Second Order Parabolic Equations with Non-local Boundary Conditions on $L^\infty(\Omega)$ and $C(\overline{\Omega})$

Generation, Regularity and Asymptotics

Vorgelegt von

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Contents

Introduction

The aim of the present thesis is to analyze second order parabolic equations with non-local boundary conditions on the spaces $L^\infty(\Omega)$ and $C(\overline{\Omega})$. More precisely we consider equations of the form

$$(NLBVP) \quad \begin{cases} \partial_t u(t,x) + \mathscr{A} u(t,x) = 0 \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega \\ u(0,x) = u_0(x) \text{ for } x \in \Omega \\ L(z)(u(t,\cdot)) = B(z)(u(t,\cdot)) \text{ for all } t \in \mathbb{R}^+ \\ \qquad \text{and almost all } z \in \partial\Omega, \end{cases}$$

where we are interested in solutions $u \in C^1(\mathbb{R}^+, C(\overline{\Omega}))$. Here $\Omega \subset \mathbb{R}^d$ is open and bounded, $\mathscr{A}$ is a strictly elliptic second order differential operator and $u_0 \in L^\infty(\Omega)$ is the initial value. The boundary conditions for $z \in \partial\Omega$ are given by the *local* functional $L(z)$ and the *non-local* functional $B(z)$ defined for certain functions on $\overline{\Omega}$ with values in $\mathbb{C}$.

We will prove well-posedness of the equation (LNBVP) and for solutions of (NLBVP) we will prove regularity results as well as give results on the asymptotic behavior as $t \to \infty$. One of our main arguments is the transference of properties of the equation with local boundary conditions, i.e. with $B(z) = 0$, to the one with non-local boundary conditions. Great care will be taken to ensure that we do not need more regularity assumptions on Ω and on the coefficients of $\mathscr{A}$ compared to the best known local results.

We will consider two kinds of local boundary conditions in this thesis: the *Dirichlet boundary condition*, i.e. $L(z)(u) = u|_{\partial\Omega}(z)$, and the *Robin boundary condition*, i.e. $L(z)(u) = \partial_\nu^{\mathscr{A}} u(z) + \beta(z)u(z)$, where $\beta \in L^\infty(\partial\Omega)$ and $\partial_\nu^{\mathscr{A}}$ is the conormal derivative to $\partial\Omega$ w.r.t. $\mathscr{A}$. The non-local part will be given by measures $\mu(z) \in \mathcal{M}(\overline{\Omega})$, $z \in \partial\Omega$, via $B(z)(u) = \int_{\overline{\Omega}} u(x)\,\mu(z)(\mathrm{d}x)$. Since we are interested in solutions $u(t,\cdot) \in C(\overline{\Omega})$, $t > 0$, it is no restriction that the non-local part is given by measures.

In the analysis of non-local Dirichlet boundary conditions we will consider elliptic operators in both divergence and non-divergence form. In the first case all coefficients of $\mathscr{A}$ are assumed to be bounded and measurable and Ω to be *Dirichlet-regular*; this very weak regularity assumption means that the classical boundary value problem

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$$

has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for all $g \in C(\partial\Omega)$ or — equivalently — that the part of the distributional Laplacian on $C_0(\Omega)$ generates a C_0-semigroup. In the second case either the coefficients of $\mathscr{A}$ are assumed to be continuous and Ω to satisfy the uniform exterior cone condition or the coefficients are assumed to be Dini-continuous and Ω to be only Dirichlet-regular.

In the treatment of non-local Robin boundary conditions we will consider elliptic operators in divergence form with measurable and bounded coefficients and Ω will be Lipschitz.

The structure of the thesis is as follows.

Chapter 1 This chapter is intended to be a soft introduction to the topic of elliptic and parabolic equations with non-local boundary conditions.

Section 1 features a short history of research spanning from the beginning of the 20th century to the latest developments. We focus on two main inspirations for the study of these equation: the more abstract research of Feller and Ventsell on the domain of generators as well as the more concrete problem introduced by Bitsadze-Samarskii.

In Section 2 we also give a mathematical motivation for the study of such boundary conditions in terms of certain Markov processes or, equivalently, of positive, contractive semigroups on $C(\overline{\Omega})$. We describe and explain Ventsell's result on the domain of the generator of such processes resp. semigroups.

As a last topic of this chapter in Section 3 we mention some applications of these non-local boundary conditions in numerics and functional equations and also show where they appear in mathematical modelling.

Chapter 2 Here we lay the groundwork to attack (NLBVP).

We want to operate in $L^\infty(\Omega)$, but by a result due to Lotz [Lot85] generators of C_0-semigroups on $L^\infty(\Omega)$ are bounded. Hence we need to consider semigroups that are not C_0, that is semigroups whose generator is not densely defined. A suitable characterization of such generators and properties of such semigroups such as contractivity, positivity, etc... is given in Section 1.

In Section 2 we give some abstract results on irreducible semigroups which will be useful for the treatment of Robin boundary conditions; in particular a comparison principle will replace the maximum principle we use in the case of Dirichlet boundary conditions. Another main result is the transference of irreducibilty of a semigroup on $L^2(\Omega)$ to its restriction on $C(\overline{\Omega})$.

We will see that the semigroup which solves the problem (NLBVP) is given by kernel operators and maps $L^\infty(\Omega)$ into $C(\overline{\Omega})$; that is the semigroup consists of *strong Feller operators*. An introduction to such operators and certain properties of these are given in Section 3. We also explain the concept of strong Feller operators on $L^\infty(\Omega)$ in contrast to the more conventional space $B_b(\overline{\Omega})$.

In Sections 4 and 5 we give properties of the local problem corresponding to (NLBVP), i.e. the one with $B(z) = 0$. In Section 4 we introduce elliptic operators in non-divergence form and the associated Dirichlet problem; Section 5 is devoted to elliptic operators in divergence form and to the associated Dirichlet and Robin boundary value problems; in particular the weak conormal derivative will be introduced. In both of these sections suitable maximum principles are proved.

As already mentioned we want to consider (NLBVP) as perturbation to the corrensponding problem with local boundary conditions. In Section 6 an abstract approach is given to such perturbations originally due to Greiner in [Gre87]. There we show generation results for the perturbed boundary conditions in two cases:

a) the original semigroup as well as the perturbation are positive and a certain related operator is power compact and

b) the perturbation itself is compact.

Furthermore we show how certain properties of the non-perturbed semigroup transfer to the perturbed one.

Chapter 3 In this chapter we treat (NLBVP) in the case of non-local Dirichlet-boundary conditions. Here we are able to treat elliptic operators in both divergence and non-divergence form at the same time. We will assume the following

(M1) $\partial\Omega \to \mathbb{C}, z \mapsto B(z)u$ is continuous for all $u \in C(\overline{\Omega})$.

(M2) $\sup_{\partial\Omega} \|\mu(z)\|_{TV} \leq 1$.

(M3) $\mu(z) \geq 0$, for all $z \in \partial\Omega$

(M4) $\mu(z)(\partial\Omega) = 0$ for all $z \in \partial\Omega$.

on the non-locality $B(z)$ resp. the measures $\mu(z)$.

Clearly assumption (M1) is necessary if we want to have a semigroup on $C(\overline{\Omega})$. Also assumption (M2) is necessary if one wants to have a contraction semigroup. The assumptions (M3) and (M4) are imposed in order to use the perturbation result in the positive case of Chapter 2 Section 6. Note however if one wants perturbations arising from probability measures, then one must assume *some* kind of restriction on the support of $\mu(z)$; in general solvability fails even for smooth coefficients and boundary $\partial\Omega$, see [Gur10, Section 26] or [Sku89]. The situation becomes much easier if one assumes the strict inequality in assumption (M2); in that case assumptions (M3) and (M4) may be dropped. Also note that assumption (M3) is necessary for the generation of a positive semigroup.

In Section 1 we prove the generation results and the strong Feller property. Furthermore we identify the space — called $C_\mu(\overline{\Omega})$ — on which the semigroup is C_0.

In Section 2 we give precise information on the asymptotics of the semigroup even if Ω consists of multiple connected components. In particular we give convenient criteria for exponential stability via graphs defined upon the connected components resp. their adjacency operator. In the case where the semigroup is not exponentially stable the semigroup converges to a finite rank operator for $t \to \infty$; we will give the precise rank of the operator in terms of the support of the $\mu(z)$ and the coefficients of $\mathscr{A}$. The main tool used in this section is the maximum principle derived in Section 3 resp. 4 of Chapter 2 combined with the assumption that $\partial\Omega$ is a null set under all $\mu(z)$; this is used to prove certain properties of the kernel of the A_μ which then imply the desired results on the asymptotics of the generated semigroup. Note that the properties of the kernel remain valid under a less restrictive assumption than (M4).

Section 3 deals with assorted topics. The first subsection is concerned with the structure of the space $C_\mu(\overline{\Omega})$; we prove an extension result for functions defined on a compact set $K \subset \Omega$ and use it to show that $C_\mu(\overline{\Omega})$ is a sublattice or a subalgebra of $C(\overline{\Omega})$ only for very specific choices of measures $\mu(z)$. In the second subsection a characterization of the irreducibility of the semigroup on $C(\overline{\Omega})$ is given in terms of the boundary perturbation $\mu(z)$. And finally in the last subsection we deal with the case we already mentioned where the measures $\mu(z)$ fulfill assumption (M1) and (M2) with strict inequality. Here we prove the generation result and that the semigroup is exponentially stable.

Chapter 4 The last chapter is devoted to non-local Robin boundary conditions and elliptic operators in divergence form. We will assume the following on the measures $\mu(z)$.

(M1) for every $u \in B_b(\overline{\Omega})$ we have that the function $z \mapsto B(z)u$ is measurable, (M2) for some $p > d-1$ we have $\int_{\partial\Omega} \|\mu(z)\|_{TV}^p \, \mathrm{d}z < \infty$ and

(M3) there exists a positive Borel measure τ on Ω such that $\mu(z)$ is absolutely continuous w.r.t. τ for all $z \in \partial\Omega$ and $L^1(\Omega, \tau)$ is separable.

Here (M1) is necessary for a generation result to hold on $L^\infty(\Omega)$. Assumptions (M2) and (M3) are made to use the perturbation result of Chapter 2 Section 6 in the case of a compact perturbation. While assumption (M3) appears to be quite abstract, it will be shown that it is fulfilled if the map $B(\cdot)u$ is sufficiently regular; more precisely if $z \mapsto B(z)u$ is *continuous* for every $u \in B_b(\overline{\Omega})$. Note that at this point no further assumption on the measures $\mu(z)$ needs to be made; in particular it will turn out that the $\mu(z)$ need not be positive (or even real) and that their support can be arbitrary for our generation result to hold.

In Section 1 we prove the aforementioned generation result and derive additional properties that imply that the semigroup is submarkovian resp. positive; here further restrictions on $\mu(z)$ (e.g. $\mu(z) \geq 0$ imply that the semigroup is positive) and on the coefficients come into play. Note that these restrictions appear to be also necessary; we will prove this later on in the special case of $\mu(z)$ being absolutely continuous w.r.t. the Lebesgue measure. This section also characterizes the Markovianess of the semigroup unconditionally.

In Section 2 results on the asymptotics of the generated semigroup are given. As in Section 2 of the previous chapter we are immediately able to prove convergence to a finite rank operator. But since we do not assume much regularity on the coefficients of $\mathscr{A}$ and on Ω we do not have precise information on the value of $\partial_\nu^{\mathscr{A}} u(z)$ at a maximum/minimum of u. Hence we do not get much use out of the maximum principle in contrast to the situation of Chapter 3. However we are able to prove irreducibility of the semigroup on $C(\overline{\Omega})$ with non-local boundary conditions the the semigroup on $C(\overline{\Omega})$ with local boundary conditions has this property. Then we are able to prove that the semigroup converges to a projection of rank one or zero using — among other results — the comparison principle of Chapter 2 Section 2. We further show a somewhat surprising result for non-local Neumann boundary conditions, that is, when we have $\beta = 0$: under these boundary conditions it is not possible that the semigroup is positive and bounded.

The results of Section 2 give us pretty precise information regarding the asymptotic behavior *if* the semigroup is irreducible on $C(\overline{\Omega})$. However this is not easy to prove in general. Hence in Section 3 we restrict ourselves to the case of $\mu(z)$ being absolutely continuous w.r.t. to the Lebesgue measure. In that case one can show more: we actually have that the semigroup extends to

$L^2(\Omega)$ and that one can use form methods. This enables us to use the transference result of Chapter 2 Section 2 to show irreducibility of the semigroup on $C(\overline{\Omega})$ and furthermore to characterize the submarkovian property of the generated semigroup. In particular if Ω is connected, this yields exponential stability resp. exponential decay to a rank 1 projection.

Finally Section 4 gives some examples of measures fulfilling the rather abstract assumptions (M1)–(M3).

After each chapter (except for the first) there is a section devoted to notes and comments where we primarily list the main sources of the preceding sections and their results.

Zusammenfassung in deutscher Sprache

In der vorliegenden Arbeit werden parabolische Differentialgleichungen zweiter Ordnung mit nicht-lokalen Randbedingungen untersucht. Es werden Gleichungen der Art

$$(NLRWP) \quad \begin{cases} \partial_t u(t,x) + \mathscr{A} u(t,x) = 0 \text{ für } (t,x) \in \mathbb{R}^+ \times \Omega \\ u(0,x) = u_0(x) \text{ für } x \in \Omega \\ L(z)(u(t,\cdot)) = B(z)(u(t,\cdot)) \text{ für alle } t \in \mathbb{R}^+ \\ \qquad \text{und fast alle } z \in \partial\Omega, \end{cases}$$

auf $L^\infty(\Omega)$ behandelt, wobei wir an Lösungen $u \in C^1(\mathbb{R}^+, C(\overline{\Omega}))$ interessiert sind. Hier sind $\Omega \subset \mathbb{R}^d$ offen und beschränkt, $\mathscr{A}$ ein gleichmäßig elliptischer Differentialoperator zweiter Ordnung und $u_0 \in L^\infty(\Omega)$ der Anfangswert. Die Randbedingungen in $z \in \partial\Omega$ sind durch das *lokale* Funktional $L(z)$ und durch das *nicht-lokale* Funktional $B(z)$ gegeben, die bestimmte auf $\overline{\Omega}$ definierte Funktionen nach $\mathbb{C}$ abbilden.

Wir werden Generatorensätze für einen zu (NLRWP) passend gewählten Operator — d.h. wir zeigen die Wohlgestelltheit der Gleichung (NLRWP) — und Regularitätsergebnisse für Lösungen von (NLRWP) beweisen sowie die Asymptotik für $t \to \infty$ dieser Lösungen behandeln. Viele Eigenschaften der Lösungen von (NLRWP) werden gezeigt, indem man sie von den Lösungen mit den entsprechenden lokalen Randbedingungen, i.e. mit $B(z) = 0$, ableitet. Dabei wird besonderen Wert darauf gelegt, dass nicht mehr Regularitätsanforderungen an die Koeffizienten von $\mathscr{A}$ und an den Rand $\partial\Omega$ gebraucht werden als die besten generellen Resultate für das lokale Problem benötigen.

Wir betrachten zwei Arten lokaler Randbedingungen in dieser Dissertation: die Dirichletrandbedingung, also $L(z)(u) = u|_{\partial\Omega}(z)$ und die Robinrandbedingung, also $L(z)(u) = \partial_\nu^{\mathscr{A}} u(z) + \beta(z)u(z)$, wobei $\partial_\nu^{\mathscr{A}}$ die zum Operator $\mathscr{A}$ gehörende Konormalenableitung an $\partial\Omega$ ist und $\beta \in L^\infty(\partial\Omega)$. Der nicht-lokale Teil ist durch Maße $\mu(z) \in \mathcal{M}(\overline{\Omega})$ für $z \in \partial\Omega$ gegeben, indem man $B(z)(u) = \int_{\overline{\Omega}} u(x)\,\mu(z)(\mathrm{d}x)$ setzt. Beachte, dass dies keine Einschränkung zu der anfangs beschriebenen Situation darstellt, da wir Lösungen $u(t,\cdot) \in C(\overline{\Omega})$, $t > 0$, betrachten und auf $C(\overline{\Omega})$ definierte Funktionale stets durch Maße gegeben sind.

Im Fall der Dirichletrandbedingung werden wir elliptische Operator sowohl in Divergenz- als auch Nichtdivergenzform behandeln. Ist $\mathscr{A}$ in Divergenzform, so nehmen wir an, dass seine Koeffizienten beschränkt und messbar sind und dass Ω *Dirichlet-regulär* ist; diese sehr schwache Regularitätsanforderung an Ω bedeutet, dass das klassische Randwertproblem

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$$

für alle $g \in C(\partial\Omega)$ genau eine Lösung $u \in C^2(\Omega) \cap C(\overline{\Omega})$ hat. Eine äquivalente Formulierung ist, dass der Teil des distributionellen Laplaceoperators in $C_0(\Omega)$ eine C_0-Halbgruppe erzeugt. Ist $\mathscr{A}$ in Nichtdivergenzform so fordern wir, dass entweder die Koeffizienten stetig sind und Ω der gleichmäßigen äußeren Kegelbedingung genügt oder dass die Koeffizienten Dini-stetig sind und Ω nur Dirichlet-regulär ist.

Bei den Robinrandbedingungen behandeln wir elliptische Operator in Divergenzform mit messbaren und beschränkten Koeffizienten und Ω wird als Lipschitz-stetig vorausgesetzt.

1 Motivation: Characterizing Feller Semigroups and Modeling

As an introduction to the topic of the present dissertation we chose to address three points of interest. Firstly we give a short outline of the history of research on non-local boundary conditions. Secondly we give a mathematical motivation to the study of non-local boundary conditions in terms of Markov processes resp. Feller semigroups. Thirdly we name some direct and indirect applications of these kind of boundary conditions in physical, biological and economic models as well as numerical computations.

1.1 A little history of non-local boundary conditions

One of the first accounts of an elliptic — but ordinary — differential equation equipped with non-local boundary conditions arose in a paper by Sommerfeld [Som09] in 1909 in the theory of turbulence. These also arise naturally as generators in the theory of one-dimensional Markov processes as seen in e.g. [Fel54]. For a historical treatise on the subject of general, that is also non-local, boundary conditions in ordinary differential equations up to the 1970s we refer to Krall [Kra75].

The first research on non-local elliptic boundary value problems in more than one dimension was conducted by Carlemann in 1932. In his lecture at the International Congress of Mathematicians in Zurich (published later in [Car33]) he investigated the problem of finding a holomorphic function in a domain $\Omega \subset \mathbb{C}$ where the values of the sought function at a boundary point z correspond to values at the point $\alpha(z)$. In this paper he considered only functions α with range in $\partial\Omega$ and with $\alpha \circ \alpha(z) = z$. Similar problems — that is problems where the non-local terms are supported on the boundary — were studied in a Hilbert space setting by Calkin [Cal39], Browder [Bro64], Beals [Bea65], Schechter [Sch66], Bade and Freeman [BF62] and others: They considered self-adjoint restrictions of the (maximal) Laplacian and other elliptic operators and mostly worked with boundary conditions involving the normal derivative.

The systematic study of elliptic and parabolic problems with non-local boundary conditions started with Feller and Venttsel in the 1950s. In his papers [Fel52, Fel57] Feller completely describes the possible boundary conditions for a second order differential equation on an interval in the following sense: He identifies and classifies the boundary conditions which make the equation solvable resp. the solutions unique; among those are also non-local terms. In particular these results completely answer the question of whether a differential operator on an interval (with given boundary conditions) generates a positive, contractive C_0-semigroup in the continuous functions on the interval. Such semigroups were later on called *Feller semigroups.*

Venttsel in [Ven59] subsequently considered the analog question on a bounded domain Ω in more than one dimension. Given a Feller semigroup — that is a positive, contractive, strongly continuous semigroup on $C(\overline{\Omega})$ — he studies the behavior of functions in the domain of the generator at the boundary $\partial\Omega$. As a result he specifies the boundary conditions that are necessary for a differential operator to be a generator of a Feller semigroup. As in the one dimensional case these boundary condition may include non-local terms which are given without loss of generality as measures operating on continuous functions. In the following section we give some more attention and details to this result.

The converse question of whether a given differential operator with boundary conditions generates a Feller semigroup has been given much attention in the following years by both analysts and probabilists. Here we only mention some papers and monographs devoted to their study: [BCP68], [Ish90], [Sku08, Sku10], [SU65], [Tai14], [Wat79]. Through these works many important facets of this problem are now well understood such as the case when the coefficients of the differential operator and the boundary of domain Ω are smooth or the case when the order of the non-locality in the boundary conditions is less than the order of the local terms. Here order is understood in the sense of order of differential operators.

Nonetheless in general the problem of identifying precisely the boundary conditions sufficient for the generation of a Feller semigroup is still open.

Parallel to this abstract approach to non-local boundary conditions a more direct line of investigation was started by Bitsadze and Samarskii [BS69] in 1969. They considered an elliptic equation with C^∞-coefficients on a smooth bounded domain $\Omega \subset \mathbb{R}^d$ with boundary conditions of the following form: $u = \lambda \cdot u \circ \omega + f_1$ on a smooth $d-1$ dimensional submanifold $M \subset \partial\Omega$ and $u = f_2$ on $\partial\Omega \setminus M$. Here f_1 and f_2 are given functions, $\lambda \in C^\infty(\mathbb{R}^d)$ and ω is a $C^\infty(\mathbb{R}^d)$-diffeomorphism mapping M into Ω (but not necessarily $\overline{M}$ into Ω!). Such boundary conditions arise in plasma theory and the paper by Bitsadze-Samarskii spurned much investigation into non-local problems; Samarskii later posed the general solvability of this problem as an open question in [Sam80].

In the following years a classification of the Bitsadze-Samarskii problem was proposed based on the structure of the support of the non-locality ω in rising difficulty: 1.) $M = \partial\Omega$ and $\omega(\partial\Omega) \subset \Omega$, 2.) $M \neq \partial\Omega$ and $\overline{\omega(M)} \cap (\overline{M} \setminus M) = \emptyset$, 3.) $M \neq \partial\Omega$ and $\overline{\omega(M)} \cap (\overline{M} \setminus M) \neq \emptyset$.

In the first case this problem reduces to the one Ventsell studied in the special case that the non-locality is given by point measures and that the coefficients and domain are smooth. The non-locality ω does not change major properties of the elliptic problem such as the smoothness of solutions, the Fredholm property of the differential operator and its index; as seen in [Kis87, KS99, Moi90, Mus90]. The second and third case add the possibility of polynomially bounded singularities near the points $\overline{M} \setminus M$. Still in the second case using appropriate weighted spaces the Fredholm property and the index do not change. In the third case the Fredholm property of the local problem may be given while that of the non-local one is not and vice versa. For a study of the second and third cases we refer to [Gur01, Gur03, Gur04, Kis88, Sku97, Zhu92].

Other properties of elliptic operators endowed with non-local boundary conditions were also studied such as their spectrum, their Fredholm index, regularity properties, Green's functions and so on. Here we mention [BAP07, Gus00, GM95, Gur03, Gur04, Gur10, Moi97, Sku08, Sku10, WI09]

We also want to mention that non-local elliptic boundary value problems of higher order than two has also received much attention from mathematicians such as Pod"yapol'skii [Pod02], Kovaleva and Skubachevskii [KS00], and Schechter [Sch66].

1.2 Generators of Feller semigroups

This dissertation is concerned with uniformly elliptic operators $\mathscr{A}$ in either divergence form formally given by

$$\mathscr{A}u = \sum_{i,j} D_i(a_{ij}D_j u) + \sum_i D_i(b_i u) + \sum_i c_i D_i u + d_0 u$$

or non-divergence form formally given by

$$\mathscr{A}u = \sum_{i,j} a_{ij} D_i D_j u + \sum_i c_i D_i u + d_0 u$$

subject to non-local boundary conditions of Dirichlet-type such as

$$u(z) = \int_{\overline{\Omega}} u(x)\, \mu(z, \mathrm{d}x) \quad \forall z \in \partial\Omega$$

or Robin-type such as

$$\partial_\nu^{\mathscr{A}} u(z) + \beta(z)u(z) = \int_{\overline{\Omega}} u(x)\, \mu(z, \mathrm{d}x) \quad \text{for almost all } z \in \partial\Omega$$

where the $\mu(z, \cdot)$ are measures on $\overline{\Omega} \subset \mathbb{R}^d$, Ω bounded and open, $\beta \in L^\infty(\Omega)$ and $\partial_\nu^{\mathscr{A}}$ is an outer derivative associated to $\mathscr{A}$.

While there are applications involving these kinds of boundary conditions (and we will mention them in the section hereafter), we now want to motivate the analysis of operators with boundary conditions of this or similar kind in a purely mathematical way, namely in terms of Feller processes or — equivalently — Feller semigroups. In the following we will skip many details and be somewhat vague in the terms used; for a detailed and comprehensive overview on Markov processes and their relation to Feller semigroups, we refer to one of the many standard works on the subject such as Lamperti [Lam77], Øksendal [Øks98] or Taira [Tai14].

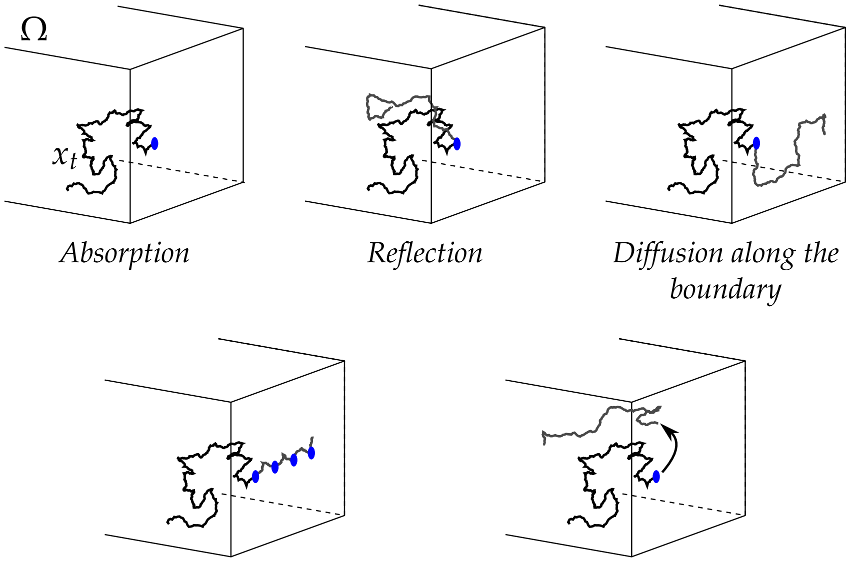
Let $\Omega \subset \mathbb{R}^d$ be open and bound and let $(\overline{\Omega}, \mathcal{F}, \mathcal{F}_t)$ be a filtrated measurable space. Let $X = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)_{t \geq 0, x \in \overline{\Omega}}$ be a family of Markov processes on that measurable space where P_x is the initial distribution of the Markov process x_t in X. Associated to X are sub-probability measures $p_t(x, \cdot)$ via $p_t(x, E) := P_x(\{x_t \in E\})$, $x \in \Omega$, $t \geq 0$, and a semigroup T on $B_b(\overline{\Omega})$ via the formula

$$(T(t)f)(x) = \int_{\overline{\Omega}} f(y)\, p_t(x, \mathrm{d}y).$$

If T leaves $C(\overline{\Omega})$ invariant, the semigroup is called *Feller semigroup* and the process X is called *Feller process*. Note that we could have started with a Feller semigroup (that is a positive, contractive semigroup on $C(\overline{\Omega})$) and constructed a Feller process X via the sub-probability measures $p_t(x, \cdot)$ in the formula above — these exist since all operators on $C(\overline{\Omega})$ are kernel operators — and the Kolmogorov Extension Theorem. One may associate to X or T an *infinitesimal generator* A which is a differential operator. The action of the operator describes the behavior of X in Ω and the domain describes the behavior at the boundary.

It is well known (see e.g. [Tai14, Theorem 9.52] or [BCP68, Théorème IX]) that the generator A is a second order elliptic operator hence the process

Illustration of the behavior at the boundary



Absorption *Reflection* *Diffusion along the boundary*

Viscosity phenomenon *Non-local phenomenon*

X is a diffusion process in Ω (possibly with jumps in Ω). Upon hitting a boundary point $z \in \partial\Omega$ many actions of X are thinkable: It may simply "die" and remain stationary indefinitely (*absorption*), it may reflect on the boundary and continue onward (*reflection*), it may diffuse along the boundary (*diffusion along the boundary*), it may stick to the boundary for some time (*viscosity phenomenon*) or it may jump either to the interior Ω or to the boundary $\partial\Omega$ (*non-local* or *jump phenomenon*).

These actions correspond to different types of boundary conditions in the generator A of T resp. X, namely for $u \in D(A)$: The *classical* ones consisting of Dirichlet boundary conditions involving $u(z)$ itself for absorption, Neumann boundary conditions involving a normal derivative $\partial_\nu u(z)$ for reflection, a new elliptic operator $\sum_{i,j=1}^{d-1} \alpha_{ij}(z) D_i D_j u(z) + \sum_{i=1}^{d} \beta_i(z) D_i u(z)$ for diffusion along the boundary and Ventsell-type boundary conditions involving $Au(z)$ for the viscosity phenomenon as well as the non-classical one with integral terms of the form $\int_{\overline{\Omega}} u(x)\,\mu(z, \mathrm{d}x)$ for the non-local phenomenon. For further information on the derivation and interpretation of the classical boundary conditions we refer to Goldstein [Gol06].

Ventsell then proved in [Ven59] that these boundary conditions resp. actions are the only ones possible for a Feller process X under slight regularity assumptions. More precisely he proved for a Feller semigroup T with generator A on $C(\overline{\Omega})$ the following: For each point $x_0 \in \partial\Omega$ let there be a varying local coordinate system $(y_1, y_2, \ldots, y_{d-1}, n)$ and a neighborhood U

such that

1. $y_i(x_0) = 0$, $i = 1\ldots,d-1$, $n(x) \geq 0$ for $x \in \overline{\Omega}$, $n(x) = 0$ for $x \in U \cap \partial\Omega$
2. y_i, $i = 1,\ldots,d-1$, and n are continuous on $\overline{\Omega}$
3. every twice continuously differentiable function on U (in the usual rectangular coordinate system) are twice continuously differentiable w.r.t $(y_1(x),\ldots,y_{d-1}(x),n(x))$ and vice versa.

Then every function $u \in D(A)$ which is twice continuously differentiable near $\partial\Omega$ fulfills

$$\begin{aligned} 0 = &q(z)u(z) - \gamma(z)\partial_n u(z) + \sigma(z)Au(z) \\ &+ \sum_{i=1}^{d-1} \beta_i(z)\partial_{y_i}u(z) - \sum_{i,j=1}^{d-1} \alpha_{ij}(z)\partial_{y_i}\partial_{y_j}u(z) \\ &+ \int_{\overline{\Omega}} u(z) - u(x) + \sum_{i=1}^{d} \partial_{y_i}u(z)y_i(x)\,\mu(z,\mathrm{d}x) \end{aligned}$$

at each point $z \in \partial\Omega$, where $q(z),\gamma(z),\sigma \geq 0$, $(\alpha_{ij}(z))_{i,j=1,\ldots,d}$ is a symmetric non-negative definite matrix and $\mu(z,\cdot)$ is a non-negative Borel measure on $\overline{\Omega}$. Note that for a different choice of the coordinate system $(y'_1, y'_2, \ldots, y'_{d-1}, n')$ in a neighborhood of the point $z \in \partial\Omega$ the coefficients $\sigma(z)$, $q(z)$ and the measures $\mu(z,\cdot)$ do not change, the rank of the matrix $(b_{ij})_{i,j}$ remains unchanged and also whether $\gamma(z)$ vanishes or not.

So in this sense the possible form of generators of Feller processes is clear. Thus if one wants to understand Feller processes resp. Feller semigroups or wants to gain insight in the phenomena they model, one has to understand elliptic operators with non-local boundary conditions.

The present dissertation deals with (part of) the reverse question, that is whether there is a Feller process or Feller semigroup with a given elliptic operator subject to boundary conditions of the above kind as generator. But we also leave this scenario slightly: Some of the generators encountered will not be densely defined, hence the generated semigroup not be C_0 on all of $C(\overline{\Omega})$. Furthermore we accept less regular domains Ω than needed in Ventsel's results. And finally we also deal with the generated semigroup on a larger space $L^\infty(\Omega)$.

1.3 Applications

In this section we want to highlight three possible applications of the theory of non-local boundary conditions: models incorporating non-local boundary conditions directly, artificial insertion of non-local boundary conditions to treat unbounded (space) domains in numerical computations, some functional differential equations may be rewritten into pure differential equations with non-local boundary conditions.

We begin with applications in numerics. When trying to approximate a solution to a differential equation on an unbounded domain Ω, it is often beneficial (or even necessary) to cut off the domain creating the smaller domain $\tilde{\Omega}$. This introduces an additional boundary where one must define some kind of boundary condition to keep the solution unique; this is called *artificial boundary condition*. Of particular interest are those artificial boundary conditions that do not change the solution; that is those boundary conditions on the additional part of the boundary $\partial\tilde{\Omega} \setminus \partial\Omega$ such that the solution of the differential equation on the truncated domain $\tilde{\Omega}$ is the restriction to $\tilde{\Omega}$ of the solution of the differential equation on the whole domain Ω. These are called *transparent* or *exact boundary conditions*. It turns out that the transparent boundary conditions are in general non-local, even if Ω is the whole space $\mathbb{R}^d$.

We will illustrate the general concept with the example of the Poisson equation on $\mathbb{R}^d$: We seek a function $u \in H^1(\mathbb{R}^d)$ with $\Delta u \in L^2(\mathbb{R}^d)$ satisfying the equation

$$\begin{cases} \Delta u = f \text{ in } \mathbb{R}^d \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.1}$$

Here $f \in L^2(\mathbb{R}^d)$ is assumed to have compact support. We want to limit the computation to a open and bounded subset $\Omega \subset \mathbb{R}^d$ with smooth boundary and $\operatorname{supp} f \subset \Omega$. To derive the transparent boundary conditions for this equation, we also consider the exterior domain $\Omega^e := \mathbb{R}^d \setminus \overline{\Omega}$. The trick to the derivation of exact boundary conditions on $\partial\Omega$ is the reformulation into an equivalent transmission problem between the domains Ω and Ω^e. On $L^2(\Omega)$ we set

$$\begin{cases} \Delta v = f \text{ in } \Omega \\ \partial_\nu v = \partial_\nu w \text{ on } \partial\Omega \end{cases} \tag{1.2}$$

and on $L^2(\Omega^e)$ we set

$$\begin{cases} \Delta w = 0 \text{ in } \Omega^e \\ w = v \text{ on } \partial\Omega \\ \lim\limits_{|x|\to\infty} w(x) = 0, \end{cases} \tag{1.3}$$

where ∂_ν denotes the (weak) outer unit normal derivative w.r.t. Ω. Clearly if u solves (1.1), then the restriction $v := u|_\Omega$ solves (1.2). Now the aim is to find the right boundary condition on $\partial\Omega$ such that we can solve (1.2) on its own.

Here the non-local boundary condition comes into play via the non-local *Dirichlet-to-Neumann operator* $D_0 \colon D(D_0) \to L^2(\partial\Omega)$ defined as follows. Let h be a function in $L^2(\partial\Omega)$ such that there is $w \in H^1(\Omega^e)$ with $\Delta w = 0$ on Ω^e, $w = h$ on $\partial\Omega$, $\lim_{|x|\to\infty} w(x) = 0$ and $\partial_\nu w$ exists as a function $L^2(\partial\Omega)$. We then set $D_0 h = \partial_\nu w$. We can use this operator to consider the equation

$$\begin{cases} \Delta v = f \text{ in } \Omega \\ \partial_\nu v = D_0(\operatorname{tr} v) \text{ on } \partial\Omega. \end{cases}$$

Then if $\partial_\nu v = D_0(\operatorname{tr} v)$ there is by definition of D_0 a function $w \in H^1(\Omega^e)$ with $\Delta w = 0$ on Ω^e, $w = v$ on $\partial\Omega$ and $\lim_{|x|\to\infty} w(x) = 0$; that is w solves (1.3) and hence with

$$\partial_\nu v = D_0(\operatorname{tr} v)$$

we have found the transparent boundary conditions on $\partial\Omega$.

We make note of some properties of the Dirichlet-to-Neumann operator. It is as already mentioned a non-local operator and can be realized as a pseudo-differential operator if $\partial\Omega$ is smooth. Furthermore one can define the Dirichlet-to-Neumann operator as the associated operator of a sesquilinear form with form domain $H^{1/2}(\partial\Omega)$. For more information and a rigorous treatment of the Dirichlet-to-Neumann operator on exterior domains we refer to [AtE15].

This basic idea to split the problem into a transmission problem and invoke a Dirichlet-to-Neumann operator is applicable for a wide array of equations, such as the heat equation with general second order elliptic operators (see [HR95]), the Schrödinger equation (see [AAB+08]), wave equations (see [HW13, Chapter 4]) and so on.

For a general treatment of the artificial boundary condition method in numerics we refer to [HW13].

We now come to applications for functional equations. Functional differential equations are one of the most applied models of physical phenomena in fields such as engineering ("elastic models", see [Sku97, Chapter 3.14–3.15], [OT95]), physics (e.g. "climate models",) and biology ("population dynamics", see [SS01]). Such equations feature a differential operator and a operator acting on the arguments of the function in the equation; therefore they are a priori more difficult to treat than pure differential equations. But some functional differential equations can be turned into pure ones albeit with a boundary condition that is more complex. As an example consider the following functional differential equation featured in [Sku97, Chapter 2.8, Example 8.4 and Chapter 3.15] on the domain $\Omega := (0,2) \times (0,1)$

$$\begin{cases} \Delta\big(u(x,y) + \gamma_1 u(x+1,y) + \gamma_2 u(x-1,y)\big) = f(x,y) \text{ for } (x,y) \in \Omega \\ u|_{\mathbb{R}^2\setminus\Omega} = 0 \end{cases} \tag{1.4}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$ and $f \in L^2(\Omega)$. We will reduce this functional partial differential equation to a pure partial differential equation with non-local boundary conditions. To that end set $T\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $Tu(x,y) := u(x,y) + \gamma_1 u(x+1,y) + \gamma_2 u(x-1,y)$, let $E_\Omega\colon L^2(\Omega) \to L^2(\mathbb{R}^2)$ be the extension operator extending a function by 0, and $R_\Omega\colon L^2(\mathbb{R}^2) \to L^2(\Omega)$ be the restriction operator and finally set $T_\Omega\colon = R_\Omega T E_\Omega\colon L^2(\Omega) \to L^2(\Omega)$.

Now let $u \in H_0^1(\Omega)$ and denote $w := T_\Omega u$. Then $w \in H^1(\Omega)$ and w satisfies the boundary conditions

$$w(\cdot,0) = w(\cdot,1) = 0,\ w(0,\cdot) = \gamma_1 u(1,\cdot),\ w(2,\cdot) = \gamma_2 u(1,\cdot)$$

in the sense of traces. But since we also have $w(1,\cdot) = u(1,\cdot)$, it follows that w satisfies a non-local boundary condition. Furthermore it was proved in [Sku97, Chapter 2.8, Theorem 8.1] that if $\gamma_1\gamma_2 \neq 1$, then the operator T_Ω is bounded and bijective as an operator

$$\begin{aligned} H_0^1(\Omega) \to \{w \in H^1(\Omega) :\ & w(\cdot,0) = w(\cdot,1) = 0, \\ & w(0,\cdot) = \gamma_1 w(1,\cdot),\ w(2,\cdot) = \gamma_2 w(1,\cdot)\}. \end{aligned}$$

Thus solving the functional differential equation (1.4) with zero boundary conditions in $L^2(\Omega)$ is equivalent to solving the differential equation with non-local boundary conditions

$$\begin{cases} \quad \Delta w = f; \\ w(x,0) = w(x,1) = 0 \text{ for almost all } 0 \le x \le 2; \\ w(0,y) = \gamma_1 w(1,y) \\ w(2,y) = \gamma_2 w(1,y) \text{ for almost all } 0 \le y \le 1 \end{cases}$$

in $L^2(\Omega)$, if we have $\gamma_1\gamma_2 \neq 1$. For a general approach see [Sku97].

Finally we present a physical and a financial model featuring a non-local boundary condition directly.

In the theory of wave scattering non-local boundary conditions enter in the following way. Suppose we have a non-relativistic particle being scattered by a potential of final range and we want to obtain its wave function with a postulated asymptotic behavior at ∞. Then one can solve the Schrödinger equation with the given potential in the interior and exterior of the volume V in which the potential is active. At the boundary surface S of

V one takes into account the continuity of the wave function and its normal derivative at S.

Pattanayak and Wolf in [PW76] derived that the time-independent part ψ of the wave function must be equal to certain integral equation at every point r inside the volume V; it is of the form

$$\psi^{(i)}(r) = -\frac{1}{4\pi}\int_S \psi(r')\partial_\nu G(r,r') - \partial_\nu\psi(r')G(r,r')\,\mathrm{d}S(r').$$

This integral involves the boundary values of ψ at S and as such it can be interpreted as non-local boundary condition. Note that in the equation above we have $\psi = \psi^{(i)} + \psi^{(s)}$ ($\psi^{(i)}$ the so-called incident wave and $\psi^{(s)}$ the scattered wave), ν is the outer unit normal at S and G is a certain green function to the Schrödinger equation.

For more details see also [Pat75] or [Hoe78].

Non-local boundary conditions also arise if one wants to model American call options on a stock with dividend rate $d > 0$ as seen in [ALY01]. Let σ be the volatility, r be the interest rate, K be the exercise prize, T_0 be the maturity date. It is then well-known (see e.g. [Mer73]) that the price $c(S,t)$ of the option at time t and stock price S can be modeled by the parabolic free boundary value

$$\begin{cases} \partial_t c + \frac{1}{2}\sigma^2 S^2 \partial_{SS} c + \\ +(r-d)S\partial c_S - rc = 0 \text{ on } (t,S) \in (0,T_0] \times (0, S_c^*(t)) \\ c(S,t) > (S-K)^+ \text{ on } (t,S) \in (0,T_0] \times (0,S_c^*(t)) \\ c(S,t) = (S-K)^+ \text{ on } (t,S) \in (0,T_0] \times [S_c^*(t),\infty) \\ c(S_c^*(t),t) = (S_c^*(t)-K)^+ \text{ on } t \in (0,T_0] \\ \partial_S c(S_c^*(t),t) = 1 \text{ on } t \in (0,T_0] \\ c(S,T_0) = (S-K)^+ \text{ on } S \in [0,\infty) \\ c(0,t) = 0 \text{ on } t \in [0,T_0]. \end{cases} \tag{1.5}$$

The free boundary $S_c^*(t)$ is called the optimal exercise prize: the option should be exercised if $S \geq S_c^*(t)$ and be held otherwise.

