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Gluing curves of genus 2 and genus 1 along their 2-torsion

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Introduction

Motivation

Algebraic curves and their arithmetic properties have been the object of much study in mathematics. They play a crucial role in solving diophantine equations, one of the main subjects in number theory, and have various applications in modern cryptography. To study complicated mathematical objects, we generally try to find invariants and associated objects with nicer properties that provide us with the tools to better understand the things we want to study. As algebraic curves do not possess a linear structure, we try to 'linearize' them. This gives rise to the following concept. To any curve C we can associate an abelian variety $\mathrm{Jac}(C)$, called the Jacobian of C. Jacobians provide us with new ways to study curves. If we restrict ourselves to algebraic curves we cannot really define something like 'the product' of two algebraic curves to create a new algebraic curve. Products do however exist in the category of abelian varieties. Moreover, decomposition of abelian varieties into simple abelian varieties is defined uniquely up to isogeny. It then makes sense to ask the following question: Given curves X and Y, does there exist a curve Z such that $\mathrm{Jac}(X) \times \mathrm{Jac}(Y)$ is isogenous to $\mathrm{Jac}(Z)$? If we can find such a curve Z we will call it a gluing of X and Y.

This decomposition is also reflected in the L-functions of the Jacobians. If A and B are two abelian varieties, $A \cong B_1 \times B_2$, then

$$L(A,s) = L(B_1,s)L(B_2,s).$$

As L-functions show up everywhere in number theory, it is important to understand the many different relationships between L-functions and their associated objects. To do this mathematicians have created an online database containing L-functions, modular forms and other relevant objects called the LMFDB (www.lmfdb.org). It is of interest to understand when we can decompose L-functions and how we can construct objects for which the L-function is the product of two other L-functions.

A first attempt to study the gluing process has been made by Frey and Kani in [10] who studied a way to glue two genus 1 curves X, Y together by finding a genus 2 curve Z and an isogeny $\phi : \mathrm{Jac}(X) \times \mathrm{Jac}(Y) \to \mathrm{Jac}(Z)$ such that $\ker\phi \subset \mathrm{Jac}(X)[2] \times \mathrm{Jac}(Y)[2]$.

Later on, Bröker, Howe, Lauter, and Stevenhagen described an explicit algorithm in [5] to construct equations for the gluing of two genus 1 curves and used this to construct curves of genus 2 with a given order and Jacobians of genus 2 with a given order.

Given a degree 2 cover $\pi : Z \to X$ where Z is a curve of genus 3 and X is a curve of genus 1, we get a Prym variety $\mathrm{Pr}(Z/X)$ of dimension 2 that is isogenous to the Jacobian of a curve Y of genus 2. In [26] Ritzenthaler and Romagny gave an explicit equation of the curve Y in terms of the equations for the curves X and Z in the case that Z is a non-hyperelliptic curve.

Goals and results

The main goal of this Ph.D.-thesis is to reverse the construction by Ritzenthaler and Romagny and to describe ways to calculate the *(2,2)-gluing* of a genus 2 curve Y_2 and a genus 1 curve X_1 along their 2-torsion. By this we mean that we want to construct a curve Z_3 such that $\mathrm{Jac}(Z_3)$ is isomorphic to $\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2)/G$ where G is a subgroup of $\mathrm{Jac}(X_1)[2] \times \mathrm{Jac}(Y_2)[2]$. We will develop and study some elementary properties of the gluing construction and we will describe two different methods to construct a (2,2)-gluing of a genus 1 curve and a genus 2 curve.

The first algorithm is purely analytical and works in the following way: We first calculate period matrices Λ_{X_1} and Λ_{Y_2} using the work of Neurohr [21] for the curves X_1 and Y_2. After that we determine a subgroup G of the two torsion of $B = \mathbb{C}^3/(\Lambda_{X_1} + \Lambda_{Y_2})$ in such a way that the Riemann surface B/G corresponds to the Jacobian of a genus 3 curve with the desired properties. Finally, we compute the Dixmier-Ohno invariants of the curve using the period matrix after which we use the algorithm by Lercier, Ritzenthaler and Sijsling [16] to find a quartic equation for the curve.

The second algorithm is algebraic and is based on the work of Bruin [6], and Ritzenthaler and Romagny [26]. The general strategy is as follows. We consider the Kummer surface $\mathrm{Kum}(Y_2) = \mathrm{Jac}(Y_2)/\langle -1 \rangle$. It can be embedded into $\mathbb{P}^3_k$ as a quartic surface with 16 singular points. If we now take a plane H in $\mathbb{P}^3_k$, its intersection with $\mathrm{Kum}(Y_2)$ will give us a curve of arithmetic genus 3. In particular, if H goes through two singular points of K, the intersection $H \cap \mathrm{Kum}(Y_2)$ is a singular curve of genus 1. Now choose H in such a way that the desingularization of $H \cap \mathrm{Kum}(Y_2)$ is isomorphic to X_1. Let $Z_3 = X_1 \times_{\mathrm{Kum}(Y_2)} \mathrm{Jac}(Y_2)$. We get the following picture:

$$\begin{tikzcd} Z_3 = X_1 \times_{\mathrm{Kum}(Y_2)} \mathrm{Jac}(Y_2) \arrow[r] \arrow[d] & \mathrm{Jac}(Y_2) \arrow[d] \\ X_1 \arrow[r] & \mathrm{Kum}(Y_2) \end{tikzcd}.$$

We will show that the curve Z_3 is a (2,2)-gluing of X_1 and Y_2.

Further research

It would be interesting to see if the algebraic construction can be applied to construct (2,2)-gluings of two genus 2 curves. Taking the intersection of a plane H that passes through one singular point of $\mathrm{Kum}(Y_2)$ will generically give us another curve $X_2 = H \cap \mathrm{Kum}(Y_2)$ of genus 2. The pullbacks of X_2 to $\mathrm{Jac}(Y_2)$ will most likely give us curves of genus 4 that are the (2,2) gluing of Y_2 and X_2. But the family of planes that pass through one singular point is of dimension 2, so this method cannot be used to construct the (2,2)-gluing of two arbitrary genus 2 curves as the moduli space of the curves of genus 2 has dimension 3. The genus 4 curves constructed in this way do have one special property however: they admit a map of degree 2 to a genus 2 curve. A natural question to ask would therefore be: Does every non-hyperelliptic genus 4 curve that admits a map to a genus 2 curve occur in this way? Another possible explanation for the difference in dimension of the two families is the fact that the moduli space of abelian varieties $\mathcal{A}_4$ of dimension 4 (which is 10-dimensional) is bigger than the moduli space of curves $\mathcal{M}_4$ of genus 4 (which is 9 dimensional).

Another open problem is to determine the exact relationship between the choice of a maximal isotropic subgroup and the choices made in the construction with the Kummer surface. Let X_1 be a curve of genus 1 and let Y_2 be a curve of genus 2. Fix a plane H that passes through the singular points P_1 and P_i and let Z_3 be the curve obtained by pulling back $H \cap \mathrm{Kum}(Y_2)$ to $\mathrm{Jac}(Y_2)$. Assume that Z_3 is a (2,2)-gluing and denote the gluing datum corresponding to Z_3 as a tuple (V, ϕ) as in Paragraph 1.3 where V is a 1-dimensional subspace of $\mathrm{Jac}(Y_2)[2]$. The choice of V is most likely related to the choice of the singular point P_i.

There might also be a way to relate ϕ to the choice of one of the six solution pairs for $j(X_1)$. Let us assume our hypothesis is correct and that choosing V as above is equivalent to choosing a family of planes that passes through P_1 and another 2-torsion point P_i. We consider the family $H_{1,i}(\lambda)$ of planes that pass through the singular points P_1 and P_i. The properties of the (16,6) configuration on a Kummer surface tell us that there exist exactly four special planes $W_1, \dots W_4$ with the property that they contain six singular points, but do not contain P_i. We observe the following: The different solution pairs for $j(X_1)$ depend on the values of the $x_i(\lambda)$ in the proof of Theorem 4.2.8. But these values are precisely determined by the intersection of the curve $H_{1,i}(\lambda)$ with the W_i. Moreover, using the (16,6)-configuration the W_i are in some sense dual to the 2-torsion points of $\mathrm{Jac}(Y_2)$ and the $x_i(\lambda)$ are closely related to the 2-torsion of $\mathrm{Jac}(X_1)$. It might therefore be the case that the gluing datum depends on the way the W_i intersect $H_{1,2}(\lambda)$.

Having a better understanding of the geometric gluing data might also give a geometric explanation for when exactly this construction fails to produce a genus 3 curve.

Finally, we can apply these gluing techniques to construct curves and Jacobians with a specific order over a finite field similar to what happens in [5] in the case of (2,2)-gluings of two genus 1 curves.

Outline

Chapter 0 In this chapter we will discuss some basis properties of abelian varieties and Jacobians.

Chapter 1 In this chapter we give the definition of an (n_1, n_2)-gluing and discuss its relationship with maximal isotropic subgroups. We then show that every (n_1, n_2)-gluing is essentially the same as an (e, e)-gluing of two isogenous curves where $e = \gcd(n_1, n_2)$. Afterwards, we will study maximal isotropic subgroups in the case of a (2,2)-gluing of a genus 1 curve X_1 and a genus 2 curve Y_2 and give an explicit description of these groups in terms of the equations of the curves. Finally, we will discuss when it is possible to define a gluing over the base field and give an explicit criterion for when a gluing over the base field exists. This criterion can be expressed in terms of the Galois groups of the cubic resolvents that are related to the equations defining the curves X_1 and Y_2.

Chapter 2 In this chapter we first study (2,2)-gluings that result into hyperelliptic curves. We give a complete description of the general form of the hyperelliptic curves that occur as a (2,2)-gluing over $\mathbb{C}$. After this we will fix a genus 2 curve Y_2 and construct a non-isotrivial family of curves whose Jacobian contains $\mathrm{Jac}(Y_2)$ as a factor. We do this by reversing the construction by Ritzenthaler-Romagny. We also show that this family can be parametrized by a $\mathbb{P}^1$.

Chapter 3 In this chapter we will give an explicit description of the gluing process over $\mathbb{C}$. For this, we start with a recap of several properties of abelian varieties over a general field and discuss how to describe these concepts over $\mathbb{C}$. We describe what the Weil-pairing looks like and give an explicit description of isogenies whose kernel is a maximal isotropic subgroup. After that we will give an algorithm for gluing principally polarized abelian varieties over $\mathbb{C}$ and discuss in which cases the glued abelian variety actually corresponds to the Jacobian of a curve. Finally, we talk about how we can reconstruct the curve from its Jacobian and describe a method to find an equation for the curve over the base field of the Jacobian (if such a curve exists).

Chapter 4 In this chapter we give an algebraic construction of a (2,2)-gluing. We start by giving the definition of a Kummer surface and give a short description of some basic properties. We discuss natural automorphisms of the Kummer surface and how we can obtain a singular genus 1 curve by intersecting the Kummer surface with a plane that passes through exactly 2 singular points. Afterwards we will describe how we can use the Kummer surface

$\mathrm{Kum}(Y_2)$ to construct a (2,2)-gluing Z_3 of a genus 2 curve Y_2 and a genus 1 curve X_1. Our curve Z_3 will be the pullback of $H \cap \mathrm{Kum}(Y_2)$ along the quotient map $\pi : \mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ where H is a plane such that $H \cap \mathrm{Kum}(Y_2) \cong X_1$

We continue by presenting the tools needed to make this construction computationally feasible. We give an explicit description of the map $\pi : \mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ and we explicitly calculate the 1-dimensional family of planes parametrized by λ passing through two singular points of the Kummer surface. We compute the j-invariant of this family of singular elliptic curves in terms of the parameter λ and we then show that this j-invariant is generically a polynomial $j(\lambda)$ of degree 12. Afterwards, we proceed by providing a method to compute the pullback of X_1 along π. As a direct computation is unfeasible, we describe a way to find a degree 2 cover of X_1 that ramifies exactly above the same points as $Z_3 = \pi^{-1}(X_1)$ over X_1 to find the curve Z_3.

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Chapter 0

Abelian varieties

Definition 0.0.1. An *abelian variety* over a field k is a connected complete group variety over k.

Definition 0.0.2. A homomorphism $\phi : A_1 \to A_2$ between two abelian varieties is called an *isogeny* if ϕ is surjective and $\ker\phi$ is finite.

Proposition 0.0.3. *Let A be an abelian variety over a field k. Then the group A^t of all line bundles on A up to linear equivalence is also an abelian variety over k and is called the dual abelian variety of A.*

Proof. See [29, Theorem 6.18] $\square$

Proposition 0.0.4. *Let A be an abelian variety over a field k and let $\mathcal{L}$ be an ample line bundle on A. Then the map $\phi_{\mathcal{L}} : A \to A^t$ given by $\phi_{\mathcal{L}}(x) = t_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$ is an isogeny. Furthermore, this map does not depend on the choice of the class of $\mathcal{L}$ modulo algebraic equivalence.*

Proof. See [29, Corollary 2.10]. $\square$

Notation 0.0.5. When two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are *algebraically equivalent*, we will write $\mathcal{L}_1 \cong_{\text{alg}} \mathcal{L}_2$.

Definition 0.0.6. Let A be an abelian variety over a field k. A *polarization* on A is the choice of the class of an ample line bundle $\mathcal{L}$ on $A_{\overline{k}}$ modulo algebraic equivalence. Equivalently, the choice of a map $\phi_{\mathcal{L}}$ as above over $\overline{k}$ also defines a polarization. If $\phi_{\mathcal{L}}$ is an isomorphism then $\mathcal{L}$ is called a *principal polarization.*

Definition 0.0.7. A tuple $(A, \mathcal{L})$ where A is an abelian variety and $\mathcal{L}$ is an ample line bundle is called a *polarized abelian variety*. If $\mathcal{L}$ is a principal polarization, then we call $(A, \mathcal{L})$ a *principally polarized abelian variety.*

Definition 0.0.8. Let $(A_1, \mathcal{L}_1), (A_2, \mathcal{L}_2)$ be two polarized abelian varieties. A map $\phi : A_1 \to A_2$ is called a *morphism of polarized abelian varieties* if there exists an integer n such that $\phi^*(\mathcal{L}_2) \cong_{\text{alg}} \mathcal{L}_1^n$.

Definition 0.0.9. Let A be an abelian variety over k and let n be a positive integer. Let μ_n be the group of the n-th roots of unity in k. We can then assign a bilinear form

$$\langle .,. \rangle_n : A[n] \times A[n] \to \mu_n \tag{1}$$

to A called the *Weil pairing*. See for example Definition 11.11 in [29].

Proposition 0.0.10. *Let C be a smooth projective curve of genus $g \geq 1$. Then there exists an abelian variety* $\mathrm{Jac}(C)$ *called the Jacobian of C, and an injection $AJ : C \to \mathrm{Jac}(C)$, called the Abel-Jacobi map, such that extending AJ linearly to divisors induces an isomorphism between* $\mathrm{Pic}^0(C)$ *and* $\mathrm{Jac}(C)$.

Proof. See [14, Theorem A.8.1.1]. □

Proposition 0.0.11. *Consider the subvariety W_{g-1} of codimension 1 of* $\mathrm{Jac}(C)$ *that is given by the image of summing $g-1$ copies of $AJ(C)$. Let $\mathcal{P}_C$ be the line bundle associated to the divisor W_{g-1}. Then* $(\mathrm{Jac}(C), \mathcal{P}_C)$ *is a principally polarized abelian variety.*

Proof. See [14, Corollary A.8.2.3]. □

Chapter 1

Gluing

1.1 Gluing curves

Definition 1.1.1. Let $(A_1, \mathcal{L}_1), (A_2, \mathcal{L}_2)$ be two principally polarized abelian varieties over a field k. Consider the product $A_1 \times A_2$ along with the projection maps $\mathrm{pr}_1 : A_1 \times A_2 \to A_1$, $\mathrm{pr}_2 : A_1 \times A_2 \to A_2$. Let n_1, n_2 be two positive integers. A triple $(\phi, B, \mathcal{M}_B)$ is called an (n_1, n_2)-*gluing* of A_1 and A_2 over k if

(D1) $(B, \mathcal{M}_B)$ is a principally polarized abelian variety over k; and

(D2) ϕ is a k-isogeny $A_1 \times A_2 \to B$, such that

$$\mathrm{pr}_1^*(\mathcal{L}_1)^{n_1} \otimes \mathrm{pr}_2^*(\mathcal{L}_2)^{n_2} \cong_{\mathrm{alg}} \phi^*(\mathcal{M}_B). \tag{1.1}$$

Definition 1.1.2. Let X, Y be two smooth curves over k. We say that a triple $(\phi, \mathrm{Jac}(Z), \mathcal{P}_Z)$ is an (n_1, n_2)-*gluing* of X and Y over k if it is an (n_1, n_2)-*gluing* of $(\mathrm{Jac}(X), \mathcal{P}_X)$ and $(\mathrm{Jac}(Y), \mathcal{P}_Y)$ over k.

Definition 1.1.3. An (n_1, n_2)-gluing of A_1 and A_2 (respectively of X and Y) over the algebraic closure $\overline{k}$ is called a *geometric* (n_1, n_2)-*gluing*.

Remark 1.1.4. In Definition 1.1.1 the class of $\phi^*(\mathcal{M}_B)$ in the Néron-Severi group of $A_1 \times A_2$ is a sum of multiples of $\mathrm{pr}_1^* \mathcal{L}_1$ and $\mathrm{pr}_2^* \mathcal{L}_2$. In general one could consider using a more general element of the Néron-Severi group, but as it always contains $\mathbb{Z}\,\mathrm{pr}_1^*(\mathcal{L}_1) \times \mathbb{Z}\,\mathrm{pr}_2^*(\mathcal{L}_2)$ and is generically equal to $\mathbb{Z} \times \mathbb{Z}$, we will only consider the (n_1, n_2) case.

In what follows, it will be important to study subgroups of $A[n]$ that are maximally isotropic with respect to the Weil pairing.

Proposition 1.1.5. *Giving a triple $(B, \phi, \mathcal{M}_B)$ satisfying (D1) and (D2) is the same as giving a maximal isotropic subgroup G of $A_1[n_1] \times A_2[n_2]$. Given such a maximal isotropic subgroup G, we have $B \cong A/G$ and* $\deg(\phi) = n_1^{d_1} n_2^{d_2}$.

Proof. Write $\mathcal{L} = \mathrm{pr}_1^*(\mathcal{L}_1)^{n_1} \otimes \mathrm{pr}_2^*(\mathcal{L}_2)^{n_2}$. It suffices to show that $K(\mathcal{L}) = A_1[n_1] \times A_2[n_2]$, as we can then use [29, Corollary 8.14] and Corollary 8.19 to obtain (D1),(D2) and $B \cong A/G$. Proposition 7.6 in loc. cit. then gives us that $\deg(\phi) = n_1^{d_1} n_2^{d_2}$. First remark that

$$\phi_{\mathrm{pr}_2^*(\mathcal{L}_2^{n_2})}|_{A_1 \times \{0\}} = 0 \text{ and } \phi_{\mathrm{pr}_1^*(\mathcal{L}_1^{n_1})}|_{\{0\} \times \{A_2\}} = 0. \tag{1.2}$$

It follows that $\phi_{\mathrm{pr}_1 \mathcal{L}_1^{n_1} \otimes \mathrm{pr}_2 \mathcal{L}_2^{n_2}} = 0$ if and only if

$$\phi_{\mathrm{pr}_1^*(\mathcal{L}_1^{n_1})}|_{A_1 \times \{0\}} = 0 \text{ or } \phi_{\mathrm{pr}_2^*(\mathcal{L}_2^{n_1})}|_{\{0\} \times \{A_2\}} = 0. \tag{1.3}$$

But

$$\begin{aligned} \phi_{\mathcal{L}_1^{n_1}}(a_1, 0) &= t_{(a_1,0)}^* \mathrm{pr}_1^*(\mathcal{L}_1^{n_1}) \otimes \mathrm{pr}_1^* \mathcal{L}_1^{-n_1} \\ &= \mathrm{pr}_1^* t_{a_1}^*(\mathcal{L}_1^{n_1}) \otimes \mathrm{pr}_1^* \mathcal{L}_1^{-n_1} = \mathrm{pr}_1^*(t_{a_1}^*(\mathcal{L}_1^{n_1}) \otimes \mathcal{L}_1^{-n_1}) \end{aligned} \tag{1.4}$$

and $\mathrm{pr}_1^*(t_{a_1}^*(\mathcal{L}_1^{n_1}) \otimes \mathcal{L}_1^{-n_1})$ is trivial if and only if $a_1 \in K(L_1^{n_1})$. Proving something similar for $\mathcal{L}_2^{n_2}$ we find that $K(\mathcal{L}) = K(\mathcal{L}_1^{n_1}) \times K(\mathcal{L}_2^{n_2})$. By [29, Proposition 8.6] we have

$$K(\mathcal{L}_1^{n_1}) \times K(\mathcal{L}_2^{n_2}) = n_1^{-1}(K(\mathcal{L}_1)) \times n_2^{-1}(K(\mathcal{L}_2)) = A_1[n_1] \times A_2[n_2]. \tag{1.5}$$

□

Remark 1.1.6. Note that the algebraic equivalence class of $\mathcal{M}_B$ is uniquely determined by the maximal isotropic subgroup G, which is also implied by [29, Corollary 7.25].

Definition 1.1.7. A maximal isotropic subgroup G of $\mathrm{Jac}(X)[n_1] \times \mathrm{Jac}(Y)[n_2]$ is called *indecomposable* if it cannot be written as the product of two isotropic subgroups of $\mathrm{Jac}(X)[n_1]$ and $\mathrm{Jac}(Y)[n_2]$. Otherwise, we will call G *decomposable*.

Definition 1.1.8. Let $(\phi, A, \mathcal{M}_A)$ be an (n_1, n_2)-gluing of two curves X and Y. If $\ker(\phi)$ is decomposable, we say that ϕ is a *decomposable gluing*.

Proposition 1.1.9. *Let $(\phi, A, \mathcal{M}_A)$ be a decomposable (n_1, n_2)-gluing of two curves X and Y over k. Then $A = A_1 \times A_2$ where A_1 is k-isogenous to* $\mathrm{Jac}(X)$ *and A_2 is k-isogenous to* $\mathrm{Jac}(Y)$.

Proof. By assumption, $\ker(\phi) = K_X \times K_Y$ where K_X is a totally isotropic subgroup of $\mathrm{Jac}(X)[n_1]$ and K_Y is a totally isotropic subgroup of $\mathrm{Jac}(Y)[n_2]$. This means we get a natural isogeny:

$$\mathrm{Jac}(X) \times \mathrm{Jac}(Y) \to \mathrm{Jac}(X)/K(X) \times \mathrm{Jac}(Y)/K(Y). \tag{1.6}$$

As this isogeny has $K(X) \times K(Y)$ as its kernel, we see that A is isogenous to the product $\mathrm{Jac}(X)/K(X) \times \mathrm{Jac}(Y)/K(Y)$ where the first term is isogenous to $\mathrm{Jac}(X)$ and the second term to $\mathrm{Jac}(Y)$. □

Theorem 1.1.10. *Let n_1, n_2 be two positive integers and let k be a field for which* $\operatorname{char} k \nmid n_1 n_2$. *Let A_1 and A_2 be abelian varieties over k. Let $e = \gcd(n_1, n_2)$. Then any (n_1, n_2)-gluing $(B, \phi, \mathcal{M})$ of A_1 and A_2 factors as $\phi = \phi_e \psi$. Here*

(i) The isogeny $\psi = \psi_1 \times \psi_2$ is a product of isogenies $\psi_i : A_i \to B_i$ for $i \in \{1, 2\}$ such that $\psi_i(\mathcal{M}_i) \sim \mathcal{L}_i^{n_i/e}$ for some algebraic equivalence class $\mathcal{M}_i$ inducing a principal polarization on B_i;

(ii) The triple $(B, \phi_e, \mathcal{M})$ is an (e, e)-gluing for the pair $((B_1, \mathcal{M}_1), (B_2, \mathcal{M}_2))$.

To prove this theorem, we consider the pairing $\langle \cdot, \cdot \rangle_{1,2}$ on $K(\mathcal{L}) = A_1[n_1] \times A_2[n_2]$ given by the product of the Weil pairings $\langle ., . \rangle_{n_1}$ and $\langle ., . \rangle_{n_2}$ on $A_1[n_1]$ and $A_2[n_2]$. It has values in $\mu_{n_1} \otimes \mu_{n_2} = \mu_{\operatorname{lcm}(n_1, n_2)}$. We need the following lemma.

Lemma 1.1.11. *Let $G \subset A_1[n_1] \times A_2[n_2]$ be maximal isotropic. Suppose that $v_p(n_1) \neq v_p(n_2)$ for some prime number p. Suppose that $v_p(n_1)$ (respectively. $v_p(n_2)$) is the larger of $\left\{v_p(n_1), v_p(n_2)\right\}$. Then G contains a group of the form $H_1 \times \{0\}$ (respectively. $\{0\} \times H_2$), where $H_i \subset A_i[p]$ is maximal isotropic.*

Proof. We may suppose that $v_p(n_1) > v_p(n_2)$. Consider the Weil pairing

$$\langle ., . \rangle_{n_1} : A_1[n_1] \times A_1[n_1] \to \mu_{n_1} \tag{1.7}$$

on the group $A_1[n_1]$. Similarly, denote the Weil pairing on $A_1[p]$ by

$$\langle ., . \rangle_p : A_1[p] \times A_1[p] \to \mu_p. \tag{1.8}$$

The pairing (1.7) induces a pairing on $Q = A_1[n_1]/pA_1[n_1]$ with values in $\mu_{n_1}/\mu_{p^{-1}n_1}$, which we denote by

$$\langle ., . \rangle_Q : Q \times Q \to \mu_{n_1}/\mu_{p^{-1}n_1}. \tag{1.9}$$

Finally, the Weil pairing $\langle ., . \rangle_{n_1}$ induces a perfect mixed pairing

$$\langle ., . \rangle_{p,Q} : A_1[p] \times Q \to \mu_p \subset \mu_{n_1}. \tag{1.10}$$

Choosing a symplectic basis B of $A_1[n_1]$ as a free module over $\mathbb{Z}/n_1\mathbb{Z}$ gives induced bases for both $A_1[p]$ and Q, the former by multiplying the elements in B with $p^{-1}n_1$ and the latter by projecting down to Q. Using these bases, all pairings above can be described by the standard symplectic matrix. Now let G be as in the Lemma.

Claim 1: The image $\overline{\operatorname{pr}_1(G)}$ of $\operatorname{pr}_1(G)$ in Q is isotropic. In particular, it has rank at most d_1.

Proof: As was mentioned before the Lemma, the Weil pairing $\langle .,. \rangle_{1,2}$ on $A_1[n_1] \times A_2[n_2]$ is given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle_{1,2} = \langle x_1, y_1 \rangle_{n_1} \langle x_2, y_2 \rangle_{n_2}, \tag{1.11}$$

where $\langle .,. \rangle_{n_2}$ is the Weil pairing on $A_2[n_2]$. If $\overline{\mathrm{pr}_1(G)}$ in Q were not isotropic, then on $G \times G$ the factor $\langle x_1, y_1 \rangle$ would attain values in $\mu_{1,2}$ of order equal to p^{n_1}. Due to our assumption at the beginning of the proof, the pairing $\langle .,. \rangle_{n_2}$ cannot attain such values. This precludes $\langle .,. \rangle_{1,2}$ from being trivial on $G \times G$ and contradicts G being isotropic.

Claim 2: The submodule $G \cap (A_1[p] \times \{0\})$ is of rank at least d_1.

Proof: Consider the orthogonal complement $\overline{\mathrm{pr}_1(G)}^{\perp}$ of $\overline{\mathrm{pr}_1(G)}$ under the mixed pairing $\langle .,. \rangle_{p,Q}$. Because the latter pairing is perfect, this is a submodule of $A_1[p]$ of rank at least d_1. By construction, $\overline{\mathrm{pr}_1(G)}^{\perp}$ has trivial pairing with the elements of $\mathrm{pr}_1(G)$. As a result, $\overline{\mathrm{pr}_1(G)}^{\perp} \times \{0\}$ has trivial pairing with the elements of $\mathrm{pr}_1(G) \times \mathrm{pr}_2(G) = G$, so $G \cup (\overline{\mathrm{pr}_1(G)}^{\perp} \times \{0\})$ is an isotropic subgroup. It follows that $\overline{\mathrm{pr}_1(G)}^{\perp} \times \{0\}$ is contained in G since G is maximal isotropic.

Claim 3: The submodule $G \cap (A_1[p] \times \{0\})$ contains a maximal isotropic submodule of $A_1[p]$ with respect to the Weil pairing $\langle .,. \rangle_p$.

Proof: This follows because after the choice of a symplectic basis above, the pairings $\langle .,. \rangle_p$ on $A_1[p]$ and $\langle .,. \rangle_Q$ on Q, as well as the mixed pairing $\langle .,. \rangle_{p,Q}$, are all given by the standard symplectic matrix. Indeed, this implies that since the image of $\mathrm{pr}_1(G)$ in Q is contained in a maximal isotropic submodule, its dual $\overline{\mathrm{pr}_1(G)}^{\perp}$ in $A_1[p]$ contains such a submodule. Additionally, $\overline{\mathrm{pr}_1(G)^{\perp}} \times \{0\} \subset G$ as we saw above.

With this series of claims, the statement of the Lemma follows by taking some maximal isotropic submodule H_1 of $G \cap (A_1[p] \times \{0\})$. $\square$

Proof of Theorem 1.1.10. The theorem follows from [29, Corollary 8.11]. Indeed, if $n_1 = n_2$, then we are done. Otherwise we can apply Lemma 1.1.11, as follows.

Let H_i be the submodule obtained by applying Lemma 1.1.11, and let $\psi_i : A_i \to A_i/H_i$ be the corresponding quotient. Suppose moreover (as we may, by symmetry) that $i = 1$. Because H_1 is maximal isotropic in $A_1[p]$, there exists a unique algebraic equivalence class $\mathcal{P}_1$ on A_1/H_1 such that $\psi_1^*(\mathcal{M}_1) \sim \mathcal{L}_1^p$, and this class $\mathcal{M}_1$ defines a principal polarization on A_1/H_1. Let $\mathcal{M}_2 = \mathcal{L}_2$.

Consider the composition

$$A_1 \times A_2 \to (A_1/H_1) \times A_2 = (A_1 \times A_2)/(H_1 \times \{0\}) \to (A_1 \times A_2)/G. \tag{1.12}$$

Let ψ_1 (respectively. ϕ) be the quotient map by H_1 (respectively. G). Then we can write $\phi = \phi_1(\psi_1 \times 1)$, and if we denote the projections of $(A_1/H_1) \times A_2$

onto its components by ρ_1 and ρ_2, then $(\psi_1 \times 1)^*(\rho_1^*(\mathcal{M}_1)) = \mathrm{pr}_1^*(\mathcal{L}_1^p)$ and $(\psi_1 \times 1)^*(\rho_1^*(\mathcal{M}_2)) = \mathrm{pr}_2^*(\mathcal{L}_2)$. This implies that

$$(\psi_1 \times 1)^*(\rho_1^*(\mathcal{M}_1^{n_1/p}) \otimes \rho_2^*(\mathcal{M}_2^{n_2})) = \mathrm{pr}_1^*(L_1^{n_1}) \otimes \mathrm{pr}_2^*(L_2^{n_2}) = L. \tag{1.13}$$

By the remark about uniqueness in Remark 1.1.6, the fact that $\mathcal{L} = \psi^*(\mathcal{M}) = (\psi_1 \times 1)^*(\phi_1^*(\mathcal{M}))$ then allows us to conclude

$$\phi_1^*(\mathcal{M}) = \rho_1^*(\mathcal{M}_1^{n_1/p}) \otimes \rho_2^*(\mathcal{M}_2^{n_2}). \tag{1.14}$$

Both $\mathcal{M}_1$ and $\mathcal{M}_2$ define principal polarizations on the corresponding factors, so that ϕ_1 is a $(p^{-1}n_1, n_2)$-gluing of the pair of principally polarized abelian varieties $((A_1/H_1, \mathcal{M}_1), (A_2, \mathcal{M}_2))$.

This process can be continued inductively. Composing all morphisms $\psi_1 \times 1$ and $1 \times \psi_2$ thus obtained, we get the Theorem. □

1.2 Maximal isotropic subgroups

As maximal isotropic subgroups play a big role in gluing, it makes sense to study them in more detail. In the most general setting one could study the maximal isotropic subgroups of $(\mathbb{Z}/k\mathbb{Z})^{2n}$ with respect to a non-degenerate symplectic paring for some positive integers k and n. As it is easier to work with vector spaces, we will only consider the case where k is a prime number p. In this case $(\mathbb{Z}/p\mathbb{Z})^{2n} \cong \mathbb{F}_p^{2n}$.

Definition 1.2.1. We define the *standard symplectic pairing* on $\mathbb{F}_p^{2n}$ to be the one given by:

$$\langle x, y \rangle_{\mathbb{F}_p^{2n}} = x^T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} y, \tag{1.15}$$

where I_n is the n-dimensional identity matrix.

Lemma 1.2.2. *Let G be a maximal isotropic subgroup of $\mathbb{F}_p^{2n}$ with respect to the standard symplectic pairing. Then $|G| = p^n$.*

Proof. See [28] 3.1 Corollary a). □

Proposition 1.2.3. *There are exactly*

$$\prod_{k=0}^{n-1} \left(p^{n-k} + 1\right) \tag{1.16}$$

maximally isotropic subgroups in $\mathbb{F}_p^{2n}$ with respect to the standard pairing.

Proof. We first count the number of ways we can construct a maximal isotropic vector space by choosing n linearly independent vectors. After that, we will divide this number by the amount of possible ways to choose a basis for a vector space of dimension n to find the number of maximally isotropic subspaces.

Let $v_1 \in \mathbb{F}_p^{2n}$ be a non-zero vector. There are $p^{2n} - 1$ possible ways to choose this vector, and $\langle v_1 \rangle$ gives us a 1-dimensional isotropic subspace of $\mathbb{F}_p^{2n}$. Assume we have already found a k-dimensional isotropic subspace V_k of $\mathbb{F}_p^{2n}$ with $k < n$. Then we need to find a vector $v_{k+1} \in V_k^{\perp} - V_k$ to construct a $k+1$-dimensional isotropic subspace. For this, we have $p^{2n-k} - p^k$ choices. So there are $\prod_{k=0}^{n-1} \left(p^{2n-k} - p^k\right)$ possible ways of constructing a basis for a maximal isotropic subspace of $\mathbb{F}_p^{2n}$.

The number of possible distinct bases for a given n-dimensional subspace of $\mathbb{F}_p^{2n}$ is equal to $\prod_{k=0}^{n-1}(p^n - p^k)$. It follows that the number of maximal isotropic subspaces is

$$\frac{\prod_{k=0}^{n-1}\left(p^{2n-k} - p^k\right)}{\prod_{k=0}^{n-1}(p^n - p^k)} = \prod_{k=0}^{n-1}\left(p^{n-k} + 1\right). \tag{1.17}$$

□

1.3 Structure of isotropic subgroups

Let X_1 be a curve of genus 1 and let Y_n be a curve of genus n. Determining a (p,p)-gluing of X_1 and Y_2 is the same as choosing a subgroup of $\mathrm{Jac}(X_1)[p] \times \mathrm{Jac}(Y_n)[p]$ that is maximally isotropic with respect to the Weil-pairing. We will therefore restrict ourselves to the following situation: Consider the vector space ${\mathbb{F}_p}^2 \times \mathbb{F}_p^{2n}$ equipped with the natural projection maps π_1 and π_2. The product pairing $\langle .,. \rangle$ on $\mathbb{F}_p^2 \times \mathbb{F}_p^{2n}$ is given by

$$\langle x, y \rangle = \langle \pi_1(x), \pi_1(y) \rangle_{\mathbb{F}_p^2} + \langle \pi_2(x), \pi_2(y) \rangle_{\mathbb{F}_p^{2n}}. \tag{1.18}$$

In the following section we will describe the structure of indecomposable maximally isotropic subgroups of ${\mathbb{F}_p}^2 \times \mathbb{F}_p^{2n}$. Similar observations were made by Mumford in [22, p. 329].

