Teichmüller curves
and
Hurwitz spaces

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Introduction

A Teichmüller curve is a curve $C \subset \mathscr{M}_g$, embedded in the moduli space of smooth projective curves of genus g, which is totally geodesic for the Teichmüller metric.

In this thesis we construct a new class of Teichmüller curves, using a characterisation due to Möller [**Möl06b**]. This involves constructing a suitable one-dimensional family of smooth projective curves parametrised by the points of a Teichmüller curve in $\mathscr{M}_g$. We show that our new Teichmüller curves are the last Teichmüller curves in a larger class of Teichmüller curves constructed in [**BM10b**].

About candidates for further Teichmüller curves not much is known. A starting point may be the following observation. The points of a Teichmüller curve embedded in $\mathscr{M}_g$ correspond to curves with real multiplication by large totally real number fields (see [**Möl06b**, Theorem 2.7] for a more precise statement). In [**Ell01**], Ellenberg constructs three one-dimensional families with this property. However, in this thesis we show that, except for some special cases, Ellenberg's families do not define Teichmüller curves. To do this, we interpret them as families over suitable Hurwitz spaces. We then describe a criterion to check whether a family of curves does not define a Teichmüller curve by studying the boundary of the associated Hurwitz space and apply this criterion for exclusion to Ellenberg's families.

We moreover show how to modify Ellenberg's families by passing to an adapted Hurwitz space in such a way that the criterion for exclusion no longer holds. It remains open whether this modification indeed produces Teichmüller curves.

Background

The first constructions of Teichmüller curves arose in the study of trajectories of billiard balls on plane polygonal billiard tables $P \subset \mathbb{R}^2 \simeq \mathbb{C}$. By gluing together suitable reflected copies of the table one may just as well study straightened non-reflected trajectories on a closed surface X. If one interprets the copies as subsets of $\mathbb{C}$, then X becomes a compact Riemann surface with a distinguished holomorphic 1-form ω induced by $\mathrm{d}z$ on $\mathbb{C}$ where $z = x + \mathrm{i}\,y$ is a coordinate on $\mathbb{C} \simeq \mathbb{R}^2$.

The group $\mathrm{SL}_2(\mathbb{R})$ naturally acts on the moduli space $\Omega\mathcal{M}_g$ of pairs of compact Riemann surfaces and distinguished non-zero holomorphic 1-forms. If the pair (X,ω) is stabilised by a lattice in $\mathrm{SL}_2(\mathbb{R})$, then the projection of the orbit of (X,ω) to the moduli space $\mathcal{M}_g$ is a Teichmüller curve. In this case the corresponding billiard table is 'dynamically optimal', i.e. the trajectories on P are either periodic or ergodic – provided that they do not end up in a corner of P. Good references for this approach to Teichmüller curves are e.g. [**McM03**], [**MT02**] and [**HS06**].

In this thesis however we use a different approach to construct Teichmüller curves. We use a criterion due to Möller [**Möl06b**]. Rather than considering an individual pair (X,ω) we consider a one-dimensional family $(\mathcal{X},\omega)\to S$ of smooth genus-g curves equipped with a holomorphic 1-form. Then we ask whether the image of the moduli map $S\to\mathcal{M}_g$, which sends a point $b\in S$ to the point of $\mathcal{M}_g$ corresponding to the fibre $\mathcal{X}_b$, is a Teichmüller curve.

The 1-form ω, which is part of the data $(\mathcal{X},\omega)$, is a section of the first relative de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/S)$. This is an $\mathcal{O}_S$-module whose fibres at $b\in S$ are the de Rham cohomology $\mathbb{C}$-vector spaces $H^1_{\mathrm{dR}}(\mathcal{X}_b)$, consisting of the closed 1-forms on the fibres $\mathcal{X}_b$ modulo exact 1-forms. The relative de Rham cohomology comes equipped with the Gauß-Manin connection ∇, whose contraction $\nabla(\partial/\partial s)$ provides a way to take the parameter derivative of sections of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/S)$ with respect to a local parameter s on S. In the case that $(\mathcal{X},\omega)\to S$ defines a Teichmüller curve, ω is a section of a flat rank-2 subbundle $\mathcal{E}\subset\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/S)$, which means that ∇ restricts to a connection on $\mathcal{E}$. Moreover, $\mathcal{E}$ is generated as $\mathcal{O}_S$-module by ω and $\nabla(\partial/\partial s)\,\omega$.

Flat rank-2 subbundles of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/S)$ are the central objects in the criterion of Möller. They have singularities in the set $\overline{S}-S$, which are divided into two types: elliptic singularities and logarithmic singularities. The conditions in the criterion of Möller are conditions regarding the nature of the singularities of $\mathcal{E}$, which can be checked in terms of a certain Fuchsian differential equation associated with $\mathcal{E}$ – the Picard-Fuchs equation. How to obtain the Picard-Fuchs equation of $\mathcal{E}$ is sketched in the example on page 3.

Theorem (Möller). *If the first relative de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/S)$ of a family $\mathcal{X}\to S$ of smooth curves of genus g contains an indigenous flat rank-2 subbundle $\mathcal{E}\subset\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/S)$ such that all points in the boundary $\overline{S}-S$ are logarithmic singularities, then the image of $S\to\mathcal{M}_g$ is a Teichmüller curve.*

We roughly explain the ingredients needed in the theorem by discussing an example, which is a special case of the families that define the new Teichmüller curves constructed in this thesis.

Example. We consider the family of smooth projective curves $\mathscr{X} \to S$ over $S = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ with coordinate s given by the affine equation

$$z^4 = x^2(x-1)^2(x-s)^3.$$

Such (families of) curves, which are cyclic covers of the projective line, are called superelliptic curves. Denote by $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ the eigenspace of $\varphi(x, z) = (x, \mathrm{i} \cdot z)$ with eigenvalue i, where i denotes the imaginary unit. More concretely, the 1-forms $\omega = \frac{z\,\mathrm{d}x}{x(x-1)(x-s)}$ and $\nabla(\partial/\partial s)\,\omega = \frac{\omega}{4(x-s)}$ are generators of $\mathscr{E}$. The section ω satisfies the relation

$$\nabla(\partial/\partial s)^2\,\omega + \frac{6s-3}{4s(s-1)}\,\nabla(\partial/\partial s)\,\omega + \frac{1}{16s(s-1)}\,\omega = \mathrm{d}\,\frac{-z}{4(x-s)^2} = 0 \in \mathscr{E}.$$

The associated ordinary linear differential operator

$$L = \left(\tfrac{\partial}{\partial s}\right)^2 + \tfrac{6s-3}{4s(s-1)} \cdot \left(\tfrac{\partial}{\partial s}\right) + \tfrac{1}{16s(s-1)}$$

on $\overline{S} = \mathbb{P}^1_{\mathbb{C}}$, the so-called Picard-Fuchs operator of ω, is a hypergeometric differential operator with singularities in $\{0, 1, \infty\} \subset \overline{S}$.

With a singularity of the differential equation one may associate a local monodromy matrix (see Appendix). In the case that the matrix is of finite order, we call the singularity an elliptic singularity; otherwise we call it a logarithmic singularity. One checks that, in our example, $b \in \{0, 1\}$ is an elliptic singularity and $b = \infty$ is a logarithmic singularity of L.

However, we may reparametrise the family (and hence the Picard-Fuchs operator) using the relation $s = \frac{1}{1-t^4}$. Then one can check that the local monodromy of the reparametrised Picard-Fuchs operator at the points corresponding to $s = 0$ resp. $s = 1$ become trivial (i.e. the local monodromy is the identity matrix). Moreover, one can check that these points are no longer singularities of the new differential operator. A translation of these two facts is that the bundle $\mathscr{E}$ becomes an indigenous bundle with only logarithmic singularities. (This is explained in more detail in Chapter 3.)

Then another key requirement in the theorem of Möller is that the reparametrised superelliptic curve can be smoothly extended over the 'removed' singularities. In the present situation, this is not the case, as the superelliptic curve degenerates at the points corresponding to $s = 0$ and $s = 1$ to a singular curve consisting of two elliptic curves intersecting at one ordinary double point. This is illustrated in Figure 0.1.

One can check that the reparametrised family admits an automorphism σ of order two which extends to the degenerate fibres, acting on them as follows. It maps the left torus in Figure 0.1 to the right one having precisely one fixed point, namely the ordinary double point. Dividing out the action of σ yields a family of elliptic curves that smoothly extends to the 'removed' singularities. Moreover, one can check that the indigenous subbundle of the relative de Rham cohomology of the

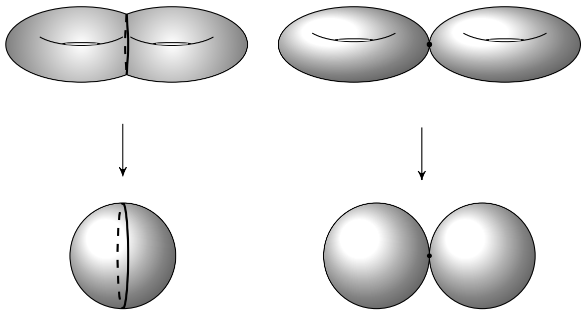


FIGURE 0.1. Contracting the loops yields the degeneration over $s = 0$ resp. $s = 1$.

reparametrised family 'descents' to the relative de Rham cohomology of the quotient family. Then the theorem of Möller implies that the image of the associated moduli map is a Teichmüller curve.

There are of course more superelliptic curves from which one can construct Teichmüller curves similar to the quotient construction explained above. In this thesis we describe all possible types of such superelliptic curves (see Proposition 3.3.2). A large class of Teichmüller curves 'coming from' these superelliptic curves is constructed in [**BM10b**] and completed by our new Teichmüller curves. The constructions used for this all fit into the following general set-up.

Given a finite group G, we denote by $\mathscr{H}_G^{(4)}$ the Hurwitz space of G-covers with ordered branch locus of cardinality 4. Let $H \subset G$ be an arbitrary subgroup. Choose a connected component $S \subset \mathscr{H}_G^{(4)}$, which in this case is an (affine) curve. Denote by $S \to \mathscr{M}_g$ the map which sends a point $b \in S$, corresponding to a G-cover $Y \to \mathbb{P}^1_{\mathbb{C}}$, to the quotient curve $X := Y/H$. Then one can ask whether the image of $S \to \mathscr{M}_g$ is a Teichmüller curve or not.

One way to answer this question is to verify the conditions of the Theorem of Möller, which involves the construction of a family $\mathscr{X} \to S$ of smooth curves over S or a suitable unramified cover of S. If the Hurwitz space $\mathscr{H}_G^{(4)}$ is a fine moduli space, then one may take the corresponding universal family; otherwise one needs to pass to a suitable unramified cover.

As already mentioned earlier, a starting point for finding candidates for families that define Teichmüller curves is the fact that they parametrise curves with real multiplication by a large field. In [**Ell01**] the following families with this property are constructed.

Theorem (Ellenberg). *(i) If p is a prime congruent to $1 \pmod 4$, then there exists a one-dimensional family $\mathcal{X}^{\mathrm{Ell},4} \to S$ of smooth $\mathbb{C}$-curves of genus $(p-1)/4$ with real multiplication by the index-4 subfield of $\mathbb{Q}(\zeta_p)$.*
(ii) If p is a prime congruent to $1 \pmod 6$, then there exists a one-dimensional family $\mathcal{X}^{\mathrm{Ell},6} \to S$ of smooth $\mathbb{C}$-curves of genus $(p-1)/6$ with real multiplication by the index-6 subfield of $\mathbb{Q}(\zeta_p)$.
(iii) If p and q are distinct odd primes, then there exists a one-dimensional family $\mathcal{X}^{\mathrm{Ell},pq} \to S$ of smooth $\mathbb{C}$-curves of genus $(p-1)(q-1)/2$ with real multiplication by the field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$.

Since Ellenberg's families parametrise curves with real multiplication and are constructed as quotients of families over a Hurwitz space of metacyclic covers (i.e. we are in the above general set-up), it is therefore natural to ask whether they define Teichmüller curves. We answer this question in the second main part of this thesis.

Results of this thesis

In the first main part of this thesis we construct the following new class of Teichmüller curves. Note that the case $m = 4$ is discussed in the example on page 3.

Theorem 3.3.8. *Let $m > 2$ be an integer and $S = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ with coordinate s. Consider the family of (smooth projective) superelliptic curves $\mathcal{Z} \to S$ given by the affine equation*

$$\begin{cases} \mathcal{Z}\colon z^{2m} = x^m(x-1)^m(x-s)^{m+2}, & \text{if } m \text{ is odd,} \\ \mathcal{Z}\colon z^m = x^{m/2}(x-1)^{m/2}(x-s)^{(m+2)/2}, & \text{otherwise.} \end{cases}$$

Then after a suitable base change $\pi\colon \overline{T} \to \overline{S} = \mathbb{P}^1_{\mathbb{C}}$ the superelliptic curve extends to a semistable curve $\mathcal{Z}_{\widetilde{T}}$ over $\widetilde{T} = \overline{T} - \pi^{-1}(\infty)$ and admits an automorphism σ of order two such that the quotient $\mathcal{X} := \mathcal{Z}_{\widetilde{T}}/\langle\sigma\rangle$ defines a Teichmüller curve, i.e. the image of $\widetilde{T} \to \mathcal{M}_g$, $b \mapsto [\mathcal{X}_b]$, is a Teichmüller curve, embedded in the moduli space of smooth curves of genus

$$g = \begin{cases} \frac{m-1}{2}, & \text{if } m \text{ is odd,} \\ \left\lfloor \frac{m}{4} \right\rfloor, & \text{otherwise.} \end{cases}$$

This is the last Teichmüller curve 'coming from' families of superelliptic curves by a certain quotient construction (see Summary 3.3.10). Together with the Teichmüller curves constructed in **[BM10b]** this provides a complete classification of such Teichmüller curves.

In the second main part of this thesis we consider the one-dimensional families from [**Ell01**]. We show that, except for some special cases, they do not define Teichmüller curves (see Theorem 4.3.5, Remark 4.3.6 and Subsection 3.3.2).

Theorem 4.3.5. *(i) Let $\mathcal{X}^{\mathrm{Ell},m} \to S$, with $m = 4$ resp. $m = 6$, be the family from part (i) resp. (ii) of the theorem of Ellenberg. Unless $p = m + 1$, the family does not define a Teichmüller curve. In the case that $p = m + 1$, the family is a non-trivial family of elliptic curves, i.e. the image of the moduli map $S \to \mathcal{M}_1$ is dense. In particular, the image is a Teichmüller curve.*
(ii) For any two distinct odd primes p and q, the family $\mathcal{X}^{\mathrm{Ell},pq} \to S$ from part (ii) of the theorem of Ellenberg does not define a Teichmüller curve.

An Ellenberg family $\mathcal{X}^{\mathrm{Ell},m} \to S$ (for $m = 4, 6$) is constructed as quotient of a family $\mathcal{Y} \to S$ of Galois covers of $\mathbb{P}^1_{\mathbb{C}}$ with metacyclic Galois group $G = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z}$. (The quotient is taken modulo $\mathbb{Z}/m\mathbb{Z}$.) In this thesis we adapt the family $\mathcal{Y} \to S$ in order to overcome the obstructions given by the boundary of the associated Hurwitz space.

Theorem 4.5.11. *There is a one-dimensional family $\widetilde{\mathcal{Y}} \to S$ of Galois covers of $\mathbb{P}^1_{\mathbb{C}}$ with semidirect Galois group $\tilde{G} = (\mathbb{Z}/p\mathbb{Z})^2 \rtimes D_m$ that factors through the family $\mathcal{Y} \to S$ with the following property. The quotient $\widetilde{\mathcal{X}}^{\mathrm{Ell},m} := \widetilde{\mathcal{Y}}/D_m \to S$ has no fibres that are singular curves of compact type.*

Moreover, we decompose the relative de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(\widetilde{\mathcal{X}}^{\mathrm{Ell},m}/S)$ into rank-2 subbundles and refine the criterion on the boundary of the Hurwitz space to exclude some of the rank-2 subbundles. The remaining bundles may be candidates for indigenous bundles to satisfy the conditions from the theorem of Möller (see Summary 4.6.8).

Overview

This thesis is organised as follows.

In Chapter 1 we gather well-known facts regarding the theory of Hurwitz spaces, mostly without giving proofs. The goal is to present (and adapt) the theory in a consistent way, just as much as we need it throughout this thesis. Hurwitz spaces are coarse moduli spaces for branched covers of the Riemann sphere. We explain the construction of the Hurwitz space $\mathcal{H}_G^{(4)}$ of Galois covers of $\mathbb{P}^1_{\mathbb{C}}$ with Galois group G and four branched points (equipped with an ordering of the branch points). It is constructed as a covering space of the space of branch loci. The monodromy representation (called Hurwitz monodromy) of the corresponding covering projection can be described by an action of the Hurwitz braid group on Nielsen tuples (see Proposition 1.3.4). The orbits of this action are in bijection with the points in the boundary $\overline{\mathcal{H}_G^{(4)}} - \mathcal{H}_G^{(4)}$ which parametrise covers between semistable curves, so-called admissible covers (see Proposition 1.4.1). The quotients of these covers by subgroups $H \subset G$ are described in Proposition 1.5.6. This description is used in Chapter 3 and Chapter 4 to check whether the H-quotient of a curve parametrised by the boundary $\overline{\mathscr{H}_G^{(4)}} - \mathscr{H}_G^{(4)}$ is smooth or not.

In Chapter 2 we study the first relative de Rham cohomology $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ of a one-dimensional family $\mathscr{Y} \to S$ of smooth genus-g curves. It comes equipped with the Gauß-Manin connection, which can be made explicit by a Fuchsian differential equation – the Picard-Fuchs equation. The main part of Chapter 2 is Section 2.3, in which we decompose $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ into subbundles that are invariant under the Gauß-Manin connection, using representation theory. We restrict to the case needed in this thesis, namely to families of Galois covers $\mathscr{Y} \to \mathbb{P}^1_S$ having a semidirect Galois group $G = A \rtimes H$ with A abelian. We moreover describe which subbundles $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ 'descent' to the relative de Rham cohomology of the quotient curve $\mathscr{Y}/H$. In other words, we describe when the Picard-Fuchs equation corresponding to $\mathscr{E}$ can be recovered in the relative de Rham cohomology of the quotient $\mathscr{Y}/H$.

In Chapter 3 we consider flat rank-2 vector bundles on a smooth projective connected $\mathbb{C}$-curve $\overline{S}$, i.e. $\mathscr{O}_{\overline{S}}$-modules $\mathscr{E}$ equipped with a logarithmic connection ∇. This is a generalisation of the notions introduced in Chapter 2. We recall some definitions (elliptic and logarithmic singularities as well as indigenous bundles), which we need to formulate the Theorem of Möller (Theorem 3.1.13). We show how one can 'remove' the elliptic singularities of a flat vector bundle by applying a suitable base change (see Lemma 3.1.8). In the case of superelliptic curves $\mathscr{Z} \to S$, we calculate the order of vanishing of the Kodaira-Spencer map associated with isotypical rank-2 subbundles $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Z}/S)$. The order of vanishing 'measures' whether $\mathscr{E}$ is indigenous (Lemma 3.3.1). In Proposition 3.3.2 we find all types of superelliptic curves whose relative de Rham cohomology contains a rank-2 subbundle that can be pulled back to an indigenous bundle. In Remark 3.3.3, Remark 3.3.5 and Remark 3.3.6 we recall from [**BM10b**] how to produce Teichmüller curves using these indigenous bundles. It turns out that there is one (and only one) case that has not been treated yet, see Proposition 3.3.7 (iii). This case yields our new class of Teichmüller curves (Theorem 3.3.8).

Using the classification of all superelliptic curves that provide indigenous bundles (given in Proposition 3.3.2), we explain in Subsection 3.3.2 why Ellenberg's pq-family $\mathscr{X}^{\mathrm{Ell},pq}$ does not define a Teichmüller curve.

Chapter 4 deals with the two remaining one-dimensional families of curves $\mathscr{X}^{\mathrm{Ell},4}$ and $\mathscr{X}^{\mathrm{Ell},6}$. They are quotients of certain metacyclic Galois covers by the second cyclic factor of the Galois group. More precisely, we consider families of Galois covers $Y \xrightarrow{\mathbb{Z}/p\mathbb{Z}} Z \xrightarrow{\mathbb{Z}/m\mathbb{Z}} \mathbb{P}^1_{\mathbb{C}}$ that are compositions of an m-cyclic cover of $\mathbb{P}^1_{\mathbb{C}}$ branched at four points and an étale p-cyclic cover. In Section 4.2 we show that $\mathscr{X}^{\mathrm{Ell},4}$ and $\mathscr{X}^{\mathrm{Ell},6}$ are essentially the only quotients of metacyclic covers of the above form such that their relative de Rham cohomology splits into rank-2 subbundles all of which carry a filtration (Proposition 4.2.1). This is a necessary condition for a flat subbundle to be indigenous. However, in Section 4.3 we prove that, except for some special cases, none of the subbundles satisfy the conditions from the theorem of Möller by studying the boundary of the Hurwitz space of metacyclic covers (Theorem 4.3.5). For this we show that the family (after an unramified base change if necessary) degenerates at some point in $\overline{S}-S$ to a singular semistable curve of compact type. Then [**Möl11**, Proposition 2.4] (resp. Lemma 4.3.1) implies that the image of $S \to \mathcal{M}_g$ is no Teichmüller curve. In Section 4.4 we describe the parameter space of Ellenberg's families more precisely via the corresponding Hurwitz monodromy. In Section 4.5 we show how to adapt Ellenberg's families such that the above criterion for exclusion on the boundary of the Hurwitz space no longer holds (Theorem 4.5.11). The fibres of the adapted families are constructed as follows. The automorphism group of Z is a dihedral group $D_m = \mathbb{Z}/m\mathbb{Z} \rtimes \langle\sigma\rangle$, but σ does not lift to Y. We lift the automorphism σ to the Galois closure $\widetilde{Y} \to \mathbb{P}^1_{\mathbb{C}}$. Our adapted family then is obtained by taking the quotient of $\widetilde{Y} \to \mathbb{P}^1_{\mathbb{C}}$ by D_m. An analogous construction was used in [**BM10b**] to construct Teichmüller curves. In our situation, it remains open whether our adaptation indeed produces Teichmüller curves. In Section 4.6 we refine the above criterion for exclusion to find rank-2 subbundles of the relative de Rham cohomology of the adapted family that may be candidates for indigenous bundles to satisfy the conditions from the theorem of Möller (see Theorem 4.6.6 and Summary 4.6.8).

Contents

Chapter 1

Hurwitz spaces

In this chapter we recall the theory of Hurwitz spaces which we adapt for our purposes, following mainly [**RW06**], [**Wew98**], [**Völ96**], [**FV91**]. We describe the boundary of Hurwitz spaces, which parametrises covers between semistable curves, so-called admissible covers (see Proposition 1.4.1). In Proposition 1.5.6 we describe the quotients of these covers. This description is needed in Chapter 3 and Chapter 4 in order to check whether a degeneration is smooth or not.

Throughout this chapter S denotes a scheme over $\mathbb{C}$ and G a finite group.

1.1. Families of covers

We introduce the notion of a family of covers over a parameter space S. We want the members of this family to have r ordered branch points. The reason for this is that – in the case that the members are branched covers of the Riemann sphere – we may describe them by ordered Nielsen tuples (cf. Section 1.2).

Definition 1.1.1. (i) A *smooth curve* $\mathscr{X}$ over S is a smooth and proper morphism $\mathscr{X} \to S$ such that the fibre $\mathscr{X}_s = \mathscr{X} \times_S \operatorname{Spec} \mathbb{C}$ over any geometric point $s\colon \operatorname{Spec} \mathbb{C} \to S$ is connected and one-dimensional.

(ii) Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth curves over S. A map $\pi\colon \mathcal{Y} \to \mathcal{X}$ is called a *cover* over S if it is a finite, flat and surjective S-morphism

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\pi"] \arrow[rr, bend right] & \mathcal{X} \arrow[r] & S. \end{tikzcd}$$

We write $\mathrm{Aut}(\pi) = \{h \in \mathrm{Aut}_S(\mathcal{Y});\ \pi \circ h = \pi\}$ for the group of automorphisms that leave π fixed.

The pullback $\pi_b\colon \mathcal{Y}_b \to \mathcal{X}_b$, given by the fibre product

$$\begin{tikzcd} \mathcal{Y}_b \arrow[r] \arrow[d, "\pi_b"'] & \mathcal{Y} \arrow[d, "\pi"] \\ \mathcal{X}_b \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "b"] & S \end{tikzcd}$$

is a cover over $\mathbb{C}$. That is why we may interpret π as a family of covers with parameter space S.

Definition 1.1.2. We say that two covers $\pi\colon \mathcal{Y} \to \mathcal{X}$ and $\pi'\colon \mathcal{Y}' \to \mathcal{X}'$ over S are *(weakly) isomorphic* if there exist S-isomorphisms $h\colon \mathcal{Y} \xrightarrow{\sim} \mathcal{Y}'$ and $m\colon \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ such that $m \circ \pi = \pi' \circ h$.

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\sim", "h"'] \arrow[d, "\pi"'] & \mathcal{Y}' \arrow[d, "\pi'"] \\ \mathcal{X} \arrow[r, "\sim", "m"'] & \mathcal{X}' \end{tikzcd}$$

Definition 1.1.3. For a geometric point $y\colon \operatorname{Spec}\mathbb{C} \to \mathcal{Y}$ the *ramification index* e_y is the order of the stabiliser of y in $\mathrm{Aut}(\pi)$. The *branch locus* of π is the smooth relative divisor $D \subset \mathcal{X}$ over S such that

(i) the natural map $D \to S$ is finite and étale,
(ii) the restriction of π to the open subset $\mathcal{X} - D$ is étale,
(iii) for every geometric point $x\colon \operatorname{Spec}\mathbb{C} \to D$ there is a geometric point $y\colon \operatorname{Spec}\mathbb{C} \to \mathcal{Y}$ with $\pi(y) = \pi \circ y = x$ and ramification index $e_y > 1$.

Remark 1.1.4. In the following we are interested in the case that the divisor $D \subset \mathcal{X}$ is *split*,. i.e. the image of r pairwise disjoint sections $x_1, \ldots, x_r\colon S \to \mathcal{X}$. In this case we write $D = \{x_1, \ldots, x_r\}$. The degree $\deg(D/S)$ of the split divisor is constant and equals r. The sections $x_i\colon S \to \mathcal{X}$ are called *branch points* of π.

Definition 1.1.5. By a G-cover of $\mathcal{X}$ over S we mean a pair (π, μ) where

- $\pi\colon \mathcal{Y} \to \mathcal{X}$ is a cover over S,
- $\mathrm{Aut}(\pi)$ acts transitively on every (geometric) fibre of π.
- $\mu\colon G \xrightarrow{\sim} \mathrm{Aut}(\pi)$ is an isomorphism.

REMARK 1.1.6. The pullback $\pi_b\colon \mathcal{Y}_b \to \mathcal{X}_b$ is a G-cover over $\mathbb{C}$ whose isomorphism $G \xrightarrow{\sim} \mathrm{Aut}(\pi_b)$ is given by restricting the automorphisms in G to $\mathcal{Y}_b$. In other words, we may interpret π as a family of G-covers with parameter space S.

In the sequel we identify $G = \mathrm{Aut}(\pi)$ and drop μ from the notation (π, μ).

DEFINITION 1.1.7. Two G-covers $\pi\colon \mathcal{Y} \to \mathcal{X}$ and $\pi'\colon \mathcal{Y}' \to \mathcal{X}'$ over S are said to be *(weakly) isomorphic* if there exist S-isomorphisms $h\colon \mathcal{Y} \xrightarrow{\sim} \mathcal{Y}'$ and $m\colon \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ such that $m \circ \pi = \pi' \circ h$ and $g \circ h = h \circ g$ for all $g \in G$.

We always consider weak isomorphisms of covers, if not stated otherwise.

DEFINITION 1.1.8. A *cover* (resp. *G-cover*) *with ordered branch locus* is a cover (resp. G-cover) whose branch locus is a split divisor $D = \{x_1, \dots, x_r\}$ together with a choice of a bijection $\alpha\colon \{1, \dots, r\} \xrightarrow{\sim} \{x_1, \dots, x_r\}$.

We usually assume that $x_i = \alpha(i)$, drop α from the notation and simply write $D = (x_1, \dots, x_r)$ for the ordered branch locus of the cover.

1.2. Nielsen tuples

Let $\pi\colon Y \to \mathbb{P}^1_{\mathbb{C}}$ be a G-cover with ordered branch locus D of cardinality $|D| = 4$. In this section we consider π as a finite non-constant holomorphic map between connected compact Riemann surfaces and D as a set of four pairwise distinct points on the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ together with a choice of a bijection $\alpha\colon \{1, \dots, 4\} \xrightarrow{\sim} D$.

In this set-up we recall from [**Völ96**, Chapter 4] a correspondence between G-covers of $\mathbb{P}^1_{\mathbb{C}}$ with ordered branch locus and so-called Nielsen tuples in

$$\mathcal{E}_r(G) := \{\mathbf{g} = (g_1, \dots, g_4) \in G^4; \quad G = \langle g_1, \cdots, g_4 \rangle,\ g_1 \cdots g_4 = 1,\ g_i \neq 1\}.$$

The restriction of π to $\pi^{-1}\left(\mathbb{P}^1_{\mathbb{C}} - D\right)$ is a covering projection with respect to the analytic topology. We fix a point $x_0 \in \mathbb{P}^1_{\mathbb{C}} - D$ and set $d := |\pi^{-1}(x_0)|$, the *degree* of π. It is well known that there exists a presentation

$$\pi_1\left(\mathbb{P}^1_{\mathbb{C}} - D, x_0\right) = \langle \gamma_1, \dots, \gamma_4; \quad \gamma_1 \cdots \gamma_4 = 1 \rangle \tag{1.2.1}$$

of the fundamental group of $\mathbb{P}^1_{\mathbb{C}} - D$, where γ_i is represented by a simple closed loop winding around the missing point $x_i = \alpha(i)$ as illustrated in Figure 1.1. In particular the order of factors in $\gamma_1 \cdots \gamma_4$ is given by the bijection α.

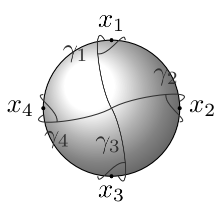


FIGURE 1.1.

We denote by $[\gamma_i]y$ the unique endpoint of the lift of γ_i to a path in Y with initial point y.

Definition 1.2.1. By $m_\pi(\gamma_i)(y) = [\gamma_i]y$ a left action

$$m_\pi \colon \pi_1\left(\mathbb{P}^1_{\mathbb{C}}-D, x_0\right) \to \mathrm{Sym}\left(\pi^{-1}(x_0)\right),$$

called the *monodromy* of π, is defined. The image $m_\pi(\gamma_i)$ is called the *local monodromy* of π at x_i. The subgroup $\mathrm{Mon}(\pi) := \langle m_\pi(\gamma_1), \ldots, m_\pi(\gamma_4)\rangle$ of $\mathrm{Sym}\left(\pi^{-1}(x_0)\right)$ is called the *monodromy group* of π.

Note that we use the convention to compose loops and permutations from left to right.

Remark 1.2.2. We will use the notion local monodromy also for covers of curves of positive genus. However we refrain from introducing a precise definition as we will only deal with conjugacy classes of the local monodromies in that case.

For every $y \in \pi^{-1}(x_0)$ there exists a unique surjective homomorphism

$$\Phi_y \colon \pi_1\left(\mathbb{P}^1_{\mathbb{C}}-D, x_0\right) \twoheadrightarrow G \tag{1.2.2}$$

such that $\Phi_y(\gamma_i) \in G = \mathrm{Aut}(\pi)$ maps $[\gamma_i]y$ to y [**Völ96**, Proposition 4.19]. Note that (1.2.2) induces a right action $G \to \mathrm{Sym}(\pi^{-1}(x_0))$ on the fibre $\pi^{-1}(x_0)$, which coincides with the opposite action of the monodromy action defined in Definition 1.2.1.

For $y' \in \pi^{-1}(x_0)$ the homomorphism $\Phi_{y'}$ is the composition of Φ_y with an inner automorphism of G. By this, we have associated with the G-cover π with ordered branch locus $D = (x_1, \ldots, x_4)$ the Nielsen tuple

$$\mathbf{g}_\pi := \left(\Phi_y(\gamma_1), \ldots, \Phi_y(\gamma_4)\right),$$

given up to uniform conjugation with elements in G.

For a fixed quadruple D of pairwise distinct points in $\mathbb{P}^1_{\mathbb{C}}$, we denote by

$$\mathscr{H}_{D,G} := \{\pi \colon Y \xrightarrow{G} \mathbb{P}^1_{\mathbb{C}};\quad D = \text{ ordered branch locus of } \pi\}/\sim$$

the set of weak isomorphism classes of G-covers $Y \to \mathbb{P}^1_{\mathbb{C}}$ with ordered branch locus D. Further, we call

$$\mathrm{Ni}_4(G) := \mathscr{E}_4(G)/G,$$

where G acts via uniform conjugation, the *inner Nielsen class* of G. For more details we refer to [**FV91**, Section 1.1].

The following proposition is well known. For a proof, see e.g. [**Völ96**].

Proposition 1.2.3. *The association $\pi \mapsto \mathbf{g}_\pi$ induces a bijection of sets*

$$\mathscr{H}_{D,G} \xrightarrow{\sim} \mathrm{Ni}_4(G).$$

Remark 1.2.4. Let $\pi\colon Y \to \mathbb{P}^1_{\mathbb{C}}$ be a G-cover with ordered branch locus $D = (x_1, \ldots, x_4)$ and associated Nielsen tuple $\mathbf{g} = (g_1, \ldots, g_4)$. Then the transitive action of the Galois group $G = \mathrm{Aut}(\pi)$ on $\pi^{-1}(x_i)$ corresponds to the natural right action

$$G \to \mathrm{Sym}(\langle g_i \rangle \backslash G) \tag{1.2.3}$$

on the right cosets in $\langle g_i \rangle \backslash G$. The reason is that the group $\langle g_i \rangle$ is the stabiliser of a point in $\pi^{-1}(x_i)$ [**Sza09**, Proposition 3.4.5].

Definition 1.2.5. If $\pi\colon Y \to \mathbb{P}^1_{\mathbb{C}}$ is a G-cover with ordered branch locus of cardinality 4 and Nielsen tuple $\mathbf{g}_\pi = (g_1, g_2, g_3, g_4)$, we call the quadruple of non-trivial conjugacy classes

$$\mathbf{C} = \big(\,\mathrm{Cl}_G(g_1), \mathrm{Cl}_G(g_2), \mathrm{Cl}_G(g_3), \mathrm{Cl}_G(g_4)\big)$$

in G the *ramification type* of π.

We write $\mathscr{H}_{D,G,\mathbf{C}} \subset \mathscr{H}_{D,G}$ for the set of weak isomorphism classes of G-covers $Y \to \mathbb{P}^1_{\mathbb{C}}$ with ordered branch locus D and ramification type $\mathbf{C}$. Moreover, we define for $\mathbf{C} = (C_1, \ldots, C_4)$ the subset

$$\mathscr{E}_4(G, \mathbf{C}) := \{\mathbf{g} = (g_1, \ldots, g_4) \in G^4; \quad G = \langle g_1, \ldots, g_4 \rangle,\, g_1 \cdots g_4 = 1,\, g_i \in C_i\}$$

of $\mathscr{E}_4(G)$. Note that we consider ordered branch loci and hence we do not allow the conjugacy classes C_i to be permuted.

Corollary 1.2.6. *The association $\pi \mapsto \mathbf{g}_\pi$ induces a bijection of sets*

$$\mathscr{H}_{D,G,\mathbf{C}} \xrightarrow{\sim} \mathrm{Ni}(G, \mathbf{C}) := \mathscr{E}_4(G, \mathbf{C})/G,$$

where G acts via uniform conjugation.

Remark 1.2.7 (Non-Galois case)**.** Similarly, for a transitive subgroup G of S_d and a quadruple $\mathbf{C} = (C_1, \ldots, C_4)$ of conjugacy classes in G, we may consider the *absolute Nielsen class*

$$\mathrm{Ni}_4^{\mathrm{abs}}(G) := \mathscr{E}_4(G, \mathbf{C})/\,\mathrm{N}_{S_d}(G),$$

where $\mathrm{N}_{S_d}(G)$ is the normaliser of G in S_d acting on $\mathscr{E}_4(G, \mathbf{C})$ by uniform conjugation. Denote by $D = (x_1, \ldots, x_4)$ a fixed tuple of four pairwise distinct points in $\mathbb{P}^1_{\mathbb{C}}$. Then $\mathrm{Ni}_4^{\mathrm{abs}}(G)$ is in one-to-one correspondence with the set of weak isomorphism classes of degree-d covers of $\mathbb{P}^1_{\mathbb{C}}$ with ordered branch locus D and monodromy group G such that the its local monodromy $\varrho_i \in G$ at x_i is an element of C_i (see [**FB82**] where we take $\mathrm{N}_{S_d}(G)$ as group $\bar{G}$ from that paper).

1.3. Hurwitz spaces

Hurwitz spaces are parameter spaces for covers of $\mathbb{P}^1_{\mathbb{C}}$. In this section we introduce Hurwitz spaces as complex manifolds that cover the space of ordered branch loci, following [**RW06**, Section 3]. Moreover, we describe the monodromy of the corresponding covering projection. We restrict to the case of four branch points.

We consider $\mathbb{P}^* := \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ and put $D_\lambda := (0, 1, \lambda, \infty)$. Then we write

$$\mathscr{H}_G^{(4)} := \coprod_{\lambda \in \mathbb{P}^*} \mathscr{H}_{D_\lambda, G}$$

for the set of weak isomorphism classes of G-covers of $\mathbb{P}^1_{\mathbb{C}}$ with ordered branch locus of cardinality 4. We call $\mathscr{H}_G^{(4)}$ the *(reduced) Hurwitz space* of G-covers of $\mathbb{P}^1_{\mathbb{C}}$ with ordered branch locus of cardinality 4.

Remark 1.3.1. We consider the map

$$\Psi : \mathscr{H}_G^{(4)} \to \mathbb{P}^*$$

that sends an isomorphism class in $\mathscr{H}_G^{(4)}$ represented by a G-cover with branch locus D_λ to its third branch point λ. We can endow the set $\mathscr{H}_G^{(4)}$ with a topology such that it is a covering space of the Riemann surface $\mathbb{P}^*$ (which may be shown analogous to [**RW06**, Proposition 3.2]). In particular, $\mathscr{H}_G^{(4)}$ is a Riemann surface.

A consequence of Proposition 1.2.3 is the following.

Corollary 1.3.2. *Any fibre of Ψ is in bijection with* $\mathrm{Ni}_4(G)$.

Definition 1.3.3. We call Ψ the *branch locus map*. The monodromy (cf. Definition 1.2.1) of Ψ is called *Hurwitz monodromy*.

By Corollary 1.3.2, the Hurwitz monodromy induces an action of the fundamental group of $\mathbb{P}^*$

$$\pi_1\big(\mathbb{P}^*, \lambda\big) \to \mathrm{Sym}\left(\mathrm{Ni}_4(G)\right) \tag{1.3.1}$$

on Nielsen tuples, which we describe more precisely, following [**FV91**, Section 1.3].

Denote by $\mathcal{Q}_i$ the standard generators of the Hurwitz braid group H_4 with four braids verifying the relations

$$\mathcal{Q}_i\mathcal{Q}_j = \mathcal{Q}_j\mathcal{Q}_i \qquad \text{for } i, j \in \{1, 2, 3\} \text{ with } |i - j| > 1, \tag{1.3.2}$$

$$\mathcal{Q}_i\mathcal{Q}_{i+1}\mathcal{Q}_i = \mathcal{Q}_{i+1}\mathcal{Q}_i\mathcal{Q}_{i+1} \qquad \text{for } i \in \{1, 2\}, \tag{1.3.3}$$

$$\mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1^2\mathcal{Q}_2\mathcal{Q}_3 = 1. \tag{1.3.4}$$

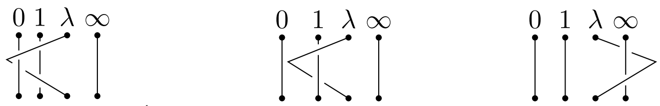
The pure Hurwitz braid group $H^{(4)}$ is the kernel of the projection $H_4 \to S_4$ that maps $\mathcal{Q}_i$ to the transposition $(i, i+1)$. It is well known that $H^{(4)}$ is the fundamental group of

$$\mathscr{U}^{(4)} = \left\{ (x_1, \ldots, x_4) \in \left(\mathbb{P}^1_{\mathbb{C}}\right)^4 \;\; x_i \neq x_j \text{ for } i \neq j \right\}.$$

Moreover, there is an isomorphism

$$H^{(4)} \xrightarrow{\sim} \pi_1(\mathbb{P}^*, \lambda) = \langle b_1, b_2, b_3;\ b_1 b_2 b_3 = 1 \rangle \tag{1.3.5}$$

given by $\mathcal{Q}_2\mathcal{Q}_1^2\mathcal{Q}_2^{-1} \mapsto b_1$, $\mathcal{Q}_2^2 \mapsto b_2$ and $\mathcal{Q}_3^2 \mapsto b_3$, where b_1, b_2, b_3 are represented by simple closed loops winding around 0, 1 and ∞, respectively (see Figure 1.2).



$b_1 = \mathcal{Q}_2\mathcal{Q}_1^2\mathcal{Q}_2^{-1}$ $\quad b_2 = \mathcal{Q}_2^2$ $\quad b_3 = \mathcal{Q}_3^2$

FIGURE 1.2.

PROPOSITION 1.3.4. *The Hurwitz monodromy* (1.3.1) *is given by*

$$[g_1, g_2, g_3, g_4] \cdot b_2 = [g_1, g_2^{g_2 g_3}, g_3^{g_2}, g_4], \qquad [g_1, g_2, g_3, g_4] \cdot b_3 = [g_1, g_2, g_3^{g_3 g_4}, g_4^{g_3}],$$

where $[\cdot]$ *expresses that we work with Nielsen tuples modulo* G *and we use the convention* $g^h = hgh^{-1}$ *for* $g, h \in G$. *Using* $b_1 = b_3^{-1}b_2^{-1}$, *we may calculate* $[g_1, g_2, g_3, g_4] \cdot b_1$.

The analogous result for unordered branch loci is proved in [**FV91**]. The proof of Proposition 1.3.4 is analogous. We sketch the idea of the proof. The braid action $H_4 \to \mathrm{Sym}(G^4)$ given by

$$(g_1, \ldots, g_4) \cdot \mathcal{Q}_i = (g_1, \ldots, g_{i-1}, g_i \cdot g_{i+1} \cdot g_i^{-1}, g_i, g_{i+2}, \ldots, g_4). \tag{1.3.6}$$

naturally induces an action on $\mathrm{Ni}_4(G)$. In [**FV91**, Section 1.4] it is shown that the braid action (1.3.6) describes the monodromy action of the branch locus map $\widetilde{\Psi}\colon \mathcal{H}^{\mathrm{in}}_{4,G} \to \mathcal{U}_4$, where $\mathcal{U}_4 = \{D \subset \mathbb{P}^1_{\mathbb{C}};\ |D| = 4\}$ and $\mathcal{H}^{\mathrm{in}}_{4,G}$ denotes the Hurwitz space of G-covers $Y \to \mathbb{P}^1_{\mathbb{C}}$ with unordered branch loci of cardinality 4, where two covers $\pi\colon Y \to \mathbb{P}^1_{\mathbb{C}}$ and $\pi'\colon Y' \to \mathbb{P}^1_{\mathbb{C}}$ are isomorphic if and only if there is an isomorphism $h\colon Y \xrightarrow{\sim} Y'$ such that $\pi = \pi' \circ h$.

The braid action (1.3.6) is illustrated in Figure 1.3 and Figure 1.4. A loop based at $D = \{x_1, \ldots, x_4\} \in \mathcal{U}_4$ is visualised by four 'simultaneous' paths in $\mathbb{P}^1_{\mathbb{C}}$ between the points $x_1, \ldots, x_4 \in \mathbb{P}^1_{\mathbb{C}}$. These are the branch points of a G-cover $\pi \in \widetilde{\Psi}^{-1}(D)$, and they move 'simultaneously' along these paths. (We have visualised the loop given by $\mathcal{Q}_2$, where the branch points x_1 and x_4 remain at their position.) Recall that we may associate with a G-cover a Nielsen tuple $\mathbf{g} = (g_1, \ldots, g_4)$ such that it holds $g_i = \Phi_y(\gamma_i)$, where the γ_i are the standard generators of $\pi_1(\mathbb{P}^1_{\mathbb{C}} - D, x_0)$, $x_0 \notin D$, from (1.2.1) and Φ_y is given as in (1.2.2). The topology on $\mathcal{H}^{\mathrm{in}}_{4,G}$ is such that the loops γ_i are deformed continuously when the points x_i move along a loop in $\mathcal{U}_4$.

As in [**RW06**, Section 2.5] there is a natural map $\mathcal{H}^{\mathrm{in}}_{4,G} \to \mathcal{H}^{\mathrm{red}}_{4,G}$ to the reduced Hurwitz space $\mathcal{H}^{\mathrm{red}}_{4,G}$ of weak isomorphism classes of G-covers with unordered branch loci of cardinality 4. The fibres of the corresponding branch locus map $\mathcal{H}^{\mathrm{red}}_{4,G} \to \mathcal{U}_4/\,\mathrm{PGL}_2(\mathbb{C})$ are still in bijection with $\mathrm{Ni}_4(G)$ and the monodromy action of $\mathcal{H}^{\mathrm{red}}_{4,G} \to \mathcal{U}_4/\,\mathrm{PGL}_2(\mathbb{C})$ is also given by (1.3.6).

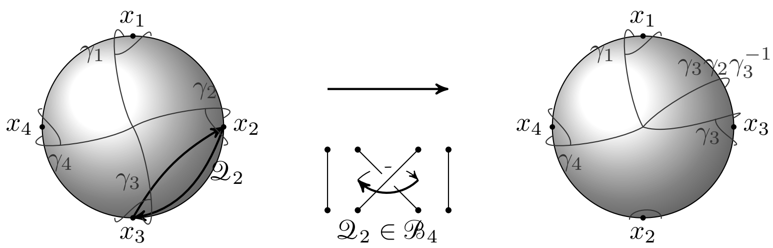


FIGURE 1.3.

Up to now we have described the situation for unordered branch loci. We now consider the (reduced) Hurwitz space of G-covers $\mathscr{H}_G^{(4)}$ with ordered branch loci. It corresponds to the reduced Hurwitz space with unordered branch loci via the following commutative diagram.

$$\begin{tikzcd}
\mathscr{H}_G^{(4)} \arrow[r] \arrow[d] & \mathscr{H}_{4,G}^{\mathrm{red}} \arrow[d] \\
\mathbb{P}^* \arrow[r] & \mathscr{U}_4/\,\mathrm{PGL}_2(\mathbb{C})
\end{tikzcd}$$

The map $\mathbb{P}^* \to \mathscr{U}/\,\mathrm{PGL}_2(\mathbb{C})$ is given by 'forgetting the order'. This shows that the monodromy action (1.3.1) in Proposition 1.3.4 is induced by the action of the generators $b_2 = \mathcal{Q}_2^2$ resp. $b_3 = \mathcal{Q}_3^2$ of the pure Hurwitz braid group $H^{(4)}$.

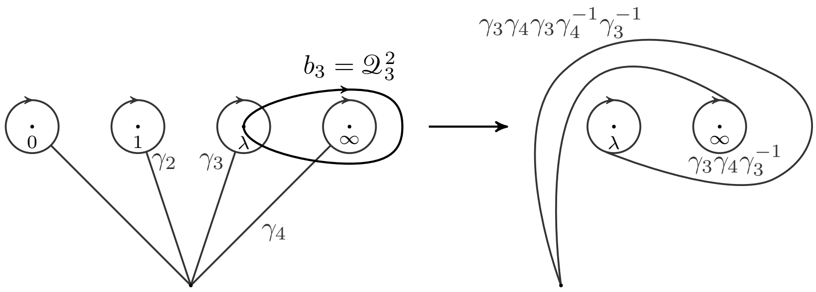


FIGURE 1.4. Note that we compose loops from left to right.

REMARK 1.3.5. Let $\mathbf{C} = (C_1, \ldots, C_4)$ be a quadruple of conjugacy classes in G. We write

$$\mathscr{H}_{G,\mathbf{C}} := \left\{\pi\colon \mathscr{Y} \xrightarrow{G} \mathbb{P}^1_{\mathbb{C}};\quad \mathbf{g}_\pi \in \mathbf{C}\right\}\Big/ \sim$$

for the subset of $\mathscr{H}_G^{(4)}$ consisting of the G-covers of $\mathbb{P}^1_{\mathbb{C}}$ with ordered branch locus of cardinality 4 and ramification type $\mathbf{C}$. Note that the braid action in Proposition 1.3.4 sends tuples in $\mathrm{Ni}_4(G)$ to tuples whose components are conjugated in G. Therefore the monodromy of the restriction $\Psi\colon \mathscr{H}_{G,\mathbf{C}} \to \mathbb{P}^*$ is the action

$$\pi_1(\mathbb{P}^*) \to \mathrm{Sym}\,(\mathrm{Ni}(G,\mathbf{C}))$$

given by the same formulas as in Proposition 1.3.4.

REMARK 1.3.6. Let A be an abelian group and $\mathbf{C}$ a quadruple of conjugacy classes in A such that $\mathrm{Ni}(A, \mathbf{C}) \neq \emptyset$. Since conjugation in abelian groups is trivial, we conclude that $\mathscr{H}_{A,\mathbf{C}} \simeq \mathbb{P}^*$.

REMARK 1.3.7 (Non-Galois case)**.** Let $G \subset S_d$ be a transitive subgroup and $\mathbf{C} = (C_1, \dots, C_4)$ a quadruple of conjugacy classes in G. Similar to Remark 1.3.1, we can introduce a branch locus map

$$\Psi' \colon \mathscr{H}_{d,G,\mathbf{C}} \to \mathbb{P}^*$$

for the Hurwitz space $\mathscr{H}_{d,G,\mathbf{C}}$ of (possibly non-Galois) of weak isomorphism classes of degree-d covers of $\mathbb{P}^1_{\mathbb{C}}$ all of which have

(i) ordered branch locus $(0, 1, \lambda, \infty)$,
(ii) monodromy group G,
(iii) and the local monodromy ϱ_i at x_i is an element of C_i.

Then the monodromy of Ψ' is the action

$$\pi_1\big(\mathbb{P}^*, \lambda\big) \to \mathrm{Sym}\left(\mathrm{Ni}_4^{\mathrm{abs}}(G)\right)$$

given by the formulas in Proposition 1.3.4, but $[\cdot]$ expresses that we work with isomorphism classes of Nielsen tuples modulo the normaliser $\mathrm{N}_{S_d}(G)$ of G in S_d (see Remark 1.2.7).

For examples of computations of Hurwitz monodromies we refer to Section 4.4.

1.3.1. Moduli of covers. We interpret the Hurwitz space $\mathscr{H}_G^{(4)}$ (and its subsets $\mathscr{H}_{G,\mathbf{C}}$) as coarse moduli space of G-covers following [**RW06**].

Denote by $H_G^{(4)}(\cdot)$ the contravariant functor from schemes over $\mathbb{C}$ to sets that sends

- a $\mathbb{C}$-scheme S to the set

$$H_G^{(4)}(S) = \left\{ \mathscr{Y} \xrightarrow{G} \mathbb{P}^1_S \quad \text{with ordered branch locus of cardinality } 4 \right\} \Big/ \sim$$

of weak isomorphism classes of G-covers,

- a morphism $\beta \colon T \to S$ to the pullback

$$\beta^* \colon H_G^{(4)}(S) \to H_G^{(4)}(T), \quad [\pi] \mapsto [\pi_T]$$

given by the fibre product

$$\begin{array}{ccc} \mathscr{Y}_T & \longrightarrow & \mathscr{Y} \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1_T & \longrightarrow & \mathbb{P}^1_S \\ \downarrow & & \downarrow \\ T & \xrightarrow{\beta} & S \end{array}$$

The functor $H_G^{(4)}(\cdot)$ is coarsely represented by a $\mathbb{C}$-scheme $\mathscr{H}$ (see [**RW06**, Theorem 2.1]). The analytification of $\mathscr{H}$ is just the Hurwitz space $\mathscr{H}_G^{(4)}$. In the case that the centre

$$Z(G) = \{g \in G; \quad gh = hg \text{ for all } h \in G\}$$

of G is trivial, i.e. $Z(G) = \{1\}$, the scheme $\mathscr{H}$ is a fine moduli space representing the functor $H_G^{(4)}(\cdot)$ ([**RW06**, Corollary 2.2]). In particular, there exists a unique G-cover

$$\mathscr{V} \xrightarrow{G} \mathbb{P}^1_{\mathscr{H}}$$

over $\mathscr{H}$ with ordered branch locus of cardinality 4 such that the fibre $\mathscr{V}_b \to \mathbb{P}^1_{\mathbb{C}}$ at any $\mathbb{C}$-rational point $b \in \mathscr{H}$ is weakly isomorphic to the G-cover parametrised by b. We call $\mathscr{V} \xrightarrow{G} \mathbb{P}^1_{\mathscr{H}}$ the *universal family* over the Hurwitz space $\mathscr{H}$.

1.4. Degeneration of covers

Let G be a finite group and $\mathbf{C}$ a quadruple of non-trivial conjugacy classes in G. Recall that $\mathscr{H}_{G,\mathbf{C}}$ denotes the Hurwitz space parametrising G-covers of $\mathbb{P}^1_{\mathbb{C}}$ branched at four ordered pairwise disjoint points $x_1 = 0$, $x_2 = 1$, $x_3 = \lambda$, $x_4 = \infty$ with ramification type $\mathbf{C}$ (see Definition 1.2.5).

The Hurwitz space $\mathscr{H}_{G,\mathbf{C}}$ admits a natural compactification $\overline{\mathscr{H}}_{G,\mathbf{C}}$ as a moduli space of admissible G-covers ([**RW06**], [**Wew98**]). In our situation $\mathscr{H}_{G,\mathbf{C}}$ is a curve and $\overline{\mathscr{H}}_{G,\mathbf{C}}$ as curve is the smooth compactification of $\mathscr{H}_{G,\mathbf{C}}$. The notion admissible covers goes back to [**HM82**]. Admissible covers over a field (here $\mathbb{C}$) are finite maps between semistable curves (i.e. projective connected reduced curves having at most ordinary double points as singularities). For our purposes we do not need the definition of admissible covers. It suffices to describe the boundary $\overline{\mathscr{H}}_{G,\mathbf{C}} - \mathscr{H}_{G,\mathbf{C}}$ in terms of degenerate Galois covers. For the definition and more details we refer to [**Wew98**], [**Wew99**] or [**RW06**, Section 5].

Recall that we have a cover, called branch locus map,

$$\Psi\colon \mathscr{H}_{G,\mathbf{C}} \to \mathscr{M}_0^{(4)} \simeq \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$$

which sends a G-cover to the branch point $x_3 = \lambda$. The cover Ψ extends to a branched cover $\Psi\colon \overline{\mathscr{H}}_{G,\mathbf{C}} \to \mathbb{P}^1_{\mathbb{C}}$. For simplicity we only describe the fibre $\Psi^{-1}(0) \subset \overline{\mathscr{H}}_{G,\mathbf{C}} - \mathscr{H}_{G,\mathbf{C}}$. The fibres above 1 and ∞ may be described similarly.

Fix $\lambda \in \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ and recall that we may associate with any point in the fibre $\Psi^{-1}(\lambda) \subset \mathscr{H}_{G,\mathbf{C}}$ a Nielsen tuple $\mathbf{g} = [g_1, g_2, g_3, g_4] \in \mathrm{Ni}(G, \mathbf{C})$, depending on the choice of generators γ_1, γ_2, γ_3, γ_4 of $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty, \lambda\}, x_0)$.

If λ moves to 0, the effect on γ_1, γ_2, γ_3, γ_4 is described by contracting a loop γ homeomorphic to $(\gamma_2\gamma_4)^{-1} = \gamma_1\gamma_2\gamma_3\gamma_2^{-1} \in \pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty, \lambda\}, x_0)$ with $0, \lambda$ in its interior and $1, \infty$ in its exterior. The contraction produces a semistable genus-0 curve $\overline{P}$ consisting of two projective lines $\overline{P}_1$ and $\overline{P}_2$ intersecting in one ordinary double point ξ such that the exterior of γ specialises to $\overline{P}_1$ (i.e. $1, \infty \in \overline{P}_1$) and the interior of γ specialises to $\overline{P}_2$ (i.e. $0, \lambda \in \overline{P}_2$). On $\overline{P}_1$ the interior of γ becomes the interior of a simple closed loop $\gamma_\xi = \gamma_1\gamma_2\gamma_3\gamma_2^{-1}$ winding around ξ. On $\overline{P}_2$ the

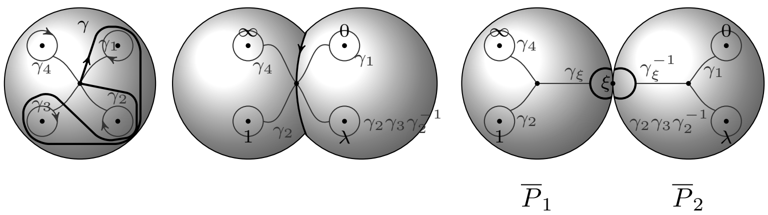


FIGURE 1.5. Degeneration.

exterior of γ becomes the interior of a simple closed loop γ_ξ^{-1} winding around ξ. In other words, for $\lambda \to 0$ the branch points 0 and λ of the cover parametrised by

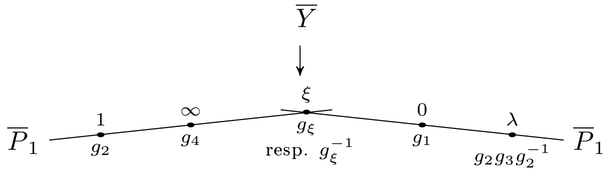


FIGURE 1.6. Degeneration.

λ coalesce in ξ on $\overline{P}_1$ with local monodromy $g_\xi = g_1g_2g_3g_2^{-1}$, whereas on $\overline{P}_2$ the branch points 1 and ∞ coalesce in ξ with local monodromy g_ξ^{-1}. This is illustrated in Figure 1.5 and Figure 1.6.

PROPOSITION 1.4.1. *(i) There is a bijection between the set of points in* $\Psi^{-1}(0) \subset \overline{\mathscr{H}}_{G,\mathbf{C}} - \mathscr{H}_{G,\mathbf{C}}$ *and the set of orbits under the action of the braid* $b_1 = \mathfrak{Q}_2\mathfrak{Q}_1^2\mathfrak{Q}_2^{-1}$ *from Proposition 1.3.4 on the set* $\mathrm{Ni}(G,\mathbf{C})$.
(ii) We may describe the bijection from (i) explicitly. Let $\mathbf{g} = [g_1, g_2, g_3, g_4] \in \mathrm{Ni}(G,\mathbf{C})$. *The admissible G-cover* $\overline{Y} \to \overline{P}$ *corresponding to the orbit of* $\mathbf{g}$ *satisfies: The restriction* $\overline{Y}|_{\overline{P}_i}$ *of* $\overline{Y}$ *to a cover over* $\overline{P}_i$ *is an induced cover* $\overline{Y}|_{\overline{P}_i} = \mathrm{Ind}_{G_i}^G \overline{Y}_i$, *where* $\overline{Y}_i$ *is a suitable connected component of* $\overline{Y}$ *and* $\overline{Y}_i \to \overline{P}_i$ *is a G_i-cover with*

$$G_1 = \langle g_2, g_4 \rangle, \quad G_2 = \langle g_1, g_2g_3g_2^{-1} \rangle$$

and ramification type

$$\mathbf{C}_1 = \big(\mathrm{Cl}_{G_1}(g_\xi),\ \mathrm{Cl}_{G_1}(g_2),\ \mathrm{Cl}_{G_1}(g_4)\big),$$
$$\mathbf{C}_2 = \big(\mathrm{Cl}_{G_1}(g_1),\ \mathrm{Cl}_{G_2}(g_2g_3g_2^{-1}),\ \mathrm{Cl}_{G_2}(g_\xi^{-1})\big).$$

Here we have $g_\xi^{-1} = g_2g_4$ *and the inertia groups of points in* $\overline{Y}$ *above* $\xi \in \overline{P}_1 \cap \overline{P}_2$ *are conjugate to* $\langle g_\xi \rangle$.

PROOF. This is shown in [**Wew98**, Proposition 4.3.2]. □

An example for calculating degenerations is given in Chapter 4, Example 4.3.3.

REMARK 1.4.2 (Induced cover)**.** We recall the notion of induced covers (following [**Sza09**, Construction 3.5.2]). Let G be a finite group and H a subgroup of

G. Let $Y \to \mathbb{P}^1_{\mathbb{C}}$ be a H-cover. We construct a cover whose covering space consists of $[G:H]$ copies of Y indexed by the left cosets of G such that the action of G on the covering space is the one that is induced by the action of H on Y. More precisely, we consider the topological product

$$\mathrm{Ind}_H^G Y := (G/H) \times Y,$$

where G/H is equipped with the discrete topology. We fix a complete system of representatives $\mathscr{R} \supset \{1\}$ for G/H and put

$$\ell Y := \{\ell\} \times Y, \qquad \text{for } \ell \in \mathscr{R}.$$

(All ℓY, $\ell \in \mathscr{R}$, are isomorphic.) By abuse of notation we often write $Y = \{1\} \times Y$. Note that for all $y \in Y$ and $\ell \in \mathscr{R}$ a point $\ell y := (\ell, y) \in \ell Y \subset \mathrm{Ind}_H^G Y$ is given. Now we define an action of G on $\mathrm{Ind}_H^G Y$. Let $\ell \in \mathscr{R}$ and $y \in Y$. For all $g \in G$ we have $g\ell \in sH$ for some $s \in \mathscr{R}$. Since $s^{-1}g\ell \in H$, by

$$g(\ell y) := g\ell y := s \cdot s^{-1}g\ell(y) \in sY$$

a faithful action of G on $\mathrm{Ind}_H^G Y$ is given and the projection

$$\mathrm{Ind}_H^G Y \to \mathrm{Ind}_H^G Y/G \simeq \mathbb{P}^1_{\mathbb{C}}$$

is called the (in general disconnected) G-cover *induced* by $Y \xrightarrow{H} \mathbb{P}^1_{\mathbb{C}}$.

1.5. Quotient covers

Let $\pi\colon Y \to \mathbb{P}^1_{\mathbb{C}}$ be a G-cover with ordered branch locus $D = (x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty)$ and associated Nielsen tuple $(g_1, g_2, g_3, g_4) \in \mathrm{Ni}_4(G)$. Let H be an arbitrary index-d subgroup of G. The cover π factors through a cover $\epsilon\colon X := Y/H \to \mathbb{P}^1_{\mathbb{C}}$.

$$\begin{tikzcd} Y \arrow[rr, "\pi"] \arrow[dr] & & \mathbb{P}^1_{\mathbb{C}} \\ & X = Y/H \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

Let $x_0 \in \mathbb{P}^1_{\mathbb{C}} - D$ and write $\pi_1(\mathbb{P}^1_{\mathbb{C}} - D, x_0) \to \mathrm{Sym}(\epsilon^{-1}(x_0))$ for the monodromy of ϵ (see Definition 1.2.1). We denote by $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in \mathrm{Sym}(\epsilon^{-1}(x_0)) = S_d$ the images of the generators $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ of $\pi_1(\mathbb{P}^1_{\mathbb{C}} - D, x_0)$ from (1.2.1), i.e. the local monodromies of ϵ in x_1, x_2, x_3, x_4.

Proposition 1.5.1. *There exists a bijection of sets $\epsilon^{-1}(x_0) \simeq G/H$ such that the permutation $\varrho_i \in \mathrm{Sym}(\epsilon^{-1}(x_0)) \simeq \mathrm{Sym}(G/H)$ maps a left coset $\ell H \in G/H$ to the left coset $g_i^{-1}\ell H$.*

Proof. Let $y \in \pi^{-1}(x_0)$. Then $\pi^{-1}(x_0) = \{gy;\ g \in G\}$ and the monodromy of π is given by

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - D, x_0) \twoheadrightarrow \mathrm{Mon}(\pi) \subset \mathrm{Sym}(\pi^{-1}(x_0)), \qquad \gamma_i \mapsto [gy \mapsto g_i^{-1}gy].$$

This implies that $\epsilon^{-1}(x_0) = \{\ell Hy;\ \ell \in G/H\}$ and the monodromy of ϵ is given by

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - D, x_0) \twoheadrightarrow \mathrm{Mon}(\epsilon) \subset \mathrm{Sym}(\epsilon^{-1}(x_0)), \qquad \gamma_i \mapsto \varrho_i = [\ell Hy \mapsto g_i^{-1}\ell Hy].$$

The proposition follows since $\epsilon^{-1}(x_0) = \{\ell Hy; \ell \in G/H\} \simeq G/H$. □

Remark 1.5.2. Let $x_i \in D$ be a ramification point. The fibre $\epsilon^{-1}(x_i)$ consists of the orbits of the H-action on $\pi^{-1}(x_i)$. Such an orbit is represented by an orbit of the right action of H on $\langle g_i \rangle \backslash G$ (see Remark 1.2.4). In other words, the fibre $\epsilon^{-1}(x_i)$ is represented by double cosets in $\langle g_i \rangle \backslash G/H$. Now let $\langle g_i \rangle \ell H \in \langle g_i \rangle \backslash G/H$ be such a representative. One easily checks that the left action of $\langle g_i \rangle$ on G/H has as stabiliser of ℓH the set $\ell H \ell^{-1} \cap \langle g_i \rangle$. Therefore the orbit of ℓH has length $e_\ell := \frac{\mathrm{ord}\, g_i}{|\ell H \ell^{-1} \cap \langle g_i \rangle|}$, and e_ℓ is the ramification index of the point represented by ℓ. With these ramification indices one could calculate the genus of Y/H.

Alternatively, one may also calculate the genus of Y/H using the following lemma.

Lemma 1.5.3 (Riemann-Hurwitz)**.** *Let $Y \to \mathbb{P}^1_{\mathbb{C}}$ be a G-cover with ordered branch locus $D = (x_1, \ldots, x_4)$ and associated Nielsen tuple $(g_1, \ldots, g_4) \in \mathrm{Ni}_4(G)$. Let H be an arbitrary subgroup of G. Then the genus of Y/H equals*

$$g(Y/H) = 1 + [G:H] - \frac{1}{2 \cdot |H|} \cdot \sum_{\substack{i=1,\ldots,4 \\ \ell \in G/\langle g_i \rangle}} |H \cap \langle \ell g_i \ell^{-1} \rangle|.$$

Proof. The stabiliser of $\ell \in G/\langle g_i \rangle$ under the left action of the group H on $G/\langle g_i \rangle$ equals $H \cap \langle \ell g_i \ell^{-1} \rangle$. Hence, by the Riemann-Hurwitz genus formula applied to $Y \to Y/H$, we have

$$2 \cdot g(Y) - 2 = (2 \cdot g(Y/H) - 2) \cdot |H| + \sum_{\substack{i=1,\ldots,4 \\ \ell \in G/\langle g_i \rangle}} (|H \cap \langle \ell g_i \ell^{-1} \rangle| - 1).$$

On the other hand, by the Riemann-Hurwitz formula applied to $Y \to \mathbb{P}^1_{\mathbb{C}}$, we have

$$2 \cdot g(Y) - 2 = -2 \cdot |G| + \sum_{i=1,\ldots,4} [G : \langle g_i \rangle] \cdot (|\langle g_i \rangle| - 1).$$

Combining the two formulas yields

$$\begin{aligned} g(Y/H) &= 1 - \tfrac{|G|}{|H|} - \tfrac{1}{2\cdot|H|} \cdot \left(\sum_{i,\ell} (|H \cap \langle \ell g_i \ell^{-1} \rangle| - 1) + \sum_{x} \tfrac{|G|}{|\langle g_i \rangle|} \cdot (|\langle g_i \rangle| - 1) \right) \\ &= 1 - [G:H] - \tfrac{1}{2\cdot|H|} \cdot \left(\sum_{x} |G| + \sum_{x,\ell} |H \cap \langle \ell g_i \ell^{-1} \rangle| \right), \end{aligned}$$

which implies the statement. Note that we have used that $\frac{|G|}{|\langle g_i \rangle|} = \sum_{\ell \in G/\langle g_i \rangle} 1$. □

Notation 1.5.4. Let $\varrho \in S_d$ be a permutation consisting of $\nu_1 + \cdots + \nu_r$ cycles. Suppose that ν_i of the cycles have length e_i $(i = 1, \ldots, r)$. Then we denote by $\mathrm{Cl}_{S_d}(e_1^{\nu_1} \cdots e_r^{\nu_r})$ the conjugacy class of ϱ in S_d.

Example 1.5.5. Let $m \geq 3$ be an odd integer and define $n := 2m$. Let $\pi\colon Y \to \mathbb{P}^1_{\mathbb{C}}$ be a G-cover with

$$G = \langle \varphi, \sigma;\ \varphi^n = \sigma^2 = 1,\ \sigma\varphi\sigma = \varphi^{-1}\rangle$$

and associated Nielsen tuple

$$(g_1, g_2, g_3, g_4) = (\varphi^{m+2}, \sigma, \varphi^m, \varphi^2\sigma) \in \mathrm{Ni}_4(G).$$

We consider the quotient cover $\epsilon\colon Y/H \to \mathbb{P}^1_{\mathbb{C}}$ with $H = \langle\sigma\rangle$ and fix a point $x_0 \in \mathbb{P}^1_{\mathbb{C}} - D$. Note that $\mathrm{Mon}(\epsilon) \simeq G$ since the normal closure of $H \subset G$ (i.e. the intersection of all normal subgroups of G containing H) is G. We make the identification $\epsilon^{-1}(x_0) = \{\varphi^k H;\ k = 1, \ldots, n\}$. One easily checks that

$$g_1^{-1}\varphi^k H = \varphi^{k+m-2}H,\ g_2^{-1}\varphi^k H = \varphi^{-1}H,\ g_3^{-1}\varphi^k H = \varphi^{k+m}H,\ g_4^{-1}\varphi^k H = \varphi^{-k+2}H.$$

The local monodromies $(\varrho_1, \varrho_2, \varrho_3, \varrho_4) \in (S_n)^4$ of ϵ at (x_1, x_2, x_3, x_4) are given by

$$\varrho_1(k) \equiv k + m - 2,\ \ \varrho_2(k) \equiv -k,\ \ \varrho_3(k) \equiv k + m,\ \ \varrho_4(k) \equiv -k + 2 \qquad (\mathrm{mod}\ n),$$

respectively. Both ϱ_2 and ϱ_4 are of order 2 with precisely two fixed points, respectively. Moreover, ϱ_1 is of order n and ϱ_3 is of order two without fixed points. In other words,

$$\varrho_1 \in \mathrm{Cl}_{S_{2n}}(n^1), \quad \varrho_2, \varrho_4 \in \mathrm{Cl}_{S_{2n}}(1^2 2^{m-1}), \quad \varrho_3 \in \mathrm{Cl}_{S_{2n}}(2^m).$$

(see Notation 1.5.4). Using the Riemann-Hurwitz genus formula, we conclude that the genus of $X = Y/H$ equals $g(X) = \frac{m-1}{2}$.

Degeneration of quotient covers. Let $Y \to \mathbb{P}^1_{\mathbb{C}}$ be a G-cover branched at four points with associated Nielsen tuple $(g_1, g_2, g_3, g_4) \in \mathrm{Ni}(G)$. Let $(\varrho_1, \varrho_2, \varrho_3, \varrho_4) \in (S_d)^4 = \mathrm{Sym}(G/H)^4$ be the quadruple of local monodromies of the quotient $X := Y/H \to \mathbb{P}^1_{\mathbb{C}}$ (Proposition 1.5.1).

As in Section 1.4, the branch locus map $\Psi\colon \mathcal{H}_G^{(4)} \to \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ extends to a branched cover $\Psi\colon \overline{\mathcal{H}}_G^{(4)} \to \mathbb{P}^1_{\mathbb{C}}$. Write

$$\overline{Y} \to \overline{P} = \overline{P}_1 \cup \overline{P}_2$$

for the admissible G-cover in $\Psi^{-1}(0) \subset \overline{\mathcal{H}}_G^{(4)} - \mathcal{H}_G^{(4)}$ corresponding to (g_1, g_2, g_3, g_4) (in the sense of Proposition 1.4.1) given by the subgroups

$$G_1 = \langle g_2, g_4\rangle, \quad G_2 = \langle g_1, g_2 g_3 g_2^{-1}\rangle$$

of G. Put $g_\xi := (g_2 g_4)^{-1}$.

We define the subgroups

$$\Sigma_1 = \langle \varrho_2, \varrho_4\rangle, \qquad \Sigma_2 = \langle \varrho_1, \varrho_2^{-1}\varrho_3, \varrho_2\rangle$$

of $\mathrm{Sym}(G/H)$ and denote by Σ_i^ℓ the group consisting of the restrictions of the permutations in Σ_i to the double cosets in $\{g\ell H;\ g \in G_i\}$.

Proposition 1.5.6. *We consider the quotient*

$$\overline{X} := \overline{Y}/H \to \overline{P} = \overline{P}_1 \cup \overline{P}_2.$$

(i) There is a bijection between the irreducible components of the restriction $\overline{X}|_{\overline{P}_i}$ and the double cosets $G_i\backslash G/H$. The irreducible component $\overline{X}_i^\ell$ corresponding to the double coset $G_i\ell H$ is a cover of $\overline{P}_i$ with monodromy group Σ_i^ℓ.
(ii) The local monodromies at three branch points of $\overline{X}_1^\ell \to \overline{P}_1$ are the restrictions of

$$\varrho_2, \varrho_4, \varrho_\xi := (\varrho_4\varrho_2)^{-1} \in \mathrm{Sym}(G/H)$$

to $\{g\ell H;\ g \in G_i\}$, where ϱ_ξ is the local monodromy at the ordinary double point $\xi \in \overline{P}_1 \cap \overline{P}_2$.
(iii) The local monodromies at the three branch points of $\overline{X}_2^\ell \to \overline{P}_1$ are the restrictions of

$$\varrho_1, \varrho_2^{-1}\varrho_3\varrho_2, \varrho_\xi^{-1} \in \mathrm{Sym}(G/H)$$

to $\{g\ell H;\ g \in G_i\}$, where ϱ_ξ^{-1} is the local monodromy at ξ.
(iv) The number of ordinary double points of $\overline{X}$ equals $|\langle g_\xi\rangle\backslash G/H|$.

Proof. This can be easily proved using the description given in Proposition 1.4.1. $\square$

Example 1.5.7. We keep the notations from Example 1.5.5 and calculate the admissible cover $\overline{X} = \overline{Y}/H \to \overline{P}_1 \cup \overline{P}_2$ corresponding to $(g_1, g_2, g_3, g_4) = (\varphi^{m+2}, \sigma, \varphi^m, \varphi^2\sigma)$. We have $G_1 = \langle g_2, g_4\rangle = \langle\varphi^2, \sigma\rangle$ and $G_2 = \langle g_1, g_2g_3g_2^{-1}\rangle = \langle\varphi\rangle$. Therefore $G_1\backslash G/H = \{G_1\varphi^\ell H;\ \ell = 1, 2\}$ and $G_2\backslash G/H = \{G_2\varphi^\ell H;\ \ell = 1\}$. We conclude that the restriction $\overline{X}|_{\overline{P}_1}$ consists of two irreducible components $\overline{X}_1^1$, $\overline{X}_1^2$, and that the restriction $\overline{X}|_{\overline{P}_2}$ consists of one irreducible component. The restriction of ϱ_2, ϱ_4 and $\varrho_\xi := (\varrho_2\varrho_4)^{-1}$ to $\{\varphi^k H;\ k \text{ odd}\}$ are the local monodromies of $\overline{X}_1^1 \to \overline{P}_1$ where ϱ_ξ is the local monodromy at the ordinary double point $\xi \in \overline{P}_1 \cap \overline{P}_2$.. Using Notation 1.5.4, the restriction of $(\varrho_4, \varrho_2, \varrho_\xi)$ is a tuple in

$$\left(\mathrm{Cl}_{S_m}(1^1 2^{(m-1)/2}),\ \mathrm{Cl}_{S_m}(1^1 2^{(m-1)/2}),\ \mathrm{Cl}_{S_m}(m^1)\right).$$

The restrictions of $(\varrho_4, \varrho_2, \varrho_\xi)$ to $\{\varphi^k H;\ k \text{ even}\}$ are the local monodromies of $\overline{X}_1^2 \to \overline{P}_1$. They form a tuple in

$$\left(\mathrm{Cl}_{S_m}(1^1 2^{(m-1)/2}),\ \mathrm{Cl}_{S_m}(1^1 2^{(m-1)/2}),\ \mathrm{Cl}_{S_m}(m^1)\right).$$

The permutations $(\varrho_\xi^{-1}, \varrho_2\varrho_3\varrho_2^{-1}, \varrho_1)$ in $\mathrm{Sym}(G/H)$ are the local monodromies of $\overline{X}_2 \to \overline{P}_2$ and form a tuple in

$$\left(\mathrm{Cl}_{S_n}(m^2), \mathrm{Cl}_{S_n}(2^m), \mathrm{Cl}_{S_n}(n^1)\right).$$

The Riemann-Hurwitz genus formula implies that $g(\overline{X}_1^1) = g(\overline{X}_1^2) = 0$ and $g(\overline{X}_2) = \frac{m-1}{2}$. Figure 1.7 illustrates the admissible cover $\overline{X} \to \overline{P}_1 \cup \overline{P}_2$. Note that we may

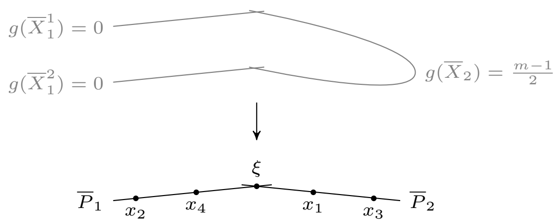


FIGURE 1.7. Degeneration of ϵ

contract the irreducible components $\overline{X}_1^1$ and $\overline{X}_1^2$ of genus 0, and obtain a smooth curve $\overline{X}_2$.

REMARK 1.5.8. The degeneration $\overline{X}$ described in Example 1.5.7 is the fibre of the curve $\mathscr{X} \to \overline{T}$ from Theorem 3.3.8 at $b = 0$.

Chapter 2

The de Rham cohomology of families of smooth curves

In this chapter we discuss a special class of flat vector bundles (a notion introduced in Chapter 3): the first de Rham cohomology of a family of smooth curves equipped with the Gauß-Manin connection. Most of the arguments are not needed in Chapter 3, but may help to understand the notion of a flat vector bundle and the calculation made in Chapter 3.

Roughly speaking, the Gauß-Manin connection 'is' the parameter derivation of 1-forms on a family of curves with respect to the parameter of this family ([**KO68**]). It can be made explicit by a Fuchsian differential equation – the Picard-Fuchs equation. This is discussed in Section 2.2. In Section 2.2.1 we describe Picard-Fuchs equations 'coming from' superelliptic curves. For more details on the classical theory of Fuchsian differential equations we refer to Appendix A.

The main part of this chapter is Section 2.3, where we decompose the de Rham cohomology of families of metacyclic covers into isotypical components. In Chapter 3 and 4 we decompose de Rham cohomologies in exactly this way.

2.1. The relative de Rham cohomology

Let $\overline{S}$ be a smooth projective connected curve over $\mathbb{C}$ and $S \subset \overline{S}$ the complement of finitely many $\mathbb{C}$-rational points. Let $f\colon \mathcal{Y} \to S$ be a smooth curve, i.e. f is a proper and smooth morphism of relative dimension 1 with connected geometric fibres. We may interpret $\mathcal{Y}$ as a family of smooth curves parametrized by S. The genus g of the fibres of f is constant since S is connected.

The higher direct image sheaf $R^1 f_* \underline{\mathbb{C}}$ can be identified with the local system on S whose fibre at $b \in S$ is the first de Rham cohomology $\mathbb{C}$-vector space $H^1_{\mathrm{dR}}(\mathcal{Y}_b)$, consisting of closed 1-forms on $\mathcal{Y}_b$ modulo exact forms. We write

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/S) := R^1 f_* \underline{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_S$$

for the associated $\mathcal{O}_S$-module, which we call the *relative de Rham cohomology* of $\mathcal{Y} \to S$. It comes equipped with the Gauß-Manin connection

$$\nabla \colon \mathcal{H}_{\mathrm{dR}}(\mathcal{Y}/S) \to \mathcal{H}_{\mathrm{dR}}(\mathcal{Y}/S) \otimes_{\mathcal{O}_S} \Omega^1_S, \qquad \nabla(\omega \otimes f) = \omega \otimes \mathrm{d}f,$$

an additive map satisfying the Leibniz rule

$$\nabla(f\omega) = \omega \otimes \mathrm{d}f + f\nabla(\omega)$$

for local sections f and ω of $\mathcal{O}_S$ and $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/S)$, respectively. For more details we refer to [**BP02**, Sections 1 and 2].

We now suppose that $\mathrm{Aut}_S(\mathcal{Y})$ contains a subgroup G such that $\mathcal{Y}/G \simeq \mathbb{P}^1_S$. We fix a quadruple $\mathbf{C} = (\mathrm{Cl}(g_1), \ldots, \mathrm{Cl}(g_4))$ of conjugacy classes in G. Moreover we suppose that $\pi \colon \mathcal{Y} \xrightarrow{G} \mathbb{P}^1_S$ is a G-cover with (ordered) branch locus $(x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty)$ such that the local monodromy of $\pi|_{\mathcal{Y}_b}$ in $x_i(b)$ is an element of $\mathrm{Cl}(g_i)$ for all $i = 1, \ldots, 4$ and $b \in S$. This is exactly the situation we consider in Chapter 3 and 4

We explain how to decompose the de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/S)$ into G-isotypical components.

Lemma 2.1.1. *Let Y and X be smooth $\mathbb{C}$-curves and $Y \to X$ a G-cover branched at 4 points with local monodromies $g_1, \ldots, g_4$ (see Remark 1.2.2). Let Ш_Y be the character of the representation $G \to \mathrm{GL}(H^1_{\mathrm{dR}}(Y))$ induced by the action of the Galois group on Y. Then*

$$\text{Ш}_Y = 2 \cdot \mathbb{1} + (2 \cdot g(X) - 2) \cdot \mathrm{Ind}^G_{\langle 1 \rangle} \mathbb{1} + \sum_{i=1}^{4} \left(\mathrm{Ind}^G_{\langle 1 \rangle} \mathbb{1} - \mathrm{Ind}^G_{\langle g_i \rangle} \mathbb{1} \right).$$

This character only depends on the conjugacy classes of $g_1, \ldots, g_4$ in G, but not on the individual representatives.

Proof. This follows completely analogous to [**Ell01**, Proposition 1.3]. $\square$

Remark 2.1.2 (Isotypical decomposition). Write $\text{Ш}_{\mathcal{Y}_b}$ for the character of the representation

$$G \to \mathrm{GL}\left(H^1_{\mathrm{dR}}(\mathcal{Y}_b)\right) \tag{2.1.1}$$

induced by the Galois action G on $\mathscr{Y}_b$. By Lemma 2.1.1 we have $\text{ш} := \text{ш}_{\mathscr{Y}_b} = \text{ш}_{\mathscr{Y}_{b'}}$ for all $b, b' \in S$ and the fibre-wise decomposition of $H^1_{\mathrm{dR}}(\mathscr{Y}_b)$ into G-isotypical components induces the decomposition

$$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S) = \bigoplus_{\theta \in \mathrm{Irr}(G)} \mathscr{E}_\theta, \tag{2.1.2}$$

into isotypical components $\mathscr{E}_\theta := \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)_\theta$, whose fibres at $b \in S$ are the θ-isotypical components of $H^1_{\mathrm{dR}}(\mathscr{Y}_b)$ (i.e. the direct summand of those irreducible subrepresentations of $H^1_{\mathrm{dR}}(\mathscr{Y}_b)$ having character θ). Here, $\mathrm{Irr}(G)$ denotes the set of irreducible G-characters. The Gauß-Manin connection restricts to a connection $\nabla \colon \mathscr{E}_\theta \to \mathscr{E}_\theta \otimes \Omega^1_S$ since the action of G on $\mathscr{Y}$ is defined over S (see [**Kat81**, Section I]).

Recall from [**Ser77**] that the number of irreducible subrepresentations of $G \to H^1_{\mathrm{dR}}(\mathscr{Y}_b))$ having character $\theta \in \mathrm{Irr}(G)$ equals

$$n_\theta := \langle \text{ш}, \theta \rangle_G,$$

where $\langle \cdot, \cdot \rangle_G$ is the scalar product on the set of G-characters given by

$$\langle v, w \rangle := |G|^{-1} \cdot \sum_{g \in G} v(g) \cdot w(g).$$

For a subgroup $H \subset G$ we write

$$(\mathscr{E}_\theta)^H \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$$

for the H-invariant submodule. The character of $G \to \mathrm{GL}(\mathscr{E}_\theta)$ is $n_\theta \cdot \theta$ and we have

$$\mathrm{rank}\,(\mathscr{E}_\theta)^H = n_\theta \cdot \langle \mathbb{1}, \mathrm{Res}_H\, \theta \rangle_H,$$

where $\mathbb{1}$ denotes the trivial character.

2.2. Picard-Fuchs equations

The goal of this section is to recall from [**BP02**, II.9] (in [**BDIP02**]) and [**Pet86**, Section 6] how a family of smooth projective curves with one parameter gives rise to a Fuchsian differential equation on the parameter space of the family, called Picard-Fuchs equation.

Let $\overline{S}$ be a smooth projective connected curve over $\mathbb{C}$ and $S \subset \overline{S}$ the complement of finitely many $\mathbb{C}$-rational points. Moreover let $f \colon \mathscr{Y} \to S$ be a smooth curve, i.e. f is a proper and smooth morphism of relative dimension 1 with connected geometric fibres. For a local parameter s at a point in $\overline{S}$ regular and non-zero on S, we denote by

$$\nabla(\tfrac{\partial}{\partial s}) \colon \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S) \xrightarrow{\nabla} \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S) \otimes \Omega^1_S \xrightarrow{1 \otimes D} \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S) \otimes \mathscr{O}_S = \mathscr{E},$$

where $D(\mathrm{d}s) = 1$, the *contraction* of ∇ against $\frac{\partial}{\partial s}$. We choose a section ω of $H^0(\mathscr{Y}, \Omega^1_{\mathscr{Y}/S})$ and let p be the smallest integer such that $\omega, \nabla(\partial/\partial s)\omega, \ldots, \nabla(\partial/\partial s)^p$

are linearly dependent in $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$. Then there exist unique sections $c_0, \dots, c_{p-1}$ of $\mathscr{O}_S$ such that $\nabla(\partial/\partial s)^p\,\omega + c_{p-1}\cdot\nabla(\partial/\partial s)\,\omega + \cdots + c_0\,\omega = 0$.

Definition 2.2.1. The corresponding ordinary differential operator

$$L := \left(\tfrac{\partial}{\partial s}\right)^p + c_{p-1}\cdot\left(\tfrac{\partial}{\partial s}\right)^{p-1} + c_1\cdot\left(\tfrac{\partial}{\partial s}\right) + \cdots + c_0$$

is called the *Picard-Fuchs operator* associated with ω.

Let $\mathscr{E}\subset\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ be the submodule generated $\omega, \nabla(\partial/\partial s)\omega, \dots, \nabla(\partial/\partial s)^{p-1}$. The Gauß-Manin connection restricts to a connection on $\mathscr{E}$. The section ω of $\mathscr{E}$ is called *cyclic section* of $\mathscr{E}$.

We describe the local system $\mathrm{Sol}(L)$ of solutions of L more precisely. Denote by $\mathscr{E}^\vee$ the dual of $\mathscr{E}$ equipped with the dual connection given by

$$\langle\nabla(\partial/\partial s)\,\omega,\gamma\rangle + \langle\omega,\nabla^\vee(\partial/\partial s)\,\gamma\rangle = \partial/\partial s\langle\,\omega,\gamma\rangle,$$

where $\langle\cdot,\cdot\rangle\colon\mathscr{E}\times\mathscr{E}^\vee\to\mathscr{O}_S$ is the corresponding pairing. In particular, for any section ω of $\mathscr{E}$ and any horizontal section γ of $\mathscr{E}^\vee$ (i.e. γ is a section of $\ker\nabla^\vee$) the derivative with respect to s of $\langle\omega,\gamma\rangle$ is

$$\tfrac{\partial}{\partial s}\,\langle\omega,\gamma\rangle = \langle\nabla(\partial/\partial s)\,\omega,\gamma\rangle. \tag{2.2.1}$$

Then one easily checks that

$$L(\omega,\gamma\rangle) = \langle\nabla(\partial/\partial s)^p\,\omega + c_{p-1}\cdot\nabla(\partial/\partial s)\;\omega + \cdots + c_0\,\omega,\ \gamma\rangle = 0.$$

Remark 2.2.2. This defines an isomorphism

$$\ker\nabla^\vee \xrightarrow{\sim} \mathrm{Sol}(L), \qquad \gamma\mapsto\langle\omega,\gamma\rangle.$$

In other words, the horizontal sections of $\mathscr{E}^\vee$ 'are' the solutions of the Picard-Fuchs differential equation $L=0$.

Let f be a nowhere zero section of $\mathscr{O}_S$. Note that a section ω is a cyclic section $\mathscr{E}$ if and only if $f\cdot\omega$ is a cyclic section of $\mathscr{E}$. If we denote by L' the Picard-Fuchs operator constructed via $f\cdot\omega$, then

$$L'\left(f\cdot\langle\omega,\gamma\rangle\right) = L'\left(\langle f\cdot\omega,\gamma\rangle\right) = 0.$$

By Appendix A, Definition A.0.13, L and L' are projectively equivalent. Therefore, in the following we speak of *the* Picard-Fuchs operator L of $\mathscr{E}$, meaning 'given up to projectively equivalence'.

Remark 2.2.3. Let L be the Picard-Fuchs operator of $\mathscr{E}$ and suppose that $p=2$. Then $\mathrm{ord}_b\,c_1(s)\geq -1$ and $\mathrm{ord}_b\,c_0(s)\geq -2$ ([**BP02**, Lemma 9.3]). Hence, the singularities of L are all regular, i.e. L is a Fuchsian differential operator (see Appendix A for the definition in case of second order differential equations).

2.2.1. An example: a class of superelliptic curves. A well-known example of a Picard-Fuchs equation is the one given by a holomorphic section of the de Rham cohomology of a superelliptic curve. This is also discussed in more detail in Chapter 3.

DEFINITION 2.2.4. We denote by $\overline{S} = \mathbb{P}^1_{\mathbb{C}}$ the projective line with coordinate s and set $S = \mathbb{P}^1_s - \{0, 1, \infty\}$. Let $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{Q}^4$ with $0 < \sigma_i < 1$ and $\sum_{i=1}^4 \sigma_i \in \mathbb{N}$ and denote by n the least common multiple of the denominators of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. We put $a_i = n{\cdot}\sigma_i$ for $i = 1, \dots, 4$ and consider the (smooth projective) *superelliptic curve* $\pi\colon \mathscr{X} \to S$ birationally determined by the affine equation

$$\mathscr{X}_s\colon z^n = x^{a_1}(x-1)^{a_2}(x-s)^{a_3}. \tag{2.2.2}$$

The tuple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is referred to as the *type* of the superelliptic curve.

Let $A \subset \mathrm{Aut}_S(\mathscr{X})$ be the cyclic subgroup of order n such that $\varphi \in A$ acts on $\mathscr{X}$ via

$$\varphi^* x = x, \quad \varphi^* z = \chi(\varphi) \cdot z, \tag{2.2.3}$$

where $\chi \in \mathrm{Hom}(A, \mathbb{C}^*)$ is a fixed injective character. Note that χ generates the group of irreducible A-characters, i.e.

$$\mathrm{Irr}(A) = \mathrm{Hom}(A, \mathbb{C}^\times) = \{\chi^0 = \mathbb{1}, \chi, \chi^2, \dots, \chi^{n-1}\}.$$

The projection

$$\mathscr{X} \to \mathbb{P}^1_S, \quad (x, z) \mapsto x$$

is an A-cover with ordered branch locus of cardinality 4 such that the fibres $\mathscr{X}_b \to \mathbb{P}^1_{\mathbb{C}}$ have ramification type

$$\mathbf{C} = (\mathrm{Cl}_A(\varphi^{a_1}), \mathrm{Cl}_A(\varphi^{a_2}), \mathrm{Cl}_A(\varphi^{a_3}), \mathrm{Cl}_A(\varphi^{a_4}))$$

for some fixed generator φ of A (see Definition 1.2.5). As an immediate consequence of Remark 2.1.2 we have a decomposition

$$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S) = \bigoplus_{k=1}^{n-1} \mathscr{E}_k, \tag{2.2.4}$$

such that the fibre of the submodule $\mathscr{E}_k \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ at any $b \in S$ is the χ^k-isotypical component of $H^1_{\mathrm{dR}}(\mathscr{X}_b)$. Note that for the trivial character χ^0 (i.e. $\ker \chi^0 = A$) we have $\mathscr{E}_0 = (\mathscr{E}_0)^A$ with fibres at $b \in S$ the relative de Rham cohomologies $H^1_{\mathrm{dR}}(\mathscr{X}_b)^A = H^1_{\mathrm{dR}}(\mathscr{Y}_b/A) = H^1_{\mathrm{dR}}(\mathbb{P}^1_{\mathbb{C}}) = \{0\}$. That is why the sum in (2.2.4) starts with $k = 1$. For $k \in \{1, \dots, n-1\}$ let $s(k)$ be the number of a_i unequal to 0 modulo $n/\gcd(k, n)$. Then $\mathrm{rank}\,\mathscr{E}_k = s(k) - 2$ ([**BM10b**, Lemma 4.1 (a)]). In particular, $\mathrm{rank}\,\mathscr{E}_k \in \{0, 2\}$.

REMARK 2.2.5. Note that $\mathscr{E}_k$ may be considered as a subbundle of the $\ker(\chi^k)$-invariant module $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)^{\ker(\chi^k)}$, which we may identify with the de

Rham cohomology of the quotient curve $\mathscr{X}/\ker(\chi^k)$. This quotient is also a cyclic cover of $\mathbb{P}^1_S$. It is therefore no restriction to describe only the components $\mathscr{E}_k$ where χ^k is an injective character. Moreover, replacing z by z^k for some k with $\gcd(k,n) = 1$ replaces χ by χ^k and the σ_i by $k\sigma_i \pmod 1$. For convenience we will in the following only consider the component $\mathscr{E}_1$. We drop the index from the notation, and write $\mathscr{E} = \mathscr{E}_1$.

LEMMA 2.2.6. *If $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 2$, then the differential form*

$$\omega := \frac{z\,\mathrm{d}x}{x(x-1)(x-s)}$$

defines a cyclic section of $\mathscr{E}$ and the associated Picard-Fuchs operator is the hypergeometric differential operator

$$L = \left(\frac{\partial}{\partial s}\right)^2 + \frac{(A+B+1)\,s - C}{s\,(s-1)} \cdot \left(\frac{\partial}{\partial s}\right) + \frac{AB}{s\,(s-1)},$$

where $A = 1 - \sigma_3$, $B = 2 - (\sigma_1 + \sigma_2 + \sigma_3)$ and $C = 2 - (\sigma_1 + \sigma_3)$.

PROOF. This is shown in [**Bou05**, Lemma 1.1.4]. The statement follows from the identity

$$\nabla(\partial/\partial s)^2\,\omega + \frac{(A+B+1)\,s - C}{s\,(s-1)}\,\nabla(\partial/\partial s)\,\omega + \frac{AB}{s\,(s-1)}\,\omega = \mathrm{d}\,\frac{x^{\sigma_1}(x-1)^{\sigma_2}}{(x-s)^{2-\sigma_3}}$$

and the fact that the right hand side is an exact 1-form on $\mathscr{X}_s$. $\square$

REMARK 2.2.7. Suppose that $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 2$. The local exponents of L at the singularities 0, 1 and ∞ is given by the following Riemann scheme (see Appendix A, Example A.0.14):

$$\begin{bmatrix} 0 & 1 & \infty \\ \hline 0 & 0 & A \\ 1-C & C-A-B & B \end{bmatrix}$$

Let $\pi\colon \overline{T} \to \overline{S} \subset S$ be a cover unbranched outside $\{0, 1, \infty\}$. Let $c \in \overline{T}$ with $\pi(c) = b = 0$ be a ramification point with ramification index e Denote by $\pi^*\mathscr{E}$ the pullback of $\mathscr{E}$ via π. If ω is a cyclic section of $\mathscr{E}$, then $\pi^*\omega$ is a cyclic section of $\pi^*\mathscr{E}$. We describe the pullback locally around the singularity $b = 0 \in S$ more precisely, in order to see how the local exponents behave under pullback. Similar calculation are made in Chapter 3 in order to 'remove' so called elliptic singularities of L (see Lemma 3.1.8). Let s be a local parameter at $b = 0$ and t a local parameter at c, satisfying $s = t^e$. Using the chain rule we conclude that

$$\begin{aligned} \pi^* \circ \tfrac{\partial}{\partial s} &= \left(\tfrac{1}{e \cdot t^{e-1}} \cdot \tfrac{\partial}{\partial t}\right) \circ \pi^*, \\ \pi^* \circ \left(\tfrac{\partial}{\partial s}\right)^2 &= \left(\tfrac{1}{e^2 \cdot t^{2e-2}} \cdot \left(\tfrac{\partial}{\partial t}\right)^2 - \tfrac{e-1}{e^2\, t^{2e-1}} \cdot \tfrac{\partial}{\partial t}\right) \circ \pi^*. \end{aligned}$$

Together with the relation $L\langle\omega,\gamma\rangle = 0$, for $\gamma \in \ker\nabla^\vee$, one easily checks that $\langle\pi^*\omega,\gamma\rangle$ is a solution of

$$\pi^*L\colon (\tfrac{\partial}{\partial t})^2 + P_k\cdot\tfrac{\partial}{\partial t} + Q_k$$

where

$$P_k = \frac{\big(1+e(A+B)\big)\cdot t^e + e\,(1-C) - 1}{t(t^e-1)},\quad Q_k = \frac{e^2AB\cdot t^{e-2}}{t^e-1}.$$

A standard calculation (e.g. using the Frobenius method [**Yos87**, I.2.5]) shows that the local exponents of π^*L at $c\in\pi^{-1}(0)$ are the ones from L at $b=0$ multiplied by e. Similar arguments apply for the singularities 1 and ∞.

2.2.2. Abelian covers. Let $\overline{T}$ be a smooth projective connected curve over $\mathbb{C}$ and $T\subset\overline{T}$ the complement of finitely many $\mathbb{C}$-rational points. Let A be an abelian group and suppose that

$$\mathscr{Y}\xrightarrow{A}\mathbb{P}^1_T$$

is an A-cover over T. As in Remark 2.1.2 we have a decomposition

$$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T) = \bigoplus_{\chi\in\mathrm{Irr}(A)}\mathscr{E}_\chi\,, \tag{2.2.5}$$

where $\mathscr{E}_\chi$ denotes the χ-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)$. We remark that $\mathrm{Irr}(A) = \mathrm{Hom}(A,\mathbb{C}^\times)$, since A is abelian.

We now describe the component $\mathscr{E}_\chi$ more precisely. We pass to the quotient $\mathscr{X}' := \mathscr{Y}/\ker\chi$ and note that $A' := A/\ker\chi \simeq \mathrm{im}\,\chi$ is a cyclic group. Given $\chi\in\mathrm{Hom}(A,\mathbb{C}^\times)$, we denote by $\bar\chi\in\mathrm{Hom}(A',\mathbb{C}^\times)$ the irreducible cyclic A'-character given by

$$\bar\chi(\varphi\cdot\ker\chi) = \chi(\varphi)$$

for all $\varphi\in A$. It is injective by construction. Note that $\big(\mathscr{E}_\chi\big)^{\ker\chi} = \mathscr{E}_\chi$ is the $\bar\chi$-isotypical component of the $\ker(\chi)$-invariant module $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)^{\ker(\chi)}$, which we may identify with $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}'/T)$.

In other words, $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)$ decomposes into isotypical components 'coming from suitable cyclic subcovers'.

Lemma 2.2.8. *Let $\chi\in\mathrm{Irr}(A)$ such that $\mathscr{X}' = \mathscr{Y}/\ker\chi$ is the pullback of a superelliptic curve*

$$\mathscr{X}\colon z^n = x^{a_1}(x-1)^{a_2}(x-s)^{a_3}$$

over $S=\mathbb{P}^1_{\mathbb{C}}-\{0,1,\infty\}$ with $n=|A'|$ via an étale cover $\pi\colon T\to S$. As in Remark 2.2.5 we may and do renormalise a_1,a_2,a_3,a_4 such that

$$\bar\varphi^*x = x,\qquad \bar\varphi^*z = \bar\chi(\bar\varphi)\cdot z$$

for $\bar\varphi\in A'\subset\mathrm{Aut}_T(\mathscr{X}')$. If $\sigma_1+\sigma_2+\sigma_3+\sigma_4=2$, then the Picard-Fuchs operator of $\mathscr{E}$ is the pullback of the hypergeometric differential operator L from Lemma 2.2.6.

PROOF. Note that $\bar{\chi}$ is injective by construction. By Lemma 2.2.6 the Picard-Fuchs operator of the $\bar{\chi}$-isotypical component of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}'/T)$ is the hypergeometric differential operator L. The rest follows from Remark 2.2.7. $\square$

2.3. Families of covers with semidirect Galois group

Let $\overline{T}$ be a smooth projective connected curve over $\mathbb{C}$ and $T \subset \overline{T}$ the complement of finitely many $\mathbb{C}$-rational points. Let $\mathcal{Y} \xrightarrow{G} \mathbb{P}^1_T$ be a G-cover and suppose that G is a semidirect product of a subgroup H by an abelian subgroup A (i.e. $G = A \cdot H$, A is normal in G and $A \cap H = \{1\}$). The goal of this section is to describe Picard-Fuchs equations on the parameter space T of the quotient curve $\mathcal{X} = \mathcal{Y}/H$. Our approach is similar to that used in [**Ell01**, Section 2] to exhibit curves with real multiplication.

As a first step we describe how H acts on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/T)$. Instead of considering the decomposition into A-isotypical components, we consider the 'coarser' decomposition

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/T) = \bigoplus_{\theta \in \mathrm{Irr}(G)} \mathcal{E}_\theta. \tag{2.3.1}$$

into G-isotypical components.

We recall from [**Ser77**, Section 8.2] the description of the irreducible characters of a semidirect product $G = A \rtimes H$ with A an abelian (normal) subgroup.

NOTATION 2.3.1. (i) For $h \in H$ and $\chi \in \mathrm{Irr}(A) = \mathrm{Hom}(A, \mathbb{C}^\times)$, we write χ^h for the irreducible A-character given by $\chi^h(a) = \chi(h^{-1}ah)$ for $a \in A$. This defines an action of H on $\mathrm{Irr}(A)$. We denote by H_χ resp. $H(\chi)$ the stabiliser resp. the orbit of $\chi \in \mathrm{Irr}(A)$.
(ii) We write $G_\chi := A \rtimes H_\chi$. For $\chi \in \mathrm{Irr}\, A$ and $\xi \in \mathrm{Irr}(H_\chi)$ we set

$$\theta_{\chi,\xi} = \mathrm{Ind}_{G_\chi}^G(\chi \cdot \xi),$$

where $(\chi \cdot \xi)(ah) := \chi(a)\xi(h)$ for $a \in A$ and $h \in H_\chi$.

The following lemma is proven in [**Ser77**, Proposition 25].

LEMMA 2.3.2. *Let $G = A \rtimes H$ be a semidirect product of H by an abelian group A. All irreducible G-characters are of the form $\theta_{\chi,\xi}$. Moreover, $\theta_{\chi,\xi} = \theta_{\chi',\xi'}$ if and only if $\chi' \in H(\chi)$ and $\xi = \xi'$.*

Using this description of the irreducible G-characters, we can write (2.3.1) as follows.

PROPOSITION 2.3.3. *Let $\mathcal{Y} \xrightarrow{G} \mathbb{P}^1_T$ be a G-cover with $G = A \rtimes H$ and A abelian. Then we have a decomposition*

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/T) = \bigoplus_{\substack{\chi \in \mathrm{Irr}(A)/H \\ \xi \in \mathrm{Irr}(H_\chi)}} \mathcal{E}_{\chi,\xi}\ ,$$

where $\mathscr{E}_{\chi,\xi}$ denotes the $\theta_{\chi,\xi}$-isotypical component. Moreover,

$$\mathscr{E}_{\chi,\mathbb{1}} = \bigoplus_{h\in H/H_\chi} \mathscr{E}_{\chi^h},$$

where $\mathscr{E}_{\chi^h}$ denotes the χ^h-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)$.

PROOF. The decomposition of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)$ follows from Lemma 2.3.2 and (2.3.1). For the decomposition of $\mathscr{E}_{\chi,\mathbb{1}}$, using [**Ser77**, Proposition 22], one checks that

$$\mathrm{Res}_A\,\theta_{\chi,\mathbb{1}} = \mathrm{Res}_A\,\mathrm{Ind}^G_{G_\chi}(\chi\cdot\mathbb{1}) = \sum_{s\in A\backslash G/G_\chi} \mathrm{Ind}^A_{A\cap sG_\chi s^{-1}}\,\chi^s = \sum_{h\in H/H_\chi} \chi^h.$$

This proves the proposition. $\square$

REMARK 2.3.4. Suppose that $\mathrm{rank}\,\mathscr{E}_\chi = 2$ and that ω is a cyclic section of $\mathscr{E}$. Then for all $h\in H$ the submodules $\mathscr{E}_{\chi^h}$ have the same Picard-Fuchs operators. The reason is that, for $h\in H_\chi$, it holds

$$\varphi h\omega = h\varphi^h\omega = h\chi^h(\varphi)\omega = \chi^h(\varphi)\cdot h\omega,$$

i.e. $h\omega$ is an eigenvector of φ with eigenvalue $\chi^h(\varphi)$; or – in other words – a section of $\mathscr{E}_{\chi^h}$. Moreover, as the automorphisms in H are defined over T it follows that

$$L(\langle h\omega,\gamma\rangle) = h\cdot L(\langle\omega,\gamma\rangle) = 0$$

(see [**Kat81**, Section I]).

We now consider the quotient curve

$$\mathscr{X} := \mathscr{Y}/H \to T$$

and describe how the decomposition in Proposition 2.3.3 behaves modulo H. Note that $(\mathscr{E}_{\chi,\xi})^H$ is a submodule of the H-invariant module $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)^H$, which we may identify with the de Rham cohomology $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/T)$.

We set

$$n_{\chi,\xi} := \langle \text{ш}, \theta_{\chi,\xi}\rangle_G,$$

where ш denotes the character of the representation $G\to\mathrm{GL}\left(H^1_{\mathrm{dR}}(\mathscr{Y}_b)\right)$, see Remark 2.1.2.

LEMMA 2.3.5. *We have*

$$\mathrm{rank}\,(\mathscr{E}_{\chi,\xi})^H = \begin{cases} n_{\chi,\xi}, & \text{if } \xi\in\mathrm{Irr}(H_\chi) \text{ is trivial},\\ 0, & \text{otherwise.}\end{cases}$$

PROOF. We have $\operatorname{rank}(\mathcal{E}_{\chi,\xi})^H = n_{\chi,\xi}\cdot\langle \mathbb{1}, \theta_{\chi,\xi}\rangle_H$ and

$$\begin{aligned}\langle \mathbb{1}, \operatorname{Res}_H \theta_{\chi,\xi}\rangle_H &= \sum_{\ell\in H\backslash G/G_\chi}\left\langle \mathbb{1}, \operatorname{Ind}^H_{H\cap \ell G_\chi \ell^{-1}}(\chi\cdot\xi)\right\rangle_H\\ &= \left\langle \mathbb{1}, \operatorname{Ind}^H_{H_\chi}\xi\right\rangle_H = \langle \mathbb{1},\xi\rangle_{H_\chi}\\ &= \begin{cases}1, \text{if } \xi=\mathbb{1},\\ 0, \text{otherwise.}\end{cases}\end{aligned}$$

Here we have used [**Ser77**, Proposition 22] for the first equality and $|H\backslash G/G_\chi| = 1$ for the second one. □

PROPOSITION 2.3.6. *Let $\mathcal{Y} \xrightarrow{G} \mathbb{P}^1_T$ be a G-cover with $G = A \rtimes H$ and A abelian. Write $\mathcal{X} = \mathcal{Y}/H$. Then*

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/T) = \mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/T)^H = \bigoplus_{\chi\in\operatorname{Irr}(A)/H} (\mathcal{E}_{\chi,\mathbb{1}})^H,$$

where $\mathcal{E}_{\chi,\mathbb{1}}$ is the $\theta_{\chi,\mathbb{1}}$-isotypical component of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/T)$.

PROOF. This follows from Lemma 2.3.5 and Proposition 2.3.3. □

REMARK 2.3.7. Let $\mathcal{E}_\chi$ be the χ-isotypical component of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/T)$. In Chapter 3 and Chapter 4 we are interested in the case that we have $\operatorname{rank}\mathcal{E}_\chi = \operatorname{rank}(\mathcal{E}_{\chi,\mathbb{1}})^H$, which is equivalent to the condition that $H_\chi = \{1\}$. In this case, suppose that ω is a cyclic section of $\mathcal{E}_\chi$. Then $(\mathcal{E}_{\chi,\mathbb{1}})^H$ and $\mathcal{E}_\chi$ have the same Picard-Fuchs operator. This may be seen as follows. The section $\eta := \sum_{h\in H} h\omega$ of $\mathcal{E}_{\chi,\mathbb{1}} = \bigoplus_{h\in H/H_\chi}\mathcal{E}_{\chi^h}$ is fixed by the action of H. Hence, it is a section of $(\mathcal{E}_{\chi,\mathbb{1}})^H$. Denote by L the Picard-Fuchs operator on $\mathcal{E}_\chi$, with solutions $\langle\omega,\gamma\rangle$, $\gamma\in\ker\nabla^\vee$. As the automorphisms in H are defined over T it follows that

$$L_\chi(\langle\eta,\gamma\rangle) = \sum_{h\in H} h\cdot L_\chi(\langle\omega,\gamma\rangle) = 0.$$

(see [**Kat81**, Section I]). If we additionally assume that there exists a character $\chi\in\operatorname{Irr}(A)$ such that $\mathcal{X}' = \mathcal{Y}/\ker(\chi)$ is the pullback of a superelliptic curve as in Lemma 2.2.8, then the Picard-Fuchs operator of $(\mathcal{E}_{\chi,\mathbb{1}})^H$ is the pullback of a hypergeometric differential operator.

In Chapter 3 we want that the type $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ of the superelliptic curve is such that we have a very special hypergeometric differential equation (see Proposition 3.3.2 or [**BM10b**, Proposition 4.2]).

2.3.1. Description of the de Rham cohomology. In this subsection we compute the ranks of the components $(\mathcal{E}_{\chi,\mathbb{1}})^H$ of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/T)$ from Proposition 2.3.6 under the following additional assumption.

We suppose that the G-cover $\mathscr{Y} \to \mathbb{P}^1_T$ as in the beginning of Section 2.3 is a cover with four branch points such that all fibres $\mathscr{Y}_b \to \mathbb{P}^1_{\mathbb{C}}$ have ramification type

$$\mathbf{C} = (\mathrm{Cl}(g_1), \mathrm{Cl}(g_2), \mathrm{Cl}(g_3), \mathrm{Cl}(g_4)), \qquad g_1 \cdots g_4 = 1,\ G = \langle g_1, \ldots, g_4 \rangle.$$

We now compute $n_{\chi,\mathbb{1}} = \mathrm{rank}(\mathscr{E}_{\chi,\mathbb{1}})^H$ for $\chi \in \mathrm{Irr}(A)$. Note that

$$n_{\chi,\mathbb{1}} = \langle \text{ш}, \theta_{\chi,\mathbb{1}} \rangle_G \cdot \langle \mathbb{1}, \mathrm{Res}\,\theta_{\chi,\mathbb{1}} \rangle_H = \langle \text{ш}, \theta_{\chi,\mathbb{1}} \rangle_G$$

(see Remark 2.1.2 and Lemma 2.3.5).

For $\chi \in \mathrm{Irr}(A)$, $i \in \{1, \ldots, 4\}$ and $\ell \in \langle g_i \rangle \backslash G / G_\chi$ we put

$$k_\chi^{i,\ell} := \begin{cases} 1, & A \cap \langle \ell g_i \ell^{-1} \rangle \subset \ker \chi, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.3.8. *It holds $n_{\mathbb{1},\mathbb{1}} = 0$ and for $\chi \neq \mathbb{1}$ we have*

$$n_{\chi,\mathbb{1}} = 2 \cdot |H(\chi)| \ - \sum_{\substack{i=1,\ldots,4 \\ \ell \in G_\chi \backslash G / \langle g_i \rangle}} k_\chi^{i,\ell}\ .$$

Proof. Lemma 2.1.1 implies that

$$n_{\chi,\mathbb{1}} = \langle \text{ш}, \theta_{\chi,\mathbb{1}} \rangle_G = 2 \cdot \langle \mathbb{1}, \theta_{\chi,\mathbb{1}} \rangle_G + 2 \cdot \langle \mathrm{Ind}_{\langle 1 \rangle}^G \mathbb{1}, \theta_{\chi,\mathbb{1}} \rangle_G - \sum_{i=1}^4 \langle \mathrm{Ind}_{\langle g_i \rangle}^G \mathbb{1}, \theta_{\chi,\mathbb{1}} \rangle_G.$$

Obviously,

$$\langle \mathbb{1}, \theta_{\chi,\mathbb{1}} \rangle_G = \begin{cases} 1, & \text{if } \chi = \mathbb{1}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\left\langle \mathrm{Ind}_{\langle 1 \rangle}^G \mathbb{1}, \theta_{\chi,\mathbb{1}} \right\rangle_G = \langle \mathbb{1}, \mathrm{Res}\,\theta_{\chi,\mathbb{1}} \rangle_{\langle 1 \rangle} = \theta_{\chi,\mathbb{1}}(1) = |H(\chi)|,$$

by [**Ser77**, Theorem 13]. Moreover, by [**Ser77**, Theorem 13, Proposition 22] it follows that

$$\left\langle \mathrm{Ind}_{\langle g_i \rangle}^G \mathbb{1}, \theta_{\chi,\mathbb{1}} \right\rangle_G = \langle \mathbb{1}, \mathrm{Res}\,\theta_{\chi,\mathbb{1}} \rangle_{\langle g_i \rangle} = \sum_{\ell \in \langle g_i \rangle \backslash G / G_\chi} \langle \mathbb{1}, \mathrm{Res}(\chi \cdot \mathbb{1}) \rangle_{G_\chi \cap \langle \ell g_i \ell^{-1} \rangle}.$$

The character $(\chi \cdot \mathbb{1}) \in \mathrm{Irr}(G_\chi)$ is trivial on $G_\chi \cap \langle \ell g_i \ell^{-1} \rangle$ if and only if

$$A \cap \langle \ell g_i \ell^{-1} \rangle \subset \ker \chi. \tag{2.3.2}$$

Hence, $\langle \mathbb{1}, \mathrm{Res}(\chi \cdot \mathbb{1}) \rangle_{G_\chi \cap \langle \ell g_i \ell^{-1} \rangle} = k_\chi^{i,\ell}$. Note that, if $\chi = \mathbb{1}$, then (2.3.2) holds for all $i = 1, \ldots, 4$ and $|H(\chi)| = 1$. □

Summary 2.3.9. Proposition 2.3.6 yields a decomposition of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/T)$ which at most consists of the H-invariants of the $\theta_{\chi,\xi}$-isotypical components with $\xi = \mathbb{1}$. With Proposition 2.3.8 one can compute the ranks of these components (some of them may equal 0). For $\chi \in \mathrm{Irr}(A)$ with $H_\chi = \{1\}$, the component $(\mathscr{E}_{\chi,\mathbb{1}})^H \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/T)$ and the χ-isotypical component $\mathscr{E}_\chi \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)$ have the same Picard-Fuchs operator (provided that a cyclic sections exists, of course).

Chapter 3

Construction of Teichmüller curves

A *Teichmüller curve* is a curve $C \subset \mathscr{M}_g$, embedded in the moduli space of curves of genus g, which is totally geodesic for the Teichmüller metric. Sometimes the normalization of C, together with the corresponding generically injective map to $\mathscr{M}_g$ is called a Teichmüller curve instead. In this chapter we do not work with the definition, but use a characterisation due to Martin Möller ([**Möl06b**]) instead. Further good references for more details are [**Möl11**] and [**Möl13**].

In [**BM10b**] a class of Teichmüller curves associated with superelliptic curves is constructed. In this chapter we review this construction, and classify all Teichmüller curves that may be constructed in this way. Indeed, we find a new class of Teichmüller curves not treated in [**BM10b**].

3.1. Flat vector bundles

In this section we recall from [**BW06**] some generalities on flat vector bundles. For more details we refer to [**Kat70**] and [**BP02**]. In Chapter 2 we have already seen an example of a flat vector bundle 'coming from geometry', namely the relative de Rham cohomology equipped with the Gauß-Manin connection. In the second part of this section we reformulate a theorem of Martin Möller [**Möl06b**, Theorem 5.3], which characterises Teichmüller curves, in the terminology of flat vector bundles.

Throughout this section let $\overline{S}$ be a smooth projective connected curve over $\mathbb{C}$ and $S \subset \overline{S}$ the complement of finitely many $\mathbb{C}$-rational points $S' = \{b_1, \ldots, b_r\}$.

Moreover we denote by
$$\Omega_S^{\log} := \Omega^1_{\overline{S}/\mathbb{C}}(S')$$
the sheaf of differential 1-forms on $\overline{S}$ with at most simple poles in the points S'.

Definition 3.1.1. A *flat vector bundle* $\mathcal{E}$ is an $\mathcal{O}_{\overline{S}}$-module together with a connection
$$\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{\overline{S}}} \Omega_S^{\log}.$$
We call the points in S' *marked points* of $\mathcal{E}$. A *flat subbundle* $\mathcal{E}' \subset \mathcal{E}$ is a submodule of $\mathcal{E}$ such that ∇ restricts to a connection $\mathcal{E}' \to \mathcal{E}' \otimes \Omega_S^{\log}$.

Recall that a connection ∇ is an additive map, satisfying the Leibniz rule
$$\nabla(f\omega) = \omega \otimes \mathrm{d}f + f\nabla(\omega),$$
for local sections f and ω of $\mathcal{O}_{\overline{S}}$ and $\mathcal{E}$, respectively.

In this section we only consider the case of flat vector bundles of rank 2.

Remark 3.1.2. Let $b \in \overline{S}$ and s a local parameter at b. In the following we identify derivations in $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{\overline{S},b})$ with homomorphisms in $\mathrm{Hom}_{\mathcal{O}_{\overline{S}}}(\Omega_S^{\log}, \mathcal{O}_{\overline{S}})$.

(i) If $b \in S$ is not a marked point, then the derivation $\partial/\partial s$ 'is' the homomorphism $D\colon (\Omega_S^{\log})_b \to \mathcal{O}_{\overline{S},b}$ with $D(\mathrm{d}s) = 1$. The so-called *contraction*
$$\nabla(\partial/\partial s)\colon \mathcal{E}_b \xrightarrow{\nabla} \mathcal{E}_b \otimes \left(\Omega_S^{\log}\right)_b \xrightarrow{1\otimes D} \mathcal{E}_b \otimes \mathcal{O}_{\overline{S},b}$$
defines a $\mathbb{C}$-linear endomorphism of the stalk $\mathcal{E}_b$ at b.

(ii) If $b \in S'$ is a marked point, then the derivation $s\partial/\partial s$ 'is' the homomorphism $D\colon (\Omega_S^{\log})_b \to \mathcal{O}_{\overline{S},b}$ with $D(\frac{\mathrm{d}s}{s}) = 1$. The contraction
$$\nabla(s\partial/\partial s)\colon \mathcal{E}_b \xrightarrow{\nabla} \mathcal{E}_b \otimes \left(\Omega_S^{\log}\right)_b \xrightarrow{1\otimes D} \mathcal{E}_b \otimes \mathcal{O}_{\overline{S},b}$$
defines a $\mathbb{C}$-linear endomorphism of the stalk $\mathcal{E}_b$ which fixes the submodule $\mathfrak{m}_b\mathcal{E}_b$. Here $\mathfrak{m}_b$ denotes the maximal ideal of the local ring $\mathcal{O}_{\overline{S},b}$.

Definition 3.1.3. Let $b \in S'$ be a marked point and s a local parameter at b.

(i) We define the *local monodromy* operator μ_b of $\mathcal{E}$ at b as the $\mathbb{C}$-linear endomorphism of $\mathcal{E}_b/\mathfrak{m}_b\mathcal{E}_b \simeq \mathbb{C}^2$ induced by $\nabla(s\partial/\partial s)$.

(ii) The *local exponents* α_b, β_b of $\mathcal{E}$ at b are defined as the eigenvalues of μ_b.

(iii) If $\mu_b = 0$ we call b a *regular point* of $(\mathcal{E}, \nabla)$. If $\mu_b \neq 0$ we call b a *regular singularity* of $(\mathcal{E}, \nabla)$ and we distinguish two cases. If μ_b is not semisimple, then $\alpha_b = \beta_b$ and μ_b is conjugate to
$$\begin{pmatrix} \alpha_b & 0 \\ 1 & \alpha_b \end{pmatrix}.$$

In this case, we say that ∇ has *logarithmic monodromy* at b, and that b is *logarithmic singularity* of $(\mathscr{E}, \nabla)$. If μ_b is semisimple, then μ_b is conjugate to

$$\begin{pmatrix} \alpha_b & 0 \\ 0 & \beta_b \end{pmatrix}.$$

In this case, we say that ∇ has *toric monodromy* at b, and that b is an *elliptic singularity* of $(\mathscr{E}, \nabla)$.

Remark 3.1.4. (i) We may represent $\nabla(s\partial/\partial s)$ on $\mathscr{E}_b$ by a 2×2-matrix $P(s) \in M_{2,2,}(\mathscr{O}_{\overline{S},b})$ with coefficients in $\mathscr{O}_{\overline{S},b}$. This is the definition of a regular singularity b of $\mathscr{E}$ in [**Kat70**, Section 11]. Then the local monodromy $\mu_b \in \mathrm{End}(\mathscr{E}_b/\mathfrak{m}_b\mathscr{E}_b)$ is represented by the 2×2-matrix $P(0)$ with coefficients in $\mathbb{C}$.
(ii) The local exponents of $\mathscr{E}$ at $b \in S'$ do not depend on the particular choice of the local parameter s at b (see [**Kat70**, Section 12]).
(iii) If $\mu_b = 0$, then

$$\nabla \colon \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{\overline{S}/\mathbb{C}}(S' - \{b\})$$

is a connection that makes $\mathscr{E}$ a flat vector bundle with markings $S' - \{b\}$. That means we can 'unmark' such points b.

In Chapter 2 we have already seen an example of a flat vector bundle: the relative the Rham cohomology of the Legendre family of elliptic curves.

Example 3.1.5. Let $f \colon \mathscr{X} \to S = \mathbb{P}^1_s - \{0, 1, \infty\}$ be the Legendre family of elliptic curves over $\mathbb{C}$ birationally determined by

$$\mathscr{X}_s \colon z^2 = x(x-1)(x-s).$$

The holomorphic 1-form

$$\omega_1 = \frac{z\,\mathrm{d}x}{x(x-1)(x-s)}$$

on $\mathscr{X}_s$ $(s \neq 0, 1, \infty)$ and the 1-form

$$\omega_2 = \frac{\omega_1}{2(x-s)}$$

on $\mathscr{X}_s$ with a pole in $x = s$ form a basis of the de Rham cohomology $\mathbb{C}$-vector space $H^1_{\mathrm{dR}}(\mathscr{X}_s)$ of dimension $\dim H^1_{\mathrm{dR}}(\mathscr{X}_s) = 2 \cdot g(\mathscr{X}_s) = 2$ ([**GH78**, Section 3.5]). The above description of ω_1 and ω_2 defines a basis of the $\mathscr{O}_S$-module $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$, the relative de Rham cohomology, with fibres

$$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S) \otimes (\mathscr{O}_{S,b}/\mathfrak{m}_b) = H^1_{\mathrm{dR}}(\mathscr{X}_b).$$

It extends to an $\mathscr{O}_{\overline{S}}$-module $\mathscr{E}$ with $\overline{S} = \mathbb{P}^1_{\mathbb{C}}$, the Deligne extension (see [**Del70**, Proposition II.5.2]), and comes equipped with the Gauß-Manin connection

$$\nabla \colon \mathscr{E} \to \mathscr{E} \otimes \Omega^{\log}_{S}$$

which satisfies $\nabla(\partial/\partial s)\,\omega_1 = \omega_2$ on S. In $\mathcal{E}$ the Picard-Fuchs equation

$$\nabla(\partial/\partial s)^2\,\omega + \frac{2s-1}{s(s-1)}\cdot\nabla(\partial/\partial s)\,\omega + \frac{1}{4s(s-1)}\,\omega = 0 \in \mathcal{E}$$

holds (see Lemma 2.2.6), where s is a local parameter at 0. Then one easily checks that the matrix of $\nabla(s\partial/\partial s)$ on the stalk $\mathcal{E}_0$ with respect to the basis $(\omega_1, \nabla(s\partial/\partial s)\,\omega_1)$ equals

$$P(s) = \begin{pmatrix} 0 & \frac{-s}{4(s-1)} \\ 1 & \frac{-s}{s-1} \end{pmatrix}.$$

Therefore

$$P(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and ∇ has a logarithmic singularity at 0. The local exponents are $\alpha_0 = \beta_0 = 0$.

Remark 3.1.6. The monodromy μ_b may be interpreted as a monodromy operator in classical sense as follows. The horizontal sections of the dual $(\mathcal{E}^\vee, \nabla^\vee)$ of $(\mathcal{E}, \nabla)$ are the solutions of the Picard-Fuchs equation on $\mathcal{E}$, which is introduced in Section 2.2 (see Remark 2.2.2). The fact that all singularities of $\mathcal{E}$ are regular yields that the corresponding Picard-Fuchs equation is Fuchsian ([**Del70**, Proposition II.5.2] or [**BP02**, Lemma 9.3]). If $b \in S'$ is a regular singularity, then the numbers $\exp(2\pi \mathrm{i}\alpha_b)$ and $\exp(2\pi \mathrm{i}\beta_b)$ are the eigenvalues of the local monodromy matrix (Definition A.0.11) of the Picard-Fuchs equation, where α_b and β_b are the local exponents of $\mathcal{E}$ at b ([**Kat70**, Remark 12.3]). Note that the notions 'local exponent', 'elliptic singularity' and 'logarithmic singularity' do also exist for Picard-Fuchs equations. However they are not completely consistent with the notions introduced in this Chapter (see Remark A.0.12 or [**Hon81**, Appendix]).

Definition 3.1.7. Let $b \in S'$ be a marked point of a flat vector bundle $(\mathcal{E}, \nabla)$. We say that the monodromy μ_b is *quasi-unipotent* (resp. *unipotent*) if the local exponents of $\mathcal{E}$ at b are rational numbers (resp. integers).

Using standard results on solutions of Fuchsian differential equations one easily shows that Definition 3.1.7 corresponds to the usual notions for the local monodromy matrices of the corresponding differential equation (from Remark 3.1.6).

The following lemma interprets the notions of Definition 3.1.7 in our context in the case of elliptic singularities. It shows how elliptic singularities may be removed by a suitable base change.

Lemma 3.1.8. *Let $b \in \overline{S} - S$ be an elliptic singularity of a flat vector bundle $(\mathcal{E}, \nabla)$ of rank 2. Assume that the monodromy at b is quasi-unipotent. Let e be the least common multiple of the denominators of the local exponents (α_b, β_b). Let $\pi : \overline{T} \to \overline{S}$ be a cover and $c \in \overline{T}$ with $\pi(c) = b$ a ramification point of π with ramification index e. Then $\pi^*(\mathcal{E}|_S)$ extends to a flat vector bundle on $\overline{T}$ which is regular at c.*

In the situation of Lemma 3.1.8, we call $e = e_b$ the *order of the local monodromy* at b.

PROOF OF LEMMA 3.1.8. Let s be a local parameter at b. The point b is a regular singularity in the terminology of [**Kat70**]. From [**Kat70**, Theorem 12.0] it follows that (after possibly modifying $\mathscr{E}_b$) there exists a basis (ω_1, ω_2) of $\mathscr{E}_b$ such that the matrix $P(s) = (p_{ij})$ of $\nabla(s\partial/\partial s)$ with respect to this basis satisfies

$$P(0) = \begin{pmatrix} \alpha_b & 0 \\ 0 & \beta_b \end{pmatrix}, \qquad 0 \leq \alpha_b \leq \beta_b < 1. \tag{3.1.1}$$

The matrix $P(0)$ is semisimple, since b is elliptic. Let t be a local parameter at c satisfying $s = t^e$. The definition of e implies that $e\alpha_b$ and $e\beta_b$ are integers. An easy calculation using the chain rule yields that the matrix of $\nabla(t\partial/\partial t)$ on the $\mathscr{O}_{\overline{T},c}$-module with basis $(t^{-e\alpha_b}\pi^*\omega_1, t^{-e\beta_b}\pi^*\omega_2)$ is

$$P^*(t) = \begin{pmatrix} e(\pi^* p_{11} - \alpha_b) & et^{e(\beta_b - \alpha_b)}\pi^* p_{12} \\ et^{e(\alpha_b - \beta_b)}\pi^* p_{21} & e(\pi^* p_{22} - \beta_b) \end{pmatrix}.$$

From (3.1.1) it follows that $P^*(0) = 0$. We extend $\pi^*(\mathscr{E}|_S)$ to a bundle $\tilde{\mathscr{E}}$ over $\pi^{-1}(S) \cup \{c\}$ by defining $\tilde{\mathscr{E}}_c$ as the $\mathscr{O}_{\overline{T}_c}$-module with basis $(t^{-e\alpha_b}\pi^*\omega_1, t^{-e\beta_b}\pi^*\omega_2)$. The fact that $P^*(0) = 0$ means exactly that $\tilde{\mathscr{E}}$ has a regular point at c. $\square$

DEFINITION 3.1.9. A *filtration* on a flat vector bundle $(\mathscr{E}, \nabla)$ of rank 2 consists of a line subbundle $\mathrm{Fil}^1 \mathscr{E} \subset \mathscr{E}$ such that $\mathrm{Gr}^1 \mathscr{E} := \mathscr{E}/\mathrm{Fil}^1 \mathscr{E}$ is also a line bundle. With such a filtration $\mathrm{Fil}^1 \mathscr{E} \subset \mathscr{E}$ we associate a *Kodaira–Spencer map*, defined as

$$\Theta : \mathrm{Fil}^1 \mathscr{E} \hookrightarrow \mathscr{E} \xrightarrow{\nabla} \mathscr{E} \otimes \Omega_S^{\log} \twoheadrightarrow \mathrm{Gr}^1 \mathscr{E} \otimes \Omega_S^{\log}.$$

REMARK 3.1.10. In the case that $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ is a flat rank-2 subbundle of a de Rham cohomology, we may consider the bundle $\mathscr{E} \cap H^0(\mathscr{Y}, \Omega^1_{\mathscr{Y}/S})$. Note that in general this does not define a filtration in the sense of Definition 3.1.9, since the rank of $\mathscr{E} \cap H^0(\mathscr{Y}, \Omega^1_{\mathscr{Y}/S})$ need not to be 1.

DEFINITION 3.1.11. An *indigenous bundle* on $\overline{S}$ with regular singularities in a finite set $\overline{S} - S$ of marked points is a flat vector bundle of rank 2 satisfying the following two conditions.

(i) There exists a filtration $\mathrm{Fil}^1 \mathscr{E}$ such that the corresponding Kodaira–Spencer map is an isomorphism.

(ii) The local monodromy at all marked points $b \in \overline{S} - S$ is non-trivial, i.e. $\mu_b \neq 0$.

REMARK 3.1.12. (i) By Lemma 3.1.8 all singularities of a flat vector bundle $\mathscr{E}$ with only unipotent monodromies are either regular points or logarithmic. If, in addition, $\mathscr{E}$ is indigenous, condition (ii) from Definition 3.1.11 implies that the set $S \subset \overline{S}$ is chosen as large as possible, i.e. the set of marked points S' is exactly the set of logarithmic singularities of $\mathscr{E}$.

(ii) In the following we always choose S as large as possible, if not stated otherwise.

(iii) The filtration in Definition 3.1.11 is unique [**BW06**, Proposition 2.2 (i)].

(iv) The local system corresponding to an indigenous bundle is called *maximal Higgs* ([**BM10b**, Section 2], [**BM10a**, Section 1]).

In the rest of this section we consider flat vector bundles 'coming from geometry'. Let $f : \mathscr{X} \to S$ be a smooth curve over S, i.e. f is a proper and smooth morphism of relative dimension 1 with connected geometric fibres. We may interpret $\mathscr{X}$ as a family of smooth curves parametrised by S. The genus g of the fibres of f is constant since S is connected.

The relative de Rham cohomology $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ comes equipped with the Gauß-Manin connection ∇. It is known that $(\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S), \nabla)$ extends to a flat vector bundle on all of $\overline{S}$, the Deligne extension (see [**Del70**, Proposition II.5.2]). All its local monodromy matrices are quasi-unipotent ([**Kat70**, Theorem 14.1]). Suppose we are given a flat subbundle $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ of rank 2 (i.e. ∇ restricts to a connection on $\mathscr{E}$). Throughout this chapter, the Deligne extension of $\mathscr{E}$ to a flat vector bundle on $\overline{S}$ is still denoted by $\mathscr{E}$. Suppose that $\mathrm{Fil}^1 \mathscr{E} := \mathscr{E}|_S \cap H^0(\mathscr{X}/S, \Omega^1_{\mathscr{X}/S})$ is a filtration in the sense of Definition 3.1.9. The construction of the Deligne extension also includes the extension of $\mathrm{Fil}^1 \mathscr{E}$ to $\overline{S}$.

We now can formulate the criterion we use to construct Teichmüller curves. The result is essentially a reformulation of [**Möl06b**, Theorem 5.3].

Theorem 3.1.13 (Möller). *Let $\overline{S}$ be a smooth projective connected $\mathbb{C}$-curve and $S \subset \overline{S}$ the complement of finitely many $\mathbb{C}$-rational points. Let $f : \mathscr{X} \to S$ be a smooth curve of genus g. Assume that there exists a flat rank-2 subbundle $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ whose Deligne extension, still denoted by $\mathscr{E}$, satisfies the following conditions.*

(i) The bundle $\mathscr{E}$ is an indigenous bundle.

(ii) All points in $\overline{S}-S$ are logarithmic singularities of $\mathscr{E}$.

Then the image of the moduli map

$$S \to \mathscr{M}_g, \qquad b \mapsto [\mathscr{X}_b],$$

which maps a point b to the moduli point of the corresponding fibre in the moduli space $\mathscr{M}_g$ of curves of genus g, is a Teichmüller curve.

Proof. All local monodromies of $\mathscr{E}$ outside the logarithmic ones are unipotent as they are regular points of $\mathscr{E}$. The condition that $\mathscr{E}$ is indigenous means that the Higgs bundle of $\mathscr{X} \to S$ has a rank-2 Higgs subbundle with maximal Higgs field as required in [**Möl06b**, Theorem 5.3]. $\square$

Suppose we are given a smooth curve $f : \mathscr{X} \to S$. Let $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ be a flat subbundle of rank 2 such that $\mathrm{Fil}^1 \mathscr{E} := \mathscr{E} \cap H^0(\mathscr{X}, \Omega^1_{\mathscr{X}/S})$ is a filtration on $\mathscr{E}$ in the sense of Definition 3.1.9. The criterion of Theorem 3.1.13 does not apply directly in the case that $\mathscr{E}$ has elliptic singularities. In this chapter we discuss possibilities to modify the family such that the criterion applies.

Let $b \in \overline{S}-S$ be an elliptic singularity. The local exponents α_b, β_b are rational numbers. Therefore Lemma 3.1.8 implies that there exists a cover $\pi : \overline{T} \to \overline{S}$, which is branched at b, such that $\pi^*(\mathscr{E}|_S)$ extends to a flat vector bundle $\tilde{\mathscr{E}}$ which has regular points above b. We put $T := \pi^{-1}(S)$. The condition that $\tilde{\mathscr{E}}$ is indigenous translates to the condition that the Kodaira-Spencer map associated with $\mathrm{Fil}^1 \mathscr{E}$ is an isomorphism at any point $c \in \overline{T}-T$ with $\pi(c) = b$. This condition may be checked in terms of the local exponents α_b, β_b of b ([**BM10b**, Proposition 3.2]).

Of course, the key requirement in the condition of Theorem 3.1.13 is that the family $\mathscr{X}_T := \mathscr{X} \otimes_S T \to T$ extends to a smooth curve over the inverse image of the elliptic points. For the families of superelliptic curves that we consider in Section 3.2 the Legendre family of elliptic curves is the only one where this is satisfied. For the other families of superelliptic curves, $\mathscr{X}_T$ extends to a family of semistable curves over $\overline{T}$. Our candidates for Teichmüller curves will be quotients of $\mathscr{X}_{\overline{T}}$ by a finite group of $\overline{T}$-automorphisms.

3.2. A class of superelliptic curves

In this section we recall some facts on flat vector bundles of rank 2 'coming from' superelliptic curves, i.e. cyclic covers of the projective line. We restrict to the case of four branch points.

We denote by $\overline{S} = \mathbb{P}^1_{\mathbb{C}}$ the projective line with coordinate s and set $S = \mathbb{P}^1_s - \{0, 1, \infty\}$. Let $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{Q}^4$ with $0 < \sigma_i < 1$ and $\sum_{i=1}^4 \sigma_i \in \mathbb{N}$ and denote by N the least common multiple of the denominators of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. We put $a_i = N \cdot \sigma_i$ for $i = 1, \ldots, 4$. Let $\mathscr{X}$ denote the smooth S-curve birationally determined by the affine equation

$$z^N = x^{a_1}(x-1)^{a_2}(x-s)^{a_3}.$$

The curve $\mathscr{X}$ is a superelliptic curve of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ (see Section 2.2.1). The corresponding N-cyclic cover

$$\mathscr{X} \to \mathbb{P}^1_S, \qquad (x, z) \mapsto x$$

is exactly branched at the sections

$$x_1 = 0,\ x_2 = 1,\ x_3 = s,\ x_4 = \infty \colon S \to \mathbb{P}^1_S.$$

The Galois group $\mathrm{Gal}(\mathscr{X}/\mathbb{P}^1_S) \simeq \mathbb{Z}/N\mathbb{Z}$ acts as $\varphi^* z = \chi_0(\varphi) z$ for $\varphi \in \mathbb{Z}/N\mathbb{Z}$, where $\chi_0 : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^*$ is an injective character.

Remark 3.2.1. The Galois group $\mathbb{Z}/N\mathbb{Z}$ acts naturally on the relative de Rham cohomology $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$. For every irreducible character $\chi \colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$, we denote by $\mathscr{E}_\chi$ the χ-isotypical part. Then $(\mathscr{E}_\chi, \nabla)$ is a flat subbundle of the flat vector bundle $(\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S), \nabla)$ ([**Kat81**, Section I]). As in Remark 2.2.5, in the following we only consider the isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ with respect to the injective character χ_0. This is no restriction as we are only interested in so-called geometrically primitive Teichmüller curves ([**Möl06a**, Definition 2.4]). We drop the index from the notation, and write $\mathscr{E} = \mathscr{E}_{\chi_0}$. We also write $\mathscr{E}$ for the Deligne extension of $\mathscr{E}$ to $\overline{S}$.

In Chapter 2 in Lemma 2.2.6 we have already given a description of the flat vector bundle $(\mathscr{E}, \nabla)$. We recall this description and give more details.

Lemma 3.2.2. *(i) We have* $\mathrm{rank}(\mathscr{E}) = 2$.
(ii) The subbundle $\mathrm{Fil}^1 \mathscr{E} := \mathscr{E} \cap H^0(\mathscr{X}, \Omega^1_{\mathscr{X}/S})$ *defines a filtration in the sense of Definition 3.1.9 if and only if* $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 2$.
(iii) If the condition from (ii) is satisfied the differential form

$$\omega := \frac{z\mathrm{d}x}{x(x-1)(x-s)}$$

defines a generically non-vanishing section of $\mathrm{Fil}^1 \mathscr{E}$. *It satisfies*

$$\nabla(\partial/\partial s)^2\, \omega + \frac{(A+B+1)s - C)}{s(s-1)} \cdot \nabla(\partial/\partial s)\, \omega + \frac{AB}{s(s-1)}\, \omega = 0$$

in $\mathscr{E}$, *where* $A = 1 - \sigma_3, B = 2 - (\sigma_1 + \sigma_2 + \sigma_3)$, *and* $C = 2 - (\sigma_1 + \sigma_3)$.
(iv) The singularities of $\mathscr{E}$ *are* $\{0, 1, \infty\} = \overline{S} - S$, *which are all quasi-unipotent. The singularity* $b \in \{0, 1, \infty\}$ *is logarithmic if and only if*

$$\begin{cases} \sigma_1 + \sigma_3 = 1 & \textit{if } b = 0, \\ \sigma_2 + \sigma_3 = 1 & \textit{if } b = 1, \\ \sigma_1 + \sigma_2 = 1 & \textit{if } b = \infty. \end{cases}$$

Proof. Parts (i), (ii), and (iii) are proved in [**Bou05**, Section 1.1]. The fact that all singularities are quasi-unipotent is a general fact (see [**Kat70**, Theorem 14.1]). In the case of superelliptic curves this may also be deduced from (iii). The rest follows from (iii) by a direct computation. □

3.3. Bouw-Möller curves

In [**BM10b**] a class of Teichmüller curves associated with superelliptic curves is constructed. In this section, we review this construction, and classify all Teichmüller curves that may be constructed in this way.

As a first step, we determine all types such that $\mathscr{E}$ from Remark 3.2.1 is an indigenous bundle. More precisely, for each type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ we consider covers $\pi\colon \overline{T} \to \overline{S}$ which are unbranched in $S = \mathbb{P}^1_s - \{0, 1, \infty\}$ such that $\pi^*\mathscr{E}$ has no elliptic singularities (see Lemma 3.1.8). We then determine for which types the pullback $\pi^*\mathscr{E}$ is indigenous, by checking whether the Kodaira-Spencer map vanishes nowhere on $\overline{T}$.

The following lemma describes the order of vanishing of the Kodaira-Spencer map. It is basically a reformulation of [**BM10b**, Proposition 3.2].

For $b \in \{0, 1, \infty\}$ we set

$$\beta_b = \begin{cases} \sigma_1 + \sigma_3 - 1 & \text{if } b = 0, \\ \sigma_2 + \sigma_3 - 1 & \text{if } b = 1, \\ \sigma_1 + \sigma_2 - 1 & \text{if } b = \infty. \end{cases} \tag{3.3.1}$$

Write $e_b \in \mathbb{N}$ for the denominator of β_b if $\beta_b \neq 0$, i.e. when b is an elliptic singularity (see Lemma 3.2.2 (iv)). It turns out that e_b is the order of the local monodromy at b.

LEMMA 3.3.1. *Assume that $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 2$, i.e. $\mathrm{Fil}^1 \mathscr{E} := \mathscr{E} \cap H^0(\mathscr{X}, \Omega^1_{\mathscr{X}/S})$ defines a filtration in the sense of Definition 3.1.9.*
(i) The Kodaira-Spencer map associated with $\mathrm{Fil}^1 \mathscr{E}$ does not vanish at the points $b \notin \{0, 1, \infty\}$ and the logarithmic singularities.
(ii) Let $b \in \{0, 1, \infty\}$ be an elliptic singularity of $\mathscr{E}$. Let $\pi : \overline{T} \to \overline{S}$ be a cover which is branched at b with ramification index e_b. Then $\pi^*(\mathscr{E}|_S)$ extends to a flat vector bundle $\tilde{\mathscr{E}}$ over $\overline{T}$ such that all points in $\pi^{-1}(b)$ are regular points of $\tilde{\mathscr{E}}$. Moreover, there exists a filtration on $\tilde{\mathscr{E}}$ such that the order of vanishing at c with $\pi(c) = b$ of the associated Kodaira-Spencer map is

$$e_b|\beta_b| - 1.$$

Note that we choose the set of marked points as small as possible (see Remark 3.1.12). Hence the regular points in $\pi^{-1}(b)$ are not contained in the set of marked points of $\tilde{\mathscr{E}}$.

PROOF OF LEMMA 3.3.1. First we assume that $b \in S$. Let ω be as in Lemma 3.2.2 (iii), and write $\omega' := \nabla(\partial/\partial s)\omega$. To prove the lemma, we use the description of the de Rham cohomology as differentials of the second kind modulo exact differentials ([**GH78**, Section 3.5]. For the case of superelliptic curves, see [**Bou05**, Section 1]). A direct calculation shows that ω' has a simple pole in $x = s$. It follows that $\boldsymbol{\omega} = (\omega, \omega')$ generate the free $\mathscr{O}_{S,b}$-module $\mathscr{E}_b$. We conclude that the Kodaira-Spencer map corresponding to the filtration $\mathrm{Fil}^1 \mathscr{E} \subset \mathscr{E}$ does not vanish at $s = b$. This proves (i) for $b \notin \{0, 1, \infty\}$. If $b = 0$ is an logarithmic singularity, then $\boldsymbol{\omega} = (\omega, \nabla(s\partial/\partial s)\,\omega)$ is a basis of $\mathscr{E}_0$, i.e. the Kodaira-Spencer map does not vanish at $b = 0$. The proof in the case that $b \in \{1, \infty\}$ is a logarithmic singularity is similar.

Now let $b \in \{0, 1, \infty\}$ be an elliptic singularity. Again we only consider the case $b = 0$ as the other cases are similar. Using Lemma 3.2.2 (ii) one easily computes that the matrix of $\nabla(s\partial/\partial s)$ acting on the $\mathscr{O}_{S,b}$-module with basis $(\omega, \nabla(s\partial/\partial s)\omega)$

equals

$$\begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} 0 & -sAB/(s-1) \\ 1 & (-(A+B)s+C-1)/(s-1) \end{pmatrix},$$

where A, B, C are given as in Lemma 3.2.2 (iii). From this it follows that the local exponents of $\mathcal{E}$ at $b=0$ are $(\alpha_b, \beta_b) = (0, 1-C) = (0, \sigma_1+\sigma_3-1)$. In particular, the denominator e_b of β_b is the order of the local monodromy at b. If $\beta_b < 0$, an easy calculation shows that the matrix of $\nabla(s\partial/\partial s)$ with respect to $(s^{-\beta_b}\omega, \nabla(s\partial/\partial s)\,(s^{-\beta_b}\omega))$ equals

$$\begin{pmatrix} 0 & -\beta_b^2 + p_2\beta_b + p_1 \\ 1 & -2\beta_b + p_2 \end{pmatrix}$$

and the local exponents are $(0, -\beta_b)$. Therefore, in the following, we assume that ω is chosen such that $\beta_b > 0$. Since $0 \le \sigma_i < 1$ it follows from (3.3.1) that $0 \le \beta_b < 1$.

Let t be a local parameter at c with $\pi(c) = b$ satisfying $s = t^e$. From Lemma 3.1.8 it follows that $\pi^*(\mathcal{E}|_S)$ extends to a flat vector bundle with a regular point at c. In the proof of Lemma 3.1.8 we have seen that a basis of the stalk $\tilde{\mathcal{E}}_c$ is given by $\eta_1 = \pi^*\omega$, $\eta_2 = t^{-e\beta}\pi^*\nabla(s\partial/\partial s)\,\omega$, where $(\mathrm{Fil}^1\,\tilde{\mathcal{E}})_b = \langle \eta_1 \rangle$. As we choose the set of marked points as small as possible, the regular points in $\pi^{-1}(b)$ are not contained in the set of marked points of $\tilde{\mathcal{E}}$. Hence, in order to calculate the order of vanishing at $c \in \pi^{-1}(b)$ of the Kodaira-Spencer map we consider the contraction against $\partial/\partial t$ (not the contraction against $t\partial/\partial t$). An easy calculation shows that $\nabla(\partial/\partial t)\eta_1 = et^{e\beta_b - 1}\eta_2$. Therefore the order of vanishing of the Kodaira-Spencer map is $e\beta_b - 1$. □

An indigenous bundle defining a Teichmüller curve in the sense of Theorem 3.1.13 has at least one logarithmic singularity. (This follows from a result of [**Vee89**], reformulated in our terminology, cf. [**McM03**, Proposition 2.2].) In the following, we therefore always assume that $\mathcal{E}$ has a logarithmic singularity in $b = \infty$. Since $0 < \sigma_i < 1$ are rational numbers such that $\sigma_1+\sigma_2+\sigma_3+\sigma_4$ is an integer, it follows from Lemma 3.2.2 (iv) that $b = \infty$ is a logarithmic singularity of $\mathcal{E}$ if and only if

$$\sigma_1 + \sigma_2 = \sigma_3 + \sigma_4 = 1. \tag{3.3.2}$$

By Lemma 3.2.2 (ii), this condition also implies that $\mathcal{E}$ admits a filtration in the sense of Definition 3.1.9.

To be able to apply the criterion of Theorem 3.1.13 we have to pass to a suitable cover $\pi : \overline{T} \to \overline{S} = \mathbb{P}^1_s$ which removes the elliptic singularities. The cover we are looking for has ramification of order e_b at the elliptic singularities $b \in \{0, 1, \infty\}$, and is unramified at the regular points of $\mathcal{E}$. At the logarithmic singularities we allow arbitrary ramification. Since we have at least one logarithmic singularity, at $b = \infty$, it is clear that such a cover exists.

Lemma 3.3.1 (ii) implies that the pullback $\pi^*(\mathscr{E}|_S)$ extends to an indigenous bundle over $\overline{T}$ all of whose marked points are logarithmic singularities if and only if the local exponents of $\mathscr{E}$ at the points $b \in \{0, 1\}$ that are elliptic singularities satisfy

$$\frac{1}{e_b} = |\beta_b| = \begin{cases} \sigma_1 + \sigma_3 - 1 & \text{if } b = 0, \\ \sigma_2 + \sigma_3 - 1 & \text{if } b = 1, \end{cases} \tag{3.3.3}$$

where e_b is the order of the local monodromy at b. The same criterion has been used in [**BM10b**]. This yields that we can restrict to the following types of superelliptic curves.

PROPOSITION 3.3.2. *A type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ satisfies* (3.3.2) *and* (3.3.3) *if and only if we are in one of the following cases, up to permuting the indices in such a way that (3.3.2) remains valid.*

(i) $\mathscr{E}$ has three logarithmic singularities:

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}).$$

(ii) $\mathscr{E}$ has two logarithmic singularities and one elliptic singularity with order of local monodromy $e = m$:

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (\tfrac{m-1}{2m}, \tfrac{m+1}{2m}, \tfrac{m-1}{2m}, \tfrac{m+1}{2m}).$$

(iii) $\mathscr{E}$ has one logarithmic singularity and two elliptic singularities with order of local monodromy $e = m$ and $e = n$, respectively:

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (\tfrac{nm-m+n}{2mn}, \tfrac{nm+m-n}{2mn}, \tfrac{nm+m+n}{2mn}, \tfrac{nm-m-n}{2mn}),$$

where $(m, n) \neq (2, 2)$.

PROOF. This is an easy calculation. □

Let $\mathscr{X}$ be the superelliptic curve over $S = \mathbb{P}^1_s - \{0, 1, \infty\}$ with one of the types from Proposition 3.3.2. Now the key requirement in Theorem 3.1.13 is that the pullback $\mathscr{X}_T$ extends smooth curve over the inverse images of the elliptic singularities.

REMARK 3.3.3. In Case (i) of Proposition 3.3.2, the curve $\mathscr{X} = \mathscr{X}_T$ is the Legendre family

$$\mathscr{X} \colon z^2 = x(x-1)(x-s)$$

of elliptic curves. (We have $\overline{T} = \overline{S}$.) The rank-2 bundle $\mathscr{E} = \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ has logarithmic singularities at all points in $\{0, 1, \infty\}$. Clearly, $\mathscr{X}$ is smooth outside the logarithmic singularities. Therefore the Legendre family defines a Teichmüller curve. This is the only superelliptic curve in our set-up (with the restrictions made in Remark 3.2.1) that defines a Teichmüller curve without further modifications.

Note that the Legendre family is semistable at $s = 0$ and $s = 1$, but not at $s = \infty$ [**Ked08**, Example 4.1.4]. The reason is that the local exponents of

$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ at $\overline{S}-S$ are not all integers. After normalising the local exponents (α_s, β_s) at $s=0$ and $s=1$ to $(\alpha_s, \beta_s) = (0,0)$, the Riemann scheme (see Appendix A) of the Picard-Fuchs operator of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ equals

$$\left[\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{array}\right].$$

For more details on the normalisation of local exponents we refer to [**Yos87**, 2.6].

For the Cases (ii) and (iii) of Proposition 3.3.2, the following lemma shows that $\mathscr{X}_T$ extends to a semistable curve over

$$\widetilde{T} := \overline{T} - \pi^{-1}(\infty),$$

but not necessarily over the logarithmic singularities in $\pi^{-1}(\infty)$.

Lemma 3.3.4. *Let $\mathscr{X}$ be a superelliptic curve over $S = \mathbb{P}^1_s - \{0,1,\infty\}$ whose type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ satisfies (3.3.2). Let $\pi \colon \overline{T} \to \overline{S} = \mathbb{P}^1_s$ be a cover that removes all elliptic singularities of $\mathscr{E}$, i.e. it has ramification of order e_b at the elliptic singularities. Then the pullback $\mathscr{X}_T$ extends to a semistable curve $\mathscr{X}_{\widetilde{T}}$ over $\widetilde{T} = \overline{T} - \pi^{-1}(\infty)$.*

Proof. Formula (3.3.1) implies that $\mathbb{C}(\overline{T}) \supset \mathbb{C}(s, s^{\sigma_1+\sigma_3}, (s-1)^{\sigma_2+\sigma_3})$. Choose $c \in \overline{T}$ with $\pi(c) = b = 0$. Let N be the least common multiple of the denominators of σ_1, σ_2, σ_3, σ_4, and put $a_i := N\sigma_i$, $i = 1, \ldots, 4$. Then the superelliptic curve $\mathscr{X}_T$ is given by the affine equation

$$\mathscr{X}_T \colon z^N = x^{a_1}(x-1)^{a_2}(x-s)^{a_3}, \tag{3.3.4}$$

where we consider $s \in \mathbb{C}(\overline{T})$ as a function on $\overline{T}$. Write $A := \widehat{\mathscr{O}}_{\overline{T},c}$. It suffices to construct an extension of $\mathscr{X}_T$ to $\operatorname{Spec} A = \operatorname{Spec} \mathbb{C}[\![t]\!]$ where t is a local parameter at c. As $\mathbb{C}(\overline{T})$ contains $s^{\sigma_1+\sigma_3}$, we can introduce new affine coordinates $\tilde{x}$ and $\tilde{z}$ with $s\tilde{x} = x$ and $s^{\sigma_1+\sigma_3}\tilde{z} = z$. In terms of these coordinates we obtain

$$\mathscr{X}_T \colon \tilde{z}^N = \tilde{x}^{a_1}(s\tilde{x}-1)^{a_2}(\tilde{x}-1)^{a_3}. \tag{3.3.5}$$

This is an alternative affine equation for $\mathscr{X}_T$, away from $s = 0$. Together we obtain a model $\mathscr{X}_{\overline{T}}$ of $\mathscr{X}_T$ over $\operatorname{Spec} A$ that consists of the equations (3.3.4) and (3.3.5) identified via the relations $s\tilde{x} = x$ and $s^{\sigma_1+\sigma_3}\tilde{z} = z$ on the generic fibre. We also obtain a finite map $\mathscr{X}_{\overline{T}} \to \mathscr{P}_{\overline{T}}$, where $\mathscr{P}_{\overline{T}}$ is given by the coordinates x and $\tilde{x}$ with the relation $s\tilde{x} = x$. The special fibre $\mathscr{X}_{\overline{T},c} = \mathscr{X}_{\overline{T}} \otimes_A A/(t)$ may be described as follows. The special fibre $\mathscr{P}_{\overline{T},c}$ consists of two irreducible components $\overline{P}^1_0$ and $\overline{P}^2_0$ which are the projective lines with coordinates x resp. $\tilde{x}$. The branch points $x = 1$ and $x = \infty$ specialise to $\overline{P}^1_0$ and the branch points $x = 0$ and $x = s$ specialise to $\overline{P}^2_0$. The components $\overline{P}^1_0$ and $\overline{P}^2_0$ intersect in a unique point ξ with coordinate $x = 0$ and $\tilde{x} = \infty$, which is an ordinary double point of $\mathscr{P}_{\overline{T},c}$. For $i = 1, 2$ let $\overline{Z}^i_0$ be the restriction of $\mathscr{X}_{\overline{T},c}$ to $\overline{P}_0^i$. Note that $\overline{Z}_0^i$ is the smooth projective, but possibly disconnected curve, given by the equations

$$\begin{aligned} \overline{Z}_0^1\colon &\quad z^N = x^{a_1+a_3}(x-1)^{a_2}, \\ \overline{Z}_0^2\colon &\quad \tilde{z}^N = (-1)^{a_2}\tilde{x}^{a_1}(\tilde{x}-1)^{a_3} \end{aligned}$$

One checks that all points of $\overline{Z}_{\bar{t},c}$ above ξ are ordinary double points as well. (The cover $\mathscr{X}_{\overline{T},c} \to \mathscr{P}_{\overline{T},c}$ is a so-called admissible cover. For more details we refer to [**Wew98**] or [**Wew99**].) The computations for the case that $b = 1$ are analogous. □

Except for Case (i) of Proposition 3.3.2, the semistable curve $\mathscr{X}_{\widetilde{T}}$ constructed in Lemma 3.3.4 has singular fibres over the 'removed' elliptic singularities and we cannot directly apply Theorem 3.1.13. In the following remarks we explain how to modify the family $\mathscr{X}_{\widetilde{T}} \to \widetilde{T}$ in order to produce Teichmüller curves.

Remark 3.3.5. In Case (ii) of Proposition 3.3.2, a quotient of $\mathscr{X}_{\widetilde{T}}$ by a suitable finite group of $\widetilde{T}$-automorphisms yields the Teichmüller curves found by Veech [**Vee89**]. For a description of this quotient construction we refer to [**BM10b**, Section 5]. In addition, we remark the following. Let

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left(\tfrac{m-1}{2m}, \tfrac{m+1}{2m}, \tfrac{m-1}{2m}, \tfrac{m+1}{2m}\right).$$

In the case that m is an even integer, this yields a superelliptic curve over $S = \mathbb{P}^1_s - \{0, 1, \infty\}$ given by the affine equation

$$\mathscr{X}\colon z^{2m} = x^{m-1}(x-1)^{m+1}(x-s)^{m-1}.$$

In the case that m is an odd integer, this yields a superelliptic curve given by the affine equation

$$\mathscr{X}\colon z^m = x^{(m-1)/2}(x-1)^{(m+1)/2}(x-s)^{(m-1)/2}. \tag{3.3.6}$$

In [**BM10b**, Section 5] the authors consider the superelliptic curve given by the affine equation

$$\mathscr{X}^{\mathrm{BM}}\colon z^{2m} = x(x-1)^{2m-1}(x-s).$$

For $i = m \pm 1$, the χ^i-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}^{\mathrm{BM}}/S)$ is an indigenous rank-2 subbundle. (Note that there is a typo in [**BM10b**, Lemma 5.1]: "$i = (n+1)/2$" should read "$i = (n+2)/2$" and "$\mathbb{L}((n-1)/2)$" should read "$\mathbb{L}((n-2)/2)$".) In the case that m is an even integer, this yields the same curve and indigenous bundle as in our construction. In the case that m is an odd integer, the character χ^i with $i = m \pm 1$ is not injective, and $\mathscr{E}_{\chi^i}$ is a subbundle of the de Rham cohomology of the quotient curve $\mathscr{X}^{\mathrm{BM}}/\ker \chi^i$. This quotient is the superelliptic curve given by (3.3.6).

Remark 3.3.6. Case (iii) of Proposition 3.3.2 is considered in [**BM10b**, Section 6]. There the authors consider Klein's four group $\tilde{H} = \langle \sigma, \tau \rangle$ acting on $\mathbb{P}^1_x \times_{\mathbb{C}} S$ by

$$\sigma(x) = \frac{s(x-1)}{x-s}, \qquad \tau(x) = \frac{s}{x}.$$

Passing to the pullback via $\pi\colon T \to S$ as in Lemma 3.3.4, the group $\tilde{H}$ lifts to the Galois closure $\mathcal{Y}$ of $\mathcal{X}_T \xrightarrow{\mathbb{Z}/N\mathbb{Z}} \mathbb{P}^1_T \xrightarrow{\tilde{H}} \mathbb{P}^1_T$ given by a commutative diagram of the following form, where $\mathcal{X}_T^\tau$ denotes the conjugate of $\mathcal{X}_T$ under τ and $\widehat{\mathcal{X}}_T$ is the maximal subextension to which τ lifts.

$$\begin{tikzcd}
& \mathcal{Y} \arrow[dl] \arrow[dr] & \\
\mathcal{X}_T \arrow[dr] \arrow[ddr, "\mathbb{Z}/N\mathbb{Z}"'] & & \mathcal{X}_T^\tau \arrow[dl] \arrow[ddl, "\mathbb{Z}/N\mathbb{Z}"] \\
& \widehat{\mathcal{X}}_T \arrow[d] & \\
& \mathbb{P}^1_T \arrow[d, "\tilde{H}"'] & \\
& \mathbb{P}^1_T &
\end{tikzcd}$$

The pullback π is exactly the one that removes the elliptic singularities of $\pi^*\mathcal{E}$. Moreover, in [**BM10b**] it is checked that the quotient $\mathcal{X} = \mathcal{Y}/\tilde{H}$ extends to a smooth curve over the removed elliptic singularities and that $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{X}/T)$ contains a flat subbundle isomorphic to $\pi^*\mathcal{E}$. Thus, $\mathcal{X}$ defines a Teichmüller curve in the sense of Theorem 3.1.13.

The following proposition shows that if (and only if) $m = n$ this construction also works for $\langle \sigma \rangle \subset \tilde{H}$ without passing to the Galois closure $\mathcal{Y}$. This produces a new class of Teichmüller curves not treated in [**BM10b**].

Proposition 3.3.7. *Let $\mathcal{X}$ be the superelliptic curve over $S = \mathbb{P}^1_s - \{0, 1, \infty\}$ with type*

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (\tfrac{nm-m+n}{2mn}, \tfrac{nm+m-n}{2mn}, \tfrac{nm+m+n}{2mn}, \tfrac{nm-m-n}{2mn}), \quad (m, n) \neq (2, 2)$$

from Case (iii) of Proposition 3.3.2, given by the affine equation

$$\mathcal{X}_s\colon z^N = x^{a_1}(x-1)^{a_2}(x-s)^{a_3},$$

where N is the least common multiple of the denominators of σ_1, σ_2, σ_3, σ_4, and $a_i = N\sigma_i$ for $i = 1, \ldots, 4$. Let $\pi\colon \overline{T} \to \overline{S} = \mathbb{P}^1_{\mathbb{C}}$ be a cover as in Lemma 3.3.4, i.e. it is unramified in S, has ramification index $e = m$ resp. $e = n$ over $b = 0$ resp.

$b=1$, and suitable ramification index over $b=\infty$. Let $\mathscr{X}_{\widetilde{T}}$ be the extension of the pullback $\mathscr{X}_T$ to a semistable curve over $\widetilde{T}=\overline{T}-\pi^{-1}(\infty)$.

(i) The group $\mathrm{Aut}_T(\mathscr{X}_T)$, where $T=\pi^{-1}(S)$, contains an automorphism σ of order 2 with

$$\sigma(x,z)=\left(\frac{s(x-1)}{x-s},\ s^{\sigma_1+\sigma_3}(s-1)^{\sigma_2+\sigma_3}\,\frac{x\,(x-1)}{z\,(x-s)}\right).$$

Fix a generator φ of the Galois group $\mathbb{Z}/N\mathbb{Z}\subset\mathrm{Aut}_T(\mathscr{X}_T)$ of the N-cyclic cover $\mathscr{X}_T\to\mathbb{P}^1_T$ given by projection onto x. Then $\sigma\varphi\sigma=\varphi^{-1}$.

(ii) The T-automorphism σ extends to an $\widetilde{T}$-automorphism of $\mathscr{X}_{\widetilde{T}}$.

(iii) The quotient

$$\mathcal{X}:=\mathscr{X}_{\widetilde{T}}/\langle\sigma\rangle\to\widetilde{T}$$

is a smooth curve (after replacing σ by $\varphi^{-1}\sigma$ if necessary) if and only if $m=n$, i.e

$$(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=(\tfrac{1}{2},\tfrac{1}{2},\tfrac{m+2}{2m},\tfrac{m-2}{2m}),\qquad m\neq 2.$$

PROOF. The existence of σ is shown in [**BM10b**, Lemma 6.4] and one easily checks that $\sigma\varphi\sigma^*z=\chi_0(\varphi^{-1})\cdot z$, where $\chi_0\colon\mathbb{Z}/N\mathbb{Z}\to\mathbb{C}^\times$ is the injective character such that $\varphi^*z=\chi_0(\varphi)z$. This proves (i).

Denote by $\mathscr{X}_{\widetilde{T}}$ the semistable curve from Lemma 3.3.4 and write $\mathcal{X}=\mathscr{X}_{\widetilde{T}}/\langle\sigma\rangle$. Denote by $\overline{Z}_b\to\overline{P}_b$ the restriction of $\mathscr{X}_{\widetilde{T}}$ to the admissible cover over $b\in\{0,1\}\subset\overline{S}-S$. Let $\overline{Z}_b^1$ be the restriction of $\overline{Z}_b$ to the irreducible genus-0 component $\overline{P}_b^1$ of $\overline{P}_b$ to which the branch points $x_1=0$ and $x_3=\lambda$ specialise, and let $\overline{Z}_b^2$ be the restriction of $\overline{Z}_b$ to the irreducible genus-0 component $\overline{P}_b^2$ of $\overline{P}_b$ to which the branch points $x_2=1$ and $x_4=\infty$ specialise.

It is easy to check that the automorphism σ extends to an automorphism of $\mathscr{X}_{\overline{T}}$ of order 2 with

$$\sigma|_{\overline{Z}_b^1}\colon\overline{Z}_b^1\xrightarrow{\sim}\overline{Z}_b^2,\qquad b\in\{0,1\}.$$

A necessary condition for $\mathcal{X}$ to be smooth over $b\in\{0,1\}$ is that $g(\overline{Z}_0^1)=g(\mathcal{X}_0)=g(\mathcal{X}_1)=g(\overline{Z}_1^1)$. The Riemann-Hurwitz genus formula applied to the cyclic covers $\overline{Z}_0^1\to\overline{P}_0^1$ and $\overline{Z}_1^1\to\overline{P}_1^1$ implies that

$$\gcd(\tfrac{N}{m},N)=\gcd(a_1+a_3,N)=\gcd(a_2+a_3,N)=\gcd(\tfrac{N}{n},N).$$

Therefore, $m=n$, i.e. the type $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ is given as stated in the proposition.

For the converse direction we suppose that $m=n\neq 2$. Let $b\in\{0,1\}$. The fibre of $\overline{Z}_b\to\overline{P}_b$ over the ordinary double point of $\overline{P}_b$ consists $N/m\leq 2$ points, where

$$N/m=\begin{cases}2, & \text{if } m \text{ is odd,}\\ 1, & \text{if } m \text{ is even.}\end{cases}$$

If the fibre consists of $N/m=1$ points, then obviously σ fixes this point. If the fibre consists of $N/m=2$ points, say $\{z_0,z_1\}$, then φ^2 fixes the points in $\{z_0,z_1\}$. Hence, either σ or $\varphi^{-1}\sigma$ fix both z_0 and z_1. After replacing σ by the $\varphi^{-1}\sigma$ if

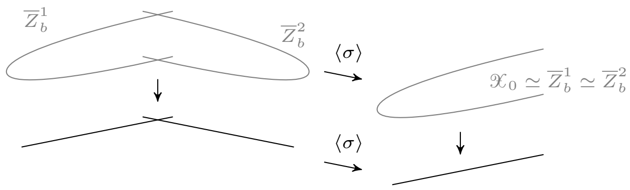


FIGURE 3.1. $m = n$ odd, i.e. $\frac{N}{m} = 2$.

necessary, we conclude that σ fixes all points in the fibre over the ordinary double point. Therefore $\mathscr{X}_b = \overline{Z}_b/\langle\sigma\rangle \simeq Z_b^1$ is smooth. Note that $g(\mathscr{X}_0) = g(\mathscr{X}_1)$ since $m = n$. This proves the proposition. □

For an odd integer m the smoothness of $\mathscr{X}_c$ with $\pi(c) = 0$ may also be deduced from Example 1.5.7.

THEOREM 3.3.8. *Let $\mathscr{Z}$ be a superelliptic curve of type*

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{m+2}{2m}, \tfrac{m-2}{2m}), \qquad m \neq 2.$$

Consider the smooth curve $\mathscr{X} = \mathscr{Z}_{\widetilde{T}}/\langle\sigma\rangle \to \widetilde{T}$ over $\widetilde{T} = \overline{T} - \pi^{-1}(\infty)$ constructed in Proposition 3.3.7. Then the image of the moduli map

$$\widetilde{T} \to \mathscr{M}_g, \quad c \mapsto [\mathscr{X}_c]$$

is a Teichmüller curve.

PROOF. To prove the theorem we apply Theorem 3.1.13, i.e. we show that the Deligne extension of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/\widetilde{T})$ contains a flat subbundle isomorphic to $\pi^*\mathscr{E}$. Note that Proposition 3.3.2 implies that

(i) $\pi^*\mathscr{E}$ is an indigenous bundle,
(ii) all points in $\pi^{-1}(\infty)$ are logarithmic singularities of $\pi^*\mathscr{E}$.

The group $G = \mathbb{Z}/N\mathbb{Z} \rtimes \langle\sigma\rangle \subset \mathrm{Aut}_T(\mathscr{Z}_T)$ is a dihedral group of order $2N$. (In particular, G is a semidirect product as required in Section 2.3.) We have chosen the flat vector bundle $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Z}/S)$ as the χ_0-isotypical component where $\chi_0\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$ is an injective irreducible character (see Remark 3.2.1). The curve $\mathscr{Z}_T$ is the pullback of a superelliptic curve over $S = \mathbb{P}^1_s - \{0, 1, \infty\}$ of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (\frac{1}{2}, \frac{1}{2}, \frac{m+2}{2m}, \frac{m-2}{2m})$ with $\sum_{i=1}^4 \sigma_i = 2$. By Proposition 2.3.6 we have a decomposition

$$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/T) = \bigoplus_{\chi \in \mathrm{Irr}(\mathbb{Z}/N\mathbb{Z})/\langle\sigma\rangle} (\mathscr{E}_{\chi,\mathbb{1}})^{\langle\sigma\rangle} \tag{3.3.7}$$

and, for $\chi = \chi_0$, the component $(\mathscr{E}_{\chi,\mathbb{1}})^{\langle\sigma\rangle}$ is isomorphic to $\pi^*\mathscr{E}$. (An isomorphism $\pi^*\mathscr{E} \xrightarrow{\sim} (\mathscr{E}_\chi, \mathbb{1})^{\langle\sigma\rangle}$ is induced by $\tilde{\omega} \mapsto \tilde{\omega} + \sigma\tilde{\omega}$.) □

Remark 3.3.9. The genus of $\mathscr{X}$ is

$$g = \begin{cases} \frac{m-1}{2}, & \text{if } m \text{ is odd,} \\ \tilde{m}, & \text{if } m = 2 + 4\tilde{m}, \\ \tilde{m}, & \text{if } m = 4\tilde{m}. \end{cases}$$

We summarise the results of this section.

Summary 3.3.10. The only indigenous bundles 'coming from superelliptic curves' are those having one of the types in Proposition 3.3.2. In Case (i) the superelliptic curve is the Legendre family of elliptic curves and defines a Teichmüller curve. In Case (ii) the only possible Teichmüller curves defined by quotients of $\mathscr{Z}_T$ are the Teichmüller curves found by Veech (see [**BM10b**, Section 5] and Remark 3.3.5). In Case (iii), the corresponding N-cyclic cover $\mathscr{Z} \to \mathbb{P}^1_S$ is branched at four points. Therefore the automorphisms of $\mathbb{P}^1_S$ that are invariant on the set of branch points of $\mathscr{Z} \to \mathbb{P}^1_S$ form a subgroup of $\tilde{H} = \langle \sigma, \tau \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$. If $m \neq n$, then one must apply the construction in [**BM10b**, Section 6]. If $m = n$, then our construction and the construction in [**BM10b**, Section 6] produce Teichmüller curves. By this we have classified all Teichmüller curves that may be produced by Theorem 3.1.13 with flat vector bundles 'coming from a superelliptic curve' (under the restrictions made in Remark 3.2.1).

3.3.1. Description of the de Rham cohomology. In this subsection we describe the decomposition

$$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/T) = \bigoplus_{\chi \in \mathrm{Irr}(\mathbb{Z}/N\mathbb{Z})/\langle \sigma \rangle} (\mathscr{E}_{\chi,\mathbb{1}})^H .$$

used in the proof of Theorem 3.3.8 in (3.3.7) more precisely. We compute

$$n_{\chi,1} := \mathrm{rank}\, (\mathscr{E}_{\chi,\mathbb{1}})^H$$

using Proposition 2.3.8.

Denote by $\mathscr{Z}$ the superelliptic curve with type given as in Theorem 3.3.8, and by φ a fixed generator of the corresponding N-cyclic cover $\mathscr{Z} \to \mathbb{P}^1_S$.

Lemma 3.3.11. *We have*

$$n_{\chi,1} = \begin{cases} 2, & \chi(\varphi^2) \neq 1 \neq \chi(\varphi^m), \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let $\chi \in \mathrm{Irr}(A) - \{1\}$. Clearly, it holds $\chi(\varphi^2) = 1$ if and only if $\chi(\varphi) = \chi(\varphi^{-1}) = \chi^\sigma(\varphi)$, which holds if and only if $H_\chi = H$, i.e. $G_\chi = G$. In this case one checks that

$$n_{\chi,1} = 2 \cdot |H(\chi)| - \sum_{i=1}^{4} k_\chi^{i,1} = 2 - 1 - 1 - 0 - 0 = 0.$$

In all other cases, we have $H_\chi = \langle 1 \rangle$, $G_\chi = A$ and

$$n_{\chi,1} = 2 \cdot |H(\chi)| - k_\chi^{1,1} - k_\chi^{2,1} - k_\chi^{3,1} - k_\chi^{3,\sigma} - k_\chi^{4,1} - k_\chi^{4,\sigma}$$

If $\chi(\varphi^m) = 1$, then one checks that $n_{\chi,1} = 4 - 1 - 1 - 1 - 1 - 0 - 0 = 0$ and if $\ker \chi \cap \{\varphi^2, \varphi^m\} = \emptyset$, then one checks that $n_{\chi,1} = 4 - 1 - 1 - 0 - 0 - 0 - 0 = 2$. $\square$

From this one may also deduce the genus of $\mathscr{X}$.

3.3.2. Ellenberg's pq-family. In Chapter 4 we consider the one-dimensional families of curves with real multiplication constructed in [**Ell01**, Corollary 4.5]. Ellenberg's construction in Case (6) of [**Ell01**, Corollary 4.5] yields a family of curves of genus $\frac{(p-1)(q-1)}{2}$ with real multiplication by $\mathbb{Q}(\zeta_{pq} + \zeta_{pq}^{-1})$. The construction in [**BM10b**, Section 6] corresponding to Case (iii) of Proposition 3.3.2 for $m = p$ and $n = q$ also yields a family of curves of genus $\frac{(p-1)(q-1)}{2}$ with real multiplication by the same field. However, this is a different one. We briefly explain why Ellenberg's family does not define a Teichmüller curve.

For distinct odd primes p and q, Ellenberg considers a family

$$\mathscr{X}^{\mathrm{Ell}} \xrightarrow{\langle \varphi \rangle} \mathbb{P}^1_T \xrightarrow{\langle \sigma \rangle} \mathbb{P}^1_T$$

of Galois covers whose Galois group is a dihedral group with reflection σ and rotation φ of order pq. Ellenberg shows that the quotient $\mathscr{X}^{\mathrm{Ell}}/\langle \sigma \rangle$ yields a family of curves with real multiplication by the field $\mathbb{Q}(\zeta_{pq} + \zeta_{pq}^{-1})$. The dihedral cover $\mathscr{X}^{\mathrm{Ell}} \to \mathbb{P}^1_T$ is branched at four points with ramification type given by the conjugacy classes

$$(\mathrm{Cl}_{D_{pq}}(\sigma), \mathrm{Cl}_{D_{pq}}(\sigma), \mathrm{Cl}_{D_{pq}}(\varphi^p), \mathrm{Cl}_{D_{pq}}(\varphi^q))$$

in $D_{pq} = \langle \varphi, \sigma \rangle$ ([**Ell01**, Section 3]). This implies that the cyclic cover $\mathscr{X}^{\mathrm{Ell}} \xrightarrow{\langle \varphi \rangle} \mathbb{P}^1_S$ has type

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (\tfrac{1}{q}, \tfrac{q-1}{q}, \tfrac{1}{p}, \tfrac{p-1}{p}).$$

This type equals none of the types in Proposition 3.3.2, even up to permutation of the σ_i's and uniform substitution of σ_i by $k \cdot \sigma_i \pmod 1$ for some integer k. This implies that, for none of the rank-2 subbundles $\mathscr{E}_\chi \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}^{\mathrm{Ell}}/S)$ where $\chi\colon \mathbb{Z}/pq\mathbb{Z} \to \mathbb{C}^\times$ is an irreducible character, the Riemann scheme of local exponents is of the form

$$\left[\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & \gamma \\ \pm 1/m & \pm 1/n & \gamma \end{array}\right]$$

for some $n, m \in \mathbb{N}$.

Chapter 4

Ellenberg's families of curves with real multiplication

In [**Ell01**, Corollary 4.1] three families of curves with real multiplication are constructed. One may ask whether the families constructed by Ellenberg define Teichmüller curves because Teichmüller curves parametrise curves with real multiplication [**Möl06b**, Theorem 2.7]. The Jacobians of the curves parametrised by Ellenberg's families have real multiplication by a field K such that $[K : \mathbb{Q}]$ equals the genus g of the parametrised curves, i.e. K is as large as possible for an abelian variety of dimension g. Ellenberg classifies families of curves with this property that can be constructed as quotients Y/H of certain metacyclic covers

$$Y \xrightarrow{A} Z \xrightarrow{H} \mathbb{P}^1_{\mathbb{C}}.$$

The two families discussed in this chapter are the only one-dimensional families from [**Ell01**] that we have not analysed in the context of Teichmüller curves yet. The third family we have already discussed in Subsection 3.3.2. In Section 4.2 we show that the de Rham cohomologies of the two remaining Ellenberg families split into flat rank-2 subbundles all of which carry a filtration in the sense of Definition

3.1.9 (page 43). This is a necessary condition for a flat subbundle to be indigenous. However, in Section 4.3 we show that none of the subbundles satisfies the conditions from Theorem 3.1.13 (page 44) by studying the boundary of a suitable Hurwitz space.

Similar to the situation in [**BM10b**] there is an automorphism of Z that does not lift to Y. In Section 4.5 we lift this automorphism to the Galois closure. An analogous construction was used in [**BM10b**] to construct Teichmüller curves. In our situation we show that by this construction the above criterion for exclusion no longer holds. Moreover, we find rank-2 subbundles of the de Rham cohomology of the adapted families that are 'good' candidates for indigenous bundles to satisfy the conditions from Theorem 3.1.13.

4.1. Ellenberg's families

In this section we introduce certain families of metacyclic covers, namely the composition of an étale cyclic cover with a cyclic cover of $\mathbb{P}^1_{\mathbb{C}}$. The two Ellenberg families mentioned in the introduction of this chapter are quotients of special families of such metacyclic covers.

Let $m \geq 2$ be an integer and $p \equiv 1 \pmod m$ a prime number. Suppose that $\alpha \in \{1, \dots, p-1\}$ represents an element in $\mathbb{F}_p^\times$ of order m. Then

$$G = G_{p,m} := \langle \varphi, \psi;\ \varphi^p = \varphi^m,\ \psi\varphi\psi^{-1} = \varphi^\alpha \rangle \tag{4.1.1}$$

is the presentation of a semidirect product $A \rtimes H$ with $A = \langle \varphi \rangle$ and $H = \langle \psi \rangle$. Such groups are called metacyclic groups. One easily verifies the relation

$$\psi^k \varphi^\ell = \varphi^{\ell\alpha^k} \psi^k \tag{4.1.2}$$

in G for all integers k and ℓ.

Definition 4.1.1. Fix a quadruple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{Q}^4$ such that $0 < \sigma_i < 1$ and $\sum_{i=1}^4 \sigma_i$ is an integer. Denote by m the least common multiple of the denominators of σ_1, σ_2, σ_3, σ_4. Put $a_i = m\sigma_i$ for $i = 1, \dots, 4$. A cover of $\mathbb{P}^1_{\mathbb{C}}$ is called *metacyclic* of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ if it is an element of the Hurwitz space $\mathscr{H}_{G,\mathbf{C}}$ of G-covers with ordered branch locus where $G = G_{p,m}$ is a metacyclic group as in (4.1.1) and

$$\mathbf{C} = \left(\mathrm{Cl}_G(\psi^{a_1}), \mathrm{Cl}_G(\psi^{a_2}), \mathrm{Cl}_(G\psi^{a_3}), \mathrm{Cl}_G(\psi^{a_4})\right)$$

a quadruple of conjugacy classes in G.

Note that $(\varphi\psi^{a_1}, \psi^{a_2}, \psi^{a_3}, \psi^{a_4}\varphi) \in \mathrm{Ni}(G, \mathbf{C})$. In particular, metacyclic covers of type $(\sigma_1, \sigma_2, \sigma_2, \sigma_3)$ exists indeed.

Proposition 4.1.2. *There exists a one-dimensional universal family*

$$\mathscr{Y} \xrightarrow{G} \mathbb{P}^1_S \to S := \mathscr{H}_{G,\mathbf{C}}$$

of metacyclic covers of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$*, i.e.* $\mathscr{Y}$ *is an* S*-scheme such that the fibre* $\mathscr{Y}_b \to \mathbb{P}^1_{\mathbb{C}}$ *at any* $\mathbb{C}$*-rational point* $b \in S$ *is isomorphic (in the sense of Definition 1.1.7) to the* G*-cover parametrised by* b*.*

Proof. Let $g = \varphi^k \psi^\ell$ be an element of the centre of G. From Relation (4.1.2) it follows that (a) $g\psi = \psi g$ if and only if $k \equiv 0 \pmod p$, and that (b) $g\varphi = \varphi g$ if and only if $\ell \equiv 0 \pmod m$. Therefore the centre of G is trivial and the Hurwitz space $\mathscr{H}_{G,\mathbf{C}}$ is a fine moduli space (see Section 1.3.1 or [**RW06**, Corollary 4.12]). $\square$

The universal family of metacyclic covers of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ factors through

$$\mathscr{Y} \xrightarrow{A} \mathscr{Z} := \mathscr{Y}/A \xrightarrow{H} \mathbb{P}^1_S,$$

where $\mathscr{Y} \xrightarrow{A} \mathscr{Z}$ is an étale cyclic A-cover and $\mathscr{Z} \xrightarrow{H} \mathbb{P}^1_S$ is a cyclic H-cover. The ramification type of the fibres $\mathscr{Z}_b \to \mathbb{P}^1_{\mathbb{C}}$ is given by the quadruple

$$\big(\mathrm{Cl}_H(\psi^{a_1}), \mathrm{Cl}_H(\psi^{a_2}), \mathrm{Cl}_H(\psi^{a_3}), \mathrm{Cl}_H(\psi^{a_4})\big)$$

of conjugacy classes in H. Throughout this chapter we keep these notations.

We now consider the smooth curve $\mathscr{X} := \mathscr{Y}/H$ over S. Note that if $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is either $(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2})$ or $(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2})$, then $\mathscr{X} \to S$ is the one-dimensional family of curves from [**Ell01**, Corollary 4.1] with real multiplication by $\mathbb{Q}(\zeta_p^{(m)})$, where m is the least common multiple of the denominators of σ_1, σ_2, σ_3, σ_4.

Definition 4.1.3. We will use the notation $\mathscr{X}^{\mathrm{Ell}} := \mathscr{Y}/H \to S$ in case that

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \big\{\big(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{1}{2}\big), \big(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{2}\big)\big\}$$

and call $\mathscr{X}^{\mathrm{Ell}}$ an *Ellenberg family.*

Note that in the following the notation $\mathscr{X} \to S$ without the superscript 'Ell' denotes the quotient of a family of metacyclic covers with arbitrary type.

4.2. The de Rham cohomology of Ellenberg's families

In this section we find all universal families of metacyclic $G_{p,m}$-covers such that the de Rham cohomology of their H-quotients splits into isotypical rank-2 bundles with a filtration (in the sense of Definition 3.1.9). It turns out that these are essentially the families considered in [**Ell01**, Corollary 4.5 (2) and (3)]. The arguments in this section are similar to the ones of [**Ell01**] in a slightly different set-up.

Let

$$\begin{tikzcd} \mathscr{Y} \arrow[rr, "G"] \arrow[dr, "A"'] & & \mathbb{P}^1_S \\ & \mathscr{Z} \arrow[ur, "H"'] & \end{tikzcd}$$

be the universal family of metacyclic covers of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ over $S = \mathscr{H}_{G,\mathbf{C}}$ with $G = G_{p,m}$ and $\mathbf{C}$ as in Section 4.1.

Recall that $G = A \rtimes H = \langle\varphi\rangle \rtimes \langle\psi\rangle$ is a metacyclic group, and that we denote by $\mathrm{Irr}(A)$ the set of of irreducible A-characters on which H acts by $\chi^{\psi}(\varphi) = \chi(\varphi^{\alpha})$ for $\chi \in \mathrm{Irr}(A)$. Moreover, we write $\mathbb{1}$ for the trivial character.

We consider the quotient

$$\mathscr{X} = \mathscr{Y}/H \to \mathbb{P}^1_S.$$

The goal of this section is to prove the following proposition.

Proposition 4.2.1. *The relative de Rham cohomology of $\mathscr{X} = \mathscr{Y}/H$ splits into flat subbundles*

$$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S) = \bigoplus_{\substack{\chi\in\mathrm{Irr}(A)/H\\ \chi\neq\mathbb{1}}} \mathscr{E}_\chi, \tag{4.2.1}$$

where every $\mathscr{E}_\chi$ is isomorphic (as a flat vector bundle with Gauß-Manin connection) to the χ-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ and

$$\operatorname{rank}\mathscr{E}_\chi = g(\mathscr{X}) - 2.$$

In particular, $\operatorname{rank}\mathscr{E}_\chi = 2$ *if and only if*

$$(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \in \left\{\left(\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{2},\tfrac{1}{2}\right),\left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2},\tfrac{1}{2}\right),\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{2}{3},\tfrac{2}{3}\right)\right\}, \tag{4.2.2}$$

up to permuting the indices. In this case $\mathscr{E}_\chi \cap H^0(\mathscr{X},\Omega^1_{\mathscr{X}/S})$ defines a filtration in the sense of Definition 3.1.9.

As in Section 2.3 the subbundles $\mathscr{E}_\chi$ in (4.2.1) will be the H-invariants of the G-isotypical decomposition of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$. Note that, in Proposition 4.2.1, $\operatorname{rank}\mathscr{E}_\chi$ does not depend on the choice of $\chi \in \mathrm{Irr}(A)$ with $\chi \neq \mathbb{1}$.

The following lemma describes the irreducible characters of the metacyclic group $G = G_{p,m}$ (cf. [**Ser77**, Proposition 25], [**Ell01**, Section 2]). We use the notation $\theta_{\chi,\xi} = \mathrm{Ind}^G_{G_\chi}(\chi\cdot\xi)$ from Notation 2.3.1.

Lemma 4.2.2. *Let θ be an irreducible character of $G = G_{p,m}$. Then one of the following cases occurs.*

(i) There exists a character $\xi \in \mathrm{Irr}(H)$ with

$$\theta(ah) = \theta_{\mathbb{1},\xi}(ah) = \xi(h) \quad \text{for } a \in A, h \in H$$

(ii) There exists a non-trivial character $\chi \in \mathrm{Irr}(A)/H$ with

$$\theta = \theta_{\chi,\mathbb{1}} = \mathrm{Ind}^G_A\,\chi$$

As an immediate consequence (see also Proposition 2.3.6), we can decompose

$$\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/T) \simeq \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)^H = \bigoplus_{\substack{\chi\in\mathrm{Irr}(A)/H\\ \chi\neq\mathbb{1}}} \left(\mathscr{E}_{\chi,\mathbb{1}}\right)^H, \tag{4.2.3}$$

into submodules, where $\mathscr{E}_{\chi,\mathbb{1}}$ is the $\theta_{\chi,\mathbb{1}}$-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/T)$. Moreover, we have that

$$\mathscr{E}_{\chi,\mathbb{1}} = \bigoplus_{h\in H} \tilde{\mathscr{E}}_{\chi^h},$$

where $\tilde{\mathscr{E}}_{\chi^h}$ denotes the χ^h-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$. (Here $\{\chi^h;\, h\in H\}$ is the H-orbit of χ.) Note that $\mathrm{rank}(\mathscr{E}_{\mathbb{1},\mathbb{1}})^H = 0$, by Proposition 2.3.8.

Lemma 4.2.3. *We have*

$$\mathrm{rank}(\mathscr{E}_{\chi,\mathbb{1}})^H = 2g(\mathscr{X}) - 2$$

for $\chi\in\mathrm{Irr}(A)$ *with* $\chi\neq\mathbb{1}$.

This could be deduced from the formulas in Chapter 2 (Proposition 2.3.8). However, in this special case we find it easier to give a direct self contained argument.

Proof of Lemma 4.2.3: Denote by $\text{ш}_{\mathscr{Y}}$ the character of the representation $G\to\mathrm{GL}(\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S))$ induced by the action of G on $\mathscr{Y}$ and put $n_{\chi,\mathbb{1}} = \langle \text{ш}_{\mathscr{Y}}, \theta_{\chi,\mathbb{1}}\rangle_G$. The character of $G\to\mathrm{GL}(\mathscr{E}_{\chi,\mathbb{1}})$ is $n_{\chi,\mathbb{1}}\cdot\theta_{\chi,\mathbb{1}}$. Therefore,

$$\mathrm{rank}(\mathscr{E}_{\chi,\mathbb{1}})^H = n_{\chi,\mathbb{1}}\cdot\langle\mathbb{1},\mathrm{Res}_H\,\theta_{\chi,\mathbb{1}}\rangle_H.$$

For $\chi\neq\mathbb{1}$ we have

$$\langle\mathbb{1},\mathrm{Res}_H\,\theta_{\chi,\mathbb{1}}\rangle_H = \langle\mathbb{1},\mathrm{Ind}^A_{H\cap A}(\chi\cdot\mathbb{1})\rangle_H = \langle\mathbb{1},\chi\rangle_{\langle 1\rangle} = 1.$$

Here we have used [**Ser77**, Proposition 22] and $|H\backslash G/A| = 1$ for the first equality. Moreover, since $\mathscr{Y}\xrightarrow{A}\mathscr{X}$ is étale, we have

$$\mathrm{Res}_A\,\text{ш}_{\mathscr{Y}} = 2\cdot\mathbb{1} + (2g(\mathscr{X})-2)\cdot\mathrm{Ind}^A_{\langle 1\rangle}\,\mathbb{1},$$

by Lemma 2.1.1. This implies that

$$\begin{aligned} n_{\chi,\mathbb{1}} &= \langle\text{ш}_{\mathscr{Y}},\theta_{\chi,\mathbb{1}}\rangle_G = \langle\mathrm{Res}_A\,\text{ш}_{\mathscr{Y}},\chi\rangle_A \\ &= 2\cdot\langle\mathbb{1},\chi\rangle_A + (2g(\mathscr{X})-2)\cdot\langle\mathrm{Ind}^A_{\langle 1\rangle}\,\mathbb{1},\chi\rangle_A = 2g(\mathscr{X})-2. \end{aligned}$$

□

We put $\mathscr{E}_\chi := (\mathscr{E}_\chi,\mathbb{1})^H$. In the following lemma we check when $\mathrm{rank}\,\mathscr{E}_\chi = 2$, i.e. $2g(\mathscr{X}) = 2$.

Lemma 4.2.4. *We have* $g(\mathscr{X}) = 2$ *if and only if we are in one of the following situations, up to permuting the indices.*

(i) $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\in(\frac14,\frac34,\frac12,\frac12)$
(ii) $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\in(\frac13,\frac23,\frac12,\frac12)$
(iii) $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\in(\frac13,\frac13,\frac23,\frac23)$

Proof. Suppose that $g(\mathscr{X}) = 2$. The Riemann-Hurwitz genus formula implies $g(\mathscr{X}) = m+1-\frac12\sum_{i=1}^4\gcd(m,a_i)$, where m is the least common multiple of the denominators of σ_1, σ_2, σ_3, σ_4. Since $0 < \sigma_i < 1$, we have $\gcd(m, a_i) \leq \frac{m}{2}$. Moreover, $\gcd(m, a_i) = \frac{m}{2}$ for at most two $i \in \{1, \ldots, 4\}$, since $\gcd(m, a_1, a_2, a_3, a_4) = 1$. Therefore $\gcd(m, a_i) \leq \frac{m}{2} + \frac{m}{2} + \frac{m}{3} + \frac{m}{3} = \frac{5m}{3}$. Thus, $g(\mathscr{X}) = 2$ implies that $m \leq 6$. Now one easily checks that the only possibilities are the ones stated in the lemma. For the converse direction one easily verifies the Riemann-Hurwitz genus formula. $\square$

Remark 4.2.5. Note that a metacyclic $G_{p,3}$-covers of type $(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ are factors of suitable metacyclic $G_{p,6}$-covers of type $(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2})$.

$$\begin{tikzcd}
\mathscr{Y} \arrow[rr, "G_{p,6}=A\rtimes\langle\psi\rangle"] \arrow[dr, "G_{p,3}=A\rtimes\langle\psi^2\rangle"'] & & \mathbb{P}^1_S \\
& \mathbb{P}^1_S \arrow[ur, "\mathbb{Z}/2\mathbb{Z}"'] &
\end{tikzcd}$$

Therefore $\mathscr{X}^{\mathrm{Ell}} = \mathscr{Y}/\langle\psi\rangle$ is a quotient of $\mathscr{Y}/\langle\psi^2\rangle$. Hence, the case that m is odd is covered by the case that m is even.

For the remainder of this section we consider the case that

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \left\{(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3})\right\}.$$

Our goal is to check whether in this case

$$\mathscr{E}_\chi = (\mathscr{E}_{\chi,\mathbb{1}})^H \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)^H \simeq \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}^{\mathrm{Ell}}/S)$$

carries a filtration in the sense of Definition 3.1.9. The key of the proof will be that $\mathscr{E}_{\chi,\mathbb{1}} = \bigoplus_{h\in H} \tilde{\mathscr{E}}_{\chi^h}$ decomposes into rank-2 χ^h-isotypical components of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ all of which carry filtration. This filtration 'descents' to $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ (see the proof of Proposition 4.2.1).

Remark 4.2.6. Let $\chi \in \mathrm{Irr}(A)$ with $\chi \neq \mathbb{1}$. Write $\tilde{\mathscr{E}}_\chi$, $H^0(\mathscr{Y}, \Omega^1_{\mathscr{Y}/S})_\chi$ resp. $H^0(\mathscr{Y}_b, \Omega^1_{\mathscr{Y}_b})_\chi$ for the χ-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$, $H^0(\mathscr{Y}, \Omega^1_{\mathscr{Y}/S})$ resp. $H^0(\mathscr{Y}_b, \Omega^1_{\mathscr{Y}_b})$. If

$$\dim H^0(\mathscr{Y}_b, \Omega^1_{\mathscr{Y}_b})_\chi = 1$$

for all $b \in S$ then $\mathrm{Fil}^1 \tilde{\mathscr{E}}_\chi := H^0(\mathscr{Y}, \Omega^1_{\mathscr{Y}/S})_\chi = \tilde{\mathscr{E}}_\chi \cap H^0(\mathscr{Y}, \Omega^1_{\mathscr{Y}/S})$ is a filtration in the sense of Definition 3.1.9, with $\mathrm{Fil}^1 \tilde{\mathscr{E}}_\chi \otimes (\mathscr{O}_S/\mathfrak{m}_b) = H^0(\mathscr{Y}_b, \Omega^1_{\mathscr{Y}_b})_\chi$.

Lemma 4.2.7. *Let $\chi \in \mathrm{Irr}(A)$ be non-trivial. Then we have*

$$\dim H^0(\mathscr{Y}_b, \Omega^1)_\chi = 1.$$

Proof. This follows from Lemma [**Bou01**, Lemma 4.3] applied to the étale cyclic cover $\mathscr{Y}_b \xrightarrow{A} \mathscr{X}_b$ where $g(\mathscr{X}_b) = 2$, by Lemma 4.2.4. $\square$

We are now ready prove Proposition 4.2.1.

PROOF OF PROPOSITION 4.2.1: Decomposition (4.2.1) is given by decomposition (4.2.3) where $\mathscr{E}_\chi = (\mathscr{E}_{\chi,\mathbb{1}})^H \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)^H$ is considered as a submodule of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$. By Lemma 4.2.4, $\operatorname{rank}\mathscr{E}_\chi = 2$ if and only if (4.2.2) holds. In this case, Lemma 4.2.7 implies that the χ-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$, which we denote by $\tilde{\mathscr{E}}_\chi$, carries the filtration $H^0(\mathscr{Y},\Omega)_\chi$ in the sense of Definition 3.1.9. Since $H_\chi = \{1\}$, we conclude that $\dim \big(H^0(\mathscr{Y},\Omega^1)_{\chi,\mathbb{1}}\big)^H = \dim H^0(\mathscr{Y},\Omega^1)_\chi = 1$, i.e. $\mathscr{E}_\chi = (\mathscr{E}_{\chi,\mathbb{1}})^H$ also carries a filtration. More precisely: let ω be a section of the filtration $H^0(\mathscr{Y},\Omega^1)_\chi \subset \tilde{\mathscr{E}}_\chi$ and consider the section

$$\eta := \sum_{h\in H} h^*\omega$$

of $\mathscr{E}_{\chi,\mathbb{1}} = \bigoplus_{h\in H} \tilde{\mathscr{E}}_{\chi^h}$. By definition, η is fixed under the action of H and hence a section of $\mathscr{E}_\chi = (\mathscr{E}_{\chi,\mathbb{1}})^H \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$. In particular, it is a section of the filtration on $\mathscr{E}_\chi = (\mathscr{E}_{\chi,\mathbb{1}})^H$. Moreover, as in Remark 2.3.7 we conclude that $\tilde{\mathscr{E}}_\chi \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ and $\mathscr{E}_\chi = (\mathscr{E}_{\chi,\mathbb{1}})^H \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}/S)$ have the same Picard-Fuchs operator. (The sections ω and η are cyclic sections of $\tilde{\mathscr{E}}_\chi$ and $\mathscr{E}_\chi$, respectively.) Therefore, $(\tilde{\mathscr{E}}_\chi,\nabla)$ and $(\mathscr{E}_\chi,\nabla)$ are isomorphic as flat vector bundles. This proves the proposition. □

4.3. Ellenberg's families do not define Teichmüller curves

Let $\mathscr{Y} \xrightarrow{G} \mathbb{P}^1_S$ be the universal family of metacyclic covers (introduced in Proposition 4.1.2) of type

$$(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \in \big\{(\tfrac14,\tfrac34,\tfrac12,\tfrac12),(\tfrac13,\tfrac23,\tfrac12,\tfrac12)\big\}$$

over the Hurwitz space $S = \mathscr{H}_{G,\mathbf{C}}$ with

$$G = G_{p,m} = \big\langle \varphi,\psi;\quad \varphi^p = \psi^m = 1,\ \psi\varphi\psi^{-1} = \varphi^\alpha \big\rangle,\quad A = \langle\varphi\rangle,\quad H = \langle\psi\rangle,$$

where m is the least common multiple of the denominators of σ_1, σ_2, σ_3, σ_4, and

$$\mathbf{C} = (\mathrm{Cl}_G(\psi^{a_1}),\mathrm{Cl}_G(\psi^{a_2}),\mathrm{Cl}({}_G\psi^{a_3}),\mathrm{Cl}_G(\psi^{a_4}))$$

with $a_i = m\sigma_i$ for $i = 1,\dots,4$. The ordered branch locus of $\mathscr{Y} \to \mathbb{P}^1_S$ is given by

$$D = (x_1 = 0, x_2 = 1, x_3, x_4 = \infty) \quad \text{with } x_i\colon S \to \mathbb{P}^1_S \text{ pairwise disjoint.}$$

Write $\overline{S} := \overline{\mathscr{H}}_{G,\mathbf{C}}$ for the Hurwitz space of admissible G-covers and $\Psi\colon \overline{S} \to \mathbb{P}^1_{\mathbb{C}}$ for the extension of the branch locus map $S \to \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$ that maps a cover having ordered branch locus $(x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty)$ to its third branch point $\lambda \in \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$.

In this section we show (in Theorem 4.3.5) that there is no flat rank-2 subbundle of the Deligne extension of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}^{\mathrm{Ell}}/S)$ that satisfies the conditions from Theorem 3.1.13. This implies that the image of the moduli map $S \mapsto \mathscr{M}_g$, $b \mapsto [\mathscr{X}^{\mathrm{Ell}}_b]$, is no Teichmüller curve (see [**Möl06b**] or [**Möl11**, Theorem 2.2]).

In order to show that there is no such subbundle, we find a point $b \in \overline{S} - S$ where $\mathcal{X}_b^{\mathrm{Ell}}$ is a singular curve, but of *compact type*, i.e. the dual graph of its irreducible components is a tree. In this case, we can apply the following lemma.

LEMMA 4.3.1. *Let $\mathcal{V} \to U$ be a semistable curve over a (possibly affine) smooth connected $\mathbb{C}$-curve U. Denote by $\tilde{U} \subset U$ the set of points where the fibres are smooth curves and suppose that $\tilde{U}$ is a dense open subset of U.*
(i) Let $b \in U$ be a point such that $\mathcal{V}_u$ is of compact type. Then b is a regular point of the Deligne extension of any flat subbundle of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{V}/\tilde{U})$.
(ii) Suppose that $\mathcal{V} \to U$ has a fibre $\mathcal{V}_b$ that is a singular curve of compact type. Then there is no flat subbundle $\mathcal{E} \subset \mathcal{H}^1_{\mathrm{dR}}(\mathcal{V}/\tilde{U})$ of rank 2 whose Deligne extension satisfies the conditions from Theorem 3.1.13.

PROOF. This is shown in [**Möl11**, Proposition 2.4]. We recall the proof in our set-up.
(i) Since $\mathcal{V}_b$ is of compact type, the generalised Jacobian of $\mathcal{V}_b$ is an abelian variety. Since the cohomology of a curve is the same as that of its Jacobian, it follows in this case that

$$H^1_{\mathrm{dR}}(\mathcal{V}_b) = H^1_{\mathrm{dR}}(\overline{V}_1) \times \cdots \times H^1_{\mathrm{dR}}(\overline{V}_\delta)$$

where $\overline{V}_1, \ldots, \overline{V}_\delta$ are the irreducible components of $\mathcal{V}_b$. This implies that b is a regular point of the Deligne extension of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{V}/\tilde{U})$, and hence of any flat subbundle, as well.
(ii) Let $\mathcal{E}$ be the Deligne extension of an arbitrary rank-2 subbundle of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{V}/\tilde{U})$. Assume that $\mathcal{E}$ satisfies the condition from Theorem 3.1.13. On the one hand, the fact that $\mathcal{V}_b$ is a singular curve implies that b is a marked point of $\mathcal{E}$ and therefore a logarithmic singularity. On the other hand, as $\mathcal{V}_b$ is of compact type, the point b must be a regular point of $\mathcal{E}$. This is a contradiction.

□

The following lemma describes the boundary $\Psi^{-1}(0) \subset \overline{S} - S$ as admissible G-covers. The fibres $\Psi^{-1}(1)$ and $\Psi^{-1}(\infty)$ may be described similarly. For our arguments, we actually do not need to describe the boundary $\overline{S} - S = \overline{\mathcal{H}}_{G,\mathbf{C}} - \mathcal{H}_{G,\mathbf{C}}$ as admissible G-covers. We only need their quotients (Lemma 4.3.4). However the description of the admissible G-covers helps to understand how to choose $b \in \overline{S} - S$. Namely, it corresponds (in the sense of Proposition 1.4.1) to the Nielsen tuple (4.3.1) on page 66.

LEMMA 4.3.2. *The admissible G-covers parametrised by the $\mathbb{C}$-rational points in $\Psi^{-1}(0) \subset \overline{S} - S$ are covers $\overline{Y} \to \overline{P}$ of a genus-0 curve which consists of two projective lines $\overline{P}_1$ and $\overline{P}_2$ intersecting in precisely one ordinary double point $\xi \in \overline{P}_1 \cap \overline{P}_2$. The restriction of $\overline{Y} \to \overline{P}$ to the component $\overline{P}_i$ ($i = 1, 2$) is the (possibly* *disconnected) G-cover induced by a G_i-cover $\overline{Y}_i \xrightarrow{G_i} \overline{P}_i$ where $\overline{Y}_i$ is a suitable connected component of $\overline{Y}$ and one of the following cases occurs.*

(i) $G_1 = G_2 = G$.

(ii) $G_1 = G$, $G_2 = H$ *or vice versa.*

The ramification type of $\overline{Y}_i \to \overline{P}_i$ $(i = 1, 2)$ is given by

$$\mathbf{C}_1 = \big(\operatorname{Cl}_{G_1}(\psi^{-1}),\ \operatorname{Cl}_{G_1}(\psi^{a_2}),\ \operatorname{Cl}_{G_1}(\psi^{a_4})\big),$$
$$\mathbf{C}_2 = \big(\operatorname{Cl}_{G_2}(\psi^{a_1}),\ \operatorname{Cl}_{G_2}(\psi^{a_3}),\ \operatorname{Cl}_{G_2}(\psi)\big),$$

where $\operatorname{Cl}_{G_1}(\psi^{-1})$ resp. $\operatorname{Cl}_{G_2}(\psi)$ describes the ramification at the ordinary double point ξ.

Proof. Every Nielsen tuple in $\mathrm{Ni}(G, \mathbf{C})$ may be represented by a tuple of the form

$$(g_1, g_2, g_3, g_4) = (\varphi^{v_1}\psi^{a_1}, \psi^{a_2}, \varphi^{v_3}\psi^{a_3}, \varphi^{v_4}\psi^{a_4})$$

after a suitable uniform conjugation. Such a tuple corresponds by Proposition 1.4.1 to an admissible G-cover. The lemma follows as one easily checks that

$$G_1 = \langle g_2, g_4\rangle = \langle \psi^{a_2}, \varphi^{v_4}\psi^{a_4}\rangle,$$
$$G_2 = \langle g_1, g_2 g_3 g_2^{-1}\rangle = \langle \varphi^{v_1}\psi^{a_1}, \varphi^{v_3\alpha^{a_2}}\psi^{a_3}\rangle.$$

Here we have used that $\alpha^{a_3} = -1$ since $a_3 = \frac{m}{2}$. In any case, G_i contains an element of order m, i.e. $G_i \supset H$. The case that $G_1 = G_2 = H$ does not occur since this would imply that $(g_1, g_2, g_3, g_4) = (\psi^{a_1}, \psi^{a_2}, \psi^{a_3}, \psi^{a_4}) \in \mathrm{Ni}(G, \mathbf{C})$, which contradicts the condition that $G = \langle g_1, g_2, g_3, g_4\rangle$. The other cases stated in the proposition occur indeed (see Example 4.3.3). □

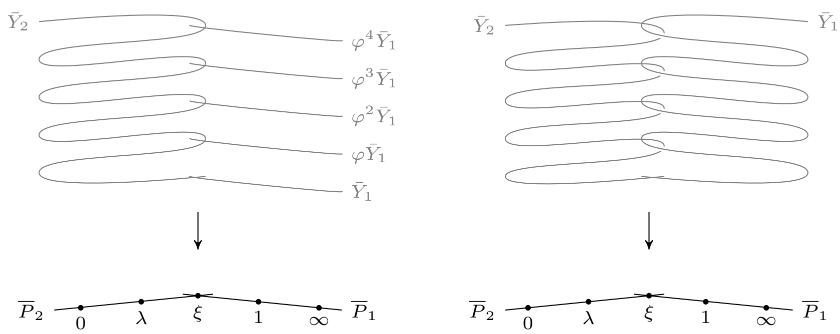


Figure 4.1. left figure: $G_1 = H$, $G_2 = G$, right figure: $G_1 = G_2 = G$.

Example 4.3.3. (i) Let $[g_1, g_2, g_3, g_4] = [\varphi\psi^{a_1}, \psi^{a_2}, \psi^{a_3}, \varphi\psi^{a_4}] \in \mathrm{Ni}(G, \mathbf{C})$. Then both $G_1 = \langle \psi^{a_2}, \varphi\psi^{a_4}\rangle$ and $G_2 = \langle \varphi\psi^{a_1}, \psi^{a_3}\rangle$ contain an element of order p, i.e.

$$G_1 = G_2 = G.$$

(ii) Let $[g_1, g_2, g_3, g_4] = [\varphi\psi^{a_1}, \psi^{a_2}, \varphi^{-1}\psi^{a_3}, \psi^{a_4}] \in \mathrm{Ni}(G, \mathbf{C})$. Then

$$G_1 = \langle \psi^{a_2}, \psi^{a_4} \rangle = H, \quad G_2 = \langle \varphi\psi^{a_1}, \varphi^{-\alpha^{a_2}}\psi^{a_3} \rangle = G.$$

(iii) Let $[g_1, g_2, g_3, g_4] = [\psi^{a_1}, \varphi^{-\alpha^{a_2}}\psi^{a_2}, \varphi^2\psi^{a_3}, \varphi\psi^{a_4}] \in \mathrm{Ni}(G, \mathbf{C})$. Then

$$G_1 = \langle \varphi^{-\alpha^{a_2}}\psi^{a_2}, \varphi\psi^{a_4} \rangle = G, \quad G_2 = \langle \psi^{a_2}, \psi^{a_3} \rangle = H.$$

For $m = 4$ and $p = 5$, Figure 4.1 shows the admissible G-covers for the Cases (ii) and (iii). For Case (ii), Figure 4.2 illustrates the process of degeneration with respect to the complex topology by contracting the inverse images of a suitable loop.

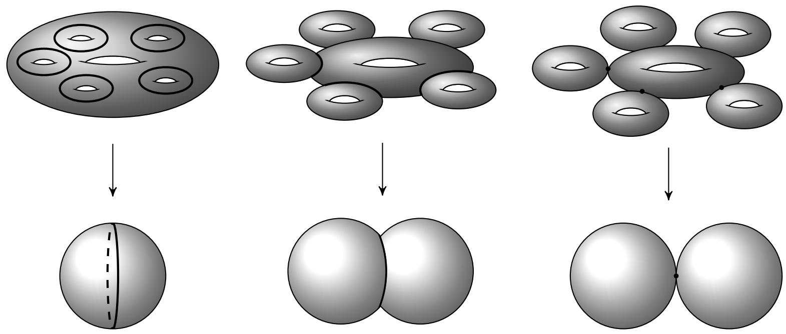


FIGURE 4.2. $G_1 = H$ and $G_2 = G$

In Case (ii) of Example 4.3.3, the following lemma describes the H-quotient

$$\overline{X} := \overline{Y}/H \to \overline{P}$$

of the admissible G-cover $\overline{Y} \to \overline{P}$ that corresponds in the sense of Proposition 1.4.1 to the Nielsen tuple

$$[g_1, g_2, g_3, g_4] = [\varphi\psi^{a_1}, \psi^{a_2}, \varphi^{-1}\psi^{a_3}, \psi^{a_4}] \in \mathrm{Ni}(G, \mathbf{C}). \tag{4.3.1}$$

LEMMA 4.3.4. *Denote by $\overline{P}_1$ the irreducible component of $\overline{P}$ to which the branch points $x_2 = 1$ and $x_4 = \infty$ specialise, and denote by $\overline{P}_2$ the irreducible component to which the branch points $x_1 = 0$ and $x_3 = \lambda$ specialise.*

(i) The restriction of $\overline{X}$ to $\overline{P}_1$ consists of $\frac{p-1}{m}$ irreducible components $\overline{X}_1^{\ell}$, $\ell = 1, \ldots, \frac{p-1}{m}$, of genus 1 and one irreducible component $\overline{X}_1^0$ of genus 0.

(ii) The restriction of $\overline{X}$ to $\overline{P}_2$ consists of one irreducible component of genus 0.

(iii) Let $\overline{X}'$ be the curve obtained by contracting all irreducible components of genus 0 which intersect the rest of $\overline{X}$ in one ordinary double point. Then $\overline{X}'$ is singular if and only if $p \neq m + 1$.

PROOF. Fix $\lambda \in \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ and let $Y \to \mathbb{P}^1_{\mathbb{C}}$ be the G-cover corresponding to the Nielsen tuple $[g_1, g_2, g_3, g_4] = [\varphi\psi^{a_1}, \psi^{a_2}, \varphi^{-1}\psi^{a_3}, \psi^{a_4}] \in \mathrm{Ni}(G, \mathbf{C})$. Let $X := Y/H \to \mathbb{P}^1_{\mathbb{C}}$ be the H-quotient. We write $(\varrho_1, \varrho_2, \varrho_3, \varrho_4) \in \mathrm{Sym}(G/H)^4$ for the local monodromies of $X \to \mathbb{P}^1_{\mathbb{C}}$ at $(x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty)$, which are described by the natural action of the local monodromies g_i^{-1} on the left cosets

$$G/H = \{\varphi^\ell H;\ \ell = 0, \dots, p-1\}$$

(see Proposition 1.5.1). More precisely, if we identify G/H with $\{1, \dots, p\}$ via the isomorphism given by $\varphi^\ell H \mapsto \ell$, then

$$\varrho_1(\ell) = (1-\ell)\alpha, \qquad \varrho_2(\ell) = -\ell\alpha^{-1},$$
$$\varrho_3(\ell) = -\ell - 1, \qquad \varrho_4(\ell) = -\ell$$

modulo p.

Using Proposition 1.5.6 with $G_1 = H$ and $G_2 = G$ we conclude that the restriction $\overline{X}_1 := \overline{X}|_{\overline{P}_1}$ consists of $|G_1\backslash G/H| = \frac{p-1}{m} + 1$ irreducible components, and the restriction $\overline{X}_2 := \overline{X}|_{\overline{P}_2}$ consists of $|G_2\backslash G/H| = 1$ irreducible component.

To calculate the local monodromies of the cover $\overline{X}_1^\ell \to \overline{P}_1 = \mathbb{P}^1_{\mathbb{C}}$ represented by a double coset $G_1\varphi^\ell H$ we proceed similar to Proposition 1.5.6, i.e we restrict ϱ_2, ϱ_4 and

$$\varrho_\xi(\ell) := (\varrho_4\varrho_2)^{-1}(\ell) = \ell\alpha \qquad (\mathrm{mod}\ p)$$

to the orbit $\{\ell\alpha^k;\ k = 0, \dots, m-1\} \subset \{0, \dots, p-1\}$ of cardinality m (if $\ell \neq 0$) resp. to the orbit $\{0\} \subset \{0, \dots, p-1\}$ of cardinality one (if $\ell = 0$). (Note that we compose the permutations ϱ_2 and ϱ_4 from right to left.)

The irreducible component $\overline{X}_1^0$ of $\overline{X}_1$ is a degree-1 cover of $\overline{P}_1 = \mathbb{P}^1_{\mathbb{C}}$ with trivial local monodromies, i.e. $g(\overline{X}_1^0) = 0$.

We now describe the irreducible components $\overline{X}_1^\ell$ of $\overline{X}_1$ represented by a double coset $G_1\varphi^\ell H$ with $\ell \neq 0$. The restrictions of ϱ_2 to $\{\ell\alpha^k;\ k = 0, \dots, m-1\}$ consists

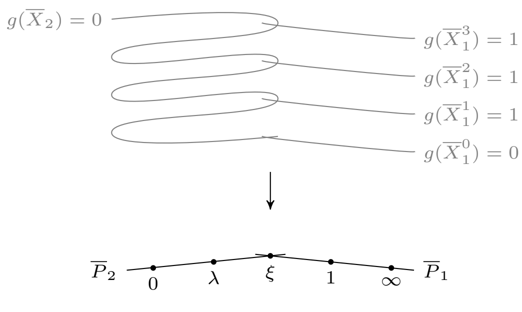


FIGURE 4.3. H-quotient for $m = 4$ and $p = 13$.

of $\gcd(m, a_1)$ orbits of length $\frac{m}{\gcd(m,a_1)}$, the restrictions of ϱ_4 consists of $\frac{m}{2}$ orbits of length 2, and the restrictions of ϱ_ξ consists of one orbit of length m. Using this description of the cycle types, the Riemann-Hurwitz genus formula implies that $g(\overline{X}_1^\ell) = 1$ for all $\ell \neq 0$.

A similar calculation shows that $g(\overline{X}_2) = 0$. Part (iii) is an immediate consequence of (i) and (ii). $\square$

After applying a suitable base change $T \to S$ we can assume that the pullback $\mathscr{Y}_T = \mathscr{Y} \times_S T$ extends to a semistable curve over $\overline{T}$ whose fibre over $\Psi^{-1}(0)$ is given as in Lemma 4.3.2. To simplify notation, we replace T by S and write $\mathscr{Y} \to \overline{S}$ for this semistable extension.

We then consider the semistable curve $\mathscr{X}^{\mathrm{Ell}} := \mathscr{Y}/H$ over $\overline{S}$. Unless $p = m+1$, in Lemma 4.3.4 we have shown that there is a singular fibre $\mathscr{X}_b^{\mathrm{Ell}}$ of compact type (i.e. the dual graph of its irreducible components is a tree). We use this fact in order to show that $\mathscr{X}^{\mathrm{Ell}}$ does not define a Teichmüller curve in the sense of Theorem 3.1.13.

THEOREM 4.3.5. *Suppose that $p \neq m+1$. Then the Deligne extension of every flat subbundle $\mathscr{E} \subset \mathscr{H}^1_{\mathrm{dR}}(\mathscr{X}^{\mathrm{Ell}}/S)$ of rank 2 has a regular point $b \in \overline{S}$ such that the fibre $\mathscr{X}_b^{\mathrm{Ell}}$ is singular. In particular, $\mathscr{X}^{\mathrm{Ell}} \to S$ does not define a Teichmüller curve in the sense of Theorem 3.1.13.*

PROOF. Choose $b \in \overline{S} - S$ as in Example 4.3.3 (ii). Lemma 4.3.4 implies that $\mathscr{X}_b^{\mathrm{Ell}}$ is a singular curve of compact type. The rest follows from Lemma 4.3.1 and [**Möl11**, Theorem 2.2]. □

REMARK 4.3.6. We now consider the case that $p = m+1$. Then $\mathscr{X}^{\mathrm{Ell}} \to \overline{S}$ is a curve of genus 1. Let $\tilde{S}$ be its smooth locus. Since $\mathscr{X}^{\mathrm{Ell}}$ has at least one singular fibre (e.g. the H-quotient of the cover on the right hand side in Figure 4.1) it follows that the image of $\tilde{S} \to \mathscr{M}_1 \simeq \mathbb{A}^1_j$ sending $b \in \tilde{S}$ to the j-invariant of $\mathscr{X}_b^{\mathrm{Ell}}$ is not a point. Hence the map is dominant and we conclude that the image is a Teichmüller curve.

4.4. The Hurwitz monodromy of Ellenberg's families

In this section we describe the Hurwitz monodromy of the branch locus map $\Psi\colon \mathscr{H}_{G,\mathbf{C}} \to \mathbb{P}^*$ (introduced in Section 1.3) for the case that $\mathscr{H}_{G,\mathbf{C}}$ is the Hurwitz space of metacyclic covers of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. We use the formulas from Section 1.3, Proposition 1.3.4. Recall that the group G and the tuple $\mathbf{C}$ are given by

$$\begin{aligned} G &= \langle \varphi, \psi; \quad \varphi^p = \psi^m = 1,\ \psi\varphi\psi^{-1} = \varphi^\alpha \rangle, \quad A = \langle \varphi \rangle, \quad H = \langle \psi \rangle, \\ \mathbf{C} &= \left(\mathrm{Cl}_G(\psi^{a_1}), \mathrm{Cl}_G(\psi^{a_2}), \mathrm{Cl}({}_G\psi^{a_3}), \mathrm{Cl}_G(\psi^{a_4})\right), \end{aligned}$$

where m is the least common multiple of the denominators of σ_1, σ_2, σ_3 and σ_4 as well as $p \equiv 1 \pmod m$ is a prime number, $\alpha \in \{1, \dots, p-1\}$ represents an element in $\mathbb{F}_p^\times$ of order m and $a_i = m\sigma_i$ for $i = 1, \dots, 4$. Our approach in Section 4.4.1 is analogous to that of [**Bou04**, Section 3].

In Section 4.4.2 we consider the Hurwitz space of (non-Galois) degree-p covers of $\mathbb{P}^1_{\mathbb{C}}$ with ordered branch locus of cardinality 4 and monodromy group G. This Hurwitz space parametrises the H-quotients of the metacyclic covers of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. Since the moduli map $S \to \mathscr{M}_g$ associated to an Ellenberg family

$\mathscr{X}^{\mathrm{Ell}} \to S$ factors through such a Hurwitz space, the description of its Hurwitz monodromy describes the image of the moduli map more precisely.

4.4.1. Hurwitz monodromy of metacyclic covers. Let $\Gamma \subset \mathrm{GL}_2(\mathbb{F}_p)$ be the subgroup generated by the matrices

$$B_2 = \begin{pmatrix} \alpha^{a_2+a_3} & 0 \\ \alpha^{a_2+a_3}(\alpha^{a_2}-1) & 1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 & \alpha^{-a_2}(\alpha^{a_4}-1) \\ 0 & \alpha^{a_3+a_4} \end{pmatrix}.$$

Put $B_1 := (B_2B_3)^{-1}$. As in [**Völ93**, Lemma 3] resp. [**Bou04**, Section 3], we show that the Hurwitz monodromy right action

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}) = \langle b_1, b_2, b_3;\ b_1b_2b_3 = 1\rangle \to \mathrm{Sym}\,(\mathrm{Ni}(G,\mathbf{C}))$$

from Proposition 1.3.4 is given by the action $\Gamma \to \mathrm{GL}_2\,(\mathbb{F}_p)$ where B_i acts on the row vectors in $\mathbb{F}_p^2$ by right multiplication.

Proposition 4.4.1. *(i) There is a bijection of sets between* $\mathrm{Ni}(G,\mathbf{C})$ *and the set*

$$W := \mathbb{F}_p^2 - \{(0,0)\}\,/\langle \alpha I\rangle$$

of equivalence classes of row vectors, where $I \in \mathrm{GL}_2(\mathbb{F}_p)$ *denotes the identity matrix, and the subgroup* $\langle \alpha I\rangle \subset \mathrm{GL}_2(\mathbb{F}_p)$ *acts by right multiplication.*
(ii) If $\mathbf{g} \in \mathrm{Ni}(G,\mathbf{C})$ *corresponds to* $(v_1,v_2) \in W$*, then* $\mathbf{g}\cdot b_i \in \mathrm{Ni}(G,\mathbf{C})$ *corresponds to* $(v_1,v_2)\cdot B_i \in W$.

Proof. (i) After a suitable uniform conjugation, every Nielsen tuple in $\mathrm{Ni}(G,\mathbf{C})$ may be represented by a tuple of the form

$$\mathbf{g} = (g_1,\ldots,g_4) = (\psi^{a_1}, \varphi^{v_1}\psi^{a_2}, \varphi^{v_3}\psi^{a_3}, \varphi^{v_2}\psi^{a_4})\,, \tag{4.4.1}$$

where v_3 is uniquely determined by v_1 and v_2 via the relation $g_1\cdots g_4 = 1$, i.e.

$$v_3 \equiv -v_1\alpha^{-a_2} - v_2\alpha^{a_3} \pmod p.$$

Nielsen tuples of the form (4.4.1) are equivalent if and only if there exits an integer k such that their components are conjugated by ψ^k, respectively. Such a uniform conjugation of a Nielsen tuple means replacing the exponents v_i by $\alpha^k v_i$. Therefore, $\mathbf{g} \mapsto (v_1,v_2)$ defines the wanted bijection. Note that $(0,0) \notin W$, as $\langle \psi^{a_1}, \psi^{a_2}, \psi^{a_3}, \psi^{a_4}\rangle \neq G$.
(ii) Let $\mathbf{g}$ be a Nielsen tuple corresponding to $(v_1,v_2) \in W$. Using (4.1.2), page 58, and the formulas from Proposition 1.3.4 one easily calculates that $\mathbf{g}\cdot b_i$ corresponds to $(v_1,v_2)\cdot B_i$. □

Example 4.4.2. (i) Suppose that

$$(\sigma_1,\sigma_2,\sigma_3,\sigma_4) = \left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{1}{2}\right),$$

i.e. $m = 4$. Then

$$B_2 = \begin{pmatrix} \alpha & 0 \\ 1-\alpha & 1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 & -2\alpha \\ 0 & 1 \end{pmatrix}, \qquad B_1 = (B_2 B_3)^{-1} = \begin{pmatrix} \alpha - 2 & 2\alpha \\ 1+\alpha & 1 \end{pmatrix}.$$

Note that B_1 and B_2 represent matrices in $\mathrm{GL}_2(\mathbb{F}_p)/\langle \alpha I\rangle$ of order 4, respectively, and B_3 represents a matrix of order p. For $i = 1, 2, 3$, let ϱ_i be the image of b_i under the Hurwitz monodromy $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}) \to \mathrm{Sym}(\mathrm{Ni}(G, \mathbf{C}))$. In the following we determine the cycle type of ϱ_3. For ϱ_1 and ϱ_2 similar calculations apply. By Proposition 4.4.1, the cycles of ϱ_3 are described by the B_3-orbits in $W = \mathbb{F}_p^2 - \{(0,0)\}/\langle \alpha I\rangle$. Since

$$B_3^\ell = \begin{pmatrix} 1 & -2\alpha\ell \\ 0 & 1 \end{pmatrix},$$

the vectors (v_1, v_2) and $(v_1, v_2) \cdot B_3^\ell$ are equivalent modulo $\langle \alpha I\rangle$ if and only if there exists an integer $k \in \{0, \ldots, 3\}$ such that

$$(\alpha^k - 1)\, v_1 \equiv 0, \qquad (\alpha^k - 1)\, v_2 \equiv -2\alpha\ell v_1 \qquad \pmod p.$$

If $v_1 \equiv 0 \pmod p$, i.e $v_2 \not\equiv 0 \pmod p$, then a solution is given by $k = 0$ and $\ell = 1$. If $v_1 \not\equiv 0 \pmod p$, we must choose $k \equiv 0 \pmod m$. Then $\ell = p$. Note that

$$w := |W| = \tfrac{p^2 - 1}{4},$$

and that the $\langle \alpha I\rangle$-orbit of $(v_1, v_2) \in \mathbb{F}_p^2 - \{(0,0)\}$ has length 4. Hence, ϱ_3 consists of $(p-1)/4$ cycles of length one and $(p-1)/4$ cycles of length p. Using Notation 1.5.4, we have

$$\varrho_3 \in \mathrm{Cl}_{S_w}\left(1^{(p-1)/4} \cdot p^{(p-1)/4}\right).$$

Similarly, one checks that

$$\varrho_1, \varrho_2 \in \mathrm{Cl}_{S_w}\left(1^{(p-1)/2} \cdot 4^{(p-1)^2/16}\right).$$

The Riemann-Hurwitz genus formula implies that

$$g(\overline{\mathscr{H}}_{G,\mathbf{C}}) = 1 + \frac{7(p-1)^2}{8}.$$

(ii) Suppose that

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left(\tfrac{2}{6}, \tfrac{4}{6}, \tfrac{1}{2}, \tfrac{1}{2}\right),$$

i.e. $m = 6$. Then

$$B_1 = \begin{pmatrix} -2-\alpha & -1-\alpha \\ 2 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \alpha^5 & 0 \\ 2\alpha & 1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 & 1+\alpha \\ 0 & 1 \end{pmatrix}$$

and one can show that

$$\varrho_1, \varrho_2 \in \mathrm{Cl}_{S_w}\left(1^{(p-1)/3} \cdot 6^{(p-1)^2/36}\right), \quad \varrho_3 \in \mathrm{Cl}_{S_w}\left(1^{(p-1)/6} \cdot p^{(p-1)/6}\right),$$

where $w = \frac{p^2-1}{6}$. The Riemann-Hurwitz genus formula implies that

$$g(\overline{\mathscr{H}}_{G,\mathbf{C}}) = 1 + \frac{11(p-1)^2}{12}.$$

Note that in both examples the Hurwitz space $\mathscr{H}_{G,\mathbf{C}}$ is also connected since p divides the order of Γ (see [**Bou04**, Proposition 3.5]).

4.4.2. Hurwitz monodromy of Ellenberg's families. We return to the general situation with arbitrary type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. Every metacyclic G-cover $Y \to \mathbb{P}^1_{\mathbb{C}}$ of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ with ordered branch locus $(x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty)$ factors through

$$Y \to X := Y/H \xrightarrow{\epsilon} \mathbb{P}^1_{\mathbb{C}}.$$

Lemma 4.4.3. *The quotient cover $\epsilon\colon X = Y/H \to \mathbb{P}^1_{\mathbb{C}}$ is a degree-p cover branched at (x_1, x_2, x_3, x_4) having monodromy group $\mathrm{Mon}(\epsilon) \simeq G$. The local monodromy of ϵ at x_i is an element of*

$$C_i^* = \mathrm{Cl}_{S_p}\left(1^1 \delta_i^{(p-1)/\delta_i}\right), \quad \delta_i = \frac{m}{\gcd(m, a_i)}.$$

Proof. The degree of ϵ is $[G : H] = p$. Since the normal closure of $H \subset G$ is G, we have an isomorphism

$$\mathrm{Mon}(\epsilon) \simeq G. \tag{4.4.2}$$

By Proposition 1.5.1, the local monodromy ϱ_i of ϵ at x_i is given by the natural left action of g_i^{-1} on G/H. It suffices to consider the case that $g_i = \psi^a$ with $a \in \{1, \ldots, m-1\}$. One checks that

$$g_i^{-1}\varphi^\ell H = \psi^{-a}\varphi^\ell H = \varphi^{\ell\alpha^{-a}} H$$

using (4.1.2). Therefore, $\varrho_i(\ell) = \ell\alpha^{-a}$ is a permutation of order $\delta = \frac{m}{\gcd(m,a)}$ with precisely one fixed point. $\square$

Remark 4.4.4. The Riemann-Hurwitz genus formula implies that

$$g(Y) = p \cdot \kappa + 1, \qquad g(Y/H) = \frac{p-1}{m} \cdot \kappa,$$

where $\kappa = m - \frac{1}{2} \cdot \sum_{i=1}^4 \gcd(m, r_i)$.

We may view the group G as a transitive subgroup of $\mathrm{Sym}(G/H) = S_p$, where $g \in G$ 'is' the permutation that sends ℓH to $g^{-1}\ell H$. Moreover we identify G with $\mathrm{Mon}(\epsilon)$ by the isomorphism (4.4.2). With these identifications made, the cover $\epsilon\colon X = Y/H \to \mathbb{P}^1_{\mathbb{C}}$ from Lemma 4.4.3 has monodromy group G and its local monodromy ϱ_i at x_i is an element of $\mathrm{Cl}_G(\psi^{a_i})$. Therefore it is parametrised by the Hurwitz space $\mathscr{H}_{d,G,\mathbf{C}}$ of weak isomorphism classes of degree-p covers of $\mathbb{P}^1_{\mathbb{C}}$ all of which have

(i) ordered branch locus of cardinality 4,

(ii) monodromy group G,

(iii) and the local monodromy ϱ_i at x_i is an element of $\mathrm{Cl}_G(\psi^{a_i})$.

As a consequence of Lemma 4.4.3 we have a map

$$\mathscr{H}_{G,\mathbf{C}} \to \mathscr{H}_{p,G,\mathbf{C}}$$

that sends a G-cover $Y \to \mathbb{P}^1_{\mathbb{C}}$ in $\mathscr{H}_{G,\mathbf{C}}$ to the branched cover $\epsilon\colon Y/H \to \mathbb{P}^1_{\mathbb{C}}$ in $\mathscr{H}_{p,G,\mathbf{C}}$ (see [**FV91**, Section 1.2]).

Our goal is to calculate the Hurwitz monodromy of the branch locus map

$$\mathscr{H}_{p,G,\mathbf{C}} \to \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\},$$

that sends a cover with ordered branch locus $(0,1,\lambda,\infty)$ to the third branch point λ. We use the formulas given in Proposition 1.3.4, but with a stronger equivalence relation for Nielsen tuples (see Remark 1.3.7).

Remark 4.4.5. There exist $\tilde{\psi} \in S_p$ and $\tilde{\alpha} \in \mathbb{F}_p^\times$ of order $p-1$ such that $\tilde{\psi}^{(p-1)/m} = \psi$ and $\tilde{\alpha}^{(p-1)/m} = \alpha$. Furthermore, the group $N := \langle \varphi, \tilde{\psi} \rangle \subset S_p$ is the normaliser of G in S_p.

As in Proposition 4.4.1 we conclude that the Hurwitz monodromy right action

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}) = \langle b_1, b_2, b_3;\ b_1 b_2 b_3 = 1 \rangle \to \mathrm{Sym}\left(\mathrm{Ni}^{\mathrm{abs}}(G,\mathbf{C})\right)$$

may still be described by the matrices $B_1, B_2, B_3 \in \mathrm{GL}_2(\mathbb{F}_p^2)$. But in comparison to Proposition 4.4.1 the equivalence relation on $\mathbb{F}_p^2$ is stronger, namely modulo $\langle \tilde{\alpha} I \rangle \supset \langle \alpha I \rangle$.

Proposition 4.4.6. *(i) There is a bijection of sets between* $\mathrm{Ni}^{\mathrm{abs}}(G,\mathbf{C})$ *and the set*

$$\tilde{W} := \mathbb{F}_p^2 - \{(0,0)\} / \langle \tilde{\alpha} I \rangle$$

of equivalence classes of row vectors, where $I \in \mathrm{GL}_2(\mathbb{F}_p)$ *denotes the identity matrix and the subgroup* $\langle \tilde{\alpha} I \rangle \subset \mathrm{GL}_2(\mathbb{F}_p)$ *acts by right multiplication.*
(ii) If $\mathbf{g} \in \mathrm{Ni}^{\mathrm{abs}}(G,\mathbf{C})$ *corresponds to* $(v_1,v_2) \in \tilde{W}$*, then* $\mathbf{g} \cdot b_i \in \mathrm{Ni}^{\mathrm{abs}}(G,\mathbf{C})$ *corresponds to* $(v_1,v_2) \cdot B_i \in \tilde{W}$*.*

Proof. The proof is essentially the same as the one for Proposition 4.4.1. The only difference is that the uniform conjugation of $\mathbf{g} = (\psi^{a_1}, \varphi^{v_1}\psi^{a_2}, \varphi^{v_3}\psi^{a_3}, \varphi^{v_2}\psi^{a_4})$ by $\tilde{\psi}$ results in replacing the exponents v_i by $\tilde{\alpha} v_i$. $\square$

Example 4.4.7. (i) Suppose that

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{1}{2}\right),$$

i.e. $m = 4$. Choose $B_1, B_2, B_3 \in \mathrm{GL}_2(\mathbb{F}_p)$ as in Example 4.4.2. For $i = 1,2,3$, let ϱ_i be the image of b_i under $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}) \to \mathrm{Sym}(\mathrm{Ni}^{\mathrm{abs}}(G,\mathbf{C}))$. By Proposition 4.4.1, the cycles of ϱ_3 are described by the B_3-orbits in $\tilde{W} = \mathbb{F}_p^2 - \{(0,0)\}/\langle \tilde{\alpha} I \rangle$, where $\tilde{\alpha} \in \mathbb{F}_p^\times$ is given as in Remark 4.4.5. The vectors (v_1, v_2) and $(v_1, v_2) \cdot B_3^\ell$ are equivalent modulo $\langle \tilde{\alpha} I \rangle$ if and only if there exists an integer $k \in \{0, \ldots, p-1\}$ such that

$$(\tilde{\alpha}^k - 1)\, v_1 \equiv 0, \quad (\tilde{\alpha}^k - 1) v_2 \equiv -2\alpha\ell v_1 \pmod p.$$

If $v_1 \equiv 0 \pmod p$, i.e. $v_2 \not\equiv 0 \pmod p$, a solution is given by $k = 0$ and $\ell = 1$. All tuples of the form $(0, v_2)$ are equivalent in $\tilde{W}$. Hence, ϱ_3 has one cycle of length one. If $v_1 \not\equiv 0 \pmod p$, we must choose $k \equiv 0 \pmod{p-1}$. In this case, $\ell \equiv 0 \pmod p$. Note that

$$\tilde{w} := |\tilde{W}| = \frac{p^2 - 1}{p - 1} = p + 1.$$

We conclude that

$$\varrho_3 \in \mathrm{Cl}_{S_{\tilde{w}}}\left(1^1 \cdot p^1\right).$$

Similar calculations imply

$$\varrho_1, \varrho_2 \in \mathrm{Cl}_{S_{\tilde{w}}}\left(1^2 \cdot 4^{(p-1)/4}\right).$$

The Riemann-Hurwitz formula implies that

$$g(\overline{\mathcal{H}}_{p,G,\mathbf{C}}) = \frac{p-1}{4} - 1$$

(ii) Suppose that

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left(\tfrac{2}{6}, \tfrac{4}{6}, \tfrac{1}{2}, \tfrac{1}{2}\right),$$

i.e. $m = 6$. With $B_1, B_2, B_3 \in \mathrm{GL}_2(\mathbb{F}_p)$ as in Example 4.4.2, one can show that

$$\varrho_1, \varrho_2 \in \mathrm{Cl}_{S_{\tilde{w}}}\left(1^2 \cdot 6^{(p-1)/6}\right), \quad \varrho_3 \in \mathrm{Cl}_{S_{\tilde{w}}}\left(1^1 \cdot p^1\right),$$

where $\tilde{w} = p + 1$. The Riemann-Hurwitz genus formula implies that

$$g(\overline{\mathcal{H}}_{p,G,\mathbf{C}}) = \frac{p-1}{3} - 1.$$

4.5. The Bouw-Möller construction for Ellenberg's families

We have shown that Ellenberg's families introduced in Section 4.1 do not define Teichmüller curves in the sense of Theorem 3.1.13 (for $p \neq m+1$). This is somewhat similar to the situation in the following remark.

Remark 4.5.1. Consider the superelliptic curve $\mathcal{Y}$ over $S = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ of type $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ as in Proposition 3.3.2 (iii) for $m \neq n$. Let $\mathcal{E}$ be the Deligne extension of the rank-2 subbundle of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{Y}/S)$ from Remark 3.2.1. After a suitable base change $T \to S$ we can assume that the elliptic singularities of $\mathcal{E}$ become regular points (see Lemma 3.1.8) and $\mathrm{Aut}_T(\mathcal{Y}_T)$ contains a dihedral group $\mathbb{Z}/N\mathbb{Z} \rtimes \langle \sigma \rangle$ with reflection σ (as in Proposition 3.3.7 (i)). For $m \neq n$ neither the superelliptic curve $\mathcal{Y}_T$ nor $\mathcal{X} := \mathcal{Y}_T / \langle \sigma \rangle$ extend to a smooth curve over all regular points of $\mathcal{E}$ (see Proposition 3.3.7 (iii)). However there is an automorphism $\tau \in \mathrm{Aut}_T(\mathbb{P}^1_T)$ that is invariant on the set of branch points of the cyclic cover $\mathcal{Y}_T \xrightarrow{\mathbb{Z}/N\mathbb{Z}} \mathbb{P}^1_T$, but does not lift to $\mathscr{Y}_T$. In [**BM10b**] the authors pass to the Galois closure $\widetilde{\mathscr{Y}}$ of $\mathscr{Y}_T \xrightarrow{\mathbb{Z}/N\mathbb{Z}} \mathbb{P}^1_T \xrightarrow{\langle\sigma,\tau\rangle} \mathbb{P}^1_T$ and show that $\widetilde{\mathscr{X}} := \widetilde{\mathscr{Y}}/\langle\sigma,\tau\rangle$ defines a Teichmüller curve.

In the situation of the present chapter, where we consider metacyclic covers $Y \xrightarrow{A} Z \xrightarrow{H} \mathbb{P}^1_{\mathbb{C}}$, the group $\mathrm{Aut}(Z)$ contains a dihedral group $H \rtimes \langle\sigma\rangle$. However, σ does not lift to an automorphism of Y. We now follow an approach which is motivated by that of [**BM10b**] discussed in Remark 4.5.1.

Let

$$\begin{tikzcd} Y \arrow[rr, "G_{p,m}"] \arrow[dr, "A"'] & & \mathbb{P}^1_{\mathbb{C}} \\ & Z \arrow[ur, "H"'] & \end{tikzcd}$$

be a metacyclic cover with Galois group

$$G_{p,m} := \langle \varphi, \psi;\, \varphi^p = \varphi^m,\, \psi\varphi\psi^{-1} = \varphi^\alpha\rangle, \quad A = \langle\varphi\rangle, \quad H = \langle\psi\rangle$$

of type

$$(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \in \left\{\left(\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{2},\tfrac{1}{2}\right),\left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2},\tfrac{1}{2}\right)\right\}$$

(see Definition 4.1.1). Recall that m is the least common multiple of the denominators of σ_1, σ_2, σ_3, σ_4, and that p is a prime with $p \equiv 1 \pmod m$.

Suppose that $(x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty)$ is the branch locus of $Y \to \mathbb{P}^1_{\mathbb{C}}$. Let $\sigma \in \mathrm{Aut}(Z)$ be a lift of the automorphism of $\mathbb{P}^1_{\mathbb{C}}$ of order 2 that is given by $x_1 \mapsto x_2$ and $x_3 \mapsto x_4$, i.e. σ is a reflection in the dihedral group $D_m = \langle\psi,\sigma\rangle = H \rtimes \langle\sigma\rangle$.

Let $\tilde{Y}$ be the Galois closure of $Y \xrightarrow{A} Z \xrightarrow{\langle\psi,\sigma\rangle} \mathbb{P}^1_{\mathbb{C}}$. If we put $\varphi_1 := \varphi$ and $\varphi_2 := \sigma\varphi_1\sigma$, then $\tilde{Y}$ is a $\tilde{G}$-cover of $\mathbb{P}^1_{\mathbb{C}}$, where $\tilde{G}$ is generated by φ_1, φ_2, ψ, σ, satisfying

$$\varphi_1^p = \varphi_2^p = \psi^m = \sigma^2 = 1, \quad \sigma\psi\sigma = \psi^{-1}, \quad \varphi_1\varphi_2 = \varphi_2\varphi_1,$$
$$\sigma\varphi_1\sigma = \varphi_2, \quad \psi\varphi_1\psi^{-1} = \varphi_1^\alpha, \quad \psi\varphi_2\psi^{-1} = \varphi_2^{\alpha^{-1}}.$$

We define

$$\tilde{A} := \langle\varphi_1,\varphi_2\rangle, \qquad \tilde{H} := \langle\psi,\sigma\rangle.$$

This construction is illustrated by the following commutative diagram, where $Y^\sigma \to \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ denotes the metacyclic cover with with Galois group $\langle\varphi_2\rangle \rtimes \langle\psi\rangle$ of type $(\sigma_2,\sigma_1,\sigma_4,\sigma_3)$. This construction is completely analogous to the construction in

[**BM10b**] on page 159.

$$\begin{tikzcd}
& \tilde{Y} \arrow[dl, "\langle\varphi_2\rangle"'] \arrow[dr, "\langle\varphi_1\rangle"] & \\
Y \arrow[dr, "\langle\varphi_1\rangle"'] & & Y^\sigma \arrow[dl, "\langle\varphi_2\rangle"] \\
& Z \arrow[d, "\langle\psi\rangle"] & \\
& \mathbb{P}^1_{\mathbb{C}} \arrow[d, "\langle\sigma\rangle"] & \\
& \mathbb{P}^1_{\mathbb{C}} &
\end{tikzcd}$$

Recall that Nielsen tuples $\tilde{\mathbf{g}} = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4)$ associated with such $\tilde{G}$-covers must satisfy $\tilde{g}_1\tilde{g}_2\tilde{g}_3\tilde{g}_4 = 1$. This implies that the ramification type of $\tilde{Y} \xrightarrow{\tilde{G}} \mathbb{P}^1_{\mathbb{C}}$ is

$$\tilde{\mathbf{C}} = \left(\mathrm{Cl}_{\tilde{G}}(\sigma),\ \mathrm{Cl}_{\tilde{G}}(\psi^{-1}\sigma),\ \mathrm{Cl}_{\tilde{G}}(\psi^{a_1}),\ \mathrm{Cl}_{\tilde{G}}(\psi^{m/2})\right),$$

where

$$a_1 = \sigma_1 \cdot m = \begin{cases} 1, & \text{if } m = 4, \\ 2, & \text{if } m = 6. \end{cases}$$

Lemma 4.5.2. *A Nielsen tuple in* $\mathrm{Ni}(\tilde{G}, \tilde{\mathbf{C}})$ *may always be represented by a tuple of the form*

$$\tilde{\mathbf{g}} = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4) = \left(\sigma,\ \varphi_1^i\varphi_2^{-i\alpha}\psi^{-1}\sigma,\ \varphi_1^k\psi^{a_1},\ \varphi_1^{(k-i)\alpha}\varphi_2^i\psi^{m/2}\right), \tag{4.5.1}$$

with $(i,k) \not\equiv (0,0) \pmod p$. *The tuple is given up to uniform conjugation with* $\psi^{m/2}$. *In particular, we have a bijection*

$$\mathrm{Ni}(\tilde{G}, \tilde{\mathbf{C}}) \simeq \left(\mathbb{F}_p^2 - \{(0,0)\}\right)/\langle -I\rangle,$$

where we divide out the action of $-I \in \mathrm{GL}_2(\mathbb{F}_p)$. *(Here* I *is the identity matrix.)*

Proof. Let $\tilde{\mathbf{g}} = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4) \in \mathrm{Ni}(\tilde{G}, \tilde{\mathbf{C}})$. After a suitable uniform conjugation we may represent the Nielsen tuple by

$$\tilde{\mathbf{g}} = (\sigma, v_2\psi^{-\ell}\sigma, v_3\psi^{\ell a_1}, v_4\psi^{m/2}),$$

where $v_2, v_3, v_4 \in \tilde{A} = \langle\varphi_1, \varphi_2\rangle$ and $\ell \in \{-1, 1\}$. Note that the centraliser $Z_{\tilde{G}}(\sigma) = \langle\sigma, \psi^{m/2}, \varphi_1\varphi_2\rangle$ is exactly the set that fixes the first component with respect to uniform conjugation. Since $\sigma \in Z_{\tilde{G}}(\sigma)$, after a further uniform conjugation by σ if necessary, we may assume that the representative is of the form

$$\tilde{\mathbf{g}} = (\sigma,\ v_2\psi^{-1}\sigma,\ v_3\psi^{a_1},\ v_4\psi^{m/2}).$$

Suppose that $v_3 = \varphi_1^x \varphi_2^y$. After another uniform conjugation by $(\varphi_1\varphi_2)^{-y(1-\alpha^{a_1})^{-1}}$ in $Z_{\tilde{G}}(\sigma)$ we find a representative of the form

$$\tilde{\mathbf{g}} = (\sigma,\ v_2\psi^{-1}\sigma,\ \varphi_1^k\psi^{a_1},\ v_4\psi^{m/2}).$$

One checks that $v_2\psi^{-1}\sigma \in \mathrm{Cl}_{\tilde{G}}(\psi^{-1}\sigma)$ if and only if $v_2 = \varphi_1^i\varphi_2^{-i\alpha}\psi^{-1}\sigma$ for some $i \in \{0, 1, \ldots, p-1\}$. The relation $\tilde{g}_1 \cdots \tilde{g}_4 = 1$ implies that $v_4 = \varphi_1^{(k-i)\alpha}\varphi_2^i\psi^{m/2}$. This implies that we may choose the representative as in (4.5.1), which is given up to the remaining conjugation with $\psi^{m/2} \in Z_{\tilde{G}}(\sigma)$. Note that $\tilde{G} = \langle \tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4\rangle$ if and only if $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4)$ is represented by a tuple of the form (4.5.1) with $(i, k) \not\equiv (0, 0) \pmod p$. This proves the lemma. $\square$

The following proposition shows the existence of a non-trivial family of $\tilde{G}$-covers with ramification type $\tilde{\mathbf{C}}$.

Proposition 4.5.3. *There exists a one-dimensional universal family*

$$\widetilde{\mathscr{Y}} \to \mathbb{P}^1_{\mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}} \to \mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}$$

of $\tilde{G}$-covers with ramification type $\tilde{\mathbf{C}}$, i.e. a scheme $\widetilde{\mathscr{Y}}$ over $\mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}$ such that the fibre $\widetilde{\mathscr{Y}}_b \to \mathbb{P}^1_{\mathbb{C}}$ at any $\mathbb{C}$-rational point $b \in \mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}$ is isomorphic (in the sense of Definition 1.1.7) to the $\tilde{G}$-cover parametrised by b.

Proof. It is easy to check that the centre of $\tilde{G}$ is trivial. Hence the Hurwitz space $\mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}$ is a fine moduli space (see [**RW06**, Corollary 4.12]). $\square$

After passing to a suitable unramified cover $S \to \mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}$, we may assume that the pullback $\widetilde{\mathscr{Y}}_S \to S$ extends to a semistable curve over $\overline{S}$. To simplify notation, we write $\widetilde{\mathscr{Y}}$ instead of $\widetilde{\mathscr{Y}}_{\overline{S}}$ for the extension of $\widetilde{\mathscr{Y}}_S$ to $\overline{S}$. Moreover we may choose the extension in such a way that the action of $\tilde{H}$ extends to $\widetilde{\mathscr{Y}}$.

In this section we consider the *adapted Ellenberg family*

$$\widetilde{\mathscr{X}}^{\mathrm{Ell}} := \widetilde{\mathscr{Y}}/\tilde{H} \to \overline{S} \qquad \text{with } \tilde{H} = \langle \psi, \sigma\rangle.$$

(Since $\widetilde{\mathscr{X}}^{\mathrm{Ell}}$ is defined as quotient of a semistable curve by a finite group, $\widetilde{\mathscr{X}}^{\mathrm{Ell}}$ is a also semistable.)

Recall that the (non-adapted) Ellenberg families $\mathscr{X}^{\mathrm{Ell}} \to \overline{S}$ discussed in Section 4.3 do not define Teichmüller curves because of the following reason. There is a point $b \in \overline{S}{-}S$ such that the fibre $\mathscr{X}_b^{\mathrm{Ell}}$ is a singular curve of compact type (see Theorem 4.3.5). In this section we show that this is not the case for the adapted Ellenberg families $\widetilde{\mathscr{X}}^{\mathrm{Ell}} \to \overline{S}$ (see Theorem 4.5.11).

As a first step we analyse the fibres of $\widetilde{\mathscr{X}}^{\mathrm{Ell}} \to \overline{S}$ more precisely. Let $b \in \overline{S}{-}S$ and $\overline{X} = \widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ the fibre at b. Write δ for the number of irreducible components and γ for the number of singularities of $\overline{X}$. In the following subsections we systematically check whether the fibres of $\widetilde{\mathscr{X}}^{\mathrm{Ell}}$ over $\overline{S}{-}S$ are

(i) of compact type, i.e. $1 - \delta + \gamma = 0$,

(ii) a Mumford curve, i.e. $1-\delta+\gamma=g(\overline{X})$,
(iii) or none of the two.

Remark 4.5.4. (i) The genus of the fibres of $\widetilde{\mathscr{X}}^{\mathrm{Ell}} \to S$ is constant and equals

$$g(\widetilde{\mathscr{X}}^{\mathrm{Ell}}) = \frac{p^2-1}{2m} - \frac{p-1}{2}.$$

This is a straight forward computation, e.g. one may use Lemma 1.5.3. For $p = m+1$ the genus equals $g(\widetilde{\mathscr{X}}^{\mathrm{Ell}}) = 1$. Otherwise, the smallest genus is given for $p = 13$ and $m = 6$. In this case, we have $g(\widetilde{\mathscr{X}}^{\mathrm{Ell}}) = 8$.
(ii) Suppose that $\overline{X}$ is the fibre of $\widetilde{\mathscr{X}}^{\mathrm{Ell}}$ over an arbitrary point $b \in \overline{S}-S$. Denote by $\overline{X}_1, \dots, \overline{X}_\delta$ the irreducible components of $\overline{X}$. Then

$$g(\overline{X}) = g(\widetilde{\mathscr{X}}^{\mathrm{Ell}}) = \sum_{i=1}^{\delta} g(\overline{X}_i) + 1 - \delta + \gamma,$$

where γ is the number of singularities of $\overline{X}$.

4.5.1. Degenerations at $\lambda \in \{0,1\}$. We denote by

$$\Psi\colon \overline{\mathscr{H}}_{\tilde{G},\tilde{\mathbf{C}}} \to \mathbb{P}^1_{\mathbb{C}}$$

the extension of the branch locus map $\mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}} \to \mathbb{P}^1_{\mathbb{C}}-\{0,1,\infty\}$ that sends a $\tilde{G}$-cover with ordered branch locus $(0,1,\lambda,\infty)$ to its third branch point λ. In this subsection we prove the following theorem.

Proposition 4.5.5. *The fibres of $\widetilde{\mathscr{X}}^{\mathrm{Ell}} \to \overline{S}$ over $\Psi^{-1}(\{0,1\})$ are either Mumford curves or smooth. In particular, there is no point $b \in \overline{S}-S$ over $\Psi^{-1}(\{0,1\})$ such that $\widetilde{\mathscr{X}}^{\mathrm{Ell}}_b$ is a singular curve of compact type.*

Remark 4.5.6. The case $\lambda = 0$ is completely analogous to the case $\lambda = 1$. In the following we only consider the case $\lambda = 1$.

We write $\overline{X} \to \overline{P}$ for the $\tilde{H}$-quotient of the admissible $\tilde{G}$-cover parametrised by points in $\overline{S}-S$ over $\Psi^{-1}(1) \subset \overline{\mathscr{H}}_{\tilde{G},\tilde{\mathbf{C}}}-\mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}$. Suppose that the admissible cover corresponds (in the sense of Proposition 1.4.1) to a Nielsen tuple

$$\tilde{\mathbf{g}} = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4) = \left(\sigma,\ \varphi_1^i \varphi_2^{-i\alpha}\psi^{-1}\sigma,\ \varphi_1^k \psi^{a_1},\ \varphi_1^{(k-i)\alpha}\varphi_2^i \psi^{m/2}\right)$$

in $\mathrm{Ni}(\tilde{G}, \tilde{\mathbf{C}})$, normalised as in (4.5.1) with $(i,k) \not\equiv (0,0) \pmod p$. Let γ be the number of singularities of $\overline{X}$, and δ the number of irreducible components of $\overline{X}$. We write $\overline{P}_1$ for the irreducible component of $\overline{P}$ to which the branch points $x_1 = 0$ and $x_4 = \infty$ specialise, and we write $\overline{P}_2$ for the irreducible component of $\overline{P}$ to which the branch points $x_2 = 1$ and $x_3 = \lambda$ specialise.

Lemma 4.5.7. *If $(k-i)\alpha \equiv i \pmod p$, then $1-\delta+\gamma = g(\overline{X})$, i.e. in this case $\overline{X}$ is a Mumford curve.*

PROOF. Since $(k-i)\alpha \equiv i \pmod p$ it follows that $\tilde{g}_\xi = \tilde{g}_2\tilde{g}_3 = \varphi_1^i\varphi_2^{(k-i)\alpha}\psi^{m/2}\sigma$ is an element of order 2. This implies that $\tilde{G}_1 = \langle\tilde{g}_1, \tilde{g}_4\rangle$ is the direct product of two cyclic groups of order 2 generated by $\tilde{g}_4$ and σ. Moreover, the group $\tilde{G}_2 = \langle\tilde{g}_2, \tilde{g}_3\rangle$ is isomorphic to a dihedral group of order $2m^*$ with

$$m^* = \frac{m}{a_1} = \begin{cases} 4, & \text{if } m = 4, \\ 3, & \text{if } m = 6, \end{cases}$$

generated by the rotation $\tilde{g}_3$ of order m^* and the reflection $\tilde{g}_2$.

Proposition 1.5.6 states that $|\tilde{G}_1\backslash\tilde{G}/\tilde{H}|$ is the number of irreducible components of $\overline{X}$ above $\overline{P}_1$, $|\tilde{G}_2\backslash\tilde{G}/\tilde{H}|$ is the number of irreducible components of $\overline{X}$ above $\overline{P}_2$ and $|\langle\tilde{g}_\xi\rangle\backslash\tilde{G}/\tilde{H}|$ is the number of singularities of $\overline{X}$.

To compute $|\tilde{G}_1\backslash\tilde{G}/\tilde{H}|$, note that the set of right cosets $\tilde{G}_1\backslash\tilde{G}$ may be represented by $\tilde{G}_1\varphi_1^x\varphi_2^y\psi^z$ with $x, y \in \{0, \ldots, p-1\}$ and $z \in \{0, \ldots, \frac{m}{2} - 1\}$. Burnside's lemma (for orbit-counting) states that

$$|\tilde{G}_1\backslash\tilde{G}/\tilde{H}| = \frac{1}{2m}\sum_{h\in\tilde{H}}\left|\left\{\tilde{G}_1\ell \in \tilde{G}_1\backslash G;\ \tilde{G}_1\ell h = \tilde{G}_1\ell\right\}\right|.$$

One checks that $h = 1 \in \tilde{H}$ fixes $\frac{p^2m}{2}$ cosets, $h = \psi^{m/2} \in \tilde{H}$ fixes $\frac{m}{2}$ cosets, $h = \psi^t\sigma \in \tilde{H}$ fixes p cosets for $t = 0, \ldots, m-1$ in case that $m = 6$ (resp. $h = \psi^t\sigma \in \tilde{H}$ fixes $2p$ cosets for $t = 0, 2$ in case that $m = 4$). The other elements of $\tilde{H}$ do not fix any coset. Hence

$$|\tilde{G}_1\backslash\tilde{G}/\tilde{H}| = \frac{(p+1)^2}{4}.$$

Similar calculations show that

$$|\langle\tilde{g}_\xi\rangle\backslash\tilde{G}/\tilde{H}| = \frac{p(p+1)}{2}, \qquad |\tilde{G}_2\backslash\tilde{G}/\tilde{H}| = \frac{p^2-1}{2m^*} + \frac{p+1}{2}.$$

By Proposition 1.5.6, it holds

$$\begin{aligned} 1 - \delta + \gamma &= 1 - |\tilde{G}_1\backslash\tilde{G}/\tilde{H}| - |\tilde{G}_2\backslash\tilde{G}/\tilde{H}| + |\langle\tilde{g}_\xi\rangle\backslash\tilde{G}/\tilde{H}| \\ &= \frac{p^2-1}{2m} - \frac{p-1}{2} = g(\widetilde{\mathscr{X}}^{\mathrm{Ell}}) = g(\overline{X}). \end{aligned}$$

□

LEMMA 4.5.8. *Suppose that $(k-i)\alpha \not\equiv i \pmod p$.*

(i) The number of irreducible components of $\overline{X}$ over $\overline{P}_2$ equals 1.

(ii) We have that $1 - \delta + \gamma = 0$.

(iii) All irreducible components of $\overline{X}$ over $\overline{P}_1$ have genus 0.

In particular, after contracting all irreducible genus-0 components of $\overline{X}$ which intersect the rest of $\overline{X}$ in one singularity, the curve $\overline{X}$ becomes a smooth curve.

PROOF. (i) One checks that $\tilde{G}_1 := \langle\tilde{g}_1, \tilde{g}_4\rangle \simeq \mathbb{F}_p \rtimes (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$ is a metabelian group with normal factor the p-cyclic group $\langle\varphi_1\varphi_2^{-1}\rangle$ and second factor the direct product of two cyclic groups generated by $\varphi_1^\kappa\varphi_2^\kappa\psi^{m/2}$ and σ, where $\kappa \equiv$

$((k-i)\alpha+i)2^{-1} \pmod p$. (We have chosen κ in such a way that $\varphi_1^\kappa\varphi_2^\kappa\psi^{m/2}$ and σ commute.) Moreover $\tilde G_2 := \langle \tilde g_2, \tilde g_3\rangle = \langle \varphi_1, \varphi_2, \psi^{a_1}, \sigma\rangle$ is either $\tilde G$ (in case that $m=4$) or an index-2 subgroup of $\tilde G$ (in case that $m=6$). In any case, we have that

$$|\tilde G_2\backslash\tilde G/\tilde H| = 1$$

which equals the number of irreducible components of $\overline{X}$ over $\overline{P}_2$ (see Proposition 1.5.6).

(ii) Similar to the proof of Lemma 4.5.7, using Burnside's lemma (for orbit-counting), one computes that

$$|\tilde G_1\backslash\tilde G/\tilde H| = |\langle \tilde g_\xi\rangle\backslash\tilde G/\tilde H| = \frac{p+1}{2}.$$

Hence

$$\begin{aligned} 1-\delta+\gamma &= 1-|\tilde G_1\backslash\tilde G/\tilde H| - |\tilde G_2\backslash\tilde G/\tilde H| + |\langle\tilde g_\xi\rangle\backslash\tilde G/\tilde H| \\ &= 1-\frac{p+1}{2}-1+\frac{p+1}{2} = 0. \end{aligned}$$

(iii) It remains to check that all irreducible components of $\overline{X}$ over $\overline{P}_1$ have genus 0. Let $\overline{Y}\to\overline{P}$ be the admissible G-cover over $\Psi^{-1}(1)\subset\overline{\mathscr{H}}_{\tilde G,\tilde{\mathbf{C}}}-\mathscr{H}_{\tilde G,\tilde{\mathbf{C}}}$ corresponding to $\tilde{\mathbf{g}}$ (as in Proposition 1.4.1). Note that we may restrict this G-cover to a $\tilde G_1$-cover $\overline{Y}_1\to\overline{P}_1\simeq\mathbb{P}^1_{\mathbb{C}}$ with ramification type $(\mathrm{Cl}_{\tilde G_1}(\tilde g_4), \mathrm{Cl}_{\tilde G_1}(\tilde g_1), \mathrm{Cl}_{\tilde G_1}(\tilde g_\xi))$ such that all irreducible components over $\overline{P}_1$ are copies of $\overline{Y}_1$. The Riemann-Hurwitz formula reads

$$\begin{aligned} 2g(\overline{Y}_1)-2 &= -2\cdot|\tilde G_1| + \tfrac{|\tilde G_1|}{\operatorname{ord}\tilde g_4}\cdot(\operatorname{ord}\tilde g_4-1) + \tfrac{|\tilde G_1|}{\operatorname{ord}\tilde g_1}\cdot(\operatorname{ord}\tilde g_1-1) \\ &\quad + \tfrac{|\tilde G_1|}{\operatorname{ord}\tilde g_\xi}\cdot(\operatorname{ord}\tilde g_\xi-1) \\ &= -2, \end{aligned}$$

since $\operatorname{ord}\tilde g_1=\operatorname{ord}\tilde g_4=2$ and $\operatorname{ord}\tilde g_\xi=2p$ in this case. Hence $g(\overline{Y}_1)=0$. The irreducible components of $\overline{X}$ over $\overline{P}_1$ are quotients of $\mathrm{Ind}_{\tilde G_1}^G(\overline{Y}_1)$ and therefore genus-0 curves.

□

PROOF OF PROPOSITION 4.5.5: Lemma 4.5.7 and Lemma 4.5.8 prove the theorem. □

4.5.2. Degenerations at $\lambda=\infty$. We now consider the fibre of $\widetilde{\mathscr{X}}^{\mathrm{Ell}}$ over $\Psi^{-1}(\infty)\subset\overline{\mathscr{H}}_{\tilde G,\tilde{\mathbf{C}}}-\mathscr{H}_{\tilde G,\tilde{\mathbf{C}}}$ corresponding (in the sense of Proposition 1.4.1) to a Nielsen tuple

$$\tilde{\mathbf{g}} = (\tilde g_1,\tilde g_2,\tilde g_3,\tilde g_4) = \left(\sigma,\ \varphi_1^i\varphi_2^{-i\alpha}\psi^{-1}\sigma,\ \varphi_1^k\psi^{a_1},\ \varphi_1^{(k-i)\alpha}\varphi_2^i\psi^{m/2}\right), \tag{4.5.2}$$

in $\mathrm{Ni}(\tilde G,\tilde{\mathbf{C}})$ with $(i,k)\not\equiv(0,0) \pmod p$.

PROPOSITION 4.5.9. *Let $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ be a fibre of $\widetilde{\mathscr{X}}^{\mathrm{Ell}}\to\overline{S}$ at a point $b\in\overline{S}-S$ over $\Psi^{-1}(\infty)$ corresponding to $\tilde{\mathbf{g}}$ as in* (4.5.2). *Write δ for the number of irreducible* *components of* $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ *and* γ *for the number of singularities of* $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$. *Then*

$$1-\delta+\gamma=\begin{cases} g(\widetilde{\mathscr{X}}^{\mathrm{Ell}})-\frac{p-1}{m}\neq 0, & \text{if } i\equiv 0 \text{ or } k+i\sum_{r=1}^{a_1}\alpha^r\equiv 0\not\equiv i \pmod p,\\ g(\widetilde{\mathscr{X}}^{\mathrm{Ell}}), & \text{otherwise.}\end{cases}$$

In particular, there is no point $b\in\overline{S}-S$ *over* $\Psi^{-1}(\{\infty\})$ *such that* $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ *is (a singular curve) of compact type.*

Proposition 4.5.9 follows from the succeeding lemma, using $\delta=|\tilde{G}_1\backslash\tilde{G}/\tilde{H}|+|\tilde{G}_2\backslash\tilde{G}/\tilde{H}|$ and $\gamma=|\langle\tilde{g}_\xi\rangle\backslash\tilde{G}/\tilde{H}|$.

Lemma 4.5.10. *(i) The elements*

$$g_\xi:=\tilde{g}_3\tilde{g}_4=\varphi_1^i\varphi_2^{-i\alpha}\psi^{-1},\qquad g_\xi^{-1}=\varphi_1^{-i\alpha}\varphi_2^i\psi$$

have order m.

(ii) The group generated by $\tilde{g}_1$ *and* $\tilde{g}_2$,

$$\tilde{G}_1:=\langle\tilde{g}_1,\tilde{g}_2\rangle=\langle\tilde{g}_\xi,\sigma\rangle\simeq D_m,$$

is a dihedral group of order $2m$ *generated by the rotation* $\tilde{g}_\xi$ *of order* m *and the reflection* σ.

(iii) The group generated by $\tilde{g}_3$ *and* $\tilde{g}_4$,

$$\tilde{G}_2:=\langle\tilde{g}_3,\tilde{g}_4\rangle=\begin{cases}\langle\varphi_2\rangle\rtimes\langle\varphi_1^{-i\alpha}\psi\rangle, & \text{if } k+i\sum_{r=1}^{a_1}\alpha^r\equiv 0\not\equiv i \pmod p\\ \langle\varphi_1\rangle\rtimes\langle\psi\rangle, & \text{if } i\equiv 0 \pmod p\\ \langle\varphi_1,\varphi_2\rangle\rtimes\langle\psi\rangle, & \text{otherwise,}\end{cases}$$

is a metabelian group.

(iv) We have

$$\begin{aligned}|\tilde{G}_1\backslash\tilde{G}/\tilde{H}| &= \frac{p+1}{2}\left(\frac{p-1}{m}+1\right).\\ |\tilde{G}_2\backslash\tilde{G}/\tilde{H}| &= \begin{cases}\frac{p-1}{m}+1, & \text{if } i\equiv 0 \text{ or } k+i\sum_{r=1}^{a_1}\alpha^r\equiv 0\not\equiv i \pmod p,\\ 1 & \text{otherwise.}\end{cases}\\ |\langle\tilde{g}_\xi\rangle\backslash\tilde{G}/\tilde{H}| &= \frac{p^2-1}{m}+1.\end{aligned}$$

Proof. The proof of the lemma is similar to the proofs of Lemma 4.5.7 and Lemma 4.5.8. □

Theorem 4.5.11. *The adapted Ellenberg family* $\widetilde{\mathscr{X}}^{\mathrm{Ell}}:=\widetilde{\mathscr{Y}}/\widetilde{H}\to\overline{S}$ *has no fibres that are singular curves of compact type.*

Proof. This follows from Proposition 4.5.5 and Proposition 4.5.9. □

4.6. The de Rham cohomology of the adapted Ellenberg families

Our next goal is to find 'good' candidates for flat rank-2 subbundles $\widetilde{\mathscr{E}}$ of the Deligne extension of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$ such that we may choose a set of marked points of $\widetilde{\mathscr{E}}$

(i) all of which are logarithmic singularities,
(ii) the curve $\widetilde{\mathscr{X}}^{\mathrm{Ell}}$ is smooth over the non-marked points,
(iii) the bundle $\widetilde{\mathscr{E}}$ is indigenous.

We have checked whether the degenerations in $\Psi^{-1}(\{0,1,\infty\})$ are Mumford curves, smooth or none of the two.

Notation 4.6.1. Write $\overline{S}^*$ for the set of points $b \in \overline{S}-S$ such that $\widetilde{\mathscr{X}}^{\mathrm{Ell}}_b$ is neither a Mumford curve nor smooth.

We have shown in Proposition 4.5.5 and Proposition 4.5.9 that the set $\overline{S}^*$ is exactly the set of points in $\overline{S}-S$ over $\Psi^{-1}(\infty) \subset \overline{\mathscr{H}}_{\tilde{G},\tilde{\mathbf{C}}}-\mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}$ corresponding (in the sense of Proposition 1.4.1) to a Nielsen tuple

$$\tilde{\mathbf{g}} = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4) = \left(\sigma,\ \varphi_1^i\varphi_2^{-i\alpha}\psi^{-1}\sigma,\ \varphi_1^k\psi^{a_1},\ \varphi_1^{(k-i)\alpha}\varphi_2^i\psi^{m/2}\right)$$

in $\mathrm{Ni}(\tilde{G},\tilde{\mathbf{C}})$ with $i \equiv 0$ or $i \not\equiv 0 \equiv k + i\sum_{r=1}^{a_1}\alpha^r \pmod p$.

Suppose that $\widetilde{\mathscr{E}}$ is a flat rank-2 subbundle of the Deligne extension of the relative de Rham cohomology $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$. Choose $b \in \overline{S}-S$ and denote by $\overline{X}_1, \dots, \overline{X}_\delta$ the irreducible components of $\widetilde{\mathscr{X}}^{\mathrm{Ell}}_b$. Write $\widetilde{\mathscr{E}}|_b = \widetilde{\mathscr{E}}_b/\mathfrak{m}_b\widetilde{\mathscr{E}}_b$ for the fibre of $\widetilde{\mathscr{E}}$ at b. Here $\mathfrak{m}_b$ denotes the maximal ideal of the local ring $\mathscr{O}_{\overline{S},b}$. We consider the submodule

$$\widetilde{\mathscr{E}}|_b^{\mathrm{comp}} := \widetilde{\mathscr{E}}|_b \cap \left(H^1_{\mathrm{dR}}(\overline{X}_1) \times \cdots \times \mathscr{H}^1_{\mathrm{dR}}(\overline{X}_\delta)\right) \subset \widetilde{\mathscr{E}}|_b$$

of the fibre of the Deligne extension of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$ at b . We call $\widetilde{\mathscr{E}}|_b^{\mathrm{comp}}$ the *component part* of $\widetilde{\mathscr{E}}$ at b.