With the standard transformations

$$T = \frac{1}{2}\sigma^2 T_0,\ \alpha = \frac{r-d}{\sigma^2},\ \beta = \alpha^2 + \frac{2r}{\sigma^2}$$
$$c(S,t) = Ke^{-\alpha x - \beta\tau}\phi(x,\tau),\ \text{where } T_0 - t = \frac{2\tau}{\sigma^2},\ S = Ke^x$$
$$g^c(x,\tau) = e^{\alpha x + \beta\tau}(e^x - 1)^+,\ x_c^*(t) = \log(\frac{1}{K}S_c^*(T_0 - \frac{2\tau}{\sigma^2}))$$

the equations (1.5) become the free boundary value problem

$$\begin{cases} \partial_\tau \phi - \partial_{xx}\phi = 0 \text{ on } (\tau, x) \in (0,T) \times (-\infty, x_c^*(t)) \\ \phi(x,\tau) > g^c(x,\tau) \text{ on } (\tau, x) \in (0,T) \times (-\infty, x_c^*(t)) \\ \phi(x,\tau) = g^c(x,\tau) \text{ on } (\tau, x) \in (0,T) \times [x_c^*(t), \infty) \\ \phi(x_c^*(\tau),\tau) = g^c(x_c^*(\tau),\tau) \text{ on } \tau \in (0,T] \\ \partial_x \phi(x_c^*(\tau),\tau) = \partial_x g^c(x_c^*(\tau),\tau) \text{ on } \tau \in (0,T] \\ \phi(x,0) = g(x,0) \text{ on } x \in \mathbb{R} \\ \lim_{x\to-\infty} e^{-\alpha x - \beta\tau}\phi(x,t) = 0 \text{ on } \tau \in [0,T]. \end{cases} \tag{1.6}$$

It can be shown that $0 \le x_c^*(t) \le X_c$ for a fixed $X_c > 0$; if we want to solve (1.6) on $[0, X_c] \times [0, T]$ we must provide a boundary condition at $x = 0$, such that the solution ϕ on the restricted domain coincides with the restricted solution on the whole domain. This can be done by simply letting $x \searrow 0$ in an appropriate representation via Green functions of the solution on the whole domain yielding the non-local boundary condition

$$\partial_x \phi(0,t) = \frac{1}{\sqrt{\pi}} \partial_t \int_0^t (t-s)^{-\frac{1}{2}} \phi(0,s)\, \mathrm{d}s.$$

For a similar derivation of the boundary condition above and a numerical method we refer to [HW03].

Preliminaries and preparations

2

Here we make the necessary preparations for our main results. The parts on non-C_0 holomorphic semigroups (Section 2.1), on a transference result on irreducibility from $L^2(\Omega)$ to $C(\overline{\Omega})$ (Proposition 2.16) on kernel operators on L^∞ rather than B_b (second half of Section 2.3), on the maximum modulus principle for elliptic operators in divergence form (Lemma 2.34 and Proposition 2.35), on the properties which translate to the perturbed semigroup (Proposition 2.52), on perturbed non-C_0-semigroups and the strong Feller property of perturbed semigroup (Section 2.6.1) and on a way to prove uniform boundedness for boundary perturbations (Section 2.6.2) may be of independent interest other than exposition and self-containment.

2.1 Holomorphic semigroups

In the present thesis we will encounter semigroups which are not strongly continuous at 0. Here we make precise what we mean by this and collect some (standard) results about this (non-standard) situation.

Definition 2.1. Let X be a Banach space. A *semigroup* on X is a strongly continuous mapping $T\colon (0,\infty) \to \mathcal{L}(X)$ such that

1. $T(t+s) = T(t)T(s)$ for all $t, s > 0$,
2. there exist constants $M > 0$, $\omega \in \mathbb{R}$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t > 0$,
3. $T(t)x = 0$ for all $t > 0$ implies $x = 0$.

If we want to emphasize that 2. holds with constants $M > 0$ and $\omega \in \mathbb{R}$, we say T is of *type* (M, ω).

If T is a semigroup of type (M, ω), there exists a unique operator A such that $(\omega, \infty) \subset \rho(A)$ and the resolvent of A is given by the Laplace transform of T, that is

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t$$

for all $x \in X$ and $\lambda > \omega$ (see [ABHN11, Proposition 3.2.4]). The operator A is called the *generator* of the semigroup T.

We note that the generator A of a semigroup T need not be densely defined; although if it is, then the semigroup T is actually a C_0-semigroup. We make note of this in the following proposition, which is taken from [ABHN11, Corollary 3.3.11]:

Proposition 2.2. *Let T be semigroup with generator A. Then T is a C_0-semigroup if and only if $D(A)$ is dense in X.*

A semigroup of type $(M, 0)$ is called *bounded*, and if in addition $M = 1$ it is called a *contraction semigroup*; the following proposition characterizes such semigroups.

Proposition 2.3. *Let T be semigroup with generator A. The following are equivalent*

i) T is a contraction semigroup

ii) $\lambda R(\lambda, A)$ is contractive for all sufficiently large λ.

Proof. $i) \Rightarrow ii)$: If $\|T(t)\| \leq 1$ for all $t > 0$ then

$$\|\lambda R(\lambda, A)x\| = \| \int_0^\infty \lambda e^{-\lambda t} T(t)x \, \mathrm{d}t\| \leq \int_0^\infty \lambda e^{-\lambda t} \|x\| \, \mathrm{d}t = \|x\|$$

for λ sufficiently large.

$ii) \Rightarrow i)$: By the Post-Widder inversion formula (see [ABHN11, Theorem 1.7.7]) we have

$$(-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} R(\frac{k}{t}, A)^{(k)} \to T(t) \text{ in } \mathcal{L}(X)$$

as $k \to \infty$ for all $t > 0$. Furthermore the derivatives of $R(\lambda, A)$ are given by

$$R(\lambda, A)^{(n)} = (-1)^n n! R(\lambda, A)^{n+1}$$

by the resolvent equation. Hence *(ii)* implies

$$\frac{1}{n!} \|\lambda^{n+1} R(\lambda, A)^{(n)}\| \leq 1.$$

for large λ and the claim follows. □

Definition 2.4. A semigroup T on a Banach space X is called *holomorphic* or *analytic*, if there exists an angle $\theta \in (0, \frac{\pi}{2}]$ such that T has a holomorphic extension to the sector

$$\Sigma_\theta := \{re^{i\theta},\, r > 0,\, |\varphi| < \theta\}$$

which is bounded on $\Sigma_\theta \cap \{z \in \mathbb{C},\, |z| \leq 1\}$. A holomorphic semigroup is called *holomorphic bounded*, if its holomorphic extension is bounded on all of Σ_θ.

We note that the semigroup law automatically holds for the holomorphic extension. Generators of holomorphic semigroups can be characterized by the following *holomorphic estimate*, cf. [ABHN11, Corollary 3.7.12 and Proposition 3.7.4].

Theorem 2.5. *An operator A on X generates a holomorphic semigroup if and only if there exists $\omega \in \mathbb{R}$ such that $\mathbb{C}^+ + \omega := \{\lambda \in \mathbb{C},\, \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and*

$$\sup_{\operatorname{Re}\lambda > \omega} \|\lambda R(\lambda, A)\| < \infty.$$

The operator A generates a bounded holomorphic semigroup if and only if ω can be chosen as 0 above.

A semigroup is called *exponentially stable*, if it is of type $(M, -\epsilon)$ for some $\epsilon > 0$. Clearly A generates an exponentially stable semigroup if and only if $A + \epsilon$ generates a bounded semigroup for some $\epsilon > 0$.

To prove exponential stability for holomorphic semigroups it suffices to study the spectrum of the generator. More precisely, we have the following.

Proposition 2.6. *Let A be the generator of a holomorphic semigroup T. Then T is exponentially stable if and only if $\sigma(A) \subset \mathbb{C}^- := \{\lambda \in \mathbb{C},\, \operatorname{Re}\lambda < 0\}$.*

Proof. Assume $\sigma(A) \subset \mathbb{C}^-$; then there is $\epsilon > 0$ such that $\rho(A) \subset \mathbb{C}^+ - \epsilon$. By [Lun95, Corollary 2.3.2] we may choose $0 < \omega' < \epsilon$ and $M' > 0$ such that T is of type $(M', -\omega')$. Hence T is exponentially stable.

The converse is obvious. □

Now let X be an ordered Banach space with order $\leq$. We recall that a semigroup T is called *positive*, if $T(t)$ is a positive operator for all $t > 0$; i.e. that we have $T(t)u \geq 0$ for all $u \geq 0$.

The following characterization of positive semigroups on Banach lattices holds.

Proposition 2.7. *Let A be the generator of a semigroup T on a Banach lattice X. Then the following are equivalent.*

i) T is positive.

ii) $R(\lambda, A)$ is positive for all sufficiently large λ.

Proof. Let T be of type (M, ω). If T is positive, then for $u \geq 0$ we also have $e^{-\lambda t}T(t)u \geq 0$ for all $t > 0$ and $\lambda > 0$. Since the positive cone of X is closed, we have

$$R(\lambda, A)u = \int_0^\infty e^{-\lambda t}T(t)u \, \mathrm{d}t \geq 0$$

for all $\lambda > \omega$.

The converse follows by the Post-Widder inversion formula, as

$$0 \leq R(\lambda, A)^{n+1}u = (-1)^n \frac{1}{n!} R(\lambda, A)^{(n)}u$$

for all $u \geq 0$. □

With (almost) the same argument the following domination result holds for positive semigroups.

Proposition 2.8. *Let S and T be positive semigroups on a Banach lattice X with generators A and B respectively. Then the following are equivalent.*

i) $S(t) \leq T(t)$ for all $t > 0$.

ii) $R(\lambda, A) \leq R(\lambda, B)$ for all sufficiently large λ.

Now let A be a generator of a positive semigroup. The *spectral bound* $s(A)$ of A is defined as

$$s(A) := \sup\{\operatorname{Re}\lambda, \, \lambda \in \sigma(A)\}.$$

In general we have $-\infty \leq s(A) < \infty$. If however $s(A) > -\infty$, then $s(A) \in \sigma(A)$ and $s(A)$ is an eigenvalue of A, see [ABHN11, Proposition 3.11.2]. We now obtain the following spectral criterion for exponential stability.

Proposition 2.9. *Let T be a positive, holomorphic semigroup on a Banach lattice X with generator A. If $[0, \infty) \subset \rho(A)$, then T is exponentially stable.*

Proof. Since $s(A) \in \sigma(A)$ when $s(A) > -\infty$, it follows from our assumption that $s(A) < 0$. Now Proposition 2.6 yields the claim. □

2.2 Irreducible semigroups

In this section we collect some known facts on positive, irreducible semigroups. In some cases we present some variations or adapt results to our special situation.

Let E be a real Banach lattice. In our context E will be $C(\overline{\Omega})$ or $L^q(\Omega)$. Let T be a strongly continuous semigroup on E which is positive, i.e. for $f \in E_+$ we have $T(t)f \in E_+$ for all $t \geq 0$. We denote the generator of T by A. The *spectral bound* of A is defined by

$$s(A) := \sup\{\operatorname{Re}\lambda : \ \lambda \in \sigma(A_{\mathbb{C}})\}$$

where $\sigma(A_{\mathbb{C}})$ is the spectrum of the generator $A_{\mathbb{C}}$ of the complexification of T. In what follows, we will not distinguish between an operator and its complexification. In particular, when we talk about the spectrum, resolvent, etc. of an operator, we always mean the spectrum/resolvent, etc. of its complexification.

By [AGG+86, C-III Theorem 1.1], $s(A) \in \sigma(A)$ whenever $\sigma(A) \neq \emptyset$. If A has compact resolvent, then $\sigma(A)$ consists of isolated points which are all eigenvalues.

Theorem 2.10. *Assume that $T(t)$ is compact for all $t > 0$ and that $s(A) = 0$. Then there exist a positive projection $P \neq 0$ of finite rank, $\epsilon > 0$ and $M \geq 0$ such that*

$$\|T(t) - P\|_{\mathscr{L}(E)} \leq Me^{-\epsilon t}$$

for all $t > 0$.

Proof. Since $T(t)$ is compact for all $t > 0$, T is immediately norm continuous and it follows from [AGG+86, C-III Corollary 2.13] that there is some $\delta > 0$ such that $\operatorname{Re}\lambda \leq -2\delta < 0$ for all $\lambda \in \sigma(A) \setminus \{0\}$. Denote by P the spectral projection with respect to 0, i.e.

$$P := \frac{1}{2\pi i} \int_{|\lambda|=\delta} R(\lambda, A)\, \mathrm{d}\lambda.$$

As $T(t)$ is compact for all $t > 0$, so is the resolvent and thus also P, whence it has finite rank. The space $F = (I - P)E$ is invariant under the semigroup and the generator A_F of the restriction has its spectrum in a strict left half plane. Since the semigroup is immediately norm continuous there exist $\epsilon > 0$, $M \geq 0$ such that $\|T(t)|_F\|_{\mathscr{L}(F)} \leq Me^{-\epsilon t}$ and hence $\|T(t) - P\|_{\mathscr{L}(E)} \leq Me^{-\epsilon t}$ for all $t \geq 0$. □

Theorem 2.10 implies in particular that there exists $u \geq 0$, $u \neq 0$, such that $T(t)u = u$ for all $t \geq 0$. Thus the Krein–Rutman Theorem which asserts that the largest eigenvalue (i.e. $s(A)$) has a positive eigenfunction is incorporated in Theorem 2.10.

We next want to investigate when P has rank one and the positive eigenfunction is strictly positive. This can be done via the notion of *irreducibility* as we will see. A subspace J of E is called an *ideal* if

(i) $u \in J$ implies $|u| \in J$ and

(ii) if $u \in J$ and $0 \leq v \leq u$ implies $v \in J$.

A positive semigroup T on E is called *irreducible* if the only if invariant closed ideals are $J = \{0\}$ and $J = E$.

If $J = C(\overline{\Omega})$ then $J \subset E$ is a closed ideal if and only if there exists a closed subset K of $\overline{\Omega}$ such that

$$J = \{f \in C(\overline{\Omega}) : \ f|_K = 0\}.$$

If $E = L^q(\Omega)$ $(1 \leq q < \infty)$ then $J \subset E$ is a closed ideal if and only if there exists a measurable subset K of Ω such that

$$J = \{f \in L^q(\Omega) : \ f|_K = 0 \text{ a.e.}\}.$$

We say that $u \in E$ is a *quasi interior point* and write $u \gg 0$ if the principal ideal

$$E_u := \{v \in E : \ \exists c > 0 \text{ such that } |v| \leq cu\}$$

is dense in E.

If $E = C(\overline{\Omega})$ then $u \gg 0$ if and only if there is $\delta > 0$ such that $u(x) \geq \delta > 0$ for all $x \in \overline{\Omega}$. In this case u is actually an inner point of the positive cone. If $E = L^p(\Omega)$ then $u \gg 0$ if and only if $u(x) > 0$ for almost every x.

For $\varphi \in E'$ we write $\varphi \gg 0$ if $\langle \varphi, f \rangle = 0$ implies $f = 0$ for all $f \in E_+$. In this case we call φ a *strictly positive functional*.

If $E = C(\overline{\Omega})$, then $\varphi \gg 0$ if and only if there exists a strictly positive Borel measure ν, i.e.$\nu(U) > 0$ for all non-empty relatively open sets $U \subset \overline{\Omega}$, such that

$$\langle \varphi, f \rangle = \int_{\overline{\Omega}} f(x) \, \mathrm{d}\nu(x).$$

If $E = L^q(\Omega)$ for $\varphi \in L^{q'}(\Omega) \simeq (L^q(\Omega))'$, where $\frac{1}{q'} + \frac{1}{q} = 1$, to say that $\varphi \gg 0$ means that $\varphi(x) > 0$ almost everywhere, so that our notation $\varphi \gg 0$ is consistent with the above.

The importance of these concepts in the study of asymptotic behavior stems from the fact that positive fixed points of positive, irreducible semigroups are *strictly positive*. More precisely, if T is a positive, irreducible, strongly continuous semigroup and $u \geq 0$, $u \neq 0$, is such that $T(t)u = u$

for all $t > 0$, then $u \gg 0$ and if $0 < \varphi \in E'$ is such that $T(t)'\varphi = \varphi$ for all $t > 0$ then φ is strictly positive. Moreover, because of irreducibility, $s(A)$ cannot be a pole of order larger than 1, see [AGG$^+$86, C-III Proposition 3.5]. This implies that $T(t)P = P$ for all $t > 0$ in the proof of Theorem 2.10 even though the semigroup is not assumed to be bounded. We thus obtain the following result on asymptotic stability.

Theorem 2.11. *Let T be a positive, irreducible strongly continuous semigroup on E with generator A. Assume that $T(t)$ is compact for $t > 0$ and $s(A) = 0$. then there exist $0 \ll u \in \ker A$, $0 \ll \varphi \in \ker A'$, $\epsilon > 0$, $M \geq 0$ such that $\langle \varphi, u \rangle = 1$ and*

$$\|T(t) - \varphi \otimes u\|_{\mathscr{L}(E)} \leq M^{-\epsilon t}$$

for all $t \geq 0$ where we have written $\varphi \otimes u$ for the projection

$$(\varphi \otimes u)(f) = \langle \varphi, f \rangle u.$$

In particular

$$\lim_{t\to\infty} T(t)f = \langle \varphi, f \rangle u,$$

i.e. the orbits of the semigroup converge to an equilibrium.

Theorems 2.10 and 2.11 lie at the heart of the Perron-Frobenius theory. We refer to [AGG$^+$86] for more information.

We shall have occasion to use the following strict monotonicity of the spectral bound resp. the comparison principle if one considers the contraposition of following statement.

Theorem 2.12. *Let S and T be strongly continuous semigroups on E with generators B and A respectively. Assume that*

(i) $0 \leq S(t) \leq T(t)$ *for all $t > 0$;*

(ii) A has compact resolvent, and

(iii) T is irreducible.

If $A \neq B$, then $s(B) < s(A)$.

Proof. This is a version of [AB92, Theorem 1.3], see also [Are05, Theorem 10.2.10] in connection with [Are05, Theorem 10.6.3 and 10.6.1]. □

Next we describe ways to prove irreducibility. On $L^2(\Omega)$ this is very easy if the semigroup is associated with a form in virtue of the Beurling–Deny–Ouhabaz criterion for the invariance of closed convex sets. In particular the following holds true (see [MVV05, Theorem 2.1] noting that for the projection on $L^2(\omega)$ for $\omega \subset \Omega$ measurable is given by $u \mapsto 1_\omega u$ or [Ouh05, Theorem 2.10]). In the following, recall that a sesquilinear form $a\colon V \times V \to \mathbb{C}$ on some closed subspace $V \subset L^2(\Omega)$ is called *elliptic*, if there is $\omega \in \mathbb{R}$ such that $a(u,u) + \omega\|u\|_{L^2(\Omega)} \geq \alpha\|u\|_{H^1(\Omega)}$ for some $\alpha > 0$.

Theorem 2.13. *Let $V \subset H^1(\Omega)$ be a closed subspace and $\Omega \subset \mathbb{R}^d$ open. Let $a\colon V \times V \to \mathbb{C}$ be an elliptic and continuous form on $L^2(\Omega)$ such that the associated semigroup T is positive. Then T is irreducible if and only if for $U \subset \Omega$ such that $1_U u \in V$ and $\operatorname{Re} a(1_U u, 1_{\Omega\setminus U} u) \geq 0$ for all $u \in V$ it follows that either $|U| = 0$ or $|\Omega \setminus U| = 0$.*

In particular the requirement $1_U u \in V$ for all $u \in V$ puts strong restrictions on the subsets U of the previous theorem, if $H_0^1(\Omega) \subset V$ (see [Are05, Proposition 11.1.2.]).

Proposition 2.14. *Let $\Omega \subset \mathbb{R}^d$ open and connected and $V \subset H^1(\Omega)$ be a closed subspace containing $H_0^1(\Omega)$. If $U \subset \Omega$ is measurable such that $1_U u \in V$ for all $u \in V$ then either $|U| = 0$ or $|\Omega \setminus U| = 0$.*

These two results give the following convenient criterion, if Ω is connected (cf also [Ouh05, Theorem 2.10]).

Corollary 2.15. *Let $V \subset H^1(\Omega)$ be a closed subspace containing $H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a connected, open set. Let $a\colon V \times V \to \mathbb{C}$ be a continuous and elliptic form such that the associated semigroup T is positive. Then T is irreducible.*

On $C(\overline{\Omega})$ irreducibility is a stronger notion than on $L^2(\Omega)$. However, sometimes the irreducibility on $C(\overline{\Omega})$ can be deduced from that on $L^2(\Omega)$.

Proposition 2.16. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and T be a positive, irreducible, strongly continuous semigroup on $L^2(\Omega)$ whose generator A has compact resolvent. Assume that T leaves $C(\overline{\Omega})$ invariant and that the restriction T^C of T to $C(\overline{\Omega})$ is strongly continuous and suppose that its generator A^C has compact resolvent. Assume that $s(A) = 0$. Then T^C is irreducible if and only if there exists $u \in \ker A \cap C(\overline{\Omega})$ such that $u(x) \geq \delta > 0$ for all $x \in \overline{\Omega}$.*

Proof. Assume that there exists $0 \ll u \in C(\overline{\Omega}) \cap \ker A$. Since T is irreducible 0 is a pole of order 1 and the residuum P is of the form

$$Pf = \Big(\int_\Omega \varphi f \,\mathrm{d}x\Big) \cdot u$$

for some $0 \ll \varphi \in L^2(\Omega)$, see [AGG+86, C-III Proposition 3.5]. Since $C(\overline{\Omega})$ is dense in $L^2(\Omega)$, it follows that the coefficients in the Laurent series expansion in $C(\overline{\Omega})$ around 0 (see [AGG+86, A-III, Equation (3.1)]) are the restriction of those in $L^2(\Omega)$. Thus 0 is also a pole of order 1 of the resolvent of A^C. The residuum

$$P^C = \frac{1}{2\pi i} \int_{|\lambda|=\epsilon} R(\lambda, A^C)\,\mathrm{d}\lambda$$

is the same, i.e. $P^C = P|_{C(\overline{\Omega})}$. Now let $J = \{f \in C(\overline{\Omega}) : f|_K = 0\}$ be an invariant ideal. Let $z \in K$, $f \in J$, $f \geq 0$. Then $(T(t)f)(z) = 0$ for all $t > 0$ and hence $(R(\lambda, A^C)f)(z) = 0$ for all $\lambda > 0$, since we suppose that $s(A) = 0$ and know that $s(A)$ is the abscissis of the Laplace transform of the semigroup [ABHN11, Theorem 5.3.1]. Thus

$$\int_\Omega f(x)\varphi(x)\,\mathrm{d}x = \lim_{\lambda\downarrow 0}(\lambda R(\lambda, A^C)f)(z) = 0.$$

Since $\varphi \gg 0$ in $L^2(\Omega)$ this implies $f = 0$. Consequently $J = 0$. This proves the sufficiency.

To show the necessity, recall that 0 is also a pole of $R(\lambda, A^C)$. It follows that $s(A^C) = 0$. It follows from Theorem 2.11 that there exists $0 \ll u \in \ker(A^C) \subset \ker(A)$. □

2.3 Kernel operators

In this section we give basic properties of kernel operators on the spaces $B_b(\overline{\Omega})$ and $L^\infty(\Omega)$ and give an example where the notion differs between the two spaces. Here we set $K = \overline{\Omega}$ where $\Omega \subset \mathbb{R}^d$ is open and bounded. We begin with the situation on $B_b(K)$.

Definition. A *kernel* on K is a map $k\colon K \times \mathscr{B}(K) \to \mathbb{C}$ such that

i) the map $K \to \mathbb{C}$, $x \mapsto k(x, A)$ is $\mathscr{B}(K)$-measurable for all $A \in \mathscr{B}(K)$,

ii) the map $\mathscr{B}(K) \to \mathbb{C}$, $A \mapsto k(x, A)$ is a (complex) Borel measure on K for all $x \in K$ and

iii) $\sup_{x\in K} |k|(x, K) < \infty$, where $|k|(x, \cdot)$ denotes the total variation of the measure $k(x, \cdot)$

We now define the term *kernel operator*, for now only for operators on $C(K)$ or $B_b(K)$.

Definition 2.17. Let $X = C(K)$ or $X = B_b(K)$. We call an operator $T \in \mathcal{L}(X)$ a *kernel operator*, if there exists a kernel on K such that

$$Tf(x) = \int_K f(y)\,k(x, \mathrm{d}y)$$

for all $f \in X$ and $x \in K$. Since there is at most one kernel satisfying the above relation, we call k the *kernel associated with* T and T the *operator associated with* k.

We note that every bounded operator on $C(K)$ is kernel operator. Indeed by the proof of [Kun11, Proposition 3.5]: Let $T \in \mathcal{L}(C(K))$ and set $k(x,\cdot) := T^*\delta_x \in \mathcal{M}(K)$, $x \in K$, where T^* denotes the adjoint operator of T and δ_x is the Dirac measure concentrated at x. By definition we have

$$(Tf)(x) = \langle Tf, \delta_x \rangle = \langle f, T^*\delta_x \rangle = \int_K f(y)\, k(x, \mathrm{d}y)$$

and also $\sup_{x\in K} |k|(x,K) \leq \|T\|_{\mathcal{L}(C(K))} < \infty$. It remains to show that $x \mapsto k(x,A)$ is measurable for all $A \in \mathscr{B}(K)$. Denote the set of sets for which the claim is true by $\mathcal{G}$ and the set of relatively open sets in K by $\mathcal{E}$. Then for each $A \in \mathcal{E}$ there is a increasing sequence $0 \leq u_n \in C(K)$ with $u_n \nearrow 1_A$ pointwise. Now the monotone convergence theorem yields

$$k(x,A) = \langle 1_A, T^*\delta_x \rangle = \lim_{n\to\infty} \langle u_n, T^*\delta_x \rangle = \lim_{n\to\infty} (Tu_n)(x)$$

for all $x \in K$. It follows that $x \mapsto k(x,A)$ is measurable as the pointwise limit of measurable functions and hence $\mathcal{E} \subset \mathcal{G}$. Using the properties of a kernel it is easy to see that $\mathcal{G}$ is a Dynkin system; the claim follows then for all measurable sets from Dynkin's π-λ theorem since $\mathcal{E}$ is closed under finite intersections.

Now let $T \in \mathcal{L}(C(K))$; as we have seen it is a kernel operator and as such it may be extended to a bounded operator $\tilde{T}$ on $B_b(K)$ by allowing bounded measurable functions in the integral. We call this extension $\tilde{T}$ the *canonical extension* of T to $B_b(K)$. We note that there may be other extension of T to $B_b(K)$ but $\tilde{T}$ is the only one which is a kernel operator.

On the other hand an operator on $B_b(K)$ need not be a kernel operator. In fact it is a kernel operator if and only if its adjoint leaves $\mathcal{M}(K)$ invariant. But for us the following characterization is more useful.

Lemma 2.18. *Let $T \in \mathcal{L}(B_b(K))$. Then the following are equivalent.*

1. *T is a kernel operator.*
2. *T is* pointwise continuous, *that is if $(f_n)_{n\in\mathbb{N}} \subset B_b(K)$ is bounded and converges pointwise to f, then $Tf_n \to Tf$ pointwise.*

If T is positive, then it suffices to consider bounded and increasing sequences in 2.

Proof. If T is a kernel operator, then the claim follows immediately from the dominated convergence theorem.

For the converse set $k(x, A) := (T1_A)(x)$ for $x \in K$ and A measurable. Since T is an operator on $B_b(K)$ the map $x \mapsto k(x, A)$ is measurable. Pointwise continuity of T yields that $k(x, \cdot)$ is a measure, thus that k is kernel. Since the step functions are dense in $B_b(K)$, we have that T is associated with k. □

An immediate consequence is the following.

Corollary 2.19. *The space of all kernel operators is norm closed in $\mathcal{L}(B_b(K))$.*

Proof. Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(B_b(K))$ be a sequence converging to $T \in \mathcal{L}(B_b(K))$ and $(f_n)_{n\in\mathbb{N}} \subset B_b(K)$ be a bounded sequence converging pointwise to f, w.l.o.g. $\|f_n\|_{B_b(K)} \leq 1$ for all $n \in \mathbb{N}$. For $\epsilon > 0$ there is $M \in \mathbb{N}$ such that $\|T_m - T\|_{\mathcal{L}(B_b(K))} \leq \epsilon$ for all $m \geq M$. Now let $m \geq M$ and $N_m \in \mathbb{N}$ be such that $|T_m f_n(x) - T_m f(x)| \leq \epsilon$ for all $n \geq N_m$. Then for $n \geq N_m$ we have

$$\begin{aligned}|Tf_n(x) - Tf(x)| \leq \\ |T_m f_n(x) - Tf_n(x)| + |T_m f_n(x) - T_m f(x)| + |T_m f(x) - Tf(x)| \\ \leq \epsilon\|f_n\|_{B_b(K)} + \epsilon + \epsilon\|f\|_{B_b(K)} \leq 3\epsilon\end{aligned}$$

proving the claim by Lemma 2.18 above. □

Definition. A *strong Feller operator* is called a kernel operator $T \in \mathcal{L}(B_b(K))$ such that $TB_b(K) \subset C(K)$. A bounded operator $T \in \mathcal{L}(C(K))$ is called a *strong Feller operator*, if its canonical extension $\tilde{T}$ is a strong Feller operator.

Obviously a kernel operator T with kernel k on $B_b(K)$ is a strong Feller operator if and only if $x \mapsto k(x, A)$ is continuous for all measurable A. By Corollary 2.19 the cone of all strong Feller operators is closed in $\mathcal{L}(B_b(K))$.

For us one important fact on strong Feller operators is the following compactness result.

Lemma 2.20. *Let $S, T \in \mathcal{L}(B_b(K))$ be strong Feller operators and S be positive. Then the composition ST is compact.*

For the proof we follow the approach in [Rev75, § 1.5].

Proof. The proof consists of two steps.

Step 1: A strong Feller operator T maps a bounded sequence of measurable functions to a pointwise converging continuous one.

We first prove that there is $\tau \in \mathcal{M}(K)$ such that $|k(x, \cdot)| \ll \tau$ for all $x \in K$. To this purpose let $(x_n)_{n\in\mathbb{N}} \subset K$ be dense and set

$$\tau := \sum_{n\in\mathbb{N}} 2^{-n}|k(x_n, \cdot)|.$$

Since k is a bounded kernel this series converges absolutely. Now let $A \subset K$ be measurable with $\tau(A) = 0$. Then it follows that for all $n \in \mathbb{N}$

$$|k|(x_n, A) \leq 2^n \tau(A) = 0.$$

This means that the continuous function $x \mapsto |k|(x, A)$ is 0 on a dense subset of K, hence identically 0 on all of K. It follows that $|k|(x, \cdot) \ll \tau$ for all $x \in K$ and hence $k(x, \cdot)$ has a density function $\varphi_x \in L^1(\Omega, \tau)$ with respect to τ for all $x \in K$. In particular this means that

$$\int_K f(y)\, k(x, \mathrm{d}y) = \int_K f(y) g_x(y)\, \tau(\mathrm{d}y)$$

for all $x \in K$ and all $f \in B_b(K)$.

Now let $(g_n)_{n\in\mathbb{N}} \subset B_b(K)$ be bounded. Since $L^1(\Omega, \tau)$ is separable in our case, we have that $g_n \to g \in L^\infty(\Omega, \tau)$ in the weak*-topology by the Banach-Alaoglu-theorem; that is

$$\int_K g_n \varphi\, \tau(\mathrm{d}x) \to \int_K g\varphi\, \tau(\mathrm{d}x), \text{ as } n \to \infty$$

for all $\varphi \in L^1(\Omega, \tau)$. Setting $\varphi = \varphi_x$ for each $x \in K$ yields the claim.

Step 2: A positive strong Feller operator S maps a pointwise converging sequence of measurable functions to a uniformly converging one.

Let $(g_n)_{n\in\mathbb{N}} \subset B_b(K)$ be a pointwise converging sequence, w. l. o. g. $g_n(x) \to 0$ for all $x \in K$. Now set $h_n(x) = \sup_{m\geq n} |g_n(x)|$. Since $|Sg_n| \leq Sh_n$ by positivity of S, it suffices to show that Sh_n converges uniformly to 0.

We have $h_n \geq h_{n+1}$ and hence $Sh_n \geq Sh_{n+1}$, i.e. $(Sh_n)_{n\in\mathbb{N}}$ is a decreasing sequence of continuous functions. Furthermore we have $\lim_{n\to\infty} h_n(x) = \limsup_{n\to\infty} |g_n(x)| = 0$ and by the monotone convergence theorem also $\lim_{n\to\infty} Sh_n(x) \to 0$ for all $x \in K$.

Applying Dini's lemma on this sequence yields the claim. □

We make note of the two steps in the proof above in a separate corollary.

Corollary 2.21. *A strong Feller operator maps a bounded sequence of functions in $B_b(K)$ into a pointwise converging one. If it is also positive, then it maps a pointwise converging sequence into a uniformly converging one.*

We now turn out attention to the situation on $L^\infty(\Omega)$. We will encounter operators $T \in \mathcal{L}(L^\infty(\Omega))$ mapping $L^\infty(\Omega)$ into $C(K)$. Thus $C(K)$ is invariant under such operators and we can consider $T|_{C(K)}$ as an operator on $C(K)$. As we have already seen this restriction is a kernel operator and as such has one canonical extension $\tilde{T}$ to $B_b(K)$. One obvious question is whether $\tilde{T} = \iota \circ T \circ \pi$ where $\pi\colon B_b(K) \to L^\infty(\Omega)$ is the canonical epimorphism, i.e.

π maps a bounded measurable function to its equivalence class modulo equality almost everywhere, and $\iota\colon L^\infty(\Omega) \to B_b(K)$ selects a representative from each equivalence class modulo equality almost everywhere such that $\iota 0 = 0$.

Unfortunately this need not be the case as the following example shows; the operator $\iota \circ T \circ \pi$ need not even be a kernel operator.

Example 2.22. Set $K = \{\infty\} \cup \mathbb{N}$ where the neighborhoods of $\infty$are the sets of the form $\{n, n+1, n+2, \ldots\}$. Then we have that $B_b(K) = \ell^\infty(K)$ and

$$C(K) = \{(x_\infty, x_1, x_2, \ldots) : \; x_n \to x_\infty\}.$$

Now pick a Banach limit φ, i.e. a functional $\varphi \in (\ell^\infty(K))^*$ with $\varphi(x) = \lim_{n\to\infty} x_n$ for all convergent sequences $x = (x_n)_{n\in\mathbb{N}}$.

Define $T \in \mathcal{L}(\ell^\infty(K))$ by

$$T(x_\infty, x_1, x_2, \ldots) := \varphi(x_1, x_2, \ldots) \cdot (1, 1, 1, \ldots).$$

Then indeed T takes values in $C(K)$. The kernel associated with $T|_{C(K)}$ is $k(x, A) = \delta_\infty(A)$. Thus the canonical extension $\tilde{T}$ evaluates a sequence $(x_\infty, x_1, x_2, \ldots) \in \ell^\infty(K)$ at the point ∞. However this need not be the value of the Banach limit $\varphi(x_1, x_2, \ldots)$ and hence $\iota \circ T \circ \pi \neq \tilde{T}$.

We now give an criterion to determine whether an operator $T \in \mathcal{L}(L^\infty(\Omega))$ is a kernel operator. It follows directly from Lemma 2.18.

Lemma 2.23. *Let $T \in \mathcal{L}(L^\infty(\Omega))$ be an operator taking values in $C(K)$. Then $\iota \circ T \circ \pi \in \mathcal{L}(B_b(K))$ is a kernel operator if and only if whenever $(f_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ is a bounded sequence of functions we have $Tf_n(x) \to Tf(x)$ for all $x \in K$ where $f(x) := \lim_{n\to\infty} f_n(x)$ for almost every $x \in \Omega$. If T is positive it suffices to consider positive and increasing sequences of functions above.*

In these cases $\iota \circ T \circ \pi$ is a strong Feller operator.

Motivated by the above result, we call an operator $T \in \mathcal{L}(L^\infty(\Omega))$ a *strong Feller operator*, if $TL^\infty(\Omega) \subset C(K)$ and $\iota \circ T \circ \pi \in \mathcal{L}(B_b(K))$ is a strong Feller operator. Let us make note of the two following facts. Firstly as with the situation on $B_b(K)$ the subspace of strong Feller operators in $\mathcal{L}(L^\infty(\Omega))$ is closed. Secondly if $A \subset \Omega$ has Lebesgue measure 0, then $\pi(1_A) = 0$ and thus $(\iota \circ T \circ \pi)1_A = 0$ in $L^\infty(\Omega)$. This implies that if $T \in \mathcal{L}(L^\infty(\Omega))$ is a strong Feller operator, then the kernel k associated to T — i.e. the kernel associated to $\iota \circ T \circ \pi$ — has that $k(x, \cdot) \in \mathcal{M}(\overline{\Omega})$ is absolutely continuous w.r.t. the Lebesgue measure on Ω for all $x \in K$.

The compactness result Lemma 2.20 also has a version in $L^\infty(\Omega)$.

Corollary 2.24. *Let $S, T \in \mathcal{L}(L^\infty(\Omega))$ be strong Feller operators and S be positive. Then T maps bounded sequences to pointwise converging ones and S maps pointwise converging sequences to uniformly converging ones. In particular ST is compact.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ be bounded and set $\tilde{f}_n := \iota f_n$, that is $\tilde{f}_n \in B_b(K)$ with $\tilde{f}_n = f_n$ almost everywhere on Ω. Then Corollary 2.21 implies that $((\iota \circ T \circ \pi)\tilde{f}_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega})$ converges pointwise. Since continuous functions which are equal almost everywhere on Ω are equal on K, we also have that $(Tf_n)_{n\in\mathbb{N}}$ converges poitwise.

By using the same argumentation on S we have Corollary 2.21 implying that S maps a poitwise converging sequence to a uniformly converging one. □

As a last topic we give an application to holomorphic semigroups.

Proposition 2.25. *Let A be the generator of a positive, analytic semigroup T on $L^\infty(\Omega)$ of type (M, ω). Assume that the operator $R(\lambda, A)$ is a strong Feller operator for each $\lambda > \omega$.*

Then for every $\lambda \in \rho(A)$ the operator $R(\lambda, A)$ is strongly Feller and compact; moreover $T(t)$ is strongly Feller and compact for $t > 0$.

Proof. The function $\rho(A) \to \mathcal{L}(L^\infty(\Omega))$, $\lambda \mapsto R(\lambda, A_\mu)$ is analytic. On (ω, ∞) it takes values in the closed cone of strong Feller operators by assumption. Since $\rho(A)$ contains a sector $\Sigma_\theta \setminus \{0\}$ for a suitable $\theta > \frac{\pi}{2}$ and the sector is connected, the uniqueness theorem for holomorphic functions (see [ABHN11, Proposition A.2] for an appropriate vector-valued formulation) implies that $R(\lambda, A)$ is a strong Feller operator for all $\lambda \in \Sigma_\theta \setminus \{0\}$. As the holomorphic semigroup T can be computed from the resolvent via a Bochner integral over a contour in $\Sigma_\theta \setminus \{0\}$, it follows that T consists also of strong Feller operators.

Now Lemma 2.24 implies that $T(t) = T(\frac{t}{2})T(\frac{t}{2})$ is compact for all $t > 0$. Consequently also the resolvent $R(\lambda, A)$ — being the Bochner integral $\int_0^\infty e^{-\lambda t}T(t)\,\mathrm{d}t$ for $\operatorname{Re}\lambda > \omega$ — is compact for all $\lambda \in \rho(A)$. In particular $\sigma(A)$ is countable and thus $\rho(A)$ connected. Invoking the uniqueness theorem for holomorphic functions a second time yields that $R(\lambda, A)$ is a strong Feller operator for all $\lambda \in \rho(A)$. □

2.4 Elliptic operators in non-divergence Form

Let $\Omega \subset \mathbb{R}^d$ be bounded and open and $\mathscr{A}\colon W^{2,d}_{\mathrm{loc}}(\Omega) \to L^d_{\mathrm{loc}}(\Omega)$, be a second order uniformly elliptic differential operator in non-divergence form; that is

$\mathscr{A}$ is given by

$$\mathscr{A}u = \sum_{i,j=1}^{d} a_{ij} D_i D_j u + \sum_{j=1}^{d} c_j D_j u + d_0 u,$$

where $a_{ij} \in C(\overline{\Omega})$ with $a_{ij} = a_{ji}$ and

$$\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2 \text{ for all } x \in \overline{\Omega},\ \xi \in \mathbb{R}^d$$

for some $\alpha > 0$ and $c_j, d_0 \in L^\infty(\Omega)$ with $d_0 \leq 0$. In this section we gather results on the realization of this elliptic operator with Dirichlet boundary conditions.

For our convenience we set

$$W(\Omega) = \bigcap_{1<p<\infty} W^{2,p}_{\text{loc}}(\Omega).$$

Define the *maximal operator* $A_{\max}$ (i.e. without boundary conditions) by

$$\begin{aligned} D(A_{\max}) &= \{u \in C(\overline{\Omega}) \cap W(\Omega) : \ \mathscr{A}u \in L^\infty(\Omega)\} \\ A_{\max} u &= \mathscr{A}u \end{aligned}$$

and A_0 as the realization with Dirichlet boundary conditions of the part of $\mathscr{A}$ in $L^\infty(\Omega)$, i.e.

$$\begin{aligned} D(A_0) &= \{u \in C_0(\Omega) \cap W(\Omega) : \ \mathscr{A}u \in L^\infty(\Omega)\} \\ A_0 u &= \mathscr{A}u. \end{aligned}$$

Note that neither the operator A_0 nor its part in $C(\overline{\Omega})$ is densely defined; but the part of $A_{\max}$ in $C(\overline{\Omega})$ is densely defined since $C^\infty(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u = v|_{\overline{\Omega}} \text{ for some } v \in C^\infty(\mathbb{R}^d)\}$ is a subset of its domain.

2.4.1 Dirichlet boundary conditions

In this subsection we collect generator results concerning A_0, results on the solvability of the elliptic problem with inhomogeneous Dirichlet boundary condition

$$\begin{aligned} -\mathscr{A}u &= f \\ u|_{\partial\Omega} &= g \end{aligned}$$

and some results on the regularity of the solutions under suitable assumptions on a_{ij} or Ω. More precisely we assume that one of the following assumptions is fulfilled:

1. the coefficients are Dini continuous for all $i, j = 1, \dots, d$ and Ω is Dirichlet regular.

2. Ω satisfies the uniform exterior cone condition.

Recall that a function g is *Dini continuous*, if its modulus of continuity

$$\omega_g(t) := \sup_{|x-y|\leq t} |g(x) - g(y)|$$

satisfies

$$\int_0^1 \frac{\omega_g(t)}{t} \, \mathrm{d}t < \infty.$$

In particular every Hölder continuous function is Dini continuous.

Furthermore we note that every Ω satisfying the exterior cone condition is Dirichlet regular and every Ω with Lipschitz boundary fulfills the exterior cone condition.

We start with a result on the solvability of the Poisson equation and on the regularity if the solution.

Proposition 2.26. *Let $\lambda \in \overline{\mathbb{C}^+}$. Then for each $f \in L^d(\Omega)$ and $\varphi \in C(\partial\Omega)$ there exists a unique $u \in C(\overline{\Omega}) \cap W^{2,d}_{loc}(\Omega)$ such that*

$$\begin{aligned} \lambda u - \mathscr{A} u &= f \\ u|_{\partial\Omega} &= \varphi. \end{aligned} \tag{2.1}$$

Moreover, if $\lambda \geq 0$, $f \geq 0$ on Ω and $\varphi \geq 0$ on $\partial\Omega$, then $u \geq 0$ on $\overline{\Omega}$. Finally, if $f \in L^\infty_{loc}(\Omega)$, then $u \in W(\Omega)$.