Lemma 1.3.1. *Let G be an indecomposable maximally isotropic subgroup of ${\mathbb{F}_p}^2 \times \mathbb{F}_p^{2n}$ with respect to the product pairing. Then π_1 is surjective.*

Proof. Assume π_1 is not surjective. Then its image will be of dimension ≤ 1. If $\pi_1(G)$ is 0-dimensional, $G \cong \{0\} \times G_2$ where G_2 is an isotropic subgroup of $\mathbb{F}_p^{2n}$. This contradicts the assumption that G is indecomposable. Now assume

$\pi_1(G)$ is a 1-dimensional vector space V. Then $\pi_1(G)$ is a maximal isotropic subspace because it is 1-dimensional. We see that

$$0 = \langle x, y\rangle = \langle \pi_1(x), \pi_1(y)\rangle_{\mathbb{F}_p^2} + \langle \pi_2(x), \pi_2(y)\rangle_{\mathbb{F}_p^{2n}} = 0 + \langle \pi_2(x), \pi_2(y)\rangle_{\mathbb{F}_p^{2n}}. \quad (1.19)$$

This shows that $\pi_2(G)$ is an isotropic subgroup of $\mathbb{F}_p^{2n}$. We have $|G| = p^{n+1}$, $|\pi_1(G)| = p$, and $|\pi_2(G)| \leq p^n$ by Lemma 1.2.2. It follows that

$$G \cong \pi_1(G) \times \pi_2(G), \quad (1.20)$$

so G is decomposable, which causes a contradiction. Hence π_1 is surjective. □

Definition 1.3.2. Let G be a group that comes equipped with a pairing $\langle \cdot, \cdot \rangle \to \mathbb{F}_p^*$ and let H be a subgroup of G. We write $H^\perp$ for the *orthogonal complement of H in G* with respect to the pairing.

Lemma 1.3.3. *Let G be an indecomposable maximally isotropic subgroup of ${\mathbb{F}_p}^2 \times \mathbb{F}_p^{2n}$ with respect to the product pairing. Let G' be the kernel of the map $\pi_1 : G \to \mathbb{F}_p^2$. Consider the subgroup $H = \pi_2(G')$ in $\mathbb{F}_p^{2n}$. Then we have the following:*

(i) The vector space $H^\perp/H$ is 2-dimensional.

(ii) There exists a symplectic pairing on $H^\perp/H$, denoted $\langle .,. \rangle_{H^\perp/H}$ that is induced by the pairing $\langle .,. \rangle_{\mathbb{F}_p^{2n}}$ on $\mathbb{F}_p^{2n}$.

(iii) We can restrict π_2 to G to get an isomorphism $\pi_2 : G \to H^\perp$.

(iv) Now let $\pi_2|_{G/G'} : G/G' \to H^\perp/H$ be the map induced by π_2 on the quotient. We similarly define $\pi_1|_{G/G'} : G/G' \to \mathbb{F}_p^2$ for the natural isomorphism induced by π_1. Then the map $\phi : \mathbb{F}_p^2 \to H^\perp/H$ given by

$$\phi = \pi_2|_{G/G'} \circ \pi_1|_{G/G'}^{-1}. \quad (1.21)$$

is an isomorphism. This morphism has the property that

$$\langle x_1, x_2\rangle_{\mathbb{F}_p^2} = -\langle \phi(x_1), \phi(x_2)\rangle_{H^\perp/H}. \quad (1.22)$$

Proof. (i) We know that G is an $(n+1)$-dimensional vector space, and that $\pi_1 : G \to \mathbb{F}_p^2$ is a surjective map. This means that $\ker \pi_1 = G'$ is an $(n-1)$-dimensional subspace. Remark that $\pi_2|_{G'}$ is injective by construction, so H is isomorphic to G' and $\dim H = n-1$. Now $H^\perp$ is a subspace of $\mathbb{F}_p^{2n}$ which is defined by $n-1$ linear equations. This shows that $H^\perp$ is an $(n+1)$-dimensional subspace. We conclude that $H^\perp/H$ has dimension 2.

(ii) We define the following pairing on $H^\perp/H \times H^\perp/H$:

$$\langle [x_1],[x_2]\rangle_{H^\perp/H} = \langle x_1,x_2\rangle_{\mathbb{F}_p^{2n}}. \tag{1.23}$$

where we use [.] to denote the class in $H^\perp/H$.

The pairing is well-defined as elements of H pair to 0 with elements of $H^\perp$ and we get $\langle x_1+h_1, x_2+h_2\rangle_{\mathbb{F}_p^{2n}} = \langle x_1,x_2\rangle_{\mathbb{F}_p^{2n}}$ for all $x_1,x_2 \in H^\perp$, $h_1,h_2 \in H$.

(iii) The restriction of the projection π_2 to G induces a morphism $G \to \mathbb{F}_p^{2n}$. We furthermore have that $\pi_2(G) \subset H^\perp$ as $H = \pi_2(G')$ and G is isotropic, so we get a map $\pi_2 : G \to H^\perp$. We claim that this map is an isomorphism. As the map is linear and both spaces have the same dimension, it is enough to show that π_2 is injective.

Claim: If π_2 is not injective, there exist $(t_1,0)$,$(s_1,s_2) \in G$ such that

$$\langle (t_1,0),(s_1,s_2)\rangle \neq 0 \tag{1.24}$$

Proof: As π_2 is not injective, we can find $0 \neq t_1 \in \mathbb{F}_p^2$ such that $(t_1,0) \in G$. As $\pi_2(G)$ is not trivial, there exists an element $(\alpha_1,\alpha_2) \in G$ with $\alpha_2 \neq 0$. Similarly, as π_1 is surjective, we find an element $(\beta_1,\beta_2) \in G$ with $\beta_1 \neq 0 \neq t_1$. Now at least one of (α_1,α_2), (β_1,β_2) and $(\alpha_1+\beta_1,\alpha_2+\beta_2)$ will have the property that the first coordinate is not equal to 0 or t_1, and that the second coordinate is non-zero. Call this element (s_1,s_2). We now find that

$$\langle (t_1,0),(s_1,s_2)\rangle = \langle t_1,s_1\rangle_{\mathbb{F}_p^2} + \langle 0,s_2\rangle_{\mathbb{F}_p^{2n}} = \langle t_1,s_1\rangle_{\mathbb{F}_p^2} \neq 0. \tag{1.25}$$

This proves the claim.

As the existence of such a pair of elements contradicts the assumption that G is isotropic, we conclude that π_2 is an isomorphism.

(iv) The isomorphism π_2 descends to a natural isomorphism on the quotient: $\pi_2|_{G/G'} : G/G' \to H^\perp/H$ and $\pi_1|_{G/G'}$ was already an isomorphism by definition. It follows that

$$\phi = \pi_2|_{G/G'} \circ \pi_1|_{G/G'}^{-1} \tag{1.26}$$

is an isomorphism. It remains to show that ϕ is pairing-reversing. By construction $\phi(t_1) = t_2 \mod H$ and $\phi(s_1) = s_2 \mod H$ for some $(t_1,t_2),(s_1,s_2) \in G$. We know that

$$0 = \langle (t_1,t_2),(s_1,s_2)\rangle = \langle t_1,s_1\rangle_{\mathbb{F}_p^2} + \langle t_2,s_2\rangle_{\mathbb{F}_p^{2n}} = \langle t_1,s_1\rangle_{\mathbb{F}_p^2} + \langle t_2,s_2\rangle_{H^\perp/H}, \tag{1.27}$$

so

$$\langle t_1,s_1\rangle_{\mathbb{F}_p^2} = -\langle \phi(t_1),\phi(s_1)\rangle_{H^\perp/H}. \tag{1.28}$$

We conclude that ϕ is a pairing-reversing isomorphism. □

Lemma 1.3.4. *Conversely, any tuple (H,ϕ) of an $(n-1)$-dimensional subspace H of $\mathbb{F}_p^{2n}$ and a pairing-reversing isomorphism $\phi : \mathbb{F}_p^2 \to H^\perp/H$ defines an indecomposable maximally isotropic subgroup G of $\mathbb{F}_p^2 \times \mathbb{F}_p^{2n}$.*

Proof. Let $G = \{(x, \phi(x) + h) | x \in \mathbb{F}_p^2,\ \ h \in H\}$. Note that $H^\perp$ is not an isotropic subgroup as it consists of p^{n+1} elements, and a maximally isotropic subgroup of $\mathbb{F}_p^{2n}$ has p^n elements. As a result, the induced pairing on $H^\perp/H$ is non-trivial, so there exist $x_1, x_2 \in \mathbb{F}_p^2$ such that $\langle \phi(x_1), \phi(x_2)\rangle_{H^\perp/H} = \langle \phi(x_1), \phi(x_2)\rangle_{\mathbb{F}_p^{2n}}$ is non-zero. So the projection of G to $\mathbb{F}_p^{2n}$ is not an isotropic subgroup of $\mathbb{F}_p^{2n}$, so G is indecomposable.

We also see that

$$\begin{aligned}\langle (x_1, \phi(x_1) + h_1), (x_2, \phi(x_2) + h_2)\rangle &= \langle x_1, x_2\rangle_{\mathbb{F}_p^2} + \langle \phi(x_1) + h_1, \phi(x_2) + h_2\rangle_{\mathbb{F}_p^{2n}} \\ &= \langle x_1, x_2\rangle_{\mathbb{F}_p^2} + \langle \phi(x_1), \phi(x_2)\rangle_{\mathbb{F}_p^{2n}} + \\ &\langle (h_1, \phi(x_2))\rangle_{\mathbb{F}_p^{2n}} + \langle \phi(x_1), h_2\rangle_{\mathbb{F}_p^{2n}} + \langle h_1, h_2\rangle_{\mathbb{F}_p^{2n}} \\ &= \langle x_1, x_2\rangle_{\mathbb{F}_p^2} + \langle \phi(x_1), \phi(x_2)\rangle_{H/H^\perp} + 0 + 0 + 0 \\ &= 0.\end{aligned} \tag{1.29}$$

So G is an isotropic subgroup. As it is of order p^{n+1}, it is maximal. □

Lemma 1.3.5. *We have that the set of symplectic automorphisms* $\mathrm{Sp}(\mathbb{F}_2^2) \cong S_3$.

Proof. Giving a symplectic morphism is the same as giving an invertible skew-symmetric 2×2 matrix. In $\mathbb{F}_2^2$ every invertible matrix is skew-symmetric, so $\mathrm{Sp}(\mathbb{F}_2^2) \cong \mathrm{GL}_2(\mathbb{F}_2^2)$. Every bijection that sends 0 to 0 is linear, so there are 6 of these maps. Using this we see that

$$\mathrm{GL}_2(\mathbb{F}_2^2) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong S_3. \tag{1.30}$$

□

Corollary 1.3.6. *There are exactly 90 distinct indecomposable maximal isotropic subgroups in* $\mathbb{F}_2^2 \times \mathbb{F}_2^4$.

Proof. A calculation shows that the above two constructions are inverse to one another. This means that an indecomposable maximally isotropic subgroup is completely determined by a choice of a tuple (H, ϕ) where H is a 1-dimensional subspace of $\mathbb{F}_2^4$, and ϕ is a pairing reversing isomorphism $\mathbb{F}_2^2 \to H^\perp/H$. As any non-zero element in $\mathbb{F}_2^4$ gives us an order 2 subgroup, there are $2^4 - 1$ possible choices for H. After fixing H, there are 6 possible pairing reversing isomorphisms $\phi : \mathbb{F}_2^2 \to H^\perp/H$. We conclude that there are $6 \cdot 15 = 90$ distinct maximally isotropic subgroups in $\mathbb{F}_2^2 \times \mathbb{F}_2^4$. □

Remark 1.3.7. Alternatively, one can use Proposition 1.2.3 to see that there are 135 maximally isotropic subgroups of $\mathbb{F}_2^6$, 15 maximally isotropic subgroups of $\mathbb{F}_2^4$ and 3 maximally isotropic subgroups of $\mathbb{F}_2^2$. It follows that there are $135 - 3 \cdot 15 = 90$ indecomposable maximally isotropic subgroups of $\mathbb{F}_2^6$.

1.4 Description in terms of roots

Let X_1 be given by the equation $y^2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$, and let Y_2 be given by the equation $y^2 = (x-\beta_1)(x-\beta_2)(x-\beta_3)(x-\beta_4)(x-\beta_5)(x-\beta_6)$. We are now going to give a more explicit description of the indecomposable maximal isotropic subgroups of $\mathrm{Jac}(X_1)\times\mathrm{Jac}(Y_2)$ in terms of the α_i and β_i.

For that we first need the following lemmas to give a different description of the structure of the 2-torsion group of hyperelliptic curves.

Lemma 1.4.1. *Let $\mathfrak{A}$ be a set with $2n+2$ elements. For any two subsets S_1, S_2 of $\mathfrak{A}$, let*

$$S_1 + S_2 = (S_1 \cup S_2)\backslash(S_1 \cap S_2). \tag{1.31}$$

Let $S^{c_{\mathfrak{A}}}$ be the complement of S in A. We define the equivalence relation $\sim$ by $S_1 \sim S_2$ if $S_2 = S_1^{c_{\mathfrak{A}}}$. Then the set

$$G_{\mathfrak{A}} = \{S \subset \mathfrak{A} | \#S \cong 0 \mod 2\}/\{\sim\} \tag{1.32}$$

forms an $(n+1)$-dimensional $\mathbb{F}_2$-vector space under the operation $+$. Furthermore, the bilinear pairing

$$\langle S_1, S_2\rangle_{G_{\mathfrak{A}}} = \#(S_1 \cap S_2) \mod 2 \tag{1.33}$$

turns $(G_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{G_{\mathfrak{A}}})$ into a symplectic vector space.

Proof. See Lemma 2.4 and Proposition 6.3 in [24]. □

Definition 1.4.2. For any set $\mathfrak{A}$, we will write $(G_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{G_{\mathfrak{A}}})$ (or simply $G_{\mathfrak{A}}$) for the corresponding symplectic vector space defined in the above lemma.

Lemma 1.4.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two sets with an even number of elements and let $f : \mathfrak{A} \to \mathfrak{B}$ be a bijective map. Then the morphism $G(f) : G_{\mathfrak{A}} \to G_{\mathfrak{B}}$ given by $G(f)([S]) = [f(S)] \in G_{\mathfrak{B}}$ is a well-defined bijective symplectic map.*

Proof. As f is injective, and

$$f(S_1)+f(S_2) = (f(S_1)\cup f(S_2))\backslash(f(S_1)\cap f(S_2)) = f(S_1\cup S_2)\backslash f(S_1\cap S_2) = f(S_1+S_2) \tag{1.34}$$

we see that $G(f)$ is linear. Furthermore, the injectivity also implies that $G(f)([S]) = [f(S)] = [f(S)^{c_{\mathfrak{B}}}] = [f(S^{c_{\mathfrak{A}}})] = G(f)([S^c])$, so $G(f)$ is well-defined. If $G(f)([S]) = \emptyset$, the injectivity gives that $S = \emptyset$, so $G(f)$ is bijective. Finally, the injectivity of f implies that f does not change the number of elements in a set, so $\langle S_1, S_2\rangle_{G_{\mathfrak{A}}} = \langle G(f)(S_1), G(f)(S_2)\rangle_{G_{\mathfrak{B}}}$. □

Lemma 1.4.4. *Let C be a hyperelliptic curve of genus g given by the equation*

$$y^2 = f(x) \tag{1.35}$$

where f is a monic polynomial of degree $2g+2$. Let $\mathfrak{A} = \{\alpha_1, \dots \alpha_{2g+2}\}$ be the set of roots of f. Let $O = (\alpha_{2g+2}, 0)$. Let $\psi : \mathfrak{A} \to \mathrm{Div}(C)$ be given by $\psi(s) = (\alpha_s, 0) - O$. Then the morphism $\phi_C : (G_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{G_{\mathfrak{A}}}) \to (\mathrm{Jac}(C)[2], \langle\cdot,\cdot\rangle_2)$ given by

$$[S] \mapsto \sum_{s \in S} [\psi(s)] \tag{1.36}$$

is an isomorphism of symplectic vector spaces.

Proof. See Lemma 2.2, Corollary 2.11 and Proposition 6.3 in [24]. □

Remark 1.4.5. A similar statement can be made in the case that $\deg f = 2g+1$. In this case we define $\mathfrak{A} = \{\alpha_1, \dots, \alpha_{2g+1}, \infty\}$.

Remark 1.4.6. The properties in Lemma 1.4.1 are defined in such a way that they imitate the addition of 2-torsion points on a hyperelliptic curve C given by an equation of the form $y^2 = f(x)$ in $\mathbb{P}^2_k$. Write every point on $\mathrm{Jac}(C)$ in the form $P + Q - 2(\alpha_{2n+1}, 0)$. Then taking the symmetric difference in the addition emulates calculating modulo $\mathrm{div}((x-\alpha_i)/(x-\alpha_{2n+2}))$ for all i where α_i is a root of f. The relationship $S \sim S^{c_{\mathfrak{A}}}$ corresponds to calculating modulo $\mathrm{div}(y/(x-\alpha_{2n+2})^{n+1})$.

Let $\mathfrak{A} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the set of roots of the equation for X_1, and let $\mathfrak{B} = \{\beta_1, \beta_2, \dots, \beta_6\}$ consist of the roots of the equation for Y_2.

Proposition 1.4.7. *Let $T \subset \mathfrak{B}$ be a set of two elements. Then $H_T = \langle [T] \rangle$ is a 1-dimensional subspace of $G_{\mathfrak{B}}$ and*

$$(H_T^{\perp}/H_T, \langle\cdot,\cdot\rangle_{H_T^{\perp}/H_T}) \cong (G_{\mathfrak{B}\setminus T}, \langle\cdot,\cdot\rangle_{G_{\mathfrak{B}}}). \tag{1.37}$$

Proof. As $[T]$ is a non-zero element of the vector space $G_{\mathfrak{B}}$, it is clear that H_T is a 1-dimensional subspace of $G_{\mathfrak{B}}$. Because $\langle [S_1], [S_2] \rangle_{G_{\mathfrak{B}}} = 0$ if and only if $\#(S_1 \cap S_2)$ is even, we see that

$$\begin{aligned} H_T^{\perp} &= \{[S] \in G_{\mathfrak{B}} | \#(S \cap T) = 0 \text{ or } \#(S \cap T) = 2\} \\ &= \{[S] \in G_{\mathfrak{B}} | S \cap T = \emptyset \text{ or } S \cap T = T\}. \end{aligned} \tag{1.38}$$

Let $[S] \in H_T^{\perp}/H_T$. We will determine all subsets in the equivalence class of S in $H_T^{\perp}/H_T$. We easily see that $S, S^{c_{\mathfrak{B}}}, S+T$ and $S^{c_{\mathfrak{B}}}+T$ are in the equivalence class of S. Assume that $S \cap T = \emptyset$. Then $S^{c_{\mathfrak{B}}} \cap T = T$ and

$$(S+T)^{c_{\mathfrak{B}}} = ((S \cup T) \setminus (S \cap T))^{c_{\mathfrak{B}}} = (S \cup T)^{c_{\mathfrak{B}}} \cup (S \cap T) = S^{c_{\mathfrak{B}}} \setminus T = S^{c_{\mathfrak{B}}} + T. \tag{1.39}$$

If $S \cap T = T$ we also get that $(S+T)^{c_{\mathfrak{B}}} = S^{c_{\mathfrak{B}}} + T$. As S was chosen arbitrarily the same holds for $(S^{c_{\mathfrak{B}}} + T)^{c_{\mathfrak{B}}}$. As a result, the equivalence class of S consists exactly of $S, S^{c_{\mathfrak{B}}}, S+T$ and $S^{c_{\mathfrak{B}}}+T$. Of these four equivalence classes there are exactly two that are contained in $B\backslash T$. When $S\cap T=\emptyset$ these are S and $S^{c_{\mathfrak{B}}}+T$. Otherwise they are $S^{c_{\mathfrak{B}}}$ and $S+T$.

Consider the set:

$$G_{\mathfrak{B}\backslash T}=\{S\subset\mathfrak{B}\backslash T|\#S\cong 0 \mod 2\}/\{S\sim S^{c_{\mathfrak{B}\backslash T}}\}. \tag{1.40}$$

Let $[S]\in H_T^{\perp}/H_T$ and let $S'\in\mathfrak{B}\backslash T$ be a subset such that $[S]=[S']$. Then by the above considerations the map $\phi: H_T^{\perp}/H_T\to G_{\mathfrak{B}\backslash T}$ given by $\phi([S])=[S']$ is a well-defined isomorphism. □

Corollary 1.4.8. *Let X_1 be given by the equation $y^2=f_1(x)$, and let Y_2 be given by the equation $y^2=f_2(x)$. Write $\mathfrak{A}$ for the set of roots of f_1 and write $\mathfrak{B}$ for the set of roots of f_2. Giving a maximally isotropic subgroup of* $\mathrm{Jac}(X_1)[2]\times\mathrm{Jac}(Y_2)[2]$ *is the same as choosing a subset $T\subset\mathfrak{B}$ with $|T|=2$ and giving a linear isomorphism $\phi: G_{\mathfrak{A}}\to G_{\mathfrak{B}\backslash T}$.*

Proof. Lemma 1.4.4 tells us that $\mathrm{Jac}(X_1)\cong G_{\mathfrak{A}}$ and $\mathrm{Jac}(Y_2)\cong G_{\mathfrak{B}}$ as symplectic vector spaces. Now Lemma 1.3.4 says that giving a maximally isotropic subgroup of $G_{\mathfrak{A}}\times G_{\mathfrak{B}}$ is the same as choosing a 1-dimensional subspace H_T of $G_{\mathfrak{B}}$ and a pairing-preserving isomorphism $H_T\to H_T^{\perp}/H_T$. Proposition 1.4.7 then says choosing H_T is equivalent to choosing a subset T of $\mathfrak{B}$ containing two elements, and that $H_T^{\perp}/H_T\cong G_{\mathfrak{B}\backslash T}$. Finally, any bijective linear map $\phi: G_{\mathfrak{A}}\to G_{\mathfrak{B}\backslash T}$ is pairing-preserving as $\mathbb{F}_2^2$ admits a unique symplectic structure. □

Corollary 1.4.9. *Let $T=\{\beta_5,\beta_6\}$ and define a function $f:\mathfrak{A}\to\mathfrak{B}\backslash T$ by $f(\alpha_i)=\beta_i$ for $i=1,\ldots 4$. Then the tuple $(H_T,G(f))$ corresponds to the maximal isotropic subgroup*

$$\begin{aligned} G=\langle &(0,[(\beta_5,0)-(\beta_6,0)]),\\ &([(\alpha_1,0)-(\alpha_4,0)],[(\beta_1,0)-(\beta_4,0)]),\\ &([(\alpha_2,0)-(\alpha_4,0)],[(\beta_2,0)-(\beta_4,0)])\rangle. \end{aligned} \tag{1.41}$$

Proof. By definition

$$H_T=\langle[T]\rangle=\langle[(\beta_5,0)-(\beta_6,0)]\rangle \tag{1.42}$$

and

$$G(f)([(0,\alpha_i)-(0,\alpha_j)])=[(0,\beta_i)-(0,\beta_j)] \tag{1.43}$$

for $i,j\in[1,\ldots 4]$. Applying the construction in Lemma 1.3.4 gives us that the corresponding maximal isotropic subgroup is defined as

$$G=\{(x,G(f)(x)+h)|x\in\mathrm{Jac}(X_1),\ \ h\in H\}. \tag{1.44}$$

We see that G is generated by the elements $(0,h)$ with $h \neq 0$ and $(x, G(f)(x))$ with $x \in \mathrm{Jac}(X_1)$. As $[(\alpha_1,0)-(\alpha_4,0)]+[(\alpha_2,0)-(\alpha_4,0)] = [(\alpha_3,0)-(\alpha_4,0)]$, we get that

$$\begin{aligned} G = \langle &(0,[(\beta_5,0)-(\beta_6,0)]), \\ &([(\alpha_1,0)-(\alpha_4,0)],[(\beta_1,0)-(\beta_4,0)]), \\ &([(\alpha_2,0)-(\alpha_4,0)],[(\beta_2,0)-(\beta_4,0)])\rangle. \end{aligned} \tag{1.45}$$

□

Example 1.4.10. Let the equation of Y_2 be given by

$$y^2 = x(x+2)(x^2-2x-2)(x^2-6) \tag{1.46}$$

and let X_1 be given by

$$y^2 = x(x-1)(x^2-2x-5). \tag{1.47}$$

Write β_i for the roots of x^2-2x-2. Let $T = \{\beta_1, \beta_2\}$ and define $f : \mathfrak{A} \to \mathfrak{B}\backslash T$ by $f(0) = 0, f(1) = -2, f(1-\sqrt{6}) = \sqrt{6}$ and $f(1+\sqrt{6}) = -\sqrt{6}$. Then the tuple $(T, G(f))$ corresponds to the maximal isotropic subgroup G of $\mathrm{Jac}(X_1)[2] \times \mathrm{Jac}(Y_2)[2]$ given by:

$$\begin{aligned} G = \langle &(0,[(\beta_1,0)-(\beta_2,0)]), \\ &([(1,0)-(0,0)],[(-2,0)-(0,0)]), \\ &([(1-\sqrt{6},0)-(0,0)],[(\sqrt{6},0)-(0,0)])\rangle. \end{aligned} \tag{1.48}$$

We can, moreover, show that this maximal isotropic subgroup is Galois-invariant.

1.5 Rationality

Given two abelian varieties A_1 of genus 1, A_2 of genus 2 over k and an indecomposable maximal isotropic subgroup G of $(A_1 \times A_2)[2]$ one can ask when $(B_G, \mathcal{M}_B)$, the (2,2)-gluing of A_1 and A_2 along G, will be isomorphic to a curve (over k). The following result provides an answer:

Proposition 1.5.1. *Let $(B, \mathcal{M}_B)$ be an absolutely indecomposable (i.e. not isomorphic over $\overline{k}$ to a product of two principally polarized abelian varieties) principally polarized abelian variety of genus 3 over k. Then there exists a curve Z_3 over k and a field extension k'/k with $[k' : k] \leq 2$, such that $(B, \mathcal{M}_B)$ is k'-isomorphic to* $\mathrm{Jac}(Z_3)$ *by a quadratic twist with respect to -1. Additionally, if Z_3 is hyperelliptic, then $k' = k$.*

Proof. See [3, Proposition 3]. ∎

Definition 1.5.2. In the situation in Proposition 1.5.1 where $(B, \mathcal{M}_B)$ is a (2,2)-gluing of $\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2)$ over k, we define Z_3 to be an arithmetic *(2,2)-gluing of X_1 and Y_2*.

Remark 1.5.3. As $(B_G, \mathcal{M}_B)$ is the quotient of an abelian variety over k by the group G, B_G is defined over k if G is Galois-invariant.

Lemma 1.5.4. *Let f be a polynomial of even degree and let $\mathfrak{A}$ be its set of roots. The action of* $\mathrm{Gal}(f) \subset S_4$ *on $\mathfrak{A}$ induces a* $\mathrm{Gal}(f)$*-action on $G_{\mathfrak{A}}$. Let $V = \langle (12)(34), (13)(24) \rangle \subset S_4$. Then* $\mathrm{Gal}(f)/(Gal(f) \cap V)$ *acts faithfully on $G_{\mathfrak{A}}$.*

Proof. Let $\sigma \in \mathrm{Gal}(f)$. As σ is injective, Lemma 1.4.3 tells us that $G(f) \in \mathrm{Sym}(G_{\mathfrak{A}})$, so $\mathrm{Gal}(f)$ acts on $G_{\mathfrak{A}}$. Now let $[S_1], [S_2] \in G_{\mathfrak{A}}$. Then $[S_1] = [S_2]$ if and only if $S_1 = S_2$ or $S_1 = S_2^c$. This implies that if $\sigma \in \mathrm{Gal}(f)$ has the property that $\sigma([S]) = [S]$ for all $[S] \in G_{\mathfrak{A}}$, then $\sigma \in V = \langle (12)(34), (13)(24) \rangle$. As a consequence, $\mathrm{Gal}(f)/V$ acts faithfully on $G_{\mathfrak{A}}$. ∎

Proposition 1.5.5. *Let X_1, Y_2, $\mathfrak{A}$ and $\mathfrak{B}$ be as above. Let $T \subset \mathfrak{B}$ with $|T| = 2$ and $\phi : G_{\mathfrak{A}} \to G_{\mathfrak{B} \setminus T}$. Then, the maximally isotropic subgroup corresponding to the tuple (H_T, ϕ) is invariant under the action of $Gal(\bar{k}/k)$ if and only if H_T is Galois-invariant, and $\phi \circ \sigma = \sigma \circ \phi$ for all $\sigma \in Gal(\bar{k}/k)$.*

Proof. Assume that the group corresponding to $G = (H_T, \phi)$ is Galois-invariant. As the elements of G are of the form (x, y) with $x \in G_{\mathfrak{A}}$ and $y \in H_T^{\perp}$ this implies that $\sigma(H_T^{\perp}) = H_T^{\perp}$ for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Let $x \in H_T^{\perp}$. We have $\langle x, y \rangle_{G_{\mathfrak{B}}} = 0$ for all $y \in H_T^{\perp}$ if and only if $x \in H_T$. As the action of $\mathrm{Gal}(\bar{k}/k)$ preserves the restriction of $\langle \cdot, \cdot \rangle_{G_{\mathfrak{B}}}$ to $H_T^{\perp}$ by Lemma 1.5.4, we get that $\sigma(H_T) = H_T$.

As H_T is Galois-invariant, it follows that T is Galois-invariant. This means that σ descends to an action on $H_T^{\perp}/H_T$. We know that

$$\sigma((x, \phi(x))) = (\sigma(x), \sigma(\phi(x))) = (\sigma(x), \phi(\sigma(x)) + h) \tag{1.49}$$

for some $h \in H_T$. It follows that $\sigma(\phi(x)) = \phi(\sigma(x)) \mod H_T$, which is what we wanted to show.

Now assume that T is Galois-invariant, and that $\phi\sigma = \sigma\phi$ for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Let $(x, \phi(x) + h) \in G$. Then

$$\sigma((x, \phi(x) + h)) = (\sigma(x), \sigma(\phi(x)) + \sigma(h)) = (\sigma(h), \phi(\sigma(x)) + h') \tag{1.50}$$

for some $h' \in H_T$. So, $\sigma((x, \phi(x) + h)) \in G$, which concludes the proof. ∎

Corollary 1.5.6. *Let X_1 be a curve of genus 1 over k, and let Y_2 be a curve of genus 2 over k. If there exists a Galois invariant maximal isotropic subgroup G of* $\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2)$*, then Y_2 has a model of the form*

$$y^2 = g(x) \tag{1.51}$$

over k where g contains a quadratic factor.

Proof. The maximally isotropic subgroup G needs to be fixed by $\mathrm{Gal}(\overline{k}/k)$. Write down some model of Y_2 of the form

$$y^2 = g(x) \tag{1.52}$$

where g is of degree 6. Assume that $g = \prod_{i=1}^{6}(x - \beta_i)$ over $\overline{k}$. Then G corresponds to some tuple (T, ϕ) $T = \{\beta_i, \beta_j\} \subset B$ with $i \neq j$. Let $\sigma \in \mathrm{Gal}(\overline{k}/k)$. As $\sigma(T) = T$ we see that $p(x) = (x - \beta_i)(x - \beta_j)$ remains fixed under all elements in $\mathrm{Gal}(\overline{k}/k)$, so $p(x)$ is a polynomial over k. This shows that $g(x)$ has a quadratic factor over k. □

Definition 1.5.7. Let X be a hyperelliptic curve over a field k and let $y^2 = f$ be an equation for X over k. Let $d_1 < d_2 < \ldots < d_n$ be positive integers. We define f to be of *type* $(d_1^{n_1} \ldots d_m^{n_m})$ if there exists a factorization $f = \prod_{i=1}^{m} \prod_{j=1}^{n_i} f_{i,j}$ into irreducible factors with $\deg(f_{i,j}) = d_i$.

Definition 1.5.8. Let g be a polynomial of degree 6. We will say that g is *gluable* over k if g contains a quadratic factor over k.

Remark 1.5.9. A polynomial is gluable over k when its type over k is equal to one of the following:

$$(1^6), (1^4 2^1), (1^2 2^2), (2^3), (1^3 3), (1^1 2^1 3^1), (1^2 4^1) \text{ or } (2^1 4^1). \tag{1.53}$$

Lemma 1.5.10. *Let $G \subset S_3$ be a group and let $\rho_i : G \to S_3$ be faithful representations for $i = 1, 2$. If $|\rho_1(G)| = |\rho_2(G)|$ then there exists $\sigma \in S_3$ such that $\rho_2 = \sigma \rho_1 \sigma^{-1}$*

Proof. For G is cyclic it suffices to remark that all cycles of the same type are conjugate to one another. Let $G = S_3$. We may assume that after conjugation with some $\sigma_1 \in S_3$ that $\sigma_1 \rho_1((123))\sigma_1^{-1} = \rho_2((123)) = (123)$. Now remark that (123) is invariant under conjugation with (123), but the orbit of (12) under conjugation with (123) consists of all 2-cycles. So we can find $\sigma_2 \in S_3$ such that conjugation with some $\sigma_1 \in S_3$ that

$$\sigma_2 \sigma_1 \rho_1(x) \sigma_1^{-1} \sigma_2^{-1} = \rho_2(x) = x. \tag{1.54}$$

Setting $\sigma = \sigma_2 \sigma_1$ proves the statement in this case. □

Definition 1.5.11. Let $f(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$ be a polynomial of degree 4. Let $\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $\beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$, $\beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$. Then the polynomial $g(x) = (x-\beta_1)(x-\beta_2)(x-\beta_3)$ is called *the cubic resolvent of f*.

Remark 1.5.12. Let L be the splitting field of a quartic polynomial f. Then $L^{V_4\cap \mathrm{Gal}(f)}$ is the splitting field of the cubic resolvent of f.

Theorem 1.5.13. *Let X_1, Y_2 be curves over k that are given by equations: $y^2 = f_1(x)$ and $y^2 = f_2(x)$ respectively. Let $\mathfrak{A}$ be the set of roots of f_1 and let $\mathfrak{B}$ be the set of roots of f_2. Assume that f_2 is gluable over k, i.e. $f_2 = g_1 g_2$ where $\deg g_1 = 4$ and $\deg g_2 = 2$. Let T be the set of roots of g_2.*

Then there exists a Galois-equivariant morphism $\phi : G_{\mathfrak{A}} \to G_{\mathfrak{B}\setminus T}$ if and only if the splitting field of the cubic resolvent of f_1 is equal to the splitting field of the cubic resolvent of g_1.

Proof. Label the elements in the sets $\mathfrak{A}$ and $\mathfrak{B}\setminus T$ with $1,2,3,4$ and let $\mathrm{Gal}(f_1)$ and $\mathrm{Gal}(g_1)$ act on them. Let L_1 be the splitting field of f_1 and let L_2 be the splitting field of g_1. Assume that $L_1^{V_4\cap\mathrm{Gal}(f_1)} = L_2^{V_4\cap\mathrm{Gal}(g_2)}$ and define L to be this field. Lemma 1.5.4 tells us we get two faithful representations $\rho_1 : \mathrm{Gal}(L/K) \to \mathrm{Sym}(G_{\mathfrak{A}}) \cong S_3$ and $\rho_2 : \mathrm{Gal}(L/K) \to \mathrm{Sym}(G_{\mathfrak{B}/T}) \cong S_3$. As $|\rho_1(\mathrm{Gal}(L/K))| = |\rho_2(\mathrm{Gal}(L/K))|$ Lemma 1.5.10 says that any two faithful representations of S_3 are conjugate and there exist bases $\mathcal{B}_A$ of $G_{\mathfrak{A}}$ and $\mathcal{B}_{B\setminus T}$ of $G_{\mathfrak{B}\setminus T}$ (as symplectic vector spaces) such that $\rho_1(x) = \rho_2(x)$ for all x with respect to these bases. Let $\phi : G_{\mathfrak{A}} \to G_{\mathfrak{B}\setminus T}$ be the morphism with the property that the matrix representation of ϕ with respect to $\mathcal{B}_A$ and $\mathcal{B}_B$ is the identity matrix. Then ϕ is Galois equivariant by construction.

