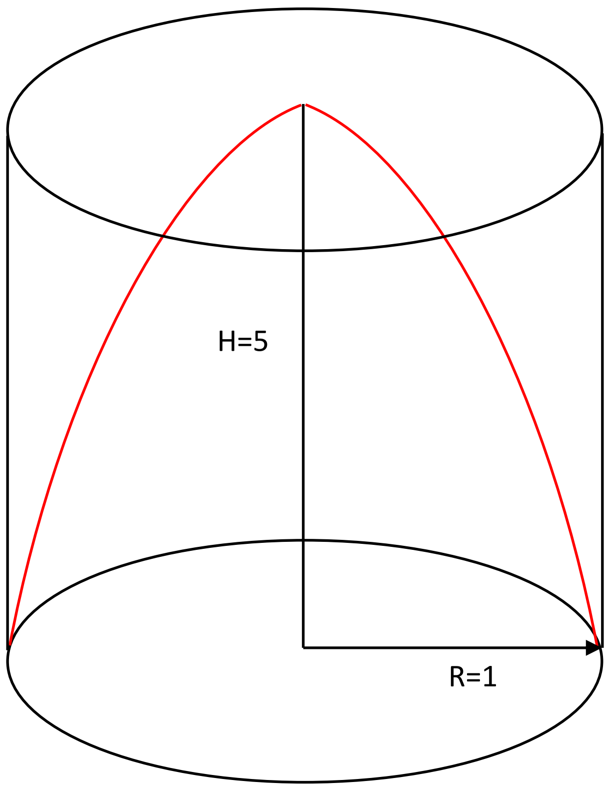
The Volume of a Sugar Loaf



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Content

1. Basics

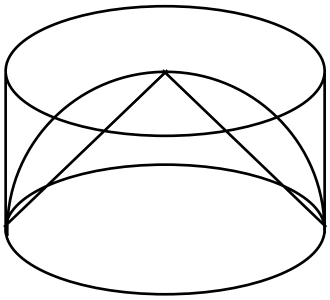
From Archimedes it is known that the size ratio of a cone, a half sphere and a circumscribed cylinder is 1:2:3 when their dimensions are as shown in Fig. 1a (NN1, NN2). But is there a rotationally symmetric body, the volume of which corresponds to half of that of the cylinder? The question is what its shape would be. While the surface line of the cone is quite straight, one can guess that the shape of the unknown body may be that of a sugar loaf. The question is whether one can find a function defining the surface line of this body for any height of the cylinder.

The city coat of arms of the district of Waghaeusel (large county seat Waghaeusel) shows three blue sugar loafs (NN3).

2. The Sugar Loaf

2.1 A Parabola as a Surface Line

Fig. 1b shows a guess of such a body. To first order of approximation it is assumed that a parabola can serve as a surface line.



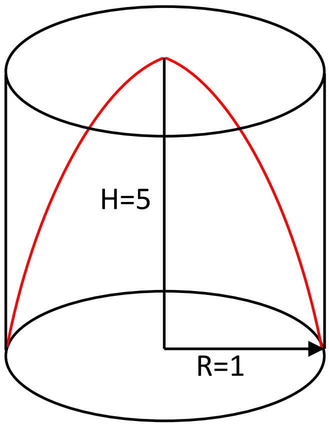


Fig 1 a (left) The size ratio of a cone, a half sphere and the circumscribed cylinder is 1:2:3.
b The presumed shape of a rotational body, its volume being half of that of the cylinder.

Tentatively, for the surface line one can take the function

$$y = R - a \cdot x^2 \qquad \text{EQ 1}$$

For y = 0 at x = H one obtains

$0 = R - a \cdot H^2$; $a \cdot H^2 = R$; $a = \frac{R}{H^2}$

The function then becomes

$$y = R - ax^2 = R - \frac{R}{H^2}x^2 = R\left(1 - \frac{x^2}{H^2}\right) \qquad \text{EQ 2}$$

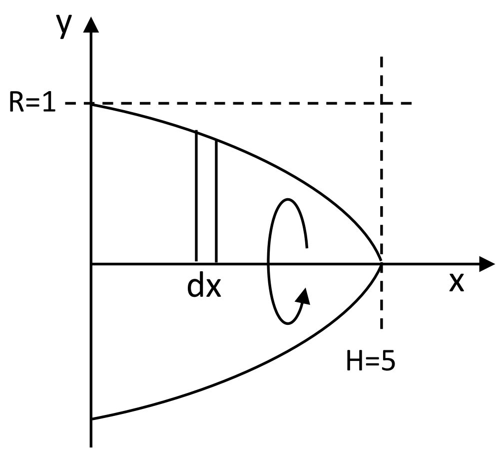


Fig 2 Calculating the volume

Rotation about the x-axis results in

$y^2\pi\,dx = \pi\left[R\left(1-\frac{x^2}{H^2}\right)\right]^2 dx = \pi R^2\left[1-\frac{2x^2}{H^2}+\frac{x^4}{H^4}\right]dx$ EQ 3

The volume of the rotation body is

$V = \int_0^H R^2\pi\left[1-\frac{2x^2}{H^2}+\frac{x^4}{H^4}\right]dx = R^2\pi\left[x-\frac{2\cdot x^3}{3\cdot H^2}+\frac{x^5}{5\cdot H^4}\right]_0^H = R^2\pi\cdot H\left[1-\frac{2}{3}+\frac{1}{5}\right]$ EQ 4

For R=1 und H=5 one obtains $\mathbf{V} = R^2\pi\cdot 5\left[1-\frac{2}{3}+\frac{1}{5}\right] = \mathbf{8.38}$

This is slightly larger than half of the cylinder´s volume which amounts to **7.85** units.

