Conservation vs. Dissipation for Weak Solutions in Fluid Dynamics

Vorgelegt von
Ibrokhimbek Akramov

aus Bukhara, Uzbekistan im Jahr 2020

Dissertation zur Erlangung des Doktorgrades Dr. rer. nat. der Fakultät für Mathematik und Wirtschaftswissenschaften der Universität Ulm

Amtierender Dekan:
Prof. Dr. Martin Müller

Gutachter:
Prof. Dr. Emil Wiedemann
Prof. Dr. Rico Zacher

Tag der Promotion:
27. Oktober 2020

Abstract

In this thesis, we consider several problems related to conservation laws of fluid dynamics. It is well-known that for classical sufficiently smooth solutions to time-dependent partial differential equations (PDEs) some quantities such as the energy are conserved, for example, for the Euler equations. However, in physical applications, solutions to the PDEs in connection with fluid dynamics are not always smooth. Therefore, we should consider weak or distributional solutions of the PDEs. Then, it is not clear whether the energy or entropy is conserved for these weak solutions.

We mainly focus our attention on the transport equation and Euler equations in this thesis which contains three main results. First, we consider the transport equation. We show that the transport equation can be renormalised with $|\cdot|^p$, even with $p < 2$. It should be noted that the second derivatives of this quantity can blow up and the well-known methods (e.g. Gwiazda et al. [44]) cannot be applied. We employ methods of complex analysis in order to obtain the conservation laws.

Secondly, we deal with energy conservation for the compressible Euler equations. By using classical commutator methods of Constantin-E-Titi, we obtain sufficient conditions under which the energy is conserved. The main problem of the considered cases is related to the physical interesting case of vacuum. This means that the density ρ can vanish on some set. Then the commutator methods can be applied if ρ satisfies some additional conditions such as quasi-nearly subharmonicity or integrability of $\frac{1}{\rho}$ near vacuum.

Finally by using convex integration methods, we show that if the density ρ satisfies some compatibility condition then the Euler equations admit infinitely many localized solutions. Moreover, the solutions that we generate satisfy the entropy condition in some finite time interval.

Acknowledgements

I would firstly like to thank my supervisor Prof. Dr. Emil Wiedemann for his support, patience and guidance. He was always approachable whenever I had a question about my research or writing. Besides, I want to express my gratitude to Prof. Dr. Rico Zacher for accepting to be the second referee of this thesis and for his useful consultations during my time in Ulm.

I am also grateful to all members of the Institute of Applied Analysis at the Ulm Univsity and all institute staff members of the Applied Mathematics at the Leibniz University Hanover for providing a conducive environment for valuable discussions. Moreover, I would like to send my thanks to Dr. Jack Skipper and Dennis Gallenmüller for very helpful conversations and valuable hints which I have profited a lot during my PhD studies.

Finally, I must express my very profound gratitude to my parents for providing me with continuous support throughout my studies.

Contents

Chapter 1

Introduction

This thesis is concerned with several partial differential equations related to fluid dynamics. The equations are mathematical models consisting essentially of evolution equations such as transport equations and Euler equations.

It can be shown that for the classical solutions (smooth solutions) of these equations some quantities are conserved at least on the physical level. Nevertheless, there are interesting questions whether the solutions, for reasonable general initial conditions, develop singularities in a finite time and what can be said about the long-time behavior of solutions in a distributional sense.

In 1949, Lars Onsager conjectured in his famous work [55] on statistical hydrodynamics that solutions of the incompressible Euler equations conserve kinetic energy if they are Hölder continuous with exponent α greater than $1/3$, whereas solutions with lower regularity might dissipate kinetic energy. This conjecture has been intensively investigated for the last two decades. The initial work towards the first part of the conjecture was by Eyink [34] where he proved for the case $\alpha > \frac{1}{2}$. Later, a complete proof was established by Constantin-E-Titi [20] under slightly weaker regularity requirements on the solution involving a similar exponent $\alpha > \frac{1}{3}$.

The second part of the Onsager conjecture turns out to be much more subtle. This part has been proved by Isett [46] (see also Buckmaster et al.[13]) where Hölder continuous weak solutions with the exponent $\alpha < \frac{1}{3}$ that dissipating energy were constructed by employing the convex integration scheme.

It is appealing to formulate an analogous conjecture for compressible and incompressible models, such as transport equation and Euler equations. Our starting two results are devoted to focus on the sufficient regularity conditions on the weak solution such that the energy conservation law holds.

Other interesting questions are how we select a physically reasonable solution of the Euler equations if the solutions to the Cauchy problem are not unique and whether there exist localized solutions to the compressible Euler equations. In our last result, we prove the existence of infinitely many localized solutions, that dissipate energy, to the compressible Euler equations for some class of density functions.

1.1 Transport Equation

The next chapter of this thesis is concerned with the transport equation

$$\partial_t \theta + u \cdot \nabla \theta = 0, \tag{1.1}$$

which describes the transport of a scalar quantity θ by a divergence-free vector field u. We consider the case of periodic boundary conditions. Let $\mathbb{T}^d$ be the flat torus $[-\pi, \pi]$ with the opposite edges identified by $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$ and consider the vector field $u\colon [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ as given. The scalar field $\theta\colon [0,T] \times \mathbb{T}^d \to \mathbb{R}$ is the unknown, which could be subject to an initial condition $\theta(0, \cdot) = \theta^0$.

The physical interpretation of the transport equations is as follows: The given field u can be thought of as the known velocity field of an incompressible flow, which mathematically forces u to be the divergence-free, for instance water on the surface of the ocean (in which case $d = 2$). The scalar θ then gives the concentration of a chemical dissolved on the ocean surface, and the chemical is transported by the given flow.

Regarding the equation (1.1), let $X(t,x)$ denote the position at time t of a fluid particle that was located at point x at time 0. For this particle trajectory map, we have

$$\dot{X}(t,x) = u(t, X(t,x)).$$

Assume that $\theta(t, X(t,x))$ is constant for all solutions $X(t,x)$ of the latter ordinary differential equations. Then, we have

$$\frac{d}{dt}[\theta(t, X(t,x))] = 0.$$

On the other hand, by using the chain rule, we obtain

$$\frac{d}{dt}\left[\theta(t, X(t,x))\right] = \partial_t \theta(t, X(t,x)) + \dot{X}(t,x) \cdot \nabla \theta(t, X(t,x)).$$

Consequently, combining the above two equations, we obtain the transport equation. So we have roughly derived the transport equation from basic physical assumptions.

1.2 Compressible Euler Equations

Another interesting model that we consider is the isentropic compressible Euler equations

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= 0,\\ \partial_t \rho + \operatorname{div}(\rho u) &= 0.\end{aligned}$$

This nonlinear partial differential equations describes the motion of a compressible inviscid fluid with velocity field $u\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and density $\rho\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, where these two equations correspond to the principle of conservation of momentum and conservation of mass respectively. Here the time T is positive, $u \otimes u$ denotes the matrix with entries $u_i u_j$, and the divergence of $u \otimes u$ is taken row-wise.

Let us assume that the density ρ is independent of time t, i.e. $\rho(t,x) = \rho(0,x) = \rho^0(x) > 0$ and denote $m := \rho u$. Then, the compressible Euler equations can be rewritten as

$$\begin{aligned}\partial_t m + \operatorname{div}_x \left(\frac{m \otimes m}{\rho}\right) + \nabla_x [p(\rho)] &= 0,\\ \operatorname{div}_x m &= 0,\end{aligned} \tag{1.2}$$

Let us consider the Cauchy problem given by $m(0,\cdot) = m^0$. We denote by $X(t,x)$ the position at time t of a fluid particle that was located at point x and time 0, i.e. $X(0,x) = x$. Suppose $m(t,x)$ is a smooth function and $X(t,x)$ satisfies

$$\rho(X(t,x))\dot{X}(t,x) = m(t, X(t,x)). \tag{1.3}$$

Further, Newton's Second Law for this fluid particle dictates

$$\rho(x)\ddot{X}(t,x) = -\nabla p(t, X(t,x))$$

under the absence of external forces. This makes sense since our model is inviscid which indicates that there are no viscosity and effects of friction.

Now if we differentiate the left hand side of equation (1.3) with respect to variable t, then we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\rho(X(t,x))\dot{X}(t,x)\Big) &= \Big(\nabla\rho(X(t,x)), \dot{X}(t,x)\Big)\,\dot{X}(t,x)\\ &\quad + \rho(X(t,x))\ddot{X}(t,x)\\ &= \left(\nabla\rho(X(t,x)), \frac{m(t,X(t,x))}{\rho(X(t,x))}\right)\frac{m(t,X(t,x))}{\rho(X(t,x))}\\ &\quad - \nabla p(t, X(t,x)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{d}{dt}m(t,X(t,x)) &= \frac{\partial m}{\partial t}(t,X(t,x)) + \nabla m(t,X(t,x))\cdot \dot{X}(t,x)\\ &= \frac{\partial m}{\partial t}(t,X(t,x)) + \nabla m(t,X(t,x))\cdot \frac{m(t,X(t,x))}{\rho(X(t,x))}.\end{aligned}$$

Note that since m is divergence-free, we have

$$\operatorname{div}\left(\frac{m\otimes m}{\rho}\right) = \left(\frac{m(t,x)}{\rho(x)},\nabla\right)m(t,x) - \frac{(\nabla\rho(x), m(t,x))}{\rho^2(x)}m(t,x).$$

Combining the last three equations, we obtain the first equation of system (1.2) which shows a derivation of the compressible Euler equations with steady density from basic physical assumptions in a non-rigorous way.

Note that we do not mention boundary conditions for the differential equations that we investigate, as we consider our equations only on the whole space or with periodic boundary conditions.

1.3 DiPerna-Lions vs. Constantin-E-Titi

Here we compare two different methods of commutator estimates for the transport equation which can also be observed in the note [65].

A remarkable property of the transport equation is that it can be renormalized: Let $\beta\colon \mathbb{R}\to\mathbb{R}$ be an arbitrary given C^1 function. Then, equation (1.1) can be multiplied by $\beta'(\theta)$ and by using the chain rule, we deduce

$$\partial_t\beta(\theta) + u\cdot\nabla\beta(\theta) = 0,$$

so that in fact arbitrary functions of a solution become a solution of the same equation. Moreover, integration in space yields

$$\frac{d}{dt}\int_{\mathbb{T}^d}\beta(\theta)\,\mathrm{d}x = 0.$$

Therefore, from this renormalization procedure we obtain infinitely many conserved quantities for equation (1.1). Nevertheless, the chain rule (we used it in the form $\partial_t\beta(\theta) = \beta'(\theta)\partial_t\theta$ and $\nabla\beta(\theta) = \beta'(\theta)\nabla\theta$) is only justified if θ is C^1, or at least Lipschitz continuous. So the natural question arises: Under what conditions on u and/or θ are solutions of the transport equation renormalized?

Before we answer this question, we introduce a distributional concept of solution, using the fact that $u\cdot\nabla\theta = \operatorname{div}(\theta u)$ by virtue of the divergence-free condition on u so that the transport equation makes sense even if θ is not a C^1 function. That is, a function $\theta\in L^\infty([0,T]\times\mathbb{T}^d)$ is called a *weak solution* of (1.1) with initial data $\theta^0 \in L^\infty(\mathbb{T}^d)$ if for every $\varphi \in C_c^1([0,T) \times \mathbb{T}^d)$ we have

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \varphi \theta + \theta \nabla \varphi \cdot u \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{T}^d} \varphi(0,x) \theta^0(x) \, \mathrm{d}x,$$

which makes sense as long as $u \in L^1_{loc}([0,T) \times \mathbb{T}^d)$.

Now we consider the famous DiPerna-Lions method [29] which imposes the regularity requirements only coefficients, in our case those of the transport equation for u. This is helpful when there are stronger a priori estimates for u than for θ. Let θ and u form a weak solution of (1.1), where u is still divergence-free and θ is bounded.

Let $\eta : \mathbb{T}^d \to \mathbb{R}$ be a standard mollifier, that is, a smooth non-negative radially symmetric function with compact support in $B_1(0)$ and

$$\int_{B_1(0)} \eta \, \mathrm{d}x = 1.$$

Set $\eta_\varepsilon(x) = \varepsilon^{-d} \eta(\frac{x}{\varepsilon})$, which is supported on $B_\varepsilon(0)$ and still has unit integral. For a function f, we write $f_\varepsilon := f * \eta_\varepsilon$.

Mollifying (1.1), we obtain (ignoring issues of time differentiability)

$$0 = \partial_t \theta_\varepsilon + \operatorname{div}(\theta u)_\varepsilon = \partial_t \theta_\varepsilon + \operatorname{div}(\theta_\varepsilon u) + R_\varepsilon, \tag{1.4}$$

where $R_\varepsilon = \operatorname{div}(\theta u)_\varepsilon - \operatorname{div}(\theta_\varepsilon u)$ is the *commutator*. Let $\beta \in C^1$ and multiply (1.4) by $\beta'(\theta_\varepsilon)$. Since θ_ε is smooth, we are allowed to use the chain rule to derive

$$\partial_t \beta(\theta_\varepsilon) + \operatorname{div}(\beta(\theta_\varepsilon) u) = -\beta'(\theta_\varepsilon) R_\varepsilon$$

note that u is divergence-free, so we have $\operatorname{div}(\theta_\varepsilon u) = u \cdot \nabla \theta_\varepsilon$. It is clear that the terms on the left hand side converge to $\partial_t \beta(\theta) + \operatorname{div}(\beta(\theta) u)$ in the sense of distributions as $\varepsilon \to 0$. Hence, since $\beta'(\theta_\varepsilon)$ is bounded uniformly in ε, it suffices to show that $R_\varepsilon \to 0$ in $L^1([0,T] \times \mathbb{T}^d)$.

Now, we compute

$$\begin{aligned}
R_\varepsilon(t,x) &= \operatorname{div}(\theta u) * \eta_\varepsilon(x) - \operatorname{div}((\theta * \eta_\varepsilon) u(x)) \\
&= -\theta u * \nabla \eta_\varepsilon(x) + \theta * \nabla \eta_\varepsilon \cdot u(x) \\
&= \int_{B_\varepsilon(x)} \theta(t,y)(u(t,x) - u(t,y)) \cdot \nabla \eta_\varepsilon(x-y) \, \mathrm{d}y \\
&= \varepsilon^{-d-1} \int_{B_\varepsilon(x)} \theta(t,y)(u(t,x) - u(t,y)) \cdot \nabla \eta(\frac{x-y}{\varepsilon}) \, \mathrm{d}y \\
&= -\int_{B_1(0)} \theta(t, x+\varepsilon z) \frac{u(t,x+\varepsilon z) - u(t,x)}{\varepsilon} \cdot \nabla \eta(z) \, \mathrm{d}z,
\end{aligned}$$

here we used the transformation $z = \frac{y-x}{\varepsilon}$. Suppose that $u \in L^1(0,T;W^{1,1}(\mathbb{T}^d))$, then, by standard difference quotient lemmas,

$$\frac{u(x+\varepsilon z,t) - u(t,x)}{\varepsilon}$$

converges to the directional derivative $\partial_z u(x)$ in $L^1((0,T)\times\mathbb{T}^d)$ for fixed z. Moreover, it is bounded in L^1 uniformly in z and ε. Since $\theta(x+\varepsilon z)$ is in L^∞, uniformly in ε and z, and converges in L^1 to $\theta(x)$ for fixed z, we obtain the strong L^1 convergence

$$\begin{aligned} R_\varepsilon(t,x) &\to -\theta(t,x)\int_{B_1(0)} \partial_z u(t,x)\cdot\nabla\eta(z)\,\mathrm{d}z \\ &= -\theta(t,x)\partial_j u_i(t,x)\int_{B_1(0)} z_j\partial_i\eta\,\mathrm{d}z \\ &= \theta(t,x)\delta_{ij}\partial_j u_i(t,x) = \theta(t,x)\operatorname{div} u(t,x) = 0, \end{aligned}$$

as desired.

So if $u \in L^1(0,T;W^{1,1}(\mathbb{T}^d))$, then every bounded weak solution of (1.1) *is renormalised.*

Let us present a commutator argument by Constantin-E-Titi that assume the regularity requirements for the coefficients and the solution itself. This is useful in the presence of advection, when the solution is transported by itself. Once again we mollify equation (1.1) in space,

$$0 = \partial_t\theta_\varepsilon + \operatorname{div}(\theta u)_\varepsilon = \partial_t\theta_\varepsilon + \operatorname{div}(\theta_\varepsilon u_\varepsilon) + S_\varepsilon,$$

with $S_\varepsilon = \operatorname{div}(\theta u)_\varepsilon - \operatorname{div}(\theta_\varepsilon u_\varepsilon)$. Note the only difference compared to (1.4) is that we choose to mollify u as well. Multiplying again by $\beta'(\theta_\varepsilon)$, we obtain (noting that u_ε is still divergence-free)

$$\partial_t\beta(\theta_\varepsilon) + \operatorname{div}(\beta(\theta_\varepsilon)u_\varepsilon) = -\beta'(\theta_\varepsilon)S_\varepsilon,$$

so we obtain renormalization provided we can show the right hand side converges to zero, in the sense of distributions, as $\varepsilon \to 0$.

Now, let $\varphi \in C_c^1((0,T)\times\mathbb{T}^d)$, so that integration by parts yields

$$\begin{aligned} -\int_0^T\int_{\mathbb{T}^d}\varphi\beta'(\theta_\varepsilon)S_\varepsilon\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{T}^d}\beta'(\theta_\varepsilon)\nabla\varphi\cdot((\theta u)_\varepsilon - \theta_\varepsilon u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\int_{\mathbb{T}^d}\varphi\beta''(\theta_\varepsilon)\nabla\theta_\varepsilon\cdot((\theta u)_\varepsilon - \theta_\varepsilon u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

We merely treat the second integral, as the first one is easier. Assume that β'' is bounded, and consider the pointwise identity

$$(\theta u)_\varepsilon - \theta_\varepsilon u_\varepsilon = -(\theta_\varepsilon-\theta)(u_\varepsilon-u) + \int_{B_\varepsilon(0)}\eta_\varepsilon(y)(\theta(\cdot-y)-\theta)(u(\cdot-y)-u)\,\mathrm{d}y.$$

Thus, one part of the desired estimate is obtained by

$$\int_0^T \int_{\mathbb{T}^d} |\nabla\theta_\varepsilon||\theta_\varepsilon - \theta||u_\varepsilon - u| \,\mathrm{d}x \,\mathrm{d}t \leq \|\nabla\theta_\varepsilon\|_{L^p} \|\theta - \theta_\varepsilon\|_{L^p} \|u - u_\varepsilon\|_{L^q}$$

for some exponents satisfying $\frac{2}{p} + \frac{1}{q} \leq 1$. The other part can be estimated in a similar way.

The question arises under what conditions these norms converge to zero as $\varepsilon \to 0$. For this reason, suppose that

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{T}^d} \int_{B_\varepsilon(0)} \frac{|\theta(x) - \theta(x-y)|^p}{\varepsilon^{d+\alpha p}} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t = 0. \tag{1.5}$$

Our aim is, under this assumption, to show $\lim_{\varepsilon\to 0} \varepsilon^{-\alpha} \|\theta - \theta_\varepsilon\|_{L^p} = 0$.

Indeed, using Jensen's inequality and the definition of η_ε,

$$\begin{aligned} \varepsilon^{-\alpha p} \|\theta - \theta_\varepsilon\|_{L^p}^p &= \int_0^T \int_{\mathbb{T}^d} \left| \int_{B_\varepsilon(0)} \frac{\theta(x) - \theta(x-y)}{\varepsilon^\alpha} \eta_\varepsilon(y) \,\mathrm{d}y \right|^p \mathrm{d}x \,\mathrm{d}t \\ &\leq \int_0^T \int_{\mathbb{T}^d} \int_{B_\varepsilon(0)} \frac{|\theta(x) - \theta(x-y)|^p}{\varepsilon^{d+\alpha p}} \eta\left(\frac{y}{\varepsilon}\right) \mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t, \end{aligned}$$

which converges to zero by virtue of assumption (1.5). Likewise, we have $\lim_{\varepsilon\to 0} \varepsilon^{-\beta} \|u - u_\varepsilon\|_{L^q} = 0$, provided

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{T}^d} \int_{B_\varepsilon(0)} \frac{|u(x) - u(x-y)|^q}{\varepsilon^{d+\beta q}} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t = 0. \tag{1.6}$$

Finally, it is not difficult to obtain the estimate $\lim_{\varepsilon\to 0} \varepsilon^{1-\alpha} \|\nabla\theta_\varepsilon\|_{L^p} = 0$. Assume now that $2\alpha + \beta \geq 1$, then

$$\begin{aligned} \|\nabla\theta_\varepsilon\|_{L^p} \|\theta - \theta_\varepsilon\|_{L^p} \|u - u_\varepsilon\|_{L^q} &\leq \varepsilon^{1-\alpha} \|\nabla\theta_\varepsilon\|_{L^p} \varepsilon^{-\alpha} \|\theta - \theta_\varepsilon\|_{L^p} \varepsilon^{-\beta} \|u - u_\varepsilon\|_{L^q} \\ &= \varepsilon^{1-2\alpha-\beta} \|\nabla\theta_\varepsilon\|_{L^p} \|\theta - \theta_\varepsilon\|_{L^p} \|u - u_\varepsilon\|_{L^q} \to 0 \end{aligned}$$

as $\varepsilon \to 0$ and so we arrive at the following result:

Theorem 1.1. *Let* $\frac{2}{p} + \frac{1}{q} \leq 1$ *and* $2\alpha + \beta \geq 1$, *and* $\theta \in L^p((0,T)\times\mathbb{T}^d)$ *and* $u \in L^q((0,T)\times\mathbb{T}^d)$ *be a weak solution of* (1.1) *and* (1.6). *If* $\beta \in W^{2,\infty}(\mathbb{R})$, *then the renormalized equation holds in the sense of distributions:*

$$\partial_t\beta(\theta) + u \cdot \nabla\beta(\theta) = 0.$$

Note that the latter result is also proved in the next chapter, nonetheless, in a slightly more restrictive functional framework. Besides, our result relaxes the assumption $\beta \in W^{2,\infty}(\mathbb{R})$.

1.4 Results and Structure of the Thesis

The thesis is concerned with various results of quasilinear partial differential equations. The results we obtain can be split into three parts, that is, the first two results are about renormalization and energy conservation for the transport equation and the compressible Euler equations respectively, whereas aim of the last result is to show non-uniqueness of admissible weak solutions for the compressible Euler equations where energy conservation fails globally. The composition of the thesis is as follows:

In Chapter 2, we present the renormalization of the transport equation and its application to active scalar equations [This is a published result, see [4]]. As we saw in the previous section, for smooth u and θ this renormalization property follows immediately from the chain rule. In a sufficiently rough setting, nevertheless, the chain rule need no longer apply, see [1, 28, 19, 5, 6, 22, 23]. We analyzed the problem of renormalization of the transport equation (1.1) with $\beta = |\cdot|^p$ and investigated the question how regular a weak solution needs to be to guarantee the conservation of every L^p norm. Whereas the classical DiPerna-Lions theory gives sufficient conditions in terms of the regularity of the coefficients, with no regularity requirement on the transported scalar, we give here sufficient conditions in terms of the combined regularities of the coefficients and the scalar. In particular, we have the following result which is a consequence of Theorem 2.1:

Theorem 1.2. *Let θ, u solve the transport equation in the sense of distributions, where $\theta \in L^\infty(0,T;\dot{B}^{\alpha}_{3,\infty}(\mathbb{T}^d))$ and $u \in L^\infty(0,T;\dot{B}^{\beta}_{3,\infty}(\mathbb{T}^d))$ with $2\alpha+\beta>1$. Then,*

$$\partial_t|\theta|^p + u\cdot\nabla|\theta|^p = 0$$

in the sense of distributions, for every $p \geq 1$ for which the equation makes sense.

In order to show that, roughly speaking, we use commutator estimates similar to those of Constantin-E-Titi in the context of Onsager's conjecture in the small neighborhood of $p>2$ in Section 2.4. Then we extend this small region, where we have this property, to a region (see Section 2.3) where the equation makes sense by using a uniqueness theorem from Complex Analysis. We will also apply this result to an active scalar equation (cf. Theorem 2.2) which is a transport equation

$$\partial_t\theta + u\cdot\nabla\theta = 0 \quad \text{on } \mathbb{T}^d,$$

together with a nonlocal coupling of the form $u = \mathcal{T}[\theta]$, where $\mathcal{T}$ is a Fourier multiplier operator of order zero.

Corollary 1.3. *Let* $\theta \in L^\infty(0,T;\dot{B}^{\alpha}_{3,\infty}(\mathbb{T}^d))$ *be a solution of an active scalar equation, where we assume* $\mathcal{T}$ *to be an* L^3*-multiplier operator and* $\alpha > \frac{1}{3}$*. Then,*

$$\partial_t|\theta|^p + \mathcal{T}[\theta]\cdot\nabla|\theta|^p = 0$$

in the sense of distributions, for every $p \geq 1$ *for which the equation makes sense.*

Chapter 3 consists of some parts of the published result [3], where we consider the compressible isentropic Euler equations on $[0,T]\times\mathbb{T}^d$ with a pressure law $p \in C^{1,\gamma-1}$, where $1 \leq \gamma < 2$. This includes all physically relevant cases, e.g. the monoatomic gas. We investigate under what conditions on its regularity a weak solution conserves the energy. Previous results (e.g. see Feireisl et al. [36]) have crucially assumed that $p \in C^2$ in the range of the density, however, for realistic pressure laws this means that we must exclude the vacuum case. Here we improve these results by giving a number of sufficient conditions for the conservation of energy, even for solutions that may exhibit vacuum. One of the main results is following (cf. Theorem 3.4):

Theorem 1.4. *Let* ρ*,* u *be a solution of* (3.1) *in the sense of distributions. Assume*

$$\begin{aligned}
&u \in B^{\alpha,\infty}_p((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B^{\beta,\infty}_q((0,T)\times\mathbb{T}^d),\\
&0 \leq \underline{\rho} \leq \rho \leq \overline{\rho} \ \textit{a.e. in}\ (0,T)\times\mathbb{T}^d,
\end{aligned}$$

for some constants $\underline{\varrho}$*,* $\overline{\varrho}$ *and* $0 \leq \alpha, \beta \leq 1$ *such that,*

$$\frac{1}{p}+\frac{2}{q} \leq 1, \qquad \frac{2}{p}+\frac{1}{q} \leq 1, \qquad p,q \geq 2, \qquad \alpha+\gamma\beta > 1 \qquad \textit{and} \qquad 2\alpha+\beta > 1.$$

Define $\mathcal{B}_{\varepsilon^\beta} := \{x : 0 < \rho^\varepsilon(x) < \varepsilon^\beta \text{ and } \rho \neq 0\}$ *and assume that*

$$\left\|\frac{\rho^\varepsilon - \rho}{\rho^\varepsilon}\right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})} \leq C(\rho),$$

where C *does not depend on* ε*. Assume further that* $p \in C^{1,(\gamma-1)}([\underline{\rho},\overline{\rho}])$*, and, in addition*

$$p'(0) = 0 \quad \textit{as soon as} \quad \underline{\rho} = 0.$$

Then the energy is locally conserved, i.e.

$$\partial_t\left(\frac{1}{2}\rho|u|^2 + P(\rho)\right) + \operatorname{div}\left[\left(\frac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\right)u\right] = 0$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$*.*

The main idea of the proof is to split the domain $(0,T) \times \mathbb{T}^d$ into three sets: the first set where the density ρ is zero; the second set is a set where ρ is close to zero, that is, $\mathcal{B}_{\varepsilon^\beta}$ defined as in the above theorem; the third set is the rest. The problem in the first set and third set can be easily solved by using preexisting results as in the paper [36]. However, we needed to assume an extra condition for ρ near zero (see Section 3.4). Then we observe that this condition can be replaced, firstly, by assuming the velocity to be a divergence-measure field (Section 3.3); secondly, imposing extra integrability on $1/\rho$ near a vacuum; thirdly, assuming ρ to be quasi-nearly subharmonic near a vacuum; and finally, by assuming that u and ρ are Hölder continuous (Section 3.4).

The purpose of Chapter 4 is to consider the existence and non-uniqueness of weak solutions to the compressible isentropic Euler equations with a steady density. To be more precise, it is concerned with the existence of compactly supported admissible solutions to the Cauchy problem for these equations. In particular, we have (cf. Theorem 4.2):

Theorem 1.5. *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a nonempty bounded open set. Assume that $\rho^0 \in C^1(\mathbb{R}^d)$ satisfies*

- $\rho^0(x) > 0$ *for any* $x \in \mathbb{R}^d$,
- $\rho^0(x) = \overline{\rho}$ *for* $x \in \mathbb{R}^d \backslash \Omega$.

Let $p \in C^1$ be a given function satisfying

$$\int_\Omega p(\rho(x))\,\mathrm{d}x = p(\overline{\rho})|\Omega|.$$

Then there exist $\Omega' \supset \Omega$, m^0 and a maximal time $\overline{T} > 0$ for which there exist infinitely many m such that $\operatorname{supp} m(t,\cdot) \subset \Omega'$ for $t \in [0,\overline{T})$ and (ρ, m) is an admissible solution of (1.2) on $[0,\overline{T}) \times \mathbb{R}^d$ with density $\rho(x) = \rho^0(x)$.

So as to prove this result we use convex integration techniques developed by De Lellis-Székelyhidi [27, 26] and Chiodaroli [17]. In the paper [27], its authors obtained infinitely many compactly supported weak solutions generated by localized initial conditions, where the density was a piecewise constant in space and independent of time. More precisely, they looked at the compressible isentropic system as a "piecewise incompressible" system and they used the result for the incompressible Euler equations to construct bounded initial density and bounded compactly supported initial momenta for which compactly supported solutions are not unique in the case $d \geq 2$.

The result developed by Chiodaroli [17] considered the case when steady density ρ is Hölder continuous in a periodic setting. Then, Chiodaroli [17] constructed for this initial periodic density infinitely many admissible weak solutions to the compressible Euler system. However, it is worth mentioning that the space periodic case is rarely an achievable physical phenomenon. Therefore, it is interesting to construct suitable initial data for, at least, steady and space non-periodic density functions for which the corresponding equations admit infinitely many weak solutions. For this reason we consider a positive density which is constant outside some bounded domain and further we assume that the function $p(\rho)$ satisfies some compatibility integral condition. Under these assumptions by using the convex integration method, we were able to construct infinitely many localized solutions to the compressible Euler equations.

It should be noted that in the paper (see p. 253, De Lellis-Székelyhidi [27]) the constants defining the pressure satisfy some condition for which they generate infinitely many localized solutions to the equations. Our compatibility integral assumption imposed to the pressure exactly corresponds to the condition to the constants demanded by De Lellis-Székelyhidi.

Very roughly, the idea goes like this, first of all, we reformulate the compressible Euler equations with steady density as the combination of the highly underdetermined linear system of partial differential equations

$$\begin{aligned} \operatorname{div}_x m &= 0, \\ \partial_t m + \operatorname{div}_x U + \nabla_x q &= 0 \end{aligned} \tag{1.7}$$

and the nonlinear pointwise constraints

$$U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{d\rho} I_d \quad \text{and} \quad q = p(\rho) + \frac{|m|^2}{d\rho} \tag{1.8}$$

so that U is a trace-free symmetric matrix. Given a triple (ρ, m, U) satisfies (1.7) for some pressure field q, we define the "generalized energy density" as

$$e(\rho, m, U) := \lambda_{\max}\left(\frac{m \otimes m}{\rho} - U\right),$$

where $\lambda_{\max}$ denotes the largest eigenvalue. It is useful to note that the generalized energy density coincides with $\frac{|m|^2}{d\rho}$ if and only if ρ, m and U satisfy (1.8). Otherwise the generalized energy density is strictly greater than $\frac{|m|^2}{d\rho}$, we state this kind of facts and notions in Section 4.3.

If $\chi(t)$ is a given smooth function, a subsolution with respect to the initial value m_0, χ and ρ_0 is defined as a field m where (m, U) solves (1.7) for $q = p(\rho_0) + \frac{\chi(t)}{d}$ and such that $m(0, \cdot) = m_0$ and $e(\rho_0(x), m(t,x), U(t,x)) < \frac{\chi(t)}{d}$ for almost every (a.e.) $x \in \mathbb{R}^d$ and $t \in (0, T)$. If X is the space of momentum fields, the value

$$\int_{\Omega'}\int_0^T \rho_0(x)\chi(t)dt\,\mathrm{d}x - \|m\|^2_{L^2([0,T]\times\Omega')}$$

on X is non-positive and will be zero if and only if (ρ_0, m) is a weak solution of the compressible Euler system with initial data and $|m(t,x)|^2 = \rho_0(x)\chi(t)\mathbb{1}_{\Omega'}$. Therefore, in order to obtain such a solution, we start with some subsolution which we construct in Section 4.6 and add highly oscillatory perturbations to this subsolution so that we will obtain a sequence for which this value tends to zero.

One of the main challenges of this scheme is that one has to make sure that after each perturbation, the subsolution is again in X so at each step of the construction one adds a highly oscillatory correction whose frequency is much larger and whose amplitude is much smaller than those of the previous corrections (Section 4.5).

Another challenge in our work is that the starting point of the construction is a subsolution with respect to the desired initial data and $\chi(t)$. To be more explicit, in order to prove the existence of infinitely many weak solutions with compact support, one initially needs to exhibit a suitable subsolution. In previous research by Wiedemann [66] and Chiodaroli [17], which involves the use of the Riesz transform to find an appropriate subsolution. Nevertheless, the Riesz transform is not local which means in our case that it does not preserve the support of functions. For this reason, under the compatibility integral condition, we consider localized solutions to a divergence type equation. Next, by using the obtained localized solutions we construct the suitable initial subsolution on non-periodic settings in contrary to the preexisting results [17, 66]. Moreover, we show that it is necessary to impose an extra compatibility condition for the pressure $p(\rho)$ to ensure the suitable initial subsolution in the two dimensional setting (Section 4.2).

One of the main aspects of our weak solutions is an admissibility property. Our admissibility condition is defined by the following *energy inequality*

$$\partial_t\left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho}\right) + \mathrm{div}_x\left[\left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho}\right)m\right] \le 0$$

(see Section 4.1). So an *admissible weak solution* refers to a weak solution satisfying the above energy inequality in the sense of distributions. This property of the solutions is ensured by the freedom to choose the smooth function χ to be sufficiently decaying and give the bound $\overline{T}$ for time t (Section 4.7).

In the conclusion, we sum up the main results of the thesis and outline future perspectives.

Chapter 2

Renormalization of Active Scalar Equations

Aim of this chapter is to investigate some renormalization properties of the transport equation and its application to active scalar equations. It consists of Sections 1, 2, 3, 4, 5 of the published result [4] with minor changes.

2.1 Introduction

Let $u : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ be a divergence-free vector field. The transport equation

$$\partial_t \theta + u \cdot \nabla \theta = 0 \tag{2.1}$$

is a fundamental ingredient in many mathematical models in fluid mechanics, for instance in the two-dimensional Euler equations (where the vorticity is transported by the flow), the surface quasi-geostrophic (SQG) and incompressible porous media (IPM) equations, which belong to the class of active scalar equations, or (when the velocity is not necessarily divergence-free) in compressible fluid dynamics, where the continuity equation represents the conservation of mass.

It is convenient to assume in addition

$$\int_{\mathbb{T}^d} u \,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^d} \theta \,\mathrm{d}x = 0, \tag{2.2}$$

which allows us to work in homogeneous function spaces.

The transport equation comes, at least formally, with infinitely many conserved quantities: Indeed, if $\beta : \mathbb{R} \to \mathbb{R}$ is any smooth function, then multiplication of (2.1) with $\beta'(\theta)$ and application of the chain rule yields

$$\partial_t \beta(\theta) + u \cdot \nabla \beta(\theta) = 0. \tag{2.3}$$

Passing from the original equation (2.1) to equation (2.3) has been known (since the seminal work [29]) as *renormalization*. It is an essential technique e.g. in the analysis of the Boltzmann equation [30] and in the theory of weak solutions of the compressible Navier-Stokes equations [52, 37].

When θ is not Lipschitz, however, the use of the chain rule is no longer justified, and many counterexamples to renormalization have been discovered [1, 28, 19, 5, 6, 22, 23].

The situation is comparable for other classes of partial differential equations: For scalar conservation laws of the form

$$\partial_t u + \operatorname{div}_x F(u) = 0,$$

if β is a smooth function, then multiplication with $\beta'(u)$ formally gives the companion law

$$\partial_t \beta(u) + \operatorname{div}_x q(u) = 0, \tag{2.4}$$

for q satisfying $q' = \beta' F'$. If β is convex, it is called an entropy, and q is the corresponding entropy flux. Also for this class of equations, the existence of nonsmooth weak solutions that fail to satisfy (2.4) is classically known (shocks).

Systems of equations typically also admit companion laws, but not an infinite number of them. For instance, the incompressible Euler equations formally satisfy the local energy equality

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}_x \left[\left(\frac{|u|^2}{2} + p \right) u \right] = 0,$$

and this equality may be violated for irregular weak solutions [61, 62].

It has therefore been of great interest to provide sufficient conditions, in terms of the regularity of solutions, that ensure the conservation of formally conserved quantities in each of the mentioned cases. One can broadly distinguish two different approaches to this problem: The theory of DiPerna-Lions for transport and continuity equations gives a Sobolev regularity condition on the transporting field u such that every solution θ of (2.1) is renormalized, even when θ itself has no regularity at all. Results of this type are extremely useful in cases where there are a priori estimates on ∇u, but not on $\nabla\theta$, as it occurs e.g. for the compressible Navier-Stokes system, or the two-dimensional Euler equations [35, 53] with vorticity in L^p.

On the other hand, in the presence of advection, the solution is transported (vaguely speaking) by itself, or by a field with similar regularity as itself. This is the case, for instance, for Euler and Navier-Stokes models, and for active scalar equations. Then one should evenly split the required regularity between the quantities appearing in the advective term. In the case of the incompressible Euler equations, this leads to the famous

Onsager exponent 1/3, and the prototypical result of this kind is in the work of Constantin-E-Titi [20] (see also Eyink [34]). There it is shown that weak solutions of the Euler equations conserve energy provided they possess fractional Besov differentiability of order greater than 1/3.

The method of Constantin-E-Titi has been refined and extended. In particular, the Besov condition has been optimized [33, 14, 39], density-dependent systems like the compressible Euler and Navier-Stokes systems have been considered [51, 36, 67, 31], general systems of conservation laws have been treated [44], and boundary effects have been taken into account [59, 60, 8, 10, 32].

Here, we consider the problem of renormalization of the transport equation (2.1) with $\beta = |\cdot|^p$, and we prove:

Theorem 2.1. *Let u be weakly divergence-free and θ a weak solution of* (2.1)*. Suppose $\theta \in L^{p_1}(0,T;\dot{B}_p^{\alpha,\infty}(\mathbb{T}^d))$ and $u \in L^{q_1}(0,T;\dot{B}_q^{\beta,\infty}(\mathbb{T}^d))$, where*

- $2\alpha + \beta > 1$;
- $\frac{2}{p} + \frac{1}{q} = 1$;
- $2 < p_1 \le \frac{pd}{(d-p\alpha)_+}, \quad 1 \le q_1 \le \frac{qd}{(d-q\beta)_+}$;
- $\frac{2}{p_1} + \frac{1}{q_1} < 1$.

Then, we have

$$\partial_t|\theta|^r + u\cdot\nabla|\theta|^r = 0$$

for every $r \ge 1$ for which $|\theta|^r u$ is locally integrable.

See the next section for definitions of the relevant function spaces. A weak solution of (2.1) is defined, as usual, in the sense of distributions, i.e. we require

$$\int_0^T \int_{\mathbb{T}^d} (\partial_t\varphi + u\cdot\nabla\varphi)\theta \,\mathrm{d}x\,\mathrm{d}t = 0$$

for every test function $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^d)$. The maximal value of r can be explicity determined in terms of $p, p_1, q, q_1, \alpha, \beta$, see Lemma (2.8) below.

The main new difficulty compared to the mentioned previous works consists of the fact that the term $|\theta|^{p-2}$, which will now appear in the commutator, is possibly unbounded when θ assumes values close to zero (when $p < 2$) or close to ∞ (when $p > 2$). This difficulty corresponds to the observation that the function $|\cdot|^p$ is not C^2 in the range of θ, and this is precisely the reason why our case is not covered by the general theory of Gwiazda et al. [44].

We specialise to the case of an active scalar equation. Such an active scalar equation is (2.1) together with a nonlocal coupling

$$u = \mathcal{T}[\theta]; \tag{2.5}$$

Here the operator $\mathcal{T}$ is represented in frequency space by a Fourier multiplier:

$$\hat{u}(\xi) = \widehat{\mathcal{T}[\theta]}(\xi) = m(\xi)\widehat{\theta}(\xi), \quad \xi \in \mathbb{Z}^d \backslash \{0\}.$$

We assume that m is an L^p multiplier (see [42, 43]). This class of equations includes, in particular, the SQG and IPM equations. Then we have:

Theorem 2.2. *Let θ be a weak solution of the active scalar equation* (2.1) *together with the coupling* (2.5). *Assume that $m(\xi)$ is an L^3 multiplier and $\theta \in L^{p_1}([0,T], \dot{B}_3^{\alpha,\infty}(\mathbb{T}^d))$ for $\frac{1}{3} < \alpha < 1$ and $p_1 \leq \frac{3d}{d-3\alpha}$. Then, we have*

$$\partial_t|\theta|^r + \mathcal{T}[\theta] \cdot \nabla|\theta|^r = 0$$

for every $r \geq 1$ for which θ is locally in $L^{r+1}((0,T);\mathbb{T}^d)$.

Note that the above results have been already mentioned in [4]. It is known that, at low regularity, the method of convex integration is applicable to certain active scalar equations, and yields non-renormalized weak solutions; see, for instance, [21, 64, 63, 48, 12]. The expected optimal exponent 1/3 has not yet been reached in the construction of non-renormalized solutions.

As a final remark, let us point out that an alternative (and arguably slightly simpler) proof than the one given here would proceed by an approximation argument, replacing $|\cdot|^p$ by a uniformly C^2 function. The advantage of our proof, however, is that it mostly does not depend on the existence of infinitely many conserved quantities (except of course for the analyticity argument), and is thus transferable to other problems.

2.2 Preliminaries

We derive some estimates in Besov spaces, which will be crucial later on. Throughout this note we denote by $\mathbb{T}^d$ the d-dimensional torus, i.e. the cube $[-\pi,\pi]^d$ with the opposite edges identified: $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$ and functions defined on the torus will thus be identified with periodic functions defined on $\mathbb{R}^d$. Next, we introduce a standard mollifier and its rescalings: Let η be a smooth function defined on $\mathbb{R}^d$ satisfying the conditions $\eta \in C^\infty(\mathbb{R}^d)$, $\eta \geq 0$,

$$\eta(x) = \begin{cases} c & \text{for} \quad |x| \leq \frac{1}{2}, \\ 0 & \text{for} \quad |x| \geq 1, \end{cases}$$

and

$$\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1.$$

Furthermore, define

$$\tilde{\eta}^\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right),$$

then clearly

$$\int_{\mathbb{R}^d} \tilde{\eta}^\varepsilon(x)\,\mathrm{d}x = 1$$

and $\mathrm{supp}(\tilde{\eta}^\varepsilon) \subset \overline{B_\varepsilon(0)} = \{|x| \leq \varepsilon\}$. We denote by η^ε the natural periodic continuation of the function $\tilde{\eta}^\varepsilon$ to $\mathbb{R}^d$. So, η^ε can be considered as it is defined on $\mathbb{T}^d$. Then for an integrable function f defined on $\Omega := [0,T] \times \mathbb{T}^d$ and for $(t,x) \in \Omega$ we denote the space mollification

$$f^\varepsilon(t,x) := (\eta^\varepsilon * f)(t,x) := \int_{B_\varepsilon(0)} \eta^\varepsilon(y) f(t, x-y) dy.$$

Obviously, f^ε is a space-periodic function defined on $[0,T] \times \mathbb{R}^d$. Thus, it can be considered as a function on Ω. Moreover, $f^\varepsilon(t,\cdot) \in C^\infty(\mathbb{T}^d)$ for a.e. $t \in [0,T]$. The following relations are obvious and well-known:

$$\begin{aligned}
&\theta^\varepsilon = \eta^\varepsilon * \theta,\\
&\mathcal{T}[\theta^\varepsilon] = \mathcal{T}[\theta]^\varepsilon,\\
&(\partial u)^\varepsilon = \partial u^\varepsilon,\\
&(\mathrm{div}_x u)^\varepsilon = \mathrm{div}_x u^\varepsilon.
\end{aligned}$$

We briefly recall some properties of Besov spaces and their equivalent definition in terms of the Littlewood-Paley decomposition. First, by assumption (2.2), for our purposes it suffices to work in the homogeneous Besov space $\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)$ ($0 < \alpha < 1$, $1 \leq p \leq \infty$), which consists of those functions $f \in L^p(\mathbb{T}^d)$ that have mean zero, and for which the norm

$$\|f\|_{\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)} = \sup_{y \in \mathbb{T}^d} \frac{\|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{T}^d)}}{|\xi|^\alpha}$$

is finite. An equivalent definition of the Besov space can be given in Fourier space in the following way: Let $\omega : \mathbb{R}^d \to \mathbb{R}$, $0 \leq \omega \leq 1$, be a smooth function such that

$$\omega(\xi) = \begin{cases} 1 & \text{for} \quad |\xi| \leq \frac{1}{2} \\ 0 & \text{for} \quad |\xi| \geq 1. \end{cases}$$

We consider a partition of unity given by ω and $\varphi(\xi) = \omega(\xi/2) - \omega(\xi)$:

$$\omega(\xi) + \sum_{\nu=0}^{\infty} \varphi(\lambda_\nu^{-1}\xi) = 1,$$

where $\lambda_\nu := 2^\nu$. The Littlewood-Paley decomposition of a function f with zero mean is then given as $f = \sum_0^\infty \Delta_k f$, where

$$\Delta_k f := \mathcal{F}^{-1}\left(\varphi\left(\lambda_k^{-1}\cdot\right)\hat{f}\right),$$

with the Fourier transform

$$\hat{f}(\xi) := \int_{\mathbb{T}^d} e^{-2\pi i(\xi,x)} f(x)\,\mathrm{d}x$$

and the inverse Fourier transform

$$\mathcal{F}^{-1}(\hat{f})(x) := \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}} e^{2\pi i(\xi,x)}\hat{f}(\xi) = f(x).$$

The Besov norm can then be equivalently characterized as

$$\|f\|_{\dot{B}_p^{\alpha,\infty}} = \sup_{k\geq 0} 2^{k\alpha}\|\Delta_k f\|_{L^p}$$

for $\alpha \in \mathbb{R}$, $1 \leq p \leq \infty$.

Let us recall the embedding theorem for Besov spaces from [7], Proposition 2.20, p. 64:

Proposition 2.3. *Let $1 \leq p_1 \leq p_2 \leq \infty$. Then, the space $\dot{B}_{p_1}^{\alpha,\infty}$ is continuously embedded in $\dot{B}_{p_2}^{\alpha-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right),\infty}$.*

Lemma 2.4. *Let $\theta \in L^p(\mathbb{T}^d)$ and $p \geq s-1 \geq 1$, then the inequality*

$$\left\||\theta^\varepsilon|^{s-1}\right\|_{L^p(\mathbb{T}^d)} \leq C(q)\varepsilon^{-\frac{d(s-2)}{p}}\|\theta\|_{L^p(\mathbb{T}^d)}^{s-1}$$

holds, where $q := \frac{p(s-1)}{2-s-p+ps}$.

Proof. $\theta^\varepsilon(\cdot)$ is a smooth function defined on $\mathbb{T}^d$. We consider an estimate for the norm of $\theta^\varepsilon(\cdot)$ by employing Young's inequality

$$\begin{aligned}\left\||\theta^\varepsilon(\cdot)|^{s-1}\right\|_{L^p(\mathbb{T}^d)} &= \left(\int_{\mathbb{T}^d}|\theta^\varepsilon(x)|^{p(s-1)}\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &= \|\theta^\varepsilon(\cdot)\|_{L^{p(s-1)}(\mathbb{T}^d)}^{s-1} \leq \|\theta(\cdot)\|_{L^p(\mathbb{T}^d)}^{s-1}\|\eta^\varepsilon\|_{L^q(\mathbb{R}^d)}^{s-1},\end{aligned}$$

where $\frac{1}{p(s-1)} = \frac{1}{p} + \frac{1}{q} - 1$. But we have

$$\begin{aligned}\|\eta^\varepsilon\|_{L^q(\mathbb{R}^d)} &= \left(\int_{B_\varepsilon(0)}\left|\frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right)\right|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &= \frac{1}{\varepsilon^d}\left(\int_{B_1(0)}\varepsilon^d\eta\left(x_1\right)^q\,\mathrm{d}x_1\right)^{\frac{1}{q}}\\ &= C(q)\varepsilon^{d\left(\frac{1}{q}-1\right)} = C(q)\varepsilon^{-\frac{d(s-2)}{p(s-1)}}\end{aligned}$$

with

$$C(q) := \left(\int_{B_1(0)} \eta(x_1)^q \, \mathrm{d}x_1 \right)^{\frac{1}{q}}.$$

In view of the above results, we achieve the desired expression. □

Lemma 2.5. *For any* $s \geq 2$, $p > 2$, $0 < \alpha < 1$, *and* $\theta \in \dot{B}_p^{\alpha,\infty}(\mathbb{T}^d)$ *the following inequality holds:*

$$\|\nabla_x \theta^\varepsilon\|_{L^{p(s-1)}(\mathbb{T}^d)} \leq C\varepsilon^{\alpha - \frac{d(s-2)}{p(s-1)} - 1} \|\theta\|_{\dot{B}_p^{\alpha,\infty}(\mathbb{T}^d)}. \tag{2.6}$$

Proof. We use the following well-known inequality for any $0 < \beta < 1$ and $1 \leq q \leq \infty$, cf. [20, 36]:

$$\|\nabla_x \theta^\varepsilon\|_{L^q(\mathbb{T}^d)} \leq C\varepsilon^{\beta - 1} \|\theta\|_{\dot{B}_q^{\beta,\infty}(\mathbb{T}^d)},$$

so that the choice $q = p(s-1)$, $\beta = \alpha - \frac{d(s-2)}{p(s-1)}$ yields

$$\|\nabla_x \theta^\varepsilon\|_{L^{p(s-1)}(\mathbb{T}^d)} \leq C\varepsilon^{\alpha - \frac{d(s-2)}{p(s-1)} - 1} \|\theta\|_{\dot{B}_{p(s-1)}^{\alpha - \frac{d(s-2)}{p(s-1)},\infty}(\mathbb{T}^d)}. \tag{2.7}$$

Then by Proposition 2.3, we have

$$\|\theta\|_{\dot{B}_{p(s-1)}^{\alpha - d\left(\frac{1}{p} - \frac{1}{p(s-1)}\right),\infty}(\mathbb{T}^d)} \leq C\|\theta\|_{\dot{B}_p^{\alpha,\infty}(\mathbb{T}^d)}.$$

This implies

$$\|\theta\|_{\dot{B}_{p(s-1)}^{\alpha - \frac{d(s-2)}{p(s-1)},\infty}(\mathbb{T}^d)} \leq C\|\theta\|_{\dot{B}_p^{\alpha,\infty}(\mathbb{T}^d)}. \tag{2.8}$$

Combining (2.7) and (2.8), we achieve the desired inequality (2.6). □

Integrating in time, we obtain

Corollary 2.6. *For any* $s \geq 2$, $1 \leq p_2 \leq \infty$, $p_1 > 2$, $0 < \alpha < 1$, *and* $\theta \in L^{p_2}(0,T; \dot{B}_{p_1}^{\alpha,\infty}(\mathbb{T}^d))$, *the following inequality*

$$\|\nabla_x \theta^\varepsilon\|_{L^{p_2}(0,T;\, L^{p_1(s-1)}(\mathbb{T}^d))} \leq C\varepsilon^{\alpha - \frac{d(s-2)}{p_1(s-1)} - 1} \|\theta\|_{L^{p_2}(0,T;\, \dot{B}_{p_1}^{\alpha,\infty}(\mathbb{T}^d))}$$

holds.

Recall that a Fourier multiplier operator acts on a function $f : \mathbb{T}^d \to \mathbb{R}$ via

$$\mathcal{T} f = \mathcal{F}^{-1}(m\hat{f}),$$

where $m : \mathbb{Z}^d \to \mathbb{C}$ is the symbol. The symbol is called an L^p-multiplier if $\mathcal{T}$ is a bounded operator from L^p to itself. From the Fourier characterization of Besov spaces, one can see that in this case $\mathcal{T}$ is also bounded from a Besov space into itself. More precisely:

Proposition 2.7. *Let $\mathcal{T} : L^p(\mathbb{T}^d) \to L^p(\mathbb{T}^d)$ be an operator with symbol $m(\xi)$. If m is an L^p- multiplier then $\mathcal{T} : \dot{B}_p^{\alpha,\infty}(\mathbb{T}^d) \to \dot{B}_p^{\alpha,\infty}(\mathbb{T}^d)$ is a bounded operator for $0 < \alpha < 1$ and $1 < p < \infty$.*

Proof. Since m is an L^p multiplier,

$$\|\mathcal{T}(2^{k\alpha}\Delta_k f)\|_{L^p} \leq C\|2^{k\alpha}\Delta_k f\|_{L^p}.$$

Therefore, we obtain

$$\sup_k \left\|\mathcal{T}(2^{k\alpha}\Delta_k f)\right\|_{L^p} \leq C \sup_k \left\|2^{k\alpha}\Delta_k f\right\|_{L^p}$$

and thus

$$\sup_k \left\|2^{k\alpha}\Delta_k \mathcal{T} f\right\|_{L^p} \leq C \sup_k \left\|2^{k\alpha}\Delta_k f\right\|_{L^p},$$

as $\mathcal{T}$ is linear and commutes with Δ_k. Hence we obtain

$$\|\mathcal{T} f\|_{\dot{B}_p^{\alpha,\infty}} \leq C\|f\|_{\dot{B}_p^{\alpha,\infty}}.$$

□

2.3 Analyticity

Here we will prove a Lemma which shows that the integral appearing in the weak formulation of the renormalized transport equation is an analytic function with respect to the renormalization exponent. For a complex number $z \in \mathbb{C}$, we write $\Re z$ for its real part. We also use the notation

$$x_+ := \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } x > 0. \end{cases}$$

Lemma 2.8. *If $\theta \in L^{p_1}_{loc}((0,T), L^p(\mathbb{T}^d))$, with $p \geq p_1$ and $u \in L^{q_1}_{loc}((0,T), L^q(\mathbb{T}^d))$, with $q \geq q_1$, $p_1 > 1$ and $\frac{1}{p_1} + \frac{1}{q_1} \leq 1$, then for any $\varphi \in C_c^\infty((0,T) \times \mathbb{T}^d)$ the function*

$$F(z) = -\int_0^T \int_{\mathbb{T}^d} |\theta|^z \left(\partial_t \varphi + u \cdot \nabla \varphi\right) \mathrm{d}x \, \mathrm{d}t$$

is an analytic function on the strip $0 < \Re z < \frac{p_1}{q_1'}$, where q_1' is the dual exponent to q_1 e.g. $\frac{1}{q_1} + \frac{1}{q_1'} = 1$. In particular, if $\theta \in L^{p_1}_{loc}(0,T; \dot{B}_p^{\alpha,\infty}(\mathbb{T}^d))$, $u \in L^{q_1}_{loc}(0,T; \dot{B}_q^{\beta,\infty}(\mathbb{T}^d))$, with $p_1 \leq \frac{pd}{(d-p\alpha)_+}$, $q_1 \leq \frac{qd}{(d-q\beta)_+}$, $\alpha > 0$, and $\frac{2}{p_1} + \frac{1}{q_1} < 1$, then there exists a positive number $\gamma > 0$ such that F is an analytic function on the strip $0 < \Re(z) < 2 + \gamma$.

Proof. We want to show that

$$F'(z) = \int_0^T \int_{\mathbb{T}^d} |\theta(t,x)|^z \log|\theta(t,x)| \left(\partial_t\varphi + u\cdot\nabla\varphi\right) \mathrm{d}x\,\mathrm{d}t$$

is a well-defined function. Note that for any fixed $\delta > 0$, there exists $C(\delta) > 0$ such that the inequality

$$|\log|y|| \le C(\delta)(1+|y|)^{2\delta}|y|^{-\delta}$$

holds for any $y \in \mathbb{R}\setminus\{0\}$. We have

$$|F'(z)| \le C\int_0^T \int_{\mathbb{T}^d} |\theta(t,x)|^{\Re z}\left(1+|\theta(t,x)|\right)^{2\delta}|\theta(t,x)|^{-\delta}|\partial_t\varphi + u\cdot\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t.$$

For $|\theta| \ge 1$, we have

$$|\theta|^{\Re z-\delta}\left(1+|\theta|\right)^{2\delta} \le 2^{2\delta}|\theta|^{\Re z+\delta},$$

while, for $|\theta| \le 1$, we obtain

$$|\theta|^{\Re z-\delta}\left(1+|\theta|\right)^{2\delta} \le 2^{2\delta}|\theta|^{\Re z-\delta},$$

further, let

$$z \in \{\Delta \le \Re z \le \frac{p_1}{q_1'} - \Delta\} \quad \text{for sufficiently small} \quad \Delta := 2\delta.$$

Using the above we have

$$\begin{aligned}
|F'(z)| &\le C\int_0^T \int_{\mathbb{T}^d} |\theta(t,x)|^{\Re z}\left(1+|\theta(t,x)|\right)^{2\delta}|\theta(t,x)|^{-\delta}|\partial_t\varphi + u\cdot\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\int_{[0,T]\times\mathbb{T}^d\cap\{|\theta|\ge1\}} |\theta(t,x)|^{\Re z+\delta}|\partial_t\varphi + u\cdot\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t \\
&+ C\int_{[0,T]\times\mathbb{T}^d\cap\{|\theta|\le1\}} |\theta(t,x)|^{\Re z-\delta}|\partial_t\varphi + u\cdot\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\int_{[0,T]\times\mathbb{T}^d\cap\{|\theta|\ge1\}} |\theta(t,x)|^{\frac{p_1}{q_1'}-\Delta+\delta}|\partial_t\varphi + u\cdot\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t \\
&+ C\int_{[0,T]\times\mathbb{T}^d\cap\{|\theta|\le1\}} |\theta(t,x)|^{\Delta-\delta}|\partial_t\varphi + u\cdot\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\int_{[0,T]\times\mathbb{T}^d} (|\theta(t,x)|^{\delta} + |\theta(t,x)|^{\frac{p_1}{q_1'}-\delta})|\partial_t\varphi + u\cdot\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t < \infty
\end{aligned}$$

and note that the last integral converges. Thus the integral for $F'(z)$ converges uniformly in $\{z \in \mathbb{C} : \Delta \le \Re z \le \frac{p_1}{q_1'} - \Delta\}$ and so there exists

$F'(z)$ for any $z \in \{0 < \Re z < \frac{p_1}{q_1'}\}$. Consequently, $F(z)$ is an analytic function on the strip $0 < \Re z < \frac{p_1}{q_1'}$.