Proof. In this proof the *Poisson operator* P on $L^d(\Omega) \oplus C(\partial\Omega)$ plays a crucial role. It is defined as

$$\begin{aligned} D(P) &= \{(u, 0) : \ u \in W^{2,d}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega}), \ \mathscr{A} u \in L^d(\Omega)\} \\ P(u, 0) &= (\mathscr{A} u, -u|_{\partial\Omega}). \end{aligned}$$

It follows from [AS14, Corollary 3.4] and the remarks thereafter that the operator P is bijective. Using the same argument for $\mathscr{A} - \lambda$, $\lambda > 0$, in the definition of P we see that $(\lambda - P)$ is bijective for all $\lambda \geq 0$. By the maximum principle of Aleksandrov (see [GT98, Theorem 9.1] or [AS14, Theorem A.1]) the inverse $(\lambda - P)^{-1}$ is a positive operator for all $\lambda \geq 0$. Thus we see that P is a resolvent positive operator in the sense of [ABHN11, Definition 3.11.1] and that $s(P) < 0$. It now follows from [ABHN11, Proposition 3.11.2] that $\lambda - P$ is invertible also for complex λ with $\operatorname{Re} \lambda \geq 0$. From [GT98, Lemma 9.16] we infer that $u \in W(\Omega)$ whenever $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for some $1 < p < \infty$ and $\mathscr{A} u \in L^\infty(\Omega)$; this proves the last assertion. □

We make note of the following further result on interior regularity which follows from the Sobolev embedding theorem.

Proposition 2.27. *Let Ω be Dirichlet regular, $U \Subset \Omega$ and $\lambda \in \overline{\mathbb{C}^+}$. Then there exists a constant $C \geq 0$ such that for all $f \in L^\infty(\Omega)$ and $\varphi \in C(\partial\Omega)$ the solution u of (2.1) is in $C^1(\overline{U})$ and satisfies the estimate*

$$\|u\|_{C^1(\overline{U})} \leq C(\|f\|_{L^\infty(\Omega)} + \|\varphi\|_{C(\partial\Omega)}).$$

Proof. As $f \in L^\infty(\Omega)$ we have $u \in W(\Omega)$ by the above proposition. By the Sobolev embedding theorem (see e.g. [GT98, Corollary 7.11]) we have $W(\Omega) \subset C^1(\Omega)$. Now the claim follows from the closed graph theorem applied to the Poisson operator P. □

We next show that A_0 generates a holomorphic semigroup on $L^\infty(\Omega)$. If the regularity assumptions on Ω or a_{ij} are fulfilled, it follows from [AS14, Theorem 4.1] that the part of A_0 in $C(\overline{\Omega})$ generates a bounded holomorphic semigroup on $C(\overline{\Omega})$ and the part in $C_0(\Omega)$ generates a C_0-semigroup. Conversely if a_{ij} are Lipschitz continuous, the assertion that A_0 generates a semigroup characterizes the Dirichlet regularity of Ω; see [AB99, Theorem 4.10].

Now we come to the generation result on $L^\infty(\Omega)$.

Theorem 2.28. *The operator A_0 generates an exponentially stable, positive, holomorphic semigroup on T_0 on $L^\infty(\Omega)$. Moreover $\|T_0(t)\| \leq 1$ for all $t > 0$.*

Proof. It follows from Proposition 2.26 that $[0, \infty) \subset \rho(A_0)$ and $R(\lambda, A_0) \geq 0$ for all $\lambda \geq 0$. We claim that $\|\lambda R(\lambda, A_0)\| \leq 1$ for all $\lambda > 0$. In light of the positivity of $R(\lambda, A_0)$ it suffices to prove $\lambda R(\lambda, A_0)1 \leq 1$. To this end put $u = R(\lambda, A_0)1$ and pick $x_0 \in \overline{\Omega}$ such that $u(x_0) = \max_{x \in \overline{\Omega}} u(x)$. If $u(x_0) = 0$ there is nothing to prove. If $u(x_0) > 0$, then $x_0 \in \Omega$. Since $\mathscr{A}u = \lambda u - 1 \in C(\overline{\Omega})$ it follows from Lemma 2.29 (proved below) that $\mathscr{A}u(x_0) \leq 0$. Thus

$$\lambda R(\lambda, A_0) = \lambda u \leq \lambda u(x_0) = \lambda u(x_0) - 1 + 1 = \mathscr{A}u(x_0) + 1 \leq 1.$$

We can now follow the proof of [AS14, Theorem 4.1] where the argumentation above replaces [AS14, Proposition 4.4] to show that A_0 generates a holomorphic semigroup T_0 on $L^\infty(\Omega)$. It follows from Proposition 2.7 that T_0 is positive and from Proposition 2.3 that T_0 is contractive. Finally, Proposition 2.6 implies that T_0 is exponentially stable. □

2.4.2 A maximum modulus principle

We now show a maximum principle suited for our purposes. We begin with the following local and complex version taken from [AS14, Lemma 4.2]:

Lemma 2.29. *Let $x_0 \in \Omega$ and $r > 0$ such that $B = B_r(x_0) \Subset \Omega$. Let $u \in W^{2,p}_{loc}(\Omega)$, $p > d$, be such that $\mathscr{A}u \in C(B)$ and $|u(x)| \leq |u(x_0)|$ for all $x \in B$. Then we have*

$$\mathrm{Re}(\overline{u(x_0)}\mathscr{A}u(x_0)) \leq 0.$$

Note that $W^{2,p}(B) \subset C(B)$ by the Sobolev embedding and hence the above (pointwise) inequality is well defined.

We now can prove our version of the maximum principle.

Proposition 2.30 (Maximum modulus principle)**.** *Let* $\mathrm{Re}\,\lambda > 0$*,* $M \geq 0$ *and $u \in C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$, $p > d$, such that $\lambda u - \mathscr{A}u = 0$. If $|u(z)| \leq M$ for all $z \in \partial\Omega$, then $|u(x)| \leq M$ for $x \in \overline{\Omega}$. If $M > 0$ and $|u(z)| \leq M$ for all $z \in \partial\Omega$, then we actually have $|u(z)| < M$ for all $z \in \Omega$. The same statement holds for $\lambda = 0$.*

Proof. Suppose $|u|$ attains its global maximum at a point $x_0 \in \Omega$ with $|u(x_0)| > 0$. By the lemma above we have $\mathrm{Re}(\overline{u(x_0)}\mathscr{A}u(x_0)) \leq 0$. Since $\lambda u = \mathscr{A}u$ it follows that

$$\mathrm{Re}\,\lambda|u(x_0)| \leq 0$$

and thus since $\mathrm{Re}\,\lambda > 0$ that $|u(x_0)| = 0$ — a contradiction to the assumption $|u(x_0)| > 0$.

For the statement in the case of $\lambda = 0$ we refer to [GT98, Theorem 9.6] applied to the real and imaginary part of u seperately. □

2.5 Elliptic operators in divergence form

Let $\Omega \subset \mathbb{R}^d$ be bounded, open and Dirichlet regular and $a_{ij}, b_i, c_i, d_0 \in L^\infty(\Omega)$, $i, j = 1, \ldots, d$ where

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\overline{\xi_j} \geq \alpha|\xi|^2$$

for all $\xi \in \mathbb{C}^d$ and almost all $x \in \Omega$. In this section we want to state results on the operator A_0 on $C(\overline{\Omega})$ and $L^\infty(\Omega)$ formally defined as

$$A_0 u = \sum_{i,j=1}^{d} D_i(a_{ij}D_j u) + \sum_{i=1}^{d} D_i(b_i u) + \sum_{i=1}^{d} c_i D_i u + d_0 u \tag{2.2}$$

with Dirichlet boundary conditions resp. Robin boundary conditions. More precisely define the sesquilinear form

$$
\begin{aligned}
a\colon H^1_{\mathrm{loc}}(\Omega) \times C_c^\infty(\Omega) &\to \mathbb{C} \\
(u,v) &\mapsto \sum_{i,j=1}^{d} \int_\Omega a_{ij} D_j u D_i \bar{v} \,\mathrm{d}x + \sum_{i=1}^{d} \int_\Omega b_i u D_i \bar{v} \,\mathrm{d}x \\
&\quad - \sum_{i=1}^{d} \int_\Omega c_i D_i u \bar{v} \,\mathrm{d}x - \int_\Omega d_0 u \bar{v} \,\mathrm{d}x
\end{aligned}
$$

and denote by $\mathscr{A}\colon H^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}(\Omega)'$ the associated *distributional operator*, that is the unique operator $\mathscr{A}$ with

$$\langle \mathscr{A} u, v \rangle = -a(u,v)$$

for all $u \in H^1_{\mathrm{loc}}(\Omega)$, $v \in C_c^\infty(\Omega)$. Note that we use the form a in the following with different domains such as $H^1(\Omega) \times H^1(\Omega)$ or $H^1_0(\Omega) \times H^1_0(\Omega)$ without giving it a different name. Also note the "$-$"-sign in the association above which follows [AB99] but is non-standard in the theory of sesquilinear forms. This makes the notation consistent with the ones in the other sections; e.g. it will always be the case that operators A generate semigroups rather than their negative $-A$.

Now denote by $A_{\max}$ the *maximal operator* (i.e. without boundary conditions) in $L^\infty(\Omega)$

$$
\begin{aligned}
D(A_{\max}) &= \{u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega),\ \mathscr{A} u \in L^\infty(\Omega)\} \\
A_{\max} u &= \mathscr{A} u.
\end{aligned}
$$

Note that $A_{\max}$ is not densely defined but the part of $A_{\max}$ in $C(\overline{\Omega})$ is. In fact: denote by B the part of $A_{\max}$ in $C(\overline{\Omega})$ and by B_0 the part of the realization of B with Dirichlet boundary conditions in $C_0(\Omega)$. Now let $v \in C(\overline{\Omega})$ and $\epsilon > 0$. The Dirichlet-regularity of Ω implies by [AB99, Theorem 4.10] (see also [LSW63] for the original result) that there is $u \in D(B)$ with $u - \mathscr{A} u = 0$ and $u = v$ on $\partial\Omega$. By [AB99, Theorem 4.4] B_0 generates a C_0-semigroup on $C_0(\Omega)$ and thus is densely defined. Hence there is $v_\epsilon \in D(B_0) \subset D(B)$ such that $\|v - u - v_\epsilon\|_{C(\overline{\Omega})} < \epsilon$. Altogether we have found a function $w := v_\epsilon + u \in D(B)$ such that $\|v - w\|_{C(\overline{\Omega})} < \epsilon$.

2.5.1 Dirichlet boundary conditions

To treat the (local) Dirichlet boundary case we define A^2 to be the realization of $\mathscr{A}$ in $L^2(\Omega)$ with Dirichlet boundary conditions, that is A^2 is the operator associated to the form a defined on $H^1_0(\Omega) \times H^1_0(\Omega)$.

An easy calculation shows that a is *elliptic*, i.e. for some $\alpha > 0$ and $\omega \in \mathbb{R}$ one has

$$a(u,u) + \omega\|u\|_{L^2(\Omega)} \geq \alpha\|u\|^2_{H^1(\Omega)}$$

for all $u \in H^1_0(\Omega)$. Thus by elementary form methods and the first Beurling-Deny criterion as in [Ouh05, Theorem 2.6] A^2 generates a positive, holomorphic C_0-semigroup T^2 on $L^2(\Omega)$ with compact resolvent as can be seen e.g. in [Ouh05, Theorem 4.7].

We now assume furthermore that $\sum_{j=1}^d D_j b_j + d_0 \leq 0$. This is to be understood in the sense of distributions, that is $\sum_{j=1}^d D_j b_j + d_0 \leq 0$ as a functional in $\mathcal{D}(\Omega)'$, i.e.

$$\sum_{j=1}^d \int_\Omega (-b_j D_j\varphi + d_0\varphi) \leq 0$$

for all $0 \leq \varphi \in C_c^\infty(\Omega)$. By density this inequality also holds for all $\varphi \in H^1_0(\Omega)$.

By the second Beurling-Deny criterion as in [MVV05, Corollary 2.8] we see that T^2 generates a submarkovian semigroup; here *submarkovian* means that T^2 is positive and L^∞-contractive. In fact using we need to show $a(u \wedge 1, (u-1)^+) \geq 0$ for all $u \in H^1_0(\Omega)$. Using that with $u \in H^1_0(\Omega)$ we also have $u \wedge 1, (u-1)^+ \in H^1_0(\Omega)$ with

$$D_j(u \wedge 1) = 1_{\{u<1\}} D_j u \quad \text{and} \quad D_j(u-1)^+ = 1_{\{u>1\}} D_j u$$

it follows that

$$a(u \wedge 1, (u-1)^+) = \int_\Omega \sum_{j=1}^d b_j D_j (u-1)^+ \,\mathrm{d}x - \int_\Omega d_0 (u-1)^+ \,\mathrm{d}x.$$

Now the assumptions imply the sought inequality.

It turns out the condition $\sum_{i=1}^d D_i b_i + d_0 \leq 0$ is also necessary as one can see as follows: Suppose that T^2 is submarkovian, that is by the second Beurling-Deny criterion we have that $a(u \wedge 1, (u-1)^+) \geq 0$ for all $u \in H^1_0(\Omega)$. It then follows that

$$\int_\Omega \sum_{j=1}^d b_j D_j (u-1)^+ \,\mathrm{d}x - \int d_0 (u-1)^+ \,\mathrm{d}x \geq 0$$

for all $u \in H^1_0(\Omega)$. Clearly if $0 \leq \varphi \in C_c^\infty(\Omega)$ there is $u \in H^1_0(\Omega)$ such that $\varphi = (u-1)^+$ which then implies that $\sum_{i=1}^d D_j b_j + d_0 \leq 0$ in $\mathcal{D}(\Omega)'$.

But more is true: Denoting by A_0 the realization in $L^\infty(\Omega)$ of $A_{\max}$ with Dirichlet boundary conditions, i.e.

$$
\begin{aligned}
D(A_0) &= \{u \in C_0(\Omega) \cap H^1_{\mathrm{loc}}(\Omega),\ \mathscr{A}u \in L^\infty(\Omega)\} \\
A_0 u &= \mathscr{A}u.
\end{aligned}
$$

we have that A_0 generates a holomorphic semigroup. In the following theorem we use the argumentation of [AB99, Corollary 4.7] but need not assume that $b_i, c_i \in W^{1,\infty}(\Omega)$, $i = 1, \ldots, d$ by considering then unknown results by Daner in [Dan00a].

Theorem 2.31. *A_0 generates a positive, contractive, holomorphic semigroup T^∞ on $L^\infty(\Omega)$. Its part in $C_0(\Omega)$ generates a C_0-semigroup.*

Proof. Denote by T^2 the C_0-semigroup generated by A_2 on $L^2(\Omega)$. By [Dan00a, Theorem 6.1] T^2 has a kernel satisfying Gaussian bounds. Hence by [AtE97, Theorem 5.4] T^2 extrapolates consistently to holomorphic C_0-semigroups T^p on $L^p(\Omega)$, $1 \leq p < \infty$ with generator A^p and to a holomorphic $\omega*$-semigroup T^∞ on $L^\infty(\Omega)$ with generator A^∞. The operators A^p are the part of A^2 in $L^p(\Omega)$,$1 \leq p \leq \infty$ and by considering $p = \infty$ we see that A_0 is the part of A^p in $C(\overline{\Omega})$ for all $1 \leq p \leq \infty$.

Furthermore by holomorphy we have

$$
\|R(\lambda, A^1)\|_{\mathcal{L}(L^1(\Omega))} \leq \frac{M}{|\lambda|} \quad \forall \operatorname{Re}\lambda > \omega
$$

for some $M, \omega \geq 0$ and by duality also

$$
\|R(\lambda, A^\infty)\|_{\mathcal{L}(L^\infty(\Omega))} \leq \frac{M}{|\lambda|} \quad \forall \operatorname{Re}\lambda > \omega.
$$

Hence $\rho(A_0) \supset \mathbb{C}^+ + \omega$ and A_0 satisfies the holomorphic estimate; by Theorem 2.5 it generates a holomorphic semigroup on $L^\infty(\Omega)$. Positivity follows from the positivity of T^2. For the generation claim on $C_0(\Omega)$ it remains to show that the part of A_0 in $C_0(\Omega)$ is densely defined. But this is a part of Theorem 4.4 in [AB99]. □

We now state results on the elliptic boundary value problem.

Proposition 2.32. *Let $\lambda \in \overline{\mathbb{C}^+}$ and $\epsilon > 0$. Then for each $f \in L^{d+\epsilon}(\Omega)$ and $\varphi \in C(\partial\Omega)$ there exists a unique $u \in C(\overline{\Omega}) \cap H^1_{loc}(\Omega)$ such that*

$$
\begin{aligned}
\lambda u - \mathscr{A}u &= f \\
u|_{\partial\Omega} &= \varphi.
\end{aligned}
\tag{2.3}
$$

Moreover, if $\lambda \geq 0$, $f \geq 0$ on Ω and $\varphi \geq 0$ on $\partial\Omega$, then $u \geq 0$ on $\overline{\Omega}$.

Proof. This proof is essentially the same as the one of Proposition 2.26. Again we use the Poisson operator P on $L^{d+\epsilon}(\Omega) \oplus C(\partial\Omega)$ defined as

$$\begin{aligned} D(P) &= \{(u,0) : \; u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega}), \; \mathscr{A}u \in L^{d+\epsilon}(\Omega)\} \\ P(u,0) &= (\mathscr{A}u, -u|_{\partial\Omega}). \end{aligned}$$

By [GT98, Theorem 8.31] the operator P is bijective. Again using the same argument for $\mathscr{A} - \lambda$, $\lambda > 0$, in the definition of P we see that $(\lambda - P)$ is bijective for all $\lambda \geq 0$. By [GT98, Theorem 8.1] the inverse $(\lambda - P)^{-1}$ is a positive operator for all $\lambda \geq 0$. Thus we see that P is a resolvent positive operator in the sense of [ABHN11, Definition 3.11.1] and that $s(P) < 0$. It now follows from [ABHN11, Proposition 3.11.2] that $\lambda - P$ is invertible also for complex λ with $\operatorname{Re} \lambda \geq 0$. □

We also need a further result on regularity.

Proposition 2.33. *Let $U \subset \Omega$ open with $\overline{U} \subset \Omega$ and $\lambda \in \overline{\mathbb{C}^+}$. Then there exist constants $C \geq 0$ and $\alpha > 0$ such that for all $f \in L^\infty(\Omega)$ and $\varphi \in C(\partial\Omega)$ the solution u of (2.1) is in $C^\alpha(\overline{U})$ and satisfies the estimate*

$$\|u\|_{C^\alpha(\overline{U})} \leq C(\|f\|_{L^\infty(\Omega)} + \|\varphi\|_{C(\partial\Omega)}).$$

Proof. By [GT98, Theorem 8.24] the solution u is a locally Hölder continuous function and the estimate follows from the closed graph theorem applied to P of the preceding proof. □

Remark (the connection between divergence and non-divergence form operators). For our treatment of the non-local Dirichlet problem we consider both divergence and non-divergence operators. Their connection is as follows. If the coefficients a_{ij} and b_i are sufficiently regular, then the informal definition of (2.2) for a divergence form operator may be taken as a formal definition and we can write

$$\mathscr{A}u = \sum_{i,j=1}^{d} a_{ij} D_i D_j u + \sum_{j=1}^{d} (\sum_{i=1}^{d} D_i a_{ij} + b_j + c_j) D_j u + (\sum_{i=1}^{d} D_i b_i + d_0) u$$

as a non-divergence form operator, whenever $u \in H^2_{\mathrm{loc}}(\Omega)$.

Now suppose that $u = R(\lambda, A_0)f$ for $f \in L^\infty(\Omega)$, as will frequently be the case in this thesis. If also the boundary $\partial\Omega$ is sufficiently smooth, then a standard regularity result for solutions of elliptic problems states that the function u suffices this regularity assumption; see [GT98, Theorem 8.12] or [Eva10, Section 6.3, Theorem 4] for a less general result. Hence in this particular situation the two different types of operators considered coincide.

2.5.2 A maximum modulus principle

In Chapter 3 of this thesis we need in addition to the local generation results a maximum principle for complex potentials as in the case of non-divergence operators. To that end we prove a Kato-inequality for the operator $\mathscr{A}$. The following lemma slightly extends [Agm82, Lemma 5.4] and [Kat72, Lemma A] to the case where the a_{ij} are not symmetric and where $b_i, c_i \neq 0$ and $d_0 \neq 0$. The proof follows [Agm82, Lemma 5.4] and remains largely the same; for more detailed computations and a more precise argumentation we refer to that proof.

Lemma 2.34. *Let $u \in H^1_{loc}(\Omega)$ be such that $\mathscr{A}u \in L^1_{loc}(\Omega)$. Then $|u| \in H^1_{loc}(\Omega)$ and*

$$\mathscr{A}|u| \geq \operatorname{Re}(\mathscr{A}u(\operatorname{sign}\bar{u}))$$

in the sense of distributions.

Here sign: $\mathbb{C} \to \mathbb{C}$ is the function $z \mapsto \frac{z}{|z|}$ for $z \neq 0$ and $0 \mapsto 0$.

Proof. For $\epsilon > 0$ consider the function $u_\epsilon := (|u|^2 + \epsilon^2)^{\frac{1}{2}}$. Then $u_\epsilon, \frac{\bar{u}}{u_\epsilon} \in H^1_{\mathrm{loc}}(\Omega)$ with

$$\begin{aligned} D_i \frac{\bar{u}}{u_\epsilon} &= \frac{D_i \bar{u}}{u_\epsilon} - \bar{u}\frac{D_i u_\epsilon}{u_\epsilon^2} \text{ and} \\ D_i u_\epsilon &= \frac{1}{2}\frac{\bar{u}}{u_\epsilon} D_i u + \frac{1}{2}\frac{u}{u_\epsilon} D_i \bar{u} = \operatorname{Re}(\frac{\bar{u}}{u_\epsilon} D_i u). \end{aligned}$$

Clearly we have $\frac{u}{u_\epsilon} \to \operatorname{sign} u$ in $L^1_{\mathrm{loc}}(\Omega)$ as $\epsilon \searrow 0$ and by the above

$$D_i u_\epsilon \to \frac{1}{2}(\operatorname{sign}\bar{u}) D_i u + \frac{1}{2}(\operatorname{sign} u) D_i \bar{u} \text{ in } L^1_{\mathrm{loc}}(\Omega).$$

It then follows that $u_\epsilon \to |u|$ not only in $L^1_{\mathrm{loc}}(\Omega)$ but also in $H^1_{\mathrm{loc}}(\Omega)$ as $\epsilon \searrow 0$.

Now consider the form a as defined on $H^1_{\mathrm{loc}}(\Omega) \times C_c^\infty(\Omega)$ and compute for $0 \leq \varphi \in C_c^\infty(\Omega)$

$$\sum_{i,j=1}^d a_{ij} D_j u_\epsilon D_i \varphi = \operatorname{Re}\big(\sum_{i,j=1}^d a_{ij} D_j u D_i(\frac{\bar{u}}{u_\epsilon}\varphi) - \varphi \sum_{i,j=1}^d a_{ij} D_j u D_i(\frac{\bar{u}}{u_\epsilon})\big).$$

A direct computation shows that

$$\operatorname{Re}\sum_{i,j=1}^d a_{ij} D_j u D_i(\frac{\bar{u}}{u_\epsilon}) \geq 0$$

which in turn shows

$$\sum_{i,j=1}^d a_{ij} D_j u_\epsilon D_i \varphi \leq \operatorname{Re}\sum_{i,j=1}^d a_{ij} D_j u D_i(\frac{\bar{u}}{u_\epsilon}\varphi).$$

Furthermore we have

$$\sum_{i=1}^{d} c_i D_i u_\epsilon \varphi = \operatorname{Re} \sum_{i=1}^{d} c_i D_i u(\frac{\bar{u}}{u_\epsilon}\varphi)$$

and since $\sum_{i=1}^{d} D_i b_i + d_0 \leq 0$

$$\begin{aligned} -\sum_{i=1}^{d} b_i u_\epsilon D_i\varphi + d_0 u_\epsilon \varphi &\leq -\sum_{i=1}^{d} b_i u \frac{\bar{u}}{u_\epsilon} D_i\varphi + d_0 u(\frac{\bar{u}}{u_\epsilon}\varphi) \\ &= \operatorname{Re}\big(-\sum_{i=1}^{d} b_i u \frac{\bar{u}}{u_\epsilon} D_i\varphi + d_0 u(\frac{\bar{u}}{u_\epsilon}\varphi)\big). \end{aligned}$$

Since $\mathscr{A}u \in L^1_{\text{loc}}(\Omega)$ an approximation argument yields

$$\begin{aligned} -\sum_{i,j=1}^{d} \int a_{ij} D_j u D_i(\frac{\bar{u}}{u_\epsilon}\varphi) - \sum_{i=1}^{d} \int b_i u \frac{\bar{u}}{u_\epsilon} D_i\varphi + \sum_{i=1}^{d} \int c_i (D_i u) \frac{\bar{u}}{u_\epsilon}\varphi + \int d_0 u \frac{\bar{u}}{u_\epsilon}\varphi \\ = \int (\mathscr{A}u) \frac{\bar{u}}{u_\epsilon}\varphi \end{aligned}$$

and in turn the inequalities listed above show

$$a(u_\epsilon, \varphi) \leq \operatorname{Re} \int (\mathscr{A}u) \frac{\bar{u}}{u_\epsilon}\varphi.$$

Since $u_\epsilon \to |u|$ in $H^1_{\text{loc}}(\Omega)$ and $\frac{\bar{u}}{u_\epsilon} \to \operatorname{sign} \bar{u}$ pointwise almost everywhere and boundedly it follows that

$$a(|u|, \varphi) \leq \int_\Omega \operatorname{Re}\big(\mathscr{A}u(\operatorname{sign}\bar{u})\big)\varphi$$

proving the lemma (note again that we associate the operator $\mathscr{A}$ via $-a(u,v) = [\mathscr{A}u, v]$!). □

Now the maximum principle can be proven as follows.

Proposition 2.35 (Maximum modulus principle). *Let* $\operatorname{Re}\lambda \geq 0$ *and* $u \in C(\overline{\Omega}) \cap H^1_{loc}(\Omega)$, *such that* $\lambda u - \mathscr{A}u = 0$. *If for some* $M \geq 0$ *we have* $|u(z)| \leq M$ *for all* $z \in \partial\Omega$, *then* $|u(x)| \leq M$ *for* $x \in \overline{\Omega}$. *If* $M > 0$ *then the inequality is strict, that is* $|u(x)| < M$ *for all* $x \in \Omega$.

Proof. Set $v = |u|$ and $\mu = \operatorname{Re}\lambda$. Since $\mathscr{A}u = \lambda u \in C(\overline{\Omega})$ we have by the lemma above

$$\mathscr{A}v \geq \operatorname{Re}\mathscr{A}u \operatorname{sign}\bar{u} = \operatorname{Re}\lambda u \operatorname{sign}\bar{u} = \mu v.$$

It follows then that

$$\mu v - \mathscr{A}v \geq 0$$

in the sense of distributions, i.e. v is a supersolution of the equation $\mu v - \mathscr{A}v = 0$. Now the claims follow by the strong maximum principle as stated in [GT98, Theorem 8.19]. □

2.5.3 Robin boundary conditions and the conormal derivative

For this subsection Ω is assumed to be Lipschitz. We begin by defining the conormal derivative of certain H^1-functions. Having the distributional operator $\mathscr{A}$ at hand this can be done as follows.

Definition. Let $u \in H^1(\Omega)$ be such that $\mathscr{A}u \in L^2(\Omega)$. For a function $h \in L^2(\partial\Omega)$ we say that h is the *weak conormal derivative of* u and write $\partial_\nu^{\mathscr{A}} u := h$ if the Green formula

$$[\mathscr{A}u, v] + a(u,v) = \int_{\partial\Omega} h \operatorname{tr} \bar{v} \, \mathrm{d}\sigma$$

holds for all $v \in H^1(\Omega)$, where σ denotes the surface measure of $\partial\Omega$.

Under our assumptions on the coefficients the weak conormal derviative, if it exists, is unique. It depends on the operator $\mathscr{A}$ only through the coefficients a_{ij} and b_j. Moreover, if the coefficients, the boundary of Ω and the function u are smooth enough, then the weak conormal derivative coincides with the usual (strong) conormal derivative given by

$$\partial_\nu^{\mathscr{A}} u = \sum_{j=1}^{d} \Big(\sum_{i=1}^{d} a_{ij} D_i u + b_j u \Big) \nu_j$$

where $\nu = (\nu_1, \dots, \nu_d)$ is the unit outer normal of Ω. For a proof of these facts and more information we refer to [Agr15, Section 8.1].

Next we endow our differential operator with Robin boundary conditions, given through a function $\beta \in L^\infty(\partial\Omega)$. To that end, we employ again the theory of sesquilinear forms, setting

$$a_\beta(u,v) := a(u,v) + \int_{\partial\Omega} \beta \operatorname{tr} u \operatorname{tr} \bar{v} \, \mathrm{d}\sigma$$

for $u, v \in H^1(\Omega)$. The associated operator A_β^2 on $L^2(\Omega)$ — again note the "$-$" below in contrast to the standard — is given by

$$D(A_\beta^2) = \{u \in H^1(\Omega) : \exists f \in L^2(\Omega) \text{ with } -a_\beta(u,v) = [f,v] \; \forall v \in H^1(\Omega)\},$$
$$A_\beta^2 u = f.$$

Testing against $v \in C_c^\infty(\Omega)$ we see that A_β^2 is a realization of $\mathscr{A}$. By the definition of the weak conormal derivative, it follows that any $u \in D(A_\beta^2)$ satisfies the Robin-boundary condition $\partial_\nu^{\mathscr{A}} u + \beta \operatorname{tr}(u) = 0$.

One can use the theory of sesquilinear forms to study the operator A_β^2 and subsequently use these results to extrapolate to the L^p-scale and to the space $C(\overline{\Omega})$ of continuous functions on $\overline{\Omega}$. There is a rich literature on these topics, we mention here [AtE97, AW03, Dan00b, Dan00a, Dan09, Nit11]. We collect the results which will be important for the rest of this article in the following theorem.

Theorem 2.36. *The operator A^2_β generates a positive, strongly continuous semigroup T^2 on $L^2(\Omega)$. Its restriction T^∞ to $L^\infty(\Omega)$ is a holomorphic semigroup on $L^\infty(\Omega)$. Each operator $T^\infty(t)$, $t > 0$, is compact and enjoys the strong Feller property. In particular, $C(\overline{\Omega})$ is invariant. The restriction T^C of T^2 to $C(\overline{\Omega})$ is a strongly continuous semigroup.*

Proof. By standard results from the theory of quadratic forms ([Ouh05, Section 1.4]) A^2_β generates a holomorphic semigroup T^2. The positivity of T^2 follows from [Ouh05, Theorem 2.6] noting that $a_\beta(u^+, u^-) = 0$ for all $u \in H^1(\Omega)$. It was proved in [Dan00a, Corollary 6.1] (see also [AtE97, Theorem 4.9]) that the semigroup T^2 has Gaussian estimates so that T^2 extrapolates to a consistent family of semigroups T^q on $L^q(\Omega)$ for $q \in [1, \infty]$. In particular, T^2 leaves the space $L^\infty(\Omega)$ invariant and restricts to a semigroup T^∞ on that space. By [AtE97, Theorem 5.3] the semigroup T^∞ is holomorphic on $L^\infty(\Omega)$. Moreover, by the proof of [Nit11, Theorem 4.3] $T^\infty(t)L^\infty(\Omega) \subset C(\overline{\Omega})$ for all $t > 0$. It was also seen in that theorem that $T^\infty(t)$ is compact for all $t > 0$. We now show that $T^\infty(t)$ is strongly Feller for $t > 0$. Since T^2 is ultracontractive by [Are02, 7.3 criterion (v)] it follows that $T^2(t)L^q(\Omega) \subset L^\infty(\Omega)$ and hence $T^2(t)L^q(\Omega) \subset T^2(t/2)L^\infty(\Omega) \subset C(\overline{\Omega})$ for some $q \in (2, \infty)$. By the closed graph theorem, $T^2(t)$ is a bounded operator from $L^q(\Omega)$ to $C(\overline{\Omega})$. Now the strong Feller property follows from the dominated convergence theorem. It follows from [Nit11, Theorem 4.3] that the restriction of the semigroup to $C(\overline{\Omega})$ is strongly continuous. □

Under additional assumptions on the coefficients we can also characterize when the semigroup T^2 is submarkovian resp. markovian. Here a semigroup is called *markovian*, if $T(t)1 = 1$ for all $t > 0$. To that end we need this auxiliary lemma.

Lemma 2.37. *Let $g \in L^2(\Omega)$ and $h \in L^2(\partial\Omega)$ be such that*

$$\int_\Omega gv \, dx + \int_{\partial\Omega} hv \, d\sigma \geq 0 \tag{2.4}$$

for all $0 \leq v \in H^1(\Omega)$. Then $g \geq 0$ a.e. on Ω and $h \geq 0$ a.e. on $\partial\Omega$. In particular, if in (2.4) identity holds for all $v \in H^1(\Omega)$, then $g = 0$ a.e. on Ω and $h = 0$ a.e. on $\partial\Omega$.

Proof. By (2.4) we have $\int_\Omega gv \, dx \geq 0$ for all $0 \leq v \in C_c^\infty(\Omega)$. Thus $g \geq 0$ almost everywhere on Ω. Given a function $\varphi \in C(\partial\Omega)$, we find a sequence

$v_n \in C^\infty(\overline{\Omega})$ such that $v_n|_{\partial\Omega} \to \varphi$ in $C(\partial\Omega)$, $0 \le v_n \le \|\varphi\|_\infty$ in Ω and such that v_n is supported in a relatively open set $U_n \subset \overline{\Omega}$ with $U_n \supset U_{n+1}$ and $\bigcap_{n\in\mathbb{N}} U_n = \partial\Omega$. Choosing $v = v_n$ in (2.4) and letting $n \to \infty$, we infer from dominated convergence that $\int_{\partial\Omega} h\varphi \, \mathrm{d}\sigma \ge 0$. As $\varphi \in C(\partial\Omega)$ was arbitrary, the claim follows. □

Proposition 2.38. *Assume in addition to the hypotheses on the coefficients that $b_j \in W^{1,\infty}(\Omega)$ for $j = 1, \dots, d$.*

(a) The semigroup T^2_β is submarkovian if and only if

$$\sum_{j=1}^d D_j b_j + d_0 \le 0 \text{ almost everywhere on } \Omega \text{ and} \tag{2.5}$$

$$\sum_{j=1}^d \operatorname{tr}(b_j)\nu_j + \beta \ge 0 \text{ almost everywhere on } \partial\Omega. \tag{2.6}$$

(b) The semigroup T^2_β is markovian if and only if

$$\sum_{j=1}^d D_j b_j + d_0 = 0 \text{ almost everywhere on } \Omega \text{ and} \tag{2.7}$$

$$\sum_{j=1}^d \operatorname{tr}(b_j)\nu_j + \beta = 0 \text{ almost everywhere on } \partial\Omega. \tag{2.8}$$

Proof. (a) The semigroup T^2_β is submarkovian if and only if the Beurling–Deny–Ouhabaz criterion holds, i.e.

$$a_\beta(u \wedge 1, (u-1)^+) \ge 0$$

for all $u \in H^1(\Omega)$, see [Ouh05, Chapter 2] and [MVV05, Corollary 2.8] or [Die16] for the case where the form is not necessarily accretive. Recall that for $u \in H^1(\Omega)$ the functions $u \wedge 1$ and $(u-1)^+$ also belong to $H^1(\Omega)$ and

$$D_j(u \wedge 1) = 1_{\{u<1\}} D_j u \quad \text{and} \quad D_j(u-1)^+ = 1_{\{u>1\}} D_j u.$$

Thus $D_i(u \wedge 1) D_j(u-1)^+ = (u-1)^+ D_j(u \wedge 1) = 0$. We see that

$$\begin{aligned} a_\beta(u \wedge 1, (u-1)^+) &= \int_\Omega \Big(\sum_{j=1}^d b_j D_j - d_0\Big)(u-1)^+ \, \mathrm{d}x + \int_{\partial\Omega} \beta (u-1)^+ \, \mathrm{d}\sigma \\ &= -\int_\Omega \sum_{j=1}^d (D_j b_j)(u-1)^+ \, \mathrm{d}x + \int_{\partial\Omega} \sum_{j=1}^d b_j \nu_j (u-1)^+ \, \mathrm{d}\sigma \\ &\qquad - \int_\Omega d_0 (u-1)^+ \, \mathrm{d}x + \int_{\partial\Omega} \beta (u-1)^+ \, \mathrm{d}\sigma. \end{aligned}$$

The latter is positive if (2.5) and (2.6) hold whence T_β^2 is submarkovian in this case. This shows sufficiency of these two conditions.

Conversely, if the semigroup T_β^2 is submarkovian, the Beurling–Deny criterion yields

$$-\int_\Omega \Big(\sum_{j=1}^d D_j b_j + d_0\Big)(u-1)^+ \,\mathrm{d}x + \int_{\partial\Omega} \Big(\sum_{j=1}^d b_j \nu_j + \beta\Big)(u-1)^+ \,\mathrm{d}\sigma \geq 0$$

for all $u \in H^1(\Omega)$. Choosing $u = 1 + v$ with $0 \leq v \in H^1(\Omega)$, Lemma 2.37 shows that (2.5) and (2.6) are valid.

(b) A markovian semigroup is in particular submarkovian whence the inequalities (2.5) and (2.6) are satisfied. If T_β^2 is submarkovian, then it is markovian if and only if $1 \in \ker(A_\beta^2)$. Note that

$$\mathscr{A}1 = \sum_{j=1}^d D_j b_j + d_0.$$

Thus (2.7) is necessary for T_β^2 to be markovian. If (2.7) holds, then for $v \in H^1(\Omega)$ we have

$$a(1,v) + [\mathscr{A}1, v] = \int_\Omega (-\sum_{j=1}^d b_j D_j \bar{v} + d_0 \bar{v}) \,\mathrm{d}x = \sum_{j=1}^d \int_{\partial\Omega} b_j \nu_j \bar{v} \,\mathrm{d}\sigma,$$

where we used an integration by parts. Thus saying $\partial_\nu^{\mathscr{A}} 1 + \beta = 0$, i.e. $1 \in D(A_\beta^2)$, is equivalent to

$$\sum_{j=1}^d \int_{\partial\Omega} b_j \nu_j \bar{v} \,\mathrm{d}\sigma = -\int_{\partial\Omega} \beta \bar{v} \,\mathrm{d}\sigma$$

for all $v \in H^1(\Omega)$ and hence to (2.8). □

To apply the abstract results on semigroups, we need some additional properties about the following elliptic problem, which were also used implicitly in Theorem 2.36.

$$\begin{cases} \lambda u - \mathscr{A}u = f \text{ on } \Omega \\ \partial_\nu^{\mathscr{A}} u + \beta u = h \text{ on } \partial\Omega. \end{cases} \tag{2.9}$$

As already seen, a_β defines a continuous sesquilinear mapping on $H^1(\Omega)$. By [Dan09, Corollary 2.5] it is also elliptic, i.e. there is some $\omega \in \mathbb{R}$ such that $a_\beta(u,u) + \omega\|u\|_{L^2(\Omega)}^2 \geq \alpha\|u\|_{H^1(\Omega)}^2$ for some $\alpha > 0$; by the same corollary we can choose $\omega > 0$. With this information at hand, one can prove existence and uniqueness of solutions to (2.9) by means of the Lax-Milgram Theorem. Indeed, considering the continuous antilinear functional F on $H^1(\Omega)$, given by $F(v) = \int_\Omega f\bar{v}\,dx + \int_{\partial\Omega} h\,\mathrm{tr}(\bar{v})\,d\sigma$, it follows from the Lax–Milgram Theorem that for $\lambda > \omega$ there is a unique $u \in H^1(\Omega)$ such that

$$a_\beta(u,v) + \lambda[u,v] = F(v)$$

for all $v \in H^1(\Omega)$. From [Nit11, Theorem 3.14 (iv)] we obtain the following result concerning regularity of the solution.

Proposition 2.39. *Fix $p > \min\{d, 2\}$ and $\lambda > \omega$. Then there exists constants $\gamma > 0$ and $C > 0$ such that whenever $f \in L^{p/2}(\Omega)$ and $h \in L^{p-1}(\partial\Omega)$ the unique solution u of (2.9) belongs to $C^\gamma(\overline{\Omega})$ and we have*

$$\|u\|_{C^\gamma(\overline{\Omega})} \le C\big(\|f\|_{L^{\frac{p}{2}}(\Omega)} + \|h\|_{L^{p-1}(\partial\Omega)}\big).$$

In what follows, we shall make use of a domination result.

Proposition 2.40. *Fix $0 \le f \in L^2(\Omega)$, $0 \le h \in L^2(\partial\Omega)$ and $0 \le \beta_1 \le \beta_2 \in L^\infty(\partial\Omega)$ and $\lambda > \omega$, where $\omega \in \mathbb{R}$ is such that $a_{\beta_1} + \omega$, $a_{\beta_2} + \omega$ and $a_0 + \omega$ are all coercive. For $j = 1, 2$, let $u_j \in H^1(\Omega)$ be the unique solution of*

$$\begin{cases} \lambda u - \mathscr{A} u = f \text{ on } \Omega \\ \partial_\nu^{\mathscr{A}} u + \beta_j u = h \text{ on } \partial\Omega. \end{cases}$$

Then $0 \le u_2 \le u_1$.

Proof. We first show positivity for weak solutions u of (2.9); to that end consider $f \le 0$ and $h \le 0$ for now. Since u solves (2.9) we have

$$\lambda[u,v] + a_\beta(u,v) = [f,v] + \int_{\partial\Omega} h\bar{v}\,\mathrm{d}\sigma$$

for all $v \in H^1(\Omega)$. Setting $v := u^+$ and noting that $a_\beta(u,u^+) = a_\beta(u^+,u^+)$ by locality of a_β, we find

$$\lambda[u^+,u^+] + a_\beta(u^+,u^+) = [f,u^+] + \int_{\partial\Omega} hu^+\,\mathrm{d}\sigma \le 0.$$

As $a_\beta + \omega$ is coercive we have that $a_\beta(u^+,u^+) \ge \alpha\|u^+\|^2_{H^1(\Omega)} - \omega\|u^+\|^2_{L^2(\Omega)}$ for some $\alpha > 0$ and together with $\lambda > \omega$ it follows that $\|u^+\|_{L^2(\Omega)} \le 0$, whence $u \le 0$.

We can prove the domination similarly. This time we fix $f \ge 0$ and $h \ge 0$ and pick ω such that $a_{\beta_1} + \omega$, $a_{\beta_2} + \omega$ and $a_0 + \omega$ are coercive. The solutions u_j, $j = 1, 2$, satisfy the equations

$$\lambda[u_j,v] + a_{\beta_j}(u_j,v) = [f,v] + \int_{\partial\Omega} f\bar{v}\,\mathrm{d}\sigma$$

for all $v \in H^1(\Omega)$. Subtracting these equations and testing against a positive v we have

$$\begin{aligned}\lambda[u_2 - u_1, v] + a_0(u_2 - u_1, v) &= \int_{\partial\Omega} (\beta_1 u_1 - \beta_2 u_2) v \, \mathrm{d}\sigma \\ &\leq \int_{\partial\Omega} \beta_2 (u_1 - u_2) v \, \mathrm{d}\sigma,\end{aligned}$$

since $u_1 \geq 0$ by the above. Testing against $v := (u_2 - u_1)^+$ and again using locality $a_0((u_2 - u_1), (u_2 - u_1)^+) = a_0((u_2 - u_1)^+, (u_2 - u_1)^+)$, we find

$$\begin{aligned}\lambda[(u_2 - u_1)^+, (u_2 - u_1)^+] &+ a_0((u_2 - u_1)^+, (u_2 - u_1)^+) \\ &\leq -\int_{\partial\Omega} \beta_2 \big((u_2 - u_1)^+\big)^2 \, \mathrm{d}\sigma \leq 0\end{aligned}$$

Using the coerciveness of $a_0 + \omega$ and $\lambda > \omega$ we deduce as above that $(u_2 - u_1)^+ = 0$, and thus that $u_2 \leq u_1$ as claimed. □

This proposition yields in particular the following monotonicity property.

