_n))(x)|\leq 1} \mathbb{1}_{|r|\leq 1} \lambda^m(dx) \nu(dr) \\ &\leq \int_{\mathbb{R}} |r|^\beta \mathbb{1}_{|r|\leq 1} \nu(dr) \|\varphi_n\|^\beta_\infty \|G_R\|^\beta_{L^\beta(\mathbb{R}^m)}.\end{aligned}$$

This shows that the term (2.12) converges to 0 as $\varphi_n \to 0$ in $\mathcal{D}(\mathbb{R}^d)$ and the rest of the proof follows with similar arguments as mentioned in the proof of Theorem 2.4. □

When $G_R \notin L^1(\mathbb{R}^m)$ we can still obtain a generalized process s under some extra conditions. Similar to Theorem 3.5 in [6] we obtain in the more general case the following result.

Theorem 2.6. (see [7], Theorem 2.) *Let $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $G_R \in L^2(\mathbb{R}^m)$, where G_R and $G(\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$ are defined as in Theorem 2.4. If the first moment of the Lévy white noise $\dot{L}$ on $\mathbb{R}^m$ with characteristic triplet (a, γ, ν) vanishes, i.e. $\mathbb{E}|\langle \dot{L}, \varphi\rangle| < \infty$ and $\mathbb{E}\langle \dot{L}, \varphi\rangle = 0$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, then $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ defined by*

$$s(\varphi) := \langle \dot{L}, G(\varphi)\rangle$$

is a generalized random process if

$$\int_{\mathbb{R}} \mathbb{1}_{|r|>1} |r| \int_{\frac{1}{|r|}}^{\infty} d_{G_R}(\alpha)\lambda^1(d\alpha)\nu(dr) < \infty \tag{2.13}$$

and

$$\int_{\mathbb{R}} \mathbb{1}_{|r|>1} |r|^2 \int_{0}^{\frac{1}{|r|}} \alpha d_{G_R}(\alpha)\lambda^1(d\alpha)\nu(dr) < \infty \tag{2.14}$$

for all $R > 0$.

Proof. This proof follows with the same arguments as in the proof of [6], Theorem 3.5, where we use $G(\varphi_n)$ instead of $G * \varphi_n$ and $\|\varphi_n\|_\infty \|G_R\|_{L^2(\mathbb{R}^m)} < \infty$ instead of $\|G * \varphi_n\|_{L^2(\mathbb{R}^d)} < \infty$. For the sake of completeness we give a detailed proof. By [40], Example 25.12, p. 163 we conclude that we need to show similar to Theorem 2.4 that (2.5), (2.6) and

$$\int_{\mathbb{R}^m} \left| \int_{\mathbb{R}} r\Big(G(\varphi_n)\Big)(x)\mathbb{1}_{|r(G(\varphi_n)(x)|>1} \right| \lambda^m(dx) \to 0, \tag{2.15}$$

are satisfied for all $(\varphi_n)_{n\in\mathbb{N}}$ converging to 0 in $\mathcal{D}(\mathbb{R}^d)$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence converging to 0 in $\mathcal{D}(\mathbb{R}^d)$ such that $\operatorname{supp} \varphi_n \subset B_R(0)$ for some $R > 0$ and all $n \in \mathbb{N}$. Considering that it holds $\int_{\mathbb{R}^m} |f(x)|\mathbb{1}_{|f(x)|>\beta}\lambda^m(dx) = \int_\beta^\infty d_f(\alpha)\lambda^1(d\alpha) + \beta d_f(\beta)$ for $\beta > 0$ and measurable f (cf. [21], Exercise 1.1.10, p.14), we estimate (2.15) together with (2.9) as follows

$$\begin{aligned}
&\int_{\mathbb{R}^m} \left| \int_{\mathbb{R}} r\Big(G(\varphi_n)\Big)(x)\mathbb{1}_{|r(G(\varphi_n)(x)|>1} \right| \lambda^m(dx) \\
\leq& \|G_R\|^2_{L^2(\mathbb{R}^m)} \|\varphi_n\|^2_\infty \int_{\mathbb{R}} \mathbb{1}_{|r|\leq 1} |r|^2 \nu(dr)
\end{aligned}$$

$$+ \int\limits_{|r|>1} \left(|r| \int\limits_{\frac{1}{|r|}}^{\infty} d_{G(\varphi_n)}(\alpha)\, \lambda^1(d\alpha) + d_{G(\varphi_n)}\left(\frac{1}{|r|}\right) \right) \nu(dr) \to 0,$$

for $n \to \infty$ by Lebesgue's dominated convergence theorem. Hence by (2.8)

$$\begin{aligned} &\int\limits_{|r|>1} \left(|r| \int\limits_{\frac{1}{|r|}}^{\infty} d_{G(\varphi_n)}(\alpha)\, \lambda^1(d\alpha) + d_{G(\varphi_n)}\left(\frac{1}{|r|}\right) \right) \nu(dr) \\ &\leq \int\limits_{|r|>1} |r| \int\limits_{\frac{1}{|r|}}^{\infty} d_{G_R}\left(\frac{\alpha}{\|\varphi_n\|_\infty}\right) \lambda^1(d\alpha)\nu(dr) + \int\limits_{|r|>1} d_{G_R}\left(\frac{1}{|r|\|\varphi_n\|_\infty}\right) \nu(dr) \\ &\leq \int\limits_{|r|>1} |r| \int\limits_{\frac{1}{|r|}}^{\infty} d_{G_R}(\alpha)\, \lambda^1(d\alpha)\nu(dr) + \int\limits_{|r|>1} d_{G_R}\left(\frac{1}{|r|}\right) \nu(dr) \end{aligned}$$

for large n and the latter is finite by (2.13), (2.14) and

$$\int\limits_0^x \alpha d_{G_R}(\alpha)\lambda^1(d\alpha) \geq d_{G_R}(x) \int\limits_0^x \alpha\lambda^1(d\alpha) = \frac{1}{2} d_{G_R}(x)x^2 \quad \text{for every } x > 0.$$

This gives (2.15). In order to control (2.5) we follow the same steps as in the proof of Theorem 2.4 and observe that we only have to show

$$\int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}} |\big(G(\varphi_n)\big)(x)|^2 |r|^2 \mathbb{1}_{|rG(\varphi_n)(x)|\leq 1} \mathbb{1}_{|r|>1} \nu(dr)\lambda^m(dx) \to 0$$

for $n \to \infty$. We see by [21], Exercise 1.1.10, p.14 and similar arguments as above that

$$\begin{aligned} &\int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}} |\big(G(\varphi_n)\big)(x)|^2 |r|^2 \mathbb{1}_{|rG(\varphi_n)(x)|\leq 1} \mathbb{1}_{|r|>1} \nu(dr)\lambda^m(dx) \\ \leq& 2 \int\limits_{\mathbb{R}} \mathbb{1}_{|r|>1} |r|^2 \int\limits_0^{\frac{1}{|r|}} \alpha d_{G(\varphi_n)}(\alpha)\lambda^1(d\alpha)\nu(dr) \to 0 \end{aligned}$$

for $n \to \infty$. Hence, we conclude that s defines a generalized process. □

Example 2.7. (see [7], Example 1.) Let $d \geq 1$, $q \in [1, 2)$ and $\frac{d}{2} < p < \frac{d}{q}$. We consider $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that it holds

$$|G(x, y)| \|x - y\|^p \leq w(y)$$

for all $x, y \in \mathbb{R}^d$, where $w \in L^{q*}_{loc}(\mathbb{R}^d)$ with $q^* = \frac{q}{q-1}$. With the Hölder's inequality we conclude for $R > 0$ and $x \in \mathbb{R}^d$

$$\begin{aligned} G_R(x) := & \int\limits_{B_R(0)} |G(x,y)|\lambda^d(dy) \\ & \leq \left(\int\limits_{B_R(0)} \|x-y\|^{-qp}\lambda^d(dy) \right)^{1/q} \left(\int\limits_{B_R(0)} |w(y)|^{q^*}\lambda^d(dy) \right)^{1/q*} \\ & \leq C(w,q,p,d,R)\min\{1, \|x\|^{-p}\}. \end{aligned} \tag{2.16}$$

We obtain that

$$\|G_R\|_{L^2(\mathbb{R}^d)} < \infty.$$

Furthermore, we observe for a Lévy white noise $\dot{L}$ with characteristic triplet (a, γ, ν) that

$$\int\limits_0^{\frac{1}{|r|}} \alpha d_{G_R}(\alpha)\lambda^1(d\alpha) \leq C \int\limits_0^{\frac{1}{|r|}} \alpha(1+\alpha^{-\frac{d}{p}})\lambda^1(d\alpha) = \tilde{C}(|r|^{-2} + |r|^{\frac{d}{p}-2})$$

and

$$\int\limits_{\mathbb{R}} \mathbb{1}_{|r|>1}|r|^2 \int\limits_0^{\frac{1}{|r|}} d_{G_R}(\alpha)\lambda^1(d\alpha)\nu(dx) \leq \int\limits_{\mathbb{R}} \mathbb{1}_{|r|>1}\tilde{C}(1+|r|^{\frac{d}{p}})\nu(dx),$$

where $\tilde{C} > 0$. If the Lévy white noise $\dot{L}$ has vanishing first moment then it follows from [40], Example 25.12 that (2.13) is satisfied. So if additionally $\dot{L}$ satisfies

$$\int\limits_{\mathbb{R}} \mathbb{1}_{|r|>1}|r|^{\frac{d}{p}}\nu(dx) < \infty$$

then it follows from Theorem 2.6 that

$$s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega),\ \varphi \mapsto s(\varphi) := \langle \dot{L}, G(\varphi) \rangle$$

defines a well-defined generalized random process.

2.2.1 Moment properties

Next we show, that if the Lévy white noise $\dot{L}$ has finite $\beta > 0$ moment, then so has the generalized random process $s(\varphi) = \langle \dot{L}, G(\varphi) \rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Theorem 2.8. (see [7], Theorem 3.) *Let $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function different from 0 and $\dot{L}$ be a Lévy white noise on $\mathbb{R}^m$ with characteristic triplet (a, γ, ν) and assume that $\langle s, \varphi \rangle := \langle \dot{L}, G(\varphi) \rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is a well-defined generalized random process. Let $\beta > 0$*

i) If $0 < \beta < 2$ assume that $G_R \in L^\beta(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$ with G_R as defined in Theorem 2.4. If $\dot{L}$ has finite β-moment, then so has s. If $\beta \geq 2$ it is sufficient to assume that $G_R \in L^\beta(\mathbb{R}^m)$.

ii) If s has finite β-moment, then $\dot{L}$ has also finite β-moment.

Proof. From [38], Theorem 2.7 we know that the Lévy measure of the random variable $\langle s, \varphi \rangle$ is given by

$$\nu_{s(\varphi)}(B) = \int_{\mathbb{R}^m} \int_{\mathbb{R}} \mathbb{1}_{B \setminus \{0\}} \left(rG(\varphi)(x) \right) \nu(dr) \lambda^m(dx).$$

Then $\langle s, \varphi \rangle$ has finite β-moment if and only if $\int_{|z|>1} |z|^\beta \nu_{s(\varphi)}(dz) < \infty$.

i) Let $\dot{L}$ have finite β-moment and assume at first that $0 < \beta < 2$. We calculate with (2.7) that

$$\begin{aligned}
\int_{|z|>1} |z|^\beta \nu_{s(\varphi)}(dz) &= \int_{\mathbb{R}} |r|^\beta \int_{|G(\varphi)(x)| > \frac{1}{|r|}} |G(\varphi)(x)|^\beta \lambda^m(dx) \nu(dr) \\
&\leq \int_{|r| \leq 1} |r|^2 \int_{|G(\varphi)(x)| > \frac{1}{|r|}} |G(\varphi)(x)|^2 \lambda^m(dx) \nu(dr) \\
&\quad + \int_{|r|>1} |r|^\beta \int_{\mathbb{R}^m} |G_R(x)|^\beta \|\varphi\|_\infty^\beta \lambda^m(dx) \nu(dr) \\
&\leq \int_{|r| \leq 1} |r|^2 \nu(dr) \|G_R\|_{L^2(\mathbb{R}^m)}^2 \|\varphi\|_\infty^2 + \|G_R\|_{L^\beta(\mathbb{R}^m)}^\beta \int_{|r|>1} |r|^\beta \nu(dr) \|\varphi\|_\infty^\beta \\
&< \infty,
\end{aligned}$$

where $R > 0$ is such that $\operatorname{supp} \varphi \subset B_R(0)$.
If $\beta \geq 2$ we obtain by similar arguments as above that

$$\int_{|z|>1} |z|^\beta \nu_{s(\varphi)}(dz) \leq \|\varphi\|_\infty^\beta \|G_R\|_{L^\beta(\mathbb{R}^m)}^\beta \int_{\mathbb{R}} |r|^\beta \nu(dr),$$

which is indeed finite.
ii) Assume that s has finite β-moment and that G is different from 0. So we know that there exists a function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |G(\varphi)(x)| \lambda^m(dx) > 0,$$

hence there exists an $r_0 > 1$ with

$$\int_{|G(\varphi)|>1/r_0} |G(\varphi)(x)|^\beta \lambda^m(dx) > 0.$$

We conclude

$$\begin{aligned}\infty > \int\limits_{|z|>1} |z|^\beta \nu_{s(\varphi)}(dz) &= \int\limits_{\mathbb{R}} |r|^\beta \int\limits_{|G(\varphi)(x)|>\frac{1}{|r|}} |G(\varphi)(x)|^\beta \lambda^m(dx)\nu(dr) \\ &\geq \int\limits_{|r|>r_0} |r|^\beta \int\limits_{|G(\varphi)(x)|>\frac{1}{|r|}} |G(\varphi)(x)|^\beta \lambda^m(dx)\nu(dr) \\ &\geq \int\limits_{|r|>r_0} |r|^\beta \nu(dr) \int\limits_{|G(\varphi)(x)|>\frac{1}{|r_0|}} |G(\varphi)(x)|^\beta \lambda^m(dx),\end{aligned}$$

hence $\int\limits_{|r|>r_0} |r|^\beta \nu(dr) < \infty$ so that $\dot{L}$ has finite β-moment. □

2.3 Second order elliptic partial differential equations in divergence form driven by Levy white noise

In this section we discuss elliptic partial differential operators of second order with variable coefficients in divergence form, i.e. partial differential operators $p(x, D)$ of the form

$$p(x, D)u = -\sum_{i,j=1}^{d} \partial_i (a_{ij}(x)\partial_j u) = -\mathrm{div}(A(x)\nabla u), \tag{2.17}$$

where $A(x) = (a_{ij}(x))_{i,j=1}^d \in C^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is a *uniformly elliptic matrix*, i.e. there exists a $C > 0$ such that

$$C^{-1}\|\xi\|^2 \leq \xi^T A(x)\xi \leq C\|\xi\|^2 \text{ for all } \xi \in \mathbb{R}^d.$$

Now let $\dot{L}$ be a Lévy white noise on $\mathbb{R}^d$ with characteristic triplet (a, γ, ν) and $p(x, D)$ be a partial differential operator (PDO) of the form (2.17). We say that a generalized stochastic process $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ is a *generalized solution* of the equation

$$p(x, D)s = \dot{L},$$

if it holds

$$\langle s, p(x, D)^*\varphi\rangle = \langle \dot{L}, \varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where $p(x,D)^*$ is the adjoint of $p(x,D)$, i.e.

$$p(x,D)^*u = -\sum_{i,j=1}^{d} \partial_i(a_{ji}(x)\partial_j u).$$

In the first theorem we derive sufficient conditions for the existence of such a solution in terms of the characteristic triplet (a,γ,ν), which is just a a simple extension of the Laplacian case. Afterwards we discuss stationarity of these generalized processes, e.g. if the coefficients are $y-$periodic for some $y \in \mathbb{R}^d$, then s is $y-$periodically stationary. We assume for the complete section that the coefficients of $p(x,D)$ are in $C^\infty(\mathbb{R}^d)$.

Theorem 2.9. (see [7], Theorem 4.) *Let $\dot{L}$ be a Lévy white noise on $\mathbb{R}^d$ with characteristic triplet (a,γ,ν) with vanishing first moment and $p(x,D)$ be a PDO of the form* (2.17). *The stochastic partial differential equation*

$$p(x,D)s = \dot{L} \tag{2.18}$$

has a generalized solution $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$, if $d \geq 5$ and

$$\int_{|r|>1} |r|^{d/(d-2)} \nu(dr) < \infty.$$

Proof. By [32], Chapter 10 there exists a locally integrable left inverse $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ of the operator $p(x,D)^*$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$$E(p(\cdot,D)^*\varphi)(x) := \int_{\mathbb{R}^d} E(x,y)p(y,D)^*\varphi(y)dy = \varphi(x) \text{ for all } x \in \mathbb{R}^d.$$

Moreover, there exists an $N \in \mathbb{N}$ such that

$$N^{-1}\|x-y\|^{2-d} \leq E(x,y) \leq N\|x-y\|^{2-d} \text{ for all } x \neq y.$$

We set

$$\langle s,\varphi\rangle := \langle \dot{L}, E(\varphi)\rangle$$

and from Example 2.7 with $w = 1$, $p = d-2$ and $q = 1$ (observe that $d \geq 5$) it follows that

$$\begin{aligned} s &: \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega), \\ &\langle s,\varphi\rangle := \langle \dot{L}, E(\varphi)\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d), \end{aligned}$$

defines a generalized process. Moreover, s is a solution of the equation (2.18), as

$$\langle s, p(x,D)^*\varphi\rangle = \langle \dot{L}, E(p(x,D)^*\varphi)\rangle = \langle \dot{L},\varphi\rangle$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$. □

The solution $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ is not unique, which is quite clear. For example, let $p(x, D) = -\Delta$ and define

$$\langle s', \varphi\rangle := \langle s, \varphi\rangle + \int_{\mathbb{R}^d} (x_1^2 - x_2^2)\varphi(x)\lambda^d(dx),$$

where s is the solution constructed in Theorem 2.9 for the equation

$$-\Delta s = \dot{L}. \tag{2.19}$$

Then it is easy to see that s' is also a solution of (2.19).

Remark 2.10. (see [7], Remark 1.) We assumed that the coefficients of the partial differential operator $p(x, D)$ are infinitely often differentiable, but this is not necessary. It would be sufficient if $a_{ij} \in C^1(\mathbb{R}^d)$ for all $i, j \in \{1, \dots, d\}$.

Remark 2.11. (see [7], Remark 2.) The method above can also be used to find solutions of SDPEs of the form

$$-\mathrm{div}(A\nabla u) + b \cdot \nabla u + Vu = \dot{L}$$

under some suitable assumptions for the functions A, b and V, as the fundamental solution E of the elliptic operator above can be bounded from above by a constant times $\|x-y\|^{d-2}$ for all $x \neq y$. For a very general result see [16]. Observe that in the most general case the fundamental solution solves the equation only in the weak sense. We will discuss in the next section what we understand under a weak solution.

As a next step we discuss stationarity properties, which depend heavily on the matrix $(a_{ij}(x))_{i,j=1}^d$. For example, if $a_{ij} : \mathbb{R}^d \to \mathbb{R}$ is constant, it is easily seen that $E(x, y) = E(x-y)$ for all $x \neq y$ and hence we observe that the constructed solution $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ in Theorem 2.9 is stationary.

Definition 2.12. (see [7], Definition 4.) A generalized process s on $\mathcal{D}(\mathbb{R}^d)$ is called *periodic with period* $l \in \mathbb{R}^d$, if $s(\cdot + l)$ has the same law as s, and *stationary* if s is periodic for every period $l \in \mathbb{R}^d$. Here, $s(\cdot + l)$ is defined by

$$\langle s(\cdot + l), \varphi\rangle := \langle s, \varphi(\cdot - l)\rangle \text{ for every } \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Remark 2.13. (see [7], Remark 3.) Let $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function which fulfills the assumptions of Theorem 2.4 with $m = d$. Assume that $G(x, y+l) = G(x+l, y)$ for all $x, y \in \mathbb{R}^d$ and for some $l \in \mathbb{R}^d$. Then it is easily seen that for $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$$(G\varphi(\cdot - l))(x) = \int_{\mathbb{R}^d} G(x, y)\varphi(y - l)\lambda^d(dy) = (G\varphi)(x + l),$$

hence the generalized process s defined in Theorem 2.4 satisfies

$$\langle s(\cdot+l),\varphi\rangle = \langle s,\varphi(\cdot-l)\rangle = \langle \dot{L}, G\varphi(\cdot-l)\rangle = \langle \dot{L}, (G\varphi)(\cdot+l)\rangle = \langle \dot{L}(\cdot-l), G\varphi\rangle.$$

Since $\dot{L} \stackrel{d}{=} \dot{L}(\cdot-l)$ it follows that in this case the process s is periodic with period l. Observe that $(s(\varphi(\cdot+ly)))_{y\in\mathbb{Z}}$ is then a stationary process for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Therefore, these models seem to be useful in statistics to model periodic processes or random fields. In the case that $G(x,y+l) = G(x+l,y)$ for all $l,x,y \in \mathbb{R}^d$, we see that s will be stationary.

Proposition 2.14. (see [7], Proposition 2.) *Let $p(x,D) : \mathcal{D}(\mathbb{R}^d) \to C(\mathbb{R}^d)$ be an elliptic partial differential operator of the form* (2.17)*, $d \geq 5$ and assume that the matrix-valued function $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is periodic with period $y \in \mathbb{R}^d$, i.e. $A(x+y) = A(x)$ for all $x \in \mathbb{R}^d$. Let $\dot{L}$ be a Lévy white noise such that it satisfies the assumption of Theorem 2.9. Then there exists a solution $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ of $p(x,D)s = \dot{L}$, which is periodically stationary with period y.*

Proof. It is enough to show that

$$E(\varphi(\cdot+y))(x) = E(\varphi)(x+y),$$

where E is again the fundamental solution of the operator $p(x,D)^*$. The assertion follows then from the stationarity of the Lévy white noise $\dot{L}$. We see that

$$p(x,D)^* E(\varphi(\cdot+y))(x) = \varphi(x+y)$$

and

$$p(x,D)^* E(\varphi)(x+y) = p(x+y,D)^* E(\varphi)(x+y) = \varphi(x+y),$$

so $E(\varphi)(\cdot+y) : \mathbb{R}^d \to \mathbb{R}$ and $E(\varphi(\cdot+y)) : \mathbb{R}^d \to \mathbb{R}$ solve the same elliptic equation. By construction it holds

$$\lim_{|x|\to\infty} |(E(\varphi)(x+y) - E(\varphi(\cdot+y))(x))| = 0 \text{ and} \tag{2.20}$$
$$p(x,D)^*(E(\varphi)(x+y) - E(\varphi(\cdot+y))(x)) = 0 \text{ for all } x \in \mathbb{R}^d.$$

where (2.20) follows from (2.7) and (2.16). By the maximum principle for uniformly elliptic equations we obtain $E(\varphi)(x+y) - E(\varphi(\cdot+y))(x) = 0$ for all $x \in \mathbb{R}^d$, hence we obtain that $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ is periodically stationary. □

From this result we can construct a stationary process on a certain group as long as the coefficients of the partial differential operator satisfy some periodicity condition.

Corollary 2.15. (see [7], Corollary 1.) Let $(\mathcal{G},+)$ be a subgroup of $(\mathbb{R}^d,+)$ and $p(x,D):\mathcal{D}(\mathbb{R}^d)\to C(\mathbb{R}^d)$ be an elliptic partial differential operator of the form (2.17) and assume that the matrix-valued function $A:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ is periodic with period $y\in\mathcal{G}$ for all $y\in\mathcal{G}$. Let $\dot{L}$ be a Lévy white noise satisfying the assumption of Theorem 2.9 and s be the generalized solution of $p(x,D)s=\dot{L}$ constructed in Theorem 2.9. Then for every $\varphi\in\mathcal{D}(\mathbb{R}^d)$ the process

$$(s_\varphi(y))_{y\in\mathcal{G}} := (\langle s,\varphi(\cdot+y)\rangle)_{y\in\mathcal{G}}$$

is a stationary process in $\mathcal{G}$.

Proof. This is a direct consequence of Proposition 2.14. □

2.3.1 The generalized and mild solutions of the electric Schrödinger equation driven by Lévy white noise

We saw in Remark 2.11 before, that we can find generalized solutions of stochastic partial differential equations given by

$$-\mathrm{div}(A(x)\nabla u)+V(x)u=\dot{L} \tag{2.21}$$

for suitable A and V by assuming that the dimension $d\geq 5$, the first moment of the Lévy white noise vanishes and under the moment condition

$$\int\limits_{|r|>1} |r|^{d/(d-2)}\nu(dr)<\infty.$$

In the case that V lies in a Reverse Hölder class these assumptions seem to be not necessary. We show that we find generalized and mild solutions in dimension 3 under much weaker conditions. At first we introduce the Reverse Hölder class $RH_p(\mathbb{R}^d)$ and if V is in this class, the moment assumption reduces to some kind of a logarithm moment condition (dependent on V), which is very similar to the case that V is a positive constant. We first define what is meant by a mild solution of (2.21).
We call $E:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ a weak fundamental solution of the generalized electric Schrödinger operator

$$-\mathrm{div}(A(x)\nabla u)+V(x)u,$$

if $E(\varphi):=\int\limits_{\mathbb{R}^d} E(x,y)\varphi(y)\lambda^d(dy)$ solves

$$-\mathrm{div}(A\nabla E(\varphi))+VE(\varphi)=\varphi$$

in the weak sense for all $\varphi\in\mathcal{D}(\mathbb{R}^d)$. We set $u(x):=\langle\dot{L},E(x,\cdot)\rangle$ to be the *mild solution* of (2.21), if $u(x)$ exists for all $x\in\mathbb{R}^d$, i.e. if $E(x,\cdot)\in D(\dot{L})$ for all $x\in\mathbb{R}^d$. Then Theorem 2.17 i) will give a sufficient condition for that to hold.

In the following we define the maximum function m and Agmon distance γ of the potential V, to apply the estimates of the fundamental solution of the generalized electric Schrödinger operator shown in [43] and [33].

Definition 2.16. (see [7], Definition 5.) Let $p \geq 1$. A function $\omega \in L^p_{loc}(\mathbb{R}^d)$ with $\omega > 0$ a.e. belongs to the *Reverse Hölder class* $RH_p(\mathbb{R}^d)$ if there exists a constant C so that for any ball $B \subset \mathbb{R}^d$,

$$\left(\frac{1}{\lambda^d(B)} \int\limits_B \omega(x)^p \lambda^d(dx)\right)^{1/p} \leq \frac{C}{\lambda^d(B)} \int\limits_B \omega(x) \lambda^d(dx).$$

Furthermore, we define for $\omega \in RH_p(\mathbb{R}^d)$ the *maximum function* $m(x, \omega)$ by

$$\frac{1}{m(x,\omega)} := \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int\limits_{B(x,r)} \omega(y) dy \leq 1 \right\} \in (0, \infty)$$

and the *distance function*

$$\gamma(x, y, \omega) := \inf_{\Gamma} \int_0^1 m(\Gamma(t), \omega) |\dot{\Gamma}(t)| \lambda^1(dt),$$

where $\Gamma : [0,1] \to \mathbb{R}^d$ is absolutely continuous and $\Gamma(0) = x$ and $\Gamma(1) = y$. Moreover, we define for $R > 0$ the ball

$$B^\omega(x, R) := \Big\{ y \in \mathbb{R}^d : \gamma(x, y, \omega) < R \Big\}.$$

The set $RH_p(\mathbb{R}^d)$ is closely connected to the space of Muckenhoupt weights A_s, $s \geq 1$, where ω measurable and non-negative is in A_s if

$$\sup_{B \text{ ball in } \mathbb{R}^d} \left(\frac{1}{\lambda^d(B)} \int\limits_B \omega(x) \lambda^d(dx)\right) \left(\frac{1}{\lambda^d(B)} \int\limits_B \omega(x)^{-s'/s} \lambda^d(dx)\right)^{s/s'} < \infty,$$

where $s' \in \mathbb{R}$ such that $\frac{1}{s} + \frac{1}{s'} = 1$. For further information see for example [44]. Especially it holds that $\omega \in A_s$ for some $s \geq 1$ if and only if there exists a $p > 1$ such that $\omega \in RH_p(\mathbb{R}^d)$. We see that the set of all positive and measurable functions bounded from above and strictly away from zero given by

$$\left\{ f : \mathbb{R}^d \to (0, \infty) : \exists C_1,\, C_2 > 0 \text{ such that } C_1 \leq f(y) \leq C_2 \text{ for all } y \in \mathbb{R}^d \right\}$$

is a subset of $RH_p(\mathbb{R}^d)$ for all $p \geq 1$. We state now an existence theorem for a mild and generalized solution of the equation

$$(-\text{div}(A\nabla) + V)s = \dot{L},$$

where V lies in $RH_{\frac{d}{2}}(\mathbb{R}^d)$ and show that under much weaker moment conditions there exists a generalized solution. We use that the weak fundamental solution E of the operator $p(x, D)$ can be bounded as follows

$$|E(x,y)| \leq C \frac{e^{-k\gamma(x,y,V)}}{\|x-y\|^{d-2}} \quad \text{for all } x, y \in \mathbb{R}^d, x \neq y, \tag{2.22}$$

where $k, C > 0$, see [33], Corollary 6.16, page 40. From now on the constant $k > 0$ is fixed and such that (2.22) is satisfied.

Theorem 2.17. (see [7], Theorem 5.) *Let $A(x) = (a_{i,j}(x))_{i,j=1}^d$ be a real, uniformly bounded and elliptic matrix and $V \in RH_{\frac{d}{2}}(\mathbb{R}^d)$. Let $\dot{L}$ be a a Lévy white noise on $\mathbb{R}^d$ with characteristic triplet (a, γ, ν) such that it holds*

$$\int\limits_{|r|>1} |r| \int\limits_0^{1/|r|} \lambda^d \left(B^V(0, -\frac{1}{k}\log(\alpha)) \right) \lambda^1(d\alpha)\nu(dr) < \infty.$$

i) If $d = 3$ then there exists a mild solution of

$$-\mathrm{div}(A\nabla u) + Vu = \dot{L},$$

which is stochastically continuous.
ii) If $d \geq 3$ then there exists a generalized solution $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ of

$$(-\mathrm{div}(A\nabla) + V)s = \dot{L}.$$

iii) Under the assumption that the first moment of the Lévy white noise exists, the mild solution u from i) gives rise to a generalized solution s of the stochastic partial differential equation $(-\mathrm{div}(A\nabla) + V)s = \dot{L}$ via

$$\langle s, \varphi \rangle := \int\limits_{\mathbb{R}^d} u(x)\varphi(x)\lambda^d(dx).$$

We will prove Theorem 2.17 in Section 2.3.3. Here we will calculate the moment condition for $\dot{L}$ for functions which are greater than a positive constant.

Example 2.18. (see [7], Example 2.) Let $d \geq 3$ and $V \in RH_{\frac{d}{2}}(\mathbb{R}^d)$ such that $V > \varepsilon$, where $\varepsilon > 0$. We observe that

$$\int\limits_0^1 m(\Gamma(t), V)|\dot{\Gamma}(t)|\lambda^1(dt) \geq C\sqrt{\varepsilon}\|y - x\| \tag{2.23}$$

for every path $\Gamma : [0,1] \to \mathbb{R}^d$ with $\Gamma(0) = x$ and $\Gamma(1) = y$ from which it follows for $0 < \alpha \leq 1$ that for fixed $k > 0$

$$\lambda^d\left(B^V(0, -\frac{1}{k}\log(\alpha))\right) \leq C_1\left(\log\left(\frac{C}{\alpha}\right)\right)^d,$$

where $C, C_1 > 0$. Since

$$\int_0^{1/r}\left(\log\left(\frac{1}{\alpha}\right)\right)^d \lambda^1(d\alpha) = \int_{\log(r)}^{\infty} \beta^d e^{-\beta}\lambda^1(d\beta) = \Gamma(d+1, \log(r)) = \frac{d!}{r}\sum_{j=0}^{d}\frac{(\log(r))^j}{j!},$$

where $\Gamma(d+1, \log(r))$ denotes the upper incomplete gamma function, this leads to

$$\begin{aligned}&\int_{|r|>1} |r| \int_0^{1/|r|} \lambda^d\left(B^V(0, -\frac{1}{k}\log(\alpha))\right)\lambda^1(d\alpha)\nu(dr)\\ \leq &\int_{|r|>1} C_2\log(|r|)^d\nu(dr) + C_3\nu(\mathbb{R}\setminus[-1,1]),\end{aligned}$$

where $C_2, C_3 > 0$. So if we assume that the Lévy white noise $\dot{L}$ with characteristic triplet (a, γ, ν) satisfies

$$\int_{|r|>1}\log(|r|)^d\nu(dr) < \infty$$

then the assumptions of Theorem 2.17 are satisfied and we obtain generalized and mild solutions, if $d \geq 3$ or $d = 3$ respectively.

2.3.2 Existence and continuity of mild solutions

In the following we give sufficient conditions for the existence and continuity of a random field $u(x) := (\langle \dot{L}, E(x,\cdot)\rangle)_{x\in\mathbb{R}^m}$, where $E : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a kernel. This will be used in the proof of Theorem 2.17, where E is the weak fundamental solution of the generalized electric Schrödinger operator.

Proposition 2.19. (see [7], Proposition 3.) *Let $\dot{L}$ be a Lévy basis on $\mathbb{R}^d$ with characteristic triplet (a, γ, ν) and let $E : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function. We define for every $x \in \mathbb{R}^m$ a function $h_x : \mathbb{R}^+ \to \mathbb{R}^+$ by*

$$h_x(r) := r\int_0^{1/r} d_{E(x,\cdot)}(\alpha)\lambda^1(d\alpha) \text{ for } r > 0.$$

i) Assume that $E(x,\cdot) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *for every* $x \in \mathbb{R}^m$ *and*

$$\int_{\mathbb{R}} \mathbb{1}_{|r|>1} h_x(|r|)\nu(dr) < \infty$$

for every $x \in \mathbb{R}^m$*. Then* $E(x,\cdot) \in D(\dot{L})$ *for every* $x \in \mathbb{R}^m$ *and hence the random field* $u = (u(x))_{x\in\mathbb{R}^m}$ *given by* $u(x) := \langle \dot{L}, E(x,\cdot)\rangle$ *for all* $x \in \mathbb{R}^m$ *exists.*

ii) Furthermore, if the function $T_E : \mathbb{R}^m \to L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *given by* $T_E(x) := E(x,\cdot)$ *is continuous in* $L^1(\mathbb{R}^d)$ *and* $L^2(\mathbb{R}^d)$ *and for every* $x \in \mathbb{R}^m$ *there exists an* $\varepsilon > 0$ *such that*

$$\sup_{x^*\in B_\varepsilon(x)} \int_{\mathbb{R}} \mathbb{1}_{|r|>1} h_{x^*}(|r|)\nu(dr) < \infty,$$

then the process $u = (u(x))_{x\in\mathbb{R}^m}$ *is stochastically continuous.*

Proof. i) Similar to the proof of Theorem 2.4 the existence of the random field u is characterised by [38], Theorem 2.7 and hence with the same calculations the result follows. ii) Using [38], Theorem 2.7 by the same reasoning as in Theorem 2.4 we have to show that

$$\int_{\mathbb{R}^d} \Big|\gamma\Big(E(x_n,y) - E(x,y)\Big) \tag{2.24}$$

$$+ \int_{\mathbb{R}} r\Big(E(x_n,y) - E(x,y)\Big)\Big(\mathbb{1}_{|r(E(x_n,y)-E(x,y))|\le 1} - \mathbb{1}_{|r|\le 1}\Big)\nu(dr)\Big|\lambda^d(dy) \to 0,$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} \min\Big(1, \Big|r\Big(E(x_n,y) - E(x,y)\Big)\Big|^2\Big)\nu(dr)\lambda^d(dy) \to 0 \quad \text{and} \tag{2.25}$$

$$a^2 \int_{\mathbb{R}^d} \Big|\Big(E(x_n,y) - E(x,y)\Big)\Big|^2 \lambda^d(dy) \to 0 \tag{2.26}$$

as $n \to \infty$, if $x_n \to x$ for $n \to \infty$. At first we observe that

$$\int_{\mathbb{R}^m} \Big|\gamma\Big(E(x_n,y) - E(x,y)\Big)\Big|\lambda^m(dy) \le |\gamma|\|E(x_n,\cdot) - E(x,\cdot)\|_{L^1(\mathbb{R}^d)} \to 0$$

as $n \to \infty$. With similar calculations as in the proof of Theorem 2.4 we can estimate the remaining term in (2.24) by

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} \Big|r\Big(E(x_n,y) - E(x,y)\Big)\Big(\mathbb{1}_{|r(E(x_n,y)-E(x,y))|\le 1} - \mathbb{1}_{|r|\le 1}\Big)\Big|\nu(dr)\lambda^d(dy)$$

$$= \int_{\mathbb{R}} |r|\mathbb{1}_{|r|\le 1} \int_{\mathbb{R}^d} |E(x_n,y) - E(x,y)|\mathbb{1}_{|E(x_n,y)-E(x,y)|>\frac{1}{|r|}}\lambda^d(dy)\nu(dr)$$

$$+\int_{\mathbb{R}} |r|\mathbb{1}_{|r|>1} \int_{\mathbb{R}^d} |E(x_n,y)-E(x,y)|\mathbb{1}_{|E(x_n,y)-E(x,y)|\leq \frac{1}{|r|}} \lambda^d(dy)\nu(dr).$$

As it holds

$$\int_{\mathbb{R}^d} |E(x_n,y)-E(x,y)|\mathbb{1}_{|E(x_n,y)-E(x,y)|> \frac{1}{|r|}} \lambda^d(dy) \leq |r| \|E(x_n,\cdot)-E(x,\cdot)\|^2_{L^2(\mathbb{R}^d)},$$

it follows from Lebesgue's dominated convergence theorem that

$$\int_{\mathbb{R}} |r|\mathbb{1}_{|r|\leq 1} \int_{\mathbb{R}^m} |E(x_n,y)-E(x,y)|\mathbb{1}_{|E(x_n,y)-E(x,y)|> \frac{1}{|r|}} \lambda^m(dy)\nu(dr) \to 0$$

as $n\to\infty$. For the last term in (2.24) we observe by [21], Prop. 1.1.3 and 1.1.4 that

$$\begin{aligned}
&\int_{\mathbb{R}^d} |E(x_n,y)-E(x,y)|\mathbb{1}_{|E(x_n,y)-E(x,y)|\leq \frac{1}{|r|}} \lambda^d(dy)\\
\leq &\int_0^{1/|r|} d_{|E(x_n,\cdot)-E(x,\cdot)|}(\alpha)\lambda^1(d\alpha)\\
\leq &\int_0^{1/|r|} d_{|E(x_n,\cdot)|}(\alpha/2)\lambda^1(d\alpha) + \int_0^{1/|r|} d_{|E(x,\cdot)|}(\alpha/2)\lambda^1(d\alpha)\\
\leq &2\left(\int_0^{\frac{1}{2|r|}} d_{|E(x_n,\cdot)|}(\alpha)\lambda^1(d\alpha) + \int_0^{\frac{1}{2|r|}} d_{|E(x,\cdot)|}(\alpha)\lambda^1(d\alpha)\right).
\end{aligned}$$

By Lebesgue's dominated convergence theorem we obtain that

$$\int_{\mathbb{R}} |r|\mathbb{1}_{|r|>1} \int_{\mathbb{R}^d} |E(x_n,y)-E(x,y)|\mathbb{1}_{|E(x_n,y)-E(x,y)|\leq \frac{1}{|r|}} \lambda^d(dy)\nu(dr) \to 0 \text{ as } n\to\infty.$$

So we showed (2.24). In order to see (2.25) observe that

$$\int_{\mathbb{R}^d} \mathbb{1}_{|r(E(x_n,y)-E(x,y))|>1}\lambda^d(dy) \leq |r|\|E(x_n,\cdot)-E(x,\cdot)\|_{L^1(\mathbb{R}^d)} \to 0 \text{ as } n\to\infty$$

and

$$\int_{\mathbb{R}^d} \mathbb{1}_{|r(E(x_n,y)-E(x,y))|>1}\lambda^d(dy) \leq d_{|E(x_n,\cdot)|}\left(\frac{1}{2|r|}\right) + d_{|E(x,\cdot)|}\left(\frac{1}{2|r|}\right).$$

Now by similar arguments as in the proof of Theorem 2.4 we see that (2.25) holds true. Furthermore, it is clear that (2.26) holds, since T_E is continuous. □

Now we state under which conditions a mild solution of a stochastic partial differential equation gives rise to a generalized solution.

Theorem 2.20. (see [7], Theorem 6.) *Let $\dot{L}$ be a Lévy white noise on $\mathbb{R}^d$ with characteristic triplet (a, γ, ν) with existing first moment and $p(x, D)$ be a partial differential operator of the form*

$$p(x, D)\varphi(x) = -\mathrm{div}(A\nabla\varphi(x)) + b(x) \cdot \nabla\varphi(x) + V(x)\varphi(x),$$

where $b \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and $V \in L^1_{loc}(\mathbb{R}^d)$ such that there exists a weak fundamental solution $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ of the equation $p(x, D)u = \delta_0$ with $E(x, \cdot) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap D(\dot{L})$ for all $x \in \mathbb{R}^d$ and

$$\int_K \|E(x, \cdot)\|^p_{L^p(\mathbb{R}^d)} \lambda^d(dx) < \infty$$

for all compact sets $K \subset \mathbb{R}^d$ for $p = 1, 2$. Then the mild solution

$$u(x) = \langle \dot{L}, E(x, \cdot)\rangle$$

of $p(x, D)u = \dot{L}$ gives rise to a generalized solution s of the stochastic partial differential equation $p(x, D)s = \dot{L}$ via

$$\langle s, \varphi\rangle := \int_{\mathbb{R}^d} u(x)\varphi(x)\lambda^d(dx), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Proof. We want to apply a stochastic Fubini theorem. Therefore we have to show that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \min\left(|rE(x, y)\varphi(x)|, |rE(x, y)\varphi(x)|^2\right) \nu(dr)\lambda^d(dy)\lambda^d(dx) < \infty. \tag{2.27}$$

We see that for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$$\begin{aligned}
&\min\left(|rE(x, y)\varphi(x)|, |rE(x, y)\varphi(x)|^2\right)\\
=&\mathbb{1}_{|rE(x,y)\varphi(x)|>1}|rE(x, y)\varphi(x)| + \mathbb{1}_{|rE(x,y)\varphi(x)|\le 1}|rE(x, y)\varphi(x)|^2\\
\le&\mathbb{1}_{|r|>1}\mathbb{1}_{|rE(x,y)\varphi(x)|>1}|rE(x, y)\varphi(x)| + \mathbb{1}_{|r|\le 1}\mathbb{1}_{|rE(x,y)\varphi(x)|>1}|rE(x, y)\varphi(x)|^2\\
&+ \mathbb{1}_{|r|\le 1}\mathbb{1}_{|rE(x,y)\varphi(x)|\le 1}|rE(x, y)\varphi(x)|^2 + \mathbb{1}_{|r|>1}\mathbb{1}_{|rE(x,y)\varphi(x)|\le 1}|rE(x, y)\varphi(x)|\\
=&\mathbb{1}_{|r|>1}|rE(x, y)\varphi(x)| + \mathbb{1}_{|r|\le 1}|rE(x, y)\varphi(x)|^2.
\end{aligned}$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that supp $\varphi \subset B_R(0)$, $R > 0$. We observe that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathbb{1}_{|r|>1}|rE(x, y)\varphi(x)|\nu(dr)\lambda^d(dy)\lambda^d(dx)$$

$$\leq \|\varphi\|_\infty \int\limits_{\mathbb{R}} \mathbb{1}_{|r|>1}|r|\nu(dr) \int\limits_{B_R(0)} \|E(x,\cdot)\|_{L^1(\mathbb{R}^d)}\lambda^d(dx) < \infty$$

and

$$\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}} \mathbb{1}_{|r|\leq 1}|rE(x,y)\varphi(x)|^2\nu(dr)\lambda^d(dy)\lambda^d(dx)$$
$$\leq \|\varphi\|_\infty^2 \int\limits_{\mathbb{R}} \mathbb{1}_{|r|\leq 1}|r|^2\nu(dr) \int\limits_{B_R(0)} \|E(x,\cdot)\|_{L^2(\mathbb{R}^d)}^2\lambda^d(dx) < \infty.$$

This shows (2.27). Since $\varphi \in \mathcal{D}(\mathbb{R}^d)$ has compact support and we have that λ^d is finite on the support of φ and with [3], Theorem 3.1 p. 926 we get that

$$\begin{aligned}\langle s,\varphi\rangle :&= \int\limits_{\mathbb{R}^d} u(x)\varphi(x)\lambda^d(dx) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} E(x,y)\varphi(x)dL(y)\lambda^d(dx)\\ &= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} E(x,y)\varphi(x)\lambda^d(dx)dL(y) \qquad \text{a.s.}\end{aligned}$$

and further it can be chosen a version of u such that $u\cdot\varphi$ is integrable with respect to λ^d. The linearity of $s:\mathcal{D}(\mathbb{R}^d)\to L^0(\Omega)$ is clear and the estimates above show that it is also continuous, hence s is a generalized random process. In order to see that $p(x,D)s=\dot{L}$, we observe that for arbitrary $f\in\mathcal{D}(\mathbb{R}^d)$

$$\begin{aligned}&\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d} E(x,y)p(x,D)^*\varphi(x)\lambda^d(dx)\right)f(y)\lambda^d(dy)\\
&= -\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d} E(x,y)f(y)\lambda^d(dy)\right)(\operatorname{div}(A^T(x)\nabla\varphi(x))\lambda^d(dx)\\
&\quad -\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d} E(x,y)f(y)\lambda^d(dy)\right)\nabla\cdot(b(x)\varphi(x))\lambda^d(dx)\\
&\quad +\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d} E(x,y)f(y)\lambda^d(dy)\right)V(x)\varphi(x)\lambda^d(dx)\\
&= \int\limits_{\mathbb{R}^d}\langle A(x)\nabla\left(\int\limits_{\mathbb{R}^d} E(x,y)f(y)\lambda^d(dy)\right),\nabla\varphi(x)\rangle\lambda^d(dx)\\
&\quad +\int\limits_{\mathbb{R}^d}(b(x)\cdot\nabla+V(x))\left(\int\limits_{\mathbb{R}^d} E(x,y)f(y)\lambda^d(dy)\right)\varphi(x)\lambda^d(dx)\\
&= \int\limits_{\mathbb{R}^d} f(x)\varphi(x)\lambda^d(dx).\end{aligned}$$

As $f \in \mathcal{D}(\mathbb{R}^d)$ was arbitrary, it follows from the fundamental lemma of calculus of variations that

$$\int_{\mathbb{R}^d} E(x,y)p(x,D)^*\varphi(x)\lambda^d(dx) = \varphi(y) \text{ a.e.}$$

Now we obtain

$$\langle s, p(x,D)^*\varphi\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} E(x,y)p(x,D)^*\varphi(x)dL(y)\lambda^d(dx) = \int_{\mathbb{R}^d} \varphi(y)dL(y) = \langle \dot{L}, \varphi\rangle,$$

so we see that s is a generalized solution. □

2.3.3 Proof of Theorem 2.17

Proof. i) Similar to [43], Remark 3.21 we observe that we can estimate the distance function γ in (2.22) and obtain for the weak fundamental solution E of the generalized electric Schrödinger operator $p(x,D)$ that it holds

$$|E(x,y)| \leq C_1 e^{-C_2(1+m(x,V)\|x-y\|)^\theta}\|x-y\|^{2-d}, \tag{2.28}$$

for some constants $C_1, C_2 > 0$ and $0 < \theta < 1$. Hence, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d} |E(x,y)|\lambda^d(dy) &\leq C_1 \int_{\mathbb{R}^d} e^{-C_2(1+m(x,V)\|z\|)^\theta}\|z\|^{2-d}\lambda^d(dz)\\ &= C_1\int_0^\infty re^{-C_2(1+m(x,V)r)^\theta}\lambda^1(dr) < \infty\end{aligned}$$

and also

$$\int_{\mathbb{R}^d} |E(x,y)|^2\lambda^d(dy) \leq C_3\int_0^\infty r^{3-d}e^{-2C_2(1+m(x,V)r)^\theta}\lambda^1(dr) < \infty,$$

where $C_3 > 0$. For $\alpha > 0$ and $x, y \in \mathbb{R}^d$, $x \neq y$ it follows with the triangular inequality that (observe that $d = 3$ and hence the Lebesgue measure of a ball with radius r is $\frac{4\pi}{3}$)

$$\begin{aligned}&\lambda^d\left(\{y \in \mathbb{R}^d : \frac{Ce^{-k\gamma(x,y,V)}}{\|x-y\|^{d-2}} > \alpha\}\right)\\ \leq&\lambda^d\left(\{y \in \mathbb{R}^d \setminus B_{e^{-k\gamma(x,0,V)/(d-2)}C^{1/(d-2)}}(x) : \frac{e^{-k\gamma(0,y,V)}}{||x-y||^{d-2}} > e^{k\gamma(x,0,V)}\alpha/C\}\right)\\ &+\frac{4\pi}{3}(e^{-k\gamma(x,0,V)/(d-2)}C^{1/(d-2)})^d\\ \leq&\lambda^d\left(\{y \in \mathbb{R}^d \setminus B_{e^{-k\gamma(x,0,V)/(d-2)}C^{1/(d-2)}}(x) : \gamma(0,y,V) \leq -\frac{1}{k}\log(\alpha)\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{4\pi}{3}(e^{-k\gamma(x,0,V)/(d-2)}C^{1/(d-2)})^d \\
\leq&\lambda^d\left(B^V(0,-\frac{1}{k}\log(\alpha))\right)+\frac{4\pi}{3}(e^{-k\gamma(x,0,V)/(d-2)}C^{1/(d-2)})^d.
\end{aligned}
$$

It follows with (2.22) that

$$
\begin{aligned}
&\int_{\mathbb{R}} |r|\mathbb{1}_{|r|>1}\int_0^{1/|r|} d_{E(x,\cdot)}(\alpha)\lambda^1(d\alpha)\nu(dr) \\
\leq&C_4(x)\left(1+\int_{|r|>1}|r|\int_0^{1/|r|}\lambda^d\left(B^V(0,-\frac{1}{k}\log(\alpha))\right)\lambda^1(d\alpha)\nu(dr)\right)<\infty
\end{aligned}
$$

by assumption, where $0 < C_4(x) < \infty$. Proposition 2.19 i) now gives the existence of a mild solution.
To show the continuity of the mild solution by the previous estimates and Proposition 2.19 ii) it is sufficient to prove that $T_E : \mathbb{R}^d \to L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $T_E(x)(\cdot) = E(x,\cdot)$, is continuous. Let $x_0 \in \mathbb{R}^d$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x_0$ as $n \to \infty$. Let $0 < 2\|x_0 - x_n\| < r_0$ for all $n \geq M$, $M \in \mathbb{N}$. We calculate that

$$
\begin{aligned}
\|E(x_0,\cdot)-E(x_n,\cdot)\|_{L^1(\mathbb{R}^d)} \leq&\|E(x_0,\cdot)-E(x_n,\cdot)\|_{L^1(B_{r_0}(x))} \\
&+\|E(x_0,\cdot)-E(x_n,\cdot)\|_{L^1(\mathbb{R}^d\setminus B_{r_0}(x))}.
\end{aligned}
$$

It was shown in [33], Lemma 3.12, page 14 that it holds for a constant $0 < \kappa < 1$

$$
m(x,V) \geq C\frac{m(0,V)}{(1+\|x\|m(0,V))^\kappa} \text{ for all } x \in \mathbb{R}^d, \tag{2.29}
$$

hence there exists an $\varepsilon > 0$ such that it follows with (2.28) that

$$
|E(x_n,y)| \leq C_1e^{-C_2(1+\varepsilon\|x_n-y\|)^\theta}\|x_n-y\|^{2-d}
$$

for every $n \in \mathbb{N}_0$. Therefore, we obtain that

$$
\|E(x_0,\cdot)-E(x_n,\cdot)\|_{L^1(B_{r_0}(x))} \leq 2\int_{B_{2r_0}(0)} C_1e^{-C_2(1+\varepsilon\|y\|)^\theta}\|y\|^{2-d}\lambda^d(dy).
$$

and

$$
\begin{aligned}
|E(x_0,y)-E(x_n,y)| \leq&C_1e^{-C_2(1+\varepsilon\|x_n-y\|)^\theta}\|x_n-y\|^{2-d} \\
&+C_1e^{-C_2(1+\varepsilon\|x_0-y\|)^\theta}\|x_0-y\|^{2-d}.
\end{aligned}
$$

As $(x_n)_{n\geq M}$ is bounded we can find an integrable majorant on $\mathbb{R}^d \setminus B_{r_0}(x)$. We know from [33], chapter 7 that E is continuous and by Lebesgue's Dominated Convergence Theorem we obtain

$$
\lim_{n\to\infty}\|E(x_0,\cdot)-E(x_n,\cdot)\|_{L^1(\mathbb{R}^d\setminus B_{r_0}(x))} = 0.
$$

We see that

$$\lim_{n\to\infty} \|E(x_0,\cdot) - E(x_n,\cdot)\|_{L^1(\mathbb{R}^d)} \leq 2\int_{B_{2r_0}(0)} C_1 e^{-C_2(1+\varepsilon\|y\|)^\theta}\|y\|^{2-d}\lambda^d(dy).$$

By letting r_0 go to 0 we obtain that $\lim_{n\to\infty}\|E(x_0,\cdot)-E(x_n,\cdot)\|_{L^1(\mathbb{R}^d)} = 0$. The same proof works for the L^2-norm.
ii) Let $\tilde{E}$ be the left inverse of $p(x,D)^*$, i.e it holds

$$\int_{\mathbb{R}^d} \tilde{E}(x,y)p(y,D)^*\varphi(y)\lambda^d(dy) = \varphi(x)$$

for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We have to show that $\tilde{E}_R \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ in order to satisfy the assumptions of Theorem 2.4. As $\tilde{E}(x,y) = E(y,x)$ we can show by a similar argument as in i) that for $R>0$

$$\tilde{E}_R(x) = \int_{B_R(0)} |\tilde{E}(x,y)|\lambda^d(dy) \leq \int_{B_R(0)} C_1 e^{-C_2(1+m(y,V)\|x-y\|)^\theta}\|x-y\|^{2-d}\lambda^d(dy).$$

By using (2.29) we obtain that

$$\tilde{E}_R(x) \leq C_R \int_{B_R(0)} e^{-kC_R^1\|x-y\|^\theta}\|x-y\|^{2-d}\lambda^d(dy) \leq \tilde{C}_R e^{-kC_R^1\|x\|^\theta}\|x\|^{2-d},$$

where $C_R, C_R^1, \tilde{C}_R > 0$. Therefore we obtain that $\|\tilde{E}_R\|_{L^1(\mathbb{R}^d)} + \|\tilde{E}_R\|_{L^2(\mathbb{R}^d)} < \infty$. We observe from (2.22) and [43], Remark 3.21 by applying the triangular inequality that

$$\begin{aligned}\tilde{E}_R(x) &\leq e^{-k\gamma(x,0,V)} \int_{B_R(0)} \frac{e^{k\gamma(y,0,V)}}{\|x-y\|^{d-2}}\lambda^d(dy)\\ &\leq C_R' e^{-k\gamma(x,0,V)} \int_{B_R(x)} \frac{1}{\|y\|^{d-2}}\lambda^d(dy)\\ &\leq C_R'' \frac{e^{-k\gamma(x,0,V)}}{\|x\|^{d-2}},\end{aligned}$$

where $C_R', C_R'' > 0$ are constants dependent on R. This leads with similar arguments as in i) to

$$\begin{aligned}&\int_{\mathbb{R}} |r|\mathbb{1}_{|r|>1} \int_0^{1/|r|} d_{\tilde{E}_R}(\alpha)\lambda^1(d\alpha)\nu(dr)\\ \leq& C_R\left(1 + \int_{|r|>1} |r| \int_0^{1/|r|} \lambda^d\left(B^V(0,-\frac{1}{k}\log(\alpha))\right)\lambda^1(d\alpha)\nu(dr)\right) < \infty,\end{aligned}$$

for a constant $C_R > 0$ dependent on $R > 0$. With Theorem 2.4 follows the existence of a generalized solution $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$.
iii) Given the mild solution from i) we obtain with (2.29) for $R > 0$ that

$$\int_{B_R(0)} \|E(x,\cdot)\|_{L^1(\mathbb{R}^d)} \lambda^d(dx) < \infty$$

and

$$\int_{B_R(0)} \|E(x,\cdot)\|_{L^2(\mathbb{R}^d)} \lambda^d(dx) < \infty.$$

Hence, we obtain the assertion by Theorem 2.20. □

3 Almost periodically stationary processes

This chapter is based on the preprint article by Berger and Mohamed [8] "Almost periodic stationary processes". In this chapter we derive a necessary and sufficient condition for stochastic processes to have almost periodic finite dimensional distributions and especially obtain characterizations for infinitely divisible processes to be almost periodic in terms of the characteristic triplets. Furthermore, we derive conditions when the process $(X_t)_{t\in\mathbb{R}}$ defined by the stochastic integral $X_t := \int_{\mathbb{R}^d} f(t,s)dL(s)$ is almost periodically stationary and also when it is almost periodic in probability, where $f(t,\cdot) \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^2(\mathbb{R}^d,\mathbb{R})$ is deterministic and L is a Lévy basis. Moreover, we discuss almost periodic Ornstein-Uhlenbeck-type processes and obtain a central limit theorem for m-dependent processes with almost periodic finite dimensional distributions.

3.1 Introduction

The study of almost periodic functions was originally motivated by considering two periodic functions with incommensurable periods. The sum of these two functions is not necessarily periodic anymore but almost periodic. In 1925 Bohr developed in [10] the theory of almost periodic functions and also showed a Fourier series representation for almost periodic functions which differs from the periodic case in so far that the summation is over periodic functions with different periods. Ever since the introduction by Bohr, almost periodic functions found wide applications in dynamic systems, representation theory, differential equations, and in various other branches of mathematics (see e.g. [9], [13], [35], [47]).

In the context of stochastic processes Gladyshev defined in [22] stochastic processes with almost periodic mean and covariance functions. Further concepts are for example almost periodicity in distribution, in probability, in quadratic mean, almost sure, etc. In [4] and [45] those different types of almost periodicity were compared and afterwards applied in order to obtain almost periodic solutions of stochastic differential equations. In particular, Morozan and Tudor gave in [34] a definition for almost periodic mappings with values in the space of probability measures. Furthermore, in [45] almost periodicity in distribution was discussed, which assumes the almost periodicity of the finite dimensional distributions of the process. Let us note that we take up the concept of almost periodic processes from

[34], [45], where we use different distance functions, which metricize the topology of weak convergence on the set of probability measures. We call such processes almost periodically stationary. The goal of this chapter is to find a characterization of stochastic processes $(X_t)_{t\in\mathbb{R}}$ to be almost periodic in distribution, especially for infinitely divisible processes via its characteristic function. As a concrete example we derive conditions for which the process defined by the stochastic integral (in the sense of Rajput and Rosinski)

$$X_t = \int_{\mathbb{R}^d} f(t,s)dL(s) \tag{3.1}$$

is almost periodic in distribution and moreover, we derive a sufficient condition for such processes to be almost periodic in probability (see [4], [37]), where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a deterministic function and L a Lévy basis. Further, we consider Ornstein-Uhlenbeck-type processes whose finite dimensional distributions are almost periodic and show the existence and uniqueness of such processes. Moreover, we derive a central limit theorem for m-dependent almost periodically stationary processes.
The chapter is organized as follows. In Section 3.2 we define almost periodically stationary processes and state general results for such processes. Afterwards, we characterize the almost periodically stationarity of stochastic processes via the characteristic function (see Theorem 3.9). This leads us in Section 3.3 to our main Theorem 3.10, which characterizes the almost periodically stationarity of infinitely divisible processes in terms of their characteristic triplets. As an application, we discuss in Section 3.4 the almost periodically stationarity of stochastic processes of the form (3.1). In Section 3.5 we derive sufficient conditions for (3.1) to be almost periodic in probability (see Theorem 3.21) and consider some examples. Afterwards, we apply our results in Section 3.6 to obtain the unique almost periodically stationary solution of the Ornstein-Uhlenbeck equation. Finally, in Section 3.7 we state the central limit theorem for m-dependent almost periodically stationary processes.

3.2 Almost periodically stationary processes

The definition of an almost periodic function was first given by Bohr [10], where he introduced almost periodicity of functions which map to the real numbers. There exist several extensions by assuming that the range of the function is contained in a Banach- or Hilbert space, or even more general, in a complete separable metric space. As we are interested in functions which map to the space of measures on a complete metric space, we need the definition of almost periodic functions in such spaces, which can be found e.g. in [34], [45].

Definition 3.1. Let (M, d) be a complete metric space and $f : \mathbb{R} \to M$ be a continuous function. We say that f is *almost periodic* if for every $\varepsilon > 0$ there exists an $L_\varepsilon > 0$ and a

$\tau = \tau(a, \varepsilon) \in [a, a + L_\varepsilon]$ for all $a \in \mathbb{R}$ such that

$$d(f(x+\tau), f(x)) < \varepsilon \text{ for all } x \in \mathbb{R}. \tag{3.2}$$

Remark 3.2. In [34], [45] it is additionally assumed that (M, d) is separable. It is however easy to see that for the following considerations, separability is not needed. This can be seen by introducing the set N as the closure of the range of the continuous function $f : \mathbb{R} \to M$ and restricting the metric d to $N \times N$. Then N is a separable complete metric space, and clearly (3.2) is not affected if f is considered as a function to M or to N, respectively. The same is true for the equivalent characterizations of almost periodicity by Bochner given below.

The definition of Bohr seems to imply that the metric is important for the definition of almost periodicity, but it turns out that only the generated topology is relevant. There exist at least two equivalent definitions, which do not include the notion of a metric, but a topology. As a consequence, we will see that we can exchange the metric as long as the different metrics generate the same topology. The first equivalent definition was given in [11], which states the following. Let $f : \mathbb{R} \to M$ be a continuous function. Then f is almost periodic if and only if the set $(f(\cdot + \tau))_{\tau \in \mathbb{R}}$ is relatively compact in the space $C(\mathbb{R}, M)$. (This property is often referred to as almost periodicity in the sense of Bochner and the equivalence with the previous definition stems from the fact that we consider functions with domain $\mathbb{R}$, see [4, p.324]). The topology on $C(\mathbb{R}, M)$ is the topology of uniform convergence on compact subsets of $\mathbb{R}$, which can be shown to be independent of the underlying metric on M, see e.g. [18, Theorem 8.2.6]. As a consequence of this result Bochner obtained another, very useful characterization of almost periodicity, the so-called *Bochner's double sequence criterion*, which states the following. A continuous function $f : \mathbb{R} \to M$ is almost periodic if and only if for every two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ there exist two subsequences $(a'_n)_{n\in\mathbb{N}}$ and $(b'_n)_{n\in\mathbb{N}}$ such that the limits $\lim_{n\to\infty} \lim_{m\to\infty} f(x + a'_n + b'_m)$ and $\lim_{n\to\infty} f(x + a'_n + b'_n)$ exist for every $x \in \mathbb{R}$ and are equal. For references, see [4], [11], [45] and [47].

In [34] Morozan and Tudor defined stochastic processes $(X_t)_{t\in\mathbb{R}}$ with *almost periodic finite dimensional distributions*. We adopt this definition, whereas we call such processes almost periodically stationary instead. An almost periodically stationary process is by definition a process such that the shift operator functions on the finite dimensional distributions are almost periodic in the suitable space.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the collection of all finite measures on the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and $\mathcal{P}(\mathbb{R}^n)$ be the collection of all probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel-σ-algebra on $\mathbb{R}^n$.

Definition 3.3. Let $X = (X_t)_{t\in\mathbb{R}}$ be a real-valued stochastic process and define for $n \in \mathbb{N}$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$ the function $\mu_{x_1,\dots,x_n} : \mathbb{R} \to \mathcal{P}(\mathbb{R}^n)$, $\mu_{x_1,\dots,x_n}(t) := \mathcal{L}(X_{x_1+t}, \dots, X_{x_n+t})$, where $\mathcal{L}(X_{x_1}, \dots, X_{x_n})$ denotes the n-dimensional distribution of X in $(x_1, \dots, x_n) \in \mathbb{R}^n$. We call X *almost periodically stationary* if for any $n \in \mathbb{N}$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$ the function $t \mapsto \mu_{x_1,\dots,x_n}(t)$ is almost periodic in $t \in \mathbb{R}$.

We did not mentioned the used metric on the space of probability measures but we always assume that the metric used is complete and induces the topology of weak convergence on $\mathcal{P}(\mathbb{R}^n)$. The following metrics satisfy this assumption and are useful for different applications. Since we will need the Prokhorov metric later on also for finite measures rather than only probability measures, we give it immediately in this context.

Definition 3.4.

a) The *Prokhorov metric* $\delta_n : \mathcal{M}(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \to [0, \infty)$ is defined as

$$\delta_n(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(\mathbb{R}^n)\},$$

where $A^\varepsilon = \bigcup_{x \in A} \{y \in \mathbb{R}^n : |y - x| < \varepsilon\}$.

b) The *bounded-Lipschitz metric* $\beta_n : \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \to [0, \infty)$ is defined as

$$\beta_n(\mu, \nu) := \sup_{\|f\|_{BL} \leq 1} \left\{ \left| \int_{\mathbb{R}^n} f \, d(\mu - \nu) \right| \right\},$$

where

$$\|f\|_{BL} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} + \sup_{x \in \mathbb{R}^n} |f(x)|,$$

for $f : \mathbb{R}^n \to \mathbb{R}$.

c) We define $\gamma_n : \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \to [0, \infty)$ as

$$\gamma_n(\mu, \nu) := \sum_{k=1}^{\infty} 2^{-k} \|\hat{\mu} - \hat{\nu}\|_{C([-k,k]^n)},$$

where $\hat{\mu}, \hat{\nu}$ are the characteristic functions of μ and ν respectively, and $C([-k, k]^n)$, $k \in \mathbb{N}$, is the space of continuous functions restricted to $[-k, k]^n$ endowed with the uniform norm.

Remark 3.5. The metrics in a) and b) are covered extensively in [17]. The metric defined in c) is indeed a complete metric which induces the topology of weak convergence on $\mathcal{P}(\mathbb{R}^n)$, and it is separable. As we did not found a proof of this result, we give a short proof here.

Proof of Remark 3.5. Clearly, γ_n is a metric on $\mathcal{P}(\mathbb{R}^n)$. Furthermore, we observe

$$\begin{aligned}\gamma_n(\mu, \nu) &= \sum_{k=1}^{\infty} 2^{-k}(1 + \sqrt{n}k) \left\| \frac{\hat{\mu} - \hat{\nu}}{1 + \sqrt{n}k} \right\|_{C([-k,k]^n)} \\ &\leq \sum_{k=1}^{\infty} 2^{-k}(1 + \sqrt{n}k) \cdot 2\beta_n(\mu, \nu)\end{aligned}$$

$$= (2 + 4\sqrt{n})\, \beta_n(\mu, \nu).$$

So if there is a sequence μ_l, $l \in \mathbb{N}$, and μ, all in $\mathcal{P}(\mathbb{R}^n)$ such that μ_l converges weakly to μ as $l \to \infty$, we see that $\lim_{l\to\infty} \gamma_n(\mu_l, \mu) = 0$.
Now take a sequence $(\mu_l)_{l\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$ such that for every $\varepsilon > 0$ there exists an m_0 such that for all $l, m > m_0$ we have $\gamma_n(\mu_l, \mu_m) < \varepsilon$. We conclude that $(\hat{\mu}_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in every space $(C([-k,k]^n), \|\cdot\|_{C([-k,k]^n)})$. Hence, there exists a bounded function $\hat{\mu} \in C(\mathbb{R})$ such that $\|\hat{\mu}_l - \hat{\mu}\|_{C([-k,k])} \to 0$ as $l \to \infty$ for every $k \in \mathbb{N}$. By Levy's continuity theorem we know that $\hat{\mu}$ is the characteristic function of a probability measure μ and the sequence μ_l converges weakly to μ, which implies that the limit exists in γ_n. This implies that the metric is indeed metricizing the weak topology on $\mathcal{P}(\mathbb{R}^n)$ and it is complete. Finally, since separability of a metric depends only on its induced topology and since δ_n is separable, so is γ_n. $\square$

We directly see that strictly stationary processes and periodic strictly stationary processes indeed satisfy the condition in Definition 3.3. Furthermore, there exists a definition of almost periodically correlated processes, i.e. a process such that its mean and covariance functions are almost periodic (see [22]).

Definition 3.6. A real-valued stochastic process $X = (X_t)_{t\in\mathbb{R}}$ with finite second moment, i.e. $\mathbb{E}X_t^2 < \infty$ for all $t \in \mathbb{R}$, is called *almost periodically correlated* if its mean and covariance functions

$$\begin{aligned} m(t) &:= \mathbb{E}X_t \quad \text{and} \\ C_a(t) &:= \mathbb{E}X_t X_{t+a} - \mathbb{E}X_t \mathbb{E}X_{t+a} \end{aligned}$$

are almost periodic functions in $t \in \mathbb{R}$ for all $a \in \mathbb{R}$.

In [4] and [45] different concepts of almost periodic stochastic processes were discussed and compared. In the following we continue their work by showing that our definition implies almost periodically correlation if the process $X = (X_t)_{t\in\mathbb{R}^d}$ is L^2-uniformly integrable. In order to do so, we shortly discuss almost periodicity in the well-known Wasserstein metric and afterwards mention further properties of almost periodically stationary processes. Let $p \in [1,\infty)$. Recall that a stochastic process $(X_t)_{t\in\mathbb{R}}$ is L^p*-uniformly integrable* if to each $\varepsilon > 0$ there exists a $k > 0$ such that $\sup_{t\in\mathbb{R}} \mathbb{E}|X_t|^p \mathbb{1}_{|X_t|>k} < \varepsilon$.