Lemma 4.6.2. *If*

$$\widetilde{\mathscr{E}}|_b^{\mathrm{comp}} = \widetilde{\mathscr{E}}|_b \tag{4.6.1}$$

then b is a regular point of $(\widetilde{\mathscr{E}}, \nabla)$.

This is a direct generalisation of Lemma 4.3.1, which we have used in Theorem 4.3.5 to show that the (non-adapted) Ellenberg families $\mathscr{X}^{\mathrm{Ell}} \to \overline{S}$ do not define Teichmüller curves.

Remark 4.6.3. Let $b \in \overline{S}^*$ and $\overline{X} = \widetilde{\mathscr{X}}^{\mathrm{Ell}}_b$ the fibre of $\widetilde{\mathscr{X}}^{\mathrm{Ell}}_b$ at b. Write $\overline{X}_1, \dots, \overline{X}_\delta$ for the irreducible components of $\overline{X}$ and γ for the number of singularities of $\overline{X}$. Let $\widetilde{\mathscr{E}}$ be an arbitrary flat rank-2 subbundle of the Deligne extension of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$

(i) If $\overline{X}$ is a Mumford curve, then $H^1_{\mathrm{dR}}(\overline{X}_1) \times \cdots \times H^1_{\mathrm{dR}}(\overline{X}_\delta) = \{0\}$. Therefore it holds $\widetilde{\mathscr{E}}|_b^{\mathrm{comp}} = \{0\}$.
(ii) If $\overline{X}$ is a smooth curve, then $\widetilde{\mathscr{E}}|_b^{\mathrm{comp}} = \widetilde{\mathscr{E}}|_b$ and b is a regular point of $\widetilde{\mathscr{E}}$.

In the following we decompose $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$ into flat rank-2 subbundles isomorphic to $\tilde{A}$-isotypical subbundles of $\mathscr{H}^1_{\mathrm{dR}}(\mathscr{Y}/S)$ and exclude those subbundles whose Deligne extension at some point $b \in \overline{S}^*$ equals its component part at b. (Since this implies that the subbundle has a regular point b whereas $\widetilde{\mathscr{X}}^{\mathrm{Ell}}_b$ is a singular curve.) The remaining subbundles are 'good' candidates to satisfy the conditions from Theorem 3.1.13.

First we describe the $\tilde{G}$-isotypical decomposition of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$. We write $\chi_{\iota,\kappa} \in \mathrm{Irr}(\tilde{A})$ for the irreducible character of $\tilde{A} = \langle \varphi_1, \varphi_2 \rangle$ given by

$$\chi_{\iota,\kappa}(\varphi_1^x \varphi_2^y) = \zeta_p^{\iota x + \kappa y}$$

and identify $\mathrm{Irr}(\tilde{A})$ with $\mathbb{F}_p^2$ via the bijection given by $\chi_{\iota,\kappa} \mapsto (\iota,\kappa)$. The group $\tilde{H} = \langle \psi, \sigma \rangle$ acts on $\mathrm{Irr}(\tilde{A})$ by

$$\psi(\iota,\kappa) = (\iota\alpha, \kappa\alpha^{-1}), \qquad \sigma(\iota,\kappa) = (\kappa,\iota).$$

This is the action introduced in Notation 2.3.1. One checks that the stabiliser $\tilde{H}_\chi$ of $\chi = \chi_{\iota,\kappa}$ is

$$\tilde{H}_\chi = \begin{cases} \tilde{H}, & \text{if } (\iota,\kappa) = (0,0), \\ \langle \psi^r \sigma \rangle, & \text{if } (\iota,\kappa) = (\iota, \iota\alpha^r) \neq (0,0), \\ \langle 1 \rangle, & \text{otherwise.} \end{cases}$$

Moreover, we write $\tilde{H}(\chi)$ for the orbit of $\chi = \chi_{\iota,\kappa}$ and define $\tilde{G}_\chi := \tilde{A} \rtimes \tilde{H}_\chi$.

As in Lemma 2.3.2 (or [**Ser77**, Section 8.2]) we conclude that all irreducible $\tilde{G}$-characters are of the form

$$\theta_{\chi,\xi} := \mathrm{Ind}_{\tilde{G}_\chi}^{\tilde{G}}(\chi \cdot \xi), \qquad \chi \in \mathrm{Irr}(\tilde{A}), \qquad \xi \in \mathrm{Irr}(\tilde{H}_\chi).$$

Moreover, $\theta_{\chi,\xi} = \theta_{\chi',\xi'}$ if and only if $\chi' \in \tilde{H}(\chi)$ and $\xi = \xi'$.

We consider the set of irreducible $\tilde{A}$-characters with trivial stabiliser, indexed by the set

$$\boldsymbol{\mu} = \left\{ (\iota,\kappa) \in \mathbb{F}_p^2; \quad \iota\alpha^r \not\equiv \kappa \pmod p \text{ for all } r = 0,\dots,m-1 \right\} / \sim,$$

where we divide out the action of $\tilde{H}$. In the following, we often write $\chi \in \boldsymbol{\mu}$ when we mean that $\chi = \chi_{\iota,\kappa} \in \mathrm{Irr}(\tilde{A})/\tilde{H}$ with $(\iota,\kappa) \in \boldsymbol{\mu}$.

Proposition 4.6.4. *The relative de Rham cohomology $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$ splits into flat subbundles*

$$\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S) = \bigoplus_{\chi \in \boldsymbol{\mu}} \widetilde{\mathscr{E}}_\chi, \tag{4.6.2}$$

where every $\widetilde{\mathscr{E}}_\chi$ has rank 2 and is isomorphic (as a flat vector bundle with Gauß-Manin connection) to the χ-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{Y}}/S)$.

PROOF. Proposition 2.3.6 yields

$$\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S) = \bigoplus_{\chi\in \mathrm{Irr}(\tilde{A})/\tilde{H}} (\widetilde{\mathscr{E}}_{\chi,\mathbb{1}})^{\tilde{H}},$$

where $\widetilde{\mathscr{E}}_{\chi,\mathbb{1}}$ denotes the isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{Y}}/S)$ with respect to $\theta_{\chi,\mathbb{1}} \in \mathrm{Irr}(\tilde{G})$. Let $\chi \in \boldsymbol{\mu} \subset \mathrm{Irr}(\tilde{A})/\tilde{H}$ and put $\widetilde{\mathscr{E}}_\chi := (\widetilde{\mathscr{E}}_{\chi,\mathbb{1}})^{\tilde{H}}$. We now compute $\operatorname{rank}\widetilde{\mathscr{E}}_\chi = \operatorname{rank}(\widetilde{\mathscr{E}}_{\chi,\mathbb{1}})^{\tilde{H}}$. We check that for all $\nu = 1,\ldots,4$ and all $\ell \in G_\chi/\langle\tilde{g}_\nu\rangle$ it holds $\tilde{A}\cap\langle\ell\tilde{g}_\nu\ell^{-1}\rangle = \{1\} \subset \ker\chi$. Moreover,

$$|\tilde{G}_\chi\backslash G/\langle\tilde{g}_2\rangle| = |\tilde{G}_\chi\backslash G/\langle\tilde{g}_1\rangle| = |\tilde{G}_\chi\backslash G/\langle\tilde{g}_4\rangle| = m,$$

$$|\tilde{G}_\chi\backslash G/\langle\tilde{g}_3\rangle| = 2a_1 = \begin{cases} 2, & \text{if } m = 4,\\ 4, & \text{if } m = 6.\end{cases}$$

Using the notation from Section 2.3.1, we compute that

$$\operatorname{rank}\left(\widetilde{\mathscr{E}}_{\chi,\mathbb{1}}\right)^{\tilde{H}} = n_{\chi,\mathbb{1}} = 2\cdot|\tilde{H}(\chi)| \quad - \sum_{\substack{\nu=1,\ldots,4\\ \ell\in G_\chi\backslash G/\langle\tilde{g}_\nu\rangle}} k_\chi^{\nu,\ell} = 2.$$

The decomposition (4.6.4) follows since one computes that

$$\tfrac{1}{2}\cdot\sum_{\chi\in\boldsymbol{\mu}}\operatorname{rank}\widetilde{\mathscr{E}}_\chi = |\boldsymbol{\mu}| = \frac{p^2-1}{2m} - \frac{p-1}{2} = g(\widetilde{\mathscr{X}}^{\mathrm{Ell}})$$

(see Remark 4.5.4 (i)).

It remains to check whether $\widetilde{\mathscr{E}}_\chi$ is isomorphic to the χ-isotypical component of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{Y}}/S)$. The cover $\widetilde{\mathscr{Y}}/\ker(\chi) \to \mathscr{Z} = \widetilde{\mathscr{Y}}/\langle\varphi_1,\varphi_2\rangle$ is an étale p-cyclic cover of a genus-2 curve. Let $\mathscr{E}_\chi$ of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{Y}}/S)$ be the χ-isotypical component. It may be considered as a subbundle of the $\ker(\chi)$-invariant module $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{Y}}/S)^{\ker(\chi)}$, which we may identify with the de Rham cohomology of the quotient curve $\widetilde{\mathscr{Y}}/\ker(\chi)$. Now we proceed as in the proof of Proposition 4.2.1. Using Lemma 4.2.7, we conclude that $\mathscr{E}_\chi \subset \mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{Y}}/S)^{\ker(\chi)}$ carries a filtration in the sense of Definition 3.1.9. Moreover, we conclude that the submodule $\widetilde{\mathscr{E}}_\chi = (\widetilde{\mathscr{E}}_{\chi,\mathbb{1}})^{\tilde{H}}$ of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{Y}}/S)^{\tilde{H}} \simeq \mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$ and the χ-isotypical component $\mathscr{E}_\chi$ of $\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{Y}}/S)$ are isomorphic as flat vector bundles. (An isomorphism $\mathscr{E}_\chi \xrightarrow{\sim} \widetilde{\mathscr{E}}_\chi = (\widetilde{\mathscr{E}}_{\chi,\mathbb{1}})^{\tilde{H}}$ is induced by sending a section ω of $\mathscr{E}_\chi$ to the section $\sum_{h\in H} h\omega$.) □

The following lemma simplifies the situation. Note that $\overline{S}^* \subset \Psi^{-1}(\infty)$.

LEMMA 4.6.5. *Let $b \in \overline{S}^*$, i.e. $\overline{X} := \widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ is neither smooth nor a Mumford curve. Write $\overline{X} \to \overline{P}$ for the corresponding admissible cover. Then the irreducible components of $\overline{X}$ over the irreducible component $\overline{P}_1$ of $\overline{P}$ to which the branch points $x_1 = 0$ and $x_2 = 1$ specialise have genus 0.*

PROOF. The cover $\overline{X} \to \overline{P}$ is the $\tilde{H}$-quotient of an admissible $\tilde{G}$-cover $\overline{Y} \to \overline{P}$. The restriction of $\overline{Y}$ to the component $\overline{P}_1$ is given by $\overline{Y}|_{\overline{P}_1} = \mathrm{Ind}_{\tilde{G}_1}^{\tilde{G}}(\overline{Y}_1)$ where $\overline{Y}_1 \to \mathbb{P}^1_{\mathbb{C}}$ is a $\tilde{G}_1$-cover of type $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_\xi)$ with

$$(\tilde{g}_1, \tilde{g}_2, \tilde{g}_\xi) = (\sigma, \varphi_1^i \varphi_2^{-i\alpha} \psi^{-1} \sigma, \varphi_1^i \varphi_2^{-i\alpha} \psi^{-1}), \qquad \tilde{G}_1 = \langle \tilde{g}_1, \tilde{g}_2 \rangle \simeq D_m,$$

where $\mathrm{ord}\, \tilde{g}_\xi = m$ (see Lemma 4.5.10). Thus, the Riemann-Hurwitz genus formula reads

$$\begin{aligned} 2g(\overline{Y}_1) - 2 &= -2 \cdot |D_m| + \frac{|D_m|}{\mathrm{ord}\, \tilde{g}_1}(\mathrm{ord}\, \tilde{g}_1 - 1) + \frac{|D_m|}{\mathrm{ord}\, \tilde{g}_2}(\mathrm{ord}\, \tilde{g}_2 - 1) \\ &\quad + \frac{|D_m|}{\mathrm{ord}\, \tilde{g}_\xi}(\mathrm{ord}\, \tilde{g}_\xi - 1) \\ &= -2, \end{aligned}$$

i.e. $g(\overline{Y}_1) = 0$. The lemma follows since the irreducible components of $\overline{X} \to \overline{P}$ over $\overline{P}_1$ are quotients of $\mathrm{Ind}_{\tilde{G}_1}^{G}(\overline{Y}_1)$. $\square$

THEOREM 4.6.6. *Let $\chi \in \boldsymbol{\mu}$, i.e. $\chi = \chi_{\iota,\kappa}$ is the irreducible A-character given by*

$$\chi(\varphi_1^x \varphi_2^y) = \zeta_p^{\iota x + \kappa y} \qquad \text{with } \iota\alpha^r \not\equiv \kappa \pmod{p} \text{ for } r = 0, \ldots, m-1.$$

Denote by $\widetilde{\mathscr{E}}_\chi \subset \mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S)$ the corresponding isotypical component from Decomposition (4.6.2) *in Proposition 4.6.4. Let $b \in \overline{S} - S$ be such that the fibre $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ is a singular curve. Then*

$$\widetilde{\mathscr{E}}_\chi|_b^{\mathrm{comp}} = \begin{cases} \{0\}, & \text{if } \iota \cdot \kappa \not\equiv 0 \pmod{p}, \\ \widetilde{\mathscr{E}}_\chi|_b, & \text{otherwise.} \end{cases}$$

PROOF. We only consider the admissible $\tilde{G}$-cover $\overline{Y} \to \overline{P}$ parametrised by points $b \in \overline{S}^*$ over $\overline{\mathscr{H}}_{\tilde{G},\tilde{\mathbf{C}}} - \mathscr{H}_{\tilde{G},\tilde{\mathbf{C}}}$ (see Notation 4.6.1). We can restrict to that case, since $\widetilde{\mathscr{E}}_\chi|_b^{\mathrm{comp}} = \{0\}$ automatically holds in the case that $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ is a Mumford curve (see Remark 4.6.3 (i)). Let $\overline{X} = \overline{Y}/\tilde{H}$. Lemma 4.6.5 implies that

$$\widetilde{\mathscr{E}}_\chi|_b^{\mathrm{comp}} = (\widetilde{\mathscr{E}}_b / \mathfrak{m}_b \widetilde{\mathscr{E}}_b) \cap H^1_{\mathrm{dR}}(\overline{X}_2),$$

where $\overline{X}_2$ is the restriction of $\overline{X}$ to the irreducible components over $\overline{P}_2$ to which the branch points $x_3 = \lambda$ and $x_4 = \infty$ specialise.

Let $\overline{Y}_2$ be the restriction of $\overline{Y}$ to the irreducible components over $\overline{P}_2$. Let $\overline{Y}_2^0$ be an irreducible component such that $\overline{Y}_2 = \mathrm{Ind}_{\tilde{G}_2}^{\tilde{G}}(\overline{Y}_2^0)$. Hence the character $\text{ш}_{\tilde{G}}$ of the representation $\tilde{G} \to \mathrm{GL}(H^1_{\mathrm{dR}}(\overline{Y}_2))$ (induced by the Galois action of $\tilde{G}$ on $\overline{Y}_2$) equals $\text{ш}_{\tilde{G}} = \mathrm{Ind}_{\tilde{G}_2}^{\tilde{G}} \text{ш}_{\tilde{G}_2}$ where

$$\text{ш}_{\tilde{G}_2} = 2 \cdot \mathbb{1} + \mathrm{Ind}_{\langle 1 \rangle}^{G} \mathbb{1} - \sum_{\nu=1}^{3} \mathrm{Ind}_{\langle h_\nu \rangle}^{G} \mathbb{1}$$

(by Lemma 2.1.1).

Similar to Section 2.3.1 we compute that

$$\operatorname{rank}\left(\widetilde{\mathscr{E}}_\chi|_b^{\mathrm{comp}}\right)^{\tilde{H}} = \langle \text{ш}_{\tilde{G}}, \theta_{\chi,\mathbb{1}}\rangle_{\tilde{G}} = 2\cdot\langle \operatorname{Ind}_{\tilde{G}_2}^{\tilde{G}}\mathbb{1}, \theta_{\chi,\mathbb{1}}\rangle_{\tilde{G}} + |\tilde{H}(\chi)| - \sum_{\substack{\nu=1,2,3\\ \ell\in\tilde{G}_\chi\backslash\tilde{G}/\langle h_\nu\rangle}} k_\chi^{\nu,\ell}$$

with

$$k_\chi^{\nu,\ell} = \begin{cases} 1, & \text{if } \tilde{A}\cap\langle \ell h_\nu \ell^{-1}\rangle \subset \ker\chi,\\ 0, & \text{otherwise.}\end{cases}$$

Note that the choice of χ implies that χ has a trivial stabiliser in $\tilde{H}$, i.e. $\tilde{G}_\chi = \tilde{A}$. Then it is easy to check that $k_\chi^{\nu,\ell} = 1$ for all $\nu = 1,2,3$ and all $\ell \in \tilde{G}_\chi\backslash\tilde{G}/\langle h_\nu\rangle$. Moreover,

$$\sum_{\nu=1}^{3} |\tilde{G}_\chi\backslash\tilde{G}/\langle h_\nu\rangle| = 2 + 2a_1 + m.$$

Then

$$|\tilde{H}(\chi)| - \sum_{\substack{\nu=1,2,3\\ \ell\in\tilde{G}_\chi\backslash\tilde{G}/\langle h_\nu\rangle}} k_\chi^{\nu,\ell} = 0.$$

Therefore

$$\tfrac{1}{2}\cdot\operatorname{rank}\left(\widetilde{\mathscr{E}}_\chi|_b^{\mathrm{comp}}\right)^{\tilde{H}} = \langle \operatorname{Ind}_{\tilde{G}_2}^{\tilde{G}}\mathbb{1}, \theta_{\chi,\mathbb{1}}\rangle_{\tilde{G}} = \langle \operatorname{Res}_{\tilde{A}}\operatorname{Ind}_{\tilde{G}_2}^{\tilde{G}}\mathbb{1}, \chi\rangle_{\tilde{A}}.$$

Proposition 22 in [**Ser77**] implies that

$$\begin{aligned}\langle \operatorname{Res}_{\tilde{A}}\operatorname{Ind}_{\tilde{G}_2}^{\tilde{G}}\mathbb{1}, \chi\rangle_{\tilde{A}} &= \sum_{\ell\in\tilde{A}\backslash\tilde{G}/\tilde{G}_2} \langle \operatorname{Ind}_{\tilde{A}\cap\ell\tilde{G}_2\ell^{-1}}^{\tilde{A}}\mathbb{1}, \chi\rangle_{\tilde{A}}\\ &= \langle \operatorname{Ind}_{\langle\varphi_1\rangle}^{\tilde{A}}\langle\mathbb{1}, \chi\rangle_{\tilde{A}} + \langle \operatorname{Ind}_{\langle\varphi_2\rangle}^{\tilde{A}}\mathbb{1}, \chi\rangle_{\tilde{A}}.\end{aligned}$$

The last equality follows by the description of $\tilde{G}_2$ in Lemma 4.5.10. (Note that $b \in \overline{S}^*$ means that $\mathbf{g} \in \operatorname{Ni}(\tilde{G}, \tilde{\mathbf{C}})$ is normalised as Lemma 4.5.2 with $i \equiv 0$ or $i \not\equiv 0 \equiv k + i\sum_{r=1}^{a_1}\alpha^r \pmod p$.) The restriction $\bar{\chi}_1 := \chi|_{\langle\varphi_1\rangle}$ is the trivial character if and only if $\iota \equiv 0 \pmod p$. Similarly, the restriction $\bar{\chi}_2 := \chi|_{\langle\varphi_2\rangle}$ is the trivial character if and only if $\kappa \equiv 0 \pmod p$. We conclude that

$$\langle \operatorname{Ind}_{\langle\varphi_1\rangle}^{\tilde{A}}\mathbb{1}, \chi\rangle_{\tilde{A}} + \langle \operatorname{Ind}_{\langle\varphi_2\rangle}^{\tilde{A}}\mathbb{1}, \chi\rangle_{\tilde{A}} = \sum_{v=1,2}\langle \mathbb{1}, \bar{\chi}_v\rangle_{\langle\varphi_v\rangle} = \begin{cases} 0, & \text{if } \iota\cdot\kappa \not\equiv 0 \pmod p,\\ 1, & \text{otherwise.}\end{cases}$$

□

Remark 4.6.7. The number of $(\iota,\kappa) \in \boldsymbol{\mu}$ with $\iota\cdot\kappa \neq 0$ equals the difference

$$g(\widetilde{\mathscr{X}}^{\mathrm{Ell}}) - (1 - \delta + \gamma) = \frac{p-1}{m}$$

(cf. Proposition 4.5.9).

We summarise the result of this section.

Summary 4.6.8. We consider the flat rank-2 vector bundles $(\widetilde{\mathscr{E}}_\chi, \nabla)$ in the decomposition

$$\mathscr{H}^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}^{\mathrm{Ell}}/S) = \bigoplus_{\chi \in \boldsymbol{\mu}} \widetilde{\mathscr{E}}_\chi$$

from Proposition 4.6.4 and assume that $p \neq m+1$. Let $\chi = \chi_{\iota,\kappa} \in \boldsymbol{\mu}$ with $\chi(\varphi_1^x \varphi_2^y) = \zeta_p^{\iota x + \kappa y}$ such that

$$\iota \cdot \kappa \equiv 0 \quad \text{and} \quad \iota \alpha^r \not\equiv \kappa \pmod{p} \text{ for } r = 0, \ldots, m-1.$$

Theorem 4.6.6 implies that the fibre of the bundle $\widetilde{\mathscr{E}}_\chi$ at a point $b \in \overline{S} - S$ where $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ is a singular curve is contained in the component part of $H^1_{\mathrm{dR}}(\widetilde{\mathscr{X}}_b^{\mathrm{Ell}})$, i.e. $\widetilde{\mathscr{E}}_\chi|_b^{\mathrm{comp}} = \widetilde{\mathscr{E}}_\chi|_b$. Lemma 4.6.2 therefore implies that b is a regular point of $\widetilde{\mathscr{E}}_\chi$, whereas $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ is a singular curve. Hence $\widetilde{\mathscr{E}}_\chi$ does not satisfy the conditions from Theorem 3.1.13.

On the other hand, all characters $\chi = \chi_{\iota,\kappa} \in \boldsymbol{\mu}$ with $\chi(\varphi_1^x \varphi_2^y) = \zeta_p^{\iota x + \kappa y}$ where

$$\iota \cdot \kappa \not\equiv 0 \quad \text{and} \quad \iota \alpha^r \not\equiv \kappa \pmod{p} \text{ for } r = 0, \ldots, m-1 \tag{4.6.3}$$

satisfy $\widetilde{\mathscr{E}}_\chi|_b^{\mathrm{comp}} = \{0\}$ for all $b \in \overline{S} - S$ where $\widetilde{\mathscr{X}}_b^{\mathrm{Ell}}$ is a singular curve.

To determine whether $\widetilde{\mathscr{X}}^{\mathrm{Ell}}$ defines a Teichmüller curve it remains to show the following.

(i) The fact that $\widetilde{\mathscr{E}}_\chi|_b^{\mathrm{comp}} = \{0\}$ implies that $\widetilde{\mathscr{E}}_\chi$ has a logarithmic singularity in b.
(ii) There exists a character $\chi \in \boldsymbol{\mu}$ which satisfies (4.6.3) such that the Kodaira-Spencer map of $\widetilde{\mathscr{E}}_\chi$ is an isomorphism.

Appendix A

Fuchsian differential equations

We briefly recall some basic facts about the classical theory of Fuchsian differential equations on $\mathbb{P}^1_{\mathbb{C}}$. For more details we refer to [**Beu07**] and [**Yos87**].

We consider the ordinary linear differential operator over $\mathbb{C}(s)$ of order 2 given by

$$L := \left(\tfrac{\partial}{\partial s}\right)^2 + c_1 \cdot \left(\tfrac{\partial}{\partial s}\right) + c_0$$

where $c_0, c_1 \in \mathbb{C}(s)$ and $\frac{\partial}{\partial s}$ is the standard derivation.

Definition A.0.9. (i) A point $b \in \overline{S} := \mathbb{P}^1_{\mathbb{C}}$ is called *singularity* of L if for some $i \in \{0, 1\}$ we have $\mathrm{ord}_b(c_i) < 0$. We write $S' \subset \overline{S}$ for the (finite) set of singular points of L and $S \subset \overline{S}$ for its complement.

(ii) We say that a singularity $b \in S'$ is *regular* if both $\mathrm{ord}_b(c_1) \geq -1$ and $\mathrm{ord}_b(c_0) \geq -2$.

(iii) We call L a *Fuchsian differential operator* if all of its singularities are regular.

(iv) The *solution sheaf* of L, denoted by $\mathrm{Sol}(L)$, is the local system on S whose fibre at $b \in S$ (with local parameter λ) is the two-dimensional $\mathbb{C}$-vector subspace of $\mathbb{C}[\![\lambda]\!]$ consisting of the Taylor series solutions in of $L = 0$ around b.

Fix $b_0 \in S$ and let $\gamma \in \pi_1(S, b_0)$. By analytical continuation of solutions along the loop γ an isomorphism $\mathrm{Sol}(L)_{b_0} \xrightarrow{\sim} \mathrm{Sol}(L)_{b_0}$ of the stalk at b_0 is induced. Let $M_\gamma \in \mathrm{GL}_2(\mathbb{C})$ be the corresponding matrix representation with respect to a fixed basis for $\mathrm{Sol}(L)_b$. Then

$$\pi_1(S, b_0) \to \mathrm{GL}_2(\mathbb{C}), \qquad \gamma \mapsto M_\gamma. \tag{A.0.4}$$

is a representation of $\pi(S, b_0)$. All choices of a basis for $\mathrm{Sol}(L)_{b_0}$ provide conjugate representations, i.e. the map (A.0.4) and its image are defined up to conjugation in $\mathrm{GL}_2(\mathbb{C})$.

Definition A.0.10. (i) The representation (A.0.4) is called *monodromy representation* of L.

(ii) The image of (A.0.4) is called *monodromy group* of L.

Definition A.0.11. Let $S' = \{b_1, \ldots, b_n\}$ be the set of singularities of L and fix a presentation

$$\pi_1(S, b) = \langle \gamma_1, \ldots, \gamma_n \,; \quad \gamma_1 \cdots \gamma_n = 1 \rangle,$$

where γ_i is represented by a simple closed loop winding around the 'missing' point b_i such that the only singularity of L contained inside of γ is b_i.

(i) The matrix M_{γ_i} is called *local monodromy* of L in b_i.

(ii) We say that b_i is an *elliptic singularity* if

$$M_{\gamma_i} \sim \begin{pmatrix} e^{2\pi \mathrm{i}\alpha} & 0 \\ 0 & e^{2\pi \mathrm{i}\beta} \end{pmatrix}.$$

(iii) We say that b_i is an *logarithmic singularity* if

$$M_{\gamma_i} \sim \begin{pmatrix} e^{2\pi \mathrm{i}\alpha} & 1 \\ 0 & e^{2\pi \mathrm{i}\alpha} \end{pmatrix}.$$

(iv) Let λ denote a local parameter at $b \in S'$ and write $c_0 = \sum_{k=-2}^{\infty} c_k^0 \cdot \lambda^k$ and $c_1 = \sum_{k=-1}^{\infty} c_k^1 \cdot \lambda^k$. The two complex solutions of the equation

$$\lambda(\lambda - 1) + \lambda \cdot c_{-1}^1 + c_{-2}^0 = 0$$

are called *local exponents* of L in b.

(v) The table

$$\left[\begin{array}{ccc} b_1 & \ldots & b_n \\ \hline e_1^{b_1} & \ldots & e_1^{b_n} \\ e_2^{b_1} & \ldots & e_2^{b_n} \end{array}\right]$$

of local exponents $e_1^{b_i}, e_2^{b_i}$ corresponding to the singularities $b_i \in S'$ is called *Riemann scheme*.

Remark A.0.12. (i) Let $b \in S'$ and denote by e_1, e_2 the local exponents of L in b. Then the (possibly equal) complex numbers $\exp(2\pi\mathrm{i} \cdot e_1)$ and $\exp(2\pi\mathrm{i} \cdot e_2)$ are the eigenvalues of the local monodromy M_γ in b.

(ii) Note that in Definition A.0.11 (ii) we allow that the local exponents are integers, i.e. M_γ is the identity matrix. In Chapter 3, Definition 3.1.3 we introduce the notion 'local exponent' for flat vector bundles, which is not completely consistent with the definition in this chapter. If the local exponents of a flat vector bundle at a point with toric monodromy both equal 0, then b is a regular point. However, if the local exponents of a Fuchsian differential equation at an elliptic singularity both equal 0, then b is a so called apparent singularity. For more details we refer to [**Yos87**, Section 3.4].

Definition A.0.13. Let L_1 and L_2 be two Fuchsian differential operators. The operators L_1 and L_2 are said to be *projectively equivalent* if there exists $f \in \mathbb{C}[s]$

such that for all $b_0 \in S$ it holds $\mathrm{ord}_{b_0}(f) = 0$ and $\mathrm{Sol}(L_2)_{b_0}$ consists of the elements of $\mathrm{Sol}(L_1)_{b_0}$ multiplied by the Taylor series expansion of f at b_0.

Example A.0.14. The *hypergeometric differential operator* on $\overline{S} := \mathbb{P}^1_{\mathbb{C}}$ is the Fuchsian differential operator

$$L = \left(\tfrac{\partial}{\partial s}\right)^2 + \frac{((A+B+1)\,s - C)}{s(s-1)}\left(\tfrac{\partial}{\partial s}\right) + \frac{AB}{s(s-1)}, \qquad A, B, C \in \mathbb{C} \tag{A.0.5}$$

with exactly three regular singularities in $S' = \{0, 1, \infty\}$ and corresponding Riemann scheme

$$\left[\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & A \\ 1-C & C-A-B & B \end{array}\right].$$

Bibliography

[BDIP02] José Bertin, Jean-Pierre Demailly, Luc Illusie, and Chris Peters. *Introduction to Hodge theory*, volume 8 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2002. Translated from the 1996 French original by James Lewis and Chris Peters.

[Beu07] Frits Beukers. Gauss' hypergeometric function. In *Arithmetic and geometry around hypergeometric functions*, volume 260 of *Progr. Math.*, pages 23–42. Birkhäuser, Basel, 2007.

[BM10a] Irene I. Bouw and Martin Möller. Differential equations associated with nonarithmetic Fuchsian groups. *J. Lond. Math. Soc. (2)*, 81(1):65–90, 2010.

[BM10b] Irene I. Bouw and Martin Möller. Teichmüller curves, triangle groups, and Lyapunov exponents. *Ann. of Math. (2)*, 172(1):139–185, 2010.

[Bou01] Irene I. Bouw. The p-rank of ramified covers of curves. *Compositio Math.*, 126(3):295–322, 2001.

[Bou04] Irene I. Bouw. Reduction of the Hurwitz space of metacyclic covers. *Duke Math. J.*, 121(1):75–111, 2004.

[Bou05] Irene I. Bouw. *Pseudo-elliptic Bundles, Deformation Data, and the Reduction of Galois Covers.* Habilitation thesis. IEM Universität Duisburg-Essen, 2005.

[BP02] Joseé Bertin and Chris Peters. Variation of Hodge structure, manifolds of Calabi–Yau, and mirror symmetry. In *Introduction to Hodge Theory*, Panoramas et synthèses. Amer. Math. Soc., 2002.

[BW06] Irene I. Bouw and Stefan Wewers. Indigenous bundles with nilpotent p-curvature. *Int. Math. Res. Not.*, pages 1–37, 2006.

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[Ell01] Jordan S. Ellenberg. Endomorphism algebras of Jacobians. *Adv. Math.*, 162(2):243–271, 2001.

[FB82] Michael Fried and R. Biggers. Moduli spaces of covers and the Hurwitz monodromy group. *J. Reine Angew. Math.*, 335:87–121, 1982.

[FV91] Michael D. Fried and Helmut Völklein. The inverse Galois problem and rational points on moduli spaces. *Math. Ann.*, 290(4):771–800, 1991.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[HM82] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton.

[Hon81] Taira Honda. Algebraic differential equations. In *Symposia Mathematica, Vol. XXIV (Sympos., INDAM, Rome, 1979)*, pages 169–204. Academic Press, London-New York, 1981.

[HS06] Pascal Hubert and Thomas A. Schmidt. An introduction to Veech surfaces. In *Handbook of dynamical systems. Vol. 1B*, pages 501–526. Elsevier B. V., Amsterdam, 2006.

[Kat70] Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, (39):175–232, 1970.

[Kat81] Nicholas M. Katz. Crystalline cohomology, Dieudonné modules, and Jacobi sums. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 165–246. Tata Inst. Fundamental Res., Bombay, 1981.

[Ked08] Kiran S. Kedlaya. p-adic cohomology: from theory to practice. In *p-adic geometry*, volume 45 of *Univ. Lecture Ser.*, pages 175–203. Amer. Math. Soc., Providence, RI, 2008.

[KO68] Nicholas M. Katz and Tadao Oda. On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.*, 8:199–213, 1968.

[McM03] Curtis T. McMullen. Billiards and Teichmüller curves on Hilbert modular surfaces. *J. Amer. Math. Soc.*, 16(4):857–885 (electronic), 2003.

[Möl06a] Martin Möller. Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmüller curve. *Invent. Math.*, 165(3):633–649, 2006.

[Möl06b] Martin Möller. Variations of Hodge structures of a Teichmüller curve. *J. Amer. Math. Soc.*, 19(2):327–344, 2006.

[Möl11] Martin Möller. Shimura and Teichmüller curves. *J. Mod. Dyn.*, 5(1):1–32, 2011.

[Möl13] Martin Möller. Teichmüller curves, mainly from the viewpoint of algebraic geometry. In *Moduli spaces of Riemann surfaces*, volume 20 of *IAS/Park City Math. Ser.*, pages 267–318. Amer. Math. Soc., Providence, RI, 2013.

[MT02] Howard Masur and Serge Tabachnikov. Rational billiards and flat structures. In *Handbook of dynamical systems, Vol. 1A*, pages 1015–1089. North-Holland, Amsterdam, 2002.

[Pet86] Chris Peters. Monodromy and Picard-Fuchs equations for families of K3-surfaces and elliptic curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 19(4):583–607, 1986.

[RW06] Matthieu Romagny and Stefan Wewers. Hurwitz spaces. In *Groupes de Galois arithmétiques et différentiels*, volume 13 of *Sémin. Congr.*, pages 313–341. Soc. Math. France, Paris, 2006.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[Sza09] Tamás Szamuely. *Galois groups and fundamental groups*, volume 117 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2009.

[Vee89] William A. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. Math.*, 97(3):553–583, 1989.

[Völ93] Helmut Völklein. Braid group action via $\mathrm{GL}_n(q)$ and $\mathrm{U}_n(q)$, and Galois realizations. *Israel J. Math.*, 82(1-3):405–427, 1993.

[Völ96] Helmut Völklein. *Groups as Galois groups*, volume 53 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1996. An introduction.

[War98] Clayton C. Ward. Calculation of Fuchsian groups associated to billiards in a rational triangle. *Ergodic Theory Dynam. Systems*, 18(4):1019–1042, 1998.

[Wew98] Stefan Wewers. *Construction of Hurwitz spaces.* PhD thesis. Preprint no. 21. IEM Essen University, 1998.

[Wew99] Stefan Wewers. Deformation of tame admissible covers of curves. In *Aspects of Galois theory (Gainesville, FL, 1996)*, volume 256 of *London Math. Soc. Lecture Note Ser.*, pages 239–282. Cambridge Univ. Press, Cambridge, 1999.

[Yos87] Masaaki Yoshida. *Fuchsian differential equations.* Aspects of Mathematics, E11. Friedr. Vieweg & Sohn, Braunschweig, 1987. With special emphasis on the Gauss-Schwarz theory.

Zusammenfassung

Teichmüller-Kurven sind Geodäten im Modulraum $\mathscr{M}_g$ der kompakten Riemannschen Flächen vom Geschlecht g. In dieser Arbeit beschäftigen wir uns mit einer Konstruktionsmethode für Teichmüller-Kurven, welche eine Charakterisierung von Möller [**Möl06b**] benutzt. Hierbei wird eine eindimensionale Familie von kompakten Riemannschen Flächen konstruiert, deren Fasern zu Punkten einer Teichmüller-Kurven in $\mathscr{M}_g$ korrespondieren. Eine solche Familie liegt vor, wenn ihre erste relative de Rham Kohomologie ein indigenes Rang-2-Unterbündel enthält, dessen Markierungen allesamt logarithmische Singularitäten sind [**Möl06b**, Theorem 5.3] .

Mit diesem Ansatz wird in der vorliegenden Arbeit eine neue Klasse von Teichmüller-Kurven konstruiert. Diese reiht sich ein in eine größere Klasse von Teichmüller-Kurven von Bouw und Möller [**BM10b**]. Wir zeigen, dass dies insgesamt eine vollständige Klassifikation aller Teichmüller-Kurven liefert, die man mittels einer gewissen Quotientenkonstruktion und mittels indigener Unterbündel der relativen de Rham Kohomologie von superelliptischen Kurven konstruieren kann (vorausgesetzt man interessiert sich nur für sogenannte primitive Teichmüller-Kurven).

Für weitere Kandidaten für Familien, die Teichmüller-Kurven liefern, gibt es nicht viele Ansatzpunkte. Ein möglicher Ansatzpunkt ist, dass die Fasern solcher Familien Kurven sind, die reelle Multiplikation mit großen total reellen Zahlkörpern besitzen. In [**Ell01**] werden von Ellenberg eindimensionale Familien mit dieser Eigenschaft konstruiert; und zwar nach dem gleichen Muster nach dem auch die oben genannten Familien konstruiert werden. Daher ist die Frage, ob diese Familien Teichmüller-Kurven liefern, naheliegend. In dieser Arbeit zeigen wir jedoch, dass dies – außer in ein paar wenigen Ausnahmefällen – nicht der Fall ist. Dazu werden die Ellenbergschen Familien als Familien über geeigneten Hurwitz-Räumen interpretiert. Hurwitz-Räume sind Modulräume für Galois-Überlagerungen der Riemann Sphäre. Wir zeigen wie man durch Untersuchen des Randes des entsprechenden Hurwitz-Raums ausschließen kann, dass eine Familie eine Teichmüller-Kurve liefert und wenden dieses Ausschlusskriterium auf die Ellenbergschen Familien an.

Des Weiteren konstruieren wir eine modifizierte Variante der Ellenbergschen Familien und zeigen, dass in diesem Fall das Ausschlusskriterium auf dem Rand des entsprechenden Hurwitz-Raums nicht erfüllt ist. Ob diese modifizierten Familien tatsächlich Teichmüller-Kurven liefern, kann jedoch nicht allein anhand des Hurwitz-Raums – also mit den Methoden, die in dieser Arbeit entwickelt werden – entschieden werden.

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