Rigid analytic curves and their Jacobians

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Glossary of Notations

We denote by
- $\mathbb{R}^+$ the positive real numbers as a multiplicative group
- $\mathbb{R}_{0}^+$ the non-negative real numbers
- $Q(R)$ the quotient field of a ring $R$
- $\hat{R}$ the subring $\{x \in R : |x| \leq 1\}$ of a normed ring $R$
- $\hat{\hat{R}}$ the ideal $\{x \in R : |x| < 1\}$ of $\hat{R}$ where $R$ is a normed ring
- $K$ a valued field
- $\hat{K}$ the ring $K$
- $k$ the field $\hat{K}/\hat{\hat{K}}$
- $\mathbb{G}_{m,K}$ the multiplicative group of $K$, seen as an analytic variety
- $\mathbb{G}_{m,K}$ the group $\{x \in K : |x| = 1\}$ seen as a formal analytic variety
- $\mathbb{G}_{m,k}$ the multiplicative group of $k$, seen as an algebraic variety
- $B^n_K$ the affinoid analytic variety $\text{Sp} K(\zeta_1, \ldots, \zeta_n)$
- $\tilde{X}, \tilde{f}, \tilde{x}$ the reductions of the corresponding formal analytic counterpart.
Introduction

In this work, we describe the general structure of a rigid analytic curve and its Jacobian.

1. The Jacobian of a curve in the complex case

The study of the Jacobian of an algebraic curve started with the research of certain integrals that appear in the calculation of the circumference of an ellipse. Niels Henrik Abel and Carl Gustav Jacob Jacobi first described the Jacobian variety around 1826. Of course, they could not formulate it in terms of algebraic curves, since it was Bernhard Riemann, a good 25 years later, who first defined the Riemann surface and thereby described algebraic curves over \( \mathbb{C} \). It took many more men and women to arrive at the modern description of this theory in the middle of the twentieth century.

The Jacobian variety of an algebraic curve of genus \( g \) is the set of line bundles of degree zero over this curve. The tensor product gives this set the natural structure of a group. A lot harder to show, but in the same way natural is its structure as an algebraic variety of dimension \( g \) itself, containing the original curve as a closed subvariety. The Jacobian variety is therefore an Abelian variety with a canonical polarization, deriving from the embedding of the curve.

Over \( \mathbb{C} \) the Jacobian variety of a curve of genus \( g \) can be described as \( H^0(X, \Omega^1_{X/\mathbb{C}}) / H_1(X, \mathbb{Z}) \), a quotient of \( \mathbb{C}^g \) by a lattice \( M \) of rank \( 2g \), which is generated by certain integrals on the curve. We can write \( M = \mathbb{Z}^g \oplus \mathbb{Z}^g \), so applying the exponential function let us write \( \mathbb{C}_{m, \mathbb{C}}^g / \exp(2\pi i \mathbb{Z}) \). So the Jacobian of a Riemann surface is the multiplicative group of \( \mathbb{C} \) to the power \( g \) modulo a lattice of rank \( g \).

2. Mumford curves and general rigid analytic curves

The complex numbers are just one possibility to create a topological and algebraic closure of \( \mathbb{Q} \). For every prime \( p \) we can define \( |x| = p^{-\nu(x)} \), where \( \nu \) is the valuation associated to \( p \). This was first described by Kurt Hensel and later refined by his student Helmut Hasse at the end of the eighteenth century. The topological closure of \( \mathbb{Q} \) with this absolute value is the field \( \mathbb{Q}_p \) of \( p \)-adic numbers. Its algebraic closure \( \mathbb{C}_p \) has infinite degree over \( \mathbb{Q}_p \). There is no equivalent of the exponential function on \( \mathbb{C}_p \) and the topology has very strange properties. Furthermore, the \( p \)-adic value gives rise to the reduction functor giving a close relation to the finite field \( \mathbb{F}_p \) and its algebraic closure.

The description of the Jacobian variety of a rigid analytic curve, i.e. a curve over \( \mathbb{Q}_p \), mainly decomposes into two parts, a combinatorial one and a so-called
formal one. Omitting the formal part, one can describe the Jacobian of Mumford curves, where the reduction has a certain simple form, as $\mathbb{G}^n_m, K / M$, where $M$ is a lattice of rank $g$, and thus showing a wonderful analogy to the complex case. Since the integral cannot be defined over the $p$-adic numbers in a meaningful way, one needs to replace the classic formulation of the Riemann relations with a more general, cohomological one.

To describe the Jacobian of a general rigid analytic curve, one needs to work with Raynaud extensions, heavily researched by Michel Raynaud, Siegfried Bosch and Werner Lütkemüller. Then one realizes that the Jacobian of a rigid analytic curve can be written as $E/M$, where $M$ is a lattice of rank $g$ and $E$ is an extension

$$0 \to \mathbb{G}^t_m, K \to E \to B \to 0$$

of a formal analytic abelian variety $B$ by the torus $\mathbb{G}^t_m, K$.

### 3. Outline of the chapters and the results of this work

In the first chapter, we will recapitulate the basic facts about rigid analytic varieties, their topology and their relationship with formal analytic schemes of locally topologically finite type. This chapter is by no means a complete introduction into the topic, we refer to [BGR84] and [Bos05] for this.

In the second chapter, we will provide some new insight into the stable reduction theorem, by refining the proofs of a few theorems of [BL85], using much shorter and less technical arguments than the original work. First, we will show that the ring $\mathcal{O}_X(X_+(\tilde{x}))$ of bounded functions on the formal fiber of a point $\tilde{x}$ of the reduction is local and henselian, by taking a close look at the normalization of $\tilde{x}$.

Secondly, we will then be able to give a much more natural proof for the Theorem 2.4.1 which describes the periphery of a formal fiber. That periphery always consists of a disjoint union of annuli, which we can equate to the structure of the normalization of the curve at the point $\tilde{x}$.

Finally, we will describe how this theorem is used to get to the stable reduction theorem.

In the third chapter, we will introduce group objects over an arbitrary category. This allows us to form a general theory describing algebraic, rigid analytic and formal analytic groups simultaneously. With this theory, we can generalize the results of [Ser88] and describe how an extension

$$0 \to \mathbb{G}^t_m, K \to E \to B \to 0$$

of an analytic or algebraic group $B$ by the torus $\mathbb{G}^t_m, K$ equals to a line bundle over $B$.

The work of the third chapter pays off in the fourth and final chapter, where we can give the explicit description of the lattice $M$ which $E$ gets divided by to form the Jacobian variety of our curve. While it was known that such a lattice exists and has full rank, we can even give a constructive formula for it. It will be shown that the formal analytic variety $B$ does not influence the absolute
value of the lattice and that one gains the explicit formula for the generators \( v_i = (v_{ij})_{j=1}^I \) of the lattice as

\[
v_{ij} = \sum_{e_n \in \gamma_i \cap \gamma_j} -d_n \cdot \log|q_n|
\]

where \( q_n \) is the height of the formal fiber of the double point corresponding to \( e_n \) and \( d_n = 1 \) if \( e_n \) has the same orientation in \( \gamma_i \) as in \( \gamma_j \) and \( d_n = -1 \) otherwise and where \( \gamma_i \) and \( \gamma_j \) are simple cycles on the curve, in close relationship to the complex case.

The lattice can be described fully by this method, but the description depends on the structure of the variety \( B \) of which little is known. We can construct \( B \) explicitly for the special family of curves \( X \), which have a reduction consisting of a rational curve and curve \( \tilde{Y} \) of genus \( g \), together with a surjective map \( \varphi: X \to Y \), with \( Y \) being a lift of \( \tilde{Y} \). Then it turns out that \( B \) is isogenic to \( \text{Jac} Y \) of the same degree as the map \( \varphi \).

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1. Some background on rigid geometry

1.1. Non-Archimedean analysis

We need the notion of a non-Archimedean normed ring.

Definition 1.1.1. A ring \(A\) together with a function \(|\cdot| : A \to \mathbb{R}_0^+\) is called a normed ring if

1. \(|x| = 0\) if and only if \(x = 0\),
2. \(|1| = 1\),
3. \(|xy| \leq |x| \cdot |y|\) for all \(x, y \in A\),
4. \(|x - y| \leq \max(|x|, |y|)\) for all \(x, y \in A\).

A normed ring is called valued if \(|xy| = |x| \cdot |y|\) for all \(x, y \in A\).

An \(A\)-module \(M\) together with a map \(\|\cdot\| : M \to \mathbb{R}_0^+\) over a valued ring is called a normed \(A\)-module if the following holds

1. \(\|x\| = 0\) if and only if \(x = 0\),
2. \(\|rx\| = |r| \cdot \|x\|\) for all \(r \in A, x \in M\),
3. \(\|x + y\| \leq \max(\|x\|, \|y\|)\) for all \(x, y \in M\).

Note that valued rings are always integral and normed modules are always torsion free. For any normed object \(T\) we denote by \(\bar{T}\) the set \(\{x \in T : |x| \leq 1\}\) and by \(\mathring{T}\) the set \(\{x \in T : |x| < 1\}\). Note that \(\bar{T}\) is again a normed ring/module and that \(\mathring{T}\) of a valued ring is a prime ideal in \(\mathring{T}\). This leads to the definition of \(\mathring{\bar{T}} = \mathring{T}/\mathring{T}\).

Definition 1.1.2. A direct sum of normed modules \(M = M_1 \oplus M_2\) is called orthonormal, denoted by \(M = M_1 \perp M_2\) if

\[\|(m_1, m_2)\| = \max(\|m_1\|, \|m_2\|)\]

If \(R\) is a valued ring, the quotient field \(Q(R)\) is also a valued ring by defining

\[\frac{a}{b} = \frac{|a|}{|b|} .\]

Proposition 1.1.3. Let \(A\) be a valued \(K\)-algebra over a valued field \(K\) with \(|A| = |K|\). Suppose \(M = M_1 \oplus M_2\) is a normed module over \(A\), with \(M_1\) and \(M_2\) being normed modules and we further have \(\|M\| = |A|\). The direct sum is orthonormal if and only if \(\mathring{M} = \mathring{M}_1 \oplus \mathring{M}_2\) over the ring \(\mathring{A}\).
1. Some background on rigid geometry

Proof. It is clear that we have $\tilde{M} = \tilde{M}_1 + \tilde{M}_2$ in any case, since the reduction is surjective. For elements $\tilde{m}_1 \in \tilde{M}_1, \tilde{m}_2 \in \tilde{M}_2$ the case $\tilde{m}_1 + \tilde{m}_2 = 0$ implies $\|m_1 + m_2\| < 1$ for any lift $m_1, m_2$, but $M = M_1 \perp M_2$. So this means we have $\max(\|m_1\|, \|m_2\|) < 1$, i.e. $\tilde{m}_1 = \tilde{m}_2 = 0$.

For the only if part of the proof look at an element $m = m_1 + m_2 \in M$ with $m_1 \in M_1$ and $m_2 \in M_2$. We can assume without restriction that $\|m_1\| \geq \|m_2\|$. Suppose at first that $\|m_1\| = 1$ and $\|m_2\| < 1$. This implies $\tilde{m}_1 \neq 0$. Thus we get $\tilde{m} \neq 0$ which means $\|m\| = 1$.

For the general case take any $a \in K$ with $|a| = \|m_1\|$. Then $\|m_1/a\| = 1$ and $\|m_2/a\| \leq 1$. So we have $\|m/a\| = 1$ which implies $\|m\| = |a|$, thus making the sum orthonormal. \(\square\)

1.2. Affinoid varieties

We have two basic approaches to formal analytic varieties which both have their merits and flaws. In the next few sections we will explain how to construct rigid analytic varieties and associate a reduction with them.

For this let $K$ be a valued field, i.e. a valued ring that is a field. The associated ring of elements with value less or equal then one, is denoted by $R = \hat{K}$ and the residue field $R/\hat{R}$ is denoted by $k$.

**Definition 1.2.1.** A power series $\sum_{k \in \mathbb{N}^n} a_k \zeta_1^{k_1} \cdots \zeta_n^{k_n}$ is called strictly convergent if $\lim_{|m|\to\infty} |a_m| = 0$. The ring of the strictly convergent power series in $n$ variables is called the Tate algebra $T_n := K(\zeta_1, \ldots, \zeta_n)$. For each $f \in T_n$ we define the Gauss norm as $|f| = \max |a_m|$.

**Definition 1.2.2.** The quotient $A := T_n/\alpha$ of $T_n$ by some ideal $\alpha$ with the reduction epimorphism $\alpha$ is called an affinoid $K$-algebra. An **affinoid variety** $\text{Sp} A$ is the pair $(\text{MaxSpec} A, A)$. On $A$ we have the residue norm

$$|\alpha(f)|_\alpha = \inf_{a \in \alpha} |f - a|$$

and the supremum norm

$$|f|_{\text{sup}} = \sup_{x \in \text{MaxSpec} A} |f(x)|,$$

where $f(x) = f \mod x$ for an maximal ideal $x \in \text{MaxSpec} A$ as in algebraic geometry.

It should be noted that there is always a finite field extension of $K$ such that there is an epimorphism $\alpha : T_n \to A$ with $|\cdot|_\alpha = |\cdot|_{\text{sup}}$.

**Definition 1.2.3.** Let $A$ be an affinoid $K$-algebra and $X := \text{Sp} A$ its affinoid variety. For $f_1, \ldots, f_r, g \in A$ without common zeroes the subset

$$X([f_1/g], \ldots, [f_r/g] \leq 1) := \{x \in X : |f_i(x)| \leq |g|, i = 1, \ldots, r\}$$

is called a rational subdomain of $X$. If $g = 1$, it is called a Weierstrass domain.

If $\varepsilon \in |K^\times|$ is a constant, we write $X(|f| \leq \varepsilon)$ or respectively $X(|g| \geq \varepsilon)$ for the corresponding rational subdomain.
Definition 1.2.4. For an affinoid variety \( X := \text{Sp} A \) a subset \( U \subset X \) is called an \textit{affinoid subdomain} of \( X \) if there is an affinoid variety \( Y := \text{Sp} B \) and a morphism \( \varphi : Y \to X \) with \( \varphi(Y) = U \) and for every morphism \( \varphi' : Y' \to X \) with \( \varphi'(X') \subset U \) there is a unique factorization \( \varphi' = \psi \circ \varphi \).

A rational subdomain is an affinoid subdomain and according to Gerritzen’s and Grauert’s theorem every affinoid subdomain is a finite union of rational subdomains.

Proposition 1.2.5. Let \( X = \text{Sp} A \) be an affinoid variety. The restriction map \( \mathcal{O}_X(X(|f| \geq \varepsilon)) \to \mathcal{O}_X(X(|f| = 1)) \) for an \( f \in \mathcal{O}_X(X) \) and any \( 0 < \varepsilon < 1 \) has a dense image.

Proof. We know that \( \mathcal{O}_X(X(|f| \geq \varepsilon)) = A[\varepsilon f^{-1}] \) and \( \mathcal{O}_X(X(|f| = 1)) = A[f^{-1}] \). By construction, the ring \( A[f, f^{-1}] \) is dense in the latter. But \( A[f, f^{-1}] \) is a subring of \( A[\varepsilon f^{-1}] \) as well, so the image of the restriction map is dense. \( \square \)

1.3. Admissible coverings and rigid analytic varieties

The topology induced by a Non-Archimedean absolute value has very bad properties. For example one can easily prove that such a topology is always completely disconnected. To counteract this behavior one needs to introduce the concept of a Grothendieck topology.

Definition 1.3.1. Let \( X \) be a set and \( \mathcal{G} \subset \mathcal{P}(X) \) a subset of the power set of \( X \). Let further be \( \{\text{Cov} U\}_{U \in \mathcal{G}} \) be a family of coverings, i.e. a family of families which satisfies \( U_i \subset U \) and \( \bigcup_{i \in I} U_i = U \) for every element \( \{U_i\}_{i \in I} \in \text{Cov}(U) \).

For a pair \( \mathcal{T} = (\mathcal{G}, \{\text{Cov} U\}_{U \in \mathcal{G}}) \) the elements of \( \mathcal{G} \) are called admissible open and the elements of \( \text{Cov}(U) \) are called admissible coverings. \( \mathcal{T} \) is called a \( \mathcal{G} \)-topology if it satisfies the following conditions.

1. \( U, V \in \mathcal{G} \Rightarrow U \cap V \in \mathcal{G} \).
2. \( U \in \mathcal{G} \Rightarrow \{U\} \in \text{Cov} U \).
3. If \( U \in \mathcal{G} \), \( \{U_i\}_{i \in I} \in \text{Cov} U \) and \( \{V_{ij}\}_{j \in J_i} \in \text{Cov} U_i \), then the covering \( \{V_{ij}\}_{i \in I, j \in J_i} \) is also admissible.
4. If \( U, V \in \mathcal{G} \) with \( U \subset V \) and \( \{V_i\}_{i \in I} \in \text{Cov} V \), then the covering \( \{V_i \cap U\}_{i \in I} \) of \( U \) is admissible.

The concept of a \( \mathcal{G} \)-topology generalizes the usual definition of topologies. Most topological concepts, like for example continuity, can directly be transferred to \( \mathcal{G} \)-topologies by replacing open sets by admissible open ones. To further add good properties one can define a unique finest slightly finer \( \mathcal{G} \)-topology of a given \( \mathcal{G} \)-topology as in [BGR84, pg. 338 et seqq.]. In addition to the stated axiom, the finest slightly finer topology will satisfy the following:

(G1) Any subset \( V \) of an admissible open set \( U \) for which an admissible covering \( \{U_i\}_{i \in I} \) of \( U \) with \( V \cap U_i \) admissible open for every \( i \in I \) exists will be admissible open.
1. Some background on rigid geometry

(G2) A covering of an admissible open set consisting of admissible open sets which has a refinement that is admissible is already admissible itself.

As a finer G-topology, the finest slightly finer G-topology will contain $X$ and $\emptyset$ as admissible open sets if the original G-topology already did.

**Definition 1.3.2.** An affinoid variety $X := \text{Sp} A$ carries the weak $G$-topology $\mathfrak{T}$ which is defined by declaring the affinoid subdomains admissible open and defining admissible coverings as finite unions of affinoid subdomains.

The finest slightly finer G-topology of $\mathfrak{T}$ is called the strong $G$-topology on $X$. Both topologies contain $\emptyset$ and $X$ as admissible open sets.

We can now define locally $G$-ringed spaces over a ring $R$ as a set $X$ with a $G$-topology and a sheaf $\mathcal{O}_X$ of algebras over $R$ in the usual way. Note that an affinoid variety with the strong topology is a locally $G$-ringed space over $K$.

**Definition 1.3.3.** A rigid analytic variety over $K$ is a locally $G$-ringed space $(X, \mathcal{O}_X)$ over $K$ if it satisfies the following conditions.

1. The $G$-topology on $X$ contains $X$ and $\emptyset$ as admissible open sets and has the properties [(G1)] and [(G2)].
2. There is an admissible covering $\{U_i\}_{i \in I}$ of $X$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affinoid variety for all $i \in I$.

1.4. The reduction of a rigid analytic variety

We now want to generalize the concept of the reduction of a normed ring for rigid analytic varieties.

**Definition 1.4.1.** Let $X = \text{Sp} A$ be an affinoid variety. The affine scheme $\tilde{X} = \text{Spec} \tilde{A}$, with $\tilde{A} = A/\{f \in A : |f| < 1\}$ is called the reduction of $X$. The map $\pi : X \to \tilde{X}$ obtained by reducing maximal ideals is called the reduction map.

An admissible open set $U$ in $X$ is called formal open if it is the preimage of a Zariski open set under $\pi$.

**Proposition 1.4.2.** Let $X = \text{Sp} A$ be an affinoid variety with irreducible reduction $\tilde{X} = \text{Spec} \tilde{A}$ and $|A| = |K|$. Then the supremum norm $|\cdot|$ is multiplicative on $A$.

*Proof.* Assume that there are elements $f, g \in A$ such that $|f \cdot g| \neq |f| \cdot |g|$. We can set without restriction $|f| = |g| = 1$, since the absolute value is always multiplicative for constants. So $f, g \in \tilde{A}$ and have non zero reductions $\tilde{f}$ and $\tilde{g}$. But $\tilde{A}$ is irreducible, so $\tilde{f} \cdot \tilde{g} \neq 0$ which means that $|f \cdot g| = 1$ in contradiction with the hypothesis. So $|\cdot|$ is multiplicative.

**Proposition 1.4.3.** Let $X = \text{Sp} A$ be an affinoid variety with reduction $\tilde{X} = \text{Spec} \tilde{A}$. Let $\tilde{Y} = \text{Spec} B$ be an algebraic variety such that $\tilde{\varphi} : \tilde{Y} \to \tilde{X}$ is a smooth morphism and has finite presentation. There is an affinoid variety $Y$ together with a morphism $\varphi : Y \to X$ reducing to $\tilde{\varphi}$.
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Proof. By our assumption, we have \( \tilde{B} = \tilde{A}[Z_1, \ldots, Z_n]/(\tilde{g}_1, \ldots, \tilde{g}_r) \). We take any lift of \( g_1, \ldots, g_r \) in \( A(\zeta_1, \ldots, \zeta_n) \) and set \( Y = \text{Sp} A(\zeta_1, \ldots, \zeta_n)/(g_1, \ldots, g_r) \).

Definition 1.4.4. An admissible open affinoid covering \( \{U_i\}_{i \in I} \) of a rigid analytic variety \( X \) is called a formal covering if \( U_i \cap U_j \) is formal open in \( U_i \) for every \( i, j \in I \).

If we are given a formal covering \( \{U_i\}_{i \in I} \) of a rigid analytic variety \( X \) we can define the reduction \( \tilde{X} \) of \( X \) by reducing the \( U_i \) and gluing along the reduction of \( U_i \cap U_j \). This reduction is not unique and depends on the used formal covering. In chapter 4 we will construct coverings such that the reduction has very simple singularities.

Proposition 1.4.5. Let \( X \) be a rigid analytic curve and \( \mathcal{U} \) a formal covering leading to the reduction \( \pi: X \to \tilde{X} \). Let \( \mathcal{U}' = (\tilde{U}'_i)_{i \in I} \) be an affine covering of \( \tilde{X} \). Then \( \mathcal{U}' = (\pi^{-1}(\tilde{U}'_i))_{i \in I} \) is again a formal covering and the associated reduction is \( \tilde{X} \).

Proof. First, we need to see that \( \mathcal{U}' \) is admissible. Take an admissible open set \( U_i \in \mathcal{U} \). Then \( \tilde{U}_i \cap \tilde{U}'_j \) is affine for every \( U'_j \in \mathcal{U}' \). Therefore there is a finite set of elements \( f_i' \in O_X(U_i) \) such that \( U_i \cap U'_j = U_i(|f_i'| \geq 1) \) so \( U_i \cap U'_j \) is a rational subdomain of \( U_i \) and as such admissible. But then \( U'_j \) is admissible by (G1). Using this construction we also see that \( U'_j \) is affinoid with reduction \( \tilde{U}'_j \). The covering \( \mathcal{U}' \) is also formal since \( U'_i \cap U'_j \) is formal by the very definition of \( \mathcal{U}' \). Since the reduction of \( U_i \cap \mathcal{U}' \) glues together to form \( \tilde{U}_i \) we get that the reduction associated to \( \mathcal{U}' \) is again \( \tilde{X} \).

1.5. Adic topology and complete rings

We now come to the second approach to formal analytic varieties.

Definition 1.5.1. Let \( R \) be a ring and \( \mathfrak{a} \subset R \) an ideal. The topology for which \( \mathfrak{a}^k \) is a basis of open neighborhoods of 0 is called the \( \mathfrak{a} \)-adic topology of \( R \).

For a module \( M \) over \( R \) we define the \( \mathfrak{a} \)-adic topology of \( M \) to be the topology with \( \mathfrak{a}^k M \) as basis for 0.

The ideal \( \mathfrak{a} \) is called the ideal of definition.

The adic topology on a ring is very similar to a norm as the following proposition illustrates.

Proposition 1.5.2. An \( \mathfrak{a} \)-adic noetherian topological ring is normed with \( \tilde{R} = \mathfrak{a} \).

Proof. Assume that \( R \) is \( \mathfrak{a} \)-adic. For \( x \in R \) we define

\[
|x| = \begin{cases} 
0 & \text{if } x = 0 \\
\exp(-n) & \text{where } n \text{ is the smallest integer with } x \notin \mathfrak{a}^{n+1} 
\end{cases}
\]
1. Some background on rigid geometry

According to Krull’s intersection theorem \[^{\text{Bos05} 2.1.2}\], we know that \(\cap a^n = \{0\}\). So this norm is well-defined and \(|x| = 0\) implies that \(x = 0\).

If \(x \in a^n\) and \(y \in a^m\) we clearly have \(xy \in a^{n+m}\) and \(x + y \in a^{\min(n,m)}\) so \(|xy| \leq |x| \cdot |y|\) and \(|x + y| \leq \max(|x|,|y|)\). Furthermore we get \(a = \bar{R}\) and for \(n\) and \(\varepsilon\) with \(\exp(-n) < \varepsilon \leq \exp(-n + 1)\) we have \(|r| < \varepsilon\) if and only if \(r \in a^n\).

So both induce the same topology. \(\square\)

The converse of this proposition is not true, as one can easily see at the example of a normed ring where \(\bar{R}\) is generated by elements of different norm.

We can, however, show that valued valuation rings are always adic, as we will do in the next few propositions.

**Proposition 1.5.3.** Let \(R\) be a ring with \(a\)-adic topology. Then

\[
\sum_{k=1}^{\infty} x_k
\]

is a Cauchy sequence if and only if \((x_k)\) is a sequence tending to zero.

**Proof.** The series \(\sum_{k=1}^{\infty} x_k\) is Cauchy if for every \(n \in \mathbb{N}\) we have \(N \in \mathbb{N}\) such that for every \(i, j > N\) the sum \(\sum_{k=i}^{j} x_k\) is in \(a^n\). The case \(i = j\) implies that \((x_k)\) is a zero sequence as always.

If \((x_k)\) tends to zero we find an \(N\) for every \(n\) such that \(x_k \in a^n\) for \(k > N\). But then \(\sum_{k=i}^{j} x_k\) is in \(a^n\) for every \(i, j > N\) as a finite sum of elements in \(a^n\). \(\square\)

**Definition 1.5.4.** An integral ring \(R\) in which for every element \(x \in Q(R)\) either \(x \in R\) or \(x^{-1} \in R\) holds is called a valuation ring.

**Proposition 1.5.5.** Let \(R\) be an integral ring. Then the following two statements are equivalent:

(i) \(R\) is a valued ring with \(R = \bar{Q}(R) = \{a/b : a, b \in Q(R) ; |a| \leq |b|\}\).

(ii) \(R\) is a valuation ring with Krull dimension 1.

**Proof.** Suppose \(R\) is a valued ring with \(R = \bar{Q}(R)\). We can extend the absolute value of \(R\) to an absolute value of \(Q(R)\) by defining

\[
\left|\frac{a}{b}\right| := \frac{|a|}{|b|}.
\]

Then if \(x \in Q(R)\) we either have \(|x| \leq 1\) which implies \(x \in R\) by our assumption or \(|x^{-1}| = |x|^{-1} < 1\) which induces \(x^{-1} \in R\).

If \(p\) is a prime ideal not equal to zero we have \(x \in p\) with \(x \neq 0\). For any \(y \in R\) we can find an \(n \in \mathbb{N}\) such that \(|y|^n < |x|\). Then \(y^n/x \in R\) and we get \(y^n \in p\) which implies \(y \in p\) since \(p\) is prime. Since \(R^\times = \{x \in R : |x| = 1\}\) by the assumption that \(R = \bar{Q}(R)\) this implies that \(p = \bar{R}\) and \(R\) has Krull dimension 1.

A valuation ring is always local according to [\(^{\text{Bos05} 2.1.6}\)]. Let \(t \in R \setminus \{0\}\) be an arbitrary non-unit. The set \(b = \{x \in R : t^{-n}x \in R\text{ for all }n \in \mathbb{Z}\}\) is
1.5. Adic topology and complete rings

a prime ideal in $R$, since $xy \in b, x \not\in b$ implies that there is an $m \in \mathbb{Z}$ with $t^{-m}x \not\in R$ and therefore $t^n x^{-1} \in R$ which implies $t^{-n}y = t^{-n-m}t^n x^{-1}xy \in R$ for all $n \in \mathbb{Z}$. Since $t \not\in b$ we get $b = 0$ by our assumption of Krull dimension 1. This means that

$$k_{m,x} = \max(n \in \mathbb{Z} : t^{-n}x^m \in R)$$

is well-defined for all $x \in R \setminus \{0\}$ and $m \geq 1$. Furthermore we get $k_{m,x} \geq 0$.

One can see directly that the limit of $k_{m,x}/m$ exists and for every $n \in \mathbb{N}$ the inequality

$$\frac{k_{n,x}}{n} \leq \lim_{m \to \infty} \frac{k_{m,x}}{m} \leq \frac{k_{n,x} + 1}{n}$$

holds. We set

$$|x| = \lim_{m \to \infty} \exp\left(-\frac{k_{m,x}}{m}\right).$$

For $x \in R^\times$ we can calculate $k_{m,x} = 0$ for all $m \in \mathbb{N}$ so $|x| = 1$. If $x$ is not a unit, then by the same argument as before there is an $m \in \mathbb{N}$ such that $t^{-x}x^m \in R$ and therefore $|x| \neq 1$. We can define $k_{m,x}$ for every $x \in Q(R)$ and one easily realizes that $|xy^{-1}| = |x| \cdot |y^{-1}|$. So $|\cdot| : Q(R) \times R^\times$ is a group morphism with kernel $R^\times$ and $x \in R$ if and only if $|x| \leq 1$.

Any valuation ring defines a valuation with the axioms of Definition [1.1.1] if one replaces $\mathbb{R}^+$ with $Q(R) \times R^\times$ imbued with the canonical ordering $\alpha \leq \beta \iff \alpha \beta^{-1} \in R$ for $\alpha, \beta \in Q(R) \times R^\times$. So $|\cdot| : Q(R) \times R^\times$ makes $R$ a valued ring with $R = Q(R)$. \qed

Proposition 1.5.6. Let $R$ be a valued ring with $R = \hat{Q}(R)$. Then the topology of $R$ is $a$-adic with $a = (t)$ for any $t \in R$ with $|t| < 1$.

Proof. The sets $B_\varepsilon := \{x \in R : |x| < \varepsilon\}$ form a basis of the open neighborhoods of 0 by the definition of the topology of a metric space. We need to show that $B_\varepsilon$ is open in the $a$-adic topology. Since $|t| < 1$ we find $n \in \mathbb{N}$ with $|t|^n < \varepsilon$. Then for every $x \in B_\varepsilon$ and every $r \in R$ we get $|x + rt^n| < \varepsilon$ and $x + a^n$ is an $a$-adic neighborhood of $x$ contained in $B_\varepsilon$.

On the other hand, let $x = rt^n \in a^n$ be any element. Choose an $\varepsilon < |t|^n$. Then $|y - x| < \varepsilon$ implies $|y/t^n - r| < 1$ and therefore $|y/t^n| \leq 1$. This means that there is an $r' \in R$ such that $y = r't^n \in a^n$ and $a^n$ is open. \qed

Remark. It should be noted that even so a valued ring $R$ with $R = \hat{Q}(R)$ is local, the ideal generating its adic topology is not the maximal ideal of $R$. Indeed if $|Q(R) \times|$ is dense in $\mathbb{R}^+$ we have $\hat{R} = \hat{R}$ for any $n \in \mathbb{N}$.

We can define a topology on every valuation ring that is not a field by using all non-zero ideals as base for the topology. This topology is Hausdorff since 0 and $t$ are separated by any ideal generated by an element smaller than $t$ in $Q(R) \times / R^\times$. Such an element exists as $t^2$ is smaller than $t$ if $t$ is not a unit and any non-unit is smaller than any unit $t$. If there is an ideal $a$ in $R$ so that this topology of $R$ is $a$-adic we call $R$ adic.
1. Some background on rigid geometry

**Proposition 1.5.7.** Let $R$ be an adic valuation ring with a finitely generated ideal of definition. Then the ideal of definition is principal and there is a non-trivial minimal prime ideal $p$ and the $t$-adic topology on $R$ coincides with the $a$-adic one for every $t \in p$.

**Proof.** We have already given the proof for the case most interesting to us, namely when $R$ is a valued ring. For arbitrary valuation ring the proof can be found in [Bos05, 2.1.7].

**Proposition 1.5.8.** Let $R$ be a Hausdorff adic ring with ideal of definition $a$. The topological ring $R$ is complete if and only if

$$R = \lim_{\to} R/a^n$$

holds where $R/a^n$ has the discrete topology.

**Proof.** Recall that we have

$$\lim_{\to} R/a^n = \{(x_n) \in \prod_{n=1}^{\infty} R/a^n : x_n \equiv x_m \mod a^n \text{ for } m \leq n\}.$$

By the definition of the inverse limit there is always a map $\varphi: R \to \lim_{\to} R/a^n$. For noetherian rings this map is injective because of Krull’s intersection theorem, otherwise this is just the assumption that $R$ is Hausdorff, so $\varphi$ is injective anyway.

If $R$ is complete and $x \in \lim_{\to} R/a^n$ is represented by a sequence $(x^n) \in \prod_{n=1}^{\infty} R/a^n$ then any sequence $(x_n) \subseteq R$ with $x_n = x^n \mod a^n$ is a Cauchy sequence in $R$ and its limit is $x$.

If $(x_k)$ is a Cauchy sequence in $\lim_{\to} R/a^n$ then for every $n$ there is an $N$ such that for every $k, l \geq N$ we have $x_k - x_l \equiv 0 \mod a^n$. We set $x^n = x_N + a^n \in R/a^n$. Then $(x^n)$ represents an element $x$ in $\lim_{\to} R/a^n$ and $(x_k) \to x$, so $\lim_{\to} R/a^n$ is complete.

**Proposition 1.5.9.** Let $R$ be an adic valuation ring with ideal of definition $a$. The ring $\hat{R} = \lim_{\to} R/a^n$ is adic if $a$ is finitely generated.

**Proof.** As said before there is an injective map $\varphi: R \to \hat{R}$. We want to show that $\hat{R}$ is adic with ideal of definition $a\hat{R}$. The topology on $\hat{R}$ is the coarsest topology on $\lim_{\to} R/a^n$ such that every projection $p_n: \hat{R} \to R/a^n$ is continuous.

This means that $(\ker p_n)_{n \in \mathbb{N}}$ is a base of neighborhoods for $0$.

Let $x \in \ker p_n$ be represented by $(x^n) \in \prod_{n=1}^{\infty} R/a^n$. Then $x^n = 0 \mod a^n$ for every $n \geq m$. If we take any lift $x_n \in R$ of $x^n$ we have $x_n \in a^n$ and $\varphi(x_n)$ converges to $x$. We have seen in Proposition [1.5.7] that we can assume $a$ to be principal, i.e. $a = (t)$. This means we can write $x_n = r_n t^m$ for some $r_n \in R$. But $x_k \equiv x_n \mod a^k$ for every $n > k$ so $r_n \equiv r_k \mod a^{k-m}$ for every $n > k \geq m$. This means $(r_k)_{k=m}^{\infty}$ represents some $r \in \hat{R}$ and $x = r \varphi(t^m)$. So $\varphi(a^n)\hat{R}$ forms a basis of the neighborhoods of $0$ and $\hat{R}$ is adic with ideal of definition $a\hat{R}$.

**Definition 1.5.10.** We call $\hat{R}$ the completion of $R$. 

8
1.6. Formal schemes

With these definitions in mind, we can now explain formal schemes.

**Definition 1.6.1.** Let $A$ be a complete Hausdorff adic ring with principal ideal of definition $a$. For a free variable $\zeta$ we call $A(\zeta) = \varprojlim A/a^n[\zeta]$ the ring of convergent power series in $\zeta$ over $A$.

**Proposition 1.6.2.** Let $A$ be a complete adic ring with principal ideal of definition $a$. The elements of the ring $A(\zeta)$ can uniquely be written as

$$\sum_{k=0}^{\infty} a_k \zeta^k$$

with a zero sequence $(a_k) \subset A$.

**Proof.** The ring $A[\zeta]$ is adic with ideal of definition $aA[\zeta]$ by the definition of the product topology. Therefore $A(\zeta)$ is the completion of $A[\zeta]$. If $a_k$ is a zero sequence then so is $a_k \zeta^k$ and therefore $\sum_{k=0}^{\infty} a_k \zeta^k$ is in $A(\zeta)$.

If $f \in A(\zeta)$ is any element, then there is a Cauchy sequence in $A[\zeta]$ converging to $f$. This sequence gives a Cauchy sequence in $A$ for every coefficient. But $A$ is complete so we can assume without restriction that the sequence for the $k$-th coefficient is constant for $n > N_k$. Therefore we can write $f = \sum_{k=0}^{\infty} a_k \zeta^k$. \qed

This proposition shows that the definition of $K(\zeta)$ given in Section 1.2 is equal to this one.

**Definition 1.6.3.** Let $A$ be a complete and Hausdorff adic ring with principal ideal of definition $a$. For $f \in A$ we call $A_f = \varprojlim A/a[f^{-1}]$ the complete localization of $A$ by $f$.

**Proposition 1.6.4.** Let $A$ be a complete and Hausdorff adic ring with principal ideal of definition $a$. Then $A_f$ is the adic completion of $A[f^{-1}]$ with respect to the ideal $aA[f^{-1}]$. There is a canonical isomorphism

$$R(\zeta)/(1 - f\zeta) \to R(f^{-1})$$

**Proof.** See [Bos05, Remark 2.1.8 and 2.1.9]. \qed

**Definition 1.6.5.** Let $A$ be a complete and Hausdorff adic ring with principal ideal of definition $a$. The affine formal scheme $\text{Spf} A$ is the space of all open prime ideals in $A$ with $\text{Spf} A(f^{-1})$ as base of open sets together with the structure sheaf $A$ where $A(\text{Spf} A(f^{-1})) = A(f^{-1})$ is obtained by complete localization.

**Remark.** A prime ideal $\mathfrak{p}$ in $A$ is open if and only if $a^n \subset \mathfrak{p}$ for some $n \in \mathbb{N}$. This implies that $\mathfrak{p}$ is open exactly if $a \subset \mathfrak{p}$ and we get a one-to-one correspondence of open prime ideals in $A$ and prime ideals in $A/a$.

One can show (see [Bos05, 2.2]) that $\text{Spf}(A)$ is indeed a locally topologically ringed space, i.e. that $A$ is a structure sheaf.
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**Definition 1.6.6.** A formal scheme is a locally topologically ringed space \((X, \mathcal{O}_X)\) such that each point \(x \in X\) has an open neighborhood \(U\) such that \((U, \mathcal{O}_X(U))\) is an affine formal scheme.

As per usual, one can construct a formal scheme by gluing affine formal schemes. With the completed tensor product, one can define a fiber product in the category of formal schemes. See, as usual [Bos05, Chapter 2] and [BGR84, Part C] for further details.

An important case of formal schemes arise as the completion of algebraic schemes. If \(X\) is any separated scheme and \(Y\) is a closed subscheme with ideal of definition \(J\) in \(\mathcal{O}_X\), then the completion of \(\mathcal{O}_X\) alongside \(J\) together with the topological space \(Y\) gives a formal scheme.

**Definition 1.6.7.** Let \(R\) be an adic valuation ring with principal ideal of definition \(a\). A topological \(R\)-algebra \(A\) is called of topologically finite type if \(A\) is isomorphic to \(R\langle \zeta_1, \ldots, \zeta_n \rangle / b\) endowed with the topology induced by \(a\).

\(A\) is called admissible, if furthermore \(b\) is finitely generated and \(a^n f = 0\) for any \(n \in \mathbb{N}\) implies \(f = 0\).

A formal scheme \(X\) is called admissible if there is an affine covering \((U_i)\) of \(X\) such that \(U_i = \text{Spf} \ A_i\) and \(A_i\) is admissible.

If \(X\) is an integral, projective, flat \(R\) scheme, where \(R\) is a valuation ring, then the completion of \(X\) along its special fiber gives an admissible formal \(R\) scheme \(\mathfrak{X}\). The scheme \(\mathfrak{X}\) is then called the analytification of \(X\).

We want to study how admissible formal schemes are connected with rigid analytic varieties with a fixed formal covering. For this, we introduce two covariant functors.

**Definition 1.6.8.** There is a functor

\[
\text{rig}: (\text{admissible formal schemes over } R) \to (\text{rigid analytic varieties over } K)
\]

defined by \(\text{rig}(\text{Spf} \ A) = \text{Sp} \ A \otimes_R K\) on the affine formal schemes, associating a formal scheme with its generic fiber, where \(K = Q(R)\).

Furthermore we have the reduction functor

\[
\text{red}: (\text{admissible formal schemes over } R) \to (\text{algebraic varieties over } k)
\]

defined via \(\text{red}(\text{Spf} \ A) = \text{Spec} \ A / a\) where \(a\) is the ideal of definition of \(R\) and \(k\) is the residue field of \(R\).

To see that these functors are well-defined we again refer to [Bos05, 2.4.]. It is easy to see that \(\text{red}(X)\) is indeed a reduction of \(\text{rig}(X)\) as this is clearly the case if \(X\) is affine and is in general is obtained by using an admissible covering of \(X\). If \(\mathfrak{X}\) is an analytification of some integral, projective, flat \(R\) scheme \(X\), then \(\text{rig}(\mathfrak{X})\) is the analytification of the generic fiber of \(X\) as we will define it in Section 1.7.

On the other hand, if we have an affinoid variety \(X_K = \text{Sp} K \langle \zeta_1, \ldots, \zeta_n \rangle / b_K\), where \(b \subset R \langle \zeta_1, \ldots, \zeta_n \rangle\) is a finitely generated ideal and \(b_K = b \otimes_R K\), then \(X = \text{Spf} R \langle \zeta_1, \ldots, \zeta_n \rangle / b\) is a formal scheme which is automatically admissible.
1.7. Analytification of an algebraic variety

by Noether normalization. The functor rig($X$) obviously yields $X_K$ and red($X$) gives the canonical reduction $\tilde{X}$ of $X$. If $X$ is a rigid analytic variety with formal covering $(U_i)_{i \in I}$, then the covering gives gluing relations for a formal scheme, using the fact that $\tilde{U}_i \cap \tilde{U}_j$ is an open subset of $\tilde{U}_i$.

These relations allow us to use admissible formal schemes and rigid analytic varieties with fixed reduction interchangeably.

1.7. Analytification of an algebraic variety

One of the most important aspects of rigid analytic varieties is the possibility to view any algebraic variety $X$ over a non-Archimedean field $K$ as a rigid analytic variety over $K$.

We only sketch the process for affine and projective varieties. See [BGR84, 9.3.4] for a more detailed discussion.

In order to construct a rigid analytic variety $X_{\text{an}}$ out of the affine algebraic variety $X = \text{Spec} K[\xi_1, \ldots, \xi_n]/a$ we take any $c \in K$ with $|c| > 1$ and set

$$T_{n,k} = K\langle c^{-k}\xi_1, \ldots, c^{-k}\xi_n \rangle.$$ 

The ring $K[\xi_1, \ldots, \xi_n]$ is part of all the $T_{n,k}$, so we get a sequence of $K$-algebra morphisms

$$T_{n,0}/aT_{n,0} \leftarrow T_{n,1}/aT_{n,1} \leftarrow \ldots,$$

which gives rise to a sequence of open immersions

$$\text{Sp} T_{n,0}/aT_{n,0} \to \text{Sp} T_{n,1}/aT_{n,1} \to \ldots.$$ 

By pasting along these maps we can construct an analytic variety $X_{\text{an}}$ with Sp $T_{n,k}/aT_{n,k}$ as admissible affinoid covering. However this covering is never formal, since Sp $T_{n,k}/aT_{n,k}$ yields merely a finite set of points in the reduction of Sp $T_{n,k+1}/aT_{n,k+1}$.

In this work the most prominent example of an analytic variety constructed this way is the variety $\mathbb{G}_{m,K}$.

**Proposition 1.7.1.** The variety $\mathbb{G}_{m,K}$ which is the analytification of the affine variety $\text{Spec} K[\zeta, \zeta^{-1}]$ is a rigid analytic group in the sense of Chapter [3]. The affinoid variety $\mathbb{G}_{m,K} = \text{Sp} K\langle \zeta, \zeta^{-1} \rangle$ is an open subgroup variety of $\mathbb{G}_{m,K}$, consisting of the elements with absolute value 1 in $K$. Its reduction is $\mathbb{G}_{m,K} = \text{Spec} k[\mathbb{Z}, \mathbb{Z}^{-1}]$ with a group structure induced by $\mathbb{G}_{m,K}$.

**Proof.** Since $\text{Spec} K[\zeta, \zeta^{-1}]$ is the algebraic description of $\mathbb{G}_{m,K}$ and morphisms carry over in the analytification process we see that $\mathbb{G}_{m,K}$ is the analytic group variety with the group structure of $K^\times$. The variety $\mathbb{G}_{m,K}$ is the first variety of the defining sequence of $\mathbb{G}_{m,K}$ and as such $\mathbb{G}_{m,K}$ is an open subgroup of $\mathbb{G}_{m,K}$.

If $X$ is projective, say $X \subset \mathbb{P}^n$, then there are $(n+1)$ affine

$$U_i = \text{Spec} K[\zeta_0/\zeta_i, \ldots, 1, \ldots, \zeta_n/\zeta_i]$$
1. Some background on rigid geometry

in \( \mathbb{P}^n \) for which we can find analytic counterparts. The gluing of the \( U_i \) is done by equating \( \zeta_i/\zeta_i \) with \( \zeta_k/\zeta_j \cdot \zeta_j/\zeta_i \) where \( \zeta_i/\zeta_j \) and \( \zeta_j/\zeta_i \) are not null. But these gluing relations are already defined for the affinoid varieties

\[
\text{Sp } K(\zeta_0/\zeta_i, \ldots, 1, \ldots, \zeta_n/\zeta_i)
\]

where ever \(|\zeta_i/\zeta_j| = 1\), so the analytification of \( \mathbb{P}^n \) and the subset \( X \) can be obtained by gluing \( n + 1 \) copies of \( \mathbb{B}^n_K \) at their borders.

Since \( \mathbb{B}^n_K \) is affinoid with reduction \( A^n_{K} \) and we glued at

\[
\text{Sp } K(\zeta_0/\zeta_i, \ldots, 1, \ldots, \zeta_0/\zeta_i, \zeta_i/\zeta_j)
\]

which reduces to the correct gluing for \( \mathbb{P}^n_K \), this also gives a formal covering of \( X \).

1.8. Proper morphisms

We want to give the definition of a proper analytic variety and show that the analytification of a projective algebraic variety is proper. To be able to do so, we need to formulate the concepts of separate morphisms and relatively compact subsets.

**Definition 1.8.1.** A morphism \( \varphi: X \to Y \) of analytic varieties is called **separated**, if the diagonal morphism \( \Delta: X \to X \times_Y X \) is a closed immersion.

One easily realizes as in [BGR84, 9.6] that morphisms of affinoid varieties and therefore affinoid varieties over \( \text{Sp } K \) are always separated. Since the definition coincides with the algebraic one, it is easy to see that analytifications of separated algebraic morphisms are again separated.

**Definition 1.8.2.** Let \( X = \text{Sp } A \) and \( Y = \text{Sp } B \) be affinoid varieties with a morphism \( \varphi: X \to Y \). An affinoid subset \( U \subset X \) is said to be **relatively compact** in \( X \) over \( Y \) if there exists an affinoid generating system \( f_1, \ldots, f_r \) of \( A \) over \( B \) such that \( U \subset X(|f_1| < 1, \ldots, |f_r| < 1) \).

The relative compactness of a subset \( U \subset X \) is equivalent of assuming that there is an \( \varepsilon \in \sqrt{|K^X|} \) with \( \varepsilon < 1 \) such that \( U \subset X(\varepsilon^{-1}f_1, \ldots, \varepsilon^{-1}f_r) \).

Now we can define proper morphisms.

**Definition 1.8.3.** A morphism \( \varphi: X \to Y \) of analytic varieties is called **proper** if \( \varphi \) is separated and if there is an admissible affinoid covering \( (Y_i)_{i \in I} \) of \( Y \) such that, for every \( i \in I \) there are two finite admissible affinoid coverings \( X_{ij} \) and \( X'_{ij} \) of \( \varphi^{-1}(Y_i) \) such that \( X_{ij} \) is relatively compact in \( X'_{ij} \) over \( Y_i \) for all indices \( i \) and \( j \).

**Proposition 1.8.4.** The analytification of a projective variety is proper.

**Proof.** The closed ball \( \mathbb{B}^n_K \) is relatively compact in every ball of greater radius. Since a projective variety allows finite admissible coverings coming from \( \mathbb{B}^n_K \) with any radius greater or equal to one we get the necessary covering for properness. \( \square \)
1.9. Étale morphisms

Definition 1.9.1. Let \( \varphi: X \to Y \) be a morphism, \( x \in X \) a point with an admissible open neighborhood \( U \) and an immersion \( \iota: U \to B^n \). The morphism \( \varphi \) is called smooth of relative dimension \( r \) at \( x \) if there are \( n - r \) sections \( g_1, \ldots, g_{n-r} \) locally at \( y = \varphi(x) \) generating the ideal defining \( U \) as a subscheme of \( B^n \) with \( dg_1, \ldots, dg_{n-r} \) being linearly independent in \( \Omega^1_{B^n/Y} \).

Furthermore \( \varphi \) is called formal smooth of relative dimension \( r \) if the differential forms \( dg_1, \ldots, dg_{n-r} \) form an orthonormal system in \( \Omega^1_{B^n/Y} \).

Definition 1.9.2. A morphism is called étale or formal étale if it is smooth or formal smooth of relative dimension 0 respectively.

Definition 1.9.3. Let \( R \) be a local ring with residue field \( k \). The ring \( R \) is called henselian if any monic polynomial \( p \in R[T] \) which admits a factorization \( \tilde{p} = \tilde{f} \cdot \tilde{g} \in k[T] \) with coprime factors \( \tilde{f} \) and \( \tilde{g} \) in the reduction has lifts of \( f \) and \( g \) in \( R[T] \) with \( p = f \cdot g \).

The smallest local ring extension \( R^h \) of \( R \) such that \( R^h \) is henselian is called the henselization of \( R \).

We mainly need the characterization of the henselization by étale morphisms. This is done by the next proposition.

Proposition 1.9.4. Let \( R \) be a local ring. The henselization \( R^h \) of \( R \) is the direct limit \( \varinjlim_i R_i \) of all isomorphism classes of \( R \)-algebras \( R_i \) which occur as local rings of some étale \( R \)-scheme at the closed point lying over the closed point of \( R \).

See [BLR90, 2.3]

The last proposition allows us to assume that the local ring \( O_{X,x} \) is henselian if one allows for étale base change.

Proposition 1.9.5. Let \( X \) be an algebraic curve over an algebraically closed field \( k \) and let \( x \in X \) a closed point. There is an étale morphism \( \hat{X} \to X \) such that \( x \) lies on \( n \) different irreducible components of \( \hat{X} \), where \( n \) is the number of points in the fiber of \( x \) in the normalization.

Proof. If \( x \) lies on \( n \) different irreducible components, then the normalization of \( X \) has \( n \) disjoint components with points lying over \( x \). Since according to [Ray70] p. 99] henselization commutes with normalization we can assume that \( x \) lies on exactly one irreducible component and show that there is only one point in the normalization of \( \hat{X} \) over \( x \).

Since we look at \( X \) up to étale morphism we can assume the local ring \( O_{X,x} \) to be henselian. Let \( g_1, \ldots, g_n \) be the points lying over \( x \) in its normalization \( X' \). There are \( g_i \in O_{X',y_i} \) such that \( p = (T - g_1) \cdots (T - g_n) \in O_{X,x}[T] \) is a monic polynomial and that \( g_i(y_j) = 1 - \delta_{ij} \). Then \( p(x) = T \cdot h \) where \( h \in k[T] \) is not divisible by \( T \) since we can calculate \( p(x) \) as \( p(y_i) \) for any \( i \) between 1 and \( n \). But \( O_{X,x} \) is henselian and any non trivial factorization of \( p \) is contrary to our assumptions, which implies \( n = 1 \).
1. Some background on rigid geometry

The lift of a morphism \( \tilde{\varphi} \) as constructed in Proposition 1.4.3 is étale if \( \tilde{\varphi} \) is étale.

**Proposition 1.9.6.** Let \( X \) and \( Y \) be affinoid varieties with associated reductions \( \pi_X : X \to \tilde{X} \) and \( \pi_Y : Y \to \tilde{Y} \). Let \( Y \) be formal smooth over \( K \). For every morphism \( \tilde{\varphi} : \tilde{X} \to \tilde{Y} \) there is a lift \( \varphi : X \to Y \) of \( \tilde{\varphi} \).

**Proof.** We set \( X = \text{Spf} \, A \) and \( Y = \text{Spf} \, B \), \( A = \lim A \otimes R_i \), \( B = \lim B \otimes R_i \), \( R = \lim R_i \) with \( R_i = R/a_i \) as before. By [BLR90 Prop. 2.2.6] we get \( \text{Hom}_R(X, Y) \to \text{Hom}_R(X_{i-1}, Y) \) since \( Y \to \text{Spf} \, R \) is smooth. Therefore the map \( \tilde{\varphi} \), which gives a map \( \varphi_i : X_i \to Y \) successively lifts to maps \( \varphi_i : X_i \to Y \) and therefore \( \varphi : X \to Y \) exists and has the proposed reduction.

**Proposition 1.9.7.** Let \( X \) and \( Y \) be affinoid varieties with reductions \( \tilde{X} \) and \( \tilde{Y} \). Let \( \varphi : X \to Y \) be a formal smooth morphism admitting a section \( \tilde{\sigma} : \tilde{Y} \to \tilde{X} \) in the reduction. There is a lift of \( \tilde{\sigma} \) which is a section of \( \varphi \).

**Proof.** Again by [BLR90 Prop. 2.2.6] we get \( \text{Hom}_Y(Y_i, X) \to \text{Hom}_Y(Y_{i-1}, X) \), which lifts \( \tilde{\sigma} \) to \( \sigma \in \text{Hom}_Y(Y, X) \). By definition of \( \text{Hom}_Y \), \( \sigma \) is a section of \( \varphi \).

1.10. Meromorphic functions

**Definition 1.10.1.** Let \( X \) be a reduced rigid analytic variety and \( U \) an open affinoid subvariety. The field of fractions \( Q(\mathcal{O}_X(U)) \) is called the **field of meromorphic function** over \( U \). It extends to a sheaf \( \mathcal{M}_X \) of meromorphic functions on \( X \). The global sections of this sheaf are called **meromorphic functions**.

**Proposition 1.10.2.** Let \( X \) be a projective rigid analytic curve over an algebraically closed field \( K \) and \( U \) be an affinoid subvariety with irreducible reduction. Then the valuation of \( \mathcal{O}_X(U) \) extends canonically to \( \mathcal{M}_X(U) \).

**Proof.** Let \( f \in \mathcal{M}_X(U) \) be a meromorphic function. There are functions \( g, h \in \mathcal{O}_X(U) \) with \( f = g/h \) by definition. We set \( |f| = |g|/|h| \). According to Proposition 1.4.2 the absolute value on \( U \) is multiplicative and as such well defined.

We can define a reduction of a meromorphic function \( f \) if \( |f| = 1 \) on \( X_i = \pi^{-1}(X_i \setminus \text{Sing} \tilde{X}) \) by setting \( \tilde{f} = \tilde{g}/\tilde{h} \) which is a rational function on \( \tilde{X}_i \).

**Proposition 1.10.3.** Let \( X \) be a projective smooth rigid analytic curve with reduction \( \tilde{X} \). Let \( \tilde{f} \in k(\tilde{X}) \) be a rational function. Then \( \tilde{f} \) has a meromorphic lift \( f \in \mathcal{M}_X(X) \).

**Proof.** Let \( \tilde{f} \) be defined on \( \tilde{U} \subset \tilde{X} \). Let \( U := \pi^{-1}(U) \), then we find a lift \( f \in \mathcal{O}_X(U) \) of \( \tilde{f} \). The algebraic functions are dense in \( \mathcal{O}_X(U) \) by definition which lets us approximate \( f \) with algebraic functions on \( U \) and therefore algebraic rational functions on \( X \). The limit of these functions therefore gives a meromorphic lift of \( f \).
1.11. Examples

Proposition 1.10.4. Let $X$ be a projective rigid analytic curve and let $f \in \mathcal{M}_X(X)$ be a non-constant meromorphic function. The set $\{x \in X : |f(x)| \leq 1\}$ is an affinoid variety.

Proof. The meromorphic function $f : X \to \mathbb{P}^1$ defines a finite covering map of $X$. Since $\mathbb{B}^1 = \{x \in \mathbb{P}^1 : |x| \leq 1\}$ and $\{x \in \mathbb{P}^1 : |x| \geq 1\}$ is an admissible affinoid covering of $\mathbb{P}^1$, the preimages of these sets are affinoid too. \qed

1.11. Examples

Example 1.11.1. The unit ball $\mathbb{B}^n_K = \text{Sp} K \langle \zeta_1, \ldots, \zeta_n \rangle = \{x \in K^n : |x_i| \leq 1\}$ is an example for an affinoid variety. The unit ball reduces to the affine space $\mathbb{A}^n_k$.

Example 1.11.2. The affine space $\mathbb{A}^n_K$ can be seen as a rigid analytic variety by covering it with balls with increasing radii. Note that any ball of strictly smaller radius in a bigger ball reduces to a point, so this covering is not formal.

Example 1.11.3. The projective space $\mathbb{P}^n_K$ can be obtained by the usual way of covering it with $n + 1$ copies of $\mathbb{A}^n_K$. One sees that covering it with $n + 1$ copies of the unit ball which are identified by their border $\{x \in K : |x| = 1\}$ is already enough. This covering is formal and the reduction obtained this way is the $\mathbb{P}^n_k$.

A projective algebraic variety given by homogeneous equations in some $\mathbb{P}^n_K$ therefore also admits a formal covering and its reduction according to this covering can be computed by reducing the equations after normalization. We denote such a rigid analytic variety as a projective analytic variety.

Example 1.11.4. Let $q \in \mathbb{G}_{m,K}$ with $|q| < 1$. The elliptic curve with parameter $q$ is defined as the quotient $X = \mathbb{G}_{m,K}/(q^2)$. We want to construct an admissible affinoid covering of $X$. Let

$$U_1 := \{x \in \mathbb{G}_{m,K} : q \leq |x| \leq \sqrt{q}\}, \quad U_2 := \{x \in \mathbb{G}_{m,K} : \sqrt{q} \leq |x| \leq 1\},$$

two annuli of height $\sqrt{q}$. We can map $U_i$ onto $X$ by taking the points modulo $q$. Both maps are analytic isomorphisms. Thus $\mathcal{U} = \{U_1, U_2\}$ is an admissible affinoid cover of $X$. Since

$$U_1 = \text{Sp} K \langle \sqrt{q}^{-1} \zeta, q \zeta^{-1} \rangle$$

and

$$U_2 = \text{Sp} K \langle \zeta, \sqrt{q} \zeta^{-1} \rangle$$

we get

$$\tilde{U}_i = \text{Spec} k[X_i, Y_i]/(X_i \cdot Y_i)$$

mapping $\sqrt{q}^{-1} \zeta$ to $X_1$, $q \zeta^{-1}$ to $Y_1$, $\zeta$ to $X_2$ and $\sqrt{q} \zeta^{-1}$ to $Y_2$. The relation $q \equiv 1$ gives us the gluing isomorphisms $X_1 = 1/Y_2$ and $X_2 = 1/Y_1$. Thus $\tilde{X}$ has two components of genus 0, meeting in two double points.
1. Some background on rigid geometry

The reduction map $\pi$ maps points with absolute value between $q$ and $\sqrt{q}$ to the double point of $\tilde{U}_1$, points with absolute value between $\sqrt{q}$ and 1 to the double point of $\tilde{U}_2$. Points with absolute value 1 are mapped onto the component $X_1 = Y_2 = 0$ and those with absolute value $\sqrt{q}$ are mapped onto the component $X_2 = Y_1 = 0$. In Chapter [3] we will show that this curve is indeed an elliptic curve.
2. The structure of a formal analytic
curve

In this chapter, we will analyze the structure of a $p$-adic curve and its dependence on the structure of its reduction.

2.1. Basic definitions

**Definition 2.1.1.** Let $K$ be an algebraically and topologically complete non-Archimedean valued field. An analytic variety $X$ over $K$ of dimension 1 with a fixed reduction $\pi: X \to \tilde{X}$ is called a formal analytic curve if $X$ is smooth over $K$. Therefore $\mathcal{O}_X(U)$ is a reduced and irreducible $K$ algebra of dimension 1.

The first important to mention fact about analytic curves is that according to [BGR84, 6.4.3/1], over an algebraically closed field $K$, we can assume that the residue norm on an affinoid subdomain of $X$ coincides with the supremum norm. The supremum norm is power-multiplicative so the reduction can not contain any nilpotent elements. Furthermore the reduction is of finite type over $k$, so the reduction is a reduced curve. This means the reduction always consists of a finite union of reduced and irreducible components.

2.2. The formal fiber of a point

Let $X/K$ be an analytic curve with reduction $\pi: X \to \tilde{X}$.

**Definition 2.2.1.** The preimage of a point $\tilde{x} \in \tilde{X}$ under $\pi$ is called the formal fiber $X_+(\tilde{x})$ of $\tilde{x}$. An admissible formal affinoid variety $U \subset X$ with $\pi^{-1}(\tilde{x}) \subset U$ is called a formal neighborhood of the formal fiber.

We want to discuss how the kind of the singularity of a point of the reduction determines the formal fiber.

**Proposition 2.2.2.** Let $X = \mathbb{B}^1 = \text{Sp } K\langle \zeta \rangle$ be the unit ball with the canonical reduction $\tilde{X} = A_k^1 = \text{Spec } k[Z]$. Then $\{x \in \mathbb{B}^1 : |\zeta(x)| < 1\}$ is the formal fiber of $0 \in \tilde{X}$. Furthermore $\mathcal{O}_X(X_+(\tilde{x}))$ is local and the reduction map extends to $\mathcal{O}_X(X_+(\tilde{x})) \to \hat{\mathcal{O}}_{\tilde{X},x}$. Furthermore for every $f \in \mathcal{O}_X$ we have $\tilde{f}(x) = \hat{f}(\tilde{x})$.

**Proof.** We write $\tilde{x}$ for the zero point of $\tilde{X} = A_k^1$. Let $x \in X_+(\tilde{x})$ be a point in the formal fiber with associated maximal ideal $m_x = (f)$ for a function

$$f = \sum_{k=0}^{\infty} a_k \zeta^k \in R(\zeta) .$$

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2. The structure of a formal analytic curve

Since \( \tilde{f} \) generates the ideal \( (Z) \) corresponding to the point \( \tilde{x} \) we can assume without restriction that \( \tilde{a}_0 = 0 \) and \( \tilde{a}_1 = 1 \) and therefore we get \( |a_0| < 1 \) and \( |a_1| = 1 \). This means that

\[
|z(x)| = \left| \frac{1}{a_1} \sum_{k=0, k \neq 1}^{\infty} a_k \zeta(x)^k \right| \leq |a_0| < 1
\]

So \( x \) is in \( \{ x \in \mathbb{B}^1 : |\zeta(x)| < 1 \} \).

On the other hand the function \( f = \zeta - x \) will reduce to \( Z \) for any \( x \) in the given set and therefore \( (f) \) will generate a maximal ideal in the formal fiber. This means that \( \mathcal{O}_X(X_+(\tilde{x})) \) is obtained from

\[
\hat{\mathcal{O}}_X(X)(\beta^{-1}\zeta) = R[\zeta]
\]

A series \( f = \sum_{k=0}^{\infty} a_k \zeta^k \) is a unit in \( R[\zeta] \) if and only if \( |a_0| = 1 \), which means that the sum of two non-units is again not a unit and that \( \mathcal{O}_X(X_+(\tilde{x})) \) is local.

Furthermore, for any \( x \in X_+(\tilde{x}) \) we have \( |f(x)| < 1 \) if and only if \( |a_0| < 1 \) and therefore \( \hat{f}(\tilde{x}) = 0 \). Since we can use a linear transformation to move any point into the origin we get \( \hat{f}(x) = \tilde{f}(\tilde{x}) \) for every \( x \in \mathbb{B}^1 \).

\textbf{Proposition 2.2.3.} Let \( X = \text{Sp} A \) be an analytic curve with smooth reduction \( \tilde{X} \). Then \( X_+(\tilde{x}) \) is isomorphic to \( \{ x \in \mathbb{B}^1 : |\zeta(x)| < 1 \} \) for any \( \tilde{x} \in \tilde{X} \). Especially \( \hat{\mathcal{O}}_X(X_+(\tilde{x})) \) is local and \( f(x) = \tilde{f}(\tilde{x}) \) for every \( f \in \hat{\mathcal{O}}_X \).

\textbf{Proof.} We adapt the proof of [BL85, Prop. 2.2] to our purposes. We choose a point \( x \in X \) with \( \pi(x) = \tilde{x} \). Since \( x \) and \( \tilde{x} \) are regular, we can localize \( A \) in a way that there is a function \( f \) with absolute value 1 which generates the maximal ideal corresponding to \( x \) and its reduction will generate the maximal ideal corresponding to \( \tilde{x} \). Now we get

\[
A = K \oplus Af \quad \text{and} \quad \tilde{A} = k \oplus \tilde{A} \tilde{f}.
\]

And therefore we get

\[
A = K \perp Af
\]

according to Proposition 1.1.3. We define the morphism \( \sigma : K(\zeta) \to A \) which maps \( \zeta \) to \( f \).

Look at

\[
\sigma : \quad K(\zeta) \xrightarrow{\zeta} A \quad , \quad \zeta \mapsto f
\]

\[
\sigma_\varepsilon : \quad K(\varepsilon^{-1}\zeta) \xrightarrow{\varepsilon^{-1}f} A(\varepsilon^{-1}f) \quad , \quad \zeta \mapsto f.
\]

The map \( \sigma_\varepsilon \) is injective for all \( \varepsilon \) since the map \( f : X \to \mathbb{B} \) has zero dimensional fibers. Furthermore for any element \( g \in \tilde{A} \) we set

\[
g_0 = g \quad \text{and} \quad g_i = g_i(x) + g_{i+1} \cdot f
\]
2.2. The formal fiber of a point

according to (2.1). So we get \( |g_i| \leq 1 \). Then the series

\[
\sum_{i=0}^{\infty} g_i(x) f^i = \sum_{i=0}^{\infty} g_i(x) \varepsilon^i (\varepsilon^{-1} f)^i
\]

converges on \( A(\varepsilon^{-1} f) \) and is in the image of \( \sigma_\varepsilon \). But then every series in \( \varepsilon^{-1} f \)

is also in the image of \( \sigma_\varepsilon \) and therefore \( \sigma_\varepsilon \) is surjective.

If we have any \( g \in \mathcal{O}_X \) such that \( \tilde{g} \in \mathfrak{m}_x \) we know that \( \tilde{g} = \tilde{h} \cdot \tilde{f} \) and therefore that \( |g - h \cdot f| < 1 \) for any lift \( h \in \mathcal{O}_X \) of \( \tilde{h} \). This means \( |g(x)| < 1 \) for any \( x \) with \( |f(x)| < 1 \) and so the formal fiber is composed of all \( x \in X \) which fulfill \( |f(x)| < 1 \).

This gives us the necessary isomorphism of the formal fiber. \( \square \)

With these two propositions, we can describe the formal fiber of a point with a function \( f \in \mathcal{O}_X \). If \( X \) is non-singular everywhere but in the interesting point \( \tilde{x} \) and \( \tilde{f} \) has a single isolated zero at \( \tilde{x} \), we now know that \( |f(y)| = 1 \) for any point outside the formal fiber. So we get \( \{ x \in X : |f(x)| < 1 \} \subset X_+(\tilde{x}) \). We want to prove that equality holds.

**Proposition 2.2.4.** Let \( X = \text{Sp } B \) be an analytic curve with reduction \( \tilde{X} \). Let further be \( f \in \mathcal{O}_X(X) \) be a function so that \( \tilde{f} \) has a single isolated zero \( \tilde{x} \) and let \( \tilde{X} \setminus \{ \tilde{x} \} \) be non-singular. Let further be \( \tilde{g} \in \mathcal{O}_X \), a function on the normalization \( \tilde{X}' \) of \( \tilde{X} \). Then there is an \( \varepsilon < 1 \) and a lift \( g \in \mathcal{O}_X(X_\varepsilon) \) where \( X_\varepsilon = X(\{ f \mid |f| \geq \varepsilon \}) \).

**Proof.** Let us first fix some notation. We set \( A := K(\zeta) \) and \( \mathbb{B}_K^1 = \text{Sp } A \). We can look at \( f \) as a function \( f : X \to \mathbb{B}_K^1 \) with a corresponding injection \( A \to B \) which maps \( \zeta \) to \( f \). Let \( \tilde{X} := \text{Spec } \tilde{B} \), \( \mathbb{A}_k := \text{Spec } \tilde{A} \) and \( X' := \text{Spec } \tilde{B}' \), the latter being the normalization. We will further need the varieties

\[ X_{\varepsilon,s} := \{ x \in X : \varepsilon \leq |f(x)|, |s(x)| = 1 \} \]

and

\[ \mathbb{B}_s^1 := \{ x \in \mathbb{B}_K^1 : \varepsilon \leq |\zeta(x)|, |s(x)| = 1 \} , \]

and their corresponding affinoid algebras \( B_{\varepsilon,s} = B(\varepsilon^{-1}, s^{-1}) \) for \( X_{\varepsilon,s} \) and \( A_{\varepsilon,s} = A(\varepsilon^{-1}, s^{-1}) \) for \( \mathbb{B}_s^1 \) where \( s \) is a function in \( \mathcal{O}_B \) with \( s(0) \neq 0 \) chosen so that \( f \) is finite.

Since \( B_{0,s} \) is finite over \( A_{0,s} \) the quotient field \( Q(B_{0,s}) \) is algebraic over \( Q(A_{0,s}) \) and \( Q(B) \) is algebraic over \( Q(A) \). Furthermore \( B_{0,s} \) is the integral closure of \( A_{0,s} \) and \( \tilde{B}_s \) is the integral closure of \( \tilde{A}_s \) in their respective fields.

Since \( K \) is algebraically closed, \( K \) is stable. Due to [BGRS1, 5.3.2 Thm. 1] \( Q(K(\zeta)) = Q(A) \) is stable. Since \( B/A \) is generically unramified, we get according to [BGRS1, 3.6 Prop. 8] that

\[ t := [B_{0,s} : A_{0,s}] = [\tilde{B}_s : \tilde{A}_s] . \tag{2.2} \]

The given element \( \tilde{g} \) induces a function on \( \tilde{X} \setminus \{ \tilde{x} \} \) so there is a lift \( g \in \mathcal{O}_X(X \setminus X_+(\tilde{x})) \). The restriction map \( \mathcal{O}_X(\varepsilon \leq |f|) \to \mathcal{O}_X(1 \leq |f|) \) has a dense image so we can choose \( g \in \mathcal{O}_X(\varepsilon \leq |f|) = B_{\varepsilon} \) without restriction.
2. The structure of a formal analytic curve

We have already seen that $B_{0,s}$ is integral over $A_{0,s}$ and therefore $B_{x f^{-1},s}$ is integral over $A_{x f^{-1},s}$. Completion gives that $B_{x,s}$ is integral over $A_{x,s}$ so there is an integral equation

$$F(T) := (-1)^t T^t + a_1 T^{t-1} + \cdots + a_t$$

of degree $t$, that annihilates $a_i \in A_{x,s}$ which is the characteristic polynomial of $g$.

Let

$$a_i = \sum_{\nu \in \mathbb{Z}} a_{i\nu} \zeta^\nu$$

be the Laurent series representation of $a_i$. Because of (2.7) the reduction $\tilde{F}$ of $F$ is the characteristic polynomial of $\tilde{g}$. Therefore we have $\tilde{a}_i \in \tilde{A}_2$ and thus $|a_{i\nu}| < 1$ for all $\nu < 0$. Furthermore since $a_i \in A_{x,s}$ the sequence $a_{i\nu} \zeta^\nu$ tends to zero for $\nu \to -\infty$. So there is an $N_0$ such that $|a_{i\nu} \zeta^\nu| < 1$ for all $\nu < N_0$. Since $|a_{i\nu}| < 1$ for all $\nu < 0$ we can find $\varepsilon_1 < 1$ such that $|a_{i\nu} \zeta^\nu| < 1$ for $N_0 \leq \nu < 0$. Thus, by adjusting $\varepsilon$, we have $|a_{i\nu} \zeta^\nu| < 1$ for all $\nu < 0$. Therefore $|a_i| < 1$ and thus $|g| < 1$ on $X_{x,s}$.

Proposition 2.2.5. Let $X, f$ and $g$ be as in Proposition 2.2.4. Denote the zero of $\tilde{f}$ by $\tilde{x}$. There is a point $x \in X_\varepsilon = X(\varepsilon \leq |f| < 1)$ such that $|g(x)| = 1$ if and only if there is a point $\tilde{y} \in \tilde{X}'$ lying over $\tilde{x}$ with $\tilde{g}(\tilde{y}) \neq 0$.

Proof. It is important to note that by Proposition 2.2.3 we know that $|f(x)| < 1$ already implies that $x$ is in the formal fiber. Furthermore we know that $|a_k(x)| < 1$ for one $x$ with $|f(x)| < 1$ implies that $|a_k(x)| < 1$ for any $x$ with $|f(x)| < 1$ since $a_k \in A_2$.

Since $\tilde{x}$ is the only zero of $\tilde{f}$ we know that $\tilde{g}$ has $m$ zeroes on the n points $\tilde{y}_1, \ldots, \tilde{y}_n$ lying over $\tilde{x}$ if and only if $\tilde{a}_k(0) = 0$ for $k < m$ and $\tilde{a}_m(0) \neq 0$. So $|a_k(x)| < 1$ for $k < m$ and $|a_m(x)| = 1$ for any $x$ in the periphery of the formal fiber. Therefore there is a point $x$ with $|g(x)| = 1$. The other implication follows by the same argument.

Lemma 2.2.6. Let $X$ be an analytic curve and $\tilde{x} \in \tilde{X}$ a point of the reduction. Then the ring $\mathcal{O}_X(X_+(\tilde{x}))$ is local with residue field $k$.

Proof. We can, without restriction, assume that $X = \text{Spec} B$ and $\tilde{X} = \text{Spec} \tilde{B}$. We can further assume that $\tilde{X}$ has no singularities beside $\tilde{x}$.

Let $h_1, h_2 \in \mathcal{O}_X(X_+(\tilde{x}))$ be two non-units. This means that there are points $x_1, x_2 \in X_+(\tilde{x})$ such that $|h_1(x_1)|, |h_2(x_2)| < 1$.

If $|h_1(x_2)| < 1$ then $|(h_1 + h_2)(x_2)| < 1$ and $h_1 + h_2$ is not a unit. Therefore we can assume that $|h_1(x_2)| = 1$.

Let us first move back in an affinoid case. Set $f = h_1 \cdot h_2$ and choose $\beta < 1$ such that $|f(x_1)|, |f(x_2)| < \beta$. We set $B_\beta = B(f/\beta)$ with reduction $\tilde{B}_\beta$. Since $h_1 \in \mathcal{O}_X(X_+(\tilde{x}))$ we get $h_1 \in \tilde{B}_\beta$. By the choice of $\beta$ the function $h_1$ is still not a unit in $B_\beta$ but $|h_1| = 1$ in $\tilde{B}_\beta$.

The function $\tilde{h}_1$ is defined on $\tilde{B}_\beta$ and therefore is defined on the normalization of $\tilde{X}_\beta$ as well. Furthermore, since $|h_1(x_1)| < 1$ we get $\tilde{h}_1(\tilde{x}) = 0$, so $\tilde{h}_1(\tilde{y}) = 0$ for
all \( \tilde{y} \) lying over \( \tilde{x} \) in the normalization. This means \( |h_1(x_2)| < 1 \) by Proposition 2.2.3 contrary to our assumption.

Therefore an element of \( h \in \mathcal{O}_X(X_+(\tilde{x})) \) is not a unit if and only if \( |h(x)| < 1 \) on every point \( x \in X_+(\tilde{x}) \). This implies that the sum of two non-units is again not a unit and \( \mathcal{O}_X(X_+(\tilde{x})) \) is local.

**Corollary 2.2.7.** Let \( X \) be an analytic curve and \( h \in \mathcal{O}_X(X_+(\tilde{x})) \) be a function. If \( |h(x)| < 1 \) for any \( x \in X_+(\tilde{x}) \) then \( |h(x)| < 1 \) for all \( x \in X_+(\tilde{x}) \).

**Corollary 2.2.8.** Let \( X \) be an analytic curve and \( f \in \mathcal{O}_X \) be a function such that \( \tilde{f} \) has a single isolated zero in the point \( \tilde{x} \). Then \( X_+(\tilde{x}) = X(\{|f| < 1\}) \).

**Proof.** As we have seen \( |f(x)| < 1 \) on every point of \( X_+(\tilde{x}) \).

**Corollary 2.2.9.** Evaluating functions and taking their reduction commutes. We get \( f(\tilde{x}) = \tilde{f}(\tilde{x}) \).

**Lemma 2.2.10.** Let \( X \) be a analytic variety and \( \tilde{x} \in \tilde{X} \) a point of the reduction. Then \( \mathcal{O}_X(X_+(\tilde{x})) \) is henselian.

**Proof.** Let us set \( \tilde{B}_+ := \mathcal{O}_X(X_+(\tilde{x})) \). Let \( P = T^n + c_{n-1}T^{n-1} + \cdots + c_0 \in \tilde{B}_+[T] \) be a monic polynomial with reduction \( \tilde{P} \in k[T] \) which has a simple zero \( \tilde{\alpha} \) in \( k \). After coordinate transformation we can assume that \( \tilde{\alpha} = 0 \). We denote the other zeroes by \( \tilde{\alpha}_i \) with \( i = 2, \ldots, n \), where \( n \) is the degree of \( \tilde{P} \).

In this setup we see that \( \tilde{c}_0 = 0 \) and therefore \( c_0 \in m \), the maximal ideal of \( \tilde{B}_+ \). Since \( \tilde{c}_i \) is the sum of every possible product of \( n - 1 \) zeroes of \( \tilde{P} \) and only one such combination, namely \( \tilde{\alpha}_2 \cdots \tilde{\alpha}_n \) differs from zero, we know that \( c_1 \not\in m \).

For every \( x \in X_+(\tilde{x}) \) and \( |\varepsilon| < 1 \) the polynomial \( P \) can be written as

\[
P = \varepsilon^n \cdot T^n/\varepsilon^n + c_{n-1}(x)\varepsilon^{n-1} \cdot T^{n-1}/\varepsilon^{n-1} + \cdots + c_1(x)\varepsilon T/\varepsilon + c_0(x)
\]

Now \( c_0 \in m \) and \( c_1 \not\in m \) imply \( |c_0(x)| = \eta < 1 \) and \( |c_1(x)\varepsilon| = |\varepsilon| \) by Corollary 2.2.7. Furthermore for \( i > 1 \) we get \( |c_i(x)\varepsilon^i| \leq |\varepsilon^i| < |\varepsilon| \), so \( P \) is \( T/\varepsilon \) distinguished of order 1 if \( \varepsilon > \eta \).

According to the Weierstraß preparation theorem [Bos05, Sect. 1.2 Cor. 9] this means there is a unit \( \omega_\varepsilon \) in \( K(\varepsilon) \) and an element \( \alpha_\varepsilon(x) \in K \) such that

\[
P(x) = (T/\varepsilon - \alpha_\varepsilon(x)) \cdot \omega_\varepsilon(x) = (T - \alpha_\varepsilon(x) \cdot \varepsilon) \cdot \omega_\varepsilon(x)/\varepsilon
\]

Since \( P \) is a polynomial and \( \omega_\varepsilon \) is a convergent power series, \( \omega_\varepsilon \) can only be a polynomial itself.

The uniqueness of Weierstraß preparation assures us that \( \alpha(x) := \alpha_\varepsilon(x) \cdot \varepsilon \) does not depend on \( \varepsilon \) and as such gives a function \( \alpha \in B_+ \). Furthermore, we know that \( |T/\varepsilon - \alpha_\varepsilon(x)| = 1 \) in the absolute value of \( K(T/\varepsilon) \), so \( |\alpha(x)| = |\alpha_\varepsilon(x) \cdot \varepsilon| \leq |\varepsilon| < 1 \) which means that \( \alpha \) is in \( m \), i.e. \( \alpha \) is the lift of \( \tilde{\alpha} \) we looked for.

**Corollary 2.2.11.** The ring of functions on a formal fiber of a point is invariant under étale base change.
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2.3. The formal fiber of regular points and double points

Proposition 2.3.1. Let $X = \text{Sp} A$ be an analytic curve with reduction $\tilde{X}$. A point $\tilde{x} \in \tilde{X}$ of the reduction is regular if and only if the formal fiber $X_+(\tilde{x})$ is isomorphic to $\{x \in B^1 : |x| < 1\}$.

Proof. We have already seen in Proposition 2.2.3 that $X_+(\tilde{x})$ is isomorphic to the open unit ball if $\tilde{x}$ is regular.

If we suppose that $X_+(\tilde{x})$ is isomorphic to $\{x \in B^1 : |x| < 1\}$ we know by Lemma 2.2.6 that $\hat{A}$ is isomorphic to $k[T]$ and therefore that $\tilde{x}$ is regular. □

Proposition 2.3.2. Let $X = \text{Sp} A$ be an analytic curve with reduction $\tilde{X}$. A point $\tilde{x} \in \tilde{X}$ of the reduction is an ordinary double point if and only if the formal fiber $X_+(\tilde{x})$ is isomorphic to $\{x \in B^1 : \varepsilon < |x| < 1\}$, an open annuli in $B^1$.

Proof. At first we want to reduce to the case that the double point lies on two different components. This can be done by applying an étale base change so this is simply a consequence of Corollary 2.2.11. However, the same result can be obtained more directly. This serves to illustrate our proof of said corollary.

Assume the maximal ideal $m_x$ corresponding to $\tilde{x}$ is generated by two functions $\tilde{g}_1, \tilde{g}_2$ such that $\tilde{f} := \tilde{g}_1 \cdot \tilde{g}_2 \in m_x^2$. This is possible since $\tilde{x}$ is a double point. Assume further that without restriction $\text{ord}_{\tilde{g}_i} \tilde{f} = 3$ on both points $\tilde{g}_i$ lying over $\tilde{x}$ in the normalization. This assumption can easily be met by having $\text{ord}_{\tilde{g}_i} \tilde{g}_i = 1$ and $\text{ord}_{\tilde{g}_i} \tilde{g}_j = 2$ for $i, j = 1, 2, i \neq j$.

Now look at $h_i := \tilde{g}_i^3/\tilde{f}$ for some lifts $g_i, f$ of $\tilde{g}_i, \tilde{f}$. Their reduction is defined on the normalization of $\tilde{X}$ and so $h_i$ is defined and of absolute value smaller than 1 on $X(|f| \geq \varepsilon^3)$ for a suitable $\varepsilon$. But we have $\tilde{h}_i(\tilde{g}_i) \neq 0$ so there is a point $x_i \in X_+(\tilde{x})$ with $|h_i(x_i)| = 1$. So we can set $\beta$ such that $|f(x_i)| = \beta^3 < 1$ and we get $|g_i(x_i)|^3 = |f(x_i)|$ which means that $|g_i/\beta| = 1$ on $X(|f| \leq \beta^3)$. Note that by adjusting $\varepsilon$ we can get values for $\beta$ arbitrarily close to 1. This means that on $A(\beta^{-3} f)$ we have $g_1/\beta \cdot g_2/\beta = f/\beta^2$. But $|f| = \beta^3$ on $A(\beta^{-3} f)$ so this reduces to $\tilde{g}_1 \cdot \tilde{g}_2 = 0$. Therefore $\tilde{x}$ joins two components of the reduction of $X(|f| \leq \beta^3)$ for any $\beta < 1$ and we only need to discuss this case.

We will use the proof as in [BLŞ5 Prop. 2.3]. Let $m_x$ be generated by $\tilde{f}$ and $\tilde{g}$ such that $\tilde{f} \cdot \tilde{g} = 0$. Assume that $f$ and $g$ are lifts chosen so that they have a common zero $x$. This can be achieved by replacing $g$ with $g - g(x)$ for a zero $x$ of $f$. Since $|g(x)| < 1$ the reduction $\tilde{g}$ is left unchanged. Therefore $m_x$ is generated by $f$ and $g$ and we get

\[ A = K \perp Af \perp Ag \]

as in the case of a regular point. We recursively define sequences $(f_i), (g_i), (h_i)$ in $A$ and $(\alpha_i)$ in $K$. For this set $f_0 = g_0 = \alpha_0 = 0$ and define

\[ h_k := \left( f - \sum_{i=0}^{k-1} f_i \right) \left( g - \sum_{i=0}^{k-1} g_i \right) - \sum_{i=0}^{k-1} \alpha_i \]

(2.5)
Then we decompose $h_k$ according to (2.4) to get
\[ h_k = \alpha_k + g_k f + f_k g \] (2.6)

Using (2.6) in (2.5) gives us
\[ h_k = f_{k-1} \sum_{i=0}^{k-2} g_i + g_{k-1} \sum_{i=0}^{k-2} f_i + f_{k-1} g_{k-1} \]

for $k \geq 2$. By our assumption of $f$ and $g$ we get $|h_1| = |f \cdot g| = \gamma < 1$. Therefore we recursively get $|f_k|, |g_k|, |\alpha_k| \leq |h_k| \leq \gamma^k$. Set
\[ f' := \left(f - \sum_{i=1}^{k-1} f_i\right) \quad \text{and} \quad g' := \left(g - \sum_{i=1}^{k-1} g_i\right) \]

Then $f' \cdot g' = \sum_{i=1}^{\infty} \alpha_i =: \alpha \in K$. Since $\tilde{f}' = \tilde{f}$ and $\tilde{g}' = \tilde{g}$ we know that $|c| < 1$. Furthermore $c \neq 0$ since $A$ is an integral domain.

We define
\[ \sigma : K(\zeta, c\zeta^{-1}) \to A \]
\[ \zeta \mapsto f' \]
\[ c\zeta^{-1} \mapsto g' \]

and again set $\sigma_\varepsilon : K(\varepsilon^{-1}\zeta, \varepsilon^{-1}c\zeta^{-1}) \to A(\varepsilon^{-1}f', \varepsilon^{-1}g')$ for the induced map. We know that $\sigma$ is injective because of (2.4). For any element $h \in A$ we can construct a series $h = h_0 + \sum_{k=0}^{\infty} h_k f^k + \sum_{k=0}^{\infty} h_k g^k$ with $h_k \in K$ by repeated application of (2.4). The series converges as long as $|\varepsilon| < 1$ so $\sigma_\varepsilon$ is an isomorphism in this case. \qed

### 2.4. The formal fiber of a general singular point

**Theorem 2.4.1.** Let $X = \text{Sp } B$ be of pure dimension 1, and let $\tilde{x}$ be a point in $\tilde{X}$. Let $U$ be a formal neighborhood of $X_+(\tilde{x})$ in $X$ and let $f$ be a function in $\mathcal{O}_X(U)$ such that $\tilde{x}$ is an isolated zero of $\tilde{f} \in \mathcal{O}_X(\tilde{U})$. Let $\tilde{x}'_1, \ldots, \tilde{x}'_n$ be the points in the normalization $\tilde{X}'$ of $\tilde{X}$ lying over $\tilde{x}$. Then, for $\varepsilon \in |K^\times|, \varepsilon < 1$ close to 1, the analytic variety $\{x \in X_+(\tilde{x}); \varepsilon \leq |f(x)| < 1\}$ decomposes into $n$ connected components $R_1, \ldots, R_n$ which are semi-open annuli.

More precisely, let $\zeta$ be a coordinate function of $B^1$, and, for $i = 1, \ldots, n$, denote by $t_i = \text{ord}_{\tilde{y}_i} (\tilde{f})$ the vanishing order of $\tilde{f}$ at $\tilde{x}'_i$. There are isomorphisms
\[ \varphi_i : R_i \to \{ z \in B^1 ; \varepsilon^{1/t_i} \leq |\zeta(z)| < 1 \} \]

such that, up to a unit in $\mathcal{O}_X(R_i)$, the function $f|_{R_i}$ equals the pullback $\varphi_i^* (\zeta_i)$. Furthermore, if the reduction $\tilde{h} \in \mathcal{O}_{\tilde{X}, \tilde{x}}$ of an element $h \in \mathcal{O}_X(X_+(\tilde{x}))$ satisfies $\tau = \text{ord}_{\tilde{x}_{i_0}'} (\tilde{h}) < \infty$ for some index $i_0$, and if $\varepsilon$ is close enough to 1, then, up to a unit in $\mathcal{O}_X(R_{i_0})$, the function $h|_{R_{i_0}}$ equals the pullback $\varphi_{i_0}^* (\zeta')$. 

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Proof. Let us first provide a proof for the simpler situation where \( f \) only has one isolated zero \( \tilde{x} \) in \( \tilde{X} \) and only one point \( \tilde{x}' \) lies above \( \tilde{x} \) in the normalization.

The first step of the proof is to use Lemma 2.2.4 on a local parameter of \( \tilde{x}' \). Since we have not assumed \( \tilde{X} \setminus \{ \tilde{x} \} \) to be non-singular, we need to adjust the proof a little.

Let us recall some notation. Let \( X = \text{Sp} B \) and \( \mathcal{B} = \text{Sp} A \) with \( A := K(\zeta) \). We have, as usual, the morphism \( f : X \to \mathcal{B} \) corresponding to an injection \( A \hookrightarrow B \) which maps \( \zeta \) to \( f \). Let \( \tilde{X} := \text{Spec} \tilde{B}, \tilde{\mathcal{A}} := \text{Spec} \tilde{A} \) and \( \tilde{X}' := \text{Spec} \tilde{B}' \), the latter being the normalization. We will further need the varieties

\[
X_{\epsilon,s} := \{ x \in X ; \epsilon \leq |f(x)|, |s(x)| = 1 \}
\]

and

\[
B_{\epsilon,s} := \{ x \in B ; \epsilon \leq |\zeta(x)|, |s(x)| = 1 \}
\]

where \( s \) is a function in \( \mathcal{O}_B \) and \( \tilde{s}(0) \neq 0 \) yet to be determined fully but chosen so that \( f \) is finite. Note that \( X_{\epsilon,s} \) and \( B_{\epsilon,s} \) only depend on \( \epsilon \) and the reduction \( \tilde{s} \) of \( s \).

Since \( K \) is algebraically closed, \( K \) is stable. Due to [BGR84, 5.3.2 Thm. 1] \( Q(K(\zeta)) = Q(A) \) is stable. Since \( B/A \) is generically unramified, we get according to [BGR84, 3.6 Prop. 8] that

\[
t := [B_{0,s} : A_{0,s}] = [\tilde{B}_{\tilde{s}} : \tilde{A}_{\tilde{s}}].
\]  

(2.7)

Let \( \tilde{g} \) be a local parameter of \( \tilde{x}' \) in \( \tilde{X}' \). Then we have \( Q(\tilde{B}') = Q(\tilde{A})[\tilde{g}] \). We can choose \( \tilde{s} \) as desired so that localizing at \( \zeta \) and \( \tilde{s} \) yields \( \tilde{B}_{\zeta,\tilde{s}} = \tilde{A}_{\zeta,\tilde{s}}[\tilde{g}] \), where \( \tilde{s}(0) \neq 0 \).

We can remove singularities by localizing, so we can apply Lemma 2.2.4 to get an \( \epsilon \) such that a lift \( g \) of \( \tilde{g} \) is defined on \( X_{\epsilon,s} \).

The polynomial \( F \) of the lemma is the minimal polynomial of \( g \) up to normalization, by our choice of assumptions.

Next, we want to show that

\[
B_{\epsilon,s} = A_{\epsilon,s}[g].
\]

Since \( Q(\tilde{B})/Q(\tilde{A}) \) is a vector space, we can adjust \( s \) in a way that \( \tilde{B}_{\tilde{s}} \) is even a finite free \( \tilde{A}_{\tilde{s}} \) module, namely

\[
\tilde{B}_{\zeta,\tilde{s}} = \tilde{A}_{\zeta,\tilde{s}}\tilde{g}^0 \oplus \cdots \oplus \tilde{A}_{\zeta,\tilde{s}}\tilde{g}^{t-1}.
\]

By applying Proposition 1.1.3 we get

\[
B_{1,s} = A_{1,s}g^0 \perp \cdots \perp A_{1,s}g^{t-1}
\]  

(2.8)

i.e. \( g^0, \ldots, g^{t-1} \) are an orthonormal system of generators of \( B_{1,s} \) over \( A_{1,s} \).

Furthermore, since \( F \) is a Weierstraß-polynomial, we can decompose \( A[\tilde{g}] \) according to [BGR84, 5.2.3 Prop. 3] so that

\[
A[\tilde{g}] = Ag^0 \oplus \cdots \oplus Ag^{t-1}
\]
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and also

\[ A_{\varepsilon,s}[g] = A_{\varepsilon,s}g^0 \oplus \cdots \oplus A_{\varepsilon,s}g^{t-1} \] (2.9)

Since \( B_{1,s} = A_{1,s}[g] \) and the support of the quotient \( B_{\varepsilon,s}/A_{\varepsilon,s}[g] \) consists of finitely many points. So there exists an \( \varepsilon < 1 \) such that \( B_{\varepsilon,s} = A_{\varepsilon,s}[g] \).

\( \tilde{A}(\zeta) \mapsto \tilde{B}'(f) \) is a finite extension of discrete valuation rings, with \( \zeta \) being a local parameter in the former and \( \tilde{g} \) in the latter. Therefore there is a unit \( \tilde{u} \) so that \( \tilde{\zeta} = \tilde{g}' \tilde{u} \). Since the residue field is algebraically closed, we can, without loss of generality, write \( \tilde{u} = 1 + \delta \) with \( \delta \in \tilde{m} \).

So we get the representation

\[ \zeta = g^{t}(1 + \delta) \] (2.10)

with \(|\delta(x)| < 1\) where for \( x \in X_{\varepsilon,s} \) and \(|\zeta(x)| < 1\).

Now \( \delta \) has a series representation

\[ \delta = \delta_0g^0 + \cdots + \delta_{t-1}g^{t-1} \]

with \( \delta_i \in A_{\varepsilon,s} \),

\[ \delta_i = \sum_{\nu \in \mathbb{Z}} d_{i,\nu} \zeta^{\nu} \cdot \]

Due to (2.8) we have \(|d_{i,\nu}| \leq 1\). Due to (2.9) we have \( d_{i,\nu} \varepsilon^{\nu} \rightarrow 0 \) for \( \nu \rightarrow -\infty \) and so there is an \( N_0 \) such that \( |d_{i,\nu} \varepsilon^{\nu}| < 1 \) for all \( \nu < N_0 \). We claim \( |d_{i,\nu}| \leq 1 \) for all \( \nu \in \mathbb{Z} \) and \( |d_{i,\nu} \varepsilon^{\nu}| < 1 \) for all \( \nu \leq 0 \).

We know \(|\delta_i| \leq 1\) and \( \delta_i \in A_{\varepsilon,s} \). Since \( \delta \in \tilde{m} \) and

\[ \tilde{B}' = \tilde{A}g^0 \oplus \cdots \oplus \tilde{A}g^{t-1} \]

we must have \( \tilde{\delta}_i \in \tilde{A}(\zeta) \subset k[\zeta] \). As in the lemma above, we can modify \( \varepsilon \) if necessary to get that \( |d_{i,\nu} \varepsilon^{\nu}| < 1 \) for all \( \nu < 0 \). The case \( \nu = 0 \) was not covered above, however, this holds since \( \delta \in \tilde{m} \). This shows that

\[ \delta \in R(\zeta, \varepsilon/\zeta) \cdot g^0 \oplus \cdots \oplus R(\zeta, \varepsilon/\zeta) \cdot g^{t-1} \] (2.11)

holds.

On the other hand, by the same reasoning, we also have a \( \gamma \in B_{\varepsilon,s} \) with

\[ g^t = \zeta(1 + \gamma) \cdot \]

Yet again, we can write

\[ \gamma = \gamma_0g^0 + \cdots + \gamma_{t-1}g^{t-1} \]

with

\[ \gamma_i = \sum_{\nu \in \mathbb{Z}} c_{i,\nu} \zeta^{\nu} \]

and the reduction \( \gamma_i \in k[\zeta] \). So we have \(|c_{i,\nu}| \leq 1 \) for all \( \nu \in \mathbb{Z} \) and \(|c_{i,\nu} \varepsilon^{\nu}| < 1 \) for \( \nu \leq 0 \). Again, this means that

\[ \gamma \in R(\zeta, \varepsilon/\zeta) \cdot g^0 \oplus \cdots \oplus R(\zeta, \varepsilon/\zeta) \cdot g^{t-1} \] (2.12)
2. The structure of a formal analytic curve

Choosing an element $\alpha \in K$ with $\alpha^t = \varepsilon$, we get

$$\varepsilon = \left( \frac{\alpha}{\gamma} \right)^t \cdot (1 + \gamma) \quad (2.13)$$

We want to show that for any arbitrary number $\beta$ with $|\beta| < 1$ we have

$$R(\langle g/\beta, \alpha/g \rangle) = R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g]$$

and that therefore $X_+(\bar{x})$ is isomorphic to a semi-open annulus.

Let $w \in R$ be a number with $|w| = \max(|\delta|, |\gamma|)$ where the absolute value of the ring $R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g]$ is used. Our calculations above have shown that

$$|w| \leq \max \left( \sup_{i=1 \ldots t-1} (d_i \nu \varepsilon^\nu), \sup_{i=1 \ldots t-1} (c_i \nu \varepsilon^\nu), \beta^t \right) < 1 .$$

By (2.10) and (2.13) we have

$$\zeta, \varepsilon/\zeta \in R(\langle g/\beta, \alpha/g \rangle) + w \cdot R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g] .$$

Since $\zeta/\beta^t, \varepsilon/\zeta$ and $g$ generate $R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g]$ this means

$$R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g] = R(\langle g/\beta, \alpha/g \rangle) + w \cdot R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g] . \quad (2.14)$$

Applying (2.14) to itself $n$-times yields

$$R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g] = R(\langle g/\beta, \alpha/g \rangle) + w^n \cdot R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g]$$

And thus, since $|w| < 1$ and $R(\langle g/\beta, \alpha/g \rangle)$ is complete, we get

$$R(\langle \zeta/\beta^t, \varepsilon/\zeta \rangle[g] = R(\langle g/\beta, \alpha/g \rangle) ,$$

which proves the theorem in this special case.

In the general case, use an étale base change on $\mathcal{A}$ so that $\mathcal{A}^{\text{âˆšt}}$ is an étale neighborhood of $\hat{x}$. Let $X^{\text{âˆšt}} := X \times \mathcal{A}^{\text{âˆšt}}$ and $\tilde{X}^{\text{âˆšt}} := \tilde{X} \times \mathcal{A}^{\text{âˆšt}}$ be the corresponding varieties for $\hat{X}$. We can chose the étale base change in such way that $X^{\text{âˆšt}}$ consists of $n$ components which decompose in $\tilde{X}^{\text{âˆšt}}$ in $n$ disjoint components, where $n$ is the number of points in the normalization $\tilde{X}$ over $\hat{x}$. In this case $\tilde{X}^{\text{âˆšt}} \setminus \{ \tilde{x} \}$ also decomposes into $n$ disjoint components.

Let $X^{\text{âˆšt}}$ be a lifting of $\tilde{X}^{\text{âˆšt}}$. In particular we have $X^{\text{âˆšt}}(\tilde{x}) \rightarrow X_+(\tilde{x})$ by Corollary 2.2.11 Since $\tilde{X}^{\text{âˆšt}} \setminus \{ \tilde{x} \}$ consists of $n$ disjoint components, $X_1 := \{ x \in X^{\text{âˆšt}} ; |f(x)| = 1 \}$ consists of $n$ disjoint components as well. Now look at $X_{\varepsilon_0} := \{ x \in X^{\text{âˆšt}} ; |f(x)| \geq \varepsilon_0 \}$ and consider a function $h \in \mathcal{O}_X(X_1)$ which behaves like a constant on any connected component with pairwise different absolute values unequal to zero on the components of $X_1$. Since the image of $O_{\tilde{X}}(X_{\varepsilon_0})$ is dense in $\mathcal{O}_X(X_1)$, we can choose a function $\tilde{h} \in \mathcal{O}_X(X_{\varepsilon_0})$ which approximates $h$ such that $|h - \tilde{h}|_{X_1} < \min\{ |h(x)| : x \in X_1 \}$. The function $|\tilde{h}| : X_{\varepsilon_0} \rightarrow \mathbb{R}$ is continuous with regard to the G-topology on $X_{\varepsilon_0}$. Since the absolute values on the components of $X_1$ are different, we can choose a $\delta$ for
2.5. Formal blow-ups

Let $i \neq j$ so that $||h_i| - |h_j|| > 2\delta > 0$, where $|\cdot|_i$ is the absolute value on the $i$-th component of $X_i$. Since $|h|$ is continuous, there is an $\varepsilon_0$ so that for every $x \in X_{\varepsilon_0}$ we get an $y \in X_1$ so that $||\tilde{h}(x)| - |h(y)|| < \delta$. By our choice of $\delta$ this is only possible if $X_{\varepsilon_0}$ also consists of $n$ disjoint components.

Now look at the $i$-th component $\hat{X}_i$ of $\tilde{X}^{\text{et}}$. Similar to the special case, choose $\tilde{g}_i \in \mathcal{O}_{\tilde{X}_i}(\tilde{X}_i)$ as a local parameter of $\tilde{x}_i'$. We can extend this function to $\tilde{X}^{\text{et}}$ by zero on the other components. Now lift $\tilde{g}_i$ to $g_i$ on $X_1$. As in the special case we see that, for $\varepsilon_i > \varepsilon_0$, we can chose this lift in $\mathcal{O}_X(X_{\varepsilon_i})$. As above, $g_i$ will, after adjusting $\varepsilon_i$, generate the algebra of $\{x \in X_1 : \varepsilon \leq |f(x)| < 1\}$. We get, as in the special case, that the $i$-th component $X_i$ of $X_{\varepsilon_0}$ will satisfy the theorem, e.g. that $\{x \in X_1 : \varepsilon \leq |f(x)| < 1\}$ is isomorphic to a semi-open annulus with local parameter $g_i$. Since $g_i$ vanishes on all the other components, choosing an $\varepsilon \geq \max_{i=0,...,n} \varepsilon_i$, we get that $\{x \in X^{\text{et}} : \varepsilon \leq |f(x)| < 1\}$ is isomorphic to a disjoint union of semi-open annuli.

Since the base change is isomorphic on the formal fiber the theorem is also proved in the general case.

For the additional assertion of the theorem, let $h \in \mathcal{O}_X(X_{\varepsilon}(\tilde{x}))$ satisfy $\tau = \text{ord}_{\tilde{x}_{i_0}}(\tilde{h}) < \infty$ for some index $i_0$. Since the base change does not change the fact that $\tilde{g}_{i_0}$ is a local parameter of the reduction of the annulus $R_{i_0}$ we have the equation

$$\tilde{h} = \tilde{u} \cdot \tilde{g}_{i_0}.$$

By lifting this equation we see that, up to a unit, $h|_{R_{i_0}}$ equals $\tilde{g}_{i_0}$, the pullback of the coordinate of $R_{i_0}$.

2.5. Formal blow-ups

In this section we want to define the concept of a formal blow-up.

**Definition 2.5.1.** Let $X$ be an analytic curve with reduction $\pi : X \to \tilde{X}$ corresponding to the admissible formal covering $\mathcal{U} = (U_i)_{i \in I}$ be a function such that $f$ has an isolated zero $\tilde{x}$ in $U_i$ so that $\tilde{x} \notin U_j$ for any $j \neq i$ and $\varepsilon \in K$ with $|\varepsilon| < 1$. The formal blow-up corresponding to $f$ and $\varepsilon$ is the admissible formal covering $\mathcal{U} \setminus U_i \cup \{U_i(\{f\} \leq \varepsilon), U_i(\{f\} \geq \varepsilon)\}$.

**Proposition 2.5.2.** Let $X$ be an analytic curve with formal covering $\mathcal{U}$. Let $\hat{\mathcal{U}}$ denote the formal blow up of $\mathcal{U}$ according to $U_i$, $f$ and $\varepsilon \in |K^\times|$ as defined above. Then $\hat{\mathcal{U}}$ is formal. The formal blow up induces a map $\hat{\varphi} : \tilde{X} \to \hat{\tilde{X}}$ which is an isomorphism on $\tilde{X} \setminus \{\tilde{x}\}$ and is the identity on $X$.

**Proof.** The covering $\hat{\mathcal{U}}$ is admissible because $U_i(\{f\} \leq \varepsilon)$ and $U_i(\{f\} \geq \varepsilon)$ are rational subdomains and as such admissible open. We have $|f| = \varepsilon$ on $U_i' := U_i(\{f\} \leq \varepsilon)$. This means that $f/\varepsilon$ is in $U_i(\{f\} \leq \varepsilon)$ and the reduction of $U_i(\{f\} = \varepsilon)$ equals to the set $\hat{U}_i''_{i,f/\varepsilon}$. Similarly $U_i(\{f\} = \varepsilon)$ reduces to $\hat{U}_i''_{i,\varepsilon}$ on $U_i'' := U_i(\{f\} \geq \varepsilon)$.

The identity on $X$ will map every point of $\hat{U}_i(|f| \leq \varepsilon)$ to the point $\tilde{x}$. On $\hat{U}_i(|f| \geq \varepsilon)$ every lift $y$ with $|f(y)| < \varepsilon$ of a point $\tilde{y}$ is in the formal fiber of $\tilde{x}$ and as such $\tilde{y}$ will be mapped on $\tilde{x}$. Every other point is left unchanged. \qed
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**Proposition 2.5.3.** Let $X$ be an analytic curve with formal covering $\mathcal{U}$. Let $\tilde{x} \in \tilde{X}$ be a point in the associated reduction. There is a formal covering $\hat{\mathcal{U}}$ which induces the same reduction $\tilde{X}$ such that $\tilde{x} \in \hat{U}_i$ for exactly one $i$.

**Proof.** For every $U_i$ with $\tilde{x} \in \hat{U}_i$ we can find an $\tilde{f}_i \in \mathcal{O}_\tilde{x}(\hat{U}_i)$ which does not vanish except in isolated points and that these points lie in another set $\hat{U}_j$ as well and $\tilde{f}_i(\tilde{x}) = 0$. Moreover we can assume that $\tilde{f}_i$ and $\tilde{f}_j$ have only $\tilde{x}$ as common zero on $\hat{U}_i \cap \hat{U}_j$. Replacing $U_i$ for all but one affected index with $U_i(|f_i| = 1)$ where $f_i$ is any lift of $f_i$ does not change the reduction associated to $\mathcal{U}$. \hfill \Box

**Definition 2.5.4.** Let $X$ be a proper analytic curve with fixed reduction $\tilde{X}$ and let $\tilde{x} \in \tilde{X}$ be a point. Let $f$ and $\varepsilon < 1$ be as in Theorem 2.4.1 so that $\{x \in X_+ (\tilde{x}) : \varepsilon \leq |f(x)| < 1\}$ consists of $n$ disjoint semi-open annuli. We define the proper curve $X^{\tilde{x}}$ by pasting $n$ discs $\mathbb{B}^1$ into the formal fiber $X_+ (\tilde{x})$ via these annuli.

To be more precise, let

$$\varphi_i : R_i \xrightarrow{-\rightarrow} \{z \in \mathbb{B}^1 : \varepsilon^{1/t_i} \leq |\zeta(z)| < 1\}$$

be the maps of Theorem 2.4.1 where $R_i$ is the $i$th connected component of $\{x \in X : \varepsilon \leq |f(x)| < 1\}$ and let $B_i = \mathbb{B}^1$ together with

$$\psi_i : B_i(|x| > \varepsilon^{1/t_i}) \xrightarrow{-\rightarrow} \{z \in \mathbb{B}^1 : \varepsilon^{1/t_i} \leq |\zeta(z)| < 1\} ; x \mapsto \varepsilon^{1/t_i}/x$$

Then $X^{\tilde{x}}$ is defined by $X_+ (\tilde{x}), B_1, \ldots, B_n$ and the gluing relations $\psi_i^{-1} \circ \varphi_i$.

**Proposition 2.5.5.** There is an canonical admissible formal covering $\hat{\mathcal{U}} = \hat{U}_i |_{i=0}^n$ of $X^{\tilde{x}}$. The associated reduction $X^{\tilde{x}}$ consists of $n$ components which are rational curves joined by a single singular point $\tilde{x}$ with the same type of singularity as $\tilde{x}$ in $\tilde{X}$.

**Proof.** By [BL85, Prop. 4.1.1] the covering consisting of $U_0 = X^{\tilde{x}} \setminus (B_1^+ \cup \cdots \cup B_n^+)$ and $U_i = B_i$ and $B_i^+$ as the open unit ball is a formal covering. The associated reduction has $n$ affine lines derived from the $B_i$ glued together with a single point $\tilde{x}$ with $X_+ (\tilde{x}) = X^{\tilde{x}}_+ (\tilde{x})$. So the type of singularity is the same according to Proposition 2.2.6. \hfill \Box

**Definition 2.5.6.** We define $g(\tilde{x}) = g(X^{\tilde{x}})$ to be the genus of $X^{\tilde{x}}$. We further set $n(\tilde{x})$ as the number of connected components of $X^{\tilde{x}} \setminus X_+ (\tilde{x})$.

**Lemma 2.5.7.** If $g(\tilde{x}) = 0$ and $n(\tilde{x}) = 1$ then $\tilde{x}$ is a regular point. If $g(\tilde{x}) = 0$ and $n(\tilde{x}) = 2$ then $\tilde{x}$ is an ordinary double point.

**Proof.** If $g(X^{\tilde{x}}) = 0$ we know that $X^{\tilde{x}}$ is a rational curve and as such isomorphic to $\mathbb{P}^1$. Therefore we know that $X_+ (\tilde{x})$ is isomorphic to $\mathbb{P}^1 \setminus \mathbb{B}^1$ if $n(\tilde{x}) = 1$ and $\mathbb{P}^1 \setminus (\mathbb{B}^1 \cup \mathbb{B}^1)$ if $n(\tilde{x}) = 2$. In the first case we get an open disc, in the second an open annulus, proving our assertion by Proposition 2.3.1 and Proposition 2.3.2. \hfill \Box

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2.6. The stable reduction theorem

**Definition 2.5.8.** Let $\tilde{X}$ be a projective connected curve over $k$ consisting of $n$ components. The cyclomatic number $z(\tilde{X})$ is the number

$$z(\tilde{X}) = \sum_{\tilde{x} \in \tilde{X}} (n(\tilde{x}) - 1) - n + 1$$

where $n(\tilde{x})$ is again the number of points lying over $\tilde{x}$ in the normalization.

*Note.* Since an ordinary double point has $n(\tilde{x}) = 2$, this definition coincides with the one we give in Chapter 4 where the cyclomatic number is defined as the number of double points minus the number of components plus one.

The main missing part of the semi-stable reduction theorem is the genus formula. We do not give a proof for this, and just state the results here.

**Theorem 2.5.9.** We get

$$g(X) = \sum_{i=1}^{n} g(\tilde{X}_i') + \sum_{\tilde{x} \in \tilde{X}} g(\tilde{x}) + z(\tilde{X})$$

**Proof.** [BL85, Section 4] □

With this formula, one can relate the genus of $X^\tilde{\pi}$ with the genus of points generated in a formal blow-up. If one chooses the function $f$ correctly, one get the following lemma, which guarantees, that the singularities will always get better by a formal blow-up.

**Lemma 2.5.10.** Let $K$ be an algebraically closed non-Archimedean field. Let $X$ be a projective analytic curve over $K$ with $g(X) \geq 1$. There is a meromorphic function $f$ and an $\varepsilon \in |K^\times|$ such that the formal covering given by $\{x \in X : |f(x)| \leq \varepsilon\}$ and $\{x \in X : |f(x)| \geq \varepsilon\}$ fulfills $g(\tilde{x}) < g(X)$ for all $\tilde{x} \in \tilde{X}$.

**Proof.** See [BL85, Lemma 7.2] □

2.6. The stable reduction theorem

**Definition 2.6.1.** A projective, connected, reduced curve $\tilde{X}/k$ that has only ordinary double points as singularities is called semi-stable. A semi-stable curve $\tilde{X}/k$ is stable if moreover every component that is isomorphic to $\mathbb{P}_k^1$ meets the rest of $\tilde{X}$ in at least three points and the arithmetic genus of $\tilde{X}$ is at least two.

**Theorem 2.6.2.** Let $X$ be a projective analytic curve with reduction $\tilde{X}$ over a field $K$ which is algebraically closed and complete. Then there is a reduction $\hat{\tilde{X}}$ generated from $\tilde{X}$ by a finite amount of formal blow-ups, that has semi-stable reduction.

**Proof.** Since $X_+(\tilde{x}) = X^\tilde{\pi}_+(\tilde{x})$ we can apply [2.5.10] to every point of $\hat{\tilde{X}}$ which is not regular or an ordinary $n$-fold point. By Induction it follows that every point is an ordinary $n$-fold point after a finite amount of formal blow-ups. Ordinary $n$-fold points can be broken down further by the methods of [BL85, Chapter 5] to get to ordinary double points which proves the assertion. □

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2. The structure of a formal analytic curve

**Proposition 2.6.3.** Let $X$ be a projective rigid analytic curve with reduction $\tilde{X}$. Let $\tilde{X}_1, \ldots, \tilde{X}_k$ be a selection of irreducible components of $\tilde{X}$ such that $\bigcup_{i=1}^k \tilde{X}_i \neq \tilde{X}$. There is a formal covering $\mathcal{U}$ of $X$ with associated reduction $\tilde{X}'$ and a finite number of point $\tilde{x}_1, \ldots, \tilde{x}_k$ such that $\tilde{X} \setminus \bigcup_{i=1}^k \tilde{X}_i = \tilde{X}' \setminus \{\tilde{x}_1, \ldots, \tilde{x}_k\}$.

**Proof.** According to Riemann–Roch there is a rational function $\tilde{f}$ such that $\tilde{f}(\tilde{x}) = 0$ if and only if $\tilde{x} \in \bigcup_{i=1}^k \tilde{X}_i$ which is not constant on every other irreducible component. A meromorphic lift $f$ of $\tilde{f}$ gives the formal covering consisting of $U_1 = \{x \in X : |f(x)| \leq 1\}$ and $U_2 = \{x \in X : |f(x)| \geq 1\}$. These sets are affinoid according to Proposition 1.10.4 and the covering is formal since the intersection $\{x \in X : |f(x)| = 1\}$ is an affinoid subvariety of both $U_1$ and $U_2$.

Take a point $\tilde{x} \in \tilde{X}_i$ with $i > k$ i.e. $\tilde{f}$ is not constant on $\tilde{X}_i$. We can assume without restriction that $\tilde{f}(\tilde{x}) = \tilde{c}$ for $c \in K$. The formal fiber of $\tilde{x}$ can be written as $X_+(\tilde{x}) = \{x \in U : |f(x) - c| < 1\}$ where $U$ is the preimage of an affine neighborhood $\tilde{U}$ of $\tilde{x}$ such that $\tilde{x}$ is the only point in $\tilde{U}$ with $\tilde{f}(\tilde{x}) = \tilde{c}$. But $\{x \in U : |f(x) - c| < 1\}$ is a formal fiber in $\{x \in X : |f(x)| = 1\}$ as well, so the identity on $X$ reduces to an isomorphism on these components. All the points of $\bigcup_{i=1}^k \tilde{X}_i$ will be mapped on a connected component of $\{x \in X : |f(x)| < 1\}$ which is a finite disjoint union of formal fibers in $\{x \in X : |f(x)| \leq 1\}$. So the components are mapped to a finite set of isolated points in $\tilde{X}'$. \qed

**Definition 2.6.4.** The reduction $\tilde{X}'$ of Proposition 2.6.3 is called the blow-down of the components $\tilde{X}_1, \ldots, \tilde{X}_k$ in $\tilde{X}$.

**Theorem 2.6.5.** Let $X$ be a projective analytic curve of genus at least two and with semi-stable reduction $\tilde{X}$, associated to the formal covering $\mathcal{U}$. There is a formal covering $\hat{\mathcal{U}}$ of $X$ inducing stable reduction.

**Proof.** Let $\hat{X}_i$ be a rational component of $\tilde{X}$ which intersects the rest of the curve in only two ordinary double points. Since the genus of $X$ is at least two, we know that $\hat{X}_i$ is not the only irreducible component of $\tilde{X}$. This means $\hat{X}_i \setminus \text{Sing} \tilde{X}$ is isomorphic to $\mathbb{P}^1$ missing two discs. But then $X_i := \pi^{-1}(\hat{X}_i \setminus \hat{X})$ is isomorphic to $K \langle \zeta, \zeta^{-1} \rangle$ with $\zeta$ defined on the formal fiber of the double points. In fact $\zeta$ is a coordinate of both these open annuli.

We blow down this component using Proposition 2.6.3. This maps the whole component to a single point $\hat{x} \in \hat{X}'$. The formal fiber of this point is the union of $\text{Sp} K \langle \zeta, \zeta^{-1} \rangle$ with two open annuli, each of which having $\zeta$ as coordinate. This results in an open annulus, again with $\zeta$ as coordinate, so $\hat{x}$ is an ordinary double point according to Proposition 2.3.2. Therefore $\hat{X}'$ is again semi-stable, but misses the component $\hat{X}_i$ of $\tilde{X}$. Repeating this process until all rational components with only two intersections with the rest of the curve are gone yields a stable reduction. \qed
2.7. Examples

Let us show the methods of this chapter in the example of hyperelliptic curves. Let \( X \) be defined by

\[
U_1 := \text{Sp} \, K \langle \xi, \eta \rangle / (\eta \xi^2 - \xi (\xi - \lambda_2) \ldots (\xi - \lambda_n))
\]
\[
U_2 := \text{Sp} \, K \langle \sigma, \tau \rangle / (\tau^2 - \sigma (1 - \sigma)(1 - \lambda_1 \sigma) \ldots (1 - \lambda_n \sigma))
\]

with \( n = 2g - 1 \) and gluing relations \( \xi = 1 / \sigma \) and \( \eta = \tau / \sigma^{n+1} \). We can assume without restriction that \( |\lambda_i| \leq 1 \) by application of an affine map. This covering is formal with associated reduction \( \tilde{X} \) given by

\[
\tilde{U}_1 := \text{Spec} \, k[X, Y] / (Y^2 - X(X - 1)(X - \tilde{\lambda}_1) \ldots (X - \tilde{\lambda}_n))
\]
\[
\tilde{U}_2 := \text{Spec} \, k[S, T] / (T^2 - S(1 - S)(1 - \tilde{\lambda}_1 S) \ldots (1 - \tilde{\lambda}_n S))
\]

and corresponding gluing relations. The curve \( \tilde{X} \) is smooth if \( \tilde{\lambda}_i \neq \tilde{\lambda}_j \) for \( i \neq j \) or equivalently if \( |\lambda_i - \lambda_j| = 1 \) for \( i \neq j \).

Otherwise we can move a singularity to the point zero so that we can assume without restriction that the function \( \tilde{f} = X \) has a single isolated zero in the offending singularity on \( \tilde{U}_1 \). We choose \( f = \xi \) as a lift. Assume that the \( \lambda_i \) are sorted by absolute value and that \( |\lambda_i| < 1 \) for \( i \leq k \) and \( |\lambda_i| = 1 \) for \( i > k \). Set

\[
\varepsilon = \max_{1 < i \leq k} |\lambda_i|.
\]

There is no branching point in \( \{ x \in U_1 : \varepsilon < |\xi(x)| < 1 \} \). So this set is composed of one annulus if \( k \) is odd and two disjoint annuli if \( k \) is even with a coordinate calculated by the covering morphism \( \eta \). Blowing \( U_1 \) up with parameters \( f \) and \( \varepsilon \) gives

\[
U_1' := \text{Sp} \, K \langle \xi, \eta, \zeta \rangle / (\xi - \varepsilon \xi, \eta \xi^2 - \xi (\xi - \lambda_2) \ldots (\xi - \lambda_n))
\]
\[
U_1'' := \text{Sp} \, K \langle \xi, \eta, \zeta \rangle / (\zeta \xi - \varepsilon, \eta \xi^2 - \xi (\xi - \lambda_2) \ldots (\xi - \lambda_n))
\]

After restructuring the ideals we get

\[
U_1' := \text{Sp} \, K \langle \xi, \eta', \zeta \rangle / (\xi - \varepsilon \xi, \eta' \xi^2 - \zeta (\zeta - \lambda_2 \varepsilon^{-1}) \ldots (\zeta - \lambda_k \varepsilon^{-1}) \cdot (\xi - \lambda_{k+1}) \ldots (\xi - \lambda_n))
\]
\[
U_1'' := \text{Sp} \, K \langle \xi, \eta'', \zeta \rangle / (\zeta \xi - \varepsilon, \eta'' \xi^2 - \zeta^j (1 - \lambda_2 \varepsilon^{-1} \zeta) \ldots (1 - \lambda_k \varepsilon^{-1} \zeta) \cdot (\xi - \lambda_{k+1}) \ldots (\xi - \lambda_n))
\]

where \( j = 0 \) if \( k \) is even and \( j = 1 \) otherwise. The new variable \( \eta' \) is just \( \eta \) modified by a constant and \( \eta'' \) is \( \eta / |\xi|^{k/2} \) also modified by a constant. This gives the reductions

\[
\tilde{U}_1' := \text{Spec} \, k[Y', Z] / \left( Y'^2 - Z (Z - \lambda_2 \varepsilon^{-1}) \ldots (Z - \lambda_k \varepsilon^{-1}) \right)
\]
\[
\tilde{U}_1'' := \text{Spec} \, k[X, Y'', Z] / \left( X Z, Y''^2 - Z (1 - \lambda_2 \varepsilon^{-1} Z) \ldots (1 - \lambda_k \varepsilon^{-1} Z) \cdot (X - \lambda_{k+1}) \ldots (X - \lambda_n) \right)
\]
2. The structure of a formal analytic curve

We see that $\tilde{U}_1''$ consists of two irreducible components, both hyperelliptic curves joint together in a single double point if $j = 1$ or two ordinary double points if $j = 0$. The variety $\tilde{U}_1'$ just closes the new curve.

The remaining singularities can be handled by the same process. Therefore the stable reduction of a hyperelliptic curve consists of hyperelliptic curves joined together by ordinary double points.
3. Group objects and Jacobians

In this chapter we want to give an overview of the theory of group objects and Jacobian varieties, which we will need in the next chapter.

3.1. Some definitions from category theory

Definition 3.1.1. A locally small category $\mathcal{C}$ is a category in which $\text{Hom}(X, Y)$ is a set for any object $X, Y \in \mathcal{C}$.

Definition 3.1.2. An object $I \in \mathcal{C}$ is called initial if the set $\text{Hom}(I, X)$ contains exactly one element for any object $X \in \mathcal{C}$. Dually an object $T$ is called terminal if $\text{Hom}(X, T)$ contains exactly one element. An object that is both initial and terminal is called zero object of $\mathcal{C}$.

Proposition 3.1.3. Initial, terminal and zero objects of a category are unique up to isomorphism.

The unique morphism in $\text{Hom}(I, X)$ and $\text{Hom}(X, T)$ is denoted by 0. If $\mathcal{C}$ has a zero object then $0 \in \text{Hom}(X, Y)$ denotes the unique morphism that factors through the zero object.

Definition 3.1.4. Let $\mathcal{C}$ be a category with a zero object. For any morphism $\varphi \in \text{Hom}(X, Y)$ we call $k \in \text{Hom}(K, X)$ the kernel of $\varphi$ if $\varphi \circ k = 0$ and if there is for any $t \in \text{Hom}(T, X)$ with $\varphi \circ t = 0$ a unique morphism $u \in \text{Hom}(T, K)$ such that $t = k \circ u$. In other words we have the universal property

$$
\begin{array}{ccc}
K & \xrightarrow{k} & X \\
\downarrow u & & \downarrow \varphi \\
T & \xrightarrow{t} & Y
\end{array}
$$

Dually we call $c \in \text{Hom}(Y, C)$ the cokernel of $\varphi$ if the following universal property

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow c & & \downarrow 0 \\
C & \xrightarrow{0} & T
\end{array}
$$

holds.
3. Group objects and Jacobians

**Definition 3.1.5.** Let $X, Y, S \in \mathcal{C}$ be objects with morphisms $\varphi \in \text{Hom}(X, S)$ and $\psi \in \text{Hom}(Y, S)$. The pullback or *fibered product* of $\varphi$ and $\psi$ is an object $P \in \mathcal{C}$ together with morphisms $p_1 \in \text{Hom}(P, X), p_2 \in \text{Hom}(P, Y)$ such that the following universal property holds.

![Diagram of pullback]

Dually an object $P$ with morphisms $i_1 \in \text{Hom}(X, P), i_2 \in \text{Hom}(Y, P)$ is called the pushout of $\varphi \in \text{Hