Portfolio Optimization with Risk Constraints

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Contents

1 Introduction
  1.1 Problem Description ........................................ 1
  1.2 Main Results ................................................ 2
  1.3 Outline ...................................................... 4
  1.4 Prior Art .................................................... 5

2 Risk Measures .................................................. 7
  2.1 Definitions .................................................. 7
     2.1.1 Fundamental Definitions and Properties ............... 7
     2.1.2 Definition of Some Important Risk Measures ........... 9
     2.1.3 Coherent Risk Measures ................................ 11
  2.2 Discussion of Several Risk Measures ......................... 11
     2.2.1 Value at Risk ........................................... 11
     2.2.2 Expected Shortfall ..................................... 12
     2.2.3 Connection Between Expected Shortfall and Conditional Value at Risk .. 13
     2.2.4 Conditional Value at Risk ............................. 14

3 Static Optimization Problems with Risk Constraints .......... 17
  3.1 Problem Statement .......................................... 17
     3.1.1 Assumptions ............................................ 17
     3.1.2 The Optimization Problems ............................. 18
     3.1.3 Some Notation .......................................... 19
  3.2 Feasibility ................................................... 20
     3.2.1 Value at Risk Problem ................................ 20
     3.2.2 Expected Shortfall Problem ............................ 21
     3.2.3 Conditional Value at Risk Problem ..................... 24
  3.3 Optimal Solutions and Their Properties ...................... 25
     3.3.1 Assumptions ............................................ 25
     3.3.2 Value at Risk Problem ................................ 27
     3.3.3 Expected Shortfall Problem ............................ 29
     3.3.4 Conditional Value at Risk Problem ..................... 34
  3.4 How to Prove Optimality ................................... 35
     3.4.1 Approach ............................................... 35
     3.4.2 Value at Risk Problem ................................ 37
     3.4.3 Expected Shortfall Problem ............................ 43
     3.4.4 Conditional Value at Risk Problem ..................... 59
## CONTENTS

4 Dynamic Optimization Problems with Risk Constraints 70
  4.1 Problem Statement ........................................... 70
    4.1.1 Market Model ........................................... 70
    4.1.2 Option Pricing and Hedging .............................. 73
    4.1.3 The Optimization Problems .............................. 74
  4.2 Optimal Strategies and Their Properties .................. 74
    4.2.1 Classical Problem without Risk Constraint ............ 75
    4.2.2 Artificial One-Stock Market ............................ 75
    4.2.3 Some Notation ......................................... 76
    4.2.4 Value at Risk Problem .................................. 76
    4.2.5 Expected Shortfall Problem ............................. 80
    4.2.6 Conditional Value at Risk Problem ...................... 90
  4.3 Equivalent Problem Statements for Other Solution Techniques 93
  4.4 Outlook ...................................................... 98
    4.4.1 Alternative Solution Methods: Hamilton-Jacobi-Bellman Approach 98
    4.4.2 Incomplete Markets ...................................... 100

5 Summary and Discussion ........................................ 101

6 Proofs .......................................................... 103
  6.1 Part 2: Risk Measures .......................................... 103
  6.2 Part 3: Static Optimization Problems with Risk Constraints 108
  6.3 Part 4: Dynamic Optimization Problems with Risk Constraints 139

Bibliography ......................................................... 150

List of Definitions ................................................ 152

List of Figures ..................................................... 155

Acknowledgements .................................................. 156

Zusammenfassung .................................................... 157
Part 1

Introduction

Imagine a world without uncertainties: Everybody would know everything, the whole history, present, and future of mankind and the universe. No decisions were to be made, nothing new to discover or experience, no surprises to happen — life would be totally boring. Luckily, the world is entirely different: Uncertainties are omnipresent and nobody knows what the future brings. Instead there is a longing for some sort of safety that created a whole industry: it is the business of insurance companies to bear financial risks. But even these specifically designed companies are unable to give a 100% guarantee, because they might find themselves in financial distress. So we live in a risky environment and there is no way to avoid risks completely. Facing this inability to get rid of all risk, we have the choice between two options: We can either try to minimize all risks to bear or we can attempt to expose ourselves only to certain acceptable risks. However, minimizing the risks is not the way people behave in many situations: For example, we travel and, even if we could reach our destination in time by train, we often choose more risky means like driving our own car. In reality, we prefer the second option: We are willing to expose ourselves to some risks, but we try to avoid taking others: If we neither mind taking the risk of traveling by car nor of using the train, our choice is based on factors like convenience, fun, or price.

1.1 Problem Description

Let us translate these fundamental observations into the world of financial mathematics: The usual subject is an investor trying to invest in a market in an optimal way. Now, we expect the investor not to be willing to take on any risk. Instead, he selects his investment strategy only among strategies that have an acceptable risk. Therefore, the investor faces an optimization problem that includes a risk constraint of some sort.

Most literature on portfolio optimization either ignores the risk completely or the risk is part of the objective function. The goal of this work is to study the effects caused by the addition of a risk constraint, which captures the idea of only choosing among strategies with acceptable risks. More precisely, we consider a class of utility based portfolio optimization problems without risk constraint and want to explore how the addition of a risk constraint affects the optimal solution.

As basis we use the following classical optimization problem: Maximize the expected terminal utility at a specific fixed time in the future subject to an initial endowment restriction. The endowment restriction is the following: The investor can only hold positions in the market that can be financed with his initial endowment, because he starts out with a certain initial amount of capital and is assumed to have no exogenous income, neither positive nor negative, i.e. funds or obligations that
are not the result of his trading in the market.

We modify that classical problem by adding an additional side condition: All admissible strategies must yield a wealth structure, which corresponds to a random variable, that fulfills a given risk restriction. In particular, we study the following type of risk constraints: we deal with upper limits on the value at risk, the expected shortfall, or the conditional value at risk of the future terminal wealth. An example of a value at risk constraint is the requirement that the 5% quantile of the wealth structure must be above a given threshold. An expected shortfall constraint limits the expectation of the portion of the wealth structure that is below a certain level. A limit on the average of the wealth structure taken over its worst for instance 3% of its outcomes is imposed by a conditional value at risk constraint.

1.2 Main Results

Our intention is to determine the optimal behavior of the investor. The method of choice is the martingale method. This approach is a two-step optimization process consisting of the search for the optimal position in the market in the first step and the derivation of its hedging strategy as the second step. It is a well-established method, which was used for instance by [Karatzas et al., 1987] and [Cox and Huang, 1989] to solve optimization problems without risk constraints in a similar setting. Motivated by this method, we investigate the optimization problems in a general static setting, first: We strip the problem to the core by getting rid of all unnecessary noise like discount factors, etc., and focus on what it boils down to: the relationship between two measures, namely the real world probability measure and a pricing measure. In this model, the search for an optimal solution corresponds to the optimal selection of a random variable describing the future terminal wealth. The objective function is the expectation of the contingent terminal utility, which is the utility of the future wealth. The expectation is taken with respect to the real world measure. The purpose of the pricing measure is to assess the present values of stochastic future wealths to verify their compliance with the initial endowment restriction. Naturally, the question arises which one of these two measures should be chosen for the additional risk constraint. If the risk constraint is chosen by the investor himself, we might argue that it should use the investor’s subjective measure, which is the probability measure. If the constraint is imposed by some external regulator or supervisor, which is not likely to rely on the subjective point of view of the investor, we are inclined to propose some neutral measure, which would be the pricing measure. Since we feel that there is no best single answer to this question, we consider both cases for each type of risk measure. A generalization left for future research is to investigate the use of a general third measure for the risk constraint.

Within the static framework, we derive conditions for the feasibility of each one of the problems. A problem is called feasible if there exists at least one random variable that fulfills the constraints imposed by the problem and results in a well-defined objective function. Such random variables are called admissible. The question of feasibility of problems under consideration is often overlooked in the literature and previous treatments of risk constraint optimization problems are no exceptions to this rule.

All non-feasible problems do not have any solution, yet it does not mean that all feasible ones automatically have one: The existence of an admissible random variable does not imply the existence of an optimal one. Thus, we also give necessary and sufficient conditions for the existence and the uniqueness of solutions. Moreover, we calculate explicit formulas for solutions for each risk constraint. For the value at risk problems and expected shortfall problems, solution candidates can be found using a Lagrange approach. However, we need a different idea for the conditional value at risk problems.
A connection between a conditional value at risk constraint and expected shortfall constraints leads to the insight that we can replace the conditional value at risk problem by a family of expected shortfall problems. To derive solutions of the conditional value at risk problem, we consider the set of solutions of this family of expected shortfall problems: Those solutions with the highest objective function value within this set are solutions of the conditional value at risk problem. Of course, we explore properties of the solutions. Besides their behavior, the probably most important property is the existence of static investment strategies that achieve the desired optimal terminal wealth. A static investment strategy is an investment strategy that can be implemented simply by buying or selling some specific instruments that might be provided by the market such as options on some underlying.

The second market model under consideration is a dynamic one: We take a look at a classical complete market model with one bond and several stocks driven by a multi-dimensional Brownian motion. The behavior in this model is characterized by a multi-dimensional stochastic process describing the invested amounts in each asset. This investment portfolio can be rearranged continuously during a finite period of time. We restrict ourselves to the power and logarithmic utility functions, because only by using an explicit utility function, we can expect an explicit formula describing the optimal behavior. Furthermore, this setup contains a unique pricing measure. Thus simply by fixing a real world measure we have an example of a model fitting into the framework of our previous general setting. Hence, we can make use of the insights gained in those static optimization problems. We are able to immediately cover the questions of feasibility of problems with risk constraints and existence of optimal strategies, as well as to calculate the optimal terminal wealths. It is well known, how to derive an investment strategy in the given dynamic market model that results in a targeted stochastic terminal wealth. Selecting an optimal terminal wealth as target and determining the corresponding investment strategy is the second phase of the martingale method. The resulting investment strategy is an optimal one.

Instead of performing tedious calculations, we introduce a specific artificial stock in the market: the optimal wealth process of the problem without risk constraint. An artificial stock is a derivative having all the characteristics of a stock price process. From the results of the static optimization problems we learn that the optimal solutions can be represented as sums of options with this artificial stock as underlying. Hence, the optimal strategies can be broken apart in the very same manner and we can rely on the well-known hedging strategies for options. This idea works for all cases, with one exception: one portion of the optimal terminal wealth of the problems with expected shortfall constraint or conditional value at risk constraint with respect to the real world measure needs a special treatment: Here, brute force calculations cannot be avoided.

Fortunately, we always get explicit forms of the optimal investment strategies. This facilitates the discussion of several of their properties a lot. We focus on the connections to the problem without risk constraint, the influence of the parameters of the risk constraints, the dependence of the initial or current wealth, and the tracking error arising if trades are only allowed at a finite number of times yet the optimal strategy of the continuous model is implemented.

For problems with value at risk constraints with respect to the real world measure and problems with expected shortfall constraints, the derivation of candidates for the optimal solutions has been done by [Basak and Shapiro, 2001] and [Gabi and Wunderlich, 2004] within their market models. They verified the optimality of their candidates, if certain Lagrange multipliers exist and the utility function is twice continuously differentiable. Neither the question of the existence of the Lagrange multipliers has been investigated by the authors nor the question of uniqueness of the optimal solutions. We are able to weaken both their assumptions in our very general static model, which is applicable in jump-diffusion markets, too, at least if the markets are complete or can be completed. In addition to the problems considered in these papers, we treat value at risk constraints with respect to the pricing measure and conditional value at risk constraints.
A very common alternative approach to derive optimal solutions is the so-called Hamilton-Jacobi-Bellman approach. We reformulate the optimization problems to change them into a form suitable for this approach or for the stochastic maximum principle. We were unable to find any previous literature treating such a reformulation. Since we can calculate the optimal strategies explicitly using the martingale method, we can easily derive solutions of the differential equations appearing in these other approaches by exploiting these explicit formulas. We content ourselves with a rough treatment of the Hamilton-Jacobi-Bellman approach in the case of a logarithmic utility function and an expected shortfall restriction with respect to the pricing measure. The exact details and the other cases are left for future research.

Finally, we try to give an answer to the question which one of the discussed risk measures is suited the most for applications in the real world. The answer is based on a comparison of their individual pros and cons with respect to some general considerations and of course on their influence on the optimal strategies and wealths.

1.3 Outline

The structure of this work is given by parts, which are divided into chapters. The last chapter of this introductory part lists earlier work on risk constraint optimization within continuous market models.

Part 2 begins with a presentation of some required notation and a list of basic properties. Subsequently, we define our notion of a risk measure in general and of several specific risk measures in particular. Next, we look at some of the implications of the definition of the value at risk, expected shortfall and conditional value at risk. Since the exact definition of the first two risk measures is rather obvious, we only try to justify our choice of a definition of the conditional value at risk by comparing it with similar risk measures found in the literature. For the solution of the upcoming optimization problems with conditional value at risk constraint, Theorem 2.9 on page 13 contains the key result: It presents a link between a conditional value at risk constraint and expected shortfall constraints.

After these preparations, we consider static optimization problems with constraints in Part 3. Only few assumptions are required to state the optimizations problems, which is done in Chapter 3.1. These assumptions, however, are not sufficient to guarantee the feasibility of the problems. Thus we list in Chapter 3.2 additional requirements for each problem type, which are both necessary and sufficient for its feasibility. The corresponding proofs are of interest, because they contain for each case an example of an admissible random variable. For each problem type, we characterize the optimal solutions and give necessary conditions for their existence. Furthermore, we look at the uniqueness of the optimal solutions and discuss several properties. All the results are presented in Chapter 3.3 and the proofs are deferred to Chapter 3.4.

In Part 4, we investigate risk constraint optimization in a classical complete financial market with deterministic coefficients. We present the market in Chapter 4.1 and discuss the corresponding optimization results in Chapter 4.2. For each type of restriction, we look at several properties of the optimal strategy. Chapter 4.3 shows that the problems with risk constraints can be reformulated such that other solution methods can be used. The details of the actual applications of other solution methods would go beyond the scope of this work, so we content ourselves with the verification of the Hamilton-Jacobi-Bellman equation in a special case to show that such an approach is feasible in principal. That verification is done in Section 4.4.1 of Chapter 4.4, which contains perspectives for future research.
A summary of our findings is given in Part 5, which focuses on the fundamental question which risk constraint should be chosen in what situation. To make the presentation clearer, several proofs are skipped throughout the work. They can be found in Part 6. Finally, the bibliography, a list of definitions as well as a list of figures, some acknowledgements, and a summary in German complete this work.

1.4 Prior Art

In the literature, several approaches to incorporate risk constraints in continuous trading models can be found. We list them in decreasing order of proximity to the present work:

[Basak and Shapiro, 2001] look at a Black-Scholes market with a finite number of stocks and constant coefficients. Given a twice continuously differentiable utility function, they look at the implications of a value at risk constraint with respect to the real world measure and an expected shortfall constraint with respect to a pricing measure within their framework. Provided that there exist Lagrange multipliers with certain properties, they establish the form of an optimal terminal wealth and its corresponding optimal investment strategy. The fundamental question of feasibility of the risk constraint problems is not raised. In a footnote on page 386, the authors comment on how a solution with an expected shortfall constraint with respect to the real world measure looks like. They state a correct optimal terminal wealth and claim that the “nature of the implications” are the same as in case of an expected shortfall constraint with respect to the pricing measure. That point of view, we only share partially: The optimal wealth can no longer be characterized as a sum of options on some underlying and in adverse states of nature the terminal value differs drastically, i.e. it is significantly lower.

[Gabih and Wunderlich, 2004] essentially try to carry out the case mentioned in that footnote in more detail. They restrict themselves to the classical Black-Scholes market with one stock and constant coefficients. In the spirit of the predecessor paper [Basak and Shapiro, 2001], they, too, prove optimality of an optimal terminal wealth given the existence of certain Lagrange multipliers. Again, the question of feasibility, the existence and computation of these multipliers, as well as the solutions in special cases are not dealt with. In addition, the authors compare their results with the results of the value at risk constraint case of [Basak and Shapiro, 2001] predominantly using numerical results.

Another approach to model risk constraints in a continuous market model with constant coefficients can be found in [Cuoco et al., 2001]: They restrict the set of admissible portfolio weights at each point in time $t$ by insisting that if the portfolio weights would remain constant over the upcoming interval from $t$ to $t + \tau$, the value at risk or tail conditional expectancy of the position at time $t + \tau$ would be below a given threshold. That hypothetical position at time $t + \tau$ has a continuous distribution, a fact that facilitates the calculation of the value at risk and tail conditional expectancy. In essence, the restrictions reduce to lower and upper bounds for the portfolio weights that only depend on the current wealth and the current time. The goal is to maximize the expected terminal utility and, in order to deal with such local constraints, they formulate a Hamilton-Jacobi-Bellman equation. In the case of the power utility function and constant bounds, i.e. bounds independent of the current wealth and time, it turns out that the optimal strategy is the known optimal strategy of the problem with power utility function yet with a different exponent and without the risk constraint. They consider other cases, too, yet have to resort to numerical computations. Note the implications of their risk restriction in the following static case: If the portfolio only consists of a derivative on
1.4. PRIOR ART

A pure stock like an option for instance, the basis for judging the risk of this investment is not its payoff structure but its hedging strategy. Now recall that for example European Binary Options have very extreme (i.e. unbounded) hedging strategies.

[Emmer et al., 2001] select an optimal fixed proportional strategy (constant over the entire horizon) given the following capital at risk constraint: The difference between the future value of a riskless investment and the $\alpha$-quantile of the future value of a portfolio with a fixed proportional strategy is restricted.

Convex duality methods are another tool to investigate portfolio optimization problems. For instance the paper [Cvitanić and Karatzas, 1992] considers proportional investment strategies that are almost in each state and point in time a member of a fixed, constant, closed and convex set. This scenario corresponds to the case of constant bounds in the paper [Cuoco et al., 2001], which we already discussed.

Finally, there is quite an extensive amount of literature dealing with risk minimization in continuous models: Some important representatives are [Browne, 1999], [Cvitanić and Karatzas, 1999], [Föllmer and Leukert, 1999], [Föllmer and Leukert, 2000], [Kohlmann and Zhou, 2000], as well as [Schied, 2004] and [Spivak and Cvitanić, 1999].

Further references to existing literature can be found throughout this text.
Part 2

Risk Measures

The main intention of this part is to define our notion of risk measures, review some specific ones and explore properties relevant for their comparison and the upcoming optimization problems. We set off by laying the groundwork, i.e. we give a short overview of extended versions of some fundamental concepts.

The proofs of all numbered statements except for the examples are deferred to Chapter 6.1.

2.1 Definitions

Our first step is to extend some classical definitions of probability theory to include finite measures that are not necessarily probability measures. These extensions are used in the sequel to facilitate the handling of pricing measures. Pricing measures are usually not normed to one, because they have to account for the time value of money. Usually, we would expect them to be normed to a value between zero and one, yet we do not need that restriction in the sequel.

2.1.1 Fundamental Definitions and Properties

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\nu\) be a finite measure on the measurable space \((\Omega, \mathcal{F})\), which means that \(\nu(\Omega) \in (0, +\infty)\).

Definition \((\mathbb{R}, \overline{\mathbb{R}})\): Let \(\mathbb{R} := (-\infty, +\infty)\) be the set of real numbers and \(\overline{\mathbb{R}} := [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}\) the extended set of real numbers.

Definition (Random Variable): A random variable is a measurable mapping from \(\Omega\) into the set of real numbers \(\mathbb{R}\).

Definition (Cumulative Distribution Function): The mapping \(F^\nu_X: \mathbb{R} \to [0, \nu(\Omega)]\) with \(x \mapsto \nu(\{\omega \in \Omega \mid X(\omega) \leq x\})\) is called the cumulative distribution function of the random variable \(X\) with respect to \(\nu\).

If \(\nu(\Omega) = 1\), \(F^\nu_X\) coincides with the classical cumulative distribution function and if \(\nu(\Omega) \in (0, 1)\), it is usually called subdistribution function.
2.1. DEFINITIONS

For our purpose, it is convenient to define the quantile function in the following way:

**Definition (Quantile Function):** The quantile function of a random variable $X$ with respect to $\nu$ is the mapping $(F^\nu_X)^{-1} : [0, \nu(\Omega)] \to \mathbb{R}$, $p \mapsto \inf \{ x \in \mathbb{R} | F^\nu_X(x) \geq p \}$. If $\nu = P$, we use the shortcut $F^{-1}_X$.

The expectation operator is simply used as an abbreviation for the Lebesgue Integral:

**Definition ($\mathbb{E}_\nu$):** For random variables $X$, we set the expectation of $X$ with respect to $\nu$ to the value $\mathbb{E}_\nu[X] := \int_\Omega X d\nu \in \mathbb{R}$ if the integral exists.

We literally take the classical definition of the conditional expectation and use it for finite measures, too:

**Definition ($\mathbb{E}_\nu [X | \mathcal{G}]$):** For random variables $X$ with $\mathbb{E}_\nu[|X|] < +\infty$, the conditional expectation with respect to some sigma-algebra $\mathcal{G} \subseteq \mathcal{F}$ is $\mathbb{E}_\nu[X | \mathcal{G}]$ with $\forall A \in \mathcal{G}: \int_A X d\nu = \int_A \mathbb{E}_\nu[X | \mathcal{G}] d\nu$.

The uniqueness of this generalized conditional expectation can be easily verified using the uniqueness of the classical definition for probability measures. However, not all properties of the classical definition are still valid: We observe that this definition of a generalized conditional expectation leads to $\mathbb{E}_\nu[X | \{\emptyset, \Omega\}] = \mathbb{E}_\nu[X]$. So the expectation of a random variable can no longer be viewed as a special case of the conditional expectation if $\nu(\Omega) \neq 1$.

**Definition (Martingale):** Let $T \in (0, +\infty)$, $(\mathcal{F}_t)_{0 \leq t \leq T}$ be a filtration on $(\Omega, \mathcal{F})$ and $(X_t)_{0 \leq t \leq T}$ be a family of random variables with $\mathbb{E}_\nu[|X_T|] < +\infty$. If for all $0 \leq s \leq t \leq T$: $\mathbb{E}_\nu[X_t | \mathcal{F}_s] = X_s$, the family $(X_t)_{0 \leq t \leq T}$ is called a martingale with respect to $(\mathcal{F}_t)_{0 \leq t \leq T}$ and $\nu$.

The definitions of submartingales and supermartingales are extended in the same way.

For the convenience of the reader, we present some properties of these fundamental definitions. We start with a short list consisting of some properties of the cumulative distribution function:

**Lemma 2.1:** Let $X$ and $Y$ be random variables with $X \leq Y$ and $x \in \mathbb{R}$. Then

(a) $\forall \beta \in \mathbb{R}: F^\nu_{X+\beta}(x) = F^\nu_X(x - \beta)$,

(b) $\forall \lambda > 0: F^\nu_{\lambda X}(x) = F^\nu_X(x/\lambda)$ and

(c) $F^\nu_X(x) \geq F^\nu_Y(x)$.

Next, let us state several features of the quantile function:

**Lemma 2.2:** Let $X$ and $Y$ be random variables with $X \leq Y$ and $p \in [0, \nu(\Omega)]$.

(a) $\nu \left( X < (F^\nu_X)^{-1}(p) \right) \leq p$.

(b) $\forall x \in \mathbb{R}: x \geq (F^\nu_X)^{-1}(p) \text{ if and only if } F^\nu_X(x) \geq p$. 

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8
2.1. DEFINITIONS

(c) $F_X^\nu((F_X^\nu)^{-1}(p)) \geq p$.

(d) $\forall \beta \in \mathbb{R}: \left( F_{X+\beta}^\nu \right)^{-1}(p) = (F_X^\nu)^{-1}(p) + \beta$.

(e) $\forall \lambda > 0: (F_{X,\lambda}^\nu)^{-1}(p) = \lambda \cdot (F_X^\nu)^{-1}(p)$.

(f) $(F_X^\nu)^{-1}(p) \leq (F_Y^\nu)^{-1}(p)$.

We are going to make use of the following continuity result:

Lemma 2.3:
Let $X$ be a random variable with continuous cumulative distribution function $F_X^\nu$.
For $n \in \mathbb{N}$ let $I_n := (a_n, b_n)$ with $a_n \in \mathbb{R}, b_n \in \mathbb{R}, a_n \leq b_n$ and $\lim_{n \to +\infty} (b_n - a_n) = 0$.
Then $\lim_{n \to +\infty} \nu(X \in I_n) = 0$.

Finally, the famous Bayes’ Rule is unaffected as well:

Lemma 2.4 (Bayes’ Rule):
Let $G \subseteq \mathcal{F}$ be a sigma-algebra. We consider two equivalent measures $\nu_1$ and $\nu_2$ on $(\Omega, \mathcal{F})$ with $\nu_1(\Omega) \in (0, +\infty)$ and $\forall A \in \mathcal{F}: \nu_1(A) = \int_A \frac{d\nu_1}{d\nu_2} d\nu_2$. Then for any random variable $Y$ satisfying $\mathbb{E}_{\nu_1}[|Y|] < +\infty$ we have $\nu_1$ and $\nu_2$-almost-everywhere Bayes’ rule:

$$
\mathbb{E}_{\nu_1}[Y|G] = \frac{1}{\mathbb{E}_{\nu_2} \left[ \frac{d\nu_1}{d\nu_2} \bigg| G \right]} \mathbb{E}_{\nu_2} \left[ Y \frac{d\nu_1}{d\nu_2} \bigg| G \right].
$$

2.1.2 Definition of Some Important Risk Measures

Having fixed the meaning of extended versions of these very common objects, we can turn our attention to the main objects of interest: the so-called risk measures. From a mathematical point of view, risk measures are $\mathbb{R}$-valued mappings with the purpose of ranking or even quantifying risks. It is customary to assign higher numbers to greater risks. In our setting, the risk is created by a random future value at a fixed future point in time. So we use the following definition:

Definition (Risk Measure): A risk measure is a mapping from the set of random variables into the extended set of real numbers.

A classical way to quantify the risk of a random variable is to calculate its variance. However, the variance does not always exist and one is probably good advised to set it to $+\infty$ in all these cases. In addition, the variance has a feature that is certainly not appealing: It punishes extremely good results. Therefore, we look at other, more attractive risk measures.

The following risk measures are very commonly used and frequently found in the literature, although their exact definition varies. We consider their extension to measures $\nu$ with $\nu(\Omega) \in (0, +\infty)$ on our measurable space $(\Omega, \mathcal{F})$.

Definition (VaR): For $\epsilon \in (0, \nu(\Omega))$, the value at risk of a random variable $X$ with respect to $\nu$ is

$$
\text{VaR}_\nu^\epsilon(X) := -\inf \{ x \in \mathbb{R} | \nu(X \leq x) > \epsilon \}.
$$
2.1. DEFINITIONS

The negative of the value at risk taken as a function of the parameter $\epsilon$ is a right-continuous inverse of the cumulative distribution function whereas our definition of the quantile function is a left-continuous inverse.

The following illustration depicts the definition of the value at risk and its connection with the quantile function for a special case.

**Definition (ES):** Given a random variable $X$ and $h \in \mathbb{R}$, the expected shortfall of the random variable $X$ to the measure $\nu$ and the level $h$ is

$$\text{ES}_h^\nu(X) := \mathbb{E}_\nu[(h - X)^+].$$

**Definition (CVaR):** We define the conditional value of risk (CVaR) of a random variable $X$ to the measure $\nu$ and the level $\alpha \in (0, \nu(\Omega))$ as follows:

$$\text{CVaR}_\nu^\alpha(X) := -\frac{\int_{\{X < (F_X^\nu)^{-1}(\alpha)\}} X d\nu + (F_X^\nu)^{-1}(\alpha) \cdot \left(\alpha - \nu\left(\{X < (F_X^\nu)^{-1}(\alpha)\}\right)\right)}{\alpha}.$$ 

In addition, $\text{CVaR}_{\nu(\Omega)}^\nu(X) := -\mathbb{E}_\nu[X] / \nu(\Omega)$ if the integral exists.

The idea behind the above definition of the conditional value at risk should become clearer by considering the following example of a random variable that $\nu$-almost-everywhere takes one out of three values:

The conditional value at risk is the negative of the average of the lowest $\alpha \cdot 100\%$ values of the random variable $X$. If the cumulative distribution function of $X$ has a jump at $(F_X^\nu)^{-1}(\alpha)$, the integral is adjusted such that the measure of the set we integrate over still equals $\alpha$.

We note that the value at risk, the expected shortfall and the conditional value at risk are well-defined for each and every random variable $X$ defined on our probability space with the exception of $\text{CVaR}_{\nu(\Omega)}^\nu(X)$. Hence they all are risk measures but $\text{CVaR}_{\nu(\Omega)}^\nu(X)$. 
2.1.3 Coherent Risk Measures

If one is asked to propose a risk measure for some purpose, the question that is raised inevitably, is what desirable properties should that risk measure possess. In the article [Artzner et al., 1999], the authors compiled a set of four axioms that are features one would naturally assume a risk measure should have. Such risk measures are called coherent by the authors. Note that we deviate a little from their original work by not introducing their so-called reference instrument, which they use for discountation purposes. The definition we give below corresponds to the case where the reference instrument is the constant 1. This point of view is shared by many authors, see for instance [Pflug, 2000] and [Föllmer and Schied, 2002].

**Definition (Coherent Risk Measure):** A risk measure $\rho$ is called coherent, if it complies with the following axioms:

- **Translation Invariance:** $\forall$ random variables $X$ and $\forall \beta \in \mathbb{R}$: $\rho(X + \beta) = \rho(X) - \beta$.
- **Subadditivity:** $\forall$ random variables $X$ and $Y$: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
- **Positive Homogeneity:** $\forall$ random variables $X$ and $\forall \lambda \geq 0$: $\rho(\lambda \cdot X) = \lambda \cdot \rho(X)$.
- **Monotonicity:** $\forall$ random variables $X$ and $Y$: $X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$.

Each one of these axioms has an intuitive interpretation. For example, the idea of the monotonicity axiom can be described as follows: An investment with a higher value (in any scenario) should have a lower risk.

A discussion of the interpretations of these axioms can be found of course in the original paper [Artzner et al., 1999].

2.2 Discussion of Several Risk Measures

Let us look at the risk measures defined in Section 2.1.2 in more depth.

2.2.1 Value at Risk

Our definition of the value at risk can be found literally for instance in [Artzner et al., 1999]. The authors show that this risk measure is not a coherent measure, because it is not subadditive. The other requirements of a coherent measure however, namely translation invariance, positive homogeneity and monotonicity, are obviously fulfilled by the value at risk.

On the plus side is the fact that the value at risk of a $\mathbb{R}$-valued random variable is always finite. The expected shortfall and conditional value at risk do not have this feature.

A negative aspect is the dependency on $\epsilon$: For a fixed random variable $X$, the mapping $\epsilon \mapsto \text{VaR}_\epsilon^\nu(X)$ is continuous from the right, yet it does not need to be left-continuous. Thus the exact choice of $\epsilon$ has the potential to be a crucial ingredient of any optimization process given some value at risk limitation, because the optimum can be very sensitive to any variation in $\epsilon$.

So let us state the value at risk limitation more precisely: We are going to study problems with a restriction of the form $\text{VaR}_\epsilon^\nu(X) \leq -q$ for some fixed $q \in \mathbb{R}$. The appearance of the infimum
operator in the definition of the value at risk is not really handy, hence, we are looking for another
way to formulate such a restriction.
The next result sheds some light on the value at risk by giving a characterization in part (a), which
could be used as an equivalent alternative definition. It inspires a different formulation of a value at
risk restriction and is presented in part (b).

Lemma 2.5:
For all random variables $X$, scalars $q \in \mathbb{R}$ and $\epsilon \in (0, \nu(\Omega))$:

(a) $\text{VaR}^\nu_\epsilon(X) = -\sup \{x \in \mathbb{R} | \nu(X < x) \leq \epsilon \}$ and

(b) $\text{VaR}^\nu_\epsilon(X) \leq -q \iff \nu(X < q) \leq \epsilon$.

Consequently, we can replace a restriction of the form $\text{VaR}^\nu_\epsilon(X) \leq -q$ by the equivalent condition
$\nu(X < q) \leq \epsilon$. The latter one can be handled easily as we will find out.

2.2.2 Expected Shortfall

The expected shortfall risk measure is not a coherent risk measure. It fulfills the monotonicity axiom
and for non-negative $h$ the subadditivity axiom, yet it violates all other axioms. Let us look at the
individual axioms in more detail:

Proposition 2.6:
Let $X$ and $Y$ be random variables.

(a) $\forall h \in \mathbb{R}, \forall \beta > 0$: $\text{ES}^\nu_h(h + \beta) = 0 \neq 0 - \beta = \text{ES}^\nu_h(h) - \beta$.
   $\forall h \in \mathbb{R}, \forall \beta < 0$: $\text{ES}^\nu_h((h - \beta) + \beta) = 0 \neq 0 - \beta = \text{ES}^\nu_h(h - \beta) - \beta$.

(b) $\forall h \geq 0$: $\text{ES}^\nu_h(X + Y) \leq \text{ES}^\nu_h(X) + \text{ES}^\nu_h(Y)$,
   $\forall h < 0$: $\text{ES}^\nu_h(h + h) = (-h) \cdot \nu(\Omega) > 0 = \text{ES}^\nu_h(h) + \text{ES}^\nu_h(h)$.

(c) $\forall h > 0$, $\text{ES}^\nu_h(2 \cdot h) = 0 \neq h \cdot \nu(\Omega) = 2 \cdot \text{ES}^\nu_h(h)$.
   $\forall h < 0$, $\text{ES}^\nu_h(2 \cdot h) = (-h) \cdot \nu(\Omega) \neq 0 = 2 \cdot \text{ES}^\nu_h(h)$.

(d) If $X \leq Y$, $\forall h \in \mathbb{R}$: $\text{ES}^\nu_h(X) \geq \text{ES}^\nu_h(Y)$.

On the one hand, the expected shortfall violates more coherence axioms than the value at risk, yet
on the other hand, it has some appealing qualities:
First, the mapping $h \mapsto \text{ES}^\nu_h(X)$ is for fixed $X$ either Lipschitz continuous with the Lipschitz constant
$\nu(\Omega)$ or it is identically $+\infty$. As such, uncertainties over the best choice of the parameter $h$ are less
grade than uncertainties about the choice of $\epsilon$ for value at risk limits.
Second, we can deal with a limitation restriction of this type in our upcoming optimization problem
directly. There is no need for any kind of transformation.
2.2. DISCUSSION OF SEVERAL RISK MEASURES

2.2.3 Connection Between Expected Shortfall and Conditional Value at Risk

Similar to the value at risk restrictions, taking the untransformed definition of the conditional value at risk for optimization purposes is not such a good idea, because we do not know how to deal with the quantile function appearing in its definition. In order to find a way to circumvent this problem, we investigate the relationship between the conditional value at risk and the expected shortfall. As a by-product, the acquired knowledge will be helpful for proving the coherence of conditional value at risk, too.

Just by looking at the definition of the conditional value at risk and the expected shortfall, one might not suspect the existence of some very useful links between them. A relatively obvious connection is that the conditional value at risk can be characterized using expected shortfalls:

Lemma 2.7:
For any \( \alpha \in (0, \nu(\Omega)) \) and random variable \( X \):

\[
\text{CVaR}_\alpha^\nu(X) = \frac{\mathbb{E}_\nu \left[ \left((F_X^\nu)^{-1}(\alpha) - X\right)^+ \right]}{\alpha} - (F_X^\nu)^{-1}(\alpha)
\]

and

\[
\text{CVaR}_\alpha^\nu(X) \leq -c \iff \mathbb{E}_\nu \left[ \left((F_X^\nu)^{-1}(\alpha) - X\right)^+ \right] \leq \left((F_X^\nu)^{-1}(\alpha) - c\right) \alpha.
\]

The usefulness of this characterization alone is somewhat limited, because instead of a fixed value \( h \), we encounter the quantile function of the random variable \( X \) which of course is not independent of \( X \).

Hence, our idea is to write \( h \) instead of \((F_X^\nu)^{-1}(\alpha)\) and take a look at what happens if we plug in different values for \( h \). The answer can be found in the next statement:

Proposition 2.8:
Let \( X \) be a random variable, \( c \in \mathbb{R} \), \( \alpha \in (0, \nu(\Omega)) \) and \( f: \mathbb{R} \to (-\infty, +\infty], h \mapsto \mathbb{E}_\nu [(h - X)^+] - (h - c) \alpha \).

Then \( f \) is non-increasing on \((-\infty, (F_X^\nu)^{-1}(\alpha)]\) and non-decreasing on \([ (F_X^\nu)^{-1}(\alpha), +\infty) \).

We realize that the function \( f \) in Proposition 2.8 takes its minimal value for \( h = (F_X^\nu)^{-1}(\alpha) \) (if \((F_X^\nu)^{-1}(\alpha) \neq +\infty\)). Therefore we wonder, if we could replace a conditional value at risk restriction by a set of expected shortfall constraints. The answer to this question is yes, and the details are as follows:

Theorem 2.9:
If \( c \in \mathbb{R} \) and \( \alpha \in (0, \nu(\Omega)) \), \( \{ X \mid \text{CVaR}_\alpha^\nu(X) \leq -c \} = \bigcup_{h \in [c, +\infty)} \{ X \mid \text{ES}_h^\nu(X) \leq (h - c) \alpha \} \).

Theorem 2.9 gives us what we want: The desired characterization of a conditional value at risk restrictions in terms of a set of expected shortfall conditions. In essence, it tells us that we can replace an optimization problem with a conditional value at risk restriction by a certain set of expected shortfall problems.
2.2. DISCUSSION OF SEVERAL RISK MEASURES

2.2.4 Conditional Value at Risk

The conditional value at risk is a coherent risk measure, because it has the following properties:

**Proposition 2.10:**
Let $X$ and $Y$ be random variables and $\alpha \in (0, \nu(\Omega))$.

(a) $\forall \beta \in \mathbb{R}$: $\text{CVaR}_\alpha^X (X + \beta) = \text{CVaR}_\alpha^X (X) - \beta$.

(b) $\text{CVaR}_\alpha^X (X + Y) \leq \text{CVaR}_\alpha^X (X) + \text{CVaR}_\alpha^Y (Y)$.

(c) $\forall \lambda \geq 0$: $\text{CVaR}_\alpha^X (\lambda \cdot X) = \lambda \cdot \text{CVaR}_\alpha^X (X)$.

(d) If $X \leq Y$, $\text{CVaR}_\alpha^X (X) \geq \text{CVaR}_\alpha^Y (Y)$.

The conditional value at risk can be viewed as a parameterized family of risk measures with parameter $\alpha$. Just as with the other families of risk measures, we look at the implications of small changes in the parameter:

**Lemma 2.11:**
The mapping $(0, \nu(\Omega)) \rightarrow \mathbb{R}$, $\alpha \mapsto \text{CVaR}_\alpha^X (X)$ is continuous for fixed $X$.

Thus small variations of the parameter $\alpha$ do not result in dramatic changes of the value of the conditional value at risk. So the conditional value at risk — like the expected shortfall — does not seem to be as vulnerable to uncertainties about the choice of the parameter as the value at risk.

The question of how to handle a conditional value at risk restriction was already highlighted in Section 2.2.3.

Let us spend a few words on our choice for the definition of the conditional value at risk: In general, the exact definition of the conditional value at risk varies from author to author and paper to paper. An alternative definition considered in an earlier version of this work was the following:

**Definition ($\tilde{\text{CVaR}}_\alpha^X (X)$):** $\forall \alpha \in (0, \nu(\Omega))$ and random variables $X$ we define the alternative conditional value at risk as $\tilde{\text{CVaR}}_\alpha^X (X) := -\inf \left\{ \int \frac{Xdu}{\nu(A)} \right\} A \in \mathcal{F} \land \nu(A) \geq \alpha$.

It is straightforward to verify that $\tilde{\text{CVaR}}_\alpha^X (X)$ is a coherent risk measure, too, due to the corresponding properties of the Lebesgue Integral and the infimum operator. However this definition is not identical to our original one:

**Example 2.12:**
Let $\Omega = \{0, 1\}$, $\mathcal{F} = \{\emptyset, \{0\}, \{1\}, \Omega\}$, $\forall A \in \mathcal{F}$: $P(A) := \frac{|A|}{\mu}$, $\alpha = \frac{3}{4}$ and $X$ be the identity mapping.

Then $\text{CVaR}_\alpha^P (X) = -\inf \left\{ \frac{\int_X XdP}{P(A)} \right\} A \in \mathcal{F} \land P(A) \geq \alpha = -\frac{1}{2}$ and $\text{CVaR}_\alpha^P (X) = -\frac{1}{2}$.

In general, the value given by the conditional value at risk is an upper bound for the value given by its alternative definition:

**Lemma 2.13:**
For all $\mathbb{R}$ valued random variables $X$ and $\alpha \in (0, \nu(\Omega))$: $\tilde{\text{CVaR}}_\alpha^X (X) \leq \text{CVaR}_\alpha^X (X)$. 

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14
If the sigma-algebra $\mathcal{F}$ of our probability space is sufficiently divisible, then both definitions of the conditional value do agree:

**Lemma 2.14:**
If $\forall A \in \mathcal{F}$, $\forall p \in [0, \nu(A)]$: $\exists B \in \mathcal{F}$: $B \subseteq A$ and $\nu(B) = p$, then for any random variable $X$ and $\alpha \in (0, \nu(\Omega))$, we have:

$$\exists B \in \mathcal{F}: B \subseteq \{ \omega \mid X(\omega) = (F_X^{-1})^{-1}(\alpha) \} \text{ and } \nu(B) = \alpha - \nu\left(\left\{ X < (F_X^{-1})(\alpha) \right\}\right) \geq 0.$$ 

Hence

$$\text{CVaR}_\alpha^{\nu}(X) = -\frac{\int_{B \cup \{ x < (F_X^{-1})^{-1}(\alpha) \}} X d\nu}{\alpha} = \text{CVaR}_\alpha^{\nu}(X).$$

We feel that in Example 2.12, CVaR$_\alpha^{\nu}(X)$ captures the idea behind a conditional value at risk more accurately than $\text{CVaR}_\alpha^{\nu}(X)$: Although the sigma-algebra is not divisible enough such that we can speak of $\alpha = 75\%$ of the worst cases, one can intuitively calculate the average value of the 75% of the worst cases by integrating over the set $\{0\}$ and taking half of the set $\{1\}$. Hence, we would expect the conditional value at risk to be $-\frac{1}{4}$. Since $\text{CVaR}_\alpha^{\nu}(X)$ gives a different result in such cases, we discard the idea of using this alternative definition.

In the paper [Artzner et al., 1999] the authors called a definition similar to $\text{CVaR}_\alpha^{\nu}(X)$ the worst conditional expectation: Their definition — again a coherent risk measure — for the constant reference instrument 1 is as follows:

**Definition (WCE$^{\alpha}_{\text{ADEH}}(X)$):** $\text{WCE}^{\alpha}_{\text{ADEH}}(X) := -\inf \left\{ \frac{\int_X X dP}{P(A)} \mid P(A) > \alpha \right\}.$

Note the usage of the requirement that $P(A)$ has to be strictly (!) greater than $\alpha$ in the definition. Hence $\text{CVaR}^{\nu}(X) \geq \text{WCE}^{\alpha}_{\text{ADEH}}(X)$. We dislike the definition of the worst conditional expectation for the same reason as our alternative definition of the conditional value at risk: It does not deliver the intuitive value in all cases. For instance in Example 2.12, $\text{WCE}^{\alpha}_{\text{ADEH}}(X) = -\frac{1}{2} \neq -\frac{1}{4}$.

Let us discuss one more possibility: In the paper [Rockafellar and Uryasev, 2000], the authors propose the following definition for the conditional value at risk:

**Definition (CVAR$_\alpha^{\text{RU}}(X)$):** $\text{CVAR}^{\alpha}_{\text{RU}}(X) := \frac{\int_{\{-X \geq F_X^{-1}(1-\alpha)\}} -X dP}{\alpha}.$

They use this definition solely for random variables with a continuous distribution function. For random variables that do not have a continuous distributions function, this definition is not suitable, because CVAR$_\alpha^{\text{RU}}$ is not a coherent risk measure in this case. For example, the monotonicity axiom is violated:

**Example 2.15 (Non-coherence of CVAR$_\alpha^{\text{RU}}$):**
Let $\Omega = \{0, 1, 2\}$, $\mathcal{F}$ be the power set of $\Omega$, $\forall A \in \mathcal{F}$: $P(A) := \frac{|A|}{|\Omega|}$, $\alpha = \frac{1}{2}$ $X$ be the identity mapping, $Y := \min \{X, 1\}$. We note that $\text{CVAR}^{\alpha}_{\text{RU}}(0) = 0$, $\text{CVAR}^{\alpha}_{\text{RU}}(X) = \int_{\{-X \geq -1\}} -X dP \cdot 2 = -\frac{2}{3}$ and $\text{CVAR}^{\alpha}_{\text{RU}}(Y) = \int_{\{-Y \geq -1\}} -Y dP \cdot 2 = -\frac{4}{3}$. Therefore $0 \leq Y \leq X$, $\text{CVAR}^{\frac{1}{2}}_{\text{RU}}(Y) < \text{CVAR}^{\frac{1}{2}}_{\text{RU}}(X) < \text{CVAR}^{\frac{1}{2}}_{\text{RU}}(0)$ and thus CVAR$_\alpha^{\text{RU}}$ is not a monotonic risk measure.

These comments illustrate that there are several approaches used in the literature to define a risk measures which captures the notion of quantifying the expected value of the worst $\alpha \cdot 100\%$. Depending on its purpose one might choose one or another.
In the paper [Acerbi and Tasche, 2002], one can find a detailed analysis of the relationship between the definitions of several of these approaches. Note that our definition of the conditional value at risk is essentially their notion of expected shortfall. They use the name conditional value at risk for another definition:

**Definition (CVAR\(_\alpha\)\(^\mathrm{AT}\) (\(X\))):** CVAR\(_\alpha\)\(^\mathrm{AT}\) (\(X\)) \(=\) \(\inf_{h \in \mathbb{R}} \left\{ \frac{\mathbb{E}_P[-(h-X)^+]}{\alpha} - h \right\}\).

Except for some basic transformations, this definition is used as well in [Pflug, 2000]. If \(\nu = P\), Corollary 4.3 of [Acerbi and Tasche, 2002] tells us that this definition is equivalent to our definition of the conditional value at risk. That corollary also affirms the validity of Theorem 2.9.
Part 3

Static Optimization Problems with Risk Constraints

Our first model is a static one: We consider an investor having the choice between different random variables. The investor's goal is to choose the random variable that has the maximal expected utility. Of course, the investor does not have unlimited funds at his disposal: Instead, he has a fixed finite initial endowment. Therefore, he can only select random variables that cost no more that his initial endowment. We assume that all random variables are priced using some pricing measure, which will be called $Q$. If the set of random variables to choose from consists of all random variables satisfying this price constraint, this model can be seen as a classic one-step model.

The new ingredient is that the investor not only faces the initial endowment constraint, but he also has to comply with some upper limit on the risk introduced by the random variable. This requirement further shrinks the set of random variables from which he has to select his preferred one.

There exists another interpretation of our market model: The investor can initially select a desired payoff for any possible outcome. The price of such a payoff profile, calculated using the pricing measure, must be not be above the initial endowment and the payoff profile must have an acceptable risk.

3.1 Problem Statement

The first step is to describe this model precisely in mathematical terms.

3.1.1 Assumptions

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $Q$ another measure on $(\Omega, \mathcal{F})$ with $Q(\Omega) \in (0, +\infty)$. The measures $P$ and $Q$ are assumed to be equivalent and we denote the Radon-Nikodym derivative of $Q$ with respect to $P$ by $\frac{dQ}{dP}$.

Additionally, we fix a measurable function $U: \mathbb{R} \to (-\infty, +\infty)$, which we call utility function, an initial endowment $x_0 \in \mathbb{R}$ as well as parameters $\nu \in \{P, Q\}$, $q \in \mathbb{R}$, $c \in (0, \nu(\Omega))$, $h \in \mathbb{R}$ and $\alpha \in (0, \nu(\Omega)]$. 

17
3.1.2 The Optimization Problems

Our objective is to solve the following three optimization problems with risk restrictions for real-valued random variables $X$:

**Definition ((P_{VaR})):**

$$
(P_{VaR}) \begin{cases}
E_P[U(X)] & \to \max \\
E_Q[X] & \leq x_0 \\
\text{Var}_r(X) & \leq -q
\end{cases}
$$

**Definition ((P_{ES})):**

$$(P_{ES}) \begin{cases}
E_P[U(X)] & \to \max \\
E_Q[X] & \leq x_0 \\
\text{ES}_h^{\nu}(X) & \leq (h - c) \cdot \alpha
\end{cases}
$$

and

**Definition ((P_{CVaR})):**

$$(P_{CVaR}) \begin{cases}
E_P[U(X)] & \to \max \\
E_Q[X] & \leq x_0 \\
\text{CVaR}_\alpha^{\nu}(X) & \leq -c
\end{cases}
$$

Accordingly, the corresponding problem without risk restriction is

**Definition ((P_0)):**

$$
(P_0) \begin{cases}
E_P[U(X)] & \to \max \\
E_Q[X] & \leq x_0
\end{cases}
$$

Let us highlight some particularities of these definitions briefly:

First of all, we recall that Lemma 2.5(b) tells us the following: Instead of working with the restriction $\text{Var}_r(X) \leq -q$ in $P_{VaR}$, we can and will use the constraint $\nu(X < q) \leq \epsilon$.

Second, one would expect that we write $\text{ES}_h^{\nu}(X) \leq l$ for some $l \in \mathbb{R}_+$ in the definition of $P_{ES}$. However, by choosing the formulation $\text{ES}_h^{\nu}(X) \leq (h - c) \cdot \alpha$, the relationship between $P_{ES}^{\nu}$ and $P_{CVaR}^{\nu}$ becomes much more transparent: In light of Theorem 2.9, $P_{CVaR}^{\nu}$ can be reformulated — for all $\alpha \in (0, \nu(\Omega))$ — as

$$
(P_{CVaR}^{\nu}) \begin{cases}
E_P[U(X)] & \to \max \\
E_Q[X] & \leq x_0 \\
\text{ES}_h^{\nu}(X) & \leq (h - c) \cdot \alpha \\
h & \in [c, +\infty)
\end{cases}
$$

Instead of inspecting $P_{CVaR}^{\nu}$ directly, we can try to find optimal solutions of $P_{ES}^{\nu}$ for all fixed $h \in [c, +\infty)$. An optimal solution with the highest objective function value, is an optimal solution of $P_{CVaR}^{\nu}$ and thus of $P_{CVaR}^{\nu}$.

In addition, the formulation $\text{ES}_h^{\nu}(X) \leq (h - c) \cdot \alpha$ is no limitation, because for a given $l \in \mathbb{R}_+$, we can always set $\alpha := \nu(\Omega)$ and $c := h - \frac{l}{\alpha}$.

Third, the usage of minus signs on some of the right-hand sides is helpful, because the definitions of the corresponding risk measures on the left-hand side of these inequalities contain them as well.
Our next step is to define the set of random variables to chose from and what we consider a solution of such an optimization problem:

**Definition (Admissibility):** A random variable $X$ is called admissible for a problem

$$\begin{align*}
\left\{ \mathbb{E}_P[U(X)] \to \max_{X \in A} \right\}
\end{align*}$$

if and only if $X \in A$ and $\mathbb{E}_P[U(X)]$ is well defined (in the extended sense, i.e. it’s value might be $+\infty$ or $-\infty$).

**Definition (Optimality):** An admissible random variable $X$ is called (optimal) solution of a problem

$$\begin{align*}
\left\{ \mathbb{E}_P[U(X)] \to \max_{X \in A} \right\}
\end{align*}$$

if and only if for all admissible $Y$:

$$\mathbb{E}_P[U(X)] \geq \mathbb{E}_P[U(Y)].$$

Several monotonicity results can be deduced immediately, for instance:

**Example 3.1:**
The set of admissible $X$ of $(PC_{\text{CVaR}}^\nu)$ increases with increasing $\alpha$ and decreases with increasing $c$. This results in a maximal expected terminal utility $\sup \{ \mathbb{E}_P[U(X)] | X \text{ is admissible} \}$ that is increasing in $\alpha$ and decreasing in $c$.

**Proof:** It is immediately clear by the definition of $(PC_{\text{CVaR}}^\nu)$ that the set of admissible $X$ decreases with increasing $c$ (i.e. decreasing $-c$).

Let $\alpha_1 \in (0, \nu(\Omega))$, $\alpha_2 \in (0, \nu(\Omega))$ with $\alpha_1 < \alpha_2$ and $X$ be admissible for $(PC_{\text{CVaR}}^\nu)$ with $\alpha = \alpha_1$. (Theorem 2.9) \Rightarrow \exists h \in [c, +\infty): \mathbb{E}_\nu[(h - X)^+] \leq (h - c) \alpha_1$.

- If $\alpha_2 \in (0, \nu(\Omega))$, Theorem 2.9 implies that $\text{CVaR}_{\nu(\Omega)}^\nu(X) \leq -c$, because $(h - c) \alpha_1 \leq (h - c) \alpha_2$.

- $\text{CVaR}_{\nu(\Omega)}^\nu(X) = -\frac{\mathbb{E}_\nu[(X - h)^+]}{\nu(\Omega)} - h + \frac{\mathbb{E}_\nu[(h - X)^+]}{\nu(\Omega)} \leq -h + (h - c) \cdot \frac{\alpha_1}{\nu(\Omega)} \leq -c.$

Thus $X$ is admissible for $(PC_{\text{CVaR}}^\nu)$ with $\alpha = \alpha_2$. \hfill \Box

### 3.1.3 Some Notation

The following definitions of $t_{\text{min}}$ and $t_{\text{max}}$ will be useful in the sequel:

**Definition ($t_{\text{min}}$):** We denote the point where the cumulative distribution function of $\frac{dQ}{d\mathbb{P}}$ leaves the value 0 for the very first time by $t_{\text{min}} := \sup \left\{ x \in \mathbb{R} \mid \frac{dQ}{d\mathbb{P}}(x) = 0 \right\} \in [0, +\infty)$.

**Definition ($t_{\text{max}}$):** We denote the point where the cumulative distribution function of $\frac{dQ}{d\mathbb{P}}$ attains the value 1 for the very first time by $t_{\text{max}} := \frac{dQ}{d\mathbb{P}}^{-1}(1) \in (0, +\infty]$.
Some implications of these definitions:

**Fact 3.2:**
(a) \( P \left( \frac{dQ}{dP} < t_{\text{min}} \right) = 0 \),
(b) \( \forall \varepsilon > 0: P \left( \frac{dQ}{dP} \in [t_{\text{min}}, t_{\text{min}} + \varepsilon] \right) > 0 \),
(c) \( P \left( \frac{dQ}{dP} > t_{\text{max}} \right) = 0 \),
(d) \( Q(\Omega) \leq t_{\text{max}} \),
(e) \( (Q(\Omega) = t_{\text{max}}) \iff \left( F_{dQ}^{P} = \mathbb{1}_{[t_{\text{max}}, +\infty]} \right) \) and
(f) \( \forall \varepsilon > 0: P \left( \frac{dQ}{dP} \in (t_{\text{max}} - \varepsilon, t_{\text{max}}] \right) > 0 \).

### 3.2 Feasibility

First of all, we are curious under which conditions the constraints imposed in \((P_{\text{VaR}})\), \((P_{\text{ES}})\) and \((P_{\text{CVaR}})\) can be fulfilled, that is we are eager to answer the question: Which conditions are necessary and sufficient for each one of the problems to have admissible points? The answer is given in the following five theorems. Examples of admissible \(X\) can be found in the respective proofs.

#### 3.2.1 Value at Risk Problem

**Theorem 3.3 (Feasibility of \((P_{\text{VaR}})\):**

The set of admissible \(X\) of the problem \((P_{\text{VaR}})\) is non-empty if and only if

- \( x_0 \geq q \cdot Q(\Omega) \) or
- \( \exists A \in \mathcal{F} \) with \( \nu(A) \in (0, \varepsilon] \).

**Proof:**

- If \( x_0 \geq q \cdot Q(\Omega) \), then \( q \) is admissible.
- For all \( A \in \mathcal{F} \) with \( \nu(A) \in (0, \varepsilon] \), \( Y \) with \( Y(\omega) := \begin{cases} \frac{x_0 - q \cdot Q(\Omega \setminus A)}{Q(A)} & \text{if } \omega \in A \\ q & \text{if } \omega \notin A \end{cases} \) is admissible.
- Case \( \exists A \in \mathcal{F} \) with \( \nu(A) \in (0, \varepsilon] \):
  - Let \( Y \) be an admissible random variable.
  - \( \nu(Y^{-1}([-\infty, q])) = \nu(Y < q) \leq \varepsilon \).
  - \( Y^{-1}([-\infty, q]) \in \mathcal{F} \Rightarrow \nu(Y^{-1}([-\infty, q])) = 0 \).
  - \( \nu \in \{P, Q\} \Rightarrow \nu \sim Q \Rightarrow Q(Y^{-1}([-\infty, q])) = 0 \).
  - \( \Rightarrow x_0 \geq \mathbb{E}_Q[Y] \geq \mathbb{E}_Q[q] = q \cdot Q(\Omega) \).

\( \square \)
3.2.2 Expected Shortfall Problem

Before stating the main result, we provide the following useful connection between different expected shortfall problems:

**Proposition 3.4:**
Let \( \lambda \in (0, 1), h_1, h_2 \in \mathbb{R}, X_1 \) be admissible for \( (P_{\text{ES}^P_{h_1}}) \) and \( X_2 \) be admissible for \( (P_{\text{ES}^P_{h_2}}) \). If \( E_P [U (\lambda X_1 + (1 - \lambda) X_2)] \) is well defined, \( \lambda X_1 + (1 - \lambda) X_2 \) is admissible for \( (P_{\text{ES}^P_{h_1 + (1 - \lambda) h_2}}) \).

**Proof:** \( \lambda X_1 + (1 - \lambda) X_2 \) fulfills the price constraint, because
\[
E_Q [\lambda X_1 + (1 - \lambda) X_2] = \lambda \cdot E_Q [X_1] + (1 - \lambda) \cdot E_Q [X_2] \leq x_0.
\]

The risk constraint is met as well:
\[
E_{\nu} [((\lambda h_1 + (1 - \lambda) h_2) - (\lambda X_1 + (1 - \lambda) X_2))^+] \leq E_{\nu} [(h_1 - X_1)^+] + E_{\nu} [(1 - \lambda) h_2 - (h_2 - c) \cdot \alpha] = ((\lambda h_1 + (1 - \lambda) h_2) - c) \cdot \alpha.
\]

**Theorem 3.5 (Feasibility of \( (P_{\text{ES}^P_{h}}) \)):**
The set of admissible \( X \) of the problem \( (P_{\text{ES}^P_{h}}) \) is non-empty if and only if
\begin{itemize}
  \item \( h = c \) and \( x_0 \geq h \cdot Q (\Omega) \) or
  \item \( h > c \) and \( x_0 > h \cdot Q (\Omega) - t_{\text{max}} \cdot (h - c) \alpha \) or
  \item \( h > c \) and \( x_0 = h \cdot Q (\Omega) - t_{\text{max}} \cdot (h - c) \alpha \) and \( P \left( \frac{dQ}{dP} = t_{\text{max}} \right) > 0 \).
\end{itemize}

Note that if \( t_{\text{max}} = +\infty, Q (\Omega) \cdot h - t_{\text{max}} \cdot (h - c) \alpha = -\infty \).

**Proof:**
\begin{itemize}
  \item Clearly, if \( h \leq c \), the set of admissible \( X \) in the problem \( (P_{\text{ES}^P_{h}}) \) is non-empty if and only if \( X = c \) is admissible.
  \item We consider the case \( h > c \) and \( x_0 > h \cdot Q (\Omega) - t_{\text{max}} \cdot (h - c) \alpha \) next:
    In this case, \( \exists \tilde{t} \in \max \left\{ 0, \frac{h \cdot Q (\Omega) - x_0}{\alpha(h-c)} \right\} \cdot t_{\text{max}} \). \( \Rightarrow Q \left( \frac{dQ}{dP} > \tilde{t} \right) = \int_{\frac{dQ}{dP} > \tilde{t}} \frac{dQ}{dP} dP \geq t_{\text{max}} \cdot P \left( \frac{dQ}{dP} > \tilde{t} \right). \)
    By defining \( l := h - \frac{h}{P \left( \frac{dQ}{dP} > \tilde{t} \right)} \cdot \alpha \leq h \), the equality \( (h - l) \cdot P \left( \frac{dQ}{dP} > \tilde{t} \right) = (h - c) \cdot \alpha \) together
\end{itemize}
with the inequality

\[
\begin{align*}
&\ h \cdot Q(\Omega) + (l - h) \cdot Q\left(\frac{dQ}{dP} > \hat{t}\right) \leq 0 \\
&\ \leq h \cdot Q(\Omega) + (l - h) \cdot P\left(\frac{dQ}{dP} > \hat{t}\right) \cdot \hat{t} \\
&\ \leq h \cdot Q(\Omega) + (l - h) \cdot P\left(\frac{dQ}{dP} > \hat{t}\right) \cdot \frac{h \cdot Q(\Omega) - x_0}{\alpha(h - c)} \\
&\ \quad = x_0
\end{align*}
\]

shows that \( X \) with \( X(\omega) := h \cdot 1_{[0,\hat{t}]}\left(\frac{dQ}{dP}(\omega)\right) + l \cdot 1_{(\hat{t},+\infty)}\left(\frac{dQ}{dP}(\omega)\right) \) is admissible for the problem \((P_{ESP})\).

\[
X(\omega)
\]

- Let \( h > c \) and \( x_0 = h \cdot Q(\Omega) - t_{\text{max}} \cdot (h - c) \) and \( P\left(\frac{dQ}{dP} = t_{\text{max}}\right) > 0 \).

We define \( l := h - \frac{h-c}{P\left(\frac{dQ}{dP} = t_{\text{max}}\right)} \cdot \alpha. \)

\[
\Rightarrow (h - l) \cdot P\left(\frac{dQ}{dP} = t_{\text{max}}\right) = (h - c) \cdot \alpha
\]

\[
\begin{align*}
&\ h \cdot Q(\Omega) + (l - h) \cdot Q\left(\frac{dQ}{dP} = t_{\text{max}}\right) \\
&\ \quad = h \cdot Q(\Omega) + (l - h) \cdot P\left(\frac{dQ}{dP} = t_{\text{max}}\right) \cdot t_{\text{max}} \\
&\ \quad = h \cdot Q(\Omega) - (h - c) \cdot \alpha \cdot t_{\text{max}} \\
&\ \quad = x_0.
\end{align*}
\]

\[
\Rightarrow X \text{ with } X(\omega) := h \cdot 1_{[0,t_{\text{max}}]}\left(\frac{dQ}{dP}(\omega)\right) + l \cdot 1_{(t_{\text{max}},+\infty)}\left(\frac{dQ}{dP}(\omega)\right) \text{ is admissible for the problem } (P_{ESP}).
\]

- Therefore we are left with the case \( h > c \) and \( x_0 \leq h \cdot Q(\Omega) - t_{\text{max}} \cdot (h - c) \). Suppose that \( X \) is admissible for the problem \((P_{ESP})\).

Since \( t_{\text{max}} > 0 \), it is immediately clear that \( x_0 < h \cdot Q(\Omega) \) and thus \( P(X < h) > 0 \).

If \( P\left(\frac{dQ}{dP} \geq t_{\text{max}}\right) = 0 \) ((\( \star \)) is a strict inequality) or \( x_0 < h \cdot Q(\Omega) - t_{\text{max}} \cdot (h - c) \alpha \) ((\( \star \star \)) is
a strict inequality) we observe a contradiction to the admissibility of \( X \):

\[
\mathbb{E}_Q[X] = \mathbb{E}_Q[h] + \mathbb{E}_Q[(X-h)^+] + \mathbb{E}_Q[-(h-X)^+] \\
\geq h \cdot Q(\Omega) + \mathbb{E}_P\left[-(h-X)^+ \frac{dQ}{dP}\right]_{\leq 0}^{\leq t_{\max}} \\
\geq h \cdot Q(\Omega) - \mathbb{E}_P[(h-X)^+] \cdot t_{\max} \geq 0 \\
\geq h \cdot Q(\Omega) - (h-c) \alpha \\
\geq x_0.
\]

\[\star\]

\[\star\star\]

\[\square\]

**Theorem 3.6 (Feasibility of \((P_{ES_Q})\)):**
The set of admissible \( X \) of the problem \((P_{ES_Q})\) is non-empty if and only if \( h \geq c \) and \( x_0 \geq h \cdot Q(\Omega) - (h-c) \alpha \).

**Proof:**
- If \( h \geq c \) and \( x_0 \geq h \cdot Q(\Omega) - (h-c) \alpha \), the constant random variable \( h - (h-c) \frac{\alpha}{Q(\Omega)} \) is admissible for \((P_{ES_Q})\), since \( \mathbb{E}_Q\left[h - (h-c) \frac{\alpha}{Q(\Omega)}\right] = h \cdot Q(\Omega) - (h-c) \alpha \leq x_0 \) and \( \mathbb{E}_Q\left[(h - (h-c) \frac{\alpha}{Q(\Omega)})^+\right] = (h-c) \alpha \).

- If \( h < c \), \( \forall X: (h-c) \alpha \frac{\alpha}{Q(\Omega)} \leq 0 \leq \mathbb{E}_Q[(h-X)^+] \).

- \( \forall X \) with \( \mathbb{E}_Q[(h-X)^+] \leq (h-c) \alpha \):
  \[
  \mathbb{E}_Q[X] = \mathbb{E}_Q[h + (X-h)^+] - (h-X)^+] = h \cdot Q(\Omega) + \mathbb{E}_Q[(X-h)^+] - \mathbb{E}_Q[(h-X)^+] \geq h \cdot Q(\Omega) - (h-c) \alpha.
  \]

\[\square\]

**Lemma 3.7:**
The set consisting of all \( h \in \mathbb{R} \) such that \((P_{ES_Q})\) is feasible, is a convex subset of \( \mathbb{R} \).

**Proof:** The proofs of Theorem 3.5 and Theorem 3.6 both yield admissible random variables \( X \) with \( \mathbb{E}_P[U(X)] < +\infty \). Hence, we can apply Proposition 3.4 and conclude the result.  

\[\square\]
3.2.3 Conditional Value at Risk Problem

**Theorem 3.8 (Feasibility of \( P_{\text{CVaR}^n} \)):**
The set of admissible \( X \) of the problem \( P_{\text{CVaR}^n} \) is non-empty if and only if
- \( x_0 \geq c \cdot Q (\Omega) \) or
- \( Q (\Omega) < t_{\text{max}} \cdot \alpha \).

Note that if \( t_{\text{max}} = +\infty \), then \( t_{\text{max}} \cdot \alpha = +\infty \).

**Proof:**
- If \( x_0 \geq c \cdot Q (\Omega) \), then \( \text{CVaR}_1^P (c) = -c \).
  Therefore \( c \) is admissible for \( (P_{\text{CVaR}^n}) \) which means that the set of admissible \( X \) of \( (P_{\text{CVaR}^n}) \) is non-empty.
- Suppose that \( Q (\Omega) < t_{\text{max}} \cdot \alpha \).
  In this case, we define \( h := \max \left\{ \frac{c+1}{C_{\text{CVaR}}} (x_0 - c \cdot t_{\text{max}} \cdot \alpha) + 1 \right\} \).
  
  \[ x_0 > \left\{ \begin{array}{ll}
  -\infty & \text{if } t_{\text{max}} = +\infty \\
  h \cdot (Q (\Omega) - t_{\text{max}} \cdot \alpha) + t_{\text{max}} \cdot c \cdot \alpha & \text{if } t_{\text{max}} < +\infty \\
  \end{array} \right. \]
  
  \[ = h \cdot Q (\Omega) - t_{\text{max}} \cdot (h - c) \cdot \alpha. \]
  
  \[ \Rightarrow (P_{\text{ES}_P}) \text{ is non-empty.} \]

If \( \alpha < 1 \), then \( (P_{\text{CVaR}^n}) \) is non-empty.

Case \( \alpha = 1 \):
We observe that the constructed admissible random variables in the proof of Theorem 3.5 are not only admissible for \( (P_{\text{ES}_P}) \), but they are integrable with respect to \( P \) (they take at most 2 distinct finite values), too. Finally, every random variable \( X \) that is integrable with respect to \( P \) and is admissible for \( (P_{\text{ES}_P}) \) is also admissible for \( (P_{\text{CVaR}^n}) \), because

\[ E_P [X] = E_P [h] + E_P [(X - h)^+] - E_P [(h - X)^+] \geq h \cdot 1 - 0 + (h - c) \cdot 1 = c \cdot 1. \]

- Let \( (P_{\text{CVaR}^n}) \) be non-empty.
  - If \( \alpha = 1 \) and \( Q (\Omega) = t_{\text{max}} \):
    \[ x_0 = \int_{\Omega} X dP = \int_{\Omega} t_{\text{max}} dP = E_P[X] \cdot t_{\text{max}} \geq c \cdot Q (\Omega). \]
  - Case \( \alpha < 1 \):
    \[ \Rightarrow \exists h \in \mathbb{R}: (P_{\text{ES}_P}) \text{ is non-empty.} \]
    - If \( h = c \), then \( (P_{\text{ES}_P}) \) being non-empty implies that \( x_0 \geq c \cdot Q (\Omega) \).
    - Let \( h > c \).
      
      \[ Q (\Omega) \geq t_{\text{max}} \cdot \alpha \text{, so } (P_{\text{ES}_P}) \text{ being non-empty results in } x_0 \geq h \cdot Q (\Omega) - t_{\text{max}} \cdot (h - c) \cdot \alpha \geq 0. \]
      Otherwise \( Q (\Omega) < t_{\text{max}} \cdot \alpha. \)

\[ \square \]

**Theorem 3.9 (Feasibility of \( P_{\text{CVaR}^\alpha} \)):**
The set of admissible \( X \) of the problem \( P_{\text{CVaR}^\alpha} \) is non-empty if and only if \( x_0 \geq c \cdot Q (\Omega) \).
Proof: For $\alpha \in \langle 0, Q(\Omega) \rangle$ it is a consequence of $h \cdot Q(\Omega) - (h - c) \alpha = h \cdot (Q(\Omega) - \alpha) + c \cdot \alpha$, Theorem 3.6 and Theorem 2.9.

For $\alpha = Q(\Omega)$, $c \cdot Q(\Omega) \leq -\text{CVaR}^\nu_{\nu(\Omega)}(X) \cdot Q(\Omega) = \mathbb{E}_\nu[X] \leq x_0 \iff x_0 \geq c \cdot Q(\Omega)$. \hfill \qed

### 3.3 Optimal Solutions and Their Properties

In this chapter, we present the set of solutions of the different problems and look at some of their properties. The rather lengthy proofs of the statements will be given separately, in Chapter 3.4.

#### 3.3.1 Assumptions

For the investigation of the feasibility of all problems, the exact definition of the function $U$ was irrelevant. For the existence and computation of optimal solutions of the problems, this is a whole different ballpark. The following assumptions are, except for Assumption 3.11 and Assumption 3.13, required for an analogous investigation of the problem without risk restriction, too.

**Assumption 3.10:**

\[
\exists u_0 \in \mathbb{R} \text{ with } \forall x \in (-\infty, u_0): U(x) = -\infty, \forall x \in (u_0, +\infty): U(x) > -\infty.
\]

**Assumption 3.11:**

$\frac{dQ}{dP}$ is continuous.

As we will see, Assumption 3.11 guarantees that the optimal solutions we derive can be written as a function of $\frac{dQ}{dP}$. It is not without drawbacks, because it prevents us from investigating the situation in certain finite financial market models like multinomial tree models and in the case $P = Q$:

**Example 3.12:**

If $P = Q$, then $\frac{dQ}{dP} = 1$, $P \left( \frac{dQ}{dP} \leq t \right) = \mathbb{1}_{\{1, +\infty\}}(t)$ and $\sigma \left( \frac{dQ}{dP} \right) = \{\emptyset, \Omega\}$.

**Assumption 3.13:**

$c \geq u_0$ and $q \geq u_0$.

If Assumption 3.13 cannot be fulfilled (even by choosing $\alpha = \nu(\Omega)$ in problem $(P_{\text{ESP}}^\nu)$), the expected shortfall restriction and the conditional value at risk restriction can be dropped for all reasonable initial endowments $x_0$. Thus the problems is reduced to the well-known classical one.

The same reasoning applies to the value at risk case.

A treatment of this classic case can be found for instance in chapter 3 of [Karatzas and Shreve, 1998].

**Assumption 3.14:**

$U$ is continuous in the interval $[u_0, +\infty)$ and features a continuous and strictly decreasing first derivative $U'$ in the interval $(u_0, +\infty)$ with values in $(0, +\infty)$. Furthermore, $\lim_{x \to u_0^+} U'(x) = +\infty$ and $\lim_{x \to +\infty} U'(x) = 0$. 

---

25
3.3. OPTIMAL SOLUTIONS AND THEIR PROPERTIES

Definition \((U_\gamma)\): For \(\gamma \in (-\infty, 1)\), we define \(U_\gamma: \mathbb{R} \rightarrow (-\infty, +\infty)\) with

- \(U_\gamma(x) := \begin{cases} \ln(x) & x > 0 \\ -\infty & x \leq 0 \end{cases}\) if \(\gamma = 0\), the ubiquitous logarithmic utility function,
- \(U_\gamma(x) := \begin{cases} \frac{x^\gamma}{\gamma} & x > 0 \\ -\infty & x \leq 0 \end{cases}\) if \(\gamma \in (-\infty, 0)\) and
- \(U_\gamma(x) := \begin{cases} \frac{x^\gamma}{\gamma} & x \geq 0 \\ -\infty & x < 0 \end{cases}\) if \(\gamma \in (0, 1)\), both known as power utility function.

Example 3.15:
Assumption 3.10 (choose \(u_0 = 0\)) and Assumption 3.14 hold for instance for \(U_\gamma\).

Definition \((U')\):
For convenience, we extend the domain of the function \(U'\) to \(\mathbb{R}\) by defining \(\forall x \leq u_0: U'(x) := +\infty\).

Definition \(((U')^{-1})\):
Let \((U')^{-1}: (0, +\infty) \rightarrow (u_0, +\infty)\) be the inverse function of \(U'\) restricted to its natural domain \((u_0, +\infty)\). We extend the definition of \((U')^{-1}\) by setting \((U')^{-1}(+\infty) := u_0\) and \(\forall x \in [-\infty, 0): (U')^{-1}(x) := +\infty\).

Example 3.16:
\(U'_\gamma(x) = \begin{cases} x^{\gamma - 1} & x > 0 \\ -\infty & x \leq 0 \end{cases}\) and \((U'_\gamma)^{-1}(x) = \begin{cases} +\infty & x \leq 0 \\ x^{\frac{1}{\gamma - 1}} & 0 < x < +\infty \\ 0 & x = +\infty \end{cases}\).

Lemma 3.17:
The mapping \((U')^{-1}\) is monotone decreasing and continuous on all of its domain \([-\infty, +\infty]\) with \((U')^{-1}([-\infty, +\infty]) = [u_0, +\infty]\).

Proof: Direct consequence of Assumption 3.14 and the definition of \((U')^{-1}\). \(\square\)

Since we are going to depend on the well-definedness of the goal function of our candidate random variables (cf. Lemma 3.36, Lemma 3.50 and Fact 3.72(b)) as well as finiteness of their prices (cf. Lemma 3.37, Lemma 3.51 and Fact 3.72(c)), we need the following assumptions:

Assumption 3.18:
Suppose that either
- \(\forall \lambda_1 \in (0, +\infty): \mathbb{E}_P \left[ U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right] < +\infty\) or
- \(\forall \lambda_1 \in (0, +\infty): \mathbb{E}_P \left[ -U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right] < +\infty\).

Assumption 3.19:
\(\forall \lambda_1 \in (0, +\infty): \mathbb{E}_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right] < +\infty\).

Now we are prepared to turn our attention to the individual problems.
3.3. OPTIMAL SOLUTIONS AND THEIR PROPERTIES

3.3.2 Value at Risk Problem

Definition ($X_{\text{VaR}}^{\lambda,J}$): For all $\lambda_1 \in (0, +\infty]$ and $J \in (0, +\infty)$, we define $X_{\text{VaR}}^{\lambda,J}$ by setting $\forall \omega \in \Omega$:

$$
X_{\text{VaR}}^{\lambda,J}(\omega) := \left\{ \begin{array}{ll}
\max\left\{ q, (U')^{-1}\left(\lambda_1 \cdot \frac{dQ}{dP}(\omega)\right) \right\} & \text{if } \frac{dQ}{dP}(\omega) \leq J \\
(U')^{-1}\left(\lambda_1 \cdot \frac{dQ}{dP}(\omega)\right) & \text{if } \frac{dQ}{dP}(\omega) > J
\end{array} \right. .
$$

Definition ($\Lambda_{x_0}^{\nu}$): For the measures $\nu \in \{P, Q\}$, $\epsilon \in (0, \nu(\Omega))$ and all initial endowments

$$
x_0 > q \cdot Q \left(\frac{dQ}{dP} \leq J\right) + u_0 \cdot Q \left(\frac{dQ}{dP} > J\right)
$$

with $J := \left(F_{\nu_{UQ}}\right)^{-1}_{\nu(\Omega) - \epsilon}$, we set $\Lambda_{x_0}^{\nu}$ to the unique value $\lambda_1 \in (0, +\infty)$ such that the equation

$$
E_Q \left[ X_{\text{VaR}}^{\lambda,J} \right] = x_0
$$

is fulfilled. Since we focus on the results here, we defer the proof of the well-definedness of $\Lambda_{x_0}^{\nu}$ to Lemma 3.38 in Chapter 3.4.

Theorem 3.20 (Optimal Solutions of ($P_{\text{VaR}}^{\nu}$)):

Let $x_0 \in \mathbb{R}$, $J := \left(F_{\nu_{UQ}}\right)^{-1}_{\nu(\Omega) - \epsilon}$ and $x_0^{\min} := q \cdot Q \left(\frac{dQ}{dP} \leq J\right) + u_0 \cdot Q \left(\frac{dQ}{dP} > J\right)$.

- If $x_0 > x_0^{\min}$, $X_{\text{VaR}}^{\Lambda_{x_0}^{\nu},J}$ is an optimal solution of ($P_{\text{VaR}}^{\nu}$).
  It is unique ($P$-almost-surely) if and only if in addition $E_P \left[ U \left( X_{\text{VaR}}^{\Lambda_{x_0}^{\nu},J} \right) \right] < +\infty$.

- If $U(u_0) > -\infty$ and $x_0 = x_0^{\min}$,
  $q \cdot 1_{[0,J]} \left(\frac{dQ}{dP}\right) + u_0 \cdot 1_{[J,+\infty)} \left(\frac{dQ}{dP}\right)$ is the ($P$-almost-surely) unique optimal solution of ($P_{\text{VaR}}^{\nu}$).

- If either $x_0 < x_0^{\min}$ or both $x_0 = x_0^{\min}$ and $U(u_0) = -\infty$:
  All admissible $X$ of ($P_{\text{VaR}}^{\nu}$) are optimal solutions of ($P_{\text{VaR}}^{\nu}$).

Comment: Let $J := \left(F_{\nu_{UQ}}\right)^{-1}_{\nu(\Omega) - \epsilon}$, $Y := (U')^{-1}\left(\lambda_1 \cdot \frac{dQ}{dP}\right)$ and $K := (U')^{-1}(\lambda_1 \cdot J)$.

Then $J > \frac{U'(q)}{\lambda_1} \iff K < q$ and

$$
X_{\text{VaR}}^{\lambda,J}(\omega) := \left\{ \begin{array}{ll}
(U')^{-1}\left(\lambda_1 \cdot \frac{dQ}{dP}(\omega)\right) & \text{if } \lambda_1 \cdot \frac{dQ}{dP}(\omega) < U'(q) \\
q & \text{if } U'(q) \leq \lambda_1 \cdot \frac{dQ}{dP}(\omega) \text{ and } \frac{dQ}{dP}(\omega) \leq J \\
(U')^{-1}\left(\lambda_1 \cdot \frac{dQ}{dP}(\omega)\right) & \text{if } \frac{dQ}{dP}(\omega) > J
\end{array} \right. ,
$$

$$
\lambda_1 \leq +\infty \left\{ \begin{array}{ll}
q + (Y(\omega) - q)^+ - (K - Y(\omega))^+ & \text{if } K < q \\
Y(\omega) & \text{if } K \geq q
\end{array} \right. .
$$

If $K < q$ and $\lambda_1 < +\infty$, $X_{\text{VaR}}^{\lambda,J}$ can be achieved by

- buying a risk free asset with payoff $q$,
- buying a European Call Option on $Y$ with strike price $q$. 

27
• selling a European Put Option on $Y$ with strike price $K$ and
• selling a European Binary Put Option on $Y$ with payoff $q - K$ if $Y < K$.

**Comment:** Let us compare the optimal solution of the problem $(P_{VaR^v})$ with the optimal solution of the same problem without the value at risk restriction $(P_0)$:

![Figure 3.1: Optimal Solution Value at Risk vs. Classic Problem $(U = U_0)$](image)

We recall that the additional restriction in $(P_{VaR^v})$ is a risk restriction. Thus one thing immediately catches our eyes when looking at the graph: All "bad" events are located right to $(F_{\nu}dQ_{\nu})^{-1}(\nu(\Omega) - \epsilon)$ and in all these events the optimal random variable of the unrestricted problem is above (strictly speaking at least not below) the optimal random variable of $(P_{VaR^v})$.

Hence, by imposing a value at risk restriction, the optimal behavior is to take even more risk — in the sense of facing a lower outcome — in this critical region. This behavior can be easily explained by the fact that the value at risk does not take into account the size of the shortfall.

It can be proven analytically quite easily, too:

Suppose that $\lambda_1^a$ and $\lambda_1^b \in (0, +\infty)$ with $E_Q[X_{VaR}^{\lambda_1^a, J}] = E_Q[(U')^{-1}(\lambda_1^a \cdot \frac{dQ}{dP})]$.

Then $\lambda_1^a \leq \lambda_1^b$, because $X_{VaR}^{\lambda_1^a, J} \geq (U')^{-1}(\lambda_1^a \cdot \frac{dQ}{dP})$ and the mapping $\lambda_1 \mapsto E_Q[(U')^{-1}(\lambda_1 \cdot \frac{dQ}{dP})]$ is strictly decreasing.

$\Rightarrow \forall \omega \in \Omega$ with $\frac{dQ}{dP}(\omega) > J$: $X_{VaR}^{\lambda_1^b, J}(\omega) = (U')^{-1}(\lambda_1^b \cdot \frac{dQ}{dP}) \leq (U')^{-1}(\lambda_1^a \cdot \frac{dQ}{dP})$.

Next, we want to investigate the stunning fact that the problems $(P_{VaR^v})$ and $(P_{VaR^Q})$ essentially have the same solution.

**Proposition 3.21:**

If $F_{\nu}P_{\nu}$ is continuous, $\epsilon_P \in [0, 1]$ and $\epsilon_Q := Q\left(\frac{dQ}{dP} > F_{\nu}^{-1}(1 - \epsilon_P)\right)$, we know that $\forall A \in \mathcal{F}$ with $P(A) \leq \epsilon_P$: $Q(A) \leq \epsilon_Q$. 

---

28
Lemma 3.22:
Let $\epsilon_P \in (0, 1)$ be fixed.
Then there exists an optimal solution of the problem $\left(P_{\text{VaR}}\right)$ with the value at risk restriction $Q(X < q) \leq \epsilon_Q := Q \left(\frac{dQ}{dP} > F_{\frac{1}{\alpha}}^{-1} (1 - \epsilon_P)\right)$ that solves $\left(P_{\text{VaR}^P}\right)$ with the value at risk restriction $P(X < q) \leq \epsilon_P$, too.

Proof: $\left(P_{\text{VaR}}\right)$ has a monotonic decreasing (in $\frac{dQ}{dP}(\omega)$) solution $X^\star$.
\[ \Rightarrow \{X^\star < q\} \subseteq \left\{ \frac{dQ}{dP} > F_{\frac{1}{\alpha}}^{-1} (1 - \epsilon_P) \right\}. \]
\[ \Rightarrow P(X^\star < q) \leq P\left(\frac{dQ}{dP} > F_{\frac{1}{\alpha}}^{-1} (1 - \epsilon_P)\right) = \epsilon_P. \]
\[ \Rightarrow X^\star \text{ is admissible for } \left(P_{\text{VaR}^P}\right). \]
(Proposition 3.21) \Rightarrow $\{X | P(X < q) \leq \epsilon_P\} \subseteq \{X | Q(X < q) \leq \epsilon_Q\}$. \(\square\)

3.3.3 Expected Shortfall Problem

In case of the expected shortfall restriction, it is no longer irrelevant what measure $\nu$ we use in its definition. Let us look at the case $\nu = P$ first.

Expected Shortfall Restriction with respect to $P$

For stating the main result in this case, we need several definitions. The well-definedness of the following expressions is either immediately clear or will be investigated in Chapter 3.4.

Definition ($\tau(h)$): Let $\tau: [c, +\infty) \to [0, +\infty]$,
\[ h \mapsto \left\{ \begin{array}{ll}
\max \{t \in [0, +\infty) \mid P\left(\frac{dQ}{dP} < t\right) = 1 - \alpha\} & \text{if } c = u_0 \\
\max \{t \in (0, +\infty) \mid P\left(\frac{dQ}{dP} < t\right) = 1 - \alpha \cdot \frac{h-c}{\lambda_2}\} & \text{if } c > u_0
\end{array} \right.. \]

Definition ($H_P, \overline{H}_P$): We set
- $H_P := \left\{ h \in [c, +\infty) \mid x_0 > Q\left(\frac{dQ}{dP} < \tau(h)\right) \cdot h + Q\left(\frac{dQ}{dP} \geq \tau(h)\right) \cdot u_0 \right\}$ and
- $\overline{H}_P := \left\{ h \in [c, +\infty) \mid x_0 \geq Q\left(\frac{dQ}{dP} < \tau(h)\right) \cdot h + Q\left(\frac{dQ}{dP} \geq \tau(h)\right) \cdot u_0 \right\}$.

Definition ($L_P$): We abbreviate $[u_0, +\infty) \times ((0, +\infty] \times [0, +\infty)) \setminus \{(+\infty, +\infty)\}$ by $L_P$.

Definition ($X^{h, \lambda_1, \lambda_2}$): We define the family of random variables $X^{h, \lambda_1, \lambda_2}$ for all $(h, \lambda_1, \lambda_2) \in L_P$ by setting for all $\omega \in \Omega$:
\[
X^{h, \lambda_1, \lambda_2}(\omega) := \left\{ \begin{array}{ll}
(U')^{-1}\left(\lambda_1 \cdot \frac{dQ}{dP}(\omega)\right) & \text{if } \lambda_1 \cdot \frac{dQ}{dP}(\omega) < U'(h) \\
h & \text{if } U'(h) \leq \lambda_1 \cdot \frac{dQ}{dP}(\omega) \leq U'(h) + \lambda_2 \\
(U')^{-1}\left(\lambda_1 \cdot \frac{dQ}{dP}(\omega) - \lambda_2\right) & \text{if } \lambda_1 \cdot \frac{dQ}{dP}(\omega) > U'(h) + \lambda_2
\end{array} \right..
\]
For all \((h, \lambda_1, \lambda_2) \in L_P\) and \(\omega \in \Omega\) the following two useful identities hold:

\[
X^{h,\lambda_1,\lambda_2}(\omega) = \min \left\{ \max \left\{ h, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\}, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) - \lambda_2 \right) \right\} = h + \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) - h \right) \right)^+ - \left( h - (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) - \lambda_2 \right) \right)^+.
\]

**Definition \((\Lambda^{h,\lambda_2})\):** For fixed \(x_0 \geq u_0 \cdot Q(\Omega)\) we define \(\Lambda^{h,\lambda_2}\) as follows:

For \((h, \lambda_2) \in [u_0, +\infty) \times \mathbb{R}_+,\) we set \(\Lambda^{h,\lambda_2} := \inf \left\{ \lambda_1 \in (0, +\infty) \left| E_Q \left[ X^{h,\lambda_1,\lambda_2} \right] \leq x_0 \right\} \in (0, +\infty) \)

and for \(h \in \mathbb{R},\) \(\Lambda^{h,+\infty} := \inf \left\{ \lambda_1 \in (0, +\infty) \left| E_Q \left[ X^{h,\lambda_1,+\infty} \right] \leq x_0 \right\} \in (0, +\infty) \).

**Definition \((L^h_2)\):** For all \(h \in \mathbb{R}\): \(L^h_2 := \left\{ \begin{array}{ll} \mathbb{R}_+ & \text{if } h \geq \frac{x_0}{\nu(\Omega)} \\ [0, +\infty] & \text{if } h < \frac{x_0}{\nu(\Omega)} \end{array} \right\}.

**Definition \((\Psi^h)\):** Let \(\Psi^h: H_P \rightarrow [0, +\infty]\) with

\[
\Psi^h := \inf \left\{ \lambda_2 \in L^h_2 \left| \mathbb{E}_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_1,\lambda_2}} \right)^+ \right] \leq (h - c) \cdot \alpha \right\}.
\]

The characterization of the optimal solutions of \((P_{ES^h_P})\) is a mixture of all previous ingredients:

**Theorem 3.23 (Optimal solutions of \((P_{ES^h_P})\)):**

Let \(x_0 \in \mathbb{R}\) and \((P_{ES^h_P})\) be feasible (cf. Theorem 3.5).

- If \(h \in H_P\):
  \(X^{h,\Lambda^{h,\Psi^h}}\) is an optimal solution of \((P_{ES^h_P})\).
  The solution is unique (\(P\)-almost-surely) if and only if \(\mathbb{E}_P \left[ U \left( X^{h,\Lambda^{h,\Psi^h}} \right) \right] < +\infty\).

- If \(U(u_0) > -\infty\) and \(h \in \overline{H_P} \setminus H_P\):
  \(h \cdot \mathbb{1}_{[0, h]}(u_0 \cdot \nu(h)) \cdot \frac{dQ}{dP} + u_0 \cdot \mathbb{1}_{[h, +\infty]}(u_0 \cdot \nu(h)) \cdot \frac{dQ}{dP}\) is the (\(P\)-almost-surely) unique optimal solution of \((P_{ES^h_P})\).

- If \(h \notin \left\{ \begin{array}{ll} \overline{H_P} & \text{if } U(u_0) > -\infty \\ H_P & \text{if } U(u_0) = -\infty \end{array} \right\}:
  \)
  All admissible \(X\) of \((P_{ES^h_P})\) are optimal solutions of \((P_{ES^h_P})\).

**Comment:** \(X^{h,\Lambda^{h,\Psi^h}}\) is a pointwise continuous function of \(h\) in the set \(H_P\) due to Lemma 3.79, Fact 3.69(d), Fact 3.69(a) and the independence of \(X^{h,\Lambda^{h,0,0}}\) from \(h\). This fact is important for the upcoming treatment of the problem with conditional value at risk constraint.

**Comment:** Again, we compare the optimal solution of the restricted problem \((P_{ES^h_P})\) with the optimal solution of the unrestricted one: The optimal solution waives part of the unrestricted optimal claim in cheap and in very expensive events to finance complete or partial insurance against small loses (i.e. values not far below \(h\)). This behavior is similar to the behavior of the optimal solution of the value at risk restriction problem \((P_{VaR^\lambda})\).
3.3. OPTIMAL SOLUTIONS AND THEIR PROPERTIES

More formally:
Consider $\lambda_1 \in (0, +\infty)$ with $x_0 = \mathbb{E}_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right]$. Then $\lambda_1 \leq \Lambda^{h,\psi_h}$ and $\forall \omega \in \Omega$ with $\Lambda^{h,\psi_h} \cdot \frac{dQ}{dP}(\omega) > U'(h) + \Psi^h$ and $\left( \Lambda^{h,\psi_h} - \lambda_1 \right) \frac{dQ}{dP}(\omega) \geq \Psi^h$.

Hence, the optimal solution of $(P_{ES_{h}})$ is less cautious in extreme states than the optimal solution of the unrestricted problem. This behavior — although not as extreme as with the value at risk restriction — might still be a concern for regulators or the like that consider to enjoin the use of a specific risk measure.

**Comment:** The optimal solution of the value at risk problem could be interpreted as a combination of a risk free investment and a collection of options on some underlying (see Section 3.3.2). A similar interpretation is not possible for the optimal solution of $(P_{ES_{h}})$, because the underlying $(U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} - \lambda_2 \right)$ does not exist for $\lambda_1 < +\infty$ and $\lambda_2 > 0$: It takes the value $+\infty$ for all $\omega \in \Omega$ with $\frac{dQ}{dP}(\omega) < \frac{\lambda_2}{\lambda_1}$ and therefore has the price $+\infty$.

However, the last part of the claim can be decomposed in the following way: Using the abbreviation $K := (U')^{-1} \left( U'(h) + \lambda_2 \right)$ we can write for all $h \geq u_0$

$$
\left( h - (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} - \lambda_2 \right) \right)^+ = \mathbb{I} \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) < K \right) \cdot h \\
- \mathbb{I} \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) < K \right) \cdot (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} - \lambda_2 \right).
$$

Hence, the claim can be composed into a risk free asset, a European Call Option, the short-selling of a European Binary Put Option and the claim $\mathbb{I} \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) < K \right) \cdot (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} - \lambda_2 \right)$. 

---

**Figure 3.2:** Optimal Solution Expected Shortfall under $P$ vs. Classic Problem ($U = U_0$)
3.3. OPTIMAL SOLUTIONS AND THEIR PROPERTIES

Expected Shortfall Restriction with respect to $Q$

Since the structure of the solutions of the problems $(P_{ES}^h)$ and $(P_{ES}^a)$ are alike, we need a similar list of definitions:

**Definition ($H_Q$):** $H_Q := \{ h \in [c, +\infty) \mid x_0 \geq h \cdot Q(\Omega) - (h - c) \cdot \alpha \}.$

**Definition ($L_Q$):** Let $L_Q := [u_0, +\infty) \times \{ (\lambda_1, \lambda_3) \in (0, +\infty] \times [0, +\infty] \mid \lambda_1 \geq \lambda_3 \}.$

**Definition ($X_Q^{h,\lambda_1,\lambda_3}$):** For all $(h, \lambda_1, \lambda_3) \in L_Q$ and $\omega \in \Omega$:

\[
X_Q^{h,\lambda_1,\lambda_3}(\omega) := \min \left\{ \max \left\{ h, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\}, (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP}(\omega) \right) \right\} = h + \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) - h \right)^+ - \left( h - (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP}(\omega) \right) \right)^+.
\]

**Definition ($\lambda^h_Q$):** Let $c \in [u_0, +\infty)$ be fixed. For all $h \in [c, +\infty)$, we define $\lambda^h_Q$ as the unique value $\lambda_3 \in \left\{ +\infty, \left[ \frac{U'(h)}{t_{\max}}, +\infty \right) \right\}$ with $E_Q \left[ \left( h - (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right) \right)^+ \right] = (h - c) \cdot \alpha.$

**Definition ($\lambda^{x_0}_Q$):** For all initial endowments $x_0 \geq u_0 \cdot Q(\Omega)$, we set $\Lambda^{x_0}_Q$ to the unique value $\lambda_3 \in (0, +\infty]$ with $E_Q \left[ (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right) \right] = x_0$.

**Definition ($\Psi^{x_0}_Q$):** Let $c \in [u_0, +\infty)$ be fixed. For all $h \in [c, +\infty)$ and $x_0 \geq u_0 \cdot Q(\Omega)$, we define $\Psi^{x_0}_Q := \min \left\{ \lambda^h_Q, \lambda^{x_0}_Q \right\} \in [0, +\infty].$

**Definition ($\Lambda^{x_0}_Q$):** Let $c \in [u_0, +\infty)$ be fixed. For all $h \in [c, +\infty)$ and initial endowments $x_0 \in [h \cdot Q(\Omega) - (h - c) \alpha, +\infty)$, we set $\Lambda^{x_0}_Q$ to the unique $\lambda_1 \in (0, +\infty] \cap \left\{ \Psi^{x_0}_Q, \max \left\{ \left. \frac{U'(h)}{t_{\min}}, \Psi^{x_0}_Q \right\} \right\}$

with $E_Q \left[ X_Q^{h,\lambda_1,\Psi^{x_0}_Q} \right] = x_0.$

The main result can be given in the following compact form:

**Theorem 3.24** (Optimal solutions of $(P_{ES}^a)$):

Let $x_0 \in \mathbb{R}$ and $(P_{ES}^a)$ be feasible (cf. Theorem 3.6).

Then $X_Q^{h,\Lambda^{x_0}_Q,\Psi^{x_0}_Q}$ is an optimal solution of $(P_{ES}^a)$.

- **Case $x_0 \in (h \cdot Q(\Omega) - (h - c) \alpha, +\infty)$:**  

  The solution is unique ($P$-almost-surely) if and only if $E_P \left[ U \left( X_Q^{h,\Lambda^{x_0}_Q,\Psi^{x_0}_Q} \right) \right] < +\infty.$

- **Case $x_0 = h \cdot Q(\Omega) - (h - c) \alpha$:**
  
  If $U(u_0) > -\infty$, $\alpha \in (0, Q(\Omega))$, $c > u_0$ or $h = c$, it is unique ($P$-almost-surely).

  Otherwise, all admissible $X$ of $(P_{ES}^a)$ are optimal solutions of $(P_{ES}^a)$.
Comment: Once more we are interested in comparing the solution with the optimal solution of the classical problem \((P_0)\). From the definition of \(\Psi^{h,x_0}_Q\) it is immediately clear that \(\Psi^{h,x_0}_Q \leq \lambda^{x_0}_0\). Combined with \(\Lambda^{h,x_0}_Q \geq \lambda^{x_0}_0\) (Fact 3.74), we can conclude that for fixed \(\omega \in \Omega\), the solution of the classical problem is above the solution of problem \((P_{ES}^h)\) if and only if it is above \(h\). This observation is also true if one replaces the word above by the word below in both occurrences.

If we compare this result with the behavior of the optimal solution of \((P_{VaR}^\nu)\) and especially with the behavior of the optimal solution of \((P_{ES}^h)\), we find that from a regulator’s perspective the usage of the expected shortfall risk measure with \(\nu = Q\) has a clear advantage over the usage of the expected shortfall risk measure with \(\nu = P\) or the value at risk measure: An agent optimizing his terminal wealth has an incentive to take less risk in all states of nature.

This interpretation can already be found in [Basak and Shapiro, 2001]. The authors only compare the changes in the optimal behavior stemming from an addition of an expected shortfall restriction with \(\nu = Q\) or a value at risk restriction with \(\nu = P\). They declare the expected shortfall restriction to be superior based on this observation and a comparison of the corresponding hedging strategies, yet do not take into account any of the coherence axioms.

For sake of completeness, we recall what we already mentioned in the introduction: A regulator might prefer a neutral measure in the definition of the risk measure to the subjective measure of the investor.

Comment: The optimal solutions possess static replication strategies, which rely on two different yet similar underlyings \(Y_1 := (U')^{-1} \left( \Lambda^{h,x_0}_Q \cdot \frac{dQ}{dP} \right)\) and \(Y_2 := (U')^{-1} \left( \Psi^{h,x_0}_Q \cdot \frac{dQ}{dP} \right)\). Depending on the Lagrange multipliers, \(X^{h,\Lambda^{h,x_0}_Q,\Psi^{h,x_0}_Q}_Q\) can be replicated using one of the following strategies:

- If \(\Psi^{h,x_0}_Q > 0\) and \(\Lambda^{h,x_0}_Q < +\infty\), the optimal claim can be achieved by
  - buying a risk free asset with payoff \(h\),
  - buying a European Call Option on \(Y_1\) with strike price \(h\) and
  - selling a European Put Option on \(Y_2\) with strike price \(h\).
If $\Psi_Q^{h,x_0} = 0$ and $\Lambda_Q^{h,x_0} < +\infty$, the optimal claim can be obtained by
- buying a risk free asset with payoff $h$
- buying a European Call Option on $Y_1$ with strike price $h$.

If $0 < \Psi_Q^{h,x_0} < +\infty = \Lambda_Q^{h,x_0}$, the optimal claim is attained by
- buying a risk free asset with payoff $h$
- selling a European Put Option on $Y_2$ with strike price $h$.

Otherwise, the optimal claim is a risk free asset with payoff $h$ if $\Psi_Q^{h,x_0} = 0$ and $\Lambda_Q^{h,x_0} = +\infty$, and payoff $u_0$ if $\Psi_Q^{h,x_0} = \Lambda_Q^{h,x_0} = +\infty$.

**Properties of the Maximal Expected Utility**

Let us take a look at few properties of the goal function given the following additional assumption:

**Assumption 3.25:**
\[
\forall \lambda_1 \in (0, +\infty): \mathbb{E}_P \left[U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right] < +\infty.
\]

**Proposition 3.26:**
Let Assumption 3.25 be fulfilled, $\nu \in \{P, Q\}$, $H := \{ h \in \mathbb{R} \mid (P_{ES^\nu}) \text{ is feasible} \}$ and for all $h \in H$, $X(h)$ be an optimal solution of $(P_{ES^\nu})$. The mapping $H \to \mathbb{R}$, $h \mapsto \mathbb{E}_P[U(X(h))]$ is
- upper semi-continuous and concave,
- continuous on $\begin{cases} H_P & \text{if } \nu = P \\ H_Q & \text{if } \nu = Q \end{cases}$ and
- strictly concave on $\{ h \in H \mid \mathbb{E}_P[U(X(h))] > -\infty \}$ if for all $h \in H$, the solution of problem $(P_0)$, $(U')^{-1} \left( \lambda_0 \cdot \frac{dQ}{dP} \right)$, is not admissible for $(P_{ES^\nu})$.

**Proof:** See page 68. \qed

An example of the function is given in Figure 4.13 on page 92.

### 3.3.4 Conditional Value at Risk Problem

In this scenario, we assume that Assumption 3.25 holds. It allows us to obtain the following results:

**Theorem 3.27 (Optimal Solution of $(P_{CVaR^\alpha})$):**
Let $x_0 \in \mathbb{R}$, $\alpha \in (0, 1)$ and $(P_{CVaR^\alpha})$ be feasible (cf. Theorem 3.8).

- Case $\overline{H}_P \neq \emptyset$:
  \[ \exists h \in \overline{H}_P \text{ such that } (P_{ES^\nu}) \text{ has optimal solutions and at the same time any such solution is an optimal solution of } (P_{CVaR^\alpha}). \]
- Case $\overline{H}_P = \emptyset$: All admissible $X$ are optimal solutions.
3.4. HOW TO PROVE OPTIMALITY

Example 3.28:
Let \( \alpha = 1 \), \( t_{\min} = 0 \) and \( x_0 > u_0 \cdot Q(\Omega) \). Then \((U')^{-1} \left( \lambda_{0}^{x_0} \cdot \frac{dQ}{dP} \right)\) is the unique optimal solution of \((P_0)\) and there exists a sequence \((X_n)\) which is admissible for \((P_{\text{CVaR}}^\alpha)\) such that \(E_P [U (X_n)] \to E_P \left[ U \left( (U')^{-1} \left( \lambda_{0}^{x_0} \cdot \frac{dQ}{dP} \right) \right) \right]\).

Therefore, if \((U')^{-1} \left( \lambda_{0}^{x_0} \cdot \frac{dQ}{dP} \right)\) is not admissible for \((P_{\text{CVaR}}^\alpha)\), we cannot expect \((P_{\text{CVaR}}^\alpha)\) to have an optimal solution.

\[\text{Theorem 3.29 (Optimal Solution of } (P_{\text{CVaR}}^\alpha)\text{):}\]
Let \(x_0 \in \mathbb{R}, \alpha \in (0, Q(\Omega))\) and \((P_{\text{CVaR}}^\alpha)\) be feasible (cf. Theorem 3.9).

- If \(\alpha = Q(\Omega)\), \((P_0)\) has optimal solutions \(X_k\) with \(E_Q [X_k] \geq c \cdot Q(\Omega)\). Any such \(X_k\) is an optimal solution of \((P_{\text{CVaR}}^\alpha)\).
- Otherwise, \(\exists h \in [c, \frac{x_0 - c \cdot Q(\Omega)}{Q(\Omega) - \alpha}]\) such that \((P_{\text{ES}}^\alpha_h)\) has optimal solutions and any one of them is an optimal solution of \((P_{\text{CVaR}}^\alpha)\).

It is important to realize that since only solutions of \((P_{\text{ES}}^\alpha_h)\) solve \((P_{\text{CVaR}}^\alpha)\), the properties found in Section 3.3.3 for the solution of \((P_{\text{ES}}^\alpha_h)\) translate directly into properties of the solution of \((P_{\text{CVaR}}^\alpha)\).

3.4 How to Prove Optimality

In this chapter, we want to take a look at the derivation of the previous main results. To make the presentation more readable, we split it into three parts. In Section 3.4.1, we present the fundamental ideas that we use to construct the optimal solutions of the static optimization problems. The remainder of the chapter tells us how these ideas translate into target milestones in each case. Rigorous proofs of most of these milestones are outsourced to Chapter 6.2.

3.4.1 Approach

We are going to construct the solutions of the problem \((P_{\text{CVaR}}^\alpha)\) for \(\alpha \in (0, \nu(\Omega))\) by reducing it to a family of problems \((P_{\text{ES}}^\alpha_h)\). Hence we are dealing with problem \((P_{\text{ES}}^\alpha_h)\) beforehand. As \((P_{\text{VaR}}^\nu)\) and \((P_{\text{ES}}^\alpha_h)\) can be solved with the same method and \((P_{\text{VaR}}^\nu)\) turns out to be the easiest one, we have naturally found our course of action.

The method of choice to derive a candidate for the optimal solution in both cases — \((P_{\text{VaR}}^\nu)\) as well as \((P_{\text{ES}}^\alpha)\) — is the Lagrange Approach from the Calculus of Variations. This approach suggests itself as \((P_{\text{VaR}}^\nu)\) can be rewritten in the form

\[
\begin{cases}
\int_{\Omega} U(X) dP \to \max \\
\int_{\Omega} \left( \frac{dQ}{dP} \cdot X \right) dP \leq x_0 \\
\int_{\Omega} 1_{(-\infty,q)}(X) dP \leq \epsilon
\end{cases}
\]
(P_{\text{VaR}}) in the form

\[
\begin{align*}
\int_\Omega U(X) dP & \quad \rightarrow \quad \max \\
\int_\Omega \left( \frac{dQ}{dP} \cdot X \right) dP & \quad \leq \quad x_0 \\
\int_\Omega \left( \frac{dQ}{dP} \cdot 1_{(-\infty, \eta)}(X) \right) dP & \quad \leq \quad \epsilon
\end{align*}
\]

(P_{\text{ES}_h}) as

\[
\begin{align*}
\int_\Omega U(X) dP & \quad \rightarrow \quad \max \\
\int_\Omega \left( \frac{dQ}{dP} \cdot X \right) dP & \quad \leq \quad x_0 \\
\int_\Omega (h - X)^+ dP & \quad \leq \quad (h - c) \cdot \alpha
\end{align*}
\]

and (P_{\text{ES}_\nu}) as

\[
\begin{align*}
\int_\Omega U(X) dP & \quad \rightarrow \quad \max \\
\int_\Omega \left( \frac{dQ}{dP} \cdot X \right) dP & \quad \leq \quad x_0 \\
\int_\Omega \left( \frac{dQ}{dP} \cdot (h - X)^+ \right) dP & \quad \leq \quad (h - c) \cdot \alpha
\end{align*}
\]

The corresponding Lagrange Functions, which are functions in \(X, \lambda_1\) and \(\lambda_2\), are

\[
\begin{align*}
\int_\Omega U(X) dP + \lambda_1 \cdot \left( x_0 - \int_\Omega \left( \frac{dQ}{dP} \cdot X \right) dP \right) + \lambda_2 \cdot \left( \epsilon - \int_\Omega 1_{(-\infty, \eta)}(X) dP \right), \\
\int_\Omega U(X) dP + \lambda_1 \cdot \left( x_0 - \int_\Omega \left( \frac{dQ}{dP} \cdot X \right) dP \right) + \lambda_2 \cdot \left( \epsilon - \int_\Omega \left( \frac{dQ}{dP} \cdot 1_{(-\infty, \eta)}(X) \right) dP \right), \\
\int_\Omega U(X) dP + \lambda_1 \cdot \left( x_0 - \int_\Omega \left( \frac{dQ}{dP} \cdot X \right) dP \right) + \lambda_2 \cdot \left( (h - c) \cdot \alpha - \int_\Omega (h - X)^+ dP \right) \quad \text{and} \\
\int_\Omega U(X) dP + \lambda_1 \cdot \left( x_0 - \int_\Omega \left( \frac{dQ}{dP} \cdot X \right) dP \right) + \lambda_2 \cdot \left( (h - c) \cdot \alpha - \int_\Omega \left( \frac{dQ}{dP} \cdot (h - X)^+ \right) dP \right).
\end{align*}
\]

Using the linearity of the Lebesgue-Integral, we put everything in a single integral — for instance

\[
\int_\Omega \left( U(X(\omega)) + \lambda_1 \cdot \left( x_0 - \left( \frac{dQ}{dP} \cdot X(\omega) \right) \right) + \lambda_2 \cdot \left( \epsilon - 1_{(-\infty, \eta)}(X(\omega)) \right) \right) P(d\omega)
\]

for problem \((P_{\text{VaR}})\) — and maximize its value by performing a pointwise maximization of the integrand as a function in \(X(\omega)\). The result is a family of random variables that are the candidates for the optimal claim and are indexed by the Lagrange multipliers \(\lambda_1\) and \(\lambda_2\). In order to ensure the existence of these pointwise maxima as well as the existence of suitable Lagrange multipliers and to have some handy properties, we need several assumptions that we formulated in Section 3.3.1. One side effect of these assumptions is that the price inequality restriction \(E_Q[X] \leq x_0\) can be replaced by the price equality restriction \(E_Q[X] = x_0\). This effect both intuitive and convenient.

The obvious next step is to find the right choice for the Lagrange multipliers. One aspirant is the combination determined by the price equality restriction and the omission of the risk restriction. This combination is a solution of the corresponding problem without risk constraint. If the random variable pertinent to this choice of multipliers fulfills the risk restriction, we are done. Otherwise we consider a choice of multipliers such that both constraints are active, i.e., fulfilled with equality.

By this two-step process we single out a candidate for the optimal solution. Note that the choice of the Lagrange multipliers is not always unique. What remains to be done is to verify that this candidate is indeed optimal and answer the question whether it is the unique solution.

In practice though, we encounter an additional nuisance: In the value at risk problem \((P_{\text{VaR}})\) and the expected shortfall problem \((P_{\text{ES}})\), the chosen parameterization of the candidates using the Lagrange multipliers cannot be extended to cover certain boundary cases. In other words, the family
of candidate random variables is not large enough. As a consequence, we have to handle these cases separately and are going to do so upfront.

3.4.2 Value at Risk Problem

In this section, we sketch a line of argument that can be used to prove Theorem 3.20. The details can be found in Chapter 6.2.

As described in the approach, we begin the investigation of the problem \((P_{\text{VaR}^\nu})\) by looking at a boundary case.

**Boundary Case**

Any admissible random variable that takes values below \(u_0\) with probability strictly greater than 0, yields an objective function value of \(-\infty\). The special case we take care of first is the point where the initial endowment equals the minimal initial wealth required to finance an admissible claim that does not exhibit this behavior. This is the point below which any admissible random variable is optimal, because all of them result in an objective function value of \(-\infty\).

**Proposition 3.30:**

Suppose that \(X\) is a random variable with values in \([u_0, +\infty)\) and \(\nu (X < q) \leq \epsilon\).

Then \(E_{\nu} [X] \geq q \cdot Q \left( \left( \frac{\partial Q}{\partial P} \right)_{\nu} \right)^{-1} (\nu(\Omega) - \epsilon) + u_0 \cdot Q \left( \left( \frac{\partial Q}{\partial P} \right)_{\nu} \right)^{-1} (\nu(\Omega) - \epsilon)\).

If in addition, \(E_{\nu} [X] = q \cdot Q \left( \left( \frac{\partial Q}{\partial P} \right)_{\nu} \right)^{-1} (\nu(\Omega) - \epsilon) + u_0 \cdot Q \left( \left( \frac{\partial Q}{\partial P} \right)_{\nu} \right)^{-1} (\nu(\Omega) - \epsilon)\), \(\nu = Q\) and \(q > u_0\) imply that \(Q (X = q) = \nu (\Omega) - \epsilon\) and \(Q (X = u_0) = \epsilon\) whereas \(\nu = P\) implies

\[ X = q \cdot \mathbb{1}_{\left( (\frac{\partial Q}{\partial P})_{Q}^{-1} (\nu(\Omega) - \epsilon) \right)} \left( \left( \frac{\partial Q}{\partial P} \right)_{Q}^{-1} (\nu(\Omega) - \epsilon) \right) + u_0 \cdot \mathbb{1}_{\left( (\frac{\partial Q}{\partial P})_{Q}^{-1} (\nu(\Omega) - \epsilon), +\infty \right)} \left( \left( \frac{\partial Q}{\partial P} \right)_{Q}^{-1} (\nu(\Omega) - \epsilon) \right) \quad (P\text{-almost-surely}). \]

We realize that in the boundary case of problem \((P_{\text{ES}_P})\), the set of admissible points consists of a single point. Hence, the optimal random variable is obviously that single point.

In case of the problem \((P_{\text{ES}_Q})\), we have to use a little more brain power to get the corresponding result:

**Proposition 3.31:**

Suppose that \(U (u_0) > -\infty\) and \(q > u_0\). The random variable

\[ X^* := q \cdot \mathbb{1}_{\left( (\frac{\partial Q}{\partial P})_{Q}^{-1} (Q(\Omega) - \epsilon) \right)} \left( \left( \frac{\partial Q}{\partial P} \right)_{Q}^{-1} (Q(\Omega) - \epsilon) \right) + u_0 \cdot \mathbb{1}_{\left( (\frac{\partial Q}{\partial P})_{Q}^{-1} (Q(\Omega) - \epsilon), +\infty \right)} \left( \left( \frac{\partial Q}{\partial P} \right)_{Q}^{-1} (Q(\Omega) - \epsilon) \right) \]

is the \((P\text{-almost-surely})\) unique solution of

\[ \begin{align*}
U (X) & \rightarrow \max \\
Q (X = q) & = Q (\Omega) - \epsilon \\
Q (X = u_0) & = \epsilon
\end{align*} \]
3.4. HOW TO PROVE OPTIMALITY

Main Case

Having dealt with the boundary case, we move on to the main case. As said previously, our approach is to maximize pointwise the integral derived from the Lagrange Function. It is convenient to drop the expressions $\lambda_1 \cdot x_0$ and $\lambda_2 \cdot \epsilon$. Being constants with respect to the maximization, this can be done without inflicting any harm.

The following mapping will be a handy tool in the discussion of the Lagrange Function.

**Definition ($\tilde{f}^{\lambda_1}$):** For $\lambda_1 \in (0, +\infty)$ we define the family of functions $\tilde{f}^{\lambda_1}: (0, +\infty) \rightarrow \mathbb{R}$, $d \mapsto U \left( (U')^{-1} (\lambda_1 \cdot d) \right) - \lambda_1 \cdot (U')^{-1} (\lambda_1 \cdot d) \cdot d + \lambda_1 \cdot q \cdot d$.

**Lemma 3.32:**
Let $\lambda_1 \in (0, +\infty)$ be fixed. The function $\tilde{f}^{\lambda_1}$ is

- is strictly decreasing on $\left(0, \frac{U'(q)}{\lambda_1}\right]$ and
- is strictly increasing on $\left[\frac{U'(q)}{\lambda_1}, +\infty\right)$.

Its minimal value $\tilde{f}^{\lambda_1} \left(\frac{U'(q)}{\lambda_1}\right)$ is $U(q)$.

Next, we define the following candidates for the second Lagrange multipliers as functions of the first one:

**Definition ($l^P_2, l^Q_2$):** Let $l^P_2: (0, +\infty) \rightarrow [0, +\infty)$,

$$
\lambda_1 \mapsto \begin{cases} 
\tilde{f}^{\lambda_1} \left( F^{-1}_{\frac{\partial q}{\partial P}} (1 - \epsilon) \right) - U(q) & \text{if } F^{-1}_{\frac{\partial q}{\partial P}} (1 - \epsilon) \geq \frac{U'(q)}{\lambda_1} \\
0 & \text{else}
\end{cases}
$$

and $l^Q_2: (0, +\infty) \rightarrow [0, +\infty)$,

$$
\lambda_1 \mapsto \begin{cases} 
\tilde{f}^{\lambda_1} \left( \left( \frac{F^Q_{\partial q}}{\partial P} \right)^{-1} (Q(\Omega) - \epsilon) \right) - U(q) & \text{if } \left( \frac{F^Q_{\partial q}}{\partial P} \right)^{-1} (Q(\Omega) - \epsilon) \geq \frac{U'(q)}{\lambda_1} \\
0 & \text{else}
\end{cases}
$$

Now we are able to discuss the integrand part of the Lagrange Function corresponding to the optimization problem with value at risk constraint with respect to $P$.

**Lemma 3.33:**
Let $\lambda_1 \in (0, +\infty)$ and $f: \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto U(x) - \lambda_1 \cdot x \cdot d - l^P_2(\lambda_1) \cdot 1_{(-\infty, q)}(x)$.

Then $x = \begin{cases} 
(U')^{-1} (\lambda_1 \cdot d) & \text{if } d < \frac{U'(q)}{\lambda_1} \\
q & \text{if } \frac{U'(q)}{\lambda_1} \leq d \leq F^{-1}_{\frac{\partial q}{\partial P}} (1 - \epsilon) \\
(U')^{-1} (\lambda_1 \cdot d) & \text{if } d > F^{-1}_{\frac{\partial q}{\partial P}} (1 - \epsilon)
\end{cases}$ maximizes $f$ over $\mathbb{R}$.

Only if $d = F^{-1}_{\frac{\partial q}{\partial P}} (1 - \epsilon)$ the maximum might not be unique.
3.4. HOW TO PROVE OPTIMALITY

Figure 3.4 depicts the behavior of $f$. If we close the gap caused by the upward jump, the function $f$ looks smooth and concave. This look is caused by the fact that $f$ is the sum of a concave function and a scaled indicator function. Hence, the point $x$ that maximizes $f$ can be either to the left of the jump, directly at the jump or to the right of the jump. Except for the second case, the value of $x$ always equals the point at which the concave part is maximized.

As usual, a thorough proof of Lemma 3.33 can be found in Chapter 6.2.

The same observation can be used for optimization problem with value at risk constraint with respect to $Q$:

**Lemma 3.34:**

Let $\lambda_1 \in (0, +\infty)$ and $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto U(x) - \lambda_1 \cdot x \cdot d - l_2^Q(\lambda_1) \cdot d \cdot \mathbb{1}_{(\infty, q]}(x)$.

Then $x = \begin{cases} (U')^{-1} (\lambda_1 \cdot d) & \text{if } d < \frac{U'(q)}{\lambda_1} \\ q & \text{if } U'(q) \leq d \leq \left( \frac{F_Q^\prime}{\mathbb{P}} \right)^{-1} (Q(\Omega) - \epsilon) \\ (U')^{-1} (\lambda_1 \cdot d) & \text{if } d > \left( \frac{F_Q^\prime}{\mathbb{P}} \right)^{-1} (Q(\Omega) - \epsilon) \end{cases}$ maximizes $f$ over $\mathbb{R}$.

Only if $d = \left( \frac{F_Q^\prime}{\mathbb{P}} \right)^{-1} (Q(\Omega) - \epsilon)$, the maximum might not be unique.

We realize that qualitatively the function shown in Figure 3.5 is the same as the function in Figure 3.4. This similarity is the reason for the stunning fact that the family of candidates $X_{\lambda_1, J}^\lambda$ for the optimal solution is the same in both cases (the definition of $X_{\lambda_1, J}^\lambda$ was already given in Section 3.3.2).

Having done the pointwise optimization and found a family of candidates, our next task is to pick the optimal member from the family: First of all, we observe that the candidate optimal claim does not explicitly depend on $\lambda_2$ and the parameter $J$ is already known. Therefore we are left with the one-dimensional problem of calculating the optimal $\lambda_1$ for a given $x_0$. This mission can be carried out rather easily. When dealing with expected shortfall or conditional value at risk restrictions however,
3.4. HOW TO PROVE OPTIMALITY

we will face the more challenging task of simultaneously calculation $\lambda_1$ and $\lambda_2$.

We list a few properties of $X_{\lambda_1,J}^{\lambda_1}$:

**Fact 3.35:**

(a) For all $\nu \in \{P, Q\}$ and $\lambda_1 \in (0, +\infty)$: $\nu \left( X_{V_{\lambda_1}} \left( \frac{F^{\nu}}{dP} \right) ^{-1} (\nu(\Omega) - \epsilon) < q \right) < \epsilon \Rightarrow l^\nu_2(\lambda_1) = 0$.

(b) $\forall \omega \in \Omega$: $(0, +\infty) \rightarrow [u_0, +\infty)$ $\lambda_1 \mapsto X_{V_{\lambda_1}}^{\lambda_1,J}(\omega)$ is monotonic decreasing and continuous.

(c) $\forall \omega \in \Omega$: $\lim_{\lambda_1 \rightarrow 0+} X_{V_{\lambda_1}}^{\lambda_1,J}(\omega) = +\infty$.

(d) $\forall \omega \in \Omega$: $X_{V_{\lambda_1}}^{\lambda_1,J}(\omega) = \{ q \text{ if } \frac{dQ}{dP}(\omega) \leq J \text{ and } u_0 \text{ if } \frac{dQ}{dP}(\omega) > J \}$.

Next, we investigate the effect and necessity of Assumption 3.18 and Assumption 3.19. These two assumptions are required in the classical problem (i.e. without risk restriction), too. Their purpose is to guarantee the existence of a Lagrange multiplier such that the resulting random variable is admissible and its price equals the initial endowment. The following analysis of these integrability conditions shows that through the addition of the value at risk restriction, the required assumptions do not change. Neither do we need to enforce stronger conditions nor can we achieve the same goal using weaker conditions.

**Lemma 3.36:**

For all $\lambda_1 \in (0, +\infty)$ and $J \in (0, +\infty)$

$$\mathbb{E}_P \left[ \left( U \left( U' \right) ^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) ^+ \right] < +\infty \iff \mathbb{E}_P \left[ \left( U \left( X_{V_{\lambda_1}}^{\lambda_1,J} \right) \right) ^+ \right] < +\infty$$
as well as
\[ E_P \left[ \left( -U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \right] < +\infty \iff E_P \left[ \left( -U \left( X_{\text{VaR}}^{\lambda_1 J} \right) \right)^+ \right] < +\infty. \]

**Lemma 3.37:**
\[ \forall \lambda_1 \in (0, +\infty) \land J \in (0, +\infty): \]
\[ E_Q \left[ X_{\text{VaR}}^{\lambda_1 J} \right] < +\infty \iff E_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right] < +\infty. \]

Lemma 3.37 ensures the validity of the following result:

**Lemma 3.38:**
Let \( J \in (0, +\infty) \) with \( P \left( \frac{dQ}{dP} > J \right) > 0 \) be fixed.

The mapping \( (0, +\infty) \to \left[ q \cdot Q \left( \frac{dQ}{dP} \leq J \right) + u_0 \cdot Q \left( \frac{dQ}{dP} > J \right), +\infty \right), \lambda_1 \mapsto E_Q \left[ X_{\text{VaR}}^{\lambda_1 J} \right] \) is strictly decreasing, continuous and surjective.

The previous result yields the well-definedness of the optimal Lagrange multiplier \( \Lambda_{\nu, u_0}^{\nu} \), given by the definition in Section 3.3.2. It is simply the inverse of the mapping at the initial wealth.

Now we are able to completely specify the candidate for the optimal solution and verify its optimality.

**Proposition 3.39:**
Let \( \nu \in \{ P, Q \} \) be fixed and \( J := \left( \frac{dQ}{dP} \right)^{-1} (\nu(\Omega) - \epsilon). \)

\[ \forall x_0 > q \cdot Q \left( \frac{dQ}{dP} \leq J \right) + u_0 \cdot Q \left( \frac{dQ}{dP} > J \right): X_{\text{VaR}}^{\Lambda_{\nu, u_0}^{\nu}} \text{ is an optimal solution of } (P_{\text{VaR}^\nu}). \]

The solution is unique \((P\text{-almost-surely})\) if and only if \( E_P \left[ \left| U \left( X_{\text{VaR}}^{\Lambda_{\nu, u_0}^{\nu}} \right) \right| \right] < +\infty. \)

**Proof:** Let \( X := X_{\text{VaR}}^{\Lambda_{\nu, u_0}^{\nu}} \) and \( Y \) be admissible for \((P_{\text{VaR}^\nu})\).

- Case \( E_{\nu} \left[ \mathbb{I}_{(-\infty, q)}(X) \right] = \epsilon: \)
  \[ E_P \left[ U \left( X \right) \right] \overset{(\ast)}{\geq} E_P \left[ U \left( Y \right) \right] + \begin{cases} \geq 0 \to \mathbb{1}_{x_0} \cdot \left( \frac{x_0}{E_Q \left[ X \right]} - \frac{x_0}{E_Q \left[ Y \right]} \right) \\
0 \to \mathbb{1}_{x_0} \cdot \left( E_{\nu} \left[ \mathbb{I}_{(-\infty, q)}(X) \right] - E_{\nu} \left[ \mathbb{I}_{(-\infty, q)}(Y) \right] \right) \end{cases} \]

- Case \( E_{\nu} \left[ \mathbb{I}_{(-\infty, q)}(X) \right] < \epsilon: \)
  (Fact 3.35) \( \Rightarrow l_2^\nu(\Lambda_{\nu, u_0}^{\nu}) = 0. \)

\[ E_P \left[ U(X) \right] \overset{(\ast\ast)}{\geq} E_P \left[ U(Y) \right] + \begin{cases} \geq 0 \to \mathbb{1}_{x_0} \cdot \left( \frac{x_0}{E_Q \left[ X \right]} - \frac{x_0}{E_Q \left[ Y \right]} \right) \\
0 \to \mathbb{1}_{x_0} \cdot \left( E_{\nu} \left[ \mathbb{I}_{(-\infty, q)}(X) \right] - E_{\nu} \left[ \mathbb{I}_{(-\infty, q)}(Y) \right] \right) \end{cases} \]

\[ \geq E_P \left[ U(Y) \right]. \]
3.4. HOW TO PROVE OPTIMALITY

(Assumption 3.11) \( P \left( \frac{dQ}{dP} = J \right) = 0. \)

Hence, if \( \mathbb{E}_P[|U(X)|] < +\infty \) and \( P(X \neq Y) > 0, (\ast) \) and (\( \ast \ast \)) are strict inequalities according to Lemma 3.33.

Necessity of \( \mathbb{E}_P[|U(X)|] < +\infty \) for uniqueness of optimal solution:

We will show that if the condition is violated, we are able to construct another different solution.

- \( \mathbb{E}_P[(U(X))^+] = +\infty: \)
  (Assumption 3.18 and Lemma 3.36) \( \Rightarrow \mathbb{E}_P[U(X)] = +\infty. \)

\( (U(q) < +\infty) \Rightarrow P(X > q) > 0. \)

\( \Rightarrow \exists q > q: P(X > q) > 0. \)

Let \( Y: \Omega \rightarrow \mathbb{R}, \omega \mapsto \begin{cases} X(\omega) & \text{if } X(\omega) \leq \tilde{q} \\ (U^{-1}) (U(X(\omega)) - U(\tilde{q}) + U(\frac{1}{2}q + \frac{1}{2}q)) & \text{if } X(\omega) > \tilde{q} \end{cases}. \)

\( \Rightarrow Y \leq X \Rightarrow \mathbb{E}_Q[Y] \leq \mathbb{E}_Q[X] = x_0. \)

\( \nu(Y < q) = \nu(X < q) \leq \epsilon. \)

\[ \mathbb{E}_P[U(Y)] = \mathbb{E}_P \left[ (U(X) - \mathbb{1}_{[q, +\infty)}(X) \cdot \left( U(\tilde{q}) - U\left( \frac{1}{2}q + \frac{1}{2}q \right) \right) \right] \]

\[ = \mathbb{E}_P[U(X)] - P(X > \tilde{q}) \cdot \mathbb{E}_\mathbb{R} \left[ \left( U(\tilde{q}) - U\left( \frac{1}{2}q + \frac{1}{2}q \right) \right) \right] \]

\[ = +\infty = \mathbb{E}_P[U(X)]. \]

\( \Rightarrow Y \) is another optimal solution.

- \( \mathbb{E}_P\left[(-U(X))^+] = +\infty: \right. \)
  (Assumption 3.18 and Lemma 3.36) \( \Rightarrow \mathbb{E}_P[U(X)] = -\infty. \)

\[ x_0 > q \cdot Q \left( \frac{dQ}{dP} \leq J \right) + u_0 \cdot Q \left( \frac{dQ}{dP} > J \right) = \mathbb{E}_Q \left[ q \cdot \mathbb{1}_{(0,J]} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{1}_{(J, +\infty)} \left( \frac{dQ}{dP} \right) \right]. \]

\( \Rightarrow P(X \neq q \cdot \mathbb{1}_{(0,J]} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{1}_{(J, +\infty)} \left( \frac{dQ}{dP} \right) ) > 0 \) and

\[ \mathbb{E}_P \left[ U \left( q \cdot \mathbb{1}_{(0,J]} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{1}_{(J, +\infty)} \left( \frac{dQ}{dP} \right) \right) \right] \]

\[ = P \left( \frac{dQ}{dP} \leq J \right) \cdot U(q) + P \left( \frac{dQ}{dP} > J \right) \cdot U(u_0) \]

\[ \geq - \infty = \mathbb{E}_P[U(X)]. \]

\( \Rightarrow q \cdot \mathbb{1}_{(0,J]} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{1}_{(J, +\infty)} \left( \frac{dQ}{dP} \right) \) is another optimal solution.

\( \square \)

Proof of the Main Result

**Proof of Theorem 3.20:** The proof is merely a summary of our findings:

- \((P_{\varnothing P^\mu})\) is feasible due to our Assumption 3.11 (cf. Theorem 3.3).

The case \( x_0 > q \cdot Q \left( \frac{dQ}{dP} \leq J \right) + u_0 \cdot Q \left( \frac{dQ}{dP} > J \right) \) is due to Proposition 3.39 and both other cases are dealt with in Proposition 3.30 and Proposition 3.31, because for all admissible \( X: \)

\( P(X < u_0) > 0 \) implies \( P(U(X) = -\infty) > 0 \) and thus \( \mathbb{E}_P[U(X)] = -\infty. \)

\( \square \)
3.4. HOW TO PROVE OPTIMALITY

3.4.3 Expected Shortfall Problem

Special Case of the Expected Shortfall Problem under $P$

We consider the following random variable $X$ in a special case of the expected shortfall problem $(P_{ES}^P)$:

\[ X(\omega) \]

\[ h \]

\[ u_0 \]

\[ \tau(h) \]

\[ \frac{dQ}{dP}(\omega) \]

The function $\tau$ was defined on page 29 and describes the point at which $X$ is discontinuous. Let us take a look at some immediate consequences of its definition:

**Lemma 3.40:**

Some basic properties of $\tau$:

(a) $\forall h \in [c, +\infty): P\left(\frac{dQ}{dP} \geq \tau(h)\right) \cdot (h - u_0) = (h - c) \cdot \alpha$,

(b) the mapping $[c, +\infty) \rightarrow [0, +\infty], h \mapsto \tau(h)$ is monotone decreasing and

(c) the mapping $[c, +\infty) \rightarrow [1 - \alpha, 1], h \mapsto P\left(\frac{dQ}{dP} < \tau(h)\right)$ is continuous.

(d) If $h = c$, $Q\left(\frac{dQ}{dP} < \tau(h)\right) \cdot h + Q\left(\frac{dQ}{dP} \geq \tau(h)\right) \cdot u_0 = Q(\Omega) \cdot h$.

**Lemma 3.41:**

The mapping $[c, +\infty) \rightarrow [0, Q(\Omega)], h \mapsto Q\left(\frac{dQ}{dP} < \tau(h)\right)$ is continuous.

The next result presents the minimal initial endowment required for the existence of a reasonable ($[u_0, +\infty)$ valued) random variable. Additionally, a solution for the case that the initial endowment equals this minimal amount is given.

**Proposition 3.42:**

Consider $h \in [u_0, +\infty)$ and a random variable $X$ with values in $[u_0, +\infty)$ that fulfills the constraint $E_P[(h - X)^+] \leq (h - c) \cdot \alpha$.

Then $E_Q[X] \geq Q\left(\frac{dQ}{dP} < \tau(h)\right) \cdot h + Q\left(\frac{dQ}{dP} \geq \tau(h)\right) \cdot u_0$.

If in addition, $E_Q[X] = Q\left(\frac{dQ}{dP} < \tau(h)\right) \cdot h + Q\left(\frac{dQ}{dP} \geq \tau(h)\right) \cdot u_0$,

\[ X = h \cdot 1_{(0, \tau(h))} \left(\frac{dQ}{dP}\right) + u_0 \cdot 1_{[\tau(h), +\infty)} \left(\frac{dQ}{dP}\right) \quad (P\text{-almost-surely}). \]

For dealing with the family of restrictions with parameter $h$ appearing in the equivalent formulation of the conditional value at risk problem $(\tilde{P}_{CVaR}^P)$, we would like to have continuity of selected optimal solution of the expected shortfall problem $(P_{ES}^P)$ in this parameter. Thus, after a tiny preparation,
we are going to show this continuity in the border case. Note that this interpretation is only allowed if we always find at least one monotonic decreasing optimal solution.

**Lemma 3.43:**
For \( n \in \mathbb{N}_0 \), let \( h_n \in [c, +\infty) \) with \( \lim_{n \to +\infty} h_n = h_0 \).

For all \( t_0 \in (0, \tau(h_0)) \) with \( P\left( \frac{dQ}{dP} \in [t_0, \tau(h_0)) \right) > 0 \), \( \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : t_0 < \tau(h_n) \).

**Lemma 3.44:**
For \( n \in \mathbb{N}_0 \), let \( h_n \in [c, +\infty) \) with \( \lim_{n \to +\infty} h_n = h_0 \). Moreover, suppose that \( \forall n \in \mathbb{N} : X_n : (0, +\infty) \to [u_0, +\infty) \) is monotonic decreasing with \( E_P\left( \left( h_n - X_n \circ \frac{dQ}{dP} \right)^+ \right) \leq (h_n - c) \cdot \alpha \).

If \( \lim_{n \to -\infty} \left( E_Q\left( X_n \circ \frac{dQ}{dP} \right) \right) = Q\left( \frac{dQ}{dP} < \tau(h_0) \right) \cdot h_0 + Q\left( \frac{dQ}{dP} \geq \tau(h_0) \right) \cdot u_0 \), \( \lim_{n \to -\infty} \left( X_n \circ \frac{dQ}{dP} \right) = h_0 \cdot 1_{(0, \tau(h_0))} \left( \frac{dQ}{dP} \right) + u_0 \cdot 1_{[\tau(h_0), +\infty)} \left( \frac{dQ}{dP} \right) \) (P-almost-surely).

It is convenient to define (see page 29) the set \( \overline{P} \) of all reasonable values for \( h \) including the border cases as well as the set \( H_P \) excluding them and look at some of their properties. We will make use of them a bit later on.

**Fact 3.45:**
\( H_P \) is relatively open in \([c, +\infty), \overline{P} \) is closed and \( \overline{P} \neq \emptyset \) implies \( x_0 \geq u_0 \cdot Q(\Omega) \).
If \( \alpha \in (0, 1), \overline{P} \) is compact.

**Expected Shortfall Restriction with respect to \( P \)**
Having dealt with the special case, we start tackling the non-degenerated cases of the problem \( (P_{\mathcal{E}P}) \).

**Assumption 3.46:**
Let \( h \in [u_0, +\infty) \) and \( x_0 \geq u_0 \cdot Q(\Omega) \) be fixed if not stated otherwise.

As before, our approach is to maximize pointwise the integral derived from the Lagrange Function. It is convenient to drop the expressions \( \lambda_1 \cdot x_0 \) and \( \lambda_2 \cdot (h - c) \cdot \alpha \).

We can see in Figure 3.6 that the integrand \( f \) of the Lagrange Function is smooth with one exception: at \( x = h \) it has a non-differentiable breaking point.

**Lemma 3.47:**
Let \( +\infty \cdot 0 := 0, h \in [u_0, +\infty), \lambda_1 \in (0, +\infty), \lambda_2 \in [0, +\infty), d \in (0, +\infty) \) and \( f : \mathbb{R} \to [-\infty, +\infty), x \mapsto U(x) - \lambda_1 \cdot x \cdot d - \lambda_2 \cdot (h - x)^+ \).

Then \( \text{argmax}_{x \in \mathbb{R}} \{ f(x) \} = \begin{cases} (U')^{-1} (\lambda_1 \cdot d) & \text{if } U'(h) - \lambda_1 \cdot d > 0 \\ h & \text{if } \lambda_1 \cdot d - \lambda_2 \leq U'(h) \leq \lambda_1 \cdot d \\ (U')^{-1} (\lambda_1 \cdot d - \lambda_2) & \text{if } U'(h) - \lambda_1 \cdot d + \lambda_2 < 0 \end{cases} \) is the unique value maximizing \( f \).

Hence, we have derived the family of candidates \( X^{h, \lambda_1, \lambda_2} \), which was defined on page 29, for the optimal solution of \( (P_{\mathcal{E}P}) \).

The next obstacle to remove is to find values of the Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \) such that
3.4. HOW TO PROVE OPTIMALITY

Figure 3.6: Integrand of Lagrange Function \( P_{ES, h} \): \( U = U_0, \lambda_1 = 1, (0, +\infty)(\lambda_1) = 3, h = 0.5 \)

the price restriction and the expected shortfall condition are met. Essentially, we have to solve a two-dimensional system of non-linear equations. We embark on this journey by looking at several properties of \( X_{h, \lambda_1, \lambda_2} \).

**Lemma 3.48:**

The mappings

- \( L_P \rightarrow [u_0, +\infty], (h, \lambda_1, \lambda_2) \mapsto X_{h, \lambda_1, \lambda_2}(\omega) \) and
- \( L_P \rightarrow [0, +\infty), (h, \lambda_1, \lambda_2) \mapsto (h - X_{h, \lambda_1, \lambda_2}(\omega))^+ \)

are continuous for fixed \( \omega \in \Omega \).

**Proof:** Since \((U')^{-1}\) is continuous (Lemma 3.17) and the maximum and minimum of two continuous functions is again continuous, the statement is obvious. \(\square\)

The expected shortfall of the candidate is a continuous function of \( h \) and the Lagrange multipliers:

**Lemma 3.49:**

The mapping \( L_P \rightarrow [0, +\infty), (h, \lambda_1, \lambda_2) \mapsto E_P \left[ (h - X_{h, \lambda_1, \lambda_2})^+ \right] \) is continuous.

**Proof:** Let \((h_n, \lambda_1^n, \lambda_2^n) \in L_P \forall n \in \mathbb{N}_0\) and let us suppose that \( \lim_{n \rightarrow +\infty} (h_n, \lambda_1^n, \lambda_2^n) = (h_0, \lambda_1^0, \lambda_2^0) \).

\(\Rightarrow \exists \max \{h_n \mid n \in \mathbb{N}_0\} =: h_{\max} \in [u_0, +\infty)\).

\(\Rightarrow \forall n \in \mathbb{N}_0 \text{ and } \forall \omega \in \Omega: 0 \leq (h_n - X_{h_n, \lambda_1^n, \lambda_2^n}(\omega))^+ \leq h_{\max} - u_0\).

Hence \(\forall n \in \mathbb{N}_0: 0 \leq E_P \left[ (h_n - X_{h_n, \lambda_1^n, \lambda_2^n})^+ \right] \leq (h_{\max} - u_0) \cdot 1 < +\infty\), so the Dominated Convergence Theorem implies that \( \lim_{n \rightarrow +\infty} E_P \left[ (h_n - X_{h_n, \lambda_1^n, \lambda_2^n})^+ \right] = E_P \left[ (h_0 - X_{h_0, \lambda_1^0, \lambda_2^0})^+ \right] \). \(\square\)

Again, we investigate the effect and necessity of the assumptions 3.18 and 3.19. Our findings are
3.4. HOW TO PROVE OPTIMALITY

the same as in the value at risk optimization problem: These classical conditions are necessary and sufficient for our integrability needs.

Lemma 3.50:
The following two statements are equivalent:

- \( \forall \lambda_1 \in (0, +\infty) : \mathbb{E}_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right] < +\infty. \)
- \( \forall (h, \lambda_1, \lambda_2) \in [u_0, +\infty) \times [0, +\infty] \text{ with } (\lambda_1, \lambda_2) \neq (+\infty, +\infty) : \mathbb{E}_Q \left[ X^{h,\lambda_1,\lambda_2} \right] < +\infty. \)

Lemma 3.51:
For all \( \forall \lambda_1 \in (0, +\infty) \)

- \( \mathbb{E}_P \left[ U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right] < +\infty \) and
- \( \forall (h, \lambda_2) \in [u_0, +\infty) \times [0, +\infty] : \mathbb{E}_P \left[ (X^{h,\lambda_1,\lambda_2})^+ \right] < +\infty 

are equivalent as well as

- \( \mathbb{E}_P \left[ -U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right] < +\infty \) and
- \( \forall (h, \lambda_2) \in [u_0, +\infty) \times [0, +\infty] : \mathbb{E}_P \left[ (-U \left( X^{h,\lambda_1,\lambda_2} \right))^+ \right] < +\infty. \)

Lemma 3.52:
The following two statements are equivalent \( \forall \lambda_1 \in (0, +\infty) : \)

- \( \mathbb{E}_P \left[ U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right] < +\infty \) and
- \( \forall (h, \lambda_2) \in [u_0, +\infty) \times [0, +\infty] : \mathbb{E}_P \left[ U \left( X^{h,\lambda_1,\lambda_2} \right) \right] < +\infty. \)

Proof: The statement is an immediate consequence of Lemma 3.51. \( \square \)

Let us continue the list of properties of \( X^{h,\lambda_1,\lambda_2}. \)

Fact 3.53:
(a) \( h \mapsto X^{h,\lambda_1,\lambda_2}(\omega) \) is monotonic increasing,
(b) \( \lambda_1 \mapsto X^{h,\lambda_1,0}(\omega) \) is strictly decreasing,
(c) \( \lambda_1 \mapsto X^{h,\lambda_1,\lambda_2}(\omega) \) is monotonic decreasing and
(d) \( \lambda_2 \mapsto X^{h,\lambda_1,\lambda_2}(\omega) \) is monotonic increasing.
(e) \( \forall \lambda_2 \in \mathbb{R}_+: \mathbb{E}_Q \left[ X^{h,+,\lambda_2} \right] = u_0 \cdot Q(\Omega). \)
(f) \( \lim_{\lambda_1 \to +\infty} \mathbb{E}_Q \left[ X^{h,\lambda_1,+\infty} \right] = h \cdot Q(\Omega). \)
(g) \( \forall \lambda_2 \in [0, +\infty]: \lim_{\lambda_1 \to 0^+} E_Q \left[ X^{h, \lambda_1, \lambda_2} \right] = +\infty. \)

**Proof:** The first five properties are due to the definition of \( X^{h, \lambda_1, \lambda_2}(\omega) \) and the fact that \((U')^{-1}\) is strictly decreasing in the interval \([0, +\infty)\). (f) and (g) can be shown by applying the Monotone Convergence Theorem. \( \square \)

**Lemma 3.54:**
Let \( A := ([u_0, +\infty) \times (0, +\infty) \times [0, +\infty)) \setminus ([u_0, +\infty) \times \{+\infty\} \times \{+\infty\}). \)
The mapping \( A \to [u_0 \cdot Q(\Omega), +\infty), (h, \lambda_1, \lambda_2) \mapsto E_Q \left[ X^{h, \lambda_1, \lambda_2} \right] \) is continuous.

**Lemma 3.55:**
Let \( h \in [u_0, +\infty) \) and \( \lambda_2 \in \mathbb{R}_+ \) be fixed.
The mapping \((0, +\infty) \to [u_0 \cdot Q(\Omega), +\infty), \lambda_1 \mapsto E_Q \left[ X^{h, \lambda_1, \lambda_2} \right] \) is surjective and \( \forall y \in (h \cdot Q(\Omega), +\infty): \exists \lambda_1 \in (0, +\infty): E_Q \left[ X^{h, \lambda_1, +\infty} \right] = y. \)

**Proof:** The surjectivity is due to its continuity (cf. Lemma 3.54), Fact 3.53(e), Fact 3.53(f) and Fact 3.53(g). \( \square \)

Let us break down the problem of finding the Lagrange multiplier \( \lambda_1 \) and \( \lambda_2 \) into two parts: The first part is to find a \( \lambda_1 \) such that the price restriction is satisfied given a fixed \( \lambda_2 \), that is to view \( \lambda_1 \) as a function of \( \lambda_2 \). According to Lemma 3.55, \( \Lambda^{h, \lambda_2} \) defined on page 30 is the wanted \( \lambda_1 \). Naturally, the second part is to find \( \lambda_2 \).

**Comment:** It does not seem to be important which operator we choose in the definition of \( \Lambda^{h, \lambda_2} \). However, as it turns out, choosing the infimum enables us to formulate Lemma 3.64, which would not be possible, if we had chosen \( \sup \{ \lambda_1 \in (0, +\infty) \mid E_Q \left[ X^{h, \lambda_1, \lambda_2} \right] \geq x_0 \} \) as definition for \( \Lambda^{h, \lambda_2} \). Consider the following example:

**Example 3.56:**
Let \( U = U_0, h = 1, x_0 = h \cdot Q(\Omega) = h \cdot 1.5, F_{\sigma h}^{U_0}(x) = \begin{cases} 0 & x \leq 1 \\ x - 1 & 1 < x < 2 \\ 1 & x \geq 2 \end{cases}, \) \((\lambda_1^q, \lambda_2^q) = (1, 1)\) and \((\lambda_1^b, \lambda_2^b) = (2, 3)\). \( X^{h, \lambda_1^q, \lambda_2} \) and \( X^{h, \lambda_1^b, \lambda_2^b} \) are depicted in Figure 3.7.

\( \Rightarrow \lambda_1^q = 1 = \sup \{ \lambda_1 \in (0, +\infty) \mid E_Q \left[ X^{h, \lambda_1^q, \lambda_2^q} \right] \geq x_0 \}, \)
\( \lambda_1^b = 2 = \sup \{ \lambda_1 \in (0, +\infty) \mid E_Q \left[ X^{h, \lambda_1^b, \lambda_2^b} \right] \geq x_0 \} \) and (cf. Lemma 3.64)
\[ E_P \left[ (h - X^{h, \lambda_1^q, \lambda_2^q})^+ \right] = 0 = E_P \left[ (h - X^{h, \lambda_1^b, \lambda_2^b})^+ \right]. \]

**Lemma 3.57:**
We observe that
(a) \( E_Q \left[ X^{h, \Lambda^{h, \lambda_2}, \lambda_2} \right] = x_0 \) and
(b) \( \inf \left\{ \Lambda^{h, \lambda_2} \mid h \in [u_0, +\infty), \lambda_2 \in [0, +\infty], \lambda_2 \neq +\infty \text{ if } h \geq \frac{x_0}{Q(\Omega)} \right\} > 0. \)

If the expected shortfall restriction is irrelevant for the optimal solution of \((P_{E h})\), we can already solve the optimization problem. We recall that \( X^{h, \Lambda^{h, 0}, 0} \) is independent of \( h \).
3.4. HOW TO PROVE OPTIMALITY

Figure 3.7: \((P_{ES_h})\): choices for the definition of \(\Lambda^{h, \lambda_2}\).

**Proposition 3.58:**
If \(E_P \left( h - X^{h, 0,0,0} \right) \leq (h - c) \cdot \alpha \), then \(X^{h, 0,0,0}\) is an optimal solution of the problems \((P_0)\), \((P_{ES_P})\) and \((P_{CVaR})\).

If \(E_P \left( |U(X^{h, 0,0,0})| \right) < +\infty\), it is not only an optimal solution but the \((P\text{-almost-surely})\) unique solution of all of the three problems.

**Proof:** Let \(Y\) be a random variable with \(E_Q [Y] \leq x_0\).

- \(\Lambda^{h,0} < +\infty\):
  (Lemma 3.47) \(\forall \omega \in \Omega:\)
  \[U \left( X^{h, 0,0,0}(\omega) \right) - \Lambda^{h,0} \cdot X^{h, 0,0,0}(\omega) \cdot \frac{dQ}{dP}(\omega) \geq U(Y(\omega)) - \Lambda^{h,0} \cdot Y(\omega) \cdot \frac{dQ}{dP}(\omega).\]

  \(\Rightarrow E_P \left( U \left( X^{h, 0,0,0} \right) \right) \geq E_P \left( U(Y) \right) + \Lambda^{h,0} \cdot \left( \underbrace{E_Q \left[ X^{h, 0,0,0} \right] - E_Q \left[ Y \right]}_{\geq 0} \right) \geq E_P \left( U(Y) \right).\)

If \(P \left( X^{h, 0,0,0} \neq Y \right) > 0\) and \(E_P \left[ U \left( X^{h, 0,0,0} \right) \right] < +\infty\), \((*)\) is a strict inequality, because \(\forall \omega \in \Omega\) with \(X^{h, 0,0,0}(\omega) \neq Y(\omega)\):

\[U \left( X^{h, 0,0,0}(\omega) \right) - \Lambda^{h,0} \cdot X^{h, 0,0,0}(\omega) \cdot \frac{dQ}{dP}(\omega) \geq U(Y(\omega)) - \Lambda^{h,0} \cdot Y(\omega) \cdot \frac{dQ}{dP}(\omega).\]

- \(\Lambda^{h,0} = +\infty:\)
  \(\Rightarrow X^{h, 0,0,0} = u_0.\)
  \(\Rightarrow x_0 = u_0 \cdot Q(\Omega).\)

If \(P \left( X^{h, 0,0,0} \neq Y \right) > 0\) and \(Y\) is admissible (this implies the existence of \(E_P \left[ U(Y) \right] \)), then
Let \( \lambda \in \mathbb{Q} \) then \( \forall \omega \in A \) if \( Q(A) > 0 \) and \( \forall \omega \in A \) \( X^{h,\lambda_1,\lambda_2}(\omega) \neq X^{h,\lambda_1^*,\lambda_2^*}(\omega) \).

\[ \Rightarrow \forall \omega \in A: X^{h,\lambda_1^*,\lambda_2^*}(\omega) > X^{h,\lambda_1,\lambda_2^*}(\omega) \]

\[ \Rightarrow \mathbb{E}_Q \left[ X^{h,\lambda_1^*,\lambda_2^*} \right] > \mathbb{E}_Q \left[ X^{h,\lambda_1,\lambda_2^*} \right] \]

\[ \Box \]

For the typical case where the expected shortfall restriction does play a role, we are going to determine the Lagrange multiplier \( \lambda_1 \). Let us recall that not all values for the and combinations of Lagrange multipliers are allowed: The possible choices of \( \lambda_1 \) can be expressed as a function of \( \lambda_2 \):

**Definition** \( (L_1^{\lambda_2}) \): For all \( \lambda_2 \in [0, +\infty], L_1^{\lambda_2} := \left\{ \begin{array}{ll} (0, +\infty) & \text{if } \lambda_2 = +\infty \\ (0, +\infty) & \text{if } \lambda_2 < +\infty \end{array} \right\} \)

In addition, we do not allow \( \lambda_2 \) to take all values of the set \([0, +\infty]\), because the choice \( \lambda_2 = +\infty \) does only make sense in some cases (cf. definition of \( L_2^b \) on page 30).

**Lemma 3.59:**

The mapping \( L_2^b \rightarrow (0, +\infty), \lambda_2 \mapsto L^{\lambda_1,\lambda_2} \) is monotonic increasing.

**Lemma 3.60:**

Let \( \lambda_2 \in [0, +\infty], \lambda_1^a \) and \( \lambda_1^b \in L_1^{\lambda_2} \) with \( \lambda_1^a < \lambda_1^b \) and \( \mathbb{E}_Q \left[ X^{h,\lambda_1^a,\lambda_2^*} \right] = \mathbb{E}_Q \left[ X^{h,\lambda_1^b,\lambda_2^*} \right] \).

Then \( X^{h,\lambda_1^a,\lambda_2^*} = X^{h,\lambda_1^b,\lambda_2^*} \) \((Q\text{-almost-everywhere})\).

**Proof:** First of all, \( \forall \omega \in \Omega \) \( X^{h,\lambda_1^a,\lambda_2^*}(\omega) \geq X^{h,\lambda_1^b,\lambda_2^*}(\omega) \).

\[ \Rightarrow \forall \omega \in A: X^{h,\lambda_1^a,\lambda_2^*}(\omega) > X^{h,\lambda_1^b,\lambda_2^*}(\omega) \]

\[ \Rightarrow \mathbb{E}_Q \left[ X^{h,\lambda_1^a,\lambda_2^*} \right] > \mathbb{E}_Q \left[ X^{h,\lambda_1^b,\lambda_2^*} \right] \]

\[ \Box \]

**Lemma 3.61:**

Let \( \lambda_2 \in [0, +\infty], \lambda_1^a \) and \( \lambda_1^d \in L_1^{\lambda_2^*} \), with \( \lambda_1^a \leq \lambda_1^d \) and \( A := \left\{ \omega \in \Omega \mid X^{h,\lambda_1^a,\lambda_2^*}(\omega) \neq X^{h,\lambda_1^d,\lambda_2^*}(\omega) \right\} \).

Then \( Q(A) = 0 \) and for all \( \lambda_1^a \) and \( \lambda_1^d \in L_1^{\lambda_2^*} \) with \( \lambda_1^a \leq \lambda_1^a \leq \lambda_1^d \):

\[ \forall \omega \in \Omega \setminus A: X^{h,\lambda_1^a,\lambda_2^*}(\omega) = X^{h,\lambda_1^d,\lambda_2^*}(\omega) \]

**Proof:** (Lemma 3.60) \( \Rightarrow Q(A) = 0 \).

Let \( \lambda_1 \in L_1^{\lambda_2^*} \) with \( \lambda_1^a \leq \lambda_1^d \).

\[ \Rightarrow \forall \omega \in \Omega: X^{h,\lambda_1^a,\lambda_2^*}(\omega) \geq X^{h,\lambda_1^a,\lambda_2^*}(\omega) \geq X^{h,\lambda_1^d,\lambda_2^*}(\omega) \]

\[ \Rightarrow \forall \omega \in \Omega \setminus A: X^{h,\lambda_1^a,\lambda_2^*}(\omega) = X^{h,\lambda_1^d,\lambda_2^*}(\omega) \]

\[ \Box \]

**Lemma 3.62:**

Let \( h \in [u_0, +\infty) \), \( \lambda_2 \in L_2^b \) and \((h_n, \lambda_2^b)\) be a sequence converging to \((h, \lambda_2)\) where without any loss of generality \( h_n \in \left\{ \begin{array}{ll} [u_0, +\infty) & \text{if } h \geq \frac{c_0}{Q(\Omega)} \\ [u_0, +\infty) & \text{if } h < \frac{c_0}{Q(\Omega)} \end{array} \right\} \) and \( \lambda_2^b \in L_2^b = L_2^b \).

Then \( \lim_{n \to \infty} X^{h_n,\lambda_1^b,\lambda_2^b}(\omega) = X^{h,\lambda_1^b,\lambda_2^b}(\omega) \) \((Q\text{-almost-everywhere})\).
3.4. HOW TO PROVE OPTIMALITY

Lemma 3.63:
Let \( A := \{ (h, \lambda_2) \mid h \in [u_0, +\infty) \land \lambda_2 \in L_2^h \} \).

The mapping \( A \rightarrow [0, +\infty), (h, \lambda_2) \mapsto E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2}} \right)^+ \right] \) is continuous.

**Proof:** \( \forall (h, \lambda_2) \in A \) and \( \forall \omega \in \Omega: 0 \leq \left( h - X^{h,\Lambda^{h,\lambda_2}}(\omega) \right)^+ \leq h - u_0. \)

\[ \Rightarrow \forall (h, \lambda_2) \in A: 0 \leq E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2}} \right)^+ \right] \leq (h - u_0) \cdot 1 < +\infty. \]

The continuity is a direct implication of \( P \sim Q, \) Lemma 3.62, the fact that every sequence converging in \([u_0, +\infty)\) (with limit in \([u_0, +\infty)\)) is bounded and the Dominated Convergence Theorem. \( \square \)

Lemma 3.64:
The mapping

- \( L_2^h \rightarrow [0, +\infty], \lambda_2 \mapsto \frac{U'(h) + \lambda_2}{\Lambda^{h,\lambda_2}} \) is monotonic increasing and
- \( L_2^h \rightarrow [0, +\infty), \lambda_2 \mapsto E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2}} \right)^+ \right] \) is monotonic decreasing.

If \( \lambda_2^0, \lambda_2^1 \in L_2^h \) with \( \lambda_2^0 < \lambda_2^1 \) and \( \Lambda^{h,\lambda_2^0} < \Lambda^{h,\lambda_2^1} \):

\[ E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2^0}} \right)^+ \right] > E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2^1}} \right)^+ \right]. \]

Lemma 3.65:
Let \( \lambda_1 := \lim_{\lambda_2 \rightarrow +\infty} \Lambda^{h,\lambda_2}. \)

If \( \lambda_1 < +\infty, \) then \( \lim_{\lambda_2 \rightarrow +\infty} E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2}} \right)^+ \right] = 0. \)

If \( \lambda_1 = +\infty, \) then \( \lim_{\lambda_2 \rightarrow +\infty} E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2}} \right)^+ \right] = (h - u_0) \cdot P \left( \frac{dQ}{dt} > \tilde{t} \right) \) for some \( \tilde{t} \in [0, +\infty] \) with \( Q \left( \frac{dQ}{dt} \leq \tilde{t} \right) \cdot \alpha + Q \left( \frac{dQ}{dt} > \tilde{t} \right) \cdot u_0 = x_0. \)

Lemma 3.66:
The set \( \left\{ h \in [u_0, +\infty) \mid E_P \left[ \left( h - X^{h,\Lambda^{h,\alpha}} \right)^+ \right] \leq (h - c) \cdot \alpha \right\} \) is closed.

Lemma 3.67:
Let \( \lambda_2^0, \lambda_2^1 \) and \( \lambda_2^2 \in L_2^h \) with \( \lambda_2^0 < \lambda_2^1 < \lambda_2^2 \).

If \( E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2^0}} \right)^+ \right] = E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2^1}} \right)^+ \right], \) then

- \( \Lambda^{h,\lambda_2^0} = \Lambda^{h,\lambda_2^2} = \Lambda^{h,\lambda_2^1}, \)
- \( X^{h,\Lambda^{h,\lambda_2^0}} = X^{h,\Lambda^{h,\lambda_2^1}} = X^{h,\Lambda^{h,\lambda_2^2}} \) (\( P\)-almost-surely),
- either \( \Lambda^{h,\lambda_2^2} = +\infty \) or \( E_P \left[ \left( h - X^{h,\Lambda^{h,\lambda_2^2}} \right)^+ \right] = 0, \)
- \( \left\{ X^{h,\Lambda^{h,\lambda_2^0}} \neq X^{h,\Lambda^{h,\lambda_2^1}} \right\} \subseteq \left\{ X^{h,\Lambda^{h,\lambda_2^2}} \neq X^{h,\Lambda^{h,\lambda_2^1}} \right\} \).
Lemma 3.68:
Let \( h \in H_P \) with \( \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] \geq (h - c) \cdot \alpha \).
Then \( \{ \lambda_2 \in L^2 \mid \mathbb{E}_P \left[ (h - X^{h,h,\psi h,\lambda_2})^+ \right] = (h - c) \cdot \alpha \} \) is non-empty and relatively closed in \( L^2 \).

Lemma 3.68 guarantees that \( \psi^h \) (see page 30) is well-defined.
Some more useful properties:

Fact 3.69:
(a) If \( \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] \leq (h - c) \cdot \alpha \), \( \psi^h = 0 \).
(b) \( \forall h \in H_P : X^{h,h,\psi h} > u_0 \).
(c) \( \left\{ h \in H_P \mid \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] < (h - c) \cdot \alpha \right\} \) is open.
(d) \( \left\{ h \in H_P \mid \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] \leq (h - c) \cdot \alpha \right\} \) is relatively closed in \( H_P \).
(e) If \( \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] \geq (h - c) \cdot \alpha \), then \( \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] = (h - c) \cdot \alpha \).

Proof:
(b) Let \( h \in H_P \).
If \( X^{h,h,\psi h} \in H_P \) (for some \( \omega \in \Omega \), we have to have that \( \Lambda^{h,\psi h} = +\infty \) and hence \( X^{h,h,\psi h} = u_0 \). This would imply that
\[
x_0 = \mathbb{E}_Q \left[ X^{h,h,\psi h} \right] = u_0 \cdot Q(\Omega)
\leq Q \left( \frac{dQ}{dP} < \tau(h) \right) \cdot h + Q \left( \frac{dQ}{dP} \geq \tau(h) \right) \cdot u_0 \leq x_0
\]
which is a contradiction.
Hence \( X^{h,h,\psi h} > u_0 \).

(c) (Lemma 3.63, \(-\infty, -c \cdot \alpha\) is open) \( \Rightarrow \) \( \left\{ h \in H_P \mid \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] < (h - c) \cdot \alpha \right\} \) is relatively open in \( H_P \).

(Fact 3.45) \( \Rightarrow \) \( \left\{ h \in H_P \mid \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] < (h - c) \cdot \alpha \right\} \) is relatively open in \([c, +\infty)\).

Finally, the result is due to the fact that \( c \notin \left\{ h \in H_P \mid \mathbb{E}_P \left[ (h - X^{h,h,\psi h})^+ \right] < (h - c) \cdot \alpha \right\} \).

The other statements are clearly due to the definition of \( \psi^h \), Lemma 3.63, Lemma 3.64 and Lemma 3.68.

Proposition 3.70:
For all \( h \in H_P : X^{h,h,\psi h} \) is an optimal solution of \( (P_{ES}^h) \).
The solution is unique (\( P \)-almost-surely) if and only if \( \mathbb{E}_P \left[ \sup \left( X^{h,h,\psi h} \right) \right] < +\infty \).
3.4. HOW TO PROVE OPTIMALITY

Proof: First of all, we observe that \( \Lambda^{h, \Psi^h} < +\infty \), because \( \Lambda^{h, \Psi^h} = +\infty \) would imply that \( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} = u_0 \) and thus \( x_0 = u_0 \cdot Q(\Omega) \leq Q\left(\frac{dQ}{dp} < \tau(h)\right) \cdot h + Q\left(\frac{dQ}{dp} \geq \tau(h)\right) \cdot u_0 \) which would contradict \( h \in H_P \).

In the case \( \mathbb{E}_P \left[ \left( h - X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right)^+ \right] \leq (h - c) \cdot \alpha \), we note that \( \Psi^h = 0 \) and hence, the optimality and the sufficiency for uniqueness of the optimal solution is a consequence of Proposition 3.58.

Case \( \mathbb{E}_P \left[ \left( h - X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right)^+ \right] > (h - c) \cdot \alpha \):

\[ \Rightarrow \mathbb{E}_P \left[ \left( h - X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right)^+ \right] = (h - c) \cdot \alpha. \]

We set \( +\infty \cdot 0 := 0 \).

For all \( Y \) admissible for \( (P_{E\text{E}_P}^p) \):

(Lemma 3.47) \( \Rightarrow \forall \omega \in \Omega: \)

\[ U \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h}(\omega) \right) - \Lambda^{h, \Psi^h} \cdot X^{h, \Lambda^{h, \Psi^h}, \Psi^h}(\omega) \cdot \frac{dQ}{dp}(\omega) - \Psi^h \cdot \left( h - X^{h, \Lambda^{h, \Psi^h}, \Psi^h}(\omega) \right)^+ \]

\[ \geq U(Y(\omega)) - \Lambda^{h, \Psi^h} \cdot Y(\omega) \cdot \frac{dQ}{dp}(\omega) - \Psi^h \cdot (h - Y(\omega))^+. \]

Note that \( \Psi^h = +\infty \) implies that \( \mathbb{E}_P \left[ \left( h - X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right)^+ \right] = 0 \) and as a consequence: \( h = c \) (Fact 3.69(e)) and \( P(Y < h) = 0 \).

Therefore

\[ \mathbb{E}_P \left[ U \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right) \right] \overset{(**) \geq 0}{\geq} \mathbb{E}_P[U(Y)] + \Lambda^{h, \Psi^h} \cdot \mathbb{E}_Q \left[ X^{h, \Lambda^{h, \Psi^h}, \Psi^h} - \mathbb{E}_Q[Y] \right] \]

\[ \overset{\geq 0}{\geq} \Psi^h \cdot \mathbb{E}_P \left[ \left( h - X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right)^+ \right] - \mathbb{E}_P \left[ (h - Y)^+ \right] \]

\[ \overset{\geq 0}{\geq} \mathbb{E}_P[U(Y)]. \]

If \( P \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \neq Y \right) > 0 \) and \( \mathbb{E}_P \left[ U \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right) \right] < +\infty \), (**) is a strict inequality, because \( \forall \omega \in \Omega \) with \( X^{h, \Lambda^{h, \Psi^h}, \Psi^h}(\omega) \neq Y(\omega) \): (**) is a strict inequality (Lemma 3.47).

Necessity of \( \mathbb{E}_P \left[ U \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right) \right] < +\infty \) for uniqueness of optimal solution:

We will show that if the condition is violated, we are able to construct another different solution.

- \( \mathbb{E}_P \left[ U \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right)^+ \right] = +\infty: \)

(Assumption 3.18 and Lemma 3.51) \( \Rightarrow \mathbb{E}_P \left[ U \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} \right) \right] = +\infty. \)

\[ (U(h) < +\infty) \Rightarrow P \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} > h \right) > 0. \]

\( \Rightarrow \exists h: P \left( X^{h, \Lambda^{h, \Psi^h}, \Psi^h} > h \right) > 0. \)
Let $Y: \Omega \to \mathbb{R}$, $\omega \mapsto$
\[
\begin{cases}
X^{h,\Lambda,\Phi_0,\Psi_0}(\omega) \\
(U^{-1}) \left( U \left( X^{h,\Lambda,\Phi_0,\Psi_0}(\omega) \right) - U(\tilde{h}) + U \left( \frac{1}{2} \tilde{h} + \frac{1}{2} h \right) \right)
\end{cases}
\]
if $X^{h,\Lambda,\Phi_0,\Psi_0}(\omega) \leq \tilde{h}$, 
if $X^{h,\Lambda,\Phi_0,\Psi_0}(\omega) > \tilde{h}$.

$\Rightarrow Y \leq X^{h,\Lambda,\Phi_0,\Psi_0} \Rightarrow \mathbb{E}_Q[Y] \leq \mathbb{E}_Q\left[ X^{h,\Lambda,\Phi_0,\Psi_0} \right] = x_0$.

$\mathbb{E}_P[(h-Y)^+] = \mathbb{E}_P\left[ (h - X^{h,\Lambda,\Phi_0,\Psi_0})^+ \right] = (h - c) \cdot \alpha$.

$\mathbb{E}_P[U(Y)] = \mathbb{E}_P\left[ U \left( X^{h,\Lambda,\Phi_0,\Psi_0} \right) - \mathbb{1}_{(h,\infty)} \left( X^{h,\Lambda,\Phi_0,\Psi_0} \right) \cdot \left( U(\tilde{h}) - U \left( \frac{1}{2} \tilde{h} + \frac{1}{2} h \right) \right) \right]$

$= \mathbb{E}_P\left[ U \left( X^{h,\Lambda,\Phi_0,\Psi_0} \right) \right] - P \left( X^{h,\Lambda,\Phi_0,\Psi_0} > \tilde{h} \right) \cdot \left( U(\tilde{h}) - U \left( \frac{1}{2} \tilde{h} + \frac{1}{2} h \right) \right)$

$= +\infty = \mathbb{E}_P\left[ U \left( X^{h,\Lambda,\Phi_0,\Psi_0} \right) \right]$.

$\Rightarrow Y$ is another optimal solution.

- $\mathbb{E}_P\left[ -U \left( X^{h,\Lambda,\Phi_0,\Psi_0} \right)^+ \right] = +\infty$.

(Assumption 3.18 and Lemma 3.51) $\Rightarrow \mathbb{E}_P \left[ U \left( X^{h,\Lambda,\Phi_0,\Psi_0} \right) \right] = -\infty$.

$x_0 \in H_F
\Rightarrow h \cdot \mathbb{1}_{(0,\tau(h))} \left( \frac{dQ}{dP} < \tau(h) \right) + h \cdot \mathbb{1}_{[\tau(h),+\infty)} \left( \frac{dQ}{dP} \right) = \mathbb{E}_Q \left[ u_0 \cdot \mathbb{1}_{(0,\tau(h))} \left( \frac{dQ}{dP} \right) \right]$.

$\Rightarrow P \left( X^{h,\Lambda,\Phi_0,\Psi_0} \neq h \cdot \mathbb{1}_{(0,\tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{1}_{[\tau(h),+\infty)} \left( \frac{dQ}{dP} \right) \right) > 0$.

$\mathbb{E}_P \left[ U \left( h \cdot \mathbb{1}_{(0,\tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{1}_{[\tau(h),+\infty)} \left( \frac{dQ}{dP} \right) \right) \right]$ $= P \left( \frac{dQ}{dP} < \tau(h) \right) \cdot U(h) + P \left( \frac{dQ}{dP} \geq \tau(h) \right) \cdot U(u_0) \geq -\infty = \mathbb{E}_P \left[ U \left( X^{h,\Lambda,\Phi_0,\Psi_0} \right) \right]$.

$\Rightarrow h \cdot \mathbb{1}_{(0,\tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{1}_{[\tau(h),+\infty)} \left( \frac{dQ}{dP} \right)$ is a different optimal solution.

Now, we have gathered enough material to prove the main result very easily:

**Proof of Theorem 3.23**: The case $h \in H_F$ is due to Proposition 3.70 and both other cases are dealt with in Proposition 3.42, because for all admissible $X$: $P(X < u_0) > 0$ implies $P(U(X) = -\infty) > 0$ and thus $\mathbb{E}_P[U(X)] = -\infty$. □
3.4. HOW TO PROVE OPTIMALITY

Expected Shortfall Restriction with respect to \( Q \)

First, we note that \( c \geq u_0 \) (Assumption 3.13), \( h \geq c \) and \( x_0 \geq h \cdot Q(\Omega) - (h - c) \alpha \) (cf. Theorem 3.6) implies that \( x_0 \geq h \cdot Q(\Omega) - (h - c) \alpha \geq h \cdot (Q(\Omega) - \alpha) + c \cdot \alpha \geq c \cdot Q(\Omega) \geq u_0 \cdot Q(\Omega) \).

Second, since we can represent all cases by choosing corresponding Lagrange multipliers, there is no special case to take care of separately. Thus calculating the pointwise maximum of the integrand appearing in the Lagrange Function is the next objective.

Lemma 3.71:
Let \( +\infty \cdot 0 := 0 \), \( h \in [u_0, +\infty) \), \( \lambda_1 \in (0, +\infty) \), \( \lambda_2^Q \in [0, +\infty) \), \( d \in (0, +\infty) \) and \( f \colon \mathbb{R} \to [-\infty, +\infty) \), \( x \mapsto U(x) - \lambda_1 \cdot x \cdot d - \lambda_2^Q \cdot d \cdot (h - x)^+ \).

Then \( \argmax_{x \in \mathbb{R}} \{ f(x) \} = \begin{cases} (U')^{-1} (\lambda_1 \cdot d) & \text{if } U'(h) - \lambda_1 \cdot d > 0 \\ h & \text{if } \lambda_1 \cdot d - \lambda_2^Q \cdot d \leq U'(h) \leq \lambda_1 \cdot d \\ (U')^{-1} (\lambda_1 \cdot d - \lambda_2^Q \cdot d) & \text{if } U'(h) - \lambda_1 \cdot d + \lambda_2^Q \cdot d < 0 \end{cases} \)

is the unique value maximizing \( f \).

Proof: We just have to set \( \lambda_2 := \lambda_2^Q \cdot d \) in Lemma 3.47.

It is convenient to replace the second Lagrange multiplier \( \lambda_2 \) by another parameter \( \lambda_3 \) connected to the original one in the following way: \( \lambda_3 = \lambda_1 - \lambda_2 \). Also, we are going to use the restriction \( \lambda_1 \geq \lambda_3 \) to capture the fact that \( \lambda_2 \) should be non-negative.

Hence, we find the definition of \( L_Q \) given on page 32 suitable for our purposes.

The family of random variables \( X_Q^{h,\lambda_1,\lambda_3} \), which are the candidates for the optimal solutions, were already defined in Subsection 3.3.3, too.

Fact 3.72:
A list of mostly implications of the definition \( X_Q^{h,\lambda_1,\lambda_3} \):

(a) For all \( (h, \lambda_1, \lambda_3) \in L_Q \) and \( \omega \in \Omega \):
\[
X_Q^{h,\lambda_1,\lambda_3}(\omega) = \begin{cases} (U')^{-1} (\lambda_1 \cdot \frac{dQ}{dP}(\omega)) & \text{if } U'(h) > \lambda_1 \cdot \frac{dQ}{dP}(\omega) \\ h & \text{if } \lambda_3 \cdot \frac{dQ}{dP}(\omega) \leq U'(h) \leq \lambda_1 \cdot \frac{dQ}{dP}(\omega) \\ (U')^{-1} (\lambda_3 \cdot \frac{dQ}{dP}(\omega)) & \text{if } U'(h) < \lambda_3 \cdot \frac{dQ}{dP}(\omega) \end{cases}
\]

(b) For all \( (h, \lambda_1, \lambda_3) \in L_Q \) we have
\[
\mathbb{E}_P \left[ \left( U \left( (U')^{-1} (\lambda_1 \cdot \frac{dQ}{dP}) \right) \right)^+ \right] < +\infty \iff \mathbb{E}_P \left[ \left( U \left( X_Q^{h,\lambda_1,\lambda_3} \right) \right)^+ \right] < +\infty,
\]

if in addition \( h = u_0 \) we get
\[
\mathbb{E}_P \left[ \left( -U \left( (U')^{-1} (\lambda_1 \cdot \frac{dQ}{dP}) \right) \right)^+ \right] < +\infty \iff \mathbb{E}_P \left[ \left( -U \left( X_Q^{h,\lambda_1,\lambda_3} \right) \right)^+ \right] < +\infty
\]

and if \( h > u_0 \) and \( \lambda_3 > 0 \) the following equivalence holds:
\[
\mathbb{E}_P \left[ \left( -U \left( (U')^{-1} (\lambda_3 \cdot \frac{dQ}{dP}) \right) \right)^+ \right] < +\infty \iff \mathbb{E}_P \left[ \left( -U \left( X_Q^{h,\lambda_1,\lambda_3} \right) \right)^+ \right] < +\infty.
\]
(c) For all \((h, \lambda_1, \lambda_3) \in L_Q\) we have \(\mathbb{E}_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right] < +\infty \iff \mathbb{E}_Q \left[ X_Q^{h,\lambda_1,\lambda_3} \right] < +\infty\).

(d) For fixed \(\omega \in \Omega\), the mapping \(L_Q \to [u_0, +\infty), (h, \lambda_1, \lambda_3) \mapsto X_Q^{h,\lambda_1,\lambda_3}(\omega)\) is continuous.

(e) The mapping \(L_Q \to [u_0 \cdot Q(\Omega), +\infty), (h, \lambda_1, \lambda_3) \mapsto \mathbb{E}_Q \left[ X_Q^{h,\lambda_1,\lambda_3} \right] \) is continuous.

(f) \(\forall h \in [u_0, +\infty), \lim_{\lambda_3 \to +\infty} \mathbb{E}_Q \left[ X_Q^{h,\lambda_1,0} \right] = +\infty\).

(g) \(\mathbb{E}_Q \left[ (h - X_Q^{h,\lambda_1,\lambda_3})^+ \right] = \mathbb{E}_Q \left[ (h - (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right))^+ \right].\)

(h) \([u_0, +\infty) \times [0, +\infty) \to [0, +\infty), (h, \lambda_3) \mapsto \mathbb{E}_Q \left[ (h - (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right))^+ \right]\) is continuous.

(i) Let \(A := \left\{ \begin{array}{ll} \{ -\infty \} & \text{if } h = u_0 \\ \frac{U'(h)}{t_{\max}}, +\infty & \text{if } h > u_0 \end{array} \right\} \). The mapping \(A \to [0, (h - u_0) \cdot Q(\Omega)], \lambda_3 \mapsto \mathbb{E}_Q \left[ (h - (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right))^+ \right] \)

is strictly increasing and surjective for fixed \(h \in [u_0, +\infty)\).

(j) The mapping \([0, +\infty) \to [0, +\infty), \lambda_3 \mapsto \mathbb{E}_Q \left[ (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right) \right] \) is a strictly decreasing and continuous function that maps onto \([u_0 \cdot Q(\Omega), +\infty]\).

(k) For all \(h\) and \(c \in [u_0, +\infty)\): \((h - c) \cdot \alpha \leq (h - u_0) \cdot Q(\Omega)\).

\(\lambda_Q^x\) is well-defined due to Fact 3.72(i) and Fact 3.72(k) (see also Fact 3.72(g)). \(\lambda_Q^{x_0}\) is well-defined due to Fact 3.72(j). Hence \(\Psi_Q^{x,x_0}\) is well-defined, too.

We continue with more observations:

**Fact 3.73:**

(a) If \(\alpha \in (0, Q(\Omega))\): \(\forall c \in [u_0, +\infty) \text{ and } \forall h \in (u_0, +\infty) \cap [c, +\infty): \lambda_Q^h < +\infty\).

(b) The mapping \([u_0 \cdot Q(\Omega), +\infty) \to [0, +\infty], x_0 \mapsto \lambda_Q^{x_0}\) is strictly decreasing and continuous.

(c) \(\forall x_0 \in [h \cdot Q(\Omega) - (h - c) \alpha, +\infty): \mathbb{E}_Q \left[ X_Q^{h,\max \left\{ \frac{U'(h)}{t_{\min}}, \Psi_Q^{h,x_0} \right\}, \Psi_Q^{h,x_0}} \right] \leq x_0.\)

(d) \(\left\{ \lambda_1 \in (0, +\infty) \cap \left[ \Psi_Q^{x_0}, \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi_Q^{h,x_0} \right\} \right] \mid \mathbb{E}_Q \left[ X_Q^{h,\lambda_1,\Psi_Q^{h,x_0}} \right] \leq x_0 \right\} \) is non-empty.

(e) The mapping \((0, +\infty) \cap \left[ \Psi_Q^{x_0}, \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi_Q^{h,x_0} \right\} \right] \to \mathbb{R}, \lambda_1 \mapsto \mathbb{E}_Q \left[ X_Q^{h,\lambda_1,\Psi_Q^{h,x_0}} \right] \) is strictly decreasing.

(f) For all initial endowments \(x_0 \geq h \cdot Q(\Omega) - (h - c) \alpha\) there exists a unique \(\lambda_1 \in (0, +\infty) \cap \left[ \Psi_Q^{x_0}, \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi_Q^{h,x_0} \right\} \right] \) with \(\mathbb{E}_Q \left[ X_Q^{h,\lambda_1,\Psi_Q^{h,x_0}} \right] = x_0.\)
Fact 3.73(f) guarantees that $\Lambda_{Q}^{h,x_{0}}$ is well-defined (see page 32).

**Fact 3.74:**
We realize that
- $\Lambda_{Q}^{h,x_{0}} \geq \lambda_{0}^{x_{0}},$
- $\mathbb{E}_{Q}\left[h - X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}\right] \leq (h - c) \alpha$ and
- $\mathbb{E}_{Q}\left[h - X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}\right] < (h - c) \alpha$ implies that $\Lambda_{Q}^{h,x_{0}} = \Psi_{Q}^{h,x_{0}} < +\infty.$

**Proposition 3.75:**
Let $c \in [u_{0}, +\infty)$ be fixed. For all $h \in [c, +\infty)$ and $x_{0} \in (h \cdot Q(\Omega) - (h - c) \alpha, +\infty)$, $X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}$ is an optimal solution of $(P_{ES_{h}})$. If and only if $\mathbb{E}_{P}\left[U\left(X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}\right)\right] < +\infty$ is the solution unique ($P$-almost-surely).

**Proof:** Let $Y$ be admissible for $(P_{ES_{h}})$.
- Case $\mathbb{E}_{Q}\left[h - X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}\right] < (h - c) \alpha$:
  
  (Fact 3.74) $\Rightarrow \Lambda_{Q}^{h,x_{0}} = \Psi_{Q}^{h,x_{0}} < +\infty.$

  (Fact 3.72(a), Lemma 3.71, $\lambda_{Q}^{h,x_{0}} := \Lambda_{Q}^{h,x_{0}} - \Psi_{Q}^{h,x_{0}} = 0$) $\Rightarrow$ For all $\omega \in \Omega$:
  
  $$U\left(X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}} (\omega)\right) - \Lambda_{Q}^{h,x_{0}} \cdot X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}} (\omega) \cdot \frac{dQ}{dP}(\omega)$$
  
  $$\geq U\left(Y(\omega)\right) - \Lambda_{Q}^{h,x_{0}} \cdot Y(\omega) \cdot \frac{dQ}{dP}(\omega).$$

  Therefore
  
  $$\mathbb{E}_{P}\left[U\left(X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}\right)\right] \overset{(**)}{=} \mathbb{E}_{P}\left[U\left(Y\right)\right] + \Lambda_{Q}^{h,x_{0}} \cdot \mathbb{E}_{Q}\left[X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}\right]_{x_{0}} \geq 0$$

  $$\geq \mathbb{E}_{P}\left[U\left(Y\right)\right].$$

  If $P\left(X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}} \neq Y\right) > 0$ and $\mathbb{E}_{P}\left[U\left(X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}\right)\right] < +\infty$, we can conclude that $P\left(\{\omega \in \Omega |(*) \text{ is a strict inequality}\}\right) > 0$ and thus (**) is a strict inequality.

- Case $\mathbb{E}_{Q}\left[h - X_{Q}^{h,\Lambda_{Q}^{h,x_{0}},\Psi_{Q}^{h,x_{0}}}\right] = (h - c) \alpha$. 


What remains to be done is to show the necessity of the solution. This can be done identically to the proof of Proposition 3.70, except that one

Proposition 3.76:

Let \( c \in [u_0, +\infty) \) and \( h \in [c, +\infty) \) be fixed and assume that \( x_0 = h \cdot Q(\Omega) - (h - c)\alpha \).

Then \( X_Q^{h,A,h,x_0,Q,h,x_0} \) is an optimal solution of \((P_{ES_h}^Q)\). It is \((P\text{-almost-surely})\) unique if at least one of the following four conditions is satisfied: \( U(u_0) > -\infty \), \( \alpha \in (0, Q(\Omega)) \), \( c > u_0 \) or \( h = c \).

Proof: Let \( Y \) be admissible for \((P_{ES_h}^Q)\).

\( \Rightarrow x_0 \geq E_Q[Y] = E_Q[(Y - h)^+] + h \cdot Q(\Omega) - E_Q[(h - Y)^+] \geq h \cdot Q(\Omega) - (h - c)\alpha = x_0. \)

\( \Rightarrow Q(Y > h) = 0. \)

- Case \( h = c \):
  \( E_Q[(h - Y)^+] \leq (h - c)\alpha = 0 \Rightarrow Q(Y < h) = 0. \)
  \( P \sim Q \Rightarrow Y = h \ (P\text{-almost-surely}). \)
  \( \Rightarrow X_Q^{h,A,h,x_0,Q,h,x_0} \) is the \((P\text{-almost-surely})\) unique admissible random variable for \((P_{ES_h}^Q)\) and thus optimal.

57
3.4. HOW TO PROVE OPTIMALITY

- Case \( h > c \) and either \( \alpha \in (0, Q(\Omega)) \) or \( c > u_0 \):
  (Fact 3.73(a)) \( \lambda_Q^h < +\infty \) and therefore \( \psi_Q^{h,x_0} < +\infty \).

\[
\forall d \in (0, +\infty), \forall \lambda_3 \in [0, +\infty); \left\{ \begin{array}{ll}
h & \text{if } \lambda_3 \cdot d \leq U'(h) \\
(U')^{-1}(\lambda_3 \cdot d) & \text{if } \lambda_3 \cdot d > U'(h)
\end{array} \right.
\]

is the unique maximum point of \((-\infty, h] \rightarrow [-\infty, +\infty) x \mapsto U(x) - \lambda_3 \cdot x \cdot d).

Since \( X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}} \) is admissible for \( (P_{ES_h^Q}) \) and \( P \sim Q, P \left( X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}} > h \right) = 0. \)

We conclude that for all \( \omega \in \Omega: \)

\[
U \left( X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}}(\omega) \right) - \psi_Q^{h,x_0} \cdot X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}}(\omega) \cdot \frac{dQ}{dP}(\omega) \]

\[
\geq U(Y(\omega)) - \psi_Q^{h,x_0} \cdot Y(\omega) \cdot \frac{dQ}{dP}(\omega).
\]

Again, this results in

\[
E_P \left[ U \left( X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}} \right) \right] \geq E_P \left[ U(Y) \right] + \psi_Q^{h,x_0} \cdot \left( E_Q \left[ X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}} \right] - E_Q [Y] \right) \]

\[
\geq E_P \left[ U(Y) \right].
\]

We note that \( E_P \left[ U \left( (h - c) \frac{Q(\Omega)}{Q(\Omega)} \right) \right] = 1 \cdot U \left( h - (h - c) \frac{Q(\Omega)}{Q(\Omega)} \right) \) is finite if one of first

three uniqueness conditions is met.

Hence \( P \left( X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}} > h \right) = 0 \) results in \( E_P \left[ U \left( X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}} \right) \right] < +\infty \), because

\( h - (h - c) \frac{Q(\Omega)}{Q(\Omega)} \) is admissible for \( (P_{ES_h^Q}) \).

As a consequence, \( (**) \) is a strict inequality if

\[
P \{ \omega \in \Omega | (*) \text{ is a strict inequality} \} = P \left( X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}} \neq Y \right) > 0.
\]

- Case \( h > c, \alpha = Q(\Omega) \) and \( c = u_0 \):
  \( \Rightarrow x_0 = h \cdot Q(\Omega) - (h - c) \alpha = u_0 \cdot Q(\Omega). \)
  \( \Rightarrow \) if \( P(Y \neq u_0) > 0, P(Y < u_0) > 0 \) and thus \( E_P [U(Y)] = -\infty. \)
  \( \Rightarrow X_Q^{h,\lambda_Q^h,\psi_Q^{h,x_0}} = X_Q^{h,+,+} = u_0 \) is an optimal solution.

If \( U(u_0) > -\infty, E_P [U(u_0)] = 1 \cdot U(u_0) > -\infty \) which ensures the uniqueness of the solution.

\( \square \)

Proof of Theorem 3.24: If \( x_0 = h \cdot Q(\Omega) - (h - c) \alpha, U(u_0) = -\infty, \alpha = Q(\Omega), c = u_0 \) and \( h > c, \)

obviously \( x_0 = u_0 \cdot Q(\Omega) \) and hence \( P(U(X) = -\infty) = P(X \leq u_0) > 0 \) yields \( E_P [U(X)] = -\infty \)

for all admissible \( X \).

All other statements are due to Proposition 3.75 and Proposition 3.76. 

\( \square \)
3.4.4 Conditional Value at Risk Problem

Our next objective is to find optimal solutions of \((P_{CVaR^{\nu}})\) for fixed \(\alpha \in (0, \nu(\Omega))\).

The fundamental idea is to find the optimal \(h\) in the reformulated problem \((\tilde{P}_{CVaR^{\nu}})\). As said previously, the reformulated problem corresponds for fixed \(h\) to a problem with expected shortfall restriction. Therefore we are able to use the previous results to solve it. Luckily, the optimal value of the objective function of the family of problems \((P_{ES_h^{\nu}})\) indexed by \(h\) is at least an upper semi-continuous function in \(h\) under an additional integrability condition (Assumption 3.25).

Furthermore, we can restrict the search for an optimal \(h\) of the reformulated problem \((\tilde{P}_{CVaR^{\nu}})\) to a non-empty compact subset of \(\mathbb{R}\). Of course, an upper semi-continuous function defined on a non-empty compact set attains its supremum at some point \(h^\ast\). Therefore \(h^\ast\) is an optimal \(h\).

As \((P_{ES_h^{\nu}})\) and \((P_{ES_0^{\nu}})\) in general have different solutions, we are going to treat the cases \(\nu = P\) and \(\nu = Q\) separately.

The approach used to derive the continuity of the optimal value of the objective function as a function in \(h\) will be to show that our optimal solutions form a continuous function of the index \(h\). After that, the integrability condition (Assumption 3.25) has to ensure that the continuity property can be preserved when dealing with the expectancy operator in the objective function.

**Continuity of our Solution of \((P_{ES_h^{P}})\) and its implications**

**Lemma 3.77:**
If \(c = u_0\), the function \(\Psi^h: \{h \in H_P | E_P \left( (h - X^{h,\Lambda,0})^+ \right) \geq (h - c) \cdot \alpha \} \to [0, +\infty]\) is monotonic non-decreasing.

**Lemma 3.78:**
Let \(\lim_{n \to +\infty} h_n = h_0 \geq \frac{x_0}{Q(\Omega)}\) with \(h_n \in H_P \forall n \in \mathbb{N}_0\). Then \(\Psi_+ := \limsup_{n \to +\infty} \Psi^{h_n} < +\infty\).

**Lemma 3.79:**
Let \(h_n \in \{h \in H_P | E_P \left( (h - X^{h,\Lambda,0})^+ \right) \geq (h - c) \cdot \alpha \}, \forall n \in \mathbb{N}_0\) with \(\lim_{n \to +\infty} h_n = h_0\) and without any loss of generality \(h_n \in \left\{u_0, \frac{x_0}{Q(\Omega)}\right\}\) if \(h_0 < \frac{x_0}{Q(\Omega)}\).

Then \(\lim_{n \to +\infty} X^{h_n,\Lambda,\nu,\Psi^{h_n}} = X^{h_0,\Lambda,\nu,\Psi^{h_0}} (P\text{-almost-surely})\).

**Proof:** Let \(L_{h_0}^{\pm} := \left\{\frac{\mathbb{R}_+}{0, +\infty} | \begin{array}{l} h_0 \geq \frac{x_0}{Q(\Omega)} \quad \text{if} \quad h_0 \leq \frac{x_0}{Q(\Omega)} \end{array}\right\}, \Psi_- := \liminf_{n \to +\infty} \Psi^{h_n}\) and \(\Psi_+ := \limsup_{n \to +\infty} \Psi^{h_n}\).

\(\Rightarrow \exists (i_n): \lim_{n \to +\infty} \Psi^{h_n} = \Psi_-\) and \(\exists (s_n): \lim_{n \to +\infty} \Psi^{h_n} = \Psi_+\).

By applying Lemma 3.78, we learn that \(h_0 \geq \frac{x_0}{Q(\Omega)} \Rightarrow \Psi_+ < +\infty\).

\(\Rightarrow \Psi_+ \in L_{h_0}^{\pm}\).

\(\Psi_- \leq \Psi_+ \Rightarrow \Psi_- \in L_{h_0}^{\pm}\).

(Lemma 3.62) \Rightarrow \lim_{n \to +\infty} X^{h_n,\Lambda,\nu,\Psi^{h_n}} = X^{h_0,\Lambda,\nu,\Psi_-}(Q\text{-almost-everywhere})\) and \(\lim_{n \to +\infty} X^{h_n,\Lambda,\nu,\Psi^{h_n}} = X^{h_0,\Lambda,\nu,\Psi_+}(Q\text{-almost-everywhere})\).
(Dominated Convergence Theorem) \( \Rightarrow \)

\[
E_P \left[ (h_0 - X^h_0, \Lambda^\phi^{0, \Psi_+, \psi_+})^+ \right] = \lim_{n \to +\infty} E_P \left[ \left( h_n - X^h_n, \Lambda^\phi^{0, \Psi_+, \psi_+} \right)^+ \right] \\
\text{Fact 3.69(e) } (h_n - c) \cdot \alpha \\
= (h_0 - c) \cdot \alpha \\
= \lim_{n \to +\infty} E_P \left[ \left( h_n - X^h_n, \Lambda^\phi^{h, \psi_+} \right)^+ \right] \\
\text{Fact 3.69(e) } (h_n - c) \cdot \alpha \\
= E_P \left[ (h_0 - X^h_0, \Lambda^\phi^{0, \Psi_+, \psi_+})^+ \right].
\]

\( h_0 \in H_P \Rightarrow x_0 > Q \left( \frac{\delta q_0}{\delta \theta} < \tau (h) \right) \cdot h_0 + Q \left( \frac{\delta q_0}{\delta \theta} \geq \tau (h) \right) \cdot u_0 \geq Q (\Omega) \cdot u_0. \)
\( \Rightarrow \Lambda^{h_0, \Psi_-} < +\infty. \)

- **Case \( \Psi_- = \Psi_+ : \)**

\( \Rightarrow \exists \lim_{n \to +\infty} \Psi^h \) and \( \lim_{n \to +\infty} \Psi^h = \Psi_- = \Psi_+ . \)

(Lemma 3.62) \( \Rightarrow \lim_{n \to +\infty} X^h_0 = X^{0, \Delta^\phi^{0, \Psi_+, \psi_+}}_{\Lambda^\phi^{0, \Psi_+, \psi_+}} (Q\text{-almost-everywhere}). \)

Since \( E_P \left[ (h_0 - X^h_0, \Lambda^\phi^{0, \Psi_+, \psi_+})^+ \right] = (h_0 - c) \cdot \alpha = E_P \left[ (h_0 - X^h_0, \Lambda^\phi^{0, \psi_0})^+ \right] \), Lemma 3.67 tells us that \( X^h_0, \Lambda^\phi^{0, \Psi_-, \psi_+} = X^h_0, \Lambda^\phi^{0, \psi_0} (P\text{-almost-surely}). \)

\( P \sim Q \Rightarrow \lim_{n \to +\infty} X^h_0, \Lambda^\phi^{h, \psi_0} = X^h_0, \Lambda^\phi^{\psi_0} (P\text{-almost-surely}). \)

- **Case \( \Psi_- < \Psi_+ : \)**

(Lemma 3.67) \( \Rightarrow E_P \left[ (h_0 - X^h_0, \Lambda^\phi^{0, \Psi_+, \psi_+})^+ \right] = 0. \)

\( \Rightarrow h_0 = c, \)

\( \Rightarrow x_0 > Q \left( \frac{\delta q_0}{\delta \theta} < \tau (h_0) \right) \cdot h_0 + Q \left( \frac{\delta q_0}{\delta \theta} \geq \tau (h_0) \right) \cdot u_0 \text{ Lemma 3.40(d) } Q (\Omega) \cdot h_0. \)

\( \Psi^{\psi_0} = 0 \) and \( t_{\text{max}} < U'(h_0) + \Psi^{\psi_0} \).

\( (U' \text{ is continuous}) \Rightarrow \exists \tilde{h} > h_0: \forall \tilde{h} \in [h_0, \tilde{h}]: t_{\text{max}} < \frac{U'(\tilde{h}) + \Psi^{\psi_0}}{\Lambda^\phi^{0, \psi_0}}. \)

\( \Rightarrow \forall \tilde{h} \in (h_0, \tilde{h}]: E_P \left[ \left( \tilde{h} - X^{\tilde{h}, \Lambda^0, 0} \right)^+ \right] = E_P \left[ \left( \tilde{h} - X^{\tilde{h}, \Lambda^0, 0} \right)^+ \right] = 0 < (\tilde{h} - c) \cdot \alpha. \)

\( \Rightarrow h_0 \text{ is an isolated point, i.e. } \exists n_0 \in \mathbb{N}: \forall n > n_0: h_n = h_0. \)

\( \Psi^{\psi_0} > 0 \) or \( t_{\text{max}} < \frac{U'(h_0) + \Psi^{\psi_0}}{\Lambda^\phi^{0, \psi_0}}, \) because:

\( \Rightarrow \Psi^{\psi_0} > 0. \)

\( \Rightarrow \lambda_2 := \max \left\{ 0, t_{\text{max}} \Lambda^\phi^{0, \psi_0} - U'(h_0) \right\} < \Psi^{\psi_0}. \)

(Lemma 3.59) \( \Lambda^{h_0, \lambda_2} \leq \Lambda^{h_0, \psi_0}. \)
Next, we assume that $h_0 = u_0$:

(Equation 3.77) $\Rightarrow \lim_{n \to +\infty} \Psi_{\text{nn}}.$

$\Rightarrow \Psi = \Psi_{+}$ contradicting $\Psi < \Psi_{+}.$

Thus we know that $h_0 > u_0.$

$\forall \varepsilon > 0:$

\[
\begin{align*}
(h_n - c) \cdot \alpha &= \mathbb{E}_P \left[ \left( h_n - X_{h_n, A_n, \Phi_{h_n, \Psi_{h_n}}} \right)^+ \right] \\
&\geq \mathbb{E}_P \left[ \left( (c - \varepsilon) - X_{h_n, A_n, \Phi_{h_n, \Psi_{h_n}}} \right)^+ \right] \\
&\quad + P \left( X_{h_n, A_n, \Phi_{h_n, \Psi_{h_n}}} < (c - \varepsilon) \right) \cdot \frac{(h_n - (c - \varepsilon))}{-\varepsilon (n \to +\infty)} \\
&\geq 0.
\end{align*}
\]

$\Rightarrow \forall \varepsilon > 0 : \lim_{n \to +\infty} P \left( X_{h_n, A_n, \Phi_{h_n, \Psi_{h_n}}} < (c - \varepsilon) \right) = 0. \tag{3.1}$

* Suppose $\exists \delta \in (0, t_{\text{max}}): \forall n_0 \in \mathbb{N}: \exists n > n_0: \frac{U'(h_n) + \Psi_{h_n}}{A_n, \Phi_{h_n}} < \delta.$

Let $\delta \in (0, t_{\text{max}})$.

$\Rightarrow \exists \delta_1 \in (\delta, t_{\text{max}})$.

$(h_0 > u_0) \Rightarrow U'(h_0) < +\infty.$

$(U')$ is continuous $\Rightarrow \exists \delta_1 > 0$ and $\exists n_0 \in \mathbb{N}: \forall n > n_0: \frac{U'(h_n - \varepsilon)}{U'(h_n)} \leq \frac{\delta_1}{\delta}.$

$\Rightarrow \forall n \geq n_0$ with $\frac{U'(h_n) + \Psi_{h_n}}{A_n, \Phi_{h_n}} < \delta$:

\[
U'(c - \varepsilon) = U'(h_0 - \varepsilon) \leq \delta_1 \cdot \frac{U'(h_n)}{\delta} \leq \delta_1 \cdot \left( A_n, \Phi_{h_n} - \Psi_{h_n} \right) \\
\leq \delta_1 \cdot A_n, \Phi_{h_n} - \Psi_{h_n}.
\]

$\Rightarrow \forall n \geq n_0$ with $\frac{U'(h_n) + \Psi_{h_n}}{A_n, \Phi_{h_n}} < \delta$: $c - \varepsilon \geq \frac{U'}{U'} \left( \frac{\delta_1}{\delta} - \Psi_{h_n} \right).$

Since $P \left( \frac{\delta_1}{\delta} \geq \delta_1 \right) > 0,$ this is a contradiction to (3.1).

$\Rightarrow \forall \delta \in (0, t_{\text{max}}): \exists n_0 \in \mathbb{N}: \forall n > n_0: \frac{U'(h_n) + \Psi_{h_n}}{A_n, \Phi_{h_n}} \geq \delta.$

(Fact 3.69(e)) $\Rightarrow \mathbb{E}_P \left[ \left( h_0 - X_{h_0, A_0, \Phi_{h_0, \Psi_{h_0}}} \right)^+ \right] = (h_0 - c) \cdot \alpha = 0,$

$\Rightarrow P \left( h_0 > X_{h_0, A_0, \Phi_{h_0, \Psi_{h_0}}} \right) = 0,$

$\Rightarrow \mathbb{E}_Q \left[ X_{h_0, A_0, \Phi_{h_0}, \Psi_{h_0}} \right] = \mathbb{E}_Q \left[ X_{h_0, A_0, \Phi_{h_0}, \Psi_{h_0}} \right] = x_0.$
As \( x_0 > Q(\Omega) \cdot h_0 \), we know that \( \forall \lambda^*_n \in (0, \Lambda^0_{h_0, \psi^{h_0}}) \) and \( \forall \lambda^*_n \in (\Lambda^0_{h_0, \psi^{h_0}}, +\infty) \):

\[
\mathbb{E}_Q \left[ X^{h_0, \lambda^*_n} \right] > \mathbb{E}_Q \left[ X^{h_0, \Lambda^0_{h_0, \psi^{h_0}}, +\infty} \right] > \mathbb{E}_Q \left[ X^{h_0, \lambda^*_n, +\infty} \right].
\]

(3.2)

* Suppose \( \Lambda^h_n, \psi^{h_n} < \Lambda^0_{h_0, \psi^{h_0}} - \varepsilon \) for infinitely many \( n \).

If \( \frac{U''(h_n) + \psi^{h_n}}{\Lambda^h_n, \psi^{h_n}} \geq \delta \):

\[
x_0 = \mathbb{E}_Q \left[ X^{h_n, \Lambda^h_n, \psi^{h_n}, \psi^{h_n}} \right]
= \int \{ \frac{dQ}{dP} < \delta \} \frac{X^{h_n, \Lambda^h_n, \psi^{h_n}, \psi^{h_n}}}{\Lambda^h_n, \psi^{h_n}} \, dQ + \int \{ \frac{dQ}{dP} \geq \delta \} \frac{X^{h_n, \Lambda^h_n, \psi^{h_n}, \psi^{h_n}}}{\Lambda^h_n, \psi^{h_n}} \, dQ
\geq \int \{ \frac{dQ}{dP} < \delta \} \max \left\{ h_n, (U')^{-1} \left( \Lambda^0_{h_0, \psi^{h_0}} - \varepsilon \right) \cdot \frac{dQ}{dP}(\omega) \right\} \, dQ + u_0 \cdot \mathbb{Q} \left( \frac{dQ}{dP} < \delta \right)
\]

This is a contradiction.

* Suppose \( \Lambda^h_n, \psi^{h_n} > \Lambda^0_{h_0, \psi^{h_0}} + \varepsilon \) for infinitely many \( n \).

\[
\Rightarrow X^{h_n, \Lambda^h_n, \psi^{h_n}, \psi^{h_n}} \bigg|_{\Omega} \leq \max \left\{ h_n, (U')^{-1} \left( \Lambda^0_{h_0, \psi^{h_0}} + \varepsilon \right) \cdot \frac{dQ}{dP}(\omega) \right\}.
\]

If \( \frac{U''(h_n) + \psi^{h_n}}{\Lambda^h_n, \psi^{h_n}} \geq \delta \):

\[
x_0 = \mathbb{E}_Q \left[ X^{h_n, \Lambda^h_n, \psi^{h_n}, \psi^{h_n}} \right]
= \int \{ \frac{dQ}{dP} < \delta \} \frac{X^{h_n, \Lambda^h_n, \psi^{h_n}, \psi^{h_n}}}{\Lambda^h_n, \psi^{h_n}} \, dQ + \int \{ \frac{dQ}{dP} \geq \delta \} \frac{X^{h_n, \Lambda^h_n, \psi^{h_n}, \psi^{h_n}}}{\Lambda^h_n, \psi^{h_n}} \, dQ
\leq \max \left\{ h_n, (U')^{-1} \left( \Lambda^0_{h_0, \psi^{h_0}} + \varepsilon \right) \cdot \frac{dQ}{dP}(\omega) \right\} \cdot \mathbb{Q} \left( \frac{dQ}{dP} < \delta \right)
\]

This is a contradiction.

\[ \lim_{n \to +\infty} \Lambda^h_n, \psi^{h_n} = \Lambda^0_{h_0, \psi^{h_0}}. \]

\[ \forall \omega \in \Omega \text{ with } \frac{dQ}{dP}(\omega) < t_{\text{max}}: \exists n_0 \in \mathbb{N}: \forall n \geq n_0, \frac{U''(h_n) + \psi^{h_n}}{\Lambda^h_n, \psi^{h_n}} \geq \frac{dQ}{dP}(\omega). \]

\[ \Rightarrow \forall \omega \in \Omega \text{ with } \frac{dQ}{dP}(\omega) < t_{\text{max}}: \exists n_0 \in \mathbb{N}: \forall n \geq n_0: \]

\[ X^{h_n, \Lambda^h_n, \psi^{h_n}, \psi^{h_n}}(\omega) = \max \left\{ h_n, (U')^{-1} \left( \Lambda^h_n, \psi^{h_n} \right) \cdot \frac{dQ}{dP}(\omega) \right\}. \]
Lemma 3.80:
Let $\lim_{n \to +\infty} h_n = h_0$ with $\forall n \in \mathbb{N}_0$: $h_n \in H_P$. Then $\{ \Lambda_{h_n} \Psi^{h_n} \mid n \in \mathbb{N}_0 \}$ is bounded.

Proof: First of all, the expected values are all well defined due to Assumption 3.25 and Lemma 3.52.
Let $\lim_{n \to +\infty} h_n = h_0$ with $\forall n \in \mathbb{N}_0$: $h_n \in H_P$.
It is sufficient to show that $\lim_{n \to +\infty} E_P \left[ U \left( X^{h_n, \Lambda_{h_n}^{0}, \Psi^0_n, \Psi^h_n} \right) \right] = E_P \left[ U \left( X^{h_0, \Lambda_0^{0}, \Psi^0_0, \Psi^h_0} \right) \right]$.
(Lemma 3.80) $\Rightarrow \exists \lambda^{\sup} < +\infty$: $\forall n \in \mathbb{N}_0$: $\Lambda_{h_n}^{\Psi^{h_n}} < \lambda^{\sup}$.
$\lambda^{\inf}_1 := \inf \{ \Lambda_{h_n}^{\Psi^{h_n}} \mid n \in \mathbb{N}_0 \}$, $h_{\text{min}} := \min \{ h_n \mid n \in \mathbb{N}_0 \}$ and $h_{\text{max}} := \max \{ h_n \mid n \in \mathbb{N}_0 \}$.

- (Lemma 3.57(b)) $\Rightarrow \lambda^{\inf}_1 > 0$.
- ($\lambda^{\sup} < +\infty$) $\Rightarrow \lambda^{\inf}_1 < +\infty$.

$\forall n \in \mathbb{N}_0$: $X^{h_{\text{min}}, \lambda^{\sup}, 0} \leq X^{h_n, \Lambda_{h_n}^{0}, \Psi^0_n, \Psi^h_n} \leq X^{h_{\text{max}}, \lambda^{\inf}_1, +\infty}$
$\Rightarrow \forall n \in \mathbb{N}_0$, $\forall \omega \in \Omega$: $U \left( X^{h_{\text{min}}, \lambda^{\sup}, 0}(\omega) \right) \leq U \left( X^{h_n, \Lambda_{h_n}^{0}, \Psi^0_n, \Psi^h_n}(\omega) \right) \leq U \left( X^{h_{\text{max}}, \lambda^{\inf}_1, +\infty}(\omega) \right)$.
By applying Lemma 3.79, $\forall n \in \mathbb{N}$: $X^{h_n, \Lambda_n^{0}, 0} = X^{h_0, \Lambda_0^{0}, 0}$, Fact 3.69(d), Lemma 3.52 and the Dominated Convergence Theorem, we can derive the desired result immediately.

Lemma 3.82:
The mapping $[c, +\infty) \to \mathbb{R}$, $h \mapsto E_P \left[ U \left( h \cdot I_{(0,\tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot I_{[\tau(h),+\infty)} \left( \frac{dQ}{dP} \right) \right) \right]$ is

- continuous in all $h \in [c, +\infty)$ if $U(u_0) > -\infty$ or $c = u_0$,
- continuous in all $h \in (c, +\infty)$,
- upper semi-continuous in $c$ if $U(u_0) = -\infty$ and $c > u_0$.

Proof: First of all,

$$E_P \left[ U \left( h \cdot I_{(0,\tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot I_{[\tau(h),+\infty)} \left( \frac{dQ}{dP} \right) \right) \right]$$

is well defined for all $h \in [c, +\infty)$.
Note that $U$ is continuous in the interval $[u_0, +\infty)$ (cf. Assumption 3.14) and Lemma 3.40(c) states that $[c, +\infty) \to [1 - \alpha, 1)$, $h \mapsto P \left( \frac{dQ}{dP} < \tau(h) \right)$ is continuous.

- Case $U(u_0) > -\infty$:
  $\Rightarrow h \mapsto E_P \left[ U \left( h \cdot I_{(0,\tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot I_{[\tau(h),+\infty)} \left( \frac{dQ}{dP} \right) \right) \right]$ is continuous.
3.4. HOW TO PROVE OPTIMALITY

- Case $U(u_0) = -\infty$ and $c = u_0$:
  \[ \forall h \in [c, +\infty): \mathbb{E}_P \left[ U \left( h \cdot \mathbb{I}_{(0, \tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{I}_{[\tau(h), +\infty)} \left( \frac{dQ}{dP} \right) \right) \right] = -\infty. \]

- Case $U(u_0) = -\infty$ and $c > u_0$:
  \[ \forall h \in (c, +\infty): \mathbb{E}_P \left[ U \left( h \cdot \mathbb{I}_{(0, \tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{I}_{[\tau(h), +\infty)} \left( \frac{dQ}{dP} \right) \right) \right] = -\infty. \]
  \[ \mathbb{E}_P \left[ U \left( c \cdot \mathbb{I}_{(0, \tau(c))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{I}_{[\tau(c), +\infty)} \left( \frac{dQ}{dP} \right) \right) \right] = U(c) > U(u_0) = -\infty. \]

Lemma 3.83:
Let $h \in (\overline{H}_P \setminus H_P), \forall n \in \mathbb{N}$: $h_n \in H_P$ and $\lim_{n \to +\infty} h_n = h$.

- If $U(u_0) > -\infty$,
  \[ \lim_{n \to +\infty} \mathbb{E}_P \left[ U \left( X_{h_n, \lambda_n}^{u_0, \Psi_n, \Phi_n} \right) \right] = \mathbb{E}_P \left[ U \left( h \cdot \mathbb{I}_{(0, \tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{I}_{[\tau(h), +\infty)} \left( \frac{dQ}{dP} \right) \right) \right], \]
  and if $U(u_0) = -\infty$,
  \[ \limsup_{n \to +\infty} \mathbb{E}_P \left[ U \left( X_{h_n, \lambda_n}^{u_0, \Psi_n, \Phi_n} \right) \right] \leq \mathbb{E}_P \left[ U \left( h \cdot \mathbb{I}_{(0, \tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{I}_{[\tau(h), +\infty)} \left( \frac{dQ}{dP} \right) \right) \right]. \]

Proof: (Lemma 3.57(b)) $\Rightarrow \lambda_{n}^{\text{inf}} := \inf \left\{ \lambda_{n} \mid n \in \mathbb{N} \right\} > 0$.
(Fact 3.69(b)) $\Rightarrow \lambda_{1}^{n} < +\infty$.

Let $h_{\text{max}} := \max \left\{ h_n \mid n \in \mathbb{N} \right\}$.

- Case $U(u_0) > -\infty$:
  Due to Lemma 3.44, $u_0 \leq X_{h_n, \lambda_n}^{u_0, \Psi_n, \Phi_n} \leq X_{h_{\text{max}}, \lambda_{1}^{\text{inf}}, +\infty}$ and the fact that both $U(u_0)$ and $U \left( X_{h_{\text{max}}, \lambda_{1}^{\text{inf}}, +\infty} \right)$ are integrable (Lemma 3.52), the result is implied by the Dominated Convergence Theorem.

- Case $U(u_0) = -\infty$:
  Due to Lemma 3.44, $X_{h_n, \lambda_n}^{\Phi_n, \Psi_n} \leq X_{h_{\text{max}}, \lambda_{1}^{\text{inf}}, +\infty}$ and the fact that $U \left( X_{h_{\text{max}}, \lambda_{1}^{\text{inf}}, +\infty} \right)$ is integrable (Lemma 3.52), the result is implied by Fatou’s Lemma.

Proposition 3.84:
The mapping
\[ \overline{H}_P \to \mathbb{R}, h \mapsto \begin{cases} \mathbb{E}_P \left[ U \left( X_{h, \lambda}^{\Phi, \Psi} \right) \right] & \text{if } h \in H_P \\ \mathbb{E}_P \left[ U \left( h \cdot \mathbb{I}_{(0, \tau(h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{I}_{[\tau(h), +\infty)} \left( \frac{dQ}{dP} \right) \right) \right] & \text{if } h \notin H_P \end{cases} \]
is

- continuous if $U(u_0) > -\infty$ and
- upper semi-continuous if $U(u_0) = -\infty$.

Proof: This proposition sums up Lemma 3.81, Lemma 3.82 and Lemma 3.83.
Now, we can wrap up the proof of Theorem 3.27:

**Proof of Theorem 3.27:**

- **Case \( \mathcal{H}_P \neq \emptyset \):**
  (Fact 3.45) \( \Rightarrow \mathcal{H}_P \) is a non-empty compact set.
  For all \( h \in \mathcal{H}_P \) (cf. Theorem 3.23):
  
  \[- h \cdot \mathbb{I}_{(0, \tau(h))} \left( \frac{dQ}{dP(H)} \right) + u_0 \cdot \mathbb{I}_{[\tau(h), +\infty)} \left( \frac{dQ}{dP(H)} \right) \]

  is admissible for \( (P_{ES, h}) \) and thus
  
  \[- (P_{ES, h}) \text{ is feasible}, \]

  \[- X^{h, \Lambda, \psi, \psi} \text{ is an optimal solution of } (P_{ES, h}) \text{ if } h \in \mathcal{H}_P \text{ and} \]

  \[- h \cdot \mathbb{I}_{(0, \tau(h))} \left( \frac{dQ}{dP(H)} \right) + u_0 \cdot \mathbb{I}_{[\tau(h), +\infty)} \left( \frac{dQ}{dP(H)} \right) \]

  is an optimal solution of \( (P_{ES, h}) \) if \( h \notin \mathcal{H}_P \).

  (Proposition 3.84) \( \Rightarrow \) The mapping

  \[ \mathcal{H}_P \to \mathbb{R}, h \mapsto \begin{cases} E_P \left[ U \left( X^{h, \Lambda, \psi, \psi} \right) \right] & \text{if } h \in \mathcal{H}_P \\ E_P \left[ U \left( h \cdot \mathbb{I}_{(0, \tau(h))} \left( \frac{dQ}{dP(H)} \right) + u_0 \cdot \mathbb{I}_{[\tau(h), +\infty)} \left( \frac{dQ}{dP(H)} \right) \right) \right] & \text{if } h \notin \mathcal{H}_P \end{cases} \]

  is upper semi-continuous which implies that it attains its supremum over the non-empty compact set \( \mathcal{H}_P \). The \( h \in \mathcal{H}_P \) attaining the supremum has the desired properties.

- **Case \( \mathcal{H}_P = \emptyset \):**
  (Theorem 2.9, Proposition 3.42) \( \Rightarrow \) For all admissible \( X \): \( P(X < u_0) > 0 \).
  \( \Rightarrow \) For all admissible \( X \): \( P(U(X) = -\infty) > 0 \) and thus \( E_P [U(X)] = -\infty \).

\( \square \)

Example 3.28 contains an example of a problem without an optimal solution. The interesting part of its proof is the explicit construction of a sequence of admissible random variable, which yield objective function values close to the optimum.

**Proof of Example 3.28:** Let \( X := (U')^{-1} \left( \lambda_0^{x_0}, +\infty \right) \).

We can safely assume that \(- c < \text{CVaR}^P(X) = - \frac{E_P[X]}{1} \) since otherwise \( X_n := X \) will do.

The mapping \( (0, +\infty) \to \left[ \frac{1}{Q\left( \sqrt{t} \right)}, +\infty \right), t \mapsto \frac{F^P_{\overline{Q}}(t)}{F^P_{\overline{Q}}(t)} \) is bijective, because \( t_{\text{min}} = 0 \).

For \( n \in \mathbb{N} \) let \( \lambda_n \in (\lambda_0^{x_0}, +\infty) \) with

\[ \frac{c - E_P[U(X)]}{x_0 - E_Q[(U')^{-1}(\lambda_n, \frac{dQ}{dP})]} \geq \frac{1}{Q(t)} \] and \( \lim_{n \to +\infty} \lambda_n = \lambda_0^{x_0} \).

\( \Rightarrow \forall n \in \mathbb{N}: \exists t_n \in (0, +\infty) \) with

\[ \frac{F^P_{\overline{Q}}(t_n)}{F^P_{\overline{Q}}(t_n)} = \frac{c - E_P[U(X)]}{x_0 - E_Q[(U')^{-1}(\lambda_n, \frac{dQ}{dP})]} \cdot \frac{c - E_P[(U')^{-1}(\lambda_n, \frac{dQ}{dP})]}{x_0 - E_Q[(U')^{-1}(\lambda_n, \frac{dQ}{dP})]} \]

We set \( X_n := (U')^{-1} \left( \lambda_n \cdot \frac{dQ}{dP} \right) + \mathbb{I}_{(0, t_n)} \left( \frac{dQ}{dP} \right) \cdot \frac{c - E_P[(U')^{-1}(\lambda_n, \frac{dQ}{dP})]}{F^P_{\overline{Q}}(t_n)} \).

Hence \( \lim_{n \to +\infty} t_n = 0 \), \( E_P[X_n] = c \), \( E_Q[X_n] = E_Q[X] \), \( \lim_{n \to +\infty} X_n(\omega) = X(\omega) \) and due to the Dominated Convergence Theorem \( \lim_{n \to +\infty} E_P [U(X_n)] = E_P [U(X)] \).

\( \square \)
### 3.4. HOW TO PROVE OPTIMALITY

**Continuity of the Solution of \((P_{\text{ES}_{\alpha}})\)**

**Fact 3.85:**
If \((P_{\text{CVaR}^{\alpha}})\) is feasible and \(\alpha \in (0, Q(\Omega))\), \(\overline{H}_Q\) equals \(c, \frac{x_0-c\alpha}{Q(\Omega)-\alpha}\) and is compact and non-empty.

**Proof:** \((P_{\text{CVaR}^{\alpha}})\) is feasible \(\text{Theorem}\ 3.9\) \(x_0 \geq c \cdot Q(\Omega) \implies c \in \overline{H}_Q\).
\(x_0 \geq h \cdot Q(\Omega) - (h-c) \cdot \alpha \iff h \leq \frac{x_0-c\alpha}{Q(\Omega)-\alpha}\).
\(\square\)

**Proposition 3.86:**
Let \(I_1 \subseteq \mathbb{R}, I_2\) be a connected subset of \(\mathbb{R}\), \(f_1: I_1 \to (-\infty, +\infty), f_2: I_1 \to (-\infty, +\infty), A := \{(x,y) | x \in I_1, y \in I_2, f_1(x) \leq y \leq f_2(x)\}\) and \(g: A \to \mathbb{R}\) with

(a) \(f_1\) as well as \(f_2\) are continuous,
(b) \(\forall x \in I_1: \{y \in I_2 | (x,y) \in A\} \to \mathbb{R}, y \mapsto g(x,y)\) is strictly increasing and
(c) \(\forall y \in I_2: \{x \in I_1 | (x,y) \in A\} \to \mathbb{R}, x \mapsto g(x,y)\) is continuous.

Then for fixed \((x,y) \in A:\)
\(\forall y_0 > y: \exists \delta_+: > 0: \forall (x, \tilde{y}) \in A \text{ with } |x-x| < \delta_+ \text{ and } \tilde{y} > y_0: |g(x, \tilde{y}) - g(x, y)| \geq \delta_+\) and
\(\forall y_0 < y: \exists \delta_- > 0: \forall (x, \tilde{y}) \in A \text{ with } |x-x| < \delta_- \text{ and } \tilde{y} < y_0: |g(x, \tilde{y}) - g(x, y)| \geq \delta_-\).

**Lemma 3.87:**
Let \(\alpha \in (0, Q(\Omega))\) and \(c \in [u_0, +\infty)\) be fixed. The mappings

(a) \((u_0, +\infty) \cap [c, +\infty) \to [0, +\infty), h \mapsto \lambda_{Q,h}^c,\)
(b) \((u_0, +\infty) \cap [c, +\infty) \to [0, +\infty), h \mapsto \Psi_{Q,h}^{u_0}\) and
(c) \((u_0, +\infty) \cap [c, +\infty) \to (0, +\infty), h \mapsto \Lambda_{Q,h}^{u_0}\)
are continuous.

**Proof:**
(a) We define the sets \(I_1 := (u_0, +\infty) \cap [c, +\infty), I_2 := [0, +\infty),\) the mappings \(f_1: I_1 \to [0, +\infty), x \mapsto \frac{f'(x)}{f''(x)}, f_2 := +\infty,\) the set \(A := \{(x, y) | x \in I_1, y \in I_2, f_1(x) \leq y \leq f_2(x)\}\) and the function \(g: I_1 \times [0, +\infty], (x, y) \mapsto E_Q \left( \left(h - \left(U^\prime \right)^{-1} \left(\lambda_3 \cdot \frac{dQ}{dP}\right) + \right) \right) \cdot (h - c) \cdot \alpha\).
Assumption 3.14 implies that Assumption (a) in Proposition 3.86 is fulfilled, Fact 3.72(i) ensures (b) and Fact 3.72(h) ensures (c).

(P) \(\Rightarrow \forall \varepsilon > 0: \exists \delta > 0: \forall \alpha h \in (u_0, +\infty) \cap [c, +\infty) \text{ with } |h-h| < \delta \text{ and } \forall \lambda_3 \in \left[\frac{U'(h)}{\max} + \infty\right] \text{ with } |\lambda_3 - \lambda_{Q,h}^c| > \varepsilon: g \left(h, \lambda_3\right) \neq 0.\)
\(\forall h \in (u_0, +\infty) \cap [c, +\infty): \lambda_{Q,h}^c < +\infty \text{ (Fact 3.73(a)) and } g \left(h, \lambda_{Q,h}^c\right) = 0.\)
\(\Rightarrow \forall \varepsilon > 0: \exists \delta > 0: \forall \alpha h \in (u_0, +\infty) \cap [c, +\infty) \text{ with } |h-h| < \delta: |\lambda_{Q,h}^c - \lambda_{Q,h}^c| \leq \varepsilon.\)
(b) \(\lambda_{Q,h}^{u_0}\) is independent of \(h\) and the minimum of a continuous function and a constant function is a continuous function.
If Assumption 3.25 is fulfilled, if we use the definitions \(I_1 := (u_0, +\infty) \cap [c, +\infty)\), \(I_2 := (0, +\infty)\), \(f_1: I_1 \to [0, +\infty), x \mapsto \Psi^{x,0}_Q\) (cf. Fact 3.73(a)), \(f_2: I_1 \to (0, +\infty], x \mapsto \max\left\{U(x), \Psi^{x,0}_Q\right\}\), \(A := \{(x, y) | x \in I_1, y \in I_2, f_1(x) \leq y \leq f_2(x)\}\) and \(g: A \to \mathbb{R}\), \((x, y) \mapsto E_Q\left[X_Q^{x,y,\Psi^{x,0}_Q}\right]\).

\[\square\]

Lemma 3.88:
For all \(\omega \in \Omega\): \(\lim_{h \to u_0^+} X_Q^{h,\Lambda^{h,\omega}_Q,\Psi^{h,\omega}_Q}(\omega) = X_Q^{u_0,\Lambda^{\omega,-\omega}_Q,\Psi^{\omega,-\omega}_Q}(\omega)\).

\textbf{Proof}: First of all, \(\lim_{h \to u_0^+} E_Q\left[\left(U'\right)^{-1}\left(\Lambda^{h,\omega}_Q \cdot \frac{dQ}{dP}\right)\right] = x_0\), due to

\[x_0 = E_Q\left[X_Q^{h,\Lambda^{h,\omega}_Q,\Psi^{h,\omega}_Q}\right] \leq E_Q\left[\left(U'\right)^{-1}\left(\Lambda^{h,\omega}_Q \cdot \frac{dQ}{dP}\right)\right] + (h - u_0) \cdot Q(\Omega)\]

\[\leq x_0 + (h - u_0) Q(\Omega).
\]

Fact 3.72(j) helps us to deduce that \(\lim_{h \to u_0^+} \Lambda^{h,\omega}_Q = \lambda^{\omega}_0\). Hence, we know that for all \(\omega \in \Omega\):

\[\lim_{h \to u_0^+} X_Q^{h,\Lambda^{h,\omega}_Q,\Psi^{h,\omega}_Q}(\omega) = \lim_{h \to u_0^+} \left[ h + \left(U'\right)^{-1}\left(\Lambda^{h,\omega}_Q \cdot \frac{dQ}{dP}(\omega)\right) - h \right]^+ + \left(h - \left(U'\right)^{-1}\left(\Psi^{h,\omega}_Q \cdot \frac{dQ}{dP}(\omega)\right)\right)^+ = (U')^{-1}\left(\lambda^{\omega}_0 \cdot \frac{dQ}{dP}(\omega)\right) = X_Q^{u_0,\Lambda^{\omega,-\omega}_Q,\Psi^{\omega,-\omega}_Q}(\omega).
\]

\[\square\]

Proposition 3.89:
If Assumption 3.25 is fulfilled, the mapping \(\{h \in \overline{\mathbb{H}}_Q | h > u_0\} \to \mathbb{R}, h \mapsto E_P\left[U\left(X_Q^{h,\Lambda^{h,\omega}_Q,\Psi^{h,\omega}_Q}\right)\right]\) is continuous.

\textbf{Proof}: Let \(\lim_{n \to +\infty} h_n = h_0\) with \(\forall n \in \mathbb{N}_0:\ h_n \in \{h \in \overline{\mathbb{H}}_Q | h > u_0\}\).

If \(\Psi^{h_n,\omega}_Q = 0\), \(E_P\left[\left(-U\left(X_Q^{h_n,\Lambda^{h_n,\omega}_Q,\Psi^{h_n,\omega}_Q}\right)\right)^+\right] \leq E_P\left[\left(-U\left(h_n\right)\right)^+\right] < +\infty.
\]

(Fact 3.72(b)) \(\Rightarrow E_P\left[U\left(X_Q^{h_n,\Lambda^{h_n,\omega}_Q,\Psi^{h_n,\omega}_Q}\right)\right] \in \mathbb{R}.
\]
\[ h_{\min} := \min_{n \in \mathbb{N}_0} \{ h_n \}, \quad h_{\max} := \max_{n \in \mathbb{N}_0} \{ h_n \}, \quad \lambda_1^{\min} := \min_{n \in \mathbb{N}_0} \{ \Lambda_{Q, h_n, x_0} \}, \quad \lambda_1^{\max} := \max_{n \in \mathbb{N}_0} \{ \Lambda_{Q, h_n, x_0} \}, \quad \lambda_3^{\min} := \min_{n \in \mathbb{N}_0} \{ \Psi_{Q, h_n, x_0} \} \text{ and } \lambda_3^{\max} := \max_{n \in \mathbb{N}_0} \{ \Psi_{Q, h_n, x_0} \} \]

are elements of \( \mathbb{R} \).

Hence \( X_{h_{\min}, \lambda_1^{\max}, \lambda_3^{\max}} \leq X_{h, \Lambda_{Q, h_n, x_0}, \Psi_{Q, h_n, x_0}} \leq X_{h_{\max}, \lambda_1^{\min}, \lambda_3^{\min}} \), Lemma 3.87 and Fact 3.72(d) allow the use of the Dominated Convergence Theorem to conclude the continuity.

\[ \square \]

**Proposition 3.90:**

If \( \alpha \in (0, Q(\Omega)) \), Assumption 3.25 is fulfilled, \( c = u_0 \) and \( \lim_{n \to +\infty} h_n = h_0 = u_0 \) with \( \forall n \in \mathbb{N}_0 \):

\[ h_n \in \overline{\mathbb{H}_Q}, \quad \text{then } \lim_{n \to +\infty} E_P \left[ U \left( X_{h_n, \Lambda_{Q, h_n, x_0}, \Psi_{h_n, x_0}} \right) \right] = E_P \left[ U \left( X_{u_0, \Lambda_{Q, u_0, x_0}, \Psi_{u_0, x_0}} \right) \right]. \]

**Proof:** If \( x_0 = u_0 \cdot Q(\Omega), \overline{\mathbb{H}_Q} = \{ u_0 \} \) (since \( \alpha < Q(\Omega) \)) and the statement is trivially true.

Hence, we can safely assume that \( x_0 > u_0 \cdot Q(\Omega) \).

Without any loss of generality we can add the restriction \( h_n < u_0 + \frac{x_0 - u_0 \cdot Q(\Omega)}{Q(\Omega) - \alpha} \) for all \( n \in \mathbb{N}_0 \).

\( \Rightarrow \forall n \in \mathbb{N}_0 \): \( h_n \cdot Q(\Omega) - (h_n - c) \cdot \alpha < x_0 \).

- If \( E_Q \left[ \left( h_n - X_{h_n, \Lambda_{Q, h_n, x_0}, \Psi_{h_n, x_0}} \right)^+ \right] = (h_n - c) \cdot \alpha \):
  \[ \Rightarrow \Psi_{h_n, x_0} = \lambda_{Q, h_n}. \]
  \[ \Rightarrow E_Q \left[ X_{h_n, \Lambda_{Q, h_n, x_0}, \Psi_{h_n, x_0}} \right] = h_n \cdot Q(\Omega) - (h_n - c) \cdot \alpha < x_0 \text{ (cf. proof of Fact 3.73(c))} \]

(Fact 3.74) \( \Rightarrow \Lambda_{h_n, x_0} < +\infty. \)

Since \( u_0 \cdot Q(\Omega) < x_0 = E_Q \left[ X_{h_n, \Lambda_{Q, h_n, x_0}, \Psi_{h_n, x_0}} \right] \leq E_Q \left[ (U')^{-1} \left( \Lambda_{Q, h_n, x_0} \cdot \frac{dQ}{dP} \right) \right] + (h_n - u_0) \cdot Q(\Omega), \]

there exist \( \lambda_1 \in (0, +\infty) \) and \( n_0 \in \mathbb{N} \) with \( \forall n \geq n_0 : \Lambda_{Q, h_n, x_0} \leq \lambda_1. \)

Let \( h_{\max} := \max_{n \in \mathbb{N}_0} \{ h_n \}. \) Due to

\[ \left( U \left( X_{h_{\max}, \Lambda_{Q, h_{\max}, x_0}, \Psi_{h_{\max}, x_0}} \right) \right)^+ \leq \left( U \left( h_{\max} \right) \right)^+ + \left( U \left( (U')^{-1} \left( \Lambda_{Q, h_{\max}, x_0} \cdot \frac{dQ}{dP} \right) \right) \right)^+ \leq \left( U \left( h_{\max} \right) \right)^+ + \left( U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \]

and

\[ \left( -U \left( X_{Q, \Lambda_{Q, h_{\max}, x_0}, \Psi_{h_{\max}, x_0}} \right) \right)^+ \leq \left( -U \left( (U')^{-1} \left( \Lambda_{Q, h_{\max}, x_0} \cdot \frac{dQ}{dP} \right) \right) \right)^+ \leq \left( -U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \]

the result is a consequence of Lemma 3.88, the continuity of \( U \), the finiteness of \( E_P \left[ (U(h_{\max}) \right]^+ \right], \]

\( E_P \left[ \left( U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \right] \) and the Dominated Convergence Theorem.

\[ \square \]

**Properties of the Maximal Expected Utility**

**Proof of Proposition 3.26:** We split the proof into several parts:
3.4. HOW TO PROVE OPTIMALITY

- The upper semi-continuity is due to Proposition 3.84, Proposition 3.89, Proposition 3.90 and the fact that \( \mathbb{E}_P[U(X(h))] = -\infty \) outside of \( \begin{cases} \overline{H}_P & \text{if } \nu = P \\ \overline{H}_Q & \text{if } \nu = Q \end{cases} \) (cf. Proposition 3.42 and the proof of Theorem 3.24).

- The continuity is implied by Lemma 3.81, Proposition 3.89 and Proposition 3.90.

- Assumption 3.25 ensures that \( \forall h \in H: \mathbb{E}_P[U(X(h))] < +\infty \):
  - If \( \nu = P \), we apply Lemma 3.51 for \( h \in \overline{H}_P \), use Theorem 3.23 if \( U(u_0) < -\infty \) and know that \( \begin{cases} P(X \leq u_0) > 0 & \text{if } U(u_0) = -\infty \\ P(X < u_0) > 0 & \text{if } U(u_0) > -\infty \end{cases} \) for all admissible \( X \) of \( (P_{ES_0}) \) in the last case.
  - If \( \nu = Q \), Fact 3.72(b) delivers the desired statement.

Let \( \lambda \in (0, 1) \) and \( h_1 \) as well as \( h_2 \in H \).

Proposition 3.4 states that \( \lambda \cdot X(h_1) + (1 - \lambda) \cdot X(h_2) \) is admissible for problem \( (P_{ES_{h_1, h_2}}) \).

Furthermore,

\[
\mathbb{E}_P[U(X(h_1) + (1 - \lambda) \cdot X(h_2))] \overset{(*)}{\geq} \lambda \cdot \mathbb{E}_P[U(X(h_1))] + (1 - \lambda) \cdot \mathbb{E}_P[U(X(h_2))],
\]

because \( U \) is concave. Hence \( H \rightarrow \overline{\mathbb{R}}, h \mapsto \mathbb{E}_P[U(X(h))] \) is concave.

- Suppose that \( h_1 \) and \( h_2 \) \( \in \{ h \in H | \mathbb{E}_P[U(X(h))] > -\infty \} \), \( h_1 < h_2 \) and that the classical solution \( (U')^{-1}\left(\lambda x_0 \cdot \frac{dQ}{dP}\right) \) is not admissible for \( (P_{ES_0}) \) \( \forall h \in H \).

Then \( P(X(h_2) = h_2) > 0 \) and, recall Assumption 3.11, \( P(X(h_1) = h_2) = 0 \), due to the fact that \( u_0 < h_2 \) and the uniqueness of the solutions of \( (P_{ES_0}) \) and \( (P_{ES_{h_1}}) \).

Hence \( P(X(h_1) \neq X(h_2)) \geq P(X(h_2) = h_2) - P(X(h_1) = h_2) > 0 \).

Thus \( U \) being strictly concave on \([u_0, +\infty)\) implies that \( (*) \) is a strict inequality.

\( \square \)

Optimal Solutions of \((P_{CVaR^Q})\)

Proof of Theorem 3.29: We have to investigate two cases:

- If \( \alpha = Q(\Omega) \), \( (U')^{-1}\left(\lambda x_0 \cdot \frac{dQ}{dP}\right) \) is an optimal solution of \((P_0)\). The definition of \( \lambda x_0 \) on page 32 yields that \( \mathbb{E}_Q[\left((U')^{-1}\left(\lambda x_0 \cdot \frac{dQ}{dP}\right)\right)] = x_0 \overset{\text{Theorem 3.9}}{\geq} c \cdot Q(\Omega) \).

Hence the set of \( X_k \) is non-empty. In addition, \( CVaR_Q^Q(X_k) = -\frac{\mathbb{E}_Q[X_k]}{Q(\Omega)} \leq -c \) implies that any \( X_k \) is admissible for \( (P_{CVaR^Q}) \) and consequently optimal.

- If \( \alpha < Q(\Omega) \), \( (\text{Fact 3.85}) \Rightarrow \overline{H}_Q \) is compact and non-empty.

(Proposition 3.89, Proposition 3.90) \( \Rightarrow \overline{H}_Q \rightarrow \mathbb{R}, h \mapsto \mathbb{E}_P[U(X_Q^{h, x_0, x_0}, h, Q)] \) is a continuous mapping defined on a non-empty and compact set attaining its supremum.

\( \square \)
Part 4

Dynamic Optimization Problems with Risk Constraints

The main topic of this part is to look at risk constraint optimization problems in a dynamic setting. Instead of just considering terminal wealths described by random variables, we now deal with investment strategies, which are stochastic processes, that lead to such terminal wealths. We choose a framework that allows us to make use of the general results for the static optimization problems of the previous part. Our primary solution technique is the martingale method, hence, these results already contain the toughest part of the derivation of the optimal dynamic investment strategies: the calculation of the optimal terminal wealth. This wealth is then treated as a hedgeable position or claim. From that point of view, the optimal investment strategy corresponds to the hedging strategy of that particular position. Thus, we determine these hedging strategies. Based on their explicit formulas, we subsequently discuss a variety of properties of the optimal solutions. Next, to gain some additional insight about the implications of a risk constraint, we investigate dynamic local restrictions for the investment strategies that ensure the fulfillment of the global constraint. These local restrictions allow the use of other solution techniques. Finally, we conclude this part by an outlook that contains ideas for future research. The proofs of most statements are collected into Chapter 6.3.

4.1 Problem Statement

4.1.1 Market Model

We adopt the idealized framework of a standard complete financial market model in continuous time over a finite time horizon $[0, T]$. The market is assumed to be frictionless, which means that there are no transaction costs. In addition, we neglect any possible effects of taxation. Within the market, besides one riskless asset, frequently called bank account or money market fund, there exist $N$ risky assets, called stocks. There are no trading limits in the market, short selling is permitted and all shares are divisible.
The Probability Space

The uncertainty inherent in the model is caused by a \( d \)-dimensional standard Brownian motion \( W_P(\cdot) := W(\cdot) = (W^1(\cdot), \ldots, W^d(\cdot))^\top \) defined on a complete probability space \((\Omega, \mathcal{F}, P)\).

At time \( t \), only the development of the Brownian motion in the interval \([0, t]\) is known. A more natural idea would be to use the development of the bank account and the stocks over \([0, t]\) in place of the Brownian motion. It is easy to verify that in our model both approaches yield the same amount of information, i.e. the same filtration.

Instead of directly working with the filtration \( \mathcal{F}^W(t) := \sigma(W(s); 0 \leq s \leq t), 0 \leq t \leq T \) generated by \( W(\cdot) \), we use the augmented filtration \( \mathcal{F} := \{ \mathcal{F}(t) \}_{0 \leq t \leq T} \) where \( \forall t \in [0, T]: \mathcal{F}(t) := \sigma(\mathcal{F}^W(t) \cup \mathcal{N}) \) and \( \mathcal{N} \) is the set of all \( P \)-null subsets of \( \mathcal{F}^W(T) \). The important technical feature of the augmented filtration is its right-continuity.

The Money Market

The price of a share in the money market is governed by the differential equation

\[
\begin{align*}
\begin{cases}
\frac{dS_0(t)}{S_0(t)} &= r(t)dt \\
S_0(0) &= 1
\end{cases}
\end{align*}
\]

where \( r \), the risk-free rate process, is assumed to be a bounded, measurable and deterministic function of time. Therefore the Lebesgue Integral \( \int_0^T r(t)dt \) is well-defined and finite and the differential equation (4.1) has the following explicit solution:

\[
S_0(t) = \exp \left[ \int_0^t r(s)ds \right], \quad 0 \leq t \leq T.
\]

The Stocks

The stocks are risky, which means their value depends on the development of the Brownian motion \( W \). Furthermore, they do not pay dividends and their prices are modeled using the following dynamics:

\[
\begin{align*}
\begin{cases}
\frac{dS_i(t)}{S_i(t)} &= \mu_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW^j(t) \\
S_i(0) &= s_i \quad \text{with} \quad s_i > 0
\end{cases}, \quad i = 1, \ldots, N.
\end{align*}
\]

The volatility matrix \( \sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \leq i, j \leq d} \) is assumed to be quadratic and invertible, implying \( d = N \). Moreover, like the risk-free rate process, the vector of stock return rates \( \mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_N(\cdot))^\top \), the volatility matrix \( \sigma(\cdot) \) as well as its inverse \( \sigma(\cdot)^{-1} \) are all assumed to be deterministic, measurable and bounded.

We abbreviate the \( i \)-th row of the matrix-valued function \( \sigma \) by \( \sigma_i(\cdot) := (\sigma_{i1}(\cdot), \ldots, \sigma_{id}(\cdot))^\top \).

The uniform boundedness of \( \sigma(\cdot)^{-1} \) in connection with the condition \( d = N \) ensures that we have a complete market model (cf. [Karatzas and Shreve, 1998, Theorem 1.6.6]).

The solution of equation (4.2) for all \( 0 \leq t \leq T \) and \( i = 1, \ldots, N \) is

\[
S_i(t) = S_i(0) \exp \left[ \int_0^t \left( \mu_i(s) - \frac{1}{2} \sum_{j=1}^d \sigma_{ij}^2(s) \right)ds + \int_0^t \sum_{j=1}^d \sigma_{ij}(s)dW^j(s) \right].
\]

This can be verified using Itô’s formula.
**Definition (1):** We set \( \underline{1} := (1, \ldots, 1)^\top \in \mathbb{R}^N \).

**Definition (θ):** The (deterministic) risk-premium process \( \theta \) is the mapping \( [0, T] \to \mathbb{R}^d, \ t \mapsto \sigma(t)^{-1} [\mu(t) - r(t)\underline{1}] \).

With \( \sigma(\cdot)^{-1}, \mu(\cdot), r(\cdot) \) being all bounded, \( \theta(\cdot) \) is bounded, too.

**Definition (WQ):** Let \( W_Q \) be the continuous process defined by \( W_Q(t) := W(t) + \int_0^t \theta(s) ds \).

**Definition (∥·∥):** For real vectors \( v \), \( \|v\|_2 := \sqrt{v^\top v} \). So \( \|v\|_2^2 = v^\top v \).

### Portfolio Processes

An investor in our market model is allowed to invest into all \( N \) available stocks and into the money market account. He is allowed to hold short positions and can change his position at any time without incurring any costs.

Let \( \Pi_i(t) \) be the total amount of money invested in the \( i \)-th stock at time \( t \) and \( \Pi_0(\cdot) \) be the process describing the amount invested in the risk-free security \( S_0 \). With \( \Pi(t) := (\Pi_1(t), \ldots, \Pi_N(t))^\top \) for \( 0 \leq t \leq T \), \( \Pi(\cdot) \) denotes an \( N \)-dimensional vector control process.

We assume that \( \Pi_0(\cdot) \) and \( \Pi(\cdot) \) are progressive measurable processes with respect to \( \mathcal{F} \) and

\[
\int_0^T \|\Pi(t)\|_2^2 \, dt < +\infty \quad (P\text{-almost-surely}).
\]

The pair \( (\Pi_0, \Pi) \) is called portfolio process or portfolio strategy. At any time \( t \in [0, T] \), the random variable describing the financial value \( X^{(\Pi_0, \Pi)}(t) \) of the portfolio process \( (\Pi_0, \Pi) \) is given by

\[
X^{(\Pi_0, \Pi)}(t) := \sum_{i=0}^N \Pi_i(t) = \Pi_0(t) + \Pi(t)^\top \underline{1}.
\]

A portfolio process is called self-financing, if \( dX^{(\Pi_0, \Pi)}(t) = \sum_{i=0}^N \frac{\Pi_i(t)}{S_i(t)} dS_i(t) \).

We are looking exclusively at self-financing strategies here, hence \( X^{(\Pi_0, \Pi)}(t) \) depends on the initial wealth \( x_0 := X^{(\Pi_0, \Pi)}(0) \) and the investments in the stocks \( \Pi_0 \) only. This allows us to replace the dependence on \( \Pi_0 \) by \( x_0 \) in the definition of the financial value and we set \( X^{x_0, \Pi}(t) := X^{(\Pi_0, \Pi)}(t) \).

We want to prohibit riskless gains, which can be created by so-called doubling strategies, thus we restrict ourselves to tame portfolio strategies:

**Definition (P):** We define the set of tame portfolios \( \mathcal{P} \) as follows:

\[
\mathcal{P} := \{ \Pi \mid \Pi \text{ is a portfolio and } \exists b \in \mathbb{R} \text{ such that } \forall s \in [0, T]: X^{x_0, \Pi}(s) \geq b \}.
\]

**Lemma 4.1:**

The discounted wealth process of a tame strategy is a supermartingale with respect to \( Q \) and for all \( h \in \mathbb{R} \) and \( \nu \in \{P, Q\} : (h - X^{x_0, \Pi}(T))^+ \) is bounded.

In the sequel, we are considering tame strategies only.
Pricing

In this market model, the unique price measure \( Q \) has the Radon-Nikodym derivative

\[
\frac{dQ}{dP} = \exp \left[ - \int_0^T r(t) dt \right] \cdot \exp \left[ - \int_0^T \theta(t)^T dW(t) - \frac{1}{2} \int_0^T \| \theta(t) \|_2^2 dt \right]
\]

\[
= \exp \left[ - \int_0^T r(t) dt \right] \cdot \exp \left[ - \int_0^T \theta(t)^T dW_Q(t) + \frac{1}{2} \int_0^T \| \theta(t) \|_2^2 dt \right].
\]

It is convenient for calculation purposes to be able to value claims before the time horizon \( T \), too:

**Definition \((\frac{dQ}{dT}(t, \omega))\):** The current (at time 0) price density of one monetary unit at time \( t \in [0, T] \) in state \( \omega \in \Omega \) is \( \frac{dQ}{dP}(t, \omega) := \exp \left[ - \int_0^t r(s) ds - \int_0^t \theta(s)^T dW_Q(s)(\omega) + \frac{1}{2} \int_0^t \| \theta(s) \|_2^2 ds \right] \).

Therefore, \( \frac{dQ}{dP} = \frac{dQ}{dT}(T, \cdot) \).

The number \( \mathbb{E}_Q [C] \) is called the price (at time 0) of a contingent claim \( C \) (a random variable) at time \( T \), if the expression is well-defined.

**Lemma 4.2:**
Assumption 3.11 is fulfilled if and only if \( \int_0^T \| \theta(t) \|_2^2 dt \neq 0 \).

### 4.1.2 Option Pricing and Hedging

**Definition \((\Phi, \varphi)\):** Let \( \Phi \) be the cumulative distribution function of the Standard Normal Distribution with respect to \( P \) and \( \varphi \) its probability density function.

The market model allows for explicit pricing and hedging of several derivative securities. We present the prices and hedging strategies of those appearing as parts of the optimal terminal wealth of the upcoming constrained dynamic optimization problems.

**Proposition 4.3:**
Let \( K > 0 \) and \( t \in [0, T] \) be fixed.

We define \( d_1(t, x) := \frac{\ln \left( \frac{S_0(t)}{K} \right) + \int_0^t r(s) ds + \frac{1}{2} \int_0^t \| \sigma(s) \|_2^2 ds}{\sqrt{\int_0^t \| \sigma(s) \|_2^2 ds}} \) and \( d_2(t, x) := \frac{\ln \left( \frac{S_0(t)}{K} \right) + \int_0^t r(s) ds - \frac{1}{2} \int_0^t \| \sigma(s) \|_2^2 ds}{\sqrt{\int_0^t \| \sigma(s) \|_2^2 ds}} \).

The random variable describing the fair price of a European Call Option on the \( i \)-th stock with strike price \( K \) at time \( t \) in this market model is

\[
\frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ \left( S_i(t) - K \right)^+ \right| \mathcal{F}(t) \right] = S_i(t) \cdot \Phi \left( d_1(t, S_i(t)) \right) - \frac{S_0(t)}{S_0(T)} \cdot K \cdot \Phi \left( d_2(t, S_i(t)) \right).
\]

Due to the Put-Call Parity, the price of a European Put Option on the \( i \)-th stock with strike price \( K \) is given by

\[
\frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ \left( K - S_i(T) \right)^+ \right| \mathcal{F}(t) \right] = \frac{S_0(t)}{S_0(T)} \cdot K \cdot \Phi \left( -d_2(t, S_i(t)) \right) - S_i(t) \cdot \Phi \left( -d_1(t, S_i(t)) \right).
\]

Finally, a fairly priced European Binary Put Option on the \( i \)-th stock with payoff \( p \geq 0 \) if \( S_i(T) < K \) sells for

\[
\frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ p \cdot 1_{(\infty, K)}(S_i(T)) \right| \mathcal{F}(t) \right] = \frac{S_0(t)}{S_0(T)} \cdot p \cdot \Phi \left( -d_2(t, S_i(t)) \right).
\]
The corresponding hedging strategies are as follows:

- The European Call Option can be hedged by rebalancing the investment in the $i$-th stock at every time $t \in [0, T)$ to $S_i(t) \cdot \Phi \left( d_1(t, S_i(t)) \right)$ and by financing the missing capital by a short position in the bond.

- The Put-Call Parity delivers the corresponding result for the European Put Option: The hedging strategy is to invest $-S_i(t) \cdot \Phi \left( -d_1(t, S_i(t)) \right)$ in the $i$-th stock — a short position — and to put all the money to work in the bond.

- The European Binary Put Option can be hedged by buying $\frac{S_0(t)}{S_0(T)} \cdot p \cdot \frac{-\varphi(-d_2(t, S_i(t)))}{\sqrt{\int_t^T \| \sigma_i(s) \|^2 ds}}$ of the $i$-th stock (again a short position) and using the proceeds as well as the initial payment as investment in the risk-free asset.

### 4.1.3 The Optimization Problems

Our objects of interest are the dynamic versions of the problems defined in Section 3.1.2. Their form does not change except that we can no longer choose any (admissible) random variable $X$ but must restrict ourselves to terminals wealths $X_{x_0, \Pi}(T)$ generated by some investment strategy $\Pi$. The price constraint can be dropped, because it is automatically fulfilled due to Lemma 4.1. Hence, $(P_0)$ translates into

$$\left\{ \mathbb{E}_P \left[ U \left( X_{x_0, \Pi}(T) \right) \right] \rightarrow \max_{\Pi \in \mathcal{P}} \right\}$$

and the other problems are modified in the same spirit. Due to the nature of a complete market model, the feasibility results of Chapter 3.2 are still valid for these transformed problems.

### 4.2 Optimal Strategies and Their Properties

We will consider utility functions $U$ of the type $U_\gamma$ and initial wealths $x_0 > 0$ only.

We have already stated that we intend to use the results obtained for the static optimization problems. Consequently, we have to verify that the assumptions required for these results are fulfilled.

**Proposition 4.4:**

$U_\gamma$ fulfills the assumptions 3.18, 3.19, 3.25 and $\forall \lambda_1 \in (0, +\infty): \mathbb{E}_Q \left[ \left( (U_\gamma')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right)^2 \right] < +\infty$.

The following assumption ensures the validity of Assumption 3.11 and will be very useful on its own:

**Assumption 4.5:**

$\forall t \in [0, T]: \theta(t) \neq 0$. 

---

74
4.2. OPTIMAL STRATEGIES AND THEIR PROPERTIES

4.2.1 Classical Problem without Risk Constraint

For \( U = U_\gamma \), problem \((P_0)\) is well understood and treated in many text books. Its optimal absolute investment strategy is

\[
\Pi^0(t) := \frac{1}{1 - \gamma} \cdot \left( \sigma(t)^\top \right)^{-1} \cdot \theta(t) \cdot X^{x_0, \Pi^0}(t).
\]

Since the wealth process \( X^{x_0, \Pi^0}(t) \) is strictly positive, this absolute strategy can be characterized as a proportional investment strategy \( \pi^0(t) := \frac{\Pi^0(t)}{X^{x_0, \Pi^0}(t)} = \frac{1}{1 - \gamma} \cdot \left( \sigma(t)^\top \right)^{-1} \cdot \theta(t) \).

4.2.2 Artificial One-Stock Market

We want to show that \( X^{x_0, \Pi^0} \) can be viewed as an artificial stock, because it fulfills a stochastic differential equation of the same type as do the stocks in our market model:

Assumption 4.5 makes sure that \( \forall t \in [0, T]: \pi^0(t) \neq 0 \) and hence \( \| \sigma(t)^\top \pi^0(t) \|_2 \neq 0 \).

Using the definitions \( \tilde{\mu}(t) := \left( 1 - \pi^0(t)^\top \right) \cdot r(t) + \pi^0(t)^\top \mu(t) \),

\[
\tilde{\sigma}(t) := \left\| \sigma(t)^\top \pi^0(t) \right\|_2 \text{ and }
\]

\[
W(t) := \int_0^t \frac{\pi^0(s)^\top \sigma(s)}{\| \sigma(s)^\top \pi^0(s) \|_2} dW(s),
\]

we observe that

\[
dX^{x_0, \Pi^0}(t) = \frac{X^{x_0, \Pi^0}(t) - \Pi^0(t)^\top \frac{1}{S_0(t)} dS_0(t) + \Pi^0(t)^\top \left[ \mu(t) dt + \sigma(t) dW(t) \right]}{S_0(t)} = X^{x_0, \Pi^0}(t) \left[ \tilde{\mu}(t) dt + \tilde{\sigma}(t) d\tilde{W}(t) \right].
\]

Lévy’s Theorem yields that \( \tilde{W}(\cdot) \) is a one-dimensional Wiener process with respect to \( P \), because

\[
\int_0^t \left( \frac{\pi^0(s)^\top \sigma(s)}{\| \sigma(s)^\top \pi^0(s) \|_2} \right) d\tilde{W}(s) = t.
\]

Hence, the stochastic differential equation (4.2) on page 71 is fulfilled if we replace \( S_i \) by \( X^{x_0, \Pi^0} \), \( \mu_i(t) \) by \( \tilde{\mu}(t) \), \( \sigma_{ij}(t) \) by the \( j \)-th entry of \( \tilde{\sigma}(t) \), \( W^j(t) \) by \( \tilde{W}(t) \), and \( s_i \) by \( X^{x_0, \Pi^0}(0) \).

Definition \((Y^\lambda(t))\): For all \( \lambda \in (0, +\infty) \) we set \( Y^\lambda(t) := \frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ \left( U' \right)^{-1} \left( \lambda \cdot \frac{dQ}{dP} \right) \left| \mathcal{F}(t) \right. \right]. \)

\( Y^\lambda(\cdot) \) is the wealth process of an artificial stock, because if \( x_0 = Y^\lambda(0), \forall t \in [0, T] \):

\[
Y^\lambda(t) = \frac{S_0(T)}{S_0(t)} \cdot \lambda^{-\frac{1}{2}} \cdot \frac{1}{\frac{dQ}{dP}(t, \cdot)} \cdot \mathbb{E}_P \left[ \frac{dQ}{dP} \left( \frac{dQ}{dP} \right)^{- \frac{1}{2}} \left| \mathcal{F}(t) \right. \right] + \mathbb{E}_P \left[ \int_0^t \left( -r(s) + \frac{1}{\gamma} \cdot \frac{\theta(s)}{\| \theta(s) \|_2} \right) d\tilde{W}(s) \right] + \int_0^t \frac{1}{\gamma - 1} \cdot \theta(s)^\top dW_Q(s) - \frac{1}{2} \int_0^t \left( \frac{1}{\gamma - 1} \right)^2 \| \theta(s) \|_2^2 ds
\]

\( = X^{x_0, \Pi^0}(t). \)
4.2. OPTIMAL STRATEGIES AND THEIR PROPERTIES

We realize that the above calculation yields a one-to-one correspondence between $Y^\lambda(t)$ and $\frac{dQ}{d\mathcal{F}}(t, \cdot)$. Since the optimal claims of Part 3 depend on the dynamics of the system only through the value of $\frac{dQ}{d\mathcal{F}}$, the corresponding investment strategies can be written in terms of $Y^\lambda(t)$, i.e. they solely invest in this artificial stock and the risk-free bond.

Another immediate consequence of the representation is that $\frac{\partial Y^\lambda(t)}{\partial \lambda} = \frac{1}{\gamma - 1} \cdot Y^\lambda(t)$.

4.2.3 Some Notation

Definition ($d_+, d_-$): As an analogy to the corresponding expressions in the option pricing formulas, we are going to use the definitions

$$
d_+(K, t, x) := \ln \left( \frac{x}{K} \right) + \int_t^T r(s) ds + \frac{1}{2} \int_t^T \|\sigma(s)^\top \pi_0(s)\|_2^2 ds
\sqrt{\int_t^T \|\sigma(s)^\top \pi_0(s)\|_2^2 ds}
\text{ and }
$$

$$
d_-(K, t, x) := \ln \left( \frac{x}{K} \right) + \int_t^T r(s) ds - \frac{1}{2} \int_t^T \|\sigma(s)^\top \pi_0(s)\|_2^2 ds
\sqrt{\int_t^T \|\sigma(s)^\top \pi_0(s)\|_2^2 ds}
$$

from now on.

With all these preparations, we can now jump directly to the treatment of the individual risk constraint problems.

4.2.4 Value at Risk Problem

We have already learned that the form of the optimal solution of the static optimization problem ($F_{\text{VaR}_\nu}$) does not depend on the choice of $\nu$. For consistency, one would expect this property to translate into an equality of the optimal investment strategies. And indeed, there are no surprises here:
Theorem 4.6 (Optimal Investment Strategy of \((P_{\text{Var}})\)):

Let \( U = U_\gamma \) for some fixed \( \gamma \in (-\infty, 1) \). If \( J := \left( \frac{E_{\text{rel}}^{\nu}}{\nu} \right)^{-1} (\nu(\Omega) - \epsilon) \), \( q > K := (U')^{-1} (\Lambda_{x_0}^\nu \cdot J) > 0 \) and \( x_0 > q \cdot \left( \frac{d_0}{d_\gamma} \right) \), then \( \frac{S_0(t)}{S_0(T)} \cdot E_Q \left[ X_{\text{Var}}^{\Lambda_{x_0}^\nu} \left| F(t) \right. \right] \) equals

\[
\frac{S_0(t)}{S_0(T)} \cdot q + \left( Y^{\Lambda_{x_0}^\nu}(t) \cdot \Phi \left( d_+ \left( q, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right) - \frac{S_0(t)}{S_0(T)} \cdot q \cdot \Phi \left( d_- \left( q, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right) \right) \]

\[
- \left( \frac{S_0(t)}{S_0(T)} \cdot K \cdot \Phi \left( d_- \left( K, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right) - Y^{\Lambda_{x_0}^\nu}(t) \cdot \Phi \left( d_+ \left( K, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right) \right)
\]

\[
- \frac{S_0(t)}{S_0(T)} \cdot (q - K) \cdot \Phi \left( d_- \left( K, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right)
\]

\[
\Pi_{\text{Var}}(t) := \pi^0(t) \cdot \left( Y^{\Lambda_{x_0}^\nu}(t) \cdot \Phi \left( d_+ \left( q, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right) + Y^{\Lambda_{x_0}^\nu}(t) \cdot \Phi \left( d_+ \left( K, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right) \right) + \frac{S_0(t)}{S_0(T)} \cdot (q - K) \cdot \frac{\varphi \left( d_- \left( K, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right)}{\sqrt{T_t \| \sigma^s \| \pi^0(s) \| 2 ds}}
\]

is an optimal investment strategy for problem \((P_{\text{Var}})\).

**Proof:** The representation of the optimal claim as a sum of derivative instruments given in Section 3.3.2 can be used to derive the optimal wealth process and optimal investment strategy by summing up the optimal wealth processes and optimal investment strategies of the derivatives.

\[
\Phi
\]

**Definition \((\pi_{\text{Var}}, p_{\text{abs}}, p_{\text{rel}})\):** Obviously, \( \Pi_{\text{Var}}(t) \) can be written as \( \pi^0(t) \cdot p_{\text{Var}}^{\text{abs}}(t) \) for some \( p_{\text{Var}}^{\text{abs}}(t) \geq 0 \) and \( \pi_{\text{Var}}(t) := \frac{S_0(t)}{S_0(T)} \cdot E_Q \left[ X_{\text{Var}}^{\Lambda_{x_0}^\nu} \left| F(t) \right. \right] = \pi^0(t) \cdot p_{\text{Var}}^{\text{rel}}(t) \) with \( p_{\text{Var}}^{\text{rel}}(t) \geq 0 \).

**Remarks 4.7:**

Some thoughts on Theorem 4.6:

- \( \Lambda_{x_0}^\nu \) is determined by the initial condition \( x_0 = \frac{S_0(0)}{S_0(T)} \cdot E_Q \left[ X_{\text{Var}}^{\Lambda_{x_0}^\nu} \left| F(0) \right. \right] \) for \( \Lambda_{x_0}^\nu \). It can be solved easily by exploiting Lemma 3.38: The function on the right-hand side of the initial condition is monotonic decreasing and continuous in \( \Lambda_{x_0}^\nu \).

- Let us consider a fixed point in time \( t \in (0, T) \). Basic limit arguments show that for any sequence \((\omega_n)\) in \( \Omega \) with limit of the underlying \( \lim_{n \to \infty} Y^{\Lambda_{x_0}^\nu}(t) (\omega_n) \) belongs to \( \{0, +\infty\} \), the quotient of current wealth and underlying converges to 1:

\[
\lim_{n \to +\infty} \frac{S_0(t)}{S_0(T)} \cdot E_Q \left[ X_{\text{Var}}^{\Lambda_{x_0}^\nu} \left| F(t) \right. \right] (\omega_n) = 1.
\]

Since

\[
\begin{align*}
\frac{\partial}{\partial Y^{\Lambda_{x_0}^\nu}(t)} \cdot E_Q \left[ X_{\text{Var}}^{\Lambda_{x_0}^\nu} \left| F(t) \right. \right] &= \Phi \left( d_+ \left( q, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right) + \Phi \left( d_- \left( K, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right) \\
&+ \frac{S_0(t)}{S_0(T)} \cdot (q - K) \cdot \frac{\varphi \left( d_- \left( K, t, Y^{\Lambda_{x_0}^\nu}(t) \right) \right)}{\sqrt{T_t \| \sigma^s \| \pi^0(s) \| 2 ds}}
\end{align*}
\]
is strictly positive, there is a one-to-one correspondence between the current wealth and $Y^{A_{x_0}}(t)$. Thus the optimal investment strategy only depends on the current time, wealth, $q$ and $\epsilon$. Furthermore, we recall that there is a one-to-one correspondence between $Y^{A_{x_0}}(t)$ and $\frac{dQ}{dP}(t, \cdot)$, too.

- We look a little closer at the strategy at a fixed time $t \in (0, T)$:

Let $(\omega_n)$ be a sequence in $\Omega$. As $\frac{Y^{A_{x_0}}(t)(\omega_n)}{Y^{A_{x_0}}(0)(\omega_n)}$ converges to 1 if $Y^{A_{x_0}}(t)(\omega_n)$ goes to either $0+$ or $+\infty$, the relative investment in the underlying $p_{\text{VaR}}(t)(\omega_n)$ goes to 1 if the current wealth at time $t$, $X^{x_0, \Pi_{\text{VaR}}}(t)(\omega_n) = \frac{S_0(t)}{S_0(0)} \cdot \mathbb{E}_Q \left[ X^{A_{x_0}}_{\text{VaR}} \left| \mathcal{F}(t) \right. \right](\omega_n)$, approaches either $0+$ or $+\infty$.

For the following illustrations, we consider a market with coefficients $r := 0.04$, $N := 1$, $\sigma_{11} := 0.4$, $\mu_{1} := 0.1$, a horizon of $T := 2$ and an investor with parameters $\gamma := 0$, $q := 1$ and $J := 1.1$.

The dependence of $p_{\text{VaR}}$ on the current wealth $X^{x_0, \Pi_{\text{VaR}}}(0+) \ (\omega)$ at time $0+$ (that is immediately after $0$) is shown in Figure 4.1 for the initial endowment $x_0 = 0.9$. The constituents of the strategy are the hedging strategies of the individual options that form the entire optimal terminal wealth. We see that the influence of each part on the total strategy varies greatly. The figure is typical for all points in time between 0 and $T$. In the discussion of Figure 3.1, it already became obvious that it is optimal to take more risks in the most adverse states. Hence, Figure 4.1 is consistent in this respect.

The impact of $J$ is discussed in Figure 4.2. Note that for fixed $\frac{dQ}{dP}(\omega)$, $X^{x_0, \Pi_{\text{VaR}}}(0+) \ (\omega)$ depends on $J$, that is it changes with $J$.

Initially however, we observe a slightly different relationship (Figure 4.3). Recall that the optimal strategy is unique just for initial endowments above a certain minimal initial endowment threshold $x_0^{\min}$ (in the example $\approx 0.70558$) that is required such that both the risk constraint can be satisfied and the expected terminal utility is not $-\infty$ (see Theorem 3.20 for details). If on the other hand the initial endowment is above an upper threshold, the optimal terminal claim is the solution of the unrestricted problem, too. Thus the optimal wealth process equals the underlying and we do not split the strategy into the option parts in this case.

![Figure 4.1: Optimal investment strategy in the value at risk problem (I)](image-url)
Suppose that the continuous trading model is used as an approximation of a market allowing a large yet finite maximal number of trades. In this scenario, implementing the investment strategy $\Pi^{VaR}$ will almost never yield exactly the optimal terminal wealth $X_{x_0,J}^{VaR}$. As a consequence, the measure of the set on which the terminal wealth is below the limit $q$ is not $\epsilon$. One might expect approximately $\epsilon + \frac{1}{2} \cdot \nu \left( X_{x_0,J}^{VaR} = q \right)$ instead of $\epsilon$.

Figure 4.4 sheds some light on this problem. It compares the probability distribution function of the terminal wealth if the portfolio adjustments according to strategy $\Pi^{VaR}(t)$ are done continuously or at fixed points in time $\frac{i}{n}T$ with $i \in \{0, \ldots, n-1\}$ and $n \in \{10, 100, 1000\}$ for the same parameters as before with the following exceptions: $\epsilon := 0.05$, $J := F_{\delta_1, \delta_2}^{-1}(1 - \delta) \approx 1.27944$.
4.2. OPTIMAL STRATEGIES AND THEIR PROPERTIES

![Graph showing dependence of $F_{X_{x_0}}^{P}(T)$ on the number of trades](image)

Figure 4.4: Dependence of $F_{X_{x_0}}^{P}(T)$ on the number of trades

and $x_0 := \frac{S_0(0)}{S_0(T)} \cdot E_Q \left[ X_{1}^{1, F}(0) \right] \approx 1.02834$. As you can see, the probability of having a terminal wealth below $q$ is contingent on the number of trades and is well above $\epsilon$.

A possibility to compensate for this effect would be to calculate the optimal strategy for a slightly higher $q$ than targeted.

- Theorem 4.6 is an extension of Proposition 3 in [Basak and Shapiro, 2001]. That proposition deals with problem $(P_{VaR})$ only and assumes constant market coefficients as well as the existence of the Lagrange multiplier. The following table is useful for comparing the results:

<table>
<thead>
<tr>
<th>trading</th>
<th>10 times</th>
<th>100 times</th>
<th>1000 times</th>
<th>continuously</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P \left( X_{x_0, VaR}^{P} (T) &lt; q \right)$, approx.</td>
<td>0.215</td>
<td>0.196</td>
<td>0.188</td>
<td>0.050</td>
</tr>
<tr>
<td>$Q \left( X_{x_0, VaR}^{P} (T) &lt; q \right)$, approx.</td>
<td>0.258</td>
<td>0.234</td>
<td>0.224</td>
<td>0.070</td>
</tr>
</tbody>
</table>

A possibility to compensate for this effect would be to calculate the optimal strategy for a slightly higher $q$ than targeted.

4.2.5 Expected Shortfall Problem

We have seen that the optimal solution of the static optimization problem $(P_{ES})$ differs greatly from the solution of its sibling $(P_{ES_{q}})$. This observation is still valid for the solutions of the dynamic versions of these problems. We treat the problem $(P_{ES_{h}})$ first.

**Definition ($\phi(a, b, c, d, \xi)$):** For $a > 0$, $b > 0$, $c \geq 0$, $d \leq \frac{\ln(a) - \ln(c)}{b}$ and $\xi \in \mathbb{R}$, we define $\phi(a, b, c, d, \xi) := \int_{-\infty}^{d} \left( \varphi(z) \cdot (a \cdot \exp[-b \cdot z] - c)^{\xi} \right) dz$. 

80
Theorem 4.8 (Optimal Investment Strategy of \((P_{\mathbb{E}})\)):

Let \(h > 0\), \(0 < \Psi < +\infty\) and \(\Lambda^{h,\Psi} < +\infty\). Then the optimal wealth \(\frac{S_0(t)}{S_0(T)} \cdot h + \left( Y^{h,\Psi} \cdot \Phi \left( d_+ \left( h, t, Y^{h,\Psi} \right) \right) - \frac{S_0(t)}{S_0(T)} \cdot h \cdot \Phi \left( \frac{1}{\gamma - 1} \right) \right) \) at time \(t \in [0, T]\) is given by

\[
\frac{S_0(t)}{S_0(T)} \cdot h + \left( Y^{h,\Psi} \cdot \Phi \left( d_+ \left( h, t, Y^{h,\Psi} \right) \right) - \frac{S_0(t)}{S_0(T)} \cdot h \cdot \Phi \left( \frac{1}{\gamma - 1} \right) \right)
\]

where \(a := \left( Y^{h,\Psi} \cdot \frac{S_0(T)}{S_0(0)} \right)^{\gamma - 1} \cdot \exp \left[ \frac{1}{\gamma - 1} \cdot \frac{1}{T} \int_t^T \|\theta(s)\|^2 ds \right], \quad b := \sqrt{\int_t^T \|\theta(s)\|^2 ds} \) and

\[
d := -d_+ \left( (h^{\gamma - 1} + \Psi^{\gamma - 1}), t, Y^{h,\Psi} \right).
\]

The corresponding investment strategy is

\[
\Pi^P(t) := \pi^0(t) \cdot \left[ Y^{h,\Psi} \cdot \Phi \left( d_+ \left( h, t, Y^{h,\Psi} \right) \right) \right] + \frac{S_0(t)}{S_0(T)} \cdot a \cdot \exp \left[ \frac{1}{2} b^2 \right] \cdot \Phi \left( a \cdot \exp \left[ b^2 \right], b, \Psi, d + b, \frac{1}{\gamma - 1} - 1 \right).
\]

Definition (\(\pi^P, p_{\text{abs}}, p_{\text{rel}}\)): The absolute investment strategy \(\Pi^P(t)\) can be split into \(\pi^0(t) \cdot p_{\text{abs}}^P(t)\) for some \(p_{\text{abs}}^P(t) \geq 0\), because \(\Phi \geq 0\). This property is passed on to the proportional investment strategy \(\pi^P(t) := \frac{S_0(t)}{S_0(T(t))} \cdot \pi_t \cdot p_{\text{rel}}^P(t)\) with \(p_{\text{rel}}^P(t) \geq 0\).

Remarks 4.9:

- Let us look at some of the implications of Theorem 4.8:

  - Once more we fix a point in time \(t \in (0, T)\) and a sequence \((\omega_n)\) in \(\Omega\).
    \[
    \Phi \left( a, b, \Psi, d, \frac{1}{\gamma - 1} \right)
    \]
    is non-negative and bounded above by \(\Phi(d) \cdot h\). Thus we realize that if the value of the underlying goes to infinity, that is \(\lim_{n \to \infty} Y^{h,\Psi} \omega_n = +\infty\), we get

    \[
    \lim_{n \to \infty} \frac{S_0(t)}{S_0(T)} \cdot \Phi \left( Y^{h,\Psi}(t) \omega_n \right) = 1.
    \]

    If the underlying approaches its lower boundary, i.e., \(\lim_{n \to +\infty} Y^{h,\Psi} \omega_n = 0\), we get the identical result:

    \[
    \lim_{n \to +\infty} \frac{S_0(t)}{S_0(T)} \cdot \Phi \left( Y^{h,\Psi}(t) \omega_n \right) = 1,
    \]

    a proof can be found in Chapter 6.3.

    Similar to the value at risk case, there is a one-to-one correspondence between the current wealth and \(Y^{h,\Psi}(t)\), because

    \[
    \frac{\partial S_0(t)}{\partial Y^{h,\Psi}(t)} \cdot \Phi \left( Y^{h,\Psi}(t) \right) + \frac{S_0(t)}{S_0(T)} \cdot a \cdot \exp \left[ \frac{1}{2} b^2 \right] \cdot \Phi \left( a \cdot \exp \left[ b^2 \right], b, \Psi, d + b, \frac{1}{\gamma - 1} - 1 \right) Y^{h,\Psi}(t)
    \]

    is strictly positive. Hence
– the asymptotic behavior of the underlying value and the current wealth are identical and
– the optimal investment strategy can be regarded as a function of the current time and
wealth only.

- The expected shortfall of the optimal claim is 
  \[ \mathbb{E}_P \left[ \left( h - X^{h, \Lambda \phi, \Psi} \right)^+ \right] = h \cdot \Phi(d) - \]
  \[ \phi \left( a, b, \Psi, \tilde{d}, \frac{1}{1 - T} \right) \] with
  \[ \tilde{a} := (Y^{h, \Lambda \phi})(t) \cdot \exp \left\{ -\frac{(2 - \gamma - 1) \cdot \frac{1}{T} \int_t^T \parallel \theta(s) \parallel^2 ds}{\gamma - 1} \right\} \]
  and
  \[ \tilde{d} := \frac{\ln \left( \frac{a \cdot \exp \left\{ \frac{1}{T} \int_t^T \parallel \theta(s) \parallel^2 ds \right\}}{\gamma - 1} \right)}{\frac{1}{T} \int_t^T \parallel \theta(s) \parallel^2 ds} \]

- A consequence of \( 0 \leq \phi \left( a \cdot \exp \left[ b^2 \right], b, \Psi, d + b, \frac{1}{\gamma - 1} - 1 \right) \leq \Phi(d + b) \cdot h^{2 - \gamma} \) is that the
  expression \( \frac{p_\text{rel}^P(t)(\omega_n)}{Y^{h, \phi}(t)(\omega_n)} \) converges to 1 if \( Y^{h, \phi}(t)(\omega_n) \to +\infty \). In Chapter 6.3 we show how
to prove that if \( Y^{h, \phi}(t)(\omega_n) \to 0 \), \( \frac{p_\text{rel}^P(t)(\omega_n)}{Y^{h, \phi}(t)(\omega_n)} \) converges to 1 as well. This implies that
  \( p_\text{rel}^P(t)(\omega_n) \) approaches 1 if \( X_{x_0, \Pi_{VaR}}^{(0+)}(t)(\omega_n) = \frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ X^{h, \phi}(t)(\omega_n) \right] \) converges
  either to 0 or to \(+\infty\).

As we can see by comparing Figure 4.5 with Figure 4.1, the optimal strategy of \( P_{ES}^h \) takes

![Figure 4.5: Optimal investment strategy in \( P_{ES}^h \) (1)](image)

even more risks in the most adverse states (i.e. low current wealth) than the optimal strategy
of \( P_{VaR} \). The market coefficients are again \( r := 0.04, N := 1, \sigma_1 := 0.4, \mu_1 := 0.1 \) and the
horizon is \( T := 2 \). The investor has the parameters \( \gamma := 0, h := 1, c := 0.9 \) and \( x_0 := 0.95 \).
Figure 4.5 is calculated for \( \alpha = 0.05 \).

Figure 4.6 illustrates the influence of different \( \alpha \). The parameter \( \alpha \) can be viewed as a measure
of the tightness of the risk restriction. The anticipation that the optimal strategy deviates more
from the classical one the more restrictive the constraint is, is somewhat confirmed. However,
for specific situations the value the optimal strategy in a restrictive situation can be identical
to the classical one even although it differs in a not as restrictive case.
4.2. OPTIMAL STRATEGIES AND THEIR PROPERTIES

The initial situation for $\alpha := 0.9$ is displayed in Figure 4.7: Although there exists an admissible strategy for any initial wealth (cf. Theorem 3.5), the solution is not always unique (for more details, see Theorem 3.23). Hence, we omitted the optimal strategy for initial wealths below $Q \left( \frac{dQ}{dt} < \tau(h) \right) - h + Q \left( \frac{dQ}{dt} \geq \tau(h) \right) \cdot u_0$.

- The optimal wealth and investment strategy at time $t$ given by Theorem 4.8 is a generalization of Proposition 4.5 in [Gabih and Wunderlich, 2004]. That proposition covers the case of the classical Black-Scholes market with constant coefficients, $\theta > 0$ and a single stock. Our
optimal investment strategy is a simpler expression, because four of the addends cancel out and [Gabih and Wunderlich, 2004] overlooked one of these cancellations. Their paper is based on the work of [Basak and Shapiro, 2001] and uses an almost identical notation. However, [Gabih and Wunderlich, 2004] confused the definition of \(d_1(z, H_t)\) and \(d_2(z, H_t)\). The correct definition is 
\[
d_2(z, H_t) := \ln \frac{z}{H_t} + \left( r - \frac{\kappa^2}{2} \right) (T - t) \kappa \sqrt{T - t} \]
and 
\[
d_1(z, H_t) := d_2(z, H_t) + \gamma \kappa \sqrt{T - t}.
\]

As [Basak and Shapiro, 2001], they do not bother to give sufficient condition for the existence of the Lagrange multipliers, either.

- Figure 4.8 depicts the influence of finite trading for the same coefficients as previously and \(\alpha := 0.05\). Although the probability of having a terminal wealth below \(h = 1\) is drastically higher with finite trading than with continuous trading, the violation of the risk restriction is not that extreme.

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>(\Lambda h, \Psi h)</th>
<th>(\Psi h)</th>
<th>(Y \Lambda h, \psi^h(t))</th>
<th>(h)</th>
<th>((h \gamma - 1 + \Psi h)^{1-\gamma})</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Gabih and Wunderlich, 2004]</td>
<td>(1 - \gamma)</td>
<td>(y_1)</td>
<td>(y_2)</td>
<td>(e^{\Lambda h, \psi^h(t)})</td>
<td>(q)</td>
</tr>
</tbody>
</table>

![Figure 4.8: Dependence of \(F^P_{X \neq h, nP(T)}\) on the number of trades](image)

Let us turn our attention to the problem \((P_{ES^Q_h})\).
4.2. OPTIMAL STRATEGIES AND THEIR PROPERTIES

**Theorem 4.10 (Optimal Investment Strategy of \((P_{ES^Q_t})\)):**

Let \( h > 0 \), \((P_{ES^Q_t})\) be feasible, \( \psi_{Q_0}^{h,x_0} > 0 \) and \( \lambda_{Q_0}^{h,x_0} < +\infty \).

Then \( \lambda_0^{x_0} = x_0^{-1} \cdot S_0(T) \cdot \exp \left[ \int_0^T \left( \frac{1}{2} \cdot \frac{1}{\gamma - 1} \cdot \|\theta(s)\|_2^2 \right) ds \right] \), the optimal wealth at time \( t \in [0,T) \)

\[
\frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ X_{Q}^{h_{x_0}^{h,x_0},\psi_{Q_0}^{h,x_0}} \bigg| \mathcal{F}(t) \right] - \frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ X_{Q}^{h_{x_0}^{h,x_0},\psi_{Q_0}^{h,x_0}} \bigg| \mathcal{F}(t) \right] \text{ is given by}
\[
\begin{align*}
\frac{S_0(t)}{S_0(T)} \cdot h + \left( Y_{Q}^{h_{x_0}^{h,x_0}}(t) \cdot \Phi \left( d_+ \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) \right) - \frac{S_0(t)}{S_0(T)} \cdot h \cdot \Phi \left( d_- \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) \\
- \left( \frac{S_0(t)}{S_0(T)} \cdot h \cdot \Phi \left( -d_- \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) - Y_{Q}^{h_{x_0}^{h,x_0}}(t) \cdot \Phi \left( -d_+ \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) \right)
\end{align*}
\]

and \( \Pi^Q \) with \( \Pi^Q(t) \) defined as

\[
\pi^0(t) \cdot \left[ Y_{Q}^{h_{x_0}^{h,x_0}}(t) \cdot \Phi \left( d_+ \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) + Y_{Q}^{h_{x_0}^{h,x_0}}(t) \cdot \Phi \left( -d_+ \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) \right]
\]

is an optimal investment strategy for problem \((P_{ES^Q_t})\).

**Proof:** \( x_0 = Y_{Q}^{h_{x_0}^{h,x_0}}(0) = \left( \lambda_0^{x_0} \right)^{-1} \cdot \exp \left[ \int_0^T \left( -r(s) + \frac{1}{2} \cdot \frac{1}{\gamma - 1} \cdot \|\theta(s)\|_2^2 \right) ds \cdot \frac{2}{\gamma - 1} \right] \). The rest of the proof can be done analogously to the proof of Theorem 4.6 using the representation of the optimal claim given in Section 3.3.3.

Like the optimal strategies of other optimization problems, the optimal strategy given by Theorem 4.10 can be characterized and split in the following way:

**Definition \((\pi^Q, P_{abs}^Q, P_{rel}^Q)\):** \( \Pi^Q(t) = \pi^0(t) \cdot P_{abs}^Q(t) \) for some \( P_{abs}^Q(t) \geq 0 \).

The optimal proportional investment strategy \( \pi^Q \) never takes a more risky position than the optimal investment strategy \( \pi^0 \) of \((P_0)\):

\[
\pi^Q(t) := \frac{\Pi^Q(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ X_{Q}^{h_{x_0}^{h,x_0},\psi_{Q_0}^{h,x_0}} \bigg| \mathcal{F}(t) \right] = \pi^0(t) \cdot P_{rel}^Q(t) \text{ for some } P_{rel}^Q(t) \in [0,1].
\]

**Proof:** \( \Pi^Q(t) \) equals

\[
\begin{align*}
\pi^0(t) \cdot \left( Y_{Q}^{h_{x_0}^{h,x_0}}(t) \cdot \Phi \left( d_+ \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) + Y_{Q}^{h_{x_0}^{h,x_0}}(t) \cdot \Phi \left( -d_+ \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) \right) \\
= \pi^0(t) \cdot \left( \frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ X_{Q}^{h_{x_0}^{h,x_0},\psi_{Q_0}^{h,x_0}} \bigg| \mathcal{F}(t) \right] \right) \\
- \frac{S_0(t)}{S_0(T)} \cdot h \cdot \left( \Phi \left( -d_- \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) - \Phi \left( -d_- \left( h, t, Y_{Q}^{h_{x_0}^{h,x_0}}(t) \right) \right) \right)
\end{align*}
\]

We feel it is now adequate to discuss properties of the optimal solution in some detail and make some further annotations on Theorem 4.10.
Remarks 4.11:

- Let \( t \in (0, T) \) be fixed. As
  \[
  \frac{S_0(t)}{S_0(T)} \mathbb{E}_Q \left[ X_{Q_h^{h,x_0},Q}^{h,A_h^{h,x_0},Q} \right] \converges to 1 \text{ if } \frac{dQ}{dT}(t)(\omega) \to 0^+, \]
  \[
  \frac{S_0(t)}{S_0(T)} \mathbb{E}_Q \left[ X_{Q_{\lambda}^{h,x_0}}^{h,A_h^{h,x_0},Q} \right] \converges to 1 \text{ if } \frac{dQ}{dT}(t)(\omega) \to +\infty, \]
  converges to 1 if \( \frac{dQ}{dT}(t)(\omega) \to +\infty \), and

\[
\frac{\partial}{\partial t} \frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ X_{Q_h^{h,x_0},Q}^{h,A_h^{h,x_0},Q} \right] = Y_{\lambda_0^{h,x_0}}(t) \cdot \Phi \left( d_+ \left( h, t, Y_{\lambda_0^{h,x_0}}(t) \right) \right) + Y_{\lambda_0^{h,x_0}}(t) \cdot \Phi \left( -d_+ \left( h, 0, Y_{\lambda_0^{h,x_0}}(0) \right) \right) \]

is strictly negative, there is a unique \( \frac{dQ}{dT}(t) \), at any fixed time \( t \), corresponding to every possible wealth > 0 and vice versa.

Therefore \( t \in [t, T] \) as well can be expressed as a function of the current wealth and the current time \( t \).

- If \( h = u_0 \), \( \lambda_0^{h} = +\infty \). Otherwise, \( \lambda_0^{h} \) is the unique solution in the set \([0, +\infty]\) of the equation

\[
\frac{S_0(0)}{S_0(T)} \cdot h \cdot \Phi \left( -d_+ \left( h, 0, Y_{\lambda_0^{h}}(0) \right) \right) - Y_{\lambda_0^{h}}(0) \cdot \Phi \left( -d_+ \left( h, 0, Y_{\lambda_0^{h}}(0) \right) \right) - (h - c) \cdot \alpha = 0
\]

(see Fact 3.72(i) for details) and

\[
\frac{\partial Y_{\lambda_0^{h}}(0)}{\partial h} = \frac{\frac{S_0(0)}{S_0(T)} \cdot \Phi \left( -d_+ \left( h, 0, Y_{\lambda_0^{h}}(0) \right) \right)}{\Phi \left( -d_+ \left( h, 0, Y_{\lambda_0^{h}}(0) \right) \right)}.\]

In any case, \( \frac{\partial Y_{\lambda_0^{h}}(0)}{\partial h} = 0 \) and \( Y_{\lambda_0^{h}}(0) = \max \left\{ Y_{\lambda_0^{h}}(0), Y_{\lambda_0^{h}}(0) \right\} \).

According to Fact 3.73, \( \Lambda_{Q_h^{h,x_0}} \) can be computed for fixed \( \Psi_{Q_h^{h,x_0}} \) as the unique root of

\[
\mathbb{E}_Q \left[ X_{Q_h^{h,x_0},Q}^{h,A_h^{h,x_0},Q} \right] - x_0 = 0
\]

with \( \Lambda_{Q_h^{h,x_0}} \in [\Psi_{Q_h^{h,x_0}}, +\infty) \setminus \{0\} \). Thus if \( \Psi_{Q_h^{h,x_0}} = \lambda_0^{h} < \lambda_0^{x_0} \):

\[
\frac{\partial Y_{\lambda_0^{h,x_0}}(0)}{\partial h} = - \frac{\partial \left( \mathbb{E}_Q \left[ X_{Q_h^{h,A_h^{h,x_0}},Q}^{h,x_0} \right] - x_0 \right)}{\partial h} \bigg|_{\lambda = \Lambda_{Q_h^{h,x_0}}}.
\]
It is straightforward to enhance Theorem 4.10 to include the cases $\Psi^{h,x_0} = 0$ and $\Lambda^{h,x_0} = +\infty$.

Furthermore, the equation

$$
\frac{\partial}{\partial h} \left( \mathbb{E}_Q \left[ X_h^{\lambda, \Psi^{h,x_0}, \psi^{h,x_0}_{Q}} \right] - x_0 \right) = \Phi \left( d_+ \left( h, 0, Y^\lambda(0) \right) \right)
$$

and

$$
\frac{\partial}{\partial h} \left( \mathbb{E}_Q \left[ X_h^{\lambda, \Psi^{h,x_0}, \psi^{h,x_0}_{Q}} \right] - x_0 \right) = \frac{S_0(0)}{S_0(T)} \left( \Phi \left( d_- \left( h, 0, Y^\psi^{h,x_0}(0) \right) \right) - \Phi \left( d_- \left( h, 0, Y^\lambda(0) \right) \right) \right) + \Phi \left( -d_+ \left( h, 0, Y^\psi^{h,x_0}(0) \right) \right) \cdot \frac{\partial Y^\psi^{h,x_0}(0)}{\partial h}.
$$

It reveals that for fixed current wealth, $p_{rel}^Q (t)$ increases with decreasing $\Lambda^{h,x_0}$ or with increasing $\Psi^{h,x_0}$. Using Fact 3.72(i), we can extend this relationship to deduce that $p_{rel}^Q (t)$ increases with increasing $\alpha$ or decreasing $c$.

As $\frac{p_{rel}^Q(t)(\omega)}{\mathbb{E}_Q[X_{h^{\lambda,x_0},\Psi^{h,x_0}_{Q}}(\omega)]}$ converges to 1 if the pricing process $\frac{dQ}{dP}(t)(\omega) \to 0+$ and $\frac{p_{rel}^Q(t)(\omega)}{\mathbb{E}_Q[X_{h^{\lambda,x_0},\Psi^{h,x_0}_{Q}}(\omega)]}$ converges to 1 if $\frac{dQ}{dP}(t)(\omega) \to +\infty$, the optimal relative investment strategy $p_{rel}^Q (t)(\omega)$ approaches 1 if $X_{x_0,0}(t)(\omega) = \frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ X_h^{\lambda, \Psi^{h,x_0}, \psi^{h,x_0}_{Q}} \right] (\omega)$ converges to either 0+ or $+\infty$.

Once more we use the market coefficients $r := 0.04$, $N := 1$, $\sigma_{11} := 0.4$, $\mu_1 := 0.1$, and the horizon $T := 2$ for the following figures. The parameters of the investor are $\gamma := 0$, $h := 1$, $c := 0.8$ and $\alpha := 0.05 \cdot Q(\Omega)$.

Figure 4.9 shows the dependence of the optimal investment strategy on the current wealth and its composition — for the initial wealth $x_0 := 1.0$ and $t = 0+$.

At the same point in time and for the same initial wealth, the effect of less stringent risk restrictions (that means higher $\alpha$) on the strategy is shown in Figure 4.10. As expected, the optimal strategy approaches the optimal strategy of the unconstrained problem.

The constellation at the outset ($\alpha$ again fixed at 0.05 $\cdot Q(\Omega)$) can be found in Figure 4.11. If the initial wealth is below $h \cdot Q(\Omega) - (h - c) \alpha$, the problem ($P_{ES}^Q$) is infeasible (recall Theorem 3.6). From a certain wealth on, the expected shortfall constraint is fulfilled automatically by the classical optimal strategy. Hence, it is optimal for the constraint case there, too.

Theorem 4.10 extends Proposition 5 found in [Basak and Shapiro, 2001] to the case of determinstic coefficients and lists sufficient conditions. Again, we specify the connection in the notation:

<table>
<thead>
<tr>
<th>present work</th>
<th>$\gamma$</th>
<th>$\Lambda^{h,x_0}$</th>
<th>$Y^{h,x_0}_{Q}(t)$</th>
<th>$\Psi^{h,x_0}_{Q}(t)$</th>
<th>$Y^{h,x_0}_{Q}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Basak and Shapiro, 2001]</td>
<td>1 - $\gamma$</td>
<td>$z_1$</td>
<td>$x(t)(z_1(\xi(t)))^\gamma$</td>
<td>$z_1 = z_2$</td>
<td>$x(t)(z_1-\zeta_2)(\xi(t))^\gamma$</td>
</tr>
</tbody>
</table>
4.2. OPTIMAL STRATEGIES AND THEIR PROPERTIES

- Once more, we consider the error that stems from an approximation of the continuous trading strategy by a finite number of trades. In Figure 4.12, we considered $r := 0.04$, $\sigma_1 := 0.4$, $\mu_1 := 0.1$, $T := 2$, $\gamma := 0$, $h := 1$, $K = 0.8$, $\Lambda_{Q}^{h,x_0} = 1$ and $\Psi_{Q}^{h,x_0} = 0.9$. Although applying the optimal strategy of the continuous market model in a finite trading setting yields a substantially higher percentage of outcomes below $h$, the expected shortfall risk constraint is not violated too badly:

<table>
<thead>
<tr>
<th>trading</th>
<th>10 times</th>
<th>100 times</th>
<th>1000 times</th>
<th>continuously</th>
</tr>
</thead>
<tbody>
<tr>
<td>expected shortfall (approx.)</td>
<td>0.0245</td>
<td>0.0229</td>
<td>0.0227</td>
<td>0.0226</td>
</tr>
</tbody>
</table>
The convergence of the expected shortfall is even faster than in the problem \( (PE_{SP}) \), as the additional small shortfalls are weighted less than higher shortfalls, because the latter occur in the more expensive states.

The optimal value of the objective function can be calculated explicitly, if the requirements of Theorem 4.10 are met. We have to differentiate the case of the logarithmic utility function from case of the power utility function. The resulting formulas are:
4.2. OPTIMAL STRATEGIES AND THEIR PROPERTIES

Corollary 4.12 (Logarithmic Utility Function):
If \( \gamma = 0 \), \( \mathbb{E}_P \left[ U_\gamma \left( X^h_{Q,\Lambda_{h_{x_0}}}, \Psi^h_{\nu_{x_0}} \right) \right] \)
equals
\[
\ln(h) + \sqrt{\int_t^T \|\theta(s)\|^2_2 \, ds} \cdot \left[ f \left( d_+ \left( h, t, Y^h_{\nu_{x_0}}(t) \right) \right) - f \left( -d_+ \left( h, t, Y^h_{\Psi_{x_0}}(t) \right) \right) \right]
\]
with \( f: x \mapsto x \cdot \Phi(x) + \varphi(x) \) and
\[
\frac{\partial \mathbb{E}_P \left[ U_\gamma \left( X^h_{Q,\Lambda_{h_{x_0}}}, \Psi^h_{\nu_{x_0}} \right) \right]}{\partial h} = \frac{1}{h} + \Phi \left( d_+ \left( h, 0, Y^h_{\nu_{x_0}}(0) \right) \right) \left( \frac{\partial Y^h_{\nu_{x_0}}(0)}{\partial h} - \frac{1}{h} \right)
+ \Phi \left( -d_+ \left( h, 0, Y^h_{\Psi_{x_0}}(0) \right) \right) \left( \frac{\partial Y^h_{\Psi_{x_0}}(0)}{\partial h} - \frac{1}{h} \right).
\]

Corollary 4.13 (Power Utility Function):
If \( \gamma \neq 0 \), \( \mathbb{E}_P \left[ U_\gamma \left( X^h_{Q,\Lambda_{h_{x_0}}}, \Psi^h_{\nu_{x_0}} \right) \right] \) is
\[
\frac{1}{\gamma} \cdot \left( \Lambda^h_{\nu_{x_0}}(S_0(T)) \cdot \exp \left[ \frac{1}{\gamma - 1} \cdot \frac{1}{\gamma} \cdot \int_0^T \|\theta(s)\|^2_2 \, ds \right] \right) \cdot \Phi \left( d\left( \Lambda^h_{\nu_{x_0}} \right) \right)
+ \frac{1}{\gamma} \cdot \left( \Phi \left( d_+ \left( \Psi^h_{\nu_{x_0}} \cdot h, 0, 1 \right) \right) - \Phi \left( d_+ \left( \Lambda^h_{\nu_{x_0}} \cdot h, 0, 1 \right) \right) \right)
+ \frac{1}{\gamma} \cdot \left( \Psi^h_{\nu_{x_0}}(S_0(T)) \cdot \exp \left[ \frac{1}{\gamma - 1} \cdot \frac{1}{\gamma} \cdot \int_0^T \|\theta(s)\|^2_2 \, ds \right] \right) \cdot \Phi \left( -d\left( \Psi^h_{\nu_{x_0}} \right) \right)
\]
with \( d(\lambda) := \frac{\ln(S_0(T)) + \frac{1 + \alpha}{\alpha} - \frac{1}{2} \int_0^T \|\theta(s)\|^2_2 \, ds}{\sqrt{\int_0^T \|\theta(s)\|^2_2 \, ds}} \).

**Proof:** The proof is an imitation of the proof of Corollary 4.12 and is omitted. \( \square \)

The derivative of the optimal value of the objective function with respect to \( h \) can be exploited for the problem \( (P_{\text{CVaR}_Q}) \), due to Proposition 3.26.

4.2.6 Conditional Value at Risk Problem

Let \( c \in [0, +\infty) \) and \( \alpha \in (0, \nu(\Omega)) \) be fixed. If \((P_{\text{CVaR}_P})\) is feasible and either \( \nu = Q \) or \( \overline{H}_P \neq \emptyset \), an optimal strategy is given by an optimal strategy of the problem \((P_{\text{ES}_P})\) with the following \( h \): If \( \nu = P \), \( h \in \overline{H}_P \) chosen as in Theorem 3.27 will do. Otherwise, Theorem 3.29 yields a suitable \( h \in \overline{H}_Q \) and Corollary 4.12 as well as Corollary 4.13 contain useful results in this respect.

We find the following two formulas useful in the upcoming example. Their calculation is straightforward.
Proposition 4.14:
For all $x_0 > 0$ and all $\alpha \in (0, Q(\Omega))$:
\[
\left( F^{Q_{Y^0}}_{Y^0}(T) \right)^{-1}(\alpha) = S_0(T) \cdot x_0 \cdot \exp \left[ -\frac{1}{2} \int_0^T \left\| \sigma(s)^\top \pi^0(s) \right\|_2^2 \, ds \right] \\
+ \Phi^{-1}(\alpha \cdot S_0(T)) \cdot \sqrt{\int_0^T \left\| \sigma(s)^\top \pi^0(s) \right\|_2^2 \, ds}
\]
and
\[
\text{CVaR}^Q_{\alpha} \left( Y^0_{\Lambda^0}(T) \right) = \frac{1}{\alpha} \cdot \mathbb{E}_Q \left[ \left( \left( F^{Q_{Y^0}}_{Y^0}(T) \right)^{-1}(\alpha) - Y^0_{\Lambda^0}(T) \right)^+ \right] - \left( F^{Q_{Y^0}}_{Y^0}(T) \right)^{-1}(\alpha)
\]
\[
= \frac{1}{\alpha} \cdot \left( \frac{1}{S_0(T)} \cdot \left( F^{Q_{Y^0}}_{Y^0}(T) \right)^{-1}(\alpha) \cdot \Phi \left( -d_+ \left( \left( F^{Q_{Y^0}}_{Y^0}(T) \right)^{-1}(\alpha), 0, x_0 \right) \right) \right) \\
- x_0 \cdot \Phi \left( -d_+ \left( \left( F^{Q_{Y^0}}_{Y^0}(T) \right)^{-1}(\alpha), 0, x_0 \right) \right)
\]
\[
- \left( F^{Q_{Y^0}}_{Y^0}(T) \right)^{-1}(\alpha).
\]

Example 4.15:
We use the market parameters $r := 0.04$, $N := d := 1$, $\mu_1 := 0.1$, $\sigma_{11} := 0.4$, the time horizon $T := 2$, and the investor’s coefficients $\gamma := 0$, $c := 0.8$, $h := 0.9$, $\alpha := \frac{0.05}{Q(\Omega)}$ as well as $x_0 := 1$ in the following illustrations. In this case, $Q(\Omega) = \frac{1}{S_0(T)} = \exp [-0.04 \cdot 2]$ and $\overline{H}_Q = \left[ c \cdot \frac{x_0 - c \cdot \alpha}{Q(\Omega) - \alpha} \right] \approx [0.8, 1.09820].$

- The unrestricted problem $(P_b)$ is solved by $(U')^{-1} \left( 1 - \frac{dQ}{d\alpha} \right)$ and

- $X_{Q, 0.9, \Lambda^0, 0.9, 0.1, \Psi_{0.9, 0.1}} \approx X_{Q, 0.9, 1, 0.01938, 0.85352}$ is the optimal solution of $(P_{ES^0_{\alpha}})$.

- In order to obtain the optimal claim of problem $(P_{CVMR^{\alpha}})$, we have to find the maximum point of the expected utility of the optimal solutions of the problems $(P_{ES^0_{\alpha}})$ as function in $h$ for $h \in \overline{H}_Q$. The relevant area of this function is depicted in Figure 4.13, which nicely shows the strict concavity of the function, which has already been mentioned in Proposition 3.26. We find that $X_{Q, h^*, \Lambda^0, h^*, 0.9, 0.1, \Psi_{0.9, 0.1}}$, which is approximately $X_{Q, 0.81310, 1, 0.00745, 0.80567}$, is the optimal claim of problem $(P_{CVMR^{\alpha}})$.

The comparison of the three different optimal terminal wealths takes center stage in Figure 4.14. Note that $h = 0.9 \neq h^*$ — otherwise the optimal claims of $(P_{ES^0_{\alpha}})$ and $(P_{CVMR^{\alpha}})$ would be identical. Both Lagrange multipliers of the optimal solution of $(P_{ES^0_{\alpha}})$ have smaller values than the Lagrange multipliers of the optimal solution of $(P_{CVMR^{\alpha}})$, i.e. $\Lambda^0, h^* \prec \Lambda^0$, $\Lambda^0, 0.9 \prec \Psi_{Q, h^*, 0.9, 0.1}$. So in this specific example we have that $X_{Q, h^*, \Lambda^0, h^*, 0.9, 0.1, \Psi_{0.9, 0.1}}(\omega) > X_{Q, 0.9, \Lambda^0, 0.9, 0.1, \Psi_{0.9, 0.1}}(\omega)$ for most of the outcomes. This observation is confirmed by the picture.

A different aspect can be learned from Figure 4.15: It discusses the influence of the initial endowment on the optimal choice for $h$. Below a certain threshold $c \cdot Q(\Omega)$ the problem is not feasible (see
4.2. OPTIMAL STRATEGIES AND THEIR PROPERTIES

Theorem 3.9), so in particular an optimal choice for \( h \) does not exist. Above that threshold and up to a certain point, there exists a unique optimal \( h \). The upper limit of that area is the point starting at which the optimal solution of the problem without risk restriction fulfills the conditional value at risk constraint. That point can be calculated by looking at Proposition 4.14 and solving the equation \( \text{CVaR}^Q \left(Y^{\lambda_0} (T) \right) = -c \) for \( x_0 \). We recall that if the risk restriction is not active, the choice is not unique, and the \( \alpha \)-quantile of the unrestricted optimal terminal wealth is a valid and optimal choice for \( h \).
4.3 Equivalent Problem Statements for Other Solution Techniques

We have seen that the martingale method can deal with the given risk constraints directly. Of course, it is not the only solution technique. Many problems in financial mathematics are solved using methods developed for controlled Markov processes like the Hamilton-Jacobi-Bellman approach or the stochastic maximum principle. These methods require a different problem description. First of all, they need a state process to describe the state of the system at any time \( t \in [0, T] \): If there were no risk constraint, a natural choice would be to take the current wealth at time \( t \).

To enforce the risk restriction, we need a characterization of the restriction at any time \( t \). With the added constraint however, the current state is not fully described by the current wealth. Hence, our idea is to describe the current state by the pair \((x_t, \epsilon_t)\) as control and use the dynamic programming approach.

Next, we have to answer the question what should be the control variable. An obvious choice is to use \( x_t \) to control the wealth process and add an additional control for \( \epsilon_t \). Since the processes \( \mathbb{E}_\nu \left[ \mathbb{1}_{(-\infty, q)}(X^{x_0, \Pi}(T)) \right] \) and \( \mathbb{E}_\nu \left[ (h - X^{x_0, \Pi}(T))^+ \right] \) are square-integrable martingales, we select a process \( \epsilon \) as control and use the dynamic \( \epsilon_t = \epsilon_0 + \int_0^t \epsilon(s)^+ dW_\nu(s) \). Using some integrability condition for \( \epsilon \), \( \epsilon_0 = \frac{(h-c)/\alpha}{\nu(X^0)} \) results in \( \mathbb{E}_\nu [\epsilon_t] \leq \epsilon \) and \( \epsilon_0 = \frac{(h-c)/\alpha}{\nu(X^0)} \) ensures \( \mathbb{E}_\nu [\epsilon_t] \leq \epsilon \).

Figure 4.15: Dependence of the optimal \( h \) in \( P_{\text{CVaR}} \) on the initial wealth

4.3 Equivalent Problem Statements for Other Solution Techniques

We have seen that the martingale method can deal with the given risk constraints directly. Of course, it is not the only solution technique. Many problems in financial mathematics are solved using methods developed for controlled Markov processes like the Hamilton-Jacobi-Bellman approach or the stochastic maximum principle. These methods require a different problem description. First of all, they need a state process to describe the state of the system at any time \( t \in [0, T] \): If there were no risk constraint, a natural choice would be to take the current wealth at time \( t \). With the added constraint however, the current state is not fully described by the current wealth.

To enforce the risk restriction, we need a characterization of the restriction at any time \( t \in [0, T] \). Once again, we can make use of the fact that the value at risk and the expected shortfall can be written using expectations: We can write

\[
\nu(X^{x_0, \Pi}(T) < q) = \mathbb{E}_\nu \left[ \mathbb{E}_\nu \left[ \mathbb{1}_{(-\infty, q)}(X^{x_0, \Pi}(T)) \right] \mathcal{F}(t) \right]
\]

and

\[
\text{ES}_h^\nu(X^{x_0, \Pi}(T)) = \mathbb{E}_\nu \left[ \mathbb{E}_\nu \left[ (h - X^{x_0, \Pi}(T))^+ \right] \mathcal{F}(t) \right].
\]

If we ensure that for some \( t \in [0, T] \), \( \mathbb{E}_\nu \left[ \mathbb{1}_{(-\infty, q)}(X^{x_0, \Pi}(T)) \right] ) \mathcal{F}(t) \] \( \leq \epsilon_t \) with \( \mathbb{E}_\nu [\epsilon_t] \leq \epsilon \), we have guaranteed that \( \nu(X^{x_0, \Pi}(T) < q) \) \( \leq \epsilon \).

Similarly, \( \mathbb{E}_\nu \left[ (h - X^{x_0, \Pi}(T))^+ \right] \mathcal{F}(t) \] \( \leq \epsilon_t \) and \( \mathbb{E}_\nu [\epsilon_t] \leq (h - c) \cdot \alpha \) yield \( \text{ES}_h^\nu(X^{x_0, \Pi}(T)) \leq (h - c) \cdot \alpha \).

Hence, our idea is to describe the current state by the pair \((x_t, \epsilon_t)\), where \( x_t \) denotes the current wealth at time \( t \) and \( \epsilon_t \) is an upper limit for the conditional expected value.

Next, we have to answer the question what should be the control variable. An obvious choice is to use \( x_t \) to control the wealth process and add an additional control for \( \epsilon_t \). Since the processes \( \mathbb{E}_\nu \left[ \mathbb{1}_{(-\infty, q)}(X^{x_0, \Pi}(T)) \right] \) and \( \mathbb{E}_\nu \left[ (h - X^{x_0, \Pi}(T))^+ \right] \) are square-integrable martingales, we select a process \( \epsilon \) as control and use the dynamic \( \epsilon_t = \epsilon_0 + \int_0^t \epsilon(s)^+ dW_\nu(s) \). Using some integrability condition for \( \epsilon \), \( \epsilon_0 = \frac{(h-c)/\alpha}{\nu(X^0)} \) results in \( \mathbb{E}_\nu [\epsilon_t] \leq \epsilon \) and \( \epsilon_0 = \frac{(h-c)/\alpha}{\nu(X^0)} \) ensures \( \mathbb{E}_\nu [\epsilon_t] \leq \epsilon \).
4.3. EQUIVALENT PROBLEM STATEMENTS FOR OTHER SOLUTION TECHNIQUES

\[(h - c) \cdot \alpha.\] Thus the complete control is the pair \((\Pi, e)\).

What remains to be done is make sure that for some choice of \(t\), \(E_\nu\left[\mathbb{1}_{(\infty, q)}(X^{x_0, \Pi}(T)) \mid \mathcal{F}(t)\right] \leq \epsilon_t\) or in the case of a expected shortfall constraint \(E_\nu\left[(h - X^{x_0, \Pi}(T))^+ \mid \mathcal{F}(t)\right] \leq \epsilon_t.\)

The easiest way is to select \(t = T\) and punish any violation by setting the utility in this state to \(-\infty.\) Thus we work with the following terminal utility functions:

**Definition \((U_{\text{VaR}})\):** Let \(U_{\text{VaR}}(x, y) :=\)

\[
\begin{cases}
U(x) & \text{if } \mathbb{1}_{(\infty, q)}(x) \leq y \\
-\infty & \text{if } \mathbb{1}_{(\infty, q)}(x) > y
\end{cases}
\]

**Definition \((U_{\text{ES}})\):** We define \(U_{\text{ES}}(x, y) :=\)

\[
\begin{cases}
U(x) & \text{if } (h - x)^+ \leq y \\
-\infty & \text{if } (h - x)^+ > y
\end{cases}
\]

So we can now state the resulting optimization problems:

**Fact 4.16:**

\((P_{\text{VaR}})^\nu\) and

\[
\begin{align*}
(P_{\text{Dynamic}})^\nu = & \min_{\epsilon_t \in \mathbb{P}, \Pi} \\
& \mathbb{E}_\nu\left[U_{\text{VaR}}(X^{x_0, \Pi}(T), \epsilon_T)\right] \rightarrow \max \\
& \epsilon_0 \\
& \epsilon_t
\end{align*}
\]

are connected in the following way:

- Any admissible strategy \(\Pi\) for \((P_{\text{VaR}})^\nu\) can be extended to an admissible strategy \((\Pi, e)\) for \((P_{\text{Dynamic}})^\nu\) and

- the first part, \(\Pi\), of any admissible strategy \((\Pi, e)\) for \((P_{\text{Dynamic}})^\nu\) with an objective function value that is not \(-\infty\) is an admissible strategy for \((P_{\text{VaR}})^\nu\).

Both transformations can be chosen such that they do not change the value of the objective function.

The definition of \((P_{\text{Dynamic}})^\nu\) allows the interpretation that the strategy at any time \(t \in [0, T)\) does only depend on

- the current time \(t,\)
- the current value of the portfolio and
- an upper limit \(\epsilon_t \cdot \nu(\Omega)\) for the shortfall probability.

This description of the required information at a given state is a generalization of the initial constellation: In the beginning we are confronted with the current time \(0\), an initial endowment \(x_0\), which equals the initial value of the portfolio, and an upper limit \(\epsilon\) for the shortfall probability.

As we have seen by investigating the solution of \((P_{\text{VaR}})^\nu\), this is not the only way to describe the dependence of the optimal strategy on the current state: Another possibility is to use the current time, the current portfolio value and the Lagrange multiplier \(\Lambda_{x_0}^{\nu}\). This second triple may contain fewer information, since the upper limit for the shortfall probability cannot be recovered in all cases.

The transformation of the expected shortfall constraint problem yields an analogous result:
Fact 4.17:
The expected shortfall problem \((P_{ES}^\nu)\) can be translated into the problem

\[
\left( P_{ES}^{Dynamic} \right) \begin{cases}
\mathbb{E}_P \left[ U_{ES} (X^{x_0,\Pi} (T), \epsilon_T) \right] & \to \max \\
\epsilon_0 & = \frac{(h-c)\alpha}{\nu(\Omega)} \\
\epsilon_t & = \epsilon_0 + \int_0^t \epsilon(s)^T dW_\nu(s) \\
\Pi & \in \mathcal{P} \\
\mathbb{E}_\nu \left[ \int_0^T \| \epsilon(s) \|^2 \right] & < +\infty
\end{cases}
\]

The relationship between \((P_{ES}^\nu)\) and \((P_{ES}^{Dynamic})\) is the same as for the corresponding problems in the value at risk case: On the one hand, any strategy \(\Pi\) that is admissible for \((P_{ES}^\nu)\) can be extended to an admissible strategy \((\Pi, \epsilon)\) of problem \((P_{ES}^{Dynamic})\) with an identical objective function value. On the other hand, for any admissible strategy \((\Pi, \epsilon)\) of \((P_{ES}^{Dynamic})\) with an objective function greater than \(-\infty\), \(\Pi\) is admissible for \((P_{ES}^\nu)\) and yields the same objective function value, too.

Therefore the optimal strategy at time \(t \in [0, T)\) depends solely on

- the current value of the portfolio and
- an upper limit \(\epsilon_t \cdot \nu(\Omega)\) for the expected shortfall given the current state.

As before, the second dependency can be replaced by the Lagrange multipliers of problem \((P_{ES}^\nu)\). A drawback of problem \((P_{ES}^{Dynamic})\) is that it does not restrict the set of admissible strategies but punishes a violation of a criterion in retrospect. What we really would like to have is some sort of dynamic confinement of the strategy such that the end criterion is fulfilled in any case. In other words we would like to use some foresight instead of being apered afterwards.

Our idea is to introduce a state restriction: We look at the process that describes the minimal amount required at the current point in time to fulfill the applicable end criterion. The current wealth must never fall below that minimal amount. Thus if the current wealth is strictly above this amount, the strategy is not delimited. As soon as the wealth hits this minimal amount however, we have to restrict the strategy to ensure that the criterion will not be violated at the end.

Since we also control the exact form of the end criterion, we cannot really speak of the end criterion. What we mean is the least strong criterion that can be achieved given the current state.

The upcoming two results are based on this idea and give another reformulation of the expected shortfall problem. It is necessary to separate the cases for different \(\nu\) and we start out with \(Q\) because its reformulation is somewhat simpler. In that case, the calculation of the minimal required wealth is based on Theorem 3.6.

Definition \((P_{ES}^{Dynamic})\) (\(Q\)): We define the hitting times

\[
t_1 (\epsilon) := \inf \{ t \in [0, T] \mid \epsilon_t = 0 \} \land T = \left\{ \begin{array}{ll}
\inf \{ t \in [0, T] \mid \epsilon_t = 0 \} & \text{if } \exists t \in [0, T]: \epsilon_t = 0 \\
T & \text{else}
\end{array} \right\},
\]

\[
t_2 (\Pi) := \inf \left\{ t \in [0, T] \mid \frac{X^{x_0,\Pi} (t)}{S_0 (t)} = \frac{h-c}{S_0 (T)} \right\} \land T \text{ for continuous processes } \epsilon \text{ and } \Pi \text{ (hence, they are}
\]

95
4.3. EQUIVALENT PROBLEM STATEMENTS FOR OTHER SOLUTION TECHNIQUES

stopping times) and the control problem

\[
\begin{aligned}
\left( \tilde{P}_{\text{Dynamic}}^{\text{ES}Q} \right) := \begin{cases}
\mathbb{E}_P \left[ U \left( X^{x_0, \Pi}(T) \right) \right] & \to \max \\
x_0 & \geq h \cdot Q (\Omega) - (h - c) \alpha \\
\epsilon_0 & = \frac{(h - c) \alpha}{Q(\Omega)} \\
\epsilon_t & = \epsilon_0 + \int_0^t e(s)^\top dW_Q(s) \\
(\Pi(t), e(t)) & \in D^Q_t \\
\Pi & \in \mathcal{P} \\
\mathbb{E}_Q \left[ \int_0^T \| e(s) \|_2^2 \, ds \right] & < +\infty
\end{cases}
\end{aligned}
\]

with

\[
D^Q_t (\omega) := \begin{cases}
\{(0, 0)\} & \text{if } (t_2 (\Pi) \lor t_1 (\epsilon_\omega)) (\omega) \leq t \\
\mathbb{R}^N \times \{0\} & \text{if } t_1 (\epsilon_\omega) (\omega) \leq t < t_2 (\Pi) (\omega) \\
\mathbb{R}^N \times \mathbb{R}^d & \text{if } t_2 (\Pi) (\omega) \leq t < t_1 (\epsilon_\omega) (\omega) \\
\mathbb{R}^N \times \mathbb{R}^d & \text{else}
\end{cases}
\]

Fact 4.18:
The relationship between \((P_{\text{ES}h}^Q)\) and \(\left( \tilde{P}_{\text{Dynamic}}^{\text{ES}Q} \right)\) is as follows: For any strategy \(\Pi\) that is admissible for \((P_{\text{ES}h}^Q)\) there exists a modification \(\tilde{\Pi}\) that yields (almost-surely) the same terminal wealth as \(\Pi\) and a process \(e\) such that \((\tilde{\Pi}, e)\) is admissible for \(\left( \tilde{P}_{\text{Dynamic}}^{\text{ES}Q} \right)\). Of course, the objective functions values are identical, too. On the other hand, for any admissible strategy \((\Pi, e)\) of \(\left( \tilde{P}_{\text{Dynamic}}^{\text{ES}Q} \right)\), \(\Pi\) is admissible for \((P_{\text{ES}h}^Q)\) with the same objective function value.

Again, \(\epsilon_t \cdot Q (\Omega)\) can be viewed as an upper limit on the expected shortfall at time \(T\) contingent on the current time \(t\) and state. The stopping time \(t_1 (\epsilon)\) ensures that this upper limit is non-negative. The other stopping time \(t_2 (\Pi)\) guarantees that there is never less expected shortfall left than we can minimally achieve from this point on.

Let us now turn our attention to the second case: For the construction of a similar control problem for \((P_{\text{ES}h}^P)\), Proposition 3.42 is the key: In general, a lower boundary for the required wealth given that one faces a terminal expected shortfall restriction with respect to \(P\) does not exist in our scenario (see Theorem 3.5). Thus we resort to the point of view that only strategies that yield a terminal wealth above or equal to \(u_0\) are of interest. With this additional restriction, Proposition 3.42 delivers a lower boundary for the required wealth.

Definition \((\tilde{P}_{\text{ES}h}^{\text{Dynamic}})\): Suppose that \(h > u_0\).

Let \(Y (t, y) := \frac{S_0(t)}{S_0(T)} \left( u_0 + \Phi \left( \Phi^{-1} (y) - \sqrt{\int_t^T \| \theta(s) \|_2^2 \, ds} \right) \cdot (h - u_0) \right)\) be the lower boundary for the required wealth at time \(t \in \mathbb{R}^d\) in state \(y\), \(t_3 (\epsilon_\omega) := \inf \{ t \in [0, T] \mid \epsilon_t = h - u_0 \} \wedge T\) and \(t_4 (\Pi) := \inf \{ t \in [0, T] \mid X_{x_0, \Pi}(t) = Y \left( t, 1 - \frac{\epsilon_t}{h - u_0} \right) \} \wedge T\).
The optimization problem \( \left( \tilde{P}_{ES}^{\rho} \right) \) is defined as

\[
\begin{align*}
\left\{ \begin{array}{ll}
\mathbb{E}_P \left[ U \left( X^{x_0, \Pi}(T) \right) \right] & \rightarrow \max \\
x_0 & \geq Y \left( 0, 1 - \alpha \cdot \frac{h - c}{h - u_0} \right) \\
\epsilon_0 & = (h - c) \cdot \alpha \\
\epsilon_t & = \epsilon_0 + \int_0^t e(s)^T dW(s) \\
\Pi & \in D_P^\rho \\
\mathbb{E}_P \left[ \int_0^T \| e(s) \|^2 ds \right] & < +\infty
\end{array} \right.
\]

with \( d_t(\epsilon_t) := (h - u_0) \cdot \left( \frac{S_0(t)}{S_0(T)} \right)^{1 - \frac{e_t - x_t}{h - u_0} - \frac{\| e_t \|^2}{\| e(\theta) \|^2}} \left( \frac{1}{\| e(\theta) \|^2} \right) \left( \frac{\sqrt{f_t^T} \| e(\theta) \|^2}{\| e(\theta) \|^2} \right) \theta(t) \)

and

\[
D_P^\rho(\omega) := \begin{cases}
\{(0, 0)\} & \text{if } (t_1(\Pi) \cup (t_1(\epsilon) \wedge t_3(\epsilon))) (\omega) \leq t \\
\mathbb{R}^N \times \{0\} & \text{if } (t_1(\epsilon) \wedge t_3(\epsilon)) (\omega) \leq t < t_4(\Pi)(\omega) \\
\{(d_t(\epsilon_t))\} & \text{if } t_4(\Pi)(\omega) \leq t < (t_1(\epsilon) \wedge t_3(\epsilon)) (\omega) \\
\mathbb{R}^N \times \mathbb{R}^d & \text{else}
\end{cases}
\]

Fact 4.19:
The problems \((P_{ES}^\rho)\) and \( \left( \tilde{P}_{ES}^{\rho} \right) \) are connected in the following way: For any strategy \( \Pi \) that is admissible for \((P_{ES}^\rho)\) and results in a terminal wealth \( X^{x_0, \Pi}(T) \geq u_0 \) (almost-surely — otherwise the objective functions value is \(-\infty\)), there exists a \( \tilde{\Pi} \) that yields (almost-surely) the same terminal wealth as \( \Pi \) and a process \( e \) such that \((\tilde{\Pi}, e)\) is admissible for \( \left( \tilde{P}_{ES}^{\rho} \right) \) and yields the identical the value of the objective function. On the other hand, for any admissible strategy \((\Pi, e)\) of \( \left( \tilde{P}_{ES}^{\rho} \right) \), \( \Pi \) is admissible for \((P_{ES}^\rho)\) with the same objective function value.

Having seen these last two variants of the expected shortfall problems, a natural question is to ask: Does this type of transformation work in the value at risk case, too? The answer however is not so clear-cut. First of all, Theorem 3.3 does not provide us with a suitable lower bound for the minimum required current wealth. As resort we look at Proposition 3.30 and consider the following example:

Example 4.20:
Consider problem \((P_{VAR}^\rho)\) with parameters \( r = 0, q > 0 = u_0 \) and \( \epsilon \in (\frac{1}{2}, 1) \). The wealth process corresponding to the initial endowment \( \frac{q}{2} \) and the investment strategy \( \Pi = 0 \), which is \( X^{\frac{q}{2}, 0}(\cdot) \), is constant and \( \nu(\Omega) = 1 \). Moreover, for all \( t \in [0, T] \),

\[
X^{\frac{q}{2}, 0}(t) = \frac{q}{2} > q \cdot (1 - \epsilon) \geq q \cdot Q \left( \frac{dQ}{dP} \right)^{-1} \left( \nu(\Omega) - \epsilon \right)
\]

and thus the wealth process fulfills the minimal required wealth criteria at all these times. However, the terminal wealth \( X^{\frac{q}{2}, 0}(T) = \frac{q}{2} \) does not fulfill the value at risk constraint, because \( \nu \left( X^{\frac{q}{2}, 0}(T) < q \right) = 1 > \epsilon \).

So we dismiss the idea of modifying the value at risk problem in the same spirit as the expected shortfall problems. At least with the given stochastic model it does not work right out of the box. A way out could be to sometimes force the investor to make a fair gamble at the end.
Our last concern is the conditional value at risk: Like we did in Section 3.1.2, we can simply use the expected shortfall problems and change $h$ from a known and given constant into a decision variable at time 0 and add the restriction $h \in [c, +\infty)$. In other words, the reduction to a family of expected shortfall problems can still be done.

4.4 Outlook

4.4.1 Alternative Solution Methods: Hamilton-Jacobi-Bellman Approach

We have seen in Chapter 4.3 that the risk restriction can be either formulated explicitly, hidden in a modified terminal utility function, or translated into restrictions for the admissible strategies. The first possibility is the natural choice for the martingale method. The others, especially the last one, are suitable for a Hamilton-Jacobi-Bellman approach or the stochastic maximum principle.

As an example, we look at the Hamilton-Jacobi-Bellman equation for problem $\tilde{p}^{\text{Dynamic ESQ}}$ in the logarithmic utility case. We do not pretend that we give a complete mathematical treatment. The purpose of this example is to give some basic understanding of how that approach could be used and point out some of the obstacles. Furthermore, it is an additional confirmation of the transformation of the risk constraint into a constraint for the set of admissible strategies.

Let $\gamma = 0$ and $h > u_0 = 0$. For all elements of $\{ (t, x, \tilde{c}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \mid x \geq h \cdot \frac{S_0(t)}{S_0(T)} - \tilde{c} \}$ we define

$$ V(t, x, \tilde{c}) := \ln(h) + \sqrt{T - t} \| \theta(s) \|_2^2 \, ds \left[ f \left( d_+ (h, t, Y^\Lambda (t, x, \tilde{c})) \right) - f \left( -d_+ (h, t, Y^\Psi (t, x, \tilde{c})) \right) \right], $$

where $f$ is the function $x \mapsto x \cdot \Phi(x) + \varphi(x)$, the value $Y^\Lambda (t, \tilde{c})$ of the underlying of the put option part of the optimal claim is the unique zero in the set $[0, +\infty]$ of the mapping

$$ y \mapsto \frac{S_0(t)}{S_0(T)} h\Phi \left( -d_- (h, t, y) \right) - y\Phi \left( -d_+ (h, t, y) \right) - \frac{S_0(t)}{S_0(T)} \tilde{c}, $$

$Y^\Psi (t, x, \tilde{c}) := \max \left\{ x, Y^\Lambda (t, \tilde{c}) \right\}$ and $Y^\Lambda (t, x, \tilde{c})$ is the zero of the mapping

$$ y \mapsto \frac{S_0(t)}{S_0(T)} h \left( y\Phi \left( d_+ (h, t, y) \right) - \frac{S_0(t)}{S_0(T)} h\Phi \left( d_- (h, t, y) \right) \right) - \frac{S_0(t)}{S_0(T)} \tilde{c} \Phi \left( -d_+ (h, t, Y^\Psi (t, x, \tilde{c})) \right) - x. $$

Furthermore, we set

$$ \Pi(t, x, \tilde{c}) := \left( \sigma(t)^{\top} \right)^{-1} \theta(t) \left[ Y^\Lambda (t, x, \tilde{c}) \Phi \left( d_+ (h, t, Y^\Lambda (t, x, \tilde{c})) \right) \right. $$

$$ \left. + Y^\Psi (t, x, \tilde{c}) \Phi \left( -d_+ (h, t, Y^\Psi (t, x, \tilde{c})) \right) \right], $$

and $e(t, x, \tilde{c}) := -\frac{S_0(T)}{S_0(0)} \theta(t) Y^\Psi (t, x, \tilde{c}) \Phi \left( -d_+ (h, t, Y^\Psi (t, x, \tilde{c})) \right)$.

If $r$ and $\theta$ are continuous function of the time, we observe the following first derivatives of $V$:

$$ \frac{\partial V}{\partial h}(t, x, \tilde{c}) = -r(t) \frac{S_0(T)}{S_0(0)} - \frac{1}{2} \| \theta(t) \|_2^2 \left[ \Phi \left( d_+ (h, t, Y^\Lambda (t, x, \tilde{c})) \right) + \Phi \left( -d_+ (h, t, Y^\Psi (t, x, \tilde{c})) \right) \right], $$

98
\[
\frac{\partial V}{\partial \tilde{t}} (t, x, \tilde{\epsilon}) = \frac{1}{Y^\lambda(t, x, \tilde{\epsilon})} \quad \text{and} \quad \frac{\partial^2 V}{\partial \tilde{t}^2} (t, x, \tilde{\epsilon}) = \frac{S_0(t)}{S_0(T)} \left( \frac{1}{Y^\lambda(t, x, \tilde{\epsilon})} - \frac{1}{Y^\lambda(t, x, \tilde{\epsilon})} \right).
\]

Selected second partial derivatives are as follows:
\[
\frac{\partial^2 V}{\partial \tilde{t}^2} (t, x, \tilde{\epsilon}) = \begin{cases} 
- \left( \frac{S_0(t)}{S_0(T)} \right)^2 \left( \frac{1}{(Y^\lambda(t, x, \tilde{\epsilon}))^2} \Phi(d_+ (h, t, Y^\lambda(t, x, \tilde{\epsilon}))) \right) + \left( \frac{1}{(Y^\lambda(t, x, \tilde{\epsilon}))^2} \Phi(\Phi(-d_+ (h, t, Y^\lambda(t, x, \tilde{\epsilon})))) \right) & \text{if } x < Y^\lambda(t, \tilde{\epsilon}) \\
0 & \text{if } x > Y^\lambda(t, \tilde{\epsilon})
\end{cases}
\]

\[
\frac{\partial^2 V}{\partial \tilde{t} \partial \epsilon} (t, x, \tilde{\epsilon}) = \begin{cases} 
\left( \frac{S_0(t)}{S_0(T)} \right)^2 \Phi(d_+ (h, t, Y^\lambda(t, x, \tilde{\epsilon}))) & \text{if } x < Y^\lambda(t, \tilde{\epsilon}) \\
0 & \text{if } x > Y^\lambda(t, \tilde{\epsilon})
\end{cases}
\]

The set of admissible actions in state \((t, x, \tilde{\epsilon})\) is defined as
\[
D (t, x, \tilde{\epsilon}) := \left\{ \begin{array}{ll}
\{ (0, 0) \} & \text{if } \frac{S_0(t)}{S_0(T)} = \frac{h-\tilde{\epsilon}}{S_0(T)} \text{ and } \tilde{\epsilon} = 0 \\
\{ (\Pi, \epsilon) \in \mathbb{R}^N \times \mathbb{R}^d \mid \epsilon = -\frac{S_0(T)}{S_0(t)} \sigma(t)^\top \Pi \} & \text{if } \frac{S_0(t)}{S_0(T)} \neq \frac{h-\tilde{\epsilon}}{S_0(T)} \text{ and } \tilde{\epsilon} = 0 \\
\mathbb{R}^N \times \mathbb{R}^d & \text{if } \frac{S_0(t)}{S_0(T)} = \frac{h-\tilde{\epsilon}}{S_0(T)} \text{ and } \tilde{\epsilon} \neq 0.
\end{array} \right.
\]

If \(x \neq Y^\lambda(t, \tilde{\epsilon})\),
\[
g (t, x, \tilde{\epsilon}, \Pi, \epsilon) := \frac{\partial V}{\partial \tilde{t}} (t, x, \tilde{\epsilon}) + \frac{\partial V}{\partial x} (t, x, \tilde{\epsilon}) \left[ x \cdot r(t) + \Pi^\top (\mu(t) - r(t) 1) \right] + \frac{\partial^2 V}{\partial \tilde{t} \partial \epsilon} (t, x, \tilde{\epsilon}) \cdot \epsilon^\top \theta(t) + \frac{1}{2} \frac{\partial^2 V}{\partial \tilde{t}^2} (t, x, \tilde{\epsilon}) \cdot \sigma(t)^\top \Pi + \frac{1}{2} \frac{\partial^2 V}{\partial \epsilon^2} (t, x, \tilde{\epsilon}) \cdot \epsilon^\top \theta(t) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} (t, x, \tilde{\epsilon}) \cdot \epsilon^\top \theta(t)
\]

is well defined and the Hamilton-Jacobi-Bellman corresponding to our problem \(\hat{P}_{ES^Q}\)
\[
0 = \sup_{(\Pi, \epsilon) \in D(t, x, \tilde{\epsilon})} \{ g (t, x, \tilde{\epsilon}, \Pi, \epsilon) \}.
\]

is fulfilled because of the following equations:
If \(x < Y^\lambda(t, \tilde{\epsilon})\),
\[
g (t, x, \tilde{\epsilon}, \Pi, \epsilon) = - \left( \frac{S_0(t)}{S_0(T)} \right)^2 \Phi(\Phi(\Phi(-d_+ (h, t, Y^\lambda(t, x, \tilde{\epsilon})))) \right) + \left( \frac{S_0(t)}{S_0(T)} \right)^2 \Phi(\Phi(-d_+ (h, t, Y^\lambda(t, x, \tilde{\epsilon}))))
\]

and \(x > Y^\lambda(t, \tilde{\epsilon})\) implies that \(g (t, x, \tilde{\epsilon}, \Pi, \epsilon) = - \frac{1}{2} \left\| \frac{\sigma(t)^\top \Pi}{x} - \theta(t) \right\|_2^2\).

In addition, the optimal strategy never crosses the set \(\{(t, x, \tilde{\epsilon}) \mid x = Y^\lambda(t, \tilde{\epsilon})\}\).

Note that we have cheated a little with the above definition of the set of admissible strategies \(D (t, x, \tilde{\epsilon})\). What we really need is to guarantee that the state constraints \(\tilde{\epsilon} \geq 0\) and \(x \leq h-\tilde{\epsilon}\) (and \(x \geq 0\)) are always fulfilled. Of course, such state constraints can be incorporated into the Hamilton-Jacobi-Bellman approach (see for instance [Øksendal, 1998]). Intuitively, this complication is not really required: We recall that in problem \(\hat{P}_{ES^Q}\) the compliance with the state constraints has been ensured using stopping times. Thus we can split up the problem into different parts.
We start out with the interval $[t, T]$ for the initial states $\{(t, x, \bar{\epsilon}) \mid \bar{\epsilon} = 0$ and $\frac{x}{S_0(t)} = \frac{h-\bar{\epsilon}}{S_0(T)}\}$, use $D = \{(0, 0)\}$ for the entire interval, and verify that $(V, (\Pi, e))$ satisfies the corresponding HJB equation. Next, we define $\tau(t) := \inf\{s \in [t, T] \mid \bar{\epsilon}_s = 0$ and $\frac{x}{S_0(t)} = \frac{h-\bar{\epsilon}_s}{S_0(T)}\}$, look at the interval $[t, \tau(t)]$ for the initial states $\{(t, x, \bar{\epsilon}) \mid \bar{\epsilon} = 0\}$, and use $D = \mathbb{R}^N \times \{0\}$ as set of admissible strategies and $V(\tau(t), x_{\tau(t)}, \bar{\epsilon}_{\tau(t)})$ as terminal payoff. Again, $(V, (\Pi, e))$ satisfies the HJB equation and we move on to the next part, and so on.

### 4.4.2 Incomplete Markets

One crucial ingredient to our treatment of the optimization problems is the existence of a unique pricing measure. This assumption is essentially a complete market model assumption. Let us consider an incomplete market with a unique probability measure and a family of pricing measures. We assume that the measure used in the risk constraint is the real world measure. The derived results are now useful in the following way:

For a fixed pricing measure taken out of the set of all pricing measures, we can calculate a candidate optimal terminal wealth in the same way as in the complete market model. However, this stochastic terminal wealth is usually not attainable by some strategy. Hence, it can only be used to identify an upper boundary for the objective function. Any admissible strategy that achieves an expected utility close to this boundary is guaranteed to be a good one.

Another way to circumvent the obstacle that not every claim is replicable is to artificially complete the market by adding additional stocks (if the market is completable). In this case the optimal solution of the completed market naturally depends on the completion, and it is sufficient to find a worst completion such that the optimal investment strategy does not invest in the additional stocks.
Part 5

Summary and Discussion

In this work, we investigated the influence of risk constraints on the optimal behavior of an agent attempting to maximize expected terminal utility. Specifically, we considered the three risk measures

- value at risk
- expected shortfall and
- conditional value at risk.

The following table lists previous literature that examines portfolio optimization with risk constraints in continuous settings. We omit the paper [Cuoco et al., 2001], because, as stated in Chapter 1.4, it deals with a different sort of risk constraint.

<table>
<thead>
<tr>
<th></th>
<th>value at risk</th>
<th>expected shortfall</th>
<th>conditional value at risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu = P )</td>
<td>[Basak and Shapiro, 2001]</td>
<td>[Gabih and Wunderlich, 2004]</td>
<td>unknown</td>
</tr>
<tr>
<td>( \nu = Q )</td>
<td>unknown</td>
<td>[Basak and Shapiro, 2001]</td>
<td>unknown</td>
</tr>
</tbody>
</table>

This work does not only fill the gaps ("unknown") in the above table. It also extends the cited literature for instance by providing necessary and sufficient conditions for the feasibility of the problems and the existence as well as the uniqueness of optimal solutions. Besides the explicit calculation of the optimal behavior, we derived several properties of the optimal solutions and compared them. Most of the mathematical energy had to be put into a Lagrange based approach to calculate the solutions of the general optimization problems of Part 3. The main tool for the dynamic continuous market model of Part 4 was the martingale method, which allowed us to make use of results of Part 3. In addition, we have pointed out how other solution techniques can be used in the dynamic model.

So what can a practitioner learn from all this? Let us put ourselves in the place of a regulator that faces the task of choosing a suitable risk measure. Naturally, we would look at different criteria, like coherence, the behavior of agents especially in adverse situations and the easiness of its calculation and implementation. So we recapitulate our results in these respects:

<table>
<thead>
<tr>
<th></th>
<th>VaR</th>
<th>ES</th>
<th>CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>coherence of risk measure</td>
<td>( \nu = P )</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>( \nu = Q )</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>optimal wealth in adverse conditions relative to optimal wealth without risk constraints</td>
<td>( \nu = P )</td>
<td>lower</td>
<td>lower</td>
</tr>
<tr>
<td>( \nu = Q )</td>
<td>lower</td>
<td>higher</td>
<td>higher</td>
</tr>
<tr>
<td>existence of static hedging strategy using options</td>
<td>( \nu = P )</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>( \nu = Q )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>calculation of risk measure requires estimation of subjective return rates (( \mu ))</td>
<td>( \nu = P )</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( \nu = Q )</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>
As a consequence of this table, we propose the use of the conditional value at risk measure with respect to the risk-neutral measure if we take a regulators view. We feel it is the best choice due to its ability to enforce more favorable behavior in adverse conditions, its independence from the investor’s subjective view of the market, and its coherence.

From a company’s perspective, the picture is not as clear-cut, because the most adverse conditions might not matter as gravely. Here the ability to estimate subjective return rates or the existence of simple hedging strategies might outweigh the other criteria. Nevertheless, the conditional value at risk measure seems again to be a good choice.
Part 6

Proofs

To make this work self-contained, we present the proofs that were omitted so far in chronological order.

6.1 Part 2: Risk Measures

Proof of Lemma 2.1:
(a) \( F_{X+\beta}^\nu(x) = \nu(\{\omega \in \Omega | X(\omega) + \beta \leq x\}) = \nu(\{\omega \in \Omega | X(\omega) \leq x - \beta\}) = F_X^\nu(x - \beta). \)
(b) \( F_{X,X}^\nu(x) = \nu(\{\omega \in \Omega | X(\omega) \leq x\}) = \nu(\{\omega \in \Omega | X(\omega) \leq \frac{x}{X}\}) = F_X^\nu\left(\frac{x}{X}\right). \)
(c) \( F_X^\nu(x) = \nu(\{\omega \in \Omega | X(\omega) \leq x\}) \geq \nu(\{\omega \in \Omega | Y(\omega) \leq x\}) = F_Y^\nu(x). \)

Proof of Lemma 2.2:
(a) \( \nu\left(X < (F_X^\nu)^{-1}(0)\right) = \nu(X < -\infty) = 0 \leq 0 \text{ and } \forall p \in (0, \nu(\Omega)): \)
\[
\nu\left(X < (F_X^\nu)^{-1}(p)\right) = \nu\left(\bigcup_{x \in \mathbb{Q}} \{X \leq x\} \right) = \lim_{x \rightarrow (F_X^\nu)^{-1}(p)^-} \nu(X \leq x) \leq p.
\]
(b) \((F_X^\nu)\) is monotone increasing
\( \Rightarrow (F_X^\nu(x) < p \Rightarrow (\forall t \leq x: F_X^\nu(t) < p \Rightarrow x \leq (F_X^\nu)^{-1}(p)) \)
\( \Rightarrow (x > (F_X^\nu)^{-1}(p) \Rightarrow F_X^\nu(x) \geq p). \)
\((F_X^\nu)\) is right-continuous
\( \Rightarrow \inf\{t \in \mathbb{R} | F_X^\nu(t) \geq p\} = \min\{t \in \mathbb{R} | F_X^\nu(t) \geq p\} \)
\( \Rightarrow (x = (F_X^\nu)^{-1}(y) \Rightarrow F_X^\nu(x) \geq p). \)
(Definition of \((F_X^\nu)^{-1}\)) \( \Rightarrow (F_X^\nu(x) \geq p \Rightarrow x \geq (F_X^\nu)^{-1}(p)) \).
\( \Rightarrow (x \geq (F_X^\nu)^{-1}(p) \iff F_X^\nu(x) \geq p). \)
(c) \((F_X^\nu)^{-1}(p) =: x \geq (F_X^\nu)^{-1}(p) \) \( \Rightarrow F_X^\nu((F_X^\nu)^{-1}(p)) = F_X^\nu(x) \geq p. \)
(d) \((F_X^\nu)^{-1}(p) = \inf\{x \in \mathbb{R} | F_X^\nu(x) \geq p\} \)
\( \Rightarrow \inf\{y + \beta | y \in \mathbb{R}, F_X^\nu(y) \geq p\} = (F_X^\nu)^{-1}(p) + \beta. \)
6.1. PART 2: RISK MEASURES

(e) \((F_{X}^{\nu})^{-1}(p) = \inf \{ x \in \mathbb{R} | F_{X}^{\nu}(x) \geq p \} \) \(\overset{Lemma \ 2.1(b)}{=} \inf \{ x \in \mathbb{R} | F_{X}^{\nu}(\frac{x}{\lambda}) \geq p \} = \inf \{ \lambda \cdot y | y \in \mathbb{R}, F_{X}^{\nu}(y) \geq p \} = \lambda \cdot (F_{X}^{\nu})^{-1}(p). \)

(f) \((F_{X}^{\nu})^{-1}(p) = \inf \{ x \in \mathbb{R} | F_{X}^{\nu}(x) \geq p \} \) \(\overset{Lemma \ 2.1(c)}{\leq} \inf \{ x \in \mathbb{R} | F_{X}^{\nu}(x) \geq p \} = (F_{X}^{\nu})^{-1}(p). \)

\[ \]

**Proof of Lemma 2.3:** Let \( \varepsilon > 0. \)
We define \( j := \lceil \frac{1}{2} \rceil + 2 \geq 3 \) and \( \delta := \min \{ (F_{X}^{\nu})^{-1} \left( \frac{i + 1}{2} \right) - (F_{X}^{\nu})^{-1} \left( \frac{i}{2} \right) | i \in \{ 1, \ldots, j - 2 \} \} > 0. \)
\( \Rightarrow \forall n \in \mathbb{N} \) with \( b_{n} - a_{n} < \delta: \nu(X \in I_{n}) \leq \frac{2}{j} < \varepsilon. \)

**Proof of Lemma 2.4:** The proof can be done in the same way as in [Karatzas and Shreve, 1997, p. 193]. For any \( A \in \mathcal{G} \) we have:

\[
\int_{A} Y \, d\nu_{1} = \int_{A} Y \, \frac{d\nu_{1}}{d\nu_{2}} \, d\nu_{2} = \int_{A} E_{\nu_{2}} \left[ Y \, \frac{d\nu_{1}}{d\nu_{2}} \right] \, d\nu_{2} = \int_{A} E_{\nu_{2}} \left[ E_{\nu_{2}} \left[ Y \, \frac{d\nu_{1}}{d\nu_{2}} \right] \right] \, d\nu_{2} = \int_{A} E_{\nu_{2}} \left[ Y \, \frac{d\nu_{1}}{d\nu_{2}} \right] \, d\nu_{2} = \int_{A} E_{\nu_{2}} \left[ Y \, \frac{d\nu_{1}}{d\nu_{2}} \right] \, d\nu_{2} = \int_{A} E_{\nu_{2}} \left[ Y \, \frac{d\nu_{1}}{d\nu_{2}} \right] \, d\nu_{2} = \int_{A} E_{\nu_{2}} \left[ Y \, \frac{d\nu_{1}}{d\nu_{2}} \right] \, d\nu_{2}.
\]

Since \( \frac{1}{E_{\nu_{2}} \left[ \frac{d\nu_{1}}{d\nu_{2}} \right] Y \, d\nu_{2} [G] \) is \( \mathcal{G} \)-measurable, the desired equation of Bayes' rule is true due to the definition of conditional expectations.

**Proof of Lemma 2.5:** \( \forall x < -\text{VaR}_{\nu}^{\nu}(X): \nu(X \leq x) \leq \varepsilon. \)
(continuity of \( \nu \) from below) \( \Rightarrow \nu(X < -\text{VaR}_{\nu}^{\nu}(X)) \leq \varepsilon. \)

(a) \( \Rightarrow: \nu(X < -\text{VaR}_{\nu}^{\nu}(X)) \leq \varepsilon \Rightarrow -\text{VaR}_{\nu}^{\nu}(X) \leq \sup \{ x \in \mathbb{R} | \nu(X < x) \leq \varepsilon \}. \)

(b) \( \Rightarrow: \sup \{ x \in \mathbb{R} | \nu(X < x) \leq \varepsilon \} \leq -\text{VaR}_{\nu}^{\nu}(X). \)

(b) \( \Rightarrow: \nu(X < q) \leq \nu(X < -\text{VaR}_{\nu}^{\nu}(X)) \leq \varepsilon. \)
Suppose \( \nu(X < q) \leq \varepsilon. \)
(continuity of \( \nu \) from below) \( \Rightarrow \nu(X \leq q) \leq \varepsilon. \)
\( \Rightarrow \text{VaR}_{\nu}^{\nu}(X) = -\inf \{ x \in \mathbb{R} | \nu(X \leq x) > \varepsilon \} = -\inf \{ x \in [q, +\infty) | \nu(X \leq x) > \varepsilon \} \leq -q. \)

**Proof of Proposition 2.6:**

(b) \( \forall h \geq 0: \)

\[
\text{ES}_{h}^{\nu}(X) + \text{ES}_{h}^{\nu}(Y) = E_{\nu} \left[ (h - X)^{+} + (h - Y)^{+} \right] \geq E_{\nu} \left[ ((h + h) - (X + Y))^{+} \right] \geq E_{\nu} \left[ (h - (X + Y))^{+} \right] = \text{ES}_{h}^{\nu}(X + Y).
\]
6.1. PART 2: RISK MEASURES

(d) \( ES_h^\nu (X) = \mathbb{E}_\nu \left[ (h - X)^+ \right] \geq \mathbb{E}_\nu [ (h - Y)^+ ] = ES_h^\nu (Y) \).

**Proof of Lemma 2.7:** We observe that

\[
\text{CVaR}_\nu^\alpha (X) = - \frac{\int_{X < (F_X^{-1})^{-1}(\alpha)}}{\alpha} X d\nu + (F_X^{-1})^{-1}(\alpha) \cdot \left( \alpha - \nu \left( \left\{ X < (F_X^{-1})^{-1}(\alpha) \right\} \right) \right)
\]

\[
= \frac{\int_{X < (F_X^{-1})^{-1}(\alpha)}}{\alpha} \left( (F_X^{-1})^{-1}(\alpha) - X \right) d\nu - (F_X^{-1})^{-1}(\alpha)
\]

and

\[
\text{CVaR}_\alpha^\nu (X) \leq -c \iff (F_X^{-1})^{-1}(\alpha) - \frac{\nu \left( (F_X^{-1})^{-1}(\alpha) - X \right)^+}{\alpha} \geq c
\]

\[
\iff (F_X^{-1})^{-1}(\alpha) \cdot \alpha - \mathbb{E}_\nu \left[ (F_X^{-1})^{-1}(\alpha) - X \right]^+ \geq c \cdot \alpha
\]

\[
\iff \mathbb{E}_\nu \left[ ((F_X^{-1})^{-1}(\alpha) - X)^+ \right] \leq (F_X^{-1})^{-1}(\alpha - c) \alpha.
\]

**Proof of Proposition 2.8:** We consider the two cases separately:

\( \forall h_1 \leq h_2 \leq (F_X^{-1})^{-1}(\alpha) \):

\[
\left[ \mathbb{E}_\nu \left[ (h_2 - X)^+ \right] - (h_2 - c) \alpha \right] - \left[ \mathbb{E}_\nu \left[ (h_1 - X)^+ \right] - (h_1 - c) \alpha \right]
\]

\[
= \mathbb{E}_\nu \left[ (h_2 - X)^+ - (h_1 - X)^+ \right] \mid_{=0 \text{ on } \{ X \geq (F_X^{-1})^{-1}(\alpha) \}} = (h_2 - h_1) \alpha
\]

\[
\leq (h_2 - h_1) \cdot \nu \left( \left\{ X < (F_X^{-1})^{-1}(\alpha) \right\} \right) \mid_{\geq 0} \leq \nu \left( \left\{ X < (F_X^{-1})^{-1}(\alpha) \right\} \right) \leq \nu \left( \left\{ X < (F_X^{-1})^{-1}(\alpha) \right\} \right) \leq \alpha
\]

\[
\leq 0,
\]
\[ \forall h_1 \geq h_2 \geq (F_X^{-1}(\alpha))^{-1}(\alpha) : \]
\[ \left[ E_\nu \left[ (h_2 - X)^+ - (h_2 - c) \alpha \right] - E_\nu \left[ (h_1 - X)^+ - (h_1 - c) \alpha \right] \right] \]
\[ = E_\nu \left[ (h_2 - X)^+ - (h_1 - X)^+ \right] \leq (h_2 - h_1) \alpha \]
\[ \leq (h_2 - h_1) \cdot \nu \left( \{ X \leq (F_X^{-1}(\alpha)) \} \right) - (h_2 - h_1) \alpha \]
\[ \leq 0. \]

**Proof of Theorem 2.9:** The statement is a consequence of the following considerations:

“\( \leq \)”: \( \text{CVaR}_\nu(X) \leq -c \) \[ \Rightarrow E_\nu \left[ ((F_X^{-1}(\alpha) - X)^+ \right] \leq ((F_X^{-1}(\alpha) - c) \alpha \]
and \( h < c \Rightarrow E_\nu \left[ (h - X)^+ \right] \geq 0 > (h - c) \alpha. \)

“\( \geq \)”: If \( E_\nu \left[ (h - X)^+ \right] \leq (h - c) \alpha \) is fulfilled for some \( h \in \mathbb{R} \), then \( E_\nu \left[ ((F_X^{-1}(\alpha) - X)^+ \right] \leq \]
\[ (F_X^{-1}(\alpha) - c) \alpha \] according to Proposition 2.8 and hence \( \text{CVaR}_\nu(X) \leq -c \) (Lemma 2.7).

**Proof of Proposition 2.10:** For \( \alpha = \nu(\Omega) \), all properties are implied by the linearity and monotonicity of the expectancy operator. Hence, it is sufficient to consider \( \alpha \in (0, \nu(\Omega)) \):

(a)
\[ \text{CVaR}_\nu(X + \beta) \]
\[ = -\frac{1}{\alpha} \cdot \left[ \int \{ X + \beta < (F_X^{-1}(\alpha)) \} \, (X + \beta) d\nu \right] \]
\[ + \left( (F_X^{-1}(\alpha)) \cdot \left( \alpha - \nu \left( \{ X + \beta < (F_X^{-1}(\alpha)) \} \right) \right) \right) \]
\[ \geq \text{CVaR}_\nu(X) - \beta. \]

(b) By investigating all four possible cases, it is an easy exercise to show that \( \forall a, c \in (-\infty, +\infty) \) and \( \forall b, d \in \mathbb{R} : (a - b)^+ + (c - d)^+ \geq ((a + c) - (b + d))^+ \).

Using \( a = (F_X^{-1}(\alpha)) \), \( b = X(\omega) \), \( c = (F_Y^{-1}(\alpha)) \) and \( d = Y(\omega) \) for all \( \omega \in \Omega \), the application of this inequality and the linearity and monotonicity of the expectancy operator yield

\[ E \left[ \left( (F_X^{-1}(\alpha)) - X \right)^+ \right] + E \left[ \left( (F_Y^{-1}(\alpha)) - Y \right)^+ \right] \]
\[ \geq E \left[ \left( (F_X^{-1}(\alpha) + (F_Y^{-1}(\alpha)) - (X + Y) \right)^+ \right]. \]
We note that
\[
E \left[ \left( (F_X^\nu)^{-1} (\alpha) + (F_Y^\nu)^{-1} (\alpha) \right) - (X + Y) \right] - \left( (F_X^\nu)^{-1} (\alpha) + (F_Y^\nu)^{-1} (\alpha) \right) \alpha \leq 0
\]

Proposition 2.8
\[
E \left[ \left( (F_{X+Y}^\nu)^{-1} (\alpha) - (X + Y) \right) \right] - \left( (F_{X+Y}^\nu)^{-1} (\alpha) \right) \alpha \leq 0
\]

Lemma 2.7
\[
\alpha \cdot \text{CVaR}_\alpha^\nu (X + Y).
\]

(Lemma 2.7) \( \Rightarrow \) \( \text{CVaR}_\alpha^\nu (X + Y) \leq \text{CVaR}_\alpha^\nu (X) + \text{CVaR}_\alpha^\nu (Y) \).

(c) If \( \lambda > 0 \),
\[
\text{CVaR}_\alpha^\nu (\lambda \cdot X) \overset{\text{Lemma 2.7}}{=} E_{\nu} \left[ \left( (F_{X+\lambda X}^\nu)^{-1} (\alpha) - \lambda \cdot X \right) \right] - (F_{\lambda \cdot X}^\nu)^{-1} (\alpha)
\]

Lemma 2.2(e)
\[
\overset{\text{Lemma 2.7}}{=} E_{\nu} \left[ \left( \lambda \cdot (F_X^\nu)^{-1} (\alpha) - \lambda \cdot X \right) \right] - \lambda \cdot (F_X^\nu)^{-1} (\alpha)
\]

Lemma 2.7
\[
\lambda \cdot \text{CVaR}_\alpha^\nu (X).
\]

If \( \lambda = 0 \):
\[
(F_0^\nu = 1_{[0, +\infty]}) \Rightarrow (F_0^\nu)^{-1} (\alpha) = 0.
\]
\( \Rightarrow \) \( \text{CVaR}_\alpha^\nu (\lambda \cdot X) = \text{CVaR}_\alpha^\nu (0) \) \( \overset{\text{Lemma 2.7}}{=} E_{\nu} \left[ \left( (F_0^\nu)^{-1} (\alpha) - 0 \right) \right] - (F_0^\nu)^{-1} (\alpha) = 0. \)

(d)
\[
\text{CVaR}_\alpha^\nu (X) \overset{\text{Lemma 2.7}}{=} E_{\nu} \left[ \left( (F_X^\nu)^{-1} (\alpha) - X \right) \right] - (F_X^\nu)^{-1} (\alpha)
\]

Proposition 2.8
\[
\overset{\text{Lemma 2.7}}{=} E_{\nu} \left[ \left( (F_Y^\nu)^{-1} (\alpha) - Y \right) \right] - (F_Y^\nu)^{-1} (\alpha) \leq 0
\]

\( \overset{\text{Lemma 2.7}}{=} \text{CVaR}_\alpha^\nu (Y) \).

Proof of Lemma 2.11: Let \( (\alpha_n) \) with \( \alpha_n \in (0, \nu (\Omega)) \) be a monotonic sequence with limit \( \alpha \in (0, \nu (\Omega)) \).

- If \( \lim_{n \to +\infty} (F_X^\nu)^{-1} (\alpha_n) = (F_X^\nu)^{-1} (\alpha) \), a straightforward application of Lemma 2.7 can be used to verify that \( \lim_{n \to +\infty} \text{CVaR}_{\alpha_n}^\nu (X) = \text{CVaR}_\alpha^\nu (X) \).

- If \( \lim_{n \to +\infty} (F_X^\nu)^{-1} (\alpha_n) \neq (F_X^\nu)^{-1} (\alpha) \), the left-continuity of the quantile function tells us that \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha \) and as a consequence \( (F_X^\nu)^{-1} (\alpha_1) \geq (F_X^\nu)^{-1} (\alpha_2) \geq \ldots \geq (F_X^\nu)^{-1} (\alpha) \).

Let \( l := \lim_{n \to +\infty} (F_X^\nu)^{-1} (\alpha_n) \). Hence \( (F_X^\nu)^{-1} (\alpha) < l \) and thus
\[
\alpha \leq F_X^\nu \left( (F_X^\nu)^{-1} (\alpha) \right) \leq \nu (X < l) \leq \nu \left( X < (F_X^\nu)^{-1} (\alpha_n) \right) \leq \alpha_n \to \alpha \ (n \to +\infty).
\]
Proof of Lemma 2.14: We start out with the obvious:

\[ \Rightarrow \nu(X < l) = F_X^\nu((F_X^\nu)^{-1}(\alpha)) = \alpha. \]

Therefore \( \mathbb{E}_\nu [(l - X)^+] = \mathbb{E}_\nu \left[ ((F_X^\nu)^{-1}(\alpha) - X)^+ \right] + (l - (F_X^\nu)^{-1}(\alpha)) \cdot \alpha. \)

Again, we can use Lemma 2.7 to verify that \( \lim_{n \to +\infty} \text{CVaR}_\nu^{(n)}(X) = \text{CVaR}_\nu^{(n)}(X). \)

\[ \square \]

Proof of Lemma 2.13: \( \text{CVaR}_\nu^{(n)}(X) = \text{CVaR}_\nu^{(n)}(X), \) because \( \Omega \in \mathcal{F} \) and the value of the integral does not change by omitting a set of measure 0.

Let us suppose that \( \alpha \in (0, \nu(\Omega)) \).

\[
\forall A \in \mathcal{F} \text{ with } \nu(A) \geq \alpha:
\]

\[
\int \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} X d\nu(A) - \int A X d\nu(A)
\]

\[
= \int \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} \left( \frac{X}{(F_X^\nu)^{-1}(\alpha)} \right) \cdot \nu(A) \cdot \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} d\nu(A)
\]

\[
- \int \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} \left( \frac{X}{(F_X^\nu)^{-1}(\alpha)} \right) \cdot \nu(A) \cdot \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} d\nu(A)
\]

\[
\leq (F_X^\nu)^{-1}(\alpha) \left[ \nu \left( A \cap \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} \right) \right] \cdot \nu(A)
\]

\[
- \left[ \nu \left( A \cap \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} \right) \right] \cdot \nu(A)
\]

and thus \( -\text{CVaR}_\nu(X) \cdot \nu(A) \leq \int A X d\nu(A). \)

\[ \square \]

Proof of Lemma 2.14: We start out with the obvious: \( B \cup \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} \) is an element of \( \mathcal{F} \) and \( \nu \left( B \cup \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} \right) = \nu(B) + \nu \left( X < (F_X^\nu)^{-1}(\alpha) \right) \).

Due to

\[
\int_{B \cup \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\}} X d\nu = \int \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\} X d\nu + (F_X^\nu)^{-1}(\alpha) \cdot \nu(B),
\]

we can conclude that

\[
\text{CVaR}_\nu(X) = -\frac{\int_{B \cup \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\}} X d\nu}{\nu(B)} \quad \text{and hence} \quad -\frac{\int_{B \cup \left\{ X < (F_X^\nu)^{-1}(\alpha) \right\}} X d\nu}{\nu(B)} \leq \text{CVaR}_\nu(X) \leq \text{CVaR}_\nu(X).
\]

\[ \square \]

6.2 Part 3: Static Optimization Problems with Risk Constraints

Proof of Fact 3.2:

(a) \( P \left( \frac{dQ}{dP} < t_{\min} \right) = \lim_{x \to t_{\min}^-} P \left( \frac{dQ}{dP} \leq x \right) = 0. \)

(b) If \( \varepsilon > 0 \) with \( P \left( \frac{dQ}{dP} \in [t_{\min}, t_{\min} + \varepsilon] \right) = 0, \) (a) results in \( 0 \leq P \left( \frac{dQ}{dP} \leq t_{\min} + \frac{1}{2} \varepsilon \right) \leq \)
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

\[ P \left( \frac{dQ}{dP} < t_{\min} + \varepsilon \right) = 0 \] and therefore \[ F_X^P \left( t_{\min} + \frac{1}{2} \varepsilon \right) = 0 \] contradicting the definition of \( t_{\min} \).

(c) \[ 0 \leq P \left( \frac{dQ}{dP} > t_{\max} \right) = 1 - F_X^P \left( F_X^{-1}^P (1) \right) \leq 1 - 1 = 0. \]

(d) \[ Q (\Omega) = \int_{\frac{dQ}{dP} \leq t_{\max}} \frac{dQ}{dP} dP \leq \int_{\frac{dQ}{dP} = t_{\max}} t_{\max} dP = t_{\max}. \]

(e) \[ P \left( \frac{dQ}{dP} < t_{\max} \right) > 0 \] implies that \( t_{\max} > \int_{\frac{dQ}{dP} \leq t_{\max}} \frac{dQ}{dP} dP = Q (\Omega) \) and \( P \left( \frac{dQ}{dP} < t_{\max} \right) = 0 \) implies that \( t_{\max} > \int_{\frac{dQ}{dP} = t_{\max}} \frac{dQ}{dP} dP = Q (\Omega). \)

(f) If \( \varepsilon > 0 \) with \( P \left( \frac{dQ}{dP} \in (t_{\max} - \varepsilon, t_{\max}) \right) = 0 \), (c) results in \( P \left( \frac{dQ}{dP} > t_{\max} - \varepsilon \right) = 0 \) and therefore \( F_X^{-1}^P (1) \leq t_{\max} - \varepsilon \) contradicting the definition of \( t_{\max}. \)

\[ \square \]

Proof of Proposition 3.21: The statement is trivially true for \( \epsilon_P = 1 \). Hence, we can assume that \( \epsilon_P < 1 \) which implies \( F_X^{-1}^P (1 - \epsilon_P) \geq 0. \)

Let \( +\infty \cdot 0 := 0. \)

\[ Q (A) - \epsilon_Q \]
\[ = \int_A \frac{dQ}{dP} dP - \int \left\{ \frac{dQ}{dP} > F_X^{-1}^P (1 - \epsilon_P) \right\} \frac{dQ}{dP} dP \]
\[ = \int_{A \cap \left\{ \frac{dQ}{dP} \leq F_X^{-1}^P (1 - \epsilon_P) \right\}} \frac{dQ}{dP} dP - \int \left\{ \frac{dQ}{dP} > F_X^{-1}^P (1 - \epsilon_P) \right\} \frac{dQ}{dP} dP \]
\[ \leq F_X^{-1}^P (1 - \epsilon_P) \cdot \left[ P \left( A \cap \left\{ \frac{dQ}{dP} \leq F_X^{-1}^P (1 - \epsilon_P) \right\} \right) - P \left( \left\{ \frac{dQ}{dP} > F_X^{-1}^P (1 - \epsilon_P) \right\} \right) \right] \]
\[ = F_X^{-1}^P (1 - \epsilon_P) \cdot \left[ \underbrace{P (A) - P \left( \left\{ \frac{dQ}{dP} > F_X^{-1}^P (1 - \epsilon_P) \right\} \right)}_{\leq \epsilon_P} \right] \left[ \underbrace{\left( \left\{ \frac{dQ}{dP} \leq F_X^{-1}^P (1 - \epsilon_P) \right\} \right)}_{\lep 0} \right] \leq 0. \]

\[ \square \]

Proof of Proposition 3.30: Let us consider the case \( q = u_0 \) first:

In this case \( X \geq q \) and therefore we get the inequality

\[ \mathbb{E}_Q [X] \geq q \cdot Q (\Omega) \]
\[ = q \cdot Q \left( \frac{dQ}{dP} \leq \left( F_X^{-1}^Q \right)^{-1} (\nu (\Omega) - \epsilon) \right) + u_0 \cdot Q \left( \frac{dQ}{dP} > \left( F_X^{-1}^Q \right)^{-1} (\nu (\Omega) - \epsilon) \right). \]

Furthermore, if \( \mathbb{E}_Q [X] = q \cdot Q (\Omega) \) in this case, we can deduce immediately that \( X = q \) \( (P\text{-almost-surely}) \). Hence, we can assume that \( q > u_0 \) for the remainder of the proof.
Case $ν = Q$: Since $Q(X < q) \leq ε$, $Q(X \geq q) \geq Q(Ω) - ε$ and thus

$$E_Q[X] \geq (q - u_0) \cdot Q(X \geq q) + u_0 \cdot Q(Ω) \geq q \cdot (Q(Ω) - ε) + u_0 \cdot ε \geq q \cdot \left(\frac{dQ}{dP}\right)^{-1}(Q(Ω) - ε) + u_0 \cdot Q \left(\frac{dQ}{dP} > \left(\frac{dQ}{dP}\right)^{-1}(Q(Ω) - ε)\right).$$

If $Q(X > q) > 0$ or $Q(u_0 < X < q) > 0$, ($*$) is a strict inequality and ($**$) is a strict inequality if $Q(X = q) > Q(Ω) - ε$. Finally, $Q(X = u_0) \leq Q(X < q) = ν(X < q) \leq ε$.

Case $ν = P$:

$$\inf_{x \in [u_0, +∞)} \left\{ \frac{dQ}{dP}(ω) \cdot x + (q - u_0) \cdot F_{ω}^{-1}(1 - ε) \cdot 1_{(−∞, q)}(x) \right\} = \min \left\{ \inf_{x \in [u_0, q)} \left\{ \frac{dQ}{dP}(ω) \cdot x + (q - u_0) \cdot F_{ω}^{-1}(1 - ε) \right\}, \inf_{x \in [q, +∞)} \left\{ \frac{dQ}{dP}(ω) \cdot x \right\} \right\}$$

Thus $\begin{cases} q & \text{if } \frac{dQ}{dP}(ω) \leq F_{ω}^{-1}(1 - ε) \\ u_0 & \text{if } \frac{dQ}{dP}(ω) > F_{ω}^{-1}(1 - ε) \end{cases}$ minimizes the function $x \mapsto \frac{dQ}{dP}(ω) \cdot x + (q - u_0) \cdot F_{ω}^{-1}(1 - ε) \cdot 1_{(−∞, q)}(x)$ over the set $[u_0, +∞)$. Note that $∀ω \in Ω$ with $\frac{dQ}{dP}(ω) \neq F_{ω}^{-1}(1 - ε)$, the minimum is attained by a unique $x \in [u_0, +∞)$.

(Assumption 3.11) $⇒ P\left(\frac{dQ}{dP} = F_{ω}^{-1}(1 - ε)\right) = 0$.

Let $Y := q \cdot 1_{(0, F_{ω}^{-1}(1 - ε))} \left(\frac{dQ}{dP}\right) + u_0 \cdot 1_{(F_{ω}^{-1}(1 - ε), +∞)} \left(\frac{dQ}{dP}\right)$.

$$⇒ E\left[\frac{dQ}{dP}(ω) \cdot Y + (q - u_0) \cdot F_{ω}^{-1}(1 - ε) \cdot 1_{(−∞, q)}(Y)\right] \leq E\left[\frac{dQ}{dP}(ω) \cdot X + (q - u_0) \cdot F_{ω}^{-1}(1 - ε) \cdot 1_{(−∞, q)}(X)\right] \leq E_Q[Y] \quad (**) \leq E_Q[X] + (q - u_0) \cdot F_{ω}^{-1}(1 - ε) \cdot P(X < q) - P(Y < q) \leq 0 \leq 0 \leq E_Q[X] = E_Q[Y].$$

If $E_Q[X] = Q\left(\frac{dQ}{dP} \leq F_{ω}^{-1}(1 - ε)\right) \cdot q + Q\left(\frac{dQ}{dP} > F_{ω}^{-1}(1 - ε)\right) \cdot u_0 = E_Q[Y]$ as well as $P(X \neq Y) > 0$, then ($*$) and ($**$) are strict inequalities yielding the contradiction $E_Q[Y] < E_Q[X] = E_Q[Y]$.

$\square$
Proof of Proposition 3.31: Let $J$ be defined as $\left(F_{\frac{Q}{P}}^{-1}(Q(\Omega) - \epsilon)\right)^{-1}$. Let $X$ be a random variable with the properties $Q(X = q) = Q(\Omega) - \epsilon$, $Q(X = w_0) = \epsilon$ and $P(X \neq X^*) > 0$. Then
\[
\mathbb{E}[U(X^*)] - \mathbb{E}[U(X)] = (U(q) - U(w_0)) \int_{\{\frac{dQ}{dP} \leq 1\cap\{X \neq q\}} - \int_{\{\frac{dQ}{dP} > 1\cap\{X = q\}}\frac{1}{dQ} dQ - \int_{\{\frac{dQ}{dP} \leq 1\cap\{X = q\}}\frac{1}{dQ} dQ}
\]
Therefore $X$ is not an optimal solution.

Proof of Lemma 3.32:

- Let $d_1 \in (0, +\infty)$ and $d_2 \in (d_1, +\infty)$.  
  
  \[\Rightarrow (U')^{-1}(\lambda_1 \cdot d_2) < (U')^{-1}(\lambda_1 \cdot d_1).\]
  
  (Mean Value Theorem) \[\Rightarrow \exists \xi \in \left((U')^{-1}(\lambda_1 \cdot d_2), (U')^{-1}(\lambda_1 \cdot d_1)\right):\]
  \[
  \lambda_1 \cdot d_1 \cdot \left((U')^{-1}(\lambda_1 \cdot d_1) - (U')^{-1}(\lambda_1 \cdot d_2)\right) < \frac{U'(\xi)}{U''(U')^{-1}(\lambda_1 \cdot d_1)} \cdot \left((U')^{-1}(\lambda_1 \cdot d_1) - (U')^{-1}(\lambda_1 \cdot d_2)\right) > 0 \]
  
  \[= U \left((U')^{-1}(\lambda_1 \cdot d_1)\right) - U \left((U')^{-1}(\lambda_1 \cdot d_2)\right) < 0.
  
  - Let $d_2 \in \left(0, \frac{U'(q)}{\lambda_1}\right]$ and $d_1 \in (0, d_2)$.  
  
  Hence
  \[\tilde{f}^{\lambda_1}(d_1) = U \left((U')^{-1}(\lambda_1 \cdot d_1)\right) + d_1 \cdot \lambda_1 \left(q - (U')^{-1}(\lambda_1 \cdot d_1)\right) \]
  
  \[> U \left((U')^{-1}(\lambda_1 \cdot d_2)\right) + d_1 \cdot \lambda_1 \left(q - (U')^{-1}(\lambda_1 \cdot d_2)\right) \leq 0 \]
  
  \[\geq U \left((U')^{-1}(\lambda_1 \cdot d_2)\right) + d_2 \cdot \lambda_1 \left(q - (U')^{-1}(\lambda_1 \cdot d_2)\right) = \tilde{f}^{\lambda_1}(d_2).
  
  - Let $d_1 \in \left[\frac{U'(q)}{\lambda_1}, +\infty\right)$ and $d_2 \in (d_1, +\infty)$. 
  
  \[\tilde{f}^{\lambda_1}(d_1) = U \left((U')^{-1}(\lambda_1 \cdot d_1)\right) + d_1 \cdot \lambda_1 \left(q - \left(U'\right)^{-1}(\lambda_1 \cdot d_1)\right) \]
  
  Therefore $X$ is not an optimal solution.
Hence
\[
\tilde{f}^\lambda_1 (d_1) = U \left( (U')^{-1} \left( \lambda_1 \cdot d_1 \right) \right) + d_1 \cdot \lambda_1 \left( q - (U')^{-1} \left( \lambda_1 \cdot d_1 \right) \right) \geq 0
\]
\[
\leq U \left( (U')^{-1} \left( \lambda_1 \cdot d_1 \right) \right) + d_2 \cdot \lambda_1 \left( q - (U')^{-1} \left( \lambda_1 \cdot d_1 \right) \right)
\]
\[
< U \left( (U')^{-1} \left( \lambda_1 \cdot d_2 \right) \right) + d_2 \cdot \lambda_1 \left( q - (U')^{-1} \left( \lambda_1 \cdot d_2 \right) \right) = \tilde{f}^\lambda_1 (d_2).
\]

\[\square\]

**Proof of Lemma 3.33:** We realize that \( \forall x < u_0: f(x) = -\infty, f \) restricted to \([u_0, +\infty) \setminus \{q\}\) is continuous and \( \forall x \in (u_0, +\infty) \setminus \{q\}: f'(x) = U'(x) - \lambda_1 \cdot d. \)

\[
\lim_{x \to u_0^+} f'(x) = +\infty \quad \text{and} \quad \lim_{x \to q^+} f'(x) = U'(q) - \lambda_1 \cdot d = \lim_{x \to q^+} f'(x).
\]

Since \( U' \) is strictly decreasing on \((u_0, +\infty), f' \) is strictly decreasing on \((u_0, +\infty) \setminus \{q\}\).

- \( U'(q) \geq \lambda_1 \cdot d; \)
  \( (U')^{-1} \left( \lambda_1 \cdot d \right) \) is the unique value maximizing \( f \).
- \( U'(q) < \lambda_1 \cdot d; \)
  The value \( \arg\max_{x \in \{ (U')^{-1} (\lambda_1 \cdot d), q \} \} \{ f(x) \} \) maximizes \( f \).
  \begin{itemize}
    \item Case \( F_{d_0}^{-1} (1 - \epsilon) < \frac{U'(q)}{\lambda_1} \):
      \[
      f \left( (U')^{-1} \left( \lambda_1 \cdot d \right) \right) = \tilde{f}^\lambda_1 (d) - \lambda_1 \cdot q \cdot d - l^P_2 (\lambda_1) = 0
      \]
      \[
      \Rightarrow \text{Only } (U')^{-1} (\lambda_1 \cdot d) \text{ minimizes } f.
      \]
    \item Case \( F_{d_0}^{-1} (1 - \epsilon) \geq \frac{U'(q)}{\lambda_1} \):
      \[
      d > F_{d_0}^{-1} (1 - \epsilon) \iff \tilde{f}^\lambda_1 (d) > \tilde{f}^\lambda_1 \left( F_{d_0}^{-1} (1 - \epsilon) \right) = l^P_2 (\lambda_1) + U(q)
      \]
      \[
      \iff f \left( (U')^{-1} \left( \lambda_1 \cdot d \right) \right) + l^P_2 (\lambda_1) + \lambda_1 \cdot q \cdot d > l^P_2 (\lambda_1) + U(q)
      \]
      \[
      \iff f \left( (U')^{-1} \left( \lambda_1 \cdot d \right) \right) > f(q).
      \]
  \end{itemize}

The same argument yields \( f \left( (U')^{-1} \left( \lambda_1 \cdot d \right) \right) = f(q) \iff d = F_{d_0}^{-1} (1 - \epsilon). \)

\[\square\]

**Proof of Lemma 3.34:** We observe that \( \forall x < u_0: f(x) = -\infty, f \) restricted to \([u_0, +\infty) \setminus \{q\}\) is continuous and \( \forall x \in (u_0, +\infty) \setminus \{q\}: f'(x) = U'(x) - \lambda_1 \cdot d. \)

\[
\lim_{x \to u_0^+} f'(x) = +\infty \quad \text{and} \quad \lim_{x \to q^+} f'(x) = U'(q) - \lambda_1 \cdot d = \lim_{x \to q^+} f'(x).
\]

Since \( U' \) is strictly decreasing on \((u_0, +\infty), f' \) is strictly decreasing on \((u_0, +\infty) \setminus \{q\}\).

- \( U'(q) \geq \lambda_1 \cdot d; \)
  \( (U')^{-1} (\lambda_1 \cdot d) \) is the unique value maximizing \( f \).
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

- $U'(q) < \lambda_1 \cdot d$:
  The value $\arg\max_{x \in \{(U')^{-1}(\lambda_1 \cdot d), q\}} \{f(x)\}$ maximizes $f$.

  - Case $\left(\frac{F_{\frac{Q}{d}}}{d^\alpha}\right)^{-1} (Q(\Omega) - \epsilon) < \frac{U'(q)}{\lambda_1}$.

    $$f \left( (U')^{-1} (\lambda_1 \cdot d) \right) = \tilde{f}^{\lambda_1} (d) - \lambda_1 \cdot q \cdot d - \ell_2^Q (\lambda_1) \cdot d$$

    Lemma 3.32
    $$\begin{align*}
    \tilde{f}^{\lambda_1} (\frac{U'(q)}{\lambda_1}) - \lambda_1 \cdot q \cdot d &= U(q) - \lambda_1 \cdot q \cdot d = f(q). \\
    \Rightarrow \text{Only } (U')^{-1} (\lambda_1 \cdot d) \text{ minimizes } f.
    \end{align*}$$

  - Case $\left(\frac{F_{\frac{Q}{d}}}{d^\alpha}\right)^{-1} (Q(\Omega) - \epsilon) \geq \frac{U'(q)}{\lambda_1}$.

    The illustration uses $U = U_0$, $\lambda_1 = 2$, $\left(\frac{F_{\frac{Q}{d}}}{d^\alpha}\right)^{-1} (Q(\Omega) - \epsilon) = 2$ and $q = 1$.

![Figure 6.1: Lagrange function $(P_{VaRQ})$: discussion of the dependence on $d$.](image)

* The mapping $d \mapsto \lambda_1 \cdot q - \lambda_1 \cdot (U')^{-1} (\lambda_1 \cdot d) - \ell_2^Q (\lambda_1)$ is continuous, strictly increasing
Proof of Fact 3.35:

- Let \( \nu \left( X_{\text{VaR}} \right) \cdot \left( F_{d^q}^{\nu} \right)^{-1}(\nu(\Omega) - \epsilon) < q \) and \( \nu(\Omega) - \epsilon < \epsilon \).

(Assumption 3.11) \( (U')^{-1} \left( \lambda_1 \cdot \left( F_{d^q}^{\nu} \right)^{-1}(\nu(\Omega) - \epsilon) \right) \geq q \).
Proof of Lemma 3.36:

- Case $E \left[ \left( U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right) \right] < +\infty$:
  
  Due to $X_{\text{Var}}^{\lambda_1, J} \leq \max \left\{ q, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right\}$,
  
  $E \left[ \left( U \left( X_{\text{Var}}^{\lambda_1, J} \right) \right) \right] \leq E \left[ \max \left\{ \left( U(q) \right)^+, \left( U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \right\} \right] \leq \underbrace{E \left[ \left( U(q) \right)^+ \right]}_{=1-\left( U(q) \right)^+ \in \mathbb{R}} + \underbrace{E \left[ \left( U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \right]}_{< +\infty} < +\infty.$

- Case $E \left[ \left( U \left( X_{\text{Var}}^{\lambda_1, J} \right) \right)^+ \right] < +\infty$:
  
  $(U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \leq X_{\text{Var}}^{\lambda_1, J} \Rightarrow E \left[ \left( U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \right] \leq E \left[ \left( U \left( X_{\text{Var}}^{\lambda_1, J} \right) \right)^+ \right] < +\infty.$

- Case $E \left[ \left( -U \left( X_{\text{Var}}^{\lambda_1, J} \right) \right)^+ \right] < +\infty$:
  
  Since $(U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \geq \min \left\{ X_{\text{Var}}^{\lambda_1, J}, (U')^{-1} \left( \lambda_1 \cdot J \right) \right\}$, we observe that
  
  $\left( -U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \leq \max \left\{ -U \left( X_{\text{Var}}^{\lambda_1, J} \right)^+, -U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot J \right) \right)^+ \right\} \leq -U \left( X_{\text{Var}}^{\lambda_1, J} \right)^+ + \left( -U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot J \right) \right) \right)^+$
  
  and therefore
  
  $E \left[ \left( -U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \right] \leq \underbrace{E \left[ \left( -U \left( X_{\text{Var}}^{\lambda_1, J} \right) \right)^+ \right]}_{< +\infty} + \underbrace{E \left[ \left( -U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot J \right) \right) \right)^+ \right]}_{=1-\left( U \left( \left( U' \right)^{-1} \left( \lambda_1 \cdot J \right) \right) \right)^+ \in \mathbb{R}} < +\infty.$
Proof of Lemma 3.37: For all $\omega \in \Omega$ we have:

$$u_0 \leq (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{d\tau}(\omega) \right) \overset{(*)}{=} X_{\text{VaR}}^{\lambda_1,J}(\omega) \leq q - u_0 + (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{d\tau}(\omega) \right).$$

In addition, $E_Q[|u_0|] = |u_0| \cdot Q(\Omega) < +\infty$.

1. $E_Q \left[ X_{\text{VaR}}^{\lambda_1,J} \right] < +\infty \Rightarrow E_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{d\tau} \right) \right] < +\infty$.

2. Suppose $E_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{d\tau} \right) \right] < +\infty$.

$$\Rightarrow E_Q \left[ q - u_0 + (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{d\tau} \right) \right] = (q - u_0) \cdot Q(\Omega) + E_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{d\tau} \right) \right] < +\infty.$$

Proof of Lemma 3.38:

1. (Fact 3.35) $\Rightarrow \forall \omega \in \Omega$: $\lambda_1 \mapsto X_{\text{VaR}}^{\lambda_1,J}(\omega)$ is monotonic decreasing.

Furthermore, $\forall \omega \in \Omega$ with $\frac{dQ}{d\tau}(\omega) > J$: $\lambda_1 \mapsto X_{\text{VaR}}^{\lambda_1,J}(\omega)$ is strictly decreasing.

Since $P \left( \frac{dQ}{d\tau} > J \right) > 0$ and $P \sim Q$: $Q \left( \frac{dQ}{d\tau} > J \right) > 0$.

$\Rightarrow \lambda_1 \mapsto E_Q \left[ X_{\text{VaR}}^{\lambda_1,J} \right]$ is strictly decreasing.

2. Because of Fact 3.35, Assumption 3.19, Lemma 3.37 and the Monotone Convergence Theorem, the mapping $\lambda_1 \mapsto E_Q \left[ X_{\text{VaR}}^{\lambda_1,J} \right]$ is continuous.

3. $E_Q \left[ X_{\text{VaR}}^{\lambda_1,J} \right] = q \cdot Q \left( \frac{dQ}{d\tau} \leq J \right) + u_0 \cdot Q \left( \frac{dQ}{d\tau} > J \right)$.

(Fact 3.35) $\Rightarrow \forall \omega \in \Omega$: $\lim_{\lambda_1 \to 0^+} X_{\text{VaR}}^{\lambda_1,J}(\omega) = +\infty$.

(Monotone Convergence Theorem) $\Rightarrow \lim_{\lambda_1 \to 0^+} E_Q \left[ X_{\text{VaR}}^{\lambda_1,J} \right] = +\infty$.

$\Rightarrow \lambda_1 \mapsto E_Q \left[ X_{\text{VaR}}^{\lambda_1,J} \right]$ is surjective.

Proof of Lemma 3.40: The statements are immediate consequences of the definition of $\tau$ and the following observations:

1. $c = u_0$:
   $\tau$ is a constant function.

2. $c > u_0$:
   $P \left( \frac{dQ}{d\tau} < \tau(h) \right) = 1 - \alpha \cdot \frac{h-c}{h-u_0} = (1 - \alpha) + \alpha \cdot \frac{c-u_0}{h-u_0}$ is continuous and monotone decreasing in $h$.

3. If $h = c$, we know that $c > u_0$ implies $\tau(h) = +\infty$ and $c = u_0$ implies $h = c = u_0$.

Proof of Lemma 3.41: If $c = u_0$, the function is constant and hence continuous. Hence, we can assume that $c > u_0$.

Let $(h_n)$ be a sequence in $[c, +\infty)$ with limit $h \in [c, +\infty)$.

Then $\exists n \in \mathbb{N}$ such that $h_n \geq h$ and $\forall n \in \mathbb{N}$: $h \geq h_n$.

(Lemma 3.40(b)) $\Rightarrow h \mapsto \tau(h)$ is monotone decreasing.
Let \((h_n)\) be a monotonic increasing sequence converging to \(h\).

\((\tau \text{ is monotonic decreasing}) \implies \tau(h_n) \text{ is monotonic decreasing, } \exists \lim_{n \to \infty} \tau(h_n) \text{ and } \lim_{n \to \infty} \tau(h_n) \geq \tau(h)\).

\((h \mapsto P\left(\frac{dQ}{dP} < \tau(h)\right) \text{ is continuous}) \implies \lim_{n \to \infty} P\left(\tau(h) \leq \frac{dQ}{dP} < \tau(h_n)\right) = 0.\)

\((\tau(h_n) \geq \lim_{n \to \infty} \tau(h_n)) \implies \left[\tau(h), \lim_{n \to \infty} \tau(h_n)\right] \subseteq [\tau(h), \tau(h_n)) \implies P\left(\tau(h), \lim_{n \to \infty} \tau(h_n)\right) = 0.\)

\(P \sim Q \implies Q\left(\left[\tau(h), \lim_{n \to \infty} \tau(h_n)\right]\right) = 0.\)

\(P \sim Q \implies Q\left(\left[\lim_{n \to \infty} \tau(h_n), \tau(h)\right]\right) = 0.\)

\((\tau \text{ is monotonic decreasing}) \implies \tau(h_n) \text{ is monotonic increasing, } \exists \lim_{n \to \infty} \tau(h_n) \text{ and } \lim_{n \to \infty} \tau(h_n) \leq \tau(h)\).

\((h \mapsto P\left(\frac{dQ}{dP} < \tau(h)\right) \text{ is continuous}) \implies \lim_{n \to \infty} P\left(\tau(h_n), \tau(h)\right) = 0.\)

\((\tau(h_n) \leq \lim_{n \to \infty} \tau(h_n)) \implies \left[\lim_{n \to \infty} \tau(h_n), \tau(h)\right] \subseteq [\tau(h_n), \tau(h)) \implies P\left(\lim_{n \to \infty} \tau(h_n), \tau(h)\right) = 0.\)

\(P \sim Q \implies Q\left(\left[\lim_{n \to \infty} \tau(h_n), \tau(h)\right]\right) = 0.\)

\(\lim_{n \to \infty} Q\left(\left[\tau(h_n), \tau(h)\right]\right) = 0 \implies \lim_{n \to \infty} Q\left(\frac{dQ}{dP} < \tau(h_n)\right) = Q\left(\frac{dQ}{dP} < \tau(h)\right).\)

Hence, the mapping \([c, +\infty) \to [0, Q(\Omega)], h \mapsto Q\left(\frac{dQ}{dP} < \tau(h)\right)\) is continuous. \(\square\)

**Proof of Proposition 3.42:** If \(\tau(h) = +\infty\), we learn from Lemma 3.40(a) that \(h = c\) which implies \(X \geq h\) (\(P\)-almost-surely) and thus \(E_Q[X] \geq Q(\Omega) \cdot h + Q(\emptyset) \cdot u_0\) as well as \(E_Q[X] = Q(\Omega) \cdot h \Rightarrow X = h\) (\(P\)-almost-surely).

Hence, we can assume that \(\tau(h) < +\infty\).

\[
\inf_{x \in [u_0, +\infty)} \left\{ \frac{dQ}{dP}(\omega) \cdot x + \tau(h) \cdot (h-x)^+ \right\} = \min \left\{ \inf_{x \in [u_0, h)} \left\{ \frac{dQ}{dP}(\omega) \cdot x + \tau(h) \cdot (h-x) \right\}, \inf_{x \in (h, +\infty)} \left\{ \frac{dQ}{dP}(\omega) \cdot x \right\} \right\}
\]

\[
= \min \left\{ \left\{ \frac{dQ}{dP}(\omega) \cdot h \right\} \begin{cases} \tau(h) \cdot h & \text{if } \frac{dQ}{dP}(\omega) < \tau(h) \\ \frac{dQ}{dP}(\omega) \cdot u_0 + \tau(h) \cdot (h-u_0) & \text{if } \frac{dQ}{dP}(\omega) = \tau(h) \\ \frac{dQ}{dP}(\omega) \cdot h & \text{if } \frac{dQ}{dP}(\omega) > \tau(h) \end{cases} \right\}
\]

117
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

\( \Rightarrow \left\{ \begin{array}{ll} h & \text{if } \frac{dQ}{dP}(\omega) < \tau (h) \\ u_0 & \text{if } \frac{dQ}{dP}(\omega) \geq \tau (h) \end{array} \right. \) minimizes \( \frac{dQ}{dP}(\omega) \cdot x + \tau (h) \cdot (h - x)^{+} \) over the set \([u_0, +\infty)\).

Note that \( \forall \omega \in \Omega \) with \( \frac{dQ}{dP}(\omega) \neq \tau (h) \), the minimum is attained by a unique \( x \in [u_0, +\infty) \).

Let \( Y := h \cdot \mathbb{1}_{(0, \tau (h))} \left( \frac{dQ}{dP} \right) + u_0 \cdot \mathbb{1}_{[\tau (h), +\infty)} \left( \frac{dQ}{dP} \right) \).

\[
\Rightarrow \mathbb{E} \left[ \frac{dQ}{dP}(\omega) \cdot Y + \tau (h) \cdot (h - Y)^{+} \right] \leq \mathbb{E} \left[ \frac{dQ}{dP}(\omega) \cdot X + \tau (h) \cdot (h - X)^{+} \right]
\]

\[
\Rightarrow \mathbb{E} [Y] \overset{(**)\text{}}{\leq} \mathbb{E} [X] + \tau (h) \left[ \mathbb{E} [X] \leq \mathbb{E} [(h - X)^{+}] \right. - \left. \mathbb{E} [(h - Y)^{+}] \right]_{\geq 0}^{\leq 0}
\]

\[\leq \mathbb{E} [X]\]

If \( \mathbb{E} [X] = Q \left( \frac{dQ}{dP} < \tau (h) \right) \cdot h + Q \left( \frac{dQ}{dP} \geq \tau (h) \right) \cdot u_0 = \mathbb{E} [Y] \) and \( P (X \neq Y) > 0 \), then (**) and (***) are strict inequalities yielding the contradiction \( \mathbb{E} [Y] < \mathbb{E} [X] = \mathbb{E} [Y] \). \( \square \)

Proof of Lemma 3.43: Suppose that \( \forall n_0 \in \mathbb{N} : \exists m(n_0) \geq n_0 : t_0 \geq \tau (h_{m(n_0)}) \).

\[
\Rightarrow P \left( \frac{dQ}{dP} < t_0 \right) < P \left( \frac{dQ}{dP} < t_0 \right) \overset{\text{Lemma 3.40(c)}}{=} \lim_{n_0 \to +\infty} P \left( \frac{dQ}{dP} < \tau (h_{m(n_0)}) \right) \leq P \left( \frac{dQ}{dP} < t_0 \right) \leq P \left( \frac{dQ}{dP} < t_0 \right)
\]

This is a contradiction. \( \square \)

Proof of Lemma 3.44:

- Let \( t_0 \in (0, +\infty) \) with \( P \left( \frac{dQ}{dP} < t_0 \right) > 0 \) and \( \varepsilon > 0 \).
  We assume that \( \forall n_0 \in \mathbb{N} : \exists m(n_0) \geq n_0 : X_{m(n_0)}(t_0) > h_{m(n_0)} + \varepsilon \).

Let \( Y_{m(n_0)}(t) := \left\{ \begin{array}{ll} h_{m(n_0)} & \text{if } t \leq t_0 \\ X_{m(n_0)} & \text{if } t > t_0 \end{array} \right. \).

Having assumed that \( X_{m(n_0)} \) is a monotonic decreasing function, we know that

\[
\begin{align*}
- \left\{ \omega \in \Omega \big| X_{m(n_0)} \left( \frac{dQ}{dP} (\omega) \right) < h_{m(n_0)} \right\} = \left\{ \omega \in \Omega \big| Y_{m(n_0)} \left( \frac{dQ}{dP} (\omega) \right) < h_{m(n_0)} \right\} \\
- \forall \omega \in \left\{ \omega \in \Omega \big| Y_{m(n_0)} \left( \frac{dQ}{dP} (\omega) \right) < h_{m(n_0)} \right\} : X_{m(n_0)} \left( \frac{dQ}{dP} (\omega) \right) = Y_{m(n_0)} \left( \frac{dQ}{dP} (\omega) \right) \end{align*}
\]
We define the mapping

\[ Y_{m(n_0)} : (0, +\infty) \to [0, +\infty), \text{ a modification of } X_{m(n_0)}, \text{ as follows:} \]

\[
Y_{m(n_0)}(t) := \begin{cases} 
0 & \text{if } t \leq \tau(h_{m(n_0)}) \land \exists t \in [0,t_1) \\
\tau(h_{m(n_0)}) & \text{if } t > \tau(h_{m(n_0)}) \land (X_{m(n_0)}(t) \leq h_{m(n_0)}) \\
X_{m(n_0)}(t) & \text{else}
\end{cases}
\]

We define the mapping:

\[ Y_{m(n_0)} : (0, +\infty) \to [u_0, +\infty), \text{ a modification of } X_{m(n_0)}, \text{ as follows:} \]

\[
Y_{m(n_0)}(t) := \begin{cases} 
u_0 & \text{if } t \in [0, \tau(h_{m(n_0)})) \\
\tau(h_{m(n_0)}) & \text{if } t > \tau(h_{m(n_0)}) \land (X_{m(n_0)}(t) \leq h_{m(n_0)}) \\
X_{m(n_0)}(t) & \text{else}
\end{cases}
\]
Proposition 3.42

\[ Y_{m(n_0)} \circ \frac{dQ}{dP} \text{ fulfills the expected shortfall condition, too:} \]

\[
E \left[ \left( h_{m(n_0)} - Y_{m(n_0)} \circ \frac{dQ}{dP} \right)^+ \right] \\
\leq E \left[ \left( h_{m(n_0)} - X_{m(n_0)} \circ \frac{dQ}{dP} \right)^+ \right] - \int_{\{ \frac{dQ}{dP} \leq 0 \}} \left( h_{m(n_0)} - X_{m(n_0)} \circ \frac{dQ}{dP} \right)^+ dP \\
- \int_{\{ \frac{dQ}{dP} \geq 0 \}} \left( u_0 - X_{m(n_0)} \circ \frac{dQ}{dP} \right) dP \\
= E \left[ \left( h_{m(n_0)} - X_{m(n_0)} \circ \frac{dQ}{dP} \right)^+ \right] \\
\leq (h_{m(n_0)} - c) \cdot \alpha.
\]

Now we can conclude that

\[
Q \left( \frac{dQ}{dP} < \tau \left( h_{m(n_0)} \right) \right) \cdot h_{m(n_0)} + Q \left( \frac{dQ}{dP} \geq \tau \left( h_{m(n_0)} \right) \right) \cdot u_0
\]

Proposition 3.42

\[
\leq E_Q \left[ Y_{m(n_0)} \circ \frac{dQ}{dP} \right] \\
= E_Q \left[ X_{m(n_0)} \circ \frac{dQ}{dP} \right] + \int_{\{ \frac{dQ}{dP} \leq 0 \}} \left[ \left( h_{m(n_0)} - X_{m(n_0)} \circ \frac{dQ}{dP} \right)^+ \right] \frac{dQ}{dP} dP \\
+ \int_{\{ \frac{dQ}{dP} \geq 0 \}} \left( u_0 - X_{m(n_0)} \circ \frac{dQ}{dP} \right) \frac{dQ}{dP} dP \\
\leq E_Q \left[ X_{m(n_0)} \circ \frac{dQ}{dP} \right] + t_1 \cdot \int_{\{ \frac{dQ}{dP} \leq 0 \}} \left( h_{m(n_0)} - X_{m(n_0)} \circ \frac{dQ}{dP} \right)^+ dP \\
+ \tau \left( h_{m(n_0)} \right) \cdot \int_{\{ \frac{dQ}{dP} \geq 0 \}} \left( u_0 - X_{m(n_0)} \circ \frac{dQ}{dP} \right) dP \\
= E_Q \left[ X_{m(n_0)} \circ \frac{dQ}{dP} \right] \\
+ (t_1 - \tau \left( h_{m(n_0)} \right)) \cdot \int_{\{ \frac{dQ}{dP} \leq 0 \}} \left( h_{m(n_0)} - X_{m(n_0)} \circ \frac{dQ}{dP} \right)^+ dP \\
\leq E_Q \left[ X_{m(n_0)} \circ \frac{dQ}{dP} \right] \\
+ (t_1 - \tau \left( h_{m(n_0)} \right)) \cdot P \left( \frac{dQ}{dP} \in [t_0, t_1] \right) \cdot \left( h_{m(n_0)} - X_{m(n_0)} \circ \frac{dQ}{dP} \right)^+ \\
\leq E_Q \left[ X_{m(n_0)} \circ \frac{dQ}{dP} \right] \left( n_0 \rightarrow +\infty \right) Q \left( \frac{dQ}{dP} < \tau \left( h_0 \right) \right) \cdot h_0 + Q \left( \frac{dQ}{dP} \geq \tau \left( h_0 \right) \right) \cdot u_0.
\]
The Squeezing/Sandwich Theorem implies that
\[
0 = (t_1 - \tau(h_{m(n_0)})) \cdot P\left(\frac{dQ}{dP} \in [t_0, t_1]\right) \cdot (h_{m(n_0)} - X_{m(n_0)}(t_0))^+ < 0
\]
which is a contradiction.

- Thus we know that \( \forall t_0 \in (0, \tau(h_0)) \) with \( P\left(\frac{dQ}{dP} \in [t_0, \tau(h_0)]\right) > 0 \):
  \[
  \lim_{n \to +\infty} X_{m(n_0)}(t_0) = h_0.
  \]
  \[
  \Rightarrow \lim_{n \to +\infty} \int_{\{\frac{dQ}{dP} < \tau(h_0)\}} \left( X_n \circ \frac{dQ}{dP} \right) dQ = Q\left(\frac{dQ}{dP} < \tau(h_0)\right) \cdot h_0.
  \]
  \[
  \Rightarrow \lim_{n \to +\infty} \int_{\{\frac{dQ}{dP} \geq \tau(h_0)\}} \left( X_n \circ \frac{dQ}{dP} \right) dQ = Q\left(\frac{dQ}{dP} \geq \tau(h_0)\right) \cdot u_0.
  \]
- Let \( t_0 \in (\tau(h_0), +\infty) \) with \( P\left(\frac{dQ}{dP} \in [\tau(h_0), t_0]\right) > 0 \).

\[
\int_{\{\frac{dQ}{dP} \geq \tau(h_0)\}} \left( X_n \circ \frac{dQ}{dP} \right) dQ - Q\left(\frac{dQ}{dP} \geq \tau(h_0)\right) \cdot u_0
\]
\[
= \int_{\{\frac{dQ}{dP} \geq \tau(h_0)\}} \left( X_n \circ \frac{dQ}{dP} - u_0 \right) dQ \geq \int_{\{\frac{dQ}{dP} \in [\tau(h_0), t_0]\}} \left( X_n \circ \frac{dQ}{dP} - u_0 \right) dQ
\]
\[
\geq Q\left(\frac{dQ}{dP} \in [\tau(h_0), t_0]\right) \cdot (X_n(t_0) - u_0) \geq 0.
\]

Applying again the Squeezing/Sandwich Theorem, we can deduce that
\[
0 = \lim_{n \to +\infty} \left( Q\left(\frac{dQ}{dP} \in [\tau(h_0), t_0]\right) \cdot (X_n(t_0) - u_0) \right),
\]
i.e. that \( \lim_{n \to +\infty} X_n(t_0) = u_0. \)

\[\square\]

**Proof of Fact 3.45:**

- (Lemma 3.41) \( \Rightarrow [c, +\infty) \rightarrow [0, Q(\Omega)], h \rightarrow Q\left(\frac{dQ}{dP} < \tau(h)\right) \) is continuous.
  \( \Rightarrow h \rightarrow Q\left(\frac{dQ}{dP} < \tau(h)\right) \cdot h + Q\left(\frac{dQ}{dP} \geq \tau(h)\right) \cdot u_0 \) is continuous.
  \(((-\infty, x_0) \) is open) \( \Rightarrow H_P \) is relatively open in \([c, +\infty).\)
  \(((-\infty, x_0) \) is closed) \( \Rightarrow \overline{H_P} \) is relatively closed in \([c, +\infty).\)
  \((c, +\infty) \) is closed) \( \Rightarrow \overline{H_P} \) is closed.
- Case \( \alpha \in (0, 1)\):
  (Lemma 3.40(c)) \( \Rightarrow P\left(\frac{dQ}{dP} < \tau(h)\right) \geq 1 - \alpha. \)
  \( \Rightarrow \left\{ \frac{dQ}{dP} < \tau(h)\right\} \supseteq \left\{ \frac{dQ}{dP} \leq F_{\frac{1}{\alpha}}^{-1}\left(1 - \alpha\right)\right\}. \)
Proof of Lemma 3.47: We note that \( \forall x < u_0: f(x) = -\infty \) and \( f \) restricted to \([u_0, +\infty)\) is continuous except for the left-continuity in the point \( h \) in the case \( \lambda_2 = +\infty \).

\( \forall x > h: f'(x) = U'(x) - \lambda_1 \cdot d \xrightarrow{x \to +\infty} -\lambda_1 \cdot d < 0 \).

Since \( U' \) is strictly decreasing on \((u_0, +\infty)\), \( f' \) is strictly decreasing on \((h, +\infty)\).

- \( \lambda_2 \in \mathbb{R}_+: \)
  \( \forall x \in (u_0, h): f'(x) = U'(x) - \lambda_1 \cdot d + \lambda_2, f' \) is strictly decreasing on \((u_0, h), \) \( \lim_{x \to u_0^+} f'(x) = +\infty \)
  and \( \lim_{x \to h^+} f'(x) = U'(h) - \lambda_1 \cdot d + \lambda_2 \geq U'(h) - \lambda_1 \cdot d = \lim_{x \to h^+} f'(x). \)

- \( \lambda_2 = +\infty: \)
  \( \forall x < h: f(x) = -\infty. \)

Thus we can study the following three cases:

- Case \( U'(h) < \lambda_1 \cdot d - \lambda_2: \)
  \( \Rightarrow (U')^{-1}(\lambda_1 \cdot d - \lambda_2) < h. \)

  \( \Rightarrow f'((U')^{-1}(\lambda_1 \cdot d - \lambda_2)) = 0. \)

- Case \( U'(h) - \lambda_1 \cdot d > 0: \)
  \( \Rightarrow (U')^{-1}(\lambda_1 \cdot d) > h. \)

  \( \Rightarrow f'((U')^{-1}(\lambda_1 \cdot d)) = 0. \)

- Case \( \lambda_1 \cdot d - \lambda_2 \leq U'(h) \leq \lambda_1 \cdot d: \)
  \( U' \) is strictly decreasing on \((u_0, +\infty)\) \( \Rightarrow \forall x > h: f'(x) < 0. \)

  If \( \lambda_2 < +\infty, (U' \) is strictly decreasing on \((u_0, +\infty)\) \( \Rightarrow \forall x \in (u_0, h): f'(x) > 0. \)

  If \( \lambda_2 = +\infty, (U'(h) \leq \lambda_1 \cdot d) \Rightarrow h \neq u_0 \Rightarrow \forall x < h: f(x) = -\infty < f(h). \)
In all three cases, the optimal value of $f$ over the set $[u_0, +\infty)$ never equals $-\infty$. 

**Proof of Lemma 3.50:** We prove both implications separately:

- If $\forall (h, \lambda_1, \lambda_2) \in [u_0, +\infty) \times (0, +\infty) \times [0, +\infty]$ with $(\lambda_1, \lambda_2) \neq (+\infty, +\infty)$: 
  
  $$
  \mathbb{E}_Q \left[ \mathcal{H}_{\lambda_1, \lambda_2} \right] < +\infty,
  $$

  then

  $$
  \forall \lambda_1 \in (0, +\infty) : X_{u_0, \lambda_1, 0} = (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right)
  $$

  implies that $\forall \lambda_1 \in (0, +\infty)$: 

  $$
  \mathbb{E}_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right] \leq \mathbb{E}_Q \left[ X_{u_0, \lambda_1, 0} \right] < +\infty.
  $$

- Let $\forall \lambda_1 \in (0, +\infty)$: 

  $$
  \mathbb{E}_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right] < +\infty.
  $$

  $X_{h, \lambda_1, \lambda_2} \geq u_0 \Rightarrow \max \left\{ 0, -X_{h, \lambda_1, \lambda_2} \right\} \leq \max \{ 0, -u_0 \}$, 

  then

  $$
  \mathbb{E}_Q \left[ \max \left\{ 0, -X_{h, \lambda_1, \lambda_2} \right\} \right] \leq \mathbb{E}_Q \left[ \max \{ 0, -u_0 \} \right] = \max \{ 0, -u_0 \} \cdot Q(\Omega) < +\infty.
  $$

  

  max \left\{ 0, X_{h, \lambda_1, \lambda_2}(\omega) \right\}

  \leq \max \left\{ 0, \max \left\{ h, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\} \right\}

  \leq \max \{ 0, h \} + \max \left\{ 0, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\}

  = \max \{ 0, h \} + (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) + \max \left\{ -(U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right), 0 \right\}

  \leq \max \{ 0, h \} + (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) + \max \{ -u_0, 0 \}

  Since $\mathbb{E}_Q \left[ (U')^{-1} \left( +\infty \cdot \frac{dQ}{dP} \right) \right] = \mathbb{E}_Q [u_0] = u_0 \cdot Q(\Omega) < +\infty$, we know that

  $$
  \mathbb{E}_Q \left[ \max \left\{ 0, X_{h, \lambda_1, \lambda_2} \right\} \right] \leq (\max \{ 0, h \} + \max \{ -u_0, 0 \}) \cdot Q(\Omega) + \mathbb{E}_Q \left[ (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right]
  $$

  < +\infty.

**Proof of Lemma 3.51:** Let $\lambda_1 \in (0, +\infty)$.

- If $\forall (h, \lambda_2) \in [u_0, +\infty) \times [0, +\infty]$:

  $$
  \mathbb{E} \left[ \left( U \left( X_{h, \lambda_1, \lambda_2} \right) \right)^+ \right] < +\infty,
  $$

  then

  $$
  X_{u_0, \lambda_1, 0} = (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right)
  $$

  implies that $\mathbb{E} \left[ \left( U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \right] < +\infty$.

- Let us suppose that $\mathbb{E} \left[ \left( U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ \right] < +\infty.$
Because of
\[
\max \left\{ 0, U \left( X^{h, \lambda_1, \lambda_2}(\omega) \right) \right\}
\leq \max \left\{ 0, U \left( \max \left\{ h, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\} \right) \right\}_{(U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \text{ if } h = u_0}
\leq \max \left\{ 0, \max \{0, U(h)\} \text{ if } h > u_0 \right\} + \max \left\{ 0, U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right) \right\}
\]
we can conclude that
\[
E \left[ \left( U \left( X^{h, \lambda_1, \lambda_2}(\omega) \right) \right)^+ \right] \leq \max \left\{ 0, \max \{0, U(h)\} \text{ if } h > u_0 \right\} \cdot 1 + E \left[ \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right)^+ \right]_{< +\infty}
\]
\[
< + \infty.
\]

- If \( \forall (h, \lambda_2) \in [u_0, +\infty) \times [0, +\infty] : E \left[ -U \left( X^{h, \lambda_1, \lambda_2} \right)^+ \right] < +\infty \), then

\[
X^{u_0, \lambda_1, 0} = (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \text{ implies that } E \left[ \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right)^+ \right] < +\infty.
\]

- Let us suppose that \( E \left[ \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right)^+ \right] < +\infty. \)

Then
\[
\max \left\{ 0, -U \left( X^{h, \lambda_1, \lambda_2}(\omega) \right) \right\}
\]
\[
= \max \left\{ 0, -U \left( \min \left\{ \max \left\{ h, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\}, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) - \lambda_2 \right) \right\} \left( U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right) \right\}
\]
\[
\leq \max \left\{ 0, -U \left( \min \left\{ \max \left\{ h, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\}, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\} \right) \right\}
\]
\[
= \max \left\{ 0, -U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right) \right\}
\]

implies
\[
E \left[ \left( -U \left( X^{h, \lambda_1, \lambda_2} \right)^+ \right) \right] \leq E \left[ \left( -U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right)^+ \right) \right] < +\infty.
\]

Proof of Lemma 3.54: Let \( A_1 := [u_0, +\infty) \times (0, +\infty] \times \mathbb{R}_+ \) and \( A_2 := [u_0, +\infty) \times (0, +\infty) \times [0, +\infty]. \)
For $i \in \{1, 2\}$:
\[ \forall n \in \mathbb{N}_0: \text{Let } (h_n, \lambda_i^0, \lambda_i^2) \in A_i \text{ with } (h_n, \lambda_i^0, \lambda_i^2) \to (h_0, \lambda_i^0, \lambda_i^2) \ (n \to \infty). \]
Then $\exists (h_{min}^i, \lambda_{min}^i, \lambda_{min}^i, \lambda_{max}^i, \lambda_{max}^i, \lambda_{max}^i) \in A_i: \forall n \in \mathbb{N}_0: h_{min}^i \leq h_n \leq h_{max}^i, \lambda_{min}^i \leq \lambda_i \leq \lambda_{max}^i \text{ and } \lambda_{min}^i \leq \lambda_i \leq \lambda_{max}^i.$
\[ \Rightarrow \forall n \in \mathbb{N}_0, \exists \omega \in \Omega: X^{h_{min}^i, \lambda_{min}^i, \lambda_{min}^i, \lambda_{max}^i, \lambda_{max}^i, \lambda_{max}^i}(\omega) \leq X^{h_{max}^i, \lambda_{min}^i, \lambda_{min}^i, \lambda_{max}^i, \lambda_{max}^i, \lambda_{max}^i}(\omega). \]
(Lemma 3.50) $\Rightarrow -\infty < E_Q \left[ X^{h_{min}^i, \lambda_{min}^i, \lambda_{min}^i, \lambda_{max}^i, \lambda_{max}^i, \lambda_{max}^i} \right] \leq E_Q \left[ X^{h_{max}^i, \lambda_{min}^i, \lambda_{min}^i, \lambda_{max}^i, \lambda_{max}^i, \lambda_{max}^i} \right] < +\infty.$
(Dominated Convergence Theorem) $\Rightarrow \lim_{n \to +\infty} E_Q \left[ X^{h_n, \lambda_{min}^i, \lambda_{min}^i} \right] = E_Q \left[ X^{h_0, \lambda_{min}^i, \lambda_{min}^i} \right]. \quad \Box$

**Proof of Lemma 3.57:** We prove both parts separately:

(a) The function $(0, +\infty) \to [h \cdot Q(\Omega), +\infty)$, $\lambda_1 \mapsto E_Q \left[ X^{h, \lambda_1, +\infty} \right]$ and $\forall \lambda_2 \in \mathbb{R}_+$ the functions $(0, +\infty) \to (u_0 \cdot Q(\Omega), +\infty)$, $\lambda_1 \mapsto E_Q \left[ X^{h, \lambda_1, \lambda_2} \right]$ are continuous (cf. Lemma 3.54), $(-\infty, x_0]$ is closed and $\forall \lambda_2 \in [0, +\infty]: \lim_{\lambda_1 \to +\infty} E_Q \left[ X^{h, \lambda_1, \lambda_2} \right] = +\infty$ (Fact 3.53(g)).

(b) Let us assume that $\exists ((h_n, \lambda_{min}^2))$ with $\lim_{n \to +\infty} A^{h_n, \lambda_{min}^2} = 0$.
\[ \Rightarrow X^{h_n, \lambda_n, \psi_n, h_n} \geq X^{h_n, \lambda_n, \psi_n, h_n, 0} = X^{h_0, \lambda_n, \psi_n, h_n, 0} \text{ and hence } x_0 = +\infty \text{ due to } \]
\[ x_0 = E_Q \left[ X^{h_n, \lambda_n, \psi_n, h_n} \right] \geq E_Q \left[ X^{h_0, \lambda_n, \psi_n, h_n, 0} \right] \text{ Fact 3.53(g) } \to +\infty \text{ (} A^{h_n, \psi_n} \to 0+. \text{)} \]

This is a contradiction to $x_0 \in \mathbb{R}$. \(\Box\)

**Proof of Lemma 3.59:** Let $\lambda_{min}^2 \in L^2$ and $\lambda_{min}^2 \in L^2$ with $\lambda_{min}^2 < \lambda_{max}^2$.
\[ \Rightarrow E_Q \left[ X^{h, \lambda_{min}^2, \lambda_{min}^2} \right] \leq E_Q \left[ X^{h, \lambda_{min}^2, \lambda_{min}^2} \right] = x_0. \]
(definition of $A^{h, \lambda_{min}^2, \lambda_{min}^2}$) $\Rightarrow A^{h, \lambda_{min}^2} \leq A^{h, \lambda_{min}^2}$. \(\Box\)

**Proof of Lemma 3.62:** Since $X^{h_n, \lambda_n, \lambda_{min}^2, \lambda_{max}^2} \geq X^{h_n, \lambda_n, \lambda_{min}^2, \lambda_{max}^2, 0} = X^{u_0, \lambda_n, \lambda_{min}^2, \lambda_{max}^2, 0}$ we know that the relationship $E_Q \left[ X^{u_0, \lambda_n, \lambda_{min}^2, \lambda_{max}^2, 0} \right] = x_0 = E_Q \left[ X^{h_n, \lambda_n, \lambda_{min}^2, \lambda_{max}^2, \lambda_{min}^2} \right] \geq E_Q \left[ X^{u_0, \lambda_n, \lambda_{min}^2, \lambda_{max}^2, 0} \right]$ holds. Hence Fact 3.53(b) implies that
\[ A^{h_n, \lambda_{min}^2} \geq A^{u_0, 0} > 0. \quad (6.1) \]

- Case $\lambda_2 = +\infty$:
\[ \Rightarrow h < \frac{x_0}{Q(\Omega)}, \]
\[ (h_n \to h) \Rightarrow h_{max} := \max \{ h_n | n \in \mathbb{N} \}, h \} < \frac{x_0}{Q(\Omega)} < +\infty. \]
Using Fact 3.53(f), $x_0 > h_{max} \cdot Q(\Omega)$ and Fact 3.53(c), we can conclude that
\[ \sup \left\{ \lambda_1 \in (0, +\infty) \left| E_Q \left[ X^{h_{max}, \lambda_1, +\infty} \right] \geq x_0 \right. \right\} < +\infty. \]
(Fact 3.53(e), $x_0 > h \cdot Q(\Omega) \geq u_0 \cdot Q(\Omega)$) $\Rightarrow A^{h, \lambda_{min}^2} \neq +\infty.$
\[ \Rightarrow X^{h_n, \lambda_n, \lambda_{min}^2, \lambda_{max}^2} \leq X^{h_n, \lambda_n, \lambda_{min}^2, +\infty} \leq X^{h_{max}, \lambda_n, \lambda_{min}^2, +\infty} \]
\[ \Rightarrow x_0 = E_Q \left[ X^{h_n, \lambda_n, \lambda_{min}^2, \lambda_{max}^2} \right] \leq E_Q \left[ X^{h_{max}, \lambda_n, \lambda_{min}^2, +\infty} \right]. \]
Hence
\[ \lambda_2 = +\infty \Rightarrow A^{h_n, \lambda_{min}^2} \leq \sup \left\{ \lambda_1 \in (0, +\infty) \left| E_Q \left[ X^{h_{max}, \lambda_1, +\infty} \right] \geq x_0 \right. \right\} < +\infty. \quad (6.2) \]
Let $\lambda_1^0 := \liminf_{n \to \infty} \{ \Lambda_{h, \lambda_2}^n \} \in (0, +\infty]$ and $\lambda_1^d := \sup_{n \to \infty} \{ \Lambda_{h, \lambda_2}^n \} \in (0, +\infty].$

$\Rightarrow \lambda_1^0 \leq \lambda_1^d.$

$\Rightarrow \exists (i_n): \lim_{n \to \infty} \Lambda_{h, \lambda_2}^{i_n} = \lambda_1^d$ and $\exists (s_n): \lim_{n \to \infty} \Lambda_{h, \lambda_2}^{s_n} = \lambda_1^d.$

$\Rightarrow \forall \omega \in \Omega: \lim_{n \to \infty} X_{h, \lambda_2}^{i_n} (\omega) = X_{h, \lambda_2} (\omega)$ and

$L_{0 0, \infty}^n (\omega)$

(Lemma 3.54) $\Rightarrow E_Q \left[ X_{h, \lambda_2} \right] = x_0$ and $E_Q \left[ X_{h, \lambda_2} \right] = x_0.$

$\Rightarrow E_Q \left[ x_{\min} \{ \lambda_1^0, \lambda_2 \} \right] = x_0 = E_Q \left[ \lambda_1^d, \lambda_2 \right].$

Let $A := \{ \omega \in \Omega: \min \{ \lambda_1^0, \lambda_2 \} \neq \lambda_2 (\omega) \}$ and let $\omega \in \Omega \setminus A$ be fixed.

We will show, that $\lim_{n \to \infty} X_{h, \lambda_2} (\omega) = X_{h, \lambda_2} (\omega)$ (cf. Lemma 3.61).

Let $\varepsilon > 0.$

Now, we apply Lemma 3.48 twice:

- First, Lemma 3.48 ensures the existence of the values $\delta_1 > 0,$ $l_1 \in (0, \min \{ \lambda_1^0, \lambda_2 \} )$ and $u_1 \in \left\{ \{ +\infty \}, \{ \min \{ \lambda_1^0, \lambda_2 \} \}, \{ +\infty \} \right\}$ as well as a neighborhood

$N_1 \subseteq \mathbb{R}$ if $\lambda_2 \neq +\infty$ of $\lambda_2$ such that:

$\forall \tilde{h} \in [u_1, +\infty)$ with $| h - \tilde{h} | < \delta_1,$ $\forall \tilde{1} \in (l_1, u_1)$ and $\forall \tilde{2} \in (\mathbb{R} \cap N_1):$

$| h_{\tilde{1}, \tilde{2}} (\omega) - h_{\min} \{ \lambda_1^0, \lambda_2 \} | < \varepsilon.$

- Similarly, Lemma 3.48 ensures the existence of the values $\delta_2 > 0,$ $l_2 \in (0, \max \{ \lambda_1^d, \lambda_2 \} )$ and

$u_2 \in \left\{ \{ +\infty \}, \{ \max \{ \lambda_1^d, \lambda_2 \} \}, \{ +\infty \} \right\}$, as well as a neighborhood

$N_2 \subseteq \mathbb{R}$ if $\lambda_2 \neq +\infty$ of $\lambda_2$ such that:

$\forall \tilde{h} \in [u_2, +\infty)$ with $| h - \tilde{h} | < \delta_2,$ $\forall \tilde{1} \in (l_2, u_2)$ and $\forall \tilde{2} \in (\mathbb{R} \cap N_2):$

$| h_{\tilde{1}, \tilde{2}} (\omega) - h_{\max} \{ \lambda_1^d, \lambda_2 \} | < \varepsilon.$

$\Rightarrow \forall \tilde{h} \in [u_0, +\infty)$ with $| h - \tilde{h} | < \min \{ \delta_1, \delta_2 \}$ and $\forall \tilde{1} \in (\mathbb{R} \cap (N_1 \cap N_2))$ with $\lambda_{h, \tilde{2}} \in (l_1, u_2):$

$| h_{\tilde{1}, \tilde{2}} (\omega) - h_{-\tilde{1}, \tilde{2}} (\omega) | < \varepsilon,$ because

- in the case $\lambda_{h, \tilde{2}} \in (l_1, \min \{ \lambda_1^0, \lambda_2 \})$ it is implied by the definition of $\delta_1,$ $l_1,$ $u_1$ and $N_1,$

- in the case $\lambda_{h, \tilde{2}} \in (\max \{ \lambda_1^d, \lambda_2 \}, u_2)$ it is immediately clear due to the definition of $\delta_2,$ $l_2,$ $u_2$ and $N_2$ and
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

in the case \( \Lambda^h, \lambda^2 \in [\min \{ \lambda^h, \Lambda^h, \lambda^2 \}, \max \{ \lambda^h, \Lambda^h, \lambda^2 \}] \) it can be seen as follows:

\[
X^{h, \Lambda^h, \lambda^2, \lambda^2}(\omega) + \varepsilon \overset{\text{Lemma 3.61}}{=} \min \{ \lambda^h, \Lambda^h, \lambda^2 \} \lambda^2(\omega) + \varepsilon
\]

\[
\geq X^{h, \Lambda^h, \lambda^2, \lambda^2}(\omega)
\]

\[
\geq X^{h, \Lambda^h, \lambda^2, \lambda^2}(\omega)
\]

\[
\geq X^{h, \max \{ \lambda^h, \Lambda^h, \lambda^2 \}, \lambda^2}(\omega) - \varepsilon
\]

Summing up, we have shown that for all \( \omega \in \Omega \setminus A \), \( \lim_{n \to \infty} X^{h, n, \Lambda^h, \lambda^2, \lambda^2}(\omega) = X^{h, \Lambda^h, \lambda^2, \lambda^2}(\omega) \) with \( Q(A) = 0 \) (cf. Lemma 3.61).

**Proof of Lemma 3.64:** Let \( \lambda^2, \lambda^2 \in L^2 \) with \( \lambda^2 < \lambda^2 \). (Lemma 3.59) \( \Rightarrow \Lambda^h, \lambda^2 \leq \Lambda^h, \lambda^2 \).

- If \( \Lambda^h, \lambda^2 = \Lambda^h, \lambda^2 \\), \( \frac{U'(h) + \lambda^2}{\Lambda^h, \lambda^2} \leq \frac{U'(h) + \lambda^2}{\Lambda^h, \lambda^2} \) and \( X^{h, \Lambda^h, \lambda^2, \lambda^2} \leq X^{h, \Lambda^h, \lambda^2, \lambda^2} \), so \( E_Q \left[ X^{h, \Lambda^h, \lambda^2, \lambda^2} \right] = \).

\[ x_0 = E_Q \left[ X^{h, \Lambda^h, \lambda^2, \lambda^2} \right] \text{ implies that } X^{h, \Lambda^h, \lambda^2, \lambda^2} = X^{h, \Lambda^h, \lambda^2, \lambda^2} \quad (Q\text{-almost-everywhere}). \]

\[ P \sim Q \Rightarrow E \left[ (h - X^{h, \Lambda^h, \lambda^2, \lambda^2})^+ \right] = E \left[ (h - X^{h, \Lambda^h, \lambda^2, \lambda^2})^+ \right]. \]

- Case \( \Lambda^h, \lambda^2 < \Lambda^h, \lambda^2 \):

\[ \forall \omega \in \Omega \text{ with } \Lambda^h, \lambda^2 \cdot \frac{dQ}{dP}(\omega) \leq U'(h) + \lambda^2: \]

\[ X^{h, \Lambda^h, \lambda^2, \lambda^2}(\omega) = \max \{ h, (U')^{-1} \left( \Lambda^h, \lambda^2 \cdot \frac{dQ}{dP}(\omega) \right) \}
\]

\[ \geq \max \{ h, (U')^{-1} \left( \Lambda^h, \lambda^2 \cdot \frac{dQ}{dP}(\omega) \right) \}
\]

\[ \geq X^{h, \Lambda^h, \lambda^2, \lambda^2}(\omega).
\]

\[ \Rightarrow 0 \geq \int_{\Lambda^h, \lambda^2 \cdot \frac{dQ}{dP} = U'(h) + \lambda^2} \left( X^{h, \Lambda^h, \lambda^2, \lambda^2} - X^{h, \Lambda^h, \lambda^2, \lambda^2} \right) dQ. \quad (6.3) \]

We note that

\[ \left\{ \frac{dQ}{dP} = \frac{\lambda^2 - \lambda^2}{\Lambda^h, \lambda^2 - \Lambda^h, \lambda^2} \right\} = \left\{ (U')^{-1} \left( \Lambda^h, \lambda^2 \cdot \frac{dQ}{dP} - \lambda^2 \right) = (U')^{-1} \left( \Lambda^h, \lambda^2 \cdot \frac{dQ}{dP} - \lambda^2 \right) \right\}. \]

- Suppose that \( X^{h, \Lambda^h, \lambda^2, \lambda^2} = X^{h, \Lambda^h, \lambda^2, \lambda^2} \) (P-almost-surely). Then

\[ 0 = P \left( X^{h, \Lambda^h, \lambda^2, \lambda^2} \neq X^{h, \Lambda^h, \lambda^2, \lambda^2} \right)
\]

\[ = P \left( \left\{ \Lambda^h, \lambda^2 \cdot \frac{dQ}{dP} < U'(h) \right\} \right.
\]

\[ \cup \left( \left\{ \Lambda^h, \lambda^2 \cdot \frac{dQ}{dP} > U'(h) + \lambda^2 \right\} \right. \]

\[ \cup \left. \left\{ \Lambda^h, \lambda^2 \cdot \frac{dQ}{dP} > U'(h) + \lambda^2 \right\} \right) \]
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

\[
P \left( \frac{dQ}{dp} = \frac{\lambda_2 - \lambda_2}{\lambda_h \cdot \lambda_2 - \lambda_h \cdot \lambda_2} \right) = 0 \Rightarrow h = X^{h, \lambda, \lambda^2, \lambda_2} = X^{h, \lambda, \lambda_2^2, \lambda_2} \text{ (P-almost-surely)}.
\]

(definition of \( L_2^0 \)) \( \Rightarrow \lambda_2^0 \in \mathbb{R}_+ \).

\[
(P \left\{ \lambda_h \cdot \frac{dQ}{dp} < U'(h) \right\}) = 0 \Rightarrow X^{h, \lambda, \lambda^2, \lambda_2} \leq h \text{ (P-almost-surely)}.
\]

\[
P \sim Q \Rightarrow \mathbb{E}_Q \left[ X^{h, \lambda, \lambda^2, \lambda_2} \right] \leq h \cdot Q(\Omega) = \mathbb{E}_Q \left[ X^{h, \lambda, \lambda^2, \lambda_2^2} \right] = x_0.
\]

(definition of \( \Lambda^{h, \lambda_2^2} \) \( \Rightarrow \Lambda^{h, \lambda^2} \geq \Lambda^{h, \lambda_2^0} \) which is a contradiction.

Therefore we know that

\[
P \left( X^{h, \lambda, \lambda^2, \lambda_2} < X^{h, \lambda, \lambda_2^2, \lambda_2} \right) > 0. \quad (6.4)
\]

As

\[\left\{ \frac{dQ}{dp} < \frac{\lambda_2 - \lambda_2}{\lambda_h \cdot \lambda_2 - \lambda_h \cdot \lambda_2} \right\} = \left\{ (U')^{-1} \left( \Lambda^{h, \lambda_2} \cdot \frac{dQ}{dp} - \lambda_2 \right) < (U')^{-1} \left( \Lambda^{h, \lambda_2^2} \cdot \frac{dQ}{dp} - \lambda_2^2 \right) \right\},\]

we can characterize the set on which \( X^{h, \lambda, \lambda^2, \lambda_2} \) and \( X^{h, \lambda, \lambda_2^2, \lambda_2} \) differ in the following way:

\[\left\{ X^{h, \lambda, \lambda^2, \lambda_2} < X^{h, \lambda, \lambda_2^2, \lambda_2} \right\} = \left\{ \Lambda^{h, \lambda_2} \cdot \frac{dQ}{dp} > U'(h) + \lambda_2 \right\} \cap \left\{ \frac{dQ}{dp} < \frac{\lambda_2 - \lambda_2^2}{\lambda_h \cdot \lambda_2 - \lambda_h \cdot \lambda_2} \right\} \text{ (inequality (6.4))} \]

\[\Rightarrow \frac{U'(h) + \lambda_2}{\Lambda^{h, \lambda_2}} < \frac{U'(h) + \lambda_2^2}{\Lambda^{h, \lambda_2^2}}.
\]

Since \( \left\{ X^{h, \lambda, \lambda_2^2, \lambda_2} > h \right\} \subset \left\{ X^{h, \lambda, \lambda_2^2, \lambda_2} > h \right\}:

\[
\mathbb{E} \left[ \left( h - X^{h, \lambda, \lambda_2^2, \lambda_2} \right)^+ \right] - \mathbb{E} \left[ \left( h - X^{h, \lambda, \lambda_2, \lambda_2} \right)^+ \right] = \int \\left\{ \Lambda^{h, \lambda_2} \cdot \frac{dQ}{dp} > U'(h) + \lambda_2 \right\} \left[ \left( h - X^{h, \lambda, \lambda_2^2, \lambda_2} \right) - \left( h - X^{h, \lambda, \lambda_2, \lambda_2} \right) \right] \frac{dP}{\Lambda^{h, \lambda_2}} = \int \\left\{ \Lambda^{h, \lambda_2} \cdot \frac{dQ}{dp} > U'(h) + \lambda_2 \right\} \left( X^{h, \lambda, \lambda_2^2, \lambda_2} - X^{h, \lambda, \lambda_2, \lambda_2} \right) \frac{1}{\frac{dQ}{dp}} \frac{dQ}{\Lambda^{h, \lambda_2}} \quad \frac{1}{\lambda_2 - \lambda_2^2} \left( \lambda_2 - \lambda_2^2 \right)
\]

\[\geq \int \frac{dQ}{\Lambda^{h, \lambda_2} \cdot \frac{dQ}{dp} > U'(h) + \lambda_2} \left( X^{h, \lambda, \lambda_2^2, \lambda_2} - X^{h, \lambda, \lambda_2, \lambda_2} \right) \frac{1}{\frac{dQ}{dp}} \frac{dQ}{\Lambda^{h, \lambda_2}} \quad \frac{1}{\lambda_2 - \lambda_2^2} \left( \lambda_2 - \lambda_2^2 \right)
\]

(6.4)

\[\geq \frac{\Lambda^{h, \lambda_2^2} - \Lambda^{h, \lambda_2}}{\lambda_2 - \lambda_2^2} \cdot \int_{U'(h) + \lambda_2} \left( X^{h, \lambda, \lambda_2^2, \lambda_2} - X^{h, \lambda, \lambda_2, \lambda_2} \right) dQ
\]

(6.3)

\[\geq 0.
\]

Proof of Lemma 3.65: \( \lim_{\lambda_2 \to +\infty} \Lambda^{h, \lambda_2} \) is well defined, because \( \lambda_2 \to \Lambda^{h, \lambda_2} \) is a monotonic increasing function according to Lemma 3.59.
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

• $\lambda_1 < +\infty$:

$$\lim_{\lambda_2 \to +\infty} X^{h,\Lambda^{h,\lambda_2},\lambda_2}(\omega)$$

$$= \lim_{\lambda_2 \to +\infty} \min \left\{ \max \left\{ h, (U')^{-1} \left( \Lambda^{h,\lambda_2} \cdot \frac{dQ}{dP}(\omega) \right) \right\}, (U')^{-1} \left( \frac{\Lambda^{h,\lambda_2}}{\lambda_2} \cdot \frac{dQ}{dP}(\omega) - \lambda_2 \right) \right\}$$

Due to $0 \leq \mathbb{E} \left[ (h - X^{h,\Lambda^{h,\lambda_2},\lambda_2})^+ \right] \leq (h - u_0) \cdot 1 < +\infty$, we can apply the Dominated Convergence Theorem to conclude

$$\lim_{\lambda_2 \to +\infty} \mathbb{E} \left[ (h - X^{h,\Lambda^{h,\lambda_2},\lambda_2})^+ \right] = \mathbb{E} \left[ (h - \max \left\{ h, (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP}(\omega) \right) \right\})^+ \right]$$

$$= 0.$$

• $\lambda_1 = +\infty$:

Let $\bar{t} := \lim_{\lambda_2 \to +\infty} \frac{U'(h) + \lambda_2}{\Lambda^{h,\lambda_2}} \in [0, +\infty]$.

$\Rightarrow \forall \omega \in \Omega$ with $\frac{dQ}{dP}(\omega) < \bar{t}$: $\exists \lambda_2' > \lambda_2$, such that $\forall \lambda_2 > \lambda_2'$:

$$\frac{dQ}{dP}(\omega) < \bar{t}$$

$\Rightarrow \forall \omega \in \Omega$ with $\frac{dQ}{dP}(\omega) > \bar{t}$ and $\forall \lambda_2 > \lambda_2'$:

$$X^{h,\Lambda^{h,\lambda_2},\lambda_2}(\omega) = \max \left\{ h, (U')^{-1} \left( \Lambda^{h,\lambda_2} \cdot \frac{dQ}{dP}(\omega) \right) \right\} \rightarrow h (\lambda_2 \to +\infty).$$

$\forall \omega \in \Omega$ with $\frac{dQ}{dP}(\omega) > \bar{t}$:

$$u_0 \leq X^{h,\Lambda^{h,\lambda_2},\lambda_2}(\omega) \leq (U')^{-1} \left( \Lambda^{h,\lambda_2} \cdot \frac{dQ}{dP}(\omega) - \lambda_2 \right)$$

$$= (U')^{-1} \left( \Lambda^{h,\lambda_2} \right) \left( \frac{dQ}{dP}(\omega) - \bar{t} \right) + \Lambda^{h,\lambda_2} \cdot \left( \lambda_2 - \bar{t} \right) \cdot \frac{dQ}{dP}(\omega) \geq \frac{dQ}{dP}(\omega)$$

$$\rightarrow u_0 (\lambda_2 \to +\infty)$$

$$\Rightarrow \lim_{\lambda_2 \to +\infty} X^{h,\Lambda^{h,\lambda_2},\lambda_2}(\omega) = \begin{cases} h & \text{if } \frac{dQ}{dP}(\omega) < \bar{t} \\ u_0 & \text{if } \frac{dQ}{dP}(\omega) > \bar{t} \end{cases}.$$
Proof of Lemma 3.66: Let \( h_n \in [u_0, +\infty) \) for \( n \in \mathbb{N}_0 \) with \( \lim_{n \to +\infty} h_n = h_0 \).

(Lemma 3.62, \( P \sim Q \)) \( \Rightarrow \lim_{n \to +\infty} X^{h_n, A^{h_n,0}} = X^{h_0, A^{h_0,0}, 0} \) (\( P \)-almost-surely).

\[ \exists h_{\text{max}} \in [u_0, +\infty) \) with \( \forall n \in \mathbb{N}_0: h_{\text{max}} \geq h_n. \]

\( \Rightarrow \forall \omega \in \Omega: 0 \leq \left( h - X^{h_n, A^{h_n,0}, 0}(\omega) \right)^+ \leq (h_{\text{max}} - u_0). \)

Since \( -\infty < 0 \leq (h_{\text{max}} - u_0) \cdot 1 < +\infty \), the Dominated Convergence Theorem yields that the mapping \([u_0, +\infty) \to \mathbb{R}_+, \ h \to E \left[ (h - X^{h, A^{h,0}, 0})^+ \right] \) is continuous.

\( \Rightarrow [u_0, +\infty) \to \mathbb{R}, \ h \to E \left[ (h - X^{h, A^{h,0}, 0})^+ \right] - h \cdot \alpha \) is continuous.

\((-\infty, -c \cdot \alpha)\) is closed and therefore its preimage is relatively closed in the closed set \([u_0, +\infty)\) and hence closed in \( \mathbb{R} \).

\[\square\]

Proof of Lemma 3.67: We show the correctness of the statements in the same order:

- Lemma 3.59 tells us that \( A^{h, \lambda_2} \leq A^{h, \lambda_2'} \leq A^{h, \lambda_2} \).

We realize that \( A^{h, \lambda_2} \leq A^{h, \lambda_2'} \) is not possible, because in this case Lemma 3.64 would result in

\[ E \left[ (h - X^{h, A^{h, \lambda_2}, \lambda_2})^+ \right] > E \left[ (h - X^{h, A^{h, \lambda_2}, \lambda_2'})^+ \right]. \]

Hence, we know that \( A^{h, \lambda_2} = A^{h, \lambda_2'} = A^{h, \lambda_2} \).

- \( \forall \omega \in \Omega \) with \( A^{h, \lambda_2} \cdot \frac{dQ}{dP}(\omega) \leq U'(h) + \lambda_2' < U'(h) + \lambda_2 < U'(h) + \lambda_2' \):

\[ X^{h, A^{h, \lambda_2}, \lambda_2}(\omega) = \max \left\{ h, (U')^{-1}(A^{h, \lambda_2} \cdot \frac{dQ}{dP}(\omega)) \right\} \]

\[ = \max \left\{ h, (U')^{-1}(A^{h, \lambda_2'} \cdot \frac{dQ}{dP}(\omega)) \right\} \]

\[ = X^{h, A^{h, \lambda_2'}, \lambda_2} (\omega) \]

\( \forall \omega \in \Omega \) with \( A^{h, \lambda_2} \cdot \frac{dQ}{dP}(\omega) > U'(h) + \lambda_2' \geq U'(h) \):

\[ X^{h, A^{h, \lambda_2'}, \lambda_2}(\omega) = (U')^{-1}(A^{h, \lambda_2} \cdot \frac{dQ}{dP}(\omega) - \lambda_2') \]

\[ \leq \min \left\{ h, (U')^{-1}(A^{h, \lambda_2'} \cdot \frac{dQ}{dP}(\omega) - \lambda_2) \right\} \]

\[ = X^{h, A^{h, \lambda_2'}, \lambda_2}(\omega) \]

\( \Rightarrow \left\{ X^{h, A^{h, \lambda_2'}, \lambda_2} \neq X^{h, A^{h, \lambda_2'}, \lambda_2'} \right\} = \left\{ X^{h, A^{h, \lambda_2}, \lambda_2'} < X^{h, A^{h, \lambda_2}, \lambda_2} \right\} \cap \left\{ X^{h, A^{h, \lambda_2'}, \lambda_2} < h \right\} \).

\[ E \left[ (h - X^{h, A^{h, \lambda_2}, \lambda_2'})^+ \right] = E \left[ (h - X^{h, A^{h, \lambda_2}, \lambda_2})^+ \right] \] implies that

\[ 0 = P \left( \left\{ X^{h, A^{h, \lambda_2}, \lambda_2'} < X^{h, A^{h, \lambda_2}, \lambda_2} \right\} \cap \left\{ X^{h, A^{h, \lambda_2'}, \lambda_2} < h \right\} \right) \]

\[ = P \left( X^{h, A^{h, \lambda_2}, \lambda_2'} \neq X^{h, A^{h, \lambda_2'}, \lambda_2} \right). \]

- Case \( A^{h, \lambda_2'} \neq +\infty \):
∀ω ∈ Ω with \( \Lambda^{h,\lambda_2} \cdot \frac{dQ}{dP}(\omega) > U'(h) + \lambda_2^2 \geq U'(h) \):

\[
X^{h,\Lambda^{h,\lambda_2},\lambda_2^2}(\omega) = (U')^{-1} \left( \Lambda^{h,\lambda_2} \cdot \frac{dQ}{dP}(\omega) - \lambda_2^2 \right)
< \min \left\{ h, (U')^{-1} \left( \Lambda^{h,\lambda_2^2} \cdot \frac{dQ}{dP}(\omega) - \lambda_2^2 \right) \right\}
= X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2}(\omega)
\]

⇒ \( P \left( X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} < h \right) = P \left( X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} \neq X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} \right) = 0. \)

⇒ \( E \left[ (h - X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2})^+ \right] = 0. \)

- Applying Fact 3.53(d), we observe that

\[
\left\{ X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} \neq X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} \right\} = \left\{ X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} < X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} \right\}
\subseteq \left\{ X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} < X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} \right\} = \left\{ X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} \neq X^{h,\Lambda^{h,\lambda_2^2},\lambda_2^2} \right\}.
\]

\( \square \)

**Proof of Lemma 3.68:**

- **Case \((h - c) \cdot \alpha = 0:\)**

  ⇒ \( h = c. \)

  (Lemma 3.40(d)) ⇒ \( Q \left( \frac{dQ}{dP} < \tau(h) \right) \cdot h + Q \left( \frac{dQ}{dP} \geq \tau(h) \right) \cdot u_0 = Q(\Omega) \cdot h. \)

  ⇒ \( x_0 > Q(\Omega) \cdot h. \)

  ⇒ \( \Lambda^{h,\infty} \) is well defined.

  ⇒ \( +\infty \in \left\{ \lambda_2 \in L^h_2 \mid E \left[ (h - X^{h,\Lambda^{h,\lambda_2},\lambda_2})^+ \right] = (h - c) \cdot \alpha \right\}. \)

- **Case \((h - c) \cdot \alpha > 0:\)**

  \((h \geq c \geq u_0) \Rightarrow h > u_0.\)

  Let \( \lambda_1 := \lim_{\lambda_2 \to +\infty} \Lambda^{h,\lambda_2}. \)

  - \( \lambda_1 < +\infty: \)

    We find out by applying Lemma 3.65 that
    \[
    \lim_{\lambda_2 \to +\infty} E \left[ (h - X^{h,\Lambda^{h,\lambda_2},\lambda_2})^+ \right] = 0 < (h - c) \cdot \alpha \leq E \left[ (h - X^{h,\Lambda^{h,\lambda_1},\lambda_1})^+ \right].
    \]

  - \( \lambda_1 = +\infty: \)

    Let \( i \in [0, +\infty] \) be defined as in Lemma 3.65.

    ⇒ \( Q \left( \frac{dQ}{dP} \leq \bar{i} \right) \cdot h + Q \left( \frac{dQ}{dP} > \bar{i} \right) \cdot u_0 = \bar{i} \cdot u_0 > Q \left( \frac{dQ}{dP} < \tau(h) \right) \cdot h + Q \left( \frac{dQ}{dP} \geq \tau(h) \right) \cdot u_0. \)

    ⇒ \( Q \left( \frac{dQ}{dP} \geq \bar{i} \right) < Q \left( \frac{dQ}{dP} \geq \tau(h) \right). \)

    ⇒ \( \{ \omega \in \Omega \mid \frac{dQ}{dP} \geq \bar{i} \} \subseteq \{ \omega \in \Omega \mid \frac{dQ}{dP} \geq \tau(h) \}. \)

    ⇒ \( Q \left( \tau(h) \leq \frac{dQ}{dP} \leq \bar{i} \right) > 0. \)
The closedness is due to Lemma 3.63, too.

Proof of Fact 3.72:

(a) Obvious consequence of Lemma 3.17.

(b) First

\[
\begin{align*}
\left( U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ & \leq \left( U \left( X_Q^{h,1,3} \right) \right)^+ \\
& \leq \left( U \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right)^+ + \left\{ \begin{array}{ll} 0 & \text{if } h = u_0 \\ (U (h))^+ & \text{if } h > u_0 \end{array} \right. \\
\end{align*}
\]

Secondly, \( X_Q^{h,1,3} = (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \) if \( h = u_0 \). Thirdly, \( \lambda_3 > 0 \) implies

\[
\begin{align*}
\left( -U \left( (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right) \right) \right)^+ & \leq \left( -U \left( X_Q^{h,1,3} \right) \right)^+ \\
& \leq \left( -U \left( (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right) \right) \right)^+ + \left( -U (h) \right)^+. \\
\end{align*}
\]

The three statements can be shown to hold true, simply by applying the expectancy operator to the respective inequalities.

(c) We can apply the same idea as in the proof of Lemma 3.37, because

\[
u_0 \leq (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \leq X_Q^{h,1,3} \leq (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) + (h - u_0).
\]

(d) Obvious consequence of Lemma 3.17.

(e) Result of Assumption 3.19, (c) and the Dominated Convergence Theorem.

(f) Implication of the Monotone Convergence Theorem.

(g) For all \((h, \lambda_1, \lambda_3) \in L_Q:\)

\[
\begin{align*}
(h - X_Q^{h,1,3})^+ &= \left( (h - (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right) \right)^+ - \left( (U')^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) - h \right)^+ \\
&= \left( h - (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right) \right)^+ \\
\end{align*}
\]

\[\square\]
(h) Due to \( 0 \leq \left( h - (U')^{-1} \left( \lambda_3 \cdot \frac{dQ}{dP} \right) \right)^+ \leq (h - u_0) \), the boundedness of converging sequences in \( \mathbb{R} \), Lemma 3.17 and the Dominated Convergence Theorem.

(i) The surjectivity is a consequence of

\[
\mathbb{E}_Q \left[ \left( h - (U')^{-1} \left( \frac{U'(h) \cdot \frac{dQ}{dP}}{t_{\text{max}}} \right) \right)^+ \right] = (h - u_0) \cdot Q(\Omega)
\]

We deduce the strict monotonicity by looking at \( \frac{U'(h)}{t_{\text{max}}} \leq y_1 < y_2 \leq +\infty, \) setting \( \varepsilon := t_{\text{max}} - \frac{U'(h)}{y_2} > 0 \) in Fact 3.2(f) and applying the strict monotonicity of \((U')^{-1}\) in \((0, +\infty]\).

(j) The function is strictly decreasing because of Lemma 3.17 and the strict monotonicity of the integral, the continuity is due to the Monotone Convergence Theorem and the surjectivity follows from the continuity and the values \( \mathbb{E}_Q \left[ (U')^{-1} \left( 0 \cdot \frac{dQ}{dP} \right) \right] = \mathbb{E}_Q[\mp \infty] = +\infty \) and \( \mathbb{E}_Q \left[ (U')^{-1} \left( +\infty \cdot \frac{dQ}{dP} \right) \right] = \mathbb{E}_Q[u_0] = u_0 \cdot Q(\Omega) \).

(k) \( (h - c) \cdot \frac{\alpha}{\alpha \geq 0} \leq (h - u_0) \cdot \frac{\alpha}{\alpha \geq 0} \leq (h - u_0) \cdot Q(\Omega) \).

Proof of Fact 3.73:

(a) If \( \alpha \in (0, Q(\Omega)) \):

\[
\mathbb{E}_Q \left[ (h - (U')^{-1} \left( +\infty \cdot \frac{dQ}{dP} \right))^{+} \right] = (h - u_0) \cdot Q(\Omega) > (h - u_0) \cdot \alpha \geq (h - c) \cdot \alpha.
\]

If \( \alpha = Q(\Omega) \): \( \mathbb{E}_Q \left[ (h - (U')^{-1} \left( +\infty \cdot \frac{dQ}{dP} \right))^{+} \right] = (h - u_0) \cdot Q(\Omega) > (h - c) \cdot \alpha. \)

Hence, in both cases: \( \lambda^h_Q \neq +\infty. \)

(b) Consequence of Fact 3.72(j).

(c) If \( \Psi^{h,x_0}_Q = \lambda^h_Q, \) \( P\left( \frac{dQ}{dP} < t_{\text{min}} \right) = 0 \) (Fact 3.2(a)) yields

\[
\mathbb{E}_Q \left[ X^h_{\text{max}} \left( \frac{t_{\text{min}}}{t_{\text{min}}, \Psi^{h,x_0}_Q} \right), \Psi^{h,x_0}_Q \right] = \mathbb{E}_Q \left[ h - (h - (U')^{-1} \left( \Psi^{h,x_0}_Q \cdot \frac{dQ}{dP} \right))^{+} \right] = h \cdot Q(\Omega) - (h - c) \cdot \alpha \leq x_0.
\]
If $\Psi^h_{Q} = \lambda^x_0$, $\mathbb{E}_Q \left[ X^h_{Q} \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi^h_{Q}, \lambda^x_0 \right\} \right] \leq \mathbb{E}_Q \left[ X^h_{Q} \lambda^x_0 \lambda^x_0 \right] = x_0$.

(d) Let $M := \left\{ \lambda_1 \in (0, +\infty) \cap \left[ \Psi^h_{Q}, \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi^h_{Q}, \lambda^x_0 \right\} \right] \right\} \mathbb{E}_Q \left[ X^h_{Q} \lambda_1 \Psi^h_{Q} \right] \leq x_0 \right\}.

\((c), \frac{U'(h)}{t_{\min}} \in (0, +\infty) \Rightarrow \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi^h_{Q} \right\} \in M, \text{ i.e. } M \text{ is non-empty.}\)

(e) Let $\lambda^h_1$ and $\lambda^h_2$ be elements of $Q$, $\left[ \Psi^h_{Q}, \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi^h_{Q}, \lambda^x_0 \right\} \right]$ with $\lambda^h_1 < \lambda^h_2$.

$\Rightarrow \lambda^h_1 < \frac{U'(h)}{t_{\min}}$ and $X^h_{Q} \lambda^h_1, \Psi^h_{Q} \geq X^h_{Q} \lambda^h_2, \Psi^h_{Q}$.

$P \left( X^h_{Q} \lambda^h_1, \Psi^h_{Q} \geq X^h_{Q} \lambda^h_2, \Psi^h_{Q} \right) \geq P \left( \frac{dQ_{\min}}{Q} \in \left( \frac{U'(h)}{\lambda^h_1} \right) \right) > 0.$

\((P \sim Q) \Rightarrow Q \left( X^h_{Q} \lambda^h_1, \Psi^h_{Q} \geq X^h_{Q} \lambda^h_2, \Psi^h_{Q} \right) > 0.$

(f) If $\Psi^h_{Q} > 0$, $\mathbb{E}_Q \left[ X^h_{Q} \Psi^h_{Q} \Psi^h_{Q} \right] \geq \mathbb{E}_Q \left[ X^h_{Q} \lambda^x_0, \lambda^x_0 \right] = x_0$.

If $\Psi^h_{Q} = 0$, Fact 3.72(f) ensures the existence of $\lambda_1 \in (0, +\infty) \cap \left[ \Psi^h_{Q}, \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi^h_{Q}, \lambda^x_0 \right\} \right]$ with $\mathbb{E}_Q \left[ X^h_{Q} \lambda_1, \Psi^h_{Q} \right] \geq x_0$.

The existence of $\lambda_1 \in (0, +\infty) \cap \left[ \Psi^h_{Q}, \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi^h_{Q}, \lambda^x_0 \right\} \right]$ with $\mathbb{E}_Q \left[ X^h_{Q} \lambda_1, \Psi^h_{Q} \right] = x_0$ is hence due to (c) and Fact 3.72(e). The uniqueness is a consequence of (e).

\[\square\]

**Proof of Fact 3.74:** First of all, $x_0 = \mathbb{E}_Q \left[ X^h_{Q} \lambda^x_0, \Psi^h_{Q} \right] \geq \mathbb{E}_Q \left[ (U')^{-1} \left( \lambda^x_0 \cdot \frac{dQ}{dP} \right) \right]$ and Fact 3.72(j) yield $\Lambda^x_0 \geq \lambda^x_0$.

Secondly,

\[\mathbb{E}_Q \left[ \left( h - X^h_{Q} \lambda^x_0, \Psi^h_{Q} \right)^+ \right] = \mathbb{E}_Q \left[ \left( h - \left( U' \right)^{-1} \left( \Psi^h_{Q}, \frac{dQ}{dP} \right) \right)^+ \right] \leq \mathbb{E}_Q \left[ \left( h - \left( U' \right)^{-1} \left( \lambda^h_0 \cdot \frac{dQ}{dP} \right) \right)^+ \right] = (h - c) \alpha.\]

Finally, $\mathbb{E}_Q \left[ \left( h - X^h_{Q} \lambda^x_0, \Psi^h_{Q} \right)^+ \right] < (h - c) \alpha$ ensures $\Psi^h_{Q} < \lambda^h_0$ which means $\Psi^h_{Q} = \lambda^x_0 > 0$. Hence $\Psi^h_{Q} \in (0, +\infty) \cap \left[ \Psi^h_{Q}, \max \left\{ \frac{U'(h)}{t_{\min}}, \Psi^h_{Q}, \lambda^x_0 \right\} \right]$ with $\mathbb{E}_Q \left[ X^h_{Q} \Psi^h_{Q} \right] = x_0$ which translates according to Fact 3.73(f) into $\Lambda^x_0 = \Psi^h_{Q}$. Now $\mathbb{E}_Q \left[ \left( h - \left( U' \right)^{-1} \left( +\infty \cdot \frac{dQ}{dP} \right) \right)^+ \right] = (h - u_0) \cdot Q (\Omega) \geq (h - c) \cdot \alpha$ ensures that $\Psi^h_{Q} < +\infty$.

\[\square\]
Proof of Lemma 3.77: For all \( h \in H_P \) with \( E \left[ \left( h - X^{h,A^{h^0}} \right)^+ \right] \geq (h - c) \cdot \alpha \):

\[
(h - c) \cdot \alpha = E \left[ \left( h - X^{h,A^{h^0} \cdot \phi^h} \right)^+ \right] \leq \left( h - u_0 \right) \cdot P \left( X^{h,A^{h^0} \cdot \phi^h} < h \right).
\]

Hence \( P \left( X^{h,A^{h^0} \cdot \phi^h} < h \right) = 0 \) implies that \( h = c \).

Therefore \( h > c \) implies that \( P \left( X^{h,A^{h^0} \cdot \phi^h} < h \right) > 0 \), thus \((*)\) is a strict inequality due to Fact 3.69(b) and hence \( P \left( X^{h,A^{h^0} \cdot \phi^h} < h \right) > \alpha \).

We distinguish the following two cases:

- \( c \in H_P \):
  \[
  \Rightarrow X^{c,A^{c^0} \cdot 0} \geq u_0.
  \]
  \[
  \Rightarrow E \left[ (c - X^{c,A^{c^0} \cdot 0})^+ \right] = E \left[ (u_0 - X^{c,A^{c^0} \cdot 0})^+ \right] = 0.
  \]
  \[
  \Rightarrow \Psi^c = 0.
  \]
  \[
  \Rightarrow \forall h \in H_P : \Psi^h \geq 0 = \Psi^c.
  \]

- Let \( h_1 \) and \( h_2 \) be elements of the set \( \{ h \in H_P \mid E \left[ \left( h - X^{h,A^{h^0}} \right)^+ \right] \geq (h - c) \cdot \alpha \} \) with \( c < h_1 < h_2 \). Then
  \[
  E \left[ \left( h_2 - X^{h_1,A^{h_1},\phi^1,\psi^1} \right)^+ \right] \geq E \left[ \left( h_1 - X^{h_1,A^{h_1},\phi^1,\psi^1} \right)^+ \right] = (h_2 - h_1) \cdot P \left( X^{h_1,A^{h_1},\phi^1,\psi^1} < h_1 \right) \]
  \[
  > (h_2 - c) \cdot \alpha.
  \]

Because of Lemma 3.64 and the definition of \( \Psi^h \), we know that \( \Psi^{h_1} \leq \Psi^{h_2} \).

Proof of Lemma 3.78: \( \exists \ (s_n) : \lim_{n \to \infty} \Psi_h^{s_n} = \Psi_+ \).

- Case \( \Psi_+ = +\infty \) and \( \exists \lambda_1 < +\infty \) such that \( \Lambda^{h^{s_n}} \cdot \Psi^{h^{s_n}} \leq \lambda_1 \) for \( \forall n \in \mathbb{N} \):
  \[
  ([0, \lambda_1] \text{ is compact} \Rightarrow \exists \text{ subsequence } (m_n) \text{ of } (s_n) \text{ and } \lambda_1^0 \in [0, \lambda_1] \text{ such that } \]
  \[
  \lim_{n \to +\infty} \Lambda^{m_n} \cdot \Psi_{h^{m_n}} = \lambda_1^0.
  \]

  (Lemma 3.49) \( \Rightarrow \lim_{n \to +\infty} E \left[ \left( h_{m_n} - X^{h_{m_n},A^{h_{m_n}} \cdot \phi^{h_{m_n}},\psi^{h_{m_n}}} \right)^+ \right] = E \left[ \left( h_0 - X^{h_0,\lambda_1^0,+,+\infty} \right)^+ \right] \).

  (\( \Psi_+ = +\infty \Rightarrow \exists \eta_0 \in \mathbb{N} : \forall n \geq \eta_0 : \Psi_{h^{m_n}} > 0. \))

  (Fact 3.69(e)) \( \Rightarrow \exists \eta_0 \in \mathbb{N} : \forall n \geq \eta_0 : E \left[ \left( h_{m_n} - X^{h_{m_n},A^{h_{m_n}} \cdot \phi^{h_{m_n}},\psi^{h_{m_n}}} \right)^+ \right] = (h_{m_n} - c) \cdot \alpha. \)

  \( \Rightarrow (h_0 - c) \cdot \alpha = \lim_{n \to +\infty} (h_{m_n} - c) \cdot \alpha = E \left[ \left( h_0 - X^{h_0,\lambda_1^0,+,+\infty} \right)^+ \right] = 0. \)

  \( \Rightarrow h_0 = c. \)

  \( \Rightarrow x_0 > Q \left( \frac{dQ}{d\mathcal{H}} < \tau (h_0) \right) \cdot h_0 + Q \left( \frac{dQ}{d\mathcal{H}} \geq \tau (h_0) \right) \cdot u_0 \text{ Lemma 3.40(d)} \)

\( = Q (\Omega) \cdot h_0 \).
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

- Case $h_0 = u_0$:
  
  $x_0 \xrightarrow{\text{HPR}} Q \left( \frac{dQ}{dP} < \tau (h_0) \right) \cdot h_0 + Q \left( \frac{dQ}{dP} \geq \tau (h_0) \right) \cdot u_0 = Q (\Omega) \cdot h_0.$

- Case $h_0 > u_0$, $\Psi_+ = +\infty$, $\lim_{n \to +\infty} \Lambda_{h_n, \Psi_{h_n}} = +\infty$:

  Without any loss of generality, we can assume that for some $\varepsilon_{\text{max}} > 0$: $\forall n \in \mathbb{N}_0$: $h_n > u_0 + \varepsilon_{\text{max}}$.

  $\Rightarrow \forall n \in \mathbb{N}_0$: $\Lambda_{h_n, \Psi_{h_n}} \in (0, +\infty)$.

  $\lim_{n \to +\infty} \Lambda_{h_n, \Psi_{h_n}} = +\infty$, Dominated Convergence Theorem $\Rightarrow \forall t \in \mathbb{R}$:

  $\lim_{n \to +\infty} \int \{ \frac{dQ}{dP} < t \} X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} \cdot dQ = h_0 \cdot Q \left( \frac{dQ}{dP} < t \right) \Rightarrow \forall n \in \mathbb{N}_0$

  We have to distinguish two cases:

  - $h_0 = c$:
    $\Rightarrow \tau (h_0) = +\infty$.

    $E_Q \left[ X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} \right] \leq E_Q \left[ X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} \right] \to h_0 \cdot Q (\Omega) (n \to +\infty)$

  - $h_0 > c$:

    Let $\varepsilon > 0$ with $\varepsilon < \varepsilon_{\text{max}}$. Hence

    $\left\{ \omega \in \Omega \mid X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} (\omega) \in (u_0 + \varepsilon, h_n) \right\}$

    $\Rightarrow \left\{ \omega \in \Omega \mid \frac{dQ}{dP} (\omega) \in \left( \frac{U' (h_n) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}}, \frac{U' (u_0 + \varepsilon) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}} \right) \right\}$.

    $\lim_{n \to +\infty} \Lambda_{h_n, \Psi_{h_n}} = +\infty \Rightarrow \lim_{n \to +\infty} \left( \frac{U' (h_n) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}} - \frac{U' (u_0 + \varepsilon) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}} \right) = 0$.

    $(F_P \frac{dQ}{dP}$ is continuous, Lemma 2.3) $\Rightarrow \lim_{n \to +\infty} P \left( \frac{dQ}{dP} \in \left( \frac{U' (h_n) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}}, \frac{U' (u_0 + \varepsilon) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}} \right) \right) = 0$.

    Since $\Psi_+ = +\infty$, we can assume without any loss of generality that $\forall n \in \mathbb{N}$: $\Psi_{h_n} > 0$.

    $\Rightarrow \forall n \in \mathbb{N}$: $E \left[ \left( h_n - X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} \right)^+ \right] = (h_n - c) \cdot \alpha.$

    $P \left( X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} < h_n \right) \cdot (h_n - u_0) \geq E \left[ \left( h_n - X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} \right)^+ \right]$

    $\Rightarrow P \left( \frac{dQ}{dP} \in \left( \frac{U' (h_n) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}}, \frac{U' (u_0 + \varepsilon) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}} \right) \right) \cdot (h_n - u_0)$

    $\geq (h_n - c) \cdot \alpha - P \left( X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} \leq u_0 + \varepsilon \right) \cdot (h_n - u_0)$.

    $\Rightarrow P \left( \frac{dQ}{dP} \in \left( \frac{U' (h_n) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}}, \frac{U' (u_0 + \varepsilon) + \Psi_{h_n}}{\Lambda_{h_n, \Psi_{h_n}}} \right) \right) \geq \frac{(h_n - c) \cdot \alpha}{h_n - u_0} - P \left( X_{h_n}^{\Lambda_{h_n, \Psi_{h_n}}, \Psi_{h_n}} \leq u_0 + \varepsilon \right).$
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

Proof of Lemma 3.80: \( \forall h \in H_P: \Lambda^h, \Psi^h \geq 0. \)

\( \forall n \in \mathbb{N}_0: \Lambda^{h_n}, \Psi^{h_n} \neq +\infty, \) because of Fact 3.69(b).

Suppose that \( \{ \Lambda^{h_n}, \Psi^{h_n} \mid n \in \mathbb{N}_0 \} \) is not bounded from above, i.e. without any loss of generality:
6.2. PART 3: STATIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

\[ \lim_{n \to +\infty} \Lambda_{h_n, \Psi^{h_n}} = +\infty. \]

From

\[ x_0 = E_Q \left[ X_{h_n, \Lambda_{h_n, \Psi^{h_n}}, \Psi^{h_n}} \right] \leq E_Q \left[ x_{h_0, \Lambda_{h_n, \Psi^{h_n}}, \Psi^+} \right] \]

\[ \leq E_Q \left[ x_{h_0, \Lambda_{h_n, \Psi^{h_n}, \Psi^+}} \right] + |h_0 - h_n| \cdot Q(\Omega) \to 0 \quad (h_n \to h_0) \]

we learn that \( h_0 \geq \frac{x_0}{Q(\Omega)} \).

(Lemma 3.78) \( \Rightarrow \lim_{n \to +\infty} \sup \Psi^{h_n} < +\infty. \)

\[ u_0 \cdot Q(\Omega) \leq Q \left( \frac{dQ}{dP} < \tau (h_0) \right) \cdot h_0 + Q \left( \frac{dQ}{dP} \geq \tau (h_0) \right) \cdot u_0 \]

\[ < x_0 = E_Q \left[ X_{h_0, \Lambda_{h_n, \Psi^{h_n}, \Psi^+}} \right] \leq E_Q \left[ X_{h_0, \Lambda_{h_n, \Psi^{h_n}, \Psi^+}} \right] + |h_0 - h_n| \cdot Q(\Omega) \to 0 \quad (h_n \to h_0) \]

This is a contradiction since \( x_0 > u_0 \cdot Q(\Omega) \) is fixed. \( \square \)

**Proof of Proposition 3.86:** Let \((x, y) \in A\) be fixed and \(y_0 > y\).

- If \( y_0 \notin I_2\), \( \{(\tilde{x}, \tilde{y}) \in A \mid \tilde{y} > y_0 \} = \emptyset \), because \( I_2 \) is connected.

- Case \( y_0 \in I_2 \) and \((x, y_0) \notin A\):
  \( f_1(x) \leq y < y_0 \) \( \Rightarrow y_0 > f_2(x) \) and \( f_2(x) \in \mathbb{R} \).
  (a) \( \Rightarrow \exists \delta^+_x > 0 : \forall \tilde{x} \in I_1, |\tilde{x} - x| < \delta^+_x \Rightarrow |f_2(\tilde{x}) - f_2(x)| < (y_0 - f_2(x)) \).
  \( \Rightarrow \forall \tilde{x} \in I_1 \) with \(|\tilde{x} - x| < \delta^+_x \): \( f_2(\tilde{x}) < y_0 \), i.e. \( \{(\tilde{x}, \tilde{y}) \in A \mid |\tilde{x} - x| < \delta^+_x, \tilde{y} > y - \varepsilon \} = \emptyset \).

- Case \( y_0 \in I_2 \) and \((x, y_0) \in A\):
  \( f_1(x) \leq y < y_0 \), (a) \( \Rightarrow \exists \delta^+_x > 0 : \forall \tilde{x} \in I_1, |\tilde{x} - x| < \delta^+_x \): \( f_1(\tilde{x}) < y_0 \).
  (b) \( \Rightarrow \delta^+_x > 0 : \forall \tilde{x} \in I_1, g(\tilde{x}, y_0) > g(x, y_0) \).
  (c) \( \Rightarrow \exists \delta^+_x > 0 : \forall \tilde{x} \in I_1 \) with \((\tilde{x}, y_0) \in A\) and \(|x - \tilde{x}| < \delta^+_x \): \( |g(\tilde{x}, y_0) - g(x, y_0)| < \delta^+_x \).
  We define \( \delta^+_x := \min \{ \delta^+_1, \delta^+_2, \delta^+_3 \} > 0 \).
  \( \forall (\tilde{x}, \tilde{y}) \in A \) with \(|\tilde{x} - x| < \delta^+_x \) and \( \tilde{y} > y_0 \):
  \( f_1(\tilde{x}) < y_0 \leq \tilde{y} \leq f_2(\tilde{x}) \Rightarrow (\tilde{x}, y_0) \in A \) and hence
  \[ g(\tilde{x}, \tilde{y}) - g(x, y) \]
  \[ > \delta^+_x \geq \delta^+_x. \]

The proof of the other statement can be done analogously. \( \square \)
6.3 Part 4: Dynamic Optimization Problems with Risk Constraints

Proof of Lemma 4.1: First of all, $S_0(\cdot)$ is bounded away from zero. Therefore the discount factor process is bounded and thus the discounted wealth process is a local martingale and bounded from below. It is well known that every local martingale bounded from below is a supermartingale. This fact can be proven by applying Fatou’s Lemma to the local martingale’s localizing sequence. Finally, $(h - X_{x_0,\Pi}(T))^+$ is bounded from below (by zero) as well as from above (by $(h - b)^+$).

Proof of Lemma 4.2: $\exp \left[ - \int_0^T r(t)dt - \frac{1}{2} \int_0^T \|\theta(t)\|^2 dt \right]$ is a constant and $\int_0^T \theta(t)\,dW(t)$ is normally distributed under $P$ with mean 0 and variance $\int_0^T \|\theta(t)\|^2 dt$. Hence $\frac{dP^*}{dP}$ is continuous $\iff \int_0^T \|\theta(t)\|^2 dt \neq 0$.

Proof of Proposition 4.3: The pricing formulae can be verified as in the classic case with constant coefficients. We will only verify the price of the European Call Option: First of all, $W_Q$ is Brownian motion with respect to $P^*$ with

$$\frac{dP^*}{dP} = \exp \left[ - \int_0^T \theta(t)^\top dW(t) - \frac{1}{2} \int_0^T \|\theta(t)\|^2 dt \right].$$

For all $t \in [0, T]$ and $\omega \in \Omega$:

$$S_i(T)(\omega) \geq K \iff S_i(t)(\omega) \cdot \exp \left[ \int_t^T \mu_i(s)ds - \frac{1}{2} \int_t^T \|\sigma_i(s)\|^2 ds + \int_t^T \sigma_i(s)^\top dW_Q(s)(\omega) \right] \geq K$$

$$\iff S_i(t)(\omega) \cdot \exp \left[ \int_t^T r(s)ds - \frac{1}{2} \int_t^T \|\sigma_i(s)\|^2 ds + \int_t^T \sigma_i(s)^\top dW_Q(s)(\omega) \right] \geq K$$

$$\iff \int_t^T \sigma_i(s)^\top dW_Q(s)(\omega) \geq \ln \left( \frac{K}{S_i(t)(\omega)} \right) - \int_t^T r(s)ds + \frac{1}{2} \int_t^T \|\sigma_i(s)\|^2 ds$$

$$\iff \frac{\int_t^T \sigma_i(s)^\top dW_Q(s)(\omega)}{\sqrt{\int_t^T \|\sigma_i(s)\|^2 ds}} \geq \frac{\ln \left( \frac{K}{S_i(t)(\omega)} \right) - \int_t^T r(s)ds + \frac{1}{2} \int_t^T \|\sigma_i(s)\|^2 ds}{\sqrt{\int_t^T \|\sigma_i(s)\|^2 ds}}.$$

Using $A(t) := \left\{ x \in \mathbb{R} \mid x \geq \frac{\ln \left( \frac{K}{S_i(t)(\omega)} \right) - \int_t^T r(s)ds + \frac{1}{2} \int_t^T \|\sigma_i(s)\|^2 ds}{\sqrt{\int_t^T \|\sigma_i(s)\|^2 ds}} \right\}$ and the insight that $\forall x \in \mathbb{R}$:
1 − Φ(x) = Φ(−x), we can compute the price of the European Call Option at time \( t \in [0, T] \):

\[
\frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ (S_t(T) - K)^+ \mid \mathcal{F}(t) \right]
\]

**Lemma 2.4**

\[
\mathbb{E}_P \left[ \exp \left[ - \int_t^T r(s) \, ds \right] \cdot \mathbb{I}_{[K, +\infty)}(S_t(T)) \cdot (S_t(T) - K) \mid \mathcal{F}(t) \right] = S_t(t) \cdot \left( \phi(x) \cdot \exp \left[ -\frac{1}{2} \int_t^T \sigma_i(s)^2 \, ds + x \cdot \sqrt{\int_t^T \|\sigma_i(s)\|^2 \, ds} \right] \right)
\]

\[
- \exp \left[ -\int_t^T r(s) \, ds \right] \cdot K \cdot \Phi \left( \frac{-\ln \left( \frac{K}{S_t(t)} \right) + \int_t^T r(s) \, ds + \frac{1}{2} \int_t^T \|\sigma_i(s)\|^2 \, ds}{\sqrt{\int_t^T \|\sigma_i(s)\|^2 \, ds}} \right)
\]

\[
= S_t(t) \cdot \Phi \left( d_1^2 \left( t, S_t(t) \right) \right) - \exp \left[ -\int_t^T r(s) \, ds \right] \cdot K \cdot \Phi \left( d_2^2 \left( t, S_t(t) \right) \right)
\]

**Derivation of the hedging strategies:**

\[
\frac{\partial d_1}{\partial t} \left( t, x \right) = -\frac{r(t)}{\sqrt{\int_t^T \|\sigma_i(s)\|^2 \, ds}} + \frac{1}{2} \cdot \frac{d_1^2 \left( t, x \right)}{\int_t^T \|\sigma_i(s)\|^2 \, ds} \cdot \|\sigma_i(t)\|^2
\]

\[
\frac{\partial d_2}{\partial t} \left( t, x \right) = -\frac{r(t) + \frac{1}{2} \|\sigma_i(t)\|^2}{\sqrt{\int_t^T \|\sigma_i(s)\|^2 \, ds}} + \frac{1}{2} \cdot \frac{d_2^2 \left( t, x \right)}{\int_t^T \|\sigma_i(s)\|^2 \, ds} \cdot \|\sigma_i(t)\|^2
\]

\[
= \frac{\partial d_1}{\partial t} \left( t, x \right) + \frac{1}{2} \cdot \frac{\|\sigma_i(t)\|^2}{\sqrt{\int_t^T \|\sigma_i(s)\|^2 \, ds}}
\]
6.3. PART 4: DYNAMIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

\[ f(t, x) := \frac{x}{S_0(t)} \cdot \Phi(d_1(t, x)) - \frac{K}{S_0(t)} \cdot \Phi(d_2(t, x)) \]

\[
\frac{\partial f}{\partial t}(t, x) = \frac{x}{S_0(t)} \cdot (-r(t)) \cdot \Phi(d_1(t, x)) + \frac{x}{S_0(t)} \cdot \varphi(d_1(t, x)) \cdot \frac{\partial d_1}{\partial t}(t, x) - \frac{K}{S_0(T)} \cdot \varphi(d_2(t, x)) \cdot \frac{\partial d_2}{\partial t}(t, x) 
\]

\[
= -\frac{\Phi(d_1(t, x))}{S_0(t)} \cdot x \cdot r(t) - \frac{K}{S_0(T)} \cdot \varphi(d_2(t, x)) \cdot \frac{1}{2} \cdot \sqrt{T \| \sigma_i(s) \|_2^2} ds 
\]

\[
+ \frac{\partial d_1}{\partial t}(t, x) \cdot \left( \frac{x}{S_0(t)} \cdot \varphi(d_1(t, x)) - \frac{K}{S_0(T)} \cdot \varphi(d_2(t, x)) \right),
\]

\[
= 0
\]

\[
\frac{\partial f}{\partial x}(t, x) = \frac{\Phi(d_1(t, x))}{S_0(t)} + \frac{\frac{x}{S_0(t)} \cdot \varphi(d_1(t, x)) - \frac{K}{S_0(T)} \cdot \varphi(d_2(t, x))}{x \cdot \sqrt{T \| \sigma_i(s) \|_2^2} ds}
\]

\[
\frac{\partial^2 f}{(\partial x)^2}(t, x) = \frac{\varphi(d_1(t, x))}{S_0(t)} \cdot \frac{1}{x \cdot \sqrt{T \| \sigma_i(s) \|_2^2} ds}.
\]

Hence, we know by Itô’s formula that

\[
df(t, S_i(t)) = \frac{\Phi(d_1(t, x))}{S_0(t)} \cdot S_i(t) \cdot \sigma_i(t)^\top dW_Q(t).
\]

\[ g \text{ with } g(t, x) := \Phi(-d_2(t, x)) \text{ has the following partial derivatives:} \]

\[
\frac{\partial g}{\partial t}(t, x) = -\varphi(-d_2(t, x)) \cdot \frac{\partial d_2}{\partial t}(t, x),
\]

\[
\frac{\partial g}{\partial x}(t, x) = \frac{-\varphi(-d_2(t, x))}{x \cdot \sqrt{T \| \sigma_i(s) \|_2^2} ds}
\]

\[
\frac{\partial^2 g}{(\partial x)^2}(t, x) = \frac{\varphi(-d_2(t, x)) \cdot \frac{d^2 d_2(t, x)}{dx^2} + \varphi(-d_2(t, x))}{x^2 \cdot \sqrt{T \| \sigma_i(s) \|_2^2} ds}.
\]

Using Itô’s formula again, we can conclude that

\[
d\frac{p}{S_0(T)} g(t, S_i(t)) = \frac{p}{S_0(T)} \cdot \frac{-\varphi(-d_2(t, S_i(t)))}{S_i(t) \cdot \sqrt{T \| \sigma_i(s) \|_2^2} ds} \cdot S_i(t) \cdot \sigma_i(t)^\top dW_Q(t).
\]

**Proof of Proposition 4.4:** \( \forall \gamma \in (-\infty, 1) : \)

\[
E_P \left[ \left( \frac{dQ}{dP} \right) \right] = \exp \left[ \int_0^T \left( -r(t) + \frac{1}{2} \cdot \frac{1}{\gamma - 1} \cdot \| \theta(t) \|_2 \right) \cdot dt \right]
\]

\[
\cdot E_P \left[ \exp \left[ \int_0^T \frac{-\gamma}{\gamma - 1} \cdot \theta(t)^\top dW(t) - \frac{1}{2} \int_0^T \| -\frac{\gamma}{\gamma - 1} \cdot \theta(t) \|_2^2 dt \right] \right]
\]

\[
< + \infty.
\]

141
Let $\lambda_1 \in (0, +\infty)$ be fixed.

$$
\Rightarrow (U'_0)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) = \left( \lambda_1 \cdot \exp \left[ - \int_0^T r(t) dt \right] \right) \frac{1}{1 - \gamma}, \text{ for all } \gamma \in (-\infty, 1) \text{ and thus}
$$

$$
\mathbb{E}_P \left[ U \left( (U'_0)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right] \leq \ln \left( \lambda_1^{-1} \right) + \left| \int_0^T r(t) dt \right| + \mathbb{E}_P \left[ \left| \int_0^T \theta(t)^\top dW(t) \right| < +\infty \right]
$$

$$
+ \frac{1}{2} \int_0^T \|\theta(t)\|_2^2 dt
$$

$$
< +\infty.
$$

In addition, $\forall \gamma \in (-\infty, 1) \setminus \{0\}$:

$$
\mathbb{E}_P \left[ U \left( (U'_\gamma)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right) \right] \leq \ln \left( \lambda_1^{-1} \right) + \mathbb{E}_P \left[ \left( \frac{dQ}{dP} \right)^{\frac{1}{\gamma}} \right] < +\infty
$$

and $\forall \gamma \in (-\infty, 1)$: $\mathbb{E}_Q \left[ (U'_\gamma)^{-1} \left( \lambda_1 \cdot \frac{dQ}{dP} \right) \right] = \lambda_1^{-1} \cdot \mathbb{E}_P \left[ \left( \frac{dQ}{dP} \right)^{\frac{1}{\gamma}} \right] < +\infty$.

Finally, $\left| (U'_\gamma)^{-1} (x) \right|^2 = (U'_\gamma)^{-1} (x)$ for $\gamma := \frac{1}{2} (\gamma + 1) \in (-\infty, 1)$ and all $x \in \mathbb{R}$. □

**Proof of Theorem 4.8:** First of all,

$$
h \geq (U'_0)^{-1} \left( A^{h, \Psi} \cdot \frac{dQ}{dP}(\omega) - \Psi^h \right)
$$

$$
\iff \frac{U'_0 (h) + \Psi^h}{\lambda_1^{\frac{1}{\gamma}} \cdot \Lambda^{h, \Psi^h}} \leq \frac{dQ}{dP}(\omega)
$$

$$
\iff \frac{\int_0^T \theta(s)^\top dW_Q(s)(\omega)}{\sqrt{\int_0^T \|\theta(s)\|_2^2 ds}} \leq \frac{\ln \left( \frac{\Lambda^{h, \Psi^h}}{h^{-1} \cdot \Psi^h} \right) - \int_0^T r(s) ds + \frac{1}{2} \int_0^T \|\theta(s)\|_2^2 ds}{\sqrt{\int_0^T \|\theta(s)\|_2^2 ds}}.
$$

Since

$$
Y^{\lambda}(t) = \lambda_1^{-\frac{1}{\gamma}} \cdot \left( \frac{dQ}{dP}(t, \cdot) \right)^{\frac{1}{\gamma - 1}} \exp \left[ \int_{\lambda}^T \left( \left[ -r(s) + \frac{1}{2} \cdot \frac{1}{\gamma - 1} \cdot \|\theta(s)\|_2^2 \right] ds \cdot \frac{\gamma}{\gamma - 1} \right) \right]
$$

and $\sqrt{\int_0^T \|\sigma(s)^\top \pi^0(s)\|_2^2} = \frac{1}{1 - \gamma} \cdot \sqrt{\int_0^T \|\theta(s)\|_2^2 ds}$, we conclude that

$$
\ln \left( \frac{\Lambda^{h, \Psi^h}}{h^{-1} \cdot \Psi^h} \right) - \int_0^T r(s) ds + \frac{1}{2} \int_0^T \|\theta(s)\|_2^2 ds
$$

$$
\frac{\sqrt{\int_0^T \|\theta(s)\|_2^2 ds}}{\int_0^T \|\theta(s)\|_2^2 ds} = -d_+ \left( \left( h^{\gamma - 1} + \Psi^h \right)^{\frac{1}{\gamma - 1}}, t, Y^{A^{h, \Psi^h}}(t) \right).
$$
Using $A(t) : = \{ x \in \mathbb{R} \mid x \leq -d (\gamma^{-1} + \Psi^h \frac{1}{\gamma-1}, t, Y^{A, \Psi^h}) \}$, we realize that

\[
\frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_Q \left[ \mathbb{1}_{[\infty, h]} \left( (U'_1)^{-1} \left( \Lambda, \Psi^h \frac{dQ}{dP} - \Psi^h \right) \right) \left( U'_1 \right)^{-1} \left( U'_1 \left( Y^{A, \Psi^h} (T) \right) - \Psi^h \right) \right] F(t)
\]

\[
= \frac{S_0(t)}{S_0(T)} \cdot \mathbb{E}_{\mathbb{P}^*} \left[ \mathbb{1}_{[U'_1(h) + \Psi^h, +\infty)]} \left( \frac{dQ}{dP} \right) \left( U'_1 \right)^{-1} \left( U'_1 \left( Y^{A, \Psi^h} (T) \right) - \Psi^h \right) \right] F(t)
\]

\[
= \frac{S_0(t)}{S_0(T)} \cdot \int_{A(t)} \varphi(x) \cdot \left( Y^{A, \Psi^h} (t) \cdot \frac{S_0(T)}{S_0(t)} \right)^{-1} \cdot \exp \left[ -x \cdot \sqrt{\int_t^T \| \theta(s) \|^2_2 \, ds} \right]
\]

\[
= \frac{S_0(t)}{S_0(T)} \cdot \phi \left( a, \sqrt{\int_t^T \| \theta(s) \|^2_2 \, ds}, \Psi^h, -d \left( \left( h^{-1} + \Psi^h \right)^{\frac{1}{\gamma-1}}, t, Y^{A, \Psi^h} \right), \frac{1}{\gamma-1} \right).
\]

The optimal investment strategy can be computed as usual by applying the following identities:

\[
\frac{\partial \phi}{\partial a} (a, b, c, d) = \int_{-\infty}^d \varphi(z) \cdot \xi \cdot (a \cdot \exp [-b \cdot z] - c)^{\xi-1} \cdot \exp [-b \cdot z] \, dz
\]

\[
= \int_{-\infty}^{d+b} \varphi(z) \cdot \exp \left[ \frac{1}{2} b^2 \right] \cdot \xi \cdot (a \cdot \exp [-b \cdot (z - b)] - c)^{\xi-1} \, dz
\]

\[
= \exp \left[ \frac{1}{2} b^2 \right], \cdot \xi \cdot \phi \left( a \cdot \exp [b^2], b, c, d + b, \xi - 1 \right).
\]

\[
\frac{\partial \phi}{\partial d} (a, b, c, d, \xi) = \varphi(d) \cdot (a \cdot \exp [-b \cdot d] - c)^{\xi}
\]

and $(a \cdot \exp [-b \cdot d] - \Psi^h)^{\frac{1}{\gamma-1}} = h$. \(\square\)

**Proof of Remarks 4.9:** We use the notation of the proof of Theorem 4.8 starting on page 142. Since

\[
\left( Y^{A, \Psi^h} (t) \cdot \frac{S_0(T)}{S_0(t)} \right)^{\gamma-1} \exp \left[ -x \cdot \sqrt{\int_t^T \| \theta(s) \|^2_2 \, ds} - \frac{1}{2 \gamma-2} \cdot \frac{1}{2} \int_t^T \| \theta(s) \|^2_2 \, ds \right] - \Psi^h \left( Y^{A, \Psi^h} (t) \right)^{\gamma-1} \frac{1}{\gamma-1}
\]

equals

\[
\left( \frac{S_0(T)}{S_0(t)} \right)^{\gamma-1} \exp \left[ -x \sqrt{\int_t^T \| \theta(s) \|^2_2 \, ds} - \frac{1}{2 \gamma-2} \cdot \frac{1}{2} \int_t^T \| \theta(s) \|^2_2 \, ds \right] - \Psi^h \left( Y^{A, \Psi^h} (t) \right)^{1-\gamma} \frac{1}{\gamma-1}
\]

for all $x \in A(t)$, it is an increasing function of $Y^{A, \Psi^h} (t)$ on the subset of $\Omega$ with $x \in A(t)$. As $A(t)$
Let \( (\omega_n) \) be a sequence in \( \Omega \) with \( \lim_{n \to +\infty} Y^{h,\phi^h}(t)(\omega_n) = 0 \).

Then \( \lim_{n \to +\infty} \frac{\phi(a,b,\Psi^h, d_t)(\omega_n)}{Y^{h,\phi^h}(t)(\omega_n)} = \frac{S_0(T)}{S_0(t)} \) and thus \( \lim_{n \to +\infty} \frac{S_0(t)}{S_0(T)} E_Q \left[ X^{h,\lambda,\phi^h,\phi^h}(t) \right] = 1 \).

The same line of argument can be used to prove that in this case \( \lim_{n \to +\infty} \frac{p_{\phi^h}(t)(\omega_n)}{p_{\phi^h}(t)(\omega_n)} = 1 \), too.

**Proof of Corollary 4.12:** The result can be deducted from

\[
\mathbb{E}_P \left[ U_\gamma \left( X^{h,\lambda,\phi^h,\phi^h}(t) \right) \right] = \ln (h) + \mathbb{E}_P \left[ \left( \ln \left( X^{h,\lambda,\phi^h,\phi^h}(t) \right) - \ln (h) \right)^+ - \ln (h) \ln \left( X^{h,\lambda,\phi^h,\phi^h}(t) \right) \right] + \mathbb{E}_P \left[ Y^{h,\phi^h}(t) \right] = \ln (h) + \sqrt{\int_t^T \|\theta(s)\|^2 ds} \cdot \mathbb{E}_P \left[ d_+ \left( h, \frac{\int_t^T \theta(s) dW(s)}{\sqrt{\int_t^T \|\theta(s)\|^2 ds}} \right) - d_- \left( h, \frac{\int_t^T \theta(s) dW(s)}{\sqrt{\int_t^T \|\theta(s)\|^2 ds}} \right) \right] \]

\[
= \ln (h) + \sqrt{\int_t^T \|\theta(s)\|^2 ds} \cdot \left[ f \left( d_+ \left( h, Y^{h,\phi^h}(t) \right) \right) - f \left( d_- \left( h, Y^{h,\phi^h}(t) \right) \right) \right].
\]

**Proof of Proposition 4.14:**

\[
\forall \lambda \in (0, +\infty), \forall x \in \mathbb{R} : F^{g_{\lambda}}(x) = \frac{S_0(0)}{S_0(T)} \cdot \mathbb{E}_Q \left[ 1 \cdot \mathbb{I}_{(-\infty,x)} \left( Y^{\lambda}(T) \right) \right] = \frac{1}{S_0(T)} \Phi \left( -d_- \left( x, 0, Y^{\lambda}(0) \right) \right).
\]
∀λ ∈ (0, +∞), ∀α ∈ (0, Q(Ω)):
\[
(F^Q_{Y^\lambda(T)})^{-1}(\alpha) = S_0(T) \cdot Y^\lambda(0) \cdot \exp\left(-\frac{1}{2} \int_0^T \left\| \sigma(s)^\top \pi^0(s) \right\|^2_2 ds \right.
\]
\[+ \Phi^{-1}(\alpha \cdot S_0(T)) \cdot \sqrt{\int_0^T \left\| \sigma(s)^\top \pi^0(s) \right\|^2_2 ds}.\]

\[\square\]

**Proof of Fact 4.16:**

- Let II be admissible for \((F_{\text{VaR}}^\nu)\).
  \[\Rightarrow \nu(X_{x_0,\Pi}(T) < q) \leq \epsilon \leq \nu(\Omega).\]
  We set
  \[\lambda := \begin{cases} \nu(\Omega) - \epsilon & \text{if } \nu(X_{x_0,\Pi}(T) < q) < \nu(\Omega) \\ 0 & \text{if } \nu(X_{x_0,\Pi}(T) < q) = \nu(\Omega) \end{cases},\]

  \[\epsilon_T := \lambda \cdot 1_{(-\infty,q)}(X_{x_0,\Pi}(T)) + (1 - \lambda) \cdot 1 \quad \text{and} \quad \epsilon_0 := \frac{\epsilon}{\nu(\Omega)} = \begin{cases} \frac{\epsilon}{\nu(\Omega)} & \text{if } \nu(X_{x_0,\Pi}(T) < q) < \nu(\Omega) \\ 1 & \text{if } \nu(X_{x_0,\Pi}(T) < q) = \nu(\Omega) \end{cases}.\]

  Note that \(\nu(X_{x_0,\Pi}(T) < q) = \nu(\Omega)\) implies that \(\epsilon = \nu(\Omega)\) and thus \(\frac{\epsilon}{\nu(\Omega)} = 1.\)

  The integral representation theorem for continuous square-integrable martingales guarantees the existence of an adapted process \(\epsilon\) with \(E_{\nu}[\int_0^T \epsilon(s)^2 ds] < +\infty\) and \(\epsilon_T = \epsilon_0 + \int_0^T \epsilon(s)^\top dW_{\nu}(s).\)

  Finally, \(1_{(-\infty,q)}(X_{x_0,\Pi}(T)) \leq \epsilon_T\) implies that \(U_{\text{VaR}}(X_{x_0,\Pi}(T), \epsilon_T) = U(X_{x_0,\Pi}(T))\).

- Let (II, e) be admissible for \((F_{\text{Dynamic}}^\nu)\) and \(E_E[U_{\text{VaR}}(X_{x_0,\Pi}(T), \epsilon_T)] > -\infty.\)

  \[\Rightarrow P(1_{(-\infty,q)}(X_{x_0,\Pi}(T)) > \epsilon_T) = 0.\]

  \[\Rightarrow E_E[U_{\text{VaR}}(X_{x_0,\Pi}(T), \epsilon_T)] = E_E[U(X_{x_0,\Pi}(T))]\]

  \[\nu(X_{x_0,\Pi}(T) < q) \leq E_E[\epsilon_T] = \epsilon_0 \cdot \nu(\Omega) = \epsilon.\]

\[\square\]

**Proof of Fact 4.17:**

- Let II be admissible for \((F_{\text{ES}}^\nu)\).

  \[\Rightarrow E_{\nu}\left[(h - X_{x_0,\Pi}(T))^+ \right] \leq (h - c) \cdot \alpha.\]

  We define for all \(t \in [0, T]:\)

  \[\epsilon_t := E_{\nu}\left[(h - X_{x_0,\Pi}(T))^+ | F(t)\right] + \left(\frac{h - c}{\nu(\Omega)} \cdot \epsilon_0 - \frac{E_{\nu}[(h - X_{x_0,\Pi}(T))^+]}{\nu(\Omega)} \right) \geq 0.\]

  \[\Rightarrow \epsilon_0 = \frac{(h - c) \cdot \alpha}{\nu(\Omega)} \quad \text{and} \quad \epsilon. \quad \text{can be chosen to be a continuous martingale with respect to} \ \nu.\]

  The integral representation theorem for continuous square-integrable martingales ensures the existence of \(\epsilon\) with \(\forall t \in [0, T]: \epsilon_t = \epsilon_0 + \int_0^t \epsilon(s)^\top dW_{\nu}(s)\) and \(E_{\nu}[\int_0^T \epsilon(s)^2 ds] < +\infty.\)

  \[(h - X_{x_0,\Pi}(T))^+ \leq \epsilon_T \Rightarrow U_{\text{ES}}(X_{x_0,\Pi}(T), \epsilon_T) = U(X_{x_0,\Pi}(T)).\]
• Let \((\Pi, \epsilon)\) be admissible for \((P^{Dynamic}_{ES^\omega})\) and \(\mathbb{E}_P \left[ U_{ES} \left( X^{x_0, \Pi}(T), \epsilon_T \right) \right] > -\infty.\)

\[\Rightarrow P \left( 1_{(-\infty, q)} (X^{x_0, \Pi}(T)) > \epsilon_T \right) = 0.\]

\[\Rightarrow \mathbb{E}_P \left[ U_{ES} \left( X^{x_0, \Pi}(T), \epsilon_T \right) \right] = \mathbb{E}_P \left[ U \left( X^{x_0, \Pi}(T) \right) \right] \quad \text{and} \quad \mathbb{E}_P \left[ (h - X^{x_0, \Pi}(T))^+ \right] \leq \mathbb{E}_\nu \left[ \epsilon_T \right] = \epsilon_0 \cdot \nu(\Omega) = (h - c) \cdot \alpha.\]

\[\square\]

**Proof of Fact 4.18:**

• Let \(\Pi\) be admissible for \((P^{ES^\omega}_h)\).

\[\Rightarrow \mathbb{E}_Q \left[ (h - X^{x_0, \Pi}(T))^+ \right] \leq (h - c) \cdot \alpha \quad \text{and} \quad x_0 \geq h \cdot Q(\Omega) - (h - c) \alpha.\]

We define for all \(t \in [0, T]:\)

\[\tilde{c}_t := \mathbb{E}_Q \left[ (h - X^{x_0, \Pi}(T))^+ \mid F(t) \right] + \frac{(h - c) \cdot \alpha - \mathbb{E}_Q \left[ (h - X^{x_0, \Pi}(T))^+ \right]}{Q(\Omega)} \geq 0.\]

\[\Rightarrow \tilde{c}_0 = \frac{(h - c) \cdot \alpha}{Q(\Omega)} \text{ and } \tilde{c}. \text{ can be chosen to be a continuous martingale with respect to } Q.\]

The integral representation theorem for continuous square-integrable martingales yields the existence of a process \(\tilde{c}\) with \(\forall t \in [0, T]: \tilde{c}_t = \tilde{c}_0 + \int_0^t \tilde{c}(s)^\top dW_Q(s)\) and \(\mathbb{E}_Q \left[ \int_0^T \|\tilde{c}(s)\|_2^2 ds \right] < +\infty.\)

- Let \(A := \{ \omega \in \Omega \mid \exists t \in [0, T]: \tilde{c}_t = 0 \} \in \mathcal{F}(t_1(\tilde{c})).\)

Hence \(\int_A (h - X^{x_0, \Pi}(T))^+ dQ \leq \int_A \tilde{c}_t(\tilde{c}) dQ = \int_A 0 dQ \quad \text{and} \quad (h - X^{x_0, \Pi}(T))^+ \geq 0\)

implies that \(Q \left( A \cap \{ X^{x_0, \Pi}(T) < h \} \right) = 0.\)

Due to \(0 = \int_A \tilde{c}_t(\tilde{c}) dQ = \int_A \tilde{c}_T dQ \geq 0, \quad Q(\tilde{c}_t(\tilde{c}) \neq \tilde{c}_T) = 0.\)

We set \(c(t)(\omega) := \tilde{c}(t)(\omega) \cdot 1_{[0, t_1(\tilde{c})]}(t)\) and \(\epsilon_t := \tilde{c}_0 + \int_0^t c(s)^\top dW_Q(s).\)

Hence \(Q(\epsilon_t(\tilde{c}) \neq \epsilon_T) = 0,\)

\[\epsilon_T = \tilde{c}_0 + \int_0^T 1_{[0, t_1(\epsilon)]}(s) e(s)^\top dW_Q(s) = \tilde{c}_0 + \int_0^{t_1(\epsilon)} e(s)^\top dW_Q(s) = \tilde{c}_t(\tilde{c})\]

and thus \(\tilde{c}_T = \epsilon_T, \quad \epsilon_t(\tilde{c}) = \tilde{c}_t(\tilde{c}) \text{ and } t_1(\epsilon) \leq t_1(\epsilon).\)

On the other hand,

\[\epsilon_{t_1(\epsilon)} = \tilde{c}_0 + \int_0^{t_1(\epsilon)} 1_{[0, t_1(\epsilon)]}(s) e(s)^\top dW_Q(s) = \tilde{c}_0 + \int_0^{t_1(\epsilon)} 1_{[0, t_1(\epsilon)]}(s) e(s)^\top dW_Q(s) = \tilde{c}_{t_1(\epsilon)},\]

so \(t_1(\tilde{c}) \leq t_1(\epsilon)\) and therefore \(t_1(\epsilon) = t_1(\tilde{c}).\)

Furthermore, \(\epsilon\) inherits the martingale property from \(\tilde{c}\).

- Let \(B := \{ \omega \in \Omega \mid \exists t \in [0, T]: \frac{X^{x_0, \Pi}(t)}{S_0(t)} = \frac{h - \epsilon_T}{S_0(T)} \} \in \mathcal{F}(t_2(\Pi)).\)

Consequently,

\[\int_B \frac{X^{x_0, \Pi}(T)}{S_0(T)} dQ \leq \int_B \frac{X^{x_0, \Pi}(t)}{S_0(t)} dQ = \int_B \frac{h - \epsilon_{t_2(\Pi)}}{S_0(T)} dQ = 0 = Q \left( B \cap \{ X^{x_0, \Pi}(T) > h \} \right) = Q \left( B \cap \{ X^{x_0, \Pi}(T) \neq h \} \right).\]

and thus \(\min \{ h, X^{x_0, \Pi}(T) \} \leq X^{x_0, \Pi}(T) \).
If \( Q (B) > 0 \),
\[
\int_B \frac{X^{x_0, \Pi} (T)}{S_0(T)} dQ = \int_B \frac{h - \epsilon_T}{S_0(T)} dQ \leq \int_B \frac{\min \{ h, X^{x_0, \Pi} (T) \}}{S_0(T)} dQ \leq \int_B \frac{X^{x_0, \Pi} (T)}{S_0(T)} dQ
\]
shows that \( \epsilon_T = (h - X^{x_0, \Pi} (T))^+ \) in this case.
In any case, \( Q (B \cap \{ X^{x_0, \Pi} (T) \neq h - \epsilon_T \}) = 0. \)

We define \( \tilde{\Pi} (t) (\omega) := \begin{cases} \Pi (t) (\omega) / S_0(t) (\sigma (t)^\top e (t) (\omega))^{-1} e (t) (\omega) & \text{if } t < t_2 (\Pi) (\omega) \\ \epsilon (t) (\omega) & \text{else} \end{cases} \) and observe that
\[
\frac{X^{x_0, \Pi} (T)}{S_0(T)} = x_0 + \int_0^{t_2 (\Pi)} \frac{\Pi (s)^\top}{S_0(s)} \sigma (s) dW_Q (s) + \int_0^T \frac{X^{x_0, \Pi} (T)}{S_0(T)} - \frac{X^{x_0, \Pi} (t_2 (\Pi))}{S_0(t_2 (\Pi))} \frac{h - \epsilon_T}{S_0(T)} dW_Q (s)
\]
implies that \( t_2 (\tilde{\Pi}) \leq t_2 (\Pi) \).

Hence
\[
\frac{X^{x_0, \Pi} (t_2 (\Pi))}{S_0(t_2 (\Pi))} = \frac{X^{x_0, \tilde{\Pi}} (t_2 (\Pi))}{S_0(t_2 (\Pi))} \quad \text{and therefore } t_2 (\Pi) \leq t_2 (\tilde{\Pi}).
\]
This means that \( t_2 (\Pi) = t_2 (\tilde{\Pi}) \).

We have shown that
\begin{itemize}
  \item the constructed pair \( (\tilde{\Pi}, \epsilon) \) is admissible for \( \tilde{P}_{Dynamic}^{E_S Q} \) and
  \item \( \tilde{\Pi} \) yields (almost-surely) the same terminal wealth as \( \Pi \).
\end{itemize}

\bullet Let \( (\Pi, \epsilon) \) be admissible for \( \tilde{P}_{Dynamic}^{E_S Q} \).

Hence
\[
X^{x_0, \Pi} (0) = x_0 \geq h \cdot Q (\Omega) - (h - c) \alpha = \frac{S_0(0)}{S_0(T)} (h - \epsilon_0) \quad \text{and} \quad \epsilon_T = \epsilon_{t_1 (\epsilon)} + \int_{t_1 (\epsilon)}^T e (s)^\top dW_Q (s) = \epsilon_{t_1 (\epsilon)} + 0 = \epsilon_{t_1 (\epsilon)} \geq 0.
\]

Since \( X^{x_0, \Pi}, S_0 \) and \( \epsilon \) are continuous functions,
\[
X^{x_0, \Pi} (T) = X^{x_0, \Pi} (t_2 (\Pi)) + \int_{t_2 (\Pi)}^T -\frac{e (s)^\top}{S_0(T)} dW_Q (s) \geq \frac{h - \epsilon_{t_2 (\Pi)}}{S_0(T)} + \left( \frac{\epsilon_T - \epsilon_{t_2 (\Pi)}}{S_0(T)} \right) = h - \epsilon_T.
\]
\[
\Rightarrow h - X^{x_0, \Pi} (T) \leq \epsilon_T.
\]
\[
\Rightarrow \mathbb{E}_Q \left[ \frac{(h - X^{x_0, \Pi} (T))^+}{h - u_0} \right] \leq \mathbb{E}_Q \left[ (\epsilon_T)^+ \right] \quad \text{for } \epsilon_T \geq 0 \quad \mathbb{E}_Q [\epsilon_T] = \epsilon_0 \cdot Q (\Omega) = (h - c) \cdot \alpha.
\]

\[\square\]

**Proof of Fact 4.19:** First of all, we realize that \( \epsilon_0 = (h - c) \cdot \alpha \leq h - u_0. \)

\bullet For \( t \in [0, T] \) let \( \epsilon_t := \epsilon_0 + \int_0^t e (s)^\top dW (s) \) with \( \int_0^T ||e (s)||_2^2 ds < +\infty \) (\( P \)-almost-surely) and
\[ \epsilon_T \in [0, h - u_0]. \text{ Then for all } 0 \leq t_1 \leq t_2 \leq T: \]
\[
\left. \frac{Y(t_2, 1 - \frac{\epsilon_{t_2}}{h-u_0})}{S_0(t_2)} - \frac{Y(t_1, 1 - \frac{\epsilon_{t_1}}{h-u_0})}{S_0(t_1)} \right|_{t_1 \wedge t_1(t_1) \wedge t_2(t_1)} = \int_{t_1 \wedge t_1(t_1) \wedge t_2(t_1)} \frac{1}{2} \sqrt{\int_T^T \frac{1}{2} \varphi \left( \Phi^{-1} \left( 1 - \frac{\epsilon_{t_2}}{h-u_0} \right) \right)^2} \, dt \]
\[
= \int_{t_1 \wedge t_1(t_1) \wedge t_2(t_1)} -1 \varphi \left( \Phi^{-1} \left( 1 - \frac{\epsilon_{t_1}}{h-u_0} \right) \right) e(s) \, dW_Q(s). \]

Thus \( \frac{Y(t_1, 1 - \frac{\epsilon_{t_1}}{h-u_0})}{S_0(t_1)} \) is a submartingale with respect to \( Q \).

- Let \( \Pi \) be admissible for \( (P_{E_S^P}) \) with \( X_{x_0,\Pi}(T) \geq u_0 \).

\[ \Rightarrow E_P \left( (h - X_{x_0,\Pi}(T))^+ \right) \leq (h - c) \cdot \alpha \text{ and } x_0 \geq Y \left( 0, 1 - \alpha \cdot \frac{h-c}{h-u_0} \right). \]

We set \( \lambda := \begin{cases} (h-u_0)-E_P\left[ (h - X_{x_0,\Pi}(T))^+ \right] & \text{if } E_P\left[ (h - X_{x_0,\Pi}(T))^+ \right] < (h - u_0) \\ 0 & \text{if } E_P\left[ (h - X_{x_0,\Pi}(T))^+ \right] = (h - u_0) \end{cases} \) \in [0, 1] \]

and define for all \( t \in [0, T] \): \( \tilde{\epsilon}_t := \lambda E_P \left[ (h - X_{x_0,\Pi}(T))^+ | F(t) \right] + (1 - \lambda)(h - u_0) \in [0, h - u_0]. \)

\( \Rightarrow \tilde{\epsilon}_0 = (h - c) \cdot \alpha \text{ and } \tilde{\epsilon} \) can be chosen to be a continuous, square integrable martingale with respect to \( P \).

The integral representation theorem for continuous square-integrable martingales guarantees the existence of an adapted process \( \tilde{\epsilon} \) with \( E_P \left[ \int_0^T \|\tilde{\epsilon}(s)^2\|_2 \, ds \right] < +\infty \) and \( \forall t \in [0, T] \): \( \tilde{\epsilon}_t = \tilde{\epsilon}_0 + \int_0^t \tilde{\epsilon}(s)^2 \, dW(s). \)

- Let \( A := \{ \omega \in \Omega | \exists t \in [0, T]: \tilde{\epsilon}_t = 0 \} \in F(t_1(\tilde{\epsilon})). \)

Using the same line of argument as in the proof of Fact 4.18, we conclude that \( \tilde{\epsilon}_{t_1(\tilde{\epsilon})} = \tilde{\epsilon}_T \).

We set \( \tilde{\epsilon}(t)(\omega) := \tilde{\epsilon}(t)(\omega) \cdot 1_{[0, t_1(\tilde{\epsilon})(\omega)]}(t) \) and \( \tilde{\epsilon}_t := \tilde{\epsilon}_0 + \int_0^t \tilde{\epsilon}(s)^2 dW(s). \)

Hence \( \tilde{\epsilon}_{t_1(\tilde{\epsilon})} \neq \tilde{\epsilon}_T = 0 \), \( \tilde{\epsilon}_{t_1(\tilde{\epsilon})} = \tilde{\epsilon}_{t_2(\tilde{\epsilon})} \) and we know that \( \tilde{\epsilon}_T = \tilde{\epsilon}_t \text{ and } t_1(\tilde{\epsilon}) \leq t_2(\tilde{\epsilon}). \)

Due to \( \tilde{\epsilon}_{t_1(\tilde{\epsilon})} = \tilde{\epsilon}_{t_2(\tilde{\epsilon})}, \) we get the analogous result: \( t_1(\tilde{\epsilon}) \leq t_1(\tilde{\epsilon}) \) and as a consequence \( t_1(\tilde{\epsilon}) = t_1(\tilde{\epsilon}). \)

- Let \( C := \{ \omega \in \Omega | \exists t \in [0, T]: \tilde{\epsilon}_t = h - u_0 \} \in F(t_3(\tilde{\epsilon})). \)

By using the equation \( \int_C (h - u_0) dP = \int_C \tilde{\epsilon}_T dP = \int_C \tilde{\epsilon}_T dP \) and the inequality \( \tilde{\epsilon}_T = \tilde{\epsilon}_T \leq h - u_0 \) we deduce \( P \left( C \cap \{ \tilde{\epsilon}_T \neq h - u_0 \} \right) = 0 \) and hence \( P \left( \tilde{\epsilon}_{t_2(\tilde{\epsilon})} \neq \tilde{\epsilon}_T \right) = 0. \)

Next, let \( \epsilon(t)(\omega) := \tilde{\epsilon}(t)(\omega) \cdot 1_{[0, t_3(\tilde{\epsilon})(\omega)]}(t) \) and \( \epsilon_0 := \tilde{\epsilon}_0 + \int_0^t \epsilon(s)^2 dW(s). \)

The above approach that led to \( \tilde{\epsilon}_T = \tilde{\epsilon}_T \text{ and } t_1(\tilde{\epsilon}) = t_1(\tilde{\epsilon}) \text{ now yields } \tilde{\epsilon}_T = \epsilon_T \text{ and } t_3(\epsilon) = t_3(\tilde{\epsilon}). \)

Finally, \( \epsilon_{t_1(\epsilon)} = E_P \left[ \epsilon_T | F(t_1(\epsilon)) \right] = E_P \left[ \epsilon_T | F(t_1(\tilde{\epsilon})) \right] = \epsilon_{t_1(\epsilon)} \text{ and } \epsilon_{t_1(\epsilon)} = E_P \left[ \epsilon_T | F(t_1(\epsilon)) \right] = E_P \left[ \epsilon_T | F(t_1(\epsilon)) \right] = \epsilon_{t_1(\epsilon)} \text{ ensure that } t_1(\epsilon) = t_1(\epsilon). \)
6.3. PART 4: DYNAMIC OPTIMIZATION PROBLEMS WITH RISK CONSTRAINTS

Let $B := \left\{ \omega \in \Omega \mid \exists t \in [0, T]: X_{x_0, t} = Y \left(t, 1 - \frac{\epsilon T}{h - u_0}\right) \right\} \in \mathcal{F}(t_4(\Pi))$. Then

$$
\int_B \frac{X_{x_0, t} \Pi(T)}{S_0(T)} dQ \leq \int_B \frac{Y \left(t_4(\Pi), 1 - \frac{\epsilon T}{h - u_0}\right)}{S_0(t_4(\Pi))} dQ = \int_B \frac{Y \left(t_4(\Pi), 1 - \frac{\epsilon T}{h - u_0}\right)}{S_0(t_4(\Pi))} dQ = \int_B \frac{h - \epsilon T}{S_0(T)} dQ = \int_B \frac{h - \epsilon T}{S_0(T)} dQ \leq \int_B \min \left\{ \frac{h, X_{x_0, t} \Pi(T)}{S_0(T)} \right\} dQ.
$$

and thus

$$
\int_B Y \left(t_4(\Pi), 1 - \frac{\epsilon T}{h - u_0}\right) \frac{S_0(t_4(\Pi))}{S_0(T)} dQ = \int_B Y \left(t_4(\Pi), 1 - \frac{\epsilon T}{h - u_0}\right) \frac{S_0(t_4(\Pi))}{S_0(T)} dQ.
$$

So

$$
\int_{t_4(\Pi)}^{t_4(\Pi)^+} \left( \frac{(h - u_0)^2 \epsilon \left(1 - \frac{\epsilon T}{h - u_0}\right)}{\int_T T \theta(s) d\phi} \right)^2 ds = 0.
$$

Due to the construction of the stochastic integral, we can therefore safely set $\epsilon(s)$ to

$$
\frac{(h - u_0)^2 \epsilon \left(1 - \frac{\epsilon T}{h - u_0}\right)}{\int_T \theta(s) d\phi} = \theta(s) \text{ without changing } \epsilon.
$$

Last of all, we set $\Pi(t)(\omega)$ to

$$
\left\{ \begin{array}{ll}
\Pi(t)(\omega) & \text{if } t < t_4(\Pi)(\omega) \\
- \frac{S_0(t)}{S_0(T)} \frac{\epsilon \left(1 - \frac{\epsilon T}{h - u_0}\right)}{\int_T \theta(s) d\phi} (\sigma(t)^{-1}e(t)(\omega)) & \text{else}
\end{array} \right.
$$

Again, an imitation of the proof of Fact 4.18 yields $X_{x_0, \Pi}(T) = X_{x_0, \tilde{\Pi}}(T)$ and $t_4(\Pi) = t_4(\tilde{\Pi})$.

• Let $(\Pi, \epsilon)$ be admissible for $\left( P_{\text{Dynamic}} \right)$. Hence $X_{x_0, t} \Pi(0) = x_0 \geq Y \left(0, 1 - \alpha \cdot \frac{h - u_0}{h - u_0}\right) = Y \left(0, 1 - \frac{\epsilon T}{h - u_0}\right)$.

$$
\epsilon T = \epsilon t_4(\epsilon) + \int_{t_4(\epsilon)}^{t_4(\Pi)} \epsilon(s) \theta(s) dW(s) = \epsilon t_4(\epsilon) + 0 = \epsilon t_4(\epsilon) \geq 0 \text{ and } \epsilon T = \epsilon t_4(\epsilon) + 0 \leq h - u_0.
$$

Since $X_{x_0, \Pi}(T) = X_{x_0, \Pi}(t_4(\Pi)) = Y \left(T, 1 - \frac{\epsilon T}{h - u_0}\right) - Y \left(t_4(\Pi) - 1 - \frac{\epsilon T}{h - u_0}\right)$ and the functions $X_{x_0, \Pi}$, $S_0$ as well as $\epsilon$ are continuous,

$$
X_{x_0, \Pi}(T) = Y \left(T, 1 - \frac{\epsilon T}{h - u_0}\right) + \left( X_{x_0, \Pi}(t_4(\Pi)) - Y \left(t_4(\Pi) - 1 - \frac{\epsilon T}{h - u_0}\right) \right) \geq Y \left(T, 1 - \frac{\epsilon T}{h - u_0}\right) = u_0 + \left( 1 - \frac{\epsilon T}{h - u_0}\right) \cdot (h - u_0) = h - \epsilon T.
$$

$\Rightarrow h - X_{x_0, \Pi}(T) \leq \epsilon T$.

$\Rightarrow E_P \left[ (h - X_{x_0, \Pi}(T))^+ \right] \leq E_P \left[ (\epsilon T)^+ \right] \geq 0 \Rightarrow E_P \left[ \epsilon T \right] = \epsilon_0 \cdot 1 = (h - c) \cdot \alpha.$
Bibliography


# List of Definitions

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>column vector of ones</td>
</tr>
<tr>
<td>Admissibility</td>
<td></td>
</tr>
<tr>
<td>Coherent Risk Measure</td>
<td></td>
</tr>
<tr>
<td>Cumulative Distribution Function</td>
<td></td>
</tr>
<tr>
<td>CVaR</td>
<td>Conditional Value at Risk</td>
</tr>
<tr>
<td>CVaR$_{\nu}$ (X)</td>
<td>Alternative Conditional Value at Risk</td>
</tr>
<tr>
<td>CVAR$_{\alpha}^{AT}$ (X)</td>
<td>Conditional Value at Risk definition used in [Acerbi and Tasche, 2002]</td>
</tr>
<tr>
<td>CVAR$_{\alpha}^{RU}$ (X)</td>
<td>Conditional Value at Risk definition used in [Rockafellar and Uryasev, 2000]</td>
</tr>
<tr>
<td>d</td>
<td>dimension of standard Brownian motion</td>
</tr>
<tr>
<td>d$^+_\nu$, d$^-\nu$</td>
<td></td>
</tr>
<tr>
<td>$\frac{d\rho}{dT}(t, \omega)$</td>
<td>price density of one monetary unit at time $t$ in state $\omega$</td>
</tr>
<tr>
<td>ES</td>
<td>Expected Shortfall</td>
</tr>
<tr>
<td>$E_{\nu}$</td>
<td>Lebesgue Integral with respect to $\nu$</td>
</tr>
<tr>
<td>$E_{\nu}[X</td>
<td>G]$</td>
</tr>
<tr>
<td>$f^{\lambda_1}$</td>
<td></td>
</tr>
<tr>
<td>$F^\nu_X$</td>
<td>Cumulative Distribution Function</td>
</tr>
<tr>
<td>$\overline{H}_P$, $\overline{H}_Q$</td>
<td></td>
</tr>
<tr>
<td>$L^\lambda_1$, $L^\lambda_2$</td>
<td></td>
</tr>
<tr>
<td>$l^P_1$, $l^Q_1$</td>
<td></td>
</tr>
<tr>
<td>$L^h_2$, $L^h_h$</td>
<td></td>
</tr>
<tr>
<td>$L_P$</td>
<td></td>
</tr>
<tr>
<td>$L_Q$</td>
<td></td>
</tr>
<tr>
<td>$\lambda_0^{\nu_0}$</td>
<td>optimal Lagrange multiplier of problem without risk constraint</td>
</tr>
<tr>
<td>Term</td>
<td>Definition</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>---------------------------------------------------------------------------</td>
</tr>
<tr>
<td>( \Lambda^{h, \lambda_2} )</td>
<td></td>
</tr>
<tr>
<td>( \lambda^h_Q )</td>
<td></td>
</tr>
<tr>
<td>( \Lambda^h_{x_0} )</td>
<td></td>
</tr>
<tr>
<td>( \Lambda^\nu_{x_0} )</td>
<td>first Lagrange multiplier value at risk problem</td>
</tr>
<tr>
<td>Martingale</td>
<td></td>
</tr>
<tr>
<td>( \mu )</td>
<td>vector of stock return rates</td>
</tr>
<tr>
<td>( N )</td>
<td>number of stocks</td>
</tr>
<tr>
<td>( | \cdot |_2 )</td>
<td>Euclidean vector norm</td>
</tr>
<tr>
<td>Optimality</td>
<td></td>
</tr>
<tr>
<td>( (P^{\text{Dynamic}}) )</td>
<td>alternative expected shortfall problem with ( \nu = P )</td>
</tr>
<tr>
<td>( (P^{\text{Dynamic}}) )</td>
<td>alternative expected shortfall problem with ( \nu = Q )</td>
</tr>
<tr>
<td>( (P_0) )</td>
<td>classic optimization problem without risk constraint</td>
</tr>
<tr>
<td>( (P_{\text{CVaR}}) )</td>
<td>optimization problem with conditional value at risk constraint</td>
</tr>
<tr>
<td>( (P_{\text{ES}}) )</td>
<td>optimization problem with an expected shortfall constraint</td>
</tr>
<tr>
<td>( \Phi, \varphi )</td>
<td>c.d.f., p.d.f of Standard Normal Distribution</td>
</tr>
<tr>
<td>( \phi(a, b, c, d, \xi) )</td>
<td></td>
</tr>
<tr>
<td>( \pi^{\text{VaR}}, p_{\text{VaR}}^{\text{VaR}}, p_{\text{rel}}^{\text{VaR}} )</td>
<td></td>
</tr>
<tr>
<td>( \pi^P, p_{\text{abs}}, p_{\text{rel}}^{P} )</td>
<td>optimal strategy expected shortfall problem with ( \nu = P )</td>
</tr>
<tr>
<td>( \pi^Q, p_{\text{abs}}, p_{\text{rel}}^{Q} )</td>
<td>optimal strategy expected shortfall problem with ( \nu = Q )</td>
</tr>
<tr>
<td>( \Pi_i )</td>
<td>total amount invested in i-th stock</td>
</tr>
<tr>
<td>( \Psi^h )</td>
<td></td>
</tr>
<tr>
<td>( \Psi^h_{x_0} )</td>
<td></td>
</tr>
<tr>
<td>( (P_{\text{VaR}}) )</td>
<td>optimization problem with value at risk constraint</td>
</tr>
<tr>
<td>( \mathbb{P} )</td>
<td>set of tame portfolios</td>
</tr>
<tr>
<td>Quantile Function</td>
<td></td>
</tr>
<tr>
<td>( r )</td>
<td>risk-free rate process</td>
</tr>
<tr>
<td>( \mathbb{R}, \overline{\mathbb{R}} )</td>
<td>set of real, extended real numbers</td>
</tr>
<tr>
<td>Random Variable</td>
<td></td>
</tr>
<tr>
<td>Risk Measure</td>
<td></td>
</tr>
<tr>
<td>( S_0 )</td>
<td>share price process of money market</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$S_i$</td>
<td>share price process of $i$-th stock</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>volatility matrix</td>
</tr>
<tr>
<td>$\tau(h)$</td>
<td>point of discontinuity special admissible random variable $ES^P$ problem</td>
</tr>
<tr>
<td>$\theta$</td>
<td>risk-premium process</td>
</tr>
<tr>
<td>$t_{\text{max}}$</td>
<td>end of support of c.d.f. of $\frac{dQ}{dP}$</td>
</tr>
<tr>
<td>$t_{\text{min}}$</td>
<td>start of support of c.d.f. of $\frac{dQ}{dP}$</td>
</tr>
<tr>
<td>$U$</td>
<td>utility function</td>
</tr>
<tr>
<td>$U'$</td>
<td>extended definition of first derivative of $U$</td>
</tr>
<tr>
<td>$(U')^{-1}$</td>
<td>extended definition of inverse of $U'$</td>
</tr>
<tr>
<td>$U_\gamma$</td>
<td>specific utility function with parameter $\gamma$</td>
</tr>
<tr>
<td>$U_{ES}$</td>
<td>modified terminal utility function expected shortfall problem</td>
</tr>
<tr>
<td>$U_{\text{VaR}}$</td>
<td>modified terminal utility function value at risk problem</td>
</tr>
<tr>
<td>VaR</td>
<td>Value at Risk</td>
</tr>
<tr>
<td>$W$</td>
<td>standard Brownian motion</td>
</tr>
<tr>
<td>$WCE_{\alpha}(X)$</td>
<td>Worst Conditional Expectation</td>
</tr>
<tr>
<td>$W_P$</td>
<td>standard Brownian motion</td>
</tr>
<tr>
<td>$W_Q$</td>
<td>modified Brownian motion</td>
</tr>
<tr>
<td>$X_{h,\lambda_1,\lambda_2}$</td>
<td>candidate for optimal terminal wealth $ES^P$ problem</td>
</tr>
<tr>
<td>$X_{h,\lambda_1,\lambda_3}$</td>
<td>candidate for optimal terminal wealth $ES^Q$ problem</td>
</tr>
<tr>
<td>$X_{\lambda_1,J}^{\text{VaR}}$</td>
<td>candidate for optimal terminal wealth value at risk problem</td>
</tr>
<tr>
<td>$x_0$</td>
<td>initial endowment</td>
</tr>
<tr>
<td>$Y^\lambda(t)$</td>
<td>value process of artificial stock</td>
</tr>
</tbody>
</table>
List of Figures

3.1 Comparison of Optimal Solution Value at Risk vs. Classic Problem . . . . . . . . . . 28
3.2 Comparison of Optimal Solution Expected Shortfall under $P$ vs. Classic Problem . . 31
3.3 Comparison of Optimal Solution Expected Shortfall under $Q$ vs. Classic Problem . . 33
3.4 The Integrand of the Lagrange Function of Problem $(P_{VaR^P})$ . . . . . . . . . . . . . . . 39
3.5 The Integrand of the Lagrange Function of Problem $(P_{VaR^Q})$ . . . . . . . . . . . . . . . 40
3.6 The Integrand of the Lagrange Function of Problem $(P_{ES^P})$ . . . . . . . . . . . . . . . 45
3.7 $(P_{ES^h})$: Choices for the Definition of $\Lambda^{h,\lambda_2}$ . . . . . . . . . . . . . . . . . 48
4.1 Optimal Investment Strategy in the Value at Risk Problem (I) . . . . . . . . . . . . 78
4.2 Optimal Investment Strategy in the Value at Risk Problem (II) . . . . . . . . . . . . 79
4.3 Optimal Initial Investment Strategy in the Value at Risk Problem . . . . . . . . . . . 79
4.4 Dependence of $F^P_{X^0,T_{VaR}(T)}$ on the Number of Trades . . . . . . . . . . . . . . . . 80
4.5 Optimal Investment Strategy in $(P_{ES^h})$ (I) . . . . . . . . . . . . . . . . . . . . . . . . 82
4.6 Optimal Investment Strategy in $(P_{ES^h})$ (II) . . . . . . . . . . . . . . . . . . . . . . . . 83
4.7 Optimal Initial Investment Strategy in Problem $(P_{ES^h})$ . . . . . . . . . . . . . . . . . . 83
4.8 Dependence of $F^P_{X^0,T_{P}(T)}$ on the Number of Trades . . . . . . . . . . . . . . . . . . 84
4.9 Optimal Investment Strategy in $(P_{ES^h})$ (I) . . . . . . . . . . . . . . . . . . . . . . . . 88
4.10 Optimal Investment Strategy in $(P_{ES^h})$ (II) . . . . . . . . . . . . . . . . . . . . . . . . 88
4.11 Optimal Initial Investment Strategy in Problem $(P_{ES^h})$ . . . . . . . . . . . . . . . . . 89
4.12 Dependence of $F^P_{X^0,T_{ES^h}(T)}$ on the Number of Trades . . . . . . . . . . . . . . . . . 89
4.13 Maximal Expected Utility of $(P_{ES^h})$ as a Function of $h$ . . . . . . . . . . . . . . . . 92
4.14 Comparison of Optimal Terminal Wealths: $(P_{VaR^Q})$, $(P_{ES^Q})$ and $(P_0)$ . . . . 92
4.15 Dependence of the Optimal $h$ in $(P_{VaR^Q})$ on the Initial Wealth . . . . . . . . . . . . . 93
4.16 The Lagrange Function of Problem $(P_{VaR^Q})$: Discussion of the Dependence on $d$ . . 113
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Zusammenfassung

In dieser Arbeit setzen wir uns mit den Auswirkungen von Risikobeschränkungen auf das optimale Verhalten eines Investors auseinander. Der zu Grunde liegende Rahmen ist ein klassisches Optimierungsproblem: Ein Investor versucht, den erwarteten Endnutzen zu einem festgelegten Zeitpunkt zu maximieren. Dabei unterliegt er folgender Beschränkung der Anfangsausstattung: Er kann nur solche Handelspositionen besitzen die durch sein Anfangsvermögen finanziert werden können, da wir annehmen, dass er weder positive noch negative externe Einkünfte besitzt und alle Vermögensveränderungen eine Folge von Handlungen im Markt sind.


Das andere Marktmodell, welches wir in dieser Arbeit betrachten, ist ein dynamisches Modell eines vollständigen Marktes mit einer risikolosen Anlageform und mehreren risikobehafteten Anlagen, deren


Abschließend versuchen wir die Frage zu beantworten, welches der untersuchten Risikomaße am besten für die Anwendung geeignet ist. Die Antwort basiert auf dem Vergleich ihrer individuellen Stärken und Schwächen und dabei insbesondere auch auf ihre jeweiligen Auswirkungen auf die optimalen Verhaltensweisen.