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Exponentialsummen mit der Möbiusfunktion

Dissertation

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Notation

The symbolism of some signs in mathematics varies depending on the used source and context. To avoid misunderstandings, here some names are listed, which are used in this thesis and are provided in the following as known.

$\mathbb{N}$	is the set of all natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	is the set of all natural numbers and zero: $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$
$\mathbb{P}$	is the set of all primes
$\mathbb{R}^+$	is the set of all positive real numbers
$\mathbb{R}_0^+$	is the set of all nonnegative real numbers
$e(z)$	$= e^{2\pi iz}$ for $z \in \mathbb{C}$
$\Re(z)$	is the real part of a complex number $z \in \mathbb{C}$
$\Im(z)$	is the imaginary part of a complex number $z \in \mathbb{C}$
$f(x) = O(g(x))$	means $\|f(x)\| \leq Cg(x)$ for $x \geq x_0$ and a certain $C > 0$. Here $f(x)$ is a complex-valued function in $x \in \mathbb{R}$ and $g(x)$ a real- valued function with only positive values for $x \geq x_0$.
$f(x) \ll g(x)$	means the same like $f(x) = O(g(x))$
$f(x) = o(g(x))$	means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
$[x]$	is for $x \in \mathbb{R}$ is the largest integer not greater than x.
$\|\|x\|\|$	means $\min_{k\in\mathbb{Z}} \|x - k\|$.
(n, m)	is the greatest common divisor of two natural numbers $n, m \in \mathbb{N}$
$[n, m]$	is the least common multiple of two natural numbers $n, m \in \mathbb{N}$
mod q	means modulo q for $q \in \mathbb{N}$
$\mathbb{C}^\times$	is the multiplicative group $\mathbb{C}\backslash\{0\}$
$(\mathbb{Z}/q\mathbb{Z})^*$	is the multiplicative group of all residue classes, which are coprime to $q \in \mathbb{N}$
$\overline{f}(z)$	$= \overline{f(z)}$. Here $f(z)$ is a complex- valued function, defined on an arbitrary set
$\mathrm{Res}_{x_0}(f)$	is the residue of f at x_0. Here f is a complex- valued function, defined on a subset $D \subset \mathbb{C}$ with $x_0 \in D$.
$\sum_n$, $\sum_p$	Sometimes the index set of the sum is omitted for reasons of space. In this case it always holds, that n runs over all natural numbers and p runs over all primes.
$\log z$	If nothing else is mentioned, $\log z$ means the main branch of the logarithm.

List of abbreviations

resp.	respectively
f.	following
RH	Riemann Hypothesis
GRH	Generalized Riemann Hypothesis
WLOG	Without loss of generality

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Chapter 1

Introduction

1.1 Motivation

There is a great interest in analytic number theory for exponential sums, whose coefficients are formed by certain number- theoretic functions. From the behavior of such sums one can draw conclusions on the distribution of prime numbers. Similar to the prime number theorem, such results on the distribution of prime numbers can be improved by the adoption of the generalized Riemann hypothesis (GRH).

In this work, the coefficients are represented by the Möbius- function. The Möbius- function is a number- theoretic function $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$, which is defined as follows:

$$\mu(n) = \begin{cases} (-1)^k & \text{for } n = p_1 \cdots p_k \text{ with } p_1, \ldots, p_k \text{ pairwise distinct} \\ 0 & \text{otherwise.} \end{cases}$$

Under certain conditions, the behavior of this sum allows conclusions on the estimation of the sum

$$S(x, \alpha) := \sum_{n \leq x} \mu(n) e(\alpha n)$$

with a real number α. This sum was introduced by Davenport, who found the estimate

$$\max_{\theta} |S(x, \alpha)| < \frac{C_1(\lambda) x}{(\log x)^{\lambda}}$$

for arbitrary $\lambda > 0$ and an arbitrary constant $C_1(\lambda) > 0$.

Hajela and Smith assumed that the Dirichlet L- series

$$L(s, \chi) = \sum_{m=1}^{\infty} \chi(m) m^s$$

have no real zeros, so- called Siegel- zeros. Under this condition they proved

$$\max_{\theta} |S(x, \theta)| < C_2 x \exp\left(-C(\log x)^{1/2}\right)$$

for $C, C_2 > 0$. Under the stronger condition

$$L(s, \chi) \neq 0 \tag{1.1}$$

for all $s \in \mathbb{C}$ with $\Re(s) > a$ for a certain $a \in [1/2, 1)$ Hajela und Smith showed the estimation

$$\max_{\theta} |S(x, \theta)| < C_3(\epsilon) x^{\frac{a+2}{3+\epsilon}}$$

for every $\epsilon > 0$.

In their joint- work "Exponential Sums formed with the Möbius Function"[1] R.C. Baker und G. Harman improved this result. Under the same premises they received for each $\epsilon > 0$ the estimate

$$\max_{\theta} |S(x,\theta)| \ll x^{b+\epsilon},$$

while $b := b(a)$ is given by

$$b := \begin{cases} a + 1/4 & \text{for} \quad 1/2 \le a < 11/20, \\ 4/5 & \text{for} \quad 11/20 \le a < 3/5, \\ (a+1)/2 & \text{for} \quad 3/5 \le a < 1. \end{cases}$$

Assuming the generalized Riemann hypothesis, also (1.1) with $a = 1/2$, they reduced the exponent of $5/6 + \epsilon$ to $3/4 + \epsilon$. The improvement is obtained by using an estimation for Gaussian sums, the mean value theorem for Dirichlet- polynomials and two lemmata of van der Corput for estimation of exponential sums.

In the paper "On an exponential sum involving the Möbius function"'[2] it is proved, that for alle α of type 1 numbers, there holds

$$S(s,\alpha) \ll x^{3/4+\epsilon}$$

for any $\epsilon > 0$ under the assumption of the average zero- density hypothesis

$$\sum_{\chi} N(\sigma, T, \chi) \ll (qT)^{2(1-\sigma)}(\log(qT))^{A}.$$

In the diploma- thesis of Claudia Fischer "Exponentialsummen, gebildet mit der Möbiusfunktion"[3] from December 2005, she took over the procedure of Baker and Harman and applied it to the calculation of sums of the form

$$S(y,\chi,\gamma) = \sum_{n \le y} \mu(n)\chi(n)e(\gamma n)$$

with a character χ mod q and $\gamma \in \mathbb{R}$. For this purpose, it is sufficient to consider sums of the form

$$\sum_{m=1}^{y-1} S(m,\chi,0)$$

and calculate them with Perron's formula. Thus, we obtain integrals of the form

$$\int_{\mathcal{C}} \frac{\sum_{m=1}^{y-1} e(\gamma m) m^s}{s \cdot L(s,\chi)} \, ds$$

with a curve that passes the critical line $\Re(s) = 1/2$ on the right, and the imaginary parts of the end points have $-T$ and T.

In order to obtain an improvement over the result of Baker and Harman, Claudia Fischer combines their technology with the procedure, used by H. Maier and H.L. Montgomery in their work "The Sum of the Möbius Function"[4], for estimating the function

$$M(x) := \sum_{n \le x} \mu(n)$$

under the Riemann hypothesis.

[1] See Baker and Harman (1)
[2] See Maier and Sankaranarayanan (15)
[3] See Fischer (9)
[4] See Maier and Montgomery (4)

The calculation can be reduced to a similar integral

$$\int_{\mathcal{C}} \frac{x^s}{s \cdot \zeta(s)}\, ds.$$

The estimate of $M(x)$ is improved by choosing the curve $\mathcal{C}$ not as a straight line, but as a piecewise linear contour, the course of which is dependent on the magnitude of the logarithm as well as the logarithmic derivative of the Riemann- ζ- function near the critical line. The distance of the contour to the critical line is chosen such that the bound that can be obtained for the integrand is approximately minimized. In the work of Claudia Fischer, the integral is estimated by obtaining an upper bound on the number of segments that have a large minimum distance from the critical line with the help of some lemmata that estimate the number of points under certain conditions and with an average value of the imaginary parts.

In my diploma- thesis of December 2007, the main focus does not lie only on an averaging over the imaginary parts, but also over the Dirichlet- characters, using the Large Sieve and the Hybrid Sieve. I considered paths $\mathcal{C}$, which are also on the right of the line $\Re(s) = 1/2$, which have to be constructed for every Dirichlet- character specifically. In order to achieve the same result, the paths must be formed on the critical line as a function of the value of $L(s, \chi)^{-1}$, however, only the paths are considered with the start and end points -1 and 1.

In the work at hand, we consider a version of the original problem by limiting the summation to integers n without large prime factors.
We treat only one aspect of this problem, namely the estimation of sums

$$\sum_{\substack{n \le x \\ p^+(n) \le x^u}} \mu(n)\chi(n),$$

where $u \in [0, 1)$ and $p^+(n)$ denotes the largest prime factor of n, averaged over all primitive characters χ mod q with $Q \le q \le 2Q$ for $q \in \mathbb{N}$.
We will also use the above tools.

1.2 Short overview of the thesis:

Chapter 2 contains the basic concepts which form the basis of this thesis. In section 2.1 we start with Abel's partial summation and in section 2.2 we consider special analytic functions. We introduce Dirichlet- series in section 2.3, characters in section 2.4 and combine these in section 2.5 to Dirichlet- L- series.
Mean value theorems and sieve methods like the Large Sieve and the Hybrid Sieve will be discussed in chapter 3.
In chapter 4 we introduce important tools for the estimation, one the one hand some monotonicity principles on horizontal lines which are parallel to the real axis in section 4.1, and on the other hand in section 4.2 estimates for the number of times, a certain function is large, using techniques of Selberg.
In chapter 5 we show the results and they will be proved as well.

Chapter 2

Basics

In this chapter, some important definitions, theorems and lemmata are given which are needed later.

2.1 Partial summation

Theorem 2.1.1. *(Abel's partial summation)*[1]
Let $a, b \in \mathbb{R}$ and $f\colon [a,b] \to \mathbb{C}$ be a continuously differentiable function. Next let $c_1, c_2, \dots$ be a sequence of complex numbers. Then it holds

$$\sum_{a<n\leq b} c_n f(n) = -\int_a^b \left(\sum_{a<n\leq t} c_n\right) f'(t)\,dt + \left(\sum_{a<n\leq b} c_n\right) f(b)$$

Proof. We have

$$\begin{aligned}\left(\sum_{a<n\leq b} c_n\right) f(b) - \sum_{a<n\leq b} c_n f(n) &= \sum_{a<n\leq b} c_n(f(b)-f(n)) = \sum_{a<n\leq b} \int_n^a c_n f'(t)\,dt \\ &= \int_a^b \left(\sum_{a<n\leq t} c_n\right) f'(t)\,dt.\end{aligned}$$

□

2.2 Analytic functions of finite order

Definition 2.2.1. (Analytic functions)
A complex- valued function $f\colon \mathbb{C} \to \mathbb{C}$ is called an analytic function, if f is holomorphic over the whole complex plane.

Definition 2.2.2. (Order of an analytic function)
Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function. If an $\alpha > 0$ exists such that

$$M_f(r) := \max\{|f(z)| : |z| \leq r\} \leq \exp(r^\alpha)$$

is true for large r, then f is called by finite order.

[1] See Lütkebohmert (3), Lemma 7.4.10

If there is no $\alpha > 0$, which satisfies this condition, then f is called of infinite order. The number

$$\alpha(f) := \inf\{\alpha > 0 : M_f(r) \leq \exp(r^\alpha)\}$$

is called the order of the analytic function f. This definition is useful because $M_f(r)$ is monotonically increasing as a function in r.

Definition 2.2.3. (Convergence exponent)
Let $s_1, s_2, \ldots \in \mathbb{C}\backslash\{0\}$ be a sequence in $\mathbb{C}$, and we have

$$0 < |s_1| \leq |s_2| \leq \ldots.$$

Under the convergence exponent of the sequence $s_1, s_2, \ldots$ one understands the lower limit of β of that $b > 0$, for which:

$$\sum_n |s_n|^{-b} < \infty \tag{2.1}$$

holds. If the sequence is finite, then we set $\beta = 0$. If there is no $b > 0$, so that (2.1) is satisfied, then set $\beta = \infty$.

Theorem 2.2.1. [2]
Let

$$e(z, k) := (1 - z) \exp\left(\sum_{n=1}^{k} \frac{z^n}{n}\right).$$

Every analytic function $f(z)$ with finite order α has a representation of the form

$$f(z) = e^{h(z)} z^m \prod_n e\left(\frac{z}{z_n}, p\right),$$

while the product runs over all zeros z_n of f with $z_n \neq 0$, which are ordered by the size of their absolute value. There exist natural numbers $\overline{p} \in \mathbb{N}_0$ with $\sum_n |z_n|^{-\overline{p}-1} < \infty$. Let p be the smallest of these numbers. Let m be the multiplicity of the zero point $z = 0$ and $h(z)$ be a polynomial of degree $g \leq \alpha$. If β is the convergence exponent of the sequence $z_1, z_2, \ldots$, we have $\alpha = \max\{g, \beta\}$. If there is no fixed $c > 0$ and sufficiently large r, such that

$$|f(s)| < \exp(cr^\alpha)$$

holds for all s with $|s| \leq r$, there is $\alpha = \beta$, and $\sum |z_n|^{-\beta}$ is divergent for $\alpha > 0$.

Proof. The proof is already found in Lütkebohmert (3) on page 111 f. □

Theorem 2.2.2. [3]
Let $0 < r < R$ and $f(s)$ holomorphic in $|s - s_0| \leq R$, while we choose $s_0 \in \mathbb{C}$ such that $f(s_0) \neq 0$. Let $s_1, \ldots s_m$ with $m \in \mathbb{N}$ be any zeros of $f(s)$ in $|s - s_0| \leq r$. Then we have

$$\left(\frac{R}{r}\right)^m \leq \frac{M(R)}{|f(s_0)|},$$

where $M(R) = \max_{|s-s_0|=R} |f(s)|$. This, especially applies for $r = R/2$ with $A := 1/\log 2$

$$m \leq A \log\left(\frac{M(R)}{|f(s_0)|}\right).$$

[2]See Prachar (6), appendix, Theorem 5.8
[3]See Prachar (6), appendix, Theorem 5.2

Proof. WLOG let $s_0 = 0$. Otherwise consider $\widetilde{f}(s) = f(s - s_0)$. The function

$$F(s) = f(s) \left(\prod_{n=1}^{m} \frac{R(s - s_n)}{R^2 - s\overline{s_n}} \right)^{-1}$$

is holomorphic in $|s| \leq R$, and it holds $|F(s)| = |f(s)|$ for $|s| = R$ since

$$\left| \frac{R(s - s_n)}{R^2 - s\overline{s_n}} \right| = \left| \frac{R(s\overline{s} - s_n\overline{s})}{\overline{s}(R^2 - s\overline{s_n})} \right| = \left| \frac{R(R^2 - s_n\overline{s})}{\overline{s}(R^2 - s\overline{s_n})} \right| = 1$$

holds for all $n = 1, \ldots, m$. So by the maximum principle it follows

$$|F(0)| = |f(0)| \prod_{n=1}^{m} \frac{R}{|s_n|} \leq \max_{|s|=R} |F(s)| = M(R).$$

Now, the first part of the assertion follows because $|s_n| \leq r$ for $n = 1, \ldots, m$, also

$$|f(0)| \left(\frac{R}{r} \right)^m \leq |f(0)| \prod_{n=1}^{m} \frac{R}{|s_n|} \leq M(R).$$

Setting $r = R/2$, the second part follows immediately. □

2.3 Dirichlet- series

Definition 2.3.1. (Dirichlet- series)
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We call

$$\sum_{n=1}^{\infty} a_n n^{-s}$$

a Dirichlet- series with $s \in \mathbb{C}$ and $n^{-s} := \exp(-s \log(n))$.
Then we call

$$\sigma_c := \inf_{\sigma\in\mathbb{R}} \left\{ \sum_{n=1}^{\infty} a_n n^{-\sigma} \text{ converges} \right\} \in [-\infty, \infty]$$

the abscissa of convergence of the Dirichlet- series $\sum_{n\in\mathbb{N}} a_n n^{-s}$ and

$$\sigma_a := \inf_{\sigma\in\mathbb{R}} \left\{ \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \text{ converges} \right\} \in [-\infty, \infty]$$

the absolute abscissa of convergence.

Lemma 2.3.1. [4]
Let $\sum_{n\in\mathbb{N}} a_n n^{-s}$ be a Dirichlet- series with coefficients $a_n \in \mathbb{C}$ und let σ_c resp. σ_a be the abscissa of convergence resp. the absolute abscissa of convergence of this Dirichlet- series. Then we have:

1. *The Dirichlet- series $\sum_{n\in\mathbb{N}} a_n n^{-s}$ converges locally uniform for $\Re(s) > \sigma_c$.*
2. *The Dirichlet- series $\sum_{n\in\mathbb{N}} a_n n^{-s}$ diverges for $\Re(s) < \sigma_c$.*
3. *The Dirichlet- series $\sum_{n\in\mathbb{N}} a_n n^{-s}$ converges absolutely for $\Re(s) > \sigma_a$.*
4. *The Dirichlet- series $\sum_{n\in\mathbb{N}} a_n n^{-s}$ does not converge absolutely for $\Re(s) < \sigma_a$.*
5. *It holds $\sigma_c \leq \sigma_a \leq \sigma_c + 1$.*
6. *The function*

$$f(s) := \sum_{n=1}^{\infty} a_n n^{-s}.$$

is holomorphic for $\Re(s) > \sigma_c$, and we have

$$f'(s) := -\sum_{n=1}^{\infty} a_n (\log n) n^{-s},$$

while this Dirichlet- series has abscissa of convergence σ_c.

Proof. 1. With Abel's partial summation (Theorem 2.1.1) we get for $s_0, s \in \mathbb{C}$ and $1 \leq m \leq n$

$$\begin{aligned}\sum_{\nu=m}^{n} a_\nu \nu^{-s} &= \sum_{\nu=m}^{n} (a_\nu \nu^{-s_0}) \nu^{s_0-s} = \left(\sum_{\nu=m}^{n} a_\nu \nu^{-s_0}\right) n^{s_0-s} - \int_m^n \left(\sum_{\nu=m}^{t} a_\nu \nu^{-s_0}\right) (t^{s_0-s})'\, dt \\ &= \left(\sum_{\nu=m}^{n} a_\nu \nu^{-s_0}\right) n^{s_0-s} + \sum_{\nu=m}^{n-1} \left(\sum_{\mu=m}^{\nu} a_\mu \mu^{-s_0}\right) (\nu^{s_0-s} - (\nu+1)^{s_0-s}). \qquad (2.2)\end{aligned}$$

Furthermore,

$$|\nu^z - (\nu+1)^z| = \left| \int_\nu^{\nu+1} z t^{z-1}\, dt \right| \leq |z| \max_{t\in[\nu,\nu+1]} t^{\Re(z)-1} \leq |z| D(z) \nu^{\Re(z)-1}$$

applies for $z \in \mathbb{C}$ und $\nu \in \mathbb{N}$, if it holds

$$D(z) := \begin{cases} 1 & \text{for } \Re(z) \leq 1, \\ 2^{\Re(z)-1} & \text{for } \Re(z) > 1. \end{cases}$$

Let

$$M_\nu := \left| \sum_{\mu=m}^{\nu} a_\mu \mu^{-s_0} \right|.$$

Then, with (2.2)

$$\left| \sum_{\nu=m}^{n} a_\nu \nu^{-s} \right| \leq M_n n^{\Re(s_0-s)} + |s_0 - s| D(s_0 - s) \sum_{\nu=m}^{n-1} M_\nu \nu^{\Re(s_0-s)-1}. \qquad (2.3)$$

[4] See Lütkebohmert (3), exercises 25 and 26, page 198 f.

A series converges locally uniform in a domain if and only if it converges compactly there. Let $\emptyset \neq K \subseteq \{s \in \mathbb{C}\colon \Re(s) > \sigma_c\}$ be compact and $\sigma := \min\{\Re(s)\colon s \in K\} > \sigma_c$. Then there exists $s_0 = \sigma_0 \in \mathbb{R}$ with $\sigma > \sigma_0 > \sigma_c$, so that the series $\sum_{n\in\mathbb{N}} a_n n^{-s_0}$ converges. Now we show that the series $\sum_{n\in\mathbb{N}} a_n n^{-s}$ converges uniformly on K. Let $C > 0$, such that

$$\left|\sum_{\nu=1}^{n} a_\nu \nu^{-s_0}\right| \leq C$$

holds for all $n \in \mathbb{N}$. Then we have

$$\left|\sum_{\nu=m}^{n} a_\nu \nu^{-s_0}\right| \leq 2C$$

for all $n, m \in \mathbb{N}$. Let $s \in K$, also $\Re(s) \geq \sigma$. Due to (2.3) we have

$$\begin{aligned}\left|\sum_{\nu=m}^{n} a_\nu \nu^{-s}\right| &\leq 2Cn^{\Re(s_0-s)} + |s_0 - s|2C\sum_{\nu=m}^{n-1}(2\nu)^{\Re(s_0-s)-1} \\ &\leq 2Cn^{\sigma_0-\sigma} + \left(|s_0| + \max_{z\in K}|z|\right)2C\sum_{\nu=m}^{n-1}(2\nu)^{\sigma_0-\sigma-1} \to 0 \qquad (2.4)\end{aligned}$$

for $n, m \to \infty$, independent of $s \in \mathbb{C}$. The statement (2.4) is also valid for all $s_0 \in \mathbb{C}$, for which $\sum_{n\in\mathbb{N}} a_n n^{-s_0}$ converges.

2. We assume that there is a point $s_0 \in \mathbb{C}$ with $\Re(s_0) < \sigma_c$, such that $\sum_{n\in\mathbb{N}} a_n n^{-s_0}$ converges. According to (2.4) the locally uniform convergence of $\sum_{n\in\mathbb{N}} a_n n^{-s}$ follows for $\Re(s) > \Re(s_0)$.

   In particular, the series $\sum_{n\in\mathbb{N}} a_n n^{-\sigma}$ would converge for every $\sigma > \Re(s_0)$, contrary to the definition of σ_c, since $\Re(s_0) < \sigma_c$. So we get the claim.

3. This follows from the first part, applied to $\sum_{n\in\mathbb{N}} |a_n| n^{-s}$.

4. This follows from the second part, applied to $\sum_{n\in\mathbb{N}} |a_n| n^{-s}$.

5. The inequality $\sigma_c \leq \sigma_a$ is obvious. Judging from the assumptions $\sigma_a > \sigma_c + 1$, there exists $\sigma \in \mathbb{R}$ with $\sigma_a > \sigma > \sigma_c + 1$, that means $\sigma - 1 > \sigma_c$. Therefore the series $\sum_{n\in\mathbb{N}} a_n n^{-(\sigma-1)}$ converges. This means that there exists $C > 0$, so that $|a_n| n^{-(\sigma-1)} \leq C$ holds for all $n \in \mathbb{N}$. It follows

$$\sum_{n=1}^{\infty} |a_n| n^{-(\sigma+\epsilon)} = \sum_{n=1}^{\infty} |a_n| n^{-(\sigma-1)} n^{(-1-\epsilon)} \leq C\sum_{n=1}^{\infty} n^{-1-\epsilon} < \infty$$

   for all $\epsilon > 0$. So we get $\sigma_a \leq \sigma + \epsilon$ for all $\epsilon > 0$. In contradiction to the above assumption we have $\sigma_a \leq \sigma$. The claim follows.

6. The holomorphy of f follows from the first part, since we may interchange summation and differentiation because of the uniform convergence and $\frac{d}{ds}(n^{-s}) = -(\log n)n^{-s}$ holds. Let σ_c be the abscissa of convergence of $\sum_{n\in\mathbb{N}} a_n n^{-s}$ and $\widehat{\sigma_c}$ the abscissa of convergence of $\sum_{n\in\mathbb{N}} a_n(\log n)n^{-s}$. Because of the first part, $\widehat{\sigma_c} \leq \sigma_c$ is obvious. Nevertheless, it still remains to show that $\widehat{\sigma_c} \geq \sigma_c$. We assume $\widehat{\sigma_c} < \sigma_c$. Let $\sigma \in \mathbb{R}$ with $\widehat{\sigma_c} < \sigma < \sigma_c$. According to the first part, the series $\sum_{n\in\mathbb{N}} a_n(\log n)n^{-s}$ converges locally uniform for $\Re(s) > \sigma$. It follows the convergence of

$$\int_{\sigma}^{\sigma+1}\left(-\sum_{n=1}^{\infty} a_n(\log n)n^{-t}\right)dt = \sum_{n=1}^{\infty} a_n\int_{\sigma}^{\sigma+1}(-\log n)n^{-t}\,dt = \sum_{n=1}^{\infty} a_n(n^{-(\sigma+1)} - n^{-\sigma}). \qquad (2.5)$$

According to the fifth, part the series $\sum_{n\in\mathbb{N}} a_n(\log n)n^{-(\sigma+1)}$ converges absolutely. Since it holds

$$\infty > \sum_{n=3}^{\infty} |a_n|(\log n)n^{-(\sigma+1)} \geq \sum_{n=3}^{\infty} |a_n|n^{-(\sigma+1)}$$

the series $\sum_{n\in\mathbb{N}} a_n n^{-(\sigma+1)}$ converges also. With (2.5) we get the convergence of $\sum_{n\in\mathbb{N}} a_n n^{-\sigma}$, in contradiction to the definition of σ_c. This suffices.

□

Theorem 2.3.1. *(Multiplication of Dirichlet- series)*[5]
Let

$$\sum_{n\in\mathbb{N}} a_n n^{-s} \quad \text{and} \quad \sum_{n\in\mathbb{N}} b_n n^{-s}$$

be two Dirichlet- series with $a_n, b_n \in \mathbb{C}$.
If $\sum_{n\in\mathbb{N}} |a_n|n^{-\sigma}$ and $\sum_{n\in\mathbb{N}} |b_n|n^{-\sigma}$ converge for $\sigma_0 < \sigma < \infty$, then it holds for $\sigma_0 < \Re(s) < \infty$

$$\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} b_n n^{-s}\right) = \sum_{n=1}^{\infty} c_n n^{-s}$$

with

$$c_n = \sum_{d|n} a_d b_{\frac{n}{d}}$$

for all $n \in \mathbb{N}$, and these series converge absolutely also in $\sigma_0 < \Re(s) < \infty$.

Proof. Since they converge absolutely, we can expand, and it holds

$$\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} b_n n^{-s}\right) = \sum_{m=1}^{\infty}\sum_{d=1}^{\infty} a_d b_m (md)^{-s} = \sum_{n=1}^{\infty}\left(\sum_{d|n} a_d b_{\frac{n}{d}}\right) n^{-s}$$

for $\Re(s) > \sigma_0$. □

Now we explain Riemann's method of complex integration, with which one can estimate Dirichlet-polynomials. It is a consequence of Perron's formula and is often also referred to as well.

Lemma 2.3.2. *(Perron's formula)*[6]
Let $b, T \in \mathbb{R}$ with $b > 1$ and $T > 0$. Then it holds for $a \in \mathbb{R}^+$

$$\left|\frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{a^s}{s}\, ds - \delta(a)\right| \leq \begin{cases} \frac{b}{\pi T} & \text{for} \quad a = 1, \\ \frac{a^b}{\pi T|\log a|} & \text{for} \quad a \neq 1. \end{cases}$$

Notice that

$$\delta(a) = \begin{cases} 1 & \text{for} \quad a > 1, \\ 1/2 & \text{for} \quad a = 1, \\ 0 & \text{for} \quad a < 1, \end{cases}$$

and $\log a$ means the main branch of the logarithm.

[5] See Prachar (6), appendix, Theorem 2.3
[6] See Lütkebohmert (3), Lemma 7.4.1

Proof. We always mean with $\log a$ the main branch of the logarithm. For $a = 1$ we have

$$\begin{aligned}\int_{b-iT}^{b+iT} \frac{1}{s}\, ds &= \log(b+iT) - \log(b-iT) = \log\left(-\frac{1-ib/T}{1+ib/T}\right) \\ &= \log(-1) + \log(1-ib/T) - \log(1+ib/T) = \pi i - 2i\arg(1+ib/T) \\ &= \pi i + R \text{ mit } |R| \leq 2b/T,\end{aligned}$$

since $|1-ib/T| = |1+ib/T|$ and $\arg(1-ib/T) = -\arg(1+ib/T)$.
In case $a > 1$ we integrate over the rectangle C with the sides

$$\begin{aligned} C_b &:= \{b+it : -T \leq t \leq T\} \\ C_{-h} &:= \{-h+it : -T \leq t \leq T\} \\ C_{+iT} &:= \{\sigma + iT : -h \leq \sigma \leq b\} \\ C_{-iT} &:= \{\sigma - iT : -h \leq \sigma \leq b\} \end{aligned}$$

and $h > b$. According to the residue theorem[7] it holds

$$\frac{1}{2\pi i}\int_C \frac{a^s}{s}\, ds = 1.$$

Now we estimate

$$\begin{aligned} \left|\int_{C_{-h}} \frac{a^s}{s}\, ds\right| &\leq \int_{-T}^{T} \frac{a^{-h}}{|-h+it|}\, dt \leq \frac{2Ta^{-h}}{h} \quad \text{and} \\ \left|\int_{C_{\pm iT}} \frac{a^s}{s}\, ds\right| &\leq \int_{-h}^{b} \frac{a^\sigma}{|\sigma \pm iT|}\, d\sigma \leq \int_{-h}^{b} \frac{a^\sigma}{T}\, d\sigma \leq \frac{a^b}{T\log(a)}. \end{aligned}$$

For $h \to \infty$ the first integral vanishes and we get the above estimation.
For $0 < a < 1$ we integrate over the rectangle C with the sides

$$\begin{aligned} C_b &:= \{b+it : -T \leq t \leq T\} \\ C_h &:= \{h+it : -T \leq t \leq T\} \\ C_{+iT} &:= \{\sigma + iT : h \geq \sigma \geq b\} \\ C_{-iT} &:= \{\sigma - iT : h \geq \sigma \geq b\} \end{aligned}$$

and also $h > b$. According to the residue theorem[8] and $b > 1$ we get

$$\frac{1}{2\pi i}\int_C \frac{a^s}{s}\, ds = 0.$$

Now we estimate analogously to the case a >1 and we also receive the claim. □

Theorem 2.3.2. *(Riemann's method of complex integration)*[9]
Let

$$f(s) := \sum_{n=1}^{\infty} a_n n^{-s}$$

be a Dirichlet- series with $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$, which converges for $\Re(s) > 1$ normally. For $\sigma \in \mathbb{R}$ and $\alpha > 0$ it holds

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} = O((\sigma-1)^{-\alpha})\ (\sigma \downarrow 1).$$

There is $|a_n| \leq A(n)$ for a monotonically increasing function A.

