Utility Maximization in Life and Pension Insurance: A Numerical Approach

Kumulative Dissertation zur Erlangung des akademischen Grades eines Doktors der Wirtschaftswissenschaften (Dr. rer. pol.) an der Fakultät für Mathematik und Wirtschaftswissenschaften der Universität Ulm

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Tag der Promotion:
01. Oktober 2021
Acknowledgments

First of all, I would like to thank my supervisor Prof. Dr. An Chen for giving me the opportunity to pursue my doctoral studies under her supervision. I am very glad for the many helpful suggestions and fruitful discussions from which I benefited a lot. And especially, that I could always knock on your door and ask for advice in case that I need one. I am very grateful for your support in challenging times.

Furthermore, I would not only like to thank Prof. Dr. Mitja Stadje for reviewing this thesis as second advisor, but also for sharing his knowledge on different mathematical questions.

I would like to thank Prof. Dr. Sandra Ludwig, thank you very much for being the chair of the final colloquium.

Special thanks go to, Silke, for all the interesting discussions we had during our coffee breaks, and my co-authors Frank, Stefan and Thai, it was a pleasure to work with you.

I would like to thank my current and former colleagues at the Institute of Insurance Science for their helpful advice in research questions as well as for all the other fun activities we had in recent years.

Finally, I would like to express my eternal gratitude to my family. All this would not have been possible without their endless support.

Ulm, November 2021

Nils Sørensen
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Overview of Research Papers

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2. Frank Bosserhoff, An Chen, Nils Sørensen and Mitja Stadje (2021). On the Investment Strategies in Occupational Pension Plans. This article has been accepted for publication in Quantitative Finance, published by Taylor & Francis. DOI: 10.2139/ssrn.3827859


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Research Context and Summary of Research Papers

1 Field of Research

This cumulative thesis contributes to the field of optimal asset allocation in the context of indifference pricing, retirement planning and retirement consumption. Further, this thesis presents new applications of numerical algorithms to optimal expected utility problems.

Optimal asset allocation problems date back to Merton (1969), who first solved this problem for the power utility in a complete market setup. Since then optimal asset allocation problems have been analyzed in numerous directions, e.g. Rouge and El Karoui (2000), Horneff et al. (2008) and many others. However, adding additional sources of risk, e.g. by untradable assets or stochastic interest rates, to these problems often lead to incomplete financial markets. Solving these problems is an ongoing challenge, as in most cases analytic solutions are not available.

Pricing of claims written on untradable assets in the context of optimal asset allocation is often considered by the utility indifference pricing principle. The risk-averse agent evaluates the price such that she is indifferent between trading and not trading the claim. Utility indifference pricing problems for European-type claims have mainly been considered from a buyer perspective under power and exponential utility, see e.g. Rouge and El Karoui (2000), Henderson (2002), Henderson and Hobson (2008), Musiela and Zariphopoulou (2004), Carmona (2008). However, when taking a seller’s perspective and having short positions pricing may not be possible due to shortcomings of the power and exponential utility. Power utility can not deal with any possibility of negative wealth, while the exponential utility is not able to cope with unbounded negative payoffs, this is also often referred to as the “short call” problem, see e.g. Henderson (2002). Therefore, Chen et al. (2011) introduce a symmetric asymptotic hyperbolic absolute risk aversion (SAHARA) utility function which allows for better analytical tractability. This utility function is defined on the whole real line and yields the highest risk-aversion at a threshold wealth. The decreasing structure of the risk-aversion at very low wealth levels allows for additional flexibility in the context of pricing claims on untradable assets.

The literature on retirement planning (accumulation of wealth) and consumption (decumulation of wealth) covers a wide range of topics within the three pillars (state pension, occupational
pension and private pension) of retirement systems, which can be found in Germany and most other developed countries. With an aging society and low interest rates, the importance of occupational and private pension plans (second and third pillar) has increased in recent years. The question how an individual should invest her wealth when saving for retirement, is discussed in many different aspects, see e.g. Benartzi and Thaler (2001), Chen et al. (2015), Menoncin and Vigna (2017), Forsyth and Vetzal (2019). With retirement, the question arises on how to use the accumulated wealth to finance consumption. Hence, individuals need to decide on how much to consume as well as how to invest the remaining wealth. In an optimal asset allocation context this has been analyzed under various model frameworks, see e.g. Dus et al. (2005), Horneff et al. (2008).

Since in most cases optimal asset allocation problems in this context do not yield analytical solutions, this thesis relies on numerical approaches to solve the underlying optimization problems. In particular, powerful numerical algorithms that can deal with multiple correlated assets are required. Further, these optimization problems can either be approached by directly approximating the resulting value function or by deriving and solving the resulting Hamilton-Jacobi-Bellmann (HJB) partial differential equation (PDE). Direct approaches use least-squares Monte Carlo methods (LSMC), see e.g. Kharroubi et al. (2014), Denault et al. (2017), while the HJB approach employs a modified finite difference scheme, see e.g. Ma and Forsyth (2016a), Ma and Forsyth (2016b). The HJB approach further needs to fulfill specific requirements to ensure convergence towards the correct solution, Barles et al. (1995). The resulting algorithms are very complex and have mostly not been applied and discussed in the literature any further.

2 Motivation and Objectives

The ongoing low interest environment and an increasing life expectancy belong to the major challenges for insurers and individuals saving for retirement in developed countries. Thus, today many insurance products are linked to investments in financial markets. This fact comes along with a couple of questions for insurers as well as individuals buying these products. For an insurer, the pricing and hedging of these products is of interest, while the individual is interested in optimizing her final outcome with respect to her risk preference.

Pricing and hedging of insurance claims is often based on an incomplete market setting and has been considered in various directions. Typical approaches are e.g. finding an equivalent martingale measure, good deal bounds or using a risk-minimizing strategy, see e.g. Föllmer and Sondermann (1986), Møller (1998), Cochrane and Saa-Requejo (2000), Young (2008). Another approach is considering the insurer as a risk-averse agent and applying utility indifference pricing, see e.g. Møller (2003). Utility indifference pricing under power and exponential utility
has been widely discussed, e.g. Henderson (2002), Young and Zariphopoulou (2002), Musiela and Zariphopoulou (2004), Davis and Yoshikawa (2016). For power utility the research is focused on the buyer perspective as any kind of negative wealth is unacceptable and, thus, the possibility of a negative payout is in many cases unbearable. The exponential utility is defined for the whole real line, but faces the so-called “short call” problem, as it is not able to price unbounded short positions under certain circumstances. Analytical solutions are mostly obtained for the exponential utility, while the power utility often does not yield an analytical solution. Further, in some settings bounds and approximations for the indifference price are derived, e.g. Henderson and Hobson (2002), Biagini et al. (2011). The SAHARA utility function has been introduced by Chen et al. (2011). This function contains both power and exponential utility as limiting cases and can overcome the limitations of these utility functions. However, it has not been applied to utility indifference pricing in incomplete markets. Hence, in this context the thesis covers the following research questions:

1. How does a SAHARA agent price claims in an utility indifference pricing framework in comparison to power and exponential utility agents?

2. How does a SAHARA agent price short positions in an utility indifference pricing framework?

In retirement planning traditionally defined benefit (DB) plans have been the leading form of occupational pension plans in developed countries. A defined benefit plan usually takes into account predefined metrics, e.g. the years of services and the wages, to determine the pension benefit. A shift to defined contribution (DC) plans has occurred in the last decades. In a DC plan, the contribution from the employee is invested into a pension fund and the payout at retirement is determined by the fund value at retirement. This shift has also been recognized by lawmakers, e.g. in the US these products are known as 401(k) plans. One of the standard investment strategies for these plans are the so-called target date funds (TDF), which have a predefined maturity date (target date). The investment strategy of a TDF often follows a so-called “glide path”, i.e. a strategy with decreasing equity holding over time. There exist various literature dealing with DC plans, e.g. Cairns et al. (2006), Gao (2009), Guan and Liang (2014), Chen et al. (2015), Li et al. (2017). In Chen et al. (2015) a constant volatility model in an expected utility framework is used, however, evidence from financial markets suggests that stochastic volatility needs to be taken into account as well, see e.g. Cont and Tankov (2012). The research questions with respect to TDFs considered in this thesis are:

3. Can a decreasing equity holding be verified in an optimal expected utility model with stochastic volatility?

4. What is the effect of the stochastic volatility in comparison to a constant volatility model?
In optimal utility frameworks it is often shown that annuitization of a substantial part of the wealth is the optimal retirement decision for an individual, see e.g. Yaari (1965). Anyhow, in practice annuitization rates observed are significantly lower than the literature suggests, referred to as the “annuity puzzle”, see e.g. Benartzi et al. (2011). For wealth that has not been annuitized, individuals have to decide on their wealth decumulation in retirement. Financial advisors usually suggest several self-managed withdrawal rules, e.g. the 4%-rule. These rules have been analyzed and discussed in different settings, see e.g. Dus et al. (2005), Horneff et al. (2008), Pfau (2010), MacDonald et al. (2013). However, most of the literature focuses on planning horizons up to an age of 100 and financial market settings with significantly higher interest rate levels than currently observable in many markets. Thus, the research questions with respect to self-managed withdrawal rules in this thesis are:

5. How do common self-managed withdrawal rules perform in a possibly long-lasting low interest environment when the planning horizon exceeds the age 100?

6. Can combinations of common self-managed withdrawal rules improve these results in an optimal utility framework or with respect to shortfall risk?

All previously mentioned research questions are based on optimal asset allocation problems in incomplete markets. The resulting optimization problems usually do not allow for any analytical solution and thus numerical methods are applied. However, numerical approaches for solving these kinds of problems are complex and have just evolved in recent years. Consequently, the literature on the application of such numerical approaches in this context is rather sparse, see e.g. Kharroubi et al. (2014), Ma and Forsyth (2016a), Ma and Forsyth (2016b), Denault et al. (2017). Further, both the HJB scheme as well as the LSMC algorithm do not give any information on numerical error or approximation bias. Thus, the objective of this thesis with respect to numerical algorithms is:

7. Compare and analyze the quality of an LSMC and an HJB approach for optimal asset allocation problems.

The next section provides detailed summaries of each of the research papers.
3 Summary of Research Papers

Research Paper 1: Indifference pricing under SAHARA utility

In the first paper, we study utility indifference pricing under SAHARA utility. Our approach is the first application of a two-dimensional finite difference scheme, developed by Ma and Forsyth (2016a), in an optimal expected utility framework. We price both long and short positions on untradable claims. Untradable claims can occur in many different settings in finance and insurance, e.g. when buying options on non-liquid assets. For long positions, we compare the results for SAHARA utility with results for power and exponential utility. Further, we analyze the optimal trading strategy for different wealth levels as well as the dependence on the correlation between the stock and the untradable asset. The paper has been presented during the Ulm Actuarial days in Ulm, Germany. This paper is joint work with An Chen and Thai Nguyen and has been accepted for publication in the Journal of Computational and Applied Mathematics.

For our analysis, we use an incomplete market setting consisting of a risky asset, a risk-less bond and a correlated untradable asset. The agent can buy a European-type claim on this untradable asset, e.g. a put or call option. The risk-averse agent aims to maximize his final utility by dynamically investing into a portfolio consisting of the risky asset and the bond. To identify a price the indifference pricing principle is used, where the price is determined such that the agent is indifferent between trading and not trading the claim. When the agent decides not to trade the claim, a complete market is obtained and the optimal expected utility can explicitly be determined. When trading the claim, an incomplete market is obtained and in general no analytic solution is attainable. From a buyer’s perspective, we show that under SAHARA utility an upper price bound exists. We determine the bound explicitly by following techniques presented in Henderson (2002) and Biagini et al. (2011).

To further price the claims under SAHARA utility, we employ an HJB approach and derive the corresponding two-dimensional HJB PDE with a cross-derivative term. To solve the resulting HJB PDE, a numerical approach is taken. Hereby we adapt a numerical algorithm presented in Ma and Forsyth (2016a) and Ma and Forsyth (2016b) to our problem. In comparison to these problems, we have a time-dependent control variable as well as an explicit dependence on the second dimension. The algorithm is a modified finite difference scheme, where the grid is rotated to obtain a positive coefficient method, which ensures convergence towards the viscosity solution.

We apply the algorithm to price different claims under SAHARA, power and exponential utility from a buyer’s perspective. To compare the results we assume that all agents start with the same initial risk-aversion. Further, for SAHARA utility we price short positions in our claims.
as well. We show that the prices are monotone in the correlation between the risky asset and the untradable claim. The optimal invested share is decreasing in initial wealth for SAHARA utility. In addition, we price a simple insurance contract under SAHARA utility.

In summary, this paper answers the research questions 1-2. We see that due to the same initial risk-aversion all utilities deliver comparable results, i.e., the price is dependent on the initial risk-aversion. Further, when taking a short position most claims can not be priced with power and exponential utility and we focus on SAHARA utility. In particular, we are able to show numerically that a risk-averse agent always charges a higher price for a short position than an agent with the same risk-aversion would be willing to pay. With our applications, we demonstrate that SAHARA utility yields additional possibilities when pricing short positions as we are not limited to bounded payoffs.

Research Paper 2: On the Investment Strategies in Occupational Pension Plans

The second paper analyzes optimal investment strategies in occupational pension plans. We consider so-called target date funds, which typically are identified with a decreasing fraction of wealth invested in equity (glide path). In our model, we focus on analyzing the effect of stochastic volatility on the investment result of a target date fund. This paper is joint work with Frank Bosserhoff, An Chen and Mitja Stadje and a revised version has been accepted for publication in Quantitative Finance.

In this paper, we study TDFs in a realistic financial market with stochastic volatility and random endowments. TDFs are DC plans with a prespecified terminal date up to which an employee contributes a share of her salary. The resulting financial market is known as a Heston model, see e.g. Heston (1993). Further, the contribution from the employee is modeled as a stochastic process itself and is correlated to the financial market. Assuming a risk-averse individual we formulate the problem as a stochastic optimization problem, where the individual chooses the investment strategy maximizing her expected terminal wealth.

To solve this optimization problem we take a simulation and regression approach. Hereby we follow an LSMC algorithm introduced by Denault et al. (2017). We analyze the stochastic volatility problem as well as the corresponding constant volatility problem. The general glide path pattern can be confirmed for the stochastic volatility model. We show that the stochastic volatility model yields additional flexibility as it can take into account the volatility level when determining the optimal strategy. Further, a sensitivity analysis with respect to the risk-aversion and the contribution process is carried out. In particular, we see that changes in the dynamics of the contribution process only have a minor impact. Further, we observe that with
decreasing risk-aversion the share invested in the risky asset increases. The initial wealth-to-contribution ratio mainly impacts the overall steepness of the glide path, i.e., if the contribution is increasing in comparison to the wealth the agent is willing to invest riskier initially. This effect decreases when moving ahead in time.

This research paper answers questions 3-4. Optimal strategies in a stochastic volatility model show a dependence on the volatility level. However, we can confirm that the more complex stochastic volatility model shows a decreasing structure of glide paths at a similar level as the constant volatility model. Further, we see that a stochastic volatility model yields a lower variance of the resulting final wealth as the additional flexibility allows to partially hedge the volatility. In addition, we conclude that the wealth-to-contribution ratio and the risk-aversion are the main determinants of the investment strategy.


The third paper investigates the performance of common withdrawal rules in a long-lasting low interest environment under increasing longevity risk. In a model with stochastic interest rates, we investigate how these rules can maintain the desired standard of living in a low interest rate environment. These rules will be studied with respect to optimal expected utility as well as the potential shortfall. Further, we introduce a mixed rule, which overall shows promising characteristics. This paper is joint work with An Chen and Stefan Schelling.

As the “annuity puzzle” indicates, many people do not buy annuities, although the literature often suggests this as being the optimal choice for risk-averse individuals. Consequently, for these people, alternative approaches to consuming and investing their wealth in retirement are needed. Financial advisors recommend different common self-managed withdrawal rules. There has been some literature discussing how these rules ensure a certain standard of living in retirement, see e.g. Dus et al. (2005), Horneff et al. (2008), MacDonald et al. (2013). Existing literature hereby typically focuses on a planning horizon until the age of 100 and uses interest rates that are significantly higher than currently observed. However, overestimation of the interest rate level combined with an underestimation of the remaining lifetime lead to a significant underestimation of the actual risk of running out of money in old age. In an expected utility framework and using a stochastic financial market, we study common withdrawal strategies in a low interest environment when the planning horizon exceeds the age of 100. We analyze the rules with respect to their payout structure and discuss their respective benefits and drawbacks. Using an LSMC algorithm we also derive the optimal consumption strategy under power utility and compare this to the other rules. We propose a mixed rule, which inherits characteristics of other withdrawal rules. In an optimal expected utility framework, we carry out an in-depth
sensitivity analysis of all rules considered. In addition, we estimate the probability of shortfall and the residual wealth at death.

Our analysis answers research questions 5 and 6. We show that common withdrawal rules yield unsatisfactory results in a low interest environment with a longer planning horizon. We show that the proposed mixed rule allows us to achieve a high expected utility and a relatively low shortfall risk as well as this rule outperforms previously recommended withdrawal rules. Further, the proposed mixed rule is easily applicable in practice.

**Research Paper 4: Utility Maximization in incomplete markets:**

**Comparison of a least-squares Monte Carlo and a two-dimensional HJB algorithm**

In the fourth paper, we investigate the quality of the LSMC algorithm used in Paper 2 and the HJB scheme used in Paper 1. These algorithms have first evolved in recent years and do not yield any information on approximation errors and estimation bias. To obtain a better understanding of their respective results, we study and compare both algorithms for different problems with respect to their optimal expected utility, their trading strategy and their order of convergence. We consider a random endowment (similar to paper 1) and a random contribution problem (similar to paper 2).

This paper compares an LSMC algorithm and an HJB scheme. The LSMC approach is a simulation and regression scheme based on the least-squares Monte Carlo method with an additional step, where the regressed function is maximized with respect to the control variable. The HJB algorithm is a modified finite difference scheme, a so-called positive coefficient method (PCM). As a detailed description of the algorithms is already found in papers 1 and 2, we analyze these with respect to different utility maximization problems in incomplete markets: a random endowment problem and a random contribution problem. In the random endowment problem, the agent receives a random endowment at terminal time. This problem is related to the problem considered in paper 1. In the random contribution setting a continuous contribution is added to the wealth process. This problem is related to the constant volatility model considered in paper 2. From a numerical perspective, the difference between the problems is that in the first problem a self-financing strategy is considered, while the second problem has a non-self-financing strategy. Consequently, both algorithms are implemented for both problems. The output of each algorithm is the optimal expected utility and the optimal trading strategy. We discuss and compare the results. Further, the algorithms are run and refined multiple times to estimate the order of convergence. For the analysis of the order of convergence, we use a proxy for the analytic solution. When deriving the proxy we rely on a reduction technique, which allows us to apply a one-dimensional algorithm and solve the problem with higher accuracy.
This research paper answers question 7. We can see that both algorithms solve both problems with satisfactory results. The optimal expected utility given by the LSMC approach is slightly above the HJB result. Further, the LSMC approach shows a fluctuating strategy, while the HJB scheme shows a convergence to the optimal strategy. Numerical experiments show that the HJB scheme has a higher order of convergence.


Research Context and Summary of Research Papers


Research Papers
1 Indifference pricing under SAHARA utility

Source:

URL: https://dx.doi.org/10.1016/j.cam.2020.113288

DOI: 10.1016/j.cam.2020.113288

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We study utility indifference pricing of untradable assets in incomplete markets using a symmetric asymptotic hyperbolic absolute risk aversion (SAHARA) utility function, both from the buyer's and seller's perspective. The use of the SAHARA utility function allows us to tackle the "short call" problem, which power and exponential utility functions are unable to solve. While no closed-form solutions are available for the indifference prices, we are able to derive some pricing bounds. Furthermore, we rely on the dynamic programming approach to solve the associated utility maximization problem, which leads to a two-dimension HJB equation. A complex algorithm discussed in Ma and Forsyth (2016) is consequently adopted to numerically solve the HJB equation. We determine utility indifference prices for options written on the untradable underlying assets and some insurance contracts.

1. Introduction

Pricing and hedging of claims on nontraded assets have been well studied in the literature (see e.g. [1–3]), but it remains one of the ongoing challenges in option pricing theory. The problem of pricing and hedging untradable risks (assets) occurs in many different settings. The untradable risks can emanate from illiquid stocks, real estate or life and pension insurance contracts whose payoffs depend on the death and survival of the insured. Typically, the inherited untradable risks can be partially eliminated by trading in correlated assets. However, a part of the risk remains and generates an incomplete market, leading to a non-unique arbitrage-free price. Pricing in incomplete markets has been considered in various directions. A typical way to do pricing in an incomplete market is to take a given equivalent martingale measure \( Q \) and determine the risk-neutral price under \( Q \), which is given by \( e^{-rT} E^Q [g(Y_T)] \). It is possible to look at the superhedging price mathematically defined by \( \sup_{Q \in \mathcal{M}} e^{-rT} E^Q [g(Y_T)] \), where \( \mathcal{M} \) is the set of equivalent martingale measures, but this often leads to an unrealistically high price. In order to obtain a reasonable arbitrage-free price interval, some upper bound can be imposed on the market price of risk process to rule out, not only the prices which violate the no-arbitrage restriction, but also those which represent deals that are too good, in the sense that the Sharpe Ratio is "too high". This good-deal bound theory was initiated by [1] and extended by e.g. [4] among many authors. Let us also mention the risk-minimizing pricing and hedging method frequently used in insurance industry. It is originally introduced in [5] and applied in pricing and hedging life insurance liabilities by [6]. The main idea of risk-minimizing is to find a self-financing strategy which minimizes the total variance of hedging errors, and the resulting variance can be used to compute the risk loading of the insurance liabilities. [7] suggests pricing life insurance liabilities via the instantaneous Sharpe Ratio.
in which risk-minimizing is also applied. In this valuation framework, a portfolio, consisting of the life insurance liability to be priced and a self-financing sub-portfolio of T-bonds and money market funds to hedge the insurance liability, is constructed. The investment strategy is determined such that the local variance of the total portfolio is minimized, and the price of the insurance liability is determined so that the instantaneous Sharpe ratio of the total portfolio equals a pre-specified value. In the present paper, we adopt utility indifference pricing, i.e. the economic agent is indifferent, in terms of expected utility, between trading and not trading the claim/contract.

There is a wide range of literature on utility indifference pricing under exponential and power utility functions, see e.g. [3,8–13]. Analytical solutions are mainly achieved for exponential functions using the Hopf–Cole transformation technique, c.f. [10,14]. Indifference pricing from the buyer’s perspective for power and exponential utility functions is considered in e.g. [9]. The latter work also provides an upper bound and an approximation of the indifference price by Taylor expansion around the number of claims. Note that exponential and power utility functions are faced with the “short call” problem, see e.g. [9]. In particular, power utility functions do not allow for negative wealth levels, i.e. pricing for short positions is impossible. While an exponential utility function is defined on the whole real line, it is still not capable of pricing unbounded short positions of a call option on a risky asset that follows a log-normal distribution, see e.g. [9].

In the current paper, we study the utility indifference pricing problem in a more general setting by considering a Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA) utility function. It has been shown in [15] that power and exponential utility functions can be seen as limiting cases of SAHARA utility functions. In addition, using SAHARA utility functions allows us to price long and short positions, meaning that the “short call” problem mentioned before can be overcome. In this sense, the SAHARA indifference pricing framework adds some additional flexibility to the existing literature, while at the same time it is able to mimic an exponential or power agent in a unified framework. However, due to the more general form of the SAHARA utility function, no analytical solutions are available for SAHARA utility indifference pricing.

We tackle the SAHARA indifference pricing problem by the dynamic programming approach and obtain a two-dimensional HJB equation with a cross-derivative term. As an explicit solution for the corresponding HJB equation is not known, we provide numerical results by adopting [16,17]. Different from [17], we have a time-dependent control variable, and in comparison with [16], our final payoff depends on the second dimension explicitly. We apply the algorithm to determine indifference prices for various contracts and discuss the results in connection with the existing literature on power and exponential utility functions. Moreover, pricing of unit-linked life-insurance contracts with untradable mortality risk is also discussed.

The main focus of this paper is to determine indifference prices both from the buyer’s and seller’s perspective of a SAHARA agent. While, from the buyer’s perspective, the SAHARA agent shows a similar pricing behavior as the power and exponential agents with the same initial risk-aversion, we mainly focus on the seller’s perspective for which we can overcome the short-call issue by using SAHARA utility functions. In particular, SAHARA utility functions lead to a monotone indifference price (from both buyer’s and seller’s perspective) in the correlation coefficient. We also observe that the greater the initial wealth, the less the optimal fraction of wealth invested in the risky asset. Finally, our numerical examples confirm that the seller’s indifference price is always higher than the buyer’s indifference price.

The remainder of the paper is structured as follows. In Section 2, we represent the financial market and the model setup. The indifference pricing framework is introduced in Section 3, where some bounds for the indifference price are also derived. In Section 4, the numerical algorithm is discussed. Numerical results, both from the buyer’s and the seller’s perspective, are represented in Section 5. Section 6 concludes the paper.

2. Financial market and SAHARA utility function

2.1. Financial market model

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a 2-dimension Brownian motion $(W, W^\perp)$, where $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ is the natural filtration of the Brownian motion, i.e. $\mathcal{F}_t = \sigma ((W_s, W^\perp_s), s \in [0, t])$, and $T \in (0, \infty)$ is a finite time horizon. We consider a financial market consisting of one risk-free asset $B$ and one risky asset $S$, whose dynamics are given as follows:

\[dB_t = rB_t dt, \quad B_0 = 1,\]

\[dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = s_0.\]

Here, the interest rate $r$, the drift $\mu > r$ and the volatility $\sigma > 0$ are constant.

Consider an economic agent who is interested in buying or selling an untradable financial European-type claim with maturity $T$. The payout of the contract is denoted by $g(y_T) \geq 0$, which is dependent on a stochastic process $y$ modeled by the following:

\[dy_t = ay_t dt + by_t \left( \rho dW_t + \sqrt{1 - \rho^2} dW_t^\perp \right), \quad y_0 = \tilde{y}_0,\]

where $\rho \in (-1, 1)$ denotes the correlation between the tradable asset $S$ and the untradable asset $y$. The drift $a$ and the volatility $b > 0$ are assumed to be constants. Note that the claim pays out $g(y_T)$ only at maturity date $T$. The process $y$
could be an external fund that is not traded in the financial market. For instance, life insurers issue diverse products with guarantees which are invested in a managed fund on a European-type equity that does not have a fully liquid market. The insurer hedges the contract by investing in liquid assets, e.g. Futures on EuroStoxx 50 (see [18]).

2.2. SAHARA utility function

SAHARA utility function first introduced by [15] incorporates constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) utility functions as the limiting cases.

Definition 2.1. A utility function \( U \) with domain \( \mathbb{R} \) is of the SAHARA class if its absolute risk aversion function \( A(x) = -U''(x)/U'(x) \) is well-defined on its entire domain and satisfies

\[
A(x) = \frac{\alpha}{\sqrt{\beta^2 + (x - d)^2}} > 0,
\]

for a given \( \beta > 0 \) (the scale parameter), \( \alpha > 0 \) (the risk-aversion parameter) and \( d \in \mathbb{R} \) (the threshold wealth).

We observe from (1) that the agent is more risk-averse when the wealth level tends towards the threshold wealth \( d \). However, \( A(d) = \frac{\alpha}{\beta} \) is still finite and at the threshold wealth level the agent is more willing to avoid risks. For simplicity we assume that the threshold wealth \( d = 0 \).

Let \( U \) be a SAHARA utility function with scale parameter \( \beta > 0 \) and risk aversion parameter \( \alpha > 0 \). Since \( A(x) \) is the derivative of \( -\ln U(x) \) one may show by simple integration that

\[
U'(x) = \tilde{c}(x + \sqrt{x^2 + \beta^2})^{-\alpha},
\]

for some constant \( \tilde{c} > 0 \) and that there exist constants \( \tilde{c}_1 \) and \( \tilde{c}_2 \) such that \( U(x) = \tilde{c}_1 + \tilde{c}_2 U(x), x \in \mathbb{R}, \) where

\[
\tilde{U}(x) = \begin{cases} 
\frac{\alpha}{\sqrt{\beta^2 - x^2}} (x + \sqrt{\beta^2 + x^2})^{-\alpha} & \alpha \neq 1, \\
\frac{1}{2} \ln(x + \sqrt{x^2 + \beta^2}) + \frac{1}{2} \beta^2 (\sqrt{x^2 + \beta^2} - x) & \alpha = 1.
\end{cases}
\]

Again, for simplicity we assume \( \tilde{c}_1 = 0 \) and \( \tilde{c}_2 = 1 \) from now on, i.e. \( U(x) = \tilde{U}(x) \). Further discussions can be found in [15]. Throughout the paper, we assume that the agent’s preference is described by SAHARA utility function.

3. Indifference pricing with SAHARA utility function

Let \( U \) be the utility function of a SAHARA agent who is endowed with an initial wealth of \( x_0 \). She dynamically invests her wealth in a self-financing portfolio of the risk-free asset and the stock \( S \). Let \( \theta_t \) be the amount invested in the risky asset at time \( t \) and suppose that the remaining amount is invested in the risk-free asset. Under the self-financing condition, the agent’s portfolio \( X^{\theta, \theta}_{0, T} \) follows the dynamics

\[
dX^{\theta, \theta}_{0, T} = (rX^{\theta, \theta}_{0, T} + \theta_t(\mu - r))dt + \theta_t\sigma dW_t, \quad X_0 = x_0.
\]

To keep the setting general, we assume that the set of admissible self-financing trading strategies is given by

\[
\Theta = \left\{ \theta : \{ \theta_t \}_{t \in [0,T]} : \theta \text{ adapted}, X_{t}^{\theta, \theta} \text{ is bounded from below}, \int_0^T \theta_t^2 ds < \infty \text{ a.s.} \right\}.
\]

The above setting with \( \theta_t \) being the amount invested in the risky asset gives flexibility to deal with negative wealth levels which are allowed in our SAHARA utility setting.\(^1\)

We are interested in finding a price \( p \) for the claim \( g(y_T) \). In our situation (incomplete market) where \( y \) is non-tradable, a risk-neutral price is not unique. In what follows, we assume the agent evaluates this contract by utility indifference principle, i.e. trading in the financial contract will leave the agent with the same expected utility as not trading. In other words, the claim price is defined as an amount \( p \) that makes the agent indifferent in terms of expected utility between trading and not trading the claim.

Regardless of buying or selling the claim the agent wants to maximize her expected utility from terminal wealth, and the no-trade case serves as a reference point. Mathematically, the buyer’s indifference price \( p \) is implicitly defined by the following:

\[
\sup_{\theta_t \in \Theta, \theta \in [0,T]} E[U(X^{\theta, \theta}_{0, T} + g(y_T))] = \sup_{\theta_t \in \Theta, \theta \in [0,T]} E[U(X^{\theta, \theta}_{0, T})].
\]

\(^1\) Note that in an asset allocation problem which requires positivity of wealth, e.g. utility maximization with a power utility function, fraction of wealth is typically considered.
The optimization problem with a seller’s position is defined by
\[
\sup_{x \in [0; T]} \mathbb{E}[U(X_{T}^{X_{0}, \theta})] = \sup_{x \in [0; T]} \mathbb{E}[U(X_{T}^{0, \theta})].
\] (6)

Note that on the right-hand-side of (5) and (6), the initial wealth is \(x_0\) as the agent makes no trade in the claim, while on the left-hand-side of (5) and (6) the agent buys/sells the contract \(g(y_T)\) and hence \(p\) monetary units is deducted/added from her initial wealth. In addition, the optimal trading strategy \(\theta\) (if exists) of the buyer, the seller and the right-hand-case in general differ.

In the case where there is no claim traded, the optimization problem dates back to [19] and the associated optimal portfolio selection problem under the SAHARA utility function has been solved in [15]. For the associated optimal control problem of the left-hand-side of (5) and of (6), an analytical solution is not available.

3.1. The merton problem for SAHARA utility function

The optimization problem of the right-hand-side of (5) and (6) can be solved analytically by looking at the indirect utility function defined by:
\[
u(t, x) := \sup_{x \in [0; T]} \mathbb{E}[U(X_{T}^{0, \theta})|X_{t}^{0, \theta} = x].
\] (7)

[15] solves the optimization problem (7) by applying the dual approach.\(^2\) We recall the result in the following proposition:

**Proposition 3.1.** For a SAHARA agent, the indirect utility function with the optimal strategy is given by the following expression:
\[
u(t, x) = \begin{cases}
\frac{e^{\frac{1}{2} \sqrt{\lambda^2 + \beta^2}(t - s)}}{1 - \alpha} \left( x + \sqrt{x^2 + \beta^2(t)} \right)^{\alpha} \left( x + \alpha \sqrt{x^2 + \beta^2(t)} \right), & \alpha \neq 1, \\
\left[ \beta \right] \frac{1}{2} \ln \left( x + \sqrt{x^2 + \beta^2(t)} \right) + \frac{1}{2} \beta \left( x \sqrt{x^2 + \beta^2(t)} - x \right), & \alpha = 1,
\end{cases}
\] where \(\lambda := (\mu - r)/\sigma\) and \(b(t) := \beta \exp \left( -r + \frac{1}{2} \lambda^2 \right) (T - t)\). The optimal amount \(\theta^*\) invested in the risky asset for initial wealth \(x_0\) is at any intermediate time point \(t \in [0, T]\) given by
\[
\theta_t^* = \frac{\lambda}{\alpha \sigma} \sqrt{\frac{X_t^{x_0, \theta^*}}{\sigma^2}},
\] where the optimal wealth \(X_t^{x_0, \theta^*}\) is given by
\[
X_t^{x_0, \theta^*} = b(t) \sinh \left( \frac{\lambda}{\alpha \sigma} \ln \left( \frac{S_t \exp \left( -r + \frac{1}{2} \lambda^2 \right) (T - t)}{S_0 (\beta \exp \left( \frac{\alpha}{\beta} \arcsinh \left( \frac{x_0}{b(t)} \right) \right) / \sigma^2) - \ln \beta} \right) \right).
\] (8)

**Proof.** See [15] \(\square\)

The optimal investment strategy given in (8) takes a hyperbolic cosine-like curve and is always positive, i.e. the agent will invest a positive amount of her wealth into the risky asset at all the times. In particular, when the portfolio value approaches zero, the optimal invested amount \(\theta_t^* \rightarrow \frac{\lambda}{\alpha \sigma} \beta \exp \left( -r + \frac{1}{2} \lambda^2 \right) (T - t)\) which differs from zero. For comparison reasons, we also recall the solutions of problems for power and exponential utility function (see e.g. [9]) which date back to [19].

**Remark 3.2.** In case of a power utility function, \(U^p(x) = \frac{x^{1-p}}{1-p}\), with a relative risk-aversion level \(\eta > 0, \eta \neq 1\), the indirect utility function is given by
\[
u(t, x) = \frac{x^{1-p}}{1-p} \exp \left( \frac{1}{2} \frac{(\mu - r)^2 (1 - \eta)}{\alpha^2} \left( T - t \right) \right).
\] (8)

\(^2\) We corrected the case \(\alpha = 1\), where the term \(1/2(r + 1/2\lambda^2)(T - t)\) was missing.
The optimal amount $\theta^p_t$ invested in the risky asset at any intermediate time point $t \in [0, T]$ is given by

$$\theta^p_t := \frac{\mu - r}{\sigma^2} X^{\rho,0}_t,$$

where $X^{\rho,0}_t$ is time-$t$ optimal wealth under power utility. The optimal relative share $\pi^p_t := \frac{\theta^p_t}{X^{\rho,0}_t}$ is known as the Merton’s constant fraction. Note that, in case of power utility, the amount invested in the risky asset goes to zero when $X^{\rho,0}_t$ approaches 0.

Remark 3.3. In case of an exponential utility function, $U^\lambda(x) = -\frac{1}{\gamma} e^{-\gamma x}$, with an absolute risk-aversion parameter $\gamma \neq 0$ the indirect utility function is given by

$$u^\lambda(t, x) := -\frac{1}{\gamma} \exp \left(-\gamma x - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} (T - t) \right).$$

The optimal amount $\theta^\lambda_t$ invested in the risky asset at any intermediate time point $t \in [0, T]$ is given by

$$\theta^\lambda_t := \frac{\mu - r}{\sigma^2 \gamma},$$

which is also known as the Merton’s constant as it is independent of wealth.

3.2. Optimal investment problem with non-traded assets

We now turn to the left-hand-side of (5) and (6). As the indifference price $p$ is not yet known, we will replace the initial wealth in both cases with $\bar{x}_0$, i.e.

$$\sup_{\pi \in \pi_{\rho} \ni \bar{x}_0} \mathbb{E}[U^\lambda(X^{\pi,0}_T + g(y_T))].$$

which can be interpreted as an optimization problem with an initial wealth $\bar{x}_0$ and a long/short position in the additional claim. It is shown e.g. in [20,21] that the duality representation of (11) is usually associated with a choice of an equivalent martingale measure and that a bound of the indifference price can be attained by carrying out a minimization over the set of equivalent martingale measures. A discussion on possible choices of equivalent martingale measures can be found in e.g. [22, Chapter 10.5]. Let $Q^0$ be a measure defined by the following density:

$$\frac{dQ^0}{dP} = \exp \left( -\frac{\mu - r}{\sigma} W_t - \frac{(\mu - r)^2}{2\sigma^2} T \right) = \exp \left( -\lambda W_t - \frac{1}{2} \lambda^2 T \right),$$

which is usually called the minimal martingale measure, e.g. [20,21]. We notice that $W^\perp$ is unaffected by the change of measure from $P$ to $Q^0$ and it is well-known that $e^{-\lambda T} S_t$ is a (super-) martingale under $Q^0$.

The following provides an upper bound for the value function of a SAHARA agent who is buying a claim.

Proposition 3.4. The expected utility of a SAHARA agent with an initial wealth $\bar{x}_0$ and a long position $g(y_T)$ (buyer’s perspective) is bounded by

$$\sup_{\pi \in \pi_{\rho}, \tilde{x}_0} \mathbb{E}[U^\lambda(X^{\pi,0}_T + g(y_T))].$$

Proof. First note that the convex dual $\tilde{U}(z) := \sup_{x \in \mathbb{R}}(U(x) - xz)$ of the SAHARA utility function is given by, see [15],

$$\tilde{U}(z) = \begin{cases} \frac{1}{2} \left( \frac{\sigma^2 + 1/\alpha}{1 + 1/\alpha} - \frac{1}{1/\alpha} \right), & \alpha \neq 1, \\ \frac{1}{2} (\beta^2 z^2 - 1) - \frac{1}{2} \ln z, & \alpha = 1. \end{cases}$$

Let $Q \in \mathcal{M}$ be an arbitrary equivalent martingale measure. It is well-known that the discounted wealth process $e^{-\gamma T} X^{\bar{x}_0,0}_t$ is a super martingale under $Q$, in particular,

$$\mathbb{E}[\Lambda e^{\gamma T} \bar{x}_0] = \kappa \bar{x}_0 \geq \mathbb{E}[\kappa e^{-\gamma T} X^{\bar{x}_0,0}_T] = \mathbb{E}[\Lambda X^{\bar{x}_0,0}_T].$$

here $\Lambda = e^{-\gamma T} \frac{x_0}{\kappa}$ is a state price deflator with a Lagrangian multiplier $\kappa > 0$. This approach is widely used in literature, see e.g. [20] and [21]. It follows that:

$$\mathbb{E}[U^\lambda(X^{\bar{x}_0,0}_T + g(y_T)) \leq \mathbb{E}[U^\lambda(X^{\bar{x}_0,0}_T + g(y_T)) - \mathbb{E}[\Lambda X^{\bar{x}_0,0}_T - \bar{x}_0 e^{\gamma T}]]
= \mathbb{E}[U^\lambda(X^{\bar{x}_0,0}_T + g(y_T)) - \Lambda (X^{\bar{x}_0,0}_T + g(y_T)) + \mathbb{E}[\Lambda (\bar{x}_0 e^{\gamma T} + g(y_T))].
The first two terms yield a convex dual formulation. The above inequality holds true for all \( \kappa > 0 \). As we are interested in the lowest possible upper bound we choose \( \Lambda \) such that

\[
\sup_{x^0} \mathbb{E} \left[ U \left( x^0 + g(y) \right) \right] \leq \inf_{\Lambda} \mathbb{E} \left[ \tilde{U}(\Lambda) + \Lambda \left( \tilde{x}_0 e^{rT} + g(y) \right) \right].
\]

The above minimization depends on \( \kappa \) and \( Q \). It is shown e.g. in [20] that minimizing over \( \Lambda \) first and then over \( \kappa \), i.e.,

\[
\sup_{x^0} \mathbb{E} \left[ U \left( x^0 + g(y) \right) \right] \leq \inf_{\kappa} \mathbb{E} \left[ \tilde{U} \left( \kappa e^{-rT} \frac{dQ}{dP} \right) + \kappa \left( \tilde{x}_0 + e^{-rT} \mathbb{E}^Q g(y) \right) \right].
\]  (12)

From [20] we know that \( Q^0 \) is the solution to the inner minimization in (12). We first consider the case \( \alpha \neq 1 \). The convex dual \( \mathbb{E} \left[ \tilde{U} \left( \kappa e^{-rT} \frac{dQ^0}{dP} \right) \right] \) can be computed explicitly as

\[
\mathbb{E} \left[ \tilde{U} \left( \kappa e^{-rT} \frac{dQ^0}{dP} \right) \right] = -\frac{\alpha b(T)}{\alpha^2 - 1} \kappa \left( \alpha \sinh(\ln(C_\kappa)) + \cosh(\ln(C_\kappa)) \right),
\]  (13)

with \( C_\kappa := \beta^{-1} \kappa^{-1/\alpha} \exp \left( -\frac{\alpha^2}{\alpha - 1} T \right) \) and \( b(t) := \beta \exp \left( -r + \frac{\beta^2}{2 \alpha^2} T \right) \), i.e. \( b(T) = \beta \).

Inserting (13) in (12) and substituting \( z := \tilde{x}_0 + e^{-rT} \mathbb{E}^Q g(y) \) we obtain

\[
\mathbb{E} \left[ \tilde{U} \left( \kappa \frac{dQ^0}{dP} \right) \right] + \kappa z = -\frac{\alpha \beta}{\alpha^2 - 1} \kappa \left( \alpha \sinh \left( \ln \left( \beta^{-1} \kappa^{-1/\alpha} \exp \left( -\frac{\alpha^2}{\alpha - 1} T \right) \right) \right) + \cosh \left( \ln \left( \beta^{-1} \kappa^{-1/\alpha} \exp \left( -\frac{\alpha^2}{\alpha - 1} T \right) \right) \right) \right) + \kappa z.
\]  (14)

The optimal \( \kappa^* \) is then obtained by taking the first order condition:

\[
\kappa^* = \left[ \beta \exp \left( -\frac{r + \frac{\beta^2}{\alpha}}{\alpha} T \right) \exp \left( \arcsinh \left( \frac{z}{\beta} \right) \right) \right]^{1/\alpha} = \exp \left( -\left( r + \frac{\beta^2}{2 \alpha} \right) (\alpha - 1)^T \right) \left( z + \sqrt{z^2 + \beta^2} \right)^{-1/\alpha}.
\]  (15)

Inserting (15) into (14) yields

\[
\mathbb{E} \left[ \tilde{U} \left( \kappa^* \frac{dQ^0}{dP} \right) \right] + \kappa^* z = -\frac{\alpha \beta}{\alpha^2 - 1} \kappa^* \left( \alpha \sinh \left( \ln \left( \beta^{-1} \kappa^{-1/\alpha} \exp \left( -\frac{\alpha^2}{\alpha - 1} T \right) \right) \right) + \cosh \left( \ln \left( \beta^{-1} \kappa^{-1/\alpha} \exp \left( -\frac{\alpha^2}{\alpha - 1} T \right) \right) \right) \right) + \kappa^* z.
\]

After simplification we obtain:

\[
\mathbb{E} \left[ \tilde{U} \left( \kappa \frac{dQ^0}{dP} \right) \right] + \kappa^* z = \frac{\left( r + \frac{\beta^2}{2 \alpha} \right)^{1-\alpha} T \left( z + \alpha \sqrt{z^2 + \beta^2} \right)^{1-\alpha}}{1 - \alpha^2} \left( z + \sqrt{z^2 + \beta^2} \right)^\alpha
\]

\[
= u(0; \tilde{x}_0 + e^{-rT} \mathbb{E}^Q g(y)).
\]

Hence,

\[
\sup_{x^0} \mathbb{E} \left[ U \left( x^0 + g(y) \right) \right] \leq u \left( 0; \tilde{x}_0 + e^{-rT} \mathbb{E}^Q g(y) \right).
\]  (16)

The case \( \alpha = 1 \) is analogous. Indeed, using the same notation as before we get

\[
\mathbb{E} \left[ \tilde{U} \left( \kappa e^{-rT} \frac{dQ^0}{dP} \right) \right] = \frac{1}{4} \left( \beta^2 \kappa^2 e^{-2rT} \mathbb{E} \left[ \frac{dQ^0}{dP} \right] \right) - \frac{1}{2} \ln \kappa - \frac{1}{2} \left[ \ln \frac{\kappa}{2} \mathbb{E} \left[ \frac{dQ^0}{dP} \right] + \frac{1}{2} r T \right]
\]

\[
= \frac{1}{4} \left( \beta \kappa^2 - 1 \right) - \frac{1}{2} \ln \kappa + \frac{1}{2} \left( r + \frac{1}{2} \lambda^2 \right) T.
\]
The first-order condition yields:
\[
0 = \frac{1}{2} \beta^2 k - \frac{1}{2} \frac{1}{k} + z = \beta \frac{1}{2} (k \beta - k^{-1} \beta^{-1}) + z \\
= -\text{sinh} \left( \ln \left( \beta^{-1} \right) \right) + z,
\]
which leads to
\[
k^* = \left[ \beta \exp \left( \text{arcsinh} \left( \frac{z}{\beta} \right) \right) \right]^{-1} = \left( z + \sqrt{z^2 + \beta^2} \right)^{-1}.
\]
(17)

After inserting (17) into (14) we obtain:
\[
\mathbb{E} \left[ \hat{U} \left( k^* \frac{dQ_0}{dP} \right) \right] + \kappa^* z \\
= \frac{1}{4} \left( \beta^2 \kappa^* - 1 \right) + \frac{1}{2} \ln \left( z + \sqrt{z^2 + \beta^2} \right) + z \left( r + \frac{1}{2} \kappa^* \right) T \\
= \frac{1}{2} \beta^{-2} z \left( \sqrt{z^2 + \beta^2} - z \right) + \frac{1}{2} \ln \left( z + \sqrt{z^2 + \beta^2} \right) + \frac{1}{2} \left( r + \frac{1}{2} \kappa^* \right) T \\
= u(0, \tilde{x}_0 + e^{-rT} \mathbb{E}^Q_0[\xi(y_T)]) \quad \square
\]

The upper bound can be seen as a shifted version of the value function in the complete market setup, where the initial wealth is \( \tilde{x}_0 + e^{-rT} \mathbb{E}^Q_0[\xi(y_T)] \).

We now derive the following upper bound for the buyer’s price, which is similar to that in [9] (for power and exponential utility).

**Lemma 3.5.** The buyer’s indifference price \( p \) under the SAHARA utility function is bounded by \( e^{-rT} \mathbb{E}^Q_0[\xi(y_T)] \).

**Proof.** From the indifference pricing equation (5) and Proposition 3.4 we obtain \( u(0, \tilde{x}_0) \leq u(0, \tilde{x}_0 + e^{-rT} \mathbb{E}^Q_0[\xi(y_T)]) \).

Since \( u \) is an increasing function, it follows that \( x_0 \leq \tilde{x}_0 + e^{-rT} \mathbb{E}^Q_0[\xi(y_T)] \). The Lemma is proven. \( \square \)

Note that the upper bound of the indirect utility level depends on the utility function, but the price bound does not. In fact, Lemma 3.5 also holds for power and exponential utility functions, see e.g. [9].

Similarly, a lower pricing bound can be determined for the seller’s indifference price. Note that discussions around bounds for short positions can be found in the literature for exponential utility, see e.g. [3,12,23]. In a general framework, [21] (Proposition 4.3) show by minimizing the dual problem for an increasing and concave utility function (SAHARA belongs to this category) that the following lower bound of the indifference price for short positions holds:
\[
p^{seller} \geq e^{-rT} \mathbb{E}^Q_0[\xi(y_T)].
\]
(18)

As both upper and lower bounds are independent of the utility function, we can conclude that the following relationship holds in our model:
\[
p^{seller} \geq e^{-rT} \mathbb{E}^Q_0[\xi(y_T)] \geq p^{buyer}.
\]
It implies that there is no trade occurring between a risk-averse buyer and a risk-averse seller when one of these above inequalities strictly holds. A similar result for exponential utility can be found in [3]. Further discussions on properties of the indifference pricing functional can be found in e.g. [3] (exponential utility) and [21] (general framework).

3.3. HJB equation formulation

Assuming a Markovian setting, we now derive the corresponding HJB equation for the optimization problem (11) which will be solved through a numerical approach. As numerical algorithms usually move backwards in time, we set \( \tau = T - t \in [0, T] \) and define the indirect utility function as follows:
\[
v(T - \tau, x, y) := \sup_{\tilde{X}_T, \tilde{Y}_T} \mathbb{E}[U(X_T, \tilde{X}_T, \tilde{Y}_T)|X_T = x, y_T = y].
\]
(19)

We use \( v \) for both the seller and buyer perspective since the only difference is the changed sign of \( g(y_T) \). For a reasonable comparison with the power utility, we assume, without loss of generality for the rest of the paper, that the wealth process at time \( t \) is non-negative a.s. for all \( t \in [0, T] \). Note that although the wealth is assumed to be non-negative, a short position in \( g(y_T) \) still can lead to a final negative payout \( X_T - g(y_T) < 0 \). By the dynamic programming principle, the indirect utility function solves the following HJB equation
\[
\sup_{\theta \in \Theta} \left[ -v_T + (r x + (\mu - r) \theta) v_x + a y v_y + \frac{1}{2} \sigma^2 \theta^2 v_{xx} + \rho b y \sigma \theta v_{xy} + \frac{1}{2} \rho^2 b^2 y^2 v_{yy} \right] = 0,
\]
(20)
on the domain \((\tau, x, y) \in [0, T] \times [0, x_{\text{max}}] \times [0, y_{\text{max}}]\). Later, for the numerical procedure, we will also assume a bounded domain for the invested amount \(\theta\). Note that a simple choice \(\theta \in [0, \theta_{\text{max}}]\) does not prevent the wealth process in (3) from going negative. This is due to the fact that an economic agent, even with a lower wealth, may still invest a considerable amount in the risky asset, leading to a negative wealth level. To achieve the positivity of the wealth, one needs to guarantee that \(\theta \to 0\) as \(x \to 0\). In particular, we assume that the agent has a restricted leverage for low wealth levels \(x \in (0, \bar{x}]\) which forces her to invest only a proportional amount of wealth into the risky asset \(\theta = \pi x\), for \(x \in (0, \bar{x}]\). The strip value \(\bar{x}\) will be chosen later as the smallest inner point on our numerical grid. For higher wealth levels we can assume our invested amount is bounded by a fixed value \(\theta_{\text{max}} > 0\). To summarize, the domain of the invested amount is set as follows:

\[
\Theta := \begin{cases} 
(0, \theta_{\text{max}}], & \text{for } (x, y) \in (\bar{x}, x_{\text{max}}] \times [0, y_{\text{max}}] \\
(0, \frac{x_{\text{max}}}{\bar{x}}), & \text{for } (x, y) \in (0, \bar{x}] \times [0, y_{\text{max}}] 
\end{cases}
\]

The upper bounds \(x_{\text{max}}, y_{\text{max}}, \theta_{\text{max}}\) will be chosen later in Section 4.4. As we aim to solve our optimization problem numerically, we already define the HJB on a bounded domain and discuss the conditions about the upper and lower boundaries in Section 3.4.

According to [24, Theorem 4.5.2], the value function \(v\) is a viscosity solution of the HJB PDE (20) with the boundary condition \(v(T, x, y) = E[U(x \pm g(y))]\). Note that the HJB equation is the same for both the seller and buyer problem but their boundary conditions are different.

Maximizing over \(\theta\), we obtain the optimal amount invested in the risky asset:

\[
\theta^*(\tau, x, y) = \frac{(\mu - r)/\sigma^2 + \rho b y / \sigma}{-v_{xx}/v_x} + \frac{\rho b y / \sigma}{v_{xy}/v_y}.
\]  

The first term in Eq. (22) is of the same form of the strategy which would be optimal when there is no unhedgeable risk. The second component accounts for the unhedgeable risk and depends on the correlation coefficient between the tradable and untradable risk. This additional hedge demand indicates how much extra capital shall be invested in the risky asset in order to behave optimally (depending on the correlation coefficient \(\rho\)). Furthermore, note that with a perfect correlation, i.e. \(\rho \in [-1, 1]\), it is possible to eliminate the entire risk and hence, (20) degenerates. In these cases, the market is complete and the indifference price equals the Black–Scholes price, see e.g. [25].

Substituting the optimal \(\theta^*\) back in (20), we have the following PDE

\[
-v_x + rnx_x + a y v_y - \frac{1}{2} \left( \frac{(\mu - r)v_x + \rho b y \sigma v_{xy}}{\sigma^2 v_x} \right)^2 + \frac{1}{2} b^2 y^2 v_{yy} = 0,
\]  

which is in general highly non-linear and no closed-form solutions for SAHARA utility functions are available. Therefore, we will rely on a numerical procedure to solve the HJB PDE (20).

3.4. Boundary conditions

As our problem is posed on a bounded domain, we need to specify what happens at the boundaries. On the lower boundaries, we do not need any special conditions. Note that with the choice of \(\Theta\) in (21) above, we enforce \(\theta \to 0\) as \(x \to 0\). Hence, the cross derivative in (20) vanishes when either \(x = 0\) or \(y = 0\). Moreover, at \((0, 0)\) the process is constant to guarantee the continuity of the operator at \((0,0)\). More specifically, the behavior of the value function at the lower boundary is specified as follows:

\[
v_r = \begin{cases} 
a y v_y + \frac{1}{2} b^2 y^2 v_{yy}, & \text{for } (x, y) \in (0) \times [0, y_{\text{max}}), \\
(r x + (\mu - r)\theta)v_x + \frac{1}{2}(\sigma \theta)^2 v_{xx}, & \text{for } (x, y) \in (0, x_{\text{max}}) \times (0), \\
0, & \text{for } (x, y) = (0, 0).
\end{cases}
\]  

For the upper boundaries, we will apply Dirichlet boundary conditions. Errors resulting from the approximate boundary conditions at \(x = x_{\text{max}}\) and \(y = y_{\text{max}}\) are small, if \(x_{\text{max}}\) and \(y_{\text{max}}\) are sufficiently large, see [26,27]. Due to the fact that the highest achievable wealth \(x_{\text{max}}\) is typically much higher than the highest value of the untradable asset \(y_{\text{max}}\), we neglect the impact of the second asset and simply add the current value of \(g(y)\) to \(x\). We can then approximate the solution by the complete market setup, i.e. \(v(\tau, x_{\text{max}}, y) = u(\tau, x_{\text{max}} + g(y))\). In case of \(y = y_{\text{max}}\), we set \(v(\tau, x, y) = u(\tau, x + \exp((a - r)(T - \tau))g(y))\).

In the next section, we will describe an implementation of the algorithm for the two-dimensional problem (20), which is initially discussed in [17] and [16]. It is a finite difference scheme with positive non-diagonal elements, also known as a positive coefficient method. It is worth emphasizing that our setting differs from the super-hedging cost problem in [17] and the mean variance problem in [16]. In particular, [17] has a constant but unknown variable as the control, while our control changes in wealth and time. In [16], the wealth process depends implicitly on the second dimension, in particular, the terminal condition is constant in the second dimension, which is not the case for our problem. Our problem has an explicit dependence on the first and the second dimension as well as a time-dependent control.
Table 1

<table>
<thead>
<tr>
<th>Notation</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{S})</td>
<td>The whole domain ([0, x_{\text{max}}] \times [0, y_{\text{max}}])</td>
</tr>
<tr>
<td>(\mathbb{S}_m)</td>
<td>All inner values ((0, x_{\text{max}}) \times (0, y_{\text{max}}))</td>
</tr>
<tr>
<td>(\mathbb{S}<em>{x</em>{\text{max}}})</td>
<td>The upper boundary in (x = x_{\text{max}}), i.e. ([x_{\text{max}}, 0) \times [0, y_{\text{max}}])</td>
</tr>
<tr>
<td>(\mathbb{S}<em>{y</em>{\text{max}}})</td>
<td>The lower boundary in (x = 0), i.e. ([0, x_{\text{max}}] \times (0, y_{\text{max}}))</td>
</tr>
<tr>
<td>(\mathbb{S}_{y})</td>
<td>The upper boundary in (y = y_{\text{max}}), i.e. ([0, x_{\text{max}}] \times [y_{\text{max}}, 0))</td>
</tr>
</tbody>
</table>

4. Numerical implementation

In this section we discuss about the algorithm used to solve our HJB PDE (20) with terminal condition \(v(T, x, y) = U(x \pm g(y))\). We now set up the problem stated in Section 3.3. Note first that as shown e.g. in [28], a numerical algorithm for an HJB PDE has to be pointwise consistent, stable and monotone to ensure convergence to the viscosity solution. It is known that for a finite difference scheme, see e.g. [29], the so-called positive coefficient method satisfies these conditions. For one-dimension HJB PDEs, finite difference scheme in principle relies on choosing between the possible different discretizations (central, forward and backward) at each node such that positive coefficients can be obtained in all non-diagonal terms. This, however, cannot be directly applied for our two-dimensional case as possible discretizations of the cross-derivative \(v_{xy}\) do not ensure the positivity for all non-diagonal coefficients. To treat this issue, [17] proposes a rotational technique. In particular, the coordinates are rotated, leading to a virtual grid on which a new value function is defined. Below we carry out a description of the algorithm, following [17].

4.1. Numerical discretization of the HJB equation

Below, we describe a discretization of the HJB equation (20) on the bounded region \((x, y) \in [0, x_{\text{max}}] \times [0, y_{\text{max}}] = \mathbb{S}\). Recall first the HJB equation

\[
\sup_{\theta \in \mathbb{S}} \left[ -v_x + (rx + (\mu - r)\theta)v_x + ayv_y + \frac{1}{2} \sigma^2 v_{xx} + \rho b y \sigma v_{xy} + \frac{1}{2} \sigma^2 b^2 v_{yy} \right] = 0.
\]

Setting the coefficients to

\[
c_{x}(x, \theta) = (rx + (\mu - r)\theta), \quad c_{xx}(x, \theta) = \frac{1}{2} \sigma^2, \quad c_{y}(y) = ay,
\]

\[
c_{yy}(y) = \frac{1}{2} \sigma^2 b^2, \quad c_{xy}(x, y, \theta) = \rho by \sigma \theta
\]

leads to the following:

\[
\sup_{\theta \in \mathbb{S}} \left[ -v_x + c_{x}(x, \theta)v_x + c_{y}(y)v_y + c_{xx}(x, \theta)v_{xx} + c_{xy}(x, y, \theta)v_{xy} + c_{yy}(y)v_{yy} \right] = 0.
\]

(25)

We discretize our HJB equation (25) over a finite grid \(N = \{x_1, x_2, \ldots, x_{N_x}\} \times \{y_1, y_2, \ldots, y_{N_y}\}\) in the plane \((x, y) \in \mathbb{S}\). We set \(x_1 = y_1 = 0, x_{N_x} = x_{\text{max}}\) and \(y_{N_y} = y_{\text{max}}\) to match the boundaries of the localized region. The number of nodes in each direction is \(N_x\) and \(N_y\), respectively. We use an equidistant timestep with step size \(\Delta \tau\), i.e. \(\tau_k = k\Delta \tau, k = 0, \ldots, N_t = \frac{T}{\Delta \tau}\) in \(\mathbb{N}\). In the following, we will use \(i \in \{1, \ldots, N_x\}\) to denote the index in the \(x\)-direction, while \(j \in \{1, \ldots, N_y\}\) denotes the index for the \(y\)-direction. Specification of the domains is summarized in Table 1. In comparison with [17], we do not consider the domain \(\mathbb{S}_{\text{out}} := \{(x, y) : x > x_{\text{max}} \lor y > y_{\text{max}}\}\), which contains points outside of \(\mathbb{S}\). In our case, points outside the domain are not used. See discussion in Section 4.4. Furthermore, we choose \(\hat{x} := x_2\) and enforce bounded leverage on the strip \([0, \hat{x}] \cup [0, y_{\text{max}}]\). Note that \([0, \hat{x}] \times [0, y_{\text{max}}] \cap N = \mathbb{S}_{\text{out}}\) only contains grid points on the boundary with \(x = 0\) where the process is assumed to be independent of \(\theta\), see the boundary conditions (24).