Corollary 2.41. *Let $\beta_1, \beta_2 \in L^\infty(\Omega)$ be such that $\beta_1 \leq \beta_2$. Then $0 \leq T^2_{\beta_2}(t) \leq T^2_{\beta_1}(t)$ for all $t \geq 0$.*

Proof. Proposition 2.40 shows that for large λ we have $0 \leq R(\lambda, A^2_{\beta_2}) \leq R(\lambda, A^2_{\beta_1})$. This implies the claim in view of Proposition 2.8. □

2.6 Perturbation of boundary conditions

In [Gre87] Greiner considers the following situation: Suppose we a have a "maximal" differential operator A, that is one that does not impose boundary conditions, on a Banach space X and some realization A_0 of A imposing boundary conditions which generates a C_0-semigroup. How can one perturb the boundary conditions of A_0 such that one ends up again with a generator A_ϕ?

At the heart of this theory lies the observation that the resolvent $R(\lambda, A_\phi)$ can be considered as perturbation of the resolvent $R(\lambda, A_0)$. More precisely under certain assumptions the equation

$$(\lambda - A_\phi)x = (\lambda - A_0)(1 - S_\lambda)x$$

holds for suitable $\lambda \in \mathbb{C}$ and $x \in X$, where S_λ is a operator associated to the perturbation ϕ. If one can assure that $1 - S_\lambda$ is invertible for enough $\lambda \in \mathbb{C}$ and the inverse obeys a uniform estimate in λ, then one can prove a generator result.

In the first subsection we will explain this generator result and give Greiner's proof of invertibility and uniform estimate where he assumed that S_λ is compact and $D := D(A) \subset X$ is dense. In fact we will also give a slight generalization of his arguments to accommodate for our situation where we want to operate in the space and $L^\infty(\Omega)$; i.e. we will give a suitable version of his theory for $\overline{D} \neq X$ and non-C_0-semigroups. The results of that subsection themselves are suitable for the case of perturbed Robin boundary condition as we will see in Chapter 4.

In the second subsection we will prove the invertibility of S_λ under the assumption that S_λ is power compact and the uniform estimate under the assumption that S_λ and the semigroup generated by A_0 are positive. It will turn out that these results are suited to the case of perturbed Dirichlet boundary conditions as will be seen in Chapter 3.

In the third subsection we consider how certain properties of the unperturbed semigroup translate to the perturbed one. There we deal with positivity and domination, irreducibility and the strong Feller property.

Before we begin we fix the notation and the assumptions for this section. Let D, X be a Banach spaces, such that D is continuously embedded in X, and $A\colon D \to X$ be a continuous operator. Let ∂X be a Banach space and $L\colon D \to \partial X$ be linear.

Denote by A_0 the operator $A|_{\ker L}$, that is

$$D(A_0) = \{x \in D : \ Lx = 0\}, \ A_0 x = Ax.$$

Denote by $\overline{D}$ the closure of D in X. Furthermore let $\phi\colon \overline{D} \to \partial X$ be linear and define the perturbed operator

$$D(A_\phi) = \{x \in D : \ Lx = \phi x\}, \ A_\phi x = Ax.$$

Now assume the following on the operators A_0, L and ϕ.

(H1) A_0 generates an analytic semigroup T_0 of type (M, ω) in the sense of Section 2.1; that is, for $\omega \in \mathbb{R}$ and $M' > 0$ we have $\mathbb{C}^+ + \omega \subset \rho(A_0)$ and for all $\lambda \in \mathbb{C}^+ + \omega$

$$\|\lambda R(\lambda, A_0)\| < M'.$$

(H2) L is continuous and surjective.

(H3) ϕ is continuous.

Further assumptions on L and ϕ will be made when appropriate.

2.6.1 Extension to non-densely defined operators

In the situation explained above Greiner proved the following result.

Theorem 2.42. *Let A_0 generate an analytic C_0-semigroup on X and ϕ be compact. Then A_ϕ also generates an analytic C_0-semigroup on X.*

We will repeat the proof given by Greiner and separate the key arguments allowing us to formulate the aforementioned generalizations.

Lemma 2.43. *Let $\lambda \in \rho(A_0)$. Then $D = \ker(\lambda - A) \oplus D(A_0)$.*

Proof. Let $x \in \ker(\lambda - A) \cap D(A_0)$. Hence x is an eigenvector of A_0 to the eigenvalue λ. But since $\lambda \in \rho(A_0)$ it follows $x = 0$.

Now let $y \in D$. Since $\lambda \in \rho(A_0)$ there is $x_1 \in D(A_0)$ such that $(\lambda - A)y = (\lambda - A_0)x_1$. Hence we have

$$y = x_1 + (y - x_1).$$

Since $A_0 \subset A$ it follows that $y - x_1 \in \ker(\lambda - A)$. □

Lemma 2.44. *Let $\lambda \in \rho(A_0)$ and $g \in \partial X$. Then there is exactly one $u \in \ker(\lambda - A)$ with $Lu = g$. Writing $L_\lambda g := u$ the induced operator $L_\lambda \colon \partial X \to \ker(\lambda - A)$ is continuous as a mapping $\partial X \to D$. Furthermore $L_\lambda L = \mathrm{id}_{\partial X}$.*

Proof. Suppose we had another function $v \in \ker(\lambda - A)$ with $Lv = g$. Then $w := u - v \in \ker(\lambda - A)$ satisfies $Lw = 0$, i.e. $w \in D(A_0)$. Since $\lambda \in \rho(A_0)$ we have $w = 0$, hence u is unique.

By the above lemma we have $D = \ker(\lambda - A) \oplus D(A_0)$. Since L is surjective and $L|_{D(A_0)} = 0$, it follow $\operatorname{im} L|_{\ker(\lambda - A)} = \partial X$ which proves the existence of u.

By that argument we have $L_\lambda = (L|_{\ker(\lambda-A)})^{-1}$. Since $L|_{\ker \lambda - A}$ is a continuous bijection between the Banach spaces $\ker(\lambda - A) \subset D$ and ∂X, we have by the open mapping theorem that L_λ is also continuous.

The remaining assertion is clear. □

Lemma 2.45. *Let $\lambda \in \rho(A_0)$. Then it holds*

$$x \in D(A_\phi) \iff (1 - L_\lambda \phi)x \in D(A_0)$$

and

$$(\lambda - A_\phi)x = (\lambda - A_0)(1 - L_\lambda \phi)x \;\forall x \in D(A_\phi).$$

Especially we have

$$R(\lambda, A_\phi) = (1 - L_\lambda \phi)^{-1} R(\lambda, A_0)$$

in the case where $1 - L_\lambda \phi \colon \overline{D} \to \overline{D}$ is invertible.

Proof. We have

$$\begin{aligned} x \in D(A_\phi) &\Leftrightarrow x \in D(A),\ Lx = \phi x \\ &\Leftrightarrow x \in D(A),\ x = L_\lambda \phi x \\ &\Leftrightarrow x \in D(A),\ (1 - L_\lambda \phi)x = 0 \\ &\Leftrightarrow (1 - L_\lambda \phi)x \in D(A),\ (1 - L_\lambda \phi)x = 0 \\ &\Leftrightarrow (1 - L_\lambda \phi)x \in D(A_0) \end{aligned}$$

and

$$\begin{aligned} (\lambda - A_0)(1 - L_\lambda \phi)x &= (\lambda - A)x - (\lambda - A)L_\lambda x \\ &= (\lambda - A_\phi)x - 0, \end{aligned}$$

since $L_\lambda x \in \ker(\lambda - A)$. □

Proposition 2.46. *Assume there exists $\omega' \geq \omega$ and $C > 0$ such that for all $\lambda \in \mathbb{C}^+ + \omega'$ the mapping $1 - L_\lambda \phi$ is invertible and*

$$\|(1 - L_\lambda \phi)^{-1}\|_{\mathcal{L}(\overline{D})} \leq C$$

holds. Then A_ϕ generates an analytic semigroup T_ϕ.

Furthermore if A_0 generates a bounded analytic semigroup and the assumption above holds for $\lambda = \omega$, then also $A_\phi - \omega$ generates a bounded analytic semigroup. If $\overline{D(A_0)} = \overline{D}$, then the part of A_ϕ in $\overline{D}$ generates an analytic C_0-semigroup on $\overline{D}$.

In particular the last assertion implies: If A_0 generates a C_0-semigroup on X, then so does A_ϕ.

Proof. Let $\lambda \in \mathbb{C}^+ + \omega'$. By hypothesis and Lemma 2.45 we have that $\lambda \in \rho(A_\phi)$ and by

$$\begin{aligned} \|\lambda R(\lambda, A_\phi)\| &\leq \|(1 - L_\lambda \phi)^{-1}\| \|\lambda R(\lambda, A_0)\| \\ &\leq CM' \end{aligned}$$

the holomorphic estimate is proven. Hence A_ϕ generates an analytic semigroup.

If the part of A_0 in $\overline{D}$ is densely defined in $\overline{D}$, so is the part of A_ϕ in $\overline{D}$ by Lemma 2.45 and A_0 as well as A_ϕ generate C_0-semigroups on $\overline{D}$ by Proposition 2.2. The claim about boundedness follows from Proposition 2.5 applied to the operator $A_\phi - \omega$. □

In light of the assumptions (H1)–(H3), the preceding lemmas and the Neumann series Greiner's Theorem is implied by the following lemma.

Lemma. *If ϕ is compact and A_0 is densely defined, then $\|L_\lambda \phi\|_{\mathcal{L}(X)} \to 0$ as* $\operatorname{Re}\lambda \to \infty$.

In fact we will prove this generalization of Theorem 2.42:

Theorem 2.47. *If ϕ is compact and $\overline{D(A_0)} = \overline{D}$, then A_ϕ generates an analytic semigroup T_ϕ on X which restricts to a strongly continuous semigroup on $\overline{D}$.*

Proof. We first show: $\|L_\lambda z\|_{\mathcal{L}(\overline{D})} \to 0$ as $\operatorname{Re}\lambda \to \infty$ for all $z \in \partial X$.

Let $\lambda \in \mathbb{C}^+ + \omega$, $z \in \partial X$ and set $x_\lambda := L_\lambda z \in D$, $x_\mu := L_\mu z \in D$, $x := x_\lambda - x_\mu \in \ker L = D(A_0) \subset D$. Since $\operatorname{im} L_\lambda \subset \ker(\lambda - A)$ we have

$$(\lambda - A_0)x = -(\lambda - A_0)x_\mu = (\mu - \lambda)x_\mu,$$

hence $x = (\mu - \lambda)R(\lambda, A_0)x_\mu$ and

$$x_\lambda = x_\mu - \lambda R(\lambda, A_0)x_\mu + \mu R(\lambda, A_0)x_\mu.$$

Since $\lambda R(\lambda, A_0)f \to f$ for all $f \in \overline{D(A_0)} = D$, the second term converges to x_μ as $\operatorname{Re}\lambda \to \infty$ and the third to 0 as $\operatorname{Re}\lambda \to \infty$; altogether we have $L_\lambda z \to 0$ as $\operatorname{Re}\lambda \to \infty$.

By [SW99, Theorem III.4.5] it follows that $L_\lambda \to 0$ as $\operatorname{Re}\lambda \to \infty$ uniformly on compact subsets. Since ϕ is compact and D is clearly dense in $\overline{D}$, we have $L_\lambda\phi \to 0$ as $\operatorname{Re}\lambda \to \infty$ uniformly on bounded subsets, i.e. $\|L_\lambda\phi\|_{\mathcal{L}(\overline{D})} \to 0$ as $\operatorname{Re}\lambda \to \infty$.

Now the theorem follows as a corollary of Proposition 2.46 by noting that in our situation of the lemma above for some $\omega' > \omega$ we have $\|L_\lambda\phi\| < \frac{1}{2}$ for all $\lambda \in \mathbb{C}^+ + \omega'$ and the inverse is given by the Neumann series

$$(1 - L_\lambda\phi)^{-1} = \sum_{n=0}^{\infty} (L_\lambda\phi)^n \text{ for all } \lambda \in \mathbb{C}^+ + \omega'.$$

This yields $\|(1 - L_\lambda\phi)^{-1}\|_{\mathcal{L}(\overline{D})} \leq 2$ for all $\lambda \in \mathbb{C}^+ + \omega'$. □

2.6.2 Uniform boundedness estimate for positive and power compact $L_\lambda\phi$

Theorem 2.42 resp. its generalization Theorem 2.47 are convenient for showing that an operator derived from the Robin boundary conditions is a generator in $L^\infty(\Omega)$ and $C(\overline{\Omega})$ but are not applicable for Dirichlet boundary conditions, since in that case neither the generator is densely defined nor is the space $D(A_0)$ dense in $\overline{D} = C(\overline{\Omega})$. Still it is possible to use Proposition 2.46, if we can prove that the invertibility and the uniform estimate there hold. A way to prove these is formulated abstractly in this subsection which will allow us to treat the two cases of $\mathscr{A}$ being in divergence and $\mathscr{A}$

being in non-divergence form simultaneously in Chapter 3. The following theorem extends the Theorem 2.42 resp. Theorem 2.47 in the sense that we do not assume $L_\lambda\phi$ to be compact but only power-compact; in our application where $L_\lambda\phi$ operates on $C(\overline{\Omega})$ this means we assume that $L_\lambda\phi$ is only weakly compact, cf. [DS88, Chapter VI].

Theorem 2.48. *Let assumptions (H1)–(H3) hold. Let X be a Banach lattice such that the positive cone $\overline{D} \cap X_+$ is generating in $\overline{D}$. If there exists $\omega' \geq \omega$ such that*

i) $1 - L_\lambda\phi$ *is injective for all* $\lambda \in \mathbb{C}^+ + \omega'$,

ii) $(L_\lambda\phi)^n$ *is compact for some* $n \in \mathbb{N}$ *and all* $\lambda \in \mathbb{C}^+ + \omega'$,

iii) $L_\lambda\phi$ *is positive for all* $\lambda > \omega'$,

iv) $L_\lambda\phi$ *is a contraction for all* $\lambda > \omega'$ *and*

v) $R(\lambda, A_0)$ *is positive for all* $\lambda > \omega'$ *(i.e.* A_0 *generates a positive semigroup),*

then A_ϕ generates a positive, holomorphic semigroup T_ϕ on X. If $\omega = \omega'$ and the assertions i)–v) also hold for $\lambda = \omega$, then the semigroup $(e^{-\omega t}T_\phi(t))_{t\geq 0}$ generated by $A_\phi - \omega$ is also exponentially stable.

For the positive cone Y_+ in an ordered Banach space Y to be *generating* means that for all $y \in Y$ there are $y_i \in Y_+$ such that $y = y_1 - y_2 + i(y_3 - y_4)$. This linear combination can be chosen such that $\max\{\|y_i\|_X, i = 1, \dots 4\} \leq \alpha\|y\|$ for a constant $\alpha > 0$ independent of y, cf. [BR84, Proposition 1.1.2]. The positive cone of a Banach lattice is always generating with constant $\alpha = 1$; furthermore the dual of a Banach lattice X is also a Banach lattice with the positive cone $X'_+ = \{\tau \in X' : \langle \tau, x\rangle \geq 0 \ \forall x \in X_+\}$. For more information on ordered Banach spaces we refer to [BR84] and for more on Banach lattices we refer to [AGG$^+$86, C-I].

Remember that assumptions (H1)–(H3) state the following: $D \subset X$ continuously embedded, $A\colon D \to X$ continuous, $L\colon D \to \partial X$ continuous and surjective, $A_0 = A|_{\ker L}$ generates a holomorphic semigroup of type (M, ω), $\phi\colon \overline{D} \to \partial X$ continuous; furthermore we defined

$$D(A_\phi) = \{x \in D : \ Lx = \phi x\}, \quad A_\phi x = Ax$$

and denoted by L_λ the operator $(L|_{\ker \lambda - A})^{-1}$ which is well-defined by Lemma 2.44.

The proof consists of a series of lemmas. In the following let X, $\overline{D}$, T, $L_\lambda\phi$ and ω' be as in Theorem 2.48. We begin with a suitable version of the Fredholm alternative. Note that the following is a special case of [Kre82, Theorem 15.4] possessing an easier proof.

Lemma 2.49. *Let Y be a Banach space and let $B \in \mathcal{L}(Y)$ such that $1 - B$ is injective. If there is $n \in \mathbb{N}$ such that B^n is compact, then $1 - B$ is invertible.*

Proof. Since B^n is compact, $\sigma(B^n)$ is countable with 0 being the only possible accumulation point and by the spectral mapping theorem the same holds for $\sigma(B)$. In particular every point in $\sigma(B) \setminus \{0\}$ is a boundary point and hence in the approximate point spectrum of B.

Now assume that $1 - B$ is not surjective, i.e. $1 \in \sigma(B)$. Then there is a sequence $(x_n)_{n\in\mathbb{N}} \subset Y$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $(1 - B)x_n \to 0$ as $n \to \infty$. Since B^n is compact, after passing to a subsequence we may assume that $B^n x_n \to y$ as $n \to \infty$ for some $y \in Y$. By continuity of B^{n-1}, we have $B^{n-1}x_n - B^n x_n = B^{n-1}(1 - B)x_n \to 0$ as $n \to \infty$. Hence it follows that $B^{n-1}x_n \to y$ as $n \to \infty$.

In particular we then have

$$By = B \lim_{n\to\infty} B^{n-1}y = \lim_{n\to\infty} B^n x_n = y.$$

But $1 - B$ is injective, hence $y = 0$ — a contradiction to $\|x_n\| = 1$ for all $n \in \mathbb{N}$. □

Now we prove the uniform estimate.

Lemma 2.50. *Let $1 - L_\lambda\phi \colon \overline{D} \to \overline{D}$ be invertible for all $\lambda \in \mathbb{C}^+ + \omega'$ for some $\omega' \geq \omega$. If T and $L_\lambda\phi$ are positive for all $\lambda \geq \omega'$ then*

$$\sup_{\operatorname{Re}\lambda \geq \omega'+\delta} \|(1 - L_\lambda\phi)^{-1}\|_{\mathcal{L}(\overline{D})} < \infty$$

for all $\delta > 0$ and $(1 - L_\lambda\phi)^{-1} \geq 0$ for $\lambda > \omega'$.

Proof. The proof consists of three steps.

Step 1: L_λ and hence $L_\lambda\phi$ are holomorphic.

Let $\lambda \in \mathbb{C}^+ + \omega'$ and $\tau \in \mathbb{C}$ such that $\lambda + \tau \in \mathbb{C}^+ + \omega'$ and $v \in \partial X$. Set $u := L_{\lambda+\tau}v$ and $w := L_\lambda v$. Then $Lu = Lw = v$ and

$$(\lambda + \tau)u - \mathscr{A}u = 0,\ \lambda w - \mathscr{A}w = 0.$$

Hence we have $(\lambda + \tau)(u - w) - A(u - w) = -\tau w$. In other words

$$\begin{aligned} & u - w = -R(\lambda + \tau, A_0)(\tau w) \\ \Rightarrow \quad & \frac{1}{\tau}(L_{\lambda+\tau} - L_\lambda)v = -R(\lambda + \tau, A_0)L_\lambda v \\ \Rightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}\lambda}L_\lambda = -R(\lambda, A_0)L_\lambda, \text{ by sending } \tau \to 0. \end{aligned}$$

Step 2: There is $C > 0$ such that $\|(1 - L_\lambda \phi)^{-1} v\|_X \le C\|v\|_X$ for all $\operatorname{Re}\lambda \ge \omega + \delta$*, $\delta > 0$,$v \in \overline{D} \cap X_+$ and $(1 - L_{\delta+\omega'}\phi) \ge 0$.*

By step 1 we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} L_\lambda = (-1)^n n! R(\lambda, A_0)^n L_\lambda.$$

Since all the operators $R(\lambda, A_0)$ and $L_\lambda \phi$ are positive for $\lambda > \omega$, it follows that $\lambda \mapsto L_\lambda \phi$ is completely monotonic on (ω', ∞).

Hence by Bernstein's Theorem — see [ABHN11, Theorem 2.7.18] for a vector-valued version (used here on the space $\mathbb{R}$) or [Wid41] for the classical version — for $0 \le v \in \overline{D}$, $0 \le \tau \in X'$ there exists an increasing function $\alpha\colon (0, \infty) \to \overline{D}$ with

$$\langle \tau, (L_\lambda \phi) v \rangle = \int_0^\infty e^{-\lambda t} \, \mathrm{d}\alpha(t)$$

for all $\lambda > \omega'$. By the uniqueness theorem for holomorphic functions this holds for all $\lambda \in \mathbb{C}^+ + \omega'$.

Hence we have for λ with $\operatorname{Re}\lambda \ge \omega' + \delta$, $\delta > 0$, and all $0 \le \tau \in X'$

$$|\langle \tau, (L_\lambda \phi) v \rangle| \le \int_0^\infty e^{-(\delta+\omega')t} \, \mathrm{d}\alpha(t) = \langle \tau, (L_{\delta+\omega'}\phi) v \rangle. \tag{2.10}$$

Since the positive cone in X' is generating, it follows that $\|(L_\lambda \phi) v\|_X$ is bounded for λ on the halfplane $\mathbb{C}^+ + \omega' + \delta$.

Since $L_{\delta+\omega'}\phi$ is positive $r(L_{\delta+\omega'}\phi) \in \sigma(L_{\delta+\omega'}\phi)$ follows and since $1 - L_{\delta+\omega'}\phi$ is invertible and $L_{\delta+\omega'}\phi$ a contraction by assumption, we have $r(L_{\delta+\omega'}) < 1$. Then the inverse of $1 - L_{\delta+\omega'}\phi$ is given by the Neumann series

$$(1 - L_{\delta+\omega'}\phi)^{-1} = \sum_{n=0}^{\infty} (L_{\delta+\omega'}\phi)^n$$

and the estimate $\|(1 - L_{\delta+\omega'}\phi)^{-1} v\|_X \le C\|v\|_X$ for some $C > 0$ follows. Since $(L_{\delta+\omega'}\phi)^n \ge 0$, we also have $(1 - L_{\delta+\omega'}\phi)^{-1} \ge 0$.

Now let λ be complex with $\operatorname{Re}\lambda \ge \delta + \omega'$. By (2.10), the fact that the positive cone of X' is generating and Lemma 2.51 below the proof we have $|(L_\lambda \phi)^n v| \le (L_{\delta+\omega'}\phi)^n v$ for all $v \ge 0$. This gives the sought estimate $\|(1 - L_\lambda \phi)^{-1} v\|_X \le C\|v\|_X$.

Step 3: The estimate for general $v \in \overline{D}$.

Now let $v \in \overline{D}$ be arbitrary. Representing $v = v_1 - v_2 + i(v_3 - v_4)$ where $v_i \cap \overline{D} \cap X_+$ such that $\max\{\|v_i\|, i = 1, \dots, 4\} \le \alpha\|v\|$ for $\alpha > 0$ independent of v we have by step 2 that $\|(L_\lambda \phi) v\|_X \le 4C\alpha\|(L_{\delta+\omega'}\phi) v\|_X$. It follows that

$$\sum_{n=0}^{\infty} (L_\lambda \phi)^n = (1 - L_\lambda \phi)^{-1}$$

and

$$\|(1 - L_\lambda \phi)^{-1}\|_X \leq 4\alpha \|(1 - L_{\delta+\omega'}\phi)^{-1}\|_X$$

proving the claim. □

The following lemma completes the proof of Lemma 2.50.

Lemma 2.51. *Let $S, T\colon X \to X$ be bounded and linear. Assume that T is positive and that $|Sx| \leq Tx$ for all $x \geq 0$. Then*

$$|Sx| \leq T|x|$$

for all $x \in X$.

Proof. By [Sch74, Proposition II.5.5] and Corollary 2 of that proposition we have that the bidual X'' of X is a Banach lattice with order continuous norm and that the evaluation map $\iota\colon X \to X''$, i.e. $\iota(x)(\psi) = \psi(x)$, is an injective lattice homomorphism, i.e. $|\iota(x)| = \iota(|x|)$.

Now let $0 \leq x \in X$ and $\theta \in \mathbb{R}$. Then

$$\mathrm{Re}(e^{i\theta}\iota(Sx)) \leq |\iota(Sx)| = \iota(|Sx|) \leq \iota(Tx).$$

It follows from [Sch74, Theorem IV.1.8] and the two lines after [Sch74, Definition IV.1.7] that $|\iota(Sx)| \leq \iota(T|x|)$ for all $x \in Y$. Hence $\iota(|Sx|) = |\iota(Sx)| \leq \iota(T|x|)$ which implies $|Sx| \leq T|x|$ as claimed. □

Now the proof of Theorem 2.48 goes as follows:

Proof of Theorem 2.48 . By Lemma 2.49 the operator $1 - L_\lambda\phi$ is invertible and by Lemma 2.50 the uniform estimate for $\|(1 - L_\lambda\phi)^{-1}\|_{\mathcal{L}(\overline{D})}$ holds. Now the claim follows by Proposition 2.46.

It remains to show, that $(e^{-\omega t}T_\phi(t))_{t>0}$ is exponentially stable, if $\omega = \omega'$ and the assertions *i)–v)* of the theorem also hold for $\lambda = \omega$.

In that case it follows from Lemma 2.49 that $1 - L_\omega\phi$ is invertible. Hence by Lemma 2.45

$$R(\omega, A_\phi) = (1 - L_\omega\phi)^{-1}R(\omega, A_0)$$

and it follows that $\omega \in \rho(A_\phi)$. Now Proposition 2.9 applied to $A_\phi - \omega$ yields the claim. □

2.6.3 Further transferred properties

To conclude this section we collect how some properties of the original semigroup conditions transfer to the semigroup with perturbed boundary conditions.

Proposition 2.52 (compactness, positivity, irreducibility). *Let the assumptions of Proposition 2.46 hold.*

i) If A_0 has compact resolvent, so has A_ϕ.

ii) Let X be a Banach lattice. If T_0 is positive, $r(L_\lambda\phi) < 1$ for $\lambda > \omega'$ and $(L_\lambda\phi)u \in X_+$ for all $u \in \overline{D} \cap X_+$ for $\lambda > \omega'$, then also T_ϕ is positive.

iii) Let X be a Banach lattice. If T_0 is irreducible, $r(L_\lambda\phi) < 1$ for $\lambda > \omega'$ and $(L_\lambda\phi)u \gg 0$ for all $0 \ll u \in \overline{D}$, $\lambda > \omega'$, then also T_ϕ is irreducible.

iv) Let X be a Banach lattice. If T_0 is positive, $r(L_\lambda\phi) < 1$ for $\lambda > \omega'$ and $(L_\lambda\phi)u \gg 0$ for all $u \in \overline{D} \cap X_+$, $u \neq 0$, $\lambda > \omega'$, then T_ϕ is irreducible.

Remember that we write $u \gg 0$, if u is a quasi interior point of X. For the proof above, we use some equivalent formulations of irreducibility as given in [AGG+86, C-III, Definition 3.1]; note that the strong continuity at 0 is not used in the equivalences there.

Proof. Claim i): This is a direct consequence of Lemma 2.45 and the ideal property of compact operators.

Claim ii): Since $r(L_\lambda\phi) < 1$, the inverse of $1 - L_\lambda\phi$ is given by $\sum_{n=0}^{\infty}(L_\lambda\phi)^n$. Since all the summands are positive for large λ by assumption, we have by Lemma 2.45 that also $R(\lambda, A_\phi)$ is positive for large λ. The claim follows by Proposition 2.7.

Claim iii): Since T_0 is irreducible, we have $R(\lambda, A_0)u \gg 0$ for all large λ, $u \geq 0$, $u \neq 0$. Furthermore with $r(L_\lambda\phi) < 1$ and $(L_\lambda\phi)u \gg 0$ for $u \gg 0$ for all large λ, we have $R(\lambda, A_\phi)u \gg 0$ for all large λ,$u \geq 0$, $u \neq 0$, since in the sum $\sum_{n=0}^{\infty}(L_\lambda\phi)^n v$ of positive operators we have the strictly positive summand $(L_\lambda\phi)v$, where $v = R(\lambda, A_0)u$. This implies the irreducibility of T_ϕ.

Claim iv): Since T_0 is positive, we have $R(\lambda, A_0) \geq 0$ for all large λ. Since $r(L_\lambda\phi) < 1$ and $(L_\lambda\phi)u \gg 0$ for all large λ, $u \geq 0$, $u \neq 0$, we have $R(\lambda, A_\phi)u \gg 0$ for all large λ,$u \geq 0$, $u \neq 0$. In fact since in the sum $\sum_{n=0}^{\infty}(L_\lambda\phi)^n v$ we have the strictly positive summand $L_\lambda\phi v$ and the rest of the summands are positive, where $v = R(\lambda, A_0)u$. This implies the irreducibility of T_ϕ. □

Similarly to the positivity above we can prove a domination result.

Proposition 2.53 (domination). *Assume that X and ∂X are Banach lattices. Moreover, assume that we are given maps $L_1, L_2 : D \to \partial X$ and $\phi_1, \phi_2 : \overline{D} \to \partial X$ such that hypothesis of Proposition 2.46 is satisfied for the operators A, L_1, Φ_1 and the operators A, L_2, Φ_2. We write $A_0^j := A|_{\ker B_j}$ and $L_\lambda^j := (L_j|_{\ker(\lambda - A)})^{-1}$ for $j = 1, 2$. Finally, we assume that*

(a) *The semigroup generated by* A_0^j *is positive for* $j = 1, 2$*;*

(b) $0 \le \Phi_1 \le \Phi_2$*;*

(c) *For all* $\lambda > \omega'$ *we have* $0 \le L_\lambda^1 \le L_\lambda^2$ *and* $r(L_\lambda^1 \phi_1) < 1$ *as well as* $r(L_\lambda^2 \phi_2) < 1$*;*

(d) *If* $u \in D$ *is positive, then* $L_2 u \le L_1 u$*.*

Then for the semigroups T_{ϕ_1} *generated by* A_{ϕ_1} *and* T_{ϕ_2} *generated by* A_{ϕ_2} *we have* $0 \le T_{\phi_1}(t) \le T_{\phi_2}(t)$ *for all* $t > 0$*.*

Proof. Let us first note that since the operators ϕ_j and L_λ^j are positive for $\lambda > \rho$ and $j = 1, 2$, it follows from the previous proposition that T_{ϕ_1} and T_{ϕ_2} are positive semigroups. It follows from (b) and (c) that

$$(I - L_\lambda^1 \phi_1)^{-1} = \sum_{n=0}^{\infty} (L_\lambda^1 \phi_1)^n \le \sum_{n=0}^{\infty} (L_\lambda^2 \phi_2)^n = (I - L_\lambda^2 \phi_2)^{-1}$$

for all $\lambda > \omega'$. Now fix $f \ge 0$ and $\lambda > \omega'$. We put $u_j := R(\lambda, A_0^j) f$. Then $(\lambda - A)(u_1 - u_2) = 0$ and $L_1 u_1 = L_2 u_2 = 0$. Using our assumption (d) and the fact that $u_1 \ge 0$, we see that

$$L_2(u_1 - u_2) = L_2 u_1 - L_1 u_1 \le 0.$$

Consequently, as $u_1 - u_2 = L_\lambda^2 (L_2(u_1 - u_2))$ and L_λ^2 is positive $u_1 - u_2 \le 0$. This proves $R(\lambda, A_0^1) \le R(\lambda, A_0^2)$. Combining this with the above and Lemma 2.45, we find

$$R(\lambda, A_{\phi_1}^1) = (I - S_\lambda^1 \phi_1)^{-1} R(\lambda, A_0^1) \le (I - S_\lambda^2 \phi_2)^{-1} R(\lambda, A_0^2) = R(\lambda, A_{\phi_2}^2)$$

for all sufficiently large λ. By Proposition 2.8 it follows that $T_1 \le T_2$. □

At last we address the strong Feller property.

Corollary 2.54. *Assume in addition to the hypothesis of Proposition 2.46 that* $X = L^\infty(\Omega)$ *and* $\overline{D} = C(\overline{\Omega})$*. If* A_0 *generates a strong Feller semigroup on* X*, then so does* A_ϕ*.*

Proof. By Proposition 2.25 it suffices to prove that for sufficiently large $\operatorname{Re}\lambda$ the operator $R(\lambda, A_\phi)$ is a strong Feller operator. But this follows from Lemma 2.45: The hypothesis implies that $R(\lambda, A_0)$ is a strong Feller operator, in particular it maps $L^\infty(\Omega)$ to $\overline{D} = C(\overline{\Omega})$. Since $U := (I - L_\lambda \phi)^{-1}$ is a bounded linear operator on $\overline{D}$ also $R(\lambda, A_\phi)$ maps $L^\infty(\Omega)$ to $C(\overline{\Omega})$. Moreover, if f_n is a bounded sequence in $L^\infty(\Omega)$ converging pointwise almost everywhere to f, then $R(\lambda, A_0) f_n$ is a bounded sequence which converges pointwise to $R(\lambda, A_0)f$ by Corollary 2.21. Since U is bounded on $C(\overline{\Omega})$ we have for $x \in \overline{\Omega}$

$$\begin{aligned}(R(\lambda, A_\phi)f_n)(x) &= \langle \delta_x, UR(\lambda, A_0)f_n\rangle = \langle U^*\delta_x, R(\lambda, A_0)f_n\rangle \\ &\to \langle U^*\delta_x, R(\lambda, A_0)f\rangle = \langle \delta_x, UR(\lambda, A_0)f\rangle = (R(\lambda, A_\phi)f)(x),\end{aligned}$$

where we have used dominated convergence. By Proposition 2.23 this implies that $R(\lambda, A_\phi)$ is a strong Feller operator. □

2.7 Notes and comments

The first section is taken from [AKK16, Section 2] which in turn is a succinct summary of results from [ABHN11, Chapter 3].

The second section is taken from [AKK, Appendix A]. Proposition 2.16 was developed in a discussion with Jochen Glück on joint train ride home from Ulm to Munich.

The third section is taken from [AKK16, Section 3] which in turn is a summary of some of the results in [Rev75], [Kun09] and [Kun11].

The fourth section is a summary of some of the results of [AS14] with slight modifications such as the formulation on $L^\infty(\Omega)$. It can also be found in [AKK16, Section 3].

The fifth section summarizes results from [AB99] for the Dirichlet problem and [Nit11] for the Robin problem. In the case of the Dirichlet problem we combine the argumentation of [ABHN11] and [AtE97] with a newer result from [Dan00a] to remove some restriction on the drift terms b_i and c_i. The maximum modulus principle in its original form is due to Kato in [Kat72, Lemma A]; here we give a slight generalization which follows a proof of Kato's result given in [Agm82, Lemma 5.4].

In the first subsection of the sixth section we summarize and dissect some of the results obtained in [Gre87]; this allows us to extract an estimate which implies a generation result even if the semigroup is not C_0 as it is assumed in Greiner's paper.

The second subsection is new as formulated for abstract operators in abstract Banach lattices X. A version of those results in the concrete setting of $L^\infty(\Omega)$ with a given differential operator can be found in [AKK16, Proposition 4.2] and the lemmas thereafter.

The results in the closing subsection on transferred properties are partly new or can be found in [AKK, Section 2].

3 Parabolic equations with non-local Dirichlet boundary conditions

In this chapter we treat second order elliptic operators $\mathscr{A}$ with non-local Dirichlet boundary conditions and show under certain conditions that they are generators of holomorphic semigroups on L^∞ and on spaces of continuous functions. We discuss equations of the form

$$\begin{cases} \partial_t u(t,x) - \mathscr{A} u(t,x) = 0 \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega \\ u(0,x) = u_0(x) \text{ for } x \in \Omega \\ u(t,z) = B(z)(u(t.\cdot)) \text{ for } (t,z) \in \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$ is open and bounded, $u_0 \in L^\infty(\Omega)$ and $B(z)\colon C(\overline{\Omega}) \to \mathbb{C}$ is a continuous functional for all $z \in \partial\Omega$. Since every continuous functional on $C(\overline{\Omega})$ is a kernel operator we may formulate the boundary condition without loss of generality as

$$u(t,z) = \int_{\overline{\Omega}} u(t,x)\, \mu(z, \mathrm{d}x) \text{ for } (t,x) \in \mathbb{R}^+ \times \partial\Omega$$

where $\mu(z,\cdot)$ is a Borel measure for all $z \in \partial\Omega$.

Having the (abstract) results of Section 2.6.2 at hand we treat elliptic operators with non-local boundary conditions in both divergence and non-divergence form at the same time. More precisely we give generation results on spaces of continuous and measurable, bounded functions as well as the strong Feller property in the first section and a detailed analysis of the asymptotics of the generated semigroup in the second section.

We now fix our assumptions on the operator and boundary conditions. As discussed above the boundary conditions are given by

$$u(z) = \int_{\overline{\Omega}} u(z)\, \mu(z, \mathrm{d}x) =: \langle \mu(z), u \rangle \quad \forall z \in \partial\Omega,$$

where $u \in C(\overline{\Omega})$ and $\mu(z)$, $z \in \partial\Omega$, is a Borel sub-probability measure on $\overline{\Omega}$; here we write $\mu(z, U) := \mu(z)(U)$ for $U \subset \overline{\Omega}$ measurable. We will impose the following assumptions on these measures.

(meas) The mapping $\partial\Omega \to \mathcal{M}(\overline{\Omega})_+$, $z \mapsto \mu(z)$ is $\sigma(\mathcal{M}(\overline{\Omega}), C(\overline{\Omega}))$-continuous, that is

$$z \mapsto \langle \mu(z), u \rangle$$

is a continuous mapping $\partial\Omega \to \mathbb{C}$ for all $u \in C(\overline{\Omega})$ and we have $\mu(z, \overline{\Omega}) \leq 1$ for all $z \in \partial\Omega$. Finally we require that the boundary $\partial\Omega$ is a null set under $\mu(z)$ for all $z \in \partial\Omega$.

The operator $\mathscr{A}$ and the domain $\Omega \subset \mathbb{R}^d$ are assumed to fulfill one of the following assumptions.

(div) The operator $\mathscr{A}\colon H^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}(\Omega)'$ is a second order differential operator in divergence form given by

$$\mathscr{A}u = \sum_{i,j=1}^{d} D_i(a_{ij} D_j u) + \sum_{i=1}^{d} D_i(b_i u) + \sum_{i=1}^{d} c_i D_i u + d_0,$$

where a_{ij}, b_i, c_i, $d_0 \in L^\infty(\Omega)$, $i, j = 1, \ldots, d$, $\sum_{i=1}^{d} D_i b_i + d_0 \leq 0$ in the sense of distributions, and

$$\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2$$

for all $\xi \in \mathbb{R}^d$ and almost all $x \in \overline{\Omega}$. The domain $\Omega \subset \mathbb{R}^d$ is bounded, open and Dirichlet regular.

When we say that an operator is in *divergence form*, then we mean that the assumption (div) is fulfilled.

(ndiv) The operator $\mathscr{A}$ is a second order differential operator in non-divergence form as in Section 2.4 given by

$$\mathscr{A}u = \sum_{i,j=1}^{d} a_{ij} D_i D_j u + \sum_{i=1}^{d} c_i D_i u + d_0 u,$$

where $u \in W^{2,d}_{\text{loc}}(\Omega)$, $a_{ij} = a_{ji} \in C(\overline{\Omega})$, $c_i, d_0 \in L^\infty(\Omega)$, $i, j = 1, \ldots, d$, $d_0 \leq 0$ and

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2$$

for all $\xi \in \mathbb{R}^d$ and all $x \in \overline{\Omega}$. Additionally one of the following regularity assumptions holds.

1. $\Omega \subset \mathbb{R}^d$ is open, bounded and satisfies the uniform exterior cone condition.
2. The coefficients a_{ij}, $i, j = 1, \ldots, d$, are Dini-continuous and $\Omega \subset \mathbb{R}^d$ is bounded, open and Dirichlet regular.

As before when we say that an operator is in *non-divergence form*, then we mean that the assumption (ndiv) is fulfilled. For the sake of consistency and for ease of notation we set $b_i = 0$, $i = 1, \ldots d$, when $\mathscr{A}$ is assumed to be in non-divergence form.

The details in these assumptions are explained in Section 2.5 resp. 2.4 of Chapter 2.

Finally in the third section we deal with assorted topics: We give a more detailed analysis of the space $C_\mu(\overline{\Omega})$ — the space where a C_0-semigroup is generated — and show several (unpleasant) properties of that space. This validates our decision to consider the semigroup on all of $C(\overline{\Omega})$ resp. $L^\infty(\Omega)$ and therefore to pay the price of losing the strong continuity at 0. Furthermore we discuss the irreducibility of the semigroup on $C(\overline{\Omega})$ and give a necessary and sufficient condition on the measures $\mu(z)$ for irreducibility to hold. As a last topic we give a generation result under a different condition on the measures: on one hand we relax the assumptions in that we simply say that all $\mu(z)$ are $\mathbb{C}$-valued measures and on the other hand we strengthen the assumptions in that we demand $|\mu(z)|(\overline{\Omega}) < 1$ for all $z \in \partial\Omega$, where $|\cdot|$ denotes the total variation of complex-valued measures. In particular we no longer assume any condition on the support of the measures.

3.1 Generation results and the strong Feller property

We now define the operator A_μ on $L^\infty(\Omega)$ with non-local boundary conditions as follows. If $\mathscr{A}$ is in non-divergence form set

$$\begin{aligned} D(A_\mu) &:= \big\{u \in C(\overline{\Omega}) \cap W(\Omega) : \mathscr{A}u \in L^\infty(\Omega), \\ &\qquad u(z) = \langle \mu(z), u \rangle \ \forall z \in \partial\Omega\big\} \\ A_\mu u &:= \mathscr{A}u, \end{aligned}$$

where was defined as $W(\Omega) = \bigcap_{1<p<\infty} W^{2,p}_{\mathrm{loc}}(\Omega)$, and if $\mathscr{A}$ is in divergence form

$$\begin{aligned} D(A_\mu) &:= \{u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega) : \ \mathscr{A}u \in L^\infty(\Omega) \\ &\qquad u(z) = \langle \mu(z), u \rangle \ \forall z \in \partial\Omega\}. \\ A_\mu u &:= \mathscr{A}u \end{aligned}$$

Remember that

$$\langle \tau, u \rangle = \int_{\overline{\Omega}} u(x)\, \tau(\mathrm{d}x)$$

for a measure $\tau \in \mathcal{M}(\overline{\Omega})$. The aim of this section is to establish generation results in the spaces $L^\infty(\Omega)$, $C(\overline{\Omega})$ and

$$C_\mu(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : \ u(z) = \langle \mu(z), u \rangle \ \forall z \in \partial\Omega\}$$

and to establish a regularity result in that the generated semigroup resp. the resolvent of A_μ is strongly Feller.

3.1.1 Generation results

For the proof that A_μ is a generator we want to use Theorem 2.48. In our case we have $X = L^\infty(\Omega)$, $D = D(A_{\max})$ and $A = A_{\max}$, where $A_{\max}$ is the maximal operator as in Section 2.4 resp. Section 2.5, $L\colon C(\overline{\Omega}) \to C(\partial\Omega)$, $Lu = u|_{\partial\Omega}$ and $\phi\colon C(\overline{\Omega}) \to C(\partial\Omega)$ with $(\phi u)(z) = \langle \mu(z), u \rangle$. Note that for ϕ to be well defined we used assumption (meas); also note that $\overline{D} = C(\overline{\Omega})$. Hence we have for $L_\lambda \phi =: S_\lambda$ that $S_\lambda\colon C(\overline{\Omega}) \to C(\overline{\Omega})$ and

$$S_\lambda f = u \ \Leftrightarrow \ \begin{cases} \lambda u - \mathscr{A}u = 0 \\ \qquad u|_{\partial\Omega} = \langle \mu(\cdot), f \rangle. \end{cases}$$

By Proposition 2.26 resp. Proposition 2.32 this is well defined for $\lambda \in \overline{\mathbb{C}^+}$. By the same propositions we have that S_λ is a positive operator for $\lambda \geq 0$. By the maximum principle of Proposition 2.30 resp. Proposition 2.35 and the assumption in (meas) of $\mu(z)$ being sub-probability measures, we have that S_λ is a contraction. We check the rest of the assumptions of the perturbation theorem in a series of lemmas.

Lemma 3.1. $1 - S_\lambda$ *is injective for* $\lambda \in \mathbb{C}^+$.

Proof. Assume that $S_\lambda u = u$ some $u \in C(\overline{\Omega})$. Then by definition we have

$$\begin{aligned} \lambda u - \mathscr{A}u &= 0 \\ u|_{\partial\Omega} &= \langle \mu(\cdot), u \rangle. \end{aligned}$$

By the maximum principle in Proposition 2.30 resp. 2.35 there is a point $z_0 \in \partial\Omega$ such that $|u(z_0)| = \|u\|_{C(\overline{\Omega})}$. Now suppose that $u \neq 0$, hence also $u(z_0) \neq 0$. By rescaling we may assume that $|u(z_0)| = 1$. Since $u(z_0) = \langle \mu(z_0), u \rangle$ we also have $\mu(z_0) \neq 0$. Hence by assumption on the measures μ we have $\mu(z_0, \Omega) > 0$ and there exists a point $x_0 \in \Omega$ and $r > 0$ such that $\epsilon := \mu(z_0, B_r(x_0)) > 0$ and $B_r(x_0) \Subset \Omega$. Since $|u(x_0)| < 1$ by the maximum principle we find $\delta > 0$ such that $|u(x)| \leq 1 - \delta$ for all $x \in B_r(x_0)$. Thus we obtain

$$
\begin{aligned}
1 = |u(z_0)| &\leq \langle \mu(z_0), |u| \rangle \\
&\leq \int_{B_r(x_0)} (1-\delta)\, \mu(z_0, \mathrm{d}x) + \int_{\overline{\Omega}\setminus B_r(x_0)} 1\, \mu(z_0, \mathrm{d}x) \\
&= (1-\delta)\mu(z_0, B_r(x_0)) + \mu(z_0, \overline{\Omega} \setminus B_r(x_0)) \\
&= \mu(z, \overline{\Omega}) - \delta\epsilon < 1
\end{aligned}
$$

— a contradiction. □

Remark. Let us note that additionally $S_\lambda f \in W(\Omega)$ if $\mathscr{A}$ is in non-divergence form by Prop. 2.26 resp. $S_\lambda f \in H^1_{\mathrm{loc}}(\Omega)$ by Proposition 2.32 for all $f \in C(\overline{\Omega})$. Hence by definition of S_λ we have for all $\lambda \in \overline{\mathbb{C}^+}$ that $1 - S_\lambda$ is injective if and only if $\lambda - A_\mu$ is injective.

Lemma 3.2. *S_λ^2 is compact for all $\lambda \in \mathbb{C}^+$ and for $\lambda = 0$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega})$ be a bounded sequence. Let $\varphi_n = \langle \mu(\cdot), u_n \rangle \in C(\partial\Omega)$. Then $v_n := S_\lambda u_n$ solves the Dirichlet problem $\lambda u - \mathscr{A} u = 0$ with boundary values φ_n. As a consequence of Proposition 2.27 resp. 2.33 the sequence $(v_n)_{n\in\mathbb{N}}$ is locally equicontinuous on Ω so that — after passing to a subsequence referred to by the same name — the sequence $(v_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of Ω to a function $u \in C_b(\Omega)$.

Now let $w_n := S_\lambda v_n = S_\lambda^2 u_n$. Then $w_n \in C(\overline{\Omega}) \cap W(\Omega)$, $\lambda w_n - \mathscr{A} w_n = 0$ and $w_n|_{\partial\Omega} = \langle \mu(\cdot), v_n \rangle =: \psi_n \in C(\partial\Omega)$ for all $n \in \mathbb{N}$. Since μ is $\sigma(\mathcal{M}(\overline{\Omega}), C(\overline{\Omega}))$-continuous and $\partial\Omega$ is a nullset for all $\mu(z)$, $z \in \partial\Omega$, μ is in particular $\sigma(\mathcal{M}(\Omega), C_b(\Omega))$-continuous. Furthermore $\partial\Omega$ is compact and hence $\{\mu(z),\, z \in \partial\Omega\}$ is $\sigma(\mathcal{M}(\Omega), C_b(\Omega))$-compact and thus by Prokhorov's Theorem (see [Bog06, Theorem 8.6.2]) *tight*, i.e. for given $\epsilon > 0$ we find a compact set $K \subset \Omega$ such that $\mu(z, \Omega \setminus K) < \epsilon$ for all $z \in \partial\Omega$. Note that the theorem requires a complete, separable metric space and that Ω is not complete with respect to its usual metric. But as open subset of $\mathbb{R}^d$ its topology is induced by a complete, separable metric. We apply Prokhorov's theorem with respect to this metric.

As $v_n \to v$ locally uniformly, the tightness of the measures $\mu(z)$, $z \in \partial\Omega$, clearly implies that $\psi_n = \langle \mu(\cdot), v_n \rangle \to \psi := \langle \mu(\cdot), v \rangle$ uniformly in $z \in \partial\Omega$.

Now follows from the maximum principle of Proposition 2.30 resp. 2.35 that $S_\lambda^2 u_n = w_n \to S_\lambda^2 u$ uniformly. □

Consequently by the two lemmas above and Lemma 2.49 we have the following corollary.

Corollary. *The operator* $1 - S_\lambda$ *is invertible for all* $\lambda \in \mathbb{C}^+$*. If* A_μ *is injective, then also* $1 - S_0$ *is invertible.*

Now we are almost ready to formulate our generation result but for contractivity of the semigroup we need the following result on the location of maximums.

Lemma 3.3. *Let* $u \in C(\overline{\Omega})$ *be real-valued such that* $u(z) \leq \langle u, \mu(z)\rangle$ *for all* $z \in \partial\Omega$*. If the maximum of* u *is positive, then it is attained in the interior* Ω *of* $\overline{\Omega}$*.*

Proof. Let $0 < c := \max_{x\in\overline{\Omega}} u(x)$. Now assume that $u(x) < c$ for all $x \in \Omega$. Then there is $z_0 \in \partial\Omega$ with $u(z_0) = c$. Since $0 < u(z_0) \leq \langle u, \mu(z_0)\rangle$, we have that $\mu(z_0) \neq 0$. Since $\operatorname{supp} \mu(z_0) \not\subset \partial\Omega$, we may take $x_0 \in \operatorname{supp} \mu(z_0) \cap \Omega$. We then find $r > 0$ and $\delta > 0$ such that $B_r(x_0) \subset \Omega$ and $u(x) \leq c - \delta$ for all $x \in B_r(x_0)$. Set $\epsilon := \mu(z_0, B_r(x_0)) > 0$. Then we have

$$\begin{aligned} c = u(z_0) &\leq \langle \mu(z_0), u\rangle \\ &= \int_{\overline{\Omega}\setminus B_r(x_0)} u(x)\, \mu(z_0, \mathrm{d}x) + \int_{B_r(x_0)} u(x)\, \mu(z_0, \mathrm{d}x) \\ &\leq c\mu(z_0, \overline{\Omega} \setminus B_r(x_0)) + (c-\delta)\mu(z_0, B_r(x_0)) \\ &= c\mu(z_0, \overline{\Omega}) - \delta\epsilon < c \end{aligned}$$

— a contradiction. Hence there is a $x_0 \in \Omega$ with $u(x_0) = c$. □

Now we can apply Theorem 2.48 and have our generation result on $L^\infty(\Omega)$.

Theorem 3.4. *Let one of the assumptions* (div) *or* (ndiv) *be fulfilled. Then* A_μ *generates a positive, contractive, holomorphic semigroup* T_μ *on* $L^\infty(\Omega)$*.*

Proof. All properties except for contractiveness follow from Theorem 2.48, the preceeding lemmas and the properties of A_0 collected in Section 2.4 resp. Section 2.5.

By Proposition 2.3 for contractiveness it suffices to show: $\|\lambda R(\lambda, A_\mu)\| \leq 1$ for all $\lambda > 0$.

Since $R(\lambda, A_\mu)$ is positive we show that $\lambda R(\lambda, A_\mu)1 \leq 1$. Set $u := R(\lambda, A_\mu)1$. By Lemma 3.3 there exists $x_0 \in \Omega$ with $u(x_0) = \max_{x\in\overline{\Omega}} u(x)$ and hence it is constant on the connected component U with $x_0 \in U$ by the maximum principle as in [GT98, Theorem 9.6] resp. [GT98, Theorem 8.19].

Let $u|_{\overline{U}} = c$. Since $\mathscr{A}u = \lambda u - 1$ it follows that $\mathscr{A}u$ has a continuous representative; in particular since $\mathscr{A}1_{\overline{\Omega}} = \sum_i D_i b_i + d_0 \in \mathcal{D}(\Omega)$, the fact that u is constant on $\overline{U}$ implies that $\sum_i D_i b_i + d_0$ is a regular distribution in $\mathcal{D}(U)'$ and has a continuous representative. Thus

$$\lambda u \leq \lambda u(x_0) = \lambda u(x_0) - 1 + 1 = \mathscr{A}u(x_0) + 1 = \underbrace{(\sum_i D_i b_i + d_0)(x_0)}_{\leq 0} + 1 \leq 1$$

yields the claim. □

Before continuing onto the strong Feller property we mention that Proposition 2.53 implies the following domination result.

Proposition 3.5. *Let $\mu_1, \mu_2 \colon \partial\Omega \to \mathcal{M}(\overline{\Omega})_+$ fulfill assumption (meas). If it holds for all $z \in \partial\Omega$ that $\mu_1(z) \leq \mu_2(z)$, i.e. for all $A \subset \overline{\Omega}$ we have $\mu_1(z, A) \leq \mu_2(z, A)$, then $0 \leq T_{\mu_1}(t) \leq T_{\mu_2}(t)$ for all $t > 0$.*

Under suitable assumptions on the mappings μ_1, μ_2 a stronger result will be proven in Section 3.2.

3.1.2 Semigroup of kernel operators and strong Feller property

We now turn to the question whether the part of A_μ in $C(\overline{\Omega})$ also generates a semigroup. Since we have already seen that A_μ generates a holomorphic semigroup on $L^\infty(\Omega)$, it remains to show that $R(\lambda, A_\mu)$ leaves $C(\overline{\Omega})$ invariant. But we show more: That $R(\lambda, A_\mu)$ is a *strong Feller operator*, and as such it maps $L^\infty(\Omega)$ to $C(\overline{\Omega})$. From this property the compactness of $R(\lambda, A_\mu)$ (and that of $T_\mu(t)$, $t > 0$) will follow even if we a priori do not even know that A_0 has compact resolvent, as in the case (ndiv) if Ω is only assumed to be Dirichlet regular.

Proposition 3.6. *The operator $R(\lambda, A_\mu)$ is a strong Feller operator for each $\lambda > 0$.*

Proof. By definition of A_μ we have that $R(\lambda, A_\mu) \in \mathcal{L}(L^\infty(\Omega))$ takes values in $C(\overline{\Omega})$ for all $\lambda > 0$. As $R(\lambda, A_\mu)$ is a positive operator by Theorem 3.4 for all $\lambda > 0$ we only need to show that $R(\lambda, A_\mu)f_n \to R(\lambda, A_\mu)f$ pointwise, whenever $(f_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ is a bounded sequence with $f_n \nearrow f$ pointwise almost everywhere.

Setting $u_n := R(\lambda, A_\mu)f_n$, the sequence $(u_n)_{n\in\mathbb{N}}$ is also increasing. Set $u(x) := \sup_{n\in\mathbb{N}} u_n(x)$ for all $x \in \overline{\Omega}$. By Proposition 2.27 resp. Proposition 2.33 the sequence u_n converges locally uniformly in Ω yielding $u \in C_b(\Omega)$. We now set $\varphi(z) := \langle \mu(z), u \rangle$; this is well defined since $\partial\Omega$ is a null set for each $\mu(z)$ by assumption. Furthermore $\varphi \in C(\partial\Omega)$ by the continuity assumption on μ. It follows from dominated convergence and the fact that $u_n \in D(A_\mu)$ that

$$\begin{aligned}\varphi(z) &= \int_\Omega u(x)\,\mu(z,\mathrm{d}x)\\ &= \lim_{n\to\infty}\int_\Omega u_n(x)\,\mu(z,\mathrm{d}x) = \lim_{n\to\infty} u_n(z) = u(z)\end{aligned} \tag{3.2}$$

for all $z \in \partial\Omega$. By Dini's lemma it follows that $u_n \to \varphi$ uniformly on $\partial\Omega$.

We now consider the Poisson operator P on $L^d(\Omega) \oplus C(\partial\Omega)$ as in the proof of Proposition 2.26 in the case of a non-divergence form operator resp. on $L^{d+\epsilon}(\Omega) \oplus C(\partial\Omega)$, $\epsilon > 0$ arbitrary, as in the proof of Proposition 2.32 in the case of a divergence form operator. In any case we then have $(u_n, 0) \in D(P)$ and $(\lambda - P)(u_n, 0) = (f_n, u_n|_{\partial\Omega})$. As $D(P) \subset C(\overline{\Omega}) \oplus \{0\}$ it follows from the closed graph theorem that $R(\lambda, P)$ is continuous from $L^d(\Omega) \oplus C(\partial\Omega)$ resp. $L^{d+\epsilon}(\Omega) \oplus C(\partial\Omega)$ to $C(\overline{\Omega}) \oplus \{0\}$. Here $C(\overline{\Omega})$ is endowed with the topology of uniform convergence on $\overline{\Omega}$. By the argumentation above $(f_n, u_n|_{\partial\Omega})$ converges to (f, φ) in $L^d(\Omega) \oplus C(\partial\Omega)$ resp. $L^{d+\epsilon}(\Omega) \oplus C(\partial\Omega)$. Thus

$$(u_n, 0) = R(\lambda, P)(f_n, u_n|_{\partial\Omega}) \to R(\lambda, P)(f, \varphi) =: (w, 0) \tag{3.3}$$

in $C(\overline{\Omega}) \oplus \{0\}$. In particular $u_n \to w$ uniformly on $\overline{\Omega}$. Since we already have $u_n \to u$ pointwise on $\overline{\Omega}$ it follows that $u = w$. By (3.3) we have that $w \in W^{2,d}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ resp. $w \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ and $\lambda w - Aw = f$. In the first case elliptic regularity as in Proposition 2.26 implies $w \in W(\Omega)$ since $f \in L^\infty(\Omega)$. In view of (3.2) it follows that $u \in D(A_\mu)$ and $\lambda u - A_\mu u = f$ in both cases.

This shows that $R(\lambda, A_\mu)f_n \to R(\lambda, A_\mu)f$ pointwise and by Lemma 2.23 that $R(\lambda, A_\mu)$ is a strong Feller operator. □

Applying this result to the semigroup T_μ generated by A_μ yields the following corollary by Proposition 2.25 and the notes after Lemma 2.23.

Corollary 3.7. *For every $\lambda \in \rho(A_\mu)$ the operator $R(\lambda, A_\mu)$ is strongly Feller and compact; moreover $T_\mu(t)$ is strongly Feller and compact for $t > 0$. In particular for the kernels k_t associated to $T_\mu(t)$ we have that $k_t(x, \cdot)$ is absolutely continuous w.r.t. the Lebesgue measure on Ω for all $x \in \overline{\Omega}$ and $t > 0$.*

3.1.3 The closure of $D(A_\mu)$ and the C_0-property

We end this section by identifying the subspace of $C(\overline{\Omega})$ whereto the restriction of T_μ is a C_0-semigroup. We define the subspace $C_\mu(\overline{\Omega})$ of $C(\overline{\Omega})$ as the space of functions fulfilling the boundary condition imposed by μ, i.e.

$$C_\mu(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u(z) = \langle \mu(z), u\rangle,\ \forall z \in \partial\Omega\}.$$

We first show the following density result.

Proposition 3.8. *The part of A_μ in $C_\mu(\overline{\Omega})$ is densely defined, that is the subspace*

$$\{u \in D(A_\mu),\ \mathscr{A}u \in C_\mu(\overline{\Omega})\}$$

is dense in $C_\mu(\overline{\Omega})$.

Proof. We first prove that the problem

$$\begin{cases} \lambda u - \mathscr{A}u = 0 \\ \qquad u|_{\partial\Omega} = \langle \mu(\cdot), u\rangle + \psi \end{cases}$$

has a unique solution $u \in C(\overline{\Omega})$ for all $\psi \in C(\partial\Omega)$ and all $\lambda \in \mathbb{C}^+$. Furthermore we show that $\|u\|_{C(\overline{\Omega})} \leq M_\lambda \|\psi\|_{C(\partial\Omega)}$ where $M_\lambda := \|(1 - S_\lambda)^{-1}\|_{\mathcal{L}(C(\overline{\Omega}))}$.

To see this note that

$$\begin{cases} \lambda u - \mathscr{A}u = 0 \\ \qquad u|_{\partial\Omega} = \langle \mu(\cdot), u\rangle + \psi \end{cases} \Leftrightarrow\ u = L_\lambda(\phi u + \psi)\ \Leftrightarrow\ (1 - L_\lambda \phi)u = L_\lambda \psi.$$

By the corollary after Lemma 3.2 the map $1 - L_\lambda\phi$ is invertible and unique solvability follows. The estimate then holds since L_λ is a contraction by the maximum principle in Proposition 2.30 resp. 2.35.

Now let $u \in C_\mu(\overline{\Omega})$ and $\epsilon > 0$. Let $\lambda > 0$ to be fixed later. Since $D(A_{\max})$ is densely defined there is $u_1 \in C(\overline{\Omega})$ with $\mathscr{A}u_1 \in C(\overline{\Omega})$ such that

$$\|u - u_1\|_{C(\overline{\Omega})} \leq \frac{\epsilon}{2M_\lambda}.$$

Hence we have

$$\begin{aligned} &\|u_1|_{\partial\Omega} - \langle \mu(\cdot), u_1\rangle\|_{C(\partial\Omega)} \\ &\quad \leq \|u - u_1\|_{C(\partial\Omega)} + \|u_1 - \langle \mu(\cdot), u_1\rangle\|_{C(\partial\Omega)} + \|\langle \mu(\cdot), u_1 - u\rangle\|_{C(\partial\Omega)} \\ &\quad \leq \frac{\epsilon}{M_\lambda}. \end{aligned}$$

Now set $f := \lambda u_1 - Au_1$ and $\psi := u_1|_{\partial\Omega} - \langle \mu(\cdot), u_1\rangle$ and $u_2 := R(\lambda, A_\mu)f$; i.e. u_2 denotes the solution of

$$\lambda u_2 - \mathscr{A}u_2 = f,\ u_2|_{\partial\Omega} = \langle \mu(\cdot), u_2\rangle.$$

Hence $w_1 := u_1 - u_2$ solves

$$\lambda w_1 - \mathscr{A}w_1 = 0,\ w_1|_{\partial\Omega} = \langle \mu(\cdot), w_1\rangle + \psi;$$

in particular by the first part of the proof we then have $\|w_1\|_{C(\overline{\Omega})} \leq M_\lambda \|\psi\|_{C(\partial\Omega)} \leq M_\lambda \epsilon / M_\lambda = \epsilon$.

Now set $u_3 := R(\lambda, A_\mu)\lambda u_2$ and $w_2 = u_2 - u_3$. We then have

$$\lambda w_2 - \mathscr{A} w_2 = -Au_2,\ w_2|_{\partial\Omega} = \langle \mu(\cdot), w_2\rangle,$$

i.e. $w_2 = -R(\lambda, A_\mu)\mathscr{A} u_2$ and hence

$$\|w_2\|_{C(\overline{\Omega})} \leq \frac{1}{\lambda}\|\mathscr{A} u_2\|_{C(\overline{\Omega})}$$

by contractivity of the generated semigroup.

Now choosing $\lambda > \|\mathscr{A} u_2\|_{C(\overline{\Omega})}/\epsilon$ yields $\|w_2\|_{C(\overline{\Omega})} \leq \epsilon$ and hence

$$\|u - u_3\|_{C(\overline{\Omega})} \leq \|u - u_1\|_{C(\overline{\Omega})} + \|w_1\|_{C(\overline{\Omega})} + \|w_2\|_{C(\overline{\Omega})} \leq 3\epsilon.$$

Since $u_3 = R(\lambda, A_\mu)\lambda u_2$ we have that $u_3 \in C_\mu(\overline{\Omega})$ and $\mathscr{A} u_3 \in C(\overline{\Omega})$. But since

$$\mathscr{A} u_3 = \mathscr{A} R(\lambda, A_\mu)\lambda u_2 = A_\mu R(\lambda, A_\mu)\lambda u_2 = R(\lambda, A_\mu)A_\mu \lambda u_2$$

it follows that $\mathscr{A} u_3|_{\partial\Omega} = \langle \mu(\cdot), \mathscr{A} u_3\rangle$ and we actually have $\mathscr{A} u_3 \in C_\mu(\overline{\Omega})$. □

Before we continue with the C_0-property we make note of the following statement we just proved.

Lemma 3.9. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Then for each $\psi \in C(\partial\Omega)$ there is exactly one solution $u \in D(A_{max})$ such that*

$$\begin{cases} \lambda u - \mathscr{A} u = 0 \text{ in } \Omega \\ \qquad u(z) = \langle \mu(z), u\rangle + \psi(z)\ \forall z \in \partial\Omega. \end{cases}$$

Furthermore we have $\|u\|_{C(\overline{\Omega})} \leq \|(1 - S_\lambda)^{-1}\|_{\mathcal{L}(C(\overline{\Omega}))}\|\psi\|_{C(\partial\Omega)}$. If A_μ is injective we also have the same result for $\lambda = 0$.

The additional claim for $\lambda = 0$ follows from the remark after Lemma 3.1 which states that $1 - S_0$ is invertible, if A_μ is injective.

Now the generation result on $C_\mu(\overline{\Omega})$ follows as a corollary.

Corollary 3.10. *The part of A_μ in $C_\mu(\overline{\Omega})$ generates a positive, holomorphic, contractive C_0-semigroup on $C_\mu(\overline{\Omega})$.*

Proof. Denote the part of A_μ in $C_\mu(\overline{\Omega})$ by B. Since we already proved that B is densely defined and also that the part of A_μ in $C(\overline{\Omega})$ generates a semigroup on $C(\overline{\Omega})$, it just remains to show that $R(\lambda, A_\mu)$ maps $C_\mu(\overline{\Omega})$ into $D(B)$.

But this clear since if we have $f \in C_\mu(\overline{\Omega})$ and $u = R(\lambda, A_\mu)f$, then in particular $u \in C_\mu(\overline{\Omega})$ and $\lambda u - \mathscr{A} u = f$. It then follows that $\mathscr{A} u = \lambda u - f \in C_\mu(\overline{\Omega})$ and hence $u \in D(B)$. □

3.2 The kernel of A_μ and asymptotics

In this section we consider the asymptotics of the semigroup T. In particular, we show that under certain conditions on μ we have exponential stability or more generally exponential decay to a finite rank projection. Furthermore we specify the rank of this projection. Such results are in part consequences of appropriate maximum principles for the operator A_μ which will also be developed in this section and in part consequences of the structure of the support of the measures $\mu(z)$. These maximum principles are derived from the maximum principle for the operator A_0 of Proposition 2.30 if $\mathscr{A}$ is in non-divergence form resp. Proposition 2.35 if $\mathscr{A}$ is in divergence form; in the following when we use "the maximum principle" we mean that we use the appropriate proposition of these two.

Before we begin we list some results of the following section; in contrast to the situation of local Dirichlet boundary conditions (i.e. $\mu = 0$) these may be surprising.

- T_μ converges to a projection; this projection may be of positive rank.
- Even though there may be infinitely many connected components of Ω this projection is of finite rank.

As we will see later on functions from the fixed space of T_μ must be constant on some connected component of Ω. For that constant to be non-zero it is necessary that every measure $\mu(z)$, where z is an element from the boundary of this connected component, is a probability measure. Nonetheless the following holds.

- The fixed space of T_μ (that is the kernel of A_μ) may contain non-locally constant functions.
- Even if there exists a connected component whose measures are all probability measures, the kernel of A_μ may be trivial; i.e. T_μ exponentially stable.

We begin by illustrating these phenomena as well as the arguments used later in two easy cases.

3.2.1 The case of one and two connected components

First let Ω consist of exactly one connected component. Then we have the following result on the kernel of A_μ.

Proposition. *We have that* $\ker A_\mu = \{0\}$ *if and only if* $\sum_i D_i b_i + d_0 \neq 0$ *on* Ω *or* $\mu(z, \overline{\Omega}) < 1$ *for at least one* $z \in \partial\Omega$.

As we will see in the subsequent sections the question of the asymptotic asymptotic behavior of the semigroup T_μ reduces largely to the question of which functions are in the kernel of A_μ. The proposition above, for example, implies that the semigroup is exponentially stable if and only if $\sum_i D_i b_i + d_0 \neq 0$ on Ω or $\mu(z, \overline{\Omega}) < 1$ for at least one $z \in \partial\Omega$. Furthermore we will see that if the semigroup is not exponentially stable then it will converge to a projection onto the constant functions.

Proof of the proposition. $\Rightarrow$: Contraposition. Let $\sum_i D_i b_i + d_0 = 0$ and $\mu(z, \overline{\Omega}) = 1$ for all $z \in \partial\Omega$. Then we have $1_{\overline{\Omega}} \in D(A_\mu)$ and $\mathscr{A} 1_{\overline{\Omega}} = 0$. Hence $\ker A_\mu \neq \{0\}$.

$\Leftarrow$: Let $u \in \ker A_\mu$. If $u = 0$, then we are done; if not, then we claim that u is constant. W.l.o.g. u has $\|u^+\|_{C(\overline{\Omega})} = 1$. We then have that either $u = 1$ on all of Ω or there is $z_0 \in \partial\Omega$ such that $u(z_0) > u(x)$ for all $x \in \Omega$ by the maximum principle; but Lemma 3.3 implies that this can actually not occur. Hence $u = 1_{\overline{\Omega}}$.

Now we consider the two possible cases: $\mu(z_0, \overline{\Omega}) < 1$ for some $z_0 \in \partial\Omega$ or $\sum_i D_i b_i + d_0 \neq 0$ in $\mathcal{D}(\Omega)'$. In the first case we have that the only constant function in $D(A_\mu)$ is 0 since otherwise we had

$$1 = 1_{\overline{\Omega}}(z_0) = \langle \mu(z_0), 1_{\overline{\Omega}} \rangle \leq \mu(z_0, \overline{\Omega}) < 1$$

— a contradiction. In the second case $\sum_i D_i b_i + d_0 \neq 0$ we have that

$$\mathscr{A} 1_{\overline{\Omega}} = \sum_{i=1}^{d} D_i b_i + d_0 \neq 0$$

as distribution implying $u \notin \ker A_\mu$. Altogether we have $\ker A_\mu = \{0\}$. □

Now let Ω consist of two connected components U_1 and U_2. For simplicity and exposition we restrict ourselves to the case where $\sum_i D_i b_i + d_0 = 0$ in $\mathcal{D}(\Omega)'$ for the rest of the examples and make use of the following easy to prove facts (which are shown subsequently in greater generality; see Lemma 3.12 and Lemma 3.16).

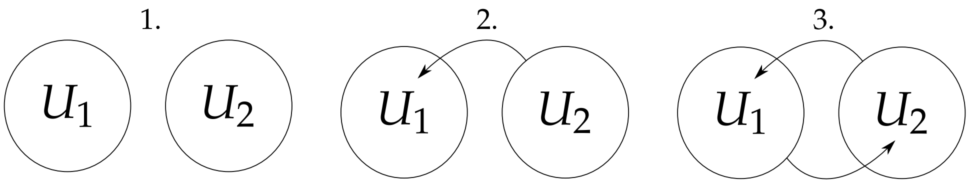
Fact 1 If a function in the kernel of A_μ attains a positive maximum in a connected component, it must be constant on that connected component.

Fact 2 Let $i, j \in \{1, 2\}$ with $i \neq j$. If a function u in the kernel of A_μ attains a positive maximum in a connected component U_i and if there is a $z_i \in \partial U_i$ with $\mu(z_i, \overline{U_j}) > 0$ then u attains this maximum also in U_j.

As illustrated before the asymptotic behavior depends on whether $\mu(z)$ is a probability measure for all $z \in \partial\Omega$ or not. But it also depends on the support of the measures, if we have more than one connected component. Here there are essentially three different situations:

1. $\mu(z_1, \overline{U_2}) = 0$ for all $z_1 \in \partial U_1$ and $\mu(z_2, \overline{U_1}) = 0$ for all $z_2 \in \partial U_2$
2. $\mu(z_1, \overline{U_2}) = 0$ for all $z_1 \in \partial U_1$ and $\mu(z_2, \overline{U_1}) > 0$ for some $z_2 \in \partial U_2$
3. $\mu(z_1, \overline{U_2}) > 0$ for some $z_1 \in \partial U_1$ and $\mu(z_2, \overline{U_1}) > 0$ for some $z_2 \in \partial U_2$

These situations may be illustrated as graphs on the connected components U_1, U_2 as below:



The first situation is essentially the case of one connected component copied. As such it is readily verified that the following holds.

Proposition. *Let* $\mu(z_1, \overline{U_2}) = 0$ *for all* $z_1 \in \partial U_1$ *and* $\mu(z_2, \overline{U_1}) = 0$ *for all* $z_2 \in \partial U_2$*. Then we have*

- $\ker A_\mu = \operatorname{lin} 1_{\overline{U_1}} \oplus \operatorname{lin} 1_{\overline{U_2}}$ *if and only if* $\mu(z_1, \overline{U_1}) = 1$ *for all* $z_1 \in \partial U_1$ *and* $\mu(z_2, \overline{U_2}) = 1$ *for all* $z_2 \in \partial U_2$*;*
- $\ker A_\mu = \operatorname{lin} 1_{\overline{U_1}} \oplus \{0_{\overline{U_2}}\}$ *if and only if* $\mu(z_1, \overline{U_1}) = 1$ *for all* $z_1 \in \partial U_1$ *and* $\mu(z_2, \overline{U_2}) < 1$ *for some* $z_2 \in \partial U_2$*, similarly for* $\ker A_\mu = \{0_{\overline{U_1}}\} \oplus \operatorname{lin} 1_{\overline{U_2}}$*;*
- $\ker A_\mu = \{0\}$ *if and only if* $\mu(z_1, \overline{U_1}) < 1$ *for some* $z_1 \in \partial U_1$ *and* $\mu(z_2, \overline{U_2}) < 1$ *for some* $z_2 \in \partial U_2$*.*

Again this proposition implies an corresponding result on the asymptotics of the semigroup.

Now onto the second one. Note that in the following result the dimension of the kernel of A_μ is at most one and independent of whether the measures $\mu(z_2)$ are probability measures for all $z_2 \in \partial U_2$ or not.

Proposition. *Let* $\mu(z_1, \overline{U_2}) = 0$ *for all* $z_1 \in \partial U_1$ *and* $\mu(z_2, \overline{U_1}) > 0$ *for some* $z_2 \in \partial U_2$*. Then we have* $\dim\ker A_\mu = 1$ *if and only if* $\mu(z_1, \overline{U_1}) = 1$ *for all* $z_1 \in \partial U_1$*. If* $\mu(z_1, \overline{U_1}) < 1$ *for some* $z \in \partial U_1$ *then we have* $\ker A_\mu = \{0\}$*.*

Proof. $\Leftarrow$: We first prove that $\dim\ker A_\mu \leq 1$ in two steps.

Step 1: Let $u \in \ker A_\mu$. We claim that it obtains its maximum on $\overline{U_1}$ and is constant there. If $u = 0$, then there is nothing to prove. If not we may assume w.l.o.g. that u is real with positive maximum 1 (otherwise consider $\operatorname{Re} u$ resp. $\operatorname{im} u$ or the negative of those and rescale). Then it obtains its maximum on at least one of $\overline{U_1}$ or $\overline{U_2}$. If it is obtained on $\overline{U_1}$ then by Fact 1 we are done. If it is obtained on $\overline{U_2}$, then by Fact 2 it is also obtained on $\overline{U_1}$; but then Fact 1 again yields the claim.

Step 2: Suppose now we have $u, v \in \ker A_\mu$, w.l.o.g. both with positive maximum 1 (otherwise again consider the real resp. the imaginary part or its negative and rescale). We claim $u = v$. By the first step it follows that u and v are constant on $\overline{U_1}$ and obtain their maximum there. But since both maxima are 1 we have that $u - v$ has maximum 0. Repeating this argument for $-u$, $-v$ yields then that $u - v = 0$.

We note here for "$\Rightarrow$" and the following proposition that only the assumption $\mu(z_2, \overline{U_1}) > 0$ for some $z_2 \in \partial U_2$ has been used in the two steps above.

Now let $\mu(z_1, \overline{U_1}) = 1$ for all $z_1 \in \partial U_1$. We now want to construct a function $u \in \ker A_\mu \setminus \{0\}$. To that end set u as 1 on $\overline{U_1}$. If $\mu(z_2, \overline{\Omega}) = 1$ for all $z \in \partial U_2$ set u as 1 on $\overline{U_2}$ and we are done.

On the other hand if we have $\mu(z_2, \overline{\Omega}) < 1$ for some $z \in \partial U_2$ set $\psi(z) = \mu(z, \overline{U_1})$ for $z \in \partial U_2$. Then we have $\psi \in C(\partial U_2)$. We now set $\mu' \colon \partial U_2 \to \mathcal{M}(\overline{U_2})$, $\mu'(z) = \mu(z)$; that is we only measure subsets of $\overline{U_2}$ with $\mu'(z)$. Now denote by $A_{\mu'}$ the elliptic differential operator on $L^\infty(U_2)$ defined analogously to A_μ on $L^\infty(\Omega)$. We have that $A_{\mu'}$ is injective. In fact functions in the kernel of A_μ are constant by Fact 1 and the only constant function in $D(A_{\mu'})$ is 0 since $\mu(z_2, \overline{\Omega}) < 1$ for some $z \in \partial U_2$. Hence we can apply Lemma 3.9 on the operator $A_{\mu'}$ and ψ as defined above yielding a function v on $\overline{U_2}$ that suffices

$$\begin{aligned} -\mathscr{A} v &= 0 \text{ on } U_2 \\ v &= \langle \mu'(\cdot), v \rangle + \psi \text{ on } \partial U_2. \end{aligned}$$

Now setting u to be v on $\overline{U_2}$ yields on the one hand $\mathscr{A} u = 0$ on all of Ω and on the other hand

$$u|_{\partial U_1} = 1 = \mu(\cdot, \overline{U_1}) = \langle \mu(\cdot), u \rangle$$

since $\mu(z_1, \overline{U_1}) = 1$ for all $z_1 \in \partial U_1$ and

$$\begin{aligned} u|_{\partial U_2} &= \langle \mu'(\cdot), v \rangle + \psi = \langle \mu'(\cdot), v \rangle + \mu(\cdot, \overline{U_1}) \\ &= \langle \mu(\cdot), v|_{\overline{U_2}} \rangle + \langle \mu(\cdot), u|_{\overline{U_1}} \rangle = \langle \mu(\cdot), u \rangle \end{aligned}$$

by construction of v and μ'. Altogether it follows that $u \in \ker A_\mu$.

$\Rightarrow$: Let $u \in \ker A_\mu \setminus \{0\}$ w.l.o.g. be real with positive maximum 1. Then $u|_{\overline{U_1}}$ is constant as seen in Step 1 of "$\Leftarrow$" and we must have $\mu(z_1, \overline{\Omega}) = 1$ for all $z_1 \in \partial U_1$. In fact if this were not the case, then we had the contradiction

$$1 = u(z_1) = \langle \mu(z_1), u \rangle \leq \mu(z_1, \overline{\Omega}) < 1$$

for $z_1 \in \partial U_1$ with $\mu(z_1, \overline{\Omega}) < 1$.

But since by assumption $\mu(z_1, \overline{U_2}) = 0$ for all $z_1 \in \partial U_1$ this implies the claim of the proposition. □

We also make note of the fact that if $\mu(z_1, \overline{U_1}) = 1$ for all $z_1 \in \partial U_1$, $\mu(z_2, \overline{U_2}) = 0$ for all $z_2 \in \partial U_2$ and $\partial U_2 \to \mathbb{R}$, $z \mapsto \mu(z, \overline{U_1})$ is not constant, then we have a non-locally constant function in the kernel of A_μ. In fact it is of the form $1_{\overline{U_1}} + v$ where the v is a solution on $\overline{U_2}$ of the elliptic problem

$$\mathscr{A} v = 0 \text{ in } U_2, \ v(z) = \mu(z, \overline{U_1}), \ z \in \partial U_2$$

which is not constant in view of the assumptions on μ above.

In the third case the situation is simpler again. The dimension of the kernel is at most one and 0 if and only if there is a point in $z \in \partial\Omega$ where $\mu(z, \overline{\Omega}) < 1$ regardless of whether $z \in \partial U_1$ or $z \in \partial U_2$.

Proposition. *Let $\mu(z_1, \overline{U_2}) > 0$ for some $z_1 \in \partial U_1$ and $\mu(z_2, \overline{U_1}) > 0$ for some $z_2 \in \partial U_2$. Then* $\ker A_\mu = \{0\}$ *if and only if there is a point in $z \in \partial\Omega$ where $\mu(z, \overline{\Omega}) < 1$. If all $\mu(z)$, $z \in \partial\Omega$, are probability measures then* $\ker A_\mu = \operatorname{lin}(1_{\overline{U_1}} + 1_{\overline{U_2}})$.

Proof. $\Leftarrow$: Assume we had $u \in \ker A_\mu$, $u \neq 0$, and w.l.o.g. that u has a positive maximum on the connected component $\overline{U_1}$. By Fact 1 we have that u is constant on $\overline{U_1}$. This leads to an immediate contradiction if for $z \in \partial U_1$ we have $\mu(z, \overline{\Omega}) < 1$, since then we must have $u|_{\overline{U_1}} = 0$.

So now assume that we have $z \in \partial U_2$ with $\mu(z, \overline{\Omega}) < 1$. Since we have $\mu(z_2, \overline{U_1}) > 0$ using Fact 2 yields that u also attains its maximum in $\overline{U_2}$; and then Fact 1 leads again to a contradiction as before.

$\Rightarrow$: Contraposition. Assume that for all $z \in \partial\Omega$ we have $\mu(z, \overline{\Omega}) = 1$. Then we have $1_{\overline{\Omega}} \in D(A_\mu)$ and $\mathscr{A} 1_{\overline{\Omega}} = 0$ yielding $\ker A_\mu \neq \{0\}$.

Arguing as in the proof of "$\Leftarrow$" in the previous proposition yields again that $\dim \ker A_\mu \leq 1$. Together with $1_{\overline{\Omega}} \in \ker A_\mu$ this implies the last claim of the proposition. □

It is clear that a listing of all the different cases as done here is infeasible for more connected components. A way to unify the different cases is returning to the idea to consider a directed graph on the connected components as in the pictures above resp. its adjacency matrix. Each point $(z_1, z_2) \in \partial U_1 \times \partial U_2$ induces a weighted directed graph $G(z_1, z_2)$ on the set of vertices $\{U_1, U_2\}$ via the relation

$$U_i \overset{\mu(z_i,\overline{U_j})}{\longrightarrow} U_j,\ i,j \in \{1,2\},\ \text{if and only if } \mu(z_i,\overline{U_j}) > 0$$

or equivalently via the adjacency matrix

$$M(z_1,z_2) = \begin{pmatrix} \mu(z_1,\overline{U_1}) & \mu(z_1,\overline{U_2}) \\ \mu(z_2,\overline{U_1}) & \mu(z_2,\overline{U_2}) \end{pmatrix}.$$

As one can directly see by the propositions above we can summarize the results on exponential stability by the following theorem.

Theorem. *If there are points $z_1 \in \partial U_1$ and $z_2 \in \partial U_2$ such that the spectral radius $r(M(z_1,z_2)) < 1$ then* $\ker A_\mu = \{0\}$.

Indeed this is immediate for the first case of two independent connected components. But also for the other two cases this remains true. In fact for non-negative matrices it holds that the spectral radius is at least the minimum of the sums of its lines. Summing up the lines of $M(z_1,z_2)$ we see that we must have $\mu(z_1,\overline{U_1}) + \mu(z_1,\overline{U_2}) < 1$ or $\mu(z_2,\overline{U_1}) + \mu(z_2,\overline{U_2}) < 1$ for some $(z_1,z_2) \in \partial U_1 \times \partial U_2$. Now the propositions proved above imply that in these cases we indeed have $\ker A_\mu = \{0\}$.

Note that we always have $r(M(z_1,z_2)) \leq 1$ due to the fact that all $\mu(z)$ are sub-probability measures.

As one can check in the propositions above the dimension of the kernel of A_μ can be estimated by the number of invariant subspaces of $M(z_1,z_2)$.

Theorem. *For all $(z_1,z_2) \in \partial U_1 \times \partial U_2$ we have*

$$\dim \ker A_\mu \leq m(z_1,z_2)$$

where $m(z_1,z_2)$ is the number of invariant, non-zero subspaces of $M(z_1,z_2)$.

Actually a much stronger result holds true in the sense that the invariant subspace resp. the vertices of the corresponding graph resp. the connected components and their associated measures must fulfill an additional property to be counted in the number $m(z_1,z_2)$ yielding an always finite upper bound.

3.2.2 Exponential stability

We now begin to extend the ideas explored in the example to the general case of Ω having an arbitrary number of connected components. The following observation reduces the question of exponential stability of the semigroup to the one of injectivity of the generator.

Lemma 3.11. *The semigroup T_μ generated by A_μ on $L^\infty(\Omega)$ is exponentially stable if and only if A_μ is injective.*

Proof. Let A_μ be injective. Then by the remark after Lemma 3.1 we have that $1 - S_0$ is injective. Exponential stability then follows from Theorem 2.48.

Now suppose A_μ is not injective, i.e. $0 \in \sigma(A_\mu)$. But then by Proposition 2.6 we have that T_μ is not exponentially stable. □

In the following stability results we assume that the coefficients b_i and d_0 fulfill

$$\sum_{i=1}^{d} D_i b_i + d_0 = 0 \text{ in } \mathcal{D}(\Omega)'$$

and formulate our results on the asymptotic behavior under that assumption. This makes the ideas we use (associated graphs and adjacency operators) clearer. Note however that the auxiliary lemmas and propositions are formulated without this restriction. In the last subsection we give more results on the asymptotics for general b_i, d_0.

Now we establish criteria whether A_μ is injective. The following lemma in combination with Lemma 3.3 is enough to establish a simple case in which we have injectivity.

Lemma 3.12. *Let $u \in \ker A_\mu$ be real-valued with positive maximum c. Then $u \equiv c$ on some connected component of Ω.*

Proof. Let $0 \neq u \in \ker A_\mu$ be real-valued and set $c := \max_{x\in\overline{\Omega}} u(x) > 0$. By assumption and Lemma 3.3, we find $x_0 \in \Omega$ with $u(x_0) = c$. But then by the maximum principle u must be constant in the connected component containing x_0. □

This is already enough for our first result on exponential stability.

Corollary 3.13. *Let $\sum_{i=1}^{d} D_i b_i + d_0 = 0$. Assume that for each connected component U of Ω we find a point $z_0 \in \partial U$ with $\mu(z_0, \overline{\Omega}) < 1$. Then we have* $\ker A_\mu = \{0\}$.

Proof. Assume we had $0 \neq u \in \ker A_\mu$; as above we may further assume $c := \max_{x\in\overline{\Omega}} u(x) > 0$. By Lemma 3.3 we find $x_0 \in \Omega$ with $u(x_0) = c$. By Lemma 3.12 we have $u|_{\overline{U}} = c1_{\overline{U}}$ on some conntected component U of Ω.