Now assume there exists a Galois-equivariant morphism $\phi : G_{\mathfrak{A}} \to G_{\mathfrak{B}\setminus T}$. Let $\sigma \in \mathrm{Gal}(\overline{k}/k)$. As ϕ is Galois-equivariant σ fixes the elements of $G_{\mathfrak{A}}$ if and only if σ fixes the elements of $G_{\mathfrak{B}}\setminus T$. As $V_4 \cap \mathrm{Gal}(f_1)$ fixes the elements of $G_{\mathfrak{A}}$ and $V_4 \cap \mathrm{Gal}(g_1)$ fixes the elements of $G_{\mathfrak{B}\setminus T}$ we find that

$$L_1^{V_4\cap\mathrm{Gal}(f_1)} = L_2^{V_4\cap\mathrm{Gal}(g_1)} \tag{1.55}$$

as we needed to show. □

Corollary 1.5.14. *Let $\rho_1 : \mathrm{Gal}(f_1) \to \mathrm{Sym}(G_{\mathfrak{A}}$ and $\rho_2 : \mathrm{Gal}(f_2) \to \mathrm{Sym}(G_{\mathfrak{B}\setminus T})$ be representations of the Galois action on $G_{\mathfrak{A}}$ and $G_{\mathfrak{B}\setminus T}$ respectively. If there exists a Galois-equivariant morphism $\phi : G_{\mathfrak{A}} \to G_{\mathfrak{B}\setminus T}$ then $|\rho_1(\mathrm{Gal}(f_1))| = |\rho_2(\mathrm{Gal}(f_2))|$.*

Remark 1.5.15. Corollary 1.5.14 gives us a necessary (but not sufficient) condition to check whether it is possible to glue a genus 1 curve and a genus 2 curve over the base field. Let X_1, Y_2, f_1 and $f_2 = g_1 g_2$ be as above. Label the roots of f_1 by $\{1,2,3,4\}$, label the roots of g_1 by $\{1,2,3,4\}$ and label the roots

Table 1.1: $|\rho_1(\mathrm{Gal}(f_1))|$ for all possible choices of $\mathrm{Gal}(f_1)$

Type of f_1	$\mathrm{Gal}(f_1)$	Structure	$\lvert\rho_A(\mathrm{Gal}(f_1))\rvert$
(1^4)	C_1	$\langle \mathrm{id}\rangle$	1
$(1^2 2)$	C_2	$\langle (12)\rangle$	2
(2^2)	C_2	$\langle (12)(34)\rangle$	1
(2^2)	V_4	$\langle (12),(34)\rangle$	2
$(1^1 3^1)$	C_3	$\langle (123)\rangle$	3
$(1^1 3^1)$	S_3	$\langle (123),(12)\rangle$	6
(4^1)	C_4	$\langle (1234)\rangle$	2
(4^1)	V_4	$\langle (12)(34),(13)(24)\rangle$	1
(4^1)	D_4	$\langle (1234),(13)\rangle$	2
(4^1)	A_4	$\langle (123),(124)\rangle$	3
(4^1)	S_4	$\langle (1234),(12)\rangle$	6

of g_2 by $\{5,6\}$. In Table 1.1 we have listed all the possibilities for $\mathrm{Gal}(f_1)$ as a subgroup of S_4 combined with the value of $\rho_1(\mathrm{Gal}(f_1))$. This was calculated by listing all possible Galois groups G (as a subgroup of S_4) for each factorization type of f_1 and then calculating the number of elements in $G \cap V_4$.

In Table 1.2 we have listed the possibilities for $\mathrm{Gal}(f_2)$ as a subgroup of S_6 and the value of $|\rho_2(\mathrm{Gal}(f_2))|$. This was calculated by listing all possible Galois groups G (as a subgroup of $S_4 \times S_2$) for each gluable factorization type of f_2, i.e. one quartic factor for which we label the roots by $1,\dots 4$ and one quadratic factor for which we denoted the roots by 5 and 6, and then calculating the number of elements in $G \cap V_4 \times S_2$. Remark that a different choice for the quadratic factor might lead to a different value for $|\rho_2(\mathrm{Gal}(f_2))|$ even though $\mathrm{Gal}(f_2)$ and the factorization type of f_2 are the same. This occurs for example when f_2 is of type $(1^4 2^1)$.

Corollary 1.5.16. *Let k be a finite field and let X_1, Y_2, f_1 and $f_2 = g_1 g_2$ be as above. Label the roots of f_1 by $\{1,2,3,4\}$, label the roots of g_1 by $\{1,2,3,4\}$ and label the roots of g_2 by $\{5,6\}$. Consider $\mathrm{Gal}(f_1,f_2)$ as a subgroup of $S_4 \times S_6$. Then Table 1.3 gives us all possibilities (for this choice of labeling of the roots) for $\mathrm{Gal}(f_1,f_2)$ for which a Galois stable maximal isotropic subgroup of $\mathrm{Jac}(X_1)[2]\times\mathrm{Jac}(Y_2)[2]$ can exist.*

Proof. A necessary condition for a gluing to exist is $|\rho_1(\mathrm{Gal}(f_1)| = |\rho_2(\mathrm{Gal}(f_2)|$. As we work over a finite field, every Galois group is cyclic. The table gives us a list of all groups of the form $\langle[\sigma_1,\sigma_2]\rangle$ for which $\mathrm{Gal}(f_1) = \langle\sigma_1\rangle$, $\mathrm{Gal}(f_2) = \langle\sigma_2\rangle$ and $|\rho_1(\mathrm{Gal}(f_1)| = |\rho_2(\mathrm{Gal}(f_2)|$ hold.

□

Table 1.2: $|\rho_2(\mathrm{Gal}(f_2))|$ for all possible choices of $\mathrm{Gal}(f_2)$

Type of f_2	$\mathrm{Gal}(f_2)$	Structure	$\lvert\rho_2(\mathrm{Gal}(f_2))\rvert$
(1^6)	C_1	$\langle \mathrm{id} \rangle$	1
$(1^4 2^1)$	C_2	$\langle (12) \rangle$	2
$(1^4 2^1)$	C_2	$\langle (56) \rangle$	1
$(1^2 2^2)$	C_2	$\langle (12)(34) \rangle$	1
$(1^2 2^2)$	C_2	$\langle (12)(56) \rangle$	2
$(1^2 2^2)$	V_4	$\langle (12),(34) \rangle$	2
$(1^2 2^2)$	V_4	$\langle (12),(56) \rangle$	2
(2^3)	C_2	$\langle (12)(34)(56) \rangle$	1
(2^3)	V_4	$\langle (12)(34),(56) \rangle$	1
(2^3)	V_4	$\langle (12),(34)(56) \rangle$	2
(2^3)	C_2^3	$\langle (12),(34),(56) \rangle$	2
$(1^3 3)$	C_3	$\langle (123) \rangle$	3
$(1^3 3)$	S_3	$\langle (123),(12) \rangle$	3
$(1^1 2^1 3^1)$	C_6	$\langle (123)(56) \rangle$	3
$(1^1 2^1 3^1)$	S_3	$\langle (123),(12) \rangle$	6
$(1^1 2^1 3^1)$	D_6	$\langle (123)(56),(12) \rangle$	6
$(1^2 4^1)$	C_4	$\langle (1234) \rangle$	2
$(1^2 4^1)$	V_4	$\langle (12)(34),(13)(24) \rangle$	1
$(1^2 4^1)$	D_4	$\langle (1234),(13) \rangle$	2
$(1^2 4^1)$	A_4	$\langle (123),(124) \rangle$	3
$(1^2 4^1)$	S_4	$\langle (1234),(12) \rangle$	6
$(2^1 4^1)$	C_4	$\langle (1234)(56) \rangle$	2
$(2^1 4^1)$	C_2^3	$\langle (12)(34),(13)(24),(56) \rangle$	2
$(2^1 4^1)$	D_4	$\langle (1234)(56),(13) \rangle$	2
$(2^1 4^1)$	D_4	$\langle (1234),(13)(56) \rangle$	2
$(2^1 4^1)$	D_4	$\langle (1234)(56),(13)(56) \rangle$	2
$(2^1 4^1)$	$C_2 \times D_4$	$\langle (1234),(13),(56) \rangle$	2
$(2^1 4^1)$	$C_2 \times A_4$	$\langle (123),(234),(56) \rangle$	3
$(2^1 4^1)$	S_4	$\langle (1234)(56),(12)(56) \rangle$	6
$(2^1 4^1)$	$S_4 \times C_2$	$\langle (1234),(12),(56) \rangle$	6

Table 1.3: Galois stable groups in the finite field case

Type of f	Type of g	Group
(1^4)	(1^6)	$\langle[\mathrm{id},\mathrm{id}]\rangle$
(1^4)	$(1^4 2^1)$	$\langle[\mathrm{id},(56)]\rangle$
(1^4)	$(1^2 2^2)$	$\langle[\mathrm{id},(12)(34)]\rangle$
(1^4)	(2^3)	$\langle[\mathrm{id},(12)(34)(56)]\rangle$
$(1^2 2^1)$	$(1^4 2^1)$	$\langle[(12),(12)]\rangle$
$(1^2 2^1)$	$(1^2 2^2)$	$\langle[(12),(12)(56)]\rangle$
$(1^2 2^1)$	$(1^2 4^1)$	$\langle[(13),(1234)]\rangle$
$(1^2 2^1)$	$(2^1 4^1)$	$\langle[(13),(1234)(56)]\rangle$
(2^2)	(1^6)	$\langle[(12)(34),\mathrm{id}]\rangle$
(2^2)	$(1^4 2^1)$	$\langle[(12)(34),(56)]\rangle$
(2^2)	$(1^2 2^2)$	$\langle[(12)(34),(12)(34)]\rangle$
(2^2)	(2^3)	$\langle[(12)(34),(12)(34)(56)]\rangle$
$(1^1 3^1)$	$(1^3 3^1)$	$\langle[(123),(123)]\rangle$
$(1^1 3^1)$	$(1^1 2^1 3^1)$	$\langle[(123),(123)(56)]\rangle$
(4^1)	$(1^4 2^1)$	$\langle[(1234),(13)]\rangle$
(4^1)	$(1^2 2^2)$	$\langle[(1234),(13)(56)]\rangle$
(4^1)	$(1^2 4^1)$	$\langle[(1234),(1234)]\rangle$
(4^1)	$(2^1 4^1)$	$\langle[(1234),(1234)(56)]\rangle$

Chapter 2

Preliminary considerations

2.1 Hyperelliptic curves

In this section we will characterize the hyperelliptic curves of genus 3 that are (2,2)-gluings. From now on we will assume that $\mathrm{char}(k) \neq 2$. We need the following proposition from [18].

Proposition 2.1.1. *Let $C : y^2 = f(x)$ and $C' : y^2 = f'(x)$ be two hyperelliptic curves of genus g over a field k. Every isomorphism $\phi : C \to C'$ is given by an expression of the form*

$$(x,y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{ey}{(cx+d)^{g+1}} \right), \tag{2.1}$$

for some $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(k)$ and $e \in k^*$. The pair (M,e) is unique up to replacement by $(\lambda M, \lambda^{g+1}e)$ for $\lambda \in k^*$. The composition of isomorphisms (M,e) and (M',e') is $(M'M, e'e)$.

In the following two propositions, we will assume that our base field is $\mathbb{C}$. A similar argument can be made for more general base fields using Tate modules.

Proposition 2.1.2. *Let p be a prime, let X, Y, and Z be curves, and assume Z is of genus g. Let $(\phi, Z, \mathcal{P}_Z)$ be the (p,p)-gluing of $(\mathrm{Jac}(X), \mathcal{P}_X)$ and $(\mathrm{Jac}(Y), \mathcal{P}_Y)$, the principally polarized abelian varieties corresponding to the curves X and Y. Write $\mathrm{pr}_1 : \mathrm{Jac}(X) \times \mathrm{Jac}(Y) \to \mathrm{Jac}(X)$ and $\mathrm{pr}_2 : \mathrm{Jac}(X) \times \mathrm{Jac}(Y) \to \mathrm{Jac}(X)$ for the two projection maps. Let $\phi^t : Z^t \to X^t \times Y^t$ be the dual morphism with respect to the polarization $\mathcal{P}_Z$ on Z and the polarization $\mathcal{P}_{X\times Y} = \mathrm{pr}_1^*((\mathcal{P}_X))^p \otimes \mathrm{pr}_2^*(\mathcal{P}_Y)^p$ on $X \times Y$. Then $\phi^t \circ \phi = [p]_{X\times Y}$.*

Proof. Let $E_{X\times Y}$ be the alternating bilinear form corresponding to the algebraic equivalence class of $P_{X\times Y}$ and let E_Z be the alternating bilinear form corresponding to the algebraic equivalence class of $\mathcal{P}_Z$. As ϕ is a (p,p)-gluing,

$\ker\phi$ is a subgroup of order p^g of $\mathrm{Jac}(X)[p]\times\mathrm{Jac}(Y)[p]$ that is maximally isotropic with respect to the pairing induced by $E_{X\times Y}$ (See Lemma 1.2.2). Now choose a basis $e_1,\dots,e_{2g}$ for $H_1(\mathrm{Jac}(X)\times\mathrm{Jac}(Y),\mathbb{Z})$ such that the matrix representation $M_{E_{X\times Y}}$ of $E_{X\times Y}$ is

$$M_{E_{X\times Y}} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}. \tag{2.2}$$

and $\ker\phi = \langle \frac{1}{p}e_1,\dots,\frac{1}{p}e_g, e_{g+1},\dots e_{2g}\rangle$. Such a basis exists because the kernel of ϕ is isotropic with respect to the Weil pairing. If we now choose $e_1,\dots,e_{2g}$ as the basis for $H_1(\mathrm{Jac}(Z),\mathbb{Z})$, the matrix representation M_ϕ of ϕ with respect to the bases chosen above will be

$$M_\phi = \begin{pmatrix} p\cdot I_g & 0 \\ 0 & I_g \end{pmatrix}. \tag{2.3}$$

By the gluing construction, we have that $\phi^*(\mathcal{P}_Z)=\mathcal{P}_{X\times Y}$. Let $\mathcal{P}_Z$ correspond to the line bundle $L(H,\alpha)$. As we will see in Lemma 3.1.15, $\phi^*(\mathcal{P}_Z)$ is given by $L(H(M_\phi\cdot,M_\phi\cdot),\alpha\circ M_\phi)$. In terms of matrices, this translates to:

$$M_\phi^t M_{E_Z} M_\phi = pM_{E_{X\times Y}}. \tag{2.4}$$

As M_{E_Z} corresponds to an alternating bilinear form coming from a principal polarization, it follows that

$$M_{E_Z} = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} \tag{2.5}$$

with respect to the basis chosen on $H_1(\mathrm{Jac}(Z),\mathbb{Z})$. In [25, II.9] we find that the isogeny induced by the the polarization, $\phi_{P_{X\times Y}}$, is given by $x\mapsto L(0,e^{2\pi i E_{X\times Y}(x,u)})$. So, fixing bases on the homology of the dual abelian varieties, we can choose to represent the polarizations $\mathrm{Jac}(X)\times\mathrm{Jac}(Y)\to(\mathrm{Jac}(X)\times\mathrm{Jac}(Y))^t$ and $\mathrm{Jac}(Z)\to\mathrm{Jac}(Z)^t$ as the matrices $M_{EX\times Y}$ and M_{E_Z} respectively. We are now ready to compute $\phi^t\circ\phi$. We have the following chain of maps:

$$\begin{tikzcd} \mathrm{Jac}(X)\times\mathrm{Jac}(Y) \arrow[r, "\phi"] & \mathrm{Jac}(Z) \arrow[r] & \mathrm{Jac}(Z)^t \arrow[dl] \\ & (\mathrm{Jac}(X)\times\mathrm{Jac}(Y))^t \arrow[r] & \mathrm{Jac}(X)\times\mathrm{Jac}(Y) \end{tikzcd} . \tag{2.6}$$

By Lemma 3.1.16 we know that taking duals corresponds to taking duals of the matrices, so we get

$$M_{E_{X\times Y}}^t (M_\phi)^t M_{E_Z} M_\phi = \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix}\begin{bmatrix} pI_g & 0 \\ 0 & I_g \end{bmatrix}\begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}\begin{bmatrix} pI_g & 0 \\ 0 & I_g \end{bmatrix} = \begin{bmatrix} pI_g & 0 \\ 0 & pI_g \end{bmatrix}, \tag{2.7}$$

which is what we wanted to show. □

Proposition 2.1.3. *Let $(\phi, Z_g, \theta_{Z_g})$ be a (p,p)-gluing of a genus 1 curve X_1 and a genus $g-1$ curve Y_{g-1}. Then there exists a degree p morphism $\pi_1 : Z_g \to X_1$.*

Proof. Let $i : Z_g \to \mathrm{Jac}(Z_g)$ be the Abel-Jacobi map and let $\mathrm{pr}_1 : \mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_{g-1}) \to \mathrm{Jac}(X_1)$ be the projection to the first component. Now the map $\pi_1 = \mathrm{pr}_1 \circ \phi^t \circ i$ gives us a morphism of curves $\pi_1 : Z_g \to X_1$. This map cannot be constant as it would contradict the fact that ϕ is an isogeny, and as π_1 maps to a connected abelian variety, it needs to be surjective. We will now determine the degree of π_1.

We first remark that $\pi_{1,*} = (\mathrm{pr}_1 \circ \phi \circ i_*)$. Indeed, let $D = \sum P_i - Q_i$ be a divisor on Z_g. Now

$$\pi_*(D) = \sum \mathrm{pr}_1 \circ \phi \circ i(P_i) - \mathrm{pr}_1 \circ \phi \circ i(Q_i) = \sum \mathrm{pr}_1 \circ \phi \circ i_*(D). \tag{2.8}$$

Choose bases for $H_0(\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_{g-1}))$ and $H_0(\mathrm{Jac}(X_1))$ such that the representation of pr_1 with respect to these bases is given by

$$M_{\mathrm{pr}_1^*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}. \tag{2.9}$$

Let M_{ϕ^t} be the matrix representation for ϕ^t with respect to the chosen basis for $H_0(\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_{g-1}))$ and some choice of basis for $H_0(\mathrm{Jac}(Z_g))$. By construction the representation for i_* is the identity map.

We now find that

$$M_{\pi_{1,*} \circ \pi_1^*} = M_{\pi_{1,*}} \cdot M_{\pi_{1,*}}^t = \left(M_{\mathrm{pr}_1} M_\phi^t\right)\left(M_\phi M_{\mathrm{pr}_1}^t\right) = M_{\mathrm{pr}_1} p \cdot \mathrm{Id}_{2g} M_{\mathrm{pr}_1}^t = p \cdot \mathrm{Id}_2. \tag{2.10}$$

where we use Proposition 2.1.2 in the third equality. This implies that

$$\pi_{1,*} \circ \pi_1^* = [p]_{\mathrm{Jac}(X_1)}, \tag{2.11}$$

so π_1 is a morphism of degree p by [19, Proposition 7.3.8].

□

Proposition 2.1.4. *Let Z_3 be a curve of genus 3 and assume that Z_3 is a (2,2)-gluing of a genus 1 curve X_1 and a genus 2 curve Y_2. Assume there exists a degree 2 morphism $\pi_2 : Z_3 \to Y_2$. Then Z_3 is a hyperelliptic curve.*

Proof. By Proposition 2.1.3 there exists a degree 2 morphism $\pi_1 : Z_3 \to X_1$. Both π_1 and π_2 give rise to involutions of the curve Z_3. Let us denote the corresponding involutions by i_1 and i_2 respectively. The involutions induce automorphisms i_1^*, i_2^* of degree 2 on the Jacobian of Z_3. As i_1 fixes the divisor classes pulled back from X_1 and i_2 fixes the divisors classes pulled back from Y_2, i_1^* and i_2^* will be represented by the matrices

$$M_{i_1^*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad M_{i_2^*} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.12}$$

in $\mathrm{End}_k^0(\mathrm{Jac}(Z_3)) \cong \mathrm{End}_k^0(\mathrm{Jac}(X_1)) \times \mathrm{End}_k^0(\mathrm{Jac}(X_2))$ after making a certain choice of bases. Now define $i_0 = i_1 \circ i_2$. We find that

$$M_{i_0^*} = M_{i_2^*} M_{i_1^*} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{2.13}$$

so multiplication with -1 comes from an automorphism of Z_3. Therefore Z_3 is a hyperelliptic curve by [17, Appendice, Théorème 3].

□

Proposition 2.1.5. *Let Z_g be a nonsingular hyperelliptic curve of genus g over an algebraically closed field k, that comes equipped with an involution that is not the hyperelliptic involution. Let $g_1 = \lfloor \frac{g}{2} \rfloor$ and $g_2 = g - g_1$. Then there exist $\alpha_1, \ldots, \alpha_{g+1} \in k$ such that we can embed Z_g into $\mathbb{A}_k^2$ with an equation of the form*

$$F: \quad y^2 = \prod_{i=1}^{g+1} (x^2 - \alpha_i) \tag{2.14}$$

and Z_g comes equipped with three involutions that are given by:

$$i_0(x,y) \mapsto (x, -y), \tag{2.15}$$
$$i_1(x,y) \mapsto (-x, y) \text{ and} \tag{2.16}$$
$$i_2 = i_0 \circ i_1(x,y) \mapsto (-x, -y). \tag{2.17}$$

Proof. Let C' be a curve, isomorphic to Z_g, that is given by an equation $F'(x', y') = 0$ in Weierstrass form. Then the hyperelliptic involution $i_0' : C' \to C'$ is given by $(x', y') \mapsto (x', -y')$. By assumption C' comes equipped with an involution $i_1' : C' \to C'$ which is not the hyperelliptic involution. According to Proposition 2.1.1 we are able to write

$$i_1'(x', y') = \left(\frac{ax' + b}{cx' + d}, \frac{ey'}{(cx' + d)^{g+1}} \right), \tag{2.18}$$

for some $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(k)$ and $e \in k^*$, so i_1' corresponds to the tuple (M, e). As i_1' is an involution, we find that $(M^2, e^2) = (\lambda^2 \mathrm{Id}, \lambda^{2g+2})$ for some $\lambda \in k^*$.

After rescaling (M,E) by an element of k^* if necessary, we can assume that $(M^2,e^2)=(I,1)$. Now choose a matrix S that diagonalizes M. In this case SMS^{-1} is of the form

$$\begin{bmatrix} -1 & 0 \\ 0 & \pm 1 \end{bmatrix} \tag{2.19}$$

as i_1' is not the identity map and not equal to i_0'.

Let $\phi_S : C' \to C$ be the isomorphism given by the tuple $(S,1)$. Then the curve C is given by the equation $F = F' \circ \phi_S^{-1}$. And i_1' lifts to an involution $\phi_S^{-1} \circ i_1' \circ \phi_S : C \to C$. W.l.o.g. we can assume the three involutions on C are given by

$$i_0(x,y) \mapsto (x,-y), \tag{2.20}$$
$$i_1(x,y) \mapsto (-x,y) \text{ and} \tag{2.21}$$
$$i_2 = i_0 \circ i_1(x,y) \mapsto (-x,-y) \tag{2.22}$$

in the new coordinates.

As F remains invariant (up to a scalar) under these involutions, it is a sum of monomials that lie in the same eigenspace of i_1. By assumption $F \in \langle 1, y^2, x, x^2, \ldots, x^{2g+2}\rangle$. The map i_1 has two eigenspaces:

$$E_1 = \langle y^2, 1, x^2, x^4, \ldots, x^{2g+2}\rangle \quad \text{and} \quad E_{-1} = \langle x, x^3, \ldots, x^{2g+1}\rangle. \tag{2.23}$$

As the latter does not contain y, any equation in it will give us a finite number of copies of a line. So, F needs to be of the form $y^2 = \prod_{i=1}^{g+1}(x^2-\alpha_i)$ for some $\alpha_i \in k$. □

Corollary 2.1.6. *In the above situation we have two degree two covers:*

$$\pi_1 : Z_g \to X_{g_1}, \quad \pi_1(x,y) \mapsto (x^2,y) \quad \text{and} \tag{2.24}$$
$$\pi_2 : Z_g \to Y_{g_2}, \quad \pi_2(x,y) \mapsto (x^2,xy), \tag{2.25}$$

where X_{g_1} and Y_{g_2} are nonsingular curves over k of genus g_1 and genus g_2 respectively. They are given by the equations:

$$X_{g_1}: \quad v^2 = \prod_{i=1}^{g+1}(u-\alpha_i), \tag{2.26}$$

$$Y_{g_2}: \quad s^2 = t\prod_{i=1}^{g+1}(t-\alpha_i). \tag{2.27}$$

Proof. Let $X_{g_1} = Z_g/\langle i_1\rangle$. The function field $K(X_{g_1})$ contains all polynomials in $K(Z_g)$ that remain invariant under i_1, and the quotient map $\pi_1 : Z_g \to X_{g_1}$

induces a natural inclusion of function fields $\pi_1^* : K(X_{g_1}) \to K(Z_g)$. As $u = x^2$ and $v = y$ are both invariant under i_1, we have that

$$L_1 = k(u,v)/(v^2 - \prod_{i=1}^{g+1}(u-\alpha_i)) \subset K(X_{g_1}). \tag{2.28}$$

We claim that $L_1 = K(X_{g_1})$. Remark that $v^2 - \prod_{i=1}^{g+1}(u-\alpha_i)$ is irreducible because it is Eisenstein in $k(v)[u]$ for the prime $(u-\alpha_1)$. This means that L_1 is a field and the map

$$u \mapsto x^2, v \mapsto y \tag{2.29}$$

is a well-defined inclusion of function fields $L_1 \hookrightarrow K(Z_g)$ of degree 2. As $[K(Z_g) : K(X_{g_1})] = 2$, it follows that $K(X_{g_1}) = L_1$. The curve X_{g_1} will then be given by the equation

$$v^2 = \prod_{i=1}^{g+1}(u-\alpha_i). \tag{2.30}$$

As the equation contains $g+1$ roots and is hyperelliptic, the curve will have genus $(g-1)/2$ if g is odd and genus $g/2$ if g is even. So $g_1 = \lfloor \frac{g}{2} \rfloor$. Now because Z_g is non-singular, the α_i have to be distinct and non-zero. This implies that X_{g_1} is also non-singular. We finally remark that the inclusion $L_1 \hookrightarrow K(Z_g)$ implies that the morphism $\pi_1 : Z_g \to X_{g_1}$ is given by $\pi_1(x,y) \mapsto (x^2, y)$.

Let $Y_{g_2} = Z_3/\langle i_2 \rangle$. Similarly, $K(Y_{g_2})$ contains the monomials xy and x^2 as they remain invariant under i_2. Setting $t = x^2, s = xy$, we find that

$$L_2 = k(s,t)/(s^2 - t\prod_{i=1}^{g+1}(t-\alpha_i)) \subset K(Y_2). \tag{2.31}$$

As before, L_2 is a field because $s^2 - t\prod_{i=1}^{g+1}(t-\alpha_i)$ is irreducible. The inclusion $L_2 \hookrightarrow K(Z_g)$ given by $t \mapsto x^2, s \mapsto xy$ is a field extension of degree 2, so $K(Y_{g_2}) = L_2$. We find that Y_{g_2} is given by the equation

$$s^2 = t\prod_{i=1}^{g+1}(t-\alpha_i). \tag{2.32}$$

This curve is also non-singular. Indeed, the α_i are distinct and non-zero as one of them being zero would imply that Z_g is of the form $y^2 = x^2\prod_{i=1}^{g}$, which is singular. As the equation contains $g+2$ roots and is hyperelliptic, the curve will have genus $(g+1)/2$ if g is odd and genus $g/2$ if g is even. So $g_2 = g - g_1$. Finally, we will give an explicit description of π_2 in terms of coordinates. Let $\langle (x-a),(y-b) \rangle$ be the maximal ideal of $(a,b) \in Z_g$ and remark that $x^2 - a^2, xy - ab \in \langle (x-a),(y-b) \rangle$. This implies that $(t-a^2),(s-ab)$ are contained in the maximal of $\pi_2(a,b)$ so the map π_2 is given explicitly by $(x,y) \mapsto (x^2, xy)$. This completes the proof. □

We are now going to show that Z_g is a (2,2)-gluing of the two other curves defined above.

Proposition 2.1.7. *The curve Z_g in Proposition 2.1.5 is a (2,2)-gluing of the curves X_{g_1} and Y_{g_2} defined in the same proposition.*

Proof. Let

$$\left\langle \frac{du}{v}, \frac{udu}{v}, \ldots \frac{u^{g_1-1}du}{v} \right\rangle, \left\langle \frac{dt}{s}, \frac{tdt}{s}, \ldots \frac{t^{g_2-1}dt}{s} \right\rangle \text{ and } \left\langle \frac{dx}{y}, \frac{xdx}{y}, \ldots \frac{x^{g-1}dx}{y} \right\rangle \tag{2.33}$$

be bases for the differential forms on X_{g_1}, Y_{g_2} and Z_g respectively. Now

$$\pi_1^*\left(\frac{u^k du}{v}\right) = \frac{x^{2k}dx^2}{y} = \frac{2x^{2k}dx}{y}, \tag{2.34}$$

$$\pi_2^*\left(\frac{t^k dt}{s}\right) = \frac{x^{2k}dx^2}{y} = \frac{x^{2k}dx}{xy} = \frac{2x^{2k-1}dx}{y}. \tag{2.35}$$

This implies that the map $\phi = \pi_1^* \times \pi_2^* : \mathrm{Jac}(X_{g_1}) \times \mathrm{Jac}(Y_{g_2}) \to \mathrm{Jac}(Z_g)$ is surjective, and therefore an isogeny. We want to show that ϕ is a (2,2)-gluing. First remark, that because both π_1 and π_2 are of degree 2, $\phi^t \circ \phi = [2]_{\mathrm{Jac}(X_{g_1}) \times \mathrm{Jac}(Y_{g_2})}$. This means that $\ker\phi \subset \mathrm{Jac}(X_{g_1})[2] \times \mathrm{Jac}(Y_{g_2})[2]$. We also know that $|\ker\phi| = |\ker\phi^t|$ and $\ker(\phi^t \circ \phi) = 2^{2g}$. It follows that $\deg\phi = 2^g$.

Let $\mathcal{P}_Z$ be the polarization on $\mathrm{Jac}(Z_g)$,$\mathcal{P}_{X_{g_1}}$ the polarization on $\mathrm{Jac}(X_{g_1})$ and $\mathcal{P}_{Y_{g_2}}$ the polarization on $\mathrm{Jac}(Y_{g_2})$. As the π_i induce morphisms of polarized abelian varieties it follows that ϕ is a morphism of polarized abelian varieties. So, $\phi^*(P_{Z_g}) = \mathrm{pr}_1^*(\mathcal{P}_{X_{g_1}})^{n_1} \otimes \mathrm{pr}_2^*(\mathcal{P}_{Y_{g_2}})^{n_2}$ for some positive n_1, n_2. The degree of the right hand side is $n_1^{2g_1} n_2^{2g_2}$. This can only be equal to 2^{2g} if $n_1 = n_2 = 2$. So $\phi^*\mathcal{P}_{\mathcal{Z}_\rangle} = p_1^*(\mathcal{P}_{X_{g_1}})^2 \otimes p_2^*(\mathcal{P}_{Y_{g_2}})^2$ and ϕ induces a (2,2)-gluing.

□

Remark 2.1.8. The structure of the kernel in Proposition 2.1.7 can be described in terms of the one in Corollary 1.4.9. We claim that the kernel of ϕ is equal to

$$\begin{aligned} G = \langle &([0], [(0,0) - \infty]), \\ &([(\alpha_i, 0) - (\alpha_{g+1}, 0)], [(\alpha_i, 0) - (\alpha_{g+1}, 0)])\rangle. \end{aligned} \tag{2.36}$$

As G is of the form given in Corollary 1.4.9 it consists of exactly $2^{g_1+g_2}$ elements.

A calculation shows that

$$\begin{aligned} \pi_1^*([(\alpha_i, 0) - (\alpha_{g+1}, 0)]) &= [(\sqrt{\alpha_i}, 0) + (-\sqrt{\alpha_i}, 0) - (\sqrt{\alpha_{g+1}}, 0) - (-\sqrt{\alpha_{g+1}}, 0)] \\ \pi_2^*([(\alpha_i, 0) - (\alpha_{g+1}, 0)]) &= [(\sqrt{\alpha_i}, 0) + (-\sqrt{\alpha_i}, 0) - (\sqrt{\alpha_{g+1}}, 0) - (-\sqrt{\alpha_{g+1}}, 0)], \end{aligned} \tag{2.37}$$

so

$$\pi_1^* \times \pi_2^*(([(\alpha_i,0)-(\alpha_{g+1},0)],[(\alpha_i,0)-(\alpha_{g+1},0)]))$$
$$= 2[(\sqrt{\alpha_i},0)+(-\sqrt{\alpha_i},0)-(\sqrt{\alpha_{g+1}},0)-(-\sqrt{\alpha_{g+1}},0)] = \mathrm{div}(\frac{(x^2-\alpha_i)}{(x^2-\alpha_{g+1})}) = [0]. \tag{2.38}$$

Furthermore,

$$\begin{aligned}\pi_2^*([(0,0)-\infty]) &= \pi_2^*\left(\left[\sum_{i=1}^{g+1}(\alpha_i,0)-(g+1)\infty\right]\right)\\ &= \left[\sum_{i=1}^{g+1}(\sqrt{\alpha_i},0)+(-\sqrt{\alpha_i},0)-(\sqrt{\alpha_{g+1}},0)-(-\sqrt{\alpha_{g+1}},0)\right]\\ &= \mathrm{div}\left(\prod_{i=1}^{g+1}(x^2-\alpha_i)\right)\\ &= [0],\end{aligned} \tag{2.39}$$

so $\pi_1^* \times \pi_2^*(([0],[(0,0)-\infty])) = [0]$. It follows that $G \subset \ker(\phi)$. Comparing cardinalities gives us equality.

Corollary 2.1.9. *Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ be distinct and non-zero. Let $H_3(t)$ be the genus 3 curve given by the equation*

$$F_3(t)(x,y) = y^2-(x^2-\alpha_1+t)(x^2-\alpha_2+t)(x^2-\alpha_3+t)(x^2-\alpha_4+t). \tag{2.40}$$

Then $t \mapsto H_3(t)$ is a non-constant family of hyperelliptic genus 3 curves for which H_3 is generically the (2,2)-gluing of $X_1 : y^2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$ and some genus 2 curve.

Proof. Let $H_1(t)$ be the genus 1 curve given by the equation

$$y^2 = (x-\alpha_1+t)(x-\alpha_2+t)(x-\alpha_3+t)(x-\alpha_4+t). \tag{2.41}$$

In Proposition 2.1.5, we showed that we always have a degree 2 map $\pi_1 : H_3(t) \to H_1(t)$. Now note that $H_1(0)$ is isomorphic to $H_1(t)$ by the isomorphism $(x,y) \mapsto (x+t,y)$. So we can assume that we always have a degree 2 map $\widetilde{\pi}_1 : H_3(t) \to H_1(0)$. According to Proposition 2.1.7 the curve $H_3(t)$ is a (2,2)-gluing of $H_1(0)$ and some genus 2 curve. It remains to be shown that $t \mapsto H_3(t)$

is a non-constant family. Putting $H_3(t)$ into Rosenhain normal form gives us the Rosenhain invariants:

$$\begin{aligned}
\lambda_1 &= \frac{2\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}-2\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}}{-\sqrt{\alpha_1+t}^2+\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}-\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}+\sqrt{\alpha_2+t}\sqrt{\alpha_2+t}},\\
\lambda_2 &= \frac{2\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}-2\sqrt{\alpha_1+t}\sqrt{\alpha_3+t}}{-\sqrt{\alpha_1+t}^2+\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}-\sqrt{\alpha_1+t}\sqrt{\alpha_3+t}+\sqrt{\alpha_2+t}\sqrt{\alpha_3+t}},\\
\lambda_3 &= \frac{-4\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}}{\alpha_1+t-2\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}+\alpha_2+t},\\
\lambda_4 &= \frac{-2\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}-2\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}}{\alpha_1+t-\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}-\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}+\sqrt{\alpha_2+t}\sqrt{\alpha_2+t}},\\
\lambda_5 &= \frac{-2\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}-2\sqrt{\alpha_1+t}\sqrt{\alpha_3+t}}{\alpha_1+t-\sqrt{\alpha_1+t}\sqrt{\alpha_2+t}-\sqrt{\alpha_1+t}\sqrt{\alpha_3+t}+\sqrt{\alpha_2+t}\sqrt{\alpha_3+t}}.
\end{aligned} \tag{2.42}$$

As the Rosenhain invariants are non-constant in t, there needs to exist at least one t for which $H_3(t)$ is a non-singular hyperelliptic curve that is not isomorphic to $H_3(0)$. Therefore $t \mapsto H_3(t)$ gives us a non-constant family of genus 3 curves with the property that all of them are a (2,2)-gluing of X_1 and some genus 2 curve.

□

Remark 2.1.10. Let Y_2 be a curve genus 2. As both the moduli space of curves of the form $y^2-(x^2-\alpha_1)(x^2-\alpha_2)(x^2-\alpha_3)(x^2-\alpha_4)$ and the moduli space of genus 2 curves have dimension 3 any dominant map between them has to have finite fibers. So there are only finitely many non-isomorphic hyperelliptic genus 3 curves that contain Y_2 as a (2,2)-gluing factor.

Corollary 2.1.11. *Let $(\phi, Z_g, \theta_{Z_g})$ be a (2,2)-gluing of a genus 1 curve X_1 and a principally polarized abelian variety of dimension g-1. If Z_g is hyperelliptic, then g has to be either 2 or 3.*

Proof. Assume Z_g is hyperelliptic. Then $g \geq 2$. Proposition 2.1.3 tells us that there exists a morphism $\pi_1 : Z_g \to X_1$ of degree 2. So Z_g comes equipped with an involution that is not the hyperelliptic involution. Now Proposition 2.1.5 gives us an explicit description of all possible involutions on Z_g and the corresponding quotient morphisms of degree 2. As π_1 needs to be one of these maps, either $\lfloor \frac{g}{2} \rfloor$ or $g - \lfloor \frac{g}{2} \rfloor$ needs to be equal to 1. This is only possible if $g \leq 3$. □

2.2 Families of non-hyperelliptic curves

Let k be a field of characteristic $\neq 2, 3$ and let Y_2 be a curve over k of genus 2. In this section, we will (under mild assumptions) construct a non-hyperelliptic curve Z_3 of genus 3 over k such that Z_3 is a $(2,2)$–gluing of Y_2 and an elliptic curve. In fact we will find a non-isotrivial infinite family of such curves.

We are going to use the following result by Ritzenthaler and Romagny [26, Theorem 1.1]:

Theorem 2.2.1. *Let Z_3 be a non-hyperelliptic curve of genus 3 over k given by the equation:*

$$Z_3 : y^4 - h(x,z)y^2 + f(x,z)g(x,z) = 0 \tag{2.43}$$

in $\mathbb{P}^2$ where

$$f = f_2x^2 + f_1xz + f_2z^2, \quad g = g_2x^2 + g_1xz + g_2z^2, \quad h = h_2x^2 + h_1xz + h_0z^2 \tag{2.44}$$

are homogeneous degree 2 polynomials. It defines a cover of the genus 1 curve

$$X_1 : y^2 - h(x,z)y + f(x,z)g(x,z) \tag{2.45}$$

in the weighted projective space $\mathbb{P}^{(1,2,1)}$. Let

$$A = \begin{bmatrix} f_2 & f_1 & f_0 \\ h_2 & h_1 & h_0 \\ g_2 & g_1 & g_0 \end{bmatrix} \tag{2.46}$$

and assume that A is invertible. Let

$$A^{-1} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}. \tag{2.47}$$

Then $\mathrm{Jac}(Z_3)$ is isogenous to $\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2)$ where Y_2 is given by the equation

$$y^2 = b \cdot (b^2 - ac) \tag{2.48}$$

in $\mathbb{P}^{(1,3,1)}$. Here

$$a = a_1 + 2a_2x + a_3x^2, \quad b = b_1 + 2b_2x + b_3x^2, \quad c = c_1 + 2c_2x + c_3x^2. \tag{2.49}$$

Proof. See [26, Theorem 1.1]. □

We will now reverse this construction. Given three quadratic polynomials

$$a = a_1 + 2a_2x + a_3x^2, \quad b = b_1 + 2b_2x + b_3x^2, \quad c = c_1 + 2c_2x + c_3x^2, \tag{2.50}$$

in $k[x]$, we can consider the curve $Y_2 : y^2 = b(b^2 - ac)$ in $\mathbb{P}^{(1,3,1)}$ and find a genus 3 curve Z_3 that is a gluing of Y_2 and an elliptic curve in the following way. We set

$$B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}. \tag{2.51}$$

and under the assumption that B is invertible, we can take

$$B^{-1} = \begin{bmatrix} f_2 & f_1 & f_0 \\ h_2 & h_1 & h_0 \\ g_2 & g_1 & g_0 \end{bmatrix}. \tag{2.52}$$

Setting

$$f = f_2x^2 + f_1xz + f_0z^2, \quad g = g_2x^2 + g_1xz + g_0z^2, \quad h = h_2x^2 + h_1xz + h_0z^2, \tag{2.53}$$

we define Z_3 to be the non-hyperelliptic curve of genus 3 over k given by the equation:

$$Z_3 : y^4 - h(x,z)y^2 + f(x,z)g(x,z) = 0 \tag{2.54}$$

in $\mathbb{P}^2$ and X_1 to be the elliptic curve X_1 in $\mathbb{P}^{(1,2,1)}$ given by

$$X_1 : y^2 - h(x,z)y + f(x,z)g(x,z) = 0. \tag{2.55}$$

Additionally, we get a degree 2 cover $Z_3 \to X_1$ given by $(x,y) \mapsto (x,y^2)$.

For the next proof we will need the following definitions

Definition 2.2.2. Let D and C be (possibly) singular curves whose only singular points are ordinary double points. A cover $\pi : D \to C$ is called *allowable* if π is unramified away from the singular locus.

Definition 2.2.3. Let $\pi : D \to C$ be an allowable cover of degree 2. We define the *Prym variety* $\mathrm{Pr}(D/C)$ as the connected component of the identity of $\ker(\pi_*)$.

These definitions are the same as the ones used in [8].

Theorem 2.2.4. *The curve Z_3 as constructed above is a $(2,2)$-gluing of Y_2 and X_1.*

Proof. Consider the degree 2-cover $\pi : Z_3 \to X_1$ given above. The map π induces an inclusion $\pi^* : (\mathrm{Jac}(X_1), \mathcal{P}_{X_1}) \to (\mathrm{Jac}(Z_3), \mathcal{P}_{Z_3})$ of polarized abelian varieties, and by [4, Lemma 12.3.1] we find that we get

$$(\pi^*)^*\mathcal{P}_{Z_3} = \mathcal{P}_{X_1}^2 \tag{2.56}$$

the pullback of π^*.

Now $\mathrm{Jac}(Y_2)$ is isomorphic to the Prym of an allowable singular cover $\pi : \widetilde{Z}_3 \to \widetilde{X}_1$ whose normalization is equal to $\pi : Z_3 \to X_1$ as is shown in the proof of Theorem 1.1 in [26].

Theorem 3.7 in [2] tells us that the princial polarization on the generalized Jacobian $\mathrm{Jac}(\widetilde{Z}_3)$ restricts to $\mathcal{P}_{Y_2}^2$ on $\mathrm{Pr}(\widetilde{Z}_3\ /\widetilde{X}_1)$ where $\mathcal{P}_{Y_2}$ is the principal polarization on $\mathrm{Pr}(\widetilde{Z}_3\ /\widetilde{X}_1) \cong \mathrm{Jac}(Y_2)$. Lemma 1 in [8] says that we get a commutative diagram of polarized abelian varieties:

$$\begin{tikzcd} \Pr(\widetilde{Z}_3/\widetilde{X}_1) \arrow[r, "\nu"] \arrow[d] & \Pr(Z_3/X_1) \arrow[d, "i"] \\ \widetilde{Z}_3 \arrow[r] & Z_3 \end{tikzcd} . \tag{2.57}$$

where ν is induced by the morphism $\widetilde{Z}_3 \to Z_3$. This implies that

$$(i \circ \nu)^*(\mathcal{P}_{Z_3}) = \mathcal{P}^2_{Y_2}. \tag{2.58}$$

Consider the map $\pi^* \times (i \circ \nu) : \mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2) \to \mathrm{Jac}(Z_3)$. As we saw above we get that $(\pi^* \times (i \circ \nu))^*(\mathcal{P}_{Z_3}) = \mathcal{P}^2_{X_1} \otimes \mathcal{P}^2_{Y_2}$, so Z_3 is a (2,2)-gluing of $\mathrm{Jac}(X_1)$ and $\mathrm{Jac}(Y_2)$. □

Remark 2.2.5. Remark that in the construction of Y_2 in Proposition 2.2.1, we need to choose a factorization $f(x,z)g(x,z)$ of the homogeneous degree 4 polynomial defining Z_3. There are three possible ways to do this.

On the other hand, the map ν in the proof of Theorem 2.2.4 induces a $2^1 1^2$ polarization $\mathcal{L}$ on $\Pr(Z_3/X_1)$. After fixing such a polarization, there are three possible ways of choosing a subvariety A of $\Pr(Z_3/X_1)$ such that the restriction of $\mathcal{L}$ to A gives us a 2^2 polarization. Indeed, after a choice of a basis v_1, v_2, v_3, v_4 for $H^0(\Pr(Z_3/X_1), \mathbb{Z})$, we may assume that the matrix representation of $\mathcal{L}$ is

$$M_{\mathcal{L}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}. \tag{2.59}$$

Choosing such a subvariety A is the same as choosing a submodule of of $H^0(\Pr(Z_3/X_1), \mathbb{Z})$, such that the restriction of $M_{\mathcal{L}}$ to this submodule is

$$M_{\mathcal{L}}|_V = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}. \tag{2.60}$$

There are only three possible ways to do this. Namely $\langle 2v_1, v_2, v_3, v_4 \rangle, G = \langle v_1, 2v_2, v_3, v_4 \rangle$ or $G = \langle v_1 + v_2, v_1 - v_2, v_3, v_4 \rangle$. These three choices correspond to the three possible ways in which we could choose the factorization of the homogeneous degree 4 polynomial $f(x,z)g(x,z)$.

Definition 2.2.6. Let

$$a = a_1 + 2a_2x + a_3x^2, b = b_1 + 2b_2 + b_3x^2, c = c_1 + 2c_2 + c_3x^2 \in k[x] \tag{2.61}$$

be quadratic polynomials such that the matrix

$$B(a,b,c) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{2.62}$$

is invertible. We will call such a triple (a,b,c) a *regular quadratic triple*, and we will write $Y_2(a,b,c)$ for the genus 2 curve given by the equation $F_2(a,b,c) = y^2 - b(b^2 - ac)$. We will similarly write $Z_3(a,b,c)$ for the genus 3 curve (2.54), $F_3(a,b,c)$ for its equation and $X_1(a,b,c)$ for the genus 1 curve (2.55). For convenience, we will also write $(f,h,g) = \iota(a,b,c)$ for the polynomials we get by inverting the matrix.

We want to construct a family of genus 3 curves with a fixed genus 2 factor. To do this we can try to find a family of regular triples $t \mapsto (a(t),b(t),c(t))$ for which all $Y_2(a(t),b(t),c(t))$ are isomorphic to one another, but for which the family $Z_3(a(t),b(t),c(t))$ is non-constant.

Lemma 2.2.7. *Let (a,b,c) be a regular triple and $\lambda \in k^*$. Then*

(i) $F_2(a,b,c) = F_2(\lambda a, b, \lambda^{-1}c)$ *and* $F_3(a,b,c) = F_3(\lambda a, b, \lambda^{-1}c)$.

(ii) $F_2(a,b,c) = F_2(c,b,a)$ *and* $F_3(a,b,c) = F_3(c,b,a)$.

(iii) Over $\overline{k}$ *we have isomorphisms* $Y_2(a,b,c) \cong Y_2(\lambda a, \lambda b, \lambda c)$ *and* $Z_3(a,b,c) \cong Z_3(\lambda a, \lambda b, \lambda c)$.

Proof. (i): We have $F_2(a,b,c) = F_2(\lambda a, b, \lambda^{-1}c)$ as $ac = \lambda a \lambda^{-1} c$. Write

$$B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{2.63}$$

and $(f,h,g) = \iota(a,b,c)$. We see that

$$B(\lambda a, b, \lambda^{-1}c) = B\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \text{ and } (B(\lambda a, b, \lambda^{-1}c))^{-1} = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} B^{-1}. \tag{2.64}$$

From this we can conclude that $\iota(\lambda a, b, \lambda^{-1}) = (\lambda^{-1} f, h, \lambda g)$ which shows that $F_3(a,b,c) = F_3(\lambda a, b, \lambda^{-1}c)$.

(ii): We get $F_2(a,b,c) = F_2(c,b,a)$ as $ac = ca$. We see that

$$B(c,b,a) = B\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } (B(c,b,a))^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} B^{-1}. \tag{2.65}$$

This gives us that $(g,h,f) = \iota(c,b,a)$, so $F_3(a,b,c) = F_3(c,b,a)$.

(iii): We have

$$F_2(a,b,c) = y^2 - b(b^2 - ac), \qquad F_2(\lambda a, \lambda b, \lambda c) = y^2 - \lambda^3 b(b^2 - ac). \tag{2.66}$$

Now the map $\phi : Y_2(a,b,c) \to Y_2(\lambda a, \lambda b, \lambda c)$ given by $(x,y) \mapsto (x, \sqrt{\lambda^{-3}}y)$ gives an isomorphism over $\overline{k}$. For the genus 3 curve, we find that

$$B(\lambda a, \lambda b, \lambda c) = B\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \text{ and } B(\lambda a, \lambda b, \lambda c)^{-1} = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} B^{-1}. \tag{2.67}$$

This tells us that

$$F_3(\lambda a, \lambda b, \lambda c) = y^4 - \lambda^{-1} h y^2 + \lambda^{-2} f g \tag{2.68}$$

where $(f,h,g) = \iota(a,b,c)$. Then the map $\phi : Z_3(a,b,c) \to Z_3(\lambda a, \lambda b, \lambda c)$ given by $(x,y,z) \mapsto (x, \sqrt{\lambda}y, z)$ gives an isomorphism over $\overline{k}$.

□

In the following lemma we will describe a way to construct a regular quadratic triple (a', b', c') out of (a,b,c) with $Y_2(a,b,c) \cong Y_2(a',b',c')$, but such that generically $Z_3(a,b,c) \ncong Z_3(a',b',c')$.

Lemma 2.2.8. *Let $\Delta \in k$ and let $(a, b = x^2 - \Delta, c)$ be a regular quadratic triple. Write*

$$b^2 - ac = p_0 x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4 = p. \tag{2.69}$$

Now let $t \in k^*$ and assume we have $a', c' \in k[x]$ such that $tb^2 - a'c' = p$. Then $Y_2(ta', tb, c')$ is isomorphic to $Y_2(a,b,c)$.

Proof. The curve $Y_2(ta', tb, c')$ is given by the equation

$$\begin{aligned} y^2 &= tb(t^2b^2 - ta'c') \\ &= t^2 b(tb^2 - a'c') \\ &= t^2 bp. \end{aligned} \tag{2.70}$$

Consider the map $Y_2(ta', tb, c') \to Y_2(a,b,c)$ given by $(x,y) \mapsto (x, yt)$. It follows the map is well-defined because

$$y^2 t^2 - t^2 bp = 0 \tag{2.71}$$

implies that

$$y^2 - bp = 0 \tag{2.72}$$

for $t \neq 0$. The map is also bijective for $t \neq 0$, so it defines an isomorphism. □

Remark 2.2.9. Lemma 2.2.7 tells us that $Y_2(a'/t, b', c')$ and $Y_2(a', b', c'/t)$ are also isomorphic to $Y_2(ta', tb', c)$ with $Z_3(a'/t, b', c')$ and $Z_3(a', b', c'/t)$ isomorphic to $Z_3(ta', tb', c)$.

Remark 2.2.10. Fixing t in Lemma 2.2.8 and searching for suitable a', c' such that $tb^2 - a'c' = p$ is the same as finding a different quadratic factorization of $a'c'$. As we will see in the following example, choosing two distinct quadratic factorizations, say

$$a = (x - \alpha_1)(x - \alpha_2), \quad c = (x - \alpha_3)(x - \alpha_4) \tag{2.73}$$

and

$$a' = (x - \alpha_1)(x - \alpha_4), \quad c' = (x - \alpha_1)(x - \alpha_3) \tag{2.74}$$

will generally give us a curve $Z_3(ta', tb', c')$ that is not isomorphic to $Z_3(ta, tb, c)$.

Example 2.2.11. Fix $t = 1$ and let $a = (x-1)(x-2), b = x^2 - 5, c = (x-3)(x-4)$. We calculate that

$$B(a,b,c) = \begin{bmatrix} 1 & 1 & 1 \\ -3/2 & 0 & -7/2 \\ 2 & -5 & 12 \end{bmatrix}, \tag{2.75}$$

so

$$B(a,b,c)^{-1} = \begin{bmatrix} -35/2 & -17 & -7/2 \\ 11 & 10 & 2 \\ 15/2 & 7 & 3/2 \end{bmatrix} \tag{2.76}$$

and we find $\iota(a,b,c) = (f,h,g)$ where

$$f = -35/2x^2 - 17xz - 7/2z^2, \quad g = 15/2x^2 + 7xz + 3/2z^2, \quad h = 11x^2 + 10xz + 2z^2. \tag{2.77}$$

This gives us the curve $X_1(a,b,c)$ with equation

$$y^2 + (-11x^2 - 10xz - 2z^2)y - 525/4x^4 - 250x^3z - 343/2x^2z^2 - 50xz^3 - 21/4z^4. \tag{2.78}$$

If we make the change of coordinates $y' = y - 1/2h$, we find a different equation for the curve, namely $y'^2 = 1/4h^2 - f \cdot g$ where the right hand side is given by a binary quartic form. The binary quartic form is given by:

$$1/4h^2 - f \cdot g = 323/2x^4 + 305x^3z + 415/2x^2z^2 + 60xz^3 + 25/4z^4, \tag{2.79}$$

which has binary quartic invariants

$$I = 1075/4, \quad J = -8800. \tag{2.80}$$

Using these, we calculate that the j-invariant of $X_1(a,b,c)$ is 127211200/193.

We now set $a' = (x-1)(x-3), b = x^2 - 5, c' = (x-2)(x-4)$. In this case, we find

$$B(a',b',c') = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & -3 \\ 3 & -5 & 8 \end{bmatrix}, \tag{2.81}$$

so

$$B(a',b',c')^{-1} = \begin{bmatrix} -15/2 & -13/2 & -3/2 \\ 7/2 & 5/2 & 1/2 \\ 5 & 4 & 1 \end{bmatrix} \tag{2.82}$$

and we find $\iota(a',b',c') = (f',h',g')$ where

$$f' = -15/2x^2 - 13/2xz - 3/2z^2, \quad g' = 5x^2 + 4xz + z^2, \quad h' = 7/2x^2 + 5/2xz + 1/2z^2. \tag{2.83}$$

This gives us the curve $X_1(a',b',c')$ with equation

$$y^2 + (-7/2x^2y + -5/2xyz - 1/2z^2y) - 75/2x^4 - 125/2x^3z - 41x^2z^2 - 25/2xz^3 - 3/2z^4. \tag{2.84}$$

If we make the change of coordinates $y' = y - 1/2h'$, we find a different equation for the curve, namely $y'^2 = 1/4h'^2 - f' \cdot g'$ where the right hand side is given by a binary quartic form. The binary quartic form is given by:

$$1/4h'^2 - f' \cdot g' = 649/16x^4 + 535/8x^3z + 695/16x^2z^2 + 105/8xz^3 + 25/16z^4, \tag{2.85}$$

which has binary quartic invariants

$$I = 3625/256, \quad J = 210475/2048. \tag{2.86}$$

Using these, we calculate that the j-invariant of $X_1(a',b',c')$ is 76215625/3088.

As the j-invariants of $X_1(a,b,c)$ and $X_1(a',b',c')$ are distinct, we see that permuting the roots of a and c gives us two non-isomorphic curves $X_3(a,b,c)$ and $X_3(a',b',c')$ that share the same genus 2 factor as $F_2(a,b,c) = F_2(a',b',c')$.

Finding tuples as in Lemma 2.2.8 is equivalent to solving the equation $tb^2 - ac = p$. Because of Lemma 2.2.7, we can assume c to be monic. Writing out the polynomials, we get

$$\begin{aligned} &p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 \\ &\quad = p \\ &\quad = tb^2 - ac \\ &\quad = (t - a_0)x^4 - (a_0c_1 + a_1)x^3 - (a_0c_2 + a_1c_1 + a_2 + 2\Delta t)x^2 \\ &\quad - (a_1c_2 + a_2c_1)x - a_2c_2 + t\Delta^2. \end{aligned} \tag{2.87}$$

Comparing coefficients gives us the following set of equations:

$$\begin{aligned} t - a_0 &= p_0 \\ -a_0c_1 - a_1 &= p_1 \\ -a_0c_2 - a_1c_1 - a_2 - 2t\Delta &= p_2 \\ -a_1c_2 - a_2c_1 &= p_3 \\ -a_2c_2 + t\Delta^2 &= p_4. \end{aligned} \tag{2.88}$$

Lemma 2.2.12. *Let p be a monic polynomial of degree 4 over k such that the curve given by $y^2 = (x^2 - \Delta)p$ is nonsingular. Let $\mathbb{A}_k^6$ be the affine space of dimension 6 with coordinates $a_0, a_1, a_2, c_1, c_2, t$. The curve $C \subset \mathbb{A}_k^6$ given by the set of equations in (2.88) is birational to the curve D given by*

$$\begin{aligned} (p_0\Delta^2 - p_4)x^3 + p_3x^2y - (2p_0\Delta + p_2)xy^2 + p_1y^3 - p_1\Delta^2x^2 + (-2p_0\Delta^2 + 2p_4)xy \\ + (2p_1\Delta - p_3)y^2 + (p_2\Delta^2 + 2p_4\Delta)x + (p_1\Delta^2 - 2p_3\Delta)y - p_3\Delta^2 \end{aligned} \tag{2.89}$$

in $\mathbb{A}_k^2$.

Proof. Let U be the open subset of $\mathbb{A}_k^6$ where $c_1^3 - 2c_1c_2 - 2c_1\Delta \neq 0$, and let V be the open subset of $\mathbb{A}_k^2$ where $x^3 - 2xy - 2x\Delta \neq 0$. We define $\phi : U \to V$ by

$$\phi(a_0, a_1, a_2, c_1, c_2, t) \to (c_1, c_2) \tag{2.90}$$

and $\psi : V \to U$ by

$$(x,y) \mapsto (f(x,y), p_1 - xf(x,y), f(x,y)(x^2 - y - 2\Delta) - xp_1 + 2p_0\Delta + p_2, x, y, p_0 - f(x,y)). \tag{2.91}$$

Here

$$f(x,y) = ((x^2 - y)p_1 - xp_2 - 2xp_0\Delta + p_3)/(x^3 - 2xy - 2x\Delta). \tag{2.92}$$

We claim that $\psi \circ \phi$ is the identity on U.

Let us first show that $\phi(C) \subset D$. We first remark that $t = a_0 + p_0$, $a_1 = -a_0c_1 - p_1$ and $a_2 = -a_1c_1 - a_0c_2 - 2t\Delta - p_2$. Combining these three gives us that

$$a_2 = (a_0c_1 + p_1)c_1 - a_0c_2 - 2(a_0 + p_0)\Delta - p_2. \tag{2.93}$$

If we then substitute a_1 and a_2 into the equation: $p_3 = -a_2c_1 - a_1c_2$ from 2.88, we get a relationship between a_0, c_1 and c_2. We find:

$$-a_0c_1^3 + 2a_0c_1c_2 + 2a_0c_1\Delta + 2c_1\Delta p_0 - (c_1^2 - c_2)p_1 + c_1p_2 - p_3 = 0. \tag{2.94}$$

Isolating a_0 gives us:

$$a_0 = ((c_1^2 - c_2)p_1 - c_1p_2 - 2c_1p_0\Delta + p_3)/(c_1^3 - 2c_1c_2 - 2c_1\Delta) = f(c_1, c_2). \tag{2.95}$$

If we now add in the last equation

$$-a_2c_2 + t\Delta^2 - p_4 \tag{2.96}$$

and substitute $t = f(c_1,c_2) + p_0$ and

$$a_2 = (f(c_1,c_2)c_1 + p + 1)c_1 - f(c_1,c_2)c_2 - 2(f(c_1,c_2) + p_0)\Delta - p_2], \tag{2.97}$$

we will get

$$\frac{F(c_1,c_2)}{(c_1^3 - 2\Delta c_1 - 2c_1c_2)} \tag{2.98}$$

where

$$\begin{aligned} F(c_1,c_2) = {} & (p_0\Delta^2 - p_4)c_1^3 + p_3c_1^2c_2 - (2p_0\Delta + p_2)c_1c_2^2 + p_1c_2^3 - p_1\Delta^2c_1^2 \\ & + (-2p_0\Delta^2 + 2p_4)c_1c_2 + (2p_1\Delta - p_3)c_2^2 + (p_2\Delta^2 + 2p_4\Delta)c_1 \\ & + (p_1\Delta^2 - 2p_3\Delta)c_2 - p_3\Delta^2, \end{aligned} \tag{2.99}$$

so $\phi(V) \subset D$. We have that $\phi \circ \psi(x,y) = (x,y)$ by construction, so it remains to check that $\psi \circ \phi = \mathrm{id}$. We have

$$\begin{aligned} & \psi \circ \phi(a_0,a_1,a_2,c_1,c_2,t) \\ & \quad = (f(c_1,c_2), p_1 - c_1f(c_1,c_2), f(c_1,c_2)(c_1^2 - c_2 - 2\Delta) - c_1p_1 + 2p_0\Delta + p_2, \\ & \qquad c_1,c_2,p_0 - f(c_1,c_2)) \\ & \quad = (a_0, p_1 - c_1a_0, a_0(c_1^2 - c_2 - 2\Delta) - c_1p_1 + 2p_0\Delta + p_2, c_1, c_2, p_0 - a_0) \\ & \quad = (a_0,a_1,a_2,c_1,c_2,t). \end{aligned} \tag{2.100}$$

Now we have shown that C is birational to D. □

Lemma 2.2.13. *The curve D as in Lemma 2.2.12 has a rational singular point in*

$$(0, -\Delta) \tag{2.101}$$

and a rational nonsingular point in

$$\left(\frac{\Delta^2p_1 + \Delta p_3}{\Delta^2p_0 - p_4}, -\Delta\right). \tag{2.102}$$

Proof. Filling in $y = -\Delta$ in the equation of D, we find:

$$\begin{aligned} (p_0\Delta^2 - p_4)x^3 & - (\Delta^2p1 + \Delta p3)x^2 \\ & + (-2p_0\Delta^3 - p_2\Delta^2 + p_2\Delta^2 + 2p_4\Delta - 2\Delta^3p_0 - 2p_4\Delta)x \\ & - p_1\Delta^3 - \Delta^2p_3 + 2\Delta^3p_1 - p_3\Delta^2 - \Delta^3p_1 + 2\Delta^2p_3 \\ & = (p_0\Delta^2 - p_4)x^3 - (\Delta^2p1 + \Delta p3)x^2 + 0 + 0. \end{aligned} \tag{2.103}$$

So now we see that we get solutions for $x = 0$ and

$$x = \left(\frac{\Delta^2 p_1 + d p_3}{\Delta^2 p_0 - p_4}\right). \tag{2.104}$$

To check whether these points are singular or not, we calculate the derivatives. We find that

$$\begin{aligned} \frac{dF}{dx} &= 3x^2(p_0\Delta^2 - p_4) + 2p_3xy - (2p_0\Delta + p_2)y^2 - 2xp_1\Delta^2 \\ &\quad - 2(p_0\Delta^2 - p_3x - p_4)y + p_2\Delta^2 + 2p_4\Delta, \\ \frac{dF}{dy} &= 3p_1y^2 + 2(2\Delta p_1 - (2\Delta p_0 + p_2)x - p_3)y\Delta^2 p_1 + p_3x^2 \\ &\quad - 2\Delta p_3 - 2(\Delta^2 p_0 - p_4)x. \end{aligned} \tag{2.105}$$

Filling in $(x, -\Delta)$ gives us

$$\begin{aligned} \frac{dF}{dx}(x,-\Delta) &= 3(\Delta^2 p_0 - p_4)x^2 - 2(\Delta^2 p_1 + \Delta p_3)x, \\ \frac{dF}{dy}(x,-\Delta) &= p_3x^2 + 2(\Delta^2 p_0 + \Delta p_2 + p_4)x. \end{aligned} \tag{2.106}$$

This shows that $(0, -\Delta)$ is a singular point.

We furthermore see that F has a singular point in x if and only if

$$x = \frac{\Delta^2 p_1 + \Delta p_3}{3(\Delta^2 p_0 - p_4)} = \frac{2(\Delta^2 p_0 + \Delta p_2 + p_4)}{p_3}. \tag{2.107}$$

Filling in $\left(\left(\frac{\Delta^2 p_1 + \Delta p_3}{\Delta^2 p_0 - p_4}\right), -\Delta\right)$ gives us

$$\frac{dF}{dx}\left(\left(\frac{\Delta^2 p_1 + \Delta p_3}{\Delta^2 p_0 - p_4}\right), -\Delta\right) = \frac{\Delta^2(\Delta p_1 + p_3)^2}{(\Delta^2 p_0 - p_4)}. \tag{2.108}$$

So $\left(\left(\frac{\Delta^2 p_1 + \Delta p_3}{\Delta^2 p_0 - p_4}\right), -\Delta\right)$ is a singular point if $\Delta p_1 + p_3 = 0$ and $\Delta^2 p_0 + \Delta p_2 + p_4 = 0$. But these two conditions imply that $p(\sqrt{\Delta}) = 0$. This contradicts the assumption that $y^2 = (x^2 - \Delta)p$ defines a nonsingular curve. □

Lemma 2.2.14. *Let D be the curve as in Lemma 2.2.12. Then D is a rational curve. More specifically, let*

$$g(z) = \frac{(\Delta^2 p_1 + (\Delta p_1 + p_3)z^2 + \Delta p_3 - (2\Delta^2 p_0 + 2\Delta p_2 + 2p_4)z)}{(p_1 z^3 - (2\Delta p_0 + p_2)z^2 + (\Delta^2 p_0 - p_4) + p_3 z)}. \tag{2.109}$$

Then the morphism: $\phi : \mathbb{A}^1_k \to D$ defined by $z \mapsto (g(z), zg(z) - \Delta)$ gives us a parametrization of D.

Proof. The curve D is given by a cubic equation and has a singular point at $(0,-\Delta)$. As D is a singular cubic, it has to be of genus 0. It moreover has a rational nonsingular point at $P = \big((p_1\Delta^2 + p_3\Delta)/(p_0\Delta^2 - p_4), -\Delta\big)$, so the curve is rational. We are going to find a rational parametrization. After substituting $y = zx - \Delta$ in the equation for D we find:

$$\begin{aligned} x^3(p_1z^3 - (2dp_0 + p_2)z^2 + (\Delta^2 p_0 - p_4) + p_3z) + x^2(\Delta^2 p_1 + (\Delta p_1 + p_3)z^2 \\ + \Delta p_3 - (2\Delta^2 p_0 + 2\Delta p_2 + 2p_4)z) = 0. \end{aligned} \tag{2.110}$$

After dividing by x^2, we can isolate the x to get the equation:

$$x = \frac{(\Delta^2 p_1 + (\Delta p_1 + p_3)z^2 + \Delta p_3 - (2\Delta^2 p_0 + 2\Delta p_2 + 2p_4)z)}{(p_1z^3 - (2dp_0 + p_2)z^2 + (\Delta^2 p_0 - p_4) + p_3z)} = g(z). \tag{2.111}$$

By construction, $y = zg(z) - \Delta$, which gives us the parametrization ϕ.

□

Theorem 2.2.15. *Let $(a, b = x^2 - \Delta, c)$ be a regular quadratic triple and let $p = b^2 - ac$ such that $Y_2(a,b,c)$ is a nonsingular curve. The curve D as in Lemma 2.2.12 is a rational curve parametrizing a non-constant family of genus 3 curves Z_3 for which Z_3 is a (2,2)-gluing of the genus 2 curve $Y_2(a,b,c)$ and some elliptic curve.*

Proof. Let $(a_0, a_1, a_2, c_1, c_2, t)$ be any set of solutions on D. Let $a' = a_0x^2 + a_1x + a_2, b' = b$ and $c' = x^2 + c_1x + c_2$. Then $tb^2 - a'c' = b^2 - ac$. Then Lemma 2.2.8 tells us that $Y_2(a,b,c)$ is isomorphic to $Y_2(ta', tb, c')$, so the curve $X_3(ta', tb, c')$ is a (2,2)-gluing of $Y_2(a,b,c)$ and $X_1(ta', tb, c')$. By Proposition 2.2.4 $Z_3(ta', tb, c')$ is a (2,2)-gluing of $Y_2(a,b,c)$ and $X_1(ta', tb, c')$. This shows that all points on D give rise to genus 3 curves that are 2,2-gluings of Y_2 and some elliptic curve.

To show that this family is non-constant, we remark that Remark 2.2.10 and Example 2.2.11 show us that (after possibly taking a field extension) there generically exists at least one solution $(a_0, a_1, a_2, c_1, c_2, 1)$ for which $X_1(a,b,c)$ is not isomorphic to $X_1(a', b, c')$. Let ϕ be the rational parametrization of D defined in Lemma 2.2.14. Let $(a_0, a_1, a_2, c_1, c_2, t) \in D$ and write $\tau(a_0, a_1, a_2, c_1, c_2, t, b) = (t(a_0x^2 + a_1x + a_2), tb, x^2 + c_1x + c_2)$. The function $J : \mathbb{A}^1_k \to k$ given by taking the j-invariant of the curve $X_1(\tau(\phi(z))$ is some polynomial function in terms of the rational parameter z. As we have shown above, there exists at least one point $(a_0, a_1, a_2, c_1, c_2, 1)$ for which $j(X_1(a,b,c))$ is not equal to $j(X_1(a', b, c'))$, so the J-function is a non-constant polynomial on a connected curve. We conclude that the curve D gives us a non-constant rational family of genus 3 curves, in which every curve is the (2,2)-gluing of $Y_2(a,b,c)$ and some elliptic curve.