2.2 Modified Surface Line

In order to get a body with a volume exactly half of that of the circumscribed cylinder, one can modify the exponent in Eq 1. In order to reduce the volume, one has to flatten the surface line by choosing an exponent smaller than 2.

With the exponent **κ=1,780 776 41** one gets

$V = \int_0^H R^2\pi\left[1-\frac{2x^\kappa}{H^\kappa}+\frac{x^{2\kappa}}{H^{2\kappa}}\right]dx = R^2\pi\left[x-\frac{2\cdot x^{\kappa+1}}{(\kappa+1)\cdot H^\kappa}+\frac{x^{2\kappa+1}}{(2\kappa+1)\cdot H^{2\kappa}}\right]_0^H =$

$= R^2\pi\left[H-\frac{2\cdot H^{\kappa+1}}{(\kappa+1)\cdot H^\kappa}+\frac{H^{2\kappa+1}}{(2\kappa+1)\cdot H^{2\kappa}}\right] = R^2\pi\cdot H\left[1-\frac{2}{2.780\,776\,41}+\frac{1}{4.561\,552\,82}\right] =$

$= \boldsymbol{R^2\pi\cdot H\cdot 0.5}$ EQ 5

which is, within the accuracy of a pocket calculator, half of the volume of the cylinder.

Table 1 Volume of the sugar loaf as a function of its height. The volume oft he cylinder is twice as large

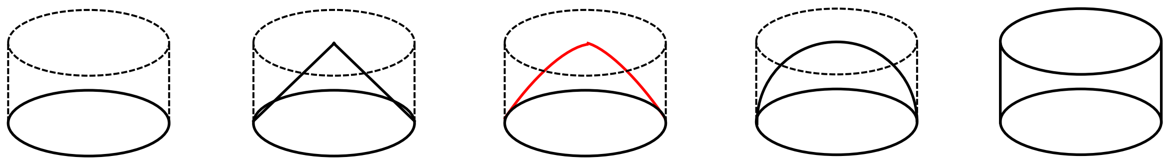
Height of cylinder	Volume of sugar loaf
1 (= radius)	1.570 796 327
2 (diameter)	3.141 592 654
5	7.853 981 634
10	15.707 963 327
30	47.123 889 8
100	157.079 632 7
300	471.238 898
1000	1570.796 327

From EQ 5 with R=H and $V = \frac{R^3\pi}{2}$ one obtains $\frac{R^3\pi}{2} = R^2\pi\left[R - \frac{2\cdot R^{\kappa+1}}{(\kappa+1)\cdot R^\kappa} + \frac{R^{2\kappa+1}}{(2\kappa+1)\cdot R^{2\kappa}}\right]$ and

κ = 1.780776406 / [-0.2807764].

The sugar loaf adds to the number of regular bodies which can be circumscribed by a cylinder with a diameter twice its height. In fractions of the volume of the cylinder there are:

0/6 (empty cylinder), 2/6 (cone), 3/6 (sugar loaf), 4/6 (half sphere) and 6/6 (cylinder).



3. Surface S of the Jacket

From Fig 2 one can calculate the jacket of the rotation body as well (R=1).

$y(x) = R - ax^\kappa$; for y = 0 at x = H: $0 = R - aH^\kappa$ -> $a = \frac{R}{H^\kappa}$

$y(x) = R - \frac{Rx^\kappa}{H^\kappa}$: $y(x) = R\left[1 - \frac{x^\kappa}{H^\kappa}\right]$

$S(x) = \int_0^H 2\,R\pi y dx = 2R^2\pi\int_0^H \left[1 - \frac{x^\kappa}{H^\kappa}\right] dx = 2R^2\pi\left[x - \frac{x^{\kappa+1}}{(\kappa+1)\cdot H^\kappa}\right]_0^H =$

$=2R^2\pi\left[H - \frac{H}{\kappa+1}\right] = 2R^2\pi H\left[1 - \frac{1}{2{,}780\,776\,41}\right]$ = 2πH · **0.640 388 203**. EQ 6

The jacket of the cylinder measures **2πH** surface units.

Table 2 Area of the jacket of the sugar loaf as a function of its height

Height of cylinder	Lateral surface of the suger loaf	Lateral surface of the cylinder
1 (= radius)	4.023 677 752	6.283 185 307
2 (=diameter)	8.047 355 504	12.566 370 61
5	20.118 388 76	31.415 926 54
10	40.236 777 52	62.831 853 07
30	120.710 332 6	188.495 559 2
100	402.367 775 2	628.318 530 7
300	1207.103 326	1884.955 592
1000	4023.677 752	6283.185 307

4. Area of the Cross Section

For R=1 and H=1, the area of the cross section A is

$$A(x) = \int_0^H 2\,y dx \;=\; 2R\int_0^H \left[1 - \frac{x^{\kappa}}{H^{\kappa}}\right] dx \;=\; 2R\left[x - \frac{x^{\kappa+1}}{(\kappa+1)\cdot H^{\kappa}}\right]_0^H = 2R\left[H - \frac{H}{\kappa+1}\right]$$

$$= 2R \cdot H\left[1 - \frac{1}{\kappa+1}\right] = 1.280\ 776\ 407 \qquad \text{EQ 7}$$

For comparison: The cross section of the cylinder measures 2 units of area, that of the half sphere 1.570 796 327 and the cross section of the cone 1 unit.

5. Other Volumes

By variation of the exponent κ of x in EQ 5 the contour line of a rotational solid of any volume between zero and that of a cylinder can be found.

References

NN1.
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NN 2. https://en.wikipedia.org/wiki/Archimedes. Seen on 2021-07-13.

NN3. https://www.waghaeusel.de/startseite/unsere+stadt/wappen+der+stadtteile.html
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