Remark 3.7. For any two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ with finite p-moment ($p \in [1,\infty)$), the p-th *Wasserstein metric* is defined as

$$W_p(\mu,\nu) := \left(\inf_{\sigma\in\Gamma(\mu,\nu)} \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^p \, d\sigma(x,y) \right)^{1/p},$$

where $\Gamma(\mu,\nu)$ denotes the set of all measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ and ν. Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures with finite p-th moment and a probability measure μ with finite p-th moment, we know that $W_p(\mu_n,\mu) \to 0$ for $n \to \infty$ is equivalent to $(\mu_n)_{n\in\mathbb{N}}$ being L^p-uniformly integrable and weakly convergent to μ (see [46, Theorem 7.12]). We observe that if a process $X = (X_t)_{t\in\mathbb{R}}$ is L^p-uniformly integrable and almost periodically stationary with respect to one of the metrics from Definition 3.4, then X is also almost periodically stationary with respect to the metric W_p, which can be seen from Bochner's double sequence criterion. Let us now summarize some results on almost periodically stationary processes.

Proposition 3.8. *Let $X = (X_t)_{t\in\mathbb{R}}$ be an almost periodically stationary process. Then the following are true:*

a) For any $n \in \mathbb{N}$ and $(x_1,\dots,x_n) \in \mathbb{R}^n$, the function $\mu_{x_1,\dots,x_n}(t)$ is uniformly continuous in $t \in \mathbb{R}$.

b) The set of probability measures $(\mathcal{L}_{X_t})_{t\in\mathbb{R}} \subset \mathcal{P}(\mathbb{R})$ is relatively compact.

c) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $g(X_t)$ is an almost periodically stationary process.

d) If X is L^2-uniformly integrable, then X is almost periodically correlated.

Proof. Statement a) follows from [31, Property 2, p.2], and b) follows from Bochner's double sequence criterion, since by choosing $x = 0$, $(b_n)_{n\in\mathbb{N}} = 0$ we find for a general sequence $(a_n)_{n\in\mathbb{N}}$ a subsequence $(a'_n)_{n\in\mathbb{N}}$ such that $(\mathcal{L}(X_{a'_n}))_{n\in\mathbb{N}}$ converges weakly, showing relative compactness of $(\mathcal{L}(X_t))_{t\in\mathbb{R}}$.
c) This is a direct consequence of Bochner's double sequence criterion and the continuous mapping theorem, see [17, Theorem 9.3.7].
d) We use Bochner's double sequence criterion. At first we know that the limits

$$\lim_{n\to\infty}\lim_{m\to\infty} \mu_{t+a_n+b_m,s+a_n+b_m} = \lim_{n\to\infty} \mu_{t+a_n+b_n,s+a_n+b_n}$$

exist in the topology of the weak convergence. By our uniform integrability condition, we conclude that the limits also exist in the Wasserstein space W_2. As a consequence we know that the limits

$$\lim_{n\to\infty}\lim_{m\to\infty} \mathbb{E}X_{t+a_n+b_m}X_{s+a_n+b_m} = \lim_{n\to\infty} \mathbb{E}X_{t+a_n+b_n}X_{s+a_n+b_n}$$

exist. By definition it follows that $\mathbb{E}X_{t+u}X_{s+u}$ is almost periodic in u for every $t,s \in \mathbb{R}$ (that $u \mapsto \mathbb{E}X_{t+u}X_{s+u}$ is continuous follows similarly from the continuity of $u \mapsto \mu_u(s,t)$ with respect to the Wasserstein metric W_2). Similarly, $u \mapsto \mathbb{E}X_{t+u}$ is almost periodic for every $t \in \mathbb{R}$. Since sums and products of almost periodic functions are again almost periodic, the claim follows.

□

With the metric γ_n from Definition 3.4 we can easily deduce a simple equivalent condition based on characteristic functions when a stochastic process is almost periodically stationary:

Theorem 3.9. *Let $(X_t)_{t\in\mathbb{R}}$ be a stochastic process. The process is almost periodically stationary if, and only if, for every $n \in \mathbb{N}$, $t_1, \dots, t_n \in \mathbb{R}$, $K \subset \mathbb{R}^n$ compact and $\varepsilon > 0$ there exists an $l > 0$ such that for every $a \in \mathbb{R}$ there exists a $\tau \in [a, a + l]$ with*

$$\sup_{x\in\mathbb{R}} \sup_{z\in K} |\widehat{\mathcal{L}}(X_{t_1+x}, \dots, X_{t_n+x})(z) - \widehat{\mathcal{L}}(X_{t_1+x+\tau}, \dots, X_{t_n+x+\tau})(z)| < \varepsilon \tag{3.3}$$

and the functions

$$\mathbb{R} \ni x \mapsto \widehat{\mathcal{L}}(X_{t_1+x}, \dots, X_{t_n+x})(z) \tag{3.4}$$

are continuous for every fixed z and vector $(t_1, \dots, t_n) \in \mathbb{R}^n$.

Proof. First observe that (3.4) is equivalent to continuity of $x \mapsto \mathcal{L}(X_{t_1+x}, \dots, X_{t_n+x})$ by Lévy's continuity theorem. The rest is a consequence of Definition 3.3 when choosing the metric γ_n: assume that the process is almost periodic. For every compact set $K \subset \mathbb{R}^n$ there exists an $m > 0$ such that $K \subset [-m, m]^n$. We see that

$$\begin{aligned}&\sup_{x\in\mathbb{R}} \sup_{z\in K} |\widehat{\mathcal{L}}(X_{t_1+x}, \dots, X_{t_n+x})(z) - \widehat{\mathcal{L}}(X_{t_1+x+\tau}, \dots, X_{t_n+x+\tau})(z)| \\ &\quad \le 2^m \sup_{x\in\mathbb{R}} \gamma_n(\mathcal{L}(X_{t_1+x}, \dots, X_{t_n+x}), \mathcal{L}(X_{t_1+x+\tau}, \dots, X_{t_n+x+\tau})),\end{aligned}$$

showing (3.3). For the converse direction fix $\varepsilon > 0$. Observe that by choosing $K = [-m, m]^d$ such that $\sum_{k=m+1}^{\infty} 2^{-k} < \varepsilon/4$ we have that

$$\begin{aligned}&\sum_{k=1}^{\infty} 2^{-k} \|\widehat{\mathcal{L}}(X_{t_1+x}, \dots, X_{t_n+x}) - \widehat{\mathcal{L}}(X_{t_1+x+\tau}, \dots, X_{t_n+x+\tau})\|_{C([-k,k]^d)} \\ \le &\|\widehat{\mathcal{L}}(X_{t_1+x}, \dots, X_{t_n+x}) - \widehat{\mathcal{L}}(X_{t_1+x+\tau}, \dots, X_{t_n+x+\tau})\|_{C([-m,m]^d)} + \varepsilon/2\end{aligned}$$

for every $x \in \mathbb{R}$. We conclude that if for given $t_1, \dots, t_n \in \mathbb{R}$ we choose τ such that

$$\|\widehat{\mathcal{L}}(X_{t_1+x}, \dots, X_{t_n+x}) - \widehat{\mathcal{L}}(X_{t_1+x+\tau}, \dots, X_{t_n+x+\tau})\|_{C([-m,m]^d)} < \varepsilon/2 \quad \text{for all } x \in \mathbb{R},$$

we get

$$\sup_{x\in\mathbb{R}} \gamma_n(\mathcal{L}(X_{t_1+x}, \dots, X_{t_n+x}), \mathcal{L}(X_{t_1+x+\tau}, \dots, X_{t_n+x+\tau})) < \varepsilon.$$

As ε was arbitrary, we showed that the process is almost periodically stationary. □

From Theorem 3.9 we see that it is indeed possible to characterize almost periodically stationary processes via the characteristic function. In many cases it is much easier to calculate the difference between characteristic functions, e.g. in the case of two infinitely divisible distributions, which we consider in the following chapter.

3.3 Almost periodically stationarity of infinitely divisible random processes

We derive sufficient and necessary conditions for an infinitely divisible process $(X_t)_{t\in\mathbb{R}}$ to be almost periodically stationary, but first we recall the definition of an infinitely divisible distribution and what an infinitely divisible process is.

A probability measure $\mu \in \mathcal{P}(\mathbb{R}^n)$ is *infinitely divisible*, if for every $m \in \mathbb{N}$ there exists $\mu_m \in \mathcal{P}(\mathbb{R}^n)$ such that $\mu = \mu_m^{*m}$ (the m-fold convolution of μ_m with itself). By the *Lévy-Khintchine formula* (see e.g. [29], [40]), $\mu \in \mathcal{P}(\mathbb{R}^n)$ is infinitely divisible if and only if there exist $\gamma \in \mathbb{R}^n$, a symmetric positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$ and a measure ν on $\mathbb{R}^n$ with

$$\int_{\mathbb{R}^n} \min(1, |x|^2)\, \nu(dx) < \infty \text{ and } \nu(\{0\}) = 0$$

(ν is called a *Lévy measure*) such that

$$\hat{\mu}(z) = \exp(\psi(z)) \tag{3.5}$$

with

$$\psi(z) = i\langle \gamma, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^n} (e^{i\langle z,y\rangle} - 1 - i\langle z, y\rangle \mathbb{1}_{|x|\leq 1})\nu(dy), \quad z \in \mathbb{R}^n \tag{3.6}$$

(here, $\langle \cdot, \cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$). We call (A, γ, ν) the *characteristic triplet* of μ, which is known to be unique, and ψ the *characteristic exponent* of μ. Further, for every $\gamma \in \mathbb{R}^n$, symmetric positive semi-definite $A \in \mathbb{R}^{n\times n}$ and Lévy measure ν, the right-hand side of (3.5), (3.6) defines the characteristic function of an infinitely divisible distribution. Instead of the indicator function $\mathbb{1}_{|x|\leq 1}$ in (3.6) one can choose a continuous representation function $c : \mathbb{R}^n \to \mathbb{R}$ which is bounded, with compact support and satisfies $c(x) = 1$ for $|x| \leq 1$ (e.g. see [26, Section VII.2a], [40, Remark 8.4]) so that the integrability with respect to ν in (3.6) is still assured. This only changes the value of γ. In this case we write $(A, \gamma^c, \nu)_c$ for the characteristic triplet of μ. A stochastic process $(X_t)_{t\in\mathbb{R}}$ is called an *infinitely divisible stochastic process* if all finite-dimensional distributions $(X_{t_1}, \dots, X_{t_n})$, with $n \in \mathbb{N}$ and $(t_1, \dots, t_n) \in \mathbb{R}^n$, are infinitely divisible. We write $(A_{t_1,\dots,t_n}, \gamma^c_{t_1,\dots,t_n}, \nu_{t_1,\dots,t_n})_c$ and $(A_{t_1,\dots,t_n}, \gamma_{t_1,\dots,t_n}, \nu_{t_1,\dots,t_n})$ when $c(x) = \mathbb{1}_{|x|\leq 1}$ for the characteristic triplet of $(X_{t_1}, \dots, X_{t_n})$, and $\psi_{t_1,\dots,t_n}$ for its characteristic exponent.

Our next theorem describes how to express almost periodically stationarity for infinitely divisible process in terms of their characteristic exponents and triplets. For a column vector $x \in \mathbb{R}^d$ we denote by x^T its transpose and for $a, b \in \mathbb{R}$ we denote by $a \wedge b$ the minimum between a and b.

Theorem 3.10. *Let $(X_t)_{t\in\mathbb{R}}$ be an infinitely divisible process and for each $d \in \mathbb{N}$ let $c_d : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support such that $c_d(x) = 1$ for $|x| \leq 1$. Denote the characteristic exponents and triplets of $(X_{t_1}, \dots, X_{t_d})$ by $\psi_{t_1,\dots,t_d}$ and $(A_{t_1,\dots,t_d}, \gamma^{c_d}_{t_1,\dots,t_d}, \nu_{t_1,\dots,t_d})_{c_d}$, respectively. Then the following assertions are equivalent:*

a) The process $(X_t)_{t\in\mathbb{R}}$ is almost periodically stationary.

b) The functions $\mathbb{R} \to C([-k,k]^d, \mathbb{C})$, $t \mapsto \psi_{t_1+t,\dots,t_d+t}\big|_{[-k,k]^d}$ (the restriction of $\psi_{t_1+t,\dots,t_d+t}$ to $[-k,k]^d$) are almost periodic for every $k \in \mathbb{N}$, $d \in \mathbb{N}$ and $(t_1, \dots, t_d) \in \mathbb{R}^d$.

c) The functions

$$\mathbb{R} \ni t \mapsto \gamma^{c_d}_{t_1+t,\dots,t_d+t}, \tag{3.7}$$

$$\mathbb{R} \ni t \mapsto A_{t_1+t,\dots,t_d+t} + \int_{\mathbb{R}^d} xx^T c_d(x)^2 \nu_{t_1+t,\dots,t_d+t}(dx), \tag{3.8}$$

$$\mathbb{R} \ni t \mapsto (|x|^3 \wedge 1)\, \nu_{t_1+t,\dots,t_d+t}(dx) \tag{3.9}$$

are almost periodic in $\mathbb{R}^d$, $\mathbb{R}^{d\times d}$ and $\mathcal{M}(\mathbb{R}^d)$, respectively, for every $d \in \mathbb{N}$ and $(t_1, \dots, t_d) \in \mathbb{R}^d$.

Remark 3.11. That $((|x|^2 \wedge 1)\, \nu_{t_1+t,\dots,t_d+t}(dx))_{t\in\mathbb{R}}$ is relatively compact follows immediately from the conditions (3.8) and (3.9), but it is not true that the function $(|x|^2 \wedge 1)\, \nu_{t_1+t,\dots,t_d+t}(dx)$ is almost periodic. The reason is that an almost periodic function needs to be continuous by definition, nevertheless $t \mapsto (|x|^2 \wedge 1)\, \nu_{t_1+t,\dots,t_d+t}(dx)$ is not necessarily continuous, which can be seen from condition (3.8). One can overcome this shortcoming in dimension one by adding the Gaussian variances to the Lévy measures as a delta-distribution at point 0, but a similar approach seems difficult in dimension greater than 1 as the Gaussian variance is then a matrix (see [26, Remark 2.10, Section VII.2].

Proof. We show a) $\Longrightarrow$ b) $\Longrightarrow$ c) $\Longrightarrow$ a). Moreover, we fix a vector $(t_1, \dots, t_d)$ and always write $\gamma_t, \nu_t, A_t, \psi_t$ and μ_t for $\gamma^{c_d}_{t+t_1,\dots,t+t_d}, \nu_{t+t_1,\dots,t+t_d}, A_{t+t_1,\dots,t+t_d}, \psi_{t+t_1,\dots,t+t_d}$ and $\mu_{t+t_1,\dots,t+t_d}$, respectively.
a) $\Longrightarrow$ b). Let $(X_t)_{t\in\mathbb{R}}$ be almost periodically stationary and $t_1, \dots, t_n \in \mathbb{R}$. If $s_n \to t$ as $n \to \infty$, then $\mathcal{L}(X_{t_1+s_n}, \dots, X_{t_d+s_n})$ converges weakly to $\mathcal{L}(X_{t_1+t}, \dots, X_{t_d+t})$ as $n \to \infty$, hence the characteristic function and hence $\psi_{t_1+s_n,\dots,t_d+s_n}$ converges locally uniformly to $\psi_{t_1+t,\dots,t_d+t}$, cf. [40, Lemma 7.7], showing continuity of $t \mapsto \psi_{t_1+t,\dots,t_d+t}|_{[-k,k]^d}$. By Theorem 3.9 we know that the functions $t \mapsto \hat{\mu}_t|_{[-k,k]^d}$ are almost periodic in the norm $\|\cdot\|_{C([-k,k]^d)}$ for every $k \in \mathbb{N}$. Observe that

$$\begin{aligned} |\hat{\mu}_t(z) - \hat{\mu}_{t+\tau}(z)| &= |\exp(\psi_t(z)) - \exp(\psi_{t+\tau}(z))| \\ &= |\exp(\psi_{t+\tau}(z))| \cdot |1 - \exp(\psi_t(z) - \psi_{t+\tau}(z))| \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}|\exp(\psi_t(z))| &= \exp(\operatorname{Re}\psi_t(z))\\ &= \exp\left(-\left(\frac{1}{2}\langle z, A_t z\rangle + \int\limits_{\mathbb{R}^d}(1-\cos(\langle x, z\rangle))\nu_t(dx)\right)\right)\\ &\geq \exp\left(-\left(\frac{1}{2}\langle z, A_t z\rangle + |z|^2\int\limits_{|x|\leq 1}|x|^2\nu_t(dx) + 2\nu_t(\{x\in\mathbb{R}^d : |x|>1\})\right)\right)\end{aligned}$$

(where we used $1-\cos(y)\leq y^2$ and the Cauchy-Schwarz inequality). Since the set $(\mu_t)_{t\in\mathbb{R}}$ is relatively compact by Proposition 3.8 b),

$$\sup_{t\in\mathbb{R}}\sup_{z\in[-k,k]^d}\frac{1}{2}\langle z, A_t z\rangle + |z|^2\int\limits_{|x|\leq 1}|x|^2\nu_t(dx) + 2\nu_t(\{x\in\mathbb{R}^d : |x|>1\}) < \infty$$

for every $k\in\mathbb{N}$ by [40, Exercise 12.5, p.66]. The previous estimates then imply that

$$\inf_{t\in\mathbb{R}}\inf_{z\in[-k,k]^d}|\exp(\psi_t(z))| > 0.$$

Since $t\mapsto \hat{\mu}_t|_{[-k,k]^d}$ is almost periodic, this together with (3.10) implies that for every $\delta>0$ there exists an $L_\delta>0$ and for each $a\in\mathbb{R}$ a $\tau=\tau(a,\delta)\in[a,a+L_\delta]$ such that $\sup_{t\in\mathbb{R}}\sup_{z\in[-k,k]^d}|1-\exp(\psi_t(z)-\psi_{t+\tau}(z))|<\delta$. Since $\mathbb{C}\ni x+iy\mapsto 1-e^{x+iy}$ is continuous with period $2\pi i$ and zero set $2\pi i\mathbb{Z}$, and since $\lim_{x\to\infty}e^x=\infty$ and $\lim_{x\to-\infty}e^x=0$, $|1-e^{x+iy}|<\delta<1$ implies $x+iy\in\bigcup_{m\in\mathbb{Z}}\{2\pi i m + w : w\in\mathbb{C}, |w|<\varepsilon\} =: D_\varepsilon$ for some $\varepsilon=\varepsilon(\delta)>0$, where $\varepsilon(\delta)\to 0$ as $\delta\to 0$. Hence $\psi_t(z)-\psi_{t+\tau}(z)\in D_{\varepsilon(\delta)}$ for every $z\in[-k,k]^d$ and $t\in\mathbb{R}$, and since $\psi_t-\psi_{t+\tau}$ is continuous with $\psi_t(0)=\psi_{t+\tau}(0)=0$, we conclude

$$\sup_{t\in\mathbb{R}}\sup_{z\in[-k,k]^d}|\psi_t(z)-\psi_{t+\tau}(z)|<\varepsilon$$

when $\varepsilon=\varepsilon(\delta)<\pi$, giving the desired almost periodicity of $t\mapsto\hat{\mu}_t|_{[-k,k]^d}$.

b) $\implies$ c). First observe that continuity of $t\mapsto\psi_t|_{[-k,k]^d}$ implies continuity of $t\mapsto\hat{\mu}_t|_{[-k,k]^d}$ which by [26, Theorem 2.9 and Remark 2.10, Section VII.2] implies continuity of the functions given in (3.7)–(3.9). Let us show their almost periodicity. Let $(a_n')_{n\in\mathbb{N}}$, $(b_m')_{m\in\mathbb{N}}\subset\mathbb{R}$ be two sequences. By Bochner's double sequence criterion and a standard diagonal subsequence argument, there exist subsequences $(a_n)_{n\in\mathbb{N}}$ and $(b_m)_{m\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\lim_{m\to\infty}\psi_{t+a_n+b_m} \text{ and } \lim_{n\to\infty}\psi_{t+a_n+b_n}$$

exist (in the sense of locally uniform convergence) and

$$\lim_{n\to\infty}\lim_{m\to\infty}\psi_{t+a_n+b_m}(z) = \lim_{n\to\infty}\psi_{t+a_n+b_n}(z) =: \psi_{t,\infty}(z)$$

for every $t \in \mathbb{R}$ and $z \in \mathbb{R}^d$. As the set of infinitely divisible distributions is closed (see [40, Lemma 7.8]), we know that $\lim_{m\to\infty} \psi_{t+a_n+b_m}$ and $\psi_{t,\infty}$ are again the characteristic exponents of infinitely divisible distributions. Denote the characteristic triplet corresponding to $\psi_{t,\infty}$ by $(\gamma_{t,\infty}, A_{t,\infty}, \nu_{t,\infty})_c$. Then by [26, Theorem 2.9 and Remark 2.10, Section VII.2]

$$\lim_{n\to\infty}\lim_{m\to\infty} \gamma_{t+a_n+b_m} = \lim_{n\to\infty} \gamma_{t+a_n+b_n} = \gamma_{t,\infty},$$
$$\lim_{n\to\infty}\lim_{m\to\infty} (|x|^3 \wedge 1)\, \nu_{t+a_n+b_m}(dx) = \lim_{n\to\infty} (|x|^3 \wedge 1)\, \nu_{t+a_n+b_n}(dx) = (|x|^3 \wedge 1)\, \nu_{t,\infty}(dx),$$

where the limits are taken weakly, and

$$\lim_{n\to\infty}\lim_{m\to\infty} A_{t+a_n+b_m} + \int_{\mathbb{R}^d} xx^T c_d(x)^2 \nu_{t+a_n+b_m}(dx)$$
$$= \lim_{n\to\infty} A_{t+a_n+b_n} + \int_{\mathbb{R}^d} xx^T c_d(x)^2 \nu_{t+a_n+b_n}(dx) = A_{t,\infty} + \int_{\mathbb{R}^d} xx^T c_d(x) \nu_{t,\infty}(dx).$$

Hence, by Bochner's double sequence criterion, the functions γ_t, $(|x|^3 \wedge 1)\nu_t(dx)$ and $A_t + \int_{\mathbb{R}^d} xx^T c(x)^2 \nu_t(dx)$ are almost periodic.

c) $\Longrightarrow$ a). Continuity of $t \mapsto \mu_t$ follows again from [26, Theorem 2.9 and Remark 2.10, Section VII.2]. For almost periodicity, we show that $t \mapsto \mu_t$ satisfies the double sequence criterion. Therefore, let (a_n') and (b_n') be sequences. Then there exist subsequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\lim_{m\to\infty} \gamma_{t+a_n+b_m} = \lim_{n\to\infty} \gamma_{t+a_n+b_n},$$
$$\lim_{n\to\infty}\lim_{m\to\infty} A_{t+a_n+b_m} + \int_{\mathbb{R}^d} xx^T c_d(x)^2 \nu_{t+a_n+b_n}(dx) = \lim_{n\to\infty} A_{t+a_n+b_n} + \int_{\mathbb{R}^d} xx^T c_d(x)^2 \nu_{t+a_n+b_n}(dx),$$
$$\lim_{n\to\infty}\lim_{m\to\infty} (|x|^3 \wedge 1)\nu_{t+a_n+b_m}(dx) = \lim_{n\to\infty} (|x|^3 \wedge 1)\nu_{t+a_n+b_n}(dx).$$

We have to show for the subsequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ that

$$\lim_{n\to\infty}\lim_{m\to\infty} \mu_{t+a_n+b_m} = \lim_{n\to\infty} \mu_{t+a_n+b_n}.$$

This follows again from [26, Theorem 2.9 and Remark 2.10, Section VII.2], if we can show that for a sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$, a sequence $(\nu_n)_{n\in\mathbb{N}}$ of Lévy measures and symmetric positive semi-definite $d \times d$-matrices $(A_n)_{n\in\mathbb{N}}$, the conditions

$$\lim_{n\to\infty} \gamma_n = \gamma, \tag{3.11}$$
$$\lim_{n\to\infty} A_n + \int_{\mathbb{R}^d} xx^T c(x)^2 \nu_n(dx) = A', \tag{3.12}$$
$$\lim_{n\to\infty} (|x|^3 \wedge 1)\nu_n(dx) = (|x|^3 \wedge 1)\nu'(dx), \tag{3.13}$$

for $\gamma \in \mathbb{R}^d$, $A' \in \mathbb{R}^{d\times d}$ and $(|x|^3 \wedge 1)\nu' \in \mathcal{M}(\mathbb{R}^d)$ imply that ν' is a Lévy measure and that $A' - \int_{\mathbb{R}^d} xx^T c_d(x)\nu'(dx)$ is symmetric and positive semi-definite. To see this, observe that (3.11) and (3.12) imply boundedness of $(\gamma_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ (observe that $\int_{\mathbb{R}^d} xx^T c_d(x)\nu_n(dx)$ is positive semi-definite), (3.12) together with $c_d(x) = 1$ for $|x| \le 1$ implies $\sup_{n\in\mathbb{N}} \int_{|x|\le 1} |x|^2\nu_n(dx) < \infty$, and (3.13) together with the fact that $(|x|^3 \wedge 1)\nu'(dx) \in \mathcal{M}(\mathbb{R}^d)$ implies $\sup_{n\in\mathbb{N}} \int_{|x|>1} \nu_n(dx) < \infty$ and $\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|>k} \nu_n(dx) = 0$. Hence, the sequence $(\rho_n)_{n\in\mathbb{N}}$ of infinitely divisible distributions with characteristic triplets (A_n, γ_n, ν_n) is tight by [40, Exercise 12.5], so that a subsequence of $(\rho_n)_{n\in\mathbb{N}}$ converges to an infinitely divisible distribution with characteristic triplet $(A_\infty, \gamma_\infty, \nu_\infty)$, say. But then $A_\infty + \int_{\mathbb{R}^d} xx^T c_d(x)\nu_\infty(dx) = A'$, $\gamma_\infty = \gamma$ and $\nu_\infty = \nu'$ by [26, Theorem 2.9 and Remark 2.10, Section VII.2], so that ν' is a Lévy measure and $A' - \int_{\mathbb{R}^d} xx^T c_d(x)\nu_\infty(dx)$ is positive semi-definite, finishing the proof.