□

Example 2.2.16. We are going to use the above described technique to construct a non-hyperelliptic genus 3 curve with QM. Let Y_2 be given by

$$y^2 = x^5 + x^4 + 4x^3 + 8x^2 + 5x + 1. \tag{2.112}$$

This curve is on `www.lmfdb.org` under label 262144.d.524288.2 and the endomorphism ring of its Jacobian is a quaternion algebra of discriminant 6 over $\overline{\mathbb{Q}}$. After the automorphism $(x,y) \mapsto \left(\frac{2x}{-x+1}, \frac{y}{(-x+1)^3}\right)$, we get the equation

$$y^2 = -25x^6 + 12x^5 + 27x^4 - 16x^3 - 3x^2 + 4x + 1. \tag{2.113}$$

and the right hand side factors as

$$-(x^2 - 1)(25x^4 - 12x^3 - 2x^2 + 4x + 1), \tag{2.114}$$

which puts it into a form we can use to describe the map ϕ in Lemma 2.2.13. Indeed, we have $p = -25x^4 + 12x^3 + 2x^2 - 4x - 1$, so $p_0 = -25, p_1 = 12, p_2 = 2, p_3 = -4, p_4 = 1$ and we find that

$$g(z) = \frac{8z^2 + 48z + 8}{12z^3 + 48z^2 - 4z - 24} \tag{2.115}$$

and $\phi(z) = (g(z), zg(z))$. Setting $z = -2$, we get $c_1 = g(-2)$ and $c_2 = -2(g(-2))$ to calculate a triple of polynomials:

$$a = 3320/147x^2 + 80/21x - 520/147, \quad b = x^2 - 1, \quad c = x^2 - 7/10x + 2/5 \tag{2.116}$$

and the variable $t = -355/147$ using the relations in (2.88). Now the curve $Z_3(a/t, b, c)$ will be given by the equation

$$\begin{aligned} -355/19208x^4 + y^4 + 1065/9604x^3 + (103x^2 + 132x + 5)/98y^2 \\ + 6745/9604x^2 + 1065/9604x - 355/19208, \end{aligned} \tag{2.117}$$

which we can simplify to

$$-x^4 + 19208/355y^4 + 6x^3 + 196/355(103x^2 + 132x + 5)y^2 + 38x^2 + 6x - 1 \tag{2.118}$$

by multiplying with 19208/355. We can further simplify this to:

$$x^4 - 12x^2y^2 - 98x^2y + 120x^2 + 1065y^4 - 3905y^2 + 2130. \tag{2.119}$$

This curve Z_3 is a (2,2)-gluing of Y_2 and some elliptic curve (Using Lemma 2.2.8 and Remark 2.2.9), and $\mathrm{End}(Z_3) \otimes \mathbb{Q} \cong \mathbb{Q} \times B$ where B is a quaternion algebra of discriminant 6.

Remark 2.2.17. In order to glue we use a factor of the form $x^2 - \Delta$ in the equation of our hyperelliptic curve. As we saw in Corollary 1.5.6 this condition is necessary in order to be able to glue over the base field.

Remark 2.2.18. We could also have chosen $b = x$ instead of $b = x^2 - d$ to find a slightly simpler family of curves.

Remark 2.2.19. It is also possible to construct a family of genus 3 curves that are (2,2)-gluings of a genus 2 curve and a fixed genus 1 factor X_1. Assume X_1 is given by the equation $y^2 - hy + fg$ for some quadratic polynomials f,g and h. Let $t \in k$ then any curve $X_{1,t} = v^2 + hv(t-1) + (t^2h^2 - 2th^2)/4 + fg$ is isomorphic to X_1 using the substitution $y = v + th/2$. Now the curve $Z_{3,t} = v^4 + hv^2(t-1) + (t^2h^2 - 2th^2)/4 + fg$ gives us a corresponding genus 3 cover of degree 2. To see that this family will generically be non-constant, remark that the curve $Z_{3,1}$ will have CM by the automorphism $(x,v) \mapsto (x,iv)$ even though for $t \neq 1$, $Z_{3,t}$ will generally not have this property.

Chapter 3

Gluing over $\mathbb{C}$

3.1 Abelian varieties over $\mathbb{C}$

Theorem 3.1.1. *Let A be an abelian variety over $\mathbb{C}$ of dimension g. Then A is analytically isomorphic to a complex torus V/Λ where V is a g-dimensional complex vector space and Λ is a discrete subgroup of V of rank $2g$.*

Proof. See [4, Lemma 1.1]. □

Definition 3.1.2. Let A be a complex torus where $A = V/\Lambda$ where V is a g-dimensional complex vector space and Λ is a discrete subgroup of V of rank $2g$. Fix a basis $\mathcal{E} = \{e_1, \dots e_g\}$ for V, and a basis $\mathcal{B} = \{\lambda_1, \dots, \lambda_{2g}\}$ of Λ. We write $\lambda_i = \sum_{j=1}^{g} \lambda_{i,j} e_j$. We will use the notation $(\lambda_i)_{\mathcal{E}}$ for the vector $(\lambda_{i,1}, \dots, \lambda_{i,g})$. Consider the $g \times 2g$ matrix

$$\Pi = \begin{bmatrix} | & \cdots & | \\ (\lambda_1)_{\mathcal{E}} & & (\lambda_{2g})_{\mathcal{E}} \\ | & \cdots & |, \end{bmatrix} \tag{3.1}$$

which has the λ_i as its column vectors with respect to $\mathcal{E}$. The matrix Π is called *the period matrix* of the complex torus A with respect to $\mathcal{E}$ and $\mathcal{B}$.

Remark 3.1.3. In the algorithm we use later on, we will calculate period matrices explicitly using the work of Neurohr and Molin [21].

Proposition 3.1.4. *Let $A = V/\Lambda$ and $B = V'/\Lambda'$ be two complex tori over $\mathbb{C}$ of dimension g and g' respectively. Let $\phi : A \to B$ be a homomorphism. Then there exists a unique $\mathbb{C}$-linear map T_ϕ such that the following diagram commutes*

$$\begin{array}{ccc} V & \xrightarrow{T_\phi} & V' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\phi} & B \end{array} . \tag{3.2}$$

Moreover, $T_\phi(\Lambda) \subset \Lambda'$ and the restriction $T_\phi|_\Lambda : \Lambda \to \Lambda'$ gives us a $\mathbb{Z}$-linear map $R_\phi : \Lambda \to \Lambda'$.

Proof. See [4, Proposition 2.1]. □

Definition 3.1.5. The map T_ϕ is called the *analytic representation* of ϕ, and R_ϕ is called the *rational representation* of ϕ.

Proposition 3.1.6. *Let $A = V/\Lambda$ and $A' = V'/\Lambda'$ be two complex tori over $\mathbb{C}$ of dimension g and g' respectively. Let $\phi : A \to A'$ be a homomorphism. Fix bases $\mathcal{E}$ and $\mathcal{E}'$ for V and V' and bases $\mathcal{B}$ and $\mathcal{B}'$ for Λ and Λ'. Let Π be the period matrix for A with respect to $\mathcal{B}$ and $\mathcal{E}$ and let Π' be the period matrix for A' with respect to $\mathcal{B}'$ and $\mathcal{E}'$. Identify R_ϕ and T_ϕ with their matrix representations with respect to the chosen bases.*

Then

$$T_\phi \Pi = \Pi' R_\phi. \tag{3.3}$$

Conversely, if we have $T \in \mathrm{Mat}_{g' \times g}(\mathbb{C})$ and $R \in \mathrm{Mat}_{2g',2g}(\mathbb{Z})$ such that

$$T\Pi = \Pi' R. \tag{3.4}$$

Then there exists some $\phi \in \mathrm{Hom}(A, A')$ such that $T_\phi = T$ and $R_\phi = R$.

Proof. See [Proposition 2.3][4]. □

Remark 3.1.7. The condition

$$T_\phi \Pi = \Pi' R_\phi \tag{3.5}$$

is equivalent to stating that $\phi(\Lambda) \subset \Lambda'$.

Not every complex torus gives rise to an abelian variety. If A is a projective group scheme over $\mathbb{C}$. Then A comes equipped with an algebraic embedding $i : A \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$. The existence of such an embedding i for a complex torus V/Λ is equivalent to the existence of an ample line bundle $\mathcal{L}$ on V/Λ. So to understand which complex tori correspond to abelian varieties we need to consider line bundles on V/Λ.

Lemma 3.1.8. *Let V be a complex vector space. There is a bijection between the Hermitian forms H on V and the real alternating forms E on V with $E(ix, iy) = E(x, y)$ given by*

$$E(x,y) = \mathrm{Im}\, H(x,y) \tag{3.6}$$
$$H(x,y) = E(ix,y) + iE(x,y) \tag{3.7}$$

Proof. See [25, p.19]. □

Definition 3.1.9. Let V/Λ be a complex torus and let H be a Hermitian form on V such that $E = \operatorname{Im} H$ with $E(\Lambda \times \Lambda) \subset \mathbb{Z}$. Let $\alpha : \Lambda \to S^1 = \{z \in \mathbb{C}^* \mid |z| = 1\}$ be a map satisfying

$$(\lambda_1 + \lambda_2) = e^{i\pi E(\lambda_1, \lambda_2)} \cdot \alpha(\lambda_1)\alpha(\lambda_2) \text{ with } \lambda_1, \lambda_2 \in \Lambda. \tag{3.8}$$

Then we define $L(H, \alpha)$ to be the line bundle given by the quotient of $V \times \mathbb{C}$ by the action of Λ on $V \times \mathbb{C}$ given by

$$f_\lambda(c, z) = (c \cdot \alpha(\lambda) \cdot e^{\pi H(z,\lambda) + \frac{1}{2}\pi H(\lambda,\lambda)}, z + \lambda). \tag{3.9}$$

The function $f_\lambda(c, z)$ is called a factor of automorphy for $L(H, \alpha)$.

Theorem 3.1.10 (Appell-Humbert)**.** *Any line bundle L on a complex torus V/Λ is isomorphic to an $L(H, \alpha)$ for a unique tuple (H, α) as above. Furthermore, the class of $L(H, \alpha)$ modulo algebraic equivalence only depends on the choice of H.*

Proof. See [25, p.20]. □

Lemma 3.1.11. *Let $A = V/\Lambda$ be a complex torus of dimension g. Let $\mathcal{E}$ be a basis for V, and let $\mathcal{B}$ be a basis for Λ. Let Π be its period matrix with respect to the chosen bases. Let H be a Hermitian form on V such that $E = \operatorname{Im} H$ with $E(\Lambda \times \Lambda) \subset \mathbb{Z}$. Write $M_E \in \operatorname{Mat}_{2g\times 2g}(\mathbb{Z})$ for the matrix representation of E with respect to $\mathcal{B}$. Then H is positive-definite if and only if*

(i) $\Pi M_E^{-1} \Pi^t = 0$,

(ii) $i\Pi M_E^{-1} \Pi^t > 0$.

Theorem 3.1.12 (Lefschetz)**.** *Let $A = V/\Lambda$ be a complex torus of dimension g. Let H be a Hermitian form on V such that $\operatorname{Im}(H)$ is integral on $\Lambda \times \Lambda$. Let α be a function as in Definition 3.1.9. Then $L(H, \alpha)$ is ample if and only if H is positive-definite.*

Proof. See [25, p.35]. □

Lemma 3.1.13 (Frobenius)**.** *Let Λ be a lattice of rank $2g$. Let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be a non-degenerate bilinear alternating form. There exist positive integers $d_1, \dots d_g$ with $d_i | d_{i+1}$ and a basis $e_1, \dots e_g, f_1, \dots f_g$ of Λ such that if we set $D = \operatorname{diag}(d_1, \dots, d_g)$ then the matrix of E with respect to this basis has the form*

$$\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}. \tag{3.10}$$

Proof. See [14, Lemma A.5.3.1] □

Definition 3.1.14. Such a basis is called a *Frobenius basis* for E, and the integers d_i are called the *invariants* of E.

Let $A = V/\Lambda$ be an abelian variety. We can explicitly construct the dual abelian variety in the following way: Let $V^t = \mathrm{Hom}_{\mathbb{C}-antilinear}(V, \mathbb{C})$ and let

$$\Lambda^t = \{w \in V^t | \mathrm{Im}\, w(\lambda) \in \mathbb{Z}, \text{ for all } \lambda \in \Lambda\}. \tag{3.11}$$

We can identify V^t/Λ^t with A^t using the isomorphism

$$w \mapsto L\Big(0, \lambda \mapsto e^{2\pi i \mathrm{Im}\, w(u)}\Big) \tag{3.12}$$

as in [25, p.86].

Lemma 3.1.15. *Let $A = V/\Lambda$ and $A' = V'/\Lambda'$ be abelian varieties and let $\phi : A \to A'$ be an isogeny. Fix bases $\mathcal{B}$ and $\mathcal{B}'$ for the lattices Λ and Λ' respectively. Let $L(H', \alpha')$ be a polarization on A'. Let E' be the alternating form corresponding to H'. Write $M_{E'}$ for the matrix representation of E' with respect to $\mathcal{B}'$. Let R_ϕ be the rational representation of ϕ with respect to Λ and Λ'. Then*

(i) $\phi^*(L(H', \alpha')) = L(H'(T_\phi, T_\phi), \alpha' \circ R_\phi)$,

(ii) The matrix representation of the alternating bilinear form $\phi^*(E)$ corresponding to $\phi^*(L(H', \alpha'))$ is given by $M_E = R_\phi^t M_{E'} R_\phi$ with respect to the basis $\mathcal{B}$.

Proof. Appendix B of [4] tells us that if $f'_\lambda(c, z)$ is a factor of automorphy for $L'(H', \alpha')$ then

$$f_\lambda(c, z) = (c \cdot \alpha'(R_\phi \lambda) \cdot e^{\pi H'(T_\phi z, R_\phi \lambda) + \frac{1}{2}\pi H'(R_\phi \lambda, R_\phi \lambda)}, T_\phi z + R_\phi \lambda) \tag{3.13}$$

is a factor of automorphy for $\phi^*(L'(H', \alpha'))$. As $T_\phi \lambda = R_\phi \lambda$ for all $\lambda \in \Lambda$, it follows that that $\phi^* L(H', \alpha') = L(H'(T_\phi, T_\phi), \alpha' \circ R_\phi)$. And we see that $\phi^*(H') = H'(T_\phi, T_\phi)$. Using once again that $T_\phi \lambda = R_\phi \lambda$ for all $\lambda \in \Lambda$, we find that the alternating bilinear form E induced by H is represented by $M_E = R_\phi^t M_{E'} R_\phi$. $\square$

Corollary 3.1.16. *Let $\phi : A = V/\Lambda \to A' = V'/\Lambda'$ be a morphism of abelian varieties. Assume that R_ϕ is the rational representation of ϕ with respect to a choice of bases on Λ and Λ'. Let $\phi^t : (A')^t \to A^t$ be the dual morphism defined by pulling back line bundles. Then ϕ^t is given by $w \mapsto R_\phi^t w$ with respect to the induced bases on the dual abelian varieties.*

Proof. Let $L(0, \alpha)$ be a line bundle. Now

$$\phi^*\Big(L\Big(0, e^{2\pi i \mathrm{Im}\, w(u)}\Big)\Big) = L\Big(0, e^{2\pi i \mathrm{Im}\, w(R_\phi u)}\Big) = L\Big(0, e^{2\pi i \mathrm{Im}\, R_\phi^t w(u)}\Big). \tag{3.14}$$

So ϕ^t is given by $w \mapsto R_\phi^t w$. $\square$

Remark 3.1.17. One can show that, given H, a compatible α as in Definition 3.1.9 always exists. See [25, p.19-20]. As we saw in Theorem 3.1.10 the polarization only depends on H or the corresponding alternating bilinear form E. If we want to study polarized abelian varieties, we can therefore restrict ourselves to studying tuples $(A = V/\Lambda, E)$ where E is an alternating bilinear form $E : \Lambda \times \Lambda \to \mathbb{Z}$ with $E(ix, iy) = E(x, y)$.

To finish, we will give an explicit description of the Weil pairing.

Proposition 3.1.18. *Let $\phi : A = V/\Lambda \to A' = V/\Lambda'$ be an isogeny and let $\phi^t : (A')^t \to A^t$ be its dual isogeny. There is an isomorphism of group schemes*

$$\beta : \ker(\phi^t) \to \ker(\phi)^D \tag{3.15}$$

explicitly given by $(0, e^{2\pi i\psi}) \to (\lambda \mapsto e^{2\pi i\psi\circ\phi(\lambda)}|_{\ker(\phi^t)})$.

Proof. A proof of the existence of this isomorphism can be found in [29, Theorem 7.5]. Let us give an explicit description of this map.

Let $\mathcal{L} \in \ker(\phi^t)$. Then $\mathcal{L} = L(0, e^{2\pi i\cdot w})$ where $w : V' \to \mathbb{C}$ is an anti-linear map that takes integer values on Λ'. Now $\phi^*(\mathcal{L}) = L(0, e^{2\pi i\cdot w\circ\phi})$ by Lemma 3.1.15. Furthermore, $\mathcal{L} \in \ker(\phi^t)$ if and only if $(0, e^{2\pi i\cdot w\circ\phi})$ is the trivial bundle, i.e. if and only if $w \circ \phi$ is the identity map on Λ. This means that the character given by $\lambda \mapsto e^{2\pi i\psi\circ\phi(\lambda)}$ on $\Lambda'/\Lambda = \ker(\phi^t)$ is well-defined. According to the proof of [29, Proposition 7.4], this character is $\beta(\mathcal{L})$. □

Definition 3.1.19. Let $n_A : A \to A$ be multiplication by n on the abelian variety A over $\mathbb{C}$. Let $\beta : \ker(f^t) \to \ker(f)^D$ be the isomorphism described in 3.1.18.

(i) Define
$$e_n : \ker(f) \times \ker(f^t) \to \mathbb{G}_{m,k} \tag{3.16}$$
to be the perfect bilinear pairing given by $e_n(x, y) = \beta(y)(x)$.

(ii) Let $\phi : A \to A^t$ be a polarization. Then the Weil pairing
$$e_n^\phi : A[n] \times A[n] \to \mu_n \tag{3.17}$$
as defined in [29] in Definition 11.11 is the bilinear pairing given by $e_n^\phi(x, y) = e_n(x, \phi(y))$.

Proposition 3.1.20. *Let $A = V/\Lambda$. Let ϕ be a polarization $A \to A^t$ and let E be the alternating bilinear form corresponding to ϕ. Then we get an explicit description for the Weil pairing:*

$$e_n^\phi\left(\frac{1}{n}\lambda_1, \frac{1}{n}\lambda_2\right) = e^{2\pi i\frac{1}{n}E(\lambda_1,\lambda_2)} \tag{3.18}$$

where $\lambda_i \in \Lambda$.

Proof. Using Proposition 3.1.18 we can describe the Weil pairing in terms of the polarization. We have

$$e_n^{\phi}(x,y) = e_n\left(x, \left(0, e^{2\pi i E(y,-)}\right)\right) = \beta\left(\left(0, e^{2\pi i E(y,-)}\right)\right)(x) = e^{2\pi i n E(y,x)}. \tag{3.19}$$

As $y \in \ker(n_X)$ we see that $y = \frac{1}{n}\lambda$ with $\lambda \in \Lambda$. So,

$$e_n^{\phi}\left(\frac{1}{n}\lambda_1, \frac{1}{n}\lambda_2\right) = e^{2\pi i n E(\frac{1}{n}\lambda_1, \frac{1}{n}\lambda_2)} = e^{2\pi i \frac{1}{n} E(\lambda_1,\lambda_2)} \tag{3.20}$$

which concludes the proof. $\square$

3.2 (2,2)-gluing over $\mathbb{C}$

Definition 3.2.1. We define the *standard symplectic matrix* $S_n \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C})$ as

$$S_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{3.21}$$

where I_n is the n-dimensional identity matrix. As before, we define the *standard symplectic pairing* on $\mathbb{C}^{2n}$ as the one induced by this matrix.

Definition 3.2.2. Let $(A = V/\Lambda, E)$ be a principally polarized abelian variety of genus g. Let Π be a period matrix of A with respect to a basis $\mathcal{B}$ of Λ and a basis $\mathcal{E}$ of V. Then Π is called *normalized* if $M_E = S_g$ where M_E is the matrix representation of E with respect to $\mathcal{B}$.

Definition 3.2.3. Let $A_1 = V_1/\Lambda_1$, $A_2 = V_2/\Lambda_2$ be abelian varieties and fix bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for Λ_1 and Λ_2. We also fix bases $\mathcal{E}_1$ and $\mathcal{E}_2$ for V_1 and V_2. Let Π_1 be the period matrix with respect to $\mathcal{B}_1$ and $\mathcal{E}_1$ and let Π_2 be the period matrix with respect to $\mathcal{B}_2$ and $\mathcal{E}_2$. Now let $V_{1,2} = V_1 \times V_2$, $\Lambda_{1,2} = \Lambda_1 \times \Lambda_2$ and $A_{1,2} = A_1 \times A_2$. Let

$$\mathcal{B}_{1,2} = \{(b_i, 0) | b_i \in \mathcal{B}_1\} \cup \{(0, b_i') | b_i' \in \mathcal{B}_2\}. \tag{3.22}$$

be a basis for $V_{1,2}$ and similarly define a basis

$$\mathcal{E}_{1,2} = \{(e_i, 0) | e_i \in \mathcal{E}_1\} \cup \{(0, e_i') | e_i' \in \mathcal{E}_2\}. \tag{3.23}$$

for $\Lambda_{1,2}$. Then the matrix $\Pi_{1,2}$ given by

$$\Pi_{1,2} = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} \tag{3.24}$$

is the period matrix for $A_{1,2}$ with respect to the bases $\mathcal{B}_{1,2}$ and $\mathcal{E}_{1,2}$. We will call it the *product period matrix* of $A_{1,2}$ for Π_1 and Π_2.

Definition 3.2.4. Let (A_1, E_1) and (A_2, E_2) be principally polarized abelian varieties, and choose bases such as above. Let M_{E_i} be the matrix representation of E_i with respect to $\mathcal{E}_i$. We define the *product polarization* $E_{1,2}$ on $A_{1,2}$ to be the one given by the matrix

$$M_{E_{1,2}} = \begin{bmatrix} M_{E_1} & 0 \\ 0 & M_{E_2} \end{bmatrix} \tag{3.25}$$

with respect to $\mathcal{E}_{1,2}$.

Lemma 3.2.5. *Let (A_1, E_1) and (A_2, E_2) be principally polarized abelian varieties over $\mathbb{C}$ of genus 1 and genus 2 respectively. Then there exist bases $\mathcal{B}_i$ for Λ_i and $\mathcal{E}_i$ for V_i such that the matrix representation of the product polarization $E_{1,2}$ with respect to $\mathcal{B}_{1,2}$ is given by*

$$M_{E_{1,2}} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}. \tag{3.26}$$

Proof. Let $A_i = V_i/\Lambda_i$ and choose a Frobenius basis $\mathcal{B}_i$ of Λ_i for E_i for $i = 1, 2$. Let $\mathcal{E}_i$ be the standard basis for $\mathbb{C}^i$ for $i = 1, 2$. Let Π_i be the period matrix of A_i with respect to $\mathcal{B}_i$ and $\mathcal{E}_i$ for $i = 1, 2$. As $\mathcal{B}_i$ is a Frobenius basis for a principal polarization, the matrix representation M_{E_i} of E_i will be the standard symplectic matrix S_i for $i = 1, 2$. Let $\Pi_{1,2}$ be the product period matrix of $A_{1,2}$ with respect to the bases $\mathcal{E}_{1,2}$ and $\mathcal{B}_{1,2}$. It follows that

$$M_{E_{1,2}} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}. \tag{3.27}$$

□

Lemma 3.2.6. *Let G be a subgroup of $A_{1,2}[2]$ that is maximally isotropic with respect to the Weil-pairing. Let $\lambda_1, \ldots, \lambda_6$ be a basis of $\Lambda_{1,2}$. Then*

$$A_{1,2}/G \cong V_{1,2}/\Lambda_G \tag{3.28}$$

where

$$\Lambda_G = \langle \mu_1, \mu_2, \mu_3, \lambda_1, \ldots, \lambda_6 \rangle \tag{3.29}$$

with $\mu_1, \mu_2, \mu_3 \in (1/2\Lambda_{1,2}) \backslash \Lambda_{1,2}$ linearly independent over $\mathbb{R}$.

Proof. We first remark that the image of the inclusion

$$(1/2\Lambda_{1,2})/\Lambda_{1,2} \hookrightarrow V_{1,2}/\Lambda_{1,2} = A_{1,2} \tag{3.30}$$

lands in $A_{1,2}[2]$. Because we have that

$$|A_{1,2}[2]| = |(1/2\Lambda_{1,2})/\Lambda_{1,2}|, \tag{3.31}$$

the map induces an isomorphism

$$\sigma : (1/2\Lambda_{1,2})/\Lambda_{1,2} \to A_{1,2}[2]. \tag{3.32}$$

A maximally isotropic subgroup of $A_{1,2}[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, so we can choose $g_1, g_2, g_3 \in G$ such that $G = \langle g_1, g_2, g_3 \rangle$. Now choose $\mu_i \in 1/2\Pi_{1,2}$ such that $\mu_i \equiv \sigma^{-1}(g_i) \mod \Pi_{1,2}$ for each $i \in \{1,2,3\}$ and define $\Lambda_G = \langle \mu_1, \mu_2, \mu_3, \lambda_1, \ldots, \lambda_6 \rangle$. Remark that Λ_G is independent of the choice of the μ_i as any two representatives of $\sigma^{-1}(g_i)$ in $1/2\Lambda_{1,2}$ differ by an element of $\Lambda_{1,2}$. By construction the kernel of

$$A_{1,2} = V_{1,2}/\Lambda_{1,2} \to V_{1,2}/\Lambda_G \tag{3.33}$$

is G, so $A_{1,2}/G \cong V_{1,2}/\Lambda_G$. □

Choose a basis $\mathcal{M} = \mu_1, \ldots, \mu_6$ for Λ_G. Then

$$\Pi_G = \begin{bmatrix} | & \cdots & | \\ (\mu_1)_{\mathcal{E}_{1,2}} & & (\mu_6)_{\mathcal{E}_{1,2}} \\ | & \cdots & | \end{bmatrix} \tag{3.34}$$

gives us a period matrix of $A_{1,2}/G$ with respect to $\mathcal{M}$ and $\mathcal{E}_{1,2}$. Let $\mu_i = \sum_{j=1}^{6} \mu_{i,j}\lambda_j$ and write $(\mu_i)_{\mathcal{B}_{1,2}} = (\mu_{i,1}, \ldots, \mu_{i,6})$. Note that $\mu_{i,j} \in 1/2\mathbb{Z}$.

Lemma 3.2.7. *Let*

$$R = \begin{bmatrix} | & \cdots & | \\ (\mu_1)_{\mathcal{B}_{1,2}} & & (\mu_6)_{\mathcal{B}_{1,2}} \\ | & \cdots & | \end{bmatrix}. \tag{3.35}$$

Then $R^{-1} = R_\phi$ is the rational representation of the quotient morphism

$$\phi_G : V_{1,2}/\Lambda_{1,2} \to V_{1,2}/\Lambda_G \tag{3.36}$$

with respect to the bases $\mathcal{B}$ and $\mathcal{M}$. In particular, $R^{-1}(\mu_i) \subset \Lambda_G$ and $\ker \phi_G = \langle \mu_i \rangle$.

Proof. As μ_i is a column vector of R, we find that $R^{-1}((\mu_i)_{\mathcal{B}_{1,2}}) = (\mu_i)_{\mathcal{M}}$. This implies that $R^{-1}(\Lambda) \subset \Lambda_G$. Proposition 3.1.6 and Remark 3.1.7 tell us there exists a morphism $\phi : V_{1,2}/\Lambda \to V_{1,2}/\Lambda_G$ for which $R_\phi = R^{-1}$. Furthermore, $v \in \ker \phi_G$ if and only if $\phi(v)$ has integer coefficients. This can only happen if v is contained in the $\mathbb{Z}$-span of the μ_i, so $\ker \phi_G = \langle \mu_i \rangle$. □

Corollary 3.2.8. *Let $(A_1, E_1), (A_2, E_2)$ be abelian varieties as above, and let G be an indecomposable maximally isotropic subgroup of $A_1[2] \times A_2[2]$ with respect to the Weil pairing. Let $B_G = V_{1,2}/\Lambda_G$ and ϕ be as above and define E_G to be the pairing corresponding to the matrix representation*

$$M_{E_G} = (R_\phi^t)^{-1} 2E_{1,2} R_\phi^{-1} \tag{3.37}$$

with respect to $\mathcal{M}$. Then (B_G, E_G) is the (2,2)-gluing of (A_1, E_1) and (A_2, E_2) along G.

Proof. By Lemma 3.2.7 the kernel of ϕ is exactly G. So Proposition 1.1.5 tells us that there exists a principal polarization E_G on B_G such that (B_G, E_G) is a (2,2)-gluing of (A_1, E_1) and $A_2, E_2)$. Using Lemma 3.1.15 we find that

$$R_\phi^t M_{E_G} R_\phi = 2E_{1,2}. \tag{3.38}$$

This proves the statement. □

Remark 3.2.9. Let X_1 and Y_2 be curves over $\mathbb{C}$ of genus 1 and genus 2 respectively. Assume that a maximally isotropic group G of $\mathrm{Jac}(X_1)[2] \times \mathrm{Jac}(Y_2)[2]$ is given in the form

$$\begin{aligned} G = \langle & (0, [(\beta_5, 0) - (\beta_6, 0)]), \\ & ([(\alpha_1, 0) - (\alpha_4, 0)], [(\beta_1, 0) - (\beta_4, 0)]), \\ & ([(\alpha_2, 0) - (\alpha_4, 0)], [(\beta_2, 0) - (\beta_4, 0)]) \rangle. \end{aligned} \tag{3.39}$$

as in Corollary 1.4.9. Let $\mathrm{Jac}(X_1) \cong V_1/\Lambda_1$ and $\mathrm{Jac}(Y_2) \cong V_2/\Lambda_2$. Choose $P_1 = (\alpha_4, 0), P_2 = (\beta_4, 0)$. Let

$$\mathrm{AJ}_g(Q) = \left(\int_{P_g}^{Q} \omega_{g,1}, \dots \int_{P_g}^{Q} \omega_{g,g} \right) \mod \Lambda_g. \tag{3.40}$$

Now $\mathrm{AJ}_1 : X_1 \to \mathrm{Jac}(X_1)$ and $\mathrm{AJ}_2 : Y_2 \to \mathrm{Jac}(Y_2)$ are explicit descriptions of the Abel-Jacobi map.

We can use the Abel-Jacobi maps to explicitly calculate coordinates for the points generating G. Let $A_i = \mathrm{AJ}_1([(\alpha_i, 0) - (\alpha_4, 0)])$ with $i = 1,2,3,4$ in $\mathrm{Jac}(X_1)$ and let $B_i = \mathrm{AJ}_2((\beta_i, 0) - (\beta_4, 0))$ with $i = 1,2,3,4$ and $T = \mathrm{AJ}_2((\beta_5, 0) - (\beta_6, 0))$. Then the points $(A_i + \Lambda_1, B_i + \Lambda_2)$ and $(0, T + \Lambda_2)$ lie in $\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2)[2]$ and generate G. The Abel-Jacobi maps can also be calculated explicitly using [21].

3.3 An algorithm for analytic gluing

We will use different algorithms depending on whether B_G is the Jacobian of a hyperelliptic curve or the Jacobian of a non-hyperelliptic curve. We will distinguish between the two cases by using theta constants.

The algorithm has been implemented in Magma and is available on [13].

Definition 3.3.1. Let $(A = V/\Lambda, E)$ be a principally polarized abelian variety and let Π be a period matrix for A. Write

$$\Pi = \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix} \tag{3.41}$$

with $\Omega_1, \Omega_2 \in \mathrm{Mat}_{g\times g}(\mathbb{C})$. Then $\tau = \Omega_1^{-1}\Omega_2$ is called a *small period matrix* of (A, E).

Definition 3.3.2. Let $\omega \in V$ and let τ be a small period matrix. We define the *theta series*

$$\theta(\omega, \tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \tau n + 2\pi i n^t \omega). \tag{3.42}$$

Definition 3.3.3. Let $\xi \in 1/2\Pi_{1,2} \subset V$. We define

$$\theta[\xi](\tau) = \exp(\pi i \xi_1^T \tau \xi_1 + 2\pi i \xi_1^T \xi_2)\theta(\xi_2 + \tau\xi_1, \tau). \tag{3.43}$$

The value $\theta[\xi](\tau)$ is called a *theta constant*. The value only depends on the equivalence class of ξ in V/Λ. (So ever ξ that gets mapped to the same 2-torsion point in $A = V/\Lambda$ gives rise to the same theta constant).

Definition 3.3.4. Let $\xi \in (1/2)\Pi$. Write $\xi = (\xi_1, \xi_2)$ where ξ_1 consists of the first g coordinates of ξ and ξ_2 consists of the last g coordinates of ξ. We say that $\theta[\xi](\tau)$ is an *even* theta constant if

$$\exp(4\pi i \xi_1^T \xi_2) = 1 \tag{3.44}$$

and $\theta[\xi](\tau)$ is called an *odd* theta constant if

$$\exp(4\pi i \xi_1^T \xi_2) = -1. \tag{3.45}$$

Definition 3.3.5. Let τ be a small period matrix abelian variety of genus 3. We define the modular form χ_{18} to be the product of all even theta constants. And we define the modular form Σ_{140} to be the thirty-fifth elementary symmetric function of the $(\theta[\xi](\tau))^8$.

Theorem 3.3.6. *Let τ be a small period matrix. The abelian variety A corresponding to τ is a non-hyperelliptic Jacobian if and only if $\chi_{18}(\tau) \neq 0$, it is a hyperelliptic Jacobian if $\chi_{18}(\tau) = 0$ and $\Sigma_{140}(\tau) \neq 0$ and it is decomposable if $\chi_{18}(\tau) = 0$ and $\Sigma_{140}(\tau) = 0$.*

Proof. See Lemma 10 and Lemma 11 in [15]. □

Remark 3.3.7. The condition that $\chi_{18}(\tau) = 0$ is equivalent to the vanishing of one even theta constant.

Algorithm 1 Calculate (2,2)-gluings of X_1 and Y_2 analytically over $\mathbb{C}$.

Input: Two curves X_1 and Y_2 and a maximally isotropic subgroup G of $\mathrm{Jac}(X_1)[2] \times \mathrm{Jac}(Y_2)[2]$.

1: Choose a homology basis $\gamma_{1,1}, \gamma_{1,2}$ for X_1, and let $\gamma_{2,1}, \dots \gamma_{2,4}$ be a homology basis for Y_2.

2: Choose a differential form $\omega_{1,1}$ on X_1, and let $\omega_{2,1}, \omega_{2,2}$ be linearly independent differential forms on Y_2.

3: Calculate

$$\lambda_{g,j} = \left(\int_{\gamma_{g,j}} \omega_{g,1}, \dots, \int_{\gamma_{g,j}} \omega_{g,g} \right) \tag{3.46}$$

for $g = 1, 2$, $j = 1, \dots, 2g$ and let $\Lambda_g = \langle \lambda_{g,j} \rangle$. Let $V_g = \mathbb{C}^g$, let $\mathcal{E}_g$ be the standard basis for $\mathbb{C}^g$ and let $\mathcal{B}_g = \lambda_{g,1}, \dots \lambda_{g,2g}$ be a basis for Λ_g. We have $\mathrm{Jac}(X_1) \cong V_1/\Lambda_1$, $\mathrm{Jac}(Y_2) \cong V_2/\Lambda_2$.

4: Let

$$\Pi_g = \begin{bmatrix} | & \cdots & | \\ \lambda_{g,1} & & \lambda_{g,2g} \\ | & \cdots & | \end{bmatrix}. \tag{3.47}$$

Now Π_1 is a period matrix for X_1 with respect to $\mathcal{E}_1$ and $\mathcal{B}_1$ and Π_2 is a period matrix for Y_2 with respect to $\mathcal{E}_1$ and $\mathcal{B}_1$.

5: Write E_1 for the alternating form induced by the natural principal polarization on $\mathrm{Jac}(X_1)$ and write E_2 for the alternating form induced by the natural principal polarization on $\mathrm{Jac}(Y_2)$. With respect to the basis $\mathcal{B}_g$ we have $M_{E_g} = S_g$ for $g = 1, 2$.

6: Let $\Pi_{1,2}$ be the product period matrix for $\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2)$ as in Definition 3.2.3 and let $V_{1,2} = V_1 \times V_2$ and $\Lambda_{1,2} = \Lambda_1 \times \Lambda_2$. for $\Lambda_{1,2}$.