[7] See Lütkebohmert (3), Theorem 5.4.4
[8] See Lütkebohmert (3), Theorem 5.4.4
[9] See Lütkebohmert (3), Theorem 7.4.2

Then it holds

$$\phi_f(x) - \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}\,ds = O\left(\frac{x(\log x)^\alpha}{T}\right) + O\left(\frac{xA(2x)(\log x + 1/\langle x\rangle)}{T}\right)$$

for $T \geq 1$ and $x > 1$. For $x > 0$ there is

$$\phi_f(x) = \begin{cases} \sum_{n<x} a_n & \text{for} \quad x \in \mathbb{R} - \mathbb{Z}, \\ \sum_{n<x} a_n + \frac{a_x}{2} & \text{for} \quad x \in \mathbb{Z}, \end{cases}$$

and $\log x$ means the main branch of the logarithm.
Next is $b = b(x) := 1 + 1/\log x$ and $\langle x\rangle = 1$ for $x \in \mathbb{Z}$ and otherwise, the distance to the nearest integer.

Proof. Since the Dirichlet- series converges normally for $Re(s) > 1$, we can integrate term by term. Applying Perron's formula (Lemma 2.3.2) we get

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}\,ds = \frac{1}{2\pi i}\sum_{n=1}^{\infty} a_n \int_{b-iT}^{b+iT}\left(\frac{x}{n}\right)^s \frac{1}{s}\,ds = \phi_f(x) + R$$

with

$$|R| \leq \frac{A(x)b}{\pi T} + \sum_{n=1, n\neq x}^{\infty} |a_n|\left(\frac{x}{n}\right)^b \frac{1}{\pi T|\log(x/n)|}.$$

There is

$$\begin{aligned} \sum_{\substack{n=1,n\neq x \\ |\log(x/n)|\geq\log(3/2)}}^{\infty} |a_n|\left(\frac{x}{n}\right)^b \frac{1}{\pi T|\log(x/n)|} &\leq \frac{x^b}{\pi T\log(3/2)}\sum_{n=1}^{\infty}|a_n|n^{-b} \\ &= O\left(\frac{x^b}{T(b-1)^\alpha}\right) = O\left(\frac{x(\log x)^\alpha}{T}\right), \end{aligned}$$

since $b = 1 + 1/\log x$. Furthermore

$$\begin{aligned} \sum_{\substack{n=1,n\neq x \\ |\log(x/n)|<\log(3/2)}}^{\infty} |a_n|\left(\frac{x}{n}\right)^b \frac{1}{\pi T|\log(x/n)|} &= \sum_{\substack{n\neq x \\ 2x/3<n<3x/2}} |a_n|\left(\frac{x}{n}\right)^b \frac{1}{\pi T|\log(x/n)|} \\ &\leq \frac{A(2x)2^b}{\pi T}\sum_{\substack{n\neq x \\ 2x/3<n<3x/2}} \frac{1}{|\log(x/n)|}. \end{aligned}$$

Therefore, it remains to show that

$$\sum_{\substack{n\neq x \\ 2x/3<n<3x/2}} \frac{1}{|\log(x/n)|} = O\left(x\log x + \frac{x}{\langle x\rangle}\right).$$

We have $1/2|z| \leq |\log(1+z)|$ for $z \in \mathbb{C}$ with $|z| < 1/2$.
Since for $2x/3 < n < x$, it holds $|(x-n)/n| < 1/2$, and we get

$$\log\left(\frac{x}{n}\right) = \log\left(1 + \frac{x-n}{n}\right) \geq \frac{x-n}{2n} \geq \frac{x-n}{2x}.$$

Now it is

$$\sum_{\substack{n\neq x\\ 2x/3<n<3x/2}} \frac{1}{|\log(x/n)|} \leq 2x \sum_{\substack{n\neq x\\ 2x/3<n<3x/2}} \frac{1}{x-n}$$
$$\leq 2x\left(\frac{1}{\langle x\rangle}+\int_{\langle x\rangle+1}^{x/3}\frac{1}{t}\,dt\right) = O\left(x\left(\log x+\frac{1}{\langle x\rangle}\right)\right).$$

For $x<n<3x/2$ we have $|(n-x)/x|<1/2$ and thus

$$\log\left(\frac{n}{x}\right)=\log\left(1+\frac{n-x}{x}\right)\geq\frac{n-x}{2x}.$$

One shows analogously here by

$$\sum_{x<n<3x/2}\frac{1}{\log(n/x)}=O\left(x\left(\log x+\frac{1}{\langle x\rangle}\right)\right).$$

This suffices. □

Definition 2.3.2. (multiplicative functions)
We call a function $f\colon \mathbb{N}\to\mathbb{C}$ <u>multiplicative</u>, if it holds $f(1)=1$ and

$$f(mn)=f(m)f(n)$$

holds for all $m,n\in\mathbb{N}$ with $(m,n)=1$. If this applies for all $m,n\in\mathbb{N}$ without restrictions, f is called <u>completely multiplicative</u>.

Notice 2.3.1. The Möbius- function μ, the function $\epsilon\colon\mathbb{N}\to\{0,1\}$ with

$$\epsilon(n):=\begin{cases}1 & \text{for}\quad n=1,\\ 0 & \text{otherwise,}\end{cases}$$

and the function $1\colon\mathbb{N}\to\{1\}$ with $1(n):=1$ for all $n\in\mathbb{N}$ are multiplicative functions.

Definition 2.3.3. (convolution)
Let $f,g\colon\mathbb{N}\to\mathbb{C}$. We define the <u>convolution</u> of f and g as follows

$$(f*g)(n):=\sum_{ab=n}f(a)g(b)$$

for all $n\in\mathbb{N}$.

Lemma 2.3.3. *(Möbius inversion formula)*
Let μ be the Möbius- function and ϵ and $1(n)$ be defined like in 2.3.1.
Then we obtain

$$\mu*1=\epsilon.$$

Proof. Since μ, ϵ and 1 are multiplicative functions, for $\mu * 1$ it holds

$$\begin{aligned}(\mu * 1)(nm) &= \sum_{ab=nm} \mu(a)1(b) = \sum_{a_1b_1=n}\sum_{a_2b_2=m} \mu(a_1a_2)1(b_1b_2) = \sum_{a_1b_1=n}\sum_{a_2b_2=m} \mu(a_1)\mu(a_2)1(b_1)1(b_2) \\ &= \left(\sum_{a_1b_1=n} \mu(a_1)1(b_1)\right)\left(\sum_{a_2b_2=m} \mu(a_2)1(b_2)\right) = (\mu * 1)(n)(\mu * 1)(m)\end{aligned}$$

for all $m, n \in \mathbb{N}$ with $(m, n) = 1$ using the multiplicativity of μ and 1.
Since ϵ is multiplicative, it suffices to show

$$(\mu * 1)(p^\alpha) = \epsilon(p^\alpha)$$

for all $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}_0$.
Let $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}_0$. Then we have

$$\begin{aligned}(\mu * 1)(p^\alpha) &= \sum_{ab=p^\alpha} \mu(a)1(b) = \sum_{a|p^\alpha} \mu(a) = \sum_{k=1}^{\alpha} \mu(p^k) \\ &= \left\{\begin{array}{ll} \mu(1) = 1 & \text{for} \quad \alpha = 0, \\ \mu(1) + \mu(p) = 1 - 1 = 0 & \text{otherwise} \end{array}\right\} = \epsilon(p^\alpha).\end{aligned}$$

□

Notice 2.3.2. We can describe the product of two Dirichlet- series by the convolution.
Let $F(s) := \sum_{n\in\mathbb{N}} f(n)n^{-s}$ and $G(s) := \sum_{n\in\mathbb{N}} g(n)n^{-s}$ be Dirichlet- series, which converge absolutely in $s_0 \in \mathbb{N}$ and let $f, g\colon \mathbb{N} \to \mathbb{C}$ be functions. According to Theorem 2.3.1 we obtain

$$F(s_0)G(s_0) = \sum_{n=1}^{\infty}\left(\sum_{d|n} f(d)g(n/d)\right) n^{-s_0} = \sum_{n=1}^{\infty}\left(\sum_{ab=n} f(a)g(b)\right) n^{-s_0} = \sum_{n=1}^{\infty}(f * g)(n)n^{-s}.$$

Dirichlet- series with multiplicative coefficients have special properties. Some of these properties will be discussed in more detail below.

Theorem 2.3.3. [10]
Let $f\colon \mathbb{N} \to \mathbb{C}$ be multiplicative and σ_a the absolute abscissa of convergence of the Dirichlet- series $F(s) = \sum_{n\in\mathbb{N}} f(n)n^{-s}$. Then it holds

$$F(s) = \prod_{p\in\mathbb{P}}\left(\sum_{\nu=0}^{\infty} f(p^\nu)p^{-\nu s}\right)$$

for $\Re(s) > \sigma_a$, where the infinite product converges normally.
If f is completely multiplicative, then we have

$$F(s) = \prod_{p\in\mathbb{P}} \frac{1}{1 - f(p)p^{-s}} \quad \text{and} \quad \frac{1}{F(s)} = \sum_{n=1}^{\infty} \mu(n)f(n)n^{-s}$$

for $\Re(s) > \sigma_a$. This representation of $F(s)$ is called Euler- product.

Proof. The proof is well- known and can be found in Fischer (9) on page 30. □

[10]See Lütkebohmert (3), exercises 28 and 39, page 199 and 202

2.4 Characters

Dirichlet- series, which represent generalizations of the Riemann- ζ- function in a certain sense, are of particular interest for the work at hand.

Definition 2.4.1. (Character modulo q)[11]
Let $q \in \mathbb{N}$. By a character modulo q we mean a function $\chi \colon \mathbb{N} \to \mathbb{C}$, which satisfies the following properties:

1. $\chi(n) \begin{cases} = 0, & \text{for} \quad (n,q) > 1, \\ \neq 0, & \text{for} \quad (n,q) = 1. \end{cases}$
2. $\chi(n) = \chi(m)$, for $m \equiv n \bmod q$.
3. $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{N}$.

The function

$$\chi_0(n) := \begin{cases} 0 & \text{for} \quad (n,q) > 1, \\ 1 & \text{for} \quad (n,q) = 1 \end{cases}$$

is called the principal character modulo q.

Because of the second property, characters can be defined on $\mathbb{Z}$ as well. If χ is a character modulo q, set $\chi(l) = \chi(n)$, if it holds $l \in \mathbb{Z}, n \in \mathbb{N}$ and $l \equiv n \bmod q$. The properties of the characters are not lost.

Obviously, products of characters are again characters, since all three properties are fulfilled. There are various ways to define the characters. In the following an equivalent definition is given, using algebraic concepts.

Definition 2.4.2. (Character modulo q)[12]
Let G be a finite abelian group. Then we call a group homomorphism $\chi \colon G \to \mathbb{C}^\times$ of G to $(\mathbb{C}^\times, \cdot)$ a character of G. For $q \in \mathbb{N}$ we call a function $\chi \colon \mathbb{N} \to \mathbb{C}$ a character modulo q, if there exists a character χ' of the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$ with

$$\chi(n) = \begin{cases} \chi'(n + q\mathbb{Z}) & \text{for} \quad (n,q) = 1, \\ 0 & \text{for} \quad (n,q) > 1. \end{cases}$$

In case $\chi' \equiv 1 \bmod q$, the character $\chi = \chi_0$ is called the principal character modulo q.

Notice 2.4.1. Definition 2.4.1 and Definition 2.4.2 are equivalent.

Lemma 2.4.1. *Let* $q \in \mathbb{N}$*. Then it holds* $|\chi(n)| = 1$ *for every character* $\chi \bmod q$ *for all* $n \in \mathbb{N}$ *with* $(n,q) = 1$.

Proof. Let χ be a character modulo q. According to Definition 2.4.2 there exists a homomorphism $\chi' \colon (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^\times$ with $\chi(n) = \chi'(n + q\mathbb{Z})$ for all $n \in \mathbb{N}$ with $(n,q) = 1$. Notice that $(\mathbb{Z}/q\mathbb{Z})^*$ is a group of order $\varphi(q)$. Thus, we have $(n + q\mathbb{Z})^{\varphi(q)} = 1 + q\mathbb{Z}$ for all $n + q\mathbb{Z} \in (\mathbb{Z}/q\mathbb{Z})^*$ and

$$(\chi(n))^{\varphi(q)} = (\chi'(n + q\mathbb{Z}))^{\varphi(q)} = \chi'((n + q\mathbb{Z})^{\varphi(q)}) = \chi'(1 + q\mathbb{Z}) = 1$$

for all $n \in \mathbb{N}$ with $(n,q) = 1$.

[11] See Prachar (6), page 102 f.
[12] See Lütkebohmert (3), exercises 29 and 30, page 200 f.

We get the claim since it holds

$$|\chi(n)| = |\chi(n)^{\varphi(q)}|^{1/\varphi(q)} = 1^{1/\varphi(q)} = 1$$

for all $n \in \mathbb{N}$ with $(n, q) = 1$. □

Theorem 2.4.1. [13]
Let $q \in \mathbb{N}$. Then there exist $\varphi(q)$ characters modulo q.

Proof. In elementary number theory we prove that there exist numbers $w, w_0, \ldots, w_r$ so that every $l \in \mathbb{N}$ with $0 \le l < q$ and $(l, q) = 1$ can be represented in exactly one way

$$l \equiv w^{\gamma} w_0^{\gamma_0} w_1^{\gamma_1} \ldots w_r^{\gamma_r}$$

while $\gamma, \gamma_0, \ldots, \gamma_r \in \mathbb{N}$ and

$$w^2 \equiv w_0^{2^{\alpha_0 - 2}} \equiv \ldots \equiv w_r^{\varphi(p_r^{\alpha_r})} \equiv 1 \bmod q.$$

The numbers $\alpha_0, \ldots, \alpha_r, p_1, p_2, \ldots, p_r \in \mathbb{N}$ and $r \in \mathbb{N}$ are given by the prime factorization of q:

$$q = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

with $p_1, p_2, \ldots, p_r \in \mathbb{P}$ pairwise different.
Owing to the second and third properties in Definition 2.4.1 we have

$$\chi(l) = \chi(w^{\gamma} w_0^{\gamma_0} w_1^{\gamma_1} \ldots w_r^{\gamma_r}) = \chi(w)^{\gamma} \chi(w_0)^{\gamma_0} \chi(w_1)^{\gamma_1} \cdot \ldots \cdot \chi(w_r)^{\gamma_r},$$

that means, χ is uniquely determined by $\chi(w), \chi(w_0), \chi(w_1) \ldots \chi(w_r)$. Because of these two properties of Definition 2.3.1 we obtain

$$\begin{aligned}
(\chi(w))^2 &= \chi\left(w^2\right) = \chi(1) = 1 \\
(\chi(w_0))^{2^{\alpha_0 - 2}} &= \chi\left(w_0^{2^{\alpha_0 - 2}}\right) = \chi(1) = 1 \\
(\chi(w_1))^{\varphi(p_1^{\alpha_1})} &= \chi\left(w_1^{\varphi(p_1^{\alpha_1})}\right) = \chi(1) = 1 \\
&\vdots \\
(\chi(w_r))^{\varphi(p_r^{\alpha_r})} &= \chi\left(w_r^{\varphi(p_r^{\alpha_r})}\right) = \chi(1) = 1
\end{aligned}$$

So we get $\chi(w) = \pm 1$, resp. $\chi(w_0), \chi(w_1), \ldots, \chi(w_r)$ are roots of unity of order $2^{\alpha_0 - 2}, \varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})$. So there are at most $2 \cdot 2^{\alpha_0 - 2} \cdot \varphi(p_1^{\alpha_1}) \cdot \ldots \cdot \varphi(p_r^{\alpha_r}) = \varphi(q)$ different options for χ. Conversely, any choice of $\chi(w), \chi(w_1), \ldots, \chi(w_r)$ as a unit root of order $2, 2^{\alpha_0 - 2}, \varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})$ leads us to a character modulo q, by setting $\chi(l) = 0$ for $(l, q) > 1$. This suffices. □

Lemma 2.4.2. [14]
Let $q \in \mathbb{N}$.

1. *For every character χ mod q it holds*

$$\sum_{l \bmod q} \chi(l) = \begin{cases} \varphi(q) & \text{for } \chi = \chi_0, \\ 0 & \text{for } \chi \neq \chi_0. \end{cases}$$

We summate over a full or reduced residue system modulo q.

[13] See Prachar (6), page 102
[14] See Lütkebohmert (3), exercise 29, page 200 and Prachar (6), page 101

2. *We have*

$$\sum_{\chi} \chi(l) = \begin{cases} \varphi(q) & \text{for} \quad l \equiv 1 \bmod q, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we summate over all $\varphi(q)$ characters modulo q.

3. *It holds for $l \in \mathbb{N}$ with $(l,q) = 1$*

$$\frac{1}{\varphi(q)} \sum_{\chi} \chi(n)\overline{\chi}(l) = \begin{cases} 1 & \text{for} \quad n \equiv l \bmod q, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. 1. It holds $\chi(n) = 0$ for every character $\chi \bmod q$ and for all $n \in \mathbb{N}$ with $(n,q) > 1$. So there is no difference, if we sum over a full or a reduced residue system. For $\chi = \chi_0$, we have

$$\sum_{l \bmod q} \chi(l) = \sum_{\substack{l \bmod q, \\ (l,q)=1}} 1 = \varphi(q).$$

For $\chi \neq \chi_0$ there exists $n \in \mathbb{N}$ with $(n,q) = 1$ and $\chi(n) \neq 1$. So we get

$$\chi(n) \sum_{l \bmod q} \chi(l) = \sum_{l \bmod q} \chi(nl) = \sum_{l \bmod q} \chi(l).$$

The last equation holds, since $(\mathbb{Z}/q\mathbb{Z})^*$ is a group under multiplication. Hence,

$$0 = (\chi(n) - 1) \sum_{l \bmod q} \chi(l) \Rightarrow 0 = \sum_{l \bmod q} \chi(l).$$

2. For $l \equiv 1 \bmod q$ we get

$$\sum_{\chi} \chi(l) = \sum_{\chi} \chi(1) = \sum_{\chi} 1 = \varphi(q),$$

since there are $\varphi(q)$ characters modulo q according to Theorem 2.4.1. Obviously, the set of all characters modulo q forms a group under multiplication: $(\chi_1\chi_2)(n) = \chi_1(n)\chi_2(n)$ for all $n \in \mathbb{N}$ and for two characters χ_1 and χ_2 modulo q. Let $l \not\equiv 1 \bmod q$. Then we have $\widetilde{\chi}$ with $\widetilde{\chi}(l) \neq 1$. Thus,

$$\widetilde{\chi}(l) \sum_{\chi} \chi(l) = \sum_{\chi} (\widetilde{\chi}\chi)(l) = \sum_{\chi} \chi(l).$$

We get

$$0 = (\widetilde{\chi}(l) - 1) \sum_{\chi} \chi(l) \Rightarrow 0 = \sum_{\chi} \chi(l).$$

3. It holds for every character $\chi \bmod q$

$$\chi(l)\overline{\chi}(l) = |\chi(l)|^2 = 1 \Rightarrow (\chi(l))^{-1} = \overline{\chi}(l)$$

for $(l,q) = 1$.
Let l' be a solution of $ll' \equiv 1 \bmod q$. Then it holds $(l',q) = 1$ and $\overline{\chi}(l) = (\chi(l))^{-1} = \chi(l')$. Thus,

$$\sum_{\chi} \chi(n)\overline{\chi}(l) = \sum_{\chi} \chi(n)\chi(l') = \sum_{\chi} \chi(nl').$$

The claim follows with the second part, since we get $n \equiv l \bmod q$ if and only if $nl' \equiv 1 \bmod q$ holds.

□

Let χ_1 and χ_2 be two characters with moduli q_1 and $q_2 \in \mathbb{N}$. There is

$$\chi_1(n) \neq 0 \text{ and } \chi_2(n) \neq 0 \Leftrightarrow (n, q_1) = (n, q_2) = 1 \Leftrightarrow (n, [q_1, q_2]) = 1$$

for $n \in \mathbb{N}$. This leads to the following definition:

Definition 2.4.3. (equivalent characters)[15]
Two characters χ_1 and χ_2 with moduli q_1 and q_2 are called equivalent, if $\chi_1(n) = \chi_2(n)$ for all $n \in \mathbb{N}$ with $(n, [q_1, q_2]) = 1$.
One say, χ_1 is induced by the modulus q_2. The modulus q_2 is called conductor of χ_1.

The characters χ_1 und χ_2 are equivalent if and only if their values are the same for all $n \in \mathbb{N}$, for which they do not vanish for both characters.

Lemma 2.4.3. [16]
One can induce every character $\chi_1 \bmod q_1$ by every multiple of q_1.

Proof. Let q_2 be a multiple of q_1. Then the function given by

$$\chi_2(n) := \begin{cases} \chi_1(n) & \text{for} \quad (n, q_2) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is a character modulo q_2 according to Definition 2.4.1, and the character χ_1 is equivalent to χ_2 since it holds $(n, q_2) = (n, [q_1, q_2])$. □

Lemma 2.4.4. [17]
Let $q_1, q_2 \in \mathbb{N}$ with $q_2 | q_1$, and let $\chi_1 \bmod q_1$ be a character modulo q_1. Then χ_1 is induced by the modulus q_2 if and only if it holds $\chi_1(n) = 1$ for all $n \in \mathbb{N}$ with $n \equiv 1 \bmod q_2$ and $(n, q_1) = 1$.

Proof. One direction follows immediately.
If χ_2 is an equivalent character to χ_1, it holds $\chi_1(n) = \chi_2(n) = \chi_2(1) = 1$ for all $n \in \mathbb{N}$ with $n \equiv 1 \bmod q$ and $(n, q_1) = 1$ since $(n, q_2) = 1$.

For the other direction we notice, that there exists $y \in \mathbb{N}$ with $(n, q_2) = 1$ for all $n \in \mathbb{N}$ such that

$$(n + q_2 y, q_1) = 1 \tag{2.6}$$

holds. Let $\widetilde{q}$ be the product of all primes, which divide q_1 but not q_2. Let $\widetilde{n} \in \mathbb{N}$ with $(\widetilde{n}, \widetilde{q}) = 1$. Then the congruence $n + q_2 y \equiv \widetilde{n} \bmod \widetilde{q}$ is solvable since $(q_2, \widetilde{q}) = 1$. If y is such a solution, then we have $(n + q_2 y, \widetilde{q}) = (\widetilde{n}, \widetilde{q}) = 1$ and also $(n + q_2 y, q_2) = (n, q_2) = 1$ for $(n, q_2) = 1$. Overall, we have $(n + q_2 y, q_1) = 1$ because of the choice of $\widetilde{q}$.
We now define χ_2 by

$$\chi_2(n) := \begin{cases} \chi_1(n + q_2 y_n) & \text{for} \quad n \in \mathbb{N} \text{ with } (n, q_2) = 1, \\ 0 & \text{for} \quad n \in \mathbb{N} \text{ with } (n, q_2) > 1. \end{cases}$$

We choose y_n for every $n \in \mathbb{N}$ with $(n, q_2) = 1$ over (2.6). Now we want to show that χ_2 is a character modulo q_2.

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[15] See Prachar (6), page 123 f.
[16] See Prachar (6), chapter IV, Lemma 6.1
[17] See Prachar (6), chapter IV, Lemmata 6.2 and 6.3

First χ_2 is well defined, that means $\chi_2(n)$ is independent of the choice of the y_n for all $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$ with $(n, q_2) = 1$ and let $y_n^{(1)}, y_n^{(2)}$, such that

$$(n + q_2 y_n^{(1)}, q_1) = (n + q_2 y_n^{(2)}, q_1) = 1$$

holds. One determines $\overline{n}$ over the congruence $n\overline{n} \equiv 1 \bmod q_2$. Then we have $(\overline{n}, q_2) = 1$ and according to (2.6) there exists $y_{\overline{n}} \in \mathbb{N}$ with $(\overline{n} + q_2 y_{\overline{n}}, q_1) = 1$. It follows

$$\chi_1(n + q_2 y_n^{(1)})\chi_1(\overline{n} + q_2 y_{\overline{n}}) = \chi_1((n + q_2 y_n^{(1)})(\overline{n} + q_2 y_{\overline{n}})) = 1,$$

since $(n + q_2 y_n^{(1)})(\overline{n} + q_2 y_{\overline{n}}) \equiv n\overline{n} \equiv 1 \bmod q_2$.
Similarly follows

$$\chi_1(n + q_2 y_n^{(2)})\chi_1(\overline{n} + q_2 y_{\overline{n}}) = 1.$$

So we have $\chi_1(n + q_2 y_n^{(1)}) = \chi_1(n + q_2 y_n^{(2)})$ since $(\overline{n} + q_2 y_{\overline{n}}, q_1) = 1$ and therefore $\chi_1(\overline{n} + q_2 y_{\overline{n}}) \neq 0$.

The first property in Definition 2.4.1 is fulfilled, since we choose $y_n \in \mathbb{N}$ for every $n \in \mathbb{N}$ such that it follows $(n + q_2 y_n, q_1) = 1$ from $(n, q_2) = 1$. So we have $\chi_1(n + q_2 y_n) \neq 0$.

To prove the second property, we choose $n, m \in \mathbb{N}$ with $n \equiv m \bmod q_2$. In the case $(m, q_2) > 1$ we have $(n, q_2) > 1$, and the second property is fulfilled. Let $(n, q_2) = (m, q_2) = 1$. One determines $\overline{n}$ over the congruence

$$m\overline{n} \equiv n\overline{n} \equiv 1 \bmod q_2$$

and chooses numbers $y_n, y_m, y_{\overline{n}}$ according to (2.6) such that

$$(m + q_2 y_m, q_1) = (n + q_2 y_n, q_1) = (\overline{n} + q_2 y_{\overline{n}}, q_1) = 1$$

holds. Then it follows

$$\chi_1(n + q_2 y_n)\chi_1(\overline{n} + q_2 y_{\overline{n}}) = \chi_1((n + q_2 y_n)(\overline{n} + q_2 y_{\overline{n}})) = \chi_1(1) = 1,$$

since $(n + q_2 y_n)(\overline{n} + q_2 y_{\overline{n}}) \equiv n\overline{n} \equiv 1 \bmod q_2$. Just apply

$$\chi_1(m + q_2 y_m)\chi_1(\overline{n} + q_2 y_{\overline{n}}) = 1.$$

Since $(\overline{n} + q_2 y_{\overline{n}}, q_1) = 1$ we have $\chi_1(\overline{n} + q_2 y_{\overline{n}}) \neq 0$, and it follows

$$\chi_1(n + q_2 y_n) = \chi_1(m + q_2 y_m),$$

in other words $\chi_2(m) = \chi_2(n)$.
The third property follows for $m, n \in \mathbb{N}$ with $(mn, q_2) = 1$ from

$$\begin{aligned}\chi_2(m)\chi_2(n) &= \chi_1(m + q_2 y_m)\chi_1(n + q_2 y_n) = \chi_1((m + q_2 y_m)(n + q_2 y_n)) \\ &= \chi_1(mn + q_2(y_n m + y_m n + q_2 y_n y_m)) = \chi_2(mn)\end{aligned}$$

for certain $y_n, y_m \in \mathbb{N}$. Here, the already shown second property was used. The third property holds for $(mn, q_2) > 1$ too.
So χ_2 is a character modulo q_2 according to Definition 2.4.1, and χ_1 and χ_2 are obviously equivalent.
□

Lemma 2.4.5. [18]
Let χ_1 and χ_2 be two equivalent characters modulo q_1 resp. modulo q_2, and let $q' = (q_1, q_2)$. Then χ_1 and χ_2 can be induced modulo q'.

Proof. We only prove the assertion for χ_1. The proof of χ_2 is analogous.
It suffices to show that it holds $\chi_1(n) = 1$ for $n \in \mathbb{N}$ with $n \equiv 1 \bmod q'$ and $(n, q_1) = 1$. Then the claim follows by applying Lemma 2.4.4. Let $n \in \mathbb{N}$, $n \equiv 1 \bmod q'$ and $(n, q_1) = 1$. Then there exists $m \in \mathbb{N}$ with $n = 1 + mq'$. Since $q' = (q_1, q_2)$ there exists $a, b \in \mathbb{Z}$ such that it holds $q' = aq_1 + bq_2$. So we have

$$n = 1 + m(aq_1 + bq_2) = 1 + maq_1 + mbq_2$$

and $1 + mbq_2$ is coprime to q_1 and q_2. Hence,

$$\chi_1(n) = \chi_1(1 + maq_1 + mbq_2) = \chi_1(1 + mbq_2) = \chi_2(1 + mbq_2) = \chi_2(1) = 1$$

due to the second property in Definition 2.4.1 and due to the equivalence of the two characters.
Possibly it holds $1 + mbq_2 \notin \mathbb{N}$. But then there exists $l \in \mathbb{N}$ with $1 + mbq_2 \equiv l \bmod q_1$, and one set $\chi_1(1 + mbq_2) = \chi(l)$. □

Theorem 2.4.2. [19]
All conductors of character χ are multiples of a unique conductor $q^* \in \mathbb{N}$.

Proof. Let q^* be the smallest conductor of χ. For $q^* = 1$ the claim follows obviously.
Let $q^* > 1$. We assume, there exists a conductor q which is not a multiple of q^*. Then we obtain $(q^*, q) < q^*$. Since (q^*, q) is a conductor of χ according to Lemma 2.4.5, this would be a contradition to the chioce of q^*. □

Thus, the following definition makes sense:

Definition 2.4.4. (primitive character)[20]
We call a character χ mod q, which is not induced by a modulus, which divides q, a primitive character.

Let χ be a character modulo q and let q^* be the smallest conductor of χ. We denote the to χ equivalent primitive character with χ^*. It is also a character modulo q and is uniquely determined because two of the characters modulo q^* do not have the same values for all $n \in \mathbb{N}$.

Two primitive characters are thus either equal or not equivalent. In contrast, two equivalent characters have the same primitive character. Generally, the set $\{n \in \mathbb{N} \colon \chi^*(n) \neq 0\}$ is larger for every character χ than the set $\{n \in \mathbb{N} \colon \chi(n) \neq 0\}$. The first mentioned set is the largest set in which χ can be continued in a certain sense, without losing the properties of the characters.

[18] See Prachar (6), chapter IV, Lemma 6.4
[19] See Prachar (6), chapter IV, Theorem 6.2
[20] See Prachar (6), page 126

2.5 L- series

Definition 2.5.1. (Dirichlet- L- series)
Let $q \in \mathbb{N}$ and χ be a character modulo q. The Dirichlet- series

$$L(s,\chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

is called (Dirichlet-) L- series.

Lemma 2.5.1. [21]
Let $q \in \mathbb{N}$ and χ be a character modulo q. Then we have:

1. *For $\chi \neq \chi_0$ the series $\sum_{n\in\mathbb{N}} \chi(n)n^{-s}$ converges for $\Re(s) > 0$. The function $L(s,\chi)$ represents an analytic function for $\Re(s) > 0$.*
2. *For $\chi = \chi_0$ the series $\sum_{n\in\mathbb{N}} \chi(n)n^{-s}$ converges for $\Re(s) > 1$. In this case $L(s,\chi)$ represents an analytic function for $\Re(s) > 1$.*
3. *The L- series $\sum_{n\in\mathbb{N}} \chi(n)n^{-s}$ converges absolutely for $\Re(s) > 1$.*

Proof. Let σ_c resp. σ_a be the abscissa of convergence resp. the abolute abscissa of convergence of $\sum_{n\in\mathbb{N}} \chi(n)n^{-s}$.

1. Let $\chi \neq \chi_0$. According to the first part of Lemma 2.4.2 we have $\sum_{n=1}^{q} \chi(n) = 0$. Thus, it follows for all $k \in \mathbb{N}$

$$\sum_{n=1}^{kq} \chi(n) = 0,$$

   that means $\sum_{n\in\mathbb{N}} \chi(n)$ converges. So we get $\sigma_c \leq 0$ and according to the first and sixth part of Lemma 2.3.1 we get the claim.

2. In the case $\chi = \chi_0$ there is

$$\left| \sum_{n=1}^{q} \chi(n) n^{-s} \right| \leq \sum_{n=1}^{q} n^{-\Re(s)}$$

   and $\sum_{n=1}^{q} n^{-\Re(s)}$ converges for $\Re(s) > 1$. The holomorphy for $\Re(s) > 1$ follows with the sixth part of Lemma 2.3.1.

3. This follows from (1), since we have $\sigma_a \leq \sigma_c + 1$ according to the fifth part of Lemma 2.3.1.

□

Notice 2.5.1. For the modulus $q = 1$ only the principal character χ_0 mod 1 exists, and the associated L- series is the Riemann- ζ- function. It holds $L(s,\chi_0) = \zeta(s)$ for $\Re(s) > 1$.

Obviously, all characters modulo q are completely multiplicative functions. From this, we obtain different representations for functions which are formed with the aid of L- series.

[21] See Lütkebohmert (3), exercise 30, page 200 f.

Lemma 2.5.2. *Let $q \in \mathbb{N}$ and let χ be a character modulo q. For $\Re(s) > 1$ it holds*

1. $L(s,\chi) = \prod_{p\in\mathbb{P}} \frac{1}{1-\chi(p)p^{-s}} = \prod_{p\in\mathbb{P}, p\nmid q} \frac{1}{1-\chi(p)p^{-s}}$.
   Particularly there is $L(s,\chi) \neq 0$ for $\Re(s) > 1$.

2. $\frac{1}{L(s,\chi)} = \sum_{n=1}^{\infty} \mu(n)\chi(n)n^{-s}$.

3. $\log L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s \log n}$, *setting* $\frac{\Lambda(1)}{\log 1} = 0$.

4. $-\frac{L'}{L}(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}$.

Proof.

1. This follows with Theorem 2.3.3, since χ is completely multiplicative, and it holds for the absolute abscissa of convergence of the L- series $\sigma_a \leq 1$.

2. This follows with Theorem 2.3.3, since χ is completely multiplicative, and it holds for the absolute abscissa of convergence of the L- series $\sigma_a \leq 1$.

3. It is well- known, that $\log(1-z)$ has the representation

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

   for $|z| < 1$. We may use the logarithm for $L(s,\chi)$ for $\Re(s) > 1$ according to the first part of this Theorem. So we get for $\Re(s) > 1$

$$\begin{aligned} \log L(s,\chi) &= \log\left(\prod_{p\in\mathbb{P}} \frac{1}{1-\chi(p)p^{-s}}\right) = -\sum_{p\in\mathbb{P}} \log(1-\chi(p)p^{-s}) \\ &= \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} \frac{(\chi(p)p^{-s})^n}{n} = \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} \frac{\chi(p^n)}{np^{sn}}. \end{aligned}$$

   There is

$$\frac{\Lambda(p^n)}{\log(p^n)} = \frac{\log p}{n \log p} = \frac{1}{n}$$

   for all $n \in \mathbb{N}$ and $p \in \mathbb{P}$. Thus,

$$\log L(s,\chi) = \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} \frac{\Lambda(p^n)\chi(p^n)}{\log(p^n)p^{-sn}} = \sum_{m=1}^{\infty} \frac{\Lambda(m)\chi(m)}{(\log m)m^{-s}}$$

   holds for $\Re(s) > 1$. The last equation is valid because of $\Lambda(m) = 0$, if m is no prime power.

4. We obtained this by differentiating of the third part and the locally uniform convergence for $\Re(s) > 1$ according to Lemma 2.3.1. Then we have

$$\frac{L'}{L}(s,\chi) = (\log L(s,\chi))' = \left(\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s \log n}\right)' = \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{\log n} \left(n^{-s}\right)' = -\sum_{n=1}^{\infty} \chi(n)\Lambda(n)n^{-s}$$

   for $\Re(s) > 1$.

□

Now we will show that all L- series are meromorphic continuable in the whole complex plane and satisfy a certain functional equation there. As a special case we obtain the functional equation of the Riemann- ζ- function.
In the following, always let $q \in \mathbb{N}$. Let $\chi \bmod q$ be a primitive character. As described above, setting $\chi(n) = \chi(m)$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $m \equiv n \bmod q$, one can interpret $\chi \bmod q$ as a function with domain $\mathbb{Z}$. Since

$$(\chi(-1))^2 = \chi((-1)^2) = \chi(1) = 1$$

there holds always $\chi(-1) = \pm 1$. So the following definition is useful:

$$a = a(\chi) := \begin{cases} 0 & \text{for} \quad \chi(-1) = 1, \\ 1 & \text{for} \quad \chi(-1) = -1. \end{cases} \tag{2.7}$$

Theorem 2.5.1. [22]
Let χ be a primitive character modulo q. Then $L(s, \chi)$ is an analytic function excluding the case $q = 1$ and $\chi = \chi_0^*$. In this case $L(s, \chi_0^*) = \zeta(s)$ is only singular at $s = 1$ with a simple pole and residue 1. It holds for all $\chi \bmod q$

$$L(1 - s, \overline{\chi}) = \epsilon_\chi 2(2\pi)^{-s} q^{s-1/2} \cos(\pi/2(s - a))\Gamma(s)L(s, \chi),$$

*where ϵ_χ is a constant depending only on χ with $|\epsilon_\chi| = 1$.
The function*

$$\xi(s, \chi) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s + a}{2}\right) L(s, \chi)$$

satisfies the equation

$$\xi(1 - s, \overline{\chi}) = \epsilon_\chi \xi(s, \chi)$$

and it is an analytic function excluding $q = 1$ and $\chi = \chi_0^*$ with a simple pole at $s = 1$. Setting

$$\xi(s) = 1/2s(s - 1)\xi(s, \chi_0^*) = 1/2s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

leads us to

$$\xi(s) = \xi(1 - s),$$

while $\xi(s)$ is an analytic function.

Proof. One can find the proof also in Fischer (9) on page 50 f. □

Theorem 2.5.2. [23]
Let χ be a character modulo q and denote χ^* as the to $\chi \bmod q$ equivalent primitive character with modulus q^*. Then it holds the following relationship between $L(s, \chi)$ and $L(s, \chi^*)$:

$$L(s, \chi^*) = \prod_{p|q, p\nmid q^*} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1} L(s, \chi).$$

In particular, $L(s, \chi)$ and $L(s, \chi^*)$ have the same zeros for $\Re(s) > 0$.

[22] See Prachar (6), chapter VII, Theorem 1.1
[23] See Prachar (6), chapter IV, Theorem 6.3

Proof. Applying Lemma 2.5.2 we get

$$\begin{aligned} L(s,\chi^*) &= \prod_{p\in\mathbb{P},p\nmid q^*} \frac{1}{1-\chi^*(p)p^{-s}} = \prod_{p\in\mathbb{P},p\nmid q} \frac{1}{1-\chi^*(p)p^{-s}} \prod_{p|q,p\nmid q^*} \frac{1}{1-\chi^*(p)p^{-s}} \\ &= \prod_{p\in\mathbb{P},p\nmid q} \frac{1}{1-\chi(p)p^{-s}} \prod_{p|q,p\nmid q^*} \frac{1}{1-\chi^*(p)p^{-s}} = L(s,\chi) \prod_{p|q,p\nmid q^*} \frac{1}{1-\chi^*(p)p^{-s}}, \end{aligned}$$

since $\chi(n) = \chi^*(n)$ holds for all $n \in \mathbb{N}$ with $(n,q) = 1$.
Thus it follows, that either $L(s,\chi)$ and $L(s,\chi^*)$ vanish for $\Re(s) > 0$ or both are different from zero, since it holds $1 - \chi^*(p)p^{-s} \neq \{0,\infty\}$ because of $|\chi(p)p^{-s}| \leq p^{-\Re(s)} < 1$ for $\Re(s) > 0$ and $p \in \mathbb{P}$. □

With this statement it follows from Theorem 2.5.1, that $L(s,\chi)$ can be analytically continued into the whole complex plane for a non- primitive character χ, except for $\chi = \chi_0$. Then $L(s,\chi_0)$ has a simple pole at $s = 1$.

In the following we study the distribution of the zeros of the L- series. It has been found that those have no zeros in the half- plane $\{s \in \mathbb{C}\colon \Re(s) > 1\}$. Because of Theorem 2.5.2 only the zeros of L- series with primitive characters are interesting. For the proof of the next theorem, however, a distinction between primitive and non- primitive characters is not necessary.

Theorem 2.5.3. [24]
Let χ be a a character modulo q. Then it holds

$$L(1+it,\chi) \neq 0$$

for all $t \in \mathbb{R}$.

Proof. One can find the proof also in Fischer (9) on page 60 f. □

Theorem 2.5.4. [25]
Let χ be a primitive character modulo q. Then it holds $\xi(s,\chi) \neq 0$ for $\Re(s) \geq 1$ and $\sigma \leq 0$.
For $\chi \neq \chi_0^*$ the series $L(s,\chi)$ has simple zeros in $\Re(s) \leq 0$ at $s = -a, -a-2, \ldots$ and no other zeros.
The function $a = a(\chi)$ is defined like in (2.7).
The Riemann- ζ- function $L(s,\chi_0^*) = \zeta(s)$ has simple zeros for $\Re(s) \leq 0$ only at $s = -2, -4, \ldots$.

Proof. We already have shown $L(s,\chi) \neq 0$ for $\Re(s) \geq 1$ in Theorem 2.5.3 and Lemma 2.5.2. According to the definition of the function $\xi(s,\chi)$ it follows $\xi(s,\chi) \neq 0$ for $\Re(s) \geq 1$, since the Gamma- function has no zeros. Since χ is a primitive character, Theorem 2.5.1 and especially the functional equation $\xi(1-s,\overline{\chi}) = \epsilon_\chi \xi(s,\chi)$ hold. Since $\overline{\chi}$ is also a primitive character, and so $\xi(s,\overline{\chi}) \neq 0$ for $\Re(s) \geq 1$, it follows $\xi(s,\chi) \neq 0$ for $\Re(s) \leq 0$. From the definition of $\xi(s,\chi)$ it follows that $L(s,\chi)$ can only have zeros where $\Gamma\left(\frac{s+a}{2}\right)$ has poles, so at the points $-a, -a-2, \ldots$.
For $\chi \neq \chi_0^*$ the function $\xi(s,\chi)$ is analytic according to Theorem 2.5.1, and therefore $\xi(s,\chi)$ has no poles for all $s \in \mathbb{C}$. So these are actually zeros of $L(s,\chi)$. For $\chi = \chi_0^*$ the point $s = 0$ is no zero of $L(s,\chi_0^*) = \zeta(s)$, since $\xi(s,\chi_0^*)$ has a simple pole at this point. It is rather with $\xi(s) = 1/2s(s-1)\xi(s,\chi_0^*)$

$$\zeta(0) = \lim_{s\to 0} 2\pi^{s/2}\xi(s)\frac{1}{s(s-1)\Gamma(s/2)} = -2\xi(0)\lim_{s\to 0}\frac{1}{s\Gamma(s/2)} = -\xi(0) = -\xi(1) = -\frac{1}{2},$$

[24]See Prachar (6), chapter IV, Theorem 4.1 and 4.2
[25]See Prachar (6), chapter VII, Theorem 1.2

because according to the Gaussian product development[26] it holds

$$\lim_{s\to 0}\frac{1}{s\Gamma(s)} = \lim_{s\to 0} e^{\gamma s/2}\frac{s}{2s}\prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)e^{-s/2n} = \frac{1}{2}.$$

The value of $\xi(1)$ is obtained by

$$\begin{aligned}\xi(1) &= \lim_{s\to 1}\frac{s(s-1)}{2}\xi(s,\chi_0^*) = 1/2\lim_{s\to 1}(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)\\ &= \frac{1}{2}\pi^{-1/2}\Gamma(1/2)\mathrm{Res}_{s=1}(\zeta(s)) = \frac{1}{2}\end{aligned}$$

with $\Gamma(1/2)=\sqrt{\pi}$. In contrast, $\xi(s,\chi_0^*)$ has no poles at $s=-2,-4,\dots$ according to Theorem 2.5.1, so these are the zeros of $\zeta(s)$. All these zeros are simple, since the gamma function has only simple poles. The Theorem is proved. □

The zeros mentioned in Theorem 2.5.4 are called trivial zeros of $L(s,\chi)$. Apart from these, all zeros of $L(s,\chi)$ are in the domain $0<\Re(s)<1$, when χ is primitive. The domain $0<\Re(s)<1$ is called the critical strip. It will be shown that $L(s,\chi)$ has infinitely many zeros in the critical strip. The function $\xi(s,\chi)$ does not disappear outside of the critical strip and has the same zeros as $L(s,\chi)$ in $0<\Re(s)<1$.

In order to derive a statement about the distribution of zeros in the critical strip, we want to apply the Theorems 2.2.1 and 2.2.2. We first examine the growth behavior of $L(s,\chi)$. We will introduce an auxiliary function.

Lemma 2.5.3. [27]
Let $\zeta(s,w)$ be given for $s\in\mathbb{C}$ with $\Re(s)\geq 1$ and for $w\in\mathbb{R}$ with $0<w\leq 1$ by

$$\zeta(s,w) := \sum_{n=0}^{\infty}(n+w)^{-s}.$$

Then $\zeta(s,w)$ can be continued analytically in $0<\Re(s)\leq 1$ and it holds

$$\zeta(s,w) = w^{-s} + O(|t|^{1/2})$$

for $\Re(s)\geq 1/2$ and $|t|\geq 2$. For $\Re(s)\geq 1/2$ and $|t|\leq 11$ it holds

$$\zeta(s,w) = w^{-s} + \frac{1}{s-1} + O(1).$$

The constant in the O-terms in both cases does not depend on w.

Proof. We first show the first assertion. Since $\zeta(\overline{s},w)=\overline{\zeta(s,w)}$ it suffices to show this for $t\geq 2$. For $\Re(s)\geq 2$ it holds

$$|\zeta(s,w)-w^{-s}| \leq \sum_{n=1}^{\infty}(n+w)^{-2} \leq \sum_{n=1}^{\infty}n^{-2} = O(1)$$

because of $w>0$. So we can assume $\Re(s)\leq 2$.

[26] See Lütkebohmert (3), Theroem 6.2.4
[27] Seee Prachar (6), chapter IV, Theorem 5.2 and 5.3

It follows with partial summation (Theorem 2.1.1) for $N \in \mathbb{N}$ and $\Re(s) > 1$

$$\begin{aligned}
\sum_{n=N+1}^{[b]} (n+w)^{-s} &= s\int_N^b ([t]-N)(t+w)^{-s-1}\,dt + ([b]-N)(b+w)^{-s} \\
&= -s\int_N^b (t-[t])(t+w)^{-s-1}\,dt - b(b+w)^{-s} + N(N+w)^{-s} \\
&\quad + \int_N^b (t+w)^{-s}\,dt + ([b]-N)(b+w)^{-s} - s\int_N^b N(t+w)^{-s-1}\,dt \\
&= -s\int_N^b (t-[t])(t+w)^{-s-1}\,dt + \int_n^b (t+w)^{-s}\,dt - (b-[b])(b+w)^{-s},
\end{aligned}$$

since it holds

$$s\int_N^b t(t+w)^{-s-1}\,dt = -b(b+w)^{-s} + N(N+w)^{-s} + \int_N^b (t+w)^{-s}\,dt$$

with $a = N$ and $b \in \mathbb{R}$. Now it follows with $b \to \infty$

$$\begin{aligned}
\zeta(s,w) - w^{-s} &= \sum_{n=1}^{N}(n+w)^{-s} - s\int_N^\infty (t-[t])(t+w)^{-s-1}\,dt + \int_N^\infty (t+w)^{-s}\,dt \\
&= \sum_{n=1}^{N}(n+w)^{-s} + \frac{(N+w)^{1-s}}{s-1} - s\int_N^\infty (t-[t])(t+w)^{-s-1}\,dt, \qquad (2.8)
\end{aligned}$$

and this leads us to the analytic continuation of $\zeta(s,w)$ for $\Re(s) > 0$, since the last integral is a holomorphic function for $\Re(s) > 0$. Moreover, it follows

$$|\zeta(s,w) - w^{-s}| \le \sum_{n=1}^{N}(n+w)^{-\sigma} + \frac{(N+w)^{1-\sigma}}{|s-1|} + \frac{|s|}{\sigma}N^{-\sigma}$$

with $\Re(s) = \sigma$, considering $0 \le t - [t] < 1$. So we get for $s = \sigma + it$ with $1/2 \le \sigma \le 2$ and $t \ge 2$

$$\zeta(s,w) - w^{-s} = O\left(\sum_{n=1}^{N} n^{-1/2} + \frac{N^{1/2}}{t} + tN^{-1/2}\right) = O\left(N^{1/2} + N^{1/2}t^{-1} + tN^{-1/2}\right) = O(t^{1/2}),$$

setting $N = [t]$. The first part of the claim is proven.

It remains to show the second part. For $\Re(s) > 2$ we have again $\zeta(s,w) = w^{-s} + O(1)$, so the assertion is true. Now let $\Re(s) \le 2$. Setting $N = 1$ in (2.8), it follows

$$\zeta(s,w) - w^{-s} = (1+w)^{-s} + \frac{(1+w)^{1-s}}{s-1} + O(1)$$

for $1/2 \le \Re(s) \le 2$ and $|t| \le 11$. In this domain it holds $(1+w)^{-s} = O(1)$, and thus

$$\begin{aligned}
\frac{(1+w)^{1-s}}{s-1} &= \frac{e^{(1-s)\log(1+w)}}{s-1} = \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{(1-s)^n(\log(1+w))^n}{n!} \\
&= \frac{1}{s-1} - \sum_{n=1}^{\infty}\frac{(1-s)^{n-1}(\log(1+w))^n}{n!} = \frac{1}{s-1} + O(1)
\end{aligned}$$

since $|s-1| = O(1)$ and $0 < \log(1+w) \le \log 2$. This suffices. □

Theorem 2.5.5. [28]
Let $q \in \mathbb{N}$ and let χ be a character modulo q. Let $s = \sigma + it$ and

$$E_0 = E_0(\chi, q) := \begin{cases} 1 & \textit{for} \quad \chi = \chi_0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then it holds

$$L(s, \chi) = O((q|t|)^{1/2})$$

for $\sigma \geq 1/2$ and $|t| \geq 2$ as well as

$$L(s, \chi) = E_0 \frac{\varphi(q) q^{-1}}{s-1} + O(q^{1/2})$$

for $\sigma \geq 1/2$ and $|t| \leq 11$.
All constants are independent of q.

Proof. Let $\zeta(s, w)$ be defined as in Lemma 2.5.3. For $\Re(s) > 1$ we have

$$L(s, \chi) = \sum_{l=1}^{q} \chi(l) \sum_{n=0}^{\infty} (qn + l)^{-s} = q^{-s} \sum_{l=1}^{q} \chi(l) \zeta(s, l/q),$$

and since $\zeta(s, w)$ has a analytic continuation in $0 < \Re(s) \leq 1$, it also holds in this domain. According to Lemma 2.5.3 it follows for $\sigma \geq 1/2$ and $|t| \geq 2$

$$\begin{aligned} L(s, \chi) &= q^{-s} \sum_{l=1}^{q} \chi(l) \left((l/q)^{-s} + O(|t|^{1/2}) \right) = \sum_{l=1}^{q} \chi(l) l^{-s} + O\left(q^{-s} \sum_{l=1}^{q} |t|^{1/2} \right) \\ &= O\left(\sum_{l=1}^{q} l^{-1/2} + (q|t|)^{1/2} \right) = O((q|t|)^{1/2}), \end{aligned}$$

where the constant in the O-term is independent of q. So the first part ist proven.
Let $\sigma \geq 1/2$ and $|t| \leq 11$. Applying Lemma 2.5.3 we get

$$L(s, \chi) = q^{-s} \sum_{l=1}^{q} \left((l/q)^{-s} + \frac{1}{s-1} + O(1) \right) = \sum_{l=1}^{q} \chi(l) l^{-s} + \frac{q^{-s}}{s-1} \sum_{l=1}^{q} \chi(l) + O(q^{1/2}). \tag{2.9}$$

It holds for $\Re(s) \geq 1/2$

$$\begin{aligned} \sum_{l=1}^{q} \chi(l) l^{-s} &= O\left(\sum_{l=1}^{q} l^{-1/2} \right) = O(q^{1/2}) \quad \text{and} \\ \sum_{l=1}^{q} \chi(l) &= E_0 \varphi(q) \end{aligned}$$

according to Lemma 2.4.2. For $|s - 1| \leq 1/2$ we have

$$\begin{aligned} \frac{q^{-s}}{s-1} &= q^{-1} \frac{e^{(1-s)\log q}}{s-1} = \frac{q^{-1}}{s-1} \sum_{n=0}^{\infty} \frac{(1-s)^n (\log q)^n}{n!} = q^{-1} \left(\frac{1}{s-1} - \sum_{n=1}^{\infty} \frac{(1-s)^{n-1} (\log q)^n}{n!} \right) \\ &= q^{-1} \left(\frac{1}{s-1} + O\left(\exp\left(\frac{\log q}{2} \right) \right) \right) = \frac{q^{-1}}{s-1} + O(q^{-1/2}), \end{aligned}$$

and it follows $q^{-s}/(s-1) = O(q^{-1/2})$ for $|s - 1| > 1/2$ and $\sigma \geq 1/2$.
Putting the last three statements in (2.9), we get the claim. □

[28] See Prachar (6), chapter IV, Theorem 5.4

Now we will show that $\xi(s,\chi)$ is an analytic function of finite order and we will determine this order as well.

Theorem 2.5.6. [29]
Let χ be a primitive character modulo q and $q \in \mathbb{N}$. The function $\xi(s,\chi)$ is analytic of order 1 for $\chi \neq \chi_0^*$. If $\rho = \rho(\chi)$ runs through the zeros of this function, then it holds for every $\alpha > 1$

$$\sum_\rho |\rho|^{-1} = \infty \quad \text{and} \quad \sum_\rho |\rho|^{-\alpha} < \infty.$$

The exponent of convergence of the sequence of the zeros of $\xi(s,\chi)$ is 1. The same applies for the function $\xi(s)$ and its zeros. In particular, these functions have infinitely many zeros in $0 < \Re(s) < 1$.

Proof. First, let $\chi \neq \chi_0^*$. Since $|\xi(1-s,\overline{\chi}| = |\xi(s,\chi)|$ and Theorem 2.2.1, it suffices to show

$$\xi(s,\chi) \ll \exp(|s|^{1+\epsilon}) \tag{2.10}$$

for $|s| \to \infty$ and $\Re(s) \geq 1/2$ as well as all $\epsilon > 0$ and to show that this is no longer true, if 1 is replaced by a smaller number.
According to Theorem 2.5.5 it holds

$$L(s,\chi) \ll q^{1/2}|s|^{1/2}$$

for $|s| \to \infty$. By Stirling's formula[30]

$$\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{1/2(s+a)} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi) \ll q^{1/2(s+a+1)} \exp\left(\frac{|s|}{2}\log|s| + O(|s|)\right)$$

follows for $|s| \to \infty$, and (2.10) is proven. On the other hand it holds for real $\sigma > 2$

$$|L(\sigma,\chi)| = \left|\sum_{n=1}^\infty \mu(n)\chi(n)n^{-\sigma}\right|^{-1} \geq \left(\sum_{n=1}^\infty n^{-\sigma}\right)^{-1} > \frac{1}{2},$$

and according to Stirling's formula[31] it follows

$$\Gamma\left(\frac{\sigma+a}{2}\right) = \exp\left(\frac{\sigma\log\sigma}{2}\right) + O(\sigma)$$

for $\sigma \to \infty$. From this it follows for sufficiently large σ

$$|\xi(s,\chi)| > 1/2\left(\frac{q}{\pi}\right)^{1/2(\sigma+a)} \exp\left(\frac{\sigma\log\sigma}{2}\right) + O(\sigma),$$

so that (2.10) certainly does not apply, if 1 is replaced by a smaller number. So $\xi(s,\chi)$ is an analytic function of order 1 for $\chi \neq \chi_0^*$, and there exists no $c > 0$, such that

$$\xi(s,\chi) \ll \exp(c|s|)$$

holds for $|s| \to \infty$.
Now we consider the case $\chi = \chi_0^*$.
For $\xi(s)$, it is the same as for $\xi(s,\chi)$ with $\chi \neq \chi_0^*$, since the factor $\frac{s(s-1)}{2}$ does not matter.
Since all zeros of $\xi(s,\chi)$ resp. $\xi(s)$ lie in $0 < \Re(s) < 1$, the Theorem is proven. □

[29]See Prachar (6), chapter VII, Theorem 2.1
[30]See Prachar (6), appendix, Theorem 6.1
[31]See Prachar (6), appendix, Theorem 6.1

For every primitive character χ the functions $\xi(s,\chi)$ and $L(s,\chi)$ have infinitely many zeros in the domain $0 < \Re(s) < 1$. To consider the distribution of zeros in the critical strip accurately, further observations are still needed.

Theorem 2.5.7. [32]
Let $q \in \mathbb{N}$ and $\sigma_0 \geq -1/2$. For every character χ mod q it holds

$$L(s,\chi) \ll q^{\sigma_0+1}(|t|+2)^{\sigma_0+1}$$

for all $s \in \mathbb{C}$ with $-\sigma_0 \leq \Re(s) \leq 1/2$. Here, the implicit constant can depend on σ_0, but not on q and $t = \Im(s)$.

Proof. First, let χ be a primitive character. Then it holds according to Theorem 2.5.1

$$|L(1-s,\overline{\chi})| = 2(2\pi)^{-\sigma}q^{\sigma-1/2}|\cos(1/2(s-a)\pi)\Gamma(s)L(s,\chi)| \tag{2.11}$$

with $\sigma = \Re(s)$.
First we consider only the case $\chi \neq \chi_0^*$. We have $-\sigma_0 \leq \Re(1-s) \leq 1/2$ for $1/2 \leq \sigma \leq \sigma_0+1$. Now it follows according to Theorem 2.5.5 for $1/2 \leq \Re(s) \leq \sigma_0+1$

$$L(s,\chi) \ll q^{1/2}(|t|+2)^{1/2} \tag{2.12}$$

and according to Stirling's formula[33] it holds in the same domain

$$|\Gamma(s)| = \exp\left(\Re((s-1/2)\log s) - \Re(s) + \frac{\log(2\pi)}{2} + O\left(\frac{1}{|t|}\right)\right)$$

for $|t| \to \infty$. Suitable estimation of $\Re((s-1/2)\log s)$ leads us to

$$\begin{aligned} |\Gamma(s)| &\leq C\exp((\Re(s)-1/2)\log|t| - \pi|t|/2 + O(|t|^{-1})) \\ &= C|t|^{\Re(s)-1/2}e^{-\pi|t|/2}(1+O(|t|^{-1}) \leq C|t|^{\sigma_0+1/2}e^{-\pi|t|/2}(1+O(|t|^{-1}) \end{aligned}$$

with a suitable constant $C > 0$, while we notice that the behaviour of $\exp(|t|^{-1})$ is for $|t| \to \infty$ the same like $1+|t|^{-1}$. So we get for $1/2 \leq \Re(|t|s) \leq \sigma_0+1$

$$|\Gamma(s)| \ll (|t|+2)^{\sigma_0+1/2}e^{-\pi|t|/2}, \tag{2.13}$$

where the constant may depend on σ_0. It follows for every $s \in \mathbb{C}$

$$|\cos(1/2(s-a)\pi)| \leq \frac{1}{2}\left(e^{\pi t/2} + e^{-\pi t/2}\right) \leq e^{\pi|t|/2}. \tag{2.14}$$

Putting (2.12), (2.13) and (2.14) in (2.11), we get for $1/2 \leq \Re(s) \leq \sigma_0+1$

$$L(1-s,\overline{\chi}) \ll q^{\sigma_0+1}(|t|+2)^{\sigma_0+1}.$$

Since with χ is also $\overline{\chi}$ primitive and not the principal character, the claim of the assertion follows by replacing $\overline{\chi}$ by χ and $1-s$ by s.
For $L(s,\chi_0^*) = \zeta(s)$ it holds $a = a(\chi_0^*) = 0$, and it follows from (2.14) and Theorem 2.5.5

$$\cos(1/2(s+a)\pi)L(s,\chi_0^*) = \cos(\pi s/2)\zeta(s) \ll (|t|+2)^{1/2}e^{\pi|t|/2}$$

for $1/2 \leq \Re(s) \leq \sigma_0+1$, since $\cos(\pi s/2)\zeta(s)$ is holomorphic at $s=1$.

[32]See Prachar (6), chapter VII, Theorem 3.1
[33]See Prachar (6), appendix, Theorem 6.1

We put this with (2.13) in (2.11), and get the claim also for $\chi = \chi_0^*$.
We assume χ is not primitive. Then let χ^* be the associated primitive character with modulus q^*. So it holds for $-\sigma_0 \leq \sigma \leq 1/2$

$$\left| \prod_{p|q, p\nmid q^*} \left(1 - \frac{\chi^*(p)}{p^s}\right) \right| \leq \prod_{p|q, p\nmid q^*} (1 + p^{\sigma_0}) \leq \prod_{p|(q/q^*)} (1 + p^{\sigma}) \ll \left(\frac{q}{q^*}\right)^{\sigma_0 + 1},$$

since for $\sigma_0 \geq 0$ we have $1 + p^{\sigma_0} \leq 2p^{\sigma_0} \leq p^{\sigma_0+1}$, against which there is for $-1/2 \leq \sigma_0 < 0$ and an arbitrary integer $m \geq 2$

$$\begin{aligned} \prod_{p|m}(1 + p^{\sigma_0}) &\leq \prod_{p \leq \log m/\log(2)} (1 + p^{\sigma}) \leq \exp\left(\sum_{p \leq 2\log m} \log(1 + p^{\sigma_0})\right) \leq \exp\left(\sum_{p \leq 2\log m} p^{\sigma_0}\right) \\ &\leq \exp\left(\sum_{n \leq 2\log m} n^{\sigma_0}\right) \leq \exp(c(\log m)^{\sigma_0+1}) \ll m^{\sigma_0+1}, \end{aligned}$$

while the constant depends on σ_0. The claim follows, since it holds according to Theorem 2.5.2

$$L(s, \chi^*) = \prod_{p|q, p\nmid q^*} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1} L(s, \chi)$$

and it is already shown that

$$L(s, \chi^*) \ll (q^*)^{\sigma_0+1}(|t| + 2)^{\sigma_0+1}$$

for primitive characters. □

Combining the Theorems 2.5.5, 2.5.6 and 2.5.7 we obtain with the functional equation for $L(s, \chi)$ the following Theorem:

Theorem 2.5.8. [34]
Let $q \in \mathbb{N}$, χ be a character modulo q and $\alpha \in \mathbb{R}$. Then it holds for $\Re(s) \geq \alpha$

$$L(s, \chi) = E_0 \frac{\varphi(q)/q}{s - 1} + O(q^c(|t| + 2)^c)$$

for a certain $c = c(\alpha)$ and E_0 defined by Theorem 2.5.5. The implicit constant in the O-term depends also on α.

Theorem 2.5.9. [35]
Let $N_\chi(T)$ denote the number of zeros of $L(s, \chi)$ in $\{s \in \mathbb{C}\colon s = \sigma + it, 0 \leq \sigma < 1, |t| \leq T\}$. It follows for $q \in \mathbb{N}$, $T \geq 0$ and an arbitrary character $\chi \bmod q$

$$N_\chi(T + 1) - N_\chi(T) \ll \log(q(T + 2)),$$

while the implicit constant is independent of q and T.

[34] See Prachar (6), chapter VII, Theorem 3.2
[35] See Prachar (6), chapter VII, Theorem 3.3

Proof. Let χ_0 be the principal character modulo q, and first, let $\chi \neq \chi_0$. We apply Theorem 2.2.2 on $L(s,\chi)$ and set $s_0 = 2 + iT$, $r = 1/2$ and R sufficiently large, such that the domain $0 \leq \Re(s) < 1$, $T < \Im(s) \leq T+1$ is entirely contained in $|s - s_0| \leq R/2$, for example $R = 6$.
According to Theorem 2.5.8 it holds $L(s,\chi) \ll q^c(|t|+2)^c$ for $|s - s_0| \leq R$. Also one has

$$|L(s,\chi)| \geq \left| \sum_{n=1}^{\infty} \mu(n) \frac{\chi(n)}{n^{s_0}} \right|^{-1} \geq \frac{1}{\zeta(2)}$$

using the formula for $1/L(s,\chi)$ in Lemma 2.5.2. The claim follows if we insert the zeros of $L(s,\chi)$ in $0 \leq \Re(s) < 1$ und $T < t \leq T+1$ for $s_1, \ldots, s_m$ in Theorem 2.2.2, since the zeros of $L(s,\chi)$ in $-T-1 \leq t < -T$ are complex conjugated to the zeros of $L(s,\overline{\chi})$ in $T < t \leq T+1$.

The claim follows also for $\chi = \chi_0$ by applying Theorem 2.2.2 in the same way on the function $(s-1)L(s,\chi_0)$. Notice, that it holds $a \leq \Re(s) \leq b$ in every strip, such that $s - 1 \ll |t| + 2$, where the implicit constant may depend on a and b. Thus, Theorem 2.5.8 leads us to $(s-1)L(s,\chi_0) \ll q^{c_1}(|t|+2)^{c_1}$ with a certain $c_1 \in \mathbb{R}^+$ in each such strip. □

Finally, we give an explicit formula:

Lemma 2.5.4. *(explicit formula)*[36]
There is

$$\sum_{n \leq x} \Lambda(n)\chi(n)n^{-it_0} = \sum_{|\Im\rho - t_0| \leq T} \frac{x^{\rho - it_0}}{\rho - it_0} + O\left(\frac{x \log^2(xt_0)}{T}\right),$$

where ρ represents the zeros of $L(s,\chi)$ in the critical strip.

Proof. This Lemma is proven similar as the formula (3.15) in Brüdern (5) on page 111. □

Definition 2.5.2. Under the <u>Riemann hypothesis (RH)</u> we understand the conjecture, that all zeros of the Riemann- ζ- function in the critical strip $0 < \Re(s) < 1$ lie on the line with $\Re(s) = 1/2$.

This is still an open question. Most mathematicians, however, assume the truth of this conjecture. There is also a conjecture to the zeros of the L- series in the critical strip.

Definition 2.5.3. The <u>generalized Riemann hypothesis (GRH)</u> states, that all zeros of $L(s,\chi)$ with all characters with the modul q with $q \geq 1$ lie in $0 < \Re(s) < 1$ on the line $\Re(s) = 1/2$.

In the following we assume the truth of the (GRH). Because of the functional equation of $\xi(s,\chi)$ (GRH) is equivalent to the statement, that there are no zeros of $L(s,\chi)$ in $1/2 < \Re(s) < 1$, since the zeros of $L(s,\chi)$ in the critical strip are symmetrical to the line $\Re(s) = 1/2$.

[36] See Brüdern (5), chapter III,page 111

Chapter 3

Mean value Theorems and the Hybrid Sieve

3.1 Introduction

In this chapter we want to estimate sums and integrals of Dirichlet- series in order to be able to calculate discrete or continuous mean values. With this tool, we can estimate exponential sums with the Möbius- function.

The idea behind this is the following one:
We consider a function that takes a very large value at a point $s_0 \in \mathbb{C}$. In a small neighborhood of the point s_0 this still applies to the said function. So we can estimate the number of points s_0 by a continuous average of this function by estimating this function with a Dirichlet- polynomial in the neighborhood of the point s_0.

3.2 The Large Sieve

We now consider the method of the Large Sieve according to Harold Davenport and P.X. Gallagher. An objective of the Large Sieve is to estimate a discrete mean by a continuous mean.
First, we consider a few lemmata.

Lemma 3.2.1. [1]
Let $u, \delta \in \mathbb{R}^+$ and a complex valued function $F(x)$ be continous in $[u-\delta/2, u+\delta/2]$ with a continous derivative in $(u-\delta/2, u+\delta/2)$. Then

$$\begin{aligned} F(u) &= \delta^{-1}\int_{u-\delta/2}^{u}\left(x-\left(u-\frac{\delta}{2}\right)\right)F'(x)\,dx+\delta^{-1}\int_{u}^{u+\delta/2}\left(x-\left(u+\frac{\delta}{2}\right)\right)F'(x)\,dx \\ &\quad +\frac{\delta}{2}\int_{u-\delta/2}^{u+\delta/2}F(x)\,dx. \end{aligned}$$

[1] See Richert (8), chapter II, page 30 f.

Proof. By partial integration we find

$$\delta^{-1}\int_{u-\delta/2}^{u}\left(x-\left(u-\frac{\delta}{2}\right)\right)F'(x)\,dx = \delta^{-1}\left[F(x)\left(x-\left(u-\frac{\delta}{2}\right)\right)\right]_{u-\delta/2}^{u} - \delta^{-1}\int_{u-\delta/2}^{u}F(x)\,dx$$
$$= \frac{F(u)}{2} - \delta^{-1}\int_{u-\delta/2}^{u}F(x)\,dx.$$

Just arises

$$\delta^{-1}\int_{u}^{u+\delta/2}\left(x-\left(u+\frac{\delta}{2}\right)\right)F'(x)\,dx = \frac{F(u)}{2} - \delta^{-1}\int_{u}^{u+\delta/2}F(x)\,dx.$$

This leads to the claim. □

Lemma 3.2.2. [2]
Let $u, \delta \in \mathbb{R}^+$ and a complex valued function $f(x)$ be continous in $[u-\delta/2, u+\delta/2]$ with a continous derivative in $(u-\delta/2, u+\delta/2)$. Then

$$|f(u)|^2 \leq \int_{u-\delta/2}^{u+\delta/2}|f(x)f'(x)|\,dx + \delta^{-1}\int_{u-\delta/2}^{u+\delta/2}|f(x)|^2\,dx.$$

Proof. Lemma 3.2.1 gives

$$|F(u)| \leq \frac{1}{2}\int_{u-\delta/2}^{u+\delta/2}|F'(x)|\,dx + \delta^{-1}\int_{u-\delta/2}^{u+\delta/2}|F(x)|\,dx.$$

We put $F(x) := f^2(x)$ and get with $F'(x) = 2f(x)f'(x)$

$$|f(u)|^2 \leq \frac{1}{2}\int_{u-\delta/2}^{u+\delta/2}2|f(x)f'(x)|\,dx + \delta^{-1}\int_{u-\delta/2}^{u+\delta/2}|f(x)|^2\,dx$$

and thus the claim. □

Theorem 3.2.1. *(The Large Sieve)*[3]
Let $a_n \in \mathbb{C}$, $M \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_R \in \mathbb{R}$. It should also be

$$S(\alpha) := \sum_{n=M+1}^{M+N} a_n e(n\alpha).$$

Then it holds

$$\sum_{r=1}^{R}|S(\alpha_r)|^2 \leq \sum_{r=1}^{R}\int_{\alpha_r-\delta/2}^{\alpha_r+\delta/2}\frac{1}{\delta}|S(\alpha)|^2 + |S(\alpha)S'(\alpha)|\,d\alpha.$$

Proof. We set $f := S(\alpha_r)$ in Lemma 3.2.2 and sum over all $r = 1, \ldots, R$. We get

$$\sum_{r=1}^{R}|S(\alpha_r)|^2 \leq \sum_{r=1}^{R}\int_{\alpha_r-\delta/2}^{\alpha_r+\delta/2}\left(\frac{1}{\delta}|S(\alpha)|^2 + |S(\alpha)S'(\alpha)|\right)d\alpha.$$

□

[2]See Richert (8), chapter II, Lemma 2.1
[3]See Davenport (2), page 156 f.

Theorem 3.2.2. [4]
For any complex numbers $a_n \in \mathbb{C}$ as well as $M \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\alpha, \alpha_1, \alpha_2, \dots, \alpha_R \in \mathbb{R}$ with $\alpha_i \neq \alpha_j$ for $i \neq j$ set

$$S(\alpha) := \sum_{n=M+1}^{M+N} a_n e(n\alpha).$$

Put

$$\delta := \min_{\substack{r,s,\\ r\neq s}} \|\alpha_r - \alpha_s\| \tag{3.1}$$

for $R \geq 2$ and set $\delta := \infty$, if $R = 1$. Then

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \leq (\pi N + \delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2.$$

The idea of the proof of Theorem 3.2.2 is contained in the following

Lemma 3.2.3. [5]
Let $N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_R \in \mathbb{R}$ with $R \geq 2$ and

$$\delta := \min_{\substack{r,s\\ r\neq s}} \|\alpha_r - \alpha_s\|.$$

Put

$$V(\alpha) := \sum_{-N/2 < m \leq N/2} b_m e(m\alpha)$$

with $b_m \in \mathbb{C}$. Then the inequality

$$\sum_{r=1}^{R} |V(\alpha_r)|^2 \leq \Delta(N, \delta) \sum_{-N/2 < m < N/2} |b_m|^2 \tag{3.2}$$

for $b_m \in \mathbb{C}$ and some positive function $\Delta(N,\delta)$ depending only on N and δ, implies

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \leq \Delta(N, \delta) \sum_{n=M+1}^{M+N} |a_n|^2$$

for any $M \in \mathbb{Z}$ and for all $a_n \in \mathbb{C}$.

Proof. We consider

$$\begin{aligned} V(\alpha) &= e\left(-\left(M + \left[\frac{N+1}{2}\right]\right)\alpha\right) S(\alpha) = \sum_{M<n\leq M+N} a_n e\left(\left(n - M - \left[\frac{N+1}{2}\right]\right)\alpha\right) \\ &= \sum_{-\left[\frac{N+1}{2}\right] < m \leq -\left[\frac{N+1}{2}\right]+N} b_m e(m\alpha) = \sum_{-N/2<m\leq N/2} b_m e(m\alpha) \end{aligned}$$

with $M = -\left[\frac{N+1}{2}\right]$ and $b_m = a_{m+M+\left[\frac{N+1}{2}\right]}$.
We get $|S(\alpha)| = |V(\alpha)|$ and the claim follows using the condition (3.2). □

[4] See Richert (8), Theorem 2.1
[5] See Richert (8), Lemma 2.2

Proof. (Proof of Theorem 3.2.2)
By Lemma 3.2.3, we shall consider V instead of S. Since $V(\alpha)$ is of period 1 we can also suppose

$$0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_R < 1.$$

By the pigeonhole- principle

$$\delta \leq \frac{1}{R} \tag{3.3}$$

follows easily. Now from Lemma 3.2.2 with $f = V$ and $u = \alpha_r$, one has

$$|V(\alpha_r)|^2 \leq \int_{I_r} |V(x)V'(x)|\, dx + \delta^{-1} \int_{I_r} |V(x)|^2\, dx$$

where $I_r := [\alpha_r - \delta/2, \alpha_r + \delta/2]$. By (3.1) we have for $r \neq s$

$$\delta \leq \|\alpha_r - \alpha_s\| \leq |\alpha_r - \alpha_s|$$

so that our intervals I_r do not overlap and their total length, i.e., length of

$$\bigcup_{r=1}^{R} I_r$$

equals $\delta R \leq 1$ on recalling (3.3).
Summing over all $r = 1 \ldots R$ and using that $V(x)$ as well as $V'(x)$ are of period 1, we can replace the integration on the right by $\int_0^1$.
Thus, we get by employing Schwarz's inequality

$$\sum_{r=1}^{R} |V(\alpha_r)|^2 \leq \left(\int_0^1 |V(x)|^2\, dx\right)^{1/2} \left(\int_0^1 |V'(x)|^2\, dx\right)^{1/2} + \delta^{-1} \int_0^1 |V(x)|^2\, dx.$$

Now it follows from

$$\int_0^1 \left|\sum_n a_n e(nx)\right|^2 dx = \sum_{n_1,n_2} a_{n_1}\overline{a}_{n_2} \int_0^1 e((n_1 - n_2)x)\, dx = \sum_n |a_n|^2$$

that

$$\begin{aligned} \int_0^1 |V(x)|^2\, dx &= \sum_{-N/2<m\leq N/2} |b_m|^2 \quad \text{and} \\ \int_0^1 |V'(x)|^2\, dx &= \sum_{-N/2<m\leq N/2} |2\pi m b_m|^2 \leq \pi^2 N^2 \sum_{-N/2<m\leq N/2} |b_m|^2 \end{aligned}$$

hold. Hence,

$$\begin{aligned} \sum_{r=1}^{R} |V(\alpha_r)|^2 &\leq \left(\sum_{-N/2<m\leq N/2} |b_m|^2\right)^{1/2} \cdot \pi N \left(\sum_{-N/2<m\leq N/2} |b_m|^2\right)^{1/2} + \delta^{-1} \sum_{-N/2<m\leq N/2} |b_m|^2 \\ &= (\pi N + \delta^{-1}) \sum_{-N/2<m\leq N/2} |b_m|^2. \end{aligned} \tag{3.4}$$

This corresponds to the condition to Lemma 3.2.2 with $\Delta(N, \delta) = \pi N + \delta^{-1}$.
With (3.4) and Lemma 3.2.2 the claim follows. □

Corollary 3.2.1. [6]
Let $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, $N \in \mathbb{N}$, $M \in \mathbb{Z}$ and $q, Q \in \mathbb{N}$. Then

$$\sum_{q \leq Q} \sum_{\substack{l=1 \\ (l,q)=1}}^{q} \left| S\left(\frac{l}{q}\right) \right|^2 \leq (\pi N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2.$$

Proof. The simplest case of an application of the Large Sieve would be an uniformly distribution of points α_r in the way

$$\alpha_r = \frac{r}{R}$$

with $1 \leq \ldots \leq R$, so that

$$\delta = \frac{1}{R}.$$

In this case, Theorem 3.2.2 leads to

$$\sum_{r=1}^{R} \left| S\left(\frac{r}{R}\right) \right|^2 \leq (\pi N + R) \sum_{n=M+1}^{M+N} |a_n|^2.$$

In the case at hand, the more interesting one, the points α_r are not uniformly distributed. Here we have

$$\alpha_r = \frac{l}{q}$$

with $1 \leq l \leq q \leq Q$ and $(l, q) = 1$.
This is the Farey- series of order Q. For two different Farey- series we get for $Q \geq 2$

$$\left\| \frac{l}{q} - \frac{l'}{q'} \right\| = \left\| \frac{lq' - ql'}{qq'} \right\| \geq \frac{1}{qq'} \geq \frac{1}{Q^2},$$

that means

$$\delta \geq \frac{1}{Q^2}.$$

So it follows from Theorem 3.2.2

$$\sum_{r=1}^{R} \left| S\left(\frac{l}{q}\right) \right|^2 \leq (\pi N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2$$

and therefore it suffices by summation over $l \leq q$ with $(l, q) = 1$ and $q \leq Q$ instead of $r = 1 \ldots R$. □

[6] See Richert (8), Theorem 2.2

3.3 Mean values of character sums

Lemma 3.3.1. [7]
It holds for every character $\chi \bmod q$

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \left| \sum_{l=1}^{q} \overline{\chi}(l) S\left(\frac{l}{q}\right) \right|^2 = \sum_{\substack{l=1 \\ (l,q)=1}}^{q} \left| S\left(\frac{l}{q}\right) \right|^2.$$

Proof. There is

$$\sum_{\chi \bmod q} \left| \sum_{l=1}^{q} \overline{\chi}(l) S\left(\frac{l}{q}\right) \right|^2 = \sum_{l_1,l_2=1}^{q} \sum_{\chi \bmod q} \overline{\chi}(l_1)\chi(l_2) S\left(\frac{l_1}{q}\right) S\left(-\frac{l_2}{q}\right).$$

According to the third part of Lemma 2.4.2 it holds

$$\sum_{\chi \bmod q} \overline{\chi}(l)\chi(n) := \begin{cases} \varphi(q) & \text{for } n \equiv l \bmod q, \\ 0 & \text{otherwise,} \end{cases}$$

and we have

$$\sum_{\chi \bmod q} \left| \sum_{l=1}^{q} \overline{\chi}(l) S\left(\frac{l}{q}\right) \right|^2 = \varphi(q) \sum_{\substack{l=1, \\ (l,q)=1}}^{q} \left| S\left(\frac{l}{q}\right) \right|^2.$$

□

Theorem 3.3.1. [8]
Let $Q \in \mathbb{N}$. *For any character* $\chi \bmod q$ *and for any complex numbers* a_n *satisfying* $a_n = 0$ *for* $(n,q) \neq 1$ *for all* $q \leq Q$ *write*

$$X(\chi) := \sum_{n=M+1}^{M+N} a_n \chi(n).$$

Then we have

$$\sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \bmod q} |\tau(\chi)|^2 \, |X(\chi)|^2 \leq (N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2$$

with

$$\tau(\chi) := \sum_{l=1}^{q} \chi(l) e\left(\frac{l}{q}\right) = \sum_{l \bmod q} \chi(l) e\left(\frac{l}{q}\right).$$

Proof. Since $(n,q) = 1$ is considered, l and also nl run through a complete residue system. Therefore

$$\tau(\overline{\chi}) a_n \chi(n) = \sum_{l=1}^{q} \overline{\chi}(nl) a_n e\left(n\frac{l}{q}\right) \chi(n) = \sum_{l=1}^{q} \overline{\chi}(l) a_n e\left(n\frac{l}{q}\right)$$

applies.

[7] See Richert (8), page 26
[8] See Richert (8), Theorem 3.1

So we get

$$\tau(\overline{\chi})X(\chi) = \sum_{l=1}^{q} \overline{\chi}(l) S\left(\frac{l}{q}\right)$$

with $S(\alpha) = \sum_{n=M+1}^{M+N} a_n e(n\alpha)$. We apply Lemma 3.3.1 and get

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} |\tau(\chi)|^2 \, |X(\chi)|^2 = \sum_{l=1}^{q} \left| S\left(\frac{l}{q}\right)\right|^2.$$

Using Corollary 3.2.1 we finish. □

Theorem 3.3.2. [9]
For any character $\chi \bmod f$, $r \in \mathbb{N}$ *and for arbitrary complex numbers* a_n *set*

$$X_r(\chi) = \sum_{n=M+1}^{M+N} a_n \chi(n) c_r(n)$$

with $c_r(n) = \sum_{l=1,(l,r)=1}^{r} e\left(n\frac{l}{r}\right)$ *for all* $r \in \mathbb{N}$ *and* $n \in \mathbb{Z}$.
Then

$$\sum_{\substack{rf \le Q,\\ (r,f)=1}} \frac{f}{\varphi(rf)} \sideset{}{^*}\sum_{\chi \bmod f} |X_r(\chi)|^2 \le (\pi N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2,$$

and $\sideset{}{^*}\sum_{\chi \bmod f}$ *denotes the sum over all primitive characters* $\chi \bmod f$.

Proof. We have for any $q \in \mathbb{N}$ by Lemma 3.3.1

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \left| \sum_{l=1}^{q} \overline{\chi}(l) S\left(\frac{l}{q}\right)\right|^2 = \sum_{\substack{l=1,\\ (l,q)=1}}^{q} \left| S\left(\frac{l}{q}\right)\right|^2.$$

Using $q = rf$ and $\chi(l) = \chi^*(l)$ for $(l,q) = 1$ and summation over $q \le Q$ gives

$$\sum_{\substack{rf \le Q,\\ (r,f)=1}} \frac{1}{\varphi(rf)} \sideset{}{^*}\sum_{\chi \bmod f} \left| \sum_{\substack{l=1,\\ (l,q)=1}}^{q} \overline{\chi}(l) S\left(\frac{l}{q}\right)\right|^2 \le \sum_{q \le Q} \sum_{\substack{l=1,\\ (l,q)=1}}^{q} \left| S\left(\frac{l}{q}\right)\right|^2. \tag{3.5}$$

We notice for any primitive character $\chi^* \bmod f$ with $q = rf$ and $(r,f) = 1$ as well as every l with $(l,q) = 1$

$$\begin{aligned} l &= \lambda r + \mu f, \\ (\lambda, f) &= 1 \text{ and } (\mu, r) = 1, \end{aligned}$$

and we will have

$$\begin{aligned} \sum_{\substack{l=1,\\ (l,q)=1}}^{q} \overline{\chi}(l) S\left(\frac{l}{q}\right) &= \sum_{n=M+1}^{M+N} a_n \sum_{\substack{\lambda=1,\\ (\lambda,f)=1}}^{f} \overline{\chi}(\lambda r) \sum_{\substack{\mu=1,\\ (\mu,r)=1}}^{r} e\left(n\left(\frac{\lambda}{f} + \frac{\mu}{r}\right)\right) \\ &= \overline{\chi}(r) \sum_{n=M+1}^{M+N} a_n \sum_{\substack{\lambda=1,\\ (\lambda,f)=1}}^{f} \overline{\chi}(\lambda) e\left(n\frac{\lambda}{f}\right) \sum_{\substack{\mu=1,\\ (\mu,r)=1}}^{r} e\left(n\frac{\mu}{r}\right) = \overline{\chi}(r)\tau(\overline{\chi})X_r(\chi), \end{aligned}$$

[9] See Richert (8), Theorem 3.2

because of the Gaussian Sum

$$\tau(\overline{\chi})\chi(n) = \sum_{l \bmod q} \overline{\chi}(l) e\left(n\frac{l}{q}\right)$$

for a primitive character $\chi^* \bmod q$ with $n \in \mathbb{Z}$. Since $(r, f) = 1$, we have by Lemma 2.4.1 that $|\overline{\chi}(r)| = 1$ and further by evaluation of Gaussian Sums[10] $|\tau(\overline{\chi})|^2 = f$. Thus

$$(3.5) \leq \sum_{\substack{rf \leq Q,\\ (r,f)=1}} \frac{1}{\varphi(rf)} \sideset{}{^*}\sum_{\chi \bmod q} |\overline{\chi}(n)|^2 |\tau(\overline{\chi})|^2 |X_r(\chi)|^2 \leq \sum_{\substack{rf \leq Q,\\ (r,f)=1}} \frac{f}{\varphi(rf)} \sideset{}{^*}\sum_{\chi \bmod q} |X_r(\chi)|^2$$

Using Corollary 3.2.1 we get the claim. □

We get the following Theorem:

Theorem 3.3.3. [11]
For any character $\chi \bmod q$ and for any complex numbers a_n, define

$$X(\chi) = \sum_{n=M+1}^{M+N} a_n \chi(n).$$

Then we have

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi \bmod f} |X(\chi)|^2 \leq (\pi N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2.$$

Proof. Since $c_1(n) = 1$, this is a special case of Theorem 3.3.2. □

3.4 Hybrid Sieve

Lemma 3.4.1. [12]
Let

$$D(t) := \sum_{\nu} c(\nu) e(\nu t),$$

where ν runs through a countable set of real numbers and the coefficients $c(\nu) \in \mathbb{C}$ are subjected to the condition

$$\sum_{\nu} |c(\nu)| < \infty. \tag{3.6}$$

Let δ and T be positive real numbers satisfying $\delta \cdot T \leq (2\pi)^{-1}$. Then, for some absolute constant c_0,

$$\int_{-T}^{T} |D(t)|^2 \, dt \leq c_0 \int_{-\infty}^{\infty} |C_\delta(y)|^2 \, dy$$

holds, where

$$C_\delta(y) := \delta^{-1} \sum_{|y-\nu| < \delta/2} c(\nu). \tag{3.7}$$

[10] See Fischer (9), page 45 f.
[11] See Richert (8), Theorem 3.3
[12] See Richert (8), Lemma 5.1

Proof. For the proof we use two results from the theory of Fourier- transforms. Introduce

$$F_\delta(y) = \begin{cases} \delta^{-1} & \text{for } |y| < \delta/2 \\ 0 & \text{otherwise,} \end{cases}$$

with

$$C_\delta(y) := \sum_\nu c(\nu) F_\delta(y-\nu).$$

In view of (3.6), $C_\delta(y)$ is a bounded integrable function and hence belongs to $L^2(-\infty,\infty)$. Therefore by Plancherel's theorem, C_δ has a Fourier- transform $\widehat{C}_\delta$ and further, by Parseval's formula, we have

$$\int_{-\infty}^{\infty} |C_\delta(y)|^2 \, dy = \int_{-\infty}^{\infty} \left|\widehat{C}_\delta(t)\right|^2 \, dt.$$

Now one has

$$\begin{aligned} \widehat{C}_\delta(t) &= \int_{-\infty}^{\infty} C_\delta(y) e(yt)\, dy = \sum_\nu c(\nu) \int_{-\infty}^{\infty} F_\delta(y-\nu) e(yt)\, dy \\ &= \sum_\nu c(\nu) e(\nu t) \int_{-\infty}^{\infty} F_\delta(x) e(xt)\, dx = D(t) \cdot \widehat{F}_\delta(t) \end{aligned}$$

on using the substitution $y - \nu = x$. Also it holds for the Fourier- transform of $F_\delta(y)$

$$\widehat{F}_\delta(t) = \int_{-\infty}^{\infty} F_\delta(x) e(xt)\, dx = \delta^{-1} \int_{-\delta/2}^{\delta/2} e(xt)\, dx = \frac{\sin(\pi\delta t)}{\pi\delta t}.$$

Thus

$$\int_{-\infty}^{\infty} |C_\delta(y)|^2 \, dy = \int_{-\infty}^{\infty} \left|\widehat{C}_\delta(t)\right|^2 \, dt = \int_{-\infty}^{\infty} \left|D(t)\cdot\widehat{F}_\delta(t)\right|^2 \, dt \geq \int_{-T}^{T} \left|D(t)\cdot\widehat{F}_\delta(t)\right|^2 \, dt.$$

For $t \leq T$ we note

$$\left|\widehat{F}_\delta(t)\right| \geq \frac{\sin(\pi\delta T)}{\pi\delta T} \geq \frac{1}{\sqrt{c_0}}$$

and this leads us to

$$\frac{1}{c_0} \int_{-T}^{T} |D(t)|^2 \, dt \leq \int_{-T}^{T} \left|D(t)\cdot\widehat{F}_\delta(t)\right|^2 \, dt \leq \int_{-\infty}^{\infty} |C_\delta(y)|^2 \, dy.$$

□

Lemma 3.4.2. [13]
For $a_n \in \mathbb{C}$, let

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Then, for $T \geq 1$

$$\int_{-T}^{T} \left|\sum_{n=1}^{\infty} a_n n^{-it}\right|^2 dt \leq c_1 T^2 \int_0^\infty \left|\sum_{x<n<xe^{1/T}} a_n\right|^2 \frac{dx}{x}$$

holds with some absolute constant c_1.

[13] See Richert (8), Lemma 5.2

Proof. In Lemma 3.4.1 we choose $\nu := -\frac{\log n}{2\pi}$ and $c(\nu) := a_n$ with $n \in \mathbb{N}$ and note that (3.6) is satisfied because of $\sum_{n=1}^{\infty} |a_n| < \infty$. Further we put $\delta := (2\pi T)^{-1}$ and

$$y := -\frac{1}{2\pi}\left(\log x + \frac{1}{2T}\right)$$

with $x > 0$, so that $\delta T \leq \frac{1}{2\pi}$ is fulfilled and the condition of summation in (3.6) reads

$$-\frac{\delta}{2} < y - \nu < \frac{\delta}{2} \Leftrightarrow -\frac{1}{4\pi T} < -\frac{1}{2\pi}\left(\log x + \frac{1}{2T}\right) + \frac{1}{2\pi}\log n < \frac{1}{4\pi T}.$$

Hence

$$0 < -\frac{1}{2\pi}\log x + \frac{1}{2\pi}\log n < \frac{1}{2\pi T} \Leftrightarrow \log x < \log n < \log x + \frac{1}{T}.$$

Therefore (3.7) yields (with the choice above)

$$\int_{-T}^{T}\left|\sum_{n=1}^{\infty} a_n n^{-it}\right|^2 dt \leq c_0 \int_0^{\infty} 2\pi T^2 \left|\sum_{x<n<xe^{1/T}} a_n\right|^2 \frac{dx}{x}.$$

This completes the proof of the Lemma. □

Now we get

Theorem 3.4.1. *(Hybrid Sieve)*[14]
For any $a_n \in \mathbb{C}$, let

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Then, for $T \geq 1$

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi \bmod q}^{*}\int_{-T}^{T}\left|\sum_{n=1}^{\infty} a_n\chi(n)n^{-it}\right|^2 dt \leq 2\pi c_1 \sum_{n=1}^{\infty}(TQ^2+n)|a_n|^2$$

holds with the constant c_1 of Lemma 3.4.2.

Proof. We use Lemma 3.4.2 with $a_n\chi(n)$ instead of a_n and then apply

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi \bmod q}^{*}$$

to the resulting. We get

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi \bmod q}^{*}\int_{-T}^{T}\left|\sum_{n=1}^{\infty} a_n\chi(n)n^{-it}\right|^2 dt \leq c_1T^2\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi \bmod q}^{*}\int_0^{\infty}\left|\sum_{x<n<xe^{1/T}} a_n\chi(n)\right|^2\frac{dx}{x}$$
$$= c_1T^2\int_0^{\infty}\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi \bmod q}^{*}\left|\sum_{x<n<xe^{1/T}} a_n\chi(n)\right|^2\frac{dx}{x}.$$

[14] See Richert (8), Theorem 5.1

We use Theorem 3.3.3 with $M := [x]$ and $N \le x(e^{1/T}-1)+1$. This leads us to

$$\begin{aligned}
\sum_{q\le Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\int_{-T}^{T}\left|\sum_{n=1}^{\infty}a_n\chi(n)n^{-it}\right|^2 dt &\le \pi c_1 T^2\int_0^\infty \left(x\left(e^{1/T}-1\right)+1+Q^2\right)\sum_{x<n<xe^{1/T}}|a_n|^2\,\frac{dx}{x}\\
&= \pi c_1 T^2\int_0^\infty \sum_{x<n<xe^{1/T}}\left(\left(e^{1/T}-1\right)+\frac{Q^2+1}{x}\right)|a_n|^2\,dx\\
&= \pi c_1 T^2\sum_{n=1}^{\infty}\int_{ne^{-1/T}}^{n}\left(\left(e^{1/T}-1\right)+\frac{Q^2+1}{x}\right)dx|a_n|^2\\
&= \pi c_1 T^2\sum_{n=1}^{\infty}\left(\left(e^{1/T}-1\right)\left(n-ne^{-1/T}\right)\right.\\
&\quad \left.+(Q^2+1)\log\left(\frac{n}{ne^{-1/T}}\right)\right)|a_n|^2\\
&= \pi c_1 T^2\sum_{n=1}^{\infty}\left(n\left(e^{1/T}-1\right)\left(1-e^{-1/T}\right)+\frac{Q^2+1}{T}\right)|a_n|^2\\
&= \pi c_1\sum_{n=1}^{\infty}\left(nT^2\left(e^{1/T}-1\right)\left(1-e^{-1/T}\right)+T(Q^2+1)\right)|a_n|^2\\
&\le 2\pi c_1\sum_{n=1}^{\infty}(n+TQ^2)|a_n|^2
\end{aligned}$$

where we have employed the estimate $T^2\left(e^{1/T}-1\right)\left(1-e^{-1/T}\right)\le 2$ for $T\ge 1$. □

3.5 Special Dirichlet- series

Definition 3.5.1. [15]
Let

$$\omega(n) := \begin{cases} 1 & \text{for } n\le u,\\ \frac{\log(uv/n)}{\log v} & \text{for } u<n\le uv,\\ 0 & \text{for } n>uv,\end{cases}$$

with $u,v\ge 1$ and $u,uv\notin\mathbb{N}$ will be determined later.

Theorem 3.5.1. [16]
For $\Re(s)\ge 1/2$ there is

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)\omega(n)}{n^s} = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s+z,\chi)\cdot\frac{u^z(v^z-1)}{z^2\log v}\,dz.$$

Proof. We have

$$\begin{aligned}
F(s,\chi) &:= -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s+z,\chi)\cdot\frac{u^z(v^z-1)}{z^2\log v}\,dz = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^{s+z}}\frac{u^z(v^z-1)}{z^2\log v}\,dz\\
&= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^s}\int_{2-i\infty}^{2+i\infty}\left(\left(\frac{uv}{n}\right)^z-\left(\frac{u}{n}\right)^z\right)\frac{1}{z^2\log v}\,dz.
\end{aligned}$$

[15] See Fischer (9), page 83
[16] See Fischer (9), page 83 f.

Partial Integration ensures

$$
\begin{aligned}
F(s,\chi) &= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^s}\left(\left[-\frac{1}{z\log v}\left(\left(\frac{uv}{n}\right)^z-\left(\frac{u}{n}\right)^z\right)\right]_{2-i\infty}^{2+i\infty}\right.\\
&\quad\left.-\int_{2-i\infty}^{2+i\infty}-\frac{1}{z\log v}\left(\left(\frac{uv}{n}\right)^z\log\left(\frac{uv}{n}\right)-\left(\frac{u}{n}\right)^z\log\left(\frac{u}{n}\right)\right)dz\right)\\
&= \sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^s}\left(\frac{1}{2\pi i}\frac{\log\frac{uv}{n}}{\log v}\cdot\int_{2-i\infty}^{2+i\infty}\left(\frac{uv}{n}\right)^z\frac{1}{z}\,dz\right.\\
&\quad\left.-\frac{1}{2\pi i}\frac{\log\frac{u}{n}}{\log v}\cdot\int_{2-i\infty}^{2+i\infty}\left(\frac{u}{n}\right)^z\frac{1}{z}\,dz+\frac{1}{2\pi i\log v}\left[\left(-\left(\frac{uv}{n}\right)^z+\left(\frac{u}{n}\right)^z\right)\frac{1}{z}\right]_{2-i\infty}^{2+i\infty}\right).
\end{aligned}
$$

We may permute sum and integral, because the series

$$\sum_{n\in\mathbb{N}}\frac{\Lambda(n)\chi(n)}{n^{z+s}}$$

is uniformly converging for $1\le\Re(z)\le 3$. The last summand disappears, and we can describe

$$
\begin{aligned}
F(s,\chi) &= \sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^s}\left(\frac{1}{2\pi i}\frac{\log\frac{uv}{n}}{\log v}\cdot\int_{2-i\infty}^{2+i\infty}\left(\frac{uv}{n}\right)^z\frac{1}{z}\,dz-\frac{1}{2\pi i}\frac{\log\frac{u}{n}}{\log v}\cdot\int_{2-i\infty}^{2+i\infty}\left(\frac{u}{n}\right)^z\frac{1}{z}\,dz\right)\\
&= \sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^s}\left(\frac{\log\frac{uv}{n}}{\log v}\cdot\lim_{U\to\infty}\frac{1}{2\pi i}\int_{2-iU}^{2+iU}\frac{\left(\frac{uv}{n}\right)^z}{z}\,dz-\frac{\log\frac{u}{n}}{\log v}\cdot\lim_{U\to\infty}\frac{1}{2\pi i}\int_{2-iU}^{2+iU}\frac{\left(\frac{u}{n}\right)^z}{z}\,dz\right)
\end{aligned}
$$

due to $u, uv\notin\mathbb{N}$ with Perron's formula (Theorem 2.3.2) to

$$
F(s,\chi) = \sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^s}\cdot\left\{\begin{array}{ll}\frac{\log\frac{uv}{n}}{\log v}-\frac{\log\frac{u}{n}}{\log v}=\frac{\log v}{\log v}=1 & \text{for } n\le u\\ \frac{\log\frac{uv}{n}}{\log v} & \text{for } u<n\le uv\\ 0-0 & \text{for } n>u\end{array}\right\}=\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)\omega(n)}{n^s}.
$$

□

Definition 3.5.2. [17]
Let $\mathcal{K}$ be a curve with $\mathcal{K}=\alpha_1\oplus\beta\oplus\alpha_2$ with $r, U\in\mathbb{R}^+$ and

- let α_1 be a linear contour, which links $2+iU$, $-1/2+iU$ and $-1/2+i$ in this order,
- let β be a linear contour, which links $-1/2+i$, $-r+i$, $-r-i$ und $-1/2-i$ in this order supposing $r=1/2-k$ with $k\in\mathbb{N}$,
- let α_2 be a linear contour, which links $-1/2-i$, $-1/2-iU$ und $2-iU$ in this order.

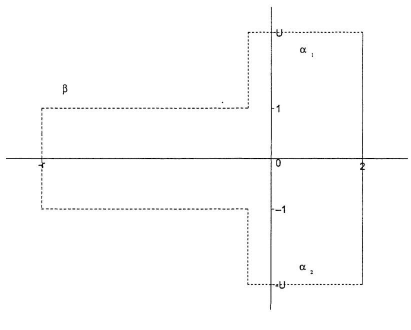

Illustration 3.1: visualisation of the curve $\mathcal{K}$

[17]See Fischer (9), page 84 f.

Theorem 3.5.2. [18]
Let $\mathcal{K}$ be the curve defined in Definition 3.5.2 with $\mathcal{K} = \alpha_1 \oplus \beta \oplus \alpha_2$.
Let $z \in \alpha_1$ or $z \in \alpha_2$, also $-1/2 \leq \Re(z) \leq 2$ and $1 \leq |\Im(z)| \leq U$. Then it holds

$$\frac{L'}{L}(z,\chi) = O(\log(q(|U|+2))).$$

Proof. In every interval of length 1, which includes U, we have $O(\log(q(U+2)))$ zeros of $L(s,\chi)$ according to Theorem 2.5.9. If a zero of $L(s,\chi)$ is located on the curve, U is increased so much that this is not the case and the distance from α_1 to the next zero does not fall below the lower limit $(c\log(q(U+2)))^{-1}$. We can choose $U > 12$, because we consider $U \to \infty$ so that we do not have to respect the pole of $\zeta(s)$ in the following proof. We set $z_0 = 2 + \Im(z)$ and consider

$$f(s) := \log\left(\frac{L(s,\chi)}{L(z_0,\chi)} \cdot \prod_{\omega} \frac{z_0-\omega}{s-\omega}\right) = \log(g(s)).$$

In this case, the branch of the logarithm is to be selected such that it holds $f(z_0) = 0$. The product runs over all zeros of $L(s,\chi)$ with $|z_0 - \omega| \leq 6$. Since $g(s)$ has neither zeros nor singularities for $|s - z_0| \leq 6$, the function $f(s)$ is uniquely defined and holomorphic.
It holds $|z_0 - \omega| \leq |s - \omega|$ for $|s - z_0| = 12$, and so we have according to Theorem 2.5.8

$$g(s) \ll q^c(|t|+2)^c$$

for a certain $c > 0$ and $|s - z_0| = 12$. Since $g(s)$ is holomorphic in $|s - z_0| \leq 12$, this also applies for $|s - z_0| \leq 6$. Thus

$$\Re(f(s)) = \log|g(s)| \ll \log(q(|\Im(s)|+2))$$

for $|s - z_0| \leq 6$ and also for $f(s) \ll \log(q(|t|+2))$, since the imaginary part of the logarithm of each branch is limited.

According to Cauchy's estimation it holds

$$f'(s) = \frac{L'}{L}(s,\chi) - \sum_{\omega} \frac{1}{s-\omega} \ll \log(q(|\Im(s)|+2))$$

for $|s - z_0| \leq 3$, also in particular for z. For this s we have

$$\sum_{|\Im(\omega)-t|>1} \frac{1}{s-\omega} \ll \log(q(|\Im(s)|+2))$$

according to Theorem 2.5.9, since it holds $|\Im(\omega) - \Im(s)| \leq 6$. Finally it follows that

$$\frac{L'}{L}(s,\chi) = \sum_{|\Im(\omega)-t|\leq 1} \frac{1}{z-\omega} + O(\log(q(|\Im(s)|+2))) = O(\log(q(U+2))),$$

according to Theorem 2.5.9, because z is not a zero of $L(s,\chi)$. □

Theorem 3.5.3. [19]
It holds for the curve $\mathcal{K}$ defined in Definition 3.5.2

$$\lim_{U\to\infty} \int_{\mathcal{K}} \frac{L'}{L}(s+z,\chi) \frac{u^z(v^z-1)}{z^2 \log v}\, dz = 0.$$

[18] See Fischer (9), page 85 f.
[19] See Fischer (9), page 86 f.

Proof. With the fundamental estimate for integrals follows

$$\begin{aligned}\left|\int_{\alpha_1}\frac{L'}{L}(s+z,\chi)\frac{u^z(v^z-1)}{z^2\log v}\,dz\right| &\leq (U+3)c\log(q(|U|+2))\frac{u^{\Re(z)}\left|v^{\Re(z)}-1\right|}{|z|^2\log v}\\ &\leq c(U+3)\log(q(U+2))\frac{u^2(|v|^2+1)}{U^2\log v}\to 0\end{aligned}$$

for $U\to\infty$. It is equally

$$\lim_{U\to\infty}\int_{\alpha_2}\frac{L'}{L}(s+z,\chi)\frac{u^z(v^z-1)}{z^2\log v}\,dz=0.$$

Now we still have to show that the integral over β tends to zero. Therefore we choose $z\in\beta$. According to the functional equation of $L(s,\chi)$ in Theorem 2.5.1 it holds

$$\begin{aligned}L(z,\chi) &= \epsilon_\chi^{-1}2^{-1}(2\pi)^z q^{-z+1/2}(\cos(\pi/2(z-a))\Gamma(z))^{-1}L(1-z,\overline{\chi})\\ &= \epsilon_\chi^{-1}\pi^{-1}(2\pi)^z q^{-z+1/2}\sin(\pi/2(z+a))\Gamma(1-z)L(1-z,\overline{\chi}),\end{aligned}$$

due to $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$ and $\frac{\sin(\pi z)}{\cos(\pi/2(z-a))}=2\sin(\pi/2(z+a))$. It follows

$$\frac{L'}{L}(z,\chi)=\log\frac{2\pi}{q}+\frac{\pi}{2}\cot(\pi/2(z+a))-\frac{\Gamma'}{\Gamma}(1-z)-\frac{L'}{L}(1-z,\overline{\chi}). \tag{3.8}$$

Since $\Re(1-z)\geq 3/2$, there is

$$\left|\frac{L'}{L}(1-z,\overline{\chi})\right|=\left|\sum_{n=1}^{\infty}\overline{\chi}(n)\Lambda(n)n^{-(1-z)}\right|\leq\sum_{n=1}^{\infty}\Lambda(n)n^{-3/2}=O(1),$$

and with Stirling's formula[20] follows

$$\frac{\Gamma'}{\Gamma}(1-z)=\log(1-z)+O\left(\frac{1}{|1-z|}\right)=O(\log(|1-z|))=O(\log(|z|+2)).$$

Finally applies

$$\cot(\pi/2(z+a))=i+\frac{2i}{e^{i\pi(z+a)}-1}=O(1),$$

because of $|e^{i\pi(z+a)}-1|>0$ since $-z\notin\mathbb{N}$. Putting all this in (3.8), it follows

$$\frac{L'}{L}(z,\chi)\ll\log(|z|+2).$$

Thus, one obtains

$$\begin{aligned}\left|\int_{\beta}\frac{L'}{L}(s+z,\chi)\frac{u^z(v^z-1)}{z^2\log v}\,dz\right| &\leq (2r+2)c\log(|z|+2)\frac{u^{\Re(z)}\left|v^{\Re(z)}-1\right|}{|z|^2\log v}\\ &\leq r\widetilde{c}\log(|z|+2)\frac{u^{-1/2}(v^{-1/2}+1)}{|r|^2\log v}\to 0\end{aligned}$$

for $r\to\infty$. So we get

$$\lim_{U\to\infty}\int_{\mathcal{K}}\frac{L'}{L}(s+z,\chi)\frac{u^z(v^z-1)}{z^2\log v}\,dz=0$$

with positive constants c and $\tilde{c}$. □

[20]See Prachar (6), appendix, Theorem 6.1

Theorem 3.5.4. [21]
Assuming GRH, there is

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)\omega(n)}{n^s} = -\frac{L'}{L}(s,\chi)-\sum_{k=0}^{\infty}\frac{u^{-a-2k-s}\left(v^{-a-2k-s}-1\right)}{(-a-2k-s)^2\log v}+E_0\frac{u^{-s}\left(v^{-s}-1\right)}{(-s)^2\log v} - \sum_{z=1/2+i\gamma}\frac{u^{z-s}\left(v^{z-s}-1\right)}{(z-s)^2\log v}+E_0\frac{u^{1-s}\left(v^{1-s}-1\right)}{(1-s)^2\log v}.$$

Proof. For the L- series of primitive characters, the following applies:

- They have no singularities except for $\chi=\chi_0^*$, where $L(s,\chi_0^*)=\zeta(s)$ has a simple pole at $s=1$.
- The zeros are for $\Re(z)\le 0$ at $z=-a,-a-2-a-4,\ldots$, where a is given by (2.7). An exception is $L(s,\chi_0^*)$, which has no zero at $z=0$.
- There is no zero of $L(z,\chi)$ for $\Re(z)\ge 1$.
- The zeros in the range $0<\Re(z)<1$ lie according to the generalized Riemann hypothesis (GRH) all on the axis $\Re(z)=1/2$.

So $\frac{L'}{L}(s+z,\chi)$ has poles of order 1 with residue 1 at $-a-2k-s$ for $k\in\mathbb{N}_0$ respectively $k\in\mathbb{N}$ for $\chi=\chi_0^*$ and further poles on the line with $\Re(s)=1/2$. The residues at these poles correspond to the multiplicity of the zero of $L(s,\chi)$ at the respective point. The singularitity at $s=1$ for $\chi=\chi_0^*$ has residue 1.
The function $\frac{u^z(v^z-1)}{z^2\log v}$ has a simple pole with residue 1 at $z=0$.
Applying the theorem of residues[22] on the right side of Theorem 3.5.1 by closing the curve $\mathcal{K}$, it follows

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)\omega(n)}{n^s} = -\frac{L'}{L}(s,\chi)-\sum_{k=0}^{\infty}\frac{u^{-a-2k-s}\left(v^{-a-2k-s}-1\right)}{(-a-2k-s)^2\log v}+E_0\frac{u^{-s}\left(v^{-s}-1\right)}{(-s)^2\log v} - \sum_{z=1/2+i\gamma}\frac{u^{z-s}\left(v^{z-s}-1\right)}{(z-s)^2\log v}+E_0\frac{u^{1-s}\left(v^{1-s}-1\right)}{(1-s)^2\log v}, \tag{3.9}$$

and the last sum runs over all zeros of $L(s,\chi)$ and $E_0=1$ holds for $\chi=\chi_0^*$ and otherwise $E_0=0$. Here we assume $s\neq 1$ and s is not a zero of $L(s,\chi)$. ☐

[21] See Fischer (9), page 87 f.
[22] See Lütkebohmert (3), Theorem 5.4.4

Chapter 4

Preparatory Lemmata

4.1 Monotonicity principles

First we prove a few monotonicity principles on horizontal lines which are parallel to the real axis. Let $q \in \mathbb{N}$ and $\chi \neq \chi_0^*$ be a primitive character modulo q.
We consider the function

$$\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) \cdot L(s,\chi),$$

from Theorem 2.5.1, while a is defined over (2.7).

Now we can deduce monotonicity principles for the function $\Re \frac{L'}{L}(s,\chi)$ and $\chi \neq \chi_0^*$ using its logarithmic derivative

$$\frac{\xi'}{\xi}(s,\chi) = (\log \xi(s,\chi))' = \frac{1}{2} \cdot \log \frac{q}{\pi} + \frac{1}{2} \cdot \frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) + \frac{L'}{L}(s,\chi).$$

Lemma 4.1.1. [1]
Let $\chi \neq \chi_0^*$ be a primitive character modulo q with $q \in \mathbb{N}$ and $T \leq t \leq 2T$.
Then

$$(\sigma - 1/2) \cdot \Re \frac{\xi'}{\xi}(\sigma + it, \chi)$$

is a strictly increasing function of σ for $1/2 \leq \sigma \leq 2$.

Proof. We proved in Theorem 2.5.6, that $\xi(s,\chi)$ is an analytic function of order 1 and

$$\sum_{\rho} |\rho|^{-1} = \infty,$$
$$\sum_{\rho} |\rho|^{-\alpha} < \infty$$

holds for every $\alpha > 1$.

[1] See Fischer (9), Lemma 19

According to Theorem 2.2.1 the function $\xi(s,\chi)$ has a representation of

$$\xi(s,\chi) = e^{a_1 s + a_0} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \cdot e^{s/\rho},$$

with $a_0, a_1 \in \mathbb{C}$ and the sum runs over all nontrivial zeros of the Dirichlet- L- function.
Thus

$$\frac{\xi'}{\xi}(s,\chi) = (\log \xi(s,\chi))' = \left(a_1 s + a_0 + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right) + \frac{s}{\rho}\right)' = a_1 + \sum_{\rho} \left(\frac{1}{\rho} - \frac{1}{\rho - s}\right).$$

According to Theorem 2.5.1 the function $\xi(s,\chi)$ fulfills

$$\xi(1-s,\overline{\chi}) = \epsilon_\chi \cdot \xi(s,\chi)$$

with a constant ϵ_χ which depends only on χ and with $|\epsilon_\chi| = 1$. For the derivation of $\xi(s,\chi)$ we get

$$\xi'(s,\chi) = \left(\frac{1}{\epsilon_\chi} \cdot \xi(1-s,\overline{\chi})\right)' = -\frac{1}{\epsilon_\chi} \cdot \xi'(1-s,\overline{\chi}).$$

So via

$$a_1 = \frac{\xi'}{\xi}(0,\chi) = \frac{-\frac{1}{\epsilon_\chi} \cdot \xi'(1,\overline{\chi})}{\frac{1}{\epsilon_\chi} \cdot \xi(1,\overline{\chi})} = -\frac{\xi'}{\xi}(1,\overline{\chi}) = -\overline{a_1} - \sum_{\rho} \left(\frac{1}{\rho} - \frac{1}{\rho - 1}\right),$$

we get $\Re(a_1)$, where the sum runs over all zeros of $L(s,\overline{\chi})$, which are identical to the zeros of $L(s,\chi)$.
So we have

$$\Re(a_1) = -\frac{1}{2} \cdot \sum_{\rho} \left(\frac{1}{\rho} - \frac{1}{\rho - 1}\right) = \frac{1}{2} \cdot \sum_{\rho} \left(\frac{1}{\rho - 1} - \frac{1}{\rho}\right).$$

Hence,

$$\begin{aligned}
\Re\frac{\xi'}{\xi}(s,\chi) &= \Re(a_1) + \sum_{\rho} \Re\left(\frac{1}{\rho} - \frac{1}{\rho - s}\right) = \frac{1}{2} \cdot \sum_{\rho} \Re\left(\frac{1}{\rho - 1} - \frac{1}{\rho}\right) + \sum_{\rho} \Re\left(\frac{1}{\rho} - \frac{1}{\rho - s}\right) \\
&= \frac{1}{2} \cdot \sum_{\rho} \Re\left(\frac{1}{\rho - 1} + \frac{1}{\rho}\right) + \sum_{\rho} \Re\left(\frac{1}{s - \rho}\right) \\
&= \frac{1}{2} \cdot \sum_{\rho = 1/2 + i\gamma} \left(\frac{1/2}{|\rho - 1|^2} + \frac{1/2}{|\rho|^2}\right) + \sum_{\rho = 1/2 + i\gamma} \frac{\sigma - 1/2}{(\sigma - 1/2)^2 + (t - \gamma)^2}.
\end{aligned}$$

Notice that all sums are converging absolutely and can be rearranged. Setting

$$c := 1/4 \sum_{\rho = 1/2 + i\gamma} \left(\frac{1}{|\rho - 1|^2} + \frac{1}{|\rho|^2}\right) \geq 0,$$

we get

$$(\sigma - 1/2) \cdot \Re\frac{\xi'}{\xi}(\sigma + it, \chi) = c(\sigma - 1/2) + \sum_{\rho = 1/2 + i\gamma} \frac{(\sigma - 1/2)^2}{(\sigma - 1/2)^2 + (t - \gamma)^2}.$$

Every summand is strictly increasing for $1/2 \leq \sigma \leq 2$, and so we get the claim. □

Lemma 4.1.2. [2]
Let $\chi \neq \chi_0^*$ be a primitive character modulo q with $q \in \mathbb{N}$ and $T \leq t \leq 2T$. Then

$$(\sigma - 1/2) \cdot \left(\Re \frac{L'}{L}(\sigma + it, \chi) - \Re \frac{\xi'}{\xi}(\sigma + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right)$$

is a strictly increasing function of σ for $1/2 \leq \sigma \leq 2$. Particulary for $1/2 \leq \sigma \leq 2$ there is

$$(\sigma - 1/2) \cdot \left(\Re \frac{L'}{L}(\sigma + it, \chi) - \Re \frac{\xi'}{\xi}(\sigma + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right) \geq 0.$$

Proof. Let $s = \sigma + it \in \mathbb{C}$ with $1/2 \leq \sigma \leq 2$ and $T \leq t \leq 2T$. According to the definition of $\xi(s, \chi)$

$$\frac{L'}{L}(s, \chi) = \frac{\xi'}{\xi}(s, \chi) - \frac{1}{2} \cdot \log \frac{q}{\pi} - \frac{1}{2} \cdot \frac{\Gamma'}{\Gamma} \left(\frac{s + a}{2} \right)$$

holds. Thus it suffices to prove

$$\begin{aligned} g(\sigma) &:= (\sigma - 1/2) \cdot \left(\frac{1}{2} \cdot \log \frac{tq}{2} - \frac{1}{2} \cdot \log \frac{q}{\pi} - \frac{1}{2} \cdot \Re \frac{\Gamma'}{\Gamma} \left(\frac{s+a}{2} \right) \right) \\ &= \frac{\sigma - 1/2}{2} \cdot \left(\log \frac{t\pi}{2} - \Re \frac{\Gamma'}{\Gamma} \left(\frac{s+a}{2} \right) \right) \geq 0 \end{aligned} \tag{4.1}$$

and the strictly increasing of this function. From Stirling's formula[3] it follows[4]

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|} \right)$$

uniformly for $|\arg s| \leq \pi - \delta$ for $\delta > 0$ and $|s| > \delta$. Applying Cauchy's integral formula[5]

$$f'(s) = \frac{1}{2\pi i} \int_{\mathcal{K}} \frac{f(z)}{(z - s)^2} \, dz$$

on the function $f(z) = \frac{\Gamma'}{\Gamma}(z) - \log z$, where $\mathcal{K}$ is a circle with center point s and radius $|s| \sin(\delta/2)$, we get

$$\left(\frac{\Gamma'}{\Gamma} \right)'(s) = \frac{1}{s} + O\left(\frac{1}{|s|^2} \right).$$

So for the derivation of (4.1) we get

$$\begin{aligned} g'(\sigma) &= \left(\frac{\sigma - 1/2}{2} \cdot \left(\log \frac{t\pi}{2} - \Re \frac{\Gamma'}{\Gamma} \left(\frac{s+a}{2} \right) \right) \right)' \\ &= \frac{1}{2} \cdot \left(\log \frac{t\pi}{2} - \Re \frac{\Gamma'}{\Gamma} \left(\frac{s+a}{2} \right) \right) + \frac{\sigma - 1/2}{2} \cdot \left(-\Re \left(\frac{\Gamma'}{\Gamma} \right)' \left(\frac{s+a}{2} \right) \right) \\ &= \frac{1}{2} \cdot \log \frac{t\pi}{2} - \frac{1}{2} \cdot \log \frac{s+a}{2} + O\left(\frac{1}{|s+a|} \right) - \frac{\sigma - 1/2}{4} \cdot \left(\Re \frac{2}{s+a} + O\left(\frac{1}{|s+a|^2} \right) \right) \\ &= \frac{1}{2} \cdot \log \pi + \frac{1}{2} \cdot \log \left(\frac{t}{2} \cdot \frac{2}{s+a} \right) + O\left(\frac{1}{|t|} \right) - \frac{\sigma - 1/2}{2} \cdot \Re \frac{1}{s+a} - \frac{\sigma - 1/2}{4} \cdot O\left(\frac{1}{|t|^2} \right) \\ &= \frac{1}{2} \cdot \log \pi + \frac{1}{2} \cdot \log \left(\frac{t}{s+a} \right) + O\left(\frac{1}{|t|} \right) - \frac{\sigma - 1/2}{2} \cdot \Re \frac{\sigma + a}{|s+a|^2} \\ &= \frac{1}{2} \cdot \log \pi + O\left(\frac{1}{|t|} \right) \geq 0 \end{aligned}$$

and the proof of both assertions is finished. □

[2]See Fischer (9), Lemma 20
[3]See Prachar (6), appendix, Theorem 6.1
[4]See Lütkebohmert (3), page 100
[5]See Lütkebohmert (3), Theorem 2.4.3

Lemma 4.1.3. [6]

Let χ be a primitive character modulo q with $q \in \mathbb{N}$ and $T \leq t \leq 2T$. Then

$$(\sigma - 1/2) \cdot \left(\Re \frac{L'}{L}(\sigma + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right)$$

is strictly increasing of σ for $1/2 \leq \sigma \leq 2$.

Proof. This follows from the Lemmata 4.1.1 and 4.1.2. □

Lemma 4.1.4. [7]

Let χ be a primitive character modulo q with $q \in \mathbb{N}$. Consequently we have for $1/2 \leq \sigma_1 \leq \sigma_2 \leq 2$ and $T \leq t \leq 2T$

$$|L(\sigma_1 + it, \chi)| \geq |L(\sigma_2 + it, \chi)| \left(\frac{\sigma_1 - 1/2}{\sigma_2 - 1/2} \right)^{(\sigma_2 - 1/2)\left(\Re \frac{L'}{L}(\sigma_2 + it, \chi) + 1/2 \cdot \log \frac{tq}{2} \right)}.$$

Proof. According to Lemma 4.1.3 we get for $1/2 \leq \sigma \leq \sigma_2$

$$(\sigma - 1/2) \cdot \left(\Re \frac{L'}{L}(\sigma + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right) \leq (\sigma_2 - 1/2) \cdot \left(\Re \frac{L'}{L}(\sigma_2 + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right)$$

and so

$$\Re \frac{L'}{L}(\sigma + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \leq \frac{\sigma_2 - 1/2}{\sigma - 1/2} \cdot \left(\Re \frac{L'}{L}(\sigma_2 + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right).$$

Integration over σ ensues

$$\int_{\sigma_1}^{\sigma_2} \Re \frac{L'}{L}(\sigma + it, \chi)\, d\sigma + \frac{(\sigma_2 - \sigma_1) \cdot \log \frac{tq}{2}}{2} \leq \int_{\sigma_1}^{\sigma_2} \frac{\sigma_2 - 1/2}{\sigma - 1/2}\, d\sigma \cdot \left(\Re \frac{L'}{L}(\sigma_2 + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right)$$

and therefore

$$\begin{aligned}
\log \left| \frac{L(\sigma_2 + it, \chi)}{L(\sigma_1 + it, \chi)} \right| &= \log |L(\sigma_2 + it, \chi)| - \log |L(\sigma_1 + it, \chi)| \\
&\leq \log |L(\sigma_2 + it, \chi)| - \log |L(\sigma_1 + it, \chi)| + \frac{(\sigma_2 - \sigma_1) \cdot \log \frac{tq}{2}}{2} \\
&= \int_{\sigma_1}^{\sigma_2} \Re \frac{L'}{L}(\sigma + it, \chi)\, d\sigma + \frac{(\sigma_2 - \sigma_1) \cdot \log \frac{tq}{2}}{2} \\
&\leq \int_{\sigma_1}^{\sigma_2} \frac{\sigma_2 - 1/2}{\sigma - 1/2}\, d\sigma \cdot \left(\Re \frac{L'}{L}(\sigma_2 + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right) \\
&= (\sigma_2 - 1/2) \cdot \log \left(\frac{\sigma_2 - 1/2}{\sigma_1 - 1/2} \right) \left(\Re \frac{L'}{L}(\sigma_2 + it, \chi) + \frac{1}{2} \cdot \log \frac{tq}{2} \right).
\end{aligned}$$

So we get the claim. □

[6] See Fischer (9), Lemma 21

[7] See Fischer (9), Lemma 22

4.2 Large value estimates

In the following let χ always denote a primitive character modulo q, while $Q \leq q \leq 2Q$ with $Q \in \mathbb{N}$.

Lemma 4.2.1. [8]
Suppose for $N \in \mathbb{N}$ and $s \in \mathbb{C}$

$$S(s,\chi) = \sum_{\substack{p \in \mathbb{P} \\ p \leq N}} a(p)\chi(p)p^{-s}$$

with $a(p) \in \mathbb{C}$. Suppose $\alpha, T, T_0 \in \mathbb{R}$ with $T \geq 2$ and $Q \in \mathbb{N}$.
For $1 \leq l \leq L$ and $1 \leq j \leq J_l$ there are pairs $(\chi_l, s_{l,j})$ consisting of primitive, pairwise different multiplicative characters $\chi_l \bmod q$ with $Q \leq q \leq 2Q$ and points $s_{l,j} \in \mathbb{C}$ with $s_{l,j} = \sigma_{l,j} + it_{l,j}$ with $\sigma_{l,j} \geq \alpha$ and $T_0 \leq t_{l,j} \leq T_0 + T$. Also, suppose that points belonging to the same character are well spaced to the extent that $|t_{l_1,j} - t_{l_2,j}| \geq \frac{1}{\log(TQ^2)}$ for $1 \leq l_{1,j} < l_{2,j} \leq L$.
If k is a positive integer such that $N^k \leq TQ^2$ then

$$\sum_{l=1}^{L}\sum_{j=1}^{J_l} |S(s_{l,j},\chi_l)|^{2k} \ll TQ^2 \log^2(TQ^2) k! \left(\sum_{\substack{p \in \mathbb{P} \\ p \leq N}} |a(p)|^2 p^{-2\alpha} \right)^k .$$

Proof. Let $D(s,\chi) = S(s,\chi)^k = \sum_{n \leq N^k} c_n \chi(n) n^{-s}$ for a certain $c_n \in \mathbb{C}$. We first show that

$$\sum_{l=1}^{L}\sum_{j=1}^{J_l} |D(s_{l,j},\chi_l)|^2 \ll TQ^2 \log^2(TQ^2) \sum_{n \leq N^k} |c_n|^2 n^{-2\alpha} \tag{4.2}$$

and prove the theorem by considering the coefficients c_n. Let

$$\mathcal{D}(a) := \left\{ z \in \mathbb{C} \colon |z-a| < \frac{1}{2\log(TQ^2)} \right\}$$

denote a disc of radius $(2\log(TQ^2))^{-1}$ centered at a. Then with the mean value theorem for holomorphic functions[9] we have

$$|D(s,\chi)|^2 = \frac{4}{\pi} \int\int_{\mathcal{D}(a)} |D(x+iy,\chi)|^2 \, dx \, dy$$

for any s. Since these discs $D(s_{l,j},\chi_l)$ are disjoint for a fixed character $\chi_l \bmod q$ and since they all lie in the half- strip $\sigma \geq \alpha - \frac{1}{\log(TQ^2)} =: \kappa$

$$\begin{aligned}
\sum_{l=1}^{L}\sum_{j=1}^{J_l} |S(s_{l,j},\chi_l)|^{2k} &= \sum_{l=1}^{L}\sum_{j=1}^{J_l} |D(s_{l,j},\chi_l)|^2 \\
&\ll \log^2(TQ^2) \sum_{Q \leq q \leq 2Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{\kappa}^{\infty} \int_{T_0-1}^{T_0+T+1} \left| \sum_{n \leq N^k} c_n \chi(n) n^{-s} \right|^2 dt \, d\sigma \\
&= \log^2(TQ^2) \int_{\kappa}^{\infty} \underbrace{\sum_{Q \leq q \leq 2Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{-(\frac{T}{2}+1)}^{\frac{T}{2}+1} \left| \sum_{n \leq N^k} c_n \chi(n) n^{-\sigma} n^{-it-i(T_0+\frac{T}{2})} \right|^2 dt}_{=:H} \; d\sigma.
\end{aligned}$$

[8] In Maier and Montgomery (4) and Fischer (9) they get in Lemma 5 resp. Lemma 23 a similar result.
[9] See Montgomery (11), Theorem 6.1

Using the Hybrid Sieve (Theorem 3.4.1), we get

$$\begin{aligned} H &= \sum_{Q\le q\le 2Q} \sum_{\chi \bmod q}^{*} \int_{-(\frac{T}{2}+1)}^{\frac{T}{2}+1} \left| \sum_{n\le N^k} c_n \chi(n) n^{-\sigma} n^{-i(T_0+\frac{T}{2})} n^{-it} \right|^2 dt \\ &\le 2\pi c_1 \sum_{n\le N^k} \left(\left(\frac{T}{2}+1\right) Q^2 + n \right) |c_n|^2 \left| n^{-i(T_0+\frac{T}{2})} \right|^2 n^{-2\sigma} \\ &= \pi c_1 T Q^2 \sum_{n\le N^k} |c_n|^2 n^{-2\sigma} + 2\pi c_1 Q^2 \sum_{n\le N^k} |c_n|^2 n^{-2\sigma} + 2\pi c_1 \sum_{n\le N^k} n|c_n|^2 n^{-2\sigma} \\ &\le \pi c_1 (TQ^2 + 2Q^2 + O(N^k)) \sum_{n\le N^k} |c_n|^2 n^{-2\sigma} \ll TQ^2 \sum_{n\le N^k} |c_n|^2 n^{-2\sigma} \end{aligned}$$

because of $N^k \le TQ^2$ and an absolute constant $c_1 \in \mathbb{R}$. Hence,

$$\begin{aligned} \sum_{l=1}^{L} \sum_{j=1}^{J_l} |S(s_{l,j}, \chi_l)|^{2k} &\ll TQ^2 \log^2(TQ^2) \int_{\kappa}^{\infty} \sum_{n\le N^k} |c_n|^2 n^{-2\sigma}\, d\sigma \\ &= TQ^2 \log^2(TQ^2) \sum_{n\le N^k} \int_{\kappa}^{\infty} |c_n|^2 n^{-2\sigma}\, d\sigma \\ &= TQ^2 \log^2(TQ^2) \sum_{n\le N^k} \left[-\frac{|c_n|^2}{\log(n)} n^{-2\sigma} \right]_{\alpha-1/\log(TQ^2)}^{\infty} \\ &= TQ^2 \log^2(TQ^2) \sum_{n\le N^k} \frac{|c_n|^2}{\log(n)} n^{-2(\alpha-1/\log(TQ^2))} \\ &\ll TQ^2 \log^2(TQ^2) \sum_{n\le N^k} |c_n|^2 n^{-2\alpha}, \end{aligned}$$

because of $\log n \ge \frac{1}{2}$ for $n > 1$ and

$$n^{2/\log(TQ^2)} \le (N^k)^{2/\log(TQ^2)} \le (TQ^2)^{2/\log(TQ^2)} = \exp(2) = O(1)$$

and we have (4.2).
With the canonical factorization $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_m^{k_m}$ with $p_1, \ldots, p_m \in \mathbb{P}$, $p_i < N$ and $k_1, \ldots, k_m \in \mathbb{N}$ with $\sum_{i=1}^{m} k_i = k$ there is

$$c_n = \binom{k}{k_1 k_2 \ldots k_m} \prod_{i=1}^{m} (a(p_i))^{k_i}.$$

If n does not have such a representation, then we have $c_n = 0$. Also we get

$$\begin{aligned} \sum_{n=1}^{N^k} |c_n|^2 n^{-2\alpha} &= \sum_{\substack{p_1,\ldots,p_m < N \\ \Sigma k_i = k}} \binom{k}{k_1 k_2 \ldots k_m}^2 \prod_{i=1}^{m} \frac{|a(p_i)|^{2k_i}}{p_i^{2k_i\alpha}} \\ &\le k! \sum_{\substack{p_1,\ldots,p_m < N \\ \Sigma k_i = k}} \binom{k}{k_1 k_2 \ldots k_m} \prod_{i=1}^{m} \frac{|a(p_i)|^{2k_i}}{p_i^{2k_i\alpha}} = k! \left(\sum_{\substack{p\in\mathbb{P} \\ p\le N}} |a(p)|^2 p^{-2\alpha} \right)^k \end{aligned}$$

with the multinomial theorem. So we get the claim. □

Lemma 4.2.2. [10]
Let $T, T_0 \in \mathbb{R}$, $q, Q \in \mathbb{N}$ and

$$\alpha \geq \frac{1}{2} + \frac{\log_2(TQ)}{\log(TQ)\log_3(TQ)}.$$

We further have

$$S(s,\chi) = \sum_{p\in\mathbb{P}} \Lambda(p)\omega(p)\chi(p)p^{-s},$$

while $\omega(\cdot)$ was defined in Definition 3.5.1, and pairs of primitive, pairwise different charakters $\chi_l \bmod q$ and points $s_{l,j} \in \mathbb{C}$ with $s_{l,j} = \sigma_{l,j} + it_{l,j}$ and $\sigma_{l,j} \geq \alpha$ and $T_0 \leq t_{l,j} \leq T_0 + T$. Then for

$$\left|\frac{L'}{L}(s_{l,j},\chi_l)\right| > \eta \log(TQ),$$

we count the number of these R pairs of points and characters for a fixed η with $0 < \eta < 1/3$, which will be determined later. Then it holds

$$R \ll TQ^2 \log^3(TQ^2) \cdot \exp\left(-(f(\eta) - \epsilon)(\alpha - 1/2)\log(TQ^2)\log((\alpha - 1/2)\log(TQ))\right)$$

with $f(\eta) = \left(\psi + \log\left(1 + \frac{1}{2\eta}\right)\right)^{-1}$ and ψ is the unique solution of the equation $\exp(-\psi) + 1 = \psi$.

Proof. We apply Lemma 4.2.1 on $S(s_{l,j},\chi_l)$, and must choose u and v of Theorem 3.5.1 sufficiently large in order to estimate $|S(s_{l,j},\chi_l)| \geq \delta \log(TQ)$ trivially down for all $j = 1,\dots,J_l$ and all $1 \leq l \leq L$, also for all pairs of characters and points, which we want to renumber with $r = 1,\dots,R$.
We estimate the summands of

$$F(s,\chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{n^s}$$

applying Theorem 3.5.1. We have

$$\begin{aligned} F(s,\chi) &= -\frac{L'}{L}(s,\chi) - \sum_{k=0}^{\infty} \frac{u^{-a-2k-s}\left(v^{-a-2k-s} - 1\right)}{(-a-2k-s)^2 \log v} + E_0 \frac{u^{-s}\left(v^{-s} - 1\right)}{(-s)^2 \log v} \\ &- \sum_{z=1/2+i\gamma} \frac{u^{z-s}\left(v^{z-s} - 1\right)}{(z-s)^2 \log v} + E_0 \frac{u^{1-s}\left(v^{1-s} - 1\right)}{(1-s)^2 \log v}. \end{aligned}$$

There is

$$\begin{aligned} \left|\sum_{z=1/2+i\gamma} \frac{u^{z-s}\left(v^{z-s} - 1\right)}{(z-s)^2 \log v}\right| &\leq \sum_{z=1/2+i\gamma} \frac{u^{1/2-\sigma}\left(v^{1/2-\sigma} + 1\right)}{|(z-s)^2 \log v|} \\ &= \frac{u^{1/2-\sigma}\left(v^{1/2-\sigma} + 1\right)}{(\sigma - 1/2)\log v} \sum_{\gamma} \frac{\sigma - 1/2}{(\sigma - 1/2)^2 + (t-\gamma)^2} \\ &\leq \frac{u^{1/2-\sigma}\left(v^{1/2-\sigma} + 1\right)}{(\sigma - 1/2)\log v} \cdot \Re\frac{\xi'}{\xi}(s,\chi) \\ &\leq \frac{u^{1/2-\alpha}\left(v^{1/2-\alpha} + 1\right)}{(\alpha - 1/2)\log v} \cdot \left(\Re\frac{L'}{L}(s,\chi) + \frac{\log(tq)}{2}\right) \end{aligned}$$

because of the Lemmata 4.1.1 and 4.1.2 for $s = \sigma + it$ with $\sigma \geq \alpha$ and $T \leq t \leq 2T$.

[10] In Maier and Montgomery (4) and Fischer (9) they get in Lemma 6 resp. Lemma 24 a similar result.

We may assume that $u, v \geq 2$ and $uv \leq TQ^2$, so we have

$$\sum_{k=0}^{\infty} \frac{u^{-a-2k-s}\left(v^{-a-2k-s}-1\right)}{(-a-2k-s)^2 \log v} = O\left(\sum_{k=0}^{\infty} \frac{2^{-k}}{T^2}\right) \ll T^{-2}$$

and

$$\frac{u^{1-s}\left(v^{1-s}-1\right)}{(1-s)^2 \log v} \ll T^{-1} \quad \text{and} \quad \frac{u^{-s}\left(v^{-s}-1\right)}{(-s)^2 \log v} \ll T^{-1}.$$

There is

$$\begin{aligned} \left|\sum_{p\in\mathbb{P}} \sum_{l=2}^{\infty} \frac{(\log p)\omega(p^l)\chi(p^l)}{p^{sl}}\right| &\ll \sum_{p\in\mathbb{P}} \frac{\log p}{p^{2\sigma}} \ll \sum_{p\in\mathbb{P}} \frac{\log p}{p^{2\alpha}} \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{2\alpha}} = -\frac{\zeta'}{\zeta}(2\alpha) \\ &\ll \frac{1}{2\alpha-1} \leq \frac{2\log(TQ)\log_3(TQ)}{\log_2(TQ)} = o(\log(TQ)) \end{aligned}$$

for $s = \sigma + it$ with $\sigma \geq \alpha$, since the Riemann- ζ- function has a simple pole at $s = 1$.
Also for all $j = 1, \ldots, J_l$ and all $1 \leq l \leq L$ there is

$$\begin{aligned} |S(s_{l,j},\chi_l)| &\geq \left|\frac{L'}{L}(s_{l,j},\chi_l)\right| - \frac{u^{1/2-\alpha}\left(v^{1/2-\alpha}+1\right)}{(\alpha-1/2)\log v}\left(\Re\frac{L'}{L}(s_{l,j},\chi_l) + \frac{1}{2}\cdot\log\frac{tq}{2}\right) \\ &\quad -O(T^{-1}) - o(\log(TQ)) \\ &> \eta\log(TQ)\left(1 - \frac{u^{1/2-\alpha}\left(v^{1/2-\alpha}+1\right)}{(\alpha-1/2)\log v}\right) - \frac{u^{1/2-\alpha}\left(v^{1/2-\alpha}+1\right)}{(\alpha-1/2)\log v}\cdot\frac{\log\frac{tQ}{2}}{2} \\ &\quad -O(T^{-1}) - o(\log(TQ)) \\ &\geq \left(\eta\left(1 - \frac{u^{1/2-\alpha}\left(v^{1/2-\alpha}+1\right)}{(\alpha-1/2)\log v}\right) - \frac{u^{1/2-\alpha}\left(v^{1/2-\alpha}+1\right)}{2(\alpha-1/2)\log v}\right)\log(TQ) - O(T^{-1}) - o(\log(TQ)), \end{aligned}$$

because of the assumption

$$\left|\frac{L'}{L}(s_{l,j},\chi_l)\right| > \eta\log(TQ)$$

for all $j = 1, \ldots, J_l$ and all $1 \leq l \leq L$. We remember that the points $s_{j,l}$ are not zeros of $L(s,\chi_l)$, because their real parts are bigger or equal to α. We get

$$|S(s_{l,j},\chi_l)| \geq 2\delta\log(TQ) - O(T^{-1}) - o(\log(TQ)) \geq \delta\log(TQ)$$

for

$$\eta\left(1 - \frac{u^{1/2-\alpha}\left(v^{1/2-\alpha}+1\right)}{(\alpha-1/2)\log(v)}\right) - \frac{1}{2}\cdot\frac{u^{1/2-\alpha}\left(v^{1/2-\alpha}+1\right)}{(\alpha-1/2)\log(v)} \geq 2\delta. \tag{4.3}$$

In the following we choose u and v so that (4.3) suffices. We want to take k in Lemma 4.2.1 as large as possible. Therefore we want the above to hold with uv as small as possible. We set

$$u = \exp\left(\frac{U}{\alpha-1/2}\right) \text{ and } v = \exp\left(\frac{V}{\alpha-1/2}\right),$$

and get for (4.3) the equivalent term

$$\begin{aligned} \eta\left(1 - \frac{e^{-U}\left(e^{-V}+1\right)}{V}\right) - \frac{e^{-U}\left(e^{-V}+1\right)}{2V} \geq 2\delta &\Leftrightarrow \eta - 2\delta \geq (\eta+1/2)\cdot\frac{e^{-U}\left(e^{-V}+1\right)}{V} \\ &\Leftrightarrow \frac{\eta-2\delta}{\eta+1/2} \geq \frac{e^{-U}\left(e^{-V}+1\right)}{V} \end{aligned} \tag{4.4}$$

with minimal $U + V$. We take U so that the above holds with equality.

Then

$$U + V = \log\left(\frac{e^{-V}+1}{V}\right) + \log\left(\frac{\eta+1/2}{\eta+2\delta}\right) + V.$$

That's minimal if and only if

$$\frac{1+e^{-V}}{V}\cdot e^V = \frac{e^V+1}{V}$$

is minimal.
This is minimized by taking $V = \psi$, where ψ is the unique real number such that $1+e^{-\psi} = \psi$. These considerations lead us to the choice

$$u = \left(\frac{\eta+1/2}{\eta-2\delta}\right)^{1/(\alpha-1/2)} \quad \text{and } v = \exp\left(\frac{\psi}{\alpha-1/2}\right).$$

For the availability of $(uv)^k \le TQ^2$ in Lemma 4.2.1, we take $k = \left[\frac{\log(TQ^2)}{\log(uv)}\right]$. Then we have

$$\begin{aligned} k+1 &> \frac{\log(TQ^2)}{\log(uv)} = \frac{\log(TQ^2)}{\frac{1}{\alpha-1/2}\cdot\log\left(\frac{\eta+1/2}{\eta-2\delta}\right)+\frac{\psi}{\alpha-1/2}} = \frac{(\alpha-1/2)\log(TQ^2)}{\log\left(1+\frac{1}{2\eta}\right)-\log\left(1-\frac{2\delta}{\eta}\right)+\psi} \\ &\ge (\alpha-1/2)\log(TQ^2)(f(\eta)-\epsilon/4), \end{aligned}$$

if $\delta = \delta(\epsilon)$ is sufficiently small. Now

$$\begin{aligned} k &\ge (\alpha-1/2)\log(TQ^2)(f(\eta)-\epsilon/4) - 1 \\ &= (\alpha-1/2)\log(TQ^2)(f(\eta)-\epsilon/4) - (\alpha-1/2)\log(TQ^2)\frac{1}{(\alpha-1/2)\log(TQ^2)} \\ &\ge (\alpha-1/2)\log(TQ^2)(f(\eta)-\epsilon/4) - (\alpha-1/2)\log(TQ^2)\frac{\log_3(TQ)}{\log_2(TQ)}\frac{\log(TQ)}{\log(TQ^2)} \\ &\ge (\alpha-1/2)\log(TQ^2)(f(\eta)-\epsilon/4) - (\alpha-1/2)\log(TQ^2)\epsilon/4 \\ &= (\alpha-1/2)\log(TQ^2)(f(\eta)-\epsilon/2) \end{aligned} \tag{4.5}$$

for T sufficiently large.

By Lemma 4.2.1 it follows with R as the number of pairs of characters and points that

$$R\cdot\delta^{2k}(\log(TQ))^{2k} \le \sum_{l=1}^{L}\sum_{j=1}^{J_l}|S(s_{l,j},\chi_l)|^{2k} \ll TQ^2\log^2(TQ^2)k!\left(\sum_{p\le uv}\frac{(\log p)^2}{p^{2\alpha}}\right)^k.$$

With (4.5) and

$$\begin{aligned} \sum_{p\in\mathbb{P}}\frac{(\log p)^2}{p^{2\alpha}} &\le \sum_{n\in\mathbb{N}}\frac{\log n\Lambda(n)}{n^{2\alpha}} = -\frac{1}{2}\cdot\left(\sum_{n\in\mathbb{N}}\frac{\Lambda(n)}{n^{2\alpha}}\right)' = -\frac{1}{2}\cdot\left(\frac{\zeta'}{\zeta}(2\alpha)\right)' = -\left(\frac{\zeta'}{\zeta}\right)'(2\alpha) \\ &\ll \frac{1}{(2\alpha-1)^2} \end{aligned}$$

it follows for a certain $c>0$ since $k! \le k^k$

$$R \ll \frac{TQ^2\log^2(TQ^2)}{\delta^{2k}(\log(TQ))^{2k}}k!\left(\frac{c}{(2\alpha-1)^2}\right)^k \le TQ^2\log^2(TQ^2)\left(\frac{ck}{\delta^2\log^2(TQ)(2\alpha-1)^2}\right)^k.$$

We have

$$\frac{k}{(2\alpha-1)\log(TQ)} \leq \frac{\log(TQ^2)}{(2\alpha-1)\log(TQ)\log(uv)} \leq \frac{1}{\log\left(\frac{\eta+1/2}{\eta-2\delta}\right)+\psi} \leq 1,$$

since $\dfrac{\eta+1/2}{\eta-2\delta} \geq 1+\dfrac{1}{2\delta}$ resp.

$$\begin{aligned} & -(\alpha-1/2)\log(TQ^2)\left((f(\eta)-\epsilon/2)\log\left(\frac{2\delta^2}{c}\right)+\frac{\epsilon}{2}\log((\alpha-1/2)\log(TQ))\right) \\ = \;& (\alpha-1/2)\log(TQ^2)\left((f(\eta)-\epsilon/2)\log\left(\frac{c}{2\delta^2}\right)-\frac{\epsilon}{2}\log((\alpha-1/2)\log(TQ))\right) \leq \log_2(TQ). \end{aligned}$$

So with (4.5) we get

$$\begin{aligned} R \;&\ll\; TQ^2\log^2(TQ^2)\left(\frac{c}{\delta^2(\log(TQ))(2\alpha-1)}\right)^k \\ &=\; TQ^2\log^2(TQ^2)\exp\left(k\log\left(\frac{c}{\delta^2\log(TQ)(2\alpha-1)}\right)\right) \\ &=\; TQ^2\log^2(TQ^2)\exp\left(-k\log\left(\frac{\delta^2\log(TQ)(2\alpha-1)}{c}\right)\right) \\ &\leq\; TQ^2\log^2(TQ^2)\exp\left(-(\alpha-1/2)\log(TQ^2)(f(\eta)-\epsilon/2)\left(\log\left(\frac{2\delta^2}{c}\right)+\log((\alpha-1/2)\log(TQ))\right)\right) \\ &=\; TQ^2\log^2(TQ^2)\exp\left(-(\alpha-1/2)\log(TQ^2)(f(\eta)-\epsilon/2)\log\left(\frac{2\delta^2}{c}\right)\right. \\ &\qquad \left.-(\alpha-1/2)\log(TQ^2)f(\eta)\log((\alpha-1/2)\log(TQ))+\frac{\epsilon}{2}(\alpha-1/2)\log(TQ^2)\log((\alpha-1/2)\log(TQ))\right) \\ &=\; TQ^2\log^2(TQ^2)\exp\left(2(\alpha-1/2)\log(TQ^2)\left((f(\eta)-\epsilon/2)\log\left(\frac{2\delta^2}{c}\right)-\frac{\epsilon}{2}\log((\alpha-1/2)\log(TQ))\right)\right. \\ &\qquad \left.+\epsilon(\alpha-1/2)\log(TQ^2)\log((\alpha-1/2)\log(TQ))-(\alpha-1/2)\log(TQ^2)f(\eta)\log((\alpha-1/2)\log(TQ))\right) \\ &\leq\; TQ^2\log^2(TQ^2)\exp\left(\log_2(TQ)-(\alpha-1/2)\log(TQ^2)\log((\alpha-1/2)\log(TQ))(f(\eta)-\epsilon)\right) \\ &\leq\; TQ^2\log^3(TQ^2)\exp\left(-(f(\eta)-\epsilon)(\alpha-1/2)\log(TQ^2)\log((\alpha-1/2)\log(TQ))\right). \end{aligned}$$

This gives the stated result. □

Lemma 4.2.3. [11]

Let $q, Q \in \mathbb{N}$. For $1 \leq j \leq J_l$ and $1 \leq l \leq L$ there are pairs of primitive pairwise different characters $\chi_l \bmod q$ and points $s_{l,j} = \sigma_{l,j} + it_{l,j}$ with

$$\sigma_{l,j} \geq \alpha \geq \frac{1}{2} + \frac{10\log\log(TQ)}{\log(TQ)}$$

and $T \leq t_{l,j} \leq 2T$ with $|t_{l_1,j} - t_{l_2,j}| \geq \frac{1}{\log(TQ^2)}$ for $l_1 \neq l_2$, if these points belong to the same character. If $|\log L(s_{l,j},\chi_l)| \geq (\alpha-1/2)\log(TQ)$ and

$$\Re\frac{L'}{L}(s_{l,j},\chi_l) \leq \frac{1}{2}\log(TQ)$$

hold, then we get for the number of pairs of characters and points

$$R \ll TQ^2\log^3(TQ^2)\exp\left(-\frac{1}{2}(\alpha-1/2)\log(TQ^2)\log\left(\frac{(\alpha-1/2)\log(TQ)}{4\log_2(TQ)}\right)\right).$$

[11] In Maier and Montgomery (4) and Fischer (9) they get in Lemma 7 resp. Lemma 26 a similar result.

Proof. By replacing s by $s+x$ in Theorem 3.5.1, where $x \in [0,1]$, we get

$$\begin{aligned} G(s,\chi) &:= \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{n^{s+x}} + \frac{L'}{L}(s+x,\chi) \\ &= -\sum_{k=0}^{\infty} \frac{u^{-a-2k-s-x}\left(v^{-a-2k-s-x}-1\right)}{(-a-2k-s-x)^2 \log v} + E_0 \frac{u^{-s-x}\left(v^{-s-x}-1\right)}{(-s-x)^2\log v} \\ &\quad - \sum_{z=1/2+i\gamma} \frac{u^{z-s-x}\left(v^{z-s-x}-1\right)}{(z-s-x)^2\log v} + E_0 \frac{u^{1-s-x}\left(v^{1-s-x}-1\right)}{(1-s-x)^2\log v}, \end{aligned} \tag{4.6}$$

if $s+x$ is neither a pole nor a zero of $L(s,\chi)$. Here γ stands for the imaginary part of all zeros of $L(s,\chi)$ in the critical strip with the real part 1/2 under GRH.
We consider the terms of the right- hand- side and get analogous to Lemma 4.2.2

$$\begin{aligned} \left| \sum_{z=1/2+i\gamma} \frac{u^{z-s-x}\left(v^{z-s-x}-1\right)}{(z-s-x)^2\log v} \right| &\leq \sum_{z=1/2+i\gamma} \frac{u^{1/2-\sigma-x}\left(v^{1/2-\sigma-x}-1\right)}{|(z-s-x)^2\log v|} \\ &= \frac{u^{1/2-\sigma-x}\left(v^{1/2-\sigma-x}+1\right)}{(\sigma-1/2)\log v} \sum_{\gamma} \frac{\sigma-1/2}{(\sigma-1/2)^2+(t-\gamma)^2} \\ &\leq \frac{u^{1/2-\sigma-x}\left(v^{1/2-\sigma-x}+1\right)}{(\sigma-1/2)\log v} \cdot \Re\frac{\xi'}{\xi}(s,\chi) \\ &\leq \frac{u^{1/2-\alpha-x}\left(v^{1/2-\alpha-x}+1\right)}{(\alpha-1/2)\log v} \cdot \left(\Re\frac{L'}{L}(s,\chi) + \frac{1}{2}\cdot\log\frac{tq}{2}\right) \\ &\leq \frac{u^{1/2-\alpha-x}\left(v^{1/2-\alpha-x}+1\right)}{(\alpha-1/2)\log v} \cdot \left(\Re\frac{L'}{L}(s,\chi) + \frac{\log(TQ)}{2}\right) \end{aligned}$$

under the Lemmata 4.1.1 and 4.1.2 for $s=\sigma+it$ with $\sigma \geq \alpha$ and $T \leq t \leq 2T$. There is

$$\begin{aligned} \left| \sum_{k=0}^{\infty} \frac{u^{-a-2k-s-x}\left(v^{-a-2k-s-x}-1\right)}{(-a-2k-s-x)^2\log v} \right| &\leq \sum_{k=0}^{\infty} \frac{u^{-a-2k-\alpha-x}\left(v^{-a-2k-\alpha-x}+1\right)}{|-a-2k-s-x|^2\log v} \\ &\leq \frac{1}{T^2}\cdot\sum_{k=0}^{\infty}(uv)^{-2k} \ll T^{-2} \end{aligned}$$

as well as

$$\frac{u^{1-s-x}\left(v^{1-s-x}-1\right)}{(1-s-x)^2\log v} \ll T^{-1} \quad \text{and} \quad \frac{u^{-s-x}\left(v^{-s-x}-1\right)}{(-s-x)^2\log v} \ll T^{-1}$$

since $u,v \geq 1$ and $uv \leq TQ^2$. We integrate over $0 \leq x \leq 1$ to see that

$$\begin{aligned} |G(s,\chi)| &= \left| \int_0^1 \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{n^{s+x}}\,dx + \int_0^1 \frac{L'}{L}(s+x,\chi)\,dx \right| \\ &\leq \int_0^1 \frac{u^{1/2-\alpha-x}\left(v^{1/2-\alpha-x}+1\right)}{(\alpha-1/2)\log v}\,dx \cdot \left(\Re\frac{L'}{L}(s,\chi)+\frac{\log(TQ)}{2}\right) + O(T^{-1}) \\ &\leq \left(\frac{u^{1/2-\alpha}}{\log u}\left(1-\frac{1}{u}\right) + \frac{(uv)^{1/2-\alpha}}{\log(uv)}\left(1-\frac{1}{uv}\right) \right) \cdot \frac{\Re\frac{L'}{L}(s,\chi)+\frac{\log(TQ)}{2}}{(\alpha-1/2)\log v} + O(T^{-1}) \\ &\leq \left(\frac{u^{1/2-\alpha}}{\log u} + \frac{(uv)^{1/2-\alpha}}{\log(uv)} \right) \cdot \frac{\Re\frac{L'}{L}(s,\chi)+\frac{\log(TQ)}{2}}{(\alpha-1/2)\log v} + O(T^{-1}), \end{aligned}$$

for $u,v \geq 1$.

On the other hand we have

$$
\begin{aligned}
|G(s,\chi)| &= \left|\int_0^1 \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{n^{s+x}}\,dx + \int_0^1 \frac{L'}{L}(s+x,\chi)\,dx\right| \\
&= \left|-\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{(\log n)n^{s+1}} + \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{(\log n)n^{s}} + \log L(s+1,\chi) - \log L(s,\chi)\right| \\
&= \left|\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{(\log n)n^{s}} - \log L(s,\chi)\right| + O(1).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left|\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{(\log n)n^{s}} - \log L(s,\chi)\right| &\le \left(\frac{u^{1/2-\alpha}}{\log u} + \frac{(uv)^{1/2-\alpha}}{\log(uv)}\right) \frac{\Re\frac{L'}{L}(s,\chi) + 1/2\log(TQ)}{(\alpha-1/2)\log v} + O(1) \\
&= \left(\frac{\alpha-1/2}{e} + \frac{\alpha-1/2}{2e^2}\right)\left(\Re\frac{L'}{L}(s,\chi) + \frac{1}{2}\log(TQ)\right) + O(1) \\
&\le \frac{9}{20}(\alpha-1/2)\log(TQ), \qquad (4.7)
\end{aligned}
$$

under $\Re\frac{L'}{L}(s,\chi) \le \frac{1}{2}\log(TQ)$ and by setting $u = v = \exp\left(\frac{1}{\alpha-1/2}\right)$.

Let the pairs of the points $s_{l,j}$ and the characters χ_l mod q be defined like in Lemma 4.2.2. We set $S(s,\chi) = \sum_{p\in\mathbb{P}} \omega(p)\chi(p)p^{-s}$ and apply Lemma 4.2.1. There is

$$
\begin{aligned}
\left|\sum_{p\in\mathbb{P}}\sum_{k=2}^{\infty} \frac{\omega(p^k)\chi(p^k)}{kp^{ks}}\right| &\le \frac{1}{2}\sum_{p\in\mathbb{P}} \frac{\omega(p^2)}{p^{2\alpha}} + \left|\sum_{p\in\mathbb{P}}\sum_{k=3}^{\infty} \frac{\omega\left(p^k\right)\chi\left(p^k\right)}{kp^{ks}}\right| \\
&\le \frac{1}{2}\sum_{p\le uv} \frac{1}{p} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = \frac{1}{2}\sum_{p\le uv} \frac{1}{p} + O(1) \\
&\le \frac{1}{2}\log_2(uv) + O(1) = \frac{1}{2}\log\left(\frac{2}{\alpha-1/2}\right) + O(1) \\
&\le \frac{1}{2}\log\left(\frac{2\log(TQ)}{10\log_2(TQ)}\right) + O(1) \\
&\le \frac{1}{2}\log_2(TQ) \le \frac{1}{20}(\alpha-1/2)\log(TQ) \qquad (4.8)
\end{aligned}
$$

with the estimation[12]

$$
\sum_{\substack{p\in\mathbb{P}\\ p\le x}} \frac{1}{p} \le \log_2(x) + O(1)
$$

for $x \in \mathbb{R}^+$. Under (4.6) and (4.7) we get

$$
\begin{aligned}
|S(s_{l,j},\chi_l)| &\ge -\left|\log L(s_{l,j},\chi_l) - \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)\omega(n)}{(\log n)n^{s_{l,j}}}\right| + |\log L(s_{l,j},\chi_l)| - \left|\sum_{p\in\mathbb{P}}\sum_{k=2}^{\infty} \frac{\omega\left(p^k\right)\chi\left(p^k\right)}{kp^{ks}}\right| \\
&\ge -\frac{9}{20}(\alpha-1/2)\log(TQ) + (\alpha-1/2)\log(TQ) - \frac{1}{20}(\alpha-1/2)\log(TQ) \\
&= \frac{(\alpha-1/2)\log(TQ)}{2}.
\end{aligned}
$$

[12]See Titchmarsh (7), page 63 ff.

Setting $k := \left[\frac{(\alpha-1/2)\log(TQ^2)}{2}\right]$ sufficiently large for the validity of $(uv)^k \leq TQ^2$ and applying Lemma 4.2.2 we get with the estimation (4.8) for the number of pairs of points and characters

$$\begin{aligned}
R\cdot(1/2(\alpha-1/2)\log(TQ))^{2k} &\leq \sum_{l=1}^{L}\sum_{j=1}^{J_l}|S(s_{l,j},\chi_l)|^2 \ll TQ^2\log^2(TQ^2)k!\left(\sum_{p\leq uv}p^{-2\alpha}\right)^k \\
&\ll TQ^2\log^2(TQ^2)k!(\log_2(uv))^k \\
&\ll TQ^2\log^2(TQ^2)k!\left(\log_2\left(\exp\left(\frac{2}{\alpha-1/2}\right)\right)\right)^k \\
&\ll TQ^2\log^2(TQ^2)k!\left(\log\left(\frac{2}{\alpha-1/2}\right)\right)^k \ll TQ^2\log^2(TQ^2)k!(\log_2(TQ))^k.
\end{aligned}$$

We have

$$\begin{aligned}
R &\ll TQ^2\log^2(TQ^2)\frac{k!(\log_2(TQ))^k}{(1/2(\alpha-1/2)\log(TQ))^{2k}} \ll TQ^2\log^2(TQ^2)\left(\frac{4k\log_2(TQ)}{(\alpha-1/2)^2\log(TQ)^2}\right)^k \\
&\ll TQ^2\log^2(TQ^2)\left(\frac{2\log(TQ^2)(\alpha-1/2)\log_2(TQ)}{(\alpha-1/2)^2\log(TQ)^2}\right)^k,
\end{aligned}$$

since $k \leq \frac{(\alpha-1/2)\log(TQ^2)}{2}$. With $\frac{\log(TQ^2)}{2\log(TQ)} = O(1)$ and $-k < -\frac{\log(TQ^2)(\alpha-1/2)}{2} + 1$ we have

$$\begin{aligned}
R &\ll TQ^2\log^2(TQ^2)\left(\frac{4\log_2(TQ)}{(\alpha-1/2)\log(TQ)}\right)^k \\
&= TQ^2\log^2(TQ^2)\left(\frac{(\alpha-1/2)\log(TQ)}{4\log_2(TQ)}\right)^{-k} \\
&\leq TQ^2\log^2(TQ^2)\left(\frac{(\alpha-1/2)\log(TQ)}{4\log_2(TQ)}\right)^{-\frac{1}{2}\log(TQ^2)(\alpha-1/2)+1} \\
&\leq TQ^2\log^2(TQ^2)\left(\frac{(\alpha-1/2)\log(TQ)}{4\log_2(TQ)}\right)^{-\frac{1}{2}\log(TQ^2)(\alpha-1/2)}\left(\frac{(\alpha-1/2)\log(TQ)}{4\log_2(TQ)}\right) \\
&\ll TQ^2\log^3(TQ^2)\exp\left(-\frac{1}{2}(\alpha-1/2)\log(TQ^2)\log\left(\frac{(\alpha-1/2)\log(TQ)}{4\log_2(TQ)}\right)\right),
\end{aligned}$$

since $\frac{(\alpha-1/2)\log(TQ)}{4\log_2(TQ)} \ll \log(TQ)$. □

Lemma 4.2.4. [13]

Let $q, Q \in \mathbb{N}$. We consider pairs of primitive pairwise different characters $\chi_l \bmod q$ and points $s_{l,j} = \sigma_{l,j} + it_{l,j}$ with $\sigma_{l,j} > 1/2$ and $T \leq t_{l,j} \leq 2T$. There exists $V \in \mathbb{R}^+$, so that

$$\begin{aligned}
V &\geq 15\log_2(TQ) \\
-V &\geq \log|L(s_{l,j},\chi_l)| \text{ and} \\
\Re\frac{L'}{L}(\sigma+it_{l,j},\chi_l) &\leq \frac{1}{2}\log(TQ)
\end{aligned}$$

holds for $\sigma_{l,j} \leq \sigma < \infty$. Thus, we get for the number of these pairs

$$R \ll TQ^2\log^3(TQ^2)\exp\left(-\frac{V}{3}\log\left(\frac{V/6}{\log_2(TQ)}\right)\right).$$

[13] In Maier and Montgomery (4) and Fischer (9) they get in Lemma 8 resp. Lemma 27 a similar result.

Proof. There is $\sigma'_{l,j} = \sigma_{l,j} + \frac{2V}{3\log(TQ)}$ and $s'_{l,j} = \sigma'_{l,j} + it_{l,j}$. Then

$$\sigma'_{l,j} \geq \frac{1}{2} + \frac{2V}{3\log(TQ)} =: \alpha \geq \frac{1}{2} + \frac{10\log_2(TQ)}{\log(TQ)},$$

and we have

$$\begin{aligned} V + \log|L(s'_{l,j}, \chi_l)| &\leq \log|L(s'_{l,j}\chi_l)| - \log|L(s_{l,j}, \chi_l)| = \int_{\sigma_{l,j}}^{\sigma'_{l,j}} \Re\frac{L'}{L}(\sigma + it_{l,j}, \chi_l)\, d\sigma \\ &\leq \frac{(\sigma'_{l,j} - \sigma_{l,j})\log(TQ)}{2} = \frac{V}{3}. \end{aligned}$$

Also there is

$$\begin{aligned} \log|L(s'_{l,j}, \chi_l)| &\leq -2V/3 \quad \text{and} \\ |\log L(s'_{l,j}, \chi_l)| &\geq -\log|L(s'_{l,j}, \chi_l)| \geq 2V/3 \geq 10\log_2(TQ) \geq (\alpha - 1/2)\log(TQ). \end{aligned}$$

So the conditions of Lemma 4.2.3 are fulfilled for the pairs of the characters $\chi_l \bmod q$ and points $s'_{l,j}$, and we get

$$\begin{aligned} R &\ll TQ^2\log^3(TQ^2)\exp\left(-\frac{1}{2}(\alpha - 1/2)\log(TQ^2)\log\left(\frac{(\alpha-1/2)\log(TQ)}{4\log_2(TQ)}\right)\right) \\ &\ll TQ^2\log^3(TQ^2)\exp\left(-\frac{V}{3}\frac{\log(TQ^2)}{\log(TQ)}\log\left(\frac{2V/3}{4\log_2(TQ)}\right)\right) \\ &\ll TQ^2\log^3(TQ^2)\exp\left(-\frac{V}{3}\log\left(\frac{V/6}{\log_2(TQ)}\right)\right). \end{aligned}$$

□

Chapter 5

The result and the proof

5.1 Introduction

Theorem 5.1.1. *Assuning GRH, let $\chi \bmod q$ be a primitive character modulo q, which is not the principal character with $Q \leq q \leq 2Q$. Let $u \in [0,1]$.*
We study the sum

$$\sum(x,u,\chi) = \sum_{\substack{n \leq x \\ p^+(n) \leq x^u}} \mu(n)\chi(n),$$

where $p^+(n)$ denotes the largest prime factor of n.
Then with positive constants A and B, which only depend on u, we have the estimation

$$\sum_{Q \leq q \leq 2Q} \sideset{}{^*}\sum_{\chi \bmod q} \left|\sum(x,u,\chi)\right| \ll x^{1/2}(\log x)^A Q^2 (\log Q)^B \exp((\log x)^{39/61}).$$

Notice 5.1.1. We do not study this for the principal character. For this, special efforts would be required.

Definition 5.1.1. [1]
Let $s \in \mathbb{C}$ and

$$\begin{aligned} M(x,\chi) &= \sum_{n \leq x} \mu(n)\chi(n) \quad \text{and} \\ M(x,\chi,s) &= \sum_{n \leq x} \mu(n)\chi(n)n^{-s}, \end{aligned}$$

and we define for $x \geq 1$ the sets

$$\begin{aligned} S(x) &= \{n \leq x \colon \mu(n) \neq 0\} \\ S_j(x) &= \{n \leq x \colon n = p_1 \cdot p_2 \cdots p_j \cdot m,\ p_1 > p_2 > \ldots > p_j > x^u,\ p^+(m) \leq x^u\} \\ T_j(x) &= \{n \leq x \colon n = p_1 \cdot p_2 \cdots p_j,\ p_1 > p_2 > \ldots > p_j > x^u\} \\ S(x,u) &= \{n \leq x \colon p^+(n) \leq x^u\}. \end{aligned}$$

[1]See Maier (12), Definition 2.1

We define

$$\sum\nolimits_j (x,\chi) = \sum_{n\in S_j(x)} \mu(n)\chi(n)$$
$$\sum\nolimits_j (x,\chi,s) = \sum_{n\in S_j(x)} \mu(n)\chi(n)n^{-s}.$$

Finally we have

$$M(x,u,\chi) = \sum_{\substack{n\le x\\ p^+(n)\le x^u}} a(n)\chi(n)n^{-s}$$

with

$$a(n) = \begin{cases} \mu(n), & \text{if } p^+(n) \le x^u \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.1.1. [2]
We have

$$\sum (x,u,\chi) = M(x,\chi) - \sum_{\substack{j\in\mathbb{N}\\ j\le\left[\frac{1}{u}\right]}} \sum\nolimits_j (x,\chi).$$

Proof. This follows, because the union

$$S(x) = \bigcup_{j\le\left[\frac{1}{u}\right]} S_j(x,u) \cup S(x,u)$$

is disjoint. □

We want to apply Perron's formula (Theorem 2.3.2).
The conditions are fulfilled since we have $|\mu(n)\chi(n)| \le 1$ and the normal convergence of the given series. So with $c = 1 + \frac{1}{\log x}$ we have

$$\sum (x,u,\chi) = \frac{1}{2\pi i}\int_{c-ix}^{c+ix} \frac{x^s}{s\cdot L(s,\chi)}\, ds - \frac{1}{2\pi i}\int_{c-ix}^{c+ix} D(s,\chi)\cdot\frac{x^s}{s}\, ds + O\left(\log x\right)$$

and so

$$\sum_{Q\le q\le 2Q}\ \sideset{}{^*}\sum_{\chi \bmod q} \left|\sum (x,u,\chi)\right| \le \sum_{Q\le q\le 2Q}\ \sideset{}{^*}\sum_{\chi \bmod q} \int_{c-ix}^{c+ix} \left|\frac{x^s}{s\cdot L(s,\chi)}\right| ds + \sum_{Q\le q\le 2Q}\ \sideset{}{^*}\sum_{\chi \bmod q} \int_{c-ix}^{c+ix} \left|D(s,\chi)\cdot\frac{x^s}{s}\right| ds + O\left(\log x\right). \quad (5.1)$$

We perform the integration of the first part by choosing suitable paths $[c-iT, c+iT]$ and certain T, using a method of Maier and Montgomery as piecewise linear contours, which depend on the size of the integrand near the critical line. Here the zeros of the Dirichlet L- function are important, since the integrand will be large at these points and in their neighbourhood.
We assume GRH:
All zeros of the Dirichlet L- function in the critical strip lie on the critical line $\Re(s) = \frac{1}{2}$.

[2]See Maier (12), Lemma 2.1

5.2 Estimation of the first part of the integral

We start with the estimation of the first term in (5.1).
We describe the paths of the integration as piecewise linear contours in the upper half- plane.

Theorem 5.2.1. [3]
For any $x \geq 2$ there are piecewise linear contours $\mathcal{C}(\chi)$ lying in the rectangle $\frac{1}{2} < \sigma < 1$ and $-x \leq t \leq x$, that link the bottom edge of the rectangle to the top, for which the following estimates with $q \in \mathbb{N}$ and $Q \leq q \leq 2Q$ and primitive characters $\chi \bmod q$ apply:

$$\sum_{Q \leq q \leq 2Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{\substack{\mathcal{C}(\chi) \\ 0 \leq t \leq 16}} \left| \frac{x^s}{L(s,\chi)} \right| |ds| \ll x^{1/2} Q^2 \log\log x. \tag{5.2}$$

For $16 \leq T \leq \exp((\log x)^{39/61})$ *we have*

$$\sum_{Q \leq q \leq 2Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{\substack{\mathcal{C}(\chi) \\ T \leq t \leq 2T}} \left| \frac{x^s}{L(s,\chi)} \right| |ds| \ll x^{1/2} T Q^2 \log(TQ) \left(\frac{e \log x}{\log(TQ)} \right)^{\frac{c \log(TQ)}{\log\log(TQ)}}. \tag{5.3}$$

For $\exp((\log x)^{39/61}) \leq T \leq x$ *we have*

$$\sum_{Q \leq q \leq 2Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{\substack{\mathcal{C}(\chi) \\ T \leq t \leq 2T}} \left| \frac{x^s}{L(s,\chi)} \right| |ds| \ll x^{1/2} (\log x)^A T Q^2 (\log(TQ^2))^B \exp\left(\left(\frac{\log x}{\log(TQ)} \right)^{39/22} \right) \tag{5.4}$$

with positive constants A and B.

Definition 5.2.1. (Definition of paths of the integration)
We define paths of the integration in three steps:

- In the first step we have a straight- line from 0 to 16.
- Then we have continous paths of the form

$$\begin{array}{cc}
\frac{1}{2} + \frac{\log(8TQ)}{\log x \log\log(8TQ)} + 16i & \frac{1}{2} + \frac{\log(16TQ)}{\log x \log\log(16TQ)} + 16i \\
\frac{1}{2} + \frac{\log(16TQ)}{\log x \log\log(16TQ)} + 32i & \cdots \\
\vdots & \vdots \\
\frac{1}{2} + \frac{\log(2^j TQ)}{\log x \log\log(2^j TQ)} + 2^{j+1}i & \frac{1}{2} + \frac{\log(2^{j+1} TQ)}{\log x \log\log(2^{j+1} TQ)} + 2^{j+1}i
\end{array}$$

  for $J = \left[\frac{(\log x)^{39/61}}{\log 2} \right]$, also $2^J \leq \exp((\log x)^{39/61}) \leq 2^{J+1}$, and for $T \leq \exp((\log x)^{39/61})$.
  The integration paths have to be determined for each character specifically. But the result varies insignificantly by using Q instead of q in the definitions of the paths.
- The third part of the paths of the integration is defined in the following way:
  Set $T = 2^J$ for every $j \geq J$ and choose for every $T \leq r \leq 2T$ the real part $\sigma_2(\chi, r)$ minimal, such that for $\eta > 0$, which will be determinded later,

$$\Re \frac{L'}{L}(s, \chi) \leq \eta \log(TQ)$$

  holds for all $s = \sigma + it$ with $\sigma \geq \sigma_2(\chi, r)$ and $r \leq t \leq r+1$.

[3] In Maier and Montgomery (4) and Fischer (9) they get in the Theorem resp. Theorem 31 similar results.

For $\chi \neq \chi_0^*$ and $s = \frac{1}{2} + \frac{1}{\log(TQ)} + i\gamma_0$ we have

$$\Re\frac{\xi'}{\xi}(s,\chi) = \frac{1}{2}\cdot\sum_{\rho=\frac{1}{2}+i\gamma}\Re\left(\frac{1}{\rho-1}+\frac{1}{\rho}\right) + \sum_{\rho=\frac{1}{2}+i\gamma}\Re\left(\frac{1}{s-\rho}\right) \geq \Re\left(\frac{1}{s-\left(\frac{1}{2}+i\gamma_0\right)}\right) = \log(TQ),$$

and we sum over all zeros of $L(s,\chi)$ in the critical strip, whose imaginary parts differ by less than 1. Then we have

$$\Re\frac{L'}{L}(s,\chi) \geq \Re\frac{\xi'}{\xi}(s,\chi) - \frac{1}{2}\log\left(\frac{tq}{2}\right) \geq \frac{1}{2}\log(TQ) > \eta\log(TQ).$$

In conclusing there is

$$\sigma_2(\chi,r) \geq \frac{1}{2} + \frac{1}{\log(TQ)}$$

for $T \leq r \leq 2T$ and T big enough.

It is obvious that $\sigma_2(\chi,r) \leq \frac{3}{4}$ holds. Now we set

$$\sigma_1(\chi,r) = \frac{1}{2} + (\sigma_2(\chi,r) - 1/2)\cdot\frac{\log(TQ)}{\log x},$$

which completes the definition of the linear contour, for which we have

$$\sigma_1(\chi,r) \geq \frac{1}{2} + \frac{1}{\log x}.$$

Proof. (Proof of Theorem 5.2.1)
We first prove the second part of Theorem 5.2.1, the inequality (5.3).
Let $16 \leq T \leq \exp((\log x)^{39/61})$ and suppose $T = 2^j$ with $4 \leq j \leq J$. According to Theorem 2.5.9 we have

$$N_\chi(T+1) - N_\chi(T) \ll \log(TQ),$$

where the implied constant is independent of q and T. Since $\frac{1}{L(s,\chi)}$ has a pole of order $\nu(\rho)$ at each zero ρ of $L(s,\chi)$ with order $\nu(\rho)$, it follows for $s = \sigma + it$ with $T \leq t \leq 2T$ and for each χ

$$\begin{aligned}\frac{1}{|L(s,\chi)|} &\ll \sum_{U=T}^{2T-1}\sum_{U\leq\tau\leq U+1}\left(\frac{1}{\left|s-\frac{1}{2}-i\tau\right|}\right)^{\nu(\frac{1}{2}+i\tau)} \ll \sum_{U=T}^{2T-1}\log(UQ)\left(\frac{\log_2(TQ)\log x}{\log(TQ)}\right)^{c'\log(UQ)}\\ &\ll T\log(TQ)\left(\frac{\log_2(TQ)\log x}{\log(TQ)}\right)^{c\log(TQ)}\end{aligned}$$

because of $\sigma \geq \frac{1}{2} + \frac{\log(TQ)}{\log_2(TQ)\log x}$. There, c and c' are positive constants independent of Q and T, and the sum runs over all zeros in the critical strip, which have real part 1/2 under GRH and imaginary part $U \leq \tau \leq U+1$. So we get

$$\sum_{Q\leq q\leq 2Q}\sideset{}{^*}\sum_{\chi \bmod q}\int_{\substack{\mathcal{C}(\chi)\\ T\leq t\leq 2T}}\left|\frac{x^s}{L(s,\chi)}\right||ds| \ll x^{1/2}TQ^2\log(TQ)\left(\frac{e\log x}{\log(TQ)}\right)^{\frac{c\log(TQ)}{\log_2(TQ)}}.$$

The first part (5.2) follows similarly to the second part using a result of Littlewood for Dirichlet L-series.[4]

[4]One can find this result for the Riemann - ζ- function in Littlewood (14).

Now we prove the third part. This part is clearly the most complex one.
There is $T = 2^j$ for $j \geq J$ and $T \leq \exp((\log x)^{39/61}) \leq x$.
For $T \leq r \leq t \leq r+1 \leq 2T$ we choose $t_1(\chi, r)$ in a way, that $L(\sigma_1(\chi, r)+it_1(\chi, r), \chi)$ has its minimum at $t = t_1(\chi, r)$. Furthermore, we have $s_1(\chi, r) = \sigma_1(\chi, r) + it_1(\chi, r)$ and

$$m(\chi, r) = \frac{1}{|L(s_1(r), \chi)|}.$$

Then it holds on the vertical parts of the linear contour

$$\int_{r \leq t \leq r+1} \left| \frac{x^s}{L(s, \chi)} \right| ds = \int_r^{r+1} \frac{x^{\sigma_1(\chi, r)}}{|L(\sigma_1(\chi, r) + it, \chi)|} dt \leq x^{\sigma_1(\chi, r)} \cdot m(\chi, r).$$

For the horizontal parts we use the monotonic growth of the function

$$\frac{x^\sigma}{|L(\sigma + it, \chi)|}$$

in σ for $\sigma \geq \min\{\sigma_1(\chi, r-1), \sigma_1(\chi, r)\}$, the validity of which we recognize, by considering their logarithmic derivative

$$\left(\log \frac{x^\sigma}{|L(\sigma + it, \chi)|} \right)' = \log x - \Re \frac{L'}{L}(\sigma + it, \chi). \tag{5.5}$$

According to the definition of $\sigma_2(r, \chi)$ it holds for $\sigma \geq \sigma_2(\chi, r)$ and $r \leq t \leq r+1$

$$\Re \frac{L'}{L}(\sigma + it, \chi) \leq \eta \log(TQ) \leq \frac{1}{2} \log(TQ) \leq \log x.$$

For $\sigma_1(\chi, r) \leq \sigma \leq \sigma_2(\chi, r)$ we have according to Lemma 4.1.3

$$\begin{aligned} \Re \frac{L'}{L}(\sigma + it, \chi) &\leq \Re \frac{L'}{L}(\sigma + it, \chi) + \frac{1}{2} \log \frac{tq}{2} \\ &\leq \frac{\sigma_2(\chi, r) - 1/2}{\sigma - 1/2} \cdot \left(\Re \frac{L'}{L}(\sigma_2(\chi, r) + it, \chi) + \frac{1}{2} \log \left(\frac{tq}{2} \right) \right) \\ &\leq \frac{\sigma_2(\chi, r) - 1/2}{\sigma - 1/2} \cdot \log(TQ) \leq \frac{\sigma_2(\chi, r) - 1/2}{\sigma_1(\chi, r) - 1/2} \cdot \log(TQ) = \log x. \end{aligned}$$

So the logarithmic derivative in (5.5) is always nonnegative.
Hence, according to Lemma 4.1.4 we have

$$\begin{aligned} m(\chi, r) &= \frac{1}{|L(\sigma_1(\chi, r) + it_1(\chi, r), \chi)|} \\ &\leq \frac{1}{|L(\sigma_2(\chi, r) + it_1(\chi, r), \chi)|} \cdot \left(\frac{\sigma_2(\chi, r) - 1/2}{\sigma_1(\chi, r) - 1/2} \right)^{(\sigma_2(\chi, r) - 1/2) \cdot \left(\Re \frac{L'}{L}(\sigma_2(\chi, r) + it_1(\chi, r), \chi) + 1/2 \log\left(\frac{t_1(\chi, r) q}{2} \right) \right)} \\ &\leq \frac{1}{|L(\sigma_2(\chi, r) + it_1(\chi, r), \chi)|} \cdot \left(\frac{\log x}{\log(TQ)} \right)^{(\sigma_2(\chi, r) - 1/2) \cdot (\eta + 1/2) \log(TQ))} \end{aligned}$$

The interval $[\sigma_1(\chi, r-1) + ir, \sigma_1(\chi, r) + ir]$ has at most length 1, so we get

$$\int_{\sigma_1(\chi, r-1)}^{\sigma_1(\chi, r)} \left| \frac{x^s}{L(s, \chi)} \right| ds = \int_{\sigma_1(\chi, r-1)}^{\sigma_1(\chi, r)} \frac{x^\sigma}{|L(\sigma + ir, \chi)|} d\sigma \leq m(\chi, r-1) x^{\sigma_1(\chi, r-1)} + m(\chi, r) x^{\sigma_1(\chi, r)}.$$

We combine this to
$$\int_T^{2T} \left| \frac{x^s}{L(s,\chi)} \right| ds \ll \sum_{r=T}^{2T} x^{\sigma_1(r)} \cdot m(\chi, r).$$
Alltogether we have
$$\sum_{Q \le q \le 2Q} \sum_{\chi \bmod q}^{*} \int_T^{2T} \left| \frac{x^s}{L(s,\chi)} \right| ds \ll x^{1/2} \cdot \sum_{Q \le q \le 2Q} \sum_{\chi \bmod q}^{*} \sum_{r=T}^{2T} n(x,T,q,r,\chi)$$
with
$$n(x,T,q,r,\chi) = \frac{\left(\frac{e^2 \log x}{\log(TQ)}\right)^{(\sigma_2(\chi,r)-1/2)\cdot(\eta+1/2)\log(TQ)}}{|L(\sigma_2(\chi,r)+it_1(\chi,r),\chi)|}$$
and a right- sliding linear contour.

To estimate the right- hand side above, we consider three types of pairs (χ, r).
Let R_1 denote the set of those pairs (χ, r), for which
$$\frac{1}{|L(\sigma_2(\chi,r)+it_1(\chi,r),\chi)|} \le \log(TQ)^{15} \cdot \exp\left(\epsilon\left(\sigma_2(\chi,r)-1/2\right)\log(TQ)\log\left((\sigma_2(\chi,r)-1/2)\log(TQ)\right)\right).$$
Let R_2 denote the set of those pairs (χ, r), where $T \le r \le 2T$, for which
$$\sigma_2(\chi,r) \ge \frac{1}{2} + \frac{C_1 \log_2(TQ)}{\log(TQ)\log_3(TQ)}$$
and
$$\frac{1}{|L(\sigma_2(\chi,r)+it_1(\chi,r),\chi)|} > \exp\left(\epsilon\left(\sigma_2(\chi,r)-1/2\right)\log(TQ)\log\left((\sigma_2(\chi,r)-1/2)\log(TQ)\right)\right).$$
Here, $C_1 = C_1(\epsilon)$ is a large constant whose value will be determined later. Finally, let R_3 denote the set of those pairs (χ, r), where $T \le r \le 2T$, for which
$$\sigma_2(\chi,r) < \frac{1}{2} + \frac{C_1 \log_2(TQ)}{\log(TQ)\log_3(TQ)}$$
and
$$\frac{1}{|L(\sigma_2(\chi,r)+it_1(\chi,r),\chi)|} > \log(TQ)^{15}.$$
Thus, every pair of (χ, r) is in at least one of the R_i.

For $(\chi, r) \in R_1$ choose $t_2(\chi, r)$, where we have $r \le t_2(\chi, r) \le r+1$, such that
$$\Re \frac{L'}{L}(\sigma_2(\chi,r)+it_2(\chi,r),\chi) = \eta \log(TQ).$$
Among the $(\chi, r) \in R_1$ we consider those pairs of (χ, r), for which $\alpha \le \sigma_2(\chi, r) \le \alpha + \delta$, where $\delta = \log^{-2}(TQ)$. Let them form the set R_1'.
We want to apply Lemma 4.2.2, which we have proven for
$$\alpha \ge \frac{1}{2} + \frac{\log_2(TQ)}{\log(TQ)\log_3(TQ)}.$$

But it is also true for

$$\alpha \leq \frac{1}{2} + \frac{\log_2(TQ)}{\log(TQ)\log_3(TQ)},$$

since we have with $\psi > 1$ at first $f(\eta) < 1$ for all $0 \leq \eta \leq 1/3$ and so on

$$\begin{aligned} & TQ^2\log^3(TQ^2)\cdot\exp\left(-(f(\eta)-\epsilon)(\alpha-1/2)\log(TQ^2)\log((\alpha-1/2)\log(TQ))\right) \\ \geq\; & TQ^2\log^3(TQ^2)\cdot\exp\left(-\frac{\log_2(TQ)}{\log_3(TQ)}\log\left(\frac{\log_2(TQ)}{\log_3(TQ)}\right)\right) \\ \geq\; & TQ^2\log^3(TQ^2)\cdot\exp\left(-\log_2(TQ)\right) = TQ^2\log^3(TQ^2)\cdot\log^{-1}(TQ) \geq TQ^2 \geq R, \end{aligned}$$

which is valid since $T \leq t_2(\chi, r) \leq 2T$ with $|t_{l_1} - t_{l_2}| \geq \frac{1}{\log(TQ^2)}$ and $Q \leq q \leq 2Q$ hold.

So by Lemma 4.2.2 the contribution of these pairs of (χ, r) to the sum is at most of the order of

$$\begin{aligned} \sum_{(\chi,r)\in R_1'} n(x,T,q,r,\chi) &= TQ^2\log^3(TQ^2)\cdot \\ & \quad \exp\left(-(f(\eta)-\epsilon)(\alpha+\delta-1/2)\log(TQ^2)\log((\alpha+\delta-1/2)\log(TQ))\right)\cdot \\ & \quad \log(TQ)^{15}\cdot\exp\left(\epsilon\,(\alpha+\delta-1/2)\log(TQ)\log\left((\alpha+\delta-1/2)\log(TQ)\right)\right)\cdot \\ & \quad \left(\frac{e^2\log x}{\log(TQ)}\right)^{(\alpha+\delta-1/2)\cdot(\eta+1/2)\log(TQ))} \\ &\ll TQ^2\log^{18}(TQ^2)\cdot\exp(g(\alpha)) \end{aligned}$$

with

$$g(\alpha) = -(f(\eta)-2\epsilon)(\alpha-1/2)\log(TQ^2)\log((\alpha-1/2)\log(TQ^2)) + (\alpha-1/2)(\eta+1/2)\log(TQ)\log\left(\frac{e^2\log x}{\log(TQ)}\right)$$

because of

$$\left(\frac{e^2\log x}{\log(TQ)}\right)^{\delta\log(TQ)} = \left(\frac{e^2\log x}{\log(TQ)}\right)^{\frac{1}{\log(TQ)}} = O(1),$$

and

$$\exp\left(-(f(\eta)-2\epsilon)\delta\log(TQ^2)\log((\alpha+\delta-1/2)\log(TQ))\right) = O(1)$$

as well as

$$\exp\left(-(f(\eta)-\epsilon)(\alpha+\delta-1/2)\log(TQ^2)\log(\delta\log(TQ))\right) = O(1).$$

The function g assumes its maximum at

$$\alpha_0 = \frac{1}{2} + \frac{1}{e\log(TQ^2)}\cdot\left(\frac{e^2\log x}{\log(TQ)}\right)^{\frac{\eta+\frac{1}{2}}{f(\eta)-2\epsilon}\cdot\frac{\log(TQ)}{\log(TQ^2)}}$$

and the maximum value attained is

$$g(\alpha_0) = \frac{f(\eta)-2\epsilon}{e}\cdot\left(\frac{e^2\log x}{\log(TQ)}\right)^{\frac{\eta+\frac{1}{2}}{f(\eta)-2\epsilon}\cdot\frac{\log(TQ)}{\log(TQ^2)}} \leq \left(\frac{e^2\log x}{\log(TQ)}\right)^{\frac{\eta+\frac{1}{2}}{f(\eta)-2\epsilon}}.$$

This motivates us to take η so as to minimize the above exponent; that is, we take η to be the unique real number such that

$$\psi + \log\left(1+\frac{1}{2\eta}\right) = \frac{1}{2\eta}.$$

Numerically, $\eta = 0,196570958763\ldots$ and $f(\eta) = 0,393141917526\ldots$. We set $\epsilon = 10^{-4}$ and observe

$$\frac{\eta + \frac{1}{2}}{f(\eta) - 2\epsilon} < \frac{39}{22}.$$

On summing over $\alpha = \frac{1}{2} + \frac{1}{\log(TQ)} + k\delta$, we conclude that the total contribution of all $(\chi, r) \in R_1$ is at most of the order of

$$\sum_{(\chi,r)\in R_1} n(x,T,q,r,\chi) \ll TQ^2 \log^{18}(TQ^2) \cdot \exp\left(\left(\frac{\log x}{\log(TQ)}\right)^{\frac{39}{22}}\right). \tag{5.6}$$

Among the $(\chi, r) \in R_2$ we first consider those $(\chi, r) \in R_2'$ for which $(\chi, r) \in R_2$, $\alpha \le \sigma_2(\chi, r) < \alpha + \delta$ with $\delta = \log^{-2}(TQ)$ and $V \le -|\log L(\sigma_2(\chi,r) + it_1(\chi,r),\chi)| < 2V$, where

$$\frac{1}{2} + \frac{C_1 \log_2(TQ)}{\log(TQ)\log_3(TQ)} \le \alpha \le 1$$

and

$$V \ge V_0(\alpha) := \epsilon(\alpha - 1/2)\log(TQ)\log((\alpha - 1/2)\log(TQ)).$$

We now take $C_1 = \frac{8}{\epsilon}\exp\left(\frac{9}{\epsilon}\right)$. This ensures that

$$\begin{aligned}
\frac{V}{6\log_2(TQ)} &\ge \frac{\epsilon(\alpha - 1/2)\log(TQ)\log((\alpha-1/2)\log(TQ))}{6\log_2(TQ)} \ge \frac{\epsilon\frac{C_1\log_2(TQ)}{\log(TQ)\log_3(TQ)}\log(TQ)\log\left(\frac{C_1\log_2(TQ)}{\log_3(TQ)}\right)}{6\log_2(TQ)} \\
&\ge \frac{\epsilon C_1}{6\log_3(TQ)} \cdot (\log_3(TQ) - \log_4(TQ)) \ge \frac{\epsilon}{6}\cdot\frac{8}{\epsilon}\cdot\exp\left(\frac{9}{\epsilon}\right)\cdot\left(1 - \frac{\log_4(TQ)}{\log_3(TQ)}\right) \ge \exp\left(\frac{9}{\epsilon}\right).
\end{aligned}$$

So we get

$$V \ge 6\exp\left(\frac{9}{\epsilon}\right)\log_2(TQ) \ge 15\log_2(TQ)$$

for a certain ϵ sufficiently small. According to the definition of $\sigma_2(\chi, r)$ and χ mod q there is

$$\Re\frac{L'}{L}(\sigma + it_1(\chi,r),\chi) \le \eta\log(Tq) < \frac{1}{2}\log(TQ)$$

for all σ with $\sigma_2(\chi,r) < \sigma < \infty$ and for all χ mod q. All conditions of Lemma 4.2.4 are fulfilled, while we get for every

$$\frac{1}{2} + \frac{C_1\log_2(TQ)}{\log(TQ)\log_3(TQ)} \le 1$$

and $V \ge V_0(\alpha)$ and

$$\frac{1}{|L(\sigma_2(\chi,r) + it_1(\chi,r),\chi)|} = \exp(-\log|L(\sigma_2(\chi,r) + it_1(\chi,r),\chi)|) \le \exp(2V)$$

the estimation

$$\begin{aligned}
\sum_{(\chi,r)\in R_2'(\alpha,V)} n(x,T,q,r,\chi) &\ll TQ^2\log^3(TQ^2)\exp\left(-\frac{V}{3}\log\left(\frac{V/6}{\log_2(TQ)}\right)\right)\exp(2V)\cdot \\
&\quad \left(\frac{e^2\log x}{\log(Tq)}\right)^{(\sigma_2(\chi,r-1/2)\cdot(\eta+1/2)\log(TQ))} \\
&\ll TQ^2\log^3(TQ^2)\exp\left(2V - \frac{V}{3}\log\left(\frac{V/6}{\log_2(TQ)}\right)\right)\cdot \\
&\quad \left(\frac{e^2\log x}{\log(Tq)}\right)^{(\alpha+\delta-1/2)\cdot(\eta+1/2)\log(TQ))} \\
&\ll TQ^2\log^3(TQ^2)\exp\left(2V - \frac{3V}{\epsilon}\right)\left(\frac{e^2\log x}{\log(TQ)}\right)^{(\alpha-1/2)\cdot(\eta+1/2)\log(TQ))}.
\end{aligned}$$

On summing over α and V we deduce

$$\sum_{(\chi,r)\in R_2} n(x,T,q,r,\chi) \ll TQ^2\log^3(TQ^2)\sum_V \exp\left(2V-\frac{3V}{\epsilon}\right)\sum_\alpha \left(\frac{e^2\log x}{\log(TQ)}\right)^{(\alpha-1/2)\cdot(\eta+1/2)\log(TQ))}$$

The first sum extends to all

$$\alpha=\frac{1}{2}+\frac{C_1\log_2(TQ)}{\log(TQ)\log_3(TQ)}+l\delta$$

with $l\in\mathbb{N}_0$ and $l\ll\log^{-2}(TQ)$ and the second sum to all $V=V_0(\alpha)\cdot 2^j$ with $j\in\mathbb{N}_0$. We get

$$\begin{aligned}\sum_V \exp\left(2V-\frac{3V}{\epsilon}\right) &= \sum_{j=0}^{\infty}\exp\left(V\cdot\frac{2\epsilon-3}{\epsilon}\right)=\sum_{j=0}^{\infty}\exp\left(V_0(\alpha)\cdot 2^j\cdot\frac{2\epsilon-3}{\epsilon}\right)\\ &\le \sum_{j=0}^{\infty}\exp\left(\frac{V_0(\alpha)}{\epsilon}\right)^{-2^j}\ll \exp\left(-\frac{V_0(\alpha)}{\epsilon}\right).\end{aligned}$$

So we get

$$\sum_{(\chi,r)\in R_2} n(x,T,q,r,\chi)\ll TQ^2\log^3(TQ^2)\cdot\exp(h(\alpha))$$

with

$$h(\alpha)=(\alpha-1/2)\cdot(\eta+1/2)\log(TQ))\log\left(\frac{e^2\log x}{\log(TQ)}\right)-(\alpha-1/2)\log(TQ)\log((\alpha-1/2)\log(TQ)).$$

This is of the same form as the function $g(\alpha)$ that arose in the preceding case. We get a maximum of $h(\alpha)$ in

$$\alpha_0=\frac{1}{2}+\frac{1}{e\log(TQ)}\cdot\left(\frac{e^2\log x}{\log(TQ)}\right)^{\eta+1/2}$$

and

$$h(\alpha_0)\le\frac{\log x}{\log(TQ)}.$$

So we get a similar result for the $(\chi,r)\in R_2$ like for the $(\chi,r)\in R_1$ but with a more favorable constant. By proceeding as in the former case, we find that the total contribution of all the $(\chi,r)\in R_2$ is at most of the order of

$$\sum_{(\chi,r)\in R_2} n(x,T,q,r,\chi)\ll TQ^2\log^3(TQ^2)\exp\left(\frac{\log x}{\log(TQ)}\right). \tag{5.7}$$

We consider the $(\chi,r)\in R_3$ and once again we sum over these pairs (χ,r) with

$$R_3'(V):=\left\{(\chi,r)\in R_3\colon V\le-\log|L(\sigma_2(\chi,r)+it_1(\chi,r),\chi)|\le\frac{51}{50}V\right\}$$

with $V\ge 15\log_2(TQ)$. There is again according to the definition of $\sigma_2(\chi,r)$

$$\Re\frac{L'}{L}(\sigma+it_1(\chi),\chi)\le\eta\log(TQ)<\frac{1}{2}\log(TQ)$$

for all σ with $\sigma_2(\chi,r)\le\sigma<\infty$ and for all χ mod q.

Applying Lemma 4.2.4 we deduce

$$\begin{aligned}
\sum_{(\chi,r)\in R_3'(V)} n(x,T,q,r,\chi) &\ll TQ^2\log^3(TQ^2)\exp\left(\frac{51}{50}V-\frac{V}{3}\log\left(\frac{V/6}{\log_2(TQ)}\right)\right)\\
&\quad\cdot\left(\frac{e^2\log x}{\log(Tq)}\right)^{(\sigma_2(\chi,r+1/2)(\eta+1/2)\log(TQ)}\\
&\le TQ^2\log^3(TQ^2)\exp\left(V\left(\frac{51}{50}-\frac{1}{3}\log\left(\frac{V/6}{\log_2(TQ)}\right)\right)\right)\cdot\\
&\quad\left(\frac{e^2\log x}{\log(TQ)}\right)^{C_1\frac{\log_2(TQ)}{\log_3(TQ)}(\eta+1/2)}.
\end{aligned}$$

On summing over $V=\left(\frac{51}{50}\right)^l 15\log_2(TQ)$ with $l\in\mathbb{N}_0$ we get

$$\begin{aligned}
\sum_V \exp\left(V\left(\frac{51}{50}-\frac{1}{3}\log\left(\frac{V/6}{\log_2(TQ)}\right)\right)\right) &= \sum_{l=0}^{\infty}\exp\left(\left(\frac{51}{50}\right)^l 15\log_2(TQ)\left(\frac{51}{50}-\frac{1}{3}\log\frac{5}{2}-\frac{l}{3}\log\frac{51}{50}\right)\right)\\
&\le 109\exp(130\log_2(TQ))+\sum_{l=109}^{\infty}\exp\left(-0,074\left(\frac{51}{50}\right)^l\log(TQ)\right)\\
&\ll \log^{130}(TQ)\ll\log^{1277}(TQ).
\end{aligned}$$

So we get

$$\sum_{(\chi,r)\in R_3} n(x,T,q,r,\chi)\ll TQ^2\log^{1280}(TQ^2)\left(\frac{e^2\log x}{\log(TQ)}\right)^{C_1\frac{\log_2(TQ)}{\log_3(TQ)}}.$$

We want to get a similar result like in the case of the $(\chi,r)\in R_1$. Considering

$$C_1\frac{\log_2(TQ)}{\log_3(TQ)}\log\left(\frac{e^2\log x}{\log(TQ)}\right)-\left(\frac{\log x}{\log(TQ)}\right)^{39/22}$$

this is maximal, if

$$\frac{\log x}{\log(TQ)}=\left(\frac{C\log_2(TQ)}{\log_3(TQ)}\right)^{22/39}$$

holds for $C>0$. There is

$$\begin{aligned}
&\exp\left(C_1\frac{\log_2(TQ)}{\log_3(TQ)}\log\left(\frac{e^2\log x}{\log(TQ)}\right)-\left(\frac{\log x}{\log(TQ)}\right)^{22/39}\right)\\
\le\ &\exp\left(C_1\frac{\log_2(TQ)}{\log(TQ)}\log\left(C'\frac{\log_2(TQ)}{\log(TQ)}\right)^{22/39}-\left(\frac{C\log_2(TQ)}{\log_3(TQ)}\right)\right)\le\exp(B_1\log_2(TQ))=\log^{B_1}(TQ)
\end{aligned}$$

for constants B_1, C, C' and C_1. Hence, the total contribution of the $(\chi,r)\in R_3$ is

$$\sum_{\chi\in R_3} n(x,T,q,r,\chi)\ll TQ^2\log^{B_2}(TQ^2)\left(\left(\frac{\log x}{\log(TQ)}\right)^{39/22}\right) \tag{5.8}$$

with a positive constant B_2, whose size is not relevant, since the exponent B_2 is not specified.

On combining this with (5.6), (5.7) and (5.8), the proof of (5.4) is complete:

$$\sum_{Q\le q\le 2Q}\ \sum_{\chi \bmod q}^{*} \int_{\substack{\mathcal{C}(\chi)\\ T\le\Im(s)\le 2T}} \left|\frac{x^s}{L(s,\chi)}\right| ds| \ll x^{1/2}(\log x)^A TQ^2(\log(TQ^2))^B \exp\left(\left(\frac{\log x}{\log(TQ)}\right)^{39/22}\right)$$

with positive constants A and B. □

Also we get for the first part of the estimation in (5.1) using (5.2), (5.3) and (5.4)

$$\begin{aligned}\sum_{Q\le q\le 2Q}\ \sum_{\chi \bmod q}^{*} \frac{1}{2\pi i}\int_{c-ix}^{c+ix} \left|\frac{x^s}{s\cdot L(s,\chi)}\right| ds &= \sum_{Q\le q\le 2Q}\ \sum_{\chi \bmod q}^{*} \int_{\substack{\mathcal{C}(\chi)\\ 0\le\Im(s)\le 16}} \left|\frac{x^s}{s\cdot L(s,\chi)}\right| ds \\ &+ \sum_{Q\le q\le 2Q}\ \sum_{\chi \bmod q}^{*} \sum_{T=2^j} \int_{\substack{\mathcal{C}(\chi)\\ T\le\Im(s)\le 2T}} \left|\frac{x^s}{s\cdot L(s,\chi)}\right| ds,\end{aligned}$$

where the sum runs over all $T = 2^j$ and $j = 4, \dots, j_{max}$ with $j_{max} \ll \log x$.
Applying Theorem 5.2.1 we get

$$\begin{aligned}\sum_{Q\le q\le 2Q}\ \sum_{\chi \bmod q}^{*} \int_{c-ix}^{c+ix} \left|\frac{x^s}{s\cdot L(s,\chi)}\right| ds &\ll x^{1/2}Q^2\Big(\log\log x + \big((\log x)^{A_1}(\log Q)^{B_1} \\ &\quad +(\log x)^{A_2}(\log Q)^{B_2}\big)\exp\left((\log x)^{39/61}\right)\Big) \\ &\ll x^{1/2}(\log x)^{A_3}Q^2(\log Q)^{B_3}\exp\left((\log x)^{39/61}\right)\end{aligned}$$

with constants A_i and B_i for $1 \le i \le 3$, if one considers that it holds

$$\exp\left(\frac{\log x}{\log(TQ)}\right)^{\frac{39}{22}} \le \exp\left((\log x)^{\left(1-\frac{39}{61}\right)\cdot\frac{39}{22}}\right) \le \exp((\log x)^{39/61})$$

for $\exp((\log x)^{39/61}) \le T \le x$. This also explains why the bound $\exp((\log x)^{39/61}) \le T$ was chosen.

5.3 Estimation of the second part of the integral

For the second part of the integral in (5.1) we first study the Dirichlet- series $D(s,\chi,u)$.
We want to estimate this Dirichlet- series and calculate the integral like in section 5.2.

Definition 5.3.1. [5]
For $1 \le A \le B$, $j \in \mathbb{N}$ and $s \in \mathbb{C}$ we set

$$\sum(A,B,j,s) = \sum_{A<p_j<\ldots<p_1<B} \chi(p_1\cdots p_j)\cdot(p_1\cdots p_j)^{-s}.$$

Lemma 5.3.1. [6]
Let $u_0 > 0$ be fixed, $u \le u_0$ and $\sigma > 1$.
Then there is a Dirichlet- series

$$D(s,\chi,u) = \sum_{n=1}^{\infty} b(n,u)\chi(n)n^{-s}$$

[5] See Maier (12), Definition 2.2
[6] See Maier (12), Lemma 2.2

of the form

$$D(s,\chi,u) = \sum_{i=1}^{I} c_i \sum_{j=1}^{R} \sum (A_i(x), B_i(x), j, s) \cdot L(s,\chi)^{-1}$$

such that $x^u < A_i(x) < B_i(x) \le x$, $b(n,u) = a(n)$ and $|b(n,u)| \le C(u_0)$ for a constant $C(u_0) > 0$, while the constants R, I, c_i and j depend only on u and are bounded by a constant depends only on u_0.

Proof. For any integer $k \ge 1$, let $\frac{1}{k} \le u$ and we prove the claim by induction on k.
Induction base: $k = 1$:
Let $n \le x$ and $p^+(n) = p > x^{1/2}$.
Then $n = p \cdot m$, where $m < x^{1/2}$ and thus $p^+(m) \le x^{1/2}$.
Thus, the Dirichlet- series

$$D(s,\chi,u) = L(s,\chi)^{-1} - \left(\sum_{x^{1/2} < p \le x} \chi(p) p^{-s} \right) \cdot L(s,\chi)^{-1}$$

satisfies

$$D(s,\chi,u) = \sum b(n,u)\chi(n)n^{-s}$$

where

$$b(n,u) = \begin{cases} \mu(n), & \text{if } p^+(n) \le x^{1/2} \\ 0, & \text{if } n \le x,\ p^+(n) > x^{1/2}. \end{cases}$$

Induction step: $k \to k+1$:
We apply Lemma 5.1.1 to obtain

$$\sum (x,u,\chi,s) = \sum_{\substack{n \le x \\ p^+(n) \le x^u}} \mu(n)\chi(n)n^{-s} = M(x,\chi,s) - \sum_{j \le k+1} \sum\nolimits_j (x,\chi,s),$$

where

$$\sum\nolimits_j (x,\chi,s) = \sum_{\substack{n:\, x^u < p_j < \ldots < p_1 \le x \\ p^+(m) \le x^u}} \mu(p_1 \cdots p_j m)\chi(p_1 \cdots p_j m)(p_1 \cdots p_j m)^{-s}. \tag{5.9}$$

From $\frac{\log x}{\log(x^u)} \le k+1$ and $p_i > x^u$ we obtain $\frac{\log x^{(1-ju)}}{\log(x^u)} \le k$ as well as $p_1 \cdots p_j m > x$, if $m \ge x^{1-ju}$. Thus we may apply the induction hypothesis, that there is a Dirichlet- series

$$D(s,\chi,u) = \sum b(n,u)\chi(n)n^{-s}$$

of the form

$$D(s,\chi,u) = \sum_{i=1}^{I} c_i \sum_{j=1}^{R} \sum (A_i(x), B_i(x), j, s) \cdot L(s,\chi)^{-1}$$

with $\sigma > 1$ such that

$$b(n,u) = \begin{cases} \mu(n), & \text{if } m \le x^{1-ju},\ p^+(m) \le x^u \\ 0, & \text{if } m \le x^{1-ju},\ p^+(m) > x^u \end{cases}$$

and $|b(n,u)| \le C(u_0)$. Insertion into (5.9) completes the induction step. $\square$

So we can use this to estimate the second part of the integral in (5.1). There is

$$\frac{1}{2\pi i}\int_{c-ix}^{c+ix} D(s,\chi)\cdot\frac{x^s}{s}\,ds = \frac{1}{2\pi i}\int_{c-ix}^{c+ix}\sum_{i=1}^{I} c_i \sum_{j=1}^{R}\sum (A_i(x),B_i(x),j,s)\cdot\frac{x^s}{s\cdot L(s,\chi)}\,ds + O\left(\log x\right).$$

We will estimate the inner sum and use the same result like in the first part otherwise. We also study the Dirichlet series such that at least one prime factor is bigger than x^u and get

$$D(s,\chi,u) = \frac{1}{j!}\cdot {\sum_j}' \chi(n)n^{-s}$$

where n runs over all numbers of the form $n = p_1\cdots p_j$ with $p_1,\ldots,p_j$ pairwise distinct.
Each n is counted with the weight $j!$, which comes from the number of different permutation of the factors $p_1,\ldots,p_j$.
In the following we eliminate the restriction $p_{l_1}\neq p_{l_2}$ by the inclusion- exclusion- principle.
For $\vec{v} = (v_1,v_2)$ with $v_1,v_2\in\{1,\ldots,j\}$ let $f(\vec{v}) = f(j,\vec{v})$ be the set of all tuplets $\vec{p} = (p_1,\ldots,p_j)$ with p_l prime, $x^u < p_l \leq x$ and $p_{v_1} = p_{v_2}$ for $v_1\neq v_2$.

Definition 5.3.2. [7]
For $\vec{p} = (p_1,\ldots,p_j)$ we set $\Pi(\vec{p}) = p_1\cdots p_j$.

We obtain

$${\sum_j}'\chi(n)n^{-s} = \sum_{x^u<p\leq x}\chi(\Pi(\vec{p}))\Pi(\vec{p})^{-s} + \sum_{w=1}^{\binom{\nu_u}{2}}(-1)^w \sum_{\vec{v}_1,\ldots,\vec{v}_w\in f(\vec{v}_1)\cap\ldots\cap f(\vec{v}_w)} \chi\left(\Pi(\vec{p})\right)\left(\Pi(\vec{p})\right)^{-s}. \quad (5.10)$$

The condition $\vec{p}\in f(\vec{v}_1)\cap\ldots\cap f(\vec{v}_w)$ is equivalent to a set of conditions of the form

$$\begin{aligned}
p_{v_1^{(1)}} &= p_{v_2^{(1)}} = \ldots = p_{v_{\kappa_1}^{(1)}},\ \mathcal{N}_1 = \{v_1^{(1)},\ldots,v_{\kappa_1}^{(1)}\}\\
p_{v_1^{(2)}} &= p_{v_2^{(2)}} = \ldots = p_{v_{\kappa_2}^{(2)}},\ \mathcal{N}_2 = \{v_1^{(2)},\ldots,v_{\kappa_2}^{(2)}\}\\
&\ \vdots\\
p_{v_1^{(\omega)}} &= p_{v_2^{(\omega)}} = \ldots = p_{v_{\kappa_\omega}^{(\omega)}},\ \mathcal{N}_\omega = \{v_1^{(\omega)},\ldots,v_{\kappa_\omega}^{(\omega)}\}.
\end{aligned}$$

This leads to

Definition 5.3.3. [8]
For $\nu_u\in\mathbb{N}$ and a tuplet $\vec{\mathcal{K}}_\omega = (\kappa_1,\ldots,\kappa_\omega)$ of natural numbers $\kappa_\omega\geq 2$ with $\kappa_1+\ldots+\kappa_\omega\leq j$ let $\mathcal{S}$ be the set of all tuplets $\vec{\mathcal{N}} = (\mathcal{N}_1,\ldots,\mathcal{N}_\omega)$ of subsets $\mathcal{N}_\varphi\subset\{1,\ldots,j\}$ with $\mathcal{N}_{\varphi_1}\cap\mathcal{N}_{\varphi_2} = \emptyset$ for $\varphi_1\neq\varphi_2$. $|\mathcal{N}_\varphi| = \kappa_\varphi$, $1\leq\varphi\leq\omega$. $sgn(\vec{\mathcal{N}})\in\{-1,1\}$ comes from the factor $(-1)^w$ in (5.10) and the condition $\vec{p}\in f(\vec{v}_1)\cap\ldots\cap f(\vec{v}_w)$.

[7] See Maier and Reck (13), Definition 6.1
[8] See Maier and Reck (13), Definition 6.2

We obtain

Lemma 5.3.2. [9]

$$j!\sum\nolimits_j^{\prime}\chi(n)n^{-s} = \left(\sum_{x^u<p\le x}\chi(p)p^{-s}\right)^j + \sum_{\substack{\vec{\mathcal{K}}_0=(\kappa_1,\dots,\kappa_\omega)\\ \kappa_1+\dots+\kappa_\omega\le j\\ \kappa_\varphi\ge 2}}\sum_{\vec{\mathcal{N}}\in\mathcal{S}} sgn(\vec{\mathcal{N}})\left(\sum_{x^u<p\le x}\chi(p)p^{-\kappa_\varphi s}\right)\cdot\left(\sum_{x^u<p\le x}\chi(p)p^{-s}\right)^{j-(\kappa_1+\dots+\kappa_\omega)}$$

Now we use Abel's partial summation (Theorem 2.1.1), estimates on the number of zeros of $L(s,\chi)$ (Theorem 2.5.9) and the explicit formula (Lemma 2.5.4) and get in intervals of the form (t_0, t_0+1),

$$\sum_{x^u<p\le x}\chi(p)p^{-s} \ll \log^3 x$$

for all $s\in\mathcal{S}$ defined in Definition 5.3.3. The other summands in Lemma 5.3.2 are bounded by a constant.

Now we use the same method we used in the first part and get an estimation for the integral

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-ix}^{c+ix} D(s,\chi)\cdot\frac{x^s}{s}\,ds &= \frac{1}{2\pi i}\int_{c-ix}^{c+ix}\sum_{i=1}^{I}c_i\sum_{j=1}^{R}\sum(A_i(x),B_i(x),j,s)\cdot\frac{x^s}{s\cdot L(s,\chi)}\,ds + O(\log x)\\ &\ll (\log^3 x)^F\cdot\frac{1}{2\pi i}\int_{c-ix}^{c+ix}\frac{x^s}{s\cdot L(s,\chi)}\,ds\\ &\ll x^{1/2}(\log x)^A Q^2(\log Q)^B\exp\left((\log x)^{39/61}\right)\end{aligned}$$

with positive constants A, B and F and we get the claim.

[9] See Maier and Reck (13), Lemma 6.1

Chapter 6

Summary

The study of sums that contain the Möbius- function has a long tradition as we have already indicated. The aim of this work was to estimate such sums, in which Dirichlet-characters modulo q occur as well and the sum runs only over those numbers that do not contain large prime factors.

The summation could be reduced by Perron's formula to an integral, while two mean value calculations were carried out, one over the imaginary parts and one over the Dirichlet- characters.

Claudia Fischer (9) has already used in her thesis an averaging over the imaginary parts. The averaging over the Dirichlet- characters was made because the path of integration is chosen specifically for each Dirichlet- character.

Perron's formula leads us to an integral with two parts.
The first part is calculated based on the method of Baker and Harman (1) combined with the path of integration of Maier and Montgomery (4) as a piecewise linear contour. Here, we use monotonicity principles on horizontal lines which are parallel to the real axis and it is examined how many times the value of the inverse of the Dirichlet- L- series exceeds a certain limit. This is determined by dividing the candidate pairs of imaginary parts and characters into three sets and estimating their contribution. In the second part we use the inclusion- exclusion principle to estimate uniformly the sum contained in the integral.

The procedure in the estimation of the second part of this integration is relatively simple and therefore does not provide optimal results.

A closer look on the exponent in the estimation would be desirable. We got the exponent $39/61$, which is smaller than 1 and fulfills the purpose. The result could, however, be improved by methods of analytic number theory.

The assumption of the validity of the generalized Riemann hypothesis is not a big restriction as it applies as hardly controversial among mathematicians.

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Ulm, January, 2014

Hans- Peter Reck

Declaration

I hereby declare that thesis was performed and written on my own and that references and resources used within this work have been explicitly indicated.

I am aware that making a false declaration may have serious consequences.

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