□

3.4 Almost periodically stationarity of $X_t = \int f(t,s)dL(s)$

In this section we derive estimates for random variables X, Y of the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \int_{\mathbb{R}^d} \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} dL(t), \tag{3.14}$$

where L is a Lévy basis and $f, g : \mathbb{R}^d \to \mathbb{R}^m$ are deterministic functions. In particular, we apply these estimates to obtain conditions for stochastic processes $(X_t)_{t\in\mathbb{R}}$ of the form

$$X_t = \int_{\mathbb{R}^d} f(t,s)dL(s), \quad t \in \mathbb{R},$$

to be almost periodically stationary.

We denote by $\mathcal{B}_b(\mathbb{R}^d)$ the set of all bounded Borel sets and by λ^d the Lebesgue measure on $\mathbb{R}^d$. We recall the definition of a Lévy basis and the Rajput-Rosinski exponent Ψ of a Lévy basis in order to obtain the above mentioned estimates.

Definition 3.12. A *Lévy basis* on $\mathbb{R}^d$ is a family $(L(A))_{A\in\mathcal{B}_b(\mathbb{R}^d)}$ of real valued random variables such that

i) $L\left(\bigcup_{n\in\mathbb{N}_0} A_n\right) = \sum_{n\in\mathbb{N}_0} L(A_n)$ a.s. for pairwise disjoint sets $(A_n)_{n\in\mathbb{N}_0} \subset \mathcal{B}_b(\mathbb{R}^d)$ with $\bigcup_{n\in\mathbb{N}_0} A_n \in \mathcal{B}_b(\mathbb{R}^d)$.

ii) $L(A_i)$ are independent for pairwise disjoint sets $A_1, \dots, A_n \in \mathcal{B}_b(\mathbb{R}^d)$ for every $n \in \mathbb{N}$.

iii) There exists $a \in [0,\infty)$, $\gamma \in \mathbb{R}$ and a Lévy measure ν on $\mathbb{R}$ such that

$$\widehat{L(A)}(z) = \mathbb{E}e^{izL(A)} = \exp(\psi(z)\lambda^d(A)), \quad A \in \mathcal{B}_b(\mathbb{R}^d),\ z \in \mathbb{R},$$

where

$$\psi_L(z) = i\gamma z - \frac{1}{2}az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbb{1}_{[-1,1]}(x))\nu(dx), \quad z \in \mathbb{R}.$$

The triplet (a, γ, ν) is called *characteristic triplet* of L and ψ_L its *characteristic exponent.*

Let L be a Lévy basis on $\mathbb{R}^d$ with characteristic triplet (a, γ, ν) and let $f : \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable function. Then f is in the *domain of* L and we write $f \in D(L)$, if the integral $\int_{\mathbb{R}^d} f(s)\, dL(s)$ can be defined as a limit in probability of integrals of suitable simple functions as outlined by Rajput and Rosinski in [38, p. 460] and we say then that the integral exists in the sense of Rajput and Rosinski. By [38, Theorem 2.7], $f \in D(L)$ if and only if

$$\int_{\mathbb{R}^d} \Psi(f(t))\, dt < \infty, \tag{3.15}$$

where

$$\Psi(z) := |U(z)| + az^2 + V(z), \quad z \in \mathbb{R},$$

is the *Rajput-Rosinski exponent of* L with

$$U(z) := \gamma z + \int_{\mathbb{R}} sz(\mathbb{1}_{|sz|\le 1} - \mathbb{1}_{|s|\le 1})\, \nu(ds),$$
$$V(z) := \int_{\mathbb{R}} \min(1, s^2 z^2)\, \nu(ds).$$

If $f = (f_1, \dots, f_m)^T$ with $f_1, \dots, f_m \in D(L)$, then

$$\int_{\mathbb{R}^d} f(s)\, dL(s) = \left(\int_{\mathbb{R}^d} f_1(s)\, dL(s), \dots, \int_{\mathbb{R}^d} f_m(s)\, dL(s) \right)^T$$

is infinitely divisible with characteristic function

$$\exp\left(\int_{\mathbb{R}^d} \psi_L(z^T f(t))\, dt \right), \quad z \in \mathbb{R}^m. \tag{3.16}$$

Definition 3.13. For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we define the *distribution function* of f as

$$d_f(\alpha) = \lambda^d(\{x \in \mathbb{R}^d : |f(x)| > \alpha\}),\ \alpha > 0.$$

We introduce a short notation for functions $f : \mathbb{R}^d \to \mathbb{R}^m$, which are in a sense good integrands for the Lévy basis L.

Definition 3.14. We say that $f = (f_1, \dots, f_m)^T \in B(L, \mathbb{R}^m)$, if the integral

$$\int\limits_{|r|>1} |r| \int\limits_0^{1/|r|} d_{f_k}(\alpha) d\alpha \nu(dr)$$

is finite for every $k \in \{1, \dots, m\}$. If the dimension m is clear, we simply write $B(L)$.

In [6, Proposition 5.2] it was shown that $B(L,\mathbb{R}) \cap L^1(\mathbb{R}^d,\mathbb{R}) \cap L^2(\mathbb{R}^d,\mathbb{R}) \subset D(L)$. We denote by f^+ and f^- the positive and negative part of a real-valued function f, respectively.

Theorem 3.15. *Let L be a Lévy basis on $\mathbb{R}^d$ with characteristic triplet (a, γ, ν) and let $(X, Y) \in \mathbb{R}^{n\times n}$ be a $2n$-dimensional infinitely divisible random vector given by*

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \int\limits_{\mathbb{R}^d} \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} dL(t),$$

where $f, g \in L^1(\mathbb{R}^d, \mathbb{R}^n) \cap L^2(\mathbb{R}^d, \mathbb{R}^n) \cap B(L)$. Let ψ_f and ψ_g be the characteristic exponents of X and Y, respectively. Then

$$\begin{aligned} &|\psi_f(z) - \psi_g(z)| \\ &\leq \int\limits_{|r|\leq 1} r^2 \nu(dr) \left(\int\limits_0^\infty \alpha(|d_{(z^Tf)^+}(\alpha) - d_{(z^Tg)^+}(\alpha)| + |d_{(z^Tf)^-}(\alpha) - d_{(z^Tg)^-}(\alpha)|) d\alpha \right) \\ &\quad + \frac{a}{2} \left| \int\limits_{\mathbb{R}^d} (z^T f(x))^2 dx - \int\limits_{\mathbb{R}^d} (z^T g(x))^2 dx \right| + \left| \gamma z^T \left(\int\limits_{\mathbb{R}^d} f(x) dx - \int\limits_{\mathbb{R}^d} g(x) dx \right) \right| \\ &\quad + \int\limits_{1<|r|\leq R} |r| \nu(dr) \left(\int_0^\infty |d_{(z^Tf)^+}(\alpha) - d_{(z^Tg)^+}(\alpha)| + |d_{(z^Tf)^-}(\alpha) - d_{(z^Tg)^-}(\alpha)| \, d\alpha \right) \\ &\quad + 3 \int\limits_{|r|>R} |r| \int_0^{1/|r|} d_{z^Tf}(\alpha) d\alpha \nu(dr) + 3 \int\limits_{|r|>R} |r| \int_0^{1/|r|} d_{z^Tg}(\alpha) d\alpha \nu(dr) \end{aligned}$$

for every $z \in \mathbb{R}^n$ and $R > 1$.

Proof. By (3.16) the characteristic exponents of X and Y can be represented by $\psi_f(z) = \int_{\mathbb{R}^d} \psi_L(z^T f(y)) dy$ and $\psi_g = \int_{\mathbb{R}^d} \psi_L(z^T g(y)) dy$. We split ψ_L into four parts

$$\psi_L(z) = \psi_1(w) + \psi_{2,R}(w) + \psi_{3,R} + \psi_4(w), \; R > 1, \; w \in \mathbb{R},$$

where

$$\psi_1(w) := \int\limits_{|r|\le 1} (e^{irw} - 1 - irw)\nu(dr),$$
$$\psi_{2,R}(w) := \int\limits_{1<|r|\le R} (e^{irw} - 1)\nu(dr),$$
$$\psi_{3,R}(w) := \int\limits_{|r|>R} (e^{irw} - 1)\nu(dr) \text{ and}$$
$$\psi_4(w) := -\frac{a}{2}w^2 + i\gamma w.$$

Denoting by L_1, $L_{2,R}$, $L_{3,R}$ and L_4 the Lévy bases with the corresponding exponents, it is easily seen that also $f, g \in B(L_1) \cap B(L_{2,R}) \cap B(L_{3,R}) \cap B(L_4)$ and hence f and g are in the corresponding domains. Clearly, $\psi_1, \psi_{2,R} \in C^\infty(\mathbb{R})$ and

$$\psi_1'(w) = \int\limits_{|r|\le 1} ir(e^{irw} - 1)\nu(dr),$$
$$\psi_1''(w) = -\int\limits_{|r|\le 1} e^{irw} r^2 \nu(dr),$$
$$\psi_{2,R}'(w) = i\int\limits_{1<|r|\le R} e^{irw} r\nu(dr).$$

Assume for the moment that $z^T f \ge 0$, then we know

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \psi_1(z^T f(x))dx + \int\limits_{\mathbb{R}^d} \psi_{2,R}(z^T f(x))dx &= \int\limits_{\mathbb{R}^d} \int_0^{z^T f(x)} \int_0^y \psi_1''(\alpha)d\alpha\, dy\, dx + \int\limits_{\mathbb{R}^d} \int_0^{z^T f(x)} \psi_{2,R}'(y)\, dy\, dx \\ &= \int_0^\infty d_{z^T f}(y) \int_0^y \psi_1''(\alpha)\, d\alpha\, dy + \int_0^\infty d_{z^T f}(y)\psi_{2,R}'(y)\, dy,\end{aligned}$$

where we used that $f \in L^1(\mathbb{R}^d, \mathbb{R}^n) \cap L^2(\mathbb{R}^d, \mathbb{R}^n)$, boundedness of ψ_1'' and $\psi_{2,R}'$ and [21, Proposition 1.1.4] in order to apply Fubini's theorem. A similar reasoning when $z^T f \le 0$ and decomposing $z^T f$ into positive and negative parts then leads to

$$\begin{aligned}\psi_f(z) = &\int\limits_0^\infty d_{(z^T f)^+}(y) \int_0^y \psi_1''(\alpha)\, d\alpha\, dy + \int\limits_0^\infty d_{(z^T f)^-}(y) \int_0^{-y} \psi_1''(\alpha)\, d\alpha\, dy \\ &+ \int_0^\infty (d_{(z^T f)^+}(y)\psi_{2,R}'(y) + d_{(z^T f)^-}(y)\psi_{2,R}'(-y))\, dy - \frac{a}{2}\int\limits_{\mathbb{R}^d} (z^T f(x))^2 dx \\ &+ i\gamma z^T \int\limits_{\mathbb{R}^d} f(x)dx + \int\limits_{\mathbb{R}^d} \psi_{3,R}(z^T f(x))dx.\end{aligned} \tag{3.17}$$

A similar formula holds for $\psi_g(z)$. Since $|\psi_1''(\alpha)| \leq \int_{|r|\leq 1} r^2\nu(dr)$ and $|\psi_{2,R}'(\alpha)| \leq \int_{1<|r|\leq R} |r|\nu(dr)$, the claim will follow if we can show

$$\left|\int_{\mathbb{R}^d} \psi_{3,R}(z^T f(x))dx - \int_{\mathbb{R}^d} \psi_{3,R}(z^T g(x))dx\right|$$
$$\leq 3 \int_{|r|>R} |r| \int_0^{1/|r|} d_{z^T f}(\alpha)d\alpha\nu(dr) + 3 \int_{|r|>R} |r| \int_0^{1/|r|} d_{z^T g}(\alpha)d\alpha\nu(dr).$$

To see this, we observe

$$\int_{|r|>R} \int_{\mathbb{R}^d} (e^{irz^T f(x)} - 1)dx\nu(dr) = \int_{|r|>R} \int_{|rz^T f(x)|\leq 1} (e^{irz^T f(x)} - 1)dx\nu(dr)$$
$$+ \int_{|r|>R} \int_{|rz^T f(x)|>1} (e^{irz^T f(x)} - 1)dx\nu(dr)$$

and it follows

$$\left|\int_{|r|>R} \int_{\mathbb{R}^d} (e^{irz^T f(x)} - 1)dx\nu(dr)\right| \leq \int_{|r|>R} |r| \int_{|rz^T f(x)|\leq 1} |z^T f(x)|dx\nu(dr) \tag{3.18}$$
$$+ 2 \int_{|r|>R} d_{z^T f}\left(\frac{1}{|r|}\right)\nu(dr)$$
$$\leq 3 \int_{|r|>R} |r| \int_0^{1/|r|} d_{z^T f}(\alpha)d\alpha\nu(dr),$$

using that $|e^{ix} - 1| = |ix \int_0^1 e^{itx}dt| \leq |x|$ for $x \in \mathbb{R}$, and applying [21, Exercise 1.1.10, p.14] on the first term. An application of Fubini's theorem finishes the proof. □

As we now obtained for processes (X, Y) of the form (3.14) an estimate for the difference of their characteristic functions in terms of the distribution functions d_f, d_g, we can derive a sufficient condition when processes $(X_t)_{t\in\mathbb{R}}$ given by $X_t = \int_{\mathbb{R}^d} f(t,s)dL(s)$ with $f(t,\cdot) \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^2(\mathbb{R}^d,\mathbb{R})$ are almost periodically stationary. To do so, we need the following lemma.

Lemma 3.16. *Let $p \in [1,\infty)$ and $f, g \in L^p(\mathbb{R}^d,\mathbb{R})$. Denote*

$$\prod(f,g) := \{(\bar{f},\bar{g}) \in L^p(\mathbb{R}^d,\mathbb{R}) \times L^p(\mathbb{R}^d,\mathbb{R}) : d_f = d_{\bar{f}} \text{ and } d_g = d_{\bar{g}}\}.$$

Then

$$\int_0^\infty \alpha^{p-1}|d_f(\alpha) - d_g(\alpha)|d\alpha \leq (\|f\|_{L^p}^{p-1} + \|g\|_{L^p}^{p-1}) \inf_{(\bar{f},\bar{g})\in\prod(f,g)} \|\bar{f} - \bar{g}\|_{L^p}.$$

Proof. Let $(\bar f, \bar g) \in \prod(f,g)$. We see

$$p\int_0^\infty \alpha^{p-1}|d_f(\alpha)-d_g(\alpha)|d\alpha = p\int_0^\infty \alpha^{p-1}|d_{\bar f}(\alpha)-d_{\bar g}(\alpha)|d\alpha$$
$$= p\int_0^\infty \alpha^{p-1}\left|\int_{\mathbb{R}^d}\left(\mathbb{1}_{|\bar f(x)|>\alpha}(x)-\mathbb{1}_{|\bar g(x)|>\alpha}(x)\right)dx\right|d\alpha.$$

Since

$$|\mathbb{1}_{|\bar f(x)|>\alpha}(x)-\mathbb{1}_{|\bar g(x)|>\alpha}(x)| = \mathbb{1}_{|\bar f(x)|\le\alpha<|\bar g(x)|}(x)+\mathbb{1}_{|\bar g(x)|\le\alpha<|\bar f(x)|}(x)$$

we obtain by the triangular inequality

$$\begin{aligned}
p\int_0^\infty \alpha^{p-1}|d_f(\alpha)-d_g(\alpha)|d\alpha \le& p\int_{\mathbb{R}^d}\int_0^\infty \mathbb{1}_{|\bar f(x)|\le\alpha<|\bar g(x)|}\alpha^{p-1}d\alpha\,dx + p\int_{\mathbb{R}^d}\int_0^\infty \mathbb{1}_{|\bar g(x)|\le\alpha<|\bar f(x)|}\alpha^{p-1}d\alpha\,dx\\
=& p\int_{\mathbb{R}^d}\mathbb{1}_{|\bar f(x)|<|\bar g(x)|}\int_{|\bar f(x)|}^{|\bar g(x)|}\alpha^{p-1}d\alpha\,dx + p\int_{\mathbb{R}^d}\mathbb{1}_{|\bar g(x)|<|\bar f(x)|}\int_{|\bar g(x)|}^{|\bar f(x)|}\alpha^{p-1}d\alpha\,dx\\
=& \int_{\mathbb{R}^d}\mathbb{1}_{|\bar f(x)|<|\bar g(x)|}(|\bar g(x)|^p-|\bar f(x)|^p)dx + \int_{\mathbb{R}^d}\mathbb{1}_{|\bar g(x)|<|\bar f(x)|}(|\bar f(x)|^p-|\bar g(x)|^p)dx\\
=& \int_{\mathbb{R}^d}\Big||\bar f(x)|^p-|\bar g(x)|^p\Big|dx.
\end{aligned}$$

We know that $\Big||a|^p-|b|^p\Big| \le p(|a|^{p-1}+|b|^{p-1})\Big||a|-|b|\Big|$ for $p>1$ and for $p=1$ we use that $\Big||a|-|b|\Big| \le |a-b|$ for every $a,b\in\mathbb{R}$. Hence, we have

$$\begin{aligned}
\int_0^\infty \alpha^{p-1}|d_f(\alpha)-d_g(\alpha)|d\alpha \le& \int_{\mathbb{R}}|\bar f(x)-\bar g(x)|(|\bar f(x)|^{p-1}+|\bar g(x)|^{p-1})dx\\
\le& \|\bar f-\bar g\|_{L^p}(\|f\|_{L^p}^{p-1}+\|g\|_{L^p}^{p-1}),
\end{aligned}$$

where we used in the last line that $\|f\|_{L^p}=\|\bar f\|_{L^p}$ and the Hölder inequality with exponent p (when $p>1$). □

Corollary 3.17. Let $f:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ be a measurable function with $f(t,\cdot)\in L^1(\mathbb{R}^d,\mathbb{R})\cap L^2(\mathbb{R}^d,\mathbb{R})$ such that the function $T_f:\mathbb{R}\to L^1(\mathbb{R}^d,\mathbb{R})\cap L^2(\mathbb{R}^d,\mathbb{R})$ given by $T_f(t):=f(t,\cdot)$ is continuous in $L^1(\mathbb{R}^d,\mathbb{R})$ and $L^2(\mathbb{R}^d,\mathbb{R})$. Furthermore, let L be a Lévy basis with characteristic triplet (a,γ,ν) such that

$$\int_{|r|>1}|r|\sup_{t\in\mathbb{R}}\int_0^{1/|r|}d_{f(t,\cdot)}(\alpha)\,d\alpha\,\nu(dr)<\infty.$$

Then $X_t := \int_{\mathbb{R}^d} f(t,s)dL(s)$ is almost periodically stationary if for every $\varepsilon > 0$ there exist L_ε and $\tau \in [a, a+L_\varepsilon]$ as well as $s_1(\tau), s_2(\tau) \in \mathbb{R}$ for all $a \in \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \|f(t,\cdot) - f(t+\tau, \cdot + s_1(\tau))\|_{L^1} + \sup_{t\in\mathbb{R}} \|f(t,\cdot) - f(t+\tau, \cdot + s_2(\tau))\|_{L^2} < \varepsilon. \tag{3.19}$$

Proof. From Proposition 2.19 it follows that $(X_t)_{t\in\mathbb{R}}$ is in the domain of L and that $(X_t)_{t\in\mathbb{R}}$ is stochastically continuous. Next, observe that continuity of $t \mapsto T_f(t)$ in $L^2(\mathbb{R}^d, \mathbb{R})$ implies boundedness of $\|T_f(t)\|_{L^2}$ for t in compacts, and using (3.19) then implies $\sup_{t\in\mathbb{R}} \|T_f(t)\|_{L^2} < \infty$, and similarly for the L^1-norm. The rest follows directly from Theorem 3.10 b), Theorem 3.15 and Lemma 3.16, by observing that for fixed $t_1, \ldots, t_d \in \mathbb{R}$, and $z \in [-k,k]^d$, $k \in \mathbb{N}$,

$$\Big(\big(f(t+t_1+\tau,\cdot), \ldots, f(t+t_d+\tau,\cdot)\big)z\Big)^+$$

and

$$\Big(\big(f(t+t_1+\tau,\cdot+s_i(\tau)), \ldots, f(t+t_d+\tau,\cdot+s_i(\tau))\big)z\Big)^+, \; i = 1,2,$$

have the same distribution function and choosing $R > 1$ large enough such that

$$\sup_{z\in[-k,k]^d} \int_{|r|>R} |r| \sup_{t\in\mathbb{R}} \int_0^{1/|r|} d_{(f(t+t_1,\cdot),\ldots,f(t+t_d,\cdot))z}(\alpha) d\alpha \nu(dr)$$

becomes sufficiently small. □

A canonical choice for $s_1(\tau)$ and $s_2(\tau)$ in (3.19) is $s_1(\tau) = s_2(\tau) = 0$, which leads to almost periodicity in probability, as shown in Theorem 3.21 below. Another canonical choice is $s_1(\tau) = s_2(\tau) = \tau$. While the choice $s_1(\tau) = s_2(\tau) = 0$ is not possible to ensure almost periodic stationarity of Ornstein-Uhlenbeck processes under suitable conditions, the choice $s_1(\tau) = s_2(\tau) = \tau$ works for this class as will be shown in Section 6.

3.5 Almost periodicity in probability

In the following, we take up the definition of stochastic processes that are almost periodic in probability (see [4], [37]). Again, the definition is independent of the choice of the metric as long as it generates the same topology. We introduce the Ky-Fan metric, which induces the topology of convergence in probability (see [17, p. 289]) and derive conditions when processes of the form $X_t = \int_{\mathbb{R}^d} f(t,s)dL(s)$ are almost periodic in probability by estimating the Ky-Fan metric. Further, we state examples for such processes $(X_t)_{t\in\mathbb{R}}$. Let $(\Omega, \mathcal{F}, P)$

be a probability space and let $L^0(\Omega, \mathbb{R})$ be the space of measurable functions from Ω to $\mathbb{R}$. We identify two random variables which coincide P-almost surely. Observe that the Ky-Fan metric is in general not separable (in [14, Section IV.3, Theorem] sufficient conditions for it to be separable are treated), but as mentioned in Remark 3.2, separability is not needed for our considerations. Specialising Definition 3.1 now to this setting, we have:

Definition 3.18. Let d be a metric on $L^0(\Omega, \mathbb{R})$ which induces the convergence in probability. Then a stochastically continuous process $X = (X_t)_{t\in\mathbb{R}}$ is *almost periodic in probability* if the function $\mathbb{R} \to L^0(\Omega, \mathbb{R})$, $t \mapsto X_t$, is almost periodic with respect to the metric d.

Again, the specific form of the metric d is irrelevant as long as it induces convergence in probability. Typical examples are

$$d(X, Y) = \mathbb{E}\left(\frac{|X - Y|}{1 + |X - Y|}\right),$$
$$d(X, Y) = \mathbb{E}\min(1, |X - Y|),$$

or the Ky-Fan metric α, defined by

$$\alpha(X, Y) := \inf\{\varepsilon \geq 0 : P(|X - Y| > \varepsilon) \leq \varepsilon\}$$

(see [17, p. 289]). Further, it is known that $X = (X_t)_{t\in\mathbb{R}}$ is almost periodic in probability if and only if X is stochastically continuous and if for any $\varepsilon > 0$ and $\eta > 0$, there exists $L_{\varepsilon,\eta} > 0$ such that any interval of length $L_{\varepsilon,\eta}$ contains at least a $\tau \in \mathbb{R}$ for which

$$P(|X_t - X_{t+\tau}| > \eta) \leq \varepsilon \quad \text{for all } t \in \mathbb{R}$$

(see [4, p. 328], [37, Definition 2.3 and Remark 2.6 I)]). As expected, almost periodicity in probability implies almost periodically stationarity, see [4, Theorem 2.14].

Next, we prove estimates of the Ky-Fan metric $\alpha(X, Y)$, where X, Y are defined as in (3.14), in terms of the characteristic triplet of L, $\|f - g\|_{L^1}$, $\|f - g\|_{L^2}$ and the distribution functions of f and g.

Lemma 3.19. *Let L be a Lévy basis on $\mathbb{R}^d$ with characteristic triplet (a, γ, ν) and let (X, Y) be a two-dimensional infinitely divisible random vector given by*

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \int_{\mathbb{R}^d} \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} dL(t),$$

where $f, g : \mathbb{R}^d \to \mathbb{R}$ are in $D(L) \cap L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$.

(i) *It holds*

$$\alpha(X,Y) \leq \min\left(1, (14 \cdot I_R(f-g))^{\frac{1}{3}}\right),$$

for every $R > 1$, *where*

$$I_R(f-g) := \frac{1}{2}\|f-g\|_{L^2}^2 \int\limits_{|r|\leq 1} r^2\nu(dr) + \frac{a}{2}\|f-g\|_{L^2}^2 + |\gamma|\,\|f-g\|_{L^1}$$
$$+ \|f-g\|_{L^1} \int\limits_{1<|r|\leq R} |r|\nu(dr) + 6 \int\limits_{|r|>R} |r| \int\limits_0^{1/|r|} (d_f(\alpha)+d_g(\alpha))d\alpha\,\nu(dr).$$

(ii) *If* L *has finite second moment and vanishing first moment, i.e.* $\sigma^2 := \mathbb{E}L([0,1]^d)^2 < \infty$ *and* $\mathbb{E}L([0,1]^d) = 0$, *then*

$$\alpha(X,Y) \leq \min\left(1, \left(\sigma^2\|f-g\|_{L^2}^2\right)^{\frac{1}{3}}\right).$$

Proof. If the assumptions in (i) hold we have $\mathbb{E}(X-Y) = 0$ and $\mathrm{Var}(X-Y) = \sigma^2\|f-g\|_{L^2}^2$ and therefore obtain with Chebyshev's inequality

$$\alpha(X,Y) \leq \min\left(1, \left(\sigma^2\|f-g\|_{L^2}^2\right)^{\frac{1}{3}}\right).$$

In order to show (ii) we first calculate with [39, Lemma 1.6.2] for $\delta > 0$,

$$P(|X-Y|>\delta) \leq 7\cdot\frac{\delta}{2}\int\limits_{-1/\delta}^{1/\delta}(1-\Re(\varphi_{X-Y}(z)))\,dz \leq 7\cdot\frac{\delta}{2}\int\limits_{-1/\delta}^{1/\delta}|\psi_{f-g}(z)|\,dz, \tag{3.20}$$

where ψ_{f-g} is the characteristic exponent of the infinitely divisible random variable $X-Y$ and $\Re(\cdot)$ denotes the real part of a complex number; the last inequality follows from

$$|1-e^{\psi(z)}| = |\psi(z)|\left|\int\limits_0^1 e^{t\psi(z)}\,dt\right| \leq |\psi(z)|\int\limits_0^1 \left|e^{t\psi(z)}\right|\,dt \leq |\psi(z)|,$$

where we used that $z \mapsto e^{t\psi(z)}$ is a characteristic function (of an infinitely divisible distribution), hence $|e^{t\psi(z)}| \leq 1$. Theorem 3.15 gives us for $R > 1$ and $z \in \mathbb{R}$

$$|\psi_{f-g}(z)| \leq \int\limits_{|r|\leq 1} r^2\nu(dr)\int\limits_0^\infty \alpha(d_{z(f-g)^+}(\alpha)+d_{z(f-g)^-}(\alpha))d\alpha + \frac{a}{2}|z|^2\|f-g\|_{L^2}^2 + |\gamma||z|\|f-g\|_{L^1}$$
$$+ \int\limits_{1<|r|\leq R} |r|\nu(dr)\int\limits_0^\infty (d_{z(f-g)^+}(\alpha)+d_{z(f-g)^-}(\alpha))d\alpha$$

$$+3 \int_{|r|>R} |r| \int_0^{1/|r|} d_{|z|(f-g)}(\alpha) d\alpha \nu(dr).$$

Using that

$$\int_0^{\frac{1}{|r|}} d_{f-g}(\alpha/|z|) d\alpha = |z| \int_0^{\frac{1}{|zr|}} d_{f-g}(\alpha) d\alpha \leq |z| \int_0^{\frac{1}{|r|}} d_{f-g}(\alpha) d\alpha \tag{3.21}$$

for $|z| \geq 1$ and $d_{f-g}(\alpha/|z|) \leq d_{f-g}(\alpha)$ for $|z| < 1$, and $d_{f-g}(\alpha) \leq d_f(\frac{\alpha}{2}) + d_g(\frac{\alpha}{2})$ (see [21, Proposition 1.1.3]), we hence obtain with [21, Proposition 1.1.4]

$$\begin{aligned} |\psi_{f-g}(z)| &\leq \frac{1}{2}|z|^2 \int_{|r|\leq 1} r^2 \nu(dr) \|f-g\|_{L^2}^2 + \frac{a}{2}|z|^2 \|f-g\|_{L^2}^2 + |\gamma||z| \, \|f-g\|_{L^1} \\ &+ |z| \, \|f-g\|_{L^1} \int_{1<|r|\leq R} |r| \nu(dr) + 3(|z|+1) \int_{|r|>R} |r| \int_0^{1/|r|} \left(d_f\left(\frac{\alpha}{2}\right) + d_g\left(\frac{\alpha}{2}\right)\right) d\alpha \, \nu(dr) \\ &\leq (1 + \max(|z|^2, |z|)) I_R(f-g), \end{aligned}$$

since $\int_0^{1/|r|} d_f(\frac{\alpha}{2}) d\alpha \leq 2 \int_0^{1/|r|} d_f(\alpha) d\alpha$. Using

$$7 \frac{\delta}{2} \int_{-1/\delta}^{1/\delta} (1 + \max(|z|^2, |z|)) dz \leq 7 \left(1 + \frac{1}{\delta^2}\right) \qquad \text{for } \delta \in (0,1],$$

we obtain from (3.20)

$$\alpha(X,Y) \leq \inf \left\{ \delta \in [0,1] : 7 \cdot I_R(f-g) \leq \frac{\delta^3}{(\delta^2+1)} \right\} \leq \inf \left\{ \delta \in (0,1] : 14 \cdot I_R(f-g) \leq \delta^3 \right\}$$

and hence $\alpha(X,Y) \leq \min\left(1, (14 \cdot I_R(f-g))^{\frac{1}{3}}\right)$, finishing the proof. □

Remark 3.20. By assuming that the Lévy basis L has finite first moment, i.e. $\int_{|r|>1} |r| \, \nu(dr) < \infty$, the term $\int_{|r|>R} |r| \int_0^{1/|r|} d_{f-g}(\alpha) d\alpha \nu(dr)$ in (3.21) can be estimated by

$$\int_{|r|>1} |r| \int_0^{\infty} d_{f-g}(\alpha) d\alpha \nu(dr) = \|f-g\|_{L^1} \int_{|r|>1} |r| \nu(dr),$$

hence we also obtain the estimate

$$\alpha(X,Y) \leq \min\left(1, (14 \cdot \tilde{I}_R(f-g))^{\frac{1}{3}}\right),$$

for every $R > 1$, where

$$\tilde{I}_R(f-g) := \frac{1}{2}\|f-g\|_{L^2}^2 \int_{|r|\leq 1} r^2\nu(dr) + \frac{a}{2}\|f-g\|_{L^2}^2 + |\gamma|\,\|f-g\|_{L^1} + \|f-g\|_{L^1} \int_{1<|r|\leq R} |r|\nu(dr) + 2\,\|f-g\|_{L^1} \int_{|r|>1} |r|\nu(dr)$$

for $f, g \in D(L) \cap L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$.

As we have proven an estimate for the Ky-Fan metric, we can now give a condition when stochastic integrals are indeed almost periodic in probability.

Theorem 3.21. *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function with $f(t, \cdot) \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ such that the function $T_f : \mathbb{R} \to L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ given by $T_f(t) := f(t, \cdot)$ is continuous in $L^1(\mathbb{R}^d, \mathbb{R})$ and $L^2(\mathbb{R}^d, \mathbb{R})$. Furthermore, let L be a Lévy basis with characteristic triplet (a, γ, ν) such that*

$$\int_{|r|>1} |r| \sup_{t\in\mathbb{R}} \int_0^{1/|r|} d_{f(t,\cdot)}(\alpha)\; d\alpha\, \nu(dr) < \infty. \tag{3.22}$$

Then $X_t := \int_{\mathbb{R}^d} f(t, s)dL(s)$ is almost periodic in probability if for every $\varepsilon > 0$ there exist L_ε and $\tau \in [a, a + L_\varepsilon]$ for all $a \in \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \|f(t,\cdot) - f(t+\tau,\cdot)\|_{L^1} + \sup_{t\in\mathbb{R}} \|f(t,\cdot) - f(t+\tau,\cdot)\|_{L^2} < \varepsilon. \tag{3.23}$$

Proof. From Proposition 2.19 it follows that $(X_t)_{t\in\mathbb{R}}$ is in the domain of L and that $(X_t)_{t\in\mathbb{R}}$ is stochastically continuous. Using the inequality in Lemma 3.19 (ii) gives a uniform estimate by taking the supremum in $t \in \mathbb{R}$ and choosing $R > 1$ large enough such that $\int_{|r|>R} |r| \sup_{t\in\mathbb{R}} \int_0^{1/|r|} d_{f(t,\cdot)}(\alpha)\; d\alpha\, \nu(dr)$ becomes sufficiently small. This implies that $(X_t)_{t\in\mathbb{R}}$ is almost periodic in probability. □

In the next example we will construct functions satisfying the conditions of Theorem 3.21. For $a, b \in \mathbb{R}$, $a \vee b$ denotes the maximum between a and b.

Example 3.22. Let L be a Lévy basis on $\mathbb{R}$ with characteristic triplet (a, γ, ν), $g \in L^1(\mathbb{R}, \mathbb{R}) \cap L^2(\mathbb{R}, \mathbb{R})$ be a function satisfying

$$\int_{|r|>1} |r| \int_0^{\frac{1}{|r|}} d_g(\alpha)\, d\alpha\nu(dr) < \infty$$

and let $u : \mathbb{R} \to \mathbb{R}$ be an almost periodic function. In the examples below the stochastic continuity of $(X_t)_{t\in\mathbb{R}}$ follows directly from the continuity of the almost periodic function u by Proposition 2.19 ii).

(i) Assume that $f(t,x) = u(t)g(x)$. We observe $\{x \in \mathbb{R} : |u(t)g(x)| > \alpha\} \subset \{x \in \mathbb{R} : |g(x)| > \alpha/(1 \vee \|u\|_\infty)\}$ and obtain

$$\int_0^{1/|r|} d_{u(t)g}(\alpha)d\alpha \leq \int_0^{1/|r|} d_g(\alpha/(1 \vee \|u\|_\infty))d\alpha \leq (1 \vee \|u\|_\infty) \int_0^{1/|r|} d_g(\alpha)d\alpha,$$

implying that condition (3.22) is satisfied. Moreover, it is clear that

$$\|f(t,\cdot) - f(t+\tau,\cdot)\|_{L^p} = |u(t) - u(t+\tau)| \cdot \|g\|_{L^p}$$

for $p = 1, 2$, so that almost periodicity of u implies (3.23).

(ii) Assume that g is differentiable satisfying $g' \in L^1(\mathbb{R},\mathbb{R}) \cap L^2(\mathbb{R},\mathbb{R})$, then $X_t := \int_{\mathbb{R}} g(u(t) + x)dL(x)$ is almost periodic in probability, as $d_{g(u(t)+\cdot)} = d_g$, hence condition (3.22) is satisfied and

$$\begin{aligned}\int_{\mathbb{R}} |g(u(t)+x) - g(u(t+\tau)+x)|dx \leq& \int_{\mathbb{R}} \left| \int_{u(t)}^{u(t+\tau)} |g'(z+x)|dz \right| dx \\ =& \|g'\|_{L^1}|u(t) - u(t+\tau)|\end{aligned}$$

and also by Jensen's inequality

$$\begin{aligned}\int_{\mathbb{R}} |g(u(t)+x) - g(u(t+\tau)+x)|^2 dx \leq& \int_{\mathbb{R}} \left(\int_{u(t)}^{u(t+\tau)} |g'(z+x)|dz \right)^2 dx \\ \leq& |u(t) - u(t+\tau)| \int_{\mathbb{R}} \left| \int_{u(t)}^{u(t+\tau)} |g'(z+x)|^2 dz \right| dx \\ =& \|g'\|_{L^2}^2 |u(t) - u(t+\tau)|^2.\end{aligned}$$

(iii) Assume that g is differentiable and for $\omega(x) := xg'(x)$ assume that

$$\|\omega\|_{L^1} + \|\omega\|_{L^2} < \infty.$$

Furthermore, let $\varepsilon > 0$ and $u : \mathbb{R} \to \mathbb{R}$ be an almost periodic function such that

$$|u(t)| > \varepsilon, \qquad \text{for all } t \in \mathbb{R}.$$

We show that $(X_t)_{t\in\mathbb{R}}$ defined by $X_t = \int_{\mathbb{R}} g(u(t)x)dL(x)$ is almost periodic in probability. To see this, observe

$$\{x \in \mathbb{R} : g(u(t)x) > \alpha\} = \left\{ \frac{1}{u(t)}x : g(x) > \alpha \right\} = \frac{1}{u(t)}\{x : g(x) > \alpha\}$$

and hence

$$d_{g(u(t)\cdot)} = \frac{1}{u(t)} d_g \leq \frac{1}{\varepsilon} d_g.$$

Therefore, we obtain that the set $(g(u(t)\cdot))_{t\in\mathbb{R}}$ satisfies condition (3.22). Assume w.l.o.g. that $u(t+\tau) > u(t)$ and calculate

$$\begin{aligned}\int_{\mathbb{R}} |g(u(t)x) - g(u(t+\tau)x)|\, dx &= \int_{\mathbb{R}} \left| \int_{[u(t),u(t+\tau)]} x g'(zx)\, dz \right| dx \\ &\leq \int_{[u(t),u(t+\tau)]} \int_{\mathbb{R}} |xg'(zx)|\, dx\, dz.\end{aligned}$$

By substitution we obtain

$$\begin{aligned}\|g(u(t)\cdot) - g(u(t+\tau)\cdot)\|_{L^1} &\leq \left| \frac{1}{u(t+\tau)} - \frac{1}{u(t)} \right| \|\omega\|_{L^1(\mathbb{R})} \\ &\leq \frac{\|\omega\|_{L^1(\mathbb{R})}}{\varepsilon^2} |u(t+\tau) - u(t)|.\end{aligned}$$

For the L^2-norm we see with Jensen's inequality

$$\begin{aligned}\int_{\mathbb{R}} |g(u(t)x) - g(u(t+\tau)x)|^2 dx &\leq \int_{\mathbb{R}} x^2 \left| \int_{u(t)}^{u(t+\tau)} g'(xz) dz \right|^2 dx \\ &\leq \int_{\mathbb{R}} x^2 |u(t+\tau) - u(t)| \int_{u(t)}^{u(t+\tau)} |g'(xz)|^2 dz dx \\ &= |u(t+\tau) - u(t)| \|\omega\|_{L^2}^2 \int_{u(t)}^{u(t+\tau)} \frac{1}{z^3} dz \\ &\leq \frac{\|u\|_\infty \|\omega\|_{L^2}^2}{\varepsilon^4} |u(t+\tau) - u(t)|^2\end{aligned}$$

and obtain that $X_t = \int_{\mathbb{R}} g(u(t)x)\, dL(x)$ is almost periodic in probability.

(iv) In this example, we will see that we can omit the assumption on g being differentiable in (ii) and (iii) above. Given $g \in L^1(\mathbb{R},\mathbb{R}) \cap L^2(\mathbb{R},\mathbb{R})$ by the denseness of $C_c^\infty(\mathbb{R})$ there exists an approximating sequence of infinitely differentiable functions with compact support $(g_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that $\|g - g_n\|_{L^1} < \varepsilon/2$ and $\|g - g_n\|_{L^2} < \varepsilon/2$ for a fixed n large enough. Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be measurable. For $p = 1, 2$, we have

$$\|g(h(t,\cdot)) - g(h(t+\tau,\cdot))\|_{L^p}$$

$$\leq 2\sup_{t\in\mathbb{R}} \|g(h(t,\cdot)) - g_n(h(t,\cdot))\|_{L^p} + \|g_n(h(t+\tau,\cdot)) - g_n(h(t,\cdot))\|_{L^p}$$

and observe for the examples (ii) and (iii) above, i.e. $h(t,x) = u(t)+x$ and $h(t,x) = u(t)x$ that the second term can be estimated with the same calculations as above, whereas now it is sufficient to assume $g \in L^1(\mathbb{R},\mathbb{R}) \cap L^2(\mathbb{R},\mathbb{R})$ to obtain that $X_t = \int_{\mathbb{R}} g(h(t,x))\, dL(x)$ is almost periodic in probability. This holds since

$$\sup_{t\in\mathbb{R}} \|g(u(t)+\cdot)\|_{L^p} = \|g\|_{L^p}$$

and

$$\sup_{t\in\mathbb{R}} \|g(u(t)\cdot)\|_{L^p} = \sup_{t\in\mathbb{R}} \frac{1}{|u(t)|^{1/p}}\|g\|_{L^p} \leq \frac{1}{\varepsilon^{1/p}}\|g\|_{L^p}.$$

Further examples such as $g(h(t,x)) = \mathbb{1}_{[0,a(t)]}(x)$, where $a : \mathbb{R} \to (0,\infty)$ is almost periodic with $\inf_{t\in\mathbb{R}} a(t) > 0$ are now also covered. To see this, we notice

$$g(h(t,x)) = \mathbb{1}_{[0,a(t)]}(x) = \mathbb{1}_{[0,1]}(u(t)x),$$

where $u(t) := \frac{1}{a(t)}$ clearly is an almost periodic function.

3.6 Almost periodically stationary Ornstein-Uhlenbeck-type processes

In this section we obtain conditions for a Lévy-driven Ornstein-Uhlenbeck process equation to have an almost periodically stationary solution. A stochastic process $(L_t)_{t\in\mathbb{R}}$ with values in $\mathbb{R}$ is called a two-sided Lévy process if it starts at zero, i.e $L_0 = 0$ a.s., has independent and stationary increments and almost surely càdlàg paths (i.e. right-continuous paths with finite left limits). It naturally induces a Lévy basis $(L(A))_{A\in\mathcal{B}_b(\mathbb{R})}$ on $\mathbb{R}$ via $L(A) = \int_{\mathbb{R}} \mathbb{1}_A(s)\, dL_s$ for $A \in \mathcal{B}_b(\mathbb{R})$. An Ornstein-Uhlenbeck process $X^{(\mu)} = (X_t^{(\mu)})_{t\in\mathbb{R}}$ driven by a Lévy process L is a solution of the stochastic differential equation $dX_t = \mu X_t dt + dL_t$, where $\mu \in \mathbb{R}$. As an extension of this model Alkadour studied in [1] a periodic version of such a process. He looked at a stochastic differential equation of the form $dX_t = \mu(t)X_t dt + dL_t$, with a periodic function μ. Let us consider an Ornstein-Uhlenbeck-type process with almost periodic coefficient function $\mu : \mathbb{R} \to \mathbb{R}$, which solves the stochastic differential equation

$$\begin{aligned} dX_t &= \mu(t)X_t dt + dL_t, \\ X_0 &= X(0), \end{aligned} \tag{3.24}$$

where L is a two-sided real-valued Lévy process and $X(0)$ some starting random variable. By (3.24) we mean that $X_{t_2} - X_{t_1} = \int_{t_1}^{t_2} \mu(s)X_s\, ds + L_{t_2} - L_{t_1}$ is satisfied whenever $t_1 < t_2$. The stochastic differential equation (3.24) is solved in the next result:

Proposition 3.23. *Let $X(0)$ be given and let $\mu : \mathbb{R} \to \mathbb{R}$ be an almost periodic function. Then the unique solution of (3.24) is given by*

$$X_t = e^{Z_t}\left(X(0) + \int_0^t e^{-Z_s} dL(s)\right), \quad t \in \mathbb{R}, \tag{3.25}$$

where $Z_t := \int_0^t \mu(s)ds$ with the interpretation $\int_0^t = -\int_t^0$ for $t < 0$.

Proof. It follows from [27, Theorem 1] that a solution to (3.24) necessarily satisfies

$$X_{t_2} = e^{Z_{t_2} - Z_{t_1}}\left(X_{t_1} + \int_{t_1}^{t_2} e^{-(Z_s - Z_{t_1})} dL(s)\right) \tag{3.26}$$

for all $t_1 \le t_2$. Choosing $(t_1, t_2) = (0, t)$ when $t \ge 0$ and $(t_1, t_2) = (t, 0)$ when $t \le 0$ gives uniqueness. On the other hand, defining X_t by (3.25), it is easily seen that X satisfies (3.26), and again by [27, Theorem 1] we see that X is a solution of (3.24). □

The proof of Proposition 3.23 actually shows that it is enough to assume that the function $\mu : \mathbb{R} \to \mathbb{R}$ is càdlàg. Now let μ be again an almost periodic function. The question now is if there exists an almost periodically stationary version of this process and if this solution is unique, i.e. if $X(0)$ can be chosen such that $(X_t)_{t\in\mathbb{R}}$ becomes almost periodically stationary and if such a choice of $X(0)$ is unique. The related question in the case when $\mu < 0$ is constant (leading to the classical Ornstein-Uhlenbeck process) and stationary solutions are looked for is well-studied, and it is well-known that then stationary solutions exist if and only if $\int_{|s|>1} \log(|s|)\, \nu(ds) < \infty$, see e.g. [25], [41], [42] or [51]. Similarly, Alkadour studied in [1] periodic stationary solutions when $t \mapsto \mu(t)$ is periodic. Returning to the question of almost periodically stationary solutions when μ is almost periodic, the usual candidate is the process of the form

$$X_t := \int_{\mathbb{R}} f(t, s) dL(s), \tag{3.27}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(t, s) := \exp\left(\int_s^t \mu(u)\, du\right) \mathbb{1}_{(-\infty, t]}(s). \tag{3.28}$$

With Z of Proposition 3.23, this is written as $X_t = \int_{-\infty}^t e^{Z_t - Z_s} dL(s)$.

Let us show that X is well-defined and almost periodically stationary under some additional conditions on μ and L. Recall that since μ is almost periodic, the limit $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^T \mu(s)\, ds$ exists in $\mathbb{R}$ (e.g. [31, Section 2.3].

Theorem 3.24. *Let $\mu : \mathbb{R} \to \mathbb{R}$ be an almost periodic function such that $C := -\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mu(s)ds > 0$ and let f be defined as in (3.28). If the Lévy basis L with characteristic triplet (a, γ, ν) satisfies*

$$\int\limits_{|s|>1} \log(|s|)\,\nu(ds) < \infty, \tag{3.29}$$

then $f(t, \cdot) \in D(L)$, and the process $X = (X_t)_{t\in\mathbb{R}}$ defined by (3.27) is the unique almost periodically stationary solution of (3.24).

Proof. As μ is almost periodic, there exists a $T_0 > 0$ such that

$$\frac{1}{T}\int\limits_{x}^{T+x} \mu(s)ds < -\frac{C}{2} \text{ for all } T \geq T_0 \text{ and } x \in \mathbb{R}, \tag{3.30}$$

see [31, Section 2.3]. Now let $s < t$ and $s \in (kT_0, (k+1)T_0]$ and $t \in (lT_0, (l+1)T_0]$, where $l, k \in \mathbb{Z}$. We calculate that

$$\begin{aligned}\int_s^t \mu(u)du &= \sum_{m=k}^{l} \int\limits_{mT_0}^{(m+1)T_0} \mu(u)du - \int_{kT_0}^{s} \mu(u)du - \int_{t}^{(l+1)T_0} \mu(u)du \\ &\leq -(l+1-k)T_0C/2 + 2T_0 \sup_{x\in\mathbb{R}} |\mu(x)| \\ &\leq -(t-s)C/2 + 2T_0 \sup_{x\in\mathbb{R}} |\mu(x)| =: C/2(s-t) + C'. \end{aligned} \tag{3.31}$$

That $f(t, \cdot) \in D(L)$ and hence that X_t given by (3.27) is indeed defined is assured by Proposition 2.19 i) as it holds for $\alpha \in (0, 1)$

$$\begin{aligned} d_{f(t,\cdot)}(\alpha) &= \lambda^1 (\{x \in (-\infty, t] : |f(t, x)| > \alpha\}) \\ &= \lambda^1 \left(\left\{x \in (-\infty, t] : \int\limits_{x}^{t} \mu(u)\,du > \log(\alpha)\right\}\right) \\ &\leq \lambda^1 \left(\left\{x \in (-\infty, t] : C/2(x-t) + C' > \log(\alpha)\right\}\right) \\ &= \frac{2\,C'}{C} - \frac{2}{C}\log(\alpha) \end{aligned}$$

and therefore

$$\int\limits_{|s|>1} |s| \sup_{t\in\mathbb{R}} \int\limits_{0}^{1/|s|} d_{f(t,\cdot)}(\alpha)\,d\alpha\,\nu(ds) \leq \frac{2}{C}(1 + C')\,\nu(\mathbb{R} \setminus [-1, 1]) + \frac{2}{C} \int\limits_{|s|>1} \log(|s|)\,\nu(ds) < \infty.$$

Let us now show that X defined by (3.27) is almost periodically stationary. As μ is almost periodic, for every $\varepsilon > 0$ there exists L_ε and $\tau = \tau(a, \varepsilon) \in [a, a + L_\varepsilon]$ for all $a \in \mathbb{R}$ such that

$$|\mu(u) - \mu(u+\tau)| < \varepsilon \text{ for all } u \in \mathbb{R}.$$

For $|t - s| \leq \frac{1}{\varepsilon}$ and $\tau = \tau(a, \varepsilon)$ we get

$$\begin{aligned}
|f(t+\tau, s+\tau) - f(t,s)| &= \mathbb{1}_{(-\infty,t]}(s) \left| \exp\left(\int_s^t \mu(u)\, du \right) - \exp\left(\int_{s+\tau}^{t+\tau} \mu(u)\, du \right) \right| \\
&= \mathbb{1}_{(-\infty,t]}(s) \exp\left(\int_s^t \mu(u)\, du \right) \left| 1 - \exp\left(\int_s^t (\mu(u+\tau) - \mu(u))\, du \right) \right| \\
&\leq \mathbb{1}_{(-\infty,t]}(s) \exp\left(\int_s^t \mu(u)\, du \right) 2\varepsilon(t-s),
\end{aligned}$$

where we used that $|1 - e^x| \leq 2|x|$ for all $|x| \leq 1$. For general $s, t \in \mathbb{R}$ we obtain by the triangular inequality and (3.31)

$$|f(t+\tau, s+\tau) - f(t,s)| \leq 2\, \mathbb{1}_{(-\infty,t]}(s) e^{C/2(s-t)+C'}.$$

Let $\varepsilon \in (0,1)$. We observe for the L_1- and L_2-norms

$$\begin{aligned}
&\int_{\mathbb{R}} |f(t+\tau, s+\tau) - f(t,s)| ds \\
=&\int_{(-\infty, t-\frac{1}{\varepsilon}]} |f(t+\tau, s+\tau) - f(t,s)| ds + \int_{(t-\frac{1}{\varepsilon}, t]} |f(t+\tau, s+\tau) - f(t,s)| ds \\
\leq& 2\, e^{C'} \int_{(-\infty, t-\frac{1}{\varepsilon}]} e^{\frac{C}{2}(s-t)} ds + 2 \int_{(t-\frac{1}{\varepsilon}, t]} \exp\left(\int_s^t \mu(u)\, du \right) \varepsilon(t-s) ds \\
\leq& 2\, e^{C'} \int_{(-\infty, -\frac{1}{\varepsilon}]} e^{\frac{C}{2}s} ds + 2\varepsilon\, e^{C'} \int_{(-\frac{1}{\varepsilon}, 0]} e^{\frac{C}{2}s} |s| ds \\
\leq& \frac{4\, e^{C'}}{C} e^{-\frac{C}{2\varepsilon}} + 2\varepsilon\, e^{C'} \int_{(-\infty, 0]} e^{\frac{C}{2}s} |s| ds
\end{aligned}$$

and with similar steps we obtain

$$\int_{\mathbb{R}} |f(t+\tau, s+\tau) - f(t,s)|^2 ds \leq C'' \left(e^{-\frac{C}{\varepsilon}} + \varepsilon \right)$$

for some constant $C'' > 0$. Therefore, the almost periodic stationarity of X follows from Corollary 3.17 with $s_1(\tau) = s_2(\tau) = \tau$ (the continuity of $t \mapsto f(t, \cdot)$ in $L^1(\mathbb{R}, \mathbb{R}) \cap L^2(\mathbb{R}, \mathbb{R})$ is easily seen). To see the uniqueness of the almost periodically stationary solution, let $Y = (Y_t)_{t \in \mathbb{R}}$ be an almost periodically stationary solution. By Proposition 3.8 b),

$(\mathcal{L}_{Y_t})_{t\in\mathbb{R}}$ is uniformly tight and with [29, Lemma 4.9] and (3.30) we observe $e^{-Z_t}Y_t \to 0$ in probability as $t \to -\infty$, where Z is defined as in Proposition 3.23. Furthermore, it holds $\int^t e^{-Z_s}\,dL_s \to \int_{-\infty}^0 e^{-Z_s}\,dL_s$ in probability as $t \to -\infty$. Hence, by (3.25), Y_0 is the probability limit of $e^{-Z_t}Y_t - \int_0^t e^{-Z_s}\,dL_s$ as $t \to \infty$, which is equal to $\int_{-\infty}^0 e^{-Z_s}\,dL(s)$. This shows that Y_0 and hence Y are unique. □

Remark 3.25. In Theorem 3.24 we saw that if the (asymptotic) mean $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T \mu(s)ds$ is negative, it is easy to construct a unique almost periodically stationary solution. In the case that the mean is positive, one can also find by the same methods a unique almost periodically stationary solution, which is then given by $X_t = -\int_t^\infty e^{Z_t - Z_s} dL(s)$ under the sufficient and necessary condition (3.29). If the mean is 0, we do not know if this implies that there exists no stationary solution, but this is the case if the function μ is periodic, see [1].

3.7 Central limit theorem for almost periodically stationary processes

In this section we prove a central limit theorem for m-dependent and L^2-uniformly integrable almost periodically stationary processes. We start with the following lemma (recall that λ^d denotes the d-dimensional Lebesgue measure).

Lemma 3.26. *Let $(X_t)_{t\in\mathbb{R}^d}$ be an L^2-uniformly integrable and jointly measurable stochastic process. Denote $Y_k := \int_{A_k} X_t\,dt$, where $(A_k)_{k\in\mathbb{Z}}$ is a sequence of Borel measurable sets with $C := \sup_{k\in\mathbb{Z}} \lambda^d(A_k) < \infty$. Then $(Y_k)_{k\in\mathbb{Z}}$ is L^2-uniformly integrable.*

The joint measurability of X implies that the paths $t \mapsto X_t(\omega)$ are Borel-measurable and the integrals $\int_{A_k} X_t\,dt$ exist almost surely pointwise, since

$$\mathbb{E}\left(\int_{A_k} |X_t|\,dt\right) = \int_{A_k} \mathbb{E}(|X_t|)\,dt \le C \sup_{s\in\mathbb{R}} \mathbb{E}(|X_s|) < \infty$$

by Fubini's theorem. Fubini's theorem then also implies that the Y_k, $k \in \mathbb{Z}$, are random variables.

Proof. At first we define $Z_k := \int_{A_k} X_t^2\,dt$ and observe by Jensen's inequality for $\varepsilon > 0$

$$\begin{aligned} Y_k^2 \mathbb{1}_{Y_k^2>\varepsilon} &\le Z_k \mathbb{1}_{|Z_k|>\frac{\varepsilon}{C}} \lambda^d(A_k) \\ &= \left(\int_{A_k} X_t^2 \mathbb{1}_{X_t^2>\frac{\varepsilon}{2C^2}}\,dt + \int_{A_k} X_t^2 \mathbb{1}_{X_t^2\le\frac{\varepsilon}{2C^2}}\,dt\right) \mathbb{1}_{|Z_k|>\frac{\varepsilon}{C}} \lambda^d(A_k) \end{aligned}$$

$$\leq \left(\int_{A_k} X_t^2 \mathbb{1}_{X_t^2 > \frac{\varepsilon}{2C^2}} \, dt + \int_{A_k} X_t^2 \mathbb{1}_{X_t^2 > \frac{\varepsilon}{2C^2}} \, dt \right) \mathbb{1}_{|Z_k| > \frac{\varepsilon}{C}} \lambda^d(A_k)$$
$$\leq 2\lambda^d(A_k) \int_{A_k} X_t^2 \mathbb{1}_{X_t^2 > \frac{\varepsilon}{2C^2}} \, dt.$$

It follows

$$\mathbb{E}(Y_k^2 \mathbb{1}_{Y_k^2 > \varepsilon}) \leq 2\lambda^d(A_k) \int_{A_k} \mathbb{E} X_t^2 \mathbb{1}_{X_t^2 > \frac{\varepsilon}{2C^2}} \, dt$$
$$\leq 2\lambda^d(A_k) C \sup_{t \in \mathbb{R}^d} \mathbb{E}(X_t^2 \mathbb{1}_{X_t^2 > \frac{\varepsilon}{2C^2}}) \to 0$$

for $\varepsilon \to \infty$, since $(X_t^2)_{t \in \mathbb{R}^d}$ is uniformly integrable. □

For $m > 0$ a stochastic process $(X_t)_{t \in \mathbb{R}}$ is called m-dependent, if $(X_t)_{t \leq u}$ and $(X_t)_{t > u+m}$ are independent for all $u \in \mathbb{R}$. In the following, we prove our central limit theorem for m-dependent almost periodically stationary processes.

Theorem 3.27. *Let $m > 0$ and $(X_t)_{t \in \mathbb{R}}$ be an m-dependent, jointly measurable, mean zero almost periodically stationary process which is L^2-uniformly integrable. Then*

$$g(s) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{E} X_{s+t} X_t \, dt$$

exists for each $s \in \mathbb{R}$ as a limit in $\mathbb{R}$, the function $g : \mathbb{R} \to \mathbb{R}$ is measurable and bounded, $\int_{-m}^{m} g(s) \, ds \geq 0$ and

$$\frac{1}{\sqrt{2T}} \int_{-T}^{T} X_t \, dt \xrightarrow{d} N\left(0, \int_{-m}^{m} g(s) \, ds\right) \quad \text{as } T \to \infty. \tag{3.32}$$

Proof. Let the process X be defined on the probability space $(\Omega, \mathcal{F}, P)$. By Proposition 3.8 d), X is almost periodically correlated, so that $t \mapsto \mathbb{E} X_{s+t} X_t$ is an almost periodic function for each $s \in \mathbb{R}$. By the mean-value property of almost periodic functions (e.g. [31, Section 2.3]), the limit $g(s) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{E} X_{s+t} X_t \, dt$ exists in $\mathbb{R}$ for each $s \in \mathbb{R}$. That g is bounded follows from the L^2-uniform integrability of X and the Cauchy-Schwarz inequality. To see that g is measurable, observe that the functions $\mathbb{R}^2 \times \Omega \to \mathbb{R}$, $(s, t, \omega) \mapsto X_s(\omega)$ and $\mathbb{R}^2 \times \Omega \to \mathbb{R}$, $(s, t, \omega) \mapsto X_t(\omega)$ are $\mathcal{B}(\mathbb{R}^2) \otimes \mathcal{F}$-measurable by joint measurability of X, hence so is $(s, t, \omega) \mapsto X_t(\omega) X_s(\omega)$. Fubini's theorem implies measurability of $\mathbb{R}^2 \ni (s, t) \mapsto \mathbb{E} X_t X_s$ (as well as the almost sure existence of pathwise integrals like $\int_{-T}^{T}\int_{-T}^{T} X_t(\omega)X_s(\omega)\,dt\,ds$), hence $\mathbb{R} \ni s \mapsto \frac{1}{2T}\int_{-T}^{T} \mathbb{E}X_{s+t}X_t\,dt$ is measurable for each $T > 0$, hence so is g as a limit of measurable functions. For $T > 0$ denote

$$V_T := \left(\mathbb{E}\left(\frac{1}{\sqrt{2T}}\int_{-T}^{T} X_t\,dt\right)^2\right)^{1/2} \geq 0.$$

As $(X_t)_{t\in\mathbb{R}}$ is m-dependent, we obtain for $T > m$

$$V_T^2 = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} \mathbb{E}X_tX_s\,dt\,ds = \frac{1}{2T}\int_{-m}^{m}\int_{-T}^{T} \mathbb{E}X_tX_{t+s}\mathbb{1}_{|t+s|\leq T}\,dt\,ds.$$

Since

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-m}^{m}\int_{-T}^{T} \mathbb{E}(X_tX_{t+s})(1-\mathbb{1}_{|t+s|\leq T})\,dt\,ds = 0$$

an application of Lebesgue's dominated convergence theorem shows

$$\lim_{T\to\infty} V_T^2 = \lim_{T\to\infty}\frac{1}{2T}\int_{-m}^{m}\int_{-T}^{T} \mathbb{E}X_{s+t}X_t\,dt\,ds = \int_{-m}^{m} g(s)\,ds =: V_\infty^2,$$

so that $V_\infty^2 = \int_{-m}^{m} g(s)\,ds \geq 0$.
To show (3.32), we distinguish the cases $V_\infty^2 = 0$ and $V_\infty^2 > 0$. If $V_\infty^2 = 0$, then the above calculations show that $\frac{1}{\sqrt{2T}}\int_{-T}^{T} X_t\,dt$ converges in L^2 to 0, hence also in distribution. Hence, we may assume $V_\infty^2 > 0$. Equation (3.32) is then equivalent to

$$\frac{1}{V_T}\frac{1}{\sqrt{2T}}\int_{-T}^{T} X_t\,dt \xrightarrow{d} N(0,1), \quad T\to\infty. \tag{3.33}$$

Since $\frac{1}{V_T}\frac{1}{\sqrt{2T}}\left(\int_{\lfloor T\rfloor}^{T} X_t\,dt + \int_{-T}^{-\lfloor T\rfloor} X_t\,dt\right)$ converges in probability to 0 as $T\to\infty$ by uniform integrability (here, $\lfloor x\rfloor$ denotes the integer part of $x \in \mathbb{R}$), it is clearly enough to prove (3.33) where T is restricted to the natural numbers. For that, we will apply a result of Heinrich [23, Theorem 2, p. 135]. Denote (for $n \in \mathbb{N}$ large enough such that $V_n > 0$)

$$\begin{aligned} W_n &:= \{-n, -n+1, \ldots, n-1\} \subset \mathbb{Z}, \\ Y_k &:= \int_k^{k+1} X_t\,dt,\ k\in\mathbb{Z}, \\ U_{n,k} &:= \frac{1}{V_n}\frac{1}{\sqrt{2n}}Y_k,\ k\in W_n \quad \text{and} \\ S_n &:= \sum_{k\in W_n} U_{n,k}. \end{aligned}$$

Then $|W_n| \to \infty$, $\mathbb{E}U_{n,k} = 0$, and $(U_{n,k})_{k\in W_n}$ is m_n-dependent with $m_n := \lfloor m \rfloor + 2$ (independent of n). Further, $S_n = \frac{1}{V_n}\frac{1}{\sqrt{2n}}\int_{-n}^{n} X_t\,dt$ satisfies $\mathbb{E}S_n^2 = 1$ by definition of V_n. Next, since $(Y_k)_{k\in\mathbb{Z}}$ is L^2-uniformly integrable by Lemma 3.26, we have

$$\sup_{n\in\mathbb{N}} \sum_{k\in W_n} \mathbb{E}U_{n,k}^2 \le \sup_{n\in\mathbb{N}} \left(\frac{1}{V_n^2}\frac{1}{2n} 2n \right) \sup_{k\in\mathbb{N}} \mathbb{E}Y_k^2 < \infty$$

and

$$\begin{aligned} \sum_{k\in W_n} \mathbb{E}\left(U_{n,k}^2 \mathbb{1}_{|U_{n,k}|\ge\varepsilon}\right) &= \frac{1}{V_n^2 2n} \sum_{k=-n}^{n-1} \mathbb{E}\left(Y_k^2 \mathbb{1}_{|Y_k|\ge \varepsilon V_n\sqrt{2n}}\right) \\ &\le \frac{1}{V_n^2} \sup_{k\in\mathbb{Z}} \mathbb{E}\left(Y_k^2 \mathbb{1}_{|Y_k|\ge \varepsilon V_n\sqrt{2n}}\right) \to 0 \end{aligned}$$

as $k \to \infty$ for every $\varepsilon > 0$. Hence, all assumptions of Theorem 2 in [23] are satisfied, yielding

$$\frac{1}{V_n}\frac{1}{\sqrt{2n}} \int_{-n}^{n} X_t\,dt = S_n \xrightarrow{d} N(0,1), n \to \infty,$$

i.e. (3.33). □

4 Generalized uniform integrability of integrals of the form $\int f(t,s)\,dL(s)$

In this chapter we derive a sufficient condition for processes $(X_t)_{t\in T}$ defined by the stochastic integral $X_t := \int_{\mathbb{R}^d} f(t,s)\,dL(s)$ to be generalized uniform integrable, where $f(t,\cdot) \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^2(\mathbb{R}^d,\mathbb{R})$ is deterministic and L is a Lévy basis. By generalized uniform integrability we mean that $(g(X_t))_{t\in T}$ is uniformly integrable for suitable submultiplicative functions g. As an application we discuss the generalized uniform integrability of Ornstein-Uhlenbeck-type processes and of the mild solution of the electric Schrödinger equation.

The concept of uniform integrability of stochastic processes or sequences is deeply connected with convergence of random variables. A classical extension of Lebesgue's dominated integration theorem is that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges to a random variable X in $L^1(\Omega)$ if and only if X_n converges in probability to X and the sequence $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable, see [50]. This result was used to derive several limit theorems for martingales and stochastic processes (see [29], [50]). Furthermore, in our central limit theorem for almost periodically stationary processes from Chapter 3 uniform integrability is assumed and uniform integrability is the missing link between convergence in distribution and convergence in the Wasserstein metrics (see [46, Theorem 7.12]). Given a submultiplicative and locally bounded function g, it is well-known that a Lévy process $(L_t)_{t\geq 0}$ with Lévy measure ν has finite g-moment for every $t > 0$ (equivalently, some $t > 0$) if and only if ν has finite g-moment, i.e $\int_{|x|>1} g(x)\,\nu(dx) < \infty$ (see [40, Theorem 25.3]). Hence, the finiteness of the g-moment is not a time dependent distributional property in the class of Lévy processes. This is also the case when considering the g-uniform integrability of sequences of infinitely divisible measures $(\mu_n)_{n\in\mathbb{N}}$, which are relatively compact, see [28, Theorem 2]. In [28] it was shown that if g is continuous and submultiplicative, and $(\mu_n)_{n\in\mathbb{N}}$ relatively compact, then $(\mu_n)_{n\in\mathbb{N}}$ is g-uniformly integrable if and only if the Lévy measure ν is g-uniformly integrable. Furthermore, a nice overview for the uniform integrability of Lévy processes $(L_t)_{t\geq 0}$ (on compact sets) can be found in [5] and for results for Lévy-type processes, see [30].

In this chapter we at first characterize the g-uniform integrability of infinitely divisible random processes in terms of their characteristic triplets, where g is a submultiplicative function. Our result follows immediately from [28, Theorem 2], where we assume additionally that our submultiplicative function g satisfies $\lim_{|x|\to\infty} g(x) = \infty$, whereas in [28, Theorem 2] relative compactness is assumed instead. As a concrete application we study stochastic integrals with deterministic kernel and prove sufficient conditions for uniform integrability of such processes. In the following, T is an arbitrary non-empty index set.

Definition 4.1. Let $g : \mathbb{R} \to [0,\infty)$ be a non-negative measurable function. A stochastic process $(X_t)_{t\in T}$ is *g-uniformly integrable* if to each $\varepsilon > 0$ there exists a $k > 0$ such that

$$\sup_{t\in T} \mathbb{E} g(X_t) \mathbb{1}_{g(X_t)>k} < \varepsilon.$$

If $g(x) = |x|^p$, we also say that $(X_t)_{t\in T}$ is *L^p-uniformly integrable*. Obviously, $(X_t)_{t\in T}$ is g-uniformly integrable if $(g(X_t))_{t\in T}$ is L^1-uniformly integrable, i.e. if $(g(X_t))_{t\in T}$ is uniformly integrable. A function g on $\mathbb{R}^d$ is called *submultiplicative* if it is non-negative, measurable and there exists a constant $C > 0$ such that

$$g(x+y) \le Cg(x)g(y) \qquad \text{for all } x, y \in \mathbb{R}^d. \tag{4.1}$$

For $p \ge 1$ the following functions are submultiplicative

$$\max(|x|^p, 1),\ \log(\max(|x|, e)),\ \exp(|x|^{1/p}).$$

If $g(x_0) = 0$ for some $x_0 \in \mathbb{R}^d$, it follows $g \equiv 0$. Henceforth, we assume $g > 0$.

We recall that for a family $(X_t)_{t\in T}$ of $\mathbb{R}$-valued infinitely divisible random variables, its *characteristic functions* φ_{X_t} are uniquely determined by the *Lévy-Khintchine formula*

$$\varphi_{X_t}(z) = \mathbb{E}e^{izX_t} = \exp\left(i\gamma_t z - \frac{1}{2}a_t z^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbb{1}_{[-1,1]}(x))\nu_t(dx) \right), \quad z \in \mathbb{R},$$

where (a_t, γ_t, ν_t) is the *characteristic triplet* of X_t, $t \in T$. Let $\mathcal{P}(\mathbb{R}^d)$ denote the collection of all probability measures on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel-σ-algebra on $\mathbb{R}^d$.
In [40, Exercise 12.5] a characterization of relative compactness of a family of infinitely divisible distributions in terms of its characteristic triplets is stated. Namely, a family of infinitely divisible distributions $(\mu_t)_{t\in T} \subset \mathcal{P}(\mathbb{R}^d)$ with characteristic triplets (a_t, γ_t, ν_t) is relatively compact (with respect to the topology induced by weak convergence) if and only if

$$\sup_{t\in T} \left(|\gamma_t| + a_t + \int_{\mathbb{R}^d} \min(|x|^2, 1)\nu_t(dx) \right) < \infty \quad \text{and} \tag{4.2}$$

$$\lim_{k\to\infty} \sup_{t\in T} \int_{|x|>k} \nu_t(dx) = 0. \tag{4.3}$$

We apply the result above to reformulate [28, Theorem 2], which characterizes the uniform integrability of processes $(g(X_t))_{t\in T}$, where $(X_t)_{t\in T}$ is infinitely divisible, $(P_{X_t})_{t\in T} \subset \mathcal{P}(\mathbb{R})$ is relatively compact and g is a continuous submultiplicative function. More precisely, their theorem reads:

Theorem 4.2. *[28, Theorem 2] Let $T = \mathbb{N}$ and $(X_t)_{t\in T}$ be a relatively compact sequence of infinitely divisible random variables on $\mathbb{R}$ with characteristic triplet (a_t, γ_t, ν_t). Let $g : \mathbb{R} \to (0, \infty)$ be a continuous submultiplicative function. Then $(X_t)_{t\in T}$ is g-uniformly integrable if and only if*

$$\lim_{k\to\infty} \sup_{t\in T} \int_{g(x)>k} g(x)\nu_t(dx) = 0. \tag{4.4}$$

In the following corollary we drop the assumption of $(P_{X_t})_{t\in T}$ being relatively compact and g being continuous and assume instead that the submultiplicative function g is locally bounded and satisfies additionally $\lim_{|x|\to\infty} g(x) = \infty$. Nevertheless, this result follows directly from Theorem 4.2.

Corollary 4.3. Let $(X_t)_{t\in T}$ be a family of infinitely divisible $\mathbb{R}$-valued random variables with characteristic triplets (a_t, γ_t, ν_t). Furthermore, let $g : \mathbb{R} \to [0, \infty)$ be a locally bounded submultiplicative function which satisfies $\lim_{|x|\to\infty} g(x) = \infty$. Then $(X_t)_{t\in T}$ is g-uniformly integrable if and only if

$$\lim_{k\to\infty} \sup_{t\in T} \int_{|x|>k} g(x)\nu_t(dx) = 0, \tag{4.5}$$

and (4.2) are satisfied.

Proof. First observe that Theorem 4.2 continues to hold for general index sets T rather than only for $\mathbb{N}$, which follows easily by considering appropriate subsequences. Secondly, the continuity assumption in Theorem 4.2 is not necessary, since by [5, Lemma 2] and local boundedness of g, we can find a submultiplicative smooth function $\tilde{g}$ and constants $c_1, c_2 > 0$ such that $c_1 g(x) \leq \tilde{g}(x) \leq c_2 g(x)$ for all $x \in \mathbb{R}^d$. Since neither the condition (4.4) nor uniform integrability of $(g(X_t))_{t\in T}$ is affected by replacing g by $\tilde{g}$, Theorem 4.2 also holds for locally bounded submultiplicative functions.
Assume now that $(g(X_t))_{t\in T}$ is L^1-uniformly integrable. Then also $(\max(1, g(X_t))_{t\in T}$ is L^1-uniformly integrable and for every $\varepsilon > 0$ there exists a $k = k(\varepsilon) > 0$ such that

$$P(g(X_t) > k) = \mathbb{E}\mathbb{1}_{\{g(X_t)>k\}} \leq \mathbb{E}\max(1, g(X_t))\mathbb{1}_{\{\max(1,g(X_t))>k\}} < \varepsilon.$$

Hence $(P_{g(X_t)})_{t\in T}$ is tight and as a consequence of $\lim_{|x|\to\infty} g(x) = \infty$, also $(P_{X_t})_{t\in T}$ is tight. By Prokhorov's theorem and [40, Exercise 12.5] it follows (4.2) and applying Theorem 4.2 gives us (4.5).
For the converse, consider the case where (4.2) and (4.5) are satisfied. Since g satisfies $\lim_{|x|\to\infty} g(x) = \infty$ it follows (4.3) from (4.5) so that $(P_{X_t})_{t\in T}$ is tight by [40, Exercise 12.5] and applying Theorem 4.2 then gives us the uniform integrability of $(g(X_t))_{t\in T}$. □

We now apply Corollary 4.3 to obtain sufficient conditions for the generalized uniform integrability of stochastic integrals $\int_{\mathbb{R}^d} f(t,x)\,dL(x)$, where L is a Lévy basis and $f: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a suitable measurable function. Next, we discuss the uniform integrability of the mild solution of the generalized electric Schrödinger equation driven by a Lévy basis (see Section 2.3.1) and of Ornstein-Uhlenbeck-type processes (see Section 3.5). In the following we denote by $\mathcal{B}_b(\mathbb{R}^d)$ the set of all Borel sets, which are bounded. We recall the definition of a Lévy basis.

Definition 4.4. A *Lévy basis* is a family $(L(A))_{A\in\mathcal{B}_b(\mathbb{R}^d)}$ of real valued random variables such that

i) $L\left(\bigcup_{n\in\mathbb{N}_0} A_n\right) = \sum_{n\in\mathbb{N}_0} L(A_n)$ a.s. for pairwise disjoint sets $(A_n)_{n\in\mathbb{N}_0} \subset \mathcal{B}_b(\mathbb{R}^d)$ with $\bigcup_{n\in\mathbb{N}_0} A_n \in \mathcal{B}_b(\mathbb{R}^d)$.

ii) $L(A_i)$ are independent for pairwise disjoint sets $A_1,\dots,A_n \in \mathcal{B}_b(\mathbb{R}^d)$ for every $n \in \mathbb{N}$.

iii) There exists $a \in [0,\infty)$, $\gamma \in \mathbb{R}$ and a Lévy measure ν on $\mathbb{R}$ such that

$$\mathbb{E}e^{izL(A)} = \exp(\psi_L(z)\lambda^d(A)), \quad A \in \mathcal{B}_b(\mathbb{R}^d),$$

where

$$\psi_L(z) = i\gamma z - \frac{1}{2}az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbb{1}_{[-1,1]}(x))\nu(dx), \quad z \in \mathbb{R}.$$

The triplet (a,γ,ν) is called *characteristic triplet* of L and ψ_L its *characteristic exponent.*

Given a Lévy basis L with characteristic triplet (a,γ,ν), where $\nu \neq 0$, and a measurable function $f: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ we consider the stochastic integral $(X_t)_{t\in\mathbb{R}^d}$ given by

$$X_t = \int_{\mathbb{R}^d} f(t,x)dL(x), \tag{4.6}$$

where the integral is in the sense of Rajput-Rosinski (see [38]). Provided the integral exists, $(X_t)_{t\in\mathbb{R}^d}$ is infinitely divisible with characteristic triplets (a_t,γ_t,ν_t), where

$$\gamma_t = \int_{\mathbb{R}^d} \left(\gamma f(t,x) + \int_{\mathbb{R}} rf(t,x)\left(\mathbb{1}_{|rf(t,x)|\le 1} - \mathbb{1}_{|r|\le 1}\right)\nu(dr)\right) dx, \tag{4.7}$$

$$a_t = a^2 \int_{\mathbb{R}^d} |f(t,x)|^2\,dx, \tag{4.8}$$

$$\nu_t(B) = (\lambda^d \otimes \nu)\left((x,r) \in \mathbb{R}^d \times \mathbb{R} : f(t,x)r \in B \setminus \{0\}\right), \quad B \in \mathcal{B}(\mathbb{R}), \tag{4.9}$$

with λ^d as the d-dimensional Lebesgue measure. By defining for a measurable function $f: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ its *distribution function* $d_{f(t,\cdot)}(\alpha) := \lambda^d(\{x \in \mathbb{R}^d : |f(t,x)| > \alpha\})$ for

$\alpha > 0$ and $t \in \mathbb{R}^d$, a sufficient condition for the integrability of $f(t,\cdot)$ with respect to L is that $f(t,\cdot) \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^2(\mathbb{R}^d,\mathbb{R})$ for all $t \in \mathbb{R}^d$ and

$$\int\limits_{|r|>1} |r| \sup_{t\in\mathbb{R}^d} \int\limits_0^{1/|r|} d_{f(t,\cdot)}(\alpha)\, d\alpha\, \nu(dr) < \infty, \tag{4.10}$$

see Proposition 2.19 i).

In the following, we derive conditions under which (4.6) is g-uniformly integrable, where $g : \mathbb{R} \to (0,\infty)$ is a submultiplicative function with further properties.

Theorem 4.5. *Let L be a Lévy basis with characteristic triplet (a,γ,ν), $\nu \neq 0$ and let $g : \mathbb{R} \to (0,\infty)$ be a submultiplicative function such that $\lim_{|x|\to\infty} g(x) = \infty$ and which satisfies*

$$g(x) \leq C(1+|x|^p), \quad x \in \mathbb{R}, \tag{4.11}$$

where $C > 0$ and $p \in [2,\infty)$. Furthermore, let $(X_t)_{t\in\mathbb{R}^d}$ be family of $\mathbb{R}$-valued infinitely divisible random variables with characteristic triplets (a_t,γ_t,ν_t) given by (4.6), where $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function with $f(t,\cdot) \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^p(\mathbb{R}^d,\mathbb{R})$ for every $t \in \mathbb{R}^d$ and

$$\sup_{t\in\mathbb{R}^d} \|f(t,\cdot)\|_{L^1} + \sup_{t\in\mathbb{R}^d} \|f(t,\cdot)\|_{L^p} < \infty. \tag{4.12}$$

If it holds (4.10) and there exists $\beta \geq 1$ such that

$$\lim_{k\to\infty} \sup_{t\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |f(t,x)|^p \mathbb{1}_{|f(t,x)|>k}\, dx = 0 \quad \text{and} \tag{4.13}$$

$$\lim_{k\to\infty} \int\limits_{|r|>\beta} \sup_{t\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} g(f(t,x)r) \mathbb{1}_{|f(t,x)r|>k}\, dx\, \nu(dr) = 0, \tag{4.14}$$

then $(X_t)_{t\in\mathbb{R}^d}$ is g-uniformly integrable.

Proof. The existence of $(X_t)_{t\in\mathbb{R}^d}$ follows directly from (4.10), and from (4.11) it follows that g is locally bounded. We observe from (4.7) and (4.8) that

$$\sup_{t\in\mathbb{R}^d} a_t = a^2 \sup_{t\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |f(t,x)|^2\, dx < \infty$$

and

$$\sup_{t\in\mathbb{R}^d} |\gamma_t| \leq \sup_{t\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \left| \gamma f(t,x) + \int\limits_{\mathbb{R}} r f(t,x) \left(\mathbb{1}_{|rf(t,x)|\leq 1} - \mathbb{1}_{|r|\leq 1} \right) \nu(dr) \right| dx$$

$$
\begin{aligned}
&\le |\gamma| \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|\,dx + \int_{|r|>1} |r| \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)| \mathbb{1}_{|f(t,x)r|\le 1}\,dx\,\nu(dr) \\
&\quad + \int_{|r|\le 1} |r| \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)| \mathbb{1}_{|f(t,x)r|>1}\,dx\,\nu(dr) \\
&\le |\gamma| \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|\,dx + \int_{|r|>1} |r| \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)| \mathbb{1}_{|f(t,x)r|\le 1}\,dx\,\nu(dr) \\
&\quad + \int_{|r|\le 1} |r|^2\,\nu(dr) \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|^2\,dx \\
&\le |\gamma| \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|\,dx + \int_{|r|>1} |r| \sup_{t\in\mathbb{R}^d} \int_0^{1/|r|} d_{f(t,\cdot)}(\alpha)\,d\alpha\,\nu(dr) \\
&\quad + \int_{|r|\le 1} |r|^2\,\nu(dr) \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|^2\,dx < \infty
\end{aligned}
$$

by (4.10) and (4.12), where we applied [21, Exercise 1.1.10] in the last line. Similarly, we have

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^d} \int_{|z|\le 1} z^2 \nu_t(dz) \\
&= \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |r|^2 |f(t,x)|^2 \mathbb{1}_{|rf(t,x)|\le 1}\,dx\,\nu(dr) \\
&\le \int_{|r|\le 1} |r|^2\,\nu(dr) \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|^2\,dx + \int_{|r|>1} |r| \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)| \mathbb{1}_{|f(t,x)r|\le 1}\,dx\,\nu(dr) \\
&\le \int_{|r|\le 1} |r|^2\,\nu(dr) \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|^2\,dx + \int_{|r|>1} |r| \sup_{t\in\mathbb{R}^d} \int_0^{1/|r|} d_{f(t,\cdot)}(\alpha)\,d\alpha\,\nu(dr) < \infty.
\end{aligned}
$$

Finally, from [21, Exercise 1.1.10] it follows

$$
\begin{aligned}
\sup_{t\in\mathbb{R}^d} \int_{|z|>1} \nu_t(dz) &= \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_{|rf(t,x)|>1}\,dx\,\nu(dr) \\
&\le \int_{|r|\le 1} |r|^2 \nu(dr) \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|^2 dx + \int_{|r|>1} \sup_{t\in\mathbb{R}^d} d_{f(t,\cdot)}\left(\frac{1}{|r|}\right) \nu(dr) \\
&\le \int_{|r|\le 1} |r|^2 \nu(dr) \sup_{t\in\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x)|^2 dx + \int_{|r|>1} |r| \sup_{t\in\mathbb{R}^d} \int_0^{1/|r|} d_{f(t,\cdot)}(\alpha)\;d\alpha\,\nu(dr) < \infty.
\end{aligned}
$$

Hence the characteristic triplets of $(X_t)_{t\in\mathbb{R}^d}$ satisfy (4.2). Further, from (4.11), (4.13) and

(4.14) it follows

$$
\begin{aligned}
\sup_{t\in\mathbb{R}^d}\int_{|z|>k} g(z)\nu_t(dz) &= \sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} g(f(t,x)r)\mathbb{1}_{|f(t,x)r|>k}\,dx\,\nu(dr)\\
&\le \int_{|r|\le\beta}\sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}^d} C(1+|f(t,x)r|^p)\mathbb{1}_{|f(t,x)r|>k}\,dx\,\nu(dr)\\
&\quad+\int_{|r|>\beta}\sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}^d} g(f(t,x)r)\mathbb{1}_{|f(t,x)r|>k}\,dx\,\nu(dr)\\
&\le \frac{C}{k^2}\int_{|r|\le\beta}|r|^2\nu(dr)\sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}^d}|f(t,x)|^2\,dx\\
&\quad+C\,\beta^{p-2}\int_{|r|\le\beta}|r|^2\,\nu(dr)\sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}^d}|f(t,x)|^p\mathbb{1}_{|f(t,x)|>\frac{k}{\beta}}\,dx\\
&\quad+\int_{|r|>\beta}\sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}^d} g(f(t,x)r)\mathbb{1}_{|f(t,x)r|>k}\,dx\,\nu(dr)\\
&\quad\to 0,
\end{aligned}
$$

as $k\to\infty$, which gives (4.5). Therefore, by Corollary 4.3 the g-uniform integrability of $(X_t)_{t\in\mathbb{R}^d}$ follows. □

Corollary 4.6. Let $p \in [2,\infty)$ and let $(X_t)_{t\in\mathbb{R}^d}$ be given by (4.6) with $f(t,\cdot)\in L^1(\mathbb{R}^d,\mathbb{R})\cap L^p(\mathbb{R}^d,\mathbb{R})$ and $\sup_{t\in\mathbb{R}^d}\|f(t,\cdot)\|_{L^1}<\infty$ for all $t\in\mathbb{R}^d$. Furthermore, let $h:\mathbb{R}^d\to[0,\infty)$, $h\in L^p(\mathbb{R}^d,\mathbb{R}^+)$ be a measurable function such that

$$d_{f(t,\cdot)}(\alpha)\le d_h(\alpha)\ \text{for all}\ \alpha>0,\ t\in\mathbb{R}^d.$$

If $\int_{|r|>1}|r|^p\,\nu(dr)<\infty$ then $(X_t)_{t\in\mathbb{R}^d}$ is L^p-uniformly integrable.

Proof. From [21, Proposition 1.1.4], we conclude

$$\int_{|r|>1}|r|\sup_{t\in\mathbb{R}^d}\int_0^{1/|r|} d_{f(t,\cdot)}(\alpha)\,d\alpha\,\nu(dr)\le\sup_{t\in\mathbb{R}^d}\|f(t,\cdot)\|_{L^1}\int_{|r|>1}|r|\,\nu(dr)<\infty,$$

and

$$\sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}^d}|f(t,x)|^p\,dx=\sup_{t\in\mathbb{R}^d}\int_0^\infty p\,\alpha^{p-1}d_{f(t,\cdot)}(\alpha)\,d\alpha\le\int_0^\infty p\,\alpha^{p-1}d_h(\alpha)\,d\alpha=\|h\|_{L^p}^p<\infty.$$

From [21, Exercise 1.1.10] and Lebesgue's dominated convergence theorem it follows

$$\int_{|r|>1}|r|^p\sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}^d}|f(t,x)|^p\mathbb{1}_{|f(t,x)r|>k}\,dx\,\nu(dr)\le\int_{|r|>1}|r|^p\int_{\mathbb{R}^d}|h(x)|^p\mathbb{1}_{|h(x)r|>k}\,dx\,\nu(dr)\to 0,$$

and

$$\sup_{t\in\mathbb{R}^d}\int_{\mathbb{R}^d} |f(t,x)|^p \mathbb{1}_{|f(t,x)|>k}\,dx \le \int_{\mathbb{R}^d} |h(x)|^p \mathbb{1}_{|h(x)|>k}\,dx \to 0$$

for $k\to\infty$. Therefore, the L^p-uniform integrability of the process $(X_t)_{t\in\mathbb{R}^d}$ follows from Theorem 4.5, where the submultiplicative function is given by $g(x) := \max\{|x|^p, 1\}$, $x \in \mathbb{R}$. □

We now state two examples.

Example 4.7. In the first example we study the uniform integrability of the mild solution of the generalized electric Schrödinger equation (2.21) driven by a Lévy basis, which we obtained in Section 2.3.1. We consider the special case discussed in Example 2.18 for $d = 3$ (with the conditions on A as stated in Theorem 2.17 and on V as stated in Example 2.18). The mild solution $(X_t)_{t\in\mathbb{R}^3}$ is given by

$$X_t = \int_{\mathbb{R}^3} f(t,x)dL(x),$$

with $f(t,\cdot) \in L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ and

$$|f(t,x)| \le \frac{C_1 e^{-C_2\|t-x\|}}{\|t-x\|}$$

for constants $C_1, C_2 > 0$ and $t, x \in \mathbb{R}^3$, $t \neq x$. This estimate follows from (2.22) and (2.23) and we showed in Example 2.18 that the existence of $(X_t)_{t\in\mathbb{R}^3}$ is assured if the Lévy basis L satisfies

$$\int_{|r|>1} \log(|r|)^3 \nu(dr) < \infty.$$

We observe

$$\sup_{t\in\mathbb{R}^3}\int_{\mathbb{R}^3} |f(t,x)|\,dx \le C_1 \sup_{t\in\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-C_2\|t-x\|}}{\|t-x\|}\,dx = C_1\int_{\mathbb{R}^3}\frac{e^{-C_2\|z\|}}{\|z\|}\,dz < \infty,$$

and by choosing $h : \mathbb{R}^3 \to (0,\infty)$, $h(x) := C_1\|x\|^{-1}e^{-C_2\|x\|}\mathbb{1}_{x\neq 0} \in L^2(\mathbb{R}^3,\mathbb{R}^+)$ we obtain by Corollary 4.6 that $(X_t)_{t\in\mathbb{R}^3}$ is L^2-uniformly integrable if we assume that the Lévy basis L satisfies

$$\int_{|r|>1} |r|^2\,\nu(dr) < \infty.$$

Example 4.8. In the second example we consider a process $(X_t)_{t\in\mathbb{R}}$ which is given by

$$X_t = \int_{\mathbb{R}} f(t,x)dL(x),$$

where

$$|f(t,x)| \leq \exp\left(-\varepsilon(t-x)\right)\mathbb{1}_{(-\infty,t]}(x) \tag{4.15}$$

for some $\varepsilon > 0$. The Ornstein-Uhlenbeck-type process discussed in Section 3.5 satisfies (4.15) if we assume that the function $\mu : \mathbb{R} \to \mathbb{R}$ satisfies $\sup_{t\in\mathbb{R}} \mu(t) < 0$. We observe for $p \geq 2$ that

$$\int_{|r|>1} |r|^p \, \nu(dr) < \infty,$$

assures us the L^p-uniform integrability of $(X_t)_{t\in\mathbb{R}}$ by Corollary 4.6, since

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} |f(t,x)| \, dx \leq \int_{-\infty}^{0} e^{\varepsilon y} \, dy < \infty,$$

and choosing $h : \mathbb{R} \to (0,\infty)$, $h(x) := e^{\varepsilon x}\mathbb{1}_{(-\infty,0)}(x) \in L^p(\mathbb{R},\mathbb{R}^+)$.

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Erklärung

Hiermit versichere ich, Farid Mohamed, dass ich die vorliegende Arbeit selbständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe. Ich erkläre außerdem, dass diese Arbeit weder im In- noch im Ausland in dieser oder ähnlicher Form in einem anderen Promotionsverfahren vorgelegt wurde.

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