4.2. Discretization of the cross derivative \(v_{xy}\)

As already mentioned, the crucial part of developing a monotone discretization scheme is a discretization of the cross-derivative \(v_{xy}\). [17] proposes two methods. The first one is to approximate \(v_{xy}\) by a seven-point stencil, which under certain circumstances leads to positive coefficients in all non-diagonal terms. However, our numerical experiments have shown that this is only applicable for a few points in our problem. Therefore, we rely on the second method where a virtual grid rotation of the Brownian motions is needed. In particular, we rotate the coordinate system such that the Brownian motions are orthogonal to each other and the cross derivative vanishes.
To this end, we introduce a new coordinate system \((z_{x,y}^1, z_{x,y}^2)\) defined by
\[
\begin{pmatrix}
    x \\
    y
\end{pmatrix} = \begin{pmatrix}
    \cos(\phi_{x,y}(\theta)) & -\sin(\phi_{x,y}(\theta)) \\
    \sin(\phi_{x,y}(\theta)) & \cos(\phi_{x,y}(\theta))
\end{pmatrix} \begin{pmatrix}
    z_{x,y}^1 \\
    z_{x,y}^2
\end{pmatrix}.
\] (26)

In Eq. (26), \(\phi_{x,y}(\theta)\) is the rotation angle and \(R^{x,y,\theta}\) is the rotation matrix. Note that the coordinates \((z_{x,y}^1, z_{x,y}^2)\), as well as the rotation angle \(\phi_{x,y}(\cdot)\) always are connected to the original point \((x, y)\).

To eliminate the cross-derivative, the rotation angle has to be chosen in such a way that there is zero correlation in the diffusion tensor of the rotated coordinate system. Let \(w\) denote the value function of \(v\) in the transformed coordinate system. It is given by the following relationship:
\[
w(t, z_{x,y}^1, z_{x,y}^2) = w(t, \cos(\phi_{x,y}(\theta))x + \sin(\phi_{x,y}(\theta))y, -\sin(\phi_{x,y}(\theta))x + \cos(\phi_{x,y}(\theta))y) := w(t, x, y),
\]
and \(w\) has to fulfill the following condition:
\[
c_{xx}(x, \theta)v_{xx} + c_{xy}(x, y, \theta)v_{xy} + c_{yy}(y)v_{yy} = \tilde{a}_{x,y}(\theta)v_{w_{1,1}^2} + \tilde{b}_{x,y}(\theta)v_{w_{2,2}^2}, \tag{27}
\]
where \(w_{1,1}^2\) and \(w_{2,2}^2\) denote the respective second-order derivatives of \(w\). Note that under the rotated coordinates, there is no cross-derivative term \(w_{1,2}^2\). Similar to [17], we can derive the rotation angle \(\phi_{x,y}(\cdot)\) which is defined by
\[
\phi_{x,y}(\theta) = \frac{1}{2} \tan^{-1} \left( \frac{\rho \sigma_y \theta}{\frac{1}{2} (\sigma^2 - 1) (by)^2} \right) = \frac{1}{2} \tan^{-1} \left( \frac{c_{yy}(x, y, \theta)}{c_{xx}(x, \theta) - c_{xy}(y)} \right), \tag{28}
\]
as well as the following coefficients for the second order derivatives in \(w\):
\[
\tilde{a}_{x,y}(\theta) := c_{xx}(x, \theta) \left( R^{x,y,\theta}_{1,1} \right)^2 + c_{xy}(x, y, \theta) R^{x,y,\theta}_{1,2} R^{x,y,\theta}_{2,1} + c_{yy}(y) \left( R^{x,y,\theta}_{2,2} \right)^2 = \frac{1}{2} (\sigma^2 - 1) (by)^2 \sin^2(\phi_{x,y}(\theta)) + \frac{1}{2} (\rho \sigma_y \theta)^2 \sin^2(\phi_{x,y}(\theta)) \geq 0,
\]
\[
\tilde{b}_{x,y}(\theta) := c_{xx}(x, \theta) \left( R^{x,y,\theta}_{1,2} \right)^2 + c_{xy}(x, y, \theta) R^{x,y,\theta}_{1,1} R^{x,y,\theta}_{2,2} + c_{yy}(y) \left( R^{x,y,\theta}_{2,1} \right)^2 = \frac{1}{2} (\sigma^2 - 1) (by)^2 \cos^2(\phi_{x,y}(\theta)) + \frac{1}{2} (\rho \sigma_y \theta)^2 \cos^2(\phi_{x,y}(\theta)) \geq 0.
\]
The indices of \(R^{x,y,\theta}\) denote the respective matrix entries. We observe that \(\tilde{a}_{x,y}(\theta) \geq 0\) due to \(|c_{yy} R^{x,y,\theta}_{2,2}| = |\rho| |2 \sqrt{c_{xx} c_{xy}} R^{x,y,\theta}_{1,1} R^{x,y,\theta}_{2,1}| \leq c_{xx}(x, \theta) \left( R^{x,y,\theta}_{1,1} \right)^2 + c_{yy}(y) \left( R^{x,y,\theta}_{2,2} \right)^2\).

### 4.3. Finite difference scheme

We now set up a finite difference scheme for (25). For the ease of notation, we will use \((k, i, j)\) to denote a specific \((t_k, x_i, y_j)\)-triplet. The corresponding value functions are denoted by \(v^{k}_{ij}\) and \(v_{ij}^{k}\) respectively. Both \(v\) and \(w\) refer to the discretized version of \(v\) and \(w\).

The HJB equation in discretized form is now given by
\[
\max_{\partial^t v_{ij}} \left[ -v_{ij} + L^d v_{ij} \right] = 0, \tag{29}
\]
where \(L^d\) is the differential operator defined by
\[
L^d v = c_{x}(x, \theta) v_{x} + c_{y}(y) v_{y} + c_{xx}(x, \theta) v_{xx} + c_{xy}(x, y, \theta) v_{xy} + c_{yy}(y) v_{yy} = c_{xx}(x, \theta) v_{x} + c_{yy}(y) v_{y} + \tilde{a}_{x,y}(\theta) v_{w_{1,1}^2} + \tilde{b}_{x,y}(\theta) v_{w_{2,2}^2}.
\] (30)

The coefficients for first-order terms \(c_x(\cdot, \cdot)\) and \(c_y(\cdot, \cdot)\) can in general attain positive and negative values. As we aim for positive coefficients in the non-diagonal terms, we need to choose between forward \((\Delta^+)^\) and backward \((\Delta^-)^\) differencing in the first-order derivatives. In particular, we denote the discretization steps on our non-uniform grid by
\[
\Delta^+ x_i = \Delta^- x_{i+1} := x_{i+1} - x_i, \quad i \in \{1, \ldots, N_x - 1\},
\]
\[
\Delta^+ y_j = \Delta^- y_{j+1} := y_{j+1} - y_j, \quad j \in \{1, \ldots, N_y - 1\}.
\]

We define the following discretization for the first-order derivatives:
\[
\nu_x(k, i, j) \approx \begin{cases} 
\frac{v_{i+1,j}^k - v_{i,j}^k}{\Delta^+ x_i} + I_{c_x(\cdot, \cdot) > 0} \cdot \frac{v_{i+1,j}^k - v_{i-1,j}^k}{\Delta^- x_{i-1}}, & \text{if } c_x(\cdot, \cdot) > 0, \\
\frac{v_{i+1,j}^k - v_{i,j}^k}{\Delta^+ x_i} + I_{c_x(\cdot, \cdot) < 0} \cdot \frac{v_{i+1,j}^k - v_{i-1,j}^k}{\Delta^- x_{i-1}}, & \text{if } c_x(\cdot, \cdot) < 0, 
\end{cases}
\]
\[
\nu_y(k, i, j) \approx \begin{cases} 
\frac{v_{i,j+1}^k - v_{i,j}^k}{\Delta^+ y_j} + I_{c_y(\cdot, \cdot) > 0} \cdot \frac{v_{i,j+1}^k - v_{i,j-1}^k}{\Delta^- y_{j-1}}, & \text{if } c_y(\cdot, \cdot) > 0, \\
\frac{v_{i,j+1}^k - v_{i,j}^k}{\Delta^+ y_j} + I_{c_y(\cdot, \cdot) < 0} \cdot \frac{v_{i,j+1}^k - v_{i,j-1}^k}{\Delta^- y_{j-1}}, & \text{if } c_y(\cdot, \cdot) < 0.
\end{cases}
\]
It can be checked that the non-diagonal term \( c_d(i, \theta)(1 - \Theta_{i,(i, \theta)}) \) is always positive. Furthermore, \( c_d \geq 0 \) is always non-negative and the second term of \( \forall \eta(k, i, j) \) can be ignored. This choice satisfies a positivity requirement.

The second-order derivatives for \( \forall \) can no longer be defined on the original grid and the virtual grid on which \( \forall \) is defined has to be used. To this extent, we assume a mesh discretization parameter \( h \) exists such that

\[
\max_{i \in \{1, \ldots, N_x - 1\}} \Delta^\tau x_i = C_l h, \quad \min_{i \in \{1, \ldots, N_x - 1\}} \Delta^\tau y_i = C_l h;
\]

\[
\max_{j \in \{1, \ldots, N_y - 1\}} \Delta^\tau y_j = C_l h, \quad \min_{j \in \{1, \ldots, N_y - 1\}} \Delta^\tau y_j = C_l h, \quad \Delta \tau = C_l h,
\]

where \( C_l, l \in \{1, 2, 3, 4, 5\} \) are constants independent of \( h \). The mesh discretization parameter \( h \) sets the points for the second-order derivatives to have a distance of \( \sqrt{h} \) to the central node \( \forall \eta |_{ij} = \forall |_{ij} \) on the virtual grid.

For the discretization of the new second-order derivatives \( w_{2,1} \) and \( w_{2,2} \), the following four points are needed:

\[
z_{i,m} = \left( \frac{x_i}{y_i} \right) + (-1)^m \sqrt{h} e_i, \quad i \in \{1, 2\}, \quad m \in \{0, 1\},
\]

where \( e_i \) denotes the standard basis of \( \mathbb{R}^2 \) in the rotated coordinate system and the last part denotes the corresponding point on the original grid. Then the second-order finite difference scheme for \( w_{2,2} \) yields

\[
w_{2,1}(|_{ij}, z_{i,m}^1, z_{i,m}^2) \approx \frac{\forall h |_{ij}^1(z_{i,m}^1 + \sqrt{h}, z_{i,m}^2) - 2\forall h |_{ij}^1(z_{i,m}^1, z_{i,m}^2) + \forall h |_{ij}^1(z_{i,m}^1 - \sqrt{h}, z_{i,m}^2)}{h} - \frac{\forall h |_{ij}^1(\theta) + \forall h |_{ij}^1(\theta)}{h},
\]

\[
w_{2,2}(|_{ij}, z_{i,m}^1, z_{i,m}^2) \approx \frac{\forall h |_{ij}^1(z_{i,m}^1 + \sqrt{h}, z_{i,m}^2) - 2\forall h |_{ij}^1(z_{i,m}^1, z_{i,m}^2) + \forall h |_{ij}^1(z_{i,m}^1 - \sqrt{h}, z_{i,m}^2)}{h} - \frac{\forall h |_{ij}^1(\theta) + \forall h |_{ij}^1(\theta)}{h},
\]

where \( \forall h \) denotes the corresponding value on the original grid, i.e. \( \forall h |_{ij}^1(\theta) \) is the value at \( \forall h |_{ij}^1(x_i + (-1)^m \sqrt{h} i, y_j + (-1)^m \sqrt{h} j) \), for \( i \in \{1, 2\} \) and \( m \in \{0, 1\} \). Each of the four points \( (x_i, y_j) + (-1)^m \sqrt{h} e_i \), do not lie on the grid \( N \) in general and the corresponding value of \( \forall h \) has to be determined by interpolating the original mesh. The value function of the off-grid points \( \forall h \) is therefore determined by a (bi-)linear interpolation between the four surrounding points on the grid. Linear interpolation is required to keep the scheme consistent, see [17]. Note that with an increasing number of grid and time points, the mesh discretization factor \( h \) decreases and approaches 0. For points inside the domain \( [h, \sqrt{h}] \times [0, x_{min}] \cup [0, x_{max}] \times [h, \sqrt{h}] \), there is a possibility that the wide stencil is below the lower boundary. To resolve this, [17] suggests truncating the stencil to the length \( h \) and using an asymptotic form for points above the upper boundary. We refer the reader to this paper for further discussions about the points outside the boundaries. However, in our setting, we have a coarse grid near the boundaries and thus the rotated points will be inside \( \mathcal{E} \) and we do not have to deal with any outside points (see Section 4.4). We furthermore remark that with the choice of \( h \) specified above, the scheme remains consistent, see [17] for further details.

Putting everything together we can write down the discretized form \( L_h \) of the linear operator \( L_0 \):

\[
L_h \Phi |_{ij}^{k+1} := l_{c_y(i, \theta)}, \quad \Delta \Phi |_{ij} \left( \forall h |_{ij}^{k+1} - \forall h |_{ij}^{k+1} \right) + \frac{c_d(i, \theta)}{\Delta \Phi |_{ij} \left( \forall h |_{ij}^{k+1} - \forall h |_{ij}^{k+1} \right)} + \frac{\forall h |_{ij}^{k} \theta}{h} \left( \forall h |_{ij}^{k+1} - \forall h |_{ij}^{k+1} \right)
\]

After rearranging the terms we obtain:

\[
L_h \Phi |_{ij}^{k+1} := l_{r_{x,i} + (\mu - r \theta) \gamma_{i+1}^{k+1} - r_{x,i} \gamma_{i+1}^{k+1}} \Delta \Phi |_{ij} \left( \forall h |_{ij}^{k+1} - \forall h |_{ij}^{k+1} \right) + \frac{\forall h |_{ij}^{k} \theta}{h} \left( \forall h |_{ij}^{k+1} - \forall h |_{ij}^{k+1} \right) + \frac{\forall h |_{ij}^{k} \theta}{h} \left( \forall h |_{ij}^{k+1} - \forall h |_{ij}^{k+1} \right)
\]

\footnote{For our setting, the use of an approximate complete solution or maybe even a linear function (for very high wealth levels), might be suitable asymptotic forms.}
Using the parameters in Table 3, we obtain approximately 770.47 for $\pi^\ast$ in the risky asset $x$ at $t = 0$ or $t = T$, depending on the sign of $(-r + \frac{\lambda^2}{\alpha^2})$. In other words,

$$\max_{t \in (0, T)} \theta^\ast_t = \begin{cases} \frac{\lambda}{\alpha \sigma} \sqrt{x_{t, \max}^2 + \beta^2 \exp \left(\left(-2r + \frac{\lambda^2}{\alpha^2}\right) \frac{T}{\sigma^2}\right)} \quad & \frac{\lambda}{\alpha^2} > 2r \\ \frac{\lambda}{\alpha} \sqrt{x_{t, \max}^2 + \beta^2} \quad & \frac{\lambda}{\alpha^2} \leq 2r \end{cases}$$

Using the parameters in Table 3, we obtain approximately 770.47 for $\theta^\ast_t$. In addition, as the impact of the untradeable claim on the optimal invested amount is expected to be low when the wealth level is high, the use of $\theta_{\max} = 800$ seems reasonable to us. In the following, we also numerically compute the indifference price and the optimal investment strategy for power utility using the same algorithm. Recall that in this case, it would be more convenient to use the share invested in the risky asset $\pi_t := \theta_t / x_t$ as the control variable. In our numerical comparison with power utility we suppose that $\pi \in [-2, 2]$ and apply a discretization stepsize $\Delta \pi = 0.01$. Unlike SAHARA and exponential utility settings, our numerical experiments for power utility have shown that the untradeable claim has a significant impact on the investment strategy $\pi$ when the wealth level is close to zero. As it can take highly positive and negative positions in the risky asset when the wealth level is close to zero, we limit the numerical result in such a bounded interval to make the comparison reasonable.

Note that the choice of the mesh discretization parameter $h$ may vary from problem to problem. In particular, for inner points $(x, y)$ which are not too close to critical points, it is intuitive, by (22), to choose $h$ such that the optimal trading
strategy $\theta^*$ at the first iteration is close to
\[
\frac{\mu - r - \frac{\partial U(x, y)}{\partial x}}{\frac{\partial^2 U(x, y)}{\partial x^2}} + \frac{\rho y}{\sigma} \frac{\frac{\partial^2 U(x, y)}{\partial x \partial y}}{\frac{\partial U(x, y)}{\partial x}}.
\]
where $U(x, y) := U(x + g(y))$.

Remark 4.1 (Consistency, Stability and Convergence). The present problem setting is different and more complex than the ones in [16,17]. The restriction on the grid we make does not affect the consistency, stability and convergence of the algorithm and only requires to increase the number of nodes in each direction. To this end, we can rely on the proofs given in [17] to conclude that their results for consistency, stability and convergence hold true for our problem as well. For further details we refer to [17].

5. Numerical analysis

The solution to the right-hand-side of Eq. (5) is obtained from the analytical results given in Section 3.1. The left-hand-side is determined by solving the HJB PDE from Section 3.2 with the numerical algorithm described in Section 4. The final step is to interpolate the numerical solution of the left-hand-side to match the analytic solution of the right-hand-side. The indifference price is then obtained by comparing the corresponding initial wealth $p := |x_0 - x_0|$.

In the following, different contracts will be analyzed. Specifically, we will look at three simple payoffs $g(y_T)$: a stock $g(y_T) = y_T$, a European put option with $g(y_T) = (K - y_T)^+$ and a European call option with $g(y_T) = (y_T - K)^+$. Unless stated otherwise we use the parameter setting stated in Table 3. The strike price is chosen such that both options are at the money initially, i.e. $y_0 = K = 100$.

Note that the structure of the overall findings stays the same, when we choose a different interest rate. The parameters for the market setup are taken from [9]. The drift $a$ of the untraded asset is chosen such that $a = \frac{\mu b}{\gamma}$.

5.1. Indifference pricing from the buyer’s perspective

We determine the price and compare the results between power, exponential and SAHARA utility function. From the buyer’s perspective, we deal with long positions. That is the reason why a power utility can also be drawn into comparison. To this end, the coefficients for each class of utility functions are chosen such that the initial risk aversion of all three utility functions is the same:
\[
y = \frac{\eta}{x_0} = \frac{\alpha}{\sqrt{\beta^2 + x_0^2}}.
\]

Given this relationship with $x_0 = 1000$, the risk-aversion parameters for the three utility functions are stated in Table 4.

We will compare the approximate indifference price comparing by the algorithm with two benchmark values. The first one is the following closed-form solution (see [10] for further details) for the indifference price of $g(y_T)$ under the exponential utility function:
\[
p^\theta := -\frac{1}{\gamma(1 - \rho^2)} e^{-\gamma T} \ln(\theta_{\alpha}^{-1}) \left[ \exp \left( -\gamma' (1 - \rho^2) g(y_T) \right) \right].
\]

The second one is the upper bound $e^{-\gamma T} \theta_{\alpha}^{-1}[g(y_T)]$ which is the risk neutral price under the minimal martingale measure (see Proposition 3.4). The price under the minimal martingale measure is determined by a simple Monte-Carlo-Simulation with $M := 10^6$ paths.

The indifference prices for the untradable asset, the European put-option as well as the European call-option are reported in Table 5 for different choices of $\rho$. We make the following main observations: All the indifference prices...
obtained are below the upper bound $e^{-rT}E^Q[g(y_T)]$, which is independent of the utility function the agent uses. Furthermore, they are very close to $p^E$, the analytic solution of the exponential utility function. These two observations show that the algorithm delivers reasonable results for all three utility functions. With a higher correlation, the indifference prices tend towards the Black–Scholes price. This is quite natural as a higher correlation value implies that most of the untradable risk can be eliminated by trading in the tradable risks, and we are closer to a complete market setting. The upper pricing bound converges towards the Black–Scholes price (BS), i.e., the indifference price if the claim is replicable, see e.g. [3]. Since the option is at-the-money, i.e., $y_0 = K$, and $r = 0$, the Black–Scholes call and put price are identical, equal to 11.92 as stated in the last column of Table 5. Moreover, the indifference price for the call is always higher than that for the put price for the same level of initial wealth, because the positive drift of the untradable asset leads to higher potentials of being in the money at maturity for the call option. These observations are aligned with the existing literature, e.g. [8,30]. Finally, for SAHARA utility function the algorithm delivers results that are close to the indifference prices obtained for power and exponential utility functions. Fig. 1 illustrates the initial optimal fraction $\frac{\Delta x}{x_0}$ invested for the three different contracts with $\rho = 0.8$ as well as for the no-contract case. It can be observed that all the investment strategies show very similar patterns, i.e., decreasing curves in initial wealth. Note that for significantly large values of the initial wealth $x_0$, the impact of the untradable asset value with a fixed initial value $y_0$ becomes less important. This explains why the optimal investment strategies for all cases are close to each other for high values of $x_0$. For low levels of initial wealth, the impact of the additional asset is more pronounced, leading to a more significant deviation from the benchmark (without contract) optimal strategy.

It is intuitive to see that the optimal asset allocation is substantially affected by the additional risk of the unhedgeable asset and the change in wealth when buying the asset. In particular, paying for the contract leaves the agent with less wealth which in turn induces an increasing fraction invested into the risky asset (compared to the no contract case). Moreover, the claim can be seen as an additional future income which reduces the optimal share invested. Although the effects of the initial payment and additional future income on the optimal trading strategy are more significant for the case $g(y_T) = y_T$, the change in optimal trading strategy is small as they compensate each other. The case of a put option $g(y_T) = (K - y_T)^+$ works like a natural hedge against unfavorable market performance, which yields an optimal investment close to the case with no claim. In contrast, holding a call $g(y_T) = (y_T - K)^+$ yields an additional risk in unfavorable market scenarios. Therefore the agent invests less risky to obtain an overall protection against adverse market outcomes.

In the following, we will address shortly the convergence issue by looking at the case $g(y_T) = y_T$, $\rho = 0.4$, for a SAHARA agent. As there is no analytical solutions for the SAHARA indifference price, we use the analytical solution $p^E = 101.7623$ resulting from the exponential case as a proxy. Note that our parameter choices match the risk aversion levels between these two utility functions, leading to very similar utility indifference prices (c.f. Table 5). In Table 6, we report the buyer’s indifference price for the case $g(y_T) = y_T$ as a function of $N_c$, $N_n$, $N_p$, $h$ and $\Delta \theta$, in order to visualize how this price changes when $\Delta r$ and $h$ turn smaller. With finer grids and time points, we observe that the prices asymptotically approach the analytical value of $p^E$. To further analyze the convergence, we can define the rate of convergence for our problem as

\[
\rho = 0.8 \\
\rho = 0.4 \\
\rho = 0.1 \\
\rho = -0.1 \\
\rho = -0.4 \\
\rho = -0.8 \\
(K - y_T)^+ \\
\rho = 0.8 \\
\rho = 0.4 \\
\rho = 0.1 \\
\rho = -0.1 \\
\rho = -0.4 \\
\rho = -0.8 
\]

\[
\begin{array}{cccccc}
\text{Power} & \text{Exponential} & \text{SAHARA} & e^{-rT}E^Q[g(y_T)] & p^E & \text{BS} \\
\hline
\rho = 0.8 & 100.45 & 100.24 & 100.32 & 100.69 & 100.56 & 100 \\
\rho = 0.4 & 101.65 & 101.64 & 101.64 & 102.07 & 101.76 & 100 \\
\rho = 0.1 & 102.71 & 102.69 & 102.70 & 103.03 & 102.69 & 100 \\
\rho = -0.1 & 103.41 & 103.39 & 103.40 & 103.75 & 103.41 & 100 \\
\rho = -0.4 & 104.41 & 104.43 & 104.43 & 104.76 & 104.47 & 100 \\
\rho = -0.8 & 105.97 & 105.98 & 105.98 & 106.19 & 106.06 & 100 \\
\hline
\rho = 0.8 & 11.60 & 11.58 & 11.53 & 11.63 & 11.60 & 11.92 \\
\rho = 0.4 & 10.97 & 10.96 & 10.94 & 11.04 & 10.97 & 11.92 \\
\rho = 0.1 & 10.30 & 10.30 & 10.29 & 10.42 & 10.35 & 11.92 \\
\rho = -0.1 & 10.03 & 10.03 & 10.03 & 10.10 & 10.04 & 11.92 \\
\rho = -0.4 & 9.59 & 9.58 & 9.57 & 9.61 & 9.55 & 11.92 \\
\rho = -0.8 & 9.01 & 8.99 & 8.96 & 9.04 & 9.01 & 11.92 \\
\hline
\rho = 0.8 & 12.27 & 12.26 & 12.23 & 12.43 & 12.26 & 11.92 \\
\rho = 0.4 & 13.02 & 13.00 & 12.98 & 13.11 & 12.96 & 11.92 \\
\rho = 0.1 & 13.32 & 13.31 & 13.31 & 13.51 & 13.34 & 11.92 \\
\rho = -0.1 & 13.72 & 13.72 & 13.71 & 13.85 & 13.58 & 11.92 \\
\rho = -0.8 & 15.18 & 15.15 & 15.10 & 15.22 & 15.16 & 11.92 \\
\end{array}
\]
follows:

$$c^q = \lim_{k \to \infty} c^q_k = \lim_{k \to \infty} \frac{|p_k - p^f|}{|p_{k-1} - p^f|^q},$$

(35)

where $q$ is the order of convergence, see [31]. Note that given $q$ being the order of convergence, the quantity $c^q$ reflects the asymptotic error, providing useful insights when using iterative methods to calculate approximations. For each refinement step, we report the estimated value $c^q$ for $q \in \{1.2, 1.0, 0.8\}$ in Table 6. It can be observed that for $q = 1.2$, $c^q$ increases in the refined step, but this effect is reversed for $q = 0.8$. For $q = 1.0$, $c^q$ remains almost constant, which numerically suggests that the convergence is approximately of order 1.

5.2. Indifference pricing from the seller’s perspective

In the following, we will discuss the results for the seller’s perspective for which we deal with the same HJB equation but with a different boundary condition. Table 7 reports the seller price for the three contracts using the SAHARA utility function. For completeness we also list the result for exponential utility function with a short put. Recall that the usage of a power utility function is no longer possible due to the short position of the contract at maturity.

Compared to the buyer prices in Table 5, the seller price for the stock and call contract is decreasing in the correlation $\rho$ and the price of the put contract is, however, increasing. In addition, it can be observed that the same dependence structure on the correlation exists as a reduced correlation leads to increased non-hedgeable risk, which is an important factor for price change. A similar behavior for a put option under the exponential utility function has been shown in [11]. Furthermore, the numerical results also confirm that the indifference price in the seller’s perspective is slightly above the lower bound stated in Eq. (18) and in particular also above the indifference price obtained in the buyer’s perspective.

The optimal trading strategies are illustrated in Fig. 2. When comparing this to the buyer’s perspective we observe that the form of the trading strategies is very similar. More precisely, the fraction invested in the stock decreases in the initial wealth level. As observed for the buyer’s strategy in Fig. 2, the impact of the additional untradable asset value, with a fixed initial value $y_0$ becomes less important for significantly large values of the initial wealth $x_0$, meaning that the optimal investment strategies for all cases are close to each other for high values of $x_0$. A deviation from the benchmark
Indifference pricing under SAHARA utility

### Table 7

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( g(y_T) )</th>
<th>SAHARA</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho = 0.8 )</td>
<td>101.05</td>
<td>12.13</td>
<td>13.10</td>
</tr>
<tr>
<td>( \rho = 0.4 )</td>
<td>102.56</td>
<td>11.66</td>
<td>14.17</td>
</tr>
<tr>
<td>( \rho = 0.1 )</td>
<td>103.59</td>
<td>10.93</td>
<td>14.29</td>
</tr>
<tr>
<td>( \rho = -0.1 )</td>
<td>104.31</td>
<td>10.68</td>
<td>14.75</td>
</tr>
<tr>
<td>( \rho = -0.4 )</td>
<td>105.38</td>
<td>10.55</td>
<td>15.74</td>
</tr>
<tr>
<td>( \rho = -0.8 )</td>
<td>106.81</td>
<td>9.91</td>
<td>17.04</td>
</tr>
</tbody>
</table>

Fig. 2. Comparison of the different optimal fraction invested for each contract type (seller perspective) for SAHARA utility function with \( \rho = 0.8 \) and fixed \( y_0 = 100 \).

(without contract) optimal strategy is also observed for low levels of initial wealth. In contrast to the buyer's perspective, we observe an opposite order when comparing the optimal strategy of different contracts. In particular, the agent for the case \( g(y) = y \) now has an optimal equity holding that is lower (higher) than the no-contract case with low (high) wealth level. Intuitively, these effects for a put and a call option are reversed, compared to the buyer's perspective.

5.3. Pricing of insurance claims

In this section we apply the SAHARA utility indifference pricing technique to price some insurance claims whose payoff substantially depends on mortality risk of the policyholder. Similar applications have been studied by e.g. [32–34]. Consider an insurance company that offers an equity-linked guarantee contract with a maturity \( T \) and a guarantee \( K \). If the policyholder is still alive at maturity she receives a payoff equal to 

\[
g^I(y_T) = \begin{cases} 
  y_T \lor K = K + (y_T - K)^+, & \zeta > T, \\
  K, & \zeta \leq T.
\end{cases}
\]

A similar setup can be found in [15]. The insurance company is interested in maximizing its utility, i.e. the indifference utility price of the contract is defined by the following optimization problem

\[
\sup_{\theta \in \Theta, s \in [0, T]} \mathbb{E} \left[ U(X_T^{K_0 + \theta, p}) - (y_T \lor K) 1_{\zeta > T} + K 1_{\zeta \leq T} \right] = \sup_{\theta \in \Theta, s \in [0, T]} \mathbb{E} [U(X_T^{K_0 + \theta})].
\]
Since the remaining life time is independent of the financial market, the left hand side optimization can be rewritten as

\[
\sup_{\mathbf{F}, \mathbf{P}, \mathbf{S}} \left( (1-q)E[U(X_{T}^{y_{0}+p, \theta} - y_{T} \lor K)] + qE[U(X_{T}^{y_{0}+p, \theta} - K)] \right). \tag{38}
\]

Thus, Eq. (20) has to be solved with terminal condition

\[
v(T, x, y) := (1-q)U(x - (y \lor K)) + qU(x - K). \]

The numerical result is reported in Table 8 for different strike prices. For comparison purpose, we also present the price for the contract without mortality \(g(y_{T}) = \max(y_{T}, K)\). As \(g(y_{T}) \leq \max(y_{T}, K)\), it is clear that the price for \(g(y_{T})\) is smaller.

6. Conclusion and outlook

We study an indifference pricing framework with a SAHARA utility function which leads to an optimization problem in an incomplete market setup. Adopting the algorithm developed in [16, 17], we are able to numerically solve a two-dimension HJB equation and compute the indifference price for various European type claims. As it is possible to use SAHARA utility function to price both long and short positions in European type claims, our framework suggests an alternative to overcome the “short call” problem. Our SAHARA utility framework also opens up a wide range of new possible pricing modelings. For example, it would be interesting to allow the contract to make payouts at intermediate times. Another extension would be to consider a pool of multiple life insurance contracts. These perspectives are left for future research.