By our assumption we find a point $z_0 \in \partial U$ with $\mu(z_0, \overline{\Omega}) < 1$. But then we have

$$c = u(z_0) = \langle u, \mu(z_0) \rangle \leq c\mu(z_0, \overline{\Omega}) < c,$$

a contradiction. Thus, we must have that $u = 0$ and $A_\mu$is injective. □

As a next step we generalize this result. To make this generalization easier to state we first define "adjacency" operators using the connected components of Ω and the measures $\mu(z)$, $z \in \partial\Omega$.

Denote by U_i, $i \in J$, the connected components of Ω, where either $J = \mathbb{N}$ or $J = \{1, \ldots, r\}$ for some $r \in \mathbb{N}$. Define for each $z = (z_i)_{i \in J} \in \prod_{i \in J} \partial U_i$ the (potentially infinite) matrix

$$M(z) := (\mu(z_i, \overline{U_j}))_{i,j \in J}.$$

Since all $\mu(z_i)$ are sub-probability measures, M is an operator on $\ell^\infty(J)$ with spectral radius $r(M(z)) \leq 1$. Now the following generalization of Corollary 3.13 holds true.

Theorem 3.14. *Let $\sum D_i b_i + d_0 = 0$. Suppose there is a $z \in \prod_{i \in J} \partial U_i$ with $r(M(z)) < 1$. Then* $\ker A_\mu = \{0\}$.

Before we start to prove this theorem in a series of lemmas, we first note that this is a generalization of Corollary 3.13 even if J is infinite. One may try to construct an example where the corollary gives injectivity but the theorem doesn't, but then one must violate the condition (meas) imposed on the measures $\mu(z)$ as the following example shows. See the remark after Theorem 3.20 for a general discussion of this topic.

Example. Let $\Omega := \bigcup_{k=1}^{\infty} (2^{-k} - 2^{-k-2}, 2^{-k} + 2^{-k-2}) \subset \mathbb{R}$, $\mu(2^{-k} - 2^{-k-2}, B) := \mu(2^{-k} + 2^{-k-2}, B) := \frac{k}{k+1} \delta_{2^{-k}}(B)$, $B \subset \mathbb{R}$. Then Corollary 3.13 shows that every $u \in C(\overline{\Omega})$ with $A_\mu u = 0$ on Ω has $u = 0$ on Ω, but $r(M(z)) = 1$ for all $z \in \prod_{k=1}^{\infty} \{2^{-k} - 2^{-k-2}, 2^{-k} + 2^{-k-2}\}$ since 1 is an approximate eigenvalue of the diagonal matrix $M(z) = \operatorname{diag}(\frac{k}{k+1}, k \in \mathbb{N})$. But in this case either $\mu \colon \partial\Omega \to \mathcal{M}(\overline{\Omega})$ is not weak*-continuous at 0 or $\mu(0) = \delta_0$ and hence $\emptyset \neq \operatorname{supp} \mu(0) \subset \partial\Omega$.

For the proof of the theorem above and also for further analysis of the kernel of A_μ in subsequent subsections it is convenient to define graphs on the connected components of Ω.

For each $z = (z_j)_{j \in J} \in \prod_{j \in J} \partial U_j$ we now interpret $M(z)$ as the adjacency matrix of a weighted, directed graph $G(z)$. More precisely it is given by $G(z) = (\mathcal{V}, E(z))$ with $\mathcal{V} = \{U_j, j \in J\}$ as the set of vertices and $E(z) = \{(U_i, U_j) \in \mathcal{V} \times \mathcal{V}, \mu(z_i, U_j) > 0\}$ as the set of (directed) edges with weights $w(z) \colon E(z) \to \mathbb{R}$, $(U_i, U_j) \mapsto \mu(z_i, U_j)$.

Recall some notions from graph theory: Let $H = (W, F)$ be a directed graph. A *path* of length n from $V_1 \in W$ to $V_n \in W$ is a tuple of vertices $(V_1, \dots, V_n) \in W^n$ with and $(V_i, V_{i+1}) \in F$ for all $i = 1, \dots, n-1$. The graph H is called *strongly connected*, if for every $S, D \in W$, $S \neq D$ there exists a path from S to D. A *strongly connected component* of a H is a subgraph of H that is strongly connected and maximal with this property. A subgraph $H' = (W', F')$ of H is called a *sink*, if there are no outgoing edges from H' into the rest of H; that is for all vertices V' in H' and edges $(V', V) \in F$ for some vertex $V \in W$, we have $V \in W'$. A *sink strongly connected component* is a subgraph of H that is both a sink and a strongly connected component.

Coming back to our particular situation we define one additional notion.

Definition 3.15. We call a subset $\mathcal{V}'$ of the connected components $\mathcal{V}$ a *mass compound*, if

- $\mathcal{V}'$ forms a sink in $G(z)$ for *all* $z \in \prod_{j \in J} \partial U_j$,
- for all pairs of distinct $U, U' \in \mathcal{V}'$ there is *some* $z \in \prod_{j \in J} \partial U_j$ such that there is a path from U to U' in $G(z)$ and
- $\mu(z, \overline{\Omega}) = 1$ for *all* $z \in \bigcup_{V \in \mathcal{V}'} \partial V$.

Equivalently one may say that $\mathcal{V}' \subset \mathcal{V}$ is a mass compound if

- $\mathcal{V}'$ forms a sink strongly connected component in the graph $(\mathcal{V}, \bigcup_{z \in \prod_{i \in J} \partial U_i} E(z))$ and
- $\mu(z, \overline{\Omega}) = 1$ for all $z \in \bigcup_{V \in \mathcal{V}'} \partial V$.

From this description it is easy to see that for two mass compounds $\mathcal{W}, \mathcal{W}'$ we either have $\mathcal{W} = \mathcal{W}'$ or $\mathcal{W} \cap \mathcal{W}' = \emptyset$.

Now we are ready to formulate suitable generalizations of Lemma 3.3 and Lemma 3.12. But first a simple but handy result.

Lemma 3.16. *Suppose that $u \in \ker A_\mu$ attains its positive maximum on $\overline{U}$ with $U \in \mathcal{V}$ and there is $U \neq W \in \mathcal{V}$ and $z_0 \in \partial U$ with $\mu(z_0, \overline{W}) > 0$. Then u attains its maximum also on W.*

Proof. Suppose $u \neq 0$; then we have w.l.o.g. that $\gamma := \max_{\overline{\Omega}} u = \max_{\overline{U}} u > 0$. Since u must be constant on $\overline{U}$ by Lemma 3.12 we then have

$$\begin{aligned}\gamma = u(z_0) &= \sum_{V \in \mathcal{V}} \langle \mu(z_0), u|_{\overline{V}} \rangle \\ &\leq \gamma \mu(z_0, \overline{\Omega} \setminus \overline{W}) + \max_{\overline{W}} u \cdot \mu(z_0, \overline{W}).\end{aligned}$$

If we had $\max_{\overline{W}} u < \gamma$, the above calculation combined with $\mu(z, \overline{W}) > 0$ would lead us to the contradiction $\gamma < \gamma$. An argument as in the proof of Lemma 3.3 gives us that the maximum is in fact attained in W. □

Remark. Stated in the language of graphs the preceding lemma reads as follows: Suppose that $u \in D(A_\mu)$ attains its maximum on $\overline{U}$ with $U \in \mathcal{V}$. If there is some $z \in \prod_{i \in J} \partial U_j$ such that there is an edge from U to V in $G(z)$, then u attains its maximum also on V.

Now for the generalization of Lemma 3.3.

Lemma 3.17. *Let $u \in \ker A_\mu$. Then for each mass compound there is a constant C such that $u|_{\overline{V}} = C$ on all connected components that are part of that mass compound.*

Proof. Let $\mathcal{V}'$ be a mass compound. Set $\Omega' := \bigcup_{U \in \mathcal{V}'} U$ and $\mu' \colon \partial\Omega' \to \mathcal{M}(\overline{\Omega'})^+, z \mapsto \mu(z)$. Since Ω' is a union of connected components it follows that μ' is weak*-continuous. Setting $v := u|_{\overline{\Omega'}}$ we have that $v \in \ker A_{\mu'}$, where $A_{\mu'}$ is the operator on $L^\infty(\Omega')$ associated to μ' defined similarily as A_μ is on $L^\infty(\Omega)$. If $v = 0$, then we are done.

Now assume that v is real-valued and $0 < \max_{\overline{\Omega'}} v =: \gamma$. By Lemma 3.12 there is a connected component U of Ω' on which v is constant. Now let $V \in \mathcal{V}'$ with $\mu(z, \overline{V}) > 0$ for some $z \in \partial U$. Then Lemma 3.16 above gives us $\max_V v = \gamma$ and the maximum principle yields $v|_{\overline{V}} = \gamma$.

Applying this argument inductively we have that $v|_{\overline{V}} = \gamma$ for all $V \in \mathcal{V}'$ for which there is a path from U in $G(z)$ for some $z \in \prod_{j \in J} \partial U_j$. But since $\mathcal{V}'$ is a mass compound this is the case for all $V \in \mathcal{V}'$.

The case for general v follows by applying the argument above to the real and imaginary parts of v or their negative separately. □

We follow up with the generalization of Lemma 3.12.

Lemma 3.18. *Let $u \in \ker A_\mu$. Then u attains its positive maximum in a connected component that is part of a mass compound.*

Proof. W.l.o.g. let $0 \neq u \in D(A_\mu)$ with $\mathscr{A} u = 0$ and $0 < \max_{\overline{\Omega}} u(x) =: \gamma$. By Lemma 3.12 there is a connected component U with $u|_{\overline{U}} = \gamma$. Now suppose U is not part of a mass compound. Then either $\mu(z_U, \overline{\Omega}) < 1$ for some $z_U \in \partial U$ or there is a connected component $V \subset \Omega$ with $\mu(z_V, \overline{\Omega}) < 1$ for some $z_V \in \partial V$ and a path in $G(z)$ from U to V for some $z \in \prod_{j \in J} \partial U_j$. The first case yields a contradiction since $\mu(z_U, \overline{\Omega}) < 1$ implies $\gamma = 0$ as in the proof of Corollary 3.13. .

Now consider the second case. Given $z \in \prod_{j\in J} \partial U_j$ set

$$\mathscr{P}(z) := \{P : \exists V, V' \in \mathcal{V} : P \text{ is a path in } G(z) \text{ from } V \text{ to } V', \\ \text{where } u|_{\overline{V}} = \gamma \text{ and } \mu(z', \overline{\Omega}) < 1 \text{ for some } z' \in \partial V'\}.$$

By assumption there is $z_0 =: (z_j)_{j\in J} \in \prod_{j\in J} \partial U_j$ such that $\mathscr{P}(z_0) \neq \emptyset$. Now let $P =: (U_{i_1}, U_{i_2}, \dots, U_{i_{k-1}}, U_{i_k})$ be a shortest path out of those in $\mathscr{P}(z_0)$. By the definition of the graph $G(z_0)$ we have $\mu(z_{i_1}, \overline{U_{i_2}}) > 0$. But by Lemma 3.16 u also attains γ in $\overline{U_{i_2}}$ which by Lemma 3.16 implies that $u|_{\overline{U_{i_2}}} = \gamma$. Hence the path $P' = (U_{i_2} \dots, U_{i_{k-1}}, U_{i_k})$ is connecting a connected component $V \in \mathcal{V}$ with $u|_{\overline{V}} = \gamma$ to a connected component $V' \in \mathcal{V}$ with $\mu(z_{U'}, \overline{\Omega}) < 1$ for some $z_{U'} \in \partial U'$, i.e. $P' \in \mathscr{P}(z_0)$ — a contradiction to P being a shortest such path. Hence we have shown our claim in both cases. □

Now Theorem 3.14 can be proved as follows:

Proof of Theorem 3.14. Contraposition. Let $0 \neq u \in \ker A_\mu$; by splitting u in its real and imaginary part and by considering $-u$ if necessary, we can assume that u is real-valued with positive maximum. Hence by Lemma 3.18 there exists a mass compound $\mathcal{U} = \{U_j,\ j \in J'\} \subset \mathcal{V}$ for some $J' \subset J$. Since $\mathcal{U}$ is a sink in $G(z)$ for all $z \in \prod_{j\in J} \partial U_j$ the subspace $\mathbb{C}^{J'} \subset \mathbb{C}^J$ is invariant under $M(z)$ for all $z \in \prod_{j\in J} \partial U_j$. Furthermore for $\xi \in \mathbb{C}^J$ with $\xi|_{J'} = 1$ and $\xi|_{J\setminus J'}$ arbitrary we have $(M(z)\xi)|_{J'} = 1$ for all $z \in \prod_{j\in J} \partial U_j$ since $\mu(z, \overline{\Omega}) = 1$ for all $z \in \partial U_{j'}$ and all $j' \in J'$.

Altogether it follows that for any $z \in \prod_{j\in J} \partial U_j$ and any $\xi \in \mathbb{C}^J$ with $\xi|_{J'} = 1$ and $\xi|_{J\setminus J'}$ arbitrary, we have $(M(z)^n\xi)|_{J'} = 1$ for all $n \in \mathbb{N}$. In particular, we have that $(M(z)^n\xi) \not\to 0$ for $n \to \infty$ and hence that $r(M(z)) \not< 1$. □

3.2.3 Asymptotics in the non-stability case and an estimate of the dimension of the kernel

From the auxiliary results in the previous section we can immediately establish convergence to a finite rank projection of the semigroup $T_\mu(t)$ for $t \to \infty$. In this subsection we are again in the general case $\sum_i D_i b_i + d_0 \leq 0$.

Theorem 3.19. *There exists a positive projection P of finite rank and constants $\epsilon > 0$, $M \geq 1$ such that*

$$\|T_\mu(t) - P\|_{\mathcal{L}(L^\infty(\Omega))} \leq M e^{-\epsilon t}$$

for all $t > 0$. If $P = 0$, then ϵ can be chosen as $0 < \epsilon < -s(A_\mu)$ and if $P \neq 0$, then ϵ can be chosen as $0 < \epsilon < -\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A_\mu) \setminus \{0\}\}$.

Proof. If A_μ is injective, we have seen in Proposition 3.11 above that T_μ is exponentially stable, i.e. the claim holds for $P = 0$.

Now let us assume that $0 \in \sigma(A_\mu)$. Since A_μ has compact resolvent by Corollary 3.7 and since the semigroup T_μ is bounded by Theorem 3.4, it follows that $R(\lambda, A_\mu)$ has a pole of order 1 at 0. Since T_μ is holomorphic it follows that $\Sigma_\theta + \omega \subset \rho(A_\mu)$ for some $\theta \in (\frac{\pi}{2}, \pi)$ and some $\omega \geq 0$, hence $\sigma(A_\mu) \cap i\mathbb{R}$ is finite. On the other hand, since T_μ is positive [AGG⁺86, Remark C-III.2.15] shows that the boundary spectrum $\sigma(A_\mu) \cap i\mathbb{R}$ is *cyclic*, i.e. if $is \in \sigma(A_\mu)$ for some $s \in \mathbb{R}$, then also $iks \in \sigma(A_\mu)$ for all $k \in \mathbb{Z}$. Consequently, this implies $\sigma(A_\mu) \cap i\mathbb{R} = \{0\}$. Together with the compactness of the resolvent this implies that $\{0\}$ is a dominating eigenvalue, i.e. there is $\epsilon > 0$ such that $\sigma(A_\mu) \setminus \{0\} \subset \mathbb{C}^- - \epsilon$.

Denote by P the residuum of $R(\lambda, A_\mu)$ at 0; this is the same as the spectral projection to the finite-dimensional eigenspace to the eigenvalue $\{0\}$. It follows from the representation of T_μ as contour integral that

$$\|T_\mu(t)(1-P)\|_{\mathcal{L}(L^\infty(\Omega))} \leq Me^{-\epsilon t}$$

for all $t > 0$ and some constants $M \geq 1$, $\epsilon > 0$ (see [ABHN11, Theorem 2.6.2 and the proof of Theorem 2.6.1]). Since on the other hand $T_\mu(t)P = P$ for all $t > 0$, we find

$$\begin{aligned}\|T(t) - P\|_{\mathcal{L}(L^\infty(\Omega))} &= \|T(t)(1-P) + T(t)P - P\|_{\mathcal{L}(L^\infty(\Omega))} \\ &= \|T(t)(1-P)\|_{\mathcal{L}(L^\infty(\Omega))} \leq Me^{-\epsilon t}\end{aligned}$$

all $t > 0$. ☐

With the tools we developed in hand we can also estimate the rank of this projection resp. the dimension of the kernel rather easily.

Theorem 3.20. *Let k be the number of distinct mass compounds in $\mathcal{V}$. Then it holds that*

$$\dim \ker A_\mu \leq k.$$

Remark. Before proving this theorem, let us remark two things.

Firstly if $\sum D_i b_i + d_0 = 0$ this estimate is sharp. The number k may be reduced in the general case and we show how to do so in the next subsection, see Theorem 3.24; this theorem also gives the precise dimension of the kernel.

Secondly this statement is never vacuous, i.e. the number k is never infinity. To see this let us assume that we had infinitely many pairwise disjoint mass compounds $\mathcal{U}_k$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ pick a boundary point z_k from a connected component in $\mathcal{U}_k$. Since Ω is bounded, there is a point $z_0 \in \operatorname{cl}(\bigcup_{k\in\mathbb{N}} \bigcup_{U \in \mathcal{U}_k} \partial U)$ and a subsequence z_{k_i}, $i \in \mathbb{N}$, converging to z_0.

Since $1_{\overline{U_j}}$ is continuous for all $j \in J$, it follows from the weak*-continuity of $z \mapsto \mu(z)$, that

$$\mu(z_{k_i}, \bigcup_{U \in \mathcal{U}_k} \overline{U}) \to \mu(z_0, \bigcup_{U \in \mathcal{U}_k} \overline{U})$$

as $i \to \infty$ for all $k \in \mathbb{N}$. Since $\mu(z_{k_i}, \bigcup_{U \in \mathcal{U}_k} \overline{U}) = 0$ for i large enough (i.e. when $k_i > k$), we have $\mu(z_0, \bigcup_{U \in \mathcal{U}_k} \overline{U}) = 0$ for all $k \in \mathbb{N}$. In particular if $z_0 \in \bigcup_{k \in \mathbb{N}} \bigcup_{U \in \mathcal{U}_k} \partial U$ we have $\mu(z_0, \overline{\Omega}) = \sum_{k \in \mathbb{N}} \mu(z_0, \bigcup_{U \in \mathcal{U}_k} \overline{U}) = 0$ — this is a contradiction to $\mathcal{U}_k$, $k \in \mathbb{N}$, being mass compounds. If $z_0 \notin \bigcup_{k \in \mathbb{N}} \bigcup_{U \in \mathcal{U}_k} \partial U$ it follows that $\mu(z_0, \partial\Omega) > 0$ — that is a contradiction to $\mu(z, \partial\Omega) = 0$ for all $z \in \partial\Omega$.

Proof of Theorem 3.20. Suppose there exist $k+1$ linear independent functions $u_1, \dots, u_{k+1} \in \ker A_\mu$; w.l.o.g. they all have a positive maximum. Since there are k mass compounds $\mathcal{U}_1, \dots, \mathcal{U}_k$ and by Lemma 3.17 the u_i are constant on the connected component in $\mathcal{U}_i$, $1 \leq i \leq k$ there are $\lambda_i \in \mathbb{R}$, $i = 1, \dots, k+1$, with at least one $\lambda_i \neq 0$ such that $v := \sum_{i=1}^k \lambda_i u_i \in \ker A_\mu$ has $v|_{\bigcup_{O \in \mathcal{U}} \overline{O}} = 0$ where $\mathcal{U} := \bigcup_{i=1}^k \mathcal{U}_i$. Now we can assume that either $v = 0$ or $\max v > 0$. The case $v = 0$ is an immediate contradiction to $u_1, \dots, u_{k+1}$ being linear independent. In the case $\max v > 0$, it follows from $v|_{\bigcup_{O \in \mathcal{U}} \overline{O}} = 0$ that v takes its positive maximum outside of a mass compound — which contradicts Lemma 3.18. □

We now give a short example illustrating that the given upper bound is sharp in general.

Example. Let $k \in \mathbb{N}$, $\Omega = \dot{\bigcup}_{1 \leq i \leq k} U_i$ and $\sum_{i=1}^d D_i b_i + d_0 = 0$. If for all $1 \leq i, j \leq k$ and all $z \in \partial U_i$ we have $\mu(z, \overline{U_j}) = \delta_{ij}$ then obviously $1_{\overline{U_i}} \in \ker A_\mu$ for all $1 \leq i \leq k$ and it follows that $\dim \ker A_\mu = k$. It is also clear that each U_i is itself a mass compound implying the sharpness of the bound in this case.

3.2.4 The case $\sum_{i=1}^d D_i b_i + d_0 \leq 0$: dimension of the kernel and characterization of exponential stability

We now come to the case where we impose no further constrictions on the coefficients b_i and d_0 other than that

$$\sum_{i=1}^d D_i b_i + d_0 \leq 0.$$

in the sense of distributions. When we say $\sum_{i=1}^d D_i b_i + d_0 \le 0$ or $\sum_{i=1}^d D_i b_i + d_0 = 0$ on some connected component V of Ω we mean that

$$-\sum_{i=1}^{d} \int_\Omega b_i D_i \varphi + \int_\Omega d_0 \varphi \le 0 \quad \forall 0 \le \varphi \in C_c^\infty(\Omega),\ \operatorname{supp} \varphi \subset V \text{ resp.}$$
$$-\sum_{i=1}^{d} \int_\Omega b_i D_i \varphi + \int_\Omega d_0 \varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega),\ \operatorname{supp} \varphi \subset V$$

and write $\sum_{i=1}^d D_i b_i + d_0 \le 0$ in $\mathcal{D}(V)'$ resp. $\sum_{i=1}^d D_i b_i + d_0 = 0$ in $\mathcal{D}(V)'$.

Here the connection between stability and the graphs on the connected components resp. their adjacency operator is somewhat muddier. In fact the exponential stability will be characterized in terms of existence of paths to connected components with certain properties; see the Proposition 3.22 and Proposition 3.23 for sufficiency and necessity respectively. But whenever $\sum_{i=1}^d D_i b_i + d_0 \ne 0$ there is no convenient analogue of Theorem 3.14.

The general result on asymptotics in the non-stability case was already formulated in Theorem 3.19 but the result on the dimension of the kernel will be improved in Theorem 3.24: we also give a lower estimate and prove that this estimate is always sharp.

We start with the generalization of Corollary 3.13.

Corollary 3.21. *Assume that for each connected component U of Ω on which $\sum_{i=1}^d D_i b_i + d_0 = 0$ we find a point $z_0 \in \partial U$ with $\mu(z_0, \overline{\Omega}) < 1$. Then we have* $\ker A_\mu = \{0\}$.

Proof. Assume we had $0 \ne u \in \ker A_\mu$; as above we may further assume $1 = \max_{x \in \overline{\Omega}} u(x) > 0$. By Lemma 3.3 we find $x_0 \in \Omega$ with $u(x_0) = 1$. Let U be the connected component containing x_0. By Lemma 3.12 we have $u|_{\overline{U}} = 1_{\overline{U}}$.

Since $\mathscr{A} u = c(\sum_i D_i b_i + d_0) = 0$ as operator in $\mathcal{D}(U)'$, it follows that $\sum_i D_i b_i + d_0 = 0$ on U. By our assumption we find a point $z_0 \in \partial U$ with $\mu(z_0, \overline{\Omega}) < 1$. But then we have

$$c = u(z_0) = \langle u, \mu(z_0) \rangle \le c\mu(z_0, \overline{\Omega}) < c,$$

a contradiction. Thus, we must have that $u = 0$ and $A_\mu$is injective. □

Before turning our attention to the estimate of the dimension of the kernel we prove a generalization of the above result.

Proposition 3.22. *Assume that for each connected component U it holds that*

- $\mu(z, \overline{\Omega}) < 1$ *for some* $z \in \partial U$

or

- $\sum_{i=1}^{d} D_i b_i + d_0 \neq 0$ *on* $\mathcal{D}(U)'$

or

- *there is* $z \in \prod_{i \in J} \partial U_i$ *such that we find a path in* $G(z)$ *from* U *to a connected component* V *where* $\mu(z, \overline{\Omega}) < 1$ *for some* $z \in \partial V$ *or* $\sum_{i=1}^{d} D_i b_i + d_0 \neq 0$ *on* V.

Then we have $\ker A_\mu = \{0\}$.

Proof. Assume to the contrary that there is a $u \in \ker A_\mu$ with

$$\gamma := \max_{x \in \overline{\Omega}} u(x) > 0.$$

Then by Lemma 3.18 u attains γ at a point $x_0 \in U$ where U is a connected component that is part of a mass compound $\mathcal{V}$. But then by definition of a mass compound we have $\mu(z, \overline{\Omega}) = 1$ for all $z \in \partial U$ and there is no path in any graph $G(z)$ to a connected component V such that $\mu(z, \overline{\Omega}) < 1$ for any $z \in \partial V$.

It remains to show that $\sum_i D_i b_i + d_0 = 0$ on U and that there is no path in any $G(z)$ to a connected component V such that $\sum_i D_i b_i + d_0 = 0$ on V. By Lemma 3.17 we have that $u = \gamma$ on all connected components that are part of the same mass compound $\mathcal{V}$ as U. But then $\mathscr{A} u = \gamma(\sum_i D_i b_i + d_0) = 0$ on $\mathcal{D}(V)'$ for all $V \in \mathcal{V}$ since $u \in \ker A_\mu$. Hence $\sum_i D_i b_i + d_0 = 0$ on V for all $V \in \mathcal{V}$. By the definition of mass compound $\mathcal{V}$ is a sink in all graphs $G(z)$ and this implies the claim. □

One perhaps surprising fact is that the existence of such paths as above actually characterizes exponential stability as seen in the following proposition.

Proposition 3.23. *Assume that* $\ker A_\mu = \{0\}$ *holds. Then for each connected component* U *we have that*

- $\mu(z, \overline{\Omega}) < 1$ *for some* $z \in \partial U$

or

- $\sum_{i=1}^{d} D_i b_i + d_0 \neq 0$ *on* U

or

- *there is $z \in \prod_{i\in J} \partial U_i$ such that we find a path in $G(z)$ from U to a connected component V where $\mu(z,\overline{\Omega}) < 1$ for some $z \in \partial V$ or $\sum_{i=1}^d D_i b_i + d_0 \neq 0$ on $\overline{V}$.*

Proof. Contraposition; that is assume there is a connected component U such that

1. $\mu(z,\overline{\Omega}) = 1$ for all $z \in \partial U$ and
2. $\sum_{i=1}^d D_i b_i + d_0 = 0$ on U and
3. for all connected components V with a path from U in $G(z)$ for some $z \in \prod_{i\in J} \partial U_i$ we have $\mu(z,\overline{\Omega}) = 1$ for all $z \in \partial V$ and $\sum_{i=1}^d D_i b_i + d_0 = 0$ on V.

We now construct a non-zero function $u \in D(A_\mu)$ with $\mathscr{A}u = 0$.

Clearly either $\{U\} =: \mathcal{U}$ is itself a mass compound or there is a mass compound $\mathcal{U}$ and a path from U to V in some $G(z)$ where $V \in \mathcal{U}$. Denote by $\mathcal{U}'$ the set of connected components which are part of some mass compound other than $\mathcal{U}$. We now set

$$u|_{\overline{V}} := 1 \text{ for all } V \in \mathcal{U}$$
$$u|_{\overline{W}} := 0 \text{ for all } W \in \mathcal{U}'.$$

It remains to fix u on $\Omega' := \bigcup_{V\in\mathcal{V}\setminus(\mathcal{U}\cup\mathcal{U}')} V$. To that end define the elliptic operator $A_{\mu'}$ on $L^\infty(\Omega')$ in the same way as A_μ on $L^\infty(\Omega)$ where $\mu'\colon \partial\Omega' \to \mathcal{M}(\overline{\Omega'})$, $\mu'(z)(A) := \mu(z,A)$. For $\psi \in C(\partial\Omega')$ with $\psi(z) := \langle \mu(z), u\rangle$ for all $z \in \partial\Omega'$ consider the problem

$$\begin{cases} -\mathscr{A}v = 0 \text{ on } \Omega' \\ \quad v(z) = \langle \mu'(z), v\rangle + \psi(z) \text{ for all } z \in \partial\Omega'. \end{cases} \tag{3.4}$$

By Lemma 3.9 this is uniquely solvable if $A_{\mu'}$ is injective, that is if $\ker A_{\mu'} = \{0\}$. But this is clear since Ω' contains no mass compounds by construction: If $w \in \ker A_{\mu'}$ then by Lemma 3.18 w cannot have a positive maximum; considering $-w$ it follows by the same lemma that w cannot have a negative minimum. Altogether we must have $w = 0$.

Now denote by v the solution to the problem (3.4) and set $u = v$ on Ω'. By construction we have $\mathscr{A}u = 0$ (since $\sum_{i=1}^d D_i b_i + d_0 = 0$ in $\mathcal{D}(V)'$ with $V \in \mathcal{U}$) and also that

$$u(z) = \langle \mu(z), u\rangle$$

for all $z \in \partial\Omega$ (since v solves (3.4) and $\mathcal{U}$ is a sink in all $G(z)$). Altogether we have $u \in \ker A_\mu$. ☐

We now improve Theorem 3.20 about the dimension of the kernel in that we give a lower upper bound and prove that it is in fact also the lower bound.

Theorem 3.24. *Let k be the number of distinct mass compounds $\mathcal{U}$ in $\mathcal{V}$ such that all connected components $V \in \mathcal{U}$ have $\sum_{i=1}^{d} D_i b_i + d_0 = 0$ on V. Then it holds that*

$$\dim\ker A_\mu = k.$$

Proof. To prove that $\dim\ker A_\mu \leq k$ in light of Theorem 3.20 it suffices to show that on a mass compound $\mathcal{U}$ with $\sum_{i=1}^{d} D_i b_i + d_0 \neq 0$ on V for some $V \in \mathcal{U}$ we have that $u|_{\overline{W}} = 0$ for all $W \in \mathcal{U}$ and all $u \in \ker A_\mu$. But by Lemma 3.17 any $u \in \ker A_\mu$ must have $u|_{\overline{W}} = \gamma$ for $W \in \mathcal{U}$ for some constant γ. Clearly if $\sum_{i=1}^{d} D_i b_i + d_0 \neq 0$ on V, then $\mathscr{A} u|_{\overline{V}} = 0$ if and only if $\gamma = 0$.

To prove that k is a lower bound on $\dim\ker A_\mu$ we construct k linearly independent functions $u_1, \ldots, u_k$ in a fashion similar to the proof of Proposition 3.23. Call $\mathcal{U}_1, \ldots, \mathcal{U}_k$ the k distinct mass compounds from the hypothesis and $\mathcal{U}$ the set of all connected components that are part of a mass compound other than $\mathcal{U}_1, \ldots, \mathcal{U}_k$. Then for $1 \leq i, j \leq k$ set

$$u_i|_{\overline{V}} := \delta_{ij} \text{ for } V \in \mathcal{U}_j.$$
$$u_i|_{\overline{W}} := 0 \text{ for } W \in \mathcal{U}.$$

We now extend these functions u_i to all of $\overline{\Omega}$ such that $u \in \ker A_\mu$. To that end define

$$\Omega' := \bigcup_{V \in \mathcal{V} \setminus (\mathcal{U}_1 \cup \ldots \cup \mathcal{U}_k \cup \mathcal{U})} V$$

and the operator $A_{\mu'}$ on Ω' in the same fashion as A_μ on Ω where $\mu' \colon \partial\Omega' \to \mathcal{M}(\overline{\Omega'})$, $\mu'(z)(A) := \mu(z, A)$. Set $\psi_i \in C(\partial\Omega')$ by $\psi_i(z) := \langle \mu(z), u_i \rangle$ for $z \in \partial\Omega'$ and $1 \leq i \leq k$ and consider the equations

$$\begin{aligned} -\mathscr{A} v_i &= 0 \text{ on } \Omega' \\ v_i(z) &= \langle \mu'(z), v_i \rangle + \psi_i(z) \text{ for all } z \in \partial\Omega'. \end{aligned} \tag{3.5}$$

for $1 \leq i \leq k$. By Lemma 3.9 these equations are uniquely solvable if $A_{\mu'}$ is injective. But since Ω' does not contain any mass compounds Lemma 3.18 implies that the only function in $\ker A_{\mu'}$ is 0.

Hence the solution v_i to equation (3.5) exists for $1 \leq i \leq k$; now set $u_i := v_i$ on $\overline{\Omega'}$. By construction and hypothesis these functions suffice $\mathscr{A} u_i = 0$ and also $u(z) = \langle \mu(z), u \rangle$ for $z \in \partial\Omega$; i.e. $u_i \in \ker A_\mu$. Clearly $u_1, \ldots, u_k$ are linearly independent and hence the theorem is proven. □

3.3 Assorted topics

Here we collect some miscellaneous results on topics surrounding the equation (3.1): We consider the space $C_\mu(\overline{\Omega})$ on which A_μ generates a C_0-semigroup and show that in some sense this space is not so convenient to work in directly. Furthermore we characterize the irreducibility of the semigroup T_μ in terms of the measures $\mu(z)$; in other words we analyze the zeroes of $R(\lambda, A_\mu)f$ where $f \geq 0$ and $f \neq 0$. Finally we show that the situation is much easier when one works with "small" measures $\mu(z)$; that is measures with $|\mu(z)|(\overline{\Omega}) < 1$ where $|\cdot|$ is the total variation. In fact under this additional condition one can allow a much wider variety of measures: they may be $\mathbb{C}$-valued and may not obey any condition on their support. We also show that the generated semigroup T_μ is still exponentially stable.

3.3.1 Analysis of the space $C_\mu(\overline{\Omega})$

We now return to our original assumption on μ and give a more detailed analysis of the space $C_\mu(\overline{\Omega})$, which is the closure of $D(A_\mu)$ and the space where the generated semigroup is C_0. In particular we show this space is a sublattice and subalgebra of $C(\overline{\Omega})$ only under very specific assumptions on μ giving further reason to our decision to consider the whole space $L^\infty(\Omega)$ resp. $C(\overline{\Omega})$ and therefore forgo the C_0-property of the generated semigroup.

Remember that the closure of $D(A_\mu)$ is given by

$$C_\mu(\overline{\Omega}) = \{u \in C(\overline{\Omega}),\ u(z) = \langle \mu(z), u\rangle\ \forall z \in \partial\Omega\}$$

as seen in Proposition 3.8.

The proof that $C_\mu(\overline{\Omega})$ fails to be a sublattice resp. subalgebra in most cases is based on the following extension result.

Proposition 3.25. *Let $K \subset \Omega$ be compact and $f \in C(K)$. Then there is a continuous extension $F \in C_\mu(\overline{\Omega})$ of f to the whole domain $\overline{\Omega}$, that is $F \in C(\overline{\Omega})$ with $F|_K = f$ and $F(z) = \langle \mu(z), F\rangle$ for all $z \in \partial\Omega$, and such that $\|F\|_{C(\overline{\Omega})} \leq \|f\|_{C(K)}$.*

The fact that $\mu(z, \partial\Omega) = 0$ plays a crucial role in the following approximation argument.

Proof. Let $K_n \subset \Omega$, $K_1 := K$, be compact subsets that exhaust Ω, that is $K_n \subset K_{n+1}$ and $\Omega = \bigcup_{n\in\mathbb{N}} K_n$. Since the set of measures $\{\mu(z),\ z \in \partial\Omega\}$ is tight as seen in the proof of Lemma 3.2 and $\mu(z, \partial\Omega) = 0$, we have

$$\forall \epsilon > 0\, \exists N \in \mathbb{N} :\ \mu(z, \overline{\Omega} \setminus K_n) \leq \epsilon\ \forall z \in \partial\Omega,\ n \geq N;$$

hence by passing to a subsequence of the K_n (which we do without changing the notation) we can arrange that

$$\mu(z, \overline{\Omega} \setminus K_n) \leq \frac{1}{2^n} \; \forall z \in \partial\Omega, \; n \in \mathbb{N}. \tag{3.6}$$

Furthermore let $\psi_n \in C_c(\Omega)$ with $1_{K_n} \leq \psi_n \leq 1_{K_{n+1}}$.

Set $\varphi_1(z) := \langle \mu(z), f \rangle$, $g_1 := 0$ and choose $f_1 \in C(\overline{\Omega})$ such that $f_1|_K = f$, $f_1|_{\partial\Omega} = \varphi_1$ and $\|f_1\|_{C(\overline{\Omega})} \leq \|f\|_{C(\overline{\Omega})}$.

We inductively set: $\varphi_{n+1}(z) := \langle \mu(z), \psi_n f_n \rangle$; $g_{n+1} \in C(\overline{\Omega})$ such that $g_{n+1}|_{K_n} = 0$, $g_{n+1}|_{\partial\Omega} = \varphi_{n+1} - \varphi_n$ and $\|g_{n+1}\|_{C(\overline{\Omega})} \leq \|\varphi_{n+1} - \varphi_n\|_{C(\partial\Omega)}$; $f_{n+1} := f_n + g_{n+1}$.

We then have $f_{n+1}|_{K_n} = f_n$, $f_{n+1}|_{\partial\Omega} = \varphi_{n+1}$, $\|f_{n+1}\|_{C(\overline{\Omega})} \leq \|f\|_{C(\overline{\Omega})}$ (since the $\mu(z)$ are sub probability measures) and $\varphi_n \in C(\partial\Omega)$.

By (3.6) it then follows that

$$\|\varphi_n - \varphi_m\|_{C(\partial\Omega)} \leq \sup_{z \in \partial\Omega} \Big| \int_{K_m \setminus K_n} \psi_n f_n - \psi_m f_m \, \mu(z, \mathrm{d}x) \Big| \leq \frac{1}{2^n} 2 \|f\|_{C(\overline{\Omega})}$$

for $n \geq m \geq N$. Thus φ_n converges uniformly to a function $\varphi \in C(\partial\Omega)$.

It is also clear that f_n converges uniformly on compact subsets of Ω to a function F which is continuous on Ω and that $F|_K = f$ by construction. For all $z \in \partial\Omega$ the function F also fulfills the boundary condition

$$F(z) = \lim_{n \to \infty} \varphi_n(z) = \lim_{n \to \infty} \langle \mu(z), \psi_{n-1} f_{n-1} \rangle = \langle \mu(z), F \rangle$$

by dominated convergence and the choice of ψ_n.

It only remains to show: $F \in C(\overline{\Omega})$, that is in our case $F(x) \to F(z)$, as $x \to z$, $x \in \Omega$, for all $z \in \partial\Omega$.

We have

$$F(x) = f_1(x) + \sum_{n=1}^{\infty} g_n(x) = f_1(x) + \int_1^{\infty} g_n(x) \, \mathrm{d}\delta(n)$$

where δ is the counting measure on $\mathbb{N}$. We then have that $g_n(x)$ is δ-integrable w.r.t. n for all $x \in \overline{\Omega}$ and $|g_n(x)| \leq \|g_n\|_{C(\overline{\Omega})}$ for all $n \in \mathbb{N}$ and $x \in \overline{\Omega}$. That is we have a δ-integrable majorant independent of x, since

$$\int_1^{\infty} \|g_n\|_{C(\overline{\Omega})} \, \mathrm{d}\delta(n) \leq \sum_{n=1}^{\infty} \|\varphi_{n+1} - \varphi_n\| \leq 2 \sum_{n=1}^{\infty} \frac{1}{2^n} = 2.$$

It then follows by dominated convergence (see e.g. [Bog06, Corollary 2.8.6] for a suitable version) that

$$\lim_{x \to z} \int_1^{\infty} g_n(x) \, \mathrm{d}\delta(n) = \int_1^{\infty} g_n(z) \, \mathrm{d}\delta(n)$$

which implies $F(x) \to F(z)$, as $x \to z$.

The extension F constructed above clearly fulfills $\|F\|_{C(\overline{\Omega})} \leq \|f\|_{C(K)}$. □

With this tool in hand we turn our attention to the main results of this subsection.

Theorem 3.26. *The space $C_\mu(\overline{\Omega})$ is a sublattice of $C(\overline{\Omega})$ if and only if for all $z \in \partial\Omega$ the measure $\mu(z)$ is a scaled Dirac measure. Furthermore $C_\mu(\overline{\Omega})$ has a unit if and only if for all $z \in \partial\Omega$ the measure $\mu(z)$ is a Dirac measure.*

In the following we make use of the notion of the *support* of a measure. Remember that for a measure $\tau \in \mathcal{M}(\overline{\Omega})$ and a point $x \in \overline{\Omega}$ one says

$$\tau = 0 \text{ in } x,$$

if there is a neighborhood U of x in $\overline{\Omega}$ such that $\langle \tau, u \rangle = 0$ for all $u \in C(\overline{\Omega})$ with $\operatorname{supp} u \subset U$. Then the *support* of τ is defined as

$$\operatorname{supp} \tau := \operatorname{cl}(\overline{\Omega} \setminus \{x \in \overline{\Omega},\ \tau = 0 \text{ in } x\}).$$

Proof of the theorem. $\Leftarrow$: Let $\mu(z)$ be a scaled Dirac measure for all $z \in \partial\Omega$, that is $\langle \mu(z), v \rangle = \lambda_z v(x_z)$ with $\lambda_z \geq 0$ and $x_z \in \overline{\Omega}$ for all $v \in B_b(\overline{\Omega})$, $z \in \partial\Omega$. Hence it follows for $u \in C_\mu(\overline{\Omega})$ and $z \in \partial\Omega$

$$\begin{aligned} u^+(z) &= \max(u(z), 0) = \max(\lambda_z u(x_z), 0) = \lambda_z \max(u(x_z), 0) \\ &= \lambda_z u^+(x_z) = \langle \mu(z), u^+ \rangle; \end{aligned}$$

i.e. $u^+ \in C_\mu(\overline{\Omega})$ and thus $C_\mu(\overline{\Omega})$ is a sublattice.

$\Rightarrow$: Assume that $C_\mu(\overline{\Omega})$ is a sublattice of $C(\overline{\Omega})$. We have to show that $|\operatorname{supp} \mu(z)| \leq 1$ for all $z \in \partial\Omega$.

For each $u \in C_\mu(\overline{\Omega})$ we also have $u^+ \in C_\mu(\overline{\Omega})$ that is we have

$$\max\Big(\int_{\overline{\Omega}} u(x)\, \mu(z, \mathrm{d}x), 0\Big) = \int_{\overline{\Omega}} \max(u(x), 0)\, \mu(z, \mathrm{d}x) \tag{3.7}$$

for all $z \in \partial\Omega$.

Suppose now we had $z_0 \in \partial\Omega$ such that $y, y' \in \operatorname{supp} \mu(z_0)$, $y \neq y'$; since $\mu(z_0, \partial\Omega) = 0$, we may assume that $y, y' \in \Omega$. Now choose $\epsilon > 0$ such that $\overline{B_\epsilon(y)}, \overline{B_\epsilon(y')} \subset \Omega$ and such that $\overline{B_\epsilon(y)} \cap \overline{B_\epsilon(y')} = \emptyset$. Choose functions $0 \leq u_y \in C(\overline{\Omega})$ supported in $B_\epsilon(y)$ and $0 \geq u_{y'} \in C(\overline{\Omega})$ supported in $B_\epsilon(y)$ such that $\langle \mu(z_0), u_y \rangle > -\langle \mu(z_0), u_{y'} \rangle > 1$.

Since $\mu(z_0, \Omega) = 0$ there is a compact subset $K' \subset \Omega$ such that

$$\mu(z_0, \overline{\Omega} \setminus K') \leq \frac{1}{4} \max(\|u_y\|_{C(\overline{\Omega})}, \|u_{y'}\|_{C(\overline{\Omega})})^{-1}.$$

We now set $K := \overline{B_\epsilon(y)} \cup \overline{B_\epsilon(y')} \cup K'$ and apply Proposition 3.25 to $C(K) \ni u := u_y|_K + u_{y'}|_K$ yielding an extension $U \in C_\mu(\overline{\Omega})$. We then have

$$
\begin{aligned}
\max\Big(\int_{\overline{\Omega}} U(x)\,\mu(z_0,\mathrm{d}x),0\Big) &= \int_{\overline{\Omega}} U(x)\,\mu(z_0,\mathrm{d}x)\\
&\le \int_{B_\epsilon(y)} u_y(x)\,\mu(z_0,\mathrm{d}x) + \int_{B_\epsilon(y')} u_{y'}(x)\,\mu(z_0,\mathrm{d}x)\\
&\quad + \mu(z_0,\overline{\Omega}\setminus K')\max(\|u_y\|_{C(\overline{\Omega})},\|u_{y'}\|_{C(\overline{\Omega})})\\
&\le \langle \mu(z_0),u_y\rangle + \langle \mu(z_0),u_{y'}\rangle + \frac{1}{4} \text{ and}\\
\int_{\overline{\Omega}} \max(u(x),0)\,\mu(z_0,\mathrm{d}x) &\ge \int_{B_\epsilon(y)} u_y(x)\,\mu(z_0,\mathrm{d}x) - \mu(z_0,\overline{\Omega}\setminus K')\|u_y\|_{C(\overline{\Omega})}\\
&\ge \langle \mu(z_0),u_y\rangle - \frac{1}{4}.
\end{aligned}
$$

By (3.7) the left sides of the equations above must be equal and hence we have

$$\frac{1}{2} \ge -\langle \mu(z_0),u_{y'}\rangle$$

contradicting the choice of $u_{y'}$.

The last claim of the theorem follows, as the non-zero constant functions are in$C_\mu(\overline{\Omega})$ if and only if all measures $\mu(z)$ are probability measures. □

In a similar fashion we can prove the following statement about $C_\mu(\overline{\Omega})$ being a subalgebra of $C(\overline{\Omega})$.

Theorem 3.27. *$C_\mu(\overline{\Omega})$ is a Banach subalgebra of $C(\overline{\Omega})$ if and only if either $\mu(z)$ is a Dirac measure for all $z \in \partial\Omega$ or $\mu(z) = 0$ for all $z \in \partial\Omega$. In the first case $C_\mu(\overline{\Omega})$ has a unit.*

Proof. For $C_\mu(\overline{\Omega})$ to be a subalgebra of $C(\overline{\Omega})$ it is necessary and sufficient (in view of our continuity assumption on $z \mapsto \mu(z)$) that for all $z \in \partial\Omega$ and all $u, v \in C_\mu(\overline{\Omega})$

$$\int_{\overline{\Omega}} u(x)v(x)\,\mu(z,\mathrm{d}x) = \int_{\overline{\Omega}} u(x)\,\mu(z,\mathrm{d}x)\int_{\overline{\Omega}} v(x)\,\mu(z,\mathrm{d}x) \tag{3.8}$$

holds. Clearly this holds when $\mu(z)$ is a Dirac measure or 0.

On the other hand suppose now that $C_\mu(\overline{\Omega})$ is a subalgebra. We first show again that $\operatorname{supp}\mu(z)$ has at most one element. Suppose now we had $z_0 \in \partial\Omega$ such that there are $y, y' \in \operatorname{supp}\mu(z_0)$ with $y \ne y'$; since $\mu(z_0,\partial\Omega) = 0$ we may again assume that $y, y' \in \Omega$. Now choose $\epsilon > 0$ and functions $u_y, u_{y'} \in C(\overline{\Omega})$ such that $\overline{B_\epsilon(y)}, \overline{B_\epsilon(y')} \subset \Omega$, $\overline{B_\epsilon(y)}, \overline{B_\epsilon(y')}$ are disjoint, $\operatorname{supp} u_y \subset B_\epsilon(y)$, $\operatorname{supp} u_{y'} \subset B_\epsilon(y')$ and $\langle \mu(z_0),u_y\rangle, \langle \mu(z_0),u_{y'}\rangle > 1$.

Since $\mu(z,\partial\Omega) = 0$, there is a compact set $K' \subset \Omega$ such that

$$\mu(z_0,\overline{\Omega}\setminus K') < \frac{1}{4}\|u_y\|^{-1}_{C(\overline{\Omega})}\|u_{y'}\|^{-1}_{C(\overline{\Omega})}.$$

We set $K := \overline{B_\epsilon(y)} \cup \overline{B_\epsilon(y')} \cup K'$ and apply Proposition 3.25 to $u_y|_K$ and to $u_{y'}|_K$ extending them to the $C_\mu(\overline{\Omega})$-functions U_y and $U_{y'}$ respectively. But since $C_\mu(\overline{\Omega})$ is a subalgebra this means that we have

$$\begin{aligned}(U_y \cdot U_{y'})(z_0) &= \int_{\overline{\Omega}} U_y U_{y'}\, \mu(z_0, \mathrm{d}x) \\ &\le \mu(z_0, \overline{\Omega} \setminus K') \|u_y\|_{C(\overline{\Omega})} \|u_{y'}\|_{C(\overline{\Omega})} \le \frac{1}{4} \text{ and} \\ U_y(z_0) U_{y'}(z_0) &= \int_{\overline{\Omega}} U_y(x)\, \mu(z_0, \mathrm{d}x) \int_{\overline{\Omega}} U_{y'}(x)\, \mu(z_0, \mathrm{d}x) \\ &\ge \left(\langle \mu(z_0), u_y \rangle - \mu(z_0, \overline{\Omega} \setminus K') \|u_y\|_{C(\overline{\Omega})}\right) \\ &\quad \cdot \left(\langle \mu(z_0), u_{y'} \rangle - \mu(z_0, \overline{\Omega} \setminus K') \|u_{y'}\|_{C(\overline{\Omega})}\right) > \frac{1}{2};\end{aligned}$$

here we also used that $\mu(z_0)$ is a subprobability measure. But by (3.8) the left sides of both equations must be equal — a contradiction.

Hence we have $|\operatorname{supp} \mu(z)| \le 1$ for all $z \in \partial\Omega$, that is for $z \in \partial\Omega$ we have $\mu(z) = \lambda \delta_y$ for some $\lambda \in \mathbb{R}$ and $y \in \Omega$. It then follows that for all $u \in C_\mu(\overline{\Omega})$

$$\begin{aligned} u^2(z) &= \langle \mu(z), u^2 \rangle = \lambda u^2(y) \text{ and} \\ u^2(z) &= \langle \mu(z), u \rangle \langle \mu(z), u \rangle = \lambda^2 u^2(y). \end{aligned}$$

Altogether it follows by choosing u to be positive at y that

$$\lambda^2 - \lambda = 0,$$

i.e. $\lambda = 1$ or $\lambda = 0$. Since $z \mapsto \mu(z)$ is weak*-continuous, this means if $\lambda = 1$ for one $z \in \partial\Omega$, then all measures $\mu(z)$ are Dirac measures and if $\lambda = 0$ for one $z \in \partial\Omega$, then all measures $\mu(z)$ are 0.

The last claim of the theorem follows as the last claim in the preceding proof. □

3.3.2 Irreducibility

In this section we want to investigate the irreducibility of the semigroup generated by the part of A_μ in $C(\overline{\Omega})$. In particular we want to apply Proposition 2.52; note however that the semigroup generated by A_0 is not irreducible on $C(\overline{\Omega})$ since clearly $C_0(\Omega)$ is an invariant closed ideal of the generated semigroup and hence we can only use part iv) of the proposition.

Remember that a positive semigroup T on a Banach lattice generated by A is called *irreducible*, if the only closed ideals that are invariant under all $T(t)$, $t > 0$, are trivial; or equivalently if the same holds for the resolvent $R(\lambda, A)$ for large $\lambda > 0$. For further properties of irreducible semigroups on spaces of continuous functions we refer to [AGG$^+$86, Section B-III].

Since the closed ideals of $C(\overline{\Omega})$ are all of the form $\{u \in C(\overline{\Omega}),\ u|_S = 0\}$ for some closed $S \subset \overline{\Omega}$, proving irreducibility is equivalent to showing that $(R(\lambda, A_\mu)f)(x) > 0$ for all $0 \le f \in C(\overline{\Omega})$, $f \ne 0$, and all $x \in \overline{\Omega}$. In order to do that we rely on the following lemma that essentially says we only have to worry about zeroes on the boundary.

Lemma 3.28. *Assume that Ω is connected. Let $\lambda > 0$, $0 \le f \in L^\infty(\Omega)$, $f \ne 0$ and $0 \le u \in D(A_{max})$ such that $\lambda u - \mathscr{A} u = f$. We then have $u(x) > 0$ for all $x \in \Omega$.*

Proof. The proof is an application of the weak Harnack inequality as seen in [GT98, Theorem 8.18] (if $\mathscr{A}$ is in divergence form) resp. [GT98, Theorem 9.22] (if $\mathscr{A}$ is in non-divergence form).

If $\mathscr{A}$ is in divergence form u is a supersolution of 0 for the operator $\mathscr{A} - \lambda$ (as defined in [GT98, (8.30)]) and hence we have

$$\|u\|_{L^p(B)} \le C \inf_{B'} u \tag{3.9}$$

where $B = B_{2R}(y)$ and $B' = B_R(y)$, $y \in \Omega$, $R > 0$, are balls such that $2B \subset \Omega$, $1 \le p < \frac{d}{d-2}$ and $C > 0$ is a constant that does not depend on u by [GT98, Theorem 8.18].

If $\mathscr{A}$ is in non-divergence form we have $(\mathscr{A} - \lambda)u \le 0$ and it follows that

$$\|u\|_{L^p(B)} \le C \inf_{B} u \tag{3.10}$$

where $B = B_R(y)$, $y \in \Omega$, $R > 0$, is a ball such that $2B \subset \Omega$ and $C > 0$ and $p \ge 1$ are constants independent of u.

Suppose now that u is 0 at a point $x \in \Omega$. Hence we may choose appropriate balls B, B' in Ω with $x \in B'$ such that (3.9) resp. (3.10) holds and it follows that $u = 0$ on B. Iterating this argument yields that $u = 0$ on all of Ω — a contradiction to $f \ne 0$. □

We now impose the following additional condition on the measures $\mu(z)$.

(spos) There is $\epsilon > 0$ such that $\mu(z, \overline{\Omega}) \ge \epsilon$ for all $z \in \partial\Omega$.

In fact in view of the compactness of $\partial\Omega$, the weak*-continuity of the map $z \mapsto \mu(z)$ and the positivity of $\mu(z)$, $z \in \partial\Omega$, it is enough to demand $\mu(z, \overline{\Omega}) \ne 0$ for all $z \in \partial\Omega$ to satisfy (spos).

In the following we consider the part of A_μ in $C(\overline{\Omega})$ without changing the notation. Let also T_μ denote the semigroup generated by A_μ on $C(\overline{\Omega})$. Clearly the condition (spos) is necessary for the semigroup T_μ to be irreducible since if there is $z_0 \in \partial\Omega$ with $\mu(z_0) = 0$ then the set $\{u \in C(\overline{\Omega}) : u(z_0) = 0\}$ is a closed invariant ideal.

But if Ω is connected, then (spos) is also sufficient as the following argument shows. Let $0 \leq f \in C(\overline{\Omega})$, $f \neq 0$, $\lambda > 0$ and $u \in D(A_\mu)$ be such that $R(\lambda, A_\mu)f =: u$. Remember that we have $R(\lambda, A_\mu) = (1 - S_\lambda)^{-1}R(\lambda, A_0)$ and set $g := R(\lambda, A_0)f$. Since $r(S_\lambda) < 1$ we a have $(1 - S_\lambda)^{-1} = \sum_{n=0}^{\infty} S_\lambda^n$, where $S_\lambda g = v$ if and only if

$$\begin{aligned} \lambda v - A_\mu v &= 0 \\ v|_{\partial\Omega} &= \langle \mu(\cdot), g \rangle. \end{aligned}$$

Since $S_\lambda$and g are positive by Proposition 2.26 resp. 2.32 we have $v \geq 0$. Now suppose we had $x_0 \in \overline{\Omega}$ such that $v(x_0) = 0$. Then by the maximum principle it follows that $x_0 \in \partial\Omega$ and hence

$$0 = v(x_0) = \langle \mu(\cdot), g \rangle.$$

Since $\mu(z, \partial\Omega) = 0$ and $\mu(z_0) \neq 0$ this means that $g = 0$ on some non-empty subset of Ω. But this contradicts Lemma 3.28 since $g = R(\lambda, A_0)f$ with f positive. Hence $v > 0$ on all of $\overline{\Omega}$. It then follows that $(1 - S_\lambda)^{-1}g = g + v + \sum_{n=2}^{\infty} S_\lambda^n g$ is strictly positive on $\overline{\Omega}$ since all other terms are positive; altogether we have that $u = R(\lambda, A_\mu)f$, $f \geq 0$, $f \neq 0$, is strictly positive. We thus have shown:

Proposition 3.29. *Let Ω be connected. Then the semigroup T_μ generated by $A_\mu$is irreducible on $C(\overline{\Omega})$ if and only if $\mu(z, \overline{\Omega}) \neq 0$ for all $z \in \partial\Omega$.*

For domains Ω with multiple connected components the condition (spos) clearly isn't enough, since then it may be that $\dim \ker A_\mu > 1$ as seen Section 3.2 (which would contradict [AGG$^+$86, B-III, Proposition 3.5. (d)].

One relatively simple criterion is that if (spos) holds and if we find $z \in \prod_{i \in J} \partial U_i$ such that the graph $G(z)$ is strongly connected, then T_μ is irreducible. Remember that strongly connected means that for every pair $U, V \in \mathcal{V}$ there is a path from U to V in $G(z)$. Since a graph is strongly connected if and only if its adjacency matrix is irreducible this result may be formulated as follows.

Corollary 3.30. *Assume that (spos) holds and that there is $z \in \prod_{i \in J} \partial U_i$ such that $M(z)$ is irreducible. Then the semigroup T_μ is irreducible on $C(\overline{\Omega})$.*

In fact this is a direct consequence of the following, more general theorem.

Theorem 3.31. *Assume that (spos) holds and that for every pair of distinct connected components U, V of Ω there is $z \in \prod_{i \in J} \partial U_i$ such that there is a path from U to V in $G(z)$. Then the semigroup T_μ is irreducible on $C(\overline{\Omega})$.*

Proof. Let $\lambda > 0$. We show that $u := R(\lambda, A_\mu)f$ is strictly positive if $f \geq 0$, $f \neq 0$. To that end set $g := R(\lambda, A_0)f$ and $v := S_\lambda g$; then

$$u = (1 - S_\lambda)^{-1} g = g + v + \sum_{n=2}^{\infty} S_\lambda^n g. \tag{3.11}$$

Since A_0 generates a positive semigroup we have that $g > 0$ on Ω and hence also $v \geq 0$, $v \neq 0$ by Lemma 3.28 and the fact that $\operatorname{supp} \mu(z) \not\subset \partial\Omega$ for all $z \in \partial\Omega$.

Assume that $\min_{\overline{\Omega}} v = 0$. By the maximum principle — note again that v suffices $\lambda v - \mathscr{A} v = 0$ — there is $z \in \partial U$ for some $U \in \mathcal{V}$ with $v(z) = 0$. Since $\mu(z) > 0$ and $\mu(z, \partial\Omega) = 0$ it follows then that $v(y) = 0$ for some $y \in V$ and some $V \in \mathcal{V}$. Again by the maximum principle we then have $v = 0$ on $\overline{V}$. Inductively by the same argument we have that $v = 0$ on all connected components for which there is a path from V in some graph $G(\mathfrak{z})$. But by assumption the existence of such a graph holds in $G(\mathfrak{z})$ for some $\mathfrak{z} \in \prod_{i \in J} \partial U_i$ and hence $v = 0$ on all of $\overline{\Omega}$ — a contradiction to $v \neq 0$.

Hence we have that $v > 0$ on all of $\overline{\Omega}$ and by 3.11 the same holds for u since all other terms are positive. □

And it turns out that the criterion established in Theorem 3.31 is also necessary as the following theorem shows.

Theorem 3.32. *Assume that (spos) does not hold or that there is a pair U, V of distinct connected components of Ω such that for every $\mathfrak{z} \in \prod_{i \in J} \partial U_i$ there is no path from U to V in $G(\mathfrak{z})$. Then T_μ is not irreducible on $C(\overline{\Omega})$.*

Proof. We already discussed the necessity of (spos).

We now prove the following statement: Let $0 \leq g \in C(\overline{\Omega})$ with $g|_{\overline{U}} = 0$ and $\lambda > 0$. If in addition $g|_{\overline{W}} = 0$ on all connected components W to which there is an edge from U in $G(\mathfrak{z})$ for some $\mathfrak{z} \in \prod_{i \in J} \partial U_i$, then $S_\lambda g = 0$ on $\overline{U}$.

In fact let $v := S_\lambda g$, that is $\lambda v - \mathscr{A} v = 0$ and $v(z) = \langle \mu(z), g \rangle$ for all $z \in \partial\Omega$ and call $\mathcal{W}$ the collection of connected components to which there is a edge from U in some $G(\mathfrak{z})$. Consider the maximum of v in $\overline{U}$. By the maximum principle it must be attained at a point $z_0 \in \partial U$ whence we have

$$\begin{aligned} v(z_0) &= \langle \mu(z_0), g \rangle \\ &= \sum_{W \in \mathcal{W}} \langle \mu(z_0), g|_{\overline{W}} \rangle = 0; \end{aligned}$$

i.e. $v = 0$ on $\overline{U}$.

Iterating the statement above yields: Let $0 \leq g \in C(\overline{\Omega})$ with $g|_{\overline{U}} = 0$ and $\lambda > 0$. If in addition $g|_{\overline{W}} = 0$ on all connected components W to which there is a path from U in $G(\mathfrak{z})$ for some $\mathfrak{z} \in \prod_{i \in J} \partial U_i$, then $S_\lambda^n g = 0$ on $\overline{U}$ for all $n \in \mathbb{N}$.

Now set $f|_{\overline{V}} = 1$, $f = 0$ everywhere else and $u := R(\lambda, A_\mu)f = (1 - S_\lambda)^{-1}g$ where $g := R(\lambda, A_0)f$. We then have $g|_V > 0$ by Lemma 3.28 and $g = 0$ everywhere else as well as $u = \sum_{n=0}^{\infty} S_\lambda^n g$. But by the above it holds that $S_\lambda^n g = 0$ on $\overline{U}$ for all $n \in \mathbb{N}_0$. We now have that $u = R(\lambda, A_\mu)f$ is not strictly positive in $\overline{\Omega}$ for a $f \in C(\overline{\Omega})$ with $f \geq 0$, $f \neq 0$ implying that T_μ is not irreducible. ☐

We thus have characterized the irreducibility of the semigroup T_μ. We note that the condition on the graphs $G(z)$ may also be phrased this way.

Corollary 3.33. *The semigroup T_μ is irreducible on $C(\overline{\Omega})$ if and only if (spos) holds and the graph $\bigcup_{z \in \prod_{j \in J} \partial U_j} G(z) = (\mathcal{V}, \bigcup_{z \in \prod_{j \in J} \partial U_j} E(z))$ is strongly connected.*

Concluding this section we want to strengthen the domination result of Proposition 3.5 under the assumption that T_{μ_1} is irreducible.

Proposition 3.34. *Let $\mu_1, \mu_2 \colon \partial\Omega \to \mathcal{M}(\overline{\Omega})_+$ fulfill assumption (meas) and for all $z \in \partial\Omega$ that $\mu_1(z) \leq \mu_2(z)$, i.e. for all $A \subset \overline{\Omega}$ we have $\mu_1(z, A) \leq \mu_2(z, A)$. If T_{μ_1} is irreducible, that is the assumptions in Corollary 3.33 hold for μ_1, and $\mu_1(z_0) < \mu_2(z_0)$ for some $z_0 \in \partial\Omega$, then $s(A_{\mu_1}) = \omega(T_{\mu_1}) < \omega(T_{\mu_2}) = s(A_{\mu_2})$.*

Proof. In view of Proposition 3.5 and Proposition 2.12 it remains to show that $T_{\mu_1} \neq T_{\mu_2}$, i.e. that $D(A_{\mu_1}) \neq D(A_{\mu_2})$. Since $\mu_2(z_0, \partial\Omega) = 0$ but $\mu_2(z_0) \neq 0$ by assumption, there is $x_0 \in \Omega$ and $r > 0$ such that $\overline{B_r(x_0)} \subset \Omega$ and $B_r(x_0) \cap \operatorname{supp} \mu(z_0) \neq \emptyset$. Now extend the function $u|_{\overline{B_r(x_0)}} = 1$ to a function $u \in C_{\mu_2}(\overline{\Omega})$ as in Proposition 3.25. But then we have

$$\begin{aligned} u(z_0) = \langle \mu_2(z_0), u \rangle &= \mu_2(z_0, \overline{B_r(x_0)}) + \int_{\overline{\Omega} \setminus \overline{B_r(x_0)}} u(z_0)\, \mu_2(z_0, \mathrm{d}x) \\ &\leq \mu_2(z_0, \overline{B_r(x_0)}) + \int_{\overline{\Omega} \setminus \overline{B_r(x_0)}} u(z_0)\, \mu_1(z_0, \mathrm{d}x) \\ &< \mu_1(z_0, \overline{B_r(x_0)}) + \int_{\overline{\Omega} \setminus \overline{B_r(x_0)}} u(z_0)\, \mu_1(z_0, \mathrm{d}x) = \langle \mu_1(z_0), u \rangle; \end{aligned}$$

that is $u \notin C_{\mu_1}(\overline{\Omega})$. In fact since u is continuous we have that the above inequality holds for a relatively open subset of $\partial\Omega$. By Proposition 3.8 this means that $\overline{D(A_{\mu_1})} \neq \overline{D(A_{\mu_2})}$ and hence $D(A_{\mu_1}) \neq D(A_{\mu_2})$.

The equality of growth and spectral bound for the semigroups T_{μ_1} resp. T_{μ_2} follows by [Lun95, Corollary 2.3.2]. ☐

3.3.3 The case $|\mu(z)|(\overline{\Omega}) < 1$

In this section we impose a different kind of assumption on the measures $\mu(z)$, $z \in \partial\Omega$: The measures may have values in $\mathbb{C}$ and the boundary need not be a null set under these measures but their total variation need to be strict sub probability measures.

More precisely let $\mathcal{M}(\overline{\Omega})$ be the Banach space of complex valued Borel measures equipped with the norm $\|\cdot\|_{\mathrm{TV}}$ and $\mu\colon \partial\Omega \to \mathcal{M}(\overline{\Omega})$, $\mu(z, A) := \mu(z)(A)$ fulfill the following assumptions:

(meas1) $z \mapsto \mu(z)$ is weak*-continuous, i.e. $z \mapsto \langle \mu(z), f\rangle$ is continuous for all $f \in C(\overline{\Omega})$.

(meas2) For all $z \in \partial\Omega$ we have $\|\mu(z)\|_{\mathrm{TV}} = |\mu(z)|(\overline{\Omega}) < 1$, where $|\nu|$ is the total variation of the (complex-valued) measure ν.

We note that by (meas1) and compactness of $\partial\Omega$ it follows that

$$q := \sup_{z\in\partial\Omega} |\mu(z)|(\overline{\Omega}) < \infty.$$

and by (meas2) that

$$q < 1.$$

Now as in the sections above if $\mathscr{A}$ is in non-divergence form set

$$\begin{aligned} D(A_\mu) &:= \big\{u \in C(\overline{\Omega}) \cap W(\Omega) : \mathscr{A}u \in L^\infty(\Omega), \\ &\qquad u(z) = \langle \mu(z), u\rangle \ \forall z \in \partial\Omega\big\} \\ A_\mu u &:= \mathscr{A}u \end{aligned}$$

and if $\mathscr{A}$ is in divergence form set

$$\begin{aligned} D(A_\mu) &:= \{u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega) : \mathscr{A}u \in L^\infty(\Omega) \\ &\qquad u(z) = \langle \mu(z), u\rangle \ \forall z \in \partial\Omega\big\}. \\ A_\mu u &:= \mathscr{A}u \end{aligned}$$

Remark. Before we begin with our analysis we want to mention that although we impose no assumption on the support of the measures $\mu(z)$, $z \in \partial\Omega$, it is prudent to assume *some* condition on it.

Illustrating this remark we consider the following situation: We assume (meas1) and (meas2) but suppose that $\mu(z, A) = \mu(z, \partial\Omega \cap A)$ for all $z \in \partial\Omega$ and all $A \subset \overline{\Omega}$ measurable. Now for $u \in D(A_\mu)$ we have for all $z \in \partial\Omega$

$$\begin{aligned} |u(z)| &= \Big|\int_{\overline{\Omega}} u(x)\, \mu(z, \mathrm{d}x)\Big| \\ &= \Big|\int_{\partial\Omega} u(x)\, \mu(z, \mathrm{d}x)\Big| \\ &\le q\|u\|_{C(\partial\Omega)}. \end{aligned}$$

Since $\partial\Omega$ is compact there is $z_0 \in \partial\Omega$ such that $|u(z_0)| = \|u\|_{C(\partial\Omega)}$. Now the calculation above shows that we must have $|u(z_0)| = 0$, i.e. $u|_{\partial\Omega} = 0$, when $q < 1$.

Hence in the case where we have $q < 1$ and the support of all measures in $\partial\Omega$, we actually just have reformulated the classical Dirichlet boundary condition in a more complicated way.

Recall that the map $S_\lambda = L_\lambda\phi\colon C(\overline{\Omega}) \to C(\overline{\Omega})$ is a continuous map defined as

$$S_\lambda f = u \;:\Leftrightarrow\; \lambda u - \mathscr{A}u = 0,\, u|_{\partial\Omega} = \langle \mu(\cdot), f\rangle.$$

This is well defined by assumption (meas1) and Proposition 2.26 resp. Proposition 2.32 for $\operatorname{Re}\lambda \geq 0$.

We have the following norm estimate for S_λ.

Lemma 3.35. *Assume (meas1). For all $\lambda \in \overline{\mathbb{C}^+}$ we have $\|S_\lambda\| \leq q$.*

Proof. Let $f \in C(\overline{\Omega})$ and set $u := S_\lambda f$. Then by the maximum principle as in Proposition 2.30 resp. Proposition 2.35 we have that there is $z_0 \in \partial\Omega$ with $\|u\|_{C(\overline{\Omega})} = |u(z_0)|$. It follows that

$$|u(z_0)| = |\langle \mu(z_0), f\rangle| \leq |\mu(z_0)|(\overline{\Omega})\|f\|_{C(\overline{\Omega})} \leq q\|f\|_{C(\overline{\Omega})}$$

proving the claim. □

Using (meas2) and the Neumann series we have the following corollary.

Corollary 3.36. *Assume (meas1) and (meas2). For all $\lambda \in \overline{\mathbb{C}^+}$ the map $1 - S_\lambda$ is invertible and its norm is bounded by*

$$\|(1 - S_\lambda)^{-1}\|_{\mathcal{L}(C(\overline{\Omega}))} \leq \frac{1}{1-q}.$$

Thus by Proposition 2.46 we have a generation result.

Theorem 3.37. *Under (meas1) and (meas2) the operator A_μ generates a bounded holomorphic semigroup T_μ on $L^\infty(\Omega)$ with compact resolvent. It maps $L^\infty(\Omega)$ to $C(\overline{\Omega})$ and the part of A_μ in $C(\overline{\Omega})$ generates the semigroup $T_\mu|_{C(\overline{\Omega})}$. Furthermore the part of A_μ in $C_\mu(\overline{\Omega})$ generates the C_0-semigroup $T_\mu|_{C_\mu(\overline{\Omega})}$.*

In fact by the definition of the operator A_μ we have that the resolvent and hence the semigroup have their range in $C(\overline{\Omega})$. The C_0-property in $C_\mu(\overline{\Omega})$ follows as in Proposition 3.8 and Corollary 3.10; note that the operator A_μ under assumptions (meas1) and (meas2) is always injective, as will be seen in the proof of Theorem 3.38 directly below. By Corollary 3.7 (applied with $\mu = 0$) we have that A_0 has compact resolvent which then implies the same for A_μ by Proposition 2.52.

We now want to investigate the asymptotic behavior of T_μ.

Theorem 3.38. *Under (meas1) and (meas2) the semigroup T_μ is exponentially stable.*

Proof. By Proposition 2.6 it remains to show that the spectrum is contained in the open left half plane $\mathbb{C}^-$. Since the semigroup is bounded we already have $\sigma(A_\mu) \subset \overline{\mathbb{C}^-}$; hence by compactness we have to show that eigenvalues on the imaginary axis cannot occur.

Let $\lambda \in \mathbb{R}$ and $u \in \ker(i\lambda - A_\mu)$.

We first treat the case of $\mathscr{A}$ being in divergence form. Set $v := |u|$. As in the proof of Proposition 2.35 we have by Lemma 2.34

$$\mathscr{A} v \geq 0.$$

Hence by the maximum principle as in [GT98, Theorem 8.19] it remains to show that $v|_{\partial\Omega} = 0$. But we have for $z \in \partial\Omega$

$$\begin{aligned} v(z)^2 &= u(z)\overline{u(z)} \\ &= \langle \mu(z), u(z)\rangle \overline{\langle \mu(z), u(z)\rangle} \\ &\leq |\mu|(z, \overline{\Omega})^2 \|u\|^2_{C(\overline{\Omega})} \\ &\leq q^2 \|v\|^2_{C(\overline{\Omega})}. \end{aligned} \tag{3.12}$$

Since $q < 1$ this yields $v(z) \leq 0$ and hence $v = 0$. Altogether this means that $\ker(i\lambda - A_\mu) = \{0\}$.

Now let $\mathscr{A}$ be in non-divergence form. Set $v := |u|^2 = u\bar{u}$. A direct calculation shows

$$\begin{aligned} D_i v &= (D_i u)\bar{u} + u D_i \bar{u} = 2\operatorname{Re}((D_i u)\bar{u}) \text{ and} \\ D_{ij} v &= (D_{ij} u)\bar{u} + D_i u D_j \bar{u} + D_i \bar{u} D_j u + u D_{ij} \bar{u}. \end{aligned}$$

Hence we have by ellipticity of $\mathscr{A}$

$$\begin{aligned} \mathscr{A} v &\geq \operatorname{Re} \sum_{i,j=1}^{d} a_{ij} (D_{ij} u)\bar{u} + \operatorname{Re} \sum_{i,j=1}^{d} a_{ij} u D_{ij} \bar{u} + 2\operatorname{Re} \sum_{i=1}^{d} c_i (D_i u)\bar{u} + c u \bar{u} \\ &\geq 2\operatorname{Re}(\mathscr{A} u \bar{u}). \end{aligned}$$

Altogether it follows that

$$\mathscr{A} v \geq 2\operatorname{Re}(\mathscr{A} u \bar{u}) = 2\operatorname{Re}(i\lambda u \bar{u}) = 0.$$

And now by the maximum principle as in [GT98, Theorem 9.1] it remains to show that $v|_{\partial\Omega} = 0$. But this follows as in equation (3.12) — here starting and ending with v instead of v^2 — and we again have $\ker(i\lambda - A_\mu) = \{0\}$. □

3.4 Notes and comments

The generation results from the first section and the results on the strong Feller property are adapted from [AKK16, Section 4 and 5] with some abstractions and modifications to treat both non-divergence and divergence form operators. The results on the closure of $D(A_\mu)$ and the C_0-property on $C_\mu(\overline{\Omega})$ are new.

The second section is new except for Corollary 3.13 and Theorem 3.19. The results on the kernel of A_μ from this section also hold (with only very minor modifications to the proofs!) under the following, less restrictive assumption on the measures $\mu(z)$:

(meas′) The mapping $\partial\Omega \to \mathcal{M}(\overline{\Omega})_+, z \mapsto \mu(z)$ is $\sigma(\mathcal{M}(\overline{\Omega}), C(\overline{\Omega}))$-continuous. We require for all $z \in \partial\Omega$ that either $\mu(z) = 0$ or $\operatorname{supp} \mu(z) \cap \Omega \neq \emptyset$.

However since we lack generation results under these assumptions, naturally no conclusion regarding asymptotics can be drawn from it.

The third section is completely new, that is the analysis of $C_\mu(\overline{\Omega})$, the irreducibility and the case $\|\mu(z)\|_{TV} < 1$. Note that the results on $C_\mu(\overline{\Omega})$ still hold if we assume instead of being sub-probability measures that the measures $\mu(z)$ are just positive and uniformly bounded. In this case one adapts Proposition 3.25 by saying that the extension F of f suffices the estimate $\|F\| \leq C\|f\|$ with $C > 0$ only depending on μ. The Theorems 3.26 and 3.27 remain valid and in the proofs only the choice of the set K' must be changed in accordance to the value C.

4 Parabolic equations with non-local Robin boundary conditions

In this chapter we turn our attention to non-local boundary conditions of Robin type. More precisely, we consider an elliptic operator $\mathscr{A}$ on a bounded Lipschitz domain Ω and a parabolic equation of the form

$$\begin{cases} \partial_t u(t,x) - \mathscr{A}u(t,x) = 0 \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega \\ u(0,x) = u_0(x) \text{ for } x \in \Omega \\ \partial_\nu^{\mathscr{A}} u(t,z) + \beta(z)u(t,z) = B(z)(u(t,\cdot)) \text{ for all } t \in \mathbb{R}^+ \\ \qquad\qquad \text{and almost all } z \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\partial_\nu^{\mathscr{A}}$ denotes the conormal derivative with respect to the operator $\mathscr{A}$, $\beta \in L^\infty(\partial\Omega)$ and $B(z)\colon C(\overline{\Omega}) \to \mathbb{C}$ is a continuous functional for all $z \in \partial\Omega$. Since any continuous functional on $C(\overline{\Omega})$ is a kernel operator the boundary condition in the last line may be written as

$$\partial_\nu^{\mathscr{A}} u(t,z) + \beta(z)u(t,z) = \langle \mu(z), u(t,\cdot)\rangle$$

where $\mu(z)$ are (complex-valued) Borel measures on $\overline{\Omega}$.

We now state our assumptions on the operator and the boundary terms.

(div) The operator $\mathscr{A}\colon H^1(\Omega) \to \mathcal{D}(\Omega)'$ is a second order differential operator in divergence form given by

$$\mathscr{A}u = \sum_{i,j=1}^d D_i(a_{ij}D_ju) + \sum_{i=1}^d D_i(b_iu) + \sum_{i=1}^d c_iD_iu + d_0u,$$

where $a_{ij}, b_i, c_i, d_0 \in L^\infty(\Omega)$, $i, j = 1, \dots, d$, and

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2$$

for all $\xi \in \mathbb{R}^d$ and almost all $x \in \overline{\Omega}$. The domain Ω is bounded and open with Lipschitz boundary.

For the details on this assumption and on the conormal derivative we refer to Section 2.5 of Chapter 2.

(meas) The map $\mu\colon \partial\Omega \to \mathcal{M}(\overline{\Omega})$ satisfies the properties

- (M1) for every $f \in B_b(\overline{\Omega})$ we have that the function $z \mapsto \langle \mu(z), f\rangle$ is measurable,
- (M2) for some $p > d-1$ we have $\int_{\partial\Omega} \|\mu(z)\|_{TV}^p \,\mathrm{d}z < \infty$ and
- (M3) there exists a positive Borel measure τ on $\overline{\Omega}$ such that $\mu(z)$ is absolutely continuous w.r.t. τ for all $z \in \partial\Omega$ and $L^1(\overline{\Omega}, \tau)$ is separable.

Note that $L^1(\overline{\Omega}, \tau)$ is separable, if τ is bounded for example.

In the first section we prove the generation results on $L^\infty(\Omega)$ and $C(\overline{\Omega})$ under assumptions (meas) and (div) as well as the contractiveness of the semigroup under additional assumptions on μ, β and the coefficients b_i, d_0. We will also see that the semigroup restricts to a C_0-semigroup on all of $C(\overline{\Omega})$. Furthermore we will show the strong Feller property of the semigroup.

In the second section we will present our results on the asymptotics of the generated semigroup. Under the additional assumption that the semigroup with classical Robin boundary conditions is irreducible we will give criteria under which the semigroup is exponentially stable or converges to a rank 1 projection. Finally we will give a blow up result for non-local Neumann boundary conditions, that is when we have $\beta = 0$.

The third section is devoted to the special case when $\mu(z)$ is absolutely continuous w.r.t. the Lebesgue measure for all $z \in \partial\Omega$. This is convenient since in that case we can prove a generation result on $L^2(\Omega)$ using form methods. These form methods allow us to easily verify whether the semigroup is irreducible on $L^2(\Omega)$. Combining this with the results of the first two sections as well as Proposition 2.16 will allow us to treat the asymptotics of the semigroup on $L^\infty(\Omega)$ resp. $C(\overline{\Omega})$. Furthermore we can actually characterize the markovian resp. sub-markovian property of the semigroup in this situation.

Finally in the short, last section we give examples of maps μ satisfying assumption (meas). This includes the rather generic case of a regularizing map μ; i.e. a map μ yielding a continuous function on the boundary from a bounded, measurable function f in $\overline{\Omega}$ via $\partial\Omega \to \mathbb{C}$, $z \mapsto \langle \mu(z), f \rangle$.

4.1 Generation results

We now prove the generation result for Robin boundary conditions. We begin by setting up the framework in which we apply Greiner's boundary perturbation. We define

$$D := \{u \in C(\overline{\Omega}) \cap H^1(\Omega) : \mathscr{A}u \in L^\infty(\Omega), \partial_\nu^{\mathscr{A}} u \in L^p(\partial\Omega)\},$$

where p is from assumption (meas) (b). Endowed with the norm

$$\|u\|_D := \|u\|_{C(\overline{\Omega})} + \|u\|_{H^1(\Omega)} + \|\mathscr{A}u\|_{L^\infty(\Omega)} + \|\partial_\nu^{\mathscr{A}} u\|_{L^p(\partial\Omega)}$$

the space D is a Banach space which is continuously embedded into $X = L^\infty(\Omega)$. Our maximal operator $A\colon D \to X$ is given by $Au := \mathscr{A}u$ which defines a continuous linear mapping from D to X. It is a consequence of Theorem 2.36 that $\overline{D} = C(\overline{\Omega})$. We set $\partial X := L^p(\partial\Omega)$ and consider the boundary operator $L : D \to \partial\Omega$ defined via $Lu = \partial_\nu^{\mathscr{A}} u + \beta u$ where $\beta \in L^\infty(\Omega)$. Finally, for μ as in (meas) the function $\phi : \overline{D} \to \partial X$ is given by

$$(\phi u)(z) := \langle \mu(z), u \rangle = \int_{\overline{\Omega}} u(x)\, \mu(z, \mathrm{d}x)$$

Making use of the results of Section 2.5 of Chapter 2 we can prove generation results for the operator $A_{\beta,\mu}$, defined by

$$\begin{aligned} D(A_{\beta,\mu}) &= \big\{u \in C(\overline{\Omega}) \cap H^1(\Omega) : \mathscr{A}u \in L^\infty(\Omega), \\ &\qquad \partial_\nu^{\mathscr{A}} u + \beta u = \langle \mu(\cdot), u \rangle \text{ on } \partial\Omega\big\} \\ A_{\beta,\mu} &= \mathscr{A}u. \end{aligned}$$

The following is the main result of this section.

Theorem 4.1. *Assume hypotheses (div) and (meas). Then the operator $A_{\beta,\mu}$ generates a holomorphic semigroup $T_{\beta,\mu}$ on $L^\infty(\Omega)$ which satisfies the strong Feller property. In particular, it leaves the space $C(\overline{\Omega})$ invariant. Its restriction to that space is a C_0-semigroup whose generator is $A^C_{\beta,\mu}$, the part of $A_{\beta,\mu}$ in $C(\overline{\Omega})$.*

Proof. Noting that the operator $A_{\beta,\mu}$ is exactly the perturbed operator A_ϕ, where A and ϕ are as defined above, the thesis follows immediately from Theorem 2.47 and Corollary 2.54 once we verified that the maps A, L and ϕ satisfy the hypotheses of Section 2.6.

(a) *The operator $L\colon D \to \partial X$ is surjective.*

Fix $\lambda > \omega$ Given $h \in \partial X = L^p(\partial\Omega)$, it follows from Proposition 2.39 that the unique solution $u \in H^1(\Omega)$ of the problem

$$\begin{cases} \lambda u + \mathscr{A} u = 0 \\ \partial_\nu^{\mathscr{A}} u + \beta u = h \end{cases}$$

belongs to $C(\overline{\Omega})$. Moreover, $\mathscr{A} u = -\lambda u \in C(\overline{\Omega}) \subset L^\infty(\Omega)$. Thus, $u \in D$ and $Lu = h$, proving that L is surjective.

(b) *The boundary perturbation ϕ is compact.*

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C(\overline{\Omega})$, say $\|u_n\|_{C(\overline{\Omega})} \leq M$ for all $n \in \mathbb{N}$. Since $\mu(z) \ll \tau$ by Hypothesis (meas) (M3), for every $z \in \partial\Omega$ we find a Radon–Nikodym density $\varphi_z \in L^1(\Omega, \tau)$ of μ_z with respect to τ, i.e. we have

$$\int_{\overline{\Omega}} f(x)\,\mu(z, \mathrm{d}x) = \int_{\overline{\Omega}} f \varphi_z \,\mathrm{d}\tau$$

for all $f \in C(\overline{\Omega})$. In particular, $(\phi u_n)(z) = \langle u_n, \varphi_z \rangle_{L^\infty(\tau), L^1(\tau)}$. Since the sequence u_n is bounded in $L^\infty(\Omega, \tau)$ and $L^1(\Omega, \tau)$ is separable, it follows from the Banach–Alaoglu theorem that we find a weak*-convergent subsequence, say $u_{n_k} \rightharpoonup^* u$ for some $u \in L^\infty(\tau)$. In particular,

$$(\phi u_{n_k})(z) = \int_{\overline{\Omega}} u_{n_k} \varphi_z \,\mathrm{d}\tau \to \int_{\overline{\Omega}} u \varphi_z \,\mathrm{d}\tau$$

for all $z \in \partial\Omega$, i.e. ϕu_n has a subsequence which converges pointwise. Note that we have

$$|(\phi u_n)(z)| \leq M \|\mu(z)\|_{\mathrm{TV}}.$$

As a consequence of Hypothesis (meas) (M2) the functions ϕu_n have a p-integrable majorant and it follows from the dominated convergence theorem that ϕu_n has a subsequence which converges in $L^p(\partial\Omega)$.

(c) The operator A_0 is exactly the part of A_β^2 in $L^\infty(\Omega)$. It follows from Theorem 2.36 that A_0 generates an analytic semigroup on $X = L^\infty(\Omega)$ which enjoys the strong Feller property and whose domain is dense in $C(\overline{\Omega})$. □

We next prove some additional properties of the semigroup $T_{\beta,\mu}$ making use of Proposition 2.52.

Proposition 4.2. *Assume hypotheses (div) and (meas) and let $T_{\beta,\mu}$ be the semigroup generated by $A_{\beta,\mu}$ according to Theorem 4.1.*

(a) $T_{\beta,\mu}$ is compact.

(b) If $\mu(z)$ is a positive measure for almost every $z \in \partial\Omega$, then the semigroup $T_{\beta,\mu}$ is positive.

Proof. (a) Follows immediately from Proposition 2.52 i), noting that the semigroup generated by A_0 is compact as a consequence of Theorem 2.36.

(b) By Theorem 2.36, the semigroup generated by A_0 is positive. If $\mu(z)$ is positive for almost every $z \in \partial\Omega$, then the map ϕ is positive. Note that for the solution map L_λ the function $L_\lambda h$ is the unique solution of the boundary value problem

$$\begin{cases} \lambda u - \mathscr{A} u = 0 \\ \partial_\nu^{\mathscr{A}} u + \beta u = h. \end{cases}$$

Thus, by Proposition 2.40, L_λ is positive for $\lambda > \omega$. Altogether $L_\lambda \phi$ is positive and it follows from Corollary 2.52ii) that $T_{\beta,\mu}$ is positive. □

Next we characterize when $T_{\beta,\mu}$ is markovian.

Proposition 4.3. *Assume in addition to hypotheses (meas) and (div) that $\mu(z)$ is a positive measure for almost every $z \in \partial\Omega$. The following are equivalent.*

(i) The semigroup $T_{\beta,\mu}$ is markovian.

(ii) We have

$$\sum_{j=1}^{d} D_j b_j + d_0 = 0 \text{ almost everywhere on } \Omega \text{ and} \tag{4.2}$$

$$\sum_{j=1}^{d} \operatorname{tr}(b_j)\nu_j + \beta = \mu(\cdot, \overline{\Omega}) \text{ almost everywhere on } \partial\Omega \tag{4.3}$$

Proof. Since $T_{\beta,\mu}$ is positive, (i) is equivalent to $1 \in \ker A_{\beta,\mu}$. Observe that $\mathscr{A}1 = \sum_{j=1}^{d} D_j b_j + d_0$. Thus $\mathscr{A}1 = 0$ if and only if (4.2) holds. In that case, integration by parts yields for $v \in H^1(\Omega)$ that

$$a(1, v) + [\mathscr{A}1, v] = \sum_{j=1}^{d} \int_\Omega b_j D_j v + d_0 v \, dx = \sum_{j=1}^{d} \int_{\partial\Omega} b_j \nu_j v \, d\sigma.$$

Thus $1 \in D(A_{\beta,\mu})$ if and only if

$$\sum_{j=1}^{d} \int_{\partial\Omega} b_j(z)\nu_j(z)v(z) \, d\sigma(z) = \int_{\partial\Omega} \big(-\beta(z) + \langle \mu(z), 1 \rangle \big) v(z) \, d\sigma(z)$$

for all $v \in H^1(\Omega)$. This is equivalent to (4.3). □

If we merely have inequalities in (4.2) and (4.3), then the semigroup is submarkovian as we show next. In the proof, we use the following monotonicity result.

Proposition 4.4. *Assume hypothesis (div) and let $\beta_1, \beta_2 \in L^\infty(\partial\Omega)$ with $\beta_2 \le \beta_1$. Moreover, let functions $\mu_1, \mu_2 : \partial\Omega \to \mathscr{M}(\overline{\Omega})$ be given such that $0 \le \mu_1(z) \le \mu_2(z)$ for almost all $z \in \partial\Omega$ and such that μ_1, μ_2 satisfy hypothesis (meas) with the same p. Then*

$$0 \le T_{\beta_1,\mu_1}(t) \le T_{\beta_2,\mu_2}(t)$$

for all $t \ge 0$.

Proof. The semigroups T_{β_1,μ_1} and T_{β_2,μ_2} are obtained from the same maximal operator A but using different boundary perturbations $\phi_j \colon u \mapsto \langle \mu_j(\cdot), u\rangle$ and boundary operators $L_j \colon u \mapsto \partial_\nu^{\mathscr{A}} u + \beta_j u$. We clearly have $L_2 u \le L_1 u$ and $0 \le \phi_1 u \le \phi_2 u$ for $u \ge 0$. Moreover, if we write $L^j_\lambda := (L_j|_{\ker(\lambda - A)})^{-1}$, then we have $S^1_\lambda \le S^2_\lambda$ by Proposition 2.40. Thus Corollary 2.53 yields the claim. □

Proposition 4.5. *Assume in addition to hypotheses (div) and (meas) also that $\mu(z)$ is positive for almost all $z \in \partial\Omega$ and that $b_j \in W^{1,\infty}(\Omega)$ for $j = 1, \dots, d$. If*

$$\sum_{j=1}^{d} D_j b_j + d_0 \le 0 \textit{ almost everywhere on } \Omega \textit{ and} \tag{4.4}$$

$$\sum_{j=1}^{d} \operatorname{tr}(b_j)\nu_j + \beta \ge \mu(\cdot, \overline{\Omega}) \textit{ almost everywhere on } \partial\Omega \tag{4.5}$$

then the semigroup $T_{\beta,\mu}$ is submarkovian.

Proof. Assume at first that $\sum_{j=1}^d D_j b_j + d_0 = 0$. Let us define $\beta_0(z) := \mu(z)(\overline{\Omega}) - \sum_{j=1}^d \operatorname{tr} b_j(z)\nu_j(z)$. By Proposition 4.3 the semigroup $T_{\beta_0,\mu}$ is markovian. As a consequence of Proposition 4.4 we have $0 \le T_{\beta,\mu}(t) \le T_{\beta_0,\mu}(t)$ for all $t > 0$ which clearly implies that $T_{\beta,\mu}$ is submarkovian. That $T_{\beta,\mu}$ is still submarkovian when $\sum_{j=1}^d D_j b_j + d_0 \le 0$ follows from a standard perturbation result as follows.

Denote by $\tilde{A}_{\beta,\mu}$ the operator where d_0 is replaced by $\tilde{d}_0 := -\sum_{j=1}^d D_j b_j$. Then the semigroup $\tilde{T}_{\beta,\mu}$ generated by $\tilde{A}_{\beta,\mu}$ is submarkovian by what has been proved so far. Note that $A_{\beta,\mu} + (d_0 - \tilde{d}_0) = \tilde{A}_{\beta,\mu}$, so that $\tilde{A}_{\beta,\mu}$ is a bounded and positive perturbation of $A_{\beta,\mu}$. Using a perturbation result for resolvent positive operators [ABHN11, Proposition 3.11.12] we find that $R(\lambda, A_{\beta,\mu}) \le R(\lambda, \tilde{A}_{\beta,\mu})$ for large enough λ and the domination of the semigroups follows from Proposition 2.8. Alternatively, the domination property can be inferred from the Dyson–Phillips formula for the perturbed semigroup, see [Kun13, Example 3.4]for a version which covers our setting. □

As a further consequence of Proposition 4.4 we have

$$0 \le T_{\beta,0}(t) \le T_{\beta,\mu}(t) \tag{4.6}$$

for all $t > 0$ in the case where $\mu(z)$ is a positive measure for almost every $z \in \partial\Omega$. It thus follows from Proposition 2.38 that condition (4.4) is necessary for $T_{\beta,\mu}$ to be submarkovian. It seems not so easy to show that also condition (4.5) is necessary for this. Also concerning the positivity of the semigroup $T_{\beta,\mu}$ it seems unclear if the condition that $\mu(z)$ is a positive measure for almost every $z \in \partial\Omega$ is necessary. However, in Section 3 we will give a proof of necessity in the special case where every measure $\mu(z)$ is absolutely continuous with respect to the Lebesgue measure.

4.2 Asymptotic behavior

The aim of this section is to describe the asymptotic behavior of the semigroup $T_{\beta,\mu}$ as $t \to \infty$. Since $T_{\beta,\mu}(t)L^\infty(\Omega) \subset C(\overline{\Omega})$ for all $t > 0$ it suffices to study $T^C_{\beta,\mu}$, the restriction to $C(\overline{\Omega})$, which is a strongly continuous semigroup. We also assume throughout that $\mu(z) \geq 0$ for almost all $z \in \partial\Omega$ so that the semigroup is positive.

For the definition of spectral bound and irreducibility we refer to Section 2.2.

The asymptotic behavior of $T^C_{\beta,\mu}$ is determined by the spectral bound $s(A^C_{\beta,\mu})$ of its generator. We first show that the spectrum is not empty.

Proposition 4.6. *One has $s(A^C_{\beta,\mu}) > -\infty$. Moreover, $s(A^C_{\beta,\mu})$ is an eigenvalue of $A^C_{\beta,\mu}$ with positive eigenfunction.*

Proof. We first show that $s(A^C_{\beta,0}) \leq s(A^C_{\beta,\mu})$.

As a consequence of Proposition 4.4 we have $0 \leq T^C_{\beta,0}(t) \leq T^C_{\beta,\mu}(t)$. Taking Laplace transforms, it follows then that $0 \leq R(\lambda, A^C_{\beta,0}) \leq R(\lambda, A^C_{\beta,\mu})$ for all large enough λ. By [ABHN11, Theorem 5.3.1] for a positive semigroup the abscissa of the Laplace transform coincides with the spectral bound. Thus, if we assume that $s(A^C_{\beta,0}) > s(A^C_{\beta,\mu})$ we have $0 \leq R(\lambda, A^C_{\beta,0}) \leq R(\lambda, A^C_{\beta,\mu})$ for all $\lambda > s(A^C_{\beta,0})$. By [ABHN11, Proposition 3.11.2] we have $s(A^C_{\beta,0}) \in \sigma(A^C_{\beta,0})$ and hence $\sup_{\lambda > s(A^C_{\beta,0})} \|R(\lambda, A^C_{\beta,0})\|_{\mathcal{L}(C(\overline{\Omega}))} = \infty$. Consequently, also $\|R(\lambda, A^C_{\beta,\mu})\|_{\mathcal{L}(C(\overline{\Omega}))}$ is unbounded as $\lambda \downarrow s(A^C_{\beta,0})$. It thus follows that $s(A^C_{\beta,0}) \in \sigma(A^C_{\beta,\mu})$, a contradiction to our assumption $s(A^C_{\beta_0}) > s(A^C_{\beta,\mu})$.

The operator $A^C_{\beta,0}$ is the part of A^2_β in $C(\overline{\Omega})$, defined before Theorem 2.36. It follows from Proposition 2.15 that the semigroup generated by A^2_β is irreducible. Since the resolvent of that operator is compact, it follows from de Pagter's Theorem (see [dP86, Theorem 3] or [AGG⁺86, C-III, Theorem 3.7.(c)]) that $s(A^2_\beta) > -\infty$. But we have $s(A^C_{\beta,0}) = s(-A^2_\beta)$ since the resolvents are compact and consistent, see [Are94, Proposition 2.6]. □

Note that the semigroup $T^C_{\beta,\mu}$ is compact and hence immediately norm continuous whence spectral bound and growth bound coincide. Thus, if $s(A_{\beta,\mu}) < 0$, then $\|T^C_{\beta,\mu}(t)\| \leq Me^{-\epsilon t}$ for all $t > 0$ and suitable constants $M \geq 0$ and $\epsilon > 0$, i.e. the semigroup is exponentially stable. If, on the other hand, $s(A^C_{\beta,\mu}) > 0$ then there exists $\epsilon > 0$ $M > 0$ such that $\|T^C_{\beta,\mu}(t)\| \geq Me^{\epsilon t}$ for all $t > 0$. Finally, if $s(A_{\beta,\mu}) = 0$ and the semigroup is bounded, then it converges. The boundedness is not easy to decide, though. However, we have a precise criterion for the semigroup to be submarkovian. In that case, we obtain the following result from Theorem 2.10.

Proposition 4.7. *Assume that* $\mu(z) \geq 0$, $b_j \in W^{1,\infty}(\Omega)$ *for* $j = 1, \ldots, d$,

$$\sum_{j=1}^{d} D_j b_j + d_0 \leq 0 \text{ almost everywhere on } \Omega \text{ and}$$

$$\sum_{j=1}^{d} \operatorname{tr}(b_j)\nu_j + \beta \geq \mu(\cdot, \overline{\Omega}) \text{ almost everywhere on } \partial\Omega.$$

Then there exist a positive projection $P \in \mathscr{L}(C(\overline{\Omega}))$ *with finite rank and* $M > 0$, $\epsilon > 0$ *such that*

$$\|T^C_{\beta,\mu}(t) - P\|_{\mathscr{L}(C(\overline{\Omega}))} \leq Me^{-\epsilon t}$$

for all $t > 0$.

In the situation of Proposition 4.7, if $s(A^C_{\beta,\mu}) = 0$, there exists a function $0 \leq u = Pu$, $u \neq 0$, i.e. a positive function in the kernel of $A^C_{\beta,\mu}$. If the semigroup is markovian, then 1 is such a function. It is interesting to know when it is the only one (up to a scalar multiple). If $T^C_{\beta,\mu}$ is irreducible, then this is the case. Unfortunately, it is not easy to prove irreducibility on $C(\overline{\Omega})$. However, it follows from the domination property (4.6) that $T^C_{\beta,\mu}$ is irreducible whenever $T^C_{\beta,0}$ is so. As for the former semigroup, a particular case will be settled in Theorem 4.14 for Ω connected resp. Theorem 4.16 and Corollary 4.17 for general Ω. We also remark that in a forthcoming paper [AtE] it will be shown that $T^C_{\beta,0}$ is irreducible whenever $b_j = 0$ and $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, d$.

Theorem 4.8. *Assume that* $\mu(z) \geq 0$, $b_j \in W^{1,\infty}(\Omega)$ *for* $j = 1, \ldots, d$,

$$\sum_{j=1}^{d} D_j b_j + d_0 = 0 \text{ almost everywhere on } \Omega \text{ and}$$

$$\sum_{j=1}^{d} \operatorname{tr}(b_j)\nu_j + \beta = \mu(\cdot, \overline{\Omega}) \text{ almost everywhere on } \partial\Omega.$$

Assume further that $T^C_{\beta,0}$ *is irreducible. Then there exists a strictly positive measure* ρ *on* $\overline{\Omega}$ *and constants* $\epsilon, M > 0$ *such that for* $P \in \mathscr{L}(C(\overline{\Omega}))$*, given by*

$$Pf = \int_{\overline{\Omega}} f \, \mathrm{d}\rho \cdot 1_{\overline{\Omega}}$$

for all $f \in C(\overline{\Omega})$*, we have*

$$\|T(t) - P\|_{\mathcal{L}(C(\overline{\Omega}))} \leq M e^{-\epsilon t}$$

for all $t > 0$*.*

Proof. By Proposition 4.2 the semigroup $T^C_{\beta,\mu}$ is markovian and hence $1_{\overline{\Omega}}$ is a fixed function of the semigroup. As a consequence of (4.6), $T^C_{\beta,\mu}$ is irreducible. Now the claim follows from Theorem 2.11. □

We next prove exponential stability in the submarkovian case.

Theorem 4.9. *Assume that* $\mu(z) \geq 0$ $b_j \in W^{1,\infty}(\Omega)$ *for* $j = 1, \ldots, d$*,*

$$\sum_{j=1}^{d} D_j b_j + d_0 \leq 0 \text{ almost everywhere on } \Omega \text{ and}$$

$$\sum_{j=1}^{d} \operatorname{tr}(b_j)\nu_j + \beta \geq \mu(\cdot, \overline{\Omega}) \text{ almost everywhere on } \partial\Omega.$$

Moreover, assume that $T^C_{\beta,0}$ *is irreducible. If one of the two inequalities above is strict on some set of positive measure, then there exist* $\epsilon, M > 0$ *such that*

$$\|T_{\beta,\mu}(t)\|_{\mathscr{L}(C(\overline{\Omega}))} \leq M e^{-\epsilon t}$$

for all $t > 0$*.*

Proof. Let us put

$$\tilde{\beta}(z) := \mu(z, \overline{\Omega}) - \sum_{j=1}^{d} \operatorname{tr} b_j(z)\nu_j(z)$$

and $\tilde{d}_0 = -\sum_{j=1}^{d} D_j b_j$. Replace d_0 with $\tilde{d}_0$ and β with $\tilde{\beta}$ and denote by $\tilde{\mathscr{A}}$ the corresponding distributional operator and $\tilde{T} = \tilde{T}^C_{\tilde{\beta},\mu}$ the corresponding semigroup on $C(\overline{\Omega})$ with generator $\tilde{A} = \tilde{A}^C_{\tilde{\beta},\mu}$. Then $0 \leq T^C_{\beta,\mu}(t) \leq \tilde{T}(t)$ for all $t > 0$, cf. the proof of Proposition 4.2. By Proposition 4.3 the semigroup $\tilde{T}$ is markovian so that its generator has spectral bound 0.

However, the generators of these two semigroups are different. To see this, let us first assume that we have$\beta \neq \tilde{\beta}$ in $L^\infty(\partial\Omega)$. Note that the conormal derivative does depend on the zero order term d_0 resp. $\tilde{d}_0$, that is we have $\partial_\nu^{\mathscr{A}} = \partial_\nu^{\tilde{\mathscr{A}}}$. We find

$$\langle \mu(z), 1 \rangle = \partial_\nu^{\tilde{\mathscr{A}}} 1 + \tilde{\beta} 1 \neq \partial_\nu^{\mathscr{A}} 1 + \beta 1.$$

Thus $1 \notin D(A^C_{\beta,\mu})$ but $1 \in D(\tilde{A})$. If, on the other hand, $\beta = \tilde{\beta}$ in $L^\infty(\partial\Omega)$, then we have $d_0 \neq \tilde{d}_0$ in $L^\infty(\Omega)$. Note that $A_{\beta,\mu} 1 = \tilde{d}_0 - d_0$. If $\tilde{d}_0 - d_0 \in C(\overline{\Omega})$, it follows that $1 \in D(A^C_{\beta,\mu})$ but $A^C_{\beta,\mu} 1 \neq \tilde{A} 1$. If $\tilde{d}_0 - d_0 \notin C(\overline{\Omega})$, then $1 \notin D(A^C_{\beta,\mu})$. In any case we have $\tilde{A} \neq A^C_{\beta,\mu}$.

Thus the claim follows from Theorem 2.12. □

We conclude this section by showing a blow-up result in the case where we perturb a markovian semigroup $T^C_{\beta,0}$ by a positive μ.

Theorem 4.10. *Assume that $b_j \in W^{1,\infty}(\Omega)$, $j = 1, \dots, d$, the identities*

$$\sum_{j=1}^{d} D_j b_j + d_0 = 0 \text{ almost everywhere on } \Omega \text{ and} \tag{4.7}$$

$$\sum_{j=1}^{d} \operatorname{tr}(b_j)\nu_j + \beta = 0 \text{ almost everywhere on } \partial\Omega. \tag{4.8}$$

and that Ω is connected. If $\mu(z) \geq 0$ for almost all $z \in \partial\Omega$ but not identically 0 almost everywhere, then there exist $\omega, M > 0$ such that

$$\| T^C_{\beta,\mu}(t) \|_{\mathcal{L}(C(\overline{\Omega}))} \geq M e^{\omega t}$$

for all $t > 0$.

Proof. The semigroup $T^C_{\beta,0}$ is markovian by Proposition 2.38 and has an extension to $L^2(\Omega)$ which is irreducible by Proposition 2.15. Now Proposition 2.16 shows that also $T^C_{\beta,0}$ is irreducible.

By Proposition 4.4 we have that $T^C_{\beta,0}(t) \leq T^C_{\beta,\mu}(t)$ for all $t > 0$. Since $\partial_\nu^{\mathscr{A}} 1 + \beta 1 = 0 < \mu(z, \overline{\Omega})$ for almost all z in a set of positive measure, it follows that $1 \notin D(A^C_{\beta,\mu})$. In particular the two semigroups are different and it follows from Theorem 2.12 that $0 = s(A^C_{\beta,0}) < s(A^C_{\beta,\mu}) =: \omega$.

Since $T^C_{\beta,\mu}$ is positive and irreducible, we have that ω is an eigenvalue of $A^C_{\beta,\mu}$ and thus there exists $u \in C(\overline{\Omega})$ with $u \geq 1$ and $A^C_{\beta,\mu} u = \omega u$. But this implies $T^C_{\beta,\mu}(t) u = e^{\omega t} u$ yielding the claim. □

Remark 4.11. A consequence of Theorem 4.10 is that the only realization of our operator with non-local *Neumann* boundary conditions, that is where $\beta = 0$, which generates a submarkovian semigroup is the one with classical (local) Neumann boundary conditions, i.e. $\beta = 0$ and $\mu = 0$.

4.3 Absolutely continuous measures and the semigroup on $L^2(\Omega)$

In this section we consider the case where all the measures $\mu(z)$ are absolutely continuous with respect to Lebesgue measure on Ω. More precisely, we assume that we are given a function $h \in L^2(\partial\Omega \times \Omega)$ such that

$$\mu(z, A) = \int_A h(z, x)\, \mathrm{d}x.$$

In this situation we can use form methods to show that the semigroup $T_{\beta,\mu}$, defined on $L^\infty(\Omega)$, has an extension to $L^2(\Omega)$. This allows us to establish irreducibility of $T^C_{\beta,\mu}$ via Propositions 2.15 and 2.16 in the markovian case, provided Ω is connected. On the other hand, we can use form methods to show that our assumptions to infer positivity resp. sub-markovianity are optimal in this situation.

We consider the form $a_{\beta,h} \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, given by

$$a_{\beta,h}(u, v) := a_\beta(u, v) - \int_{\partial\Omega} \int_\Omega h(z, x) u(x)\, \mathrm{d}x\, \overline{v(z)}\, \mathrm{d}\sigma(z).$$

Then the form $a_{\beta,h}$ is elliptic and continuous. Denote by $A^2_{\beta,h}$ the associated operator on $L^2(\Omega)$; remember that in contrast to the norm in the theory of sesquilinear forms we associate via

$$[A^2_{\beta,h} u, v] = -a_{\beta,h}(u, v)$$

for suitable functions $u, v \in H^1(\Omega)$. Then $A^2_{\beta,h}$ generates a holomorphic, strongly continuous semigroup $T^2_{\beta,h}$ on $L^2(\Omega)$. It is easy to see that if in addition

$$\int_{\partial\Omega} \Big(\int_\Omega |h(z, x)|\, \mathrm{d}x \Big)^p \mathrm{d}\sigma(z) < \infty \tag{4.9}$$

for some $p > \max(2, d-1)$, then the measures $\mu(z) = h(z, x)\, \mathrm{d}x$ satisfy hypothesis (meas) whence we obtain a semigroup $T_{\beta,\mu}$ on $L^\infty(\Omega)$ with generator $A_{\beta,\mu}$ by Theorem 4.1. Using the definition of the co-normal derivative one sees that the part of $A^2_{\beta,h}$ in $L^\infty(\Omega)$ is precisely the operator $A_{\beta,\mu}$. It follows that $T^2_{\beta,h}$ leaves the space $L^\infty(\Omega)$ invariant and the restriction of that semigroup to $L^\infty(\Omega)$ is $T_{\beta,\mu}$.

Proposition 4.12. *With the notation above, we have:*

(a) The semigroup $T^2_{\beta,h}$ is positive if and only if $h \geq 0$ almost everywhere.

(b) Assume $b_j \in W^{1,\infty}(\Omega)$ for $j = 1, \dots, d$. Then the semigroup $T^2_{\beta,h}$ is sub-markovian if and only if and $\sum_{j=1}^d D_j b_j + d_0 \leq 0$ almost everywhere in Ω,

$h \geq 0$ almost everywhere , $\int_\Omega h(z,x)\,\mathrm{d}x \leq \beta(z) + \sum_{j=1}^d \operatorname{tr} b_j(z)\nu_j(z)$ for almost every $z \in \partial\Omega$ and .

Proof. (a) By the first Beurling–Deny crition [Ouh05, Theorem 2.6] $T^2_{\beta,\mu}$ is positive if and only if $a_{\beta,\mu}(u^+, u^-) \leq 0$ for all $u \in H^1(\Omega)$. If $h \geq 0$ almost everywhere this is clearly fulfilled.

Conversely assume that $T^2_{\beta,\mu}(t) \geq 0$ for all $t > 0$. Then

$$\int_{\partial\Omega}\int_\Omega h(z,x)u^+(x)\,\mathrm{d}x\,u^-(z)\,\mathrm{d}\sigma(z) = -a_{\beta,h}(u^+,u^-) \geq 0$$

for all $u \in H^1(\Omega)$. Now let functions $0 \leq v \in C_c^\infty(\Omega)$ and $0 \leq \varphi \in C(\partial\Omega)$ be given. We find a sequence $w_n \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq w_n \leq \|\varphi\|_\infty$ such that $\operatorname{supp} w_n \cap \operatorname{supp} v = \emptyset$ and $w_n(z) \to \varphi(z)$ for all $z \in \partial\Omega$. Inserting $u = v - w_n$ in the above inequality and using dominated convergence, we obtain that

$$\int_{\partial\Omega}\int_\Omega h(z,x)v(x)\,\mathrm{d}x\,\varphi(z)\,\mathrm{d}\sigma(z) \geq 0$$

As $0 \leq \varphi \in C(\partial\Omega)$ was arbitrary, we conclude that

$$\int_\Omega h(z,x)v(x)\,\mathrm{d}x \geq 0$$

for almost all $z \in \partial\Omega$. As $0 \leq v \in C_c^\infty(\Omega)$ was arbitrary, it follows that for almost all $z \in \partial\Omega$ we have $h(z,x) = 0$ for almost all $x \in \Omega$. Now Fubini's theorem implies that $h \geq 0$ with respect to the product measure, proving the necessity of the condition.

(b) The sufficiency of the inequality above was already established in Proposition 4.5(b).

Now we prove its necessity. If the semigroup is sub-markovian, it is positive and thus $h \geq 0$ almost everywhere by (a).

By the Beurling–Deny–Ouhabaz criterion [MVV05, Corollary 2.8], for $u \in H^1(\Omega)$ we have

$$\begin{aligned}
0 &\leq a_{\beta,h}(u \wedge 1, (u-1)^+) \\
&= \sum_j \int_\Omega (D_j b_j)(u-1)^+\,\mathrm{d}x + \int_\Omega d_0 (u-1)^+\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega}\Big(\sum_j b_j\nu_j(u-1)^+ + \beta - \int_\Omega (u\wedge 1)(x)h(\cdot,x)\,\mathrm{d}x\Big)(u-1)^+\,\mathrm{d}\sigma.
\end{aligned}$$

Now let $v \in H^1(\Omega)$ such that $v \geq 0$. Inserting $u = v + 1$ in the above inequality, the desired inequalities follow from Lemma 2.37.

We have already noted after Proposition 4.5 that the condition $\sum_{j=1}^d D_j b_j + d_0 \leq 0$ almost everywhere in Ω is necessary for $T_{\beta,\mu}$ to be sub-markovian. □

We now consider the case where the semigroup is submarkovian. Then we can prove irreducibility via Corollary 2.15 and exponential stability via Theorem 4.9.

Theorem 4.13. *Assume that Ω is connected, $b_j \in W^{1,\infty}(\Omega)$ for $j = 1, \dots, d$, $h \in L^p(\partial\Omega; L^2(\Omega))$ with $p > \max\{d-1, 2\}$, $h \geq 0$ and that $\sum_{j=1}^d D_j b_j + d_0 \leq 0$ almost everywhere on Ω and*

$$\sum_{j=1}^{d} b_j(z)\nu_j(z) + \beta(z) \geq \int_\Omega h(z,x)\,\mathrm{d}x$$

almost everywhere on $\partial\Omega$. If one of the two inequalities above is strict on some set of positive measure, then there exist $\epsilon, M > 0$ such that

$$\|T_{\beta,\mu}(t)\|_{\mathscr{L}(C(\overline{\Omega}))} \leq Me^{-\epsilon t}$$

for all $t > 0$ where $\mu(z) = h(z,x)\,\mathrm{d}x$.

In the same vein one can deduce convergence of the semigroup to an equilibrium using Theorem 4.8.

Theorem 4.14. *Assume that Ω is connected, $b_j \in W^{1,\infty}(\Omega)$ for $j = 1, \dots, d$, $h \in L^p(\partial\Omega; L^2(\Omega))$ with $p > \max\{d-1, 2\}$, $h \geq 0$ almost everywhere and that $\sum_{j=1}^d D_j b_j + d_0 = 0$ almost everywhere on Ω and*

$$\sum_{j=1}^{d} b_j(z)\nu_j(z) + \beta(z) = \int_\Omega h(z,x)\,\mathrm{d}x$$

almost everywhere on $\partial\Omega$. Then the semigroup $T^C_{\beta,\mu}$ on $C(\overline{\Omega})$ is irreducible and markovian where $\mu(z) = h(z,x)\,\mathrm{d}x$. Consequently, there exist $0 \ll \varphi \in L^2(\Omega)$ such that $\int_\Omega \varphi(x)\,\mathrm{d}x = 1$ and constants $\epsilon, M > 0$ such that

$$\|T^C_{\beta,\mu}(t) - \varphi \otimes 1\|_{\mathscr{L}(C(\overline{\Omega})} \leq Me^{-\epsilon t}$$

for all $t > 0$.

But actually it is possible to extend this result to the general case of Ωbeing not connected. We first want to derive conditions which imply that $T^2_{\beta,\mu}$ is irreducible using Theorem 2.13.

Assume that $h \geq 0$ and denote by U_j, $j \in J$, J at most countable, the pairwise disjoint connected components of Ω. Let $\omega \subset \Omega$ be measurable such that $1_\omega u \in H^1(\Omega)$ for all $u \in H^1(\Omega)$. Then Proposition 2.14 implies that ω must be a union of connected components of Ω except for null sets; we assume now w.l.o.g. that $\omega = \bigcup_{j \in J'} U_j$ for some $J' \subset J$. Suppose now that $J' \neq \emptyset$ and $J' \neq J$ and set $U := \bigcup_{j\in J'} U_j$ and $V := \Omega \setminus U$. We want to construct a function $u \in H^1(\Omega)$ such that

$$a_{\beta,h}(1_\omega u, 1_{\Omega\setminus\omega} u) = \int_{\partial V}\int_U u(x)h(z,x)\,\mathrm{d}x\,u(z)\,\mathrm{d}\sigma(z) < 0.$$

Clearly this is possible if and only if $u \in H^1(\Omega)$ with $u|_V = -1$ and $u|_U = 1$ does the job since we assumed that $h \geq 0$. We then have $a_{\beta,h}(1_\omega u, 1_{\Omega\setminus\omega}u) < 0$ if and only if the projection of h to $L^2(\partial V \times U)$ is not zero, that is $1_{\partial V\times U} h \neq 0$. Hence we have proved the following characterization of irreducibility for the semigroup $T^2_{\beta,h}$ on $L^2(\Omega)$.

Proposition 4.15. *Let $h \geq 0$ and U_j, $j \in J$, J at most countable, be the pairwise disjoint connected components of Ω. Then the positive semigroup $T^2_{\beta,h}$ is irreducible on $L^2(\Omega)$ if and only if for all $J' \subset J$ with $J' \neq 0$, $J' \neq J$ we have*

$$1_{\partial V(J')\times U(J')} h \neq 0,$$

where $U(J') := \bigcup_{j\in J'} U_j$ and $V(J') := \Omega \setminus U(J')$.

Now of course the semigroup $T^C_{\beta,0}$ with local boundary conditions is not irreducible if Ω is not connected and hence Theorem 4.8 is not usable directly. But if one inspects the proof we only need the irreducibility of $T^C_{\beta,0}$ to prove the one of $T^C_{\beta,\mu}$; in this case here we have $\mu(z) = h(z,x)\,\mathrm{d}x$.

Now in the markovian case, that is assuming that $b_j \in W^{1,\infty}(\Omega)$ for $j = 1,\dots,d$, $h \in L^p(\partial\Omega; L^2(\Omega))$ with $p > \max\{d-1, 2\}$, $h \geq 0$ almost everywhere and that $\sum_{j=1}^d D_j b_j + d_0 = 0$ almost everywhere on Ω and

$$\sum_{j=1}^d b_j(z)\nu_j(z) + \beta(z) = \int_\Omega h(z,x)\,\mathrm{d}x$$

almost everywhere on $\partial\Omega$, we can use Proposition 2.16 to prove irreducibility instead since then we have $\operatorname{lin}\{1_{\overline{\Omega}}\} = \ker A^C_{\beta,\mu}$. The following extension of Theorem 4.14 to Ω being unconnected summarizes the considerations above.

Theorem 4.16. *Let U_j, $j \in J$, J at most countable, be the pairwise disjoint connected components of Ω. Assume that $b_j \in W^{1,\infty}(\Omega)$ for $j = 1,\dots,d$, $h \in L^p(\partial\Omega; L^2(\Omega))$ with $p > \max\{d-1, 2\}$, $h \geq 0$ almost everywhere and that $\sum_{j=1}^d D_j b_j + d_0 = 0$ almost everywhere on Ω and*

$$\sum_{j=1}^d b_j(z)\nu_j(z) + \beta(z) = \int_\Omega h(z,x)\,\mathrm{d}x$$

almost everywhere on $\partial\Omega$.

Then the semigroup $T^C_{\beta,\mu}$, where $\mu(z) = h(z,x)\,\mathrm{d}x$, is irreducible on $C(\overline{\Omega})$ if and only if h is such that for all $J' \subset J$ with $J' \neq 0$, $J' \neq J$ we have

$$1_{\partial V(J')\times U(J')} h \neq 0,$$

where $U(J') := \bigcup_{j\in J'} U_j$ and $V(J') := \Omega \setminus U(J')$.

In that case the semigroup $T^C_{\beta,\mu}$ is markovian and consequently, there exist $0 \ll \varphi \in L^2(\Omega)$ such that $\int_\Omega \varphi(x)\,\mathrm{d}x = 1$ and constants $\epsilon, M > 0$ such that

$$\|T^C_{\beta,\mu}(t) - \varphi \otimes 1\|_{\mathscr{L}(C(\overline{\Omega}))} \leq Me^{-\epsilon t}$$

for all $t > 0$.

The condition on h characterizing the irreducibility may seem a bit unwieldy but in fact it is almost the same condition that characterizes irreducibility for the non-local Dirichlet boundary condition. To see this define the graph $G := (\mathcal{V}, E)$ on the connected components U_j, $j \in J$, of Ω as

$$\begin{aligned} \mathcal{V} &= \{U_j,\ j \in J\} \\ E &= \{(U_i, U_j) \in V \times V,\ \int_{\partial U_i}\int_{U_j} h(z,x)\,\mathrm{d}x\,\mathrm{d}\sigma(z) > 0\} \end{aligned}$$

Then we can phrase the condition on h in terms of this graph as follows: We have

$$\forall J' \subset J,\ J' \neq 0,\ J' \neq J:\ 1_{\partial V(J')\times U(J')} h \neq 0,$$

where $U(J') := \bigcup_{j\in J'} U_j$ and $V(J') := \Omega \setminus U(J')$, if and only if we have

$$\forall J' \subset J,\ J' \neq 0,\ J' \neq J:\ 1_{\partial V\times U} h \neq 0,$$

for a particular pair $V \in V(J'), U \in U(J')$. This second formulation is clearly equivalent to the statement

$$\forall \mathcal{W} \subset \mathcal{V},\ \mathcal{W} \neq \emptyset,\ \mathcal{W} \neq \mathcal{V}\ \exists (U,V) \in E:\ U \in \mathcal{W} \text{ and } V \in \mathcal{V} \setminus \mathcal{W};$$

that is, if we partition the graph into two parts, we can always find an edge between the two parts. Finally an induction argument shows that this is equivalent to the graph G being strongly connected. Summarizing the thoughts above we have the following result on the irreducibility of the semigroup phrased in the same vein as Corollary 3.33.

Corollary 4.17. *Let $h \geq 0$. Then the following holds:*

(a) The semigroup $T^2_{\beta,h}$ is irreducible on $L^2(\Omega)$ if and only if the graph G is strongly connected.

(b) *Assume that* $b_j \in W^{1,\infty}(\Omega)$ *for* $j = 1, \dots, d$, $h \in L^p(\partial\Omega; L^2(\Omega))$ *with* $p > \max\{d-1, 2\}$, $\sum_{j=1}^d D_j b_j + d_0 = 0$ *almost everywhere on* Ω *and* $\sum_{j=1}^d b_j(z)\nu_j(z) + \beta(z) = \int_\Omega h(z,x)\,\mathrm{d}x$ *almost everywhere on* $\partial\Omega$. *Then the semigroup* $T^C_{\beta,\mu}$ *with* $\mu(z) = h(z,x)\,\mathrm{d}x$ *is irreducible on* $C(\overline{\Omega})$ *if and only if G is strongly connected.*

Note that in contrast to the situation in Chapter 3, we do not require that $\mu(z)$ is strictly positive for (almost) all $z \in \partial\Omega$.

Unfortunately we are unable to generalize Theorem 4.13 in a similar way, since in the submarkovian case it seems not so easy to provide a strictly positive eigenfunction of $A^C_{\beta,\mu}$ and thus also Proposition 2.16 cannot be applied.

Remark. Corollary 4.17 (b) also lets us extend Theorem 4.10 to the case where Ω is not connected. In fact let the assumptions of Corollary 4.17 (b) hold such that $T^C_{\beta,\mu}$ is irreducible. Now set $\mu'(z) = h(z,x)\,\mathrm{d}x + h'(z,x)\,\mathrm{d}x$ where $0 \le h' \in L^p(\partial\Omega; L^2(\Omega))$ with $h'(z,x) > 0$ on a set of positive measure. Since $1 \in D(A^C_{\beta,\mu})$ but $1 \notin D(A^C_{\beta,\mu'})$ using Theorem 2.12 yields that $0 = s(A^C_{\beta,\mu}) < s(A^C_{\beta,\mu'}) =: \omega$. Repeating the final step of the proof of Theorem 4.10 again yields the claim about blow-up.

4.4 Measures $\mu(z)$ satisfying (meas)

In this brief section we give some examples of maps μ for which hypothesis (meas) is satisfied.

Example 4.18. Assume that for every $B \in \mathscr{B}(\overline{\Omega})$ the complex-valued map $\partial\Omega \to \mathbb{C}$, $z \mapsto \mu(z,B)$ is continuous. Then μ satisfies conditions (M1), (M2) and (M3) in hypothesis (meas).

Proof. It is obvious that (M1) holds. As for (M2), we note that by continuity and compactness of $\partial\Omega$ we have $\sup_{z\in\partial\Omega} |\mu(z,B)| < \infty$ for every $B \in \mathscr{B}(\overline{\Omega})$. Now [Bog06, Corollary 4.6.4] yields $\sup_{z\in\partial\Omega} \|\mu(z)\| < \infty$. To prove (M3), pick a dense sequence z_n in $\partial\Omega$. We set

$$\tau := \sum_{n\in\mathbb{N}} \frac{1}{2^n} |\mu(z_n)|,$$

where $|\mu(z)|$ denotes the total variation of $\mu(z)$. Then τ is a finite positive measure and we have $\mu(z_n) \ll \tau$ for every $n \in \mathbb{N}$. Let $B \in \mathscr{B}(\overline{\Omega})$ with $\tau(B) = 0$ be given. Consider the function $\varphi(z) := \mu(z,B)$. By the above $\varphi(z_n) = 0$ for all $n \in \mathbb{N}$. Moreover, φ is continuous by assumption. Thus $\varphi \equiv 0$, proving that in fact $\mu(z) \ll \tau$ for all $z \in \partial\Omega$. □

Remark. Since the simple functions are dense in $B_b(\overline{\Omega})$, it is clear that the assumption in the example above is fulfilled if and only if $\partial\Omega \to \mathbb{C}$, $z \mapsto \langle \mu(z), f \rangle$ is continuous for all $f \in B_b(\overline{\Omega})$. If one extends the concept of strong Feller operators to operators between the bounded and measurable functions on two different measure spaces, one may say the following: the assumptions on (meas) are fulfilled, if $B_b(\overline{\Omega}) \to B_b(\partial\Omega)$, $f \mapsto \langle \mu(\cdot), f \rangle$ is strongly Feller.

Similarly, we can consider maps μ which only take countably many values.

Example 4.19. Assume that $\mu(z) = \sum_{n\in J} 1_{A_n}(z)\mu_j$ where $(A_n)_{j\in J} \subset \mathscr{B}(\partial\Omega)$, $(\mu_j)_{j\in J} \subset \mathscr{M}(\overline{\Omega})$ and J is a finite or countably infinite index set. Then μ satisfies hypothesis (meas) provided $\sum_{n\in J} \sigma(A_n)|\mu_n|(\overline{\Omega})^p < \infty$ where p is as in hypothesis (meas) (M2).

Proof. Part (M1) is obvious and (M2) was assumed. Part (M3) is fulfilled with $\tau = \sum_{n\in J} 2^{-n}|\mu_n|$. □

The same argument holds, if μ is allowed to scale these countably many measures.

Example 4.20. Assume that $\mu(z) = \sum_{n\in J} \lambda(z) 1_{A_n}(z)\mu_j$ where $\lambda \in B_b(\partial\Omega)$, $(A_n)_{j\in J} \subset \mathscr{B}(\partial\Omega)$, $(\mu_j)_{j\in J} \subset \mathscr{M}(\overline{\Omega})$ and J is a finite or countably infinite index set. Then μ satisfies hypothesis (meas) provided $\sum_{n\in J} \sigma(A_n)|\mu_n|(\overline{\Omega})^p < \infty$ where p is as in hypothesis (meas) (M2).

4.5 Notes and comments

The first section is from [AKK, Section 4]. A slight generalization is given here in that our assumption (meas) (M3) does not suppose that τ is bounded, but instead that $L^1(\Omega, \tau)$ is separable. Now if one is interested in non-local boundary conditions given by point measures (as e.g. in the classical problem Bitsadze and Samarskii considered in [BS69]), then one may be tempted to remove the restriction of $L^1(\Omega, \tau)$ being separable and use the counting measure on $\overline{\Omega}$ as τ. But then not only does the given generation proof fail, but also the statement of the boundary perturbation ϕ being compact is wrong. Indeed if $\partial\Omega$ is smooth and $\mu(z) = \delta_{\omega(z)}$ with $\omega\colon \partial\Omega \to \omega(\partial\Omega) \subset \Omega$ a diffeomorphism, then any bounded sequence of continuous periodic functions on $[0,1]^{d-1} \subset \mathbb{R}^{d-1}$ such that no subsequence converges in $L^p([0,1]^{d-1})$ gives rise to a counterexample to the compactness of $\phi\colon C(\overline{\Omega}) \to L^p(\partial\Omega)$ (which in this case is defined as $(\phi u)(z) = u(\omega(z))$).

The results on the asymptotic behavior conditional on the irreducibility of the semigroup with local boundary conditions of the second section is taken from [AKK, Section 5].

The third section on the non-locality given by absolutely continuous measures (w.r.t. the Lebesgue measure) is adapted from [AKK, Section 6]. The characterization of the irreducibility of $T^2_{\beta,h}$ and result on the exponential decay of $T^C_{\beta,\mu}$, $\mu(z) = h(z,x)\,\mathrm{d}x$, to an equilibrium in the case of Ω being not connected are new.

The last section is from [AKK, Section 7], except for Example 4.20.

Notation Index

$\mathbb{C}^+$	open right half-plane of $\mathbb{C}$
$\mathbb{C}^-$	open left half-plane of $\mathbb{C}$
$\mathbb{R}^+$	the interval $(0, \infty)$
u^+	maximum of u and 0 in a vector lattice
u^-	minimum of u and 0 in a vector lattice
V_+	positive cone of the ordered vector space V
$\mathscr{B}(K)$	Borel σ-algebra of a topological measure space K
$B_b(K)$	bounded and measurable functions on the metric measuere space K
$C(K)$	continuous functions on the topological space K
$\mathcal{M}(K)$	complex Borel measures on the measure space K
$\rho(A)$	resolvent set of the operator A
$\sigma(A)$	spectrum of the operator A
$\sigma_p(A)$	point spectrum of the operator A
$s(A)$	spectral bound of the operator A
$R(\lambda, A)$	resolvent in $\lambda \in \rho(A)$ of the operator A
$\omega(T)$	growth bound of the semigroup T
X′	Dual space of the Banach space X
$\langle T, x \rangle$	duality bracket for $T \in X'$, $x \in X$, i.e. $T(x)$
$[u, v]$	scalar product on the Hilbert space H

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