For the "in particular" part of the lemma, observe that by Besov embedding (Proposition 2.3), $\dot{B}^{\alpha}_{p,\infty}$ embeds continuously into L^r with $r := \frac{pd}{(d-p\alpha)_+}$, and likewise $\dot{B}^{\beta}_{q,\infty}$ embeds continuously into L^s with $s := \frac{qd}{(d-q\beta)_+}$ (with the convention $r = \infty$ if the denominator is zero); by assumption, these exponents are not smaller than p_1 and q_1, respectively, and so the first part of the Lemma applies. Since

$$\frac{2}{p_1} + \frac{1}{q_1} < 1,$$

we obtain $\frac{p_1}{q_1'} > 2$, and the conclusion follows.

□

2.4 Commutator Estimates

In this section, we derive the below theorem, which is also stated in [4].

Theorem 2.9. *Let (θ, u) be a weak solution of (2.1) on $(0,T) \times \mathbb{T}^d$. Assume that $\theta \in L^{p_1}(0,T;\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d))$, $u \in L^{p_2}(0,T;\dot{B}^{\beta,\infty}_q(\mathbb{T}^d))$ with $2 < p_1 \leq \frac{pd}{(d-p\alpha)_+}$, $p_2 \leq \frac{qd}{(d-q\beta)_+}$, $\frac{2}{p} + \frac{1}{q} = 1$ and $\frac{2}{p_1} + \frac{1}{p_2} < 1$, where $1 \leq p, q \leq \infty$ and $0 < \alpha, \beta < 1$. Let $\gamma > 0$ and $2\alpha + \beta > 1 + \frac{d\gamma}{p}$. Then, for any $z \in S := \{z \in \mathbb{C} : 2 < \Re z < 2 + \gamma\}$, the relation*

$$\partial_t |\theta|^z + \operatorname{div}_x(|\theta|^z u) = 0$$

holds in the sense of distributions on $(0,T) \times \mathbb{T}^d$.

Proof. By mollifying (2.1) with respect to spatial variables, we obtain

$$\partial_t \theta^\varepsilon + \operatorname{div}_x(\theta u)^\varepsilon = 0. \tag{2.9}$$

If θ^ε is a weak solution to equation (2.9) then it has the weak derivative with respect to t. Moreover, $\partial_t \theta^\varepsilon$ belongs to $L^{\frac{p_1 p_2}{p_1+p_2}}([0,T] \times \mathbb{T}^d)$ under the conditions of Theorem 2.9.

Note that if $\Re z > 2$, then $|\theta^\varepsilon(t,\cdot)|^{z-1} \operatorname{sgn} \theta^\varepsilon(t,\cdot)$ is a continuously differentiable function on $\mathbb{T}^d$ for a.e. $t \in (0,T)$.

Let $\varphi \in C_c^\infty\left((0,T) \times \mathbb{T}^d\right)$. The multiplication of $\partial_t \theta^\varepsilon$ with $z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi$ is well-defined in the sense of distributions, owing to the condition $\frac{2+\delta}{p_1} + \frac{1}{p_2} \leq 1$ for some positive number $\delta > 0$. Thus multiplication of (2.9) with $z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi$ and integration over time and space gives

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \theta^\varepsilon z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \operatorname{div}_x(\theta u)^\varepsilon z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Here we take $\varepsilon > 0$ small enough so that $\varepsilon_1 \geq \varepsilon$, where ε_1 is chosen such that $\operatorname{supp}\varphi \subset (\varepsilon_1, T-\varepsilon_1) \times \mathbb{T}^d$. The previous equation can be written as

$$\int_{\varepsilon_1}^{T-\varepsilon_1} \int_{\mathbb{T}^d} \partial_t \theta^\varepsilon z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_{\varepsilon_1}^{T-\varepsilon_1} \int_{\mathbb{T}^d} \operatorname{div}_x(\theta^\varepsilon u^\varepsilon) z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_{\varepsilon_1}^{T-\varepsilon_1} \int_{\mathbb{T}^d} \operatorname{div}_x \left(\theta^\varepsilon u^\varepsilon - (\theta u)^\varepsilon\right) z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t =: R^\varepsilon.$$

Our goal is to prove that

$$\lim_{\varepsilon \to 0} R^\varepsilon = 0.$$

Fix $t \in (0,T)$ and consider the integral defined by

$$R_1^\varepsilon(t) := \int_{\mathbb{T}^d} \operatorname{div}_x \left(\theta^\varepsilon u^\varepsilon - (\theta u)^\varepsilon\right) z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x.$$

We can rewrite it as

$$R_1^\varepsilon(t) = \int_{\mathbb{T}^d} \operatorname{div}_x \left((\theta^\varepsilon(t,x) - \theta(t,x))(u^\varepsilon(t,x) - u(t,x))\right) z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi(t,x) \, \mathrm{d}x$$
$$- \int_{\mathbb{T}^d} \operatorname{div}_x \Bigg(\int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)[\theta(t,x-\xi) - \theta(t,x)]$$
$$\times [u(t,x-\xi) - u(t,x)] d\xi \Bigg) z|\theta^\varepsilon(t,x)|^{z-1} \operatorname{sgn} \theta^\varepsilon(t,x) \varphi(t,x) \, \mathrm{d}x =: I(t) + J(t),$$

and estimate

$$\begin{aligned} |I(t)| &= \left| \int_{\mathbb{T}^d} \operatorname{div}_x \left((\theta^\varepsilon - \theta)(u^\varepsilon - u)\right) z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \right| \\ &= \left| \int_{\mathbb{T}^d} \left((\theta^\varepsilon - \theta)(u^\varepsilon - u)\right) \cdot \nabla \left(z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi\right) \, \mathrm{d}x \right| \\ &\leq \left| \int_{\mathbb{T}^d} \left((\theta^\varepsilon - \theta)(u^\varepsilon - u)\right) \cdot \left(z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \nabla\varphi\right) \, \mathrm{d}x \right| \\ &\quad + \left| \int_{\mathbb{T}^d} \left((\theta^\varepsilon - \theta)(u^\varepsilon - u)\right) \cdot \nabla \left(z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon\right) \varphi \, \mathrm{d}x \right| \\ &=: |I_1(t)| + |I_2(t)|. \end{aligned}$$

We estimate $I_1(t)$ by the generalized Hölder inequality

$$|I_1(t)| \leq \left| \int_{\mathbb{T}^d} \left((\theta^\varepsilon - \theta)(u^\varepsilon - u)\right) \cdot \left(z|\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \nabla\varphi\right)(t,x) \, \mathrm{d}x \right|$$
$$\leq |z| \|\varphi(t,\cdot)\|_{C^1} \|\theta^\varepsilon(t,\cdot) - \theta(t,\cdot)\|_{L^p(\mathbb{T}^d)} \|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^q(\mathbb{T}^d)} \left\| |\theta^\varepsilon(t,\cdot)|^{\Re z-1} \right\|_{L^p(\mathbb{T}^d)}.$$

Now, we use the inequality

$$\|\theta^\varepsilon(t,\cdot) - \theta(t,\cdot)\|_{L^p(\mathbb{T}^d)} \leq C\varepsilon^\alpha \|\theta(t,\cdot)\|_{\dot{B}_p^{\alpha,\infty}(\mathbb{T}^d)}, \tag{2.10}$$

which is stated e.g. in the paper [20]. Similarly, we have

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^q(\mathbb{T}^d)} \leq C\varepsilon^\beta \|u(t,\cdot)\|_{\dot{B}^{\beta,\infty}_q(\mathbb{T}^d)}. \tag{2.11}$$

In view of Lemma 2.4, (2.10) and (2.11), we obtain

$$\begin{aligned}
&|I_1(t)| \\
&\leq C|z|\|\varphi(t,\cdot)\|_{C^1}\varepsilon^\alpha \|\theta(t,\cdot)\|_{\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)}\varepsilon^\beta \|u(t,\cdot)\|_{\dot{B}^{\beta,\infty}_q(\mathbb{T}^d)}\varepsilon^{-\frac{d(\Re z-2)}{p}}\|\theta(t,\cdot)\|^{\Re z-1}_{\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)} \\
&\leq C|z|\|\varphi(t,\cdot)\|_{C^1}\|\theta(t,\cdot)\|^{\Re z}_{\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)}\|u(t,\cdot)\|_{\dot{B}^{\beta,\infty}_q(\mathbb{T}^d)}\varepsilon^{\alpha+\beta-\frac{d(\Re z-2)}{p}}.
\end{aligned}$$

Now, we integrate the last inequality over $(0,T)$. Since $\operatorname{supp}\|\varphi(t,\cdot)\|_{C^1} \subset (\varepsilon_1, T-\varepsilon_1)$, we have

$$\int_0^T |I_1(t)|\,\mathrm{d}t \leq C|z|\|\varphi\|_{C^1}\varepsilon^{\alpha+\beta-\frac{d(\Re z-2)}{p}}\int_{\varepsilon_1}^{T-\varepsilon_1}\|\theta(t,\cdot)\|^{\Re z}_{\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)}\|u(t,\cdot)\|_{\dot{B}^{\beta,\infty}_q(\mathbb{T}^d)}\,\mathrm{d}t.$$

Since $2 < \Re(z) < 2+\gamma$ and $\frac{2+\gamma}{p_1} + \frac{1}{p_2} \leq 1$, then by Hölder's inequality we get

$$\begin{aligned}
\int_0^T |I_1(t)|\,\mathrm{d}t \leq C|z|\|\varphi\|_{C^1}\varepsilon^{\alpha+\beta-\frac{d(\Re z-2)}{p}}\|\theta\|^{\Re z}_{L^{p_1}(\varepsilon_1,T-\varepsilon_1;\,\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d))} \\
\times \|u\|_{L^{p_2}(\varepsilon_1,T-\varepsilon_1;\,\dot{B}^{\beta,\infty}_q(\mathbb{T}^d))}.
\end{aligned}$$

Now, we estimate $I_2(t)$ for fixed t:

$$\begin{aligned}
|I_2(t)| &\leq |z^2-z|\|\varphi(t,\cdot)\|_C \int_{\mathbb{T}^d} |(\theta^\varepsilon-\theta)(t,x)||(u^\varepsilon-u)(t,x)||\theta^\varepsilon(t,x)|^{\Re z-2}| \\
&\times |\nabla\theta^\varepsilon(t,x)|\,\mathrm{d}x \leq |z^2-z|\|\varphi(t,\cdot)\|_C\|(\theta^\varepsilon-\theta)(t,\cdot)\|_{L^p(\mathbb{T}^d)} \\
&\times \|(u^\varepsilon-u)(t,\cdot)\|_{L^q(\mathbb{T}^d)}\left\||\theta^\varepsilon|^{\Re z-2}|\nabla\theta^\varepsilon|\right\|_{L^p(\mathbb{T}^d)}.
\end{aligned}$$

By Hölder's inequality we obtain

$$\begin{aligned}
\left\||\theta^\varepsilon(t,\cdot)|^{\Re z-2}|\nabla\theta^\varepsilon(t,\cdot)|\right\|_{L^p(\mathbb{T}^d)} &= \left(\int_{\mathbb{T}^d}|\theta^\varepsilon(t,x)|^{p(\Re z-2)}|\nabla\theta^\varepsilon(t,x)|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} \leq \\
&\left(\left\||\theta^\varepsilon(t,\cdot)|^{p(\Re z-2)}\right\|_{L^{\frac{\Re z-1}{\Re z-2}}(\mathbb{T}^d)}\||\nabla\theta^\varepsilon(t,\cdot)|^p\|_{L^{\Re z-1}(\mathbb{T}^d)}\right)^{\frac{1}{p}} \\
&= \|\theta^\varepsilon(t,\cdot)\|^{\Re z-2}_{L^{p(\Re z-1)}(\mathbb{T}^d)}\|\nabla\theta^\varepsilon(t,\cdot)\|_{L^{p(\Re z-1)}(\mathbb{T}^d)}.
\end{aligned}$$

We take $s := \Re z$ in Lemma 2.5 to deduce

$$\|\theta^\varepsilon(t,\cdot)\|^{\Re z-2}_{L^{p(\Re z-1)}(\mathbb{T}^d)} \leq C(p)\varepsilon^{-\frac{d(\Re z-2)^2}{p(\Re z-1)}}\|\theta(t,\cdot)\|^{\Re z-2}_{L^p(\mathbb{T}^d)}. \tag{2.12}$$

Combining (2.6) and (2.12), we obtain

$$\left\| |\theta^\varepsilon(t,\cdot)|^{\Re z-2} |\nabla \theta^\varepsilon(t,\cdot)| \right\|_{L^p(\mathbb{T}^d)} \leq C(p)\varepsilon^{\alpha-1-\frac{d(\Re z-2)}{p}} \|\theta(t,\cdot)\|^{\Re z-1}_{\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)}.$$

Thus, combining all obtained estimates we get

$$\begin{aligned} |I_2(t)| &\leq C|z||z-1|\|\varphi\|_C \varepsilon^\alpha \varepsilon^\beta \|u(t,\cdot)\|_{\dot{B}^{\beta,\infty}_q(\mathbb{T}^d)} \varepsilon^{\alpha-1-\frac{d(\Re z-2)}{p}} \|\theta(t,\cdot)\|^{\Re z}_{\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)} \\ &= C|z||z-1|\|\varphi\|_C \|\theta(t,\cdot)\|^{\Re z}_{\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d)} \|u(t,\cdot)\|_{\dot{B}^{\beta,\infty}_q(\mathbb{T}^d)} \varepsilon^{2\alpha+\beta-1-\frac{d(\Re z-2)}{p}}. \end{aligned}$$

Finally, integrating the last inequality over $(0,T)$ similarly to the estimate for I_1, we arrive at

$$\begin{aligned} \int_0^T |I_2(t)|\,\mathrm{d}t \leq C|z||z-1|\|\varphi\|_C \varepsilon^{2\alpha+\beta-1-\frac{d(\Re z-2)}{p}} \|\theta\|^{\Re z}_{L^{p_1}(\varepsilon_1,T-\varepsilon_1;\,\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d))} \\ \times \|u\|_{L^{p_2}(\varepsilon_1,T-\varepsilon_1;\,\dot{B}^{\beta,\infty}_q(\mathbb{T}^d))}. \end{aligned}$$

Hence we obtain the estimate for $I(t)$:

$$\begin{aligned} \int_0^T |I(t)|\,\mathrm{d}t \leq C(z)\|\varphi\|_{C^1} \varepsilon^{2\alpha+\beta-1-\frac{d(\Re z-2)}{p}} \\ \times \|\theta\|^{\Re z}_{L^{p_1}([\varepsilon_1,T-\varepsilon_1],\,\dot{B}^{\alpha,\infty}_p(\mathbb{T}^d))} \|u\|_{L^{p_2}([\varepsilon_1,T-\varepsilon_1],\,\dot{B}^{\beta,\infty}_q(\mathbb{T}^d))}. \end{aligned}$$

Now, we consider an estimate for the second term $J(t)$:

$$\begin{aligned} |J(t)| = \Bigg| \int_{\mathbb{T}^d} &\mathrm{div}_x \bigg(\int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)[\theta(t,x-\xi)-\theta(t,x)] \\ &\times [u(t,x-\xi)-u(t,x)]d\xi \bigg) z|\theta^\varepsilon(t,x)|^{z-1}\,\mathrm{sgn}\,\theta^\varepsilon(t,x)\varphi(t,x)\,\mathrm{d}x \Bigg| \\ = \Bigg| &\int_{\mathbb{T}^d}\int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)[\theta(t,x-\xi)-\theta(t,x)] \\ &\times [u(t,x-\xi)-u(t,x)]d\xi \cdot \nabla_x \left(z|\theta^\varepsilon(t,x)|^{z-1}\,\mathrm{sgn}\,\theta^\varepsilon(t,x)\varphi(t,x)\right)\,\mathrm{d}x \Bigg| \\ \leq \Bigg| &\int_{\mathbb{T}^d}\int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)[\theta(t,x-\xi)-\theta(t,x)] \\ &\times [u(t,x-\xi)-u(t,x)]d\xi \cdot \left(z|\theta^\varepsilon|^{z-1}\,\mathrm{sgn}\,\theta^\varepsilon \nabla_x\varphi\right)\,\mathrm{d}x \Bigg| \\ + \Bigg| &\int_{\mathbb{T}^d}\int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)[\theta(t,x-\xi)-\theta(t,x)] \\ &\times [u(t,x-\xi)-u(t,x)]d\xi \cdot \left(\nabla_x z|\theta^\varepsilon|^{z-1}\,\mathrm{sgn}\,\theta^\varepsilon\right)\varphi\,\mathrm{d}x \Bigg| \\ =: &\,|J_1(t)| + |J_2(t)|. \end{aligned}$$

We estimate $J_1(t)$ by using the Cauchy-Schwarz and generalized Hölder inequalities:

$$\begin{aligned} |J_1(t)| \leq |z| \|\varphi(t,\cdot)\|_{C^1} \int_{\mathbb{T}^d} \Big| \int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)[\theta(t,x-\xi)-\theta(t,x)] \\ \times [u(t,x-\xi)-u(t,x)]d\xi \Big| |\theta^\varepsilon(x,t)|^{z-1}\,\mathrm{d}x \\ \leq |z|\|\varphi(t,\cdot)\|_{C^1} \int_{\mathbb{T}^d} \int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)|\theta(t,x-\xi)-\theta(t,x)| \\ \times |u(t,x-\xi)-u(t,x)|d\xi|\theta^\varepsilon(t,x)|^{\Re z-1}\,\mathrm{d}x \\ \leq |z|\|\varphi(t,\cdot)\|_{C^1} \int_{\mathbb{T}^d} \eta^\varepsilon(\xi)\|\theta(t,\cdot-\xi)-\theta(t,\cdot)\|_{L^p(\mathbb{T}^d)} \\ \times \|u(t,\cdot-\xi)-u(t,\cdot)\|_{L^q(\mathbb{T}^d)} \Big\| |\theta^\varepsilon(t,\cdot)|^{\Re z-1}\Big\|_{L^p(\mathbb{T}^d)} d\xi. \end{aligned}$$

Since $\theta \in \dot{B}_p^{\alpha,\infty}$, $u \in \dot{B}_q^{\beta,\infty}$ for a.e. $t \in [0,T]$, we have

$$\|\theta(t,\cdot-\xi)-\theta(t,\cdot)\|_{L^p(\mathbb{T}^d)} \leq |\xi|^\alpha \|\theta(t,\cdot)\|_{\dot{B}_p^{\alpha,\infty}(\mathbb{T}^d)}$$

and

$$\|u(t,\cdot-\xi)-u(t,\cdot)\|_{L^q(\mathbb{T}^d)} \leq |\xi|^\beta \|u(t,\cdot)\|_{\dot{B}_p^{\beta,\infty}(\mathbb{T}^d)}.$$

Observe that

$$\int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)|\xi|^{\alpha+\beta}d\xi = C\varepsilon^{\alpha+\beta}$$

as $\operatorname{supp}\eta^\varepsilon \subset \{\xi \in \mathbb{R}^d : |\xi| < \varepsilon\}$. Hence, we have

$$|J_1(t)| \leq C|z|\|\varphi(t,\cdot)\|_{C^1}\|\theta\|^{\Re z}_{\dot{B}_p^{\alpha,\infty}}\|u\|_{\dot{B}_q^{\beta,\infty}}\varepsilon^{\alpha+\beta-\frac{d(\Re z-2)}{p}}.$$

Then we integrate the last inequality over $[0,T]$:

$$\int_0^T |J_1(t)|\,\mathrm{d}t \leq C|z|\|\varphi\|_{C^1}\varepsilon^{\alpha+\beta-\frac{d(\Re z-2)}{p}} \int_{\varepsilon_1}^{T-\varepsilon_1} \|\theta(t,\cdot)\|^{\Re z}_{\dot{B}_p^{\alpha,\infty}(\mathbb{T}^d)}\|u(t,\cdot)\|_{\dot{B}_q^{\beta,\infty}(\mathbb{T}^d)}\,\mathrm{d}t.$$

The last integral converges. Similarly, we estimate $J_2(t)$:

$$|J_2(t)| = \Big| \int_{\mathbb{T}^d} \int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)[\theta(t,x-\xi) - \theta(t,x)] \times [u(t,x-\xi) - u(t,x)] d\xi \cdot \left(z(z-1)|\theta^\varepsilon|^{z-2} \nabla_x \theta^\varepsilon(t,x) \right) \varphi \, \mathrm{d}x \Big|$$

$$\le |z||z-1| \|\varphi\|_C \int_{\mathbb{T}^d} \Big| \int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi)[\theta(t,x-\xi) - \theta(t,x)] \times [u(t,x-\xi) - u(t,x)] d\xi \cdot \nabla \theta^\varepsilon(t,x) |\theta^\varepsilon(t,x)|^{z-2} \Big| \, \mathrm{d}x$$

$$\le |z||z-1| \|\varphi\|_C \int_{\mathbb{T}^d} \int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi) |\theta(t,x-\xi) - \theta(t,x)| \times |u(t,x-\xi) - u(t,x)| d\xi |\theta^\varepsilon(t,x)|^{\Re z - 2} |\nabla \theta^\varepsilon(t,x)| \, \mathrm{d}x$$

$$\le |z||z-1| \|\varphi\|_C \int_{[-\varepsilon,\varepsilon]^d} \eta^\varepsilon(\xi) \|\theta(t,\cdot-\xi) - \theta(t,\cdot)\|_{L^p(\mathbb{T}^d)} \times \|u(t,\cdot-\xi) - u(t,\cdot)\|_{L^q(\mathbb{T}^d)} \Big\| |\theta^\varepsilon(t,\cdot)|^{\Re z-2} |\nabla \theta^\varepsilon(t,\cdot)| \Big\|_{L^p(\mathbb{T}^d)} \, \mathrm{d}\xi.$$

Using the previous inequalities and Lemma 2.5, we obtain

$$|J_2(t)| \le C|z||z-1| \|\varphi\|_C \, \|\theta(t,\cdot)\|^{\Re z}_{\dot{B}^{\alpha,\infty}_p} \, \|u(t,\cdot)\|_{\dot{B}^{\beta,\infty}_q} \varepsilon^{2\alpha+\beta-1-\frac{d(\Re z-2)}{p}}.$$

Hence,

$$\int_0^T |J_2(t)| \, \mathrm{d}t \le C|z||z-1| \|\varphi\|_C \, \|\theta\|^{\Re z}_{L^{p_1}([\varepsilon_1, T-\varepsilon_1], \dot{B}^{\alpha,\infty}_p(\mathbb{T}^d))} \times \|u\|_{L^{p_2}([\varepsilon_1, T-\varepsilon_1], \dot{B}^{\beta,\infty}_q(\mathbb{T}^d))} \varepsilon^{2\alpha+\beta-1-\frac{d(\Re z-2)}{p}}.$$

Thus

$$\int_0^T |J(t)| \, \mathrm{d}t \le C(z) \|\varphi\|_{C^1} \, \|\theta\|^{\Re z}_{L^{p_1}([\varepsilon_1, T-\varepsilon_1], \dot{B}^{\alpha,\infty}_p(\mathbb{T}^d))} \times \|u\|_{L^{p_2}([\varepsilon_1, T-\varepsilon_1], \dot{B}^{\beta,\infty}_q(\mathbb{T}^d))} \varepsilon^{2\alpha+\beta-1-\frac{d(\Re z-2)}{p}}.$$

In other words,

$$\left| \int_0^T \int_{\mathbb{T}^d} \partial_t \theta^\varepsilon z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi - \operatorname{div}_x(\theta^\varepsilon u^\varepsilon) z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t \right|$$

$$\le C(z) \|\varphi\|_{C^1} \, \|\theta\|^{\Re z}_{L^{p_1}([\varepsilon_1, T-\varepsilon_1], \dot{B}^{\alpha,\infty}_p(\mathbb{T}^d))} \|u\|_{L^{p_2}([\varepsilon_1, T-\varepsilon_1], \dot{B}^{\beta,\infty}_q(\mathbb{T}^d))} \varepsilon^{2\alpha+\beta-1-\frac{d(\Re z-2)}{p}}.$$

In particular, since $\Re z < 2 + \gamma$, then

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \theta^\varepsilon z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi - \operatorname{div}_x(\theta^\varepsilon u^\varepsilon) z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t = o(1)$$

as $\varepsilon \to +0$, whenever $2\alpha + \beta > 1 + \frac{d\gamma}{p}$. Since $\theta^\varepsilon(t,\cdot)$, $u^\varepsilon(t,\cdot) \in C^\infty(\mathbb{T}^d)$ for a.e. $t \in [0,T]$ and $\operatorname{div}_x(u^\varepsilon(t,\cdot)) = 0$, we have

$$\operatorname{div}_x(\theta^\varepsilon u^\varepsilon) z |\theta^\varepsilon|^{z-1} \operatorname{sgn}(\theta^\varepsilon) = \operatorname{div}_x(|\theta^\varepsilon|^z u^\varepsilon).$$

Consequently, integration by parts yields

$$\begin{aligned} \int_{\mathbb{T}^d} \operatorname{div}_x(\theta^\varepsilon u^\varepsilon) z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x &= \int_{\mathbb{T}^d} \operatorname{div}_x(|\theta^\varepsilon|^z u^\varepsilon) \varphi \, \mathrm{d}x \\ &= -\int_{\mathbb{T}^d} |\theta^\varepsilon|^z u^\varepsilon \cdot \nabla \varphi \, \mathrm{d}x. \end{aligned}$$

On the other hand, by the chain rule it holds that

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \theta^\varepsilon z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{\mathbb{T}^d} \partial_t |\theta^\varepsilon|^z \varphi \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Therefore,

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d} \partial_t \theta^\varepsilon z |\theta^\varepsilon|^{z-1} \operatorname{sgn}(\theta^\varepsilon) \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \operatorname{div}_x(\theta^\varepsilon u^\varepsilon) z |\theta^\varepsilon|^{z-1} \operatorname{sgn} \theta^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = \int_0^T \int_{\mathbb{T}^d} \partial_t |\theta^\varepsilon|^z \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \operatorname{div}_x(|\theta^\varepsilon|^z u^\varepsilon) \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = -\int_0^T \int_{\mathbb{T}^d} |\theta^\varepsilon|^z \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{\mathbb{T}^d} |\theta^\varepsilon|^z u^\varepsilon \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = -\int_0^T \int_{\mathbb{T}^d} |\theta^\varepsilon|^z \left(\partial_t \varphi + u^\varepsilon \cdot \nabla \varphi \right) \mathrm{d}x \, \mathrm{d}t =: F_\varepsilon(z). \end{aligned}$$

From the above estimates that we obtained, we have

$$\lim_{\varepsilon \to 0} F_\varepsilon(z) = 0.$$

On the other hand, as $\varepsilon \to 0$, $|\theta^\varepsilon|^z \to |\theta|^z$ and $u^\varepsilon \to u$, and thus

$$\lim_{\varepsilon \to 0} F_\varepsilon(z) = F(z),$$

where

$$F(z) = -\int_0^T \int_{\mathbb{T}^d} |\theta|^z \left(\partial_t \varphi + u \cdot \nabla \varphi \right) \mathrm{d}x \, \mathrm{d}t.$$

□

Note that if we set the test function $\varphi(t,x) = \varphi(t)$ to depend only on the variable t, then we obtain the following corollary on the *global* conservation of L^p norms:

Corollary 2.10. *Let a pair* θ *be a weak solution of the transport problem* (2.1) *on* $(0,T)\times\mathbb{T}^d$. *Assume that*

$$\theta \in \dot{B}^{\alpha,\infty}_p, \quad u \in \dot{B}^{\beta,\infty}_q \quad \text{for some} \quad 1 \le p, q \le \infty \quad \text{and} \quad 0 < \alpha, \beta < 1$$

such that $\frac{2}{p}+\frac{1}{q}=1$. *Let* $\gamma > 0$ *and assume* $2\alpha+\beta > 1+\frac{d\gamma}{p}$. *Then, for any* $z \in \{z \in \mathbb{C}: \quad 2 < \Re z < 2+\delta\}$, *the total conservation of* L^z *norm of* θ *is valid, that is,*

$$\int_0^T \varphi'(t) \int_{\mathbb{T}^d} |\theta|^z \,\mathrm{d}x\,\mathrm{d}t = 0.$$

2.5 Conclusion of Main Results

Using Lemma 2.8, we know that

$$F(z) := \int_0^T \int_{\mathbb{T}^d} (\partial_t\varphi(t) + u\cdot\nabla\varphi)|\theta|^z \,\mathrm{d}x\,\mathrm{d}t$$

is an analytic function on $1 < \Re z < r$, where r is any exponent for which $|\theta|^r u$ is locally integrable in $(0,T)\times\mathbb{T}^d$. By Theorem 2.9, $F=0$ on an open set, so that Theorem 2.1 follows from the unique continuation principle for analytic functions.

Theorem 2.2 then immediately follows by Proposition 2.7.

Chapter 3

Energy Conservation for the Compressible Euler Equations with Vacuum

In this chapter, we consider the compressible Euler equations with vacuum. Our purpose is to investigate its regularity assumptions on a weak solution to conserve the energy. It mainly contains Section 1, Section 2, Section 3, Section 4 of the published paper [3] with minor modifications.

3.1 Introduction

In recent years some substantial effort has been directed towards investigating the relation between energy (or, more generally, entropy) conservation and regularity of weak solutions to a given physical system of equations.

Onsager's conjecture states that a weak solution of the (three- dimensional) incompressible Euler system will conserve energy if it is Hölder regular with exponent greater than 1/3. Otherwise it is possible for solutions to exist where anomalous dissipation of energy occurs. First results towards energy conservation for weak solutions are due to Eyink [34] and Constantin, E, Titi [20], see also [33]. The sharpest results in optimal Besov spaces are due to Cheskidov et al. [14] and Fjordholm-Wiedemann [39]. Further, Bardos and Titi [8], Bardos-Titi-Wiedemann [10], and Drivas-Nguyen [32] have extended these results to consider solutions on a bounded domain.

Investigating the possibility of analogous statements for other systems has become another lively direction of research. Sufficient regularity conditions for the energy to be conserved were studied for a number of models: inhomogeneous incompressible Euler [16] and Navier-Stokes [51], compressible Euler [36], the full Euler system [31], compressible Navier-Stokes [67], or Euler-Korteweg [25]. A general class of first-order conservation laws was considered in [44], and in [11] on bounded domains.

Another direction of research was aimed towards the construction of $(1/3-\varepsilon)-$Hölder continuous solutions to the incompressible Euler system that do *not* conserve energy. With the application, and further refinements, of the method of convex integration this was achieved recently by Isett [46] and by Buckmaster et al. [13]. Thus the famous conjecture of Lars Onsager for the incompressible Euler equations is fully resolved.

One of the major differences between incompressible and compressible fluid dynamics is the possible formation of *vacuum* in the latter case. This means that the density of the fluid may become zero in some region. More precisely, consider the isentropic compressible Euler system

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 0, \\ \partial_t \rho + \operatorname{div}(\rho u) = 0, \end{aligned} \tag{3.1}$$

where u denotes the velocity and ρ the density of the fluid. We will specify the constitutive pressure law $p = p(\rho)$ later. It is classically known that conservation laws like (3.1) may develop singularities (shocks) in finite time, which prohibits the use of a smooth notion of solution. Rather, one works with solutions in the sense of distributions, which may be very rough. Suppose now the density were initially bounded away from zero, $\rho^0 \geq c > 0$. If the solution were smooth, then from the continuity equation $\partial_t \rho + \operatorname{div}(\rho u) = 0$ it would easily follow (cf. equation (7) in [29]) that ρ remains bounded away from zero for all times. More precisely, this requires u to have bounded divergence. However, there seems to be no way to guarantee that the velocity component of a weak solution of (3.1) has bounded divergence, and thus it can not be excluded that the solution spontaneously develops vacuum in finite time. In fact, to our knowledge it remains an outstanding open question whether this can actually occur for the compressible Euler or even Navier-Stokes equations.

The formation of vacuum constitutes a degeneracy that, in many situations, vastly complicates the mathematical analysis of compressible models. For instance, the compressible Euler equations cease to be strictly hyperbolic in vacuum regions. In the context of the current contribution, densities close to zero invalidate the methods and results from previous works like [36, 44, 11]: There, it is a crucial assumption that the nonlinearities depend on the dependent variables in a twice continuously differentiable fashion, in order to treat them like a quadratic expression in the commutator estimates. For the system (3.1), a typical and physically reasonable pressure law would be the polytropic one, i.e. $p(\rho) = \rho^\gamma$ with $\gamma > 1$. The second derivative, however, is of order $\rho^{\gamma-2}$ and thus blows up at zero, at least if $\gamma < 2$. But the regime $1 < \gamma < 2$ is precisely the relevant one (for instance, a monoatomic gas has $\gamma = 5/3$).

The starting point of our current work is the result of Feireisl, Gwiazda, Świerczewska-Gwiazda, Wiedemann for the compressible Euler system [36], which we quote below. It gives sufficient conditions, in terms of Besov regularity of a weak solution, for energy conservation, but only as long as vacuum is excluded. In the presence of vacuum, the relevant commutator estimate involving the pressure completely breaks down, and it turns out that substantially new techniques are required to fix this. To our knowledge, the only other result on energy conservation for non-C^2 nonlinearities is the one on active scalar equations [4], using however different techniques.

In the current article, we give a number of sufficient conditions to ensure energy conservation even after possible formation of vacuum.

First (Section 3.3), we consider the condition that the velocity be a so-called divergence-measure field; this notion is well-known in geometric measure theory and hyperbolic conservation laws, but it may seem a bit unmotivated to consider in the present situation. However, justification comes from the compressible Navier-Stokes system, whose a priori estimates ensure this condition.

In Section 3.4, we identify as a sufficient condition for energy conservation an estimate for the quotient between the density and its mollification, see equation (3.14). This, in itself, may seem rather artificial, and we go on to identify more natural conditions that will ensure (3.14) to hold. Arguably, our strongest result is Corollary 3.7: Under the slightly stronger assumption of Hölder (instead of Besov) regularity, but with the expected exponents, we can show energy conservation *no matter how the density behaves near vacuum.* It is surprising that this result is completely agnostic to the way that ρ approaches zero. It crucially relies on a new measure-theoretic observation (Lemma 3.6) that may be of independent interest.

If one does want to assume only Besov regularity, then one needs to make further assumptions on the density near vacuum; we show that energy is conserved provided the density descends into vacuum sufficiently fast (Corollary 3.10) or sufficiently slowly (Corollary 3.14).

3.1.1 The Result of Feireisl et al.

To formulate the local or global energy equality for (3.1) it is useful to define the so-called pressure potential by

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2} \, \mathrm{d}r.$$

The following theorem was proven in [36, Theorem 4.1].

Theorem 3.1. *Let ρ, u be a solution of* (3.1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\varrho} \le \rho \le \overline{\rho}$$

a.e. in $(0,T)\times\mathbb{T}^d$, *for some constants* $\underline{\varrho}$, $\overline{\varrho}$, *and* $0 \le \alpha, \beta \le 1$ *such that*

$$\beta > \max\left\{1 - 2\alpha; \frac{1-\alpha}{2}\right\}.$$

Assume further that $p \in C^2[\underline{\varrho}, \overline{\varrho}]$, *and, in addition*

$$p'(0) = 0 \text{ as soon as } \underline{\varrho} = 0.$$

Then the energy is locally conserved, i.e.

$$\partial_t\left(\frac{1}{2}\rho|u|^2 + P(\rho)\right) + \operatorname{div}\left[\left(\frac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\right)u\right] = 0 \tag{3.2}$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$.

Our aim in the current note is to improve the above theorem by relaxing the C^2 assumption on the pressure. This will allow, for instance, to apply the theorem in the physically relevant case of the isentropic pressure law $p(\rho) = \kappa\rho^\gamma$ with the adiabatic coefficient $\gamma \in (1,2)$, without excluding vacuum.

3.2 Preliminaries

3.2.1 Function Spaces

For $\Omega := (0,T)\times\mathbb{T}^d$ we recall the Besov spaces $B_p^{\alpha,\infty}(\Omega)$ which is the space of tempered distributions w for which the norm

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)} := \|w\|_{L^p(\Omega)} + \sup_{\xi\in\Omega} \frac{\|w(\cdot+\xi) - w\|_{L^p(\Omega\cap(\Omega-\xi))}}{|\xi|^\alpha} \tag{3.3}$$

is finite. The above norm provides a control over shifts of the distribution w, making Besov spaces a convenient environment for our analysis, as it relies on convolutions with a mollifying kernel.

Let $\eta \in C_c^\infty\left(\mathbb{R}^N\right)$ be a positive, radial function of integral 1 with

$$\eta(x) = \begin{cases} 1 & \text{for} \quad |x| \le \frac{1}{3}, \\ 0 & \text{for} \quad |x| \ge 1, \end{cases}$$

and for $N = 1 + d$ set

$$\eta^\varepsilon(x) = \frac{1}{\varepsilon^N}\eta\left(\frac{x}{\varepsilon}\right).$$

We define the notation $w^\varepsilon := \eta^\varepsilon * w$. For any function w, w^ε is well-defined on $\Omega^\varepsilon = \{x \in \Omega : \mathrm{d}(x, \partial\Omega) > \varepsilon\}$.

It is then easy to check that the definition of the Besov spaces implies

$$\|w^\varepsilon - w\|_{L^p(\Omega^\varepsilon)} \leq C\varepsilon^\alpha \|w\|_{B_p^{\alpha,\infty}(\Omega)}$$

and

$$\|\nabla w^\varepsilon\|_{L^p(\Omega^\varepsilon)} \leq C\varepsilon^{\alpha-1} \|w\|_{B_p^{\alpha,\infty}(\Omega)}.$$

By $\mathcal{M}(\Omega)$ we denote the space of signed Radon measures equipped with the total variation norm

$$\|\mu\|_{TV} := \int_\Omega \mathrm{d}|\mu|.$$

3.2.2 Derivation of the Local Energy Equality

The starting point in the proof of Theorem 3.1, as well as all our results, is to mollify the Euler equations, then derive the local energy equality for the regularized quantities, and finally estimate commutator errors generated by nonlinear terms. As this strategy is a common part in the proofs of our theorems, we devote this section to the said derivation, omitting the details of passing to the limit under the assumptions of Theorem 3.1.

We begin by mollifying the momentum equation in time and space to obtain

$$\partial_t(\rho u)^\varepsilon + \operatorname{div}(\rho u \otimes u)^\varepsilon + \nabla p^\varepsilon(\rho) = 0, \tag{3.4}$$

or, in terms of commutators

$$\begin{aligned}\partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon) + \nabla p(\rho^\varepsilon) = \partial_t(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon) + \operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon \\ - (\rho u \otimes u)^\varepsilon) + \nabla\left(p(\rho^\varepsilon) - p^\varepsilon(\rho)\right).\end{aligned} \tag{3.5}$$

Making use of the following identity

$$\operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon) = u^\varepsilon \operatorname{div}(\rho u)^\varepsilon + ((\rho u)^\varepsilon \cdot \nabla)u^\varepsilon,$$

we can see that multiplying (3.5) by u^ε yields

$$\rho^\varepsilon \partial_t\left(\frac{1}{2}|u^\varepsilon|^2\right) + ((\rho u)^\varepsilon \cdot \nabla)\frac{1}{2}|u^\varepsilon|^2 + \rho^\varepsilon u^\varepsilon \nabla\left(P'(\rho^\varepsilon)\right) = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon, \tag{3.6}$$

where

$$\begin{aligned} r_1^\varepsilon &= \partial_t(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)\cdot u^\varepsilon, \\ r_2^\varepsilon &= \operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon - (\rho u \otimes u)^\varepsilon)\cdot u^\varepsilon, \\ r_3^\varepsilon &= \nabla\left(p(\rho^\varepsilon) - p^\varepsilon(\rho)\right)\cdot u^\varepsilon. \end{aligned}$$

Using the mollified continuity equation

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho u)^\varepsilon = 0, \tag{3.7}$$

multiplied by $\frac{1}{2}|u^\varepsilon|^2$, we can rewrite (3.6) as

$$\partial_t \left(\frac{1}{2}\rho^\varepsilon |u^\varepsilon|^2\right) + \operatorname{div}\left((\rho u)^\varepsilon \, \frac{1}{2}|u^\varepsilon|^2\right) + \rho^\varepsilon u^\varepsilon \, \nabla \left(P'(\rho^\varepsilon)\right) = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon. \tag{3.8}$$

On the other hand writing (3.7) in the form

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon),$$

and multiplying by $P'(\rho^\varepsilon)$ we get

$$\partial_t \left(P(\rho^\varepsilon)\right) + \operatorname{div}(\rho^\varepsilon u^\varepsilon) P'(\rho^\varepsilon) = \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon) \; P'(\rho^\varepsilon). \tag{3.9}$$

Combining (3.8) and (3.9) we obtain

$$\begin{aligned} \partial_t \left(\frac{1}{2}\rho^\varepsilon |u^\varepsilon|^2 + P(\rho^\varepsilon)\right) + \operatorname{div}\left((\rho u)^\varepsilon \, \frac{1}{2}|u^\varepsilon|^2 + \rho^\varepsilon u^\varepsilon P'(\rho^\varepsilon)\right) \\ = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon + s^\varepsilon, \end{aligned} \tag{3.10}$$

where we set

$$s^\varepsilon := \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon) \; P'(\rho^\varepsilon).$$

The proof of Theorem 4.1 in [36] shows that when ρ, u are Besov regular and p is of class C^2, then the left-hand side of (3.10) converges to the left-hand side of (3.2) and each term on the right-hand side of (3.10) converges to zero, each convergence in the sense of distributions.

3.3 Energy Conservation Assuming the Divergence of Velocity is a Bounded Measure

Our first result establishes local energy conservation for weak solutions of (3.1) under the additional assumption that the velocity field u is a divergence-measure field. Note that the below result can be also seen in [3].

Remark 3.2. See [15], and references therein, for details on the role of divergence-measure fields in the theory of hyperbolic conservation laws.

Theorem 3.3. *Let ρ, u be a solution of* (3.1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \overline{\rho}$$

a.e. in $(0,T)\times\mathbb{T}^d$*, for some constants* $\underline{\varrho}$*,* $\overline{\varrho}$*, and* $0\le\alpha,\beta\le 1$ *such that*

$$\beta > \max\left\{1-2\alpha; \frac{1-\alpha}{2}\right\}.$$

Assume further that

$$\operatorname{div} u \in \mathcal{M}((0,T)\times\mathbb{T}^d), \qquad \text{and} \qquad p\in C[\underline{\varrho},\overline{\varrho}].$$

Then the energy is locally conserved, i.e.

$$\partial_t\left(\frac{1}{2}\rho|u|^2+P(\rho)\right)+\operatorname{div}\left[\left(\frac{1}{2}\rho|u|^2+p(\rho)+P(\rho)\right)u\right]=0$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$*.*

Proof. Take a sequence $p^\delta\in C^2[\underline{\rho},\overline{\rho}]$ that converges uniformly to $p\in C[\underline{\rho},\overline{\rho}]$, that is, for each $\delta>0$

$$\|p-p^\delta\|_{L^\infty}\le\delta.$$

Then using p^δ in (3.4) we have

$$\partial_t(\rho u)^\varepsilon+\operatorname{div}(\rho u\otimes u)^\varepsilon+\nabla(p^\delta(\rho))^\varepsilon=\nabla[(p^\delta(\rho))^\varepsilon-p^\varepsilon(\rho)]. \tag{3.11}$$

Now the left-hand side of the last equality satisfies all the conditions of Theorem 3.1, so for each fixed $\delta>0$ we have, in the limit as $\varepsilon\to 0$,

$$\partial_t\left(\frac{1}{2}\rho|u|^2+P^\delta(\rho)\right)+\operatorname{div}\left[\left(\frac{1}{2}\rho|u|^2+p^\delta(\rho)+P^\delta(\rho)\right)u\right], \tag{3.12}$$

where

$$P^\delta(\rho):=\rho\int_1^\rho\frac{p^\delta(r)}{r^2}dr.$$

We will now show that (3.12) converges as $\delta\to 0$ in the sense of distributions on $(0,T)\times\mathbb{T}^d$ to

$$\partial_t\left(\frac{1}{2}\rho|u|^2+P(\rho)\right)+\operatorname{div}\left[\left(\frac{1}{2}\rho|u|^2+p(\rho)+P(\rho)\right)u\right].$$

Let $\varphi\in C_c^\infty((0,T)\times\mathbb{T}^d)$. From the choice of p^δ we have

$$\left|\int_0^T\int_{\mathbb{T}^d}\nabla\varphi\cdot(p^\delta(\rho)-p(\rho))u\,dxdt\right|\le C\|\varphi\|_{C^1}\|p^\delta-p\|_{L^\infty}\|u\|_{L^3}\le C(\varphi,u)\delta.$$

For the terms containing $P^\delta(\rho)$ notice that

$$
\begin{aligned}
|P^\delta(\rho) - P(\rho)| &\le \rho \int_1^\rho \frac{|p^\delta(r) - p(r)|}{r^2}\,\mathrm{d}r \\
&\le \|p^\delta - p\|_{L^\infty}\, \rho \left| \int_1^\rho \frac{1}{r^2}\,\mathrm{d}r \right| \\
&\le (1+\rho)\|p^\delta - p\|_{L^\infty}.
\end{aligned}
$$

Hence we can estimate

$$
\left| \int_0^T \int_{\mathbb{T}^d} \partial_t \varphi \left(P^\delta(\rho) - P(\rho)\right) dx\,\mathrm{d}t \right| \le C\, \|\varphi\|_{C^1}\, (1 + \|\rho\|_{L^1})\delta \le C(\varphi)\delta,
$$

and similarly for the divergence term. It follows that both terms of (3.12) containing P^δ converge as $\delta \to 0$ to the corresponding terms for P.

The final step of the proof is to consider the term coming into (3.10) from the right-hand side of (3.11). We need to show that

$$
\nabla[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)] \cdot u^\varepsilon
$$

converges to zero in the sense of distributions on $(0,T) \times \mathbb{T}^d$ as first ε and then δ tend to zero. Multiplying by $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^d)$, integrating over time and space, and integrating by parts we obtain

$$
\begin{aligned}
\int_0^T \int_{\mathbb{T}^d} \nabla[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\varphi u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t = &- \int_0^T \int_{\mathbb{T}^d} [(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\varphi \operatorname{div} u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T \int_{\mathbb{T}^d} [(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\nabla\varphi \cdot u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.13}
$$

For the second term on the right-hand side of the last equality we see that

$$
\begin{aligned}
\left| \int_0^T \int_{\mathbb{T}^d} [(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\nabla\varphi \cdot u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t \right| &= \left| \int_0^T \int_{\mathbb{T}^d} [p^\delta(\rho) - p(\rho)]^\varepsilon \nabla\varphi \cdot u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\le C\|\varphi\|_{C^1} \|(p^\delta - p)^\varepsilon\|_{L^\infty} \|u\|_{L^3} \\
&\le C\|\varphi\|_{C^1} \|p^\delta - p\|_{L^\infty} \|u\|_{L^3} \le C\delta.
\end{aligned}
$$

Finally, for the first term on the right-hand side of (3.13) we invoke the assumption that $\operatorname{div} u$ is a bounded Radon measure to see that

$$\left|\int_0^T\int_{\mathbb{T}^d}\varphi[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\operatorname{div} u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| = \left|\int_0^T\int_{\mathbb{T}^d}\varphi[p^\delta(\rho)-p(\rho)]^\varepsilon(\operatorname{div} u)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right|$$

$$\leq \|\varphi\|_{C^0}\|(p^\delta - p)^\varepsilon\|_{L^\infty}\|(\operatorname{div} u)^\varepsilon\|_{L^1}$$

$$\leq \|\varphi\|_{C^0}\|p^\delta - p\|_{L^\infty}\|\operatorname{div} u\|_{TV} \leq C\delta$$

and so we are done. □

3.4 Energy Conservation Assuming Hölder Continuity of the Pressure

For the next result we fix $1 < \gamma < 2$ and we will assume that the pressure p is of class $C^{1,(\gamma-1)}$, thus relaxing the regularity assumption of Theorem 3.1. The expense of this relaxation is that we require $\alpha + \gamma\beta > 1$ where before we only needed $\alpha + 2\beta > 1$.

Theorem 3.4. *Let* ρ, u *be a solution of* (3.1) *in the sense of distributions. Assume*

$$u \in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d),$$
$$0 \leq \underline{\rho} \leq \rho \leq \overline{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d,$$

for some constants $\underline{\varrho}$, $\overline{\varrho}$ *and* $0 \leq \alpha, \beta \leq 1$ *such that,*

$$\frac{1}{p} + \frac{2}{q} \leq 1, \quad \frac{2}{p} + \frac{1}{q} \leq 1, \quad p, q \geq 2, \quad \alpha + \gamma\beta > 1 \quad \text{and} \quad 2\alpha + \beta > 1.$$

Define $\mathcal{B}_{\varepsilon^\beta} := \{x : 0 < \rho^\varepsilon(x) < \varepsilon^\beta \text{ and } \rho \neq 0\}$ *and assume that*

$$\left\|\frac{\rho^\varepsilon - \rho}{\rho^\varepsilon}\right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})} \leq C(\rho), \tag{3.14}$$

where C *does not depend on* ε. *Assume further that* $p \in C^{1,(\gamma-1)}([\underline{\rho},\overline{\rho}])$, *and, in addition*

$$p'(0) = 0 \quad \text{as soon as} \quad \underline{\rho} = 0.$$

Then the energy is locally conserved, i.e. (3.2) *holds in the sense of distributions on* $(0,T)\times\mathbb{T}^d$.

A large part of the proof of this theorem is identical to the proof of Theorem 3.1. In particular we regularize the balance equations to derive an energy balance for the smooth functions ρ^ε and u^ε. Then we need to show that the corresponding commutator errors vanish in the limit $\varepsilon \to 0$. This is done in the same way as in [36], the only difference being in the terms involving the pressure. In particular, we will have to estimate an appropriate norm of the difference $p(\rho)^\varepsilon - p(\rho^\varepsilon)$. This will be done by means of the following lemma, which is an adaptation to our present case of the argument in [36, p. 10], see also [44, Lemma 3.1].

Lemma 3.5. *Let $\gamma \in (1,2)$ and $p \in C^{1,\gamma-1}([a,b])$. If $\rho \in B^{\beta,\infty}_{\gamma q}(\Omega;[a,b])$, then*

$$\|p^\varepsilon(\rho) - p(\rho^\varepsilon)\|_{L^q} \leq C\varepsilon^{\gamma\beta}\,\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q}}$$

Proof. First we note that by the fundamental theorem of calculus

$$\begin{aligned} p(s) - p(s_0) = \int_{s_0}^s p'(t)\,\mathrm{d}t &= \int_{s_0}^s p'(s_0)\,\mathrm{d}t + \int_{s_0}^s p'(t) - p'(s_0)\,\mathrm{d}t \\ &= p'(s_0)(s-s_0) + \int_{s_0}^s p'(t) - p'(s_0)\,\mathrm{d}t. \end{aligned}$$

Since $p' \in C^{0,\gamma-1}$, we have

$$\begin{aligned} \left|\int_{s_0}^s p'(t) - p'(s_0)\,\mathrm{d}t\right| &\leq \int_{s_0}^s |p'(t) - p'(s_0)|\,\mathrm{d}t \\ &\leq C\int_{s_0}^s \mathrm{d}t \sup_{t\in[s_0,s]} |t-s_0|^{\gamma-1} \\ &\leq C|s-s_0|^\gamma. \end{aligned}$$

Thus,

$$|p(s) - p(s_0) - p'(s_0)(s-s_0)| \leq C|s-s_0|^\gamma.$$

As the constant C is independent of s, s_0 we see that

$$|p(\rho^\varepsilon) - p(\rho) - p'(\rho)(\rho^\varepsilon - \rho)| \leq C|\rho - \rho^\varepsilon|^\gamma, \tag{3.15}$$

and similarly,

$$|p(\rho(y)) - p(\rho(x)) - p'(\rho(x))(\rho(y) - \rho(x))| \leq C|\rho(x) - \rho(y)|^\gamma. \tag{3.16}$$

Applying convolution against the function η^ε with respect to y in (3.16) and using Jensen's inequality we obtain

$$|p^\varepsilon(\rho) - p(\rho) - p'(\rho)(\rho^\varepsilon - \rho)| \leq C|\rho - \rho(\cdot)|^\gamma *_y \eta^\varepsilon. \tag{3.17}$$

Combining (3.15) and (3.17) we get

$$|p^\varepsilon(\rho) - p(\rho^\varepsilon)| \leq C|\rho - \rho^\varepsilon|^\gamma + C|\rho - \rho(\cdot)|^\gamma *_y \eta^\varepsilon. \tag{3.18}$$

Taking the L^q norm of both sides of (3.18) for the first term on the right-hand side we see that

$$C\||\rho - \rho^\varepsilon|^\gamma\|_{L^q} = C\|\rho - \rho^\varepsilon\|_{L^{\gamma q}}^\gamma.$$

Finally, for the L^q norm of (3.18) for the second term on the right-hand side by Jensen's inequality and Fubini's theorem we have

$$\begin{aligned}
C\||\rho - \rho(\cdot)|^\gamma *_y \eta^\varepsilon\|_{L^q} &\leq C\left(\int\int |\rho(x) - \rho(x-y)|^{\gamma q}\,\mathrm{d}x\eta_\varepsilon(y)\,\mathrm{d}y\right)^{1/q} \\
&= C\left(\int \|\rho(\cdot) - \rho(\cdot - y)\|_{L^{\gamma q}}^{\gamma q}\eta_\varepsilon(y)\,\mathrm{d}y\right)^{1/q} \\
&\leq C\sup_y |\eta_\varepsilon(y)|^{1/q}\left(\int_{\operatorname{supp}\eta_\varepsilon} \|\rho(\cdot) - \rho(\cdot - y)\|_{L^{\gamma q}}^{\gamma q}\,\mathrm{d}y\right)^{1/q} \\
&\leq C\sup_{y\in\operatorname{supp}\eta_\varepsilon} \|\rho(\cdot) - \rho(\cdot - y)\|_{L^{\gamma q}}^\gamma.
\end{aligned}$$

Finally, we use the definition of the Besov norm and (3.3) to write

$$\begin{aligned}
\|p^\varepsilon(\rho) - p(\rho^\varepsilon)\|_{L^q} &\leq C\left(\|\rho^\varepsilon - \rho\|_{L^{\gamma q}}^\gamma + \sup_{s\in\operatorname{supp}\eta^\varepsilon} \|\rho(\cdot) - \rho(\cdot - s)\|_{L^{\gamma q}}^\gamma\right) \\
&\leq C\varepsilon^{\gamma\beta}\|\rho\|_{B_{\gamma q}^{\beta,\infty}}^\gamma + \sup_{s\in\operatorname{supp}\eta^\varepsilon} |s|^{\gamma\beta}\|\rho\|_{B_{\gamma q}^{\beta,\infty}}^\gamma \\
&\leq C\varepsilon^{\gamma\beta}\|\rho\|_{B_{\gamma q}^{\beta,\infty}}^\gamma
\end{aligned}$$

and we are done. □

Proof of Theorem 3.4. As remarked above the only novelty needed to establish the desired result is to estimate commutator errors due to nonlinearity of the pressure. Precisely, we need to show that the local version of r_3^ε and s^ε, which we will denote R^ε and S^ε, of equation (3.10) converge to zero as $\varepsilon \to 0$. For a test function $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^d)$ we denote

$$R^\varepsilon := \int_0^T \int_{\mathbb{T}^d} \nabla(p(\rho^\varepsilon) - p(\rho)^\varepsilon)\cdot\varphi u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t, \tag{3.19}$$

and

$$S^\varepsilon := \int_0^T \int_{\mathbb{T}^d} \varphi\,\mathrm{div}[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\; P'(\rho^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.$$

Integrating (3.19) by parts and using Lemma 3.5 we obtain the following estimate.

$$|R^\varepsilon| \leq \|\varphi\|_{\mathcal{C}^1} \int_0^T \int_{\mathbb{T}^d} |p(\rho)^\varepsilon - p(\rho)^\varepsilon| (|\nabla u^\varepsilon| + |u^\varepsilon|) \; dx$$

$$dt$$

$$\leq C \|\varphi\|_{\mathcal{C}^1} \|p(\rho^\varepsilon) - p(\rho)^\varepsilon\|_{L^{\frac{q}{2}}} (\|\nabla u^\varepsilon\|_{L^p} + \|u^\varepsilon\|_{L^p})$$
$$\leq C(\varepsilon^{\gamma\beta+(\alpha-1)} + \varepsilon^{\gamma\beta+\alpha}) \|\rho\|^\gamma_{B^{\beta,\infty}_{\frac{\gamma q}{2}}} \|u\|_{B^{\alpha,\infty}_p}$$
$$\leq C(\varepsilon^{\gamma\beta+(\alpha-1)} + \varepsilon^{\gamma\beta+\alpha}) \|\rho\|^\gamma_{B^{\beta,\infty}_q} \|u\|_{B^{\alpha,\infty}_p}$$

where for the last inequality we used that $\frac{\gamma q}{2} < q$, so we can embed $B^{\beta,\infty}_q$ into $B^{\beta,\infty}_{\frac{\gamma q}{2}}$.

We now investigate the term S^ε and see that we can integrate by parts to obtain

$$\begin{aligned}|S^\varepsilon| = &\left| \int_0^T \int_{\mathbb{T}^d} \varphi \operatorname{div}[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon] \; P'(\rho^\varepsilon) \, \mathrm{d}x \, \mathrm{d}t \right| \\ &\leq \int_0^T \int_{\mathbb{T}^d} |\nabla\varphi \cdot [\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon] \; P'(\rho^\varepsilon)| \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^T \int_{\mathbb{T}^d} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)] \cdot \nabla P'(\rho^\varepsilon)| \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{3.20}$$

We make note of the following pointwise identity

$$\begin{aligned}\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon &= (\rho^\varepsilon - \rho)(u^\varepsilon - u) \\ &- \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{T}^d} \eta^\varepsilon(\tau,\xi)(\rho(t-\tau, x-\xi) - \rho(t,x))(u(t-\tau,x-\xi) - u(t,x)) \, \mathrm{d}\xi \, \mathrm{d}\tau \end{aligned} \tag{3.21}$$

and using (3.21) allows us to split first term on the right-hand side of (3.20) into two terms. Here again we focus on the first of these terms only, as the other one produces the same estimates, after applying Fubini's theorem, as seen in [36]. We see that

$$\int_0^T \int_{\mathbb{T}^d} |\nabla\varphi \cdot (\rho^\varepsilon - \rho)(u^\varepsilon - u) P'(\rho^\varepsilon)| \, \mathrm{d}x \, \mathrm{d}t \leq \|\varphi\|_{C^1} \varepsilon^\beta \|\rho\|_{B^{\beta,\infty}_q} \varepsilon^\alpha \|u\|_{B^{\alpha,\infty}_p} \|P'(\rho^\varepsilon)\|_{L^\infty}.$$

We will now focus on the second term on the right-hand side of (3.20), namely,

$$\int_0^T \int_{\mathbb{T}^d} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)] \cdot \nabla P'(\rho^\varepsilon)| \, \mathrm{d}x \, \mathrm{d}t.$$

and by letting $y = (t, x)$ we split $(0, T) \times \mathbb{T}^d$ into two disjoint domains $\mathcal{A} := \{y : \rho^\varepsilon(y) = 0\}$ and $\mathcal{A}^c$ and see that trivially on $\mathcal{A}$ that $\rho(y) = 0$

a.e.. For the integral over $\mathcal{A}$ we note that $\nabla P'(\rho^\varepsilon)$ is a distribution that may have a singular part but we see that $\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]$ is smooth and equals zero on $\mathcal{A}$ and so any singular part vanishes. Thus we are left with

$$\int_{\mathcal{A}^c} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\nabla P'(\rho^\varepsilon)| \,\mathrm{d}x\,\mathrm{d}t$$

and using again the identity (3.21) we obtain

$$\int_{\mathcal{A}^c} |\varphi[(\rho^\varepsilon - \rho)(u^\varepsilon - u)]\nabla P'(\rho^\varepsilon)| \,\mathrm{d}x\,\mathrm{d}t.$$

For the integral over $\mathcal{A}^c$ we see that

$$\int_{\mathcal{A}^c} |\varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)\nabla P'(\rho^\varepsilon)| \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{A}^c} |\varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)P''(\rho^\varepsilon)\cdot\nabla\rho^\varepsilon| \,\mathrm{d}x\,\mathrm{d}t$$

and we observe that by the definition of P we have $\rho^\varepsilon P''(\rho^\varepsilon) = p'(\rho^\varepsilon)$, and by assumption p' is bounded. Therefore we have the following bound

$$\int_{\mathcal{A}^c} \left|\varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)P''(\rho^\varepsilon)\nabla\rho^\varepsilon\right| \,\mathrm{d}x\,\mathrm{d}t \\ \leq \int_{\mathcal{A}^c} \left|\varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)p'(\rho^\varepsilon)\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\right| \,\mathrm{d}x\,\mathrm{d}t.$$

We have assumed that $p'(0) = 0$ and $p' \in C^{0,\gamma-1}$ and so take any ρ_1, ρ_2 such that $p'(\rho_2) = 0$ and we obtain that

$$|p'(\rho_1)| = |p'(\rho_1) - p'(\rho_2)| \leq C|\rho_1 - \rho_2|^{\gamma-1} \leq C|\rho_1|^{\gamma-1}$$

using the definition of Hölder continuity. Thus letting $\rho_1 = \rho^\varepsilon(x)$ for each x we see that $|p'(\rho^\varepsilon)(x)| \leq C|\rho^\varepsilon|^{\gamma-1}(x)$ and so we obtain

$$\int_{\mathcal{A}^c} \left|\varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)p'(\rho^\varepsilon)\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\right| \,\mathrm{d}x\,\mathrm{d}t \\ \leq C\int_{\mathcal{A}^c} \left|\varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\right| \,\mathrm{d}x\,\mathrm{d}t.$$

We will split the integral over $\mathcal{A}^c$ further into different disjoint domains, $\mathcal{B}_{\varepsilon^\beta} := \{y : 0 < \rho^\varepsilon(y) < \varepsilon^\beta\}$ and $\mathcal{C}_{\varepsilon^\beta} := \{y : \rho^\varepsilon(y) \geq \varepsilon^\beta\}$.

For the integral over $\mathcal{B}_{\varepsilon^\beta}$ we see that

$$\begin{aligned}\left|\int_{\mathcal{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon} \,\mathrm{d}x\,\mathrm{d}t\right| \\ \leq& C\|\varphi\|_{C^0}\varepsilon^{\beta-1+\alpha}\|\rho\|_{B^{\beta,\infty}_q}\|u\|_{B^{\alpha,\infty}_p} \\ &\times \|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathcal{B}_{\varepsilon^\beta})}\left\|\frac{\rho^\varepsilon - \rho}{\rho^\varepsilon}\right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})} \\ \leq& C\|\varphi\|_{C^0}\varepsilon^{\gamma\beta-1+\alpha}\|\rho\|_{B^{\beta,\infty}_q}\|u\|_{B^{\alpha,\infty}_p}\left\|\frac{\rho^\varepsilon - \rho}{\rho^\varepsilon}\right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})},\end{aligned}$$

where for the last line as $\rho^\varepsilon(y) \leq \varepsilon^\beta$ so $(\rho^\varepsilon(y))^{\gamma-1} \leq \varepsilon^{\beta(\gamma-1)}$ as $\gamma - 1 > 0$. We also have the assumption that $\left\|\frac{\rho^\varepsilon-\rho}{\rho^\varepsilon}\right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})} \leq C$ and so we have the bound $C\varepsilon^{\gamma\beta-1+\alpha}$ as wanted.

We are left with the integral over $\mathcal{C}_{\varepsilon^\beta}$ and see that

$$\left|\int_{\mathcal{C}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| \leq C\|\varphi\|_{C^0}\varepsilon^{\beta-1+\alpha}\|\rho\|_{B_q^{\beta,\infty}}\|u\|_{B_p^{\alpha,\infty}}\left\|\frac{\rho^\varepsilon-\rho}{(\rho^\varepsilon)^{2-\gamma}}\right\|_{L^q(\mathcal{C}_{\varepsilon^\beta})}.$$

As $\rho^\varepsilon \geq \varepsilon^\beta$ thus $(\rho^\varepsilon)^{-1} \leq \varepsilon^{-\beta}$ and so $(\rho^\varepsilon)^{\gamma-2} \leq \varepsilon^{\beta(\gamma-2)}$ we obtain

$$\begin{aligned}\left\|\frac{\rho^\varepsilon-\rho}{(\rho^\varepsilon)^{2-\gamma}}\right\|_{L^q(\mathcal{C}_{\varepsilon^\beta})} &\leq \|\rho^\varepsilon - \rho\|_{L^q(\mathcal{C}_{\varepsilon^\beta})}\,\varepsilon^{\beta(\gamma-2)}\\ &\leq C\varepsilon^\beta\|\rho\|_{B_q^{\beta,\infty}}\varepsilon^{\beta(\gamma-2)}\\ &\leq C\varepsilon^{\beta(\gamma-1)}.\end{aligned}$$

We are thus done as obtain convergence to zero as long as $\gamma\beta + \alpha > 1$. We have thus shown that, under the assumptions of the theorem, we have $R^\varepsilon, S^\varepsilon \to 0$. The result follows. □

We have written Theorem 3.4 in the most general from but observe that the condition

$$\left\|\frac{\rho^\varepsilon-\rho}{\rho^\varepsilon}\right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})} \leq C$$

feels rather artificial and is not in the $p \in C^2$ result from [36]. We will now focus on finding conditions on ρ for different L^q norms that will control this term.

Our first result will show that when we assume that $q = 1$ and so u, ρ are Hölder continuous, not just Besov functions, then we can control this term directly as expected and not have to ask for any special extra conditions.

Lemma 3.6. *Let $w \in L^1(\Omega)$ be non-negative, where $\Omega \subset (0,T) \times \mathbb{T}^d$ satisfies $|\Omega| \neq 0$ and $w^\varepsilon|_\Omega > 0$. Then $\left\|\frac{w^\varepsilon-w}{w^\varepsilon}\right\|_{L^1(\Omega)} \leq C$, where C does not depend on ε but may depend on w and Ω.*

Proof. It suffices to show that $\left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)} \leq C$. Indeed, since $|\Omega| \leq C$,

$$\left\|\frac{w^\varepsilon - w}{w^\varepsilon}\right\|_{L^1(\Omega)} \leq \|1\|_{L^1(\Omega)} + \left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)} = C + \left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)}.$$

Fix $\varepsilon > 0$, $N = d+1$ and let $\{Q_j\}_{j=1}^n$ be a partition of $(0,T) \times \mathbb{T}^d$ into disjoint cubes with side length ε/C_N, where C_N is a constant depending only on the dimension, and select the cubes such that $|\Omega \cap Q_j| \neq 0$. Decomposing w as $w = \sum_{j=1}^n w\chi_{Q_j}$, we see that

$$\left\| \frac{w}{w^\varepsilon} \right\|_{L^1(\Omega)} = \left\| \frac{\sum_{j=1}^n w\chi_{Q_j}}{\sum_{k=1}^n (w\chi_{Q_k})^\varepsilon} \right\|_{L^1(\Omega)} \leq \sum_{j=1}^n \left\| \frac{w\chi_{Q_j}}{(w\chi_{Q_j})^\varepsilon} \right\|_{L^1(\Omega)}.$$

We now want to bound $(w\chi_{Q_j})^\varepsilon$ from below. Recalling from Section 3.2 that $\eta = 1$ for $|x| < 1/3$, we have, for $x \in Q_j$, that

$$\begin{aligned}
(w\chi_{Q_j})^\varepsilon(x) &\geq \frac{1}{\varepsilon^N} \int_{\{|\frac{x-y}{\varepsilon}| \leq \frac{1}{3}\}} \eta\left(\frac{x-y}{\varepsilon}\right) (w\chi_{Q_j})(y)\,\mathrm{d}y \\
&= \frac{1}{\varepsilon^N} \int_{B_{\frac{\varepsilon}{3}}(x)} (w\chi_{Q_j})(y)\,\mathrm{d}y \\
&= \frac{\omega_N}{|B_\varepsilon|} \int_{B_{\frac{\varepsilon}{3}}(x)} (w\chi_{Q_j})(y)\,\mathrm{d}y \\
&\geq \frac{\omega_N}{|B_\varepsilon|} \int_{Q_j} w(y)\,\mathrm{d}y,
\end{aligned}$$

where we obtain the last inequality provided C_N is large enough so that $B_{\frac{\varepsilon}{3}}(x) \supset Q_j$ for all $x \in Q_j$. Thus we obtain

$$\begin{aligned}
\left\| \frac{w}{w^\varepsilon} \right\|_{L^1(\Omega)} &\leq \sum_{j=1}^n \left\| \frac{w\chi_{Q_j}}{(w\chi_{Q_j})^\varepsilon} \right\|_{L^1(\Omega)} \\
&\leq \sum_{j=1}^n \frac{|B_\varepsilon|}{\omega_N} \frac{\int_{Q_j} w\,\mathrm{d}x}{\int_{Q_j} w\,\mathrm{d}x} \\
&\leq C \sum_{j=1}^n |Q_j| \leq C
\end{aligned} \tag{3.22}$$

where we have used a dimensional constant to relate the measure of the balls to the associated cubes. □

As a consequence we obtain the following corollary where by assuming Hölder continuity of u and ρ we obtain a natural extension of 3.1 to the case where $p \in C^{1,\gamma-1}$.

Corollary 3.7. *Let ρ, u be a solution of* (3.1) *in the sense of distributions. Assume*

$$\begin{aligned}
&u \in C^\alpha((0,T) \times \mathbb{T}^d), \quad \rho, \rho u \in C^\beta((0,T) \times \mathbb{T}^d), \\
&0 \leq \underline{\rho} \leq \rho \leq \overline{\rho} \ \textit{a.e. in}\ (0,T) \times \mathbb{T}^d,
\end{aligned}$$

for some constants $\underline{\varrho}$, $\overline{\varrho}$ *and* $0 \leq \alpha, \beta \leq 1$ *such that,*

$$\alpha + \gamma\beta > 1 \qquad \text{and} \qquad 2\alpha + \beta > 1.$$

Assume further that $p \in C^{1,(\gamma-1)}([\underline{\rho}, \overline{\rho}])$, *and, in addition*

$$p'(0) = 0 \quad \text{as soon as} \quad \underline{\rho} = 0.$$

Then the energy is locally conserved, i.e. (3.2) *holds in the sense of distributions on* $(0,T) \times \mathbb{T}^d$.

Proof. For the integral over $\mathcal{B}_{\varepsilon^\beta}$, in the proof of Theorem 3.4, we see that

$$\begin{aligned}
&\left| \int_{\mathcal{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1} \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon} \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\qquad \leq C \|\varphi\|_{C^0} \varepsilon^{\beta - 1 + \alpha} \|\rho\|_{\mathcal{C}^\beta} \|u\|_{\mathcal{C}^\alpha} \|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathcal{B}_{\varepsilon^\beta})} \left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^1(\mathcal{B}_{\varepsilon^\beta})} \\
&\qquad \leq C \|\varphi\|_{C^0} \varepsilon^{\gamma\beta - 1 + \alpha} \|\rho\|_{\mathcal{C}^\beta} \|u\|_{\mathcal{C}^\alpha}.
\end{aligned}$$

For the other bounds as we are on a domain with finite measure so we can bound the Besov norms by the Hölder norms. □

Remark 3.8. Notice that the condition $u \in C^\alpha((0,T) \times \mathbb{T}^d)$ and $\rho \in C^\beta((0,T) \times \mathbb{T}^d)$ implies that $\rho u \in C^{\min(\alpha,\beta)}((0,T) \times \mathbb{T}^d)$. Therefore, if one has $\alpha \geq \beta$, then the requirement that ρu be in $C^\beta((0,T) \times \mathbb{T}^d)$ can be dropped. See also Remark 3.2 (2) in [36].

When we still want to consider Besov spaces for ρ and u we have to consider extra conditions on ρ in order to control the term $\left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})}$. Our first method will be to ask for an integrability condition on $\frac{1}{\rho}$.

Lemma 3.9. *Assume that* $\frac{1}{w} \in L^p$ *and* $w \in L^q$. *Then*

$$\left\| \frac{w^\varepsilon - w}{w^\varepsilon} \right\|_{L^r} \leq C \qquad \text{for} \qquad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{r}$$

and in fact if $r < \infty$

$$\lim_{\varepsilon \to 0} \left\| \frac{w^\varepsilon - w}{w^\varepsilon} \right\|_{L^r} = 0 \qquad \text{for} \qquad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{r}.$$

Proof. Using Hölder's inequality and then Jensen's inequality, as the integral of the mollifier is one and $1/x$ is a convex function we get that $\left\|\frac{1}{w^\varepsilon}\right\| \leq \left\|(\frac{1}{w})^\varepsilon\right\| \leq \left\|\frac{1}{w}\right\|$ and so

$$\left\| \frac{w^\varepsilon - w}{w^\varepsilon} \right\|_{L^r} \leq \|w^\varepsilon - w\|_{L^q} \left\| \frac{1}{w^\varepsilon} \right\|_{L^p} \leq \|w^\varepsilon - w\|_{L^q} \left\| \frac{1}{w} \right\|_{L^p} \leq C.$$

As long as $q < \infty$ we see that this, in fact, converges to zero. □

We now obtain the following corollary adding this condition into Theorem 3.4. We note that when $p = q = 3$ then we obtain the best result with the weakest integrability assumption in the Besov norms.

Corollary 3.10. *Let ρ, u be a solution of (3.1) in the sense of distributions. Assume*

$$u \in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d),$$
$$0 \le \underline{\rho} \le \rho \le \overline{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d,$$

for some constants $\underline{\varrho}$, $\overline{\varrho}$ and $0 \le \alpha, \beta \le 1$ such that,

$$\frac{1}{p} + \frac{2}{q} \le 1, \quad \frac{2}{p} + \frac{1}{q} \le 1, \quad p, q \ge 2, \quad \alpha + \gamma\beta > 1 \quad \text{and} \quad 2\alpha + \beta > 1.$$

Define $\mathcal{E} := \{x : \rho \neq 0\}$ and assume that

$$\frac{1}{\rho} \in L^q(\mathcal{E}).$$

Assume further that $p \in C^{1,(\gamma-1)}([\underline{\rho}, \overline{\rho}])$, and, in addition

$$p'(0) = 0 \quad \text{as soon as} \quad \underline{\rho} = 0.$$

Then the energy is locally conserved, i.e. (3.2) holds in the sense of distributions on $(0,T)\times\mathbb{T}^d$.

Proof. For the integral over $\mathcal{B}_{\varepsilon^\beta}$, in the proof of Theorem 3.4, we see that as $\rho \in L^\infty$ and $\varepsilon^\beta \ge \rho^\varepsilon$ then

$$\begin{aligned}
&\left| \int_0^T \int_{\mathcal{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)(\rho^\varepsilon)^{\gamma-1} \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon} \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\qquad \le C\|\varphi\|_{C^0} \varepsilon^{\beta-1+\alpha} \|\rho\|_{B_q^{\beta,\infty}} \|u\|_{B_p^{\alpha,\infty}} \\
&\qquad\qquad \times \|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathcal{B}_{\varepsilon^\beta})} \left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})} \\
&\qquad \le C\|\varphi\|_{C^0} \varepsilon^{\gamma\beta-1+\alpha} \|\rho\|_{B_q^{\beta,\infty}} \|u\|_{B_p^{\alpha,\infty}} \left\| \frac{1}{\rho} \right\|_{L^q(\mathcal{E})}
\end{aligned}$$

and so we are done using lemma 3.9 for the final step. □

Remark 3.11. Even though we have written $\frac{1}{\rho} \in L^q(\mathcal{E})$ we can fix some $\delta > 0$ and only need this condition on some $\mathcal{B}_\delta$, as for $\varepsilon^1 > \varepsilon^2$, then $\mathcal{B}_{\varepsilon^2} \subset \mathcal{B}_{\varepsilon^1}$ and so when $\varepsilon^\beta < \delta$, then $\mathcal{B}_{\varepsilon^\beta} \subset \mathcal{B}_\delta$.

One can see that the condition $\frac{1}{\rho} \in L^q(\mathcal{B}_\delta)$ is quite a strong assumption and requires a quick approach of the function to the null set. Above we used conventional bounds to obtain a general integral result but do not consider the local structure of the function.

We notice that a point-wise estimate $\rho \leq C\rho^\varepsilon$ would allow to control the L^q norm of $\frac{\rho^\varepsilon - \rho}{\rho^\varepsilon}$ and, though convexity of ρ would do, we will now show a nice link between this and quasi-nearly subharmonic functions which are much more general functions than subharmonic, quasi-subharmonic and nearly subharmonic functions [57]. The main motivation for the study of this notion in this work is that it happens, as will be shown below, to be equivalent to the L^∞-boundedness of our problem term $\frac{\rho^\varepsilon - \rho}{\rho^\varepsilon}$.

Definition 3.12. Let $X \subset \mathbb{R}^d$ be a set and $u : X \to [0, +\infty)$ be Borel measurable. Then u is quasi-nearly subharmonic on X, that is $u \in QNS(X)$, if there is a constant $\varepsilon_0 = \varepsilon_0(u)$, $0 < \varepsilon_0 < 1$, such that for each open set $O \subset X$, $O \neq X$, for each $x \in O$ and each r, $0 < r \leq \varepsilon_0 \delta^O(x)$, one has $u \in L^1(B_r(x))$ and

$$u(x) \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} u(y)\,\mathrm{d}y \quad \text{for some constant} \quad C \geq 1, \tag{3.23}$$

where C is independent of r, $|B_r(x)| = \omega_d r^d$ is the volume of the ball and

$$\delta^O(x) = \operatorname{dist}(x, O^c) \quad \text{for the complement } O^c \text{ of } O \text{ in } X.$$

Lemma 3.13. *Let $u : X \to [0, +\infty)$ be a Borel measurable function. Then u is quasi-nearly subharmonic if and only if for every $O \subset\subset X$ there exist M, ε_0 such that for any $0 < \varepsilon < \varepsilon_0$*

$$u(x) \leq M u^\varepsilon(x) \quad \textit{for any} \quad x \in O.$$

Proof. Let $u : X \to [0, +\infty)$ be a quasi-nearly subharmonic function. Then for any $\varepsilon < \operatorname{dist}(O, \partial X)$, u^ε is a well-defined smooth function on O. Suppose that $O \subset\subset X$ is a precompact set. Then $\delta_0 = \operatorname{dist}(O, \partial X)$ is a positive number and for $\varepsilon < \delta_0$,

$$O \subset \{x : \ \operatorname{dist}(x, \partial X) > \varepsilon\}$$

and u^ε is well-defined on O. We prove that there exist M and ε_0 such that

$$u(x) \leq M u^\varepsilon(x) \quad \text{for any} \quad x \in O, \quad 0 < \varepsilon < \varepsilon_0.$$

Indeed, we have

$$u^\varepsilon(x) = \frac{1}{\varepsilon^d} \int_X \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\ \mathrm{d}y.$$

Note that $y \in X$ for $x \in O$ and $|x-y| < \varepsilon$. Since $u \geq 0$ and recalling that from definition of η, we know that $\eta = 1$ for $|x| < 1/3$, we have

$$\begin{aligned} u^\varepsilon(x) &\geq \frac{1}{\varepsilon^d} \int_{\{|\frac{x-y}{\varepsilon}| \leq \frac{1}{3}\}} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\ \mathrm{d}y \\ &= \frac{1}{\varepsilon^d} \int_{\{|\frac{x-y}{\varepsilon}| \leq \frac{1}{3}\}} u(y)\ \mathrm{d}y \\ &= \frac{\omega_d}{3^d \left|B_{\frac{\varepsilon}{3}}(x)\right|} \int_{B_{\frac{\varepsilon}{3}}(x)} u(y)\ \mathrm{d}y \\ &\geq \frac{\omega_d u(x)}{3^d C} \end{aligned}$$

for sufficiently small ε. Therefore, we obtain

$$u(x) \leq \frac{3^d C u^\varepsilon(x)}{\omega_d} \quad \text{for sufficiently small} \quad \varepsilon \leq \varepsilon_0 \delta^O(x).$$

On the other hand, if $u(x) \leq M u^\varepsilon(x)$, then we have

$$\begin{aligned} u(x) &\leq \frac{M}{\varepsilon^d} \int_X \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\ \mathrm{d}y \\ &= \frac{M\omega_d}{\omega_d \varepsilon^d} \int_{|x-y|\leq\varepsilon} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\ \mathrm{d}y \\ &\leq \frac{M\omega_d}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\ \mathrm{d}y. \end{aligned} \tag{3.24}$$

Hence we deduce

$$u(x) \leq \frac{C}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\ \mathrm{d}y.$$

This completes the proof of the lemma. □

From this point-wise control showing that $\rho(x) \leq M\rho^\varepsilon(x)$ we obtain another corollary to our main result.

Corollary 3.14. *Let ρ, u be a solution of* (3.1) *in the sense of distributions. Assume*

$$\begin{aligned} &u \in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d), \\ &0 \leq \underline{\rho} \leq \rho \leq \overline{\rho} \ \textit{a.e. in} \ (0,T)\times\mathbb{T}^d, \end{aligned}$$

for some constants $\underline{\varrho}$, $\overline{\varrho}$ and $0 \leq \alpha, \beta \leq 1$ such that,

$$\frac{1}{p} + \frac{2}{q} \leq 1, \quad \frac{2}{p} + \frac{1}{q} \leq 1, \quad p, q \geq 2, \quad \alpha + \gamma\beta > 1 \quad \textit{and} \quad 2\alpha + \beta > 1.$$

Assume that $\rho \in QNS(\mathcal{B}_\delta)$ *for some* $\delta > 0$ *and* $p \in \mathcal{C}^{1,(\gamma-1)}([\underline{\rho}, \overline{\rho}])$ *with*

$$p'(0) = 0 \quad \textit{as soon as} \quad \underline{\rho} = 0.$$

Then the energy is locally conserved, i.e. (3.2) *holds in the sense of distributions on* $(0,T) \times \mathbb{T}^d$.

Proof. For the integral over $\mathcal{B}_{\varepsilon^\beta}$, in the proof of Theorem 3.4, we see that

$$\left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})} \leq \left\| \frac{\rho^\varepsilon + C\rho^\varepsilon}{\rho^\varepsilon} \right\|_{L^q(\mathcal{B}_{\varepsilon^\beta})} \leq C \quad \text{for} \quad \varepsilon^\beta < \delta$$

and so we are done.

□

Remark 3.15. (i) The condition $\rho \in QNS(\mathcal{B}_\delta)$ deals with the $\rho, \rho_\varepsilon = 0$ without splitting into cases and so using this condition the proof is simplified.

(ii) We note that this condition is weaker than local convexity of ρ on $\mathcal{B}_\delta$ which would also give the same result.

(iii) In view of Lemma 3.13, it is essentially a matter of taste if one prefers to formulate Corollary 3.14 in terms of quasi-nearly subharmonicity, or directly under the assumption $\rho \leq C\rho^\varepsilon$.

3.4.1 Counterexample for the L^p Case

We indicate in this subsection why Lemma 3.6 is no longer true when the L^1-norm is replaced with the L^p-norm for a $p > 1$. This shows that the Hölder assumption of Corollary 3.7 cannot easily be relaxed.

We can see $\rho^\varepsilon(x)$ is like a weighted average of ρ over the ball $B_\varepsilon(x)$ and so heuristically we can see that

$$\frac{\rho - \rho^\varepsilon}{\rho^\varepsilon} \simeq \frac{\rho(x) - \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} \rho(y)\,\mathrm{d}y}{\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} \rho(y)\,\mathrm{d}y}$$

(which is rigorous for $\eta_\varepsilon = \frac{1}{|B_\varepsilon|}\chi_{B_\varepsilon(0)}(x)$) and assuming the right hand side is bounded and rearranging gives the condition (3.23). We see that looking at a condition of the form

$$\left\| \frac{\rho(\cdot) - \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(\cdot)} \rho(y)\,\mathrm{d}y}{\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(\cdot)} \rho(y)\,\mathrm{d}y} \right\|_{L^p} < C,$$

in a sense a "relatively weighted L^p mean oscillation condition", could potentially be the weakest condition to control (3.14).

We notice that for the L^1 norm we obtain perfect cancellation in the fraction when calculating (3.22), as a mollifier acts like a local weighted average. However, when we perform the calculation in (3.22), but in L^p, then instead we obtain

$$\sum_{j=1}^{n} \frac{|B_\varepsilon|}{\omega_N} \frac{\|w\chi_{Q_j}\|_{L^p}}{\int_{Q_j} w \, \mathrm{d}x} = \sum_{j=1}^{n} \frac{|B_\varepsilon|}{\omega_N} \frac{\left(\int_{Q_j} w^p \, \mathrm{d}x\right)^{1/p}}{\int_{Q_j} w \, \mathrm{d}x}$$

and if we assume that the $w = 1$ then we get $\sum_{j=1}^{n} \frac{|B_\varepsilon|}{\omega_N} |Q_j|^{1/p-1}$. As $1/p - 1 < 0$ then for certain functions this term could blow up.

In fact if one chooses a function made of separated spikes where the supports get smaller and smaller then we can show this blow up. We will formulate a simple counterexample so that it is in one dimension, non-continuous and non-negative though more regular counter examples can be constructed in higher dimensions, that are for instance, even smooth and strictly positive.

Firstly, note that if we show that $\left\| \frac{f}{f^\varepsilon} \right\|_{L^p}$ blows up as $\varepsilon \to 0$ then $\left\| \frac{f}{f^\varepsilon} - \frac{f^\varepsilon}{f^\varepsilon} \right\|_{L^p}$ will also blow up. We can take $x \in \mathbb{T}$ and define our counter example

$$f(x) := \sum_{i=1}^{\infty} \chi_{\left[\frac{1}{i}, \frac{1}{i} + \frac{1}{2^i}\right]}(x).$$

It is easy to see that $f \in B_p^{\alpha,\infty}(\mathbb{T})$ for $p > 1$ and any $0 < \alpha < 1 - \frac{1}{p}$ by regularizing and using Lemma 2.49 from [7]. So that we have the sum of separated spikes so they are further than $\frac{1}{i^2}$ apart yet have supports of size $\frac{1}{2^i}$. Let $\varepsilon = \frac{1}{2i^2}$ and see that as f is non-negative we can bound the sum below by just the ith spike and see that as mollification only acts locally, so the value on the denominator is only dependent on the ith spike, thus we obtain

$$\left\| \frac{f}{f^\varepsilon} \right\|_{L^p(\mathbb{T})} \geq \left\| \frac{1}{f^\varepsilon} \right\|_{L^p\left(\frac{1}{i}, \frac{1}{i} + \frac{1}{2^i}\right)} = \left\| \left(\left(\chi_{\left[\frac{1}{i}, \frac{1}{i} + \frac{1}{2^i}\right]} \right)^{\frac{1}{2i^2}} \right)^{-1} \right\|_{L^p\left(\frac{1}{i}, \frac{1}{i} + \frac{1}{2^i}\right)}. \tag{3.25}$$

We can then the bound mollification of $\chi_{\left[\frac{1}{i}, \frac{1}{i} + \frac{1}{2^i}\right]}$ in a similar method to (3.24) but in one dimension and so we can bound (3.25) below by

$$\left\| \frac{f}{f^\varepsilon} \right\|_{L^p(\mathbb{T})} \geq C \frac{2^i}{2i^2} \|1\|_{L^p\left(\frac{1}{i}, \frac{1}{i} + \frac{1}{2^i}\right)} = C \frac{2^i}{2i^2} 2^{-i/p} = C \frac{2^{i(1-1/p)}}{2i^2}.$$

As f is the sum of infinitely many spikes there will exist an appropriate spike for any ε_i and thus we can send $i \to \infty$ and, as $1 - 1/p > 0$, so $C \frac{2^{i(1-1/p)}}{2i^2} \to \infty$ which implies that $\left\| \frac{f}{f^\varepsilon} \right\|_{L^p(\mathbb{T})} \to \infty$.

Chapter 4

Non-Unique Admissible Weak Solutions of the Compressible Euler Equations with Compact Support in Space

This chapter is concerned with the existence and non-uniqueness of compactly supported admissible solutions to the Cauchy problem for the compressible Euler equations with a steady density. The material presented in Sections 4.1, 4.2, 4.3, 4.4, 4.6, 4.7 stems from the recent prepint [2] with minor changes, whereas Section 4.5 gives a proof of perturbation properties.

4.1 Introduction

In this chapter, we consider the isentropic compressible Euler system consisting of $(d+1)$ equations

$$\begin{cases} \partial_t \rho + \operatorname{div}_x m = 0, \\ \partial_t m + \operatorname{div}_x \left(\frac{m \otimes m}{\rho} \right) + \nabla_x [p(\rho)] = 0, \\ \rho(0, \cdot) = \rho^0, \quad m(0, \cdot) = m^0, \end{cases} \tag{4.1}$$

where m is the momentum and ρ is the density of a gas. The pressure p is a function of ρ, which is determined from the constitutive thermodynamic relations of the gas and is assumed continuously differentiable on $(0, \infty)$ with $p'(\rho) > 0$ throughout the chapter. The latter condition makes the system strictly hyperbolic on the set of admissible values

$\{\rho > 0\}$ (cf. [24]). Furthermore, thermodynamically admissible processes must satisfy an additional constraint given by the energy inequality

$$\partial_t \left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho} \right) + \operatorname{div}_x \left[\left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \right] \leq 0, \tag{4.2}$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2\varepsilon'(r)$.

Let T be a fixed positive time. By a bounded weak solution of (4.1) we mean a pair $(\rho, m) \in L^\infty((0,T)\times\mathbb{R}^d)$ such that the following identities hold for every test function $\psi \in C_c^\infty([0,T); C_c^\infty(\mathbb{R}^d))$, and for any vector field $\varphi \in C_c^\infty([0,T); C_c^\infty(\mathbb{R}^d))$:

$$\int_0^T \int_{\mathbb{R}^d} [\rho\partial_t\psi + m \cdot \nabla_x\psi] \; \mathrm{d}x \; \mathrm{d}t + \int_{\mathbb{R}^d} \rho^0(x)\psi(0,x) \; \mathrm{d}x = 0$$

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} \left[m\partial_t\varphi + \left\langle \frac{m \otimes m}{\rho}, \nabla_x\varphi \right\rangle + p(\rho) \operatorname{div}_x \varphi \right] \; \mathrm{d}x \; \mathrm{d}t \\ + \int_{\mathbb{R}^d} m^0\varphi(0,x) \; \mathrm{d}x = 0. \end{aligned} \tag{4.3}$$

It is tacitly assumed, as part of the definition, that all the integrals are well-defined (if ρ is bounded below by a positive constant, this will automatically be the case).

Weak solutions satisfying (4.2) in the sense of distributions represent a special case of *entropy solutions*, as have been studied for decades in the theory of hyperbolic conservation laws. They were long viewed as the solution paradigm for conservation laws, giving rise to a very satisfactory well-posedness theory at least in the scalar case [50]. However, the expected uniqueness of entropy solutions for the Cauchy problem for systems of conservation laws was disproved in the groundbreaking work of De Lellis-Székelyhidi [27], who gave examples of non-unique entropy solutions for (4.1) with piecewise constant density. Their convex integration scheme was later refined, e.g., in [17, 18, 38, 54] to yield larger classes of non-unique solutions, particularly for the Riemann problem.

Our aim in this note is to construct non-unique compactly supported solutions to (4.1), in the sense that the momentum m has compact support in space for every time, and the density ρ is constant outside a compact set in space for every time. More precisely, we construct such solutions which are *semi-stationary*, i.e., ρ is independent of time. The semi-stationary system has been studied in [17], but only under periodic boundary conditions. Although the specific examples of [27] already have compact support in the mentioned sense, we extend the range of compactly supported non-unique solutions to a much wider class of (smooth) initial densities. Note that this allows us to view our solutions either as solutions on all of $\mathbb{R}^d$ or as solutions on any bounded domain containing the support of m and $\rho - \overline{\rho}$. Our work can thus be seen as a proper extension of the results in [17].

More precisely, our results are as follows:

Theorem 4.1. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^d$ a bounded open set, $T > 0$, and $\Omega' \supset\supset \Omega$ locally Lipschitz. Assume that $\rho^0 \in C^1(\mathbb{R}^d)$ is a positive function satisfying $\rho^0(x) = \overline{\rho} > 0$ for $x \in \mathbb{R}^d \backslash \Omega$ and the pressure $p \in C^1(\mathbb{R}^d)$ with $\int_\Omega p(\rho_0(x))\,\mathrm{d}x = p(\overline{\rho})|\Omega|$. There exists a bounded initial momentum m^0 with $\operatorname{supp}(m^0) \subset \Omega'$ for which there are infinitely many weak solutions $(\rho, m) \in C^1(\mathbb{R}^d) \times C([0,T); H_w(\mathbb{R}^d))$ of*

$$\begin{cases} \operatorname{div}_x m = 0, \\ \partial_t m + \operatorname{div}_x \left(\frac{m \otimes m}{\rho}\right) + \nabla_x[p(\rho)] = 0, \\ m(0, \cdot) = m^0 \end{cases} \tag{4.4}$$

on $[0,T) \times \mathbb{R}^d$ with density $\rho(x) = \rho^0(x)$. Moreover, the obtained weak solutions m satisfy

$$\begin{aligned} |m(t,x)|^2 &= \rho^0(x)\chi(t)\mathbb{1}_{\Omega'} \quad \text{a.e.} \quad \text{in} \quad [0,T) \times \mathbb{R}^d, \\ |m^0(x)|^2 &= \rho^0(x)\chi(0)\mathbb{1}_{\Omega'} \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^d \end{aligned} \tag{4.5}$$

for some smooth function $\chi : \mathbb{R} \to \mathbb{R}$.

Theorem 4.2. *Under the same assumptions of Theorem 4.1, there exists a maximal time $\overline{T} > 0$ such that the weak solutions (ρ, m) of (4.4) (coming from Theorem 4.1) satisfy the admissibility condition:*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \left[\left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho} \right) \partial_t \varphi + \left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \cdot \nabla_x \varphi \right] \mathrm{d}x\, \mathrm{d}t \\ &+ \int_{\mathbb{R}^d} \left(\rho^0 \varepsilon(\rho^0) + \frac{1}{2}\frac{|m^0|^2}{\rho} \right) \varphi(0, \cdot)\, \mathrm{d}x \geq 0 \end{aligned} \tag{4.6}$$

for every nonnegative $\varphi \in C_c^\infty([0,T); C_c^\infty(\mathbb{R}^d))$ with $T \leq \overline{T}$.

Corollary 4.3. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a nonempty bounded open set. Assume that $\rho^0 \in C^1(\mathbb{R}^d)$ satisfies*

- *$\rho^0(x) > 0$ for any $x \in \mathbb{R}^d$,*
- *$\rho^0(x) = \overline{\rho}$ for $x \in \mathbb{R}^d \backslash \Omega$.*

Let $p \in C^1$ be given function such that $\int_\Omega p(\rho(x))\,\mathrm{d}x = p(\overline{\rho})|\Omega|$. Then there exist $\Omega' \supset \Omega$, m^0 and a positive time $\overline{T}$ such that $\operatorname{supp} m^0 \subset \Omega'$, $\operatorname{div}_x m^0 = 0$ for which there exist infinitely many m such that $\operatorname{supp} m(t,\cdot) \subset \Omega'$ for $t \in [0, \overline{T})$ and (ρ, m) is an admissible solution of (4.1) on $[0,\overline{T}) \times \mathbb{R}^d$ with $\rho(t,x) = \rho_0(x)\mathbb{1}_{[0,\overline{T})}(t) \in C^1([0,\overline{T}) \times \mathbb{R}^d)$.

The results of this chapter are partially motivated by possible applications to the 3D axisymmetric incompressible Euler equations (work in progress). Let us mention that, in the incompressible context, spatially compactly supported solutions can readily be constructed in L^∞, but for (Hölder-)continuous solutions this question has apparently only been dealt with in [47]. In the compressible setting, continuous non-unique solutions are not yet available. To our knowledge, the present note yields the first examples of compactly supported non-unique solutions to (4.1) whose density is not piecewise constant.

Methodologically, the general convex integration scheme employed here follows the ones in [27, 17]. However, in the absence of periodic boundary conditions, we can no longer use the subsolutions constructed in [17]. Instead, we use elliptic theory to obtain compactly supported solutions for a Poisson equation whose right-hand side has a specific structure (Lemma 4.5), and from this we build our subsolutions. We can then show, albeit with more effort than in [17], that these subsolutions give rise to weak solutions satsfying the energy inequality (4.2).

The chapter is organized as follows. Section 4.2 contains some notions and facts that we will need later on. In Section 4.3, we recall some known results from convex integration theory and adapt some of them to our settings. The purpose of Section 4.4 is to provide a general criterion on the existence of weak solutions to (4.1) for a given initial data. Section 4.7 proves the non-uniqueness results by mainly employing Proposition 4.17 and Proposition 4.25.

4.2 Preliminaries

In this section we establish some facts and notions which will be useful later on. For any set $\Omega \subset \mathbb{R}^d$ and $\varepsilon > 0$, define $\Omega^\varepsilon := \{y \in \mathbb{R}^d : \text{dist}(y, \Omega) < \varepsilon\}$.

For a domain $\Omega \subset \mathbb{R}^d$, we denote by $L^2_w(\Omega)$ the space of measurable, square integrable functions equipped with the weak topology. By $H(\Omega)$, we denote the space of solenoidal L^2-vectorfields $\Omega \to \mathbb{R}^d$, and $H_w(\Omega)$ is the same space but with the weak topology.

Next, recall the Paley-Wiener-Schwartz Theorem (for more details, see [58, 45]):

Theorem 4.4 (Paley-Wiener-Schwartz). *If u is a distribution of order N with support contained in a closed ball $\overline{B_r(0)} \subset \mathbb{R}^d$, then its Fourier transform $\widehat{u}$ can be extended to an entire function in $\mathbb{C}^d$ satisfying*

$$|\widehat{u}(\xi)| \leq C(1 + |\xi|)^N e^{r|Im\ \xi|}, \quad \xi \in \mathbb{C}^d.$$

Conversely, an entire function in $\mathbb{C}^d$ meeting such an estimate is the Fourier transform of a distribution of order N supported inside $\overline{B_r(0)} \subset \mathbb{R}^d$.

Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a smooth spherically symmetric function satisfying the following conditions: $w(x)$ is constant for $|x| \leq \frac{1}{2}$, $\operatorname{supp}(\omega) \subset B_1(0)$, $\omega(x) \geq 0$ and

$$\int_{\mathbb{R}^d} \omega(x)\,\mathrm{d}x = 1.$$

Denote $\omega^\varepsilon(x) = \dfrac{1}{\varepsilon^d}\omega\left(\dfrac{x}{\varepsilon}\right)$ for $\varepsilon > 0$. Since $\rho^0(x) = \overline{\rho}$ for $x \notin \Omega$, we have $p(\rho^0) = p(\overline{\rho}) = \overline{p}$. Let $p_1(x) = p(\rho^0(x)) - \overline{p}$. Then $\operatorname{supp} p_1 \subset \Omega$ and

$$\int_{\mathbb{R}^d} p_1(x)\,\mathrm{d}x = \int_{\Omega} p_1(x)\,\mathrm{d}x = 0.$$

The main point about the following lemma is that, for right hand sides of a specific form, Poisson's equation admits solutions with compact support.

Lemma 4.5. *Let p^ε be defined by $p^\varepsilon(x) := p_1(x) - p_1 * \omega^\varepsilon(x)$, so that $\operatorname{supp}(p^\varepsilon) \subset \Omega^\varepsilon$. Then, there exists $u \in C_c^{2,\alpha}(\mathbb{R}^d)$ for every $0 < \alpha < 1$ such that*

$$\Delta u = p^\varepsilon \quad \textit{and} \quad \operatorname{supp} u \subset \overline{\Omega^\varepsilon}.$$

Proof. By the Paley-Wiener-Schwartz Theorem (Theorem 4.4), the Fourier transform $\widehat{p^\varepsilon}$ is an analytic function, and we have

$$\widehat{p^\varepsilon} = \widehat{p_1} - \widehat{p_1}\widehat{\omega^\varepsilon} = \widehat{p_1}(1 - \widehat{\omega^\varepsilon}).$$

Note that $\widehat{\omega^\varepsilon}(0) = 1$. Since ω^ε is a spherically symmetric function, so is $\widehat{\omega^\varepsilon}(\xi)$. So there exists a single variable analytic function g on $\mathbb{C}$ such that

$$\widehat{\omega^\varepsilon}(\xi) = g(|\xi|^2), \quad \text{where} \quad |\xi|^2 = \xi_1^2 + \cdots + \xi_d^2.$$

Again due to the Paley-Wiener-Schwartz Theorem, as $\operatorname{supp} \omega^\varepsilon \subset B_\varepsilon(0)$, we have the estimate

$$|\widehat{\omega^\varepsilon}(\xi)| \leq C \exp(\varepsilon |\mathrm{Im}\ \xi|) \quad \text{for} \quad \xi \in \mathbb{C}^d.$$

Moreover, since $g(0) = 1$, there exists an analytic function g_1 such that $1 - g(z) = -z g_1(z)$. Therefore,

$$1 - g(|\xi|^2) = -|\xi|^2 g_1(|\xi|^2),$$

and so we have

$$\left||\xi|^2 g_1(|\xi|^2)\right| \leq C \exp(\varepsilon |\mathrm{Im}\ \xi|).$$

We claim that

$$\left|g_1(|\xi|^2)\right| \le C_2 \exp(\varepsilon|\text{Im }\xi|).$$

Indeed, $g_1(z)$ is bounded for $|z| \le 1$, say $|g_1(z)| \le A$ for $|z| \le 1$. If $||\xi|^2| \ge 1$, then obviously we have

$$\left|g_1(|\xi|^2)\right| \le C \exp(\varepsilon|\text{Im }\xi|).$$

Thus, combining the two obtained bounds we get

$$\left|g_1(|\xi|^2)\right| \le \max\{A, C\} \exp(\varepsilon|\text{Im }\xi|) \quad \text{for all} \quad \xi \in \mathbb{C}^d.$$

Invoking once again the Paley-Wiener-Schwartz Theorem, the inverse Fourier transform $\breve{g_1}$ is concentrated in $\overline{B_\varepsilon(0)}$, i.e., $\operatorname{supp} \breve{g_1} \subset \overline{B_\varepsilon(0)}$. Hence, we obtain

$$\widehat{p^\varepsilon} = -|\xi|^2 \widehat{p_1} g_1(|\xi|^2).$$

Let u be a distribution defined by its Fourier transform

$$\widehat{u}(\xi) = \widehat{p_1} g_1(|\xi|^2).$$

Then $u = p_1 * \breve{g_1}$, and from $\operatorname{supp} \breve{g_1} \subset \overline{B_\varepsilon(0)}$ and $\operatorname{supp} p_1 \subset \Omega$ it follows that $\operatorname{supp} u \subset \overline{\Omega^\varepsilon}$. Moreover, we have

$$\widehat{\Delta u} = -|\xi|^2 \widehat{u}(\xi) = -|\xi|^2 \widehat{p_1} g_1(|\xi|^2) = \widehat{p^\varepsilon}$$

and thus $\Delta u = p^\varepsilon$ as desired. Finally, the regularity $u \in C^{2,\alpha}$ follows from standard Schauder theory, as p^ε is continuously differentiable.

The lemma is proved. □

Remark 4.6. Note that, generally, $p \in C_c^\infty(\mathbb{R}^d)$ does not imply that the unique decaying solution u of $\Delta u = p$ is compactly supported. For instance, if $p \ge 0$ is not identically zero, the the corresponding solution u does not have compact support. Indeed, any compactly supported subharmonic function vanishes identically by the maximum principle.

Lemma 4.7. *Let $d \ge 2$. Let Ω' be a bounded locally Lipschitz domain containing $\overline{\Omega^\varepsilon} \subset \mathbb{R}^d$. If $p \in C_c^\infty(\Omega^\varepsilon)$ satisfies the compatibility condition*

$$\int_{\Omega'} p(x)\,\mathrm{d}x = 0,$$

then there exists $(\varphi_j)_{j=1}^d \subset C_c^\infty(\Omega')$ such that

$$\sum_{j=1}^d \partial_j \varphi_j = p.$$

This lemma is an immediate consequence of [40, Theorem III.3.3].

Remark 4.8. Note that the regularity assumption on Ω' can be further weakened, for instance, to a bounded domain of $\mathbb{R}^d$, such that $\Omega' = \cup_{k=1}^N \Omega'_k$, $N \geq 1$, where each Ω'_k is star-shaped with respect to some open ball B_k with $\overline{B_k} \subset \Omega'_k$ and

$$\int_{\Omega'_k} p(x)\,\mathrm{d}x = 0, \quad \text{for any} \quad k = 1, \dots, N$$

(see Theorem III3.1 [40]).

Let $\mathcal{S}_d$ be the space of symmetric $d \times d$ matrices and let $\mathcal{S}_0^d$ be the subspace of $\mathcal{S}_d$ with null trace. Further we denote by I_d the $d \times d$ identity matrix.

Proposition 4.9. *Let $p \in C_c^\infty(\Omega^\varepsilon)$ such that $\int_{\Omega'} p(x)\,\mathrm{d}x = 0$, with Ω' being given as in Lemma 4.7. Then there exists a pair (m, U) of a vector field and a matrix field with values in $\mathcal{S}_0^d$ satisfying the following conditions:*

(i) $(m(t), U) \in C_c^\infty(\Omega')$ *for each* $t \in \mathbb{R}$,

(ii) $\operatorname{div} m = 0$,

(iii) $\partial_t m + \operatorname{div} U + \nabla p = 0$,

(iv) m *is linear in* t, *and* U *does not depend on* t.

Proof. Define the matrix field A by

$$A := \left[\frac{d}{1-d}\left(\partial_i \varphi_j - \frac{p}{d}\delta_{ij}\right)\right]_{ij}, \quad i, j = 1, \dots, d,$$

where $(\varphi_j)_{j=1}^d$ are chosen as in Lemma 4.7 and δ_{ij} is the Kronecker delta. We have

$$\operatorname{tr}(A) = \frac{d}{1-d}\left(\sum_{i=1}^d \partial_i \varphi_i - p\right) = \frac{d}{1-d}(p - p) = 0.$$

We write $A = U + V$, where $U = \frac{1}{2}(A^t + A)$, $V = \frac{1}{2}(A - A^t)$, and A^t is the transpose of A. Then $U \in \mathcal{S}_0^d$, and V is a skew-symmetric matrix, i.e., $V^t = -V$. Note that if V is a skew-symmetric matrix field, then $\operatorname{div}\operatorname{div} V = 0$. Indeed,

$$\operatorname{div}\operatorname{div} V = \sum_{i,j=1} \partial_i \partial_j V_{ij} = -\sum_{i,j=1}^d \partial_j \partial_i V_{ij}.$$

Further, we have

$$\begin{aligned}(\operatorname{div} A)_i &= \frac{d}{1-d}\left(\sum_{j=1}^{d}\partial_i\partial_j\varphi_j - \frac{\partial_i p}{d}\right)\\ &= \frac{d}{1-d}\partial_i\left(\sum_{j=1}^{d}\partial_j\varphi_j - \frac{p}{d}\right) = \left(\frac{d}{1-d}\frac{d-1}{d}\right)\frac{\partial p}{\partial x_i}\\ &= -\partial_i p.\end{aligned}$$

Thus, we get $\operatorname{div} A = -\nabla p$. Take $m = t \operatorname{div} V$. Then,

$$\operatorname{div} m = t \operatorname{div}\operatorname{div} V = 0$$

as well as

$$\partial_t m + \operatorname{div} U = \operatorname{div} V + \operatorname{div} U = \operatorname{div} A = -\nabla p.$$

The proposition is proved. □

Remark 4.10. Let $p \in C_c^\infty(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} p(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \neq 0$. Then there are no functions $U_1, U_2 \in C_c^\infty(\mathbb{R}^2)$ satisfying

$$\begin{cases}\partial_1 U_1 + \partial_2 U_2 = \partial_1 p,\\ \partial_1 U_2 - \partial_2 U_1 = \partial_2 p.\end{cases}$$

Indeed, if this were the case, then $\Delta U_2 = 2\partial_1\partial_2 p$, or equivalently $-|\xi|^2\widehat{U_2} = -2\xi_1\xi_2\widehat{p}$, where $\widehat{p}(0) \neq 0$. Then $\widehat{U_2}$ and $\widehat{p}$ would be analytic functions in virtue of the compact support of U_2 and p (see Theorem 4.4). However, $|\xi|^2\widehat{U_2} = 2\xi_1\xi_2\widehat{p}$ implies that $|\xi|^2$ divides $\widehat{p}$, in contradiction to $\widehat{p}(0) \neq 0$.

This consideration shows that our compatibility assumption is necessary in the proof of Proposition 4.9, as otherwise it would be impossible to construct a symmetric traceless matrix field A with compact support such that $\operatorname{div} A = -\nabla p$.

4.3 Geometric Setup

We formulate the Euler equations as a differential inclusion and recall some well-known tools, thereby closely following [17, 27].

Lemma 4.11. *Let $m \in L^\infty((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$, $U \in L^\infty((0,T) \times \mathbb{R}^d; \mathcal{S}_0^d)$ and*
$q \in L^\infty((0,T) \times \mathbb{R}^d; \mathbb{R})$ such that

$$\begin{aligned}\operatorname{div}_x m &= 0,\\ \partial_t m + \operatorname{div}_x U + \nabla_x q &= 0.\end{aligned} \tag{4.7}$$

If (m, U, q) *solve* (4.7) *and in addition there exists* $\rho \in L^\infty(\mathbb{R}^d; \mathbb{R}^+)$ *such that*

$$\begin{aligned} U &= \frac{m \otimes m}{\rho} - \frac{|m|^2}{d\rho} I_d \quad a.e.\ in \quad [0,T] \times \mathbb{R}^d, \\ q &= p(\rho) + \frac{|m|^2}{d\rho} \quad a.e.\ in \quad [0,T] \times \mathbb{R}^d, \end{aligned} \tag{4.8}$$

then m *and* ρ *solve* (4.4) *distributionally. Conversely, if* m *and* ρ *are weak solutions of* (4.4)*, then* m*,* $U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{d\rho} I_d$ *and* $q = p(\rho) + \frac{|m|^2}{d\rho}$ *satisfy* (4.7) *and* (4.8)*.*

The above Lemma is based on Lemma 3.1 in [17].

Next, for any given $\rho \in (0, \infty)$, we define the graph

$$K_\rho := \left\{ (m, U, q) \in \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R}^+ : U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{d\rho} I_d, \ \ q = p(\rho) + \frac{|m|^2}{d\rho} \right\}.$$

In the present setting, it is convenient to consider "slices" of the graph K_ρ as in [17]. For any given $\chi \in \mathbb{R}^+$, we thus define

$$\begin{aligned} K_{\rho,\chi} := \Bigg\{ (m, U, q) \in \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R}^+ : U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{d\rho} I_d, \\ q = p(\rho) + \frac{|m|^2}{d\rho}, \ \ |m|^2 = \rho\chi \Bigg\}. \end{aligned}$$

Consider the $(d+1) \times (d+1)$ symmetric matrix in block form

$$M = \begin{pmatrix} U + qI_d & m \\ m & 0 \end{pmatrix}.$$

Note that, with the new coordinates $y = (t, x) \in \mathbb{R}^d$, the system (4.7) can be easily rewritten as $\operatorname{div}_y M = 0$. Thus, the wave cone associated with the system (4.7), i.e., the set of all states (m, U, q) such that there exists $\xi \in \mathbb{R}^{d+1} \setminus \{0\}$ such that $(m, U, q) h(y \cdot \xi)$ satisfies (4.7) for every profile $h : \mathbb{R} \to \mathbb{R}$, is equal to

$$\Lambda = \left\{ (m, U, q) \in \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R}^+ : \det \begin{pmatrix} U + qI_d & m \\ m & 0 \end{pmatrix} = 0 \right\}.$$

For any $S \in \mathcal{S}^d$ let $\lambda_{\max}(S)$ denote the largest eigenvalue of S.

Lemma 4.12. *For* $(\rho, m, U) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{S}_0^d$ *let*

$$e(\rho, m, U) := \lambda_{\max}\left(\frac{m \otimes m}{\rho} - U \right).$$

Then, for any given $\rho, \chi \in \mathbb{R}^+$*,* *(i)* $e(\rho,\cdot,\cdot)\colon \mathbb{R}^d \times \mathcal{S}_0^d \to \mathbb{R}$ *is convex;*

(ii) $\frac{|m|^2}{d\rho} \le e(\rho, m, U)$, *with equality if and only if* $U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{d\rho} I_d$*;*

(iii) $\|U\|_\infty \le (d-1)e(\rho, m, U)$, *where* $\|U\|_\infty$ *is the operator norm of* U*;*

(iv) the $\frac{\chi}{d}-$*sublevel set of* e *is the convex hull of* $K_{\rho,\chi}$*, namely,*

$$K^{co}_{\rho,\chi} = \left\{ (m, U, q) \in \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R}^+ : e(\rho, m, U) \le \frac{\chi}{d}, q = p(\rho) + \frac{\chi}{d} \right\}$$

and $K_{\rho,\chi} = K^{co}_{\rho,\chi} \cap \{|m|^2 = \rho\chi\}$.

For the proof of the Lemma we refer to [27, Lemma 3] and [17, Lemma 3.2].

Finally, for any $\rho, \chi \in \mathbb{R}^+$, we define the hyperinterior of $K^{co}_{\rho,\chi}$:

$$\text{hint } K^{co}_{\rho,\chi} := \left\{ (m, U, q) \in \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R}^+ : e(\rho, m, U) < \frac{\chi}{d},\ \ q = p(\rho) + \frac{\chi}{d} \right\}.$$

Lemma 4.13 (Lemma 3.3 in [17])**.** *Let* $a, b \in \mathbb{R}^d$ *with* $|a| = |b|$ *and* $a \ne b$*, and let* $\rho \in \mathbb{R}^+$*. Then* $\left(a - b, \frac{a\otimes a}{\rho} - \frac{b\otimes b}{\rho}, 0\right) \in \Lambda$.

Definition 4.14. Given $\rho, \chi \in \mathbb{R}^+$ we call σ an *admissible segment* for (ρ, χ) if σ is a line segment in $\mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R}^+$ satisfying the following conditions:

- σ is contained in the hyperinterior of $K^{co}_{\rho,\chi}$;
- σ is parallel to $\left(a - b, \frac{a\otimes b}{\rho} - \frac{b\otimes b}{\rho}, 0\right)$ for some $a, b \in \mathbb{R}^d$ with $|a|^2 = |b|^2 = \rho\chi$ and $a \ne \pm b$.

Lemma 4.15 (Lemma 3.5 in [17])**.** *There exists a constant* $F = F(n) > 0$ *such that for any* $\rho, \chi \in \mathbb{R}^+$ *and for any* $z = (m, U, q) \in$ *hint* $K^{co}_{\rho,\chi}$ *there exists an admissible line segment for* (ρ, χ)

$$\sigma = [(m, U, q) - (\overline{m}, \overline{U}, 0), (m, U, q) + (\overline{m}, \overline{U}, 0)]$$

such that

$$|\overline{m}| \ge \frac{F}{\sqrt{\rho\chi}} \left(\rho\chi - |m|^2\right).$$

Proposition 4.16 (Proposition 3.6 in [17])**.** *For all given* $\rho, \chi \in \mathbb{R}^+$*, the* $\Lambda-$*convex hull of* $K_{\rho,\chi}$ *coincides with the convex hull of* $K_{\rho,\chi}$*.*

4.4 A Criterion for the Existence of Admissible Solutions

In this section we give some criteria to recognize initial data m^0 for which there exist infinitely many weak admissible solutions to (4.1). Again, this section closely follows [27] and [17]. It can be also observed in [2].

Proposition 4.17. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $\rho_0 \in C^1(\mathbb{R}^d)$ be a given density function with $\rho_0(x) = \overline{\rho} = const$ for $x \in \mathbb{R}^d \setminus \Omega$ and let $T > 0$ be any finite time and $\Omega' \supset \Omega$ be a bounded open set. Assume that there exist (m_0, U_0, q_0) continuous solutions of*

$$\begin{cases} \operatorname{div}_x m_0 = 0, \\ \partial_t m_0 + \operatorname{div}_x U_0 + \nabla_x q_0 = 0 \quad \text{on} \quad (0,T) \times \mathbb{R}^d \end{cases}$$

with $m_0 \in C([0,T]; H_w(\mathbb{R}^d))$, $\operatorname{supp}(m_0(t,\cdot), U_0(t,\cdot)) \subset\subset \Omega'$ for all $t \in (0,T)$, and a function $\chi \in C^\infty([0,T]; \mathbb{R}^+)$ such that

$$e\left(\rho_0(x), m_0(t,x), U_0(t,x)\right) < \frac{\chi(t)}{d}$$

for all $(t,x) \in (0,T) \times \mathbb{R}^d$,

$$q_0(t,x) = p(\rho_0(x)) + \frac{\chi(t)}{d}$$

for all $(t,x) \in (0,T) \times \mathbb{R}^d$. Then there exist infinitely many weak solutions (ρ, m) of the system (4.4) in $[0,T) \times \mathbb{R}^d$ with density $\rho(x) = \rho_0(x)$ and such that

$$\begin{aligned} & m \in C([0,T]; H_w(\mathbb{R}^d)), \\ & m(t,\cdot) = m_0(t,\cdot) \quad \text{for} \quad t = 0, T \quad \text{and for a.e. } x \in \mathbb{R}^d, \\ & |m(t,x)|^2 = \rho_0(x)\chi(t)\mathbb{1}_{\Omega'} \quad \text{for a.e} \quad (t,x) \in (0,T) \times \mathbb{R}^d. \end{aligned} \tag{4.9}$$

4.4.1 The Space of Subsolutions

Let m_0 be a vector field as in Proposition 4.17 with associated modified pressure q_0,

$$q_0 = p(\rho_0) + \frac{\chi(t)}{d},$$

where ρ_0 and χ are given functions as in the assumptions of Proposition 4.17. Consider momentum fields $m\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfy

$$\operatorname{div} m = 0, \tag{4.10}$$

the initial conditions

$$\begin{aligned} &m(0,x) = m_0(0,x),\\ &m(t,x) = m_0(T,x),\\ &\operatorname{supp} m(t,\cdot) \subset \Omega' \quad \text{for all} \quad t \in (0,T) \end{aligned} \tag{4.11}$$

and such that there exists a continuous matrix field $U \colon \mathbb{R}^d \times (0,T) \to \mathcal{S}_0^d$ with

$$\begin{aligned} &e\left(\rho_0(x), m(t,x), U(t,x)\right) < \frac{\chi(t)}{d} \quad \text{for all} \quad (t,x) \in (0,T) \times \Omega',\\ &\operatorname{supp}\left(U(t,\cdot)\right) \subset \Omega' \quad \text{for all} \quad t \in (0,T),\\ &\partial_t m + \operatorname{div} U + \nabla q_0 = 0 \quad \text{in} \quad [0,T] \times \mathbb{R}^d. \end{aligned} \tag{4.12}$$

Definition 4.18. Let X_0 be the set of such momentum fields:

$$X_0 = \Big\{ m \in C^0\left((0,T); C_c(\mathbb{R}^d)\right) \cap C\left([0,T]; H_w(\mathbb{R}^d)\right) : \\ (4.10), (4.11), (4.12) \text{ are satisfied} \Big\}$$

and let X be the closure of X_0 in $C\left([0,T]; H_w(\mathbb{R}^d)\right)$. Then X_0 is called the space of strict subsolutions.

Let

$$G = \sup_{t\in[0,T]} \chi(t) \int_{\Omega'} \rho_0(x)\, \mathrm{d}x.$$

Since for any $m \in X_0$ with associated matrix field U, we have that (see Lemma 4.12 (ii))

$$\begin{aligned} \int_{\mathbb{R}^d} |m(t,x)|^2\, \mathrm{d}x &= \int_{\Omega'} |m(t,x)|^2\, \mathrm{d}x\\ &\le \int_{\Omega'} d\rho_0(x) e\left(\rho_0(x), m(t,x), U(t,x)\right)\, \mathrm{d}x\\ &\le \chi(t) \int_{\Omega'} \rho_0(x)\, \mathrm{d}x \le G \quad \text{for all } t \in [0,T]. \end{aligned}$$

We can observe that X_0 consists of functions $m \colon [0,T] \to H(\mathbb{R}^d)$ taking values in a bounded subset $B = B_G(0)$ of $H(\mathbb{R}^d)$. Without loss of generality, we can assume that B is weakly closed. Then B is metrizable in its weak topology and, if we let d_B be a metric on B inducing the weak topology, we have that (B, d_B) is a compact metric space. Moreover, we can define on $Y := C([0,T], (B, d_B))$ a metric d naturally induced by d_B via

$$d(f_1, f_2) = \max_{t\in[0,T]} d_B\left(f_1(t,\cdot), f_2(t,\cdot)\right).$$

Note that the topology induced on Y by d is equivalent to the topology of Y as a subset of $C([0,T];H_w)$. In addition, the space (Y,d) is complete. Finally, X is the closure in (Y,d) of X_0 and hence (X,d) is as well a complete metric space. If $m \in X$ then $\operatorname{supp}(m(t,\cdot)) \subset \overline{\Omega'}$ for any $t \in [0,T]$.

Lemma 4.19. *If $m \in X$ is such that $|m(t,x)|^2 = \rho_0(x)\chi(t)\mathbb{1}_{\Omega'}$ for a.e. $(t,x) \in (0,T) \times \mathbb{R}^d$, then the pair (ρ_0, m) is a weak solution of (4.4) in $[0,T) \times \mathbb{R}^d$ satisfying (4.9).*

Proof. Let $m \in X$ be such that

$$|m(t,x)|^2 = \rho_0(x)\chi(t) \quad \text{for a.e.} \quad (t,x) \in (0,T) \times \Omega'.$$

By density of X_0, there exists a sequence $\{m_k\} \subset X_0$ such that $m_k \xrightarrow{d} m$ in X. For any $m_k \in X_0$, let U_k be the associated smooth matrix field enjoying the properties (4.12). By using Lemma 4.12 (iii) and

$$e\left(\rho_0(x), m_k(t,x), U_k(t,x)\right) < \frac{\chi(t)}{d},$$

the following pointwise estimate holds for the sequence $\{U_k\}$:

$$|U_k(t,x)| \le (d-1)e\left(\rho_0(x), m_k(t,x), U_k(t,x)\right) < \frac{(d-1)\chi(t)}{d}.$$

Consequently,

$$\|U_k\|_\infty \le (d-1)\left\|e\left(\rho_0(\cdot), m_k(t,\cdot), U_k(t,\cdot)\right)\right\|_\infty < \frac{(d-1)\chi(t)}{d}.$$

As a consequence, $\{U_k\}$ is uniformly bounded in $L^\infty((0,T)\times\mathbb{R}^d)$, and by possibly extracting a subsequence, we have

$$U_k \overset{*}{\rightharpoonup} U \quad \text{in} \quad L^\infty((0,T)\times\mathbb{R}^d).$$

Following [17], $\overline{\text{hint } K^{co}_{\rho_0,\chi}} = K^{co}_{\rho_0,\chi}$ is a convex and compact set by Lemma 4.12 (i), (ii), (iii). Hence $m \in X$ with associated matrix field U solves

$$\begin{aligned} &\operatorname{div}_x m = 0, \\ &\partial_t m + \operatorname{div}_x U + \nabla_x q_0 = 0 \quad \text{on} \quad [0,T]\times\mathbb{R}^d \end{aligned}$$

and (m, U, q_0) takes values in $K^{co}_{\rho_0,\chi}$ almost everywhere. If, in addition, $|m(t,x)|^2 = \rho_0\chi(t)\mathbb{1}_{\Omega'}$, then $(m,U,q_0)(t,x) \in K_{\rho,\chi}$ a.e. in $[0,T]\times\mathbb{R}^d$ (because $K^{co}_{\rho,\chi} \cap \left\{|m|^2 = \rho\chi\right\} = K_{\rho,\chi}$). Lemma 4.11 allows us to conclude that (ρ_0, m) is a weak solution of (4.4). Finally, since $m_k \to m$ in $C([0,T];H_w(\mathbb{R}^d))$ and $|m(t,x)|^2 = \rho_0(x)\chi(t)\mathbb{1}_{\Omega'}$ for almost every $(t,x) \in (0,T)\times\mathbb{R}^d$, we see that m satisfies also (4.9). Lemma 4.19 is proved. □

Lemma 4.20. *The identity map* $I\colon (X,d) \to L^2([0,T];H)$ *defined by* $m \mapsto m$ *is a Baire-1 map, and therefore the set of points of continuity is residual in* (X,d).

Proof. Let $\varphi(t,x) \geq 0$, $\varphi \in C^\infty(\mathbb{R}^{d+1})$, $\operatorname{supp}\varphi \subset \left\{|x|^2 + t^2 \leq 1\right\}$ and

$$\int_{|x|^2+t^2\leq 1} \varphi(t,x)\ \mathrm{d}x\ \mathrm{d}t = 1.$$

Let $\varphi_r(t,x) = r^{-(d+1)}\varphi(rx,rt)$ be a convolution kernel. For each fixed $m \in X$, we have

$$\varphi_r * m \to m \quad \text{strongly in} \quad L^2([0,T];H)$$

as $r \to 0+$. Indeed,

$$\varphi_r * m(t,x) = \int_{\mathbb{R}^{d+1}} m(y,\tau)\varphi_r(x-y,t-\tau)\ \mathrm{d}y d\tau.$$

If $m \in L^2(H)$ then it is well-known that

$$\|\varphi_r * m - m\|_{L^2([0,T];H)} \to 0 \quad \text{as}\, r \to 0+\,.$$

On the other hand, for each $r > 0$ and $m_k \in X$, $m_k \xrightarrow{d} m$ implies

$$\varphi_r * m_k \to \varphi_r * m \quad \text{in} \quad L^2([0,T];H).$$

Therefore, each map $I_r\colon (X,d) \to L^2([0,T];H)$, $m \mapsto \varphi_r * m =: I_r(m)$ is continuous, and $I(m) = \lim_{r\to 0+} I_r(m) = m$ for all $m \in X$. This shows that $I\colon (X,d) \to L^2([0,T];H)$ is pointwise limit of continuous maps. Hence, it is a Baire-1 map. As a consequence, the set of points of continuity of I is residual in (X,d). It means that I is continuous except at a set of points of the first category (see Theorem 7.3 in [56]). The first category set is the set which can be represented as a countable union of nowhere dense sets. So if (X,d) contains continuum points then the set of points of continuity I is infinite. □

4.4.2 Proof of Proposition 4.17

We aim to show that all points of continuity of the identity map correspond to solutions of (4.4) satisfying the requirements of Proposition 4.17. Continuity of the identity map (see Lemma 4.20) will then allow us to prove Proposition 4.17, once we know that the cardinality of X is infinite. In light of Lemma 4.19, for our purpose it suffices to prove the following claim.

Claim: If $m \in X$ is a point of continuity of I, then

$$|m(t,x)|^2 = \rho_0(x)\chi(t)\mathbb{1}_{\Omega'} \text{ for a.e } (t,x) \in (0,T) \times \mathbb{R}^d. \tag{4.13}$$

As in [17], (4.13) is equivalent to

$$\|m\|_{L^2([0,T]\times\Omega')} = \left(\int_{\Omega'}\int_0^T \rho_0(x)\chi(t)\ \mathrm{d}t\ \mathrm{d}x\right)^{\frac{1}{2}},$$

since for any $m \in X$, we have

$$|m(t,x)|^2 \leq \rho_0(x)\chi(t)\mathbb{1}_{\Omega'}(x)$$

for almost all $(t,x) \in (0,T)\times\Omega'$. Thanks to this remark, the claim is reduced to the following lemma.

Lemma 4.21. *Let ρ_0, χ be given functions as in Proposition 4.17. Then there exists a constant $\beta = \beta(d) > 0$ such that, given $m \in X_0$, there exists a sequence $\{m_k\} \subset X_0$ with the following properties:*

$$\|m_k\|_{L^2([0,T]\times\Omega')} \geq \|m\|_{L^2([0,T]\times\Omega')} + \beta\left(\int_{\Omega'}\int_0^T \rho_0(x)\chi(t)dt\ \mathrm{d}x - \|m\|^2_{L^2([0,T]\times\Omega')}\right)^2 \tag{4.14}$$

and $m_k \to m$ in $C([0,T], H_w(\Omega'))$.

Before we give the proof of the above Lemma, let us show how Lemma 4.21 implies the claim. Let $m \in X$ be a point of continuity of I. Owing to the density of X_0 in X and Lemma 4.21, there exist sequences $\{m_k\}, \{\widetilde{m}_k\} \subset X_0$ such that $m_k \xrightarrow{d} m$, $\widetilde{m}_k \xrightarrow{d} m$, and

$$\liminf_{k\to\infty} \|\widetilde{m}_k\|^2_{L^2([0,T]\times\Omega')} \geq \liminf_{k\to\infty}\left(\|m_k\|^2_{L^2([0,T]\times\Omega')} + \beta\left(\int_{\Omega'}\int_0^T \rho_0(x)\chi(t)\ \mathrm{d}t\ \mathrm{d}x - \|m_k\|^2_{L^2([0,T]\times\Omega')}\right)^2\right).$$

By the assumption, I is continuous at m, which implies that both m_k and $\widetilde{m}_k$ converge strongly to m and

$$\|m\|_{L^2([0,T]\times\Omega')} \geq \|m\|_{L^2([0,T]\times\Omega')} + \beta\left(\int_{\Omega'}\int_0^T \rho_0(x)\chi(t)\ \mathrm{d}t\ \mathrm{d}x - \|m\|^2_{L^2([0,T]\times\Omega')}\right)^2.$$

Therefore $\|m\|^2_{L^2([0,T]\times\Omega')} = \int_{\Omega'}\int_0^T \rho_0(x)\chi(t)\ \mathrm{d}t\ \mathrm{d}x$ and the claim is proved. □

4.5 Perturbation Property

The sake of this section is to prove Lemma 4.21. To do so, first, we will recall some of the facts from [17, 27]. As mentioned before in Section 4.3, the system (4.7) is equivalent to $\operatorname{div}_y M = 0$ in the variables $y = (t, x) \in \mathbb{R}^{d+1}$, where $M \in \mathcal{S}^{d+1}$ is defined via the linear map

$$\mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R} \ni (m, U, q) \mapsto M = \begin{pmatrix} U + qI_d & m \\ m & 0 \end{pmatrix}$$

More precisely, this map builds an identification between the set of solutions (m, U, q) to (4.7) and the set of symmetric $(d+1) \times (d+1)$ matrices M with $M_{(d+1)(d+1)} = 0$ and $tr(M) = q$. Therefore, solutions of (4.7) with $q \equiv 0$ correspond to matrix fields $M\colon \mathbb{R}^{d+1} \to \mathbb{R}^{(d+1)\times(d+1)}$ such that

$$\operatorname{div}_y M = 0, \quad M^T = M, \quad M_{(d+1)(d+1)} = 0, \quad tr(M) = 0. \tag{4.15}$$

Moreover, given a density ρ and two states $(a, U_a, q_a), (b, U_b, q_b) \in K_\rho$ with non collinear momentum vector fields a and b having same magnitude ($|a| = |b|$), and hence same pressure ($q_a = q_b$), then the corresponding matrices M_a and M_b have the following form

$$M_a = \begin{pmatrix} \frac{a\otimes a}{\rho} + p(\rho)I_d & a \\ a & 0 \end{pmatrix} \quad \text{and} \quad M_b = \begin{pmatrix} \frac{b\otimes b}{\rho} + p(\rho)I_d & b \\ b & 0 \end{pmatrix}$$

satisfy

$$M_a - M_b = \begin{pmatrix} \frac{a\otimes a - b\otimes b}{\rho} & a-b \\ a-b & 0 \end{pmatrix}.$$

Finally note that $tr(M_a - M_b) = 0$ and $M_a - M_b \in \Lambda$ corresponds to a special direction. The following Proposition provides a potential for solutions of (4.7) oscillating between two states M_a and M_b at constant pressure.

Proposition 4.22. *Let $a, b \in \mathbb{R}^d$ such that $|a| = |b|$ and $a \neq b$. Let also $\rho \in \mathbb{R}$. Then there exists a matrix-valued, constant coefficient, homogeneous linear differential operator of order 3*

$$A(\partial)\colon C_c^\infty(\mathbb{R}^{d+1}) \to C_c^\infty(\mathbb{R}^{d+1}; \mathbb{R}^{(d+1)\times(d+1)})$$

such that $M = A(\partial)\varphi$ satisfies (4.15) for all $\varphi \in C_c^\infty(\mathbb{R}^{d+1})$. Moreover there exists $\eta \in \mathbb{R}^{d+1}$ such that

- *η is not parallel to e_{d+1};*
- *if $\varphi(y) = \psi(y \cdot \eta)$, then*

$$A(\partial)\varphi(y) = (M_a - M_b)\psi'''(y \cdot \eta).$$

Lemma 4.23. *Let $\eta \in \mathbb{R}^{d+1}$ be a vector which is not parallel to e_{d+1}. Then for any bounded open set $B \subset \mathbb{R}^d$*

$$\lim_{N\to\infty} \int_B \sin^2(N\eta \cdot (t,x)) \, \mathrm{d}x = \frac{1}{2}|B|$$

uniformly in $t \in \mathbb{R}$.

In the following subsection, we will show the perturbation lemma for our scheme of convex integration.

4.5.1 Proof of Lemma 4.21

For given $m \in X_0$, let U be associated matrix field. We fix the domain $D := [0,T] \times \Omega'$. We search for a sequence $\{m_k\} \subset X_0$ with associated matrix fields $\{U_k\}$ with m satisfying (4.14) and it has the following form:

$$(m_k, U_k) = (m, U) + \sum_j \left(\widetilde{m}_{k,j}, \widetilde{U}_{k,j}\right), \tag{4.16}$$

where each $z_{k,j} := \left(\widetilde{m}_{k,j}, \widetilde{U}_{k,j}\right)$ is compactly supported in some suitable ball $B_{k,j}(x_{k,j}, t_{k,j}) \subset D$. Next we continue as follows.

Step 1. Let $m \in X_0$ with associated matrix field U. Using Lemma 3.5 from [17], we can find for any $(t,x) \in D$ a line segment

$$\sigma_{(t,x)} = \left[(m(t,x), U(t,x), q_0(x)) - \lambda_{(t,x)}, (m(t,x), U(t,x), q_0(x)) + \lambda_{(t,x)}\right]$$

admissible for $(\rho_0(x), \chi(t))$ and with direction

$$\lambda_{(t,x)} := \left(\overline{m}(t,x), \overline{U}(t,x), 0\right)$$

such that

$$|\overline{m}(t,x)| \geq \frac{F}{\sqrt{\rho_0(x)\chi(t)}} \left(\rho_0(x)\chi(t) - |m(t,x)|^2\right). \tag{4.17}$$

Since $z := (m, U)$ and $K^{co}_{\rho_0,\chi}$ are uniformly continuous in (t,x), there exists an $\varepsilon > 0$ such that for any $(t,x), (t_0, x_0) \in D$ with $|x - x_0| + |t - t_0| < \varepsilon$, we have

$$(z(t,x), q_0(x)) \pm \left(\overline{m}(t_0,x_0), \overline{U}(t_0,x_0), 0\right) \subset \text{hint } K^{co}_{\rho_0,\chi}. \tag{4.18}$$

Step 2. We fix $(t_0, x_0) \in D$ for the moment. Now, let $0 \leq \varphi_{r_0} \leq 1$ be a smooth cutoff function on D with $\operatorname{supp} \varphi_{r_0} \subset B_{r_0}(t_0, x_0) \subset D$ for some $r_0 > 0$, $\varphi_{r_0} \equiv 1$ on $B_{r_0/2}(t_0, x_0)$ and $\varphi_r < 1$ outside. Due to Proposition 5.1 from [17] and the identification $(m, U, q) \mapsto M$, for the admissible line segment $\sigma_{(t_0,x_0)}$, there exist an operator A_0 and a direction $\eta_0 \in \mathbb{R}^{d+1}$ not parallel to e_{d+1}, such that for any $k \in \mathbb{N}$

$$A_0\left(\frac{\cos\left(k\eta_0\cdot(t,x)\right)}{k^3}\right) = \lambda_{(t_0,x_0)}\sin\left(k\eta_0\cdot(t,x)\right),$$

and such that

$$\left(\widetilde{m}_{k,0},\widetilde{U}_{k,0}\right) := A_0\left(\varphi_{r_0}(t,x)k^{-3}\cos\left(k\eta_0\cdot(t,x)\right)\right)$$

satisfies

$$\begin{aligned} \operatorname{div}_x m &= 0,\\ \partial_t m + \operatorname{div}_x U + \nabla_x q &= 0 \end{aligned}$$

with $q \equiv 0$. e.g. $\operatorname{div}_x M = 0$ is equivalent to

$$\begin{cases} \operatorname{div}_x m = 0,\\ \partial_t m + \operatorname{div}_x U = 0, \end{cases}$$

$$\begin{aligned} \left\|\left(\widetilde{m}_{k,0},\widetilde{U}_{k,0}\right) - \varphi_{r_0}\left(\overline{m}(t_0,x_0),\overline{U}(t_0,x_0)\sin(k\eta_0\cdot(t,x))\right)\right\|_\infty &\\ \leq C(A_0,\eta_0,\|\varphi_0\|_{C^3})\frac{1}{k}, & \end{aligned} \tag{4.19}$$

where $C = C(A_0,\eta_0,\|\varphi_0\|_{C^3})$ is constant. Since A_0 is a linear differential operator of homogeneous degree 3. Moreover, for all $(t,x) \in B_{r_0/2}(t_0,x_0)$, we have

$$|\widetilde{m}_{k,0}(t,x)|^2 = |\overline{m}(t_0,x_0)|^2\sin^2\left(k\eta_0\cdot(t,x)\right),$$

because for $(t,x) \in B_{r_0/2}(t_0,x_0)$, $\varphi_{r_0}(t,x) = 1$. Since η_0 is not parallel to e_{n+1}, we can use Lemma 5.2 from [17] to see that

$$\lim_{k\to\infty}\int_{B_{r_0/2}(t_0,x_0)} |\widetilde{m}_{k,0}(t,x)|^2 \,\mathrm{d}x = \frac{1}{2}\int_{B_{r_0/2}(t_0,x_0)} |\overline{m}(t_0,x_0)|^2 \,\mathrm{d}x$$

uniformly in $t \in [0,T]$. Particularly, applying (4.17), we derive

$$\begin{aligned} \lim_{k\to\infty}\int_{B_{r_0/2}(t_0,x_0)} |\widetilde{m}_{k,0}(t,x)|^2 \,\mathrm{d}x \geq &\\ \frac{F^2}{2\rho_0(x_0)\chi(t_0)}\left(\rho_0(x_0)\chi(t_0) - |m(t_0,x_0)|^2\right)^2 |B_{\frac{r_0}{2}(t_0,x_0)}|. & \end{aligned} \tag{4.20}$$

Step 3. Now, notice that uniform continuity of m implies that there exists an $\overline{r} > 0$ such that for any $r < \overline{r}$ there exists a finite family of pairwise disjoint balls $B_{r_j}(t_j, x_j) \subset D$ with $r_j < \overline{r}$ such that

$$\begin{aligned}\int_D \left(\rho_0(x)\chi(t) - |m(t,x)|^2\right)^2 \, \mathrm{d}x\,\mathrm{d}t \le \\ 2\sum_j \left(\rho_0(x_j)\chi(t_j) - |m(t_j,x_j)|^2\right)^2 |B_{r_j}(t_j,x_j)|.\end{aligned} \tag{4.21}$$

Fix $s > 0$ with $s < \min(\overline{r}, \varepsilon)$ and select a finite family of pairwise disjoint balls $B_{r_j}(t_j, x_j) \subset D$ with $r_j < s$ such that (4.21) holds. In order to obtain a pair $\left(\widetilde{m}_{k,j}, \widetilde{U}_{k,j}\right)$ for each $k \in \mathbb{N}$, we apply the construction of Step 2 in each $B_{2r_j}(t_j, x_j)$.

Final step. By letting (m_k, U_k) be as in(4.16), one can observe that the sum therein consists of finitely many terms. Thus, in view of (4.18) and (4.19), we infer that there exists $k_0 \in \mathbb{N}$ such that $m_k \in X_0$ for all $k \ge k_0$. Further, due to (4.20) and (4.21) one can write

$$\begin{aligned}\lim_{k\to\infty}\int_D |m_k(t,x) - m(t,x)|^2 \, \mathrm{d}x\,\mathrm{d}t &= \sum_j \lim_{k\to\infty}\int_D |\widetilde{m}_{k,j}(t,x)|^2 \, \mathrm{d}x\,\mathrm{d}t \\ &\ge \sum_j \frac{F^2}{2\rho_0(x_j)\chi(t_j)} \left(\rho_0(x_j)\chi(t_j) - |m(t_j,x_j)|^2\right)^2 |B_{r_j}(t_j,x_j)| \\ &\ge C\int_D \left(\rho_0(x)\chi(t) - |m(t,x)|^2\right)^2 \, \mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.22}$$

In view of $m_k \xrightarrow{d} m$ and (4.22), we get

$$\begin{aligned}\liminf_{k\to\infty} \|m_k\|^2_{L^2(D)} &= \|m\|_2^2 + \liminf_{k\to\infty}\|m_k - m\|_2^2 \\ &\ge \|m\|_2^2 + C\int_D \left(\rho_0(x)\chi(t) - |m(t,x)|^2\right)^2 \, \mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

which gives (4.14) with $\beta = \beta(d) = \beta(F(d))$. □

4.6 Construction of Suitable Initial Data

The aim of this section is to prove the existence of a subsolution in the sense of Definition 4.18 for which we apply Proposition 4.17 to generate infinitely many solutions. The material stated here can be also seen in [2].

Proposition 4.24. *Let $\rho_0 \in C^1(\mathbb{R}^d; \mathbb{R}^+)$ be a function satisfying the conditions $\rho_0 > 0$ on $\mathbb{R}^d$ and $\rho_0(x) = \overline{\rho}$ constant on $\mathbb{R}^d \backslash \Omega$. Let $p(\rho_0)$ be C^1 function such that*

$$\int_\Omega p(\rho_0)\, \mathrm{d}x = p(\overline{\rho})|\Omega|,$$

$T > 0$, and Ω' a bounded locally Lipschitz domain with $\Omega' \supset\supset \Omega$.

Then there exist $\widetilde{U} \colon \mathbb{R}^d \to \mathcal{S}_0^d$ and $\widetilde{m}(t,\cdot) \colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\partial_t \widetilde{m} + \operatorname{div}_x \widetilde{U} + \nabla_x q_0 = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^d,$$

$$\operatorname{supp}(\widetilde{m}(t,\cdot), \widetilde{U}(\cdot)) \subset \Omega' \quad \text{for any} \quad t \in [0,T],$$

$$e(\rho_0(x), \widetilde{m}(t,x), \widetilde{U}(t,x)) < \frac{\chi(t)}{d} \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}^d, \tag{4.23}$$

for any continuous function $\chi : [0,T) \to \mathbb{R}$ such that

$$\chi(t) > d\lambda(t) := d \left\| e(\rho_0(\cdot)), \widetilde{m}(t,\cdot), \widetilde{U}(\cdot) \right\|_{L^\infty(\Omega')}$$

for every $t \in [0,T)$ and for

$$q_0(t,x) := p(\rho_0(x)) + \frac{\chi(t)}{d} \quad \text{for all } x \in \mathbb{R} \times \mathbb{R}^d.$$

Proof. Let $\varepsilon > 0$ be so small that $\overline{\Omega^\varepsilon} \subset \Omega'$ and p^ε be given as in Lemma 4.5, i.e., $p^\varepsilon = p(\rho_0) - p(\rho_0) * \omega^\varepsilon$. Then, this Lemma implies that there exists $u \in C_c^{2,\alpha}(\Omega')$ such that $\Delta u = p^\varepsilon$ and $\operatorname{supp} u \subset \overline{\Omega^\varepsilon}$. Now, we define a matrix field by

$$U_{ij}^{(1)} = -\frac{d}{d-1} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad i \neq j,$$

$$U_{ii}^{(1)} = -\frac{d}{d-1} \frac{\partial^2 u}{\partial x_i^2} + \frac{p^\varepsilon}{d-1}.$$

Then $U^{(1)}$ is obviously symmetric and

$$\operatorname{tr}(U^{(1)}) = -\frac{d}{d-1}(\Delta u - p^\varepsilon) = 0.$$

We show that

$$\operatorname{div} U^{(1)} = -\nabla p^\varepsilon.$$

Indeed, for fixed i, we have

$$\begin{aligned} -\operatorname{div}_x U_{i\cdot}^{(1)} &= \frac{d}{d-1} \frac{\partial^3 u}{\partial x_i^3} - \frac{\frac{\partial}{\partial x_i} p^\varepsilon}{d-1} + \frac{d}{d-1} \frac{\partial}{\partial x_i} \Delta u - \frac{d}{d-1} \frac{\partial^3 u}{\partial x_i^3} \\ &= \frac{\partial}{\partial x_i} p^\varepsilon. \end{aligned}$$

Next, note that $p_1 * \omega^\varepsilon$ is a smooth function with compact support satisfying the condition of Proposition 4.9, i.e.,

$$\int_{\mathbb{R}^d} p_1 * \omega^\varepsilon(x) \, \mathrm{d}x = 0.$$

So Proposition 4.9 implies that we can find a matrix field $U^{(2)} \in \mathcal{S}_0^d$ and a vector field $\widetilde{m}$ satisfying the following conditions:

- $(\widetilde{m}(t), U^{(2)}) \in C_c^\infty(\Omega')$ for every $t \in \mathbb{R}$,
- $\operatorname{div}_x \widetilde{m} = 0$,
- $\partial_t \widetilde{m} + \operatorname{div}_x U^{(2)} + \nabla(p_1 * \omega^\varepsilon) = 0$.

Next, we define $\widetilde{U} := U^{(1)} + U^{(2)}$, then we have

$$\operatorname{supp}(\widetilde{m}(t,\cdot), \widetilde{U}(\cdot)) \subset \Omega' \quad \text{for every} \quad t \in \mathbb{R}.$$

Besides, recalling

$$\lambda(t) = \left\| e(\rho_0(\cdot)), \widetilde{m}(t,\cdot), \widetilde{U}(\cdot) \right\|_{L^\infty(\Omega')} = \left\| \lambda_{\max}\left(\frac{\widetilde{m} \otimes \widetilde{m}}{\rho_0} - \widetilde{U} \right) \right\|_{L^\infty(\Omega')},$$

then since $\widetilde{m}$ is linearly t-dependent and $\widetilde{U}$ is independent of t, we have

$$|\lambda(t)| \le C_1 t^2 + C_2 \quad \text{for some} \quad C_1, C_2 > 0 \quad \text{for any} \quad t \in \mathbb{R}.$$

This indicates that we can choose any continuous function χ on $\mathbb{R}$ satisfying $\chi(t) > d\lambda(t)$ to ensure (4.23). So we get $(\widetilde{m}, \widetilde{U})$ satisfying the required conditions, thereby completing the proof of Proposition 4.24. □

Proposition 4.25. *Let ρ_0, p be continuously differentiable functions as in Proposition 4.24 and $\Omega' \supset\supset \Omega$ bounded Lipschitz. Also, let $T > 0$ be any given time and $(\widetilde{m}, \widetilde{U}, q_0)$ and χ be as in Proposition 4.24. Then there exists a pair (m_0, U_0) solving the system*

$$\begin{aligned} \operatorname{div}_x m_0 &= 0 \\ \partial_t m_0 + \operatorname{div}_x U_0 + q_0 &= 0 \end{aligned} \tag{4.24}$$

distributionally on $(0,T)\times\mathbb{R}^d$ enjoying the following properties: (m_0, U_0, q_0) is continuous in $(0,T] \times \mathbb{R}^d$ and $m_0 \in C([0,T]; H_w(\mathbb{R}^d))$,

$$\operatorname{supp}(m_0(0,\cdot), U_0(0,\cdot)) \subset \overline{\Omega'}, \tag{4.25}$$

$$\operatorname{supp}(m_0(t,\cdot), U_0(t,\cdot)) \subset\subset \Omega' \quad \textit{for all } t \neq 0, \tag{4.26}$$

$$q_0(t,x) = p(\rho_0(x)) + \frac{\chi(t)}{d} \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^d, \tag{4.27}$$

$$e(\rho_0(x), m_0(t,x), U_0(t,x)) < \frac{\chi(t)}{d} \quad \textit{for all } (t,x) \in (0,T] \times \mathbb{R}^d. \tag{4.28}$$

Furthermore,

$$|m_0(0,x)|^2 = \rho_0(x)\chi(0) \quad a.e \textit{ in} \quad \Omega'. \tag{4.29}$$

Proof. Identity (4.27) is already satisfied by definition of q_0.

In analogy with Definition 4.18 (see also [27] and [17]), we consider the space X_0 defined as the set of continuous vector fields $m\colon [0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ to which there exists a continuous matrix field $U : [0,T) \times \mathbb{R}^d \to \mathcal{S}_0^d$ such that

$$\operatorname{div}_x m = 0$$
$$\partial_t m + \operatorname{div}_x U + q_0 = 0$$
$$\operatorname{supp}(m - \widetilde{m}) \subset \left[0, \frac{T}{2}\right) \times \Omega'$$
$$U(t,\cdot) = \widetilde{U}(\cdot) \quad \text{for all } t \in \left[\frac{T}{2}, T\right)$$

and

$$e(\rho_0(x), m(t,x), U(t,x)) < \frac{\chi(t)}{n} \text{ for all } (t,x) \in [0,T) \times \Omega'. \tag{4.30}$$

Note that $\widetilde{m} \in X_0$, where $\widetilde{m}$ is the vector field given by Proposition 4.24. As before, we set d to be a metrization of the convergence in $C([0,T); L^2_w(\Omega'))$, and X to be the closure of X_0 w.r.t. this topology.

Now following [17, 27] we use the following claim which can be verified by minor modifications in the proof of Lemma 4.21:

Claim: Let $\emptyset \neq \Omega_0 \subset\subset \Omega'$ be a given domain and $\delta > 0$. For every $\alpha > 0$ there exists $\beta > 0$ such that the following holds: Let $m \in X_0$ with associated matrix field U be such that

$$\int_{\Omega_0} \left[|m(0,x)|^2 - (\rho_0(x)\chi(0))\right] \,\mathrm{d}x < -\alpha.$$

Then, there exists a sequence $m_k \in X_0$ with associated matrix field U_k such that

$$\operatorname{supp}(m_k - m, U_k - U) \subset [0,\delta] \times \Omega_0,$$
$$m_k \xrightarrow{d} m,$$

and

$$\liminf_{k\to\infty} \int_{\Omega_0} |m_k(0,x)|^2 \,\mathrm{d}x \geq \int_{\Omega_0} |m(0,x)|^2 \,\mathrm{d}x + \beta\alpha^2.$$

Next, fix an exhausting sequence of bounded open subsets $\Omega_k \subset \Omega_{k+1} \subset \Omega'$, each compactly contained in Ω', and such that $|\Omega_{k+1} \setminus \Omega_k| \leq 2^{-k}$. Let also η_ε be a standard mollifying kernel in $\mathbb{R}^d$ with $\operatorname{supp} \eta_\varepsilon \subset B_\varepsilon(0)$. In view of the claim above, we construct inductively a sequence of momentum fields $m_k \in X_0$, associated matrix fields U_k and a sequence of numbers $\gamma_k < 2^{-k}$ as follows.

Firstly let $m_1(t,x) = \widetilde{m}(t,x)$, $U_1(t,x) = \widetilde{U}(x)$ for all $(t,x) \in [0,T) \times \mathbb{R}^d$. After obtaining $(m_1, U_1), \ldots, (m_k, U_k)$ and $\gamma_k, \ldots, \gamma_{k-1}$, we choose $\gamma_k < 2^{-k}$ in such a way that

$$\sup_{t \in [0,T)} \|m_k - m_k * \eta_{\gamma_k}\|_{L^2(\Omega')} < 2^{-k}. \tag{4.31}$$

Next, we set

$$\alpha_k = -\int_{\Omega_k} \left[|m_k(0,x)|^2 - \rho_0(x)\chi(0) \right] \,\mathrm{d}x.$$

Note that because of (4.30) we have $\alpha_k > 0$. Then we apply the claim with Ω_k, $\alpha = \alpha_k$ and $\delta = 2^{-k}T$ to obtain $m_{k+1} \in X_0$ and an associated smooth matrix field U_{k+1} such that

$$\operatorname{supp}(m_{k+1} - m_k, U_{k+1} - U_k) \subset [0, 2^{-k}T] \times \Omega_k, \tag{4.32}$$

$$d(m_{k+1}, m_k) < 2^{-k}, \tag{4.33}$$

$$\int_{\Omega_k} |m_{k+1}(0,x)|^2 \,\mathrm{d}x \geq \int_{\Omega_k} |m_k(0,x)|^2 \,\mathrm{d}x + \beta\alpha_k^2. \tag{4.34}$$

Since d induces the topology of $C([0,T); L^2_w(\Omega'))$ we can additionally prescribe that

$$\|(m_k - m_{k+1}) * \eta_{\gamma_j}\|_{L^2(\Omega')} < 2^{-k} \quad \text{for all } j \leq k \text{ for } t = 0, \tag{4.35}$$

because

$$\|(m_k - m_{k+1}) * \eta_{\gamma_j}(t=0)\|_{L^2(\Omega')} \leq d(m_{k+1}, m_k) < 2^{-k}.$$

In view of (4.33), we derive the existence of a function $m_0 \in C([0,T); H_w(\Omega'))$ such that

$$m_k \xrightarrow{d} m_0.$$

From (4.32) we see that for any compact subset A of $(0,T) \times \Omega'$ there exists k_0 such that $(m_k, U_k)|_A = (m_{k_0}, U_{k_0})|_A$ for all $k > k_0$. So (m_k, U_k) converges in $C_{loc}((0,T) \times \Omega')$ to a continuous pair (m_0, U_0) solving equation (4.24) in $(0,T) \times \mathbb{R}^d$ and satisfying (4.25),(4.26),(4.27),(4.28). In order to show that (4.29) also holds for m_0, we observe that (4.34) yields

$$\alpha_{k+1} \leq \alpha_k - \beta\alpha_k^2 + C|\Omega_{k+1} \backslash \Omega_k| \leq \alpha_k - \beta\alpha_k^2 + C2^{-k},$$

which implies that $\alpha_k \to 0$ as $k \to \infty$. Furthermore,

$$\begin{aligned} 0 &\geq \int_{\Omega'} \left[|m_k(0,x)|^2 - \rho_0(x)\chi(0) \right] \,\mathrm{d}x \\ &\geq -(\alpha_k + C_1|\Omega' \backslash \Omega_k|) \\ &\geq -(\alpha_k + C_1 2^{-(k-1)}), \end{aligned}$$

because $|\Omega'\backslash\Omega_k| = \cup_{j=k}^{\infty}|\Omega_{j+1}\backslash\Omega_j| = \sum_{j=k}^{\infty} 2^{-j} = 2^{-(k-1)}$. The latter two observations imply that

$$\lim_{k\to\infty}\int_{\Omega'}\left[|m_k(0,x)|^2 - \rho_0(x)\chi(0)\right]\,\mathrm{d}x = 0. \tag{4.36}$$

On the other hand, owing to (4.31) and (4.35) we can write for $t = 0$ and for every k

$$\|m_k - m_0\|_{L^2} \le \|m_k - m_k * \eta_{\gamma_k}\|_{L^2} + \|m_k * \eta_{\gamma_k} - \overline{m} * \eta_{\gamma_k}\|_{L^2} + \|\overline{m} * \eta_{\gamma_k} - \overline{m}\|_{L^2}\,.$$

By construction, $\|m_k - m_k * \eta_{\gamma_k}\|_{L^2} < 2^{-k}$ and

$$\begin{aligned}\|m_k * \eta_{\gamma_k} - m_0 * \eta_{\gamma_k}\|_{L^2} &\le \sum_{j=0}^{\infty}\|m_{k+j} * \eta_{\gamma_k} - m_{k+j+1} * \eta_{\gamma_k}\|_{L^2}\\ &\le \sum_{j=0}^{\infty} 2^{-(k+j)} = 2^{-(k-1)}.\end{aligned}$$

Thus we have

$$\|m_k - m_0\|_{L^2} \le 2^{-k} + 2^{-(k-1)} + \|m_0 * \eta_{\gamma_k} - m_0\|_{L^2}\,.$$

Hence $\|m_k - m_0\|_{L^2} \to 0$ as $k \to \infty$, i.e., $m_k(0,\cdot) \to m_0(0,\cdot)$ strongly in $H(\Omega')$ as $k \to \infty$. Combining this with (4.36) implies

$$|m_0(0,x)|^2 = \rho_0(x)\chi(0) \quad \text{for a.e. } x \in \Omega'$$

which completes the proof of Proposition 4.25. □

4.7 Proof of the Main Results

In this section we give the proofs of our results, which can also glimpsed in our article [2].

4.7.1 Proof of Theorem 4.1

Let T be any finite positive time and $\rho_0 \in C^1(\mathbb{R}^d)$ be a given density function as in Theorem 4.1. Further assume that (m_0, U_0, q_0) and χ is given by Proposition 4.25. Then, (m_0, U_0, q_0, χ) fulfills the assumptions of Proposition 4.17. Thus, there exist infinitely many solutions $m \in C([0,T), H_w(\mathbb{R}^d))$ of (4.4) in $[0,T) \times \mathbb{R}^d$ with density ρ_0 such that

$$m(0,x) = m_0(0,x) \quad \text{for a.e. } x \in \Omega'$$

and

$$|m(t,x)|^2 = \rho_0(x)\chi(t)\mathbb{1}_{\Omega'} \quad \text{a.e. in } (0,T) \times \mathbb{R}^d.$$

Since $|m_0(0,x)|^2 = \rho_0(x)\chi(0)$ a.e. in Ω' as well, it is enough to define $m^0(x) = m_0(0,x)$ to satisfy (4.5) and hence conclude the proof. □

4.7.2 Proof of Theorem 4.2

Under the assumptions of Theorem 4.1, we have shown the existence of a bounded initial momentum m^0 allowing for infinitely many solutions $m \in C([0,T];H_w(\mathbb{R}^d))$ of (4.4) on $[0,T) \times \mathbb{R}^d$ with density ρ_0. Moreover, according to Proposition 4.24, for an arbitrary continuous function $\chi\colon \mathbb{R} \to \mathbb{R}^+$ with $\chi > d\lambda > 0$, we have the following equalities:

$$|m(t,x)|^2 = \rho_0(x)\chi(t)\mathbb{1}_{\Omega'} \quad \text{a.e. in } [0,T) \times \mathbb{R}^d, \tag{4.37}$$

and in particular

$$|m^0(x)|^2 = \rho_0(x)\chi(0) \quad \text{a.e. in } \Omega'. \tag{4.38}$$

Now, we claim that there exist constants $C_1, C_2 > 0$ such that choosing the function $\chi(t) > d\lambda$ on $[0,T)$ among solutions of the differential inequality

$$\chi'(t) \leq -C_1\chi^{\frac{1}{2}}(t) - C_2\chi^{\frac{3}{2}}(t)$$

yields weak solutions (ρ_0, m) of (4.4) (obtained through Theorem 4.1) that will also satisfy the admissibility condition (4.6) on $[0,T) \times \mathbb{R}^d$.

Suppose for the moment this claim is true. Then, one may simply choose χ to be the solution of the ordinary differential equation

$$\chi'(t) = -C_1\chi^{\frac{1}{2}}(t) - C_2\chi^{\frac{3}{2}}(t)$$

with initial condition $\chi(0) = \chi^0$ sufficiently large so that χ which will remain greater than $n\lambda$ up to some positive time $\overline{T}$.

Finally, we aim to prove the claim. Since $m \in C([0,T];H_w(\mathbb{R}^d))$ is divergence free and fulfills (4.37),(4.38) and ρ_0 is time independent, (4.6) reduces to the following inequality

$$\frac{1}{2}\chi'(t) + m \cdot \nabla\left(\varepsilon(\rho_0) + \frac{p(\rho_0)}{\rho_0}\right) + \frac{\chi(t)}{2} m \cdot \nabla\left(\frac{1}{\rho_0}\right) \leq 0 \tag{4.39}$$

intended in the sense of distributions on $[0,T) \times \mathbb{R}^d$. As $\rho_0 \in C^1(\mathbb{R}^d)$ is bounded, there exists a constant C_0^2 with $\rho_0 \leq C_0^2$ on $\mathbb{R}^d$, whence (see (4.37),(4.38))

$$|m(t,x)| \leq C_0\sqrt{\chi(t)} \quad \text{a.e. on } [0,T) \times \Omega'. \tag{4.40}$$

Analogously we can find constants $c_1, c_2 > 0$ with

$$\left|\nabla\left(\varepsilon(\rho_0) + \frac{p(\rho_0)}{\rho_0}\right)\right| \leq c_1 \quad \text{a.e. in } \Omega \tag{4.41}$$

$$\left|\nabla\left(\frac{1}{\rho_0}\right)\right| \leq c_2 \quad \text{a.e. in } \Omega. \tag{4.42}$$

As a consequence of (4.40),(4.41) and (4.42), (4.39) holds as soon as χ satisfies

$$\chi'(t) \leq -2c_1 C_0 \chi^{\frac{1}{2}}(t) - c_2 C_0 \chi^{\frac{3}{2}}(t) \quad \text{on } [0, T).$$

Therefore, by choosing $C_1 := 2c_1 C_0$ and $C_2 := c_2 C_0$ we can conclude the proof of the claim. □

4.7.3 Proof of Corollary 4.3

In analogy with [17] note that the proof of Corollary 4.3 relies on Theorems 4.1-4.2. Given a continuously differentiable initial density ρ^0 we apply Theorems 4.1-4.2 for $\rho_0(x) := \rho^0(x)$ thus obtaining a positive time $\overline{T}$ (depending on $\left\|\rho^0\right\|_{C^1}$) and a bounded initial momentum m^0 for which there exist infinitely many solutions $m \in C([0,T]; H_w(\mathbb{R}^d))$ of (4.4) on $[0, \overline{T}) \times \mathbb{R}^d$ with density ρ^0 and, additionally, the following holds:

$$|m(t,x)|^2 = \rho_0(x)\chi(t)\mathbb{1}_{\Omega'} \quad \text{a.e. in } [0, \overline{T}) \times \mathbb{R}^d,$$

$$|m^0(x)|^2 = \rho_0(x)\chi(0)\mathbb{1}_{\Omega'} \quad \text{a.e. in } \mathbb{R}^d$$

for a suitable smooth function $\chi : [0, \overline{T}] \to \mathbb{R}^+$. Now, define $\rho(t,x) = \rho_0(x)\mathbb{1}_{[0,\overline{T})}(t)$. This indicates that (4.3) holds. Similarly to [17] we observe that ρ is independent of t and m is weakly divergence-free for almost every $0 < t < \overline{T}$. Therefore, the pair (ρ, m) is a compactly supported weak solution of (4.1) with initial data (ρ^0, m^0). In the end, note that each solution obtained is also admissible. In fact, for $\rho(t,x) = \rho_0(x)\mathbb{1}_{[0,\overline{T})}(t)$, (4.6) is assured by Theorem 4.2. Corollary 4.3 is proved. □

Conclusion

The various topics have been discussed in this thesis. Certainly, there are numerous open problems in this field. We would like to point out briefly some of them.

In the beginning of the thesis, we have seen the renormalization of the transport equation, with slightly relaxed functional settings for any renormalizing function $\beta \in W^{2,\infty}(\mathbb{R})$ (Theorem 1.1). Moreover, in the course of chapter 2, we have observed the relaxation of this regularity assumption on β to the particular function given by $|\cdot|^p$ (Theorem 2.1). However, it would be interesting to see the renormalization for a more general class of functions than this function.

Chapter 3 has been concerned with the regularity assumptions in order to conserve energy with the case of vacuum for the compressible Euler equations on periodic settings. Actually, the results we have obtained can be extended to bounded domains with C^2 boundary. Furthermore, the methods we have employed are transferable to the compressible Navier–Stokes equations, for more details see [3], where we have also compared our results with existing work [16] for Navier–Stokes.

In Chapter 4, we have shown the existence and non-uniqueness of admissible weak solutions with compact support to the compressible Euler equations with a steady density under the compatibility integral condition for the pressure $p = p(\rho)$ (Theorem 1.5). Moreover, we have seen under these settings, the condition is necessary in two dimensions. Nevertheless, it is appealing to weaken the condition so that the statement is valid for more general density functions. As a result, it can be applied to 3–dimensional axisymmetric incompressible Euler equations without swirl to draw similar conclusions, where the equations are given by

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0, \\ \operatorname{div} u &= 0. \end{aligned}$$

It is notable that these 3D equations modeling axisymmetric flow in the cylindrical system can be rewritten as

$$\begin{aligned} \partial_t u^r + u^r \partial_r u^r + u^z \partial_z u^r + \partial_r p = 0, \\ \partial_t u^z + u^r \partial_r u^z + u^z \partial_z u^z + \partial_z p = 0, \\ \partial_r (r u^r) + \partial_z (r u^z) = 0, \end{aligned}$$

where $u(t,x) = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z$, $u^\theta(r,z,t) \equiv 0$, $p(t,x) = p(r,z,t)$. Here e_r, e_θ and e_z, form an orthogonal basis of the cylindrical coordinates (see e.g. [49]). If the pressure $p = p(r,t)$ is independent of z, then the Euler equations with axisymmetric solutions can be expressed as 2D compressible Euler equations (3.1) with an artificial pressure function of a "density" (we take formally $\rho(r,z) = r$) after employing a suitable change of variables. Currently, we can not directly use our methods to obtain infinitely many solutions to these equations. Because, in order to have a solution with compact support, the initial density must be positive and the pressure must satisfy the compatibility condition. In the axisymmetric case, our artificial pressure does not satisfy our compatibility condition and the density blows up at some point.

Bibliography

[1] M. Aizenman. A sufficient condition for the avoidance of sets by measure preserving flows in $\mathbb{R}^n$. *Duke Math. J.*, **45** (1978), 809–814.

[2] I. Akramov, E. Wiedemann. Non-Unique Admissible Weak Solutions of the Compressible Euler Equations with Compact Support in Space. Preprint 2020, arXiv:2003.13287.

[3] I. Akramov, T. Debiec, J.W.D. Skipper, E. Wiedemann. Energy Conservation for the Compressible Euler and Navier-Stokes Equations with Vacuum. *Analysis & PDE*, **13-3** (2020), 789–811.

[4] I. Akramov and E. Wiedemann. Renormalization of active scalar equations. *Nonlinear Analysis*, **179** (2019), 254–269.

[5] G. Alberti, S. Bianchini, G. Crippa. Structure of level sets and Sard-type properties of Lip-schitz maps. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **12** (2013), 863–902.

[6] G. Alberti, S. Bianchini, G. Crippa A uniqueness result for the continuity equation in two dimensions. *J. Eur. Math. Soc. (JEMS)*, *16* (2014), 201–234.

[7] H. Bahouri, J.Y. Chemin, and R. Danchin. Fourier analysis and nonlinear partial differential equations. *Springer Science & Business Media*, **343**, 2011.

[8] C. Bardos, and E. S. Titi. Onsager's Conjecture for the incompressible Euler equations in bounded domains. *Arch. Ration. Mech. Anal.*, **228** (2018), 197–207.

[9] B. Bardos, L. Székelyhidi, and E. Wiedemann. Non-uniqueness for the Euler equations: the effect of the boundary. *Russian Mathematical Surveys*, **69** (2014), no. 2, 189–207.

[10] C. Bardos, E. S. Titi, and E. Wiedemann. Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit. *Comm. Math. Phys.*, **370** (2019), no. 1, 291–310.

[11] C. Bardos, P. Gwiazda, A. Świerczewska-Gwiazda, E. S. Titi, and E. Wiedemann. On the extension of Onsager's conjecture for general conservation laws. *J. Nonlinear Sci.*, **29** (2019), no. 2, 501–510.

[12] T. Buckmaster, S. Shkoller, and V. Vicol. Nonuniqueness of weak solutions to the SQG equation. *Comm. Pure Appl. Math.*, **72** (2019), 1809–1874.

[13] T. Buckmaster, C. De Lellis, L. Székelyhidi, and V. Vicol. Onsager's conjecture for admissible weak solutions. *Comm. Pure Appl. Math.*, **72** (2019), 229–274.

[14] A. Cheskidov, P. Constantin, S. Friedlander, R. Shvydkoy. Energy conservation and Onsager's conjecture for the Euler equations. *Nonlinearity*, **21** (2008), 1233–1252.

[15] G. Q. Chen, M. Torres. Divergence-Measure Fields, Sets of Finite Perimeter and Conservation Laws. *Arch. Ration. Mech. Anal.*, **175** (2005), 245–267.

[16] R. M. Chen, C. Yu. Onsager's energy conservation for inhomogeneous Euler equations. *J. Math. Pures Appl. (9)* 131 (2019), 1–16.

[17] E. Chiodaroli. A counterexample to well-posedness of entropy solutions to the compressible Euler system. *J. Hyperbolic Differ. Equ.*, **11** (2014), 493–519.

[18] E. Chiodaroli, C. De Lellis, O. Kreml. Global ill-posedness of the isentropic system of gas dynamics. *Comm. Pure Appl. Math.*, **68** (2015), 1085–1283.

[19] F. Colombini, T. Luo and J. Rauch. Uniqueness and nonuniqueness for nonsmooth divergence free transport. *Seminaire: Équations aux Dérivées Partielles*, 2002-2003, Sémin. Équ. Dériv. Partielles, Exp. No. XXII, *École Polytech., Palaiseau*, 1–21, 2003.

[20] P. Constantin, W. E, and E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Comm. Math. Phys.*, **165** (1994), 207–209.

[21] D. Córdoba, D. Faraco, F. Gancedo. Lack of uniqueness for weak solutions of the incompressible porous media equation. *Arch. Ration. Mech. Anal.*, **200** (2011), 725–746.

[22] G. Crippa, N. Gusev, S. Spirito, E. Wiedemann. Failure of the chain rule for the divergence of bounded vector fields. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **17** (2017), 1–18.

[23] G. Crippa, N. Gusev, S. Spirito, E. Wiedemann. Non-uniqueness and prescribed energy for the continuity equation. *Commun. Math. Sci.* **13** (2015), 1937–1947.

[24] C. M. Dafermos. Hyperbolic conservation laws in continuum physics, vol. 325 of Grundleheren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. *Springer, Berlin*, 2000.

[25] T. Debiec, P. Gwiazda, A. Świerczewska-Gwiazda, A. Tzavaras. Conservation of energy for the Euler-Korteweg equations. *Calc. Var. Partial Differential Equations*, **57** (2018) Art. 160.

[26] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math. (2)* **170** (2009), 1417–1436.

[27] C. De Lellis and L. Székelyhidi. On admissibility criteria for weak solutions of the Euler equations. *Arch. Rational Mech. Anal.*, **195** (2010), 225–260.

[28] N. Depauw. Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan. *C.R. Math. Acad. Sci. Paris* **337**:4 (2003), 249–252.

[29] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98** (1989), 511–547.

[30] R. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, **130** (1989), 321–366.

[31] T. D. Drivas and G. L. Eyink. An Onsager singularity theorem for turbulent solutions of compressible Euler equations. *Comm. Math. Phys.*, **359**(2018), 733–763.

[32] T. D. Drivas and H. Nguyen. Onsager's conjecture and anomalous dissipation on domains with boundary. *SIAM J. Math. Anal.*, **50** (2018), 4785-4811.

[33] J. Duchon and R. Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity* **13**:1(2000), 249–255.

[34] G. L. Eyink. Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. *Phys. D*, **78**:3-4 (1994), 222–240.

[35] G. L. Eyink. Dissipation in turbulent solutions of $2D$ Euler equations. *Nonlinearity* **14**:4 (2001), 787–802.

[36] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann. Regularity and Energy Conservation for the Compressible Euler Equations. *Arch. Ration. Mech. Anal.*, **223**:3 (2017), 1375–1395.

[37] E. Feireisl. On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable. *Comment. Math. Univ. Carol.* **42** (2001), 83–98.

[38] E. Feireisl. Weak solutions to problems involving inviscid fluids. *Mathematical fluid dynamics, present and future*, 377–399, Springer Proc. Math. Stat., 183, *Springer, Tokyo,* 2016.

[39] U. S. Fjordholm and E. Wiedemann. Statistical solutions and Onsager's conjecture. *Phys. D*, **376** (2018), 259–265.

[40] G. P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Springer Monographs in Mathematics. *Springer-Verlag, New York,* 2011.

[41] D. Gilbarg, and N. Trudinger. Elliptic partial differential equations of second order. *Springer*, 2015.

[42] L. Grafakos. Classical Fourier Analysis. Third Edition, Graduate Texts in Math., no 249, *Springer, New York*, 2014.

[43] L. Grafakos. Modern Fourier Analysis. Third Edition, Graduate Texts in Math., no 250, *Springer, New York*, 2014.

[44] P. Gwiazda, M. Michálek, A. Świerczewska-Gwiazda. A note on weak solutions of conservation laws and energy/entropy conservation. *Arch. Ration. Mech. Anal.*, **229**:3 (2018), 1223–1238.

[45] L. Hörmander. The analysis of linear partial differential operators. *I, Springer-Verlag,* 1990.

[46] P. Isett. A proof of Onsager's conjecture. *Ann. Math.*, **188**:3(2018), 871–963.

[47] P. Isett and S.-J. Oh. On nonperiodic Euler flows with Hölder regularity. *Arch. Ration. Mech. Anal.*, **221** (2016), 725–804.

[48] P. Isett, V. Vicol. Hölder continuous solutions of active scalar equations. *Ann. PDE*, **1** (2015), 1–77.

[49] Q. Jiu, J. Wu, W. Yang. Viscous approximation and weak solutions of the 3D axisymmetric Euler equations. *Math. Methods Appl. Sci.*, **38** (2015), 548–558.

[50] S. N. Kruzhkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.),* **81(123)** 1970, 228–255.

[51] T. M. Leslie and R. Shvydkoy. The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations. *J. Differ. Eq.* **261**:6 (2016), 3719–3733.

[52] P.-L. Lions. Mathematical Topics in Fluid Mechanics. Vol. 2, Compressible Models, Clarendon Press, *Oxford Science Publications, Oxford*, 1998.

[53] M. Lopes Filho, A. Mazzucato, H. Nussenzveig Lopes. Weak solutions, renormalized solutions and enstrophy defects in $2D$ turbulence. *Arch. Ration. Mech. Anal.*, **179** (2006), 353–387.

[54] S. Markfelder and C. Klingenberg. The Riemann problem for the multidimensional isentropic system of gas dynamics is ill-posed if it contains a shock. *Arch. Ration. Mech. Anal.,* **227** (2018), 967–994.

[55] L. Onsager. Statistical hydrodynamics. *Nuovo Cimento (9)* **6** (1949), no. Suppl, Supplemento, no. 2, 279–287.

[56] J. C. Oxtoby. Measure and category, second ed., vol.2 of Graduate Texts in Mathematics. *Springer-Verlag, New York,* 1990.

[57] M. Pavlović, J. Riihentaus. Quasi-nearly subharmonic functions in locally uniformly homogeneous spaces. Positivity **15**:1 (2011), 1–10.

[58] M. Reed and B. Simon. Methods of modern mathematical physics. II. Fourier analysis, self-adjointness. *Academic Press, New York-London,* 1975.

[59] J. C. Robinson, J. L. Rodrigo, J. W. D. Skipper. Energy conservation in the $3D$ Euler equations on $\mathbb{T}^2 \times \mathbb{R}^+$. *Partial differential equations in fluid mechanics*, 224–251, London Math. Soc. Lecture Note Ser., 452, *Cambridge Univ. Press, Cambridge,* 2018.

[60] J. C. Robinson, J. L. Rodrigo, J. W. D. Skipper. Energy conservation in the $3D$ Euler equations on $\mathbb{T}^2 \times \mathbb{R}^+$ for weak solutions defined without reference to the pressure. *Asymptot. Anal.*, **110** (2018), 185–202.

[61] V. Scheffer. An inviscid flow with compact support in space-time. *J. of Geom. Anal.,* **3** (1993), 343–401.

[62] A. Shnirelman. On the nonuniqueness of weak solutions of the Euler equations. *Comm. Pure Appl. Math.,* **50** (1997), 1261–1286.

[63] R. Shvydkoy. Convex integration for a class of active scalar equations. *J. Amer. Math. Soc.*, **24**:4 (2011), 1159–1174.

[64] L. Székelyhidi, Jr. Relaxation of the incompressible porous media equation. *Ann. Sci. Éc. Norm. Supér.*, **45** 2012, 491–509.

[65] E. Wiedemann. Conserved Quantities and Regularity in Fluid Dynamics. Preprint 2020, arXiv:2003.07807.

[66] E. Wiedemann. Existence of weak solutions for the incompressible Euler equations. *In Annales de l'Institut Henri Poincare (C) Non Linear Analysis* **28:5** (2011), 727–730. Elsevier Masson.

[67] C. Yu. Energy conservation for the weak solutions of the compressible Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, **225**:3 (2017), 1073–1087.

Overview of Research Papers

Numerous results in this dissertation are already published with minor modifications in the following scientific papers:

1. I. Akramov, E. Wiedemann. Non-Unique Admissible Weak Solutions of the Compressible Euler Equations with Compact Support in Space. Preprint 2020, arXiv:2003.13287.

2. I. Akramov, T. Debiec, J.W.D. Skipper, E. Wiedemann. Energy Conservation for the Compressible Euler and Navier-Stokes Equations with Vacuum. *Analysis & PDE*, **13-3** (2020), 789–811.

3. I. Akramov and E. Wiedemann. Renormalization of active scalar equations. *Nonlinear Analysis*, **179** (2019), 254–269.