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1 Indifference pricing under SAHARA utility

A. Chen, T. Nguyen and N. Sørensen

Journal of Computational and Applied Mathematics 388 (2021) 113288

2 On the Investment Strategies in Occupational Pension Plans

Source:
Frank Bosserhoff, An Chen, Nils Sørensen and Mitja Stadje (2021). On the Investment Strategies in Occupational Pension Plans. This article has been accepted for publication in Quantitative Finance, published by Taylor & Francis.

URL: http://dx.doi.org/10.2139/ssrn.3827859

DOI: 10.2139/ssrn.3827859
ON THE INVESTMENT STRATEGIES IN OCCUPATIONAL PENSION PLANS

FRANK BOSSERHOFF*, AN CHEN*, NILS SØRENSEN*, MITJA STADJE*

ABSTRACT. Demographic changes increase the necessity to base the pension system more and more on the second and the third pillar, namely the occupational and private pension plans; this paper deals with Target Date Funds (TDFs), which are a typical investment opportunity for occupational pension planners. TDFs are usually identified with a decreasing fraction of wealth invested in equity (a so-called glide path) as retirement comes closer, i.e., wealth is invested more risky the younger the saver is. We investigate whether this is actually optimal in the presence of non-tradable income risk in a stochastic volatility environment. The retirement planning procedure is formulated as a stochastic optimization problem. We find it is the (random) contributions that induce the optimal path exhibiting a glide path structure, both in the constant and stochastic volatility environment. Moreover, the initial wealth and the initial contribution made to a retirement account strongly influence the fractional amount of wealth to be invested in risky assets. The risk aversion of an individual mainly determines the steepness of the glide path.

1. INTRODUCTION

One of the major societal challenges in various countries is the changing demographics in the sense of an aging society. While birth rates remain low, life expectancy has been increasing continuously for several decades, leading to high costs for social security systems. Consequently, a shrinking working population has to support pensions of a growing retiring population, destroying an effective functioning of the statutory pay-as-you-go system. Hence, the role of the second and the third pillar, namely the occupational and private pension plans, is expected to gain more and more importance in the future. This paper is concerned with mechanisms ensuring a functioning second pillar.

Traditionally, Defined Benefit (DB) plans were the leading form of occupational retirement plans in developed countries. In a DB plan, the employee’s pension benefit is determined by a formula taking into account years of service for the employer and wages or salary. In the last decades, most industrial countries have experienced the conversion of DB plans to so-called Defined Contribution (DC) plans (see e.g. Broeders and Chen (2010)). In a DC plan, sponsoring companies (and often also their employees) pay a promised contribution to an external pension fund which invests the contributions in financial assets. The pension payment is then simply determined as the market value of the backing assets. The fundamental advantages of DC plans are twofold. First of all, a DC plan allows pension beneficiaries to invest more freely and participate in the higher stock market returns which in particular in

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Date: April 18, 2021.
2010 Mathematics Subject Classification. 91B15; 91G10; 91G15; 91G60.
Key words and phrases. Optimal Asset Allocation, Defined Contribution Plans, Target Date Funds.
Frank Bosserhoff acknowledges support from Deutscher Verein für Versicherungswissenschaften e.V.
times of low interest rates seems necessary to obtain sufficient overall returns in the long run. Second, again in environments of falling interest rates, it is increasingly burdensome for DB pension funds or insurance companies to provide guarantees which require building up high and costly reserves in order to maintain them. Thus, in all OECD countries DC plans have played an increasingly important role, see also the discussion in Aaronson and Coronado (2005) for further reasons for the transition from DB towards DC plans. Needless to say, in a DC plan the employees bear the entire investment risk. For instance, DC beneficiaries in the U.S. manage the investment risk by so called “Individual Retirement Accounts”, or more frequently by making contributions to so-called 401(k) plans, see Copeland (2011) for details on these plans. For a 401(k) DC beneficiary, one of the default investment strategies is a Target Date Fund (TDF). TDFs are investment funds with a pre-specified maturity (target date). This target date is usually the time point of retirement of some individual or a group of individuals. Because of their structure, these funds place themselves in the category of “life-cycle” funds, rather than in the category of “life-style” funds where the risk profile of the investor is the investment paradigm. These funds are the investment side of a DC plan and directly coupled with the planned retirement year of the DC plan beneficiaries. They have the advantage that the pension beneficiaries do not have to choose a number of investments but only a single fund. The main mechanism behind these TDFs is that those who retire later shall invest more in equity, while those who retire earlier shall invest less in equity. In other words, equity holdings in TDFs shall decrease in age. Therefore, TDFs are usually identified by practitioners with “glide paths”, i.e., a decreasing curve of the equity holding (as a fraction of wealth) over time. Assets in TDFs have grown from 71 billion U.S. dollars in 2005 to 2.2 trillion U.S. dollars at the beginning of 2020 (Wallace (2021)).

In this paper we investigate whether the TDF is a reasonable choice for the individual in a realistic financial market with a stochastic volatility and random endowment environment. We consider assets more complex than the usual log-normal case by modeling the instantaneous variance of the risky asset through a stochastic process. This way, the empirically verified fact that asset prices exhibit stochastic volatility is accounted for (see Cont and Tankov (2004) and references therein). The resulting financial market is known as Heston Model (Heston (1993)). Moreover, we model the contribution the employee makes to the investment fund as stochastic process as well. Since we explicitly exclude perfect correlation between the latter and the financial market, the contribution risk is not tradeable. From a practical point of view, it is reasonable to consider random contributions due to unforeseen changes in wages or salary, unemployment or reduced working hours, which has for example been faced by many workers due to the covid pandemic; however, a perfect correlation to risky assets seems unrealistic. In such a stochastic environment, we approach then the question from a mathematical point of view in the sense that we formulate the retirement saving problem as a stochastic optimization problem; if TDFs are a reasonable choice, the solution to such a stochastic optimization problem will display the aforementioned glide path structure. We aim at finding the optimal investment strategy for each pension beneficiary with different risk aversion and income levels (leading to different contributions to the pension fund).

Undoubtedly, one of the core mathematical foundations of the current project is optimal dynamic asset allocation. The optimal consumption and asset allocation problem (utility maximization problem) in a continuous-time setting dates back to Merton (1969, 1971), where analytic solutions for the case of utility functions exhibiting hyperbolic absolute risk aversion are obtained. Merton’s pioneering work has been extended in numerous directions. A second core foundation to this project is Heston model which is an example of a so-called stochastic...
volatility model. Originally, this has been introduced in Heston (1993) for option pricing and as possible explanation of the volatility smile. An application of this stochastic volatility model to utility maximization is found e.g. in Kallsen and Muhle-Karbe (2010), Kraft and Steffensen (2013), and Chen et al. (2018).

There is some academic literature dealing with DC plans: Cairns et al. (2006) consider the optimization problem under a power utility function which uses the plan member’s salary as numeraire. A guarantee is incorporated by Guan and Liang (2014), who allow for stochastic interest rates and stochastic volatility while assuming that the salary risk is fully hedgeable by trading in the risky assets. The problem of maximizing the expected utility at the time of retirement using a constant elasticity of variance model for the stock price dynamics, which is essentially an extension of the classical geometric Brownian motion, is discussed in Gao (2009). The utility is measured in terms of mean-variance efficiency in Guan and Liang (2015) and a game-theoretic formulation is found in Li et al. (2017). All these approaches to determining the optimal investment strategy are fundamentally different from ours as, from a mathematical point of view, we allow for additional randomness and incorporate non-tradeable risk simultaneously. Working in a complex market environment comes at the expense of a three-dimensional value function, that is, the optimization problem induced is challenging from a technical point of view. We solve it by means of a Least-Squares Monte Carlo (LSMC) approach following Denault et al. (2017). We remark that the LSMC methods have not been widely applied in insurance, or specifically retirement planning problems. It allows us to explicitly investigate whether the glide path structure, that is characteristic for TDFs in practice, is still optimal when random volatility and/or not-tradeable salary risk are present. Moreover, the LSMC algorithm is very flexible in the sense that we can directly draw comparisons with the solution obtained in a corresponding constant volatility environment, i.e., the effect of the stochastic volatility on retirement planning is illustrated in detail.

In our paper it is shown that in a stochastic volatility environment accounting for contribution risk the glide path structure is still optimal. More specifically, we find it is the (stochastic) contribution that causes the glide path structure. Particular attention is paid to comparisons of the constant and the stochastic volatility case; the respective strategies show a similar qualitative behaviour, but the variance of the optimal terminal wealth in the constant volatility model is higher than in the random volatility environment. The risk aversion of an individual mainly determines the steepness of the glide path, while the ratio of the initial value of the pension income account and the initial contribution level severely impact the fraction of total wealth invested in the risky asset. Moreover, we illustrate that the drift and volatility of the contribution process only have a minor impact on the optimal investment strategy; this implies that the same TDF can be used for individuals with different contribution parameters, which is an important fact from a practical point of view. The structure of this paper is as follows: Section 2 introduces the stochastic volatility model for the financial market, the contribution process and the pension account. The latter is then used to formulate the stochastic optimization problem reflecting the retirement planning problem. Section 3 explains the numerical procedure applied in detail. Furthermore, a discussion of parameter values is provided. Section 4 contains our main results. It starts with a brief review of well-known stochastic optimization problems that are actually special cases of the problems ultimately considered in this paper. Section 5 succinctly summarizes our main findings. Table 1.1 contains abbreviations frequently used in the remainder of this paper.
2. Model Setup

2.1. Financial Market. Let $T > 0$ be the deterministic retirement time point of some individual. Clearly, the optimal trading strategy of a retirement planner might vary considerably depending on the market model presumed. In the sequel, we consider a financial market modeling the volatility of the risky asset as a stochastic process as well as a model assuming the volatility parameter to be constant. To this end, we work on a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This probability space is equipped with a three-dimensional standard Brownian motion $W = (W^{(1)}, W^{(2)}, W^{(3)})^\top$. Let $(\mathcal{F}_t)_{t \in [0,T]}$ be the right-continuous completion of the filtration generated by $W$. On this space, both financial markets are defined. It is supposed that they are frictionless and two financial assets can be traded continuously at arbitrary quantities. As usual, the pension beneficiary aims at gainfully investing in these assets to maximize her wealth at the time point of retirement. In addition, it is assumed that a constant share of the pension beneficiary’s labor income is continuously invested into the pension income account. As the contribution made to the investment fund is the quantity relevant for retirement planning, we directly consider the contribution as a stochastic process. We next introduce the two optimization problems considered in this paper.

2.2. The Stochastic Volatility Model. For some deterministic interest rate $r \geq 0$, denote the price process of the risk-free asset, for example a bank account, by $B = (B_t)_{t \in [0,T]}$ such that $B$ solves

$$dB_t = rB_t \, dt,$$

with $B_0 = 1$. Regarding the risky asset $S = (S_t)_{t \in [0,T]}$, we assume an evolution according to the Heston Model as originally suggested in Heston (1993), i.e.

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW^{(1)}_t,$$

$$d\nu_t = \lambda (\theta - \nu_t) \, dt + \sigma \sqrt{\nu_t} \left( \rho S \, dW^{(1)}_t + \sqrt{1 - \rho^2} \, dW^{(2)}_t \right),$$

with $S_0 > 0, \nu_0 = x > 0$. The process $\nu = (\nu_t)_{t \in [0,T]}$ is the instantaneous variance of the stock and modeled as Cox-Ingersoll-Ross (CIR) process. The parameters $\mu, \lambda, \theta$ and $\sigma_\nu$ are positive constants; in this case, $\mu$ is the instantaneous drift of $S$, the parameter $\lambda$ is the rate at which $\nu_t$ reverts to $\theta$, which is the long-term variance of $S$, and $\sigma_\nu$ is the volatility of the volatility (also known as 'vol of vol') and therefore determines the variance of $\nu_t$. Furthermore, we assume

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Term</th>
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<tr>
<td>CE</td>
<td>Certainty Equivalent</td>
</tr>
<tr>
<td>CRRA</td>
<td>Constant Relative Risk Aversion</td>
</tr>
<tr>
<td>CVM</td>
<td>Constant Volatility Model</td>
</tr>
<tr>
<td>CVMRCP</td>
<td>Constant Volatility Model Random Contribution Problem</td>
</tr>
<tr>
<td>DC</td>
<td>Defined Contribution</td>
</tr>
<tr>
<td>LSMC</td>
<td>Least-Squares Monte Carlo</td>
</tr>
<tr>
<td>RRA</td>
<td>Relative Risk Aversion</td>
</tr>
<tr>
<td>SVM</td>
<td>Stochastic Volatility Model</td>
</tr>
<tr>
<td>SVMRCP</td>
<td>Stochastic Volatility Model Random Contribution Problem</td>
</tr>
<tr>
<td>TDF</td>
<td>Target Date Fund</td>
</tr>
</tbody>
</table>

Table 1.1. Table of Frequently Used Abbreviations
that the *Feller condition* $2\lambda \theta > \sigma^2_v$ is satisfied ensuring $\nu_t > 0$ for all $t$. Since $W^{(1)}$ and $W^{(2)}$ are by definition two independent standard Brownian motions, the instantaneous variance consists of two parts, one correlated and one independent to the risky asset. If $\rho_S = \pm 1, \nu$ is perfectly positively or negatively correlated to $S$. In case $\rho_S = 0$, $\nu$ is independent of $S$. Note that the risk induced by the variance is fully hedgeable through trading in the stock only if $\rho_S \in \{-1, 1\}$; since this is not the case, the market defined through (2.1) - (2.3) is *incomplete*. We henceforth refer to this market as Stochastic Volatility Model (SVM).

Denote the aforementioned contribution process of some representative individual by $C = (C_t)_{t \in [0,T]}$ and assume that its dynamics are given by

$$dC_t = \mu_C C_t \, dt + \sigma_C C_t \left( \rho_C \, dW^{(1)}_t + \sqrt{1 - \rho^2_C} \, dW^{(3)}_t \right), \quad (2.4)$$

with $C_0 = c > 0$, constants $\mu_C \in \mathbb{R}, \sigma_C > 0$ and a correlation coefficient $\rho_C \in (-1, 1)$ allowing for the same interpretation as $\rho_S$ above. Hence, the contribution risk is correlated to the investment risk, however, the Brownian motion $W^{(3)}$ resembles shocks to the contributions being independent from any movements at the financial market; in particular, the contribution risk cannot be hedged through trading the risky asset.

In a DC pension plan, the changes in the individuals’ retirement account stem from trading gains, changes in the underlying price processes as well as continuous contributions made over time. More precisely, denote by $\pi = (\pi_t)_{t \in [0,T]}$ the proportion of the total wealth invested by the fund manager in the stock $S$, accordingly $1 - \pi$ displays the fraction of wealth put into the bank account. The allocation considered in this paper is given by $(1 - \pi, \pi)$ and if not explicitly stated otherwise, we always refer to the fraction of total wealth invested whenever mentioning the trading strategy in the sequel. Denote the pension wealth process induced by $P^\pi_t = (P^\pi_t)_{t \in [0,T]}$. It is assumed to solve the SDE

$$dP^\pi_t = P^\pi_t \pi_t \frac{dS_t}{S_t} + P^\pi_t (1 - \pi_t) \frac{dB_t}{B_t} + C_t \, dt$$

$$= (P^\pi_t r + P^\pi_t \pi_t (\mu - r) + C_t) \, dt + P^\pi_t \pi_t \sqrt{\nu_t} \, dW^{(1)}_t, \quad (2.5)$$

$P^\pi_0 = p > 0$. For $C_t \equiv 0$ for all $t$, the wealth process would be self-financing; hence, the form of (2.5) shows that the only cash injections stem from additional contribution income. Next, we define the set of admissible trading strategies; for notational convenience we write $\mathcal{R}_+ := \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$. Let $A \subseteq \mathbb{R}$ be a fixed given closed, convex set, which can express plausible trading restrictions. For instance $A = [0, 1]$ corresponds to the prohibition of short selling and borrowing.

**Definition 2.1.**

- A *progressively measurable trading strategy* $\pi$ is called admissible if it takes values in $A$, and if for any point $(t, p) \in [0,T] \times \mathbb{R}_+$, there exists a unique càdlàg adapted strong solution $P^\pi_t$ to (2.5) starting from $p$ at $s = t$ fulfilling $\mathbb{E}[|P^\pi_s|^2] < \infty$ as well as $P^\pi_s > 0$ for all $s > t$.

- An admissible trading strategy $\pi$ in the form $\pi_s = \pi(s, P^\pi_s, \nu_s, C_s)$ for some measurable function $\pi : [0, T] \times \mathcal{R}_+ \rightarrow A$ is called Markovian feedback strategy.

- We denote the set of admissible trading strategies of Markovian feedback type by $\Pi$. 

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Unless stated otherwise, in the remainder of the paper we always implicitly refer to strategies of Markovian feedback type.

Since DC pension plans typically pay out a lump-sum instead of annuities to the beneficiaries, we consider this case in the sequel. A natural target of the DC plan investor is then the maximization of the expected utility of the terminal wealth. Due to its analytical tractability we consider the power utility function, that is

\[ U : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad z \mapsto z^{1-\gamma} / (1-\gamma), \quad \gamma \geq 0, \gamma \neq 1, \tag{2.6} \]

which assumes that the investor has a constant relative risk aversion (CRRA) given by \( \gamma \).

The long-term behavior of the economy suggests that the long-term relative risk aversion cannot strongly depend on wealth, see e.g. Campbell et al. (2002); thus, as retirement planning is rather a long-term investment, using a utility function with CRRA is well-motivated economically.

We write \( \mathbb{E}_{t,p,x,c}[\cdot] = \mathbb{E}[\cdot | P_t^\pi = p, \nu_t = x, C_t = c] \) for the conditional expectation given the quadruple \((t, p, x, c) \in [0, T] \times \mathbb{R}_+\). The attractiveness of some trading strategy \( \pi \) given the aforementioned quadruple is then evaluated using the following functional:

**Definition 2.2.** The gain function \( J : [0, T] \times \mathbb{R}_+ \times \Pi \rightarrow \mathbb{R}_+ \) is defined by

\[ J(t, p, x, c, \pi) := \mathbb{E}_{t,p,x,c}[U(P_T^\pi)], \]

The natural goal of a retirement planner is the maximization of the gain function, i.e., the following stochastic optimization problem has to be solved:

\[ v(t, p, x, c) = \sup_{\pi \in \Pi} J(t, p, x, c, \pi). \tag{SVMRCP} \]

We call \( v \) the associated value function and name the problem SVMRCP (Stochastic Volatility Model Random Contribution Problem). Consequently, a control \( \pi^* \) is called optimal for SVMRCP if for the initial condition \((t, p, x, c) \in [0, T] \times \mathbb{R}_+\), it holds that \( v(t, p, x, c) = J(t, p, x, c, \pi^*) \).

2.3. The Constant Volatility Model. The focus of the paper is to analyze the suitability of TDFs for long-term retirement goals under realistic market conditions like stochastic volatility. To outline the impact of such a fluctuating volatility, a comparison with the constant volatility case will be drawn. Replacing the random instantaneous volatility in (2.2) by the constant \( \sigma_S > 0 \) yields the following dynamics for the risky asset:

\[ dS_t = \mu S_t \, dt + \sigma_S S_t \, dW_t^{(1)}, \tag{2.7} \]

with \( S_0 > 0 \). Obviously, the solution to (2.7) is the time-homogeneous geometric Brownian motion. The combination of the riskless asset \( B \) described through (2.1) and (2.7) is henceforth called Constant Volatility Model (CVM).

Assuming the random contribution is still described by (2.4), the portfolio process induced by the CVM is different from (2.5) through the substitution of \( \sqrt{\nu_t} \) by \( \sigma_S \) only. Sticking to the notation \( P^\pi \) for the portfolio process in the CVM, the definition of an admissible trading strategy in the CVM is readily deduced from Definition 2.1. In particular, an admissible trading strategy of Markovian feedback type is given in the form \( \pi_s = \pi(s, P_s^\pi, C_s) \) for some measurable function \( \pi : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow A \). Note that \( \pi_s \) does not depend on \( \nu_s \) anymore.
Using the power utility function (2.6) again, a suitably adapted form of the gain functional leads to the following stochastic optimization problem:

\[ v(t, p, c) = \sup_{\pi \in \Pi} J(t, p, c, \pi). \]  

(CVMRCP)

By a slight abuse of notation, we again use \( v \) as the associated value function and name the occurring problem CVMRCP (Constant Volatility Model Random Contribution Problem). Consequently, a control \( \pi^* \) is called optimal for CVMRCP if, for the initial condition \((t, p, c) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \), it holds that \( v(t, p, c) = J(t, p, c, \pi^*) \).

**Remark 2.3.**

(i) The CVM is obviously a special case of the SVM that is obtained by replacing the CIR process through a constant. The random contribution is assumed to be the same in both optimization problems. The fundamental difference in the optimization problems SVMRCP and CVMRCP is the omission of the random volatility in the latter. Consequently, the value function corresponding to the CVMRCP is reduced by one dimension. The inclusion of a third dimension in the value function can raise significant numerical problems that are discussed in the next section.

(ii) In practice, a target date fund is typically composed of a basket of risky assets, i.e., (2.2) or (2.7) would be a multi-dimensional process and (2.5) would change accordingly. However, as the focus of the paper is to study the impact of the realistic financial market setting like stochastic volatility and the untradable salary risk on the optimal pension asset allocation, we just consider one representative risky asset. In this sense, the focus of this paper is not the optimal allocation of money among a collection of risky securities as the point of retirement comes closer, but a change in the category of assets money is allocated to, namely from risky to conservative assets.

(iii) In practice, the fraction of wealth invested in risky assets by a TDF amounts to 40-75 % (Holt et al. (2016)). In our analyses below we mostly obtain higher values. This stems from the omission of any shortselling and regulatory constraints which are typically present in practice.

### 3. Algorithm and Data

**3.1. A Least-Squares Monte Carlo Approach.** It is a well known fact that (stochastic) optimization problems typically increase in terms of complexity, the more dimensions the corresponding value function has. As the CVMRCP is a particular case of the SVMRCP, if we can find a functioning numerical procedure which handles the SVMRCP properly, it can also be applied to the CVMRCP. It is not unusual in this context to resort to dynamic programming techniques that ultimately lead to solving a Hamilton-Jacobi-Bellmann (HJB) equation when approaching a stochastic optimization problem, see e.g. Pham (2009) for a comprehensive overview. However, a major disadvantage of the HJB approach is the so-called *curse of dimensionality*, i.e., the running time is increasing significantly when considering a higher-dimensional problem. According to Broadie and Glasserman (2004) and Andreasson and Shevchenko (2019), the HJB approach actually becomes impractical when two dimensions are exceeded. Noting that the SVMRCP has a three-dimensional value function gives rise to the consideration of alternatives such as Monte Carlo methods. The Least-Squares Monte Carlo (LSMC) approach to stochastic optimization problems is capable of handling multiple risk sources without suffering from the curse of dimensionality. In addition, this approach is useful when some flexibility regarding the dynamics of the underlying stochastic processes is required; this is particularly important for us as we consider for example constant and...
stochastic volatility. The ability of the LSMC technique to find the optimal control in an expected utility problem is shown in Kharroubi et al. (2013), for background information on LSMC techniques see e.g. Longstaff and Schwartz (2001) and Clément et al. (2002) and references therein. In order to solve the SVMRCP and the CVMRCP, we use an LSMC algorithm following Denault et al. (2017) adapted to our setting. This algorithm is a simulation-and-regression procedure that regresses on decision variables (the admissible trading strategy) and the exogenous state variables (portfolio value, contribution and possibly the volatility). The algorithm is designed for discrete time models, so we decide on \( N \in \mathbb{N} \) intermediate time steps per year from \( t = 0 \) to \( t = T \), i.e. \( \Delta t = 1/N \), and we observe the realizations of our stochastic processes at times \( t \in \Xi := \{0, \Delta t, 2\Delta t, \ldots, T\} \). The algorithm consists of three parts, namely the introduction of prerequisites, the backward step and the forward step. We write down the algorithm for the SVMRCP and remark that a reduction to the CVMRCP is immediate by dropping the stochastic volatility component.

Prerequisites:

1. **Gridpoints for the share invested**: Let \( n_{\pi} \in \mathbb{N} \) be the number of grid points and \( \{\pi_i\}_{i=1}^{n_{\pi}} \) the corresponding grid of possible proportions of total wealth invested in the risky asset. In our case we allocate the \( n_{\pi} \) points uniformly on the set of admissible trading strategies \( A = [\pi, \bar{\pi}], -\infty < \pi < \bar{\pi} < \infty \).

2. **Generation of paths**: Let \( n_r \in \mathbb{N} \) denote the number of realizations considered. Simulate and store \( n_r \) possible paths of the discretized two-dimensional state process \( \{(C_t(j), \nu_t(j))\}_{t \in \Xi, j=1}^{n_r} \).

3. **Basis functions**: Define the vector of basis functions \( B \), which in our case consists of monomials of the control \( \pi \) and the state variables influencing the optimal strategy; mixed terms are included as well:

\[
B(\pi, c, \nu) = [1 \pi \pi^2 c c^2 \nu \nu^2 \pi c \pi \nu \nu c]^T.
\]

4. **Grid of possible wealth levels**: A grid of possible wealth levels at each point in time has to be computed and stored. This grid should be wide enough to cover the range of realistic values of wealth, but not so large such that its simulation induces computational problems. To obtain in-between points and simultaneously account for the fact that bounds of the wealth levels become wider as time progresses due to the contribution and trading gains, a fixed amount of in-between points is impractical. Instead, we follow a suggestion in Denault et al. (2017) for such kind of situations. Empirical evidence suggests that the following methodology leads to a realistic and efficient grid: simulate \( n_r \) wealth paths using the values of the state process from Step 2 plugged in into (2.5), and invest the fixed share \( \bar{\pi} \) in the risky asset. For each time point \( t \in \Xi \setminus \{0, T\} \), the \( q_1 \)- and \( 1 - q_2 \)-quantile are determined and chosen as the lower bound \( P_{t,\min} \) and the upper bound \( P_{t,\max} \), respectively. The quantiles \( q_1, q_2 \) can freely be chosen to fit the concrete problem. Further, define a fixed step-size \( \Delta P \), which is calculated after the first iteration as follows:

\[
\Delta P := \frac{P_{t,\max} - P_{t,\min}}{n_p},
\]

whereby \( n_p \) is the number of in-between points after one time step; it is chosen by the user. The number of grid points at a certain time point is given by \( N_{p,t} = \lceil (P_{t,\max} - \ldots \rceil \)
\[ P_{t,\min}/\Delta P \], \( t \in \Xi \setminus \{0, T\} \), where \( \lceil \cdot \rceil \) denotes the ceil function. The grid points are then defined by \( P_{t,k} = P_{t,\min} + k\Delta P \), \( k \in \{0, 1, \ldots, N_{p,t}\} \). Note that \( P_{t,\max} \) is in general not a grid point. For time \( t = 0 \) the only gridpoint is the initial wealth, i.e. \( P_{0,0} = p \).

The Backward Step:

For the ease of notation, the indices \( k, i, j \) always refer to the corresponding point on the sets discussed above. Further, a subscript always denotes the dependence on a grid point while a superscript refers to a path/choice dependence.

For each \( t = T - \Delta t \) to 0 and for each \( k = 1 \) to \( N_{p,t} \) do:

**STEP 1:** Given some portfolio value \( P_{t,k} \) (this is the \( k \)-th possible value at time \( t \)), generate all possible wealth levels at time \( t + \Delta t \) by combining every possible path of the return with each allocation point (in total, there are \( n_\pi \cdot n_r \) possibilities):

\[ P_{t+\Delta t}^{(k,i,j)} := \mathcal{T} \left( P_{t,k}, \pi_t, c_t^{(j)}, \nu_t^{(j)} \right), \quad i = 1, \ldots, n_\pi, \text{ and } j = 1, \ldots, n_r. \quad (3.1) \]

The function \( \mathcal{T} \) is the rebalancing function for the wealth process, i.e., in our case the corresponding Euler-Maruyama scheme applied to (2.5).

**STEP 2:** For each \( P_{t+\Delta t}^{(k,i,j)} \) generate a corresponding value of the value function \( v_{t+\Delta t}^{(k,i,j)} \). Note that for each fixed \( k \in \{1, \ldots, N_{p,t}\} \), there are exactly \( n_\pi \cdot n_r \) combinations.

(a) If \( t = T - \Delta t \), it is the final time that money is re-allocated. The value function \( v_T \) simply coincides with the utility (of the terminal wealth):

\[ v_T^{(k,i,j)} = u \left( P_{T}, \pi_T, c_T^{(j)}, \nu_T^{(j)} \right) = \mathcal{T} \left( P_{T}, \pi_T, c_T^{(j)}, \nu_T^{(j)} \right). \]

(b) If \( t < T - \Delta t \), the value \( v_{t+\Delta t}^{(k,i,j)} \) is computed by interpolation. The rationale is as follows: Since the collection of pairs \( \{v_{t+\Delta t,k}, P_{t+\Delta t,k}\}_{k=1}^{N_{p,t+\Delta t}} \) is known from STEP 6 (see below) in the previous time step, interpolation is used as follows (for fixed \( j \) and \( i \)): For each \( P_{t+\Delta t}^{(k,i,j)} \) estimate \( v_{t+\Delta t}^{(k,i,j)} \) by linear interpolation on the set \( \{v_{t+\Delta t,k}, P_{t+\Delta t,k}\}_{k=1}^{N_{p,t+\Delta t}} \).

During the interpolation, \( v_{t+\Delta t,k}^{(j)} \) is transformed by the inverse utility function \( u^{-1}(x) = ((1 - \gamma) x)^{1/(1 - \gamma)} \). It is common to use transformation functions before and after interpolation to reduce a potential interpolation bias; see e.g. Andreasson and Shevchenko (2019), Carroll and Ruppert (1988) and in particular Denault et al. (2017) for our case.

**STEP 3:** To each combination \( (\pi_t, c_t^{(j)}, \nu_t^{(j)}) \), associate the basis vector \( B_t^{(i,j)} := B \left( \pi_t, c_t^{(j)}, \nu_t^{(j)} \right) \).

Note that there are \( n_r \cdot n_\pi \) such combinations.

**STEP 4:** Regress the dependent values \( v_{t+\Delta t}^{(k,i,j)} \) on the independent basis vectors \( B_t^{(i,j)} \) and obtain the corresponding vector of regression coefficients by \( \beta_{t,k} \):

\[ \beta_{t,k} = \arg \min_{\beta \in \mathbb{R}^{10}} \sum_{j=1}^{n_r} \sum_{i=1}^{n_\pi} \left[ \beta^T B \left( \pi_t, c_t^{(j)}, \nu_t^{(j)} \right) - v_{t+\Delta t}^{(k,i,j)} \right]^2. \quad (3.2) \]

**STEP 5:** Optimize the regression surface w.r.t. the position \( \pi \): for each \( j = 1, \ldots, n_r \), the value \( (c_t^{(j)}, \nu_t^{(j)}) \) is known, so the following quantity gives the optimal portfolio weight:

\[ \hat{\pi}_{t,k}^{(j)} = \arg \max_{\pi \in A} \beta_{t,k}^T B \left( \pi, c_t^{(j)}, \nu_t^{(j)} \right), \quad j = 1, \ldots, n_r. \quad (3.3) \]
This means that for every wealth level \( P_{t,k} \), we have a collection \( \{ \hat{\pi}(j)_{t,k} \}_{j=1}^{n_r} \) of optimal portfolio weights.

**STEP 6:** Compute the values \( \tilde{v}_{t,k}^{(j)} \) associated with optimal position \( \hat{\pi}_{t,k}^{(j)} \). Note that this step belongs to the method called **realized values** described in Denault et al. (2017).

(a) If \( t = T - \Delta t \), then
\[
\tilde{v}_{t,k}^{(j)} = u \left( T \left( P_{t,k}, \hat{\pi}_{t,k}^{(j)}, c_t^{(j)}, \nu_t^{(j)} \right) \right).
\]

(b) If \( t < T - \Delta t \), then compute the realized wealth
\[
P_{t+\Delta t}^{(k,j)} = T \left( P_{t,k}, \hat{\pi}_{t,k}^{(j)}, c_t^{(j)}, \nu_t^{(j)} \right),
\]
and interpolate those realized wealths through the \( \{ \tilde{v}_{t+\Delta t,k}^{(j)}, P_{t+\Delta t,k} \}_{k=1}^{n_p} \) pairs. Use the inverse utility interpolation method described in **STEP 2b**.

The Forward Step:

With the choice of wealth gridpoints, the backward step becomes independent of concrete wealth developments. To obtain paths consistent with the optimal strategy, a forward simulation is needed. Again \( n_r \) new scenarios are simulated. For time \( t = 0 \), there is only one wealth node which coincides with the initial wealth \( p \). The optimal strategy for all \( n_r \) paths is then given by:
\[
\hat{\pi}_0^{(j)} = \arg \max_{\pi \in A} \beta_0^T \beta \left( \pi, c, \nu_0 \right), \quad j = 1, \ldots, n_r.
\]

For \( t \in \{ \Delta t, \ldots, T - \Delta t \} \), the optimal strategy has to be interpolated between the wealth points, i.e., for \( P_t^{(j)} \in [P_{t,\text{min}}, P_{t,\text{min}} + N_{p\Delta} \Delta P] \), choose \( \tilde{k}_t^{(j)} := \lfloor (P_t^{(j)} - P_{t,\text{min}})/\Delta P \rfloor \), such that \( P_{t,\tilde{k}_t^{(j)}} \) is the closest grid point below \( P_t^{(j)} \). Then use (3.3) with \( \tilde{k}_t^{(j)} \) and \( \tilde{k}_t^{(j)} + 1 \) (closest point above \( P_t^{(j)} \)) to obtain \( \hat{\pi}_{t,\tilde{k}_t^{(j)}}^{(j)} \) and \( \hat{\pi}_{t,\tilde{k}_t^{(j)} + 1}^{(j)} \). By linear interpolation the optimal strategy is finally obtained:
\[
\hat{\pi}_t^{(j)} = (1 - \omega_t^{(j)}) \hat{\pi}_{t,\tilde{k}_t^{(j)}}^{(j)} + \omega_t^{(j)} \hat{\pi}_{t,\tilde{k}_t^{(j)} + 1}^{(j)},
\]
where \( \omega_t^{(j)} := (P_t^{(j)} - P_{t,\tilde{k}_t^{(j)}})/\Delta P \) is the weighting factor obtained from the linear interpolation.

For \( P_t^{(j)} \notin [P_{t,\text{min}}, P_{t,\text{min}} + N_{p\Delta} \Delta P] \), we simply take the closest point and use (3.3) to get the optimal strategy. Note that since we do not extrapolate the wealth and the strategy outside of our gridded domains it is in particular important to choose \( A, q_1 \) and \( q_2 \) such that only few paths are affected.

### 3.2. Parameter Specification

In order to learn about the effect the fluctuation of the volatility has on the solution of the retirement planning problem, parameters have to be chosen such that the differences in the solutions of the SVMRCP and the CVMRCP can be ascribed to the presence of the random volatility. Moreover, in order to obtain practically relevant results, the parameters should be consistent with observations at the financial market.

For the SVM, we borrow the parameters from Liu and Pan (2003), where a sound estimation procedure has been used. We summarize them in Table 3.1. It is well known that considerable sales of a stock lead to price decreases and increasing uncertainty that is reflected by a rising volatility. Our SVM captures this phenomenon and for that reason the correlation between the Brownian motions driving the risky asset and the volatility, respectively, is negative.
\[ S_0 = 1, \quad \nu_0 = 0.0169, \quad \mu = 0.06, \quad r = 0.02, \]
\[ \sigma_\nu = 0.25, \quad \rho_\nu = -0.4, \quad \theta = 0.0169, \quad \lambda = 5 \]

Table 3.1. Input Parameters for Stochastic Volatility Model

In order to enable the aforementioned comparability between the results of the SVMRCP and the CVMRCP, certain care needs to be taken when specifying the parameters of the CVM. In the SVM, there are two market prices of risk, namely the market price of asset risk and the market price of volatility risk, while the latter is of course not present in the CVM. In order to capture and illustrate the effect of the stochastic volatility, the market prices of asset risk should coincide. This is approximately achieved by setting the volatility parameter in the CVM equal to the long-term volatility in the SVM, so we let \( \sigma_S = \sqrt{\theta} = 0.13 \). Furthermore, the parameters \( \mu \) and \( r \) as specified in Table 3.1 are used in the CVMRCP as well.

The specification of the parameters of the contribution process is borrowed from Chen et al. (2015) and summarized in Table 3.2. We see that this leads to a frequently observed Sharpe ratio of 20\%. Moreover, the correlation with the risky asset is rather small. This stems from the fact that the development of a risky asset typically does not have a strong impact on the income or contribution stream of an individual and vice versa. Nevertheless, interpreting the development of a risky asset as a proxy for the overall economic situation, it is reasonable that the correlation coefficient is slightly positive.

\[ c = 1, \quad \mu_C = 0.04, \quad \sigma_C = 0.1, \quad \rho_C = 0.05 \]

Table 3.2. Input Parameters for Random Contribution

We need to assign values to the initial wealth level, the retirement planning horizon and the relative risk aversion (RRA). This is done in Table 3.3. The initial wealth level is five times as much as the initial contribution and seems reasonable for someone starting saving for retirement. The sensitivity of the optimal strategy w.r.t. changes in the initial wealth is examined later on. Note that the choice \( T = 10 \) means that the individual under consideration retires in ten years from now; later we investigate the sensitivity of the optimal strategy w.r.t. a prolongation of the planning horizon. A constant RRA of three is a common choice in the literature; the effect of a change is also considered later on. Finally, we need to specify

\[ p = 5, \quad T = 10, \quad \gamma = 3 \]

Table 3.3. Input Parameters for the Fund

the parameters of the LSMC algorithm, which is done in Table 3.4. The choice is based on the replication of solutions to well-investigated stochastic optimization problems under the constraint that the running time is reasonable. In particular, using the parameters specified in Table 3.4, the Merton problem explained below is easily solved; moreover, in Kallsen and Muhle-Karbe (2010) the value function of an expected utility optimization problem in a stochastic volatility model (without any contribution) similar to ours is explicitly calculated and our algorithm yields the same value of the value function (evaluated at time zero) up to several digits. This shows that the parameter choice is meaningful.
4. Numerical Results

4.1. Review. We start the discussion of the numerical results with a brief review of the Merton Problem, which corresponds to the optimization problem in the CVM without any contribution, and the corresponding counterpart in the SVM, which we call Heston Problem for the sake of convenience. Dropping the contribution, the respective value functions are reduced by one dimension and the portfolio process is self-financing.

The solution to the Merton Problem, also known as Merton Ratio, is given by \((\mu - r)/\sigma^2\gamma\) and labeled Merton Strategy in Figure 4.1(A). It refers to the analytical \(\pi^*\). The numerically determined LSMC solution is called Mean Optimal Strategy. It corresponds to the average of optimal strategies determined by the LSMC algorithm; here and in the sequel, we only consider such averages. Obviously, the analytical and the numerical solution are relatively close to each other, thereby among others showing that the LSMC algorithm yields highly satisfactory results in this case. The optimal strategy depends on the market parameters and the risk aversion only. In particular, it is independent of time and wealth. The expected wealth induced is also depicted. The terminal value amounts to about 8.2, which corresponds to an average increase of 5\% per year. Thus, on average the Merton portfolio outperforms an investment in the riskless asset solely. Similar remarks apply in the case of the Heston Problem, which is graphically depicted in Figure 4.1(B). The strategy seems also independent of time. We see that it is fluctuating around the optimal fraction to be invested in the Merton Problem with partly larger spikes than in the Merton numerical solution. However, as the volatility in the CVM is chosen equal to the long-term volatility in the CIR-process, the mean-reversion property of the latter causes the solution in the Heston problem to be relatively close to the Merton Ratio. The terminal expected wealth nearly coincides with the one generated by the Merton portfolio in the CVM.

4.2. The CVMRCP and the SVMRCP. In this subsection we present the solutions to the CVMRCP and the SVMRCP, respectively, using the parameters specified in Section 3.2.
above. The numerical results are displayed in Figure 4.2. In the left panel we see the optimal fraction to be invested in equity in the CVM, the wealth induced and the random contribution (all quantities are again to be understood as averages). We note that the fraction of wealth invested in the risky asset is - up to minor local movements - decreasing over time. The decay is obviously larger at the beginning than towards maturity. For example, in the first five years we observe a reduction of more than 100 percentage points while in the second five years the decrease amounts to about 20 percentage points. Moreover, the spikes are more visible in the first period mentioned. Hence, the investment behaviour becomes less risky as time progresses. Over ten years, the optimal expected wealth increases by roughly 380 %, which corresponds to an average annual growth (return of currently wealth plus contribution from income) of more than 16 %. Note that the annual growth in percentage in general is higher in the early phase of the investment because the financial returns are supplemented with contributions from income which take a relative large share. In the right panel of Figure 4.2 the solution to the SVMRCP is displayed. Overall, we can see several similarities to the constant volatility case of the left panel: the optimal trading strategy is decreasing over time and the decay is stronger in the first half of the savings period than in the second half. The development of the expected terminal wealth and therefore the part that can be ascribed to the investment in the risky asset is nearly the same. Measuring the difference of wealth invested in the risky asset, only negligible differences can be observed.

In order to gain further insight into the effect the inclusion of the stochastic volatility has on the optimal wealth process, Table 4.1 contains Mean, Variance and Certainty Equivalent (CE)\(^1\) of the optimal terminal wealth; they are to be understood as empirical quantities. Likewise, we investigate the effect of a change of the correlation coefficient between the Brownian motions driving the risky asset and the stochastic variance. Obviously, the expected values in the CVM and the SVM for all levels of correlation considered nearly coincide. Regarding the variation, we see that the variance of the terminal wealth in the SVM is increasing as \(\rho_v\) is decreasing - this effect can be explained as follows: an increasing correlation coefficient \(\rho_v\) leads to an increasing variance of \(S_t\) for any \(t \in (0, T]\). A higher variance is identified with a higher riskiness of the investment and consequently leads to a reduction in the fraction of wealth invested in \(S\). At first sight it is not clear which effect dominates. Indeed, considering the original Merton problem (set the contribution to zero in the constant volatility case) with optimal strategy \((\mu - r)/(\sigma_S^2 \gamma)\) and optimal portfolio process

\[
dP^{\text{Merton}}_t = P^{\text{Merton}}_t \left( r + \frac{(\mu - r)^2}{\gamma \sigma_S^2} \right) dt + P^{\text{Merton}}_t \frac{(\mu - r)}{\gamma \sigma_S} dW^{(1)}_t,
\]

in the denominator of the resulting stochastic integral there is \(\sigma_S\) as multiplicative factor. That is, an increase of the volatility of the stock leads to a decreasing variance of the portfolio process. Our results show that the same logic applies in the SVMRCP. Thus, a low correlation, which leads to a lower variance of the risky asset, induces a relatively higher investment. Table 4.1 then shows that the effect of the variance of \(S\) on the optimal trading strategy is so strong that the variance of the optimal terminal wealth is finally the highest for the originally least risky case of \(\rho_v = -0.9\), and that it is the lowest for the actually riskiest case in which \(\rho_v = 0.9\). Hence, the optimal trading strategy counteracts the impact of the correlation coefficient on the variance of the final wealth. The same remarks apply to the behaviour of the CE considered

\[^{1}\text{The non-random amount yielding the same level of expected utility as the optimal terminal wealth of the portfolio: } CE = u^{-1} \left( E_{t,p,x,c}(P^{\pi^*_T}_T) \right). \text{ If not explicitly stated otherwise, we refer to the CE seen from time point } t = 0.\]

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in dependence on $\rho_\nu$. Furthermore, we see that the variance of the optimal wealth in the CVM is higher than in the SVM for all levels of correlation considered. Using the log-normal return distribution, it is straightforward to calculate that the variance of the stock in the CVM is achieved in the SVM for a correlation around 0.2. The values of the variance shown in Table 4.1 clearly imply that for this value of the correlation the variance in the SVM is smaller than in the CVM. A possible explanation for the higher variance in the CVM is that an investor with a risk aversion of $\gamma = 3$ perceives the CVM as relatively safe, leading to a higher fraction of wealth invested in the risky asset. This higher fraction of wealth then counteracts the safety of a constant volatility environment.

Reasoning from the similarities of the optimal strategies in Figure 4.2(A) and Figure 4.2(B) and the almost coincidence of the expected values in the two models in Table 4.1, the question arises whether there is a strong effect when applying for instance the solution of the CVMRCP in the SVMRCP. This corresponds to the situation that an investor presumes the constant volatility setting and determines her optimal strategy, while the instantaneous variance actually develops according to the CIR-process. This yields an expected value of the terminal wealth of 26.51, a variance of 81.71 and the CE then amounts to 22.20. Hence, drawing comparisons with the column showing the case $\rho_\nu = -0.4$ of Table 4.1, the erroneously followed strategy yields a marginally lower expected payoff and CE, while the variance increases by roughly 10.5%. Hence, although the strategies show a very similar qualitative behaviour, ignoring the presence of random volatility leads to a higher risk exposure.
So far, our analysis has focused on the average of optimal strategies over one million paths. In order to further learn about the impact of stochastic volatility, we now broaden our perspective. In Figure 4.3, the time point is fixed to five years after the start of the planning horizon and the x-axis shows different possible wealth levels. For each path, after observing the corresponding wealth level at time \( t = 5 \), we plot the respective optimal fraction to be invested. The left panel thereby corresponds to the SVM and the right panel to the CVM. It is obvious that for the same wealth levels the amplitude of the optimal fraction of wealth to be invested in the risky asset is much larger in the SVM than in the CVM. Thus, the stochastic volatility explains a considerably larger range of possible investment fractions. The middle panel resembles the case that the stochastic volatility lies between 10% and 20%, i.e., the stochastic volatility is in a narrow neighborhood of the constant one. For these values of the volatility, we see that the strategy is closer to the optimal one determined in the CVM. It also shows a similar dependence on the wealth level. Hence, the large range of strategies explained by the SVM yields average results that nearly coincide with the CVM.

The omission of the random contribution reduces the CVMRCP to the Merton Problem and the SVMRCP to the Heston Problem. We see that the contribution’s inclusion in either case induces an investment strategy that is decreasing over time with a considerably higher fraction of wealth to be invested in equity at the beginning. Put differently, the investment behaviour is initially much more risk-affine. This is due to the fact that the investor knows fully well that potential losses might be offset by the continuous (future) contribution stream which makes her accept a higher level of riskiness. As the time point of retirement approaches, the future contribution stream becomes relatively low and the risk exposure is diminished through a reduction of the fraction of wealth invested in the risky asset.

4.3. Impact of Initial Wealth and Initial Contribution. We next investigate the effect the initial wealth and contribution levels have on the optimal strategies. Since the observations in the CVM and the SVM were very similar so far, we only consider the SVM in this subsection and remark that analogue results are found for the CVM. The sensitivity analysis with respect to the initial wealth and the human capital (the initial value of the total future contributions) is important, as various combinations of the initial wealth and human capital can represent pension beneficiaries with different ages. For instance, a young beneficiary typically has a lower initial wealth, but a higher human capital. On the contrary, an older beneficiary probably
Figure 4.4. Impact of Initial Wealth (left) and Initial Contribution (right), Stochastic Volatility Model

has accumulated quite some wealth so far and will start with a higher initial wealth level, but carry less human capital, as she will work shorter than the younger one. These analyses help us understand the investment behavior of pension beneficiaries with different ages.

In the left panel of Figure 4.4 the behavior of the optimal investment strategy for different initial wealth levels is depicted. Hereby, we have assumed that the initial contribution $C_0$ stays unchanged, i.e., the initial value of total future contributions (human capital) does not change. Reasoning from the graphic, this implies that the pension beneficiary who owns less initial wealth $P_0$ (upper curve), but the same level of human capital, invests a higher fraction of wealth in the risky asset. A higher initial wealth goes along with a higher certainty of ending up with a large final wealth. Consequently, the necessity to take on risk to generate money for retirement is reduced and an investor can follow a more conservative strategy. As time progresses, the strategies starting with different wealth levels become closer to each other. This is a consequence of using a utility function with CRRA which is common in economics as, we recall, the latter implies that the long-term behaviour of economic agents (in this case the retirement planner) is independent of wealth.

In the right panel of Figure 4.4 the optimal strategies for different initial levels of contribution are displayed. Hereby, we hold the initial wealth $P_0$ fixed and vary the initial contribution $C_0$. The higher $C_0$, the more human capital the pension beneficiary owns. An increase of $C_0$, i.e. an increase in the human capital, fosters investment in the risky asset (upper curve). To some level, it can be seen as the pension beneficiaries tend to borrow the future income to invest more in the risky asset. In both panels, we observe that the higher the ratio $P_0/C_0$ (realized by a lower initial contribution level $C_0$ or a higher initial wealth level $P_0$), the lower the resulting optimal fraction invested in the risky asset.

4.4. Effect of Risk Aversion. A close look at the Merton Ratio reveals that the fractional amount to be invested in equity is decreasing as the RRA is increasing. In this section we investigate whether this effect can also be observed if a random contribution is present and when stochastic volatility is included. In Figure 4.5 we depict the optimal investment strategy over time for different levels of risk aversion. With increasing RRA, the investment in the risky asset is reduced. Moreover, we observe that as $\gamma$ increases, the shape of the optimal
strategy changes from the glidepath type to a horizontal line as in the Heston solution shown in Figure 4.1(B). This means that the impact of the contribution causing the shape of the glidepath is diminished when the RRA is becoming large.

In Table 4.2 the CE, Mean and Variance of the optimal terminal wealth is depicted for different levels of risk aversion in the SVM. We observe that the CE, Mean and Variance are decreasing when the RRA is rising. This is a consequence of the reduction of the fraction of wealth to be invested in equity. Looking again at the lines in Figure 4.5, it is apparent that the curve resembling the strategy for $\gamma = 0.5$ is considerably higher than all the others, and the marginal difference between this one and the curve showing the strategy for $\gamma = 1.5$ is the largest. This observation is also numerically depicted in Table 4.2 as the marginal gap between CE, Mean and Variance is the biggest when increasing the RRA from 0.5 to 1.5. Thus, the effect on the strategy when changing from a risk-affine investor (corresponding to the case $\gamma < 1$) to a risk-averse investor, say a value of $\gamma = 1.5$, is larger than the effect of becoming even more risk-averse.

![Figure 4.5. Optimal Trading Strategy for Different Levels of Relative Risk Aversion, Stochastic Volatility Model](image)

<table>
<thead>
<tr>
<th>RRA</th>
<th>CE</th>
<th>$E[P^*_T]$</th>
<th>$\text{Var}[P^*_T]$</th>
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</thead>
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<tr>
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<td>103.11</td>
<td>173305.67</td>
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</tr>
<tr>
<td>6</td>
<td>19.70</td>
<td>22.94</td>
<td>33.21</td>
</tr>
</tbody>
</table>

**Table 4.2. Comparison of Different Levels of Relative Risk Aversion, Stochastic Volatility Model**

4.5. **Impact of the Contribution’s Parameters.** We recall the previous result that a low wealth-to-contribution ratio leads to a rather risky investment strategy. In this subsection we further investigate the marginal impact of the contribution on the optimal share to be
invested in equity. To this end, we vary the parameters in the contribution process. First, we let the contribution’s drift $\mu_C$ change, which only has a relatively small impact on the optimal strategy, see Figure 4.6. Overall we find that a higher drift makes the agent willing to invest more riskily initially. This result is consistent with our previous findings as a higher drift leads in general to a higher level of contribution, which in turn decreases the aforementioned wealth-to-contribution ratio. However, this effect gradually declines over time and the optimal strategies depicted in Figure 4.6 show a nearly identical behaviour at maturity. This is a consequence of the fact that the wealth is dominating the wealth-to-contribution ratio as time progresses and therefore the effect of a marginally higher contribution vanishes. In Table 4.3 an analysis of the final wealth’s CE, Mean and Variance is performed for different levels of $\mu_C$ presuming a deterministic contribution stream (upper panel) and the stochastic contribution process (middle and lower panel); as the CVM and the SVM show similar results in this case, we restrict the analysis to the latter. Note that $\mu_C = \sigma_C \equiv 0$ means that the contribution is kept at the initial level $C_0$ and for every period of length $\Delta$, the amount $C_0 \cdot \Delta$ is contributed to the pension fund. For the situation that the contribution is deterministic, an increase of the contribution’s drift naturally leads to a higher CE, Mean and Variance. Looking at the lower panel of Table 4.3, we keep $\sigma_C$ at a value of 0.1 and vary $\mu_C$. The same effects as in the previously discussed deterministic contribution case are observed.

Similar to the drift, we analyze the impact of the volatility of the income process $\sigma_C$. Comparing the deterministic with the stochastic income process (both for the value $\mu_C = 0.04$), we see that the deterministic stream yields a higher CE, a negligibly higher expected wealth and a slightly lower variance. Thus, the presence of randomness in the contribution process only adds slightly to the riskiness of the investment procedure.

In the lower panel of Table 4.3, we vary the value of $\sigma_C$ while keeping $\mu_C$ constant. The CE and the expected terminal wealth are not very sensitive to changes in the volatility of the contribution process. However, a higher volatility of the contribution process adds some variation to the final wealth. Figure 4.7 contains a graphical illustration. Overall we see that with increasing volatility the agent invests slightly less risky in the early stages of the contract while the strategies for different values of $\sigma_C$ show a very similar behaviour as maturity approaches. However, the overall effect is small. Our results show that the main factor of the contribution process effecting the optimal strategy is the initial level.

<table>
<thead>
<tr>
<th>$\sigma_C$</th>
<th>$\mu_C$</th>
<th>CE</th>
<th>$\mathbb{E}[P_T^\pi]$</th>
<th>Var[$P_T^\pi$]</th>
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<td>22.86</td>
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<td>74.19</td>
</tr>
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<tr>
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<td>0.04</td>
<td>22.23</td>
<td>26.34</td>
<td>83.66</td>
</tr>
</tbody>
</table>

Table 4.3. Sensitivity to Varying Parameters of the Contribution Process, Stochastic Volatility Model

4.6. **Prolongation of the Planning Horizon.** So far we have considered a planning horizon of $T = 10$ years. This usually corresponds to a retirement planner in the second half of his
fifties. As more and more people start saving for their pension when they are younger, we consider the case $T = 30$ in this subsection. This comes along with a significant increase of the computational effort stemming from the increased number of wealth nodes. For longer time horizons such as $T = 30$, we therefore use a slightly modified version for the calculation of the step-size which is time-dependent and recalculated on an annual basis (using the notation introduced in the Prerequisites of the LSMC algorithm above):

$$
\Delta P_t := \frac{P_{\Delta t+[t], \max} - P_{\Delta t+[t], \min}}{n_p + \lfloor t \rfloor}.
$$

Figure 4.8 shows the corresponding optimal strategies, whereby the left panel treats the CV case and the right panel the SV case. For both models, the same qualitative conclusions as drawn for the case $T = 10$ apply. However, looking at the fraction to be invested in the
In Figure 4.8 we have worked with the parameter values introduced in Section 3.2. Thereby we implicitly assumed that these values remain constant over a time period of 30 years. However, since the parameters might change due to various reasons, it is necessary to investigate whether sticking to a previously computed strategy while parameter values actually changed does highly effect the turnout of the investment. To this end, we consider exemplarily the scenario that the long-term mean of the volatility $\sqrt{\theta}$ rises to 18% after 15 years. We calculate the optimal strategy when the long-term mean does not change and apply this strategy in a market where there is actually a change. The results are summarized in Table 4.4, where the Mean, Variance and CE are listed. The column labeled SVM thereby corresponds to the case that the long-term mean remains at 13% and the right column shows the result for a market when the long-run mean increases. Obviously, the expected wealth levels nearly coincide, while the variance significantly increases when the strategy is not adjusted for an increased long-term volatility. Likewise, the CE decreases when the strategy is erroneously not adjusted. Hence, when considering such a long-term investment, a fund manager needs to account for changed market parameters; the results in Table 4.4 show that ignoring such kind of changes may lead to severe disadvantages for the retirement planner.
A remark on the absolute value of the variance for the column labeled SVM in Table 4.4 is due. Comparing this value with the corresponding variance of the optimal terminal wealth for the case $\rho_{\nu} = -0.4$ in Table 4.1, a sharp increase is observed. A reason for this is the extension of the planning horizon by 20 years and the higher fractional amount of total wealth to be invested in equity. However, this does not mean that an investment over 30 years is disproportionately more risky. This can for instance be seen from a comparison of the respective coefficient of variation (CV). The coefficient of variation is defined as the ratio of the standard deviation to the mean, that is $\sqrt{\text{Var}[P^*_T]} / \mathbb{E}[P^*_T]$. As such, it indicates the degree of variability (measured by the standard deviation) in relation to the mean. Calculating the CV for the column $\rho_{\nu} = -0.4$ in Table 4.1 yields about 0.32, and the corresponding value in the SVM case in Table 4.4 amounts to 0.89. Hence, the value for the 30-year case is slightly less than three times the value for the 10-year case, so we observe an increase that is approximately proportional to the extension of the investment period. To sum up, a savings period of 30 years leads to a higher amount of total wealth invested in the risky asset, but the overall risk exposure is not disproportionately higher.

### 5. Conclusion

In this paper we study the optimal trading strategy of a Target Date Fund when stochastic volatility is included and random contribution risk is present. Using tools from stochastic and numerical optimization we show that the glide path structure that TDFs are frequently identified with is still optimal in such a complex market environment. Given the fact that more and more people use TDFs to save for their pension, our result is of high practical relevance. We show that qualitatively the strategies obtained when presuming a constant volatility do not differ substantially from the ones achieved when stochastic volatility is accounted for, however, the strategies are not simply interchangeable. In the SVM a lower variance of the optimal terminal wealth is generated than in the CVM.

We also outline that a main factor determining the fraction of wealth to be invested in equity is the ratio of initial wealth to initial contribution. Marginally, a low initial wealth induces a more risky strategy. Conversely, a high initial contribution, which is actually the initial value of the accumulated future contributions and therefore allows for the interpretation of human capital, also leads to a riskier investment strategy. As long as the ratio of initial wealth-to-contribution does not change, the strategy is unaffected.

It is illustrated that the risk aversion plays an important role for the determination of the optimal trading strategy. A higher risk aversion clearly goes along with a more conservative investment behaviour. Furthermore, the risk aversion determines the shape of the investment strategy plotted as a function of time, i.e., it effects the steepness of the curve. Our sensitivity analysis also shows that differences in the drift $\mu_C$ and the volatility $\sigma_C$ of the contribution process only have negligible impacts on the resulting optimal glide path. From a practical point
of view, it is therefore possible to offer the same TDF product to a group with homogeneous risk preferences without having perfect knowledge of the contribution dynamics of each individual.

REFERENCES


3 On the Impact of Low Interest Rates on Common Withdrawal Rules in Old Age

Source:
An Chen, Stefan Schelling and Nils Sørensen (2021). On the Impact of Low Interest Rates on Common Withdrawal Rules in Old Age. A version of this article has been submitted (under review) to The European Journal of Finance, published by Taylor & Francis.
ON THE IMPACT OF LOW INTEREST RATES ON COMMON WITHDRAWAL RULES IN OLD AGE

AN CHEN∗, STEFAN SCHELLING‡, AND NILS SØRENSEN§

Abstract. Increasing life expectancy cause major challenges for retirement planning. Ensuring a desired standard of living in the retirement period is additionally challenged by the current and possibly long-lasting low interest environment. According to the literature, the annuitization of wealth should be a vital part of retirement planning in this context. However, many people prefer to manage a substantial part of their wealth as well as how to withdraw it during the retirement period by themselves. In this context, several easily applicable self-managed withdrawal rules are commonly recommended by financial advisors. In this paper, we show that increasing life expectancy and the current low interest environment cast some doubt on the viability of these self-managed withdrawal rules. Further, we propose a mixed rule which combines the fixed fraction and remaining lifetime rule and can significantly improve the retirees’ welfare. The results provide important insights for revising common recommendations by financial advisors, but also for retirement product design and regulation.

Keywords: Retirement Savings, Withdrawal Rules, Optimal Consumption Strategy, Optimal Investment Strategy, Least-squares Monte Carlo

JEL-Codes: G22, G23, J26, J32

Date: June 13, 2021. Preliminary version.