7: Using the implementation of the Abel-Jacobi map from [21] and Remark 3.2.9 we calculate μ_1, μ_2, μ_3 as in Lemma 3.2.6 such that

$$\langle \mu_1, \mu_2, \mu_3, (\lambda_{1,1}, 0), (\lambda_{1,2}, 0), (0, \lambda_{2,1}), \dots, (0, \lambda_{2,4}) \rangle = \Lambda_G. \tag{3.48}$$

8: Calculate basis $\nu_1, \nu_2 \dots, \nu_6$ for Λ_G.

9: Use Lemma 3.2.7 with the basis we calculated in Step 7 to find a rational representation R_ϕ of the quotient map $\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2) \to B_G$.

10: Calculate M_{E_G} using Corollary 3.2.8.

11: Let S be a symplectic matrix such that $(S^t)^{-1} M_{E_G} S^{-1} = S_3$. As a consequence,

$$(S^{-1})^t (R_\phi^t)^{-1} 2E_{1,2} R_\phi^{-1} S^{-1} = S_3. \tag{3.49}$$

12: **return** The normalized period matrix of $V_{1,2}/\Lambda_G$ given by

$$\Pi_G = (\Pi_{1,2}) R_\phi^{-1} S^{-1}. \tag{3.50}$$

Algorithm 2 Calculate the genus 3 curve corresponding to a principally polarized abelian variety of genus 3 if it exists.

Input: The normalized period matrix of an indecomposable principally polarized abelian variety of genus 3 in normalized form.

```
1:  Calculate the theta constants for Π_G.
2:  if χ_18 ≠ 0 then
3:      The curve Z_3 is non-hyperelliptic and we calculate the Dixmier-Ohno
        invariants of Π_G over ℂ.
4:      Try to recognize the Dixmier-Ohno invariants as elements of Q̄.
5:      Reconstruct a quartic plane model of Z_3 that has the same Dixmier-
        Ohno invariants as Π_G.
6:      return The quartic equation of a genus 3 curve Z_3 in P²_ℂ such that
        Jac(Z_3) is isomorphic over ℂ to an abelian variety with period matrix
        Π_G.
7:  else
8:      if Σ_140 ≠ 0 then
9:          the glued curve Z_3 is hyperelliptic and we compute the Rosenhain
            invariants of Z_3 over ℂ.
10:         return The equation of a curve Z_3 in P²_ℂ such that Jac(Z_3) is isomor-
            phic over ℂ to an abelian variety with period matrix Π_G.).
11:     else
12:         return "The abelian variety is decomposable".
13:     end if
14: end if
```

Remark 3.3.8. For Steps 3-5 we use the methods developed by Kilicer, Labrande, Lercier, Ritzenthaler, Sijsling, and Streng in [16].

Remark 3.3.9. For Step 10 we use the algorithm developed by Balakrishnan, Ionica, Lauter, and Vincent in [1].

Remark 3.3.10. We can combine Algorithm 1 and 2 to find an explicit equation of a curve Z_3 that is the (2,2)-gluing of X_1 and Y_2. But as multiple steps in the algorithms are approximative, it is necessary to verify that the calculated curve is indeed what we want it to be. For this we can use an implementation of the construction of Ritzenthaler and Romagny described in Theorem 2.2.1.

3.4 (2,2)-Gluing over $\mathbb{Q}$

Let X_1 and Y_2 be two curves over $\mathbb{Q}$ of genus 1 and genus 2 respectively and let G be a Galois-invariant maximal isotropic subgroup. Then the (2,2)-gluing of X_1 and Y_2 along G will be a principally polarized abelian variety B over $\mathbb{Q}$.

Assume that it is indecomposable. In this case Proposition 1.5.1 tells us there exists a curve Z_3 such that B is isomorphic to $\mathrm{Jac}(Z_3)$ over a quadratic field extension of $\mathbb{Q}$. We would like to construct this curve.

Algorithm 1 gives us a way to find a period matrix of the (2,2)-gluing of X_1 and Y_2 along G and Algorithm 2 allows us to find a curve Z_3' that is isomorphic to Z_3 over $\mathbb{C}$. Even though this curve is not necessarily defined over $\mathbb{Q}$, we can use it to find an equation for Z_3 over $\mathbb{Q}$.

Definition 3.4.1. Let C be a curve of genus 3 and let $F(x:y:z) = 0$ be an equation defining C in $\mathbb{P}^3_k$. For the purposes of this section we will call

$$\omega_1 = \frac{xdx}{dF/dy}, \omega_2 = \frac{ydx}{dF/dy}, \omega_3 = \frac{zdx}{dF/dy} \tag{3.51}$$

the *basis of differentials* induced by F.

Proposition 3.4.2. *Let*

$$T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{3.52}$$

be an invertible matrix over k. Let $\phi_T : \mathbb{P}^2_k \to \mathbb{P}^2_k$ be the induced isomorphism given by

$$\phi_T(x:y:z) = (a_1x + a_2y + a_3z : b_1x + b_2y + b_3z : c_1x + c_2y + c_3z). \tag{3.53}$$

Let C and D be quartic curves of genus 3 in $\mathbb{P}^2_k$ where C is given by the equation $F_C(x:y:z) = 0$ and D is given by the equation

$$F_D = F_C \circ \phi(x:y:z) = 0. \tag{3.54}$$

Let

$$\omega_{C,1}, \omega_{C,2}, \omega_{C,3} \tag{3.55}$$

be the basis of differentials induced by F_C for C and let Π_C be the period matrix of C calculated with respect to this basis. Let

$$\omega_{D,1}, \omega_{D,2}, \omega_{D,3} \tag{3.56}$$

be the basis of differentials induced by F_D for D and let Π_D be the period matrix of D calculated with respect to this basis. Then there exists a non-zero constant $\mu \in k$ such that

$$\Pi_D = (\mu T)^{-1}\Pi_C. \tag{3.57}$$

Proof. The differentials $\omega_{D,i}$ form a basis for the global sections of the canonical sheaf $H^0(\Omega_D)$ and the corresponding canonical embedding $D \to \mathbb{P}^2_k$ given by $P \mapsto (x(P) : y(P) : z(P))$ is the embedding of D into $\mathbb{P}^2_k$ as the zero set of F_D. The morphism ϕ_T maps D to C as

$$0 = F_D(P) = F_C \circ \phi_T(P). \tag{3.58}$$

As T is an invertible matrix we find that ϕ_T is an isomorphism. The differentials $\phi^*(\omega_{C,i})$ build a basis for $H^0(\Omega_D)$. So ϕ^* is an invertible linear map, and we find that

$$\phi^*(\omega_{C,1}) = \alpha_1\omega_{D,1} + \alpha_2\omega_{D,2} + \alpha_3\omega_{D,3}, \tag{3.59}$$

$$\phi^*(\omega_{C,2}) = \beta_1\omega_{D,1} + \beta_2\omega_{D,2} + \beta_3\omega_{D,3}, \tag{3.60}$$

$$\phi^*(\omega_{C,3}) = \gamma_1\omega_{D,1} + \gamma_2\omega_{D,2} + \gamma_3\omega_{D,3} \tag{3.61}$$

for certain $\alpha_i, \beta_i, \gamma_i$. Note that

$$\phi^*\left(\frac{\omega_{C,2}}{\omega_{C,3}}\right) = \phi^*(y/z) = \frac{b_1x + b_2y + b_3z}{c_1x + c_2y + c_3z}. \tag{3.62}$$

This means that

$$\phi^*(\omega_{C,2}) = \frac{b_1x + b_2y + b_3z}{c_1x + c_2y + c_3z}\phi^*(\omega_{C,3}). \tag{3.63}$$

As $\phi^*(\omega_{C,2})$ does not have any poles, it follows that $\gamma_i = \mu c_i$ for $i = 1,2,3$ and for some constant $\mu \in k$. We also get that $\beta_i = \mu b_i$ for $i = 1,2,3$. Using furthermore, that

$$\phi^*(\omega_{C,1}) = \frac{a_1x + a_2y + a_3z}{c_1x + c_2y + c_3z}\phi^*(\omega_{C,3}), \tag{3.64}$$

we also have $\alpha_i = \mu a_i$ for $i = 1,2,3$ and the same constant μ. This implies that if we take the $\omega_{C,i}$ as a basis on Ω_C and the $\omega_{D,i}$ as a basis on Ω_D, the linear map $\phi^* : H^0(\Omega_C) \to H^0(\Omega_D)$ is represented by the matrix μT^t with respect to these bases. So the dual map $H^0(\Omega_D)^\vee \to H^0(\Omega_C)^\vee$ is represented by the matrix $(\mu T^t)^t$ with respect to the natural dual bases. This gives us that $(\mu T)\Pi_D = \Pi_C$. So the curve D has period matrix $\Pi_D = (\mu T)^{-1}\Pi_C$ with respect to the basis of differentials $\omega_{D,i}$. □

Corollary 3.4.3. *Let B be a principally polarized abelian variety over k with period matrix Π_B and let Z_3' be a curve over $\overline{k}$ given by an equation*

$$F(x : y : z) = 0 \tag{3.65}$$

in $\mathbb{P}^2_{\overline{k}}$ such that $\mathrm{Jac}(Z_3')$ is isomorphic to B over $\overline{k}$. Let $\Pi_{Z_3'}$ be the period matrix of $\mathrm{Jac}(Z_3')$ with respect to the basis of differential forms induced by F. Let T be the *analytic representation of the isomorphism $B \to \mathrm{Jac}(Z_3')$, i.e. $T\Pi_B = \Pi_{Z_3'}$. Then there exists $\lambda \in \overline{k}$ such that*

$$\lambda \cdot F \circ \phi_T(x:y:z) = 0 \tag{3.66}$$

is an equation of a curve Z_3 that is isomorphic to Z_3' with coefficients in k.

Proof. Proposition 3.4.2 tells us there exists a constant μ such that $T\Pi_{Z_3'} = \mu\Pi_B$. Now the curve given by the equation

$$F \circ \phi_T(x:y:z) = 0 \tag{3.67}$$

has period matrix $\mu\Pi_B$. Let $\sigma \in Gal(\overline{k}/k)$. As B is defined over k it has a basis of global differential forms over k and the period matrix of $\sigma(F \circ \phi_T(x:y:z))$ has the same period lattice as the one induced by Π_B up to multiplication with a constant. This means that the basis of differential forms induced by F on $\mathrm{Jac}(Z_3')$ and the basis of differential forms induced by $\sigma(F)$ only differ up to multiplication with a constant. As a consequence, the polynomials $F \circ \psi$ and $\sigma(F \circ \psi)$ (that are given by the embeddings belonging to the above choices of bases for differential forms) have the same solution set in $\mathbb{P}^2_{\overline{k}}$. This solution set is therefore defined over k. So there exists a constant $\lambda \in \overline{k}$ such that

$$\lambda \cdot F \circ \phi_T(x:y:z) = 0 \tag{3.68}$$

is an equation over k. $\square$

Remark 3.4.4. It suffices to divide by an element of $\overline{k}$ such that one of the coefficients of the equation is defined over k. One can, for example, normalize the equation such that the coefficient in front of x^4 is 1.

3.5 Examples

A first example

Let X_1 be the curve given by the equation

$$y^2 = 4(x^3 - x^2 - 2x - 1) + x^2. \tag{3.69}$$

This curve is isomorphic to $X_0(49)$. Let Y_2 be the genus 2 curve given by

$$y^2 = x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6. \tag{3.70}$$

Then the genus 3 curve given by the equation

$$32x^4 + 11x^2y^2 - 454x^2yz - 59x^2z^2 + 92y^4 - 248y^3z + 34y^2z^2 + 200yz^3 - 76z^4 = 0 \tag{3.71}$$

is a geometric (2,2)-gluing of the above two curves. This curve attains its endomorphisms over a field of degree 16, whereas the original genus 2 curve attains them over a field extension of degree 24.

Arithmetically gluing the same two curves gives us the genus 3 curve Z_3 given by the equation

$$x^4 + 48x^2yz - 288y^4 + 288y^2z^2 - 8z^4 = 0. \tag{3.72}$$

In this case Z_3 is an arithmetic gluing of X_1 and Y_2 with twist $\sqrt{3}$.

(3,3,3)-gluing

A slight adaption of the algorithm allows us to glue along 3-torsion. Let

$$C_i : y^2 = x^3 + r_i \tag{3.73}$$

be three curves of genus 1 where the r_i are distinct roots of $r_i^3 + r_i - 1 = 0$. As before, we calculate period matrices Π_i for each of the three curves and we consider the product period matrix of these three to get a period matrix of $\mathrm{Jac}(C_1)\times\mathrm{Jac}(C_2)\times\mathrm{Jac}(C_3)$. After that we randomly picked maximally isotropic subgroups G of $\mathrm{Jac}(C_1)\times\mathrm{Jac}(C_2)\times\mathrm{Jac}(C_3)$ and calculated the geometric (3,3,3)-gluing along G until we found a curve that is defined over $\mathbb{Q}$. A genus 3 curve over $\mathbb{Q}$ whose Jacobian is isomorphic to a geometric (3,3,3)-gluing of the C_i is given by the equation:

$$-24x^3z + 9x^2y^2 - 30xyz^2 + 10y^3z - 75z^4 \tag{3.74}$$

in $\mathbb{P}^2_k$.

70-torsion point

Let X_1 be the genus 1 curve given by

$$y^2 = 4x^3 + 5x^2 - 98x + 157. \tag{3.75}$$

This curve is isomorphic to the curve with label 118.c1 in the LMFDB and it has a torsion point of order 5. Let Y_2 be the genus 2 curve given by

$$y^2 = 4x^5 + 17x^4 + 22x^3 + 15x^2 + 6x + 1. \tag{3.76}$$

This curve is isomorphic to the curve with label 295.a.295.1 in the LMFDB and it has a torsion point of order 14.

We see that

$$4x^5 + 17x^4 + 22x^3 + 15x^2 + 6x + 1 = (x^2 + 3x + 1)(4x^3 + 5x^2 + 3x + 1). \tag{3.77}$$

As the equation for Y_2 contains a quadratic factor, it is gluable. Furthermore, one can show that $4x^3+5x^2+3x+1$ and $4x^3+5x^2-98x+157$ have the same splitting field, so the necessary conditions for the existence of an arithmetic $(2,2)$-gluing over $\mathbb{Q}$ are satisfied.

Gluing these two curves along a suitable choice of a maximally isotropic subgroup G using the above algorithms gives us a genus 3 curve Z_3 with the following equation in $\mathbb{P}^2_{\mathbb{Q}}$:

$$-32x^4-43x^2y^2+104x^2yz+332x^2z^2+2y^4-12y^3z-28y^2z^2-112yz^3-48z^4=0. \tag{3.78}$$

The curve Z_3 has the property that $\mathrm{Jac}(Z_3)$ is isomorphic to $\mathrm{Jac}(X_1)\times\mathrm{Jac}(Y_2)/G$ over a quadratic field extension of $\mathbb{Q}$, so it is an arithmetic (2,2)-gluing of X_1 and Y_2. A calculation shows that this is the case over $\mathbb{Q}(\sqrt{5})$.

By construction, $\mathrm{Jac}(X_1)\times\mathrm{Jac}(Y_2)$ contains a torsion point of order 14 and a torsion point of order 5. As G is contained in the 2-torsion, we therefore know that $\mathrm{Jac}(X_1)\times\mathrm{Jac}(Y_2)/G$ contains a torsion point of order 35. But a priori some of the 2-torsion could have disappeared after taking the quotient by G. Examining the action of the Galois group on the quotient $\mathrm{Jac}(X_1)[2]\times\mathrm{Jac}(Y_2)[2]/G$ however shows that there exists at least one non-trivial point in $\mathrm{Jac}(X_1)[2]\times\mathrm{Jac}(Y_2)[2]/G$ that is Galois-invariant. This means that $\mathrm{Jac}(X_1)\times\mathrm{Jac}(Y_2)/G$ also contains a 2-torsion point, and it therefore contains a 70-torsion point.

Chapter 4

Algebraic gluing

4.1 Kummer surfaces

Definition 4.1.1. Let S be a surface over a field k and let P be a singular point of S. We say that P is a *node* of S if

$$\widehat{\mathcal{O}}_{S,P} \otimes \overline{k} \cong \frac{\overline{k}[[x,y,z]]}{(x^2+y^2+z^2)}. \tag{4.1}$$

Definition 4.1.2. A *Kummer surface K* is a reduced irreducible quartic surface in $\mathbb{P}^3_k$ with 16 nodes and no other singular points.

Proposition 4.1.3. *Let k be a field and let A be an indecomposable principally polarized abelian variety of dimension 2. Then the surface $A/\langle -1\rangle$ is a Kummer surface.*

Proof. See Proposition 4.23 in [11]. □

Definition 4.1.4. Let k be a field and let Y_2 be a curve of genus 2. We define $\mathrm{Kum}(Y_2) = \mathrm{Jac}(Y_2)/\langle -1\rangle$ to be the Kummer surface associated to Y_2.

Proposition 4.1.5. *Let K be a Kummer surface over k. There exist A,B,C,D, $a,b,c,d \in \overline{k}$ with*

$$ad \neq \pm bc, \quad ac \neq \pm bd, \quad ab \neq \pm cd, \quad a^2+b^2+c^2+d^2 \neq 0,$$

$$a^2+d^2 \neq b^2+c^2, \quad a^2+c^2 \neq b^2+d^2, \quad a^2+b^2 \neq c^2+d^2.$$

such that we can realize $K_{\overline{k}}$ as a quartic equation in $\mathbb{P}^3_{\overline{k}}$ of the form

$$K(x,y,z,t) = x^4+y^4+z^4+t^4+2Dxyzt+A(x^2t^2+y^2z^2)+B(y^2t^2+x^2z^2)+C(z^2t^2+x^2y^2) \tag{4.2}$$

and such that its 16 singular points are given by:

$$(a,b,c,d), (d,-c,b,-a), (d,c,-b,-a), (c,d,-a,-b),$$

$$(-c,d,a,-b),(-b,a,d,-c),(b,-a,d,-c),(d,c,b,a),$$

$$(c,d,a,b),(b,a,d,c),(a,-b,-c,d),(-a,b,-c,d),$$

$$(-a,-b,c,d),(d,-c,-b,a),(-c,d,-a,b),(-b,-a,d,c).$$

Proof. See Corollary 2.18 and Theorem 2.20 in [11]. □

Definition 4.1.6. A *(16,6)- configuration* is a set of 16 planes and 16 points in $\mathbb{P}^3_k$ such that every plane contains exactly 6 points and every point lies on exactly 6 planes. We will also call these planes *special planes*.

Definition 4.1.7. A (16,6)-configuration is called *non-degenerate* if every two special planes contain exactly two points in the configuration (or equivalently that every pair of points is contained in exactly two special planes)

Proposition 4.1.8. *Let K be a Kummer surface over k. Then there exist 16 planes such that the set of these planes together with the 16 singular points on K form a non-degenerate (16,6)-configuration.*

Proof. See [11, Proposition 2.16]. □

Lemma 4.1.9. *Assume we have an equation for the Kummer surface K as in Proposition 4.1.5. Then there exists a group $K(2)$ consisting of linear automorphisms of K that is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. It is generated by the maps $\alpha,\beta,\alpha',\beta'$ where*

- $\alpha(x,y,z,t)=(t,z,y,x)$,
- $\beta(x,y,z,t)=(z,t,x,y)$,
- $\alpha'(x,y,z,t)=(x,-y,-z,t)$,
- $\beta'(x,y,z,t)=(-x,y,-z,t)$.

Proof. This follows from the properties of (16,6)-configurations described in Paragraph 1 of [11]. □

Corollary 4.1.10. *Let K be a Kummer surface and let P_i,P_j,P_k be three distinct nodes on K. Let H be a plane going through P_j and P_k. There always exists a linear automorphism $\sigma\in K(2)$ such that the plane $\sigma(H)$ goes through P_i.*

Proof. This follows from Lemma 4.1.9 and Proposition 4.1.5. □

4.2 Algebraic gluing

Proposition 4.2.1. *Let K be a Kummer surface and let P_i, P_j be a pair of singular points on K. Take the 1-dimensional family $H_{i,j}(\lambda)$ of planes passing through P_i and P_j.*

(i) The family of curves

$$H_{i,j}(\lambda) \cap K \tag{4.3}$$

consists generically of genus 1 curves that have exactly two nodes.

(ii) There are exactly six planes in $H_{i,j}(\lambda)$ that pass through more than two nodes of K. Six of them pass through exactly 3 singular points of K and two of them intersect in exactly six singular points of K.

Proof. For (i): The family $H_{i,j}(\lambda) \cap \mathrm{Kum}(Y_2)$ consists of curves C that pass through exactly two nodes of K. A generic member C of this family is irreducible outside of the two singular points because of the Theorem of Bertini. Proposition 2.20 in [11] tells us that the arithmetic genus of C is 1.

We will now show that the singular points of C are generically nodes. As remarked in [11, Lemma 2.5] the point P_i is a node of C if and only if H intersects the tangent cone of P_i in two distinct lines. To show that P_i is a node of a generic C, it suffices to show that there exists a plane that passes through P_i and P_j and intersects the tangent cone of P_i in two lines. Assume that every plane that intersects both P_i and P_j intersects the tangent cone of P_i in a single line L. This implies that P_j lies on the tangent cone of P_i. This is impossible however as the intersection product of L with P_i on K is ≥ 3 and the intersection product of L with P_j is ≥ 2 and the intersection product of L with K is 4. As a consequence, a plane that intersects both P_i and P_j will generically intersect the tangent cones of both P_i and P_j in two distinct lines. So, P_i and P_j are nodes of C.

For (ii): Assume $H_{i,j}(\lambda)$ passes through a third singular point P_k of K. Lemma 4.42 in [11] says that $|H_{i,j}(\lambda) \cap \mathrm{Sing}(K)|$ is either 3 or 6. As the (16,6)-configuration is non-degenerate, there exist exactly two distinct special planes H and H' that contain both P_i and P_j. As $|(H_1 \cup H_2) \cap \mathrm{Sing}(K)| = 10$ there are exactly six points P_k for which the plane through P_i, P_j and P_k is not a special plane. $\square$

Theorem 4.2.2. *Let k be a field, and let Y_2 be a curve of genus 2. Let $\mathrm{Kum}(Y_2) = \mathrm{Jac}(Y_2)/\langle -1 \rangle \subset \mathbb{P}^3_k$ be the Kummer surface associated to Y_2. Let H be a plane in $\mathbb{P}^3_k$ that passes through exactly two singular points P, Q, such that $\widetilde{X_1} = H \cap K$ is isomorphic to a genus 1 curve with two nodes. Write $i_1 : \widetilde{X_1} \to \mathrm{Kum}(Y_2)$ for the inclusion map, and let $\tau_1 : X_1 \to \widetilde{X_1}$ be the desingularization. Now let $Z_3 = X_1 \times_{\mathrm{Kum}(Y_2)} \mathrm{Jac}(Y_2)$ be the pullback. We get the following diagram:*

$$\begin{tikzcd} Z_3 = X_1 \times_{\mathrm{Kum}(Y_2)} \mathrm{Jac}(Y_2) \arrow[r, "\tau_3"] \arrow[d, "\overline{\pi}"] & \widetilde{Z}_3 = \widetilde{X}_1 \times_{\mathrm{Kum}(Y_2)} \mathrm{Jac}(Y_2) \arrow[r, hook, "i_3"] \arrow[d, "\widetilde{\pi}"] & \mathrm{Jac}(Y_2) \arrow[d, "\pi"] \\ X_1 \arrow[r, "\tau_1"] & \widetilde{X}_1 \arrow[r, hook, "i_1"] & \mathrm{Kum}(Y_2) \end{tikzcd} . \tag{4.4}$$

Then Z_3 is an irreducible curve of genus 3.

Proof. First note that because i_1 is a closed immersion, the morphism $i_3 : \widetilde{Z}_3 \to \mathrm{Jac}(Y_2)$ is also a closed immersion. Now the morphism $\widetilde{\pi} : \widetilde{Z}_3 \to \widetilde{X}_1$ is finite because the morphism $\mathrm{Jac}(X_2) \to \mathrm{Kum}(Y_2)$ is finite of degree 2. It follows that $\widetilde{Z}_3$ is of dimension 1, and that the morphism $\widetilde{Z}_3 \to \widetilde{X}_1$ is also of degree 2. The map $\widetilde{\pi} : \widetilde{Z}_3 \to \widetilde{X}_1$ is ramified above $i_1^{-1}(P)$ and $i_1^{-1}(Q)$ as the branch locus of π consists of the singular points of $\mathrm{Kum}(Y_2)$. As $\overline{\pi} : Z_3 \to X_1$ is the desingularization of $\widetilde{\pi}$, it is a degree 2 cover of a genus 1 curve. The points $i_1^{-1}(P)$ and $i_1^{-1}(Q)$ are nodes because P and Q were nodes and i is an embedding, so $\overline{\pi}$ is ramified above the points $P_1, P_2 \in p_1^{-1}(\{i^{-1}(P)\})$ and $Q_1, Q_2 \in \tau_1^{-1}(\{i^{-1}(P)\})$. Using the Riemann-Hurwitz formula, it follows that Z_3 is an irreducible curve of genus 3.

□

Remark 4.2.3. If we assume that $\mathrm{Jac}(Y_2)$ is isomorphic to $\mathrm{Pr}(\widetilde{Z}_3)/\widetilde{X}_1)$ then the argument in Theorem 2.2.4 shows that Z_3 is a (2,2)-gluing of X_1 and Y_2.

Lemma 4.2.4. *Let K be a Kummer surface over $\overline{k}$ and let $\sigma \in K(2)$. Let $H \subset \mathbb{P}^3$ be a plane going through two singular points P_i and P_j. Then σ maps H to another plane $\sigma(H)$ going through two singular points. If $\sigma(H) = H$, then either σ swaps P_i and P_j or $\sigma = \mathrm{Id}$.*

Proof. As σ is an automorphism of K it maps singular points of K to singular points of K. Now let H be a plane such that $\sigma(H) = H$. If H only contains two singular points, it is clear that σ either swaps P_i and P_j or $\sigma = \mathrm{Id}$. If H contains exactly three singular points then $\sigma = \mathrm{id}$ as $\sigma \in K(2)$ has no fix points in $\mathrm{Sing}(K)$ and σ has order 2.

If H contains six singular points, it is a special plane and there are only two special planes that contain both P_1 and P_2. So either σ is the identity map or σ is the map that swaps the two planes.

We may assume without loss of generality that $P_i = (d, -c, b, -a)$ and $P_j = (d, c, -b, -a)$. Then the two special planes H', H'' containing both P_i and P_j are given by

$$\begin{aligned} H'(x:y:z:t) &= ax + by + cz + dt, \\ H''(x:y:z:t) &= ax - by - cz + dt. \end{aligned} \tag{4.5}$$

The automorphism that maps H' to H'' is given by $(x:y:z:t) \mapsto (x:-y:-z:t)$ which is exactly the map that swaps P_i and P_j.

$\square$

Lemma 4.2.5. *Let K be a Kummer surface over k and assume we have an equation for K as in Proposition 4.1.5 over k. Let P_i, P_j be singular points of K and let a, b, c, d be as in the proposition. Assume that the set $\{P_i, P_j\}$ is Galois-invariant. Then a, b, c and d are defined over (at most) a quadratic extension of k.*

Proof. Let $\sigma \in \mathrm{Gal}(\overline{k}/k)$. Then σ induces an automorphism of K, so it can only send singular points to singular points. Without loss of generality we may assume that $P_i = (d, -c, b, -a)$ and $P_j = (d, c, -b, -a)$. As the set $\{P_i, P_j\}$ is Galois-invariant, it follows that either $\sigma(a) = a$, $\sigma(d) = d, \sigma(b) = b, \sigma(c) = c$ or $\sigma(a) = a$, $\sigma(d) = d, \sigma(b) = -b, \sigma(c) = -c$. So a, b, c and d are defined over at most a quadratic extension of k. $\square$

Consider the family $H_{i,j}(\lambda)$ of planes passing through two singular points P_i, P_j of a Kummer surface K. Given a curve of genus 1 X_1, we want to be able to find a λ_0 such that $H_{i,j}(\lambda_0)$ is isomorphic to X_1. In order to do this, we will describe a way to find an expression for the j-invariant of $H_{i,j}(\lambda)$ in terms of λ.

Lemma 4.2.6. *Let k be a field, and let $K \subset \mathbb{P}^3$ be a Kummer surface with singular points $P_1, \ldots, P_{16}$. Let $H_{1,2}(\lambda)$ be the family of planes going through P_1 and P_2. Fix $\lambda_0 \in k$. Let $U \cong \mathbb{A}^2_k$ be an affine open of $H_{1,2}(\lambda_0)$ containing both P_1 and P_2. Let $\widetilde{C}_{\lambda_0} = U \cap \mathrm{Kum}(Y_2)$. Let (x_i, y_i) be the coordinates of P_i in U. Define the function $\widetilde{g}: \widetilde{C}_{\lambda_0} \backslash \{P_1\} \to k$ in the following way:*

$$\widetilde{g}((x,y)) = \left(\frac{y_1 - y}{x_1 - x}\right). \tag{4.6}$$

Then $\widetilde{g}$ extends to a function

$$g : C_{\lambda_0} \to \mathbb{P}^1_k$$

of degree 2 where C_{λ_0} is the normalization of $\widetilde{C}_{\lambda_0}$.

Proof. As U is a plane and K is a quartic surface we find that $\widetilde{C}_{\lambda_0}$ is a quartic plane curve. Now let $Q_1 = (x, y) \in \widetilde{C}_{\lambda_0} \backslash \{P_1\}$. Then the line L through Q_1 and P_1 is the unique line with slope $(y_1 - y)/(x_1 - x)$ through P_1. Note that the intersection number of $\widetilde{C}_{\lambda_0}$ and L is 4. Now L intersects $\widetilde{C}_{\lambda_0}$ in P_1 and Q_1. As P_1 is a node of $\widetilde{C}_{\lambda_0}$ its contribution to the intersection number is 2, we find that the line L will generically intersect $C_{\lambda_0} \cap U$ in a fourth point Q_2. This means that generically $|\widetilde{g}^{-1}(c)| = 2$ for $c \in k$. As $\widetilde{g}$ is a rational map of degree 2 between two affine curves, it extends to a function $g : C_{\lambda_0} \to \mathbb{P}^1_k$ of degree 2. $\square$

Corollary 4.2.7. *Let E be an elliptic curve and let $g : E \to \mathbb{P}^1_k$ be a map of degree 2. Let x_1, x_2, x_3, x_4 be the x-coordinates of the ramification points of g. Let c be the cross-ratio of x_1, x_2, x_3 and x_4. Then the j-invariant of E is*

$$\frac{(c^2-c+1)^3}{c^2(c-1)^2}. \tag{4.7}$$

Proof. See [27] Chapter III Proposition 1.7. □

Theorem 4.2.8. *Let k be a field and let $K \subset \mathbb{P}^3$ be a Kummer surface over k. Assume we have an equation for K as in Proposition 4.1.5 over k. Let P_1 and P_2 be two singular points on K such that the set $\{P_1, P_2\}$ is defined over k. Let $H_{1,2}(\lambda)$ be the family of planes going through P_1 and P_2. Then the j-invariant of the family $H_{1,2}(\lambda)$, is a rational function $j(H(\lambda)) \in k(\lambda)$ of degree at most 12.*

Proof. Assume that K is given by the homogeneous polynomial

$$\kappa(x,y,z,t) = x^4+y^4+z^4+t^4+2Dxyzt+A(x^2t^2+y^2z^2)+B(y^2t^2+x^2z^2)+C(z^2t^2+x^2y^2) \tag{4.8}$$

in $\mathbb{P}^3_k$ with singular points $P_1 = (d,-c,b,-a)$ and $P_2 = (d,c,-b-a)$. In this case the family of planes going through P_1 and P_2 is given by

$$H_{1,2}(\lambda) = ax+by+cz+dt+\lambda(ax-by-cz+dt). \tag{4.9}$$

From Proposition 4.1.5 it follows that only one of a,b,c,d can be 0. Without loss of generality we assume that $b,d \neq 0$ and we let U be the affine open subset of $H_{1,2}(\lambda)$ that we get by setting $z = 1$ to get a plane that contains both P_1 and P_2. Let $\widetilde{C}_{\lambda_0} = U \cap K$. It follows that we can describe $\widetilde{C}_{\lambda_0}$ as a curve in $\mathbb{A}^2_k$ given by the equation $F(x,y) = 0$ where

$$F(x,y) = \kappa\left(x,y,1,\frac{(1+\lambda)ax+(1-\lambda)(by+c)}{d(-1-\lambda)}\right) \tag{4.10}$$

and define an isomorphism $\phi : \widetilde{C}_{\lambda_0} \to K \cap U$ by

$$\phi(x,y) = \left(x,y,1,\frac{(1+\lambda)ax+(1-\lambda)(by+c)}{d(-1-\lambda)}\right). \tag{4.11}$$

Using this isomorphism we get $\phi^{-1}(d,-c,b,-a) = (d/b,-c/b)$. Let

$$g : (U \cap K)\backslash\{(d/b,-c/b)\} \to k \tag{4.12}$$

be the function defined by mapping a point P to the slope of the line passing through $(d/b,-c/b)$ and P as in Lemma 4.2.6. We will find the ramification points of g.

A line in U with slope μ passing through $(d/b, -c/b)$ satisfies the equation

$$y = \mu x - c/b - \mu d/b. \tag{4.13}$$

Consider the polynomial

$$F(x, \mu x - c/b - \mu d/b) \tag{4.14}$$

in $k(\lambda,\mu)[x]$.

Let $D(\mu) \in k(\lambda)$ be the discriminant of $F/((x-d/b)^2)$ with respect to x. Solving $D(\mu) = 0$ gives us the values of μ for which the intersection number of L with $\widetilde{C}_{\lambda_0}$ is greater than 2. We divided by $(x-d/b)^2$ to exclude the case where L intersects P_1.

A calculation shows that the zeroes of $D(\mu)$ are:

$$0, \tag{4.15}$$
$$x_1(\lambda) = ((ab\lambda + ab + cd\lambda + cd)/(b^2\lambda - b^2 - d^2\lambda - d^2)), \tag{4.16}$$
$$x_2(\lambda) = ((ab\lambda + ab - cd\lambda - cd)/(b^2\lambda - b^2 + d^2\lambda + d^2)), \tag{4.17}$$
$$x_3(\lambda) = ((-ac\lambda - ac - bd\lambda - bd)/(ad\lambda + ad - bc\lambda + bc)), \tag{4.18}$$
$$x_4(\lambda) = ((-ac\lambda - ac + bd\lambda + bd)/(ad\lambda + ad - bc\lambda + bc)). \tag{4.19}$$

The 0 coincides with the horizontal line that passes through P_2. All the other values gives us the branch points of the map g. All of these elements are rational functions of degree 1 in λ. To compute the j-invariant of the normalization of $\widetilde{C}_{\lambda_0}$ we compute the cross-ratio of $x_1(\lambda), x_2(\lambda), x_3(\lambda), x_4(\lambda)$. It is equal to

$$c(\lambda) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} = \tag{4.20}$$

$$\frac{\begin{array}{r}(-a^4\lambda^2 + b^4\lambda^2 - 2b^2c^2\lambda^2 + c^4\lambda^2 + 2a^2d^2\lambda^2 - d^4\lambda^2 - 2a^4\lambda - 2b^4\lambda + 4b^2c^2\lambda \\ - 2c^4\lambda + 4a^2d^2\lambda - 2d^4\lambda - a^4 + b^4 - 2b^2c^2 + c^4 + 2a^2d^2 - d^4)\end{array}}{\begin{array}{r}(-a^4\lambda^2 + b^4\lambda^2 + 2b^2c^2\lambda^2 + c^4\lambda^2 - 2a^2d^2\lambda^2 - d^4\lambda^2 - 2a^4\lambda - 2b^4\lambda - 4b^2c^2\lambda \\ - 2c^4\lambda - 4a^2d^2\lambda - 2d^4\lambda - a^4 + b^4 + 2b^2c^2 + c^4 - 2a^2d^2 - d^4)\end{array}} \tag{4.21}$$

which is a rational function of degree at most 2. Remark that it is also invariant under the action of $\mathrm{Gal}(\overline{k}/k)$ as a, b, c and d only occur as squares in $c(\lambda)$. It follows that the j-invariant

$$j(\lambda) = \frac{(c(\lambda)^2 - c(\lambda) + 1)^3}{c(\lambda)^2(c(\lambda) - 1)^2}. \tag{4.22}$$

is a rational function in $k(\lambda)$ of degree at most 12.

□

Lemma 4.2.9. *If K is given as in Proposition 4.1.5 then $j(H(\lambda)) = j(H(1/\lambda))$.*

Proof. Let $P_1 = (d, -c, b, -a), P_2 = (d, c, -b, -a)$ and define

$$H : ax + by + cz + dt = 0, \tag{4.23}$$
$$H' : ax - by - cz + dt = 0. \tag{4.24}$$

Let σ be the automorphism that swaps P_1 and P_2. In this case, $\sigma = \alpha'$. We see that σ maps the plane $H + \lambda H'$ to $H' + \lambda H$. The latter equation is equivalent to

$$H + 1/\lambda H'. \tag{4.25}$$

As σ is an automorphism, this implies that the curves $H_{1,2}(\lambda) \cap K$ and $H_{1,2}(1/\lambda) \cap \mathrm{Kum}(Y_2)$ have the same j-invariant. Now fix $c \in k$ and let α be a root of $j(H(\lambda)) - c = 0$. Then by the above, $1/\alpha$ will also be a root of $j(H(\lambda)) - c$. So $j(H(\lambda)) = j(H(1/\lambda))$.

□

Definition 4.2.10. Let K be a Kummer surface and let $H_{i,j}(\lambda)$ be the family of planes that intersect P_i and P_j. Let $\lambda, \mu \in \overline{k}$ such that the automorphism that swaps P_i and P_j maps the curve $H_{i,j}(\lambda)$ isomorphically to $H_{i,j}(\mu)$. Let $c = j(\lambda) = j(\mu)$. Then we call (λ, μ) a *solution pair for c*.

4.3 Explicit Jacobians

Proposition 4.3.1. *Let k be a field, and let Y_2 be a curve of genus 2. Let $P \in \mathrm{Jac}(Y_2)[2]$. Then P gives rise to an element $\sigma_P \in \mathrm{Kum}(Y_2)(2)$ in the following way:*

$$\begin{tikzcd} \mathrm{Jac}(Y_2) \arrow[r, "x \mapsto x+P"] \arrow[d, "\pi"] & \mathrm{Jac}(Y_2) \arrow[d, "\pi"] \\ \mathrm{Kum}(Y_2) \arrow[r, "\sigma_P"] & \mathrm{Kum}(Y_2) \end{tikzcd} . \tag{4.26}$$

Furthermore, $\mathrm{Kum}(Y_2)(2) \cong \langle \sigma_P | P \in \mathrm{Jac}(Y_2) \rangle$.

Proof. This follows from Proposition 4.15 in [11]. □

In order to give an explicit algorithm to construct the gluing of a genus 2 curve Y_2 and a genus 1 curve X_1 over k, we will need an explicit description of the quotient map $\mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ where $\mathrm{Kum}(Y_2)$ is the Kummer surface of Y_2.

In order to do this we will first write down an equation for an affine open subset of the Jacobian using the ideas by Cantor in [7].

Proposition 4.3.2. *Let Y_2 be a smooth curve of genus 2 over a field k given by the equation $y^2 = f(x)$ in $\mathbb{P}_k^2$. Let $i : Y_2 \to Y_2$ be the involution given by $(x,y) \mapsto (x,-y)$. There exists a bijection between the set*

$$\mathcal{S} = \{(P,Q) = ((x_1,y_2),(x_2,y_2)) \in \mathrm{Sym}^2(Y_2) | x_1 \neq x_2\} \tag{4.27}$$

and the set

$$\begin{aligned} \mathcal{P} = \{&(a(x),b(x)) | a(x) = x^2 + a_1 x + a_2 b(x) = b_1 x + b_2, \\ &\textit{where } a(x) \textit{ is a separable polynomial and } \left(b(x)^2 - f(x)\right) \equiv 0 \mod a(x)\}. \end{aligned} \tag{4.28}$$

Proof. Let $P = (x_1,y_1), Q = (x_2,y_2) \in Y_2$ with $(P,Q) \in \mathcal{S}$. Let

$$b_{P,Q}(x) = \frac{y_2 - y_1}{x_2 - x_1} x - \frac{y_2 - y_1}{x_2 - x_1} x_1 + y_1 \tag{4.29}$$

be an equation for the y coefficient of the line through P and Q depending on x and define

$$a_{P,Q}(x) = (x - x_1)(x - x_2) \tag{4.30}$$

to be the quadratic polynomial that has x_1 and x_2 as its roots. As $x_1 \neq x_2$ the polynomial $b_{P,Q}$ is well-defined. Now $(b_{P,Q}(x)^2 - f(x)) = 0$ if and only if x is in the intersection of Y_2 and the line defined by $y = b(x)$. It follows that $(b_{P,Q}(x)^2 - f(x)) \equiv 0 \mod a_{P,Q}(x)$. This gives us a map in one direction.

Now assume that we have an element $(a,b) \in \mathcal{P}$. Let x_1, x_2 be the roots of a. Then

$$(x - x_1)(x - x_2) \mid \left(b(x)^2 - f(x)\right), \tag{4.31}$$

and x_1 and x_2 are the x-coordinates of the intersection points of Y_2 with the curve defined by the line $y = b(x)$. As the line given by $y = b(x)$ is not a vertical line, we find that $x_1 \neq x_2$. It follows that the tuple

$$(P,Q)_{a,b} = ((x_1, b(x_1)),(x_2, b(x_2))) \tag{4.32}$$

is an element of $\mathcal{S}$. Define $\phi : \mathcal{S} \to \mathcal{P}$ by

$$\phi((P,Q)) = (a_{P,Q}, b_{P,Q}) \tag{4.33}$$

and $\psi : \mathcal{P} \to \mathcal{S}$ by

$$\psi(a,b) = (P,Q)_{a,b}. \tag{4.34}$$

A calculation shows that

$$\begin{aligned} \psi \circ \phi((x_1,y_1),(x_2,y_2)) &= \psi\left((x - x_1)(x - x_2), \frac{y_2 - y_1}{x_2 - x_1} x - \frac{y_2 - y_1}{x_2 - x_1} x_1 + y_1\right) \\ &= ((x_1,y_1),(x_2,y_2)), \end{aligned} \tag{4.35}$$

so the above maps are inverse to one another and we have found a bijection between the sets $\mathcal{S}$ and $\mathcal{P}$. □

Corollary 4.3.3. *Let Y_2 be a curve of genus 2 over a field k given by the equation $y^2 = f(x)$ in $\mathbb{P}_k^2$. Let g_1 and g_2 be polynomials in $k[a_1, a_2, b_1, b_2]$ such that*

$$g_1(a_1, a_2, b_1, b_2)x + g_0(a_1, a_2, b_1, b_2) \equiv \left(b(x)^2 - f(x)\right) \mod a(x). \tag{4.36}$$

Then the system of equations

$$g_1(a_1, a_2, b_1, b_2) = 0, \tag{4.37}$$

$$g_2(a_1, a_2, b_1, b_2) = 0 \tag{4.38}$$

describes an affine open subset U of $\mathrm{Jac}(Y_2)$ *in* $\mathbb{A}_k^4$.

Proof. The variety $\mathrm{Sym}^2(Y_2)$ is isomorphic to $\mathrm{Jac}(Y_2)$ after blowing down the line E consisting of points of the form $(P, i(P))$. See e.g. [23]. As $\mathcal{S} = \mathrm{Sym}^2(Y_2)\backslash E$ this implies that $\mathcal{S}$ is isomorphic to an affine open subset U of $\mathrm{Jac}(Y_2)$. By Proposition 4.3.2 the variety $\mathcal{S}$ is isomorphic to the variety $\mathcal{P}$ and the latter consists exactly of the points (a_1, a_2, b_1, b_2) satisfying $g_1 = 0$ and $g_2 = 0$. This concludes the proof. □

We will now combine this with Müller's description of the Kummer surface in [20] to give an affine equation for the Kummer surface $\mathrm{Jac}(Y_2)$ in $\mathbb{P}_k^3$ and an explicit description of the map $\mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$.

Proposition 4.3.4. *Let Y_2 be a curve of genus 2 over a field k given by the equation*

$$y^2 = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + f_6 x^6 \tag{4.39}$$

in $\mathbb{A}_k^2$. Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points on Y_2 and let $P + Q \in U \subset$ $\mathrm{Jac}(Y_2)$ *where U is as in Corollary 4.3.3. Let*

$$\begin{aligned}\kappa_1 &= 1,\\ \kappa_2 &= x_1 + x_2,\\ \kappa_3 &= x_1 x_2,\\ \kappa_4 &= \frac{F_0(x_1, x_2) - 2y_1 y_2}{(x_1 - x_2)^2},\end{aligned} \tag{4.40}$$

where

$$F_0(x, y) = 2f_0 + f_1(x+y) + 2f_2(xy) + f_3(x+y)xy + 2f_4(xy)^2 + f_5(x+y)(xy)^2 + 2f_6(xy)^3. \tag{4.41}$$

Then we can define a map $\pi : U \to$ $\mathrm{Kum}(Y_2)$*given by $(P, Q) \mapsto (\kappa_1 : \kappa_2 : \kappa_3 : \kappa_4)$ such that π is equal to the quotient morphism* $\mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ *restricted to U.*

The functions $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ satisfy the quartic equation

$$K(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = K_2(\kappa_1, \kappa_2, \kappa_3)\kappa_4^2 + K_1(\kappa_1, \kappa_2, \kappa_3)\kappa_4 + K_0(\kappa_1, \kappa_2, \kappa_3) = 0 \tag{4.42}$$

and this equation gives us a projective embedding of $\mathrm{Kum}(Y_2)$ *in* $\mathbb{P}^3_k$.

Here

$$K_2(\kappa_1,\kappa_2,\kappa_3) = \kappa_2^2 - 4\kappa_1\kappa_3 \tag{4.43}$$

$$\begin{aligned} K_1(\kappa_1,\kappa_2,\kappa_3) = &-4\kappa_1^3 f_0 - 2\kappa_1^2\kappa_2 f_1 - 4\kappa_1^2\kappa_3 f_2 - 2\kappa_1\kappa_2\kappa_3 f_3 \\ &- 4\kappa_1\kappa_3^2 f_4 - 2\kappa_2\kappa_3^2 f_5 - 4\kappa_3^3 f_6 \end{aligned} \tag{4.44}$$

$$\begin{aligned} K_0(\kappa_1,\kappa_2,\kappa_3) = &-4\kappa_1^4 f_0 f_2 + \kappa_1^4 f_1^2 - 4\kappa_1^3\kappa_2 f_0 f_3 - 2\kappa_1^3\kappa_3 f_1 f_3 \\ &- 4\kappa_1^2\kappa_2^2 f_0 f_4 + 4\kappa_1^2\kappa_2\kappa_3 f_0 f_5 - 4\kappa_1^2\kappa_2\kappa_3 f_1 f_4 - 4\kappa_1^2\kappa_3^2 f_0 f_6 \\ &+ 2\kappa_1^2\kappa_3^2 f_1 f_5 - 4\kappa_1^2\kappa_3^2 f_2 f_4 + \kappa_1^2\kappa_3^2 f_3^2 - 4\kappa_1\kappa_2^3 f_0 f_5 \\ &+ 8\kappa_1\kappa_2^2\kappa_3 f_0 f_6 - 4\kappa_1\kappa_2^2\kappa_3 f_1 f_5 + 4\kappa_1\kappa_2\kappa_3^2 f_1 f_6 \\ &- 4\kappa_1\kappa_2\kappa_3^2 f_2 f_5 - 2\kappa_1\kappa_3^3 f_3 f_5 - 4\kappa_2^4 f_0 f_6 - 4\kappa_2^3\kappa_3 f_1 f_6 \\ &- 4\kappa_2^2\kappa_3^2 f_2 f_6 - 4\kappa_2\kappa_3^3 f_3 f_6 - 4\kappa_3^4 f_4 f_6 + \kappa_3^4 f_2 \end{aligned} \tag{4.45}$$

Proof. See [20, Paragraph 2]. □

Remark 4.3.5. If we want to put a Kummer surface given by the equation in Proposition 4.3.4 into the standard form as described in Theorem 4.1.5 we might need to take a field extension.

Corollary 4.3.6. *Let* U *be an affine open subset of the Jacobian in* $\mathbb{A}^4 = k[a_1,a_2,b_1,b_2]$ *given by the system of equations* $g_1 = 0, g_2 = 0$ *as in Corollary 4.3.3. Then the map* $U \to \mathrm{Kum}(Y_2)$ *from Proposition 4.3.4 can be explicitly described by*

$$(a_1,a_2,b_1,b_2) \mapsto \left(1 : -a_1 : a_2 : \frac{\widetilde{F}_0(-a_1,a_2) - 2(b_1^2 a_2 - b_1 b_2 a_1 + b_2^2)}{a_1^2 - 4a_2}\right) \tag{4.46}$$

where

$$\widetilde{F}_0(x,y) = 2f_0 + f_1 x + 2f_2 y + f_3 xy + 2f_4 y^2 + f_5 xy^2 + 2f_6 y^3. \tag{4.47}$$

Proof. The isomorphism in 4.3.3 maps $(P,Q) = ((x_1,y_1),(x_2,y_2))$ to

$$(a(x),b(x)) = (x^2 - (x_1+x_2) + x_1x_2, \frac{y_2-y_1}{x_2-x_1}x - \frac{y_2-y_1}{x_2-x_1}x_1 + y_1). \tag{4.48}$$

This implies that $a_1 = -x_1 - x_2$ and $a_2 = x_1x_2$. As a result, $(x_1-x_2)^2 = a_1^2 - 4a_2$. Furthermore,

$$y_1y_2 = (b_1x_1 + b_2)(b_1x_2 + b_2) = (b_1^2 a_2 - b_1 b_2 a_1 + b_2^2). \tag{4.49}$$

Substituting these identities into F_0 from 4.3.4 gives us the map. □

Lemma 4.3.7. *The point* $(0:0:0:1)$ *is always a singular point on the projective embedding of* $\mathrm{Kum}(Y_2)$ *in* $\mathbb{P}^3_k$ *given by equation (4.3.4).*

Proof. A calculation shows that

$$\begin{aligned}
\frac{\partial K}{\partial \kappa_1}(0,0,0,1) &= \frac{\partial K_2}{\partial \kappa_1}(0,0,0)\cdot 1^2 + \frac{\partial K_1}{\partial \kappa_1}(0,0,0)\cdot 1 + \frac{\partial K_0}{\partial \kappa_1}(0,0,0) = 0+0+0 = 0,\\
\frac{\partial K}{\partial \kappa_2}(0,0,0,1) &= \frac{\partial K_2}{\partial \kappa_2}(0,0,0)\cdot 1^2 + \frac{\partial K_1}{\partial \kappa_2}(0,0,0)\cdot 1 + \frac{\partial K_0}{\partial \kappa_2}(0,0,0) = 0+0+0 = 0,\\
\frac{\partial K}{\partial \kappa_3}(0,0,0,1) &= \frac{\partial K_2}{\partial \kappa_3}(0,0,0)\cdot 1^2 + \frac{\partial K_1}{\partial \kappa_3}(0,0,0)\cdot 1 + \frac{\partial K_0}{\partial \kappa_3}(0,0,0) = 0+0+0 = 0,\\
\frac{\partial K}{\partial \kappa_4}(0,0,0,1) &= 2K_2(0,0,0)\cdot 1 + K_1(0,0,0) = 0+0 = 0.
\end{aligned} \tag{4.50}$$

□

Lemma 4.3.8. *Let Y_2 be a genus 2 curve over k given by the equation $y^2 = f$ in $\mathbb{P}^2_k$ and let $\pi : \mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ be the quotient map. Assume that f is gluable, i.e. $f = (x^2+ux+v)g$.*

Then $\mathrm{Sing}(\mathrm{Kum}(Y_2))$ *contains the point*

$$\pi(P) = \left(1, u, v, \frac{\widetilde{F}_0(u,v)}{u^2-4v}\right) \tag{4.51}$$

where

$$\widetilde{F}_0(u,v) = 2f_0 + f_1u + 2f_2v + f_3uv + 2f_4v^2 + f_5uv^2 + 2f_6v^3. \tag{4.52}$$

Proof. Assume that $(x^2-ux+v) = (x-\beta_5)(x-\beta_6)$ with $\beta_5, \beta_6 \in \overline{k}$. This means that $P = (\beta_5, 0) + (\beta_6, 0)$ is a 2-torsion point in $\mathrm{Jac}(Y_2)$. As $\pi(P)$ is the image of a 2-torsion point it will be a rational singular point in $\mathrm{Kum}(Y_2)$. □

Lemma 4.3.9. *The morphism $\pi : \mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ induces an inclusion of function fields $\phi : K(\mathrm{Kum}(Y_2) \to K(\mathrm{Jac}(Y_2))$. Then*

(i) There exist $\alpha_i, \beta_j \in k(a_1, a_2)$ such that

$$b_1b_2 = \alpha_1(a_1,a_2) + \alpha_2(a_1,a_2)b_1^2, \tag{4.53}$$
$$b_2^2 = \beta_1(a_1,a_2) + \beta_2(a_1,a_2)b_1^2. \tag{4.54}$$

(ii) Let

$$h = \frac{(\kappa_2^2 - 4\kappa_3)\kappa_4 - \widetilde{F}_0(\kappa_2,\kappa_3) + 2\kappa_2\alpha_1(-\kappa_2,\kappa_3) + 2\beta_1(-\kappa_2,\kappa_3)}{-2\kappa_3 - 2\kappa_2\alpha_2(-\kappa_2,\kappa_3) - 2\beta_2(-\kappa_2,\kappa_3)} \tag{4.55}$$

Proof. We will start by proving (i). Note that the polynomials g_1 and g_2 from Corollary 4.3.3 used to define $\mathrm{Jac}(Y_2)$ are elements of $k(a_1, a_2)[b_1^2, b_2^2, b_1 b_2]$. We can therefore write

$$g_1 = \lambda_0 + \lambda_{1,1} b_1^2 + \lambda_{1,2} b_1 b_2 + \lambda_{2,2} b_2^2, \tag{4.56}$$

$$g_2 = \mu_0 + \mu_{1,1} b_1^2 + \mu_{1,2} b_1 b_2 + \mu_{2,2} b_2^2 \tag{4.57}$$

with $\lambda_i, \mu_i \in k(a_1, a_2)$. Consider this as a set of linear equations with variables $1, b_1^2, b_1 b_2$ and b_2^2. Now we can use these equations to find expressions for $b_1 b_2$ and b_2^2 in terms of a_1, a_2 and b_1^2 to find $\alpha_1, \alpha_2, \beta_1, \beta_2$ with the required properties.

We now prove (ii):

We know that $\phi(\kappa_2) = -a_1, \phi(\kappa_3) = a_2$ and

$$\phi(\kappa_4) = \frac{\widetilde{F}_0(-a_1, a_2) - 2(b_1^2 a_2 - b_1 b_2 a_1 + b_2^2)}{a_1^2 - 4a_2}. \tag{4.58}$$

After substituting $b_1 b_2$ and b_2^2 for the terms calculated in (i) we get

$$\begin{aligned}
\phi(\kappa_4) &= \frac{\widetilde{F}_0(-a_1, a_2)}{a_1^2 - 4a_2} - \frac{2(b_1^2 a_2 - (\alpha_1(a_1, a_2) + \alpha_2(a_1, a_2) b_1^2) a_1 + \beta_1(a_1, a_2) + \beta_2(a_1, a_2) b_1^2)}{a_1^2 - 4a_2} \\
&= \frac{\widetilde{F}_0(\phi(\kappa_2), \phi(\kappa_3))}{\phi(\kappa_2)^2 - 4\phi(\kappa_3)} - 2\frac{(b_1^2 \phi(\kappa_3) + \alpha_1(-\phi(\kappa_2), \phi(\kappa_3))\phi(\kappa_2)}{\phi(\kappa_2)^2 - 4\phi(\kappa_3)} \\
&\quad - 2\frac{\alpha_2(-\phi(\kappa_2), \phi(\kappa_3)) b_1^2)\phi(\kappa_2) + \beta_1(-\phi(\kappa_2), \phi(\kappa_3)) + \beta_2(-\phi(\kappa_2), \phi(\kappa_3)) b_1^2)}{\phi(\kappa_2)^2 - 4\phi(\kappa_3)}.
\end{aligned} \tag{4.59}$$

It follows that

$$b_1^2 = \frac{(\phi(\kappa_2)^2 - 4\phi(\kappa_3)) \cdot \phi_4(\kappa_4) - F_0(\phi(\kappa_2), \phi(\kappa_3)) + 2(\alpha_1(-\phi(\kappa_2), \phi(\kappa_3)) + 2\beta_1(-\phi(\kappa_2), \phi(\kappa_3)))}{-2\phi(\kappa_3) - 2\alpha_2(-\phi(\kappa_2), \phi(\kappa_3)))\phi(\kappa_2) - 2\beta_2(-\phi(\kappa_2), \phi(\kappa_3))}. \tag{4.60}$$

So we can define

$$h = \frac{(\kappa_2^2 - 4\kappa_3)\kappa_4 - \widetilde{F}_0(\kappa_2, \kappa_3) + 2\kappa_2 \alpha_1(-\kappa_2, \kappa_3) + 2\beta_1(-\kappa_2, \kappa_3)}{-2\kappa_3 - 2\kappa_2 \alpha_2(-\kappa_2, \kappa_3) - 2\beta_2(-\kappa_2, \kappa_3)}. \tag{4.61}$$

□

Corollary 4.3.10. *Let π, ϕ and h be as in Lemma 4.3.9. Then we can extend ϕ to a morphism $\overline{\phi} : K(\mathrm{Kum}(Y_2))[\sqrt{h}] \to K(\mathrm{Jac}(Y_2))$ such that $\overline{\phi}$ is an isomorphism. Furthermore, let C be a curve on $\mathrm{Kum}(Y_2)$ and let $K(C)$ be the function field of C. Then $K(C)[\sqrt{h}]$ is the function field of $\pi^{-1}(C)$ in $\mathrm{Jac}(Y_2)$.*

Proof. Define $\overline{\phi}(x) = \phi(x)$ for $x \in K(\mathrm{Kum}(Y_2)$ and $\overline{\phi}(\sqrt{h}) = b_1$. It suffices to show that a_1, a_2, b_1 and b_2 are in the image of $\overline{\phi}$. We already have $\overline{\phi}(\kappa_2) = -a_1, \overline{\phi}(\kappa_3) = a_2$ and $\overline{\phi}(\sqrt{h}) = b_1$. As

$$\begin{aligned}
\overline{\phi}(\kappa_4) &= \frac{F_0(-a_1, a_2) - 2(b_1^2 a_2 - b_1 b_2 a_1 + b_2^2)}{a_1^2 - 4a_2} \\
&= \frac{F_0(-a_1, a_2) - 2(b_1^2 a_2 - b_1 b_2 a_1 + \beta_1(a_1, a_2) + \beta_2(a_1, a_2) b_1^2)}{a_1^2 - 4a_2},
\end{aligned} \tag{4.62}$$

it follows that $b_1 b_2$ and therefore b_2 are also in the image of $\overline{\phi}$, which is what we wanted to show. The statement that $K(C)[\sqrt{h}]$ is the function field of $\pi^{-1}(C)$ follows automatically from the fact that $\mathrm{Kum}(Y_2)[\sqrt{h}]$ is the function field of $\mathrm{Jac}(Y_2)$. □

Theorem 4.3.11. *If X_1, Y_2 and Z_3 are curves as in Proposition 2.2.1 and $\overline{\pi} : Z_3 \to X_1$ is the degree 2 cover in the same proposition then there exist injective rational maps $j_3 : Z_3 \to \mathrm{Jac}(Y_2)$ and $j_1 : X_1 \to \mathrm{Kum}(Y_2)$ such that $j_1(X_1)$ is the intersection of $\mathrm{Kum}(Y_2)$ with a plane that passes through two singular points. We get the following commutative diagram:*

$$\begin{array}{ccc} Z_3 & \xrightarrow{j_3} & \mathrm{Jac}(Y_2) \\ \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \pi} \\ X_1 & \xrightarrow{j_1} & \mathrm{Kum}(Y_2). \end{array} \tag{4.63}$$

Proof. In the code in https://github.com/JRSijsling/gen3deg2prym Sijsling constructs an explicit rational map $Z_3 \dashrightarrow \mathrm{Jac}(Y_2)$. After a change of coordinates we may assume that Z_3 has an affine open V of the form

$$v^4 + v^2 g(u) + u h(u) \tag{4.64}$$

where $g(u) = g_2 u^2 + g_1 u + g_0$ and $h(u) = h_2 u^2 + h_1 u + h_0$. We calculate an equation for Y_2 using Theorem 2.2.1 and use this equation to construct the affine open $U \subset \overline{k}[a_1, a_2, b_1, b_2]$ of the Jacobian $\mathrm{Jac}(Y_2)$ given by the equations in Corollary 4.3.3.

Let

$$\begin{aligned} \alpha(u,v) &= (g_2 h_0 - g_0 h_2) v^2 + (g_2^2 h_0 - g_2 g_0 h_2) u^2 + (g_2 g_1 h_0 - g_2 g_0 h_1 - h_2 h_0) u, \\ \beta(u,v) &= g_2^2 h_0 - g_2 g_0 h_2 v^3 + (g_2^3 h_0 - g_2^2 g_0 h_2) u^2 \\ &\quad + ((g_2^2 g_1 h_0 - g_2 g_1 g_0 h_2 - g_2 h_2 h_0 + g_0 h_2^2) u + g_2^2 g_0 h_0 - g_2 g_0^2 h_2) v \\ N(u,v) &= (g_2^2 h_1 - g_2 g_1 h_2 + h_2^2) u + g_2^2 h_0 - g_2 g_0 h_2. \end{aligned} \tag{4.65}$$

Then the map $j_3 : V \to U$ is explicitly given by

$$(u,v) \mapsto (\alpha(u,v)/N(u,v), 0, \beta(u,v)/(N(u,v)), \beta(u,v)/(uN(u,v))). \tag{4.66}$$

In the code it is shown that the image of j_3 is contained in U and that the map given by α generically has degree 4. To show that j_3 is generically injective it suffices to show that it is injective in one single case. Indeed, if we can prove that j_3 is injective in a single point for a single curve for which the degree of α is maximal, it will be injective on an open subset, so it will be generically true. More precisely, it suffices to show that there exist g, h and distinct points $P_1, \dots P_4$ such that $\alpha(P_i) = \alpha(P_j)$ for all i and j, but $\beta(P_i) \neq \beta(P_j)$ when $i \neq j$. We consider the case where $g(u) = u^2 - 4u - 6$ and $h(u) = u^2 - 4u - 5$ over $\mathbb{Q}$. Define

$$\begin{aligned} &P_1 = (3/2 : -5/2 : 1), \quad P_2 = (6/5 : -8/5 : 1), \\ &P_3 = (-6/5 : -8/5 : 1), \quad P_4 = (-3/2 : -5/2 : 1). \end{aligned} \tag{4.67}$$

Then a calculation shows that

$$\begin{aligned} &j_3(P_1) = (4, 0, -15, 6), \quad j_3(P_1) = (4, 0, -12, 15/2), \\ &j_3(P_1) = (4, 0, 12, -15/2), \quad j_3(P_1) = (4, 0, 15, -6) \end{aligned} \tag{4.68}$$

which shows that there exist four distinct points with $\alpha(u,v)/N(u,v) = 4$. We conclude that j_3 is generically injective.

Let $\pi : U \to \mathrm{Kum}(Y_2)$ be the map given in Corollary 4.3.6. As $j_3(Z_3)$ is contained in the plane given by $a_2 = 0$, it follows that $\pi(j_3(Z_3))$ is a curve contained in the plane H defined by $\kappa_3 = 0$. This means we have a rational map $Z_3 \dashrightarrow \pi(j_3(Z_3))$ of degree 2. We claim that the curve $\pi(j_3(Z_3))$ is of genus 1. Indeed, if $\pi(j(Z_3))$ is not of genus 1 then it will either be of genus 2 or of genus 0. But then Z_3 is a hyperelliptic curve. Indeed, if $j(Z_3)$ has genus 2 then Proposition 2.1.4 tells us that Z_3 is a hyperelliptic curve. In the second case we have a degree 2 cover from Z_3 to a genus 0 curve, so the statement follows by definition. As Z_3 is non-hyperelliptic by assumption this leads to a contradiction. So we conclude that $\pi(j(Z_3))$ is of genus 1. As any plane section of a quartic surface in $\mathbb{P}^3$ has arithmetic genus 3 this means that the plane H has to intersect $\mathrm{Kum}(Y_2)$ in two singular points. Finally, it remains to be shown that the above diagram commutes. Let $i : Z_3 \to Z_3$ be the involution $(u,v) \mapsto (u, -v)$ that corresponds to the degree 2 cover $Z_3 \to X_1$. Then

$$\begin{aligned} j_3(i(u,v)) &= j_3((u,-v)) \\ &= (\alpha(u,-v)/N(u,-v), 0, \beta(u,-v)/(N(u,-v)), \beta(u,-v)/(-vN(u,-v))) \\ &= (\alpha(u,v)/N(u,v), 0, -\beta(u,v)/(N(u,v)), -\beta(u,v)/(uN(u,v))). \end{aligned} \tag{4.69}$$

Now the map $(a_1, a_2, b_1, b_2) \mapsto (a_1, a_2, -b_1, -b_2)$ sends the divisor $P + Q$ corresponding to the equations $x^2 + a_1 x + a_2$ and $y = b_1 x + b_2$ to the divisor $P' + Q'$ corresponding to the equations $x^2 + a_1 x + a_2$ and $y = -b_1 x - b_2$. We see that $j_3 \circ (i)$ is multiplication by -1 on $\mathrm{Jac}(Y_2)$ and we conclude that we have found a commutative diagram as in (4.63).

□

Remark 4.3.12. Concretely this means that every (2,2)-gluing Z_3 of X_1 and Y_2 that can be written as a degree 2 cover as in Theorem 2.2.1 occurs as the desingularization of the pullback of a plane section of $\mathrm{Jac}(Y_2)$.

Remark 4.3.13. A more elegant prove for the generic injectivity of the map $j_3 : Z_3 \to \mathrm{Jac}(Y_2)$ due to D. Lombardo is the following: Assume that j_3 is not injective. If $j(Z_3)$ is of genus 2, then Z_3 would be hyperelliptic , which gives us a contradiction, so $j_3(Z_3)$ is either of genus 1 or of genus 0 because if $j_3(Z_3)$ is of genus 2, then Z_3 would be hyperelliptic. It is impossible for $j_3(Z_3)$ to be of genus 0 as the map j_3 is not constant. On the other hand, if $j_3(Z_3)$ is a curve of genus 1 then $\mathrm{Jac}(Y_2)$ would be isogenous to the product of two elliptic curves, which cannot be true generically.

4.4 Degree 2 covers

Proposition 4.4.1. *Let C be a genus 1 curve over an algebraically closed field k with $\mathrm{char}(k) \neq 2$, and let P_1, P_2, P_3, P_4 be distinct points in C. Then there are exactly four distinct covers of degree 2 that are ramified above the P_i and unramified everywhere else.*

Proof. We define the divisor Q by

$$Q \cong \frac{P_1 + P_2 + P_3 + P_4}{4} \tag{4.70}$$

in $\mathrm{Pic}(C)$. Then $P_1 + P_2 + P_3 + P_4 - 4Q$ is a principal divisor. Let f be a function such that

$$\mathrm{div}(f) = P_1 + P_2 + P_3 + P_4 - 4Q. \tag{4.71}$$

Let X be the curve with function field $K(C)[\sqrt{f}]$. Then the natural inclusion $K(C) \to K(X)$ is a field extension of degree 2 that is ramified exactly above the P_i. It follows that this field extension induces a cover

$$D \to C \tag{4.72}$$

of degree 2 that is ramified exactly above the P_i.

Now assume that $\pi' : X' \to C$ is another cover of degree 2 that is ramified over the P_i and unramified outside of the P_i. Then $K(X') = K(C)[\sqrt{f'}]$ for some function f' with

$$\mathrm{div}(f') = P_1 + P_2 + P_3 + P_4 - 2D \tag{4.73}$$

where D is a divisor of degree 2. We get that $\mathrm{div}(f/f') = 2D - 4Q$. Write $D \cong 2Q + D_0$ for a divisor D_0 of degree 0.

In $\mathrm{Pic}(C)$ we have

$$P_1 + P_2 + P_3 + P_4 - 2D \cong P_1 + P_2 + P_3 + P_4 - 4Q + 2D_0. \tag{4.74}$$

As $P_1 + P_2 + P_3 + P_4 - 4Q$ is principal this implies that $2D_0$ is a principal divisor, so $D_0 \in \mathrm{Pic}(C)[2]$. Now let $K(C)[\sqrt{f'}]$ and $K(C)[\sqrt{f''}]$ be the function fields of two degree 2 covers of C that are ramified in P_i and unramified outside the P_i. Assume that

$$\begin{aligned} \operatorname{div}(f') &= P_1 + P_2 + P_3 + P_4 - 4Q + 2D_0', \\ \operatorname{div}(f'') &= P_1 + P_2 + P_3 + P_4 - 4Q + 2D_0'' \end{aligned} \tag{4.75}$$

with $D_0', D_0'' \in \operatorname{Pic}(C)[2]$. Now $K(C)[\sqrt{f'}]$ and $K(C)[\sqrt{f''}]$ define the same cover if and only if f/f' is a square in $K(C)$. We will show that f/f' is a square if and only if $D_0' - D_0''$ is principal. If $f/f' = g^2$. Then $\operatorname{div}(g) = D_0' - D_0''$. On the other hand if there exists a function g such that $\operatorname{div}(g) = D_0' - D_0''$ then $f/f' = g^2$. It follows that the two covers are distinct if and only if $D_0' - D_0'' \not\cong 0$. As $D_0' - D_0'' \in \operatorname{Pic}(C)[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ there are at most four distinct covers that are ramified above the P_i. Let D_T be the divisor

$$P_1 + P_2 + P_3 + P_4 - 4Q + 2T \tag{4.76}$$

with $T \in \operatorname{Pic}(C)[2]$. There exists a function f_T such that $\operatorname{div}(f_T) = D_T$ as D_T is principal. Now we can construct the cover corresponding to the inclusion $K(C) \to K(C)[\sqrt{f_T}]$ for every $T \in \operatorname{Pic}(C)[2]$, so there also exist four distinct covers.

$\square$

Corollary 4.4.2. *Let C be a curve of genus 1 over $\overline{k}$ given in homogeneous coordinates by the equation*

$$C : y^2 - h(x,z)y + f(x,z) = 0 \tag{4.77}$$

in $\mathbb{P}_k^{1,2,1}$ where h is a homogeneous polynomial of degree 2 and f is a homogeneous polynomial of degree 4 with four distinct roots. Let P_1, P_2, P_3, P_4 be the points on C that intersect with the line $y = 0$. Write $p(x,z) = f(x,z) - 1/4h(x,z)^2$ and assume that

$$p(x,z) = (x - \alpha_1 z)(x - \alpha_2 z)(x - \alpha_3 z)(x - \alpha_4 z). \tag{4.78}$$

Define

$$T_j = (\alpha_j : \frac{h(\alpha_j, 1)}{2} : 1). \tag{4.79}$$

We let

$$v_j = \frac{y}{(x - \alpha_1 z)(x - \alpha_j z)}, \qquad L_j = K(E)(\sqrt{v_j}) \tag{4.80}$$

for $j = 1, 2, 3, 4$. Let X_j be the curve with function field L_j. Then the X_j are distinct and they come equipped with a degree 2 cover $\pi_j : X_j \to C$ that is ramified over the P_i and unramified outside of the P_i.