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1. **Introduction**

In an aging society, the necessity of individual retirement planning and old-age provision becomes increasingly important. The ongoing low interest environment further increases the risk of individuals to outlive their savings in old age. Although most literature indicates that annuitization of a significant fraction of wealth is optimal for an expected utility maximizer (e.g., [Yaari, 1965] and [Davidoff et al., 2005]), in reality we observe that many people prefer not to annuitize or annuitize only a significantly lower fraction of wealth (e.g., [Brown et al., 2008]). This discrepancy is often referred to as the “annuity puzzle”, cf., [Benartzi et al., 2011]. While possible objective reasons like home ownership, existing pensions, costs, or bequests might influence individual’s annuitization decision, literature indicates that these are not sufficient to explain the reluctance to annuities. Behavioral aspects seem to be more decisive, such as present bias, mental accounting. Unfortunately, it is difficult for people to overcome the influences of such cognitive biases on the annuitization decision. Therefore, in order to maintain their standard of living in old age, it is crucial for many people to appropriately draw down their wealth during retirement. This enhances the need for self-managed withdrawal rules that provide appealing benefit characteristics, and are simultaneously easily applicable in practice. Various recommendations from financial advisors for self-managed withdrawal rules have been discussed and analyzed in the literature. Some of these rules have even been considered by legislators (e.g., the default pattern under US tax law mimics the “remaining lifetime rule”). The methodologies used and the results of this stream of literature are diverse, see e.g. [MacDonald et al., 2013] for an extensive overview. However, when focusing on easily applicable self-managed withdrawal rules, in particular the following have been considered: the fixed benefit approach, the fixed percentage rule, the maximal age rule, and the remaining lifetime rule. Some of them have even found especially attractive to the retirees (cf.,[Dus et al., 2005] and [Horneff et al., 2008]).

The objective of this paper is to examine the extent to which often recommended self-managed withdrawal rules (some of these are even used as statutory default payout plans) can provide appealing benefit characteristics in a long-lasting low interest rate environment and under increasing longevity risk. Most existing literature focuses on a planning horizon until age 100 and assume a significantly higher interest rate level than we are currently observing in many countries, e.g., [Horneff et al., 2008]. By doing so, they significantly underestimate the risk of running out of money in old age. We analyze how common self-managed withdrawal rules perform in a possibly long-lasting low interest environment, if the planning horizon possibly exceeds the age 100. We conduct our analysis within an expected utility framework taking
into account a stochastic financial market. As a benchmark, we consider an annuity as well as the optimal withdrawal plan maximizing the expected lifetime utility. Additionally, we analyze the expected present value of the shortfall, benefits and residual wealth of these rules. We discuss advantages and disadvantages of common self-managed withdrawal rules in this setting and propose an easily applicable mixed rule that provides favorable benefit characteristics. The proposed mixed rule is a simple combination of the fixed fraction rule and the remaining lifetime rule. We also analyze the impact of an existing pension in terms of social security benefits on withdrawals and investment strategies.

Firstly, by analyzing the benefit over time of the common self-managed withdrawal rules we find that a low interest rate environment typically leads to (significantly) decreasing benefits over time, and in particular for ages above 100 there is a high risk of wealth being totally depleted. Secondly, when analyzing the expected utility for different levels of risk aversion, we particularly find that the remaining life time rule performs worst among all rules in almost all cases, which is in sharp contrast towards existing literature dealing with shorter planning horizons and higher interest rate levels, e.g., [Horneff et al., 2008]. Also, the optimal equity share is lower compared to higher interest rate assumptions. Due to the unsatisfactory results in the optimal expected utility analysis we propose a mixed rule, which displays promising characteristics. For instance, the simple mixed rule provides an almost constant median benefit for the first 25 years of retirement. Further, we are able to show that, in most cases, the simple mixed rule outperforms all other rules and performs only slightly worse than the corresponding optimal strategy. An in-depth analysis even shows that the result holds for reasonable choices of the underlying fixed fraction rule and that the mixed rule can be designed such that its certainty equivalent remains almost constant for different risk aversion levels. This makes the mixed rule easily applicable for individuals in practice where the risk preferences are unknown. Further, a sensitivity analysis is carried out for different planning horizons, interest rate levels and market risk premia.

Compared to annuities, we find that mostly that constant annuities deliver a higher expected lifetime utility than the simple withdrawal rules. This result is, at the first sight, surprising, as the upside movement potential from the financial market can improve the expected lifetime utility of the retirees. In order to gain more insights about this result, we carry out an analysis about expected shortfall, benefits and residual wealth. We observe that over the lifetime the self-managed withdrawal rules show a relatively high amount of shortfall, and at the same time a part of the wealth
is most likely not consumed due to early death.

We also discuss the case of social security benefit and show that the presence of social security benefit make the withdrawal strategies play a less relevant rule in the individuals’ expected utility, and consequently the rules are less distinguishable. In particular, if the social security benefit is sufficiently high, it allows individuals to consume their wealth freely as they do not suffer, when the wealth is depleted. This holds for both normal and low interest rate environment. For low risk-aversion levels it is even possible to achieve a higher expected utility than when buying an annuity.

Overall, the results show that in a long-lasting low interest environment and for long planning horizons frequently recommended self-withdrawal rules provide unsatisfactory benefit characteristics and often bear a significant risk of running out of money. In contrast, the proposed mixed rule shows that a simple modification of the rules results in more favorable benefit characteristics and can significantly increase the expected utility for most individuals compared to other common strategies. These results seem to be particularly important for individuals and financial advisors, but also to product providers (retirement savings product design), regulators and lawmakers when it comes to implemented default patterns and (tax) incentives as well as state subsidies.

The remainder of the paper is structured as follows: Section 2 introduces the model framework and the rules considered. Section 3 introduces the expected utility problem. Section 4 carries out the numerical analysis and discusses the main findings. We analyze the strategies are in terms of their expected utility as well as their expected shortfall. Finally, Section 5 concludes the paper.

## 2. Model Framework

In this section, we introduce the model framework. We consider an individual at retirement age $x$ (in full years) at time $t = 0$, and denote the remaining lifetime of the individual by $\tau < \omega - x$ (in full years) where $\omega$ specifies the limiting age. For the further specifications in this section, we assume that $t \in \{0, \ldots, T\}$ with $T \in \mathbb{N}$ denoting the terminal date of the considered time horizon (planning horizon). Hereby, the planning horizon $T$ may differ from the limiting age $\omega$, as different maturity dates are addressed in various withdrawal rules. For the $t$-year survival probability we use

\[ \text{For the sake of simplicity we focus on the description of the model framework used in the base case. Extensions will be discussed and explained in the respective sections.} \]
the common actuarial notation \( t p_x := P(\tau \geq t) \).

We assume that the policyholder under consideration receives a predefined annual constant lifelong social security benefit \( A^{soc} \geq 0 \). In addition to the social security benefit, the individual is endowed with an initial wealth \( x_0 \), which is invested in an investment fund or used to buy a private life annuity with constant annual payments \( A \). For both, the payments from the life annuity and the social security benefit, we assume that they are consumed at the beginning of the same year and not reinvested. Further, the individual can withdraw from the investment fund to meet their additional consumption needs. As a consequence, the overall consumption in a certain year \( t \) consists of two parts: the social security benefit and the payment from the life annuity or the withdrawn amount from the investment fund.\(^2\)

Next, we introduce the investment fund and the considered simple withdrawal rules. We denote the wealth process of the individual from the investment fund by \( X = (X_t)_{t \geq 0} \subset [0, T] \). We assume that the individual withdraws a fraction of her wealth \( c = (c_t)_{t \geq 0} \subset [0, 1] \) at the beginning of each year and invests the remaining part further in the investment fund for future consumption. We use both terminologies “withdrawal rule” and “consumption rule” to describe \( (c_t) \). Consequently, the wealth process can be described by

\[
X_{t+1} := X_t (1 - c_t) I_{t+1} \quad \forall t \geq 0,
\]

where \( I_t \) denotes the return of the investment fund between \( t - 1 \) and \( t \), in which the wealth is invested. The fund investment comes with an arbitrary equity ratio \( \pi = (\pi_t)_{t \geq 0} \subset [0, 1] \). The remaining part \( (1 - \pi_t) \) is invested in a bond fund. A detailed description of the capital market and the definition of the investment fund is provided in Section 4.1.

As we aim to analyze withdrawal rules which are applicable in practice, we focus on simple self-managed withdrawal rules which are based on common recommendations (for example by financial advisers). Apart from that we also derive (numerically) as a benchmark the optimal consumption and investment strategies, cf., Section 3. The self-managed withdrawal rules provide an annual benefit \( B_t := c_t X_t \), where \( c_t \) depends on the rule. We use the term “simple withdrawal rules” or “self-managed

\(^2\)This assumption is particularly reasonable for individuals with middle wealth, since social security benefits are often not fully sufficient to finance the desired standard of living. Further, in this setting consumption is only determined by income and liquid wealth and hence independent of illiquid assets (which are therefore not explicitly modeled in the framework), cf., [Skinner, 1996] or also [Levin, 1998] for empirical evidence.
withdrawal rule” for the rest of the paper to refer to the rules discussed below.

The fixed benefit approach pays out a fixed benefit \( B \) at each payment date until the wealth \( X_t \) is depleted or until death of the policyholder under consideration, whichever comes first, i.e., \( B_t = \min(B, X_t) \) and \( c_t = \min(1, B/X_t) \), respectively. One natural possibility is to set \( B_t \) equal to the annual payment of a lifelong annuity. Hence, this mimics the life annuity as long as withdrawals are possible. However, as wealth is invested in an investment fund, adverse market developments can result in situations where wealth is insufficient to cover the fixed benefit and hence drops subsequently to zero.

The fixed percentage rule defines the benefit as a fixed percentage \( q \in (0, 1) \) of the current wealth \( X_t \), that is, \( c_t = q \) and hence \( B_t = q \cdot X_t \), as long as the policyholder is alive. A well-known special case of this rule is the so-called 4% rule, where \( q = 0.04 \) and which is often named and discussed in the context of self-managed withdrawal strategies, cf., [Scott et al., 2009]. In contrast to the fixed benefit rule, the benefit varies over time and the wealth will never be totally depleted when applying this rule.

The maximal age rule considers a maximal age \( \zeta \) (in full years) which is not necessarily equal to the limiting age \( \omega \) or in line with the planning horizon \( T \). The fraction of consumption is given by \( c_t = \min(1, 1/(\zeta - x - t)) \) which results in a benefit \( B_t = X_t/(\zeta - x - t) \), as long as the policyholder is alive and has not reached age \( \zeta \), otherwise \( B_t = 0 \). A common example is the so-called 100 minus age rule, where the maximal age is set to \( \zeta = 100 \). Applying this rule, a 65-year old individual withdrawals in the first year \( 1/35 \cdot X_0 \), in the second year \( 1/34 \cdot X_1 \), etc. While this strategy starts very conservatively, the fraction of wealth that is withdrawn increases heavily in very old age until the wealth is completely depleted when reaching the maximal age (but never before the maximal age).

The (expected) remaining lifetime rule defines the benefit by \( B_t = \min (1, 1/\mathbb{E} [\tau | \tau \geq t]) \cdot X_t \), as long as the policyholder is still alive. This rule takes into account the expected remaining life expectancy at each point in time. The fraction of wealth that is withdrawn increases over time. Further, the wealth will be completely depleted as soon as the expected remaining life expectancy is lower than one year.\(^3\)

\(^3\)In fact, typically this applies to pensioners older than 100 years (105 under our model assumptions used in Section 4).
Lastly, besides the simple withdraw rules above, we suggest an additional self-managed withdrawal rule, which is easily applicable, and named by us as the mixed rule as it combines characteristics of the fixed percentage rule as well as of the remaining lifetime rule. Under the mixed rule, the benefit is given by a weighted average of these two rules. In detail, we consider \( c_t = \min(1, \left( a \cdot c^{rlr}_t a + b \cdot q \right) / (a + b)) \) with \( q \) denoting the fixed percentage and \( c^{rlr}_t = 1 / \mathbb{E}\tau | \tau \geq t \). For simplicity we only allow for \( a, b \in \mathbb{N}_0 \) (and consider the ratio \( a : b \)). The benefit is then given by \( B_t = (a \cdot c^{rlr}_t + b \cdot q)X_t / (a + b) \). The motivation behind this choice is that, the remaining life time rule consumes much in early stages and the wealth will be depleted in very old ages, while the fixed fraction rule on the other hand provides a conservative withdrawal behavior and the wealth is never depleted.

Besides applying the different withdrawal strategies introduced above, we additionally allow the individual to annuitize her initial wealth at time \( t = 0 \). The annuity\(^4\) pays a constant annual payment at the beginning of each year as long as the individual is alive. The actuarially fair annuity factor for an annuity at time \( t = 0 \) is defined by \( \tilde{a}_x := \sum_{k=0}^{\omega - x} k p_x \cdot P(0, k) \) with \( P(s, d) \) denoting the fair price of a zero bond with maturity \( d \) at time \( s \leq d \), with \( P(d, d) = 1 \).\(^5\) We assume that effects of adverse selection and other market incompleteness are covered by an expense factor \( e^{ann} \). Therefore, the applied annuity factor can be calculated by \( \tilde{\tilde{a}}_x := (1 + e^{ann}) \cdot \tilde{a}_x \).

Consequently, the constant annual annuity payment is given by \( \tilde{\tilde{a}}^{-1}_x \) per one monetary unit premium, and hence for a premium of \( x_0 \) the annual annuity payment is given by \( A := x_0 \tilde{\tilde{a}}^{-1}_x \). We do not consider partial annuitization.

3. Expected Utility and Utility Maximization

In this section, we describe at first the expected remaining lifetime utility framework that is used to evaluate the strategies from an individual’s point of view. In addition, we discuss how to derive optimal withdrawal and investment strategies within this framework, which will be used in the later analysis as benchmark.

We assume that individuals are risk-averse, with risk-aversion coefficient \( \gamma \), and endowed with an additive time-separable utility function

\[
u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma > 0 \text{ and } \gamma \neq 1, \\ \ln(x), & \gamma = 1. \end{cases}
\]

\(^4\)If not stated otherwise, we use the term “annuity” for life annuity.
\(^5\)We use fair prices of zero bonds which are in line with the financial market, cf., Appendix 4.
The power utility function is of the class of constant relative risk aversion (CRRA). The individual evaluates her expected remaining lifetime utility by

$$E \left[ \sum_{t=0}^{T-1} \mathbb{1}_{\{r_t>t\}} e^{-\delta t} u (B_t + A^{soc}) \right] = E \left[ \sum_{t=0}^{T-1} \mathbb{1}_{\{r_t>t\}} e^{-\delta t} u (B_t + A^{soc}) \right].$$

(2)

The parameter $\delta$ denotes a constant subjective discounting factor of the individual. The left hand side of equation (2) can be derived from the assumption that mortality risk and financial market risk are assumed to be independent. Similar models have been used in the literature, e.g., by [Brown and Poterba, 2000], [Dushi and Webb, 2004], and [Horneff et al., 2008].

To determine the optimal investment strategy, we assume, that the individual can change the equity ratio on an annual basis (that is, $I_{t+1}$ in Equation (1) is given by the return of an investment fund with a constant equity ratio during the year, while the equity ratio may change for different years). Hence, the problem becomes an optimal control problem with control variables $(c, \pi)$ and the time-$t$ value function, if the individual is still alive at time $t$:

$$v(t, x, r) = \max_{(c, \pi) \in [0,1]^2} E \left[ \sum_{l=t}^{T-1} l p_{x+t} e^{-\delta(l-t)} u (c_l X_l + A^{soc}) \right] X_t = x, r_t = r.$$  

(3)

To solve the optimal withdrawal and investment problem we adapt the least-squares Monte Carlo (LSMC) approach discussed in [Andreasson and Shevchenko, 2019], [Denault et al., 2017] and [Denault and Simonato, 2017] to our more complex case.

Note that we consider the same expected remaining lifetime utility framework when deriving the optimal result for the simple withdrawal rules introduced in Section 2. However, in those cases, the problem simplifies significantly as the consumption rule is either fixed (the remaining lifetime rule), or specified by a single parameter, e.g., the percentage $q$ in the case of the fixed percentage rule. We optimize over the free parameter, i.e., the optimal choice of $c$ under a specific self-managed withdrawal rule in optimization problem (3) is achieved by optimizing the corresponding parameter instead: $B$ (fixed amount rule), $q$ (fixed fraction rule), $\zeta$ (maximal age rule), $b$ (mixed rule). For the remaining lifetime, no additional optimization parameter exists and for the mixed rule we only optimize over $b$ to keep this rule simple. Further, for the simple withdrawal rules we only allow fixed constant equity ratios, i.e. $\pi_t^* \equiv \pi_0^*$ for all $t \in [0,1, \cdots , T)$. In other words, the optimal design of the simple withdrawal rules is determined by the optimal choice of the respective parameter $(B, \zeta, q, b)$ in combination with the initially chosen equity ratio $\pi_0^*$. 


4. Numerical Analysis

In this section we will carry out the numerical analysis and present our findings. First, in Section 4.1 we introduce the financial market model and the parameterization used in the numerical analysis. Section 4.2 presents descriptive results of the benefit and wealth developments of the considered self-managed withdrawal rules as well as the optimal strategy. Next, in Section 4.3.1 we present the results of the expected utility analysis. Further, the section discusses the optimal design of the considered rules. Lastly, we provide an analysis on the expected present value of shortfall and the residual wealth (wealth at death) in Section 4.4.

4.1. Setting. In this section we describe the base case setting which is considered if not explicitly stated otherwise. For the analysis we consider survival probabilities $p_x$ with $x = 65$ based on the German Federal Statistical Office’s cohort mortality tables with trend (V2) for females$^6$ born in the year 1952 (cf., [Federal Statistical Office, 2017]). For the simulation we consider $\omega = 110$. The mortality tables of the Federal Statistical Office have a cut-off age of 100 years. Hence, we extrapolated the mortality tables until an age of 110 years using a Kannisto model approach starting at an age of 80 years (cf., [Wilmoth et al., 2007]).

For the financial market model we assume that equity is modeled by a geometric Brownian motion $S$ and interest rates are driven by a Vasicek short rate model $r$ ([Vasicek, 1977]). Further, the bond fund $R$ is modeled by means of a “rolling” bond investment which continuously reinvests in zero bonds with constant maturity $T_B$ (cf., [Rutkowski, 1999] or [Bielecki et al., 2005]). More precisely, for the financial market we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ under the real-world measure $\mathbb{P}$ satisfying the usual conditions. $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ with $\sigma$-algebra $\mathcal{F}_t$ containing the available information at time $t$. The dynamics$^7$ are then given by

\[ dS_t = S_t \left( (r_t + \lambda_S)dt + \sigma_S dW^S_t \right) \]
\[ dr_t = \kappa (\xi - r_t)dt + \sigma_r dW^r_t \]
\[ dR_t = R_t \left( r_t - \frac{\lambda_r \sigma_r}{\kappa} \left( 1 - e^{-\kappa T_B} \right) \right) dt - \frac{\sigma_r}{\kappa} \left( 1 - e^{-\kappa T_B} \right) dW^r_t \]


$^7$Note that we can derive closed formulas for the dynamics of all described processes and the price of zero bonds, where $\exp \{- \int_0^t r_u du \}$ is used as numeraire, cf., [Brigo and Mercurio, 2007]. The price of zero bonds with different maturities can be used to determine the term structure.
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Table 1. Equity fund annual return (log(\(S_t/S_0\))) and bond fund annual return (log(\(R_t+1/R_t\))) characteristics (mean, standard deviation, 10%-quantile, median, 90%-quantile) in % as well as annuity characteristics \((\bar{a}_x,100/\bar{a}_x)\), \((\tilde{a}_x,100/\tilde{a}_x)\), the second value within each bracket corresponds to the fixed nominal annuity payouts at the beginning of each year per 100 monetary units premium) under different parameter settings.

<table>
<thead>
<tr>
<th>(\lambda^S/\xi)</th>
<th>1%</th>
<th>3%</th>
<th>5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>equity 2%</td>
<td>(3.1, 21.0, -22.0, 1.0, 30.8)</td>
<td>(5.2, 21.4, -20.4, 3.0, 33.5)</td>
<td>(7.3, 21.9, -18.8, 5.1, 36.2)</td>
</tr>
<tr>
<td>equity 4%</td>
<td>(5.2, 21.4, -20.4, 3.0, 33.5)</td>
<td>(7.3, 21.9, -18.8, 5.1, 36.2)</td>
<td>(9.4, 22.3, -17.2, 7.3, 38.9)</td>
</tr>
<tr>
<td>equity 6%</td>
<td>(7.3, 21.9, -18.8, 5.1, 36.2)</td>
<td>(9.4, 22.3, -17.2, 7.3, 38.9)</td>
<td>(11.7, 22.8, -15.5, 9.4, 41.7)</td>
</tr>
</tbody>
</table>

Table 1 provides an overview of return characteristics of equity as well as the bond fund under the different settings.

with \(S_0, R_0 > 0, r_0 \in \mathbb{R}, \sigma_S, \kappa, \xi, \sigma_r > 0\) and \(dW^S_t dW^r_t = \eta \in (-1,1)\), that is, \(W^S_t = \eta W^r_t + \sqrt{1-\eta^2} W^*_t\) with \(W^*\) and \(W^r\) independent Brownian motions\(^8\) under \(\mathbb{P}\). Moreover, \(\lambda_S > 0\) denotes the constant equity risk premium. The simulation of the financial market is done on a daily basis assuming 252 days per year. We also apply a daily rebalancing between equity and bond fund of the considered investment funds which are used to calculate the annual returns of the investment fund \(I_t\) for \(t \in \{1, \ldots, T\}\). The parameters have been chosen in accordance with the European money market and recent literature (cf., [Hieber et al., 2019]), that is, \(\sigma_S = 20\%, \sigma_r = 1.5\%, \lambda_r = -23\%, \kappa = 30\%, \eta = 15\%\) and \(T_B = 10\). To analyze the impact of a long-lasting low interest rate environment, we use for the base case a mean reversion level for the short rate of \(\xi = 1\%\). For the sake of comparison, we also consider higher values, which have been used in past studies, e.g., [Horneff et al., 2008]. Hence, \(\xi \in \{1\%, 3\%, 5\%\}\) and we assume that \(r_0 = \xi\). Further, in the base case we set the equity risk premium \(\lambda_S = 4\%\). However, as the equity risk premium possibly has a decisive impact, we analyze the results for lower and higher values, i.e., \(\lambda_S \in \{2\%, 4\%, 6\%\}\). Table 1 provides an overview of return characteristics of equity as well as the bond fund under the different settings.

To calculate the annuity factors, we use the average survival probabilities for females and fair prices of zero bonds which are in line with the financial market model. Further, we assume that the applied annuity factors are reduced by the expense factor \(c_{ann} = 15\%\) which captures also adverse selection effects.\(^9\) Table 1 provides an overview of the fair annuity factors for a life annuity \(\bar{a}_x\) as well as the applied annuity

\(^8\)Note that the random remaining lifetime is assumed to be independent of the financial market.

\(^9\)This value has been chosen such that the annuity payments are in line with annuity rates in the German annuity market in 2017 (under the assumption that surplus participation remains stable) for \(\xi = 3\%\).
factor $\tilde{a}_x$ and the corresponding yearly fixed nominal annuity payout at the beginning
of each year per 100 monetary units premium under the different settings.

We consider individuals with risk aversion coefficients $\gamma \in \{1, \ldots, 6\}$ and discounting factor $e^{-\delta} = 0.97$, which is in line with the literature, cf., [Blake et al., 2003]. To calculate the expected remaining lifetime utility, we consider different planning horizons ($T = 35$ and 45)$^{10}$ and different assumptions on the existing social security benefits ($A^{soc} = 0$ and $0.1 \cdot \tilde{x}_0$). As we are interested in mimicking simple and pragmatic strategy recommendations, we restrict the analysis of the self-managed withdrawal rules to constant equity ratios (equidistantly distributed between 0 and 1 with step size 2.5%). Further, for the sake of simplicity, we assume that for these strategies the individual will never spend more than 90% of her wealth within one year.$^{11}$

4.2. Descriptive Results. At first we discuss the typical developments of benefits and wealth over time resulting from the different strategies in a long-lasting low interest-rate environment. To this end, we focus in this part on the case, where the interest rate is on a low long term mean reversion level of $\xi = 1\%$. We fix $\lambda_S = 4\%$, $T = 45$, $B = A$ for the fixed benefit rule, $q = 4.6\%$ for the fixed percentage rule$^{12}$ and $\zeta = 100$ for the maximal age rule. Further, for the self-managed withdrawal rules we consider a stock ratio of 25%.$^{13}$ All displayed results in this part are based on 20,000 simulation paths. As benchmark strategy we consider the optimal strategy providing the maximal expected remaining lifetime utility for $\gamma = 3$ according to Formula 3 in case without social security.

Figure 1 displays medians and percentiles of the benefits over time in case of the fixed benefit rule (panel (a)), the fixed percentage rule (panel (b)), and the maximal age rule (panel (c)). We observe very different developments of the benefits over time for the different strategies. The fixed benefit rule provides no upside potential. However, in unfavorable capital market scenarios there is a rather high probability to run out of money in old ages (e.g., roughly 5% (respectively 25%) at the age 90 (respectively 95)). The maximal age rule has a lower initial benefit which then typically increases over time. Hence, such a strategy could be worthwhile if increasing expenses are

$^{10}$Hence, in the case $T = 35$ the upper bound of the summation in Formula (2) is $\min(T - 1, \omega - x - 1)$. In this case the individual is assumed to simply ignore ages beyond 100 years.

$^{11}$Note that without this or a similar assumption, in many cases the expected utility of self-managed withdrawal rules which allow for a complete depletion of wealth would not be defined. While 90% represents an arbitrary choice, the choice of this boundary (assumed to be smaller than 1 but significantly larger than 0) has only a negligible impact on the results.

$^{12}$The value has been chosen to match the annuity payment in the first year.

$^{13}$This value has been chosen to be in line with the optimal stock for most rules ratio for an expected utility maximizer for $\gamma = 3$ in the case without social security, cf., Section 4.3.
expected. However, further results show that in most cases a large part of wealth (the median wealth at age 90 is 60% of the initial wealth) has not been depleted until the age of 90. The fixed percentage rule results in a (almost linearly) decreasing median of the benefits starting at 4.6 and decreasing to 2.0 at age 110. The strategy provides some upside potential in favorable capital market scenarios, but also the risk for significantly lower benefits in unfavorable capital market scenarios (e.g., at age 90 the benefit has almost halved with a probability of roughly 25%). Further results show that the chosen fraction of 4.6% is rather conservative as the median wealth at age 85 (respectively 110) is almost 70% (respectively 45%) of the initial wealth and even the 1% percentile at this age is roughly 40% (respectively 17%) of the initial wealth.

Next, we display in Figure 2 the benefits over time for the remaining lifetime rule and the optimal utility rule (panels (a) and (c)) as well as their corresponding wealth developments (panels (b) and (d)). The results for the benefits of the remaining lifetime rule show that the median benefit slightly increases from 4.6 to 5.5 at age 80 and decreases subsequently to almost zero at age 100 (since wealth is completely depleted at this age, cf., panel (b)). Moreover, the median as well as the higher percentiles are humped-shaped which indicates a significant upside potential in case of positive capital market scenarios in middle ages (with the peak being reached in the mid-eighties). Finally, the benefits of the optimal strategy (panel (c)) have an almost linearly decreasing median (similar to the fixed percentage rule) while at the same time the higher percentiles are slightly humped shaped (similar but less pronounced than for the remaining lifetime rule). The depletion of wealth has a similar structure than for the remaining lifetime rule (panel (d)). However, in contrast to

\[\text{However, it is higher than for the frequently recommended 4\% rule.}\]
the remaining lifetime rule, wealth is only depleted totally at the maximal age.

The results above suggest that the optimal withdrawal strategy (at least in this setting) combines characteristics of the fixed percentage rule as well as of the remaining lifetime rule. To this end, we additionally analyze the mixed rule where the benefit is given by a weighted average of these two rules. Figure 3 displays the withdrawal fraction $c_t$ for the fixed fraction rule ($q = 4.6\%$), the expected remaining lifetime rule as well as for the mixed rule with ratio 1:2 and $q = 4.6\%$ and the optimal strategy. It is striking that the fraction of the mixed rule is very close to the fraction of the optimal strategy (except for the very last years). Further, Figure 4 displays the structure of the benefits for the mixed strategy with ratio 1:2 and $q = 4.6\%$. We observe that the median shows a slight decrease ($\approx 4.7 - 4.2$) until an age of 90. Subsequently
4.3. Expected Utility Analysis. Next, we analyze the expected utility of the self-managed withdrawal rules and the annuity as well as the optimal investment and consumption strategy under different settings (as described in Section 4.1).

4.3.1. The Case Without Social Security. First we consider the case without social security ($A_{soc} = 0$) for a planning horizon $T = 45$ and a risk-premium $\lambda_S = 4\%$. Figure 5 displays the certainty equivalent (CE) benefits resulting from different withdrawal strategies (different lines) as well as from an annuity product in case of full annuitization at age 65 (indicated by the gray area where the upper bound displays the CE of a fair annuity without charges and the lower bound of an annuity with charges $c_{ann} = 15\%$). The CE’s are displayed in terms of the annual payment of an annuity (with charges), i.e. $A = 4.6$ for an interest level of 1\%, $A = 5.7$ for 3\% and $A = 6.9$.

![Figure 3. Fraction of wealth withdrawn over time. In case of optimal utility the mean is shown.](image-url)
for 5%, see Table 1. Further, the results show, for each risk aversion and each withdrawal rule, the CE for the optimal design. That is, the optimal stock ratio as well as the optimal choice of \( B \in (0, x_0] \) for the fixed benefit rule, \( q \in (0, 1) \) for the fixed percentage rule, \( \zeta \in \{65, \ldots, 110\} \) for the maximal age rule. For the sake of simplicity, we restrict the analysis of the mixed rule to \( a = 1 \) and \( b \in \mathbb{N}_0 \) as well as \( q = 4.6\% \), i.e., we only vary the combination specified by the ratio \( a : b \) and not the underlying fixed percentage. In the base case (low mean reversion level \( \xi = 1\% \) displayed in panel (a)) we observe that

1. the fair annuity and even annuities with charges provide in almost all cases a higher CE than self-managed withdrawal rules (including the optimal strategy). In particular, for higher risk aversions (\( \gamma \geq 3 \)) annuities perform significantly better;
2. the remaining lifetime rule performs dramatically worse than all other optimized withdrawal-rules for \( \gamma \geq 2 \) as benefits reduce to almost zero for high ages (cf., Section 4.2); and
3. the optimal fixed percentage rule outperforms all other self-managed withdrawal rules for \( \gamma \geq 2 \) (except for the optimal mixed rule and the optimal strategy); and
4. it is striking that the optimal mixed rule provides a CE that is almost identical to the CE of the optimal strategy. This supports the suggestion from Section 4.2 that a simple mixed strategy can be highly beneficial.

\(^{15}\)For example, \( CE = 2 \) means that the CE is twice the annual payment of an annuity with charges.

\(^{16}\)We discussed the impact of \( q \) in a sensitivity analysis.
Panels (b) and (c) of Figure 5 display the same results for higher mean reversion levels (\( \xi = 3\% \) and \( 5\% \)). Almost all self-managed withdrawal rules perform better compared to the previous case (except for the maximum age rule). While the fair annuity (upper bound of the gray area) is still superior in all cases, for \( \xi = 5\% \) the mixed rule and also the fixed percentage rule are above the lower bound of the annuities for rather small risk aversion and only slightly below for higher risk aversions.
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Figure 6. CE (in terms of the annual payment of an annuity with charges) for different withdrawal strategies as well as for an annuity (indicated by the gray area where the upper bound displays the CE of a fair annuity without charges and the lower bound of an annuity with charges). In the case $A^{\text{soc}} = 0$, $T = 45$, $\xi = 1\%$ for different levels of equity risk premium $\lambda_S$.

The later is also true for the fixed benefit rule).

Overall, the results illustrate that while annuities (with charges) perform typically only slightly better than self-managed withdrawal rules (at least in their optimal specification) in high interest rate environments (panel (c)), the difference is significantly higher in a low interest rate environment (panel (a)). Also, the difference between the optimal simple self-managed withdrawal rule (except for the mixed rule) and the optimal strategy is significantly higher in case of a low interest rate environment. This indicates the increasing importance of a properly designed withdrawal strategy in the current low interest rate environment. The relatively high lifetime utility from the annuity compared to the self-managed withdrawal rules is driven by the longevity risk in combination with the low interest rate environment and will further be analyzed in the upcoming Section 4.4.

Additionally, we have analyzed the impact of the risk premium $\lambda_S$. Figure 6 shows the results for $\lambda_S = 2\%$ (panel (a)) respectively $\lambda = 6\%$ (panel (b)) and all other parameters as in the base case. Overall, the results are very similar to in the base case. A higher risk premium particularly improves the result for the optimal fixed percentage rule, the optimal mixed rule and the optimal strategy.

Lastly, we have analyzed the impact of a shorter planning horizon $T = 35$, cf., Figure 7. While the self-managed withdrawal rules perform slightly better in this case, we
observe that even in this case the annuity performs significantly better (even with charges) for $\gamma \geq 2$. It is worth emphasizing that also in this case the remaining lifetime rule performs significantly worse than all other strategies for $\gamma \geq 3$. This result further questions the (sole) use of the remaining lifetime rule in a low interest rate environment, cf., [Dus et al., 2005] or [Horneff et al., 2008]. Again, it is striking that the optimal mixed strategy can significantly increase the expected utility and is very close to the optimal strategy.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{CE (in terms of the annual payment of an annuity with charges) for different withdrawal strategies as well as for an annuity (indicated by the gray area where the upper bound displays the CE of a fair annuity without charges and the lower bound of an annuity with charges). In the case $A^{soc} = 0, T = 35, \xi = 1\%, \lambda_S = 4\%$.}
\end{figure}

### 4.3.2. The Case With Social Security.
In this section we assume $A^{soc} = 0.1 \tilde{x}_0$. Figure 8 illustrates the results in the case $\xi = 1\%$. We find that the self-managed withdrawal rules perform significantly better compared to the annuity than in the case without social security since total wealth depletion in old age is less problematic. We even observe that for (very) low values of $\gamma$ all self-managed withdrawal rules provide a higher CE than the annuity. In contrast to the case without social security we find that the remaining lifetime rule is in almost all cases preferred over all other self-managed withdrawal rules (except for the optimal strategy). Further analyses show that the results are very similar for other choices of the mean reversion level $\xi$, the risk premium $\lambda_S$ as well as for the planning horizon. Overall, we find that the choice of the self-managed withdrawal rule is less crucial if the social security benefit covers a main part of consumption and additional consumption only provides little
marginal utility (which is indeed only the case if the social security benefit is able to maintain the desired standard of living in old age).

4.3.3. Characteristics of the Optimal Self-Managed Withdrawal Rules. In this section we restrict our analysis to the case without social security and investigate some aspects of the optimal self-managed withdrawal rules in more detail:

- Optimal Equity Ratio: Firstly, the optimal equity ratio is the same for all variable self-managed withdrawal rules\(^{17}\) and independent of the considered mean reversion level. Figure 9 displays the optimal equity ratio for the simple withdrawal rules (except for the fixed benefit rule) for different levels of the risk premium \(\lambda_S \in \{2\%, 4\%, 6\%\}\). Overall, the optimal equity ratio is decreasing in \(\gamma\) as well as in \(\lambda_S\). We observe that for less risk averse agents, the stock ratio heavily depends on the risk premium (e.g., for \(\gamma = 1\) the optimal stock ratio equals 100\% in case of \(\lambda_S = 6\%\), 72.5\% for \(\lambda_S = 4\%\), and only 27.5\% for \(\lambda_S = 2\%\)). For more risk averse agents (\(\gamma \geq 3\)), the optimal equity ratio is always below 40\% and depends less on \(\lambda_S\). Interestingly, we find that in most cases it is significantly below the commonly recommended 60/40\% equity/bond allocation (e.g., 25\% in the base case for \(\gamma = 3\)), cf., [Horneff

\(^{17}\)That is, for all rules except for the fixed benefit rule. We refrain from discussing the optimal equity ratio for the fixed benefit rule as it is always outperformed by other rules.
et al., 2008]. Secondly, while for the optimal strategy the optimal equity ratio changes over time (dependent on $r_t$), it is very close to the optimal (constant) equity ratio for the variable self-managed withdrawal rules.\(^\text{18}\)

- Optimal Fixed Percentage Rule: We find that the optimal fixed percentage increases only slightly for increasing $\lambda_S$. In the base case ($\xi = 1\%, \lambda_S = 4\%$) the optimal percentage is given by 8.75\% for $\gamma = 1$, 6\% for $\gamma = 3$, and 4.75\% for $\gamma = 6$. Increasing the mean reversion level results typically in a higher optimal percentage (at least for $\gamma \geq 2$), e.g., for $\xi = 5\%$ (ceteris paribus) the optimal percentage is 7.5\% for $\gamma = 3$ and 6.75\% for $\gamma = 6$ (while it is still 8.75\% for $\gamma = 1$). Overall, we find that the optimal fixed percentage is in all considered cases (significantly) above the recommended 4\%-rule.\(^\text{19}\)

- Optimal Mixed Rule: Figure 10 shows the results for different ratios $a : b$ in the base case. Overall, we find that mixing the remaining lifetime rule and the fixed percentage rule can significantly increase the expected utility for $\gamma \geq 2$. We observe that the higher $\gamma$, the lower the optimal fraction of the remaining lifetime rule. For $\gamma = 1$ the remaining lifetime rule provides the highest CE. Further, we also find that the CE’s of the mixed rule are rather similar for $a = 1$ and $b \in \{3, 4, 5\}$ for the considered risk aversions ($\gamma \in [1, 6]$). Further, these results are very similar for other choices of $\xi$, $\lambda_S$, $T$ as well as for reasonable choices of $q$ used for the fixed percentage rule. This indicates that a mixed strategy with one of these parameter combinations can be beneficial for a wide range of investors and independent of the current capital market environment.

4.4. Analysis of the Expected Present Value of the Shortfall, Benefits, and Residual Wealth. In this section we analyze the expected present value of the shortfall, the benefits as well as of the residual wealth for the fixed percentage rule (with $q = 6\%$), the remaining lifetime rule and the mixed rule (with ratio 1:2). We always consider the optimal setting in the case of $\gamma = 3$ and compare it with the result for the optimal strategy in this case.\(^\text{20}\)

Firstly, we define shortfall at time $t$ in our setting as the event that the payout at this time point $t$ is below a certain threshold value. More specifically, we compare the payout with the annuity payment $A$, and thus the shortfall at time point $t$ is given

\(^{18}\)For the case with social security benefit, we observe a similar structure depending on $\gamma$ for the optimal equity ratio, however, on a higher level.

\(^{19}\)One possible reason is that we do not consider a bequest motive.

\(^{20}\)Note that the benefit and wealth patterns over time are very similar as in the cases discussed in Section 4.2 (in particular, Figures 1 (c), 2 and 4).
by $\max(A - B_t, 0)$. Apart from the fixed percentage rule, the benefits of the other rules typically increase in the first years such that most of the wealth is depleted at a certain point in time. Consequently, for very advanced ages the benefits are rather low and the shortfall probabilities are very high in these cases, e.g., after surviving 35 years the payout of all three rules have a shortfall probability of more than 99%.
To take the size of the shortfall into account, we analyze the expected present value of the shortfall (EPVSF) which is defined as follows\(^{21}\)

\[
EPVSF := \sum_{t=0}^{T} \tilde{p}_x \mathbb{E}^Q \left( \exp \left( - \int_0^t r_s ds \right) \max(A - B_t, 0) \right).
\]

where \(Q\) is the equivalent martingale measure\(^{22}\) and \(\tilde{p}_x\) the survival probability under \(Q\) taking into account the safety loading.

Secondly, we analyze the expected present value of the residual wealth, that has not been depleted at time of death. This figure provides an impression on how good a strategy utilizes the initial wealth. The expected present value of residual wealth (EPVRW) is defined by

\[
EPVRW := \sum_{t=0}^{T} t-1 \tilde{p}_x \mathbb{E}^Q \left( \exp \left( - \int_0^t r_s ds \right) X_t \right).
\]

Further, we also report the expected present value of the benefits (EPVB), which is closely related to the EPVRW:

\[
EPVB := \sum_{t=0}^{T} t-1 \tilde{p}_x \mathbb{E}^Q \left( \exp \left( - \int_0^t r_s ds \right) B_t \right).
\]

Recall that in the optimization problem stated in equation (3), we just assume that the individual starts with an identical initial wealth level, independent of what withdrawal strategies are applied. In particular, we do not assume that the today’s market value of the benefits the individual obtains from applying various withdrawal strategies is identical. In other words, the initial market value of the individual’s retirement benefits differs from strategy to strategy, and very likely does not correspond to the initial wealth invested. The two magnitudes EPVRW and EPVB help us gain more insights, and particularly understand better why the regular annuities, even with rather large loadings, can outperform the other withdrawal investment strategies.

Table 2 reports the above three magnitudes: EPVSF, EPVRW and EPVB. Due to the low interest environment, the gains of the financial market are not sufficient to

\(^{21}\)This is in line with [Dus et al., 2005].

\(^{22}\)The corresponding Wiener processes under \(Q\) satisfy \(d\tilde{W}^S_t = dW^S_t + \frac{\lambda}{\tilde{g}_S} dt\) and \(d\tilde{W}^r_t = dW^r_t + \lambda_t dt\).

\(^{23}\)We parameterize the survival probability with loadings according to the Gompertz law, i.e. \(\tilde{p}_x := \exp \left( \exp \left( x - \tilde{m}_b \right) \left( 1 - \exp(\tilde{b}) \right) \right)\) and calibrate the modal age \(\tilde{m}\) and dispersion coefficient \(\tilde{b}\) such that \(\tilde{m} = \sum_{i=0}^{\infty} \tilde{p}_x \cdot P(0, t)\) under the constraint that we minimize the sum of the squared error between the fair survival probabilities \(\tilde{p}_x\) and the survival probabilities \(\tilde{p}_x\), i.e. \(\min_{\tilde{m}, \tilde{b}} \sum_{i=0}^{\infty} (\tilde{p}_x - \tilde{p}_x)^2\).

We obtain \(\tilde{m} = 95.551\) and \(\tilde{b} = 6.245\).
maintain the wealth level over time and thus in most cases the benefits of the self-
managed withdrawal strategies drop to very low levels in advanced ages (c.f. the
results from Section 4.2). This is also reflected by the relatively high EPVSF of the
withdrawal strategies. On the other hand, the EPVRW, see the second column of Ta-
ble 2, indicates that a considerable amount is left untouched, e.g., due to early death.
Therefore, the annuity leads to the highest expected present value of the benefits, see
the last column of Table 2. In other words, the annuity does not only eliminate the
shortfall risk but also yields the highest expected payout. This explains the relatively
high expected lifetime utility of the annuity in comparison to the withdrawal rules.

Comparing the four withdrawal strategies we find that the fixed fraction rule yields
the highest EPVSF as well as the highest EPVRW. This is because the fixed fraction
rule with \( q = 6\% \) results in rather high absolute benefits in the first years which
decrease at a diminishing rate over time. The remaining lifetime rule depletes most
of the wealth in the first 20 years and thus the EPVRW is significantly lower. If the
retiree is older than 90 years, she will most likely suffer from shortfall in all scenar-
ios. However, since there is already a rather high probability to die at a younger
age, the EPVSF is the lowest among all rules. Although this rule shows the highest
expected benefits and the lowest expected shortfall, keep in mind that the wealth
in all cases is almost completely depleted at the age of around 100 and thus the
retiree’s benefits drop to very low wealth levels for older ages, which explains the
unsatisfactory result in an expected utility framework for \( \gamma > 1 \), see also Figure 2
(a) and (b). The mixed rule with a ratio of 1:2 has an EPVSF slightly above the
remaining lifetime rule while yielding a relatively high EPVRW, which is only slightly
below the EPVRW of the fixed fraction rule. This emphasizes the advantages of the
mixed rule as the benefit is also viable in advanced ages, while the shortfall risk in
the early years of retirement is rather low, due to a more sustainable payout strategy.

The result for the optimal strategy is slightly above the mixed rule for the EPVSF,
while the EPVRW is considerably lower in comparison to the EPVRW of the mixed
rule. This is in line with the results from the previous section, where the mixed rule

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Rule} & \text{EPVSF} & \text{EPVRW} & \text{EPVB} \\
\hline
\text{annuity (with loadings)} & 0 & 0 & 100 \\
\text{fixed fraction rule} & 25.33 & 20.36 & 79.73 \\
\text{remaining lifetime rule} & 17.57 & 8.99 & 91.12 \\
\text{mixed rule 1:2} & 18.08 & 16.91 & 83.19 \\
\text{optimal rule} & 19.59 & 12.69 & 87.40 \\
\hline
\end{array}
\]

Table 2. Expected present value of shortfall and death benefit
was always slightly below the optimal rule: in early years of retirement the benefit patterns are very similar, for very old ages the optimal rule will withdrawal slightly more such that less wealth remains at death.

5. Conclusion

In this paper, we have analyzed common self-managed withdrawal rules for the retirement period in a possibly long-lasting low interest environment. Our results show, that frequently recommended self-managed withdrawal rules often result in unfavorable benefit characteristics and carry a significant shortfall risk in very old ages. While these easy applicable self-managed withdrawal rules are always dominated (in terms of expected utility) by annuities (which, however, are not in high demand) and more complex withdrawal strategies (which, however, can only hardly be implemented in reality), they lead to considerable utility losses, especially in a low interest rate environment. In particular, for individuals with a longer planning horizon, these self-managed withdrawal rules perform very poorly in an expected utility framework. This also includes frequently recommended self-managed withdrawal rules like the remaining lifetime rule and the fixed percentage rule. The results clearly show that under these (currently prevailing) conditions, a carefully designed withdrawal rule is more important than ever to ensure the desired standard of living in old age.

We propose an easily applicable mixed rule, which combines characteristics of the remaining lifetime rule and the fixed percentage rule. It provides a decent benefit structure in the early years of retirement and reduces the risk of running out of money in very old ages (compared to other commonly recommended self-managed withdrawal rules). Further, we observe that the mixed rule provides only a slightly lower expected utility level than the optimal strategy for a wide range of risk aversions, but much easier to be implemented.

The presented findings can help financial advisors to provide more suitable recommendations, but also product designers to revise their product designs for the retirement period. Further, the results are also of interest for regulators and lawmakers, e.g., in the context of defining (tax) incentives and default options in pension plans.
On the Impact of Low Interest Rates on Common Withdrawal Rules in Old Age

REFERENCES


4 Utility Maximization in incomplete markets: Comparison of a least-squares Monte Carlo and a two-dimensional HJB algorithm

Source:
Nils Sørensen. Utility Maximization in incomplete markets: Comparison of a least-squares Monte Carlo and a two-dimensional HJB algorithm. Working paper
UTILITY MAXIMIZATION IN INCOMPLETE MARKETS:
COMPARISON OF A LEAST-SQUARES MONTE CARLO AND A
TWO-DIMENSIONAL HJB ALGORITHM

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Abstract. Expected utility problems in incomplete markets do not yield analytic solutions in general. New numerical algorithms have been developed to solve these kinds of problems. Although these algorithms show promising results, no information on approximation errors is given. We analyze and compare the results of a least-squares Monte Carlo algorithm and a two-dimension finite difference scheme on different utility maximization problems. We determine and compare the optimal utility level, the optimal strategy obtained as well as the order of convergence.

Keywords: utility maximization, finite difference scheme, incomplete markets, least-squares Monte Carlo, HJB

JEL-Codes: G11 G22 D52 C61 C63

1. Introduction

Dealing with untradable assets in utility maximization problems is one of the ongoing challenges. Such assets are found in many different settings in finance and insurance, when individuals e.g. trade in European-type options or save for retirement. Utility maximization problems date back to Merton (1971), who solves the problem analytically in a complete market framework. However, adding an untradable asset to the problem leads to an incomplete market and does in most cases not allow analytic solutions. To solve these problems powerful numerical approaches, that can deal with multiple correlated assets, are required. A numerical scheme can approach this problem in two ways, either deriving a direct solution of the expected value problem or solving the resulting Hamilton-Jacobi-Bellmann (HJB) PDE/BSDE. A direct application of standard numerical methods is not possible as they in general cannot deal with the optimality requirement. In particular, the optimization problem has to be solved at each time step and leads to additional complexity of the numerical approach. Further, if an HJB approach is taken, the numerical algorithm needs to meet

Date: June 13, 2021.

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further requirements to ensure convergence to the correct solution, see e.g. Barles and Souganidis (1991). Thus numerical algorithms that can solve optimal expected utility problems in a multidimensional framework are very rare and have first evolved in recent years, see e.g. Ma and Forsyth (2016) and Denault et al. (2017). In Denault et al. (2017) a least-squares Monte Carlo algorithm (LSMC) is taken, which directly approximates the expected value problem, while Ma and Forsyth (2016) derives the resulting HJB scheme and uses a modified finite difference scheme. To the best of our knowledge, these two approaches are the only general algorithms available to solve these kinds of problems. Still, a few other algorithms exist, but most of them are closely related, see e.g. Kharroubi et al. (2014) for the LSMC and e.g. Debrabant and Jakobsen (2013) for the HJB approach.

This paper aims to analyze and compare the performance of the LSMC algorithm and the HJB scheme. We will rely on implementations from Chen et al. (2021) and Bosserhoff et al. (2021) as they can directly be applied to the problems considered in this paper. Both algorithms shall be compared in terms of optimal expected utility, the optimal strategy and the order of convergence. The comparison of the utility and the strategy is mostly quantitative. The order of convergence is a metric to measure the speed of convergence when refining. In particular, a higher order of convergence yields a better approximation with an increasing number of grid points.

Although both algorithms aim to solve the optimal expected utility problem the overall structure is quite different. The LSMC approach is based on simulation and regression. First, a forward simulation of the wealth and the untradable asset are carried out to obtain possible outcomes of the expected utility at the final time point. Hereby a naive strategy is used. This result is optimized by regression backward in time. In particular, the algorithm only considers simulation outcomes for its result. The HJB scheme is an analytic approach and approximates the resulting HJB PDE. Both the wealth and the untradable asset is gridded up to some maximal value. The expected utility at the terminal time is derived for all combinations of grid points. A modified finite difference scheme is used to derive the optimal strategy and calculate the conditional expected utility values backward in time. The optimal time-0 result is obtained for all combinations of grid points.

In this paper, these algorithms will be compared for an agent with power utility and two different example problems, a random endowment and a random contribution setting. In the random endowment problem, the agent has a self-financing strategy, which leads to an almost constant optimal strategy. On the other hand, the random contribution setting leads to a non self-financing strategy and a decreasing optimal strategy over time. This difference might, in particular, affect the HJB result as
strategy points are only allowed on a grid and thus the algorithm will overestimate some points, while others are underestimated. Both algorithms do not give any error estimates or information on the approximation bias. To obtain a better overview of the quality of the numerical results obtained in the above-mentioned papers we will compare both algorithms. First, the algorithms are compared based on their output of the expected utility. Then a new time-0 value function based on the optimal strategy is derived and compared. Consequently, a superior strategy leads to a higher utility. Further, the order of convergence is estimated. Here we will rely on a reduced problem (both problems can be reduced by one dimension) and apply a result obtained from a one-dimensional algorithm. For the random endowment problem, we will see, that the LSMC approach yields a slightly higher certainty equivalent (CE). The HJB algorithm yields a stable strategy, which shows a slightly increasing behavior. For the LSMC it is not possible to conclude if the strategy is increasing or constant due to the fluctuations that come with the use of Monte Carlo methods. The convergence is sublinear for both algorithms. The results for the random contribution problem show similar characteristics. The certainty equivalent for the LSMC is again higher. For both algorithms, the optimal strategy is decreasing over time. For coarse grids, the HJB approach shows deviations from the optimal strategy, but this effect is eliminated when refining the grid. The order of convergence in the random contribution setting is linear for the HJB approach, while the LSMC again is sublinear.

Overall, it can be concluded, that both algorithms deliver very good results for both problems and that the differences between the results are rather small. The LSMC approach leads to slightly higher CE, while the HJB approach yields a stable optimal trading strategy and shows a higher order of convergence. The low order of convergence for the endowment problem may be caused by some specific grid choice. Further, we will also see, that both are capable of handling self-financing as well as non self-financing strategies.

The remainder of the paper is structured as follows. In Section 2 the financial market setting is introduced. Section 3 gives a short introduction to the algorithms. Section 4 discusses the random endowment problem and analyzes the respective convergence results. Section 5 considers the random contribution problem and Section 6 concludes the paper.

2. Financial market and utility functions

Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with a 2-dimension Brownian motion \((W, W^\perp)\), where \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}\) is the natural filtration of the Brownian motion, i.e. \(\mathcal{F}_t = \sigma \left( (W_s, W^\perp_s), \ s \in [0,t] \right)\), and \(T \in (0, \infty)\) is a finite time horizon. We consider
a financial market consisting of one risk-free asset $B$ and one risky asset $S$, whose dynamics are given as follows:

$$\begin{align*}
    dB_t &= rB_t dt,
    & B_0 = 1, \\
    dS_t &= \mu S_t dt + \sigma S_t dW_t,
    & S_0 = s_0.
\end{align*}$$

Here, the interest rate $r$, the drift $\mu > r$ and the volatility $\sigma > 0$ are constant.

Furthermore, an untradable asset exists, whose dynamics are given as a correlated stochastic process $y$ and modelled as follows:

$$dy_t = ay_t dt + by_t \left( \rho dW_t + \sqrt{1 - \rho^2} dW_t^\perp \right), \quad y_0 = \bar{y}_0,$$

where $\rho \in (-1, 1)$ denotes the correlation between the tradable asset $S$ and the untradable asset $y$. The drift $a$ and the volatility $b > 0$ are assumed to be constants. Untradable assets occur in many different situations in finance and insurance, e.g. claims on non-liquid assets or in occupational pensions plans, where a regular contribution is paid to the fund. As we aim to analyze different problems we will not specify the untradable asset any further here. 

For our analysis, we assume that the risk-averse economic agent acts according to a utility function $U$ and is endowed with some initial wealth $x_0$. She aims to maximize her utility over the observation period $[0, T]$. The objective function of the utility maximization problem will be fixed when we come to concrete problems later.

The utility function $U$ used by the agent is power utility, i.e.

$$U(x) = \frac{x^{1-\eta}}{1-\eta}, \text{ with a relative risk-aversion level } \eta > 0, \eta \neq 1.$$ 

Often power utility is also referred to as a constant relative risk aversion (CRRA) utility function.

### 3. Numerical Approach

As we aim to compare the results of a least-squares Monte Carlo algorithm and an HJB scheme we will shortly outline both algorithms. The LSMC algorithm used in this paper was first described by Denault et al. (2017) and is applied to a target date fund setting in Bosserhoff et al. (2021). The two-dimensional HJB algorithm originates from Ma and Forsyth (2016) and an implementation for an indifference pricing framework is found in Chen et al. (2021). For the technical details, we refer the reader to the mentioned papers. We will first specify the optimization problem in the next section when discussing the concrete optimization problem. For the description of the algorithms, it is sufficient to assume, that an indirect utility
function $v$ is given. The value function $v$ depends on the time point $t$, the wealth $X$, the untradable asset $y$ as well as an investment strategy $\pi$.

3.1. Least-squares Monte Carlo. The least-squares Monte Carlo approach was first introduced by Longstaff and Schwartz (2001) for pricing of American style options. The idea is to approximate the conditional expectation of a future payout at a time $t$ by regressing on a chosen set of basis functions. It is often called a simulation and regression approach, a forward simulation is used to simulate the path of the asset and the regression is used to approximate the conditional expectation. Given the regressed solution at a time point, it can be used to calculate the expected value, i.e. the fair price of the American option. While LSMC approaches are found in many different settings, the extension to optimal control theory is first done by Kharroubi et al. (2014). With a randomized control variable and an additional optimization step on the regressed function, they showed the convergence of their regression scheme to the optimal expected utility solution. The algorithm used in this paper was originally discussed by Denault et al. (2017) for different asset allocation problems.

As usual in these kind of problems the algorithm is moving backwards in time starting with terminal time $T$, where the value function equals the utility. The main steps of the algorithm are:

1. Prerequisites: Generation of $n_r$ paths of $X$ and $y$. As no optimal strategy is known yet a forward simulation with a naive strategy is used. In the cases considered in this paper the naive strategy will always invest 100% into the risky asset. Based on this result at each time point a grid of wealth levels is generated. Note that, the strategy is chosen such that the resulting grid covers reasonable wealth levels, see e.g. Bosserhoff et al. (2021) for further details.

2. Regression step: For each time step $t < T$ and each wealth grid node $X$ the value function $v(t + \Delta t, X_t + \Delta t, y_t + \Delta t)$ is determined and regressed on the set $(\pi, y_t)$. Here $\pi$ is a possible strategy which is chosen from a fixed set of $n_\pi$ strategies. The value function $v(t + \Delta t, X_t + \Delta t, y_t + \Delta t)$ is known from the previous time step. A set of regression coefficients $\beta_{t,X}$ is obtained by:

$$
\beta_{t,X} = \arg \min_{\beta \in \mathbb{R}^6} \sum_{j=1}^{n_\pi} \sum_{i=1}^{n_r} \left[ B(\pi_j, y_t^{(i)}) - v(t + \Delta t, X_t^{(i,j)} + \Delta t, y_t^{(i)}) \right]^2,
$$

where $B(\pi, y) := [1, \pi, \pi^2, y, y^2, y\pi]$ is the set of basis functions and the superscript indicates the path dependence. As we use power utility, it is often considered that monomials are a good choice, see e.g. Bosserhoff et al. (2021).
(3) Optimization step: For each of the $n_r$ paths the corresponding regressed function is maximized with respect to $\pi$. The result is not limited to the previously chosen grid.

(4) Before proceeding to the next time step, the value function $v(t, X, y_t)$ is derived based on a new forward simulation using the optimal strategy $\pi^*(t, X, y_t)$.

The output of the algorithm is the following:

- The expected maximal utility $v(0, x_0, y_0)$ is obtained through the backward analysis.
- A set of regression parameters $\beta_{t,X}$. Note that $\beta_{t,X}$ is dependent on the time and wealth grid $(t, X)$.

Note that due to the wealth grid there are no concrete scenarios with optimal forward strategies given. However, optimal paths can be generated by a new forward simulation, where the optimal strategy is derived from the regression parameters at each time step. In particular, the time-0 value function can also be derived from a simple Monte Carlo forward simulation with new paths and the optimal strategy.

3.2. **Finite difference HJB scheme.** The HJB scheme relies on the dynamic programming principle and aims to optimize the HJB PDE at each time step. In comparison to classical PDEs not only do the optimization steps lead to additional complexity to the problem, but also one has to guarantee that the resulting approximated value function convergences to the viscosity solution. A numerical scheme has to fulfill certain requirements to ensure this convergence. More specifically a scheme has to be monotone, point-wise consistent and $l_\infty$-stable, see Barles and Souganidis (1991). It is known that for finite difference schemes a positive coefficient method (PCM) fulfills these conditions. A finite difference scheme is a PCM scheme if it yields a positive definite matrix. For a one-dimensional problem this can be achieved when choosing between forward, backward and central differencing at each node such that all non-diagonal terms are negative, see Wang and Forsyth (2008). For a two-dimensional problem, it is in general not possible to achieve negative coefficients in all non-diagonal nodes when a cross-derivative is present in the HJB equation. To deal with this issue Ma and Forsyth (2016) proposes a local coordinate rotation. The rotation yields a new virtual grid on which the negativity can be ensured.

Different from the LSMC approach, no simulated paths are given and all relevant dimensions $(t, x, y, \pi)$ have a predefined grid. The resulting grid is a tradeoff between numerical accuracy and computational time. When applying the algorithm, we assume that an HJB PDE can be derived from the indirect utility $v$.

The algorithm takes the following steps at each time point:
(1) For each node rotate the coordinate system such that it yields a positive coefficient method. A objective function $F$, which incorporates the finite difference coefficients as well as the corresponding values on our value function is obtained for each grid node $(x, y)$. Note that the rotated grid defines are new virtual grid and for the differencing a predefined mesh discretization parameter $h$ is used, see e.g. Chen et al. (2021) or Ma and Forsyth (2016).

(2) Maximize the function $F$ with respect to the strategy $\pi$. This has to be done by a line search, since the discretized optimal solution may differ from the analytic optimum. Note that the rotation angle depends on the strategy $\pi$ as well as the grid points $(x, y)$. The final output is the maximized finite difference scheme.

(3) Iterate the finite difference scheme until a certain level of convergence is reached and then proceed to the next time step. Here an implicit time stepping is used to guarantee an unconditional convergence.

In particular the second step is very time-consuming.

The output of the algorithm is the following:

- The expected optimal utility $v(0, x_0, y_0)$ obtained through the backward analysis.
- The optimal strategy $\pi^*(t, x, y)$ at each node $(t, x, y)$.

Similar to the LSMC algorithm new forward paths have to be generated to simulate the forward evolution of a scenario. Again this can be solved by a Monte Carlo forward simulation, where the optimal strategy is interpolated between the nodes.

The main difference between both numerical approaches is that the HJB scheme is an analytic approach that approximates the resulting HJB PDE while the LSMC approach is based on simulation and regression, where it is assumed that it converges towards the correct solution due to the law of large numbers. Further, the HJB scheme convergences towards the viscosity solution and it has to be shown it coincides with the value function. In the context of LSMC problems with heteroskedasticity and approximation bias arise, see e.g. Andreasson and Shevchenko (2019) for further discussion on this topic. As we aim for a numerical analysis we will not further discuss these issues.

3.3. Comparison of the algorithms. Both algorithms take a different approach to solving the problems discussed and thus to the best of our knowledge there is no established way of how these can be compared. The HJB algorithm works on fixed grid $(t, x, y, \pi)$ and with each refinement all step sizes are halved and for the LSMC algorithm the time steps and paths generated are doubled with each refinement. Both
algorithms deliver a time-0 utility value \( v(0, x_0, y_0) \) as well as the optimal strategy \( \pi^* \) dependent on \( (t, x, y) \).\(^1\)

For simplicity, we will compare all results using the certainty equivalent (CE) defined by: \( CE = U^{-1}(v_0) \), where \( v_0 \) is a time-0 solution of the value function.

With no information on error estimates and approximation bias, a natural approach is to compare the time-0 value function \( v(0, x_0, y_0) \) and the higher utility value yields a better result. However, if e.g. one algorithm systematically overestimates the result it would yield a higher utility and a wrong conclusion is drawn. In other words, this approach is flawed and a comparison independent of the algorithm used is required.

To this end, we rely on the output for the optimal strategies and generate \( 10^6 \) new Monte Carlo paths using the optimal strategies obtained. At each time point \( t \) the optimal strategy \( \pi^* \) is chosen based on \( (X^*_t, y_t) \). At terminal date, the optimal wealth \( X^*_T \) is given and one can calculate the corresponding time-0 value function \( \tilde{v}_0 \) as well as the corresponding CE. These results are independent of the initial algorithm and thus become comparable. In particular, if an algorithm yields an optimal strategy that is superior to the other approach this leads to a higher utility, i.e. it is a better approximation of the optimization problem. Besides that, this result allows us further to validate the value function obtained from the algorithm.

For the remainder of the paper, we will always refer to the value obtained from the new Monte Carlo simulation as the forward result and denote the value with \( CE^{FW} \). Consequently, the output of the algorithm is the backward result and the corresponding certainty equivalent is denoted by \( CE^{BW} \).

Further the order of convergence of each algorithm is analyzed. To this extent the algorithm is run 4 times and refined with each run, i.e. a sequence of \( (CE^{BW}_k)_{1 \leq k \leq 4} \) is obtained. With each refinement the result \( CE^{BW}_k \) should asymptotically approach the corresponding analytic solution \( CE^a \). The order of convergence is defined by the following equation:

\[
 c^q := \lim_{k \to \infty} c^q_k = \lim_{k \to \infty} \frac{|CE^{BW}_k - CE^a|}{|CE^{BW}_{k-1} - CE^a|^q},
\]  

(2)

where \( q \) is the order of convergence and \( c^q \) is the asymptotic error constant, see e.g. Schatzman (2002). To estimate the order of convergence we will report the result calculating \( c^q_k \) for different candidates \( q \in \{0.5, 1.0, 1.5\} \). This choice is motivated by convergences orders for standard numerical schemes for finite differences \( (q \geq 1) \) and Monte Carlo methods \( (q = 0.5) \), see e.g. Stoer and Bulirsch (2013), Caflisch et al. (1998). The order \( q = 1.0 \) corresponds to linear convergence, while \( q = 0.5 \) and \( q = 1.5 \) yields sub-/super-linear convergence. If a constant \( c^q \) exists such that all

\(^1\)The LSMC does not give the optimal strategy. The strategy is derived from optimizing the regressed function, when talking about the optimal strategy it is assumed that this has been done.
lim_{k \to \infty} c^q_k = c^q\) we conclude that \(q\) is the order of convergence with the asymptotic error constant \(c^q\). See also Chen et al. (2021) for a similar analysis.

However, for the problems considered in the following an analytical solution is not known and we will therefore rely on a reduction of the original problem by one dimension. This allows us to use an one-dimensional algorithm to solve this problem with a much higher accuracy and we use this result as a proxy for the unknown analytical solution \(CE^a\).

4. Problem 1: Random Endowment at maturity \(T\)

4.1. Setting. In the first problem we consider that the agent receives a random endowment \(y_T\) at maturity \(T\). She dynamically invests her wealth in a self-financing portfolio of the risk-free asset and the stock \(S\). Let \(\pi_t\) be the fraction invested in the risky asset at time \(t\) and suppose that the remaining fraction is invested in the risk-free asset. Under the self-financing condition, the agent’s portfolio \(X^{x_0,\pi}\) follows the dynamics

\[
dX^{x_0,\pi}_t = \left( rX^{x_0,\pi}_t + \pi_t X^{x_0,\pi}_t (\mu - r) \right) dt + \pi_t X^{x_0,\pi}_t \sigma dW_t, \quad X_0 = x_0. \tag{3}
\]

To keep the setting general, we assume that the set of admissible self-financing trading strategies is given by

\[
\Pi = \left\{ \pi = (\pi_t)_{t \in [0,T]} : \pi \text{ adapted}, X^{x_0,\pi}_t \geq 0, \int_0^T \pi_s^2 ds < \infty \text{ a.s.} \right\}. \tag{4}
\]

Note that the wealth process at time \(t\) is non-negative a.s. for all \(t \in [0,T]\), i.e. we implicitly assume that \(\pi_t \to 0\) as \(X^{x_0,\pi}_T \to 0\). This assumption is reasonable as in general unrestricted lending and borrowing is not possible/allowed. The random endowment is paid out at maturity date \(T\) to the agent. This kind of product could e.g. be an external fund which is not liquid traded or an endowment insurance contract.

The agent aims to maximize her expected utility and her optimization problem can be stated as follows:

\[
\sup_{\pi_u \in \Pi, \quad s \in [0,T]} E\left[ U\left( X^{x_0,\pi}_T + y_T \right) \right], \tag{5}
\]

s.t. \(dX^{x_0,\pi}_t = \left( rX^{x_0,\pi}_t + \pi_t X^{x_0,\pi}_t (\mu - r) \right) dt + \pi_t X^{x_0,\pi}_t \sigma dW_t, \quad X_0 = x_0,\)

\[
dy_t = ay_t dt + by_t \left( \rho dW_t + \sqrt{1 - \rho^2} dW_t^\perp \right), \quad y_0 = \bar{y}_0.
\]

We point out that no explicit solution to this problem is known. However in the literature an upper bound is found, see e.g. Henderson (2002), Chen et al. (2021):
Proposition 4.1. The expected utility of a risk-averse agent with an initial wealth $x_0$ and a random endowment $y_T$ is bounded by

$$\sup_{\pi_s \in \Pi, s \in [0,T]} \mathbb{E} \left[ U(X_T^{x_0, \pi, y_T}) \right] \leq \sup_{\pi_s \in \Pi, s \in [0,T]} \mathbb{E} \left[ U(X_T^{x_0 + e^{-rT} \mathbb{E}^Q_0[y_T], \pi}) \right] = \frac{(x_0 + e^{-rT} \mathbb{E}^Q_0[y_T])^{1-\eta}}{1-\eta} \exp \left( r + \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} (1 - \eta)T \right),$$

where the right-hand-side is the indirect utility function of the optimal complete problem with initial wealth $x_0 + e^{-rT} \mathbb{E}^Q_0[y_T]$ and $Q_0$ is the minimal martingale measure defined by:

$$\frac{dQ_0}{dP} = \exp \left( -\frac{\mu - r}{\sigma} W_T - \frac{(\mu - r)^2}{2\sigma^2} T \right).$$

This upper bound connects the incomplete problem to a problem in a complete market with no additional asset available.

To solve this problem numerically we take two different approaches. For the first approach we estimate the expected utility (5) and try to optimize the trading strategy directly. For the second approach we aim to maximize the corresponding HJB equation at each time step. To this extent we assume that we have a Markovian setting and define the indirect utility function as follows:

$$v(T - \tau, x, y) := \sup_{\pi_s \in \Pi, s \in [\tau, T]} \mathbb{E}[U(X_T^{x_0, \pi}) | X_T^{x_0} = x, y_T = y].$$

(6)

Note that as numerical algorithms usually move backwards in time, we incorporate this by using $\tau = T - t \in [0, T]$.

By the dynamic programming principle, the indirect utility function is the solution to the following two-dimensional HJB equation

$$\sup_{\pi_s \in \Pi} \left[ -v_x + (rx + (\mu - r)\pi x)v_x + ayv_y + \frac{1}{2}(\sigma^2\pi x)^2v_{xx} + \rho\sigma b\pi xyv_{xy} + \frac{1}{2}b^2y^2v_{yy} \right] = 0,$$

(7)

on the domain $(\tau, x, y) \in [0, T] \times [0, x_{\text{max}}] \times [0, y_{\text{max}}]$. As we aim to solve this numerically we already here state the HJB equation on a bounded domain where $x_{\text{max}}$ and $y_{\text{max}}$ are interpreted as being some maximum wealth/endowment. When $x_{\text{max}}$ and $y_{\text{max}}$ are chosen reasonably high, the error due to this restriction is small in area of interest, see e.g. Ma and Forsyth (2016).

Further, we point out that this specific problem, can be reduced by one dimension by substituting $z := X/y$ and factoring out $y^{1-\eta}$, see e.g. Henderson and Hobson (2002). Without proof the one-dimensional problem can be stated as follows:
Remark 4.2. Set \( z = \frac{X}{y} \) and rewrite the value function to:

\[
v(t, X, y) = y^{1 - \eta} \bar{v} \left( t, \frac{X}{y} \right), \quad \forall y > 0. \tag{8}
\]

Then \( \bar{v}(t, z) \) is a reduced value function, which is the solution of the following HJB equation:

\[
\sup_{\pi \in \Pi} \left\{ -v_{r} + \left( (1 - \eta)a - \frac{1}{2} \eta(1 - \eta)b^2 \right) v + \left( (r - a)z + \eta b^2 z + \pi((\mu - r) - \eta \rho b \sigma) z \right) v_z \\
+ \frac{1}{2} \left( \pi^2 \sigma^2 z^2 + b^2 z^2 - 2 \pi \rho b \sigma z^2 \right) v_{zz} \right\} = 0.
\]

See e.g. Henderson and Hobson (2002) for further details on reduction of this utility function.

For our purpose a major benefit of the reduced problem is, that we can use fast and efficient algorithms to solve this problem with much higher precision in comparison with numerical algorithms which use the corresponding two-dimensional problems directly. More specifically, we can use this reduced result as proxy for the analytic solution.

4.2. Parameter choice. For the numerical analysis the market parameters used are reported in Table 1. As the focus relies on comparing the results from the LSMC and HJB algorithm the parameter choice is purely academic.

<table>
<thead>
<tr>
<th>Market parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = 0.02 )</td>
<td>( \mu = 0.06 )</td>
</tr>
<tr>
<td>( \sigma = 0.2 )</td>
<td>( \rho = 0.1 )</td>
</tr>
<tr>
<td>( a = 0.04 )</td>
<td>( b = 0.1 )</td>
</tr>
<tr>
<td>( \eta = 3 )</td>
<td>( T = 1 )</td>
</tr>
<tr>
<td>( x_0 = 100 )</td>
<td>( \bar{y}_0 = 50 )</td>
</tr>
</tbody>
</table>

Table 1. Parameters for the financial market.

In Table 2 the parameter setting used for LSMC algorithm is stated. The number of grid points \( n_{\pi} \) and \( n_{x} \) for possible trading strategies \( \pi \) and wealth levels \( X \) are fixed. The upper and lower bounds of the trading strategy \( \bar{\pi} \) and \( \bar{\pi} \) are determined by numerical tests to be a reasonable choice. The parameters \( q_1 = q_2 \) denote the quantiles which determine the highest and lowest wealth grid point \( X \) at each time step, see e.g. Bosserhoff et al. (2021) for further discussion. The number of nodes are controlled by \( n_{x} \), which specifies the amount of nodes at time point \( \Delta t \). The parameters \( n_{\pi} \) and \( n_{x} \) are fixed and will not be part of the analyses, as numerical tests show, that their impact on the convergence for the problems considered is negligible and they do not affect the first six decimal digits of the result. The convergence
test will double the time points $N_t$ and number of paths generated $n_r$ with each refinement.

<table>
<thead>
<tr>
<th>Parameters LSMC algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_\pi = 11, \quad n_x = 5, \quad \pi = 0, \quad \bar{\pi} = 1,$</td>
</tr>
<tr>
<td>$q_1 = 0.1, \quad q_2 = 0.1.$</td>
</tr>
</tbody>
</table>

Table 2. Input parameters for the LSMC algorithm.

The parameter set used for the HJB scheme is shown in Table 3. The upper bounds for $x_{max}$ and $y_{max}$ are both four times the initial value. Numerical tests with higher bounds have shown, that a higher bound of $x_{max}$ and $y_{max}$ does not change the result within six digits for the finest grid (see next section). As all dimensions are gridded the step size in each dimension is halved when the grid is refined. In particular, this affects time points $N_t$, the wealth points $N_x$, the points for the untradable asset $N_y$, the step size for the strategy $\Delta \pi$, as well as the mesh discretization parameter $h$.

Further, we point out that the bounds on $\pi$ differ between the LSMC and the HJB approach. This is due to the fact, that the LSMC algorithm has a forward simulation which defines values for $X$ and $y$. The HJB does not have this and takes all possible combinations of $X$ and $y$, which leads to extreme stock positions for very low wealth levels. These positions are not considered in the LSMC as they have an extremely low probability and thus do not occur in the forward simulation. Note that, although this might seem like a drawback from the HJB scheme, the HJB algorithm delivers a result for all combinations of grid points $(X, y)$, while the LSMC algorithm requires a new simulation when the initial values $(x_0, y_0)$ are changed.

<table>
<thead>
<tr>
<th>Parameters 2D HJB algorithm</th>
</tr>
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<tr>
<td>$x_{min} = 0, \quad x_{max} = 400, \quad y_{min} = 0, \quad y_{max} = 200,$</td>
</tr>
<tr>
<td>$\pi_{min} = -2, \quad \pi_{max} = 2.$</td>
</tr>
</tbody>
</table>

Table 3. Input parameters for the HJB scheme.

As this problem can be reduced by one dimension (1D) an algorithm from Wang and Forsyth (2008) is used to derive the solution and optimal strategy for the one-dimensional problem stated in Remark 4.2. The parameters used in this algorithm are stated in Table 4. The term 1D is used when referring to this result.

4.3. Numerical analysis. This section analyzes the results of boths algorithms. For the HJB scheme the implementation used here is discussed in Chen et al. (2021) and can directly be applied to this problem. For the LSMC part the implementation in Bosserhoff et al. (2021) is used. However, some slight adaptions are required:
Parameters 1D HJB algorithm
\(z_{\text{min}} = 0, \quad z_{\text{max}} = 1000, \quad \pi_{\text{min}} = -2, \quad \pi_{\text{max}} = 2.\)
\(\Delta \pi = 10^{-4}, \quad N_z = 240000, \quad \Delta \pi = 10^{-4},\)

Table 4. Input parameters for the PCM 1D scheme.

- For all forward simulations of the wealth process the dynamics of Equation (3) is used.
- As the random endowment is paid out to the agent at the final time point this step has to consider \(y_T\) as well, i.e. the boundary condition at \(t = T\) is \(U(X + y_T)\).

<table>
<thead>
<tr>
<th>(N_t)</th>
<th>(n_r)</th>
<th>(CE^{FW})</th>
<th>(CE^{BW})</th>
</tr>
</thead>
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</table>

Table 5. LSMC convergence result Problem 1.

<table>
<thead>
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<th>(N_x)</th>
<th>(N_y)</th>
<th>(h)</th>
<th>(\Delta \pi)</th>
<th>(CE^{FW})</th>
<th>(CE^{BW})</th>
</tr>
</thead>
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<td>0.02</td>
<td>154.7252</td>
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<tr>
<td>64</td>
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<td>801</td>
<td>0.1</td>
<td>0.01</td>
<td>154.7312</td>
<td>154.7135</td>
</tr>
</tbody>
</table>

Table 6. HJB convergence result Problem 1.

We report the numerical results for the LSMC approach in Table 5 and for the HJB scheme in Table 6. The columns \(CE^{FW}\) and \(CE^{BW}\) state the CE result for the forward and the backward approach.

The LSMC backward result \(CE^{BW}\) shows a monotone increase for the time steps \(N_t = 16, N_t = 32\) and \(N_t = 64\). The result \(N_t = 8\) does not fit to the other results. However, as Monte Carlo schemes first become stable with increasing number of paths, this result is still within normal fluctuations. On the other hand, the LSMC forward result \(CE^{FW}\) does not yield a clear pattern, which again can be traced back to the increased volatility of the results for low number of paths. The results for \(N_t = 32\) and \(N_t = 64\) differ less than \(10^{-3}\) in comparison to their backward results, i.e. a connection between the forward and the backward result is present when the number of paths is sufficiently large.
The backward result for the HJB scheme shows a monotone increase in certainty equivalent. For the forward result, the three finer grids show a monotone behavior. The result for $N_t = 8$ behaves differently due to the coarse grid. Comparing both algorithms one can see, that for the fine grids $N_t = 32$ and $N_t = 64$ the HJB scheme shows a higher $CE^{FW}$ while on the other hand, the $CE^{BW}$ is lower. All values are below the upper bound from Equation (4.1), which we calculate to a CE of 154.9529. In absolute numbers, all values differ less than $10^{-1}$, i.e. the results are comparable and we can already achieve good results on coarse grids.

Further, in Figure 1 we depict the resulting mean optimal strategies of the forward simulation. The LSMC curves seem to fluctuate around an (almost) constant strategy, while the HJB curves convergence to a slightly increasing fraction invested. We point out, that the HJB approach can only result in optimal strategies on its grid, thus the curves for $N_t = 8$ and $N_t = 16$ show a decreasing behavior, while the curves for $N_t = 32$ and $N_t = 64$ are increasing over time. The curve for $N_t = 32$ is decreasing up to $t = 0.2$ due to this grid dependence. Interestingly the LSMC strategy does not show a less volatile optimal strategy with increasing number paths, however, the impact of each time step decreases and overall the $CE$ still shows a convergence behavior. Both curves indicate an optimal fraction invested of around $47 - 48\%$, and the HJB scheme shows an increasing strategy over time. This result can be confirmed by the one-dimensional solution (dashed line in panel (b)). This supports the fact, that the HJB scheme yields the superior trading strategy, which leads to a higher $CE^{FW}$.

Note that in the $N_t = 8$ case the last optimization happens at $t = 0.875$ and thus the curve ends at this time point.
To determine the order of convergence the reduced problem with only one dimension will be utilized. The benefit of having only one dimension is that much fewer operations are needed for the optimization in each time step and thus the scaling potential of a one-dimensional algorithm allows us to take finer grids, which lets us achieve higher levels of accuracy. Further, it has better coverage of possible outcomes. Taking $z = X/y$, then the LSMC scheme considers values within the range $z \in [0.9729, 3.1315]$. The HJB algorithm yields a much wider range of $z \in [0, 625]^3$, but it also only considers $\approx 124000$ different values of $z$.

Taking all this into account and comparing this to the parameter choice of the 1D algorithm in Table 4 we can assume that the resulting $CE_{1D}^{BW}$ is sufficiently good to serve as a proxy for the unknown solution.

For Problem 1 the one-dimensional algorithm yields a result of $CE_{1D}^{BW} = 154.7241$. To analyze the order of convergence see Table 7. For each refinement step $N_t$ we report the estimated value $c^q$ for $q \in \{0.5, 1.0, 1.5\}$ for the LSMC and the HJB algorithm. If the values reported in one column are approximately constant or show some kind of convergence behavior this indicates, that $q$ is the corresponding order of convergence, see e.g. Chen et al. (2021). For the LSMC we can see that for $q = 1.0$ and $q = 1.5$ the value for $N_t = 32$ is always the smallest and thus no convergence is seen. For $q = 0.5$ the values show a monotone decrease from 0.0921 at $N_t = 16$ to 0.0443 at $N_t = 32$ and 0.0443 at $N_t = 64$. From this, we can conclude, that the column $c^{0.5}$ is converging and that the LSMC has the order of convergence $q = 0.5$.

For the HJB scheme, we can see that for $q = 0.5$ all values lie within a range of less than 10$^{-2}$ but also here $N_t = 32$ is the lowest. For $q = 1.0$ and $q = 1.5$ the values show a monotone increase with each refinement, both sequences also show no sign of convergence as the difference is increasing in refinement steps. Although no clear convergence can be seen for $q = 0.5$ all values in the $c^{0.5}$-column are very close to each other and we can assume that the differences are due to minor numerical inaccuracies. Thus also here we conclude that $q = 0.5$ is the order of convergence.

---

3 We excluded the case $y = 0$.
4 Before counting the values $z$ was rounded to 10$^{-4}$.
5 The parameters $n_r, N_x, N_y, h$ and $\Delta \pi$ are the same as in Table 5 and 6.
Taking everything into account we can conclude that, the HJB scheme tends to approximate the optimal strategy slightly better and therefore also leads to a higher $CE^{FW}$. On the other hand, the LSMC approach yields a higher $CE$ in the backward result. Further, we point out that in absolute numbers with a difference of $\approx 10^{-2}$ towards the “optimal” result both approaches yield a very good result for the finest grid. Both algorithms show an order of convergence of $q = 0.5$. The biggest difference seen is the approximation of the optimal strategy and here the HJB approach clearly delivers the more satisfying result.

5. Problem 2: Random contribution

5.1. Setting. In this setting we assume that our agent receives a random continuous income stream into her portfolio. Besides that, she dynamically invests her wealth in a portfolio of the risk-free asset and the stock $S$, i.e. we can write down the wealth process as follows:

$$\begin{align*}
dX^{x_0, y, \pi}_t &= (rX^{x_0, y, \pi}_t + \pi_t X^{x_0, y, \pi}_t(\mu - r))dt + \pi_t X^{x_0, y, \pi}_t \sigma dW_t + y_t dt, \quad X_0 = x_0.
\end{align*}$$

(9)

Note that in this setting the wealth process is not self-financing due to the strictly positive income stream. The set of admissible trading strategies $\Pi$ does not change from Problem 1, i.e. it is given by Equation (4) as well. In particular, we point out that from a numerical perspective the change from a self-financing to a non self-financing strategy is important. An example of these kinds of problems is e.g. a defined contribution (DC) product. The agent pays a share of her salary into the DC plan and aims to maximize her final wealth at retirement date (terminal date). Again we write down the optimization problem:

$$\begin{align*}
\sup_{\pi_s \in \Pi, \ s \in [0, T]} & \mathbb{E}[U(X^{x_0, y, \pi}_T)], \\
\text{s.t.} & \begin{align*}
dX^{x_0, y, \pi}_t &= (rX^{x_0, y, \pi}_t + \pi_t X^{x_0, y, \pi}_t(\mu - r))dt + \pi_t X^{x_0, y, \pi}_t \sigma dW_t + y_t dt, \quad X_0 = x_0, \\
dy_t &= a y_t dt + b y_t \left(p dW_t + \sqrt{1 - \rho^2} dW_t^{\perp}\right), \quad y_0 = y_0.
\end{align*}
\end{align*}$$

(10)

Under the same assumptions as for Problem 1 we derive the HJB equation from the indirect utility function given by:

$$\begin{align*}
v(T - \tau, x, y) := \sup_{\pi_s \in \Pi, \ s \in [0, T]} & \mathbb{E}[U(X^{x_0, y, \pi}_T)|X^{x_0, \pi}_t = x, y_t = y], \\
\text{s.t.} & \begin{align*}
dX^{x_0, y, \pi}_t &= (rX^{x_0, y, \pi}_t + \pi_t X^{x_0, y, \pi}_t(\mu - r))dt + \pi_t X^{x_0, y, \pi}_t \sigma dW_t + y_t dt, \quad X_0 = x_0, \\
dy_t &= a y_t dt + b y_t \left(p dW_t + \sqrt{1 - \rho^2} dW_t^{\perp}\right), \quad y_0 = y_0.
\end{align*}
\end{align*}$$

(11)
By the dynamic programming principle, the indirect utility function is the solution to the following HJB equation

\[
\sup_{\pi \in \Pi} \left[ -v_\tau + (rx + (\mu - r)\pi x + y)v_x + ayv_y + \frac{1}{2}(\sigma x)^2v_{xx} + \rho \sigma b x y v_{xy} + \frac{1}{2}y^2b^2v_{yy} \right] = 0, \\
\]

on the domain \((\tau, x, y) \in [0, T] \times [0, x_{\max}] \times [0, y_{\max}]\). Again, the bounds are chosen for the same reason as before. This problem can be reduced by one dimension as well. We only state the result and refer to Chen et al. (2014) for further details:

**Remark 5.1.** Set \(z = \frac{X}{y}\) and the value function can be rewritten to:

\[
v(t, X, y) = y^{1-\eta} \bar{v} \left( t, \frac{X}{y} \right), \quad \forall y > 0. \tag{13}
\]

The reduced value function \(\bar{v}(t, z)\) is the solution of the following HJB equation:

\[
\sup_{\pi \in \Pi} \left[ -\bar{v}_\tau + \left( (1 - \eta)a - \frac{1}{2}\eta(1 - \eta)b^2 \right) \bar{v} + \frac{1}{2} \left( \pi^2 \sigma^2 z^2 + b^2 z^2 - 2\pi \rho b \sigma z^2 \right) \bar{v}_{zz} + \left( (r - a)z + \eta b^2 z + \pi((\mu - r) - \eta \rho b)z + 1 \right) \bar{v}_z \right] = 0.
\]

5.2. **Numerical analysis.** We now turn to the results of the algorithms. The implementation for the LSMC used here can be found in Bosserhoff et al. (2021) for a target date fund setting, which corresponds to our random contribution problem. The HJB scheme is again taken from Chen et al. (2021), some adaptations are needed:

- The HJB equation (12) is used.
- The boundary condition at \(t = T\) is \(U(X)\).
- The boundary condition at upper boundaries \(X = x_{\max}\) or \(y = y_{\max}\) is:
  
  \[
v(\tau, X, y) = U(e^{c(T-\tau)}(X + (T - \tau)y)) \exp \left( \left( r + \frac{1}{2} \frac{(\mu - r)}{\sigma^2 \eta} \right) (1 - \eta)T \right).\]

This condition corresponds to the assumption, that the current contribution is constant and paid for the remaining time. This choice is a straightforward adaptation of the boundary condition in Chen et al. (2021) for this problem. For further discussion on boundary conditions see e.g. Chen et al. (2021) or Ma and Forsyth (2016).

Again, for the one-dimensional problem an algorithm from Wang and Forsyth (2008) and Chen et al. (2015) is used to derive the solution and optimal strategy for the one-dimensional problem stated in Remark 5.1. The parameters used are the same.
as in Problem 1, i.e. see Section 4.2. The term 1D is used when referring to this result. The same arguments as before can also be applied here.

<table>
<thead>
<tr>
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<th>$CE_{FW}$</th>
<th>$CE_{BW}$</th>
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Table 8. LSMC convergence result Problem 2.

<table>
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<th>$N_x$</th>
<th>$N_y$</th>
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Table 9. HJB convergence result Problem 2.

The numerical results for Problem 2 are listed in Table 8 and 9. For both the LSMC and the HJB case the CE show a monotone increase for the forward as well as the backward approach. Comparing the forward with the backward result of the LSMC approach it only differs by around $10^{-3}$ on the finest grid. For the HJB algorithm on the other hand the difference is around 0.5. Similar to Problem 1, the LSMC yields the higher backward result while the HJB scheme has the higher forward result on the finest grid $N_t = 64$.

Comparing the obtained trading strategies depicted in Figure 2 we see a declining strategy over time. For the LSMC algorithm again the strategies do fluctuate around the possible optimum. The HJB strategies are again close to each other. Only the coarse grid shows a high deviation, which is simply due to the coarse grid step size of 0.08. This effect was not seen in Problem 1 since 0.48 is a grid point. Overall we can draw a similar conclusion as for Problem 1 and the one-dimensional strategy (dashed line in panel (b)) is again supporting the optimal strategy of the HJB scheme.

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$c^{0.5}$</th>
<th>$c^{1.0}$</th>
<th>$c^{1.5}$</th>
<th>$c^{0.5}$</th>
<th>$c^{1.0}$</th>
<th>$c^{1.5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>16</td>
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<td>0.3054</td>
<td>0.8960</td>
<td>1.0149</td>
<td>0.4864</td>
<td>0.2331</td>
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<tr>
<td>32</td>
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<td>0.4961</td>
<td>2.6341</td>
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<td>0.4941</td>
<td>0.3395</td>
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<tr>
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<td>5.7485</td>
<td>0.4922</td>
<td>0.4811</td>
<td>0.4704</td>
</tr>
</tbody>
</table>

Table 10. Order of convergence analysis Problem 2.
The one-dimensional algorithm reports a $CE^{BW}$ of 154.4047, which again is slightly above the backward result for both algorithms. The rate of convergence for the LSMC and the HJB approach is reported in Table 10. For the LSMC algorithm we see that for $q = 1.0$ and $q = 1.5$ the values show a monotone increase and no convergence. For $q = 0.5$ the values differ less that $10^{-1}$, from this we conclude a convergence order of around 0.5 for the LSMC algorithm. For the HJB method the $c^{0.5}$-column show a monotone decrease, while $c^{1.5}$-column show a monotone increase. Further, both columns do not show any convergence behavior. Here the values reported for $q = 1.0$ differ less than $10^{-1}$ and we can conclude that the rate of convergence is approximately of order 1.

Overall, we can conclude that both algorithms deliver good results, but we also point out that the backward result from the HJB scheme is around 0.5 below the expected result. Again the HJB approach show a very good approximation of the optimal strategy while the LSMC show a strategy fluctuating around the decreasing pattern. Further, the HJB approach show a higher order of convergence and is thus the preferred choice.

6. Conclusion

In this paper, we compare an LSMC approach and a numerical HJB scheme for different two-dimensional optimization problems. In comparison with the HJB scheme, the LSMC algorithm yields a slightly higher backward result in all cases. However, when analyzing the trading strategies we see, that indeed the HJB algorithm shows a stable strategy, which for fine grids outperforms the LSMC strategy. For the LSMC approach, we can report an order of convergence of around 0.5. This result is comparable to convergence orders for standard Monte Carlo methods. The order of
convergence for the HJB scheme is ambiguous, Problem 1 shows a convergence order of 0.5, while Problem 2 clearly reports 1.0. However, due to the rotation technique the convergence might not be equally good at all nodes e.g. due to minor approximation errors when interpolating on the grid. We point out that Chen et al. (2021) also concludes an order of 1.0. However, Chen et al. (2021) use a lower risk-aversion and the untradable asset has a much lower impact, as it is only small in comparison to the wealth. These two aspects might also have an impact on the order of convergence and are left for future research. Using the optimal strategy obtained from the algorithms we see an expected structure, i.e. the strategy in Problem 1 is slightly increasing, while in Problem 2 it is decreasing over time. The difference from a numerical point of view was the self-financing strategy in Problem 1 and the non self-financing strategy in Problem 2. Here one can point out, that the HJB scheme in terms of $CE^{BW}$ performs slightly worse for Problem 2. However, taking everything into account both algorithms already deliver very good results and the differences are only small. The higher order of convergence benefits the HJB scheme for fine grids.
References


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