Proof. We will calculate $\mathrm{div}(v_j)$. The zeroes of the function $\frac{y}{(x-\alpha_1 z)(x-\alpha_j z)}$ are given by the points P on C for which $y(P) = 0$. By assumption these points are P_1, P_2, P_3 and P_4. The poles of $\frac{y}{(x-\alpha_1 z)(x-\alpha_j z)}$ are given by the points P for which $(x-\alpha_1 z)(x-\alpha_j z) = 0$. Assume that $P = (x : y : 1)$. As P is a zero of $x - \alpha_k z$ for $k = 1, j$ this implies that $x = \alpha_k$. Filling in P in the equation for C gives us $(y - \frac{h(\alpha_k,1)}{2})^2 = 0$. This implies that v_j has a double pole in T_1 and a double pole in T_j. We claim that the function v_j has no other poles. Indeed, a calculation shows that if we assume that $P = (x : y : 0)$ is a zero of $(x - \alpha_k z)$ then both x and y have to be 0, which is impossible. It follows that

$$\mathrm{div}(v_j) = \mathrm{div}\left(\frac{y}{(x-\alpha_1 z)^2}\right) = P_1 + P_2 + P_3 + P_4 - 2T_1 - 2T_j. \tag{4.81}$$

Now note that we have

$$\mathrm{div}\left(\frac{x-\alpha_k z}{x-\alpha_j z}\right) = 2T_k - 2T_j. \tag{4.82}$$

for $k, j \in 1,2,3,4$. As $T_k - T_j \not\cong 0$ when $k \neq j$ this implies that $T_k - T_j$ is a 2-torsion point for all k, j. So $\mathrm{div}(v_j) - \mathrm{div}(v_k) = 2T_k - 2T_j = 2T$ where $T \in \mathrm{Pic}(C)[2]$. As we have seen in the proof of Proposition 4.4.1 this means that the fields L_j and L_k are distinct for $j \neq k$, so the field extensions $K(E) \to L_j$ correspond to four distinct covers of degree 2 that are ramified above the P_i and are unramified outside of the P_i. □

Corollary 4.4.3. *Let $\pi_i : C(L_i) \to E$ be the degree 2 cover over $\overline{k}$ as in Corollary 4.4.2. There exist functions u, v in the Riemann-Roch space $L(4T_1 + 4T_i)$ such that*

(i) The curve C has an equation of the form

$$v^2 + vh(u) + f(u) = 0 \tag{4.83}$$

where h is a polynomial of degree 2 and f is a polynomial of degree 4.

(ii) The curve X_i has an equation over $\overline{k}$ of the form

$$t^4 + t^2 h(s) + f(s) = 0 \tag{4.84}$$

where h is a polynomial of degree 2 and f is a polynomial of degree 4.

(iii) The cover $\pi_i : X_i \to C$ is explicitly given by $\pi_i(s,t) = (s, t^2)$.

Proof. By the theorem of Riemann-Roch we have $l(D) = \deg(D)$ for all divisors D with $\deg(D) \geq 1$. This implies that $l(T_1 + T_2) = 2$ and we can find a basis

$1, u \in L(T_1 + T_2)$. We also have that $l(2T_1 + 2T_i) = 4$ and we claim that the functions

$$1, u, u^2 \text{ and } v = \frac{y}{(x - \alpha_1 z)(x - \alpha_i z)} \tag{4.85}$$

form a basis of $L(2T_1 + 2T_i)$. It is clear that $1, u$ and u^2 are linearly independent. Now if $v = a_2 u^2 + a_1 u + a_0$, we could substitute $u' = au + b$ for a suitable choice of a and b to ensure that $v = u'^2$. This cannot occur however as v has four distinct zeroes. So, $1, u, u^2$ and v are linearly independent.

Similarly we have $l(4T_1 + 4T_i) = 8$, so there exists a linear dependence between the functions

$$1, u, u^2, u^3, u^4, v, uv, u^2 v \text{ and } v^2. \tag{4.86}$$

As the divisor $l(4T_1 + 4T_2)$ is very ample the linear dependence between these functions in the Riemann-Roch space gives us an equation of the form

$$v^2 + v(b_0 u + b_1) = a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0 \tag{4.87}$$

for the curve C where $a_0, \ldots a_4, b_0, b_1 \in \overline{k}$ in $\mathbb{A}^2_{\overline{k}}$. Consider the curve X_i in $\mathbb{A}^2_{\overline{k}}$ given by

$$t^4 + t^2(b_0 s + b_1) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0. \tag{4.88}$$

Then the map $\pi_i : X_i \to C$ given by $(s,t) \mapsto (s, t^2)$ corresponds to the natural inclusion of function fields $K(C) \to K(C)(\sqrt{v_i}) = K(C)[z]/(z^2 - v_i) \cong K(X_i)$. As

$$\operatorname{div}(v) = P_1 + P_2 + P_3 + P_4 - 2T_1 - 2T_i \tag{4.89}$$

this is exactly the cover from Corollary 4.4.2. □

Remark 4.4.4. In [9, Theorem 1.1] it was already stated that there exist exactly four distinct covers with a given ramification locus in the case where the base field is $\mathbb{C}$. They furthermore give a similar equation for these types of covers.

4.5 An algorithm for algebraic gluing

Let X_1 be a genus 1 curve over $\overline{k}$, and let Y_2 be a genus 2 curve over $\overline{k}$. In this section, we will combine the above results to describe an algorithm to construct all possible (2,2)-gluings of X_1 and Y_2 over $\overline{k}$. The algorithm has been implemented in Magma and is available on [12].

Theorem 4.5.1. *Let X_1 be a curve of genus 1 over k and let Y_2 be a curve of genus 2 over k. Then:*

(i) Every (2,2)-gluing of X_1 and Y_2 that is a non-hyperelliptic curve over $\overline{k}$ can be found using Algorithm 3.

(ii) *Generically there is a bijection between the indecomposable maximal isotropic subgroups of* $\mathrm{Jac}(X_1)[2] \times \mathrm{Jac}(Y_2)[2]$ *and tuples* $(P_i, (\lambda, \mu))$ *where*

$$P_i \in \mathrm{Sing}(\mathrm{Kum}(Y_2)) \backslash \{1 : 0 : 0 : 0)\} \tag{4.90}$$

and (λ, μ) *is a solution pair for* $j(X_1)$.

Proof. Any non-hyperelliptic genus 3 curve Z_3 admits a quartic equation in $\mathbb{P}^2_k$. If it is the (2,2)-gluing of a genus 1 curve X_1 and a genus 2 curve Y_2, we get a double cover $\pi_1 : Z_3 \to X_1$ as in Theorem 4.2.2. After a change of coordinates we may assume that Z_3 and X_1 are given as in (2.43) and (2.45). By Remark 2.2.5 we may also assume that Y_2 is given as in (2.48). For any such double cover $\pi : Z_3 \to X_1$ we can use Theorem 4.3.11 to find an embedding $i : Z_3 \to \mathrm{Jac}(Y_2)$ such that $p(i(Z_3)$ is the intersection of a plane with $\mathrm{Kum}(Y_2)$ passing through two singular points and such that the desingularization of the singular cover $i(Z_3) \to p(i(Z_3))$ is $\pi : Z_3 \to X_1$. We conclude that all possible (2,2)-gluings can be constructed using Algorithm 3.

Corollary 4.1.10 says that any embedding of one the above double covers can be mapped isomorphically to one that goes through P_1. This gives us 15 distinct choices for tuples of the form (P_1, P_i) with $i \neq 1$. For each of these tuples we consider the family $H_{P_i}(\lambda)$ and look for λ with the property that $H_{P_i}(\lambda) = j(X_1)$. According to Theorem 4.2.8 this polynomial will generically be of degree 12. Using Lemma 4.2.9 we see that this generically give us 6 solution pairs. This implies that the Algorithm will generically give us 90 curves that correspond to the 90 indecomposable maximal isotropic subgroups. □

Remark 4.5.2. In Step 7 of Algorithm 3 we need to calculate the value of $j(H_{P_i}(\lambda))$. We can try to calculate $j(H_{P_i}(\lambda))$ in Magma by considering $H_{P_i}(\lambda)$ as a curve E over $k(\lambda)$, turning E into an elliptic curve by choosing a rational point and calculating the j-invariant of E. But in the implementation we calculate $j(\lambda)$ in the same way as described in Theorem 4.2.8 as this turns out to be considerably faster.

Calculating the curve Z_3 in Magma by taking the pullback of $\widetilde{X}_1(\lambda_0)$ takes rather long in practice. We therefore use the following (faster) algorithm that uses the methods discussed in Paragraph 4.4.

Algorithm 3 Calculate (2,2)-gluings of X_1 and Y_2

Input: X_1 and Y_2

1: Initialize an empty list L.
2: Calculate an affine model for $\mathrm{Jac}(Y_2)$ as in Corollary 4.3.3.
3: Calculate $j(X_1)$.
4: Calculate a model for $\mathrm{Kum}(Y_2)$ and the projection map $\mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ as in Proposition 4.3.4.
5: Calculate a function h with the property that $K(\mathrm{Kum}(Y_2))[\sqrt{h}] \cong K(\mathrm{Jac}(Y_2))$ as in Lemma 4.3.9.
6: **for** $P_i \in \mathrm{Sing}(\mathrm{Kum}(Y_2)) \backslash \{(1:0:0:0)\}$ **do**
7:   Calculate the 1-dimensional family $H_{1,i}(\lambda)$ of planes that pass through $(1:0:0:0)$ and P_i.
8:   Calculate the set $\Lambda(X_1)$ of all λ such that $j(H_{P_i}(\lambda)) = j(X_1)$.
9:   **for** $\lambda_0 \in \Lambda(X_1)$ **do**
10:     Determine the singular genus 1 curve $\widetilde{X}_1(\lambda_0) = H_{1,i}(\lambda_0) \cap \mathrm{Kum}(Y_2)$
11:     Calculate the curve Z_3 with function field $K(\widetilde{X}_1(\lambda_0))[\sqrt{h}]$ using Algorithm 4.
11:     Add the gluing Z_3 to L.
12:   **end for**
13:   **return** L.
14: **end for**

Remark 4.5.3. In Step 7 we use the condition that $D_i - D = 2T$ for some principal divisor T is equivalent to stating that the function fields of the corresponding covers are identical. This was also used in the proof of Proposition 4.4.1.

Remark 4.5.4. Algorithm 3 and Algorithm 4 do not work purely over the base field. In general one needs to take field extensions to construct the glued curve Z_3.

Algorithm 4 Calculate an equation for the curve Z_3 with function field $K(\widetilde{X}_1(\lambda_0))[\sqrt{h}]$

Input: Equation for the curve $\widetilde{X}_1(\lambda_0)$ and h as in Algorithm 3.

1: Use Lemma 4.2.6 to compute the branch points $\alpha_1, \dots \alpha_4$ of the map $g : \widetilde{X}_1(\lambda_0) \to \mathbb{P}^1_k$ that maps a point P to the slope of the line that connects P with the singular point $(1:0:0:0)$.
2: Define the curve C by the equation $y^2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$ and compute a birational map $\tau : C \to \widetilde{X}_1(\lambda_0)$.
3: Let Q, R be the singular points on $\widetilde{X}_1(\lambda_0)$ and calculate $\tau^{-1}(Q) = \{Q_1, Q_2\}$ and $\tau^{-1}(R) = \{R_1, R_2\}$.
4: Calculate the divisor D of the image of h in $K(C)$.
5: Find divisors D_1, D_2, D_3, D_4 that correspond to the four distinct degree 2 covers with ramification points Q_1, Q_2, R_1, R_2 as in Corollary 4.4.2.
6: **for** $i \in \{1, \dots, 4\}$ **do**
7:   **if** there exists a principal divisor T such that $D_i - D = 2T$ **then**
8:     Calculate the degree 2 cover $X_3 \to X_1$ corresponding to D_i as in Corollary 4.4.3.
9:     **return** A quartic equation of X_3 with a degree 2 cover to X_1.
10:   **end if**
11: **end for**

4.6 Examples

An example over $\mathbb{Q}$

Let X_1 be the curve given by

$$y^2 = x^4 + 2x^3 - x^2 - 2x \tag{4.91}$$

and let Y_2 be the curve given by

$$y^2 = x^6 - 2x^5 - 10x^4 + 20x^3 + 9x^2 - 18x \tag{4.92}$$

over $\mathbb{Q}$. Using the algorithm we find that an affine open of $\mathrm{Jac}(Y_2)$ is given by the following system of equations in $\mathbb{Q}[a_1, a_2, a_3, a_4]$.

$$\begin{aligned} -a_1^4 a_2 - 2a_1^3 a_2 + 3a_1^2 a_2^2 + 10a_1^2 a_2 + 4a_1 a_2^2 + 20a_1 a_2 - a_2^3 - 10a_2^2 \\ + a_2 b_1^2 - 9a_2 - b_2^2 = 0, \\ -a_1^5 - 2a_1^4 + 4a_1^3 a_2 + 10a_1^3 + 6a_1^2 a_2 + 20a_1^2 - 3a_1 a_2^2 - 20a_1 a_2 \\ + a_1 b_1^2 - 9a_1 - 2a_2^2 - 20a_2 - 2b_1 b_2 - 18 = 0. \end{aligned} \tag{4.93}$$

The projective equation for $\mathrm{Kum}(Y_2)$ in $\mathbb{P}^2_{\mathbb{Q}}$ is given by

$$\begin{aligned} &324x_1^4 + 720x_1^3x_3 - 720x_1^2x_2x_3 - 144x_1x_2^2x_3 + 72x_2^3x_3 + 832x_1^2x_3^2 \\ &\quad - 36x_2^2x_3^2 + 80x_1x_3^3 - 80x_2x_3^3 + 44x_3^4 + 36x_1^2x_2x_4 - 36x_1^2x_3x_4 \\ &\quad - 40x_1x_2x_3x_4 + 40x_1x_3^2x_4 + 4x_2x_3^2x_4 - 4x_3^3x_4 + x_2^2x_4^2 \\ &\quad - 4x_1x_3x_4^2 = 0 \end{aligned} \tag{4.94}$$

and the morphism $\pi : \mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ is explicitly given by

$$\pi(a_1, a_2, b_1, b_2) = \left[1 : -a_1 : a_2 : \frac{2a_1a_2^2 - 20a_1a_2 + 2a_1b_1b_2 + 18a_1 + 2a_2^3 - 20a_2^2 - 2a_2b_1^2 + 18a_2 - 2b_2^2}{a_1^2 - 4a_2}\right]. \tag{4.95}$$

The 16 singular points of $\mathrm{Kum}(Y_2)$ are

$$\begin{aligned} &(1:0:0:0), \quad (-1/6:1/3:1/2:1), \quad (-1/6:-1/2:0:1), \\ &(-1/9:-2/9:0:1), \quad (-1/10:0:1/10:1), \quad (-1/18:-1/18:0:1), \\ &(-1/22:-1/22:1/11:1), \quad (-1/30:-1/6:-1/5:1), \\ &(-1/30:-1/15:1/10:1), (-1/42:1/42:1/7:1), \\ &(-1/42:2/21:-1/14:1), \quad (-1/90:0:1/10:1), \\ &(1/18:-1/18:0:1), \quad (1/14:3/14:1/7:1), \\ &(1/6:-1/2:0:1), \quad \text{and} \quad (1/6:2/3:1/2:1). \end{aligned} \tag{4.96}$$

We consider the family of planes $H_{1,2}(\lambda)$ passing through $P = (0:0:0:1)$ and $Q = (-1/6:1/3:1/2:1)$. After calculating an equation for the family $H_{P,Q}(\lambda)$ and substituting this into $\mathrm{Kum}(Y_2)$ we get the following affine equation for $H_{P,Q}(\lambda) \cap \mathrm{Kum}(Y_2)$:

$$\begin{aligned} &\lambda^4x^4 + (-8/3\lambda^4 + 8/9\lambda^3 - 20/9\lambda^2 - 4/9\lambda + 2/9)x^3y + 1/9\lambda^2x^3 \\ &\quad + (8/3\lambda^4 - 16/9\lambda^3 + 322/81\lambda^2 + 16/9\lambda + 1/27)x^2y^2 \\ &\quad + (-7/27\lambda - 16/81)\lambda x^2y + 1/324x^2 \\ &\quad + (-32/27\lambda^4 + 32/27\lambda^3 - 568/243\lambda^2 - 452/243\lambda - 40/81)xy^3 \\ &\quad + (16/81\lambda^2 + 80/243\lambda + 16/243)xy^2 - 1/81\lambda xy \\ &\quad + (16/81\lambda^4 - 64/243\lambda^3 + 328/729\lambda^2 + 424/729\lambda + 196/729)y^4 \\ &\quad + (-4/81\lambda^2 - 32/243\lambda - 16/243)y^3 + (2/243\lambda + 1/243)y^2 = 0 \end{aligned} \tag{4.97}$$

Considering this as an elliptic curve over $\mathbb{Q}(\lambda)$ we calculate the j-invariant of this curve:

$$j(\lambda) = \frac{N(\lambda)}{D(\lambda)} \tag{4.98}$$

where

$$N(\lambda) = \frac{65597103937}{63504}\lambda^{12} - \frac{16021112665}{5292}\lambda^{11} + \frac{40655765575}{10584}\lambda^{10} - \frac{43725731107}{15876}\lambda^{9} + \frac{26001691661}{21168}\lambda^{8} - \frac{947478545}{2646}\lambda^{7} + \frac{1139864011}{15876}\lambda^{6} - \frac{9439439}{882}\lambda^{5} + \frac{2922317}{2352}\lambda^{4} - \frac{62653}{588}\lambda^{3} + \frac{2901}{392}\lambda^{2} - \frac{9}{28}\lambda + \frac{9}{784} \tag{4.99}$$

and

$$D(\lambda) = \lambda^{12} - \frac{125}{21}\lambda^{11} + \frac{12937}{1764}\lambda^{10} + \frac{4058}{441}\lambda^{9} - \frac{6583}{441}\lambda^{8} + \frac{620}{441}\lambda^{7} + \frac{5515}{882}\lambda^{6} - \frac{1714}{441}\lambda^{5} + \frac{436}{441}\lambda^{4} - \frac{17}{147}\lambda^{3} + \frac{1}{196}\lambda^{2} \tag{4.100}$$

The j-invariant of X_1 is 35152/9. The numerator of $j(\lambda) - 35152/9$ factors as

$$(\lambda - 9/23)(\lambda - 1/11)(\lambda^2 - 38/67\lambda - 9/67)(\lambda^2 - 98/193\lambda - 3/193)\cdot (\lambda^2 - 42/85\lambda + 1/85)(\lambda^2 - 22/47\lambda + 3/47)(\lambda^2 - 2/5\lambda + 1/5). \tag{4.101}$$

A calculation shows that the roots of the linear factors form a solution pair for $j(X_1)$ (and so does every pair of roots of any of the quadratic factors).

We will construct the degree 2 cover above $\widetilde{X}_1(9/23) = H_{P,Q}(9/23) \cap \mathrm{Kum}(Y_2)$. The singular elliptic curve E is given by the equation

$$x^4 - \frac{758638}{59049}x^3y + \frac{529}{729}x^3 + \frac{9802687}{177147}x^2y^2 - \frac{294653}{59049}x^2y + \frac{279841}{2125764}x^2 - \frac{34825060}{531441}xy^3 + \frac{15294448}{1594323}xy^2 - \frac{12167}{59049}xy + \frac{113058100}{4782969}y^4 - \frac{8495740}{1594323}y^3 + \frac{498847}{1594323}y^2 = 0 \tag{4.102}$$

in $\mathbb{A}^2_{\mathbb{Q}}$.

A calculation gives us that image of the function h mentioned in Step 4 in Algorithm 3 in the function field of $\widetilde{X}_1(9/23)$ is given by

$$\frac{-108505270}{718449183}x^3y - \frac{83441}{8869743}x^3 + \frac{6042040789}{2155347549}x^2y^2 + \frac{146387545}{718449183}x^2y - \frac{13367}{1062882}x^2 - \frac{4203449668}{6466042647}xy^3 - \frac{9737297495}{19398127941}xy^2 + \frac{26734}{1358127}xy - \frac{45573331130}{58194383823}y^4 + \frac{5805288791}{19398127941}y^3 - \frac{1096094}{36669429}y^2. \tag{4.103}$$

To compute $K(\widetilde{X}_1(9/23))/(t^2 - h)$ we proceed as in Algorithm 4. The branch points of the degree 2 map $g : \widetilde{X}_1(9/23) \to \mathbb{Q}$ are

$$(-115/132, -23/220), \quad (-529/1458, 0), \quad (-95/714, 5/119), \text{ and } (-287/10974, 164/1829). \tag{4.104}$$

We can use this to calculate a curve $\widetilde{X}_{1,\mathrm{leg}}$ with an equation in Legendre form that is isomorphic to $\widetilde{X}_1(9/23)$). The curve $\widetilde{X}_{1,\mathrm{leg}}$ is given by the equation:

$$y^2 = x^3 - 5/4x^2 + 1/4x.$$

The isomorphism between $\widetilde{X}_1(9/23)$) and $\widetilde{X}_{1,\mathrm{leg}}$ is defined over a quadratic extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$. Here α is a root of

$$t^2 - \frac{1560266582250435577102214 01}{3430827991390870996885220 8000}. \tag{4.105}$$

Using this, the image of the function h in the function field of $K(\widetilde{X}_{1,\mathrm{leg}}) \otimes \mathbb{Q}(\alpha)$ can be calculated explicitly and is given by a rational function of degree 14 with rather large coefficients. We will call it h_{leg}.

Let P_1, P_2 be the two points you get when you desingularize P and let Q_1, Q_2 be the two points that come from desingularizing Q. It turns out the divisor $P_1+P_2+Q_1+Q_2$ is defined over the field $\mathbb{Q}(\beta,\gamma)$ where β is a root of $t^2+3/32$ and γ is a root of $t^2 - 327/250t + 4761/10000$. We now fix choose 2-torsion points $T_1, \dots T_4$ and calculate functions f_i with $\operatorname{div} f_i = P_1 + P_2 + Q_1 + Q_2 - 2T_i - 2T_1$. Checking if $\operatorname{div}(f_i/h_{\mathrm{leg}})$ is a square for all i will give us the right function f_i. Using Riemann-Roch as in Corollary 4.4.3 gives us the equation

$$\begin{aligned} u^4 &- \frac{244312307247680}{12491063134299}\alpha u^3 + \left(\frac{286830015625}{36438849216}\beta\gamma - \frac{250115773625}{48585132288}\beta\right)u^2v^2 \\ &+ \frac{5876}{8855}u^2 + \left(\frac{-5050078616774562500 0}{1338579798660883737}\alpha\beta\gamma\right. \\ &\left.+ \frac{1100917138456854625 0}{446193266220294579}\alpha\beta\right)uv^2 - \frac{83804221642880}{37473189402897}\alpha u \\ &- \frac{1044509681265625}{171408346712064}v^4 + \left(\frac{52518171875}{63767986128}\beta\gamma - \frac{45795845875}{85023981504}\beta\right)v^2 \\ &+ \frac{1460}{111573} = 0. \end{aligned} \tag{4.106}$$

for the curve Z_3 over $\mathbb{Q}(\alpha,\beta,\gamma)$ that is a (2,2)-gluing of X_1 and Y_2. Calculating its Dixmier-Ohno invariants $I_3, I_6, I_9, J_9, I_{12}, J_{12}, I_{15}, J_{15}, I_{18}, J_{18}, I_{21}, J_{21}$ and I_{27} normalized in their respective projective spaces gives us:

$$\begin{aligned}
&[1, -\frac{4710901289284}{3628465988415}, -\frac{3357064198133902069 1035}{107075523332888 2719}, \\
&-\frac{170253359117663776 3761}{118972803703209191}, -\frac{265903004894909499810 5841500}{17554353905445923885 6163}, \\
&-\frac{30132846826773454489723 8663629}{15798918514901331497 05467}, \frac{1116844499003104295234870156025513200}{77703978921024367715278954 0077}, \\
&\frac{2927022757173291765113456671013195312}{6475331576752030642939912833975}, \\
&\frac{1368739358122733014115458337930845109478 28000}{1146516769701190073933200896856832961}, \\
&\frac{20809640572620673758718938657804157183 07264}{424635840630070397753037369206 23443}, \\
&\frac{39488514974778161321445251409062667639936987839 6756}{16916774680767180457579203909059790361 24773}, \\
&\frac{2890807348027939553764130322607710887083661031921 7768}{281946244679453007626320065150996506020 7955}, \\
&\frac{4771483514821686208217895701735808004026618537717273264384000000}{168400388375428624223601640850577404681639052255 4871}].
\end{aligned} \tag{4.107}$$

A simplified equation of this curve over $\mathbb{Q}$ is

$$\begin{aligned}
&12x^4 - 111x^2y^2 + 478x^2yz - 577x^2z^2 - 533y^4 + 948y^3z \\
&\quad - 2574y^2z^2 + 2196yz^3 - 2277z^4 = 0.
\end{aligned} \tag{4.108}$$

An example over $\mathbb{F}_{19}$

Let X_1 be the curve given by

$$y^2 = x^4 + 9x^3 + 4x^2 + 15x \tag{4.109}$$

and let Y_2 be the curve given by

$$y^2 = x^6 + 17x^4 + 2x^3 + 8x^2 + 5x \tag{4.110}$$

over $\mathbb{F}_{37}$. Using the algorithm we find that an affine open of $\mathrm{Jac}(Y_2)$ is given by the following system of equations in $\mathbb{F}_{37}[a_1, a_2, a_3, a_4]$:

$$\begin{aligned}
&-a_1^4a_2 + 3a_1^2a_2^2 + 2a_1^2a_2 + 2a_1a_2 + -a_2^3 - 2a_2^2 + a_2b_1^2 + 11a_2 - b_2^2 = 0, \\
&-a_1^5 + 4a_1^3a_2 + 2a_1^3 + 2a_1^2 - 3a_1a_2^2 - 4a_1a_2 + a_1b_1^2 + 11a_1 - 2a_2 - 2b_1b_2 + 5 = 0.
\end{aligned} \tag{4.111}$$

The projective equation for $\mathrm{Kum}(Y_2)$ in $\mathbb{P}^2_{\mathbb{F}_{19}}$ is given by

$$\begin{aligned}
&6x_1^4 - x_1^3x_3 + 2x_1^2x_2x_3 - x_2^3x_3 + 11x_1^2x_3^2 + x_1x_2x_3^2 + 6x_2^2x_3^2 + 11x_2x_3^3 \\
&\quad + 8x_3^4 + 9x_1^2x_2x_4 + 6x_1^2x_3x_4 - 4x_1x_2x_3x_4 + 8x_1x_3^2x_4 - 4x_3^3x_4 \\
&\quad + x_2^2x_4^2 - 4x_1x_3x_4^2 = 0
\end{aligned} \tag{4.112}$$

and the morphism $\pi : \mathrm{Jac}(Y_2) \to \mathrm{Kum}(Y_2)$ is explicitly given by

$$\begin{aligned}&\pi(a_1, a_2, b_1, b_2) = \\ &\left[1 : -a_1 : a_2 : (17a_1a_2 + 2a_1b_1b_2 + 14a_1 + 2a_2^3 - 4a_2^2 - 2a_2b_1^2 - 3a_2 - 2b_2^2)/(a_1^2 - 4a_2)\right].\end{aligned} \tag{4.113}$$

The 16 singular points of $\mathrm{Kum}(Y_2)$ are

$$\begin{aligned}&(0:0:0:1), (3:8:6:1), (4:7:8:1), (6:9:-4:1),\\ &(6:-2:5:1), (7:-2:0:1), (10:6:0:1), (11:16:0:1),\\ &(13:4:2:1), (13:10:-4:1), (13:-3:6:1), (14:1:4:1),\\ &(14:7:6:1), (14:11:0:1), (-4:1:14:1), (-4:4:0:1)\end{aligned} \tag{4.114}$$

We consider the family of planes $H_{1,2}(\lambda)$ passing through $P = (0:0:0:1)$ and $Q = (3:8:6:1)$. After calculating an equation for the family $H_{P,Q}(\lambda)$ and substituting this into $\mathrm{Kum}(Y_2)$ we get the following affine equation for $H_{P,Q}(\lambda) \cap \mathrm{Kum}(Y_2)$:

$$\begin{aligned}&x^4 + (\lambda^4 + 5\lambda^3 + 13\lambda^2 + 3x^3y + 11\lambda^2)x^3\\ &\quad + (-2\lambda^4 - \lambda^3 + 8\lambda^2 + 10\lambda + 1)x^2y^2 + (16\lambda^2 + 4\lambda)x^2y - 3x^2\\ &\quad + (6\lambda^4 + 14\lambda^3 + 17\lambda^2 + -4\lambda + 2)xy^3\\ &\quad + (16\lambda + 4)xy^2 + 12\lambda xy + (-2\lambda^4 - 2\lambda^3 + 9\lambda^2 - 4\lambda + 5)y^4\\ &\quad + (6\lambda^2 - \lambda + 5)y^3 + (3\lambda + 6)y^2 = 0\end{aligned} \tag{4.115}$$

Considering this as an elliptic curve over $\mathbb{F}_{19}(\lambda)$ we calculate the j-invariant of this curve:

$$j(\lambda) = \frac{\begin{aligned}15\lambda^{12} + 12\lambda^{11} + 7\lambda^{10} - \lambda^9 + 4\lambda^8 + 7\lambda^7 + 3\lambda^6\\ + 5\lambda^5 + 2\lambda^4 + 9\lambda^3 + 14\lambda^2 + 13\lambda + 9\end{aligned}}{\begin{aligned}\lambda^{12} - 4\lambda^{11} + 6\lambda^{10} + 8\lambda^9 + 11\lambda^8 + 12\lambda^7 + 10\lambda^6\\ + 6\lambda^5 + 13\lambda^4 - 3\lambda^3 - 3\lambda^2\end{aligned}} \tag{4.116}$$

The j-invariant of X_1 is 35152/9. The numerator of $j(\lambda) - 35152/9$ factors as

$$(x+2)^2(x+3)(x+6)(x+7)(x+12)(x+15)(x^2+11x+8)(x^2+15x+13). \tag{4.117}$$

Let $\mathbb{F}_{19^2} = \mathbb{F}_{19}(\alpha)$ where α is a root of x^2-x+2. Then α^{59} is a root of $x^2+15x+13$. We will construct the degree 2 cover above $\widetilde{X}_1(\alpha^{59}) = H_{P,Q}(\alpha^{59}) \cap \mathrm{Kum}(Y_2)$. As we are working over a finite field, the computations are much easier and it is computationally feasible to explicitly calculate the pullback $\widetilde{X}_1(\alpha^{59})_{\mathrm{Kum}(Y_2)} \times \mathrm{Jac}(Y_2)$ in a reasonable amount of time. (Algorithm 4 is still much faster however.) The pullback is given by the equations:

$$\begin{aligned}&\alpha^{59}a_1^3 + \alpha^{46}a_1^2a_2 + a_1^2 + \alpha^{279}a_1a2 + \alpha^{266}a_2^2 + 15a_2 = 0,\\ &-a_1^4a_2 + 3a_1^2a_2^2 + 2a_1^2a_2 + 2a_1a_2 + -a_2^3 - 2a_2^2 + a_2b_1^2 + 11a_2 - b_2^2 = 0,\\ &-a_1^5 + 4a_1^3a_2 + 2a_1^3 + 2a_1^2 - 3a_1a_2^2 - 4a_1a_2 + a_1b_1^2\\ &+ 11a_1 - 2a_2 - 2b_1b_2 + 5 = 0.\end{aligned} \tag{4.118}$$

This scheme consists of two irreducible components. One component is a curve of genus 0, and the other component is a curve of genus 3. The genus 3 component is a curve Z_3 over $\mathbb{F}_{19^2}$ in $\mathbb{P}^2_{\mathbb{F}_{19^2}}$ defined by

$$x^4+\alpha^{228}x^3y+\alpha^{270}x^2y^2+\alpha^{133}xy^3+\alpha^{47}y^4+6x^2z^2+\alpha^{22}xyz^2+\alpha^{118}y^2z^2+17z^4=0 \tag{4.119}$$

and it is the (2,2)-gluing of X_1 and Y_2. Calculating its Dixmier-Ohno invariants $I_3, I_6, I_9, J_9, I_{12}, J_{12}, I_{15}, J_{15}, I_{18}, J_{18}, I_{21}, J_{21}$ and I_{27} normalized in their respective projective spaces gives us:

$$[1, 17, 9, 16, 0, 3, 2, 13, 13, 13, 2, 12, 8]. \tag{4.120}$$

A simplified equation of this curve over $\mathbb{F}_{19}$ is

$$x^4+5x^2y^2-x^2yz-x^2z^2+8y^4+8y^3z+9y^2z^2-5yz^3+2z^4. \tag{4.121}$$

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Zusammenfassung in deutscher Sprache

Überblick

Diese Arbeit befasst sich mit der Bestimmung einer $(2,2)$-Verklebung einer Kurve X_1 vom Geschlecht 1 mit einer Kurve Y_2 vom Geschlecht 2. Damit ist gemeint, dass wir eine Kurve Z_3 vom Geschlecht 3 mit der Eigenschaft, dass $\mathrm{Jac}(Z_3)$ isomorph zu $\mathrm{Jac}(X_1) \times \mathrm{Jac}(Y_2)/G$ für eine Untergruppe G von $\mathrm{Jac}(X_1)[2] \times \mathrm{Jac}(Y_2)[2]$ ist, suchen. Dieses Verfahren stellt die Umkehrung einer Konstruktion von Ritzenthaler und Romagny aus [26] dar. Wir werden zwei unterschiedliche Algorithmen zur Bestimmung der Verklebung vorstellen.

Inhalt

Wir fangen mit der Beschreibung einiger Basiseigenschaften abelscher Varietäten an. Danach definieren wir (n_1, n_2)-Verklebungen und betrachten den Zusammenhang zwischen Verklebungen und maximalen isotropen Untergruppen. Anschließend beschränken wir uns auf den Fall $n_1 = n_2 = 2$ und geben eine explizite Beschreibung von Gruppen dieser Art. Dazu betrachten wir auch unter welchen Bedingungen Verklebungen über dem Grundkörper definiert sind.

Wir beschäftigen uns kurz mit dem Fall, dass unsere Kurve vom Geschlecht 3 hyperelliptisch ist und fangen danach mit der Beschreibung des Verklebungsprozesses über $\mathbb{C}$ an. Wir erklären, wie die Objekte und Abbildungen, die man für die Verklebung über $\mathbb{C}$ braucht explizit aussehen und beschreiben danach einen Algorithmus, mit welchem man eine Verklebung mittels analytischer Methoden explizit berechnen kann. Außerdem erklären wir auch, wie wir eine (2,2)-Verklebung Z_3 über dem Grundkörper konstruieren können, falls eine solche Kurve existiert.

Im weiteren Teil der Arbeit beschreiben wir eine algebraische Konstruktion für das (2,2)-Verkleben. Dazu definieren wir eine Kummer-Fläche und besprechen, wie man eine Geschlecht 1 Kurve als den Schnitt einer Kummer-Fläche mit einer Fläche bekommen kann. Im Anschluss zeigen wir, wie man eine (2,2)-Verklebung Z_3 von einer Kurve vom Geschlecht 1 X_1 und einer Kurve vom Geschlecht 2 Y_2 aus der Kummer-Fläche $K = \mathrm{Jac}(Y_2)/(-1)$ und einer Fläche H erhalten kann. Dabei ist H so gewählt, dass die Desingularisierung von $H \cap K$ isomorph zu X_1 ist. Es stellt sich heraus, dass die Desingularisierung des Pullback von $H \cap K$ entlang der Quotientenabbildung $\pi : \mathrm{Jac}(Y_2) \to K$ eine (2,2)-Verklebung ist. Wir beschreiben einen Algorithmus und alle Hilfsmittel, die man braucht um diese Konstruktion explizit ausführen zu können.

Ehrenwörtliche Erklärung

Ich versichere hiermit, dass ich die Arbeit selbstständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe.

Ulm, den

. . . .

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Graduated cum laude. I took courses in Funtional Programming, Algorithmics, Cryptography, Concurrency, Data Analysis/Retrieveal and Image processing. Instead of a Bachelor thesis I helped build the website `http://www.annotatedbooksonline.com/` as a Software Project together with a group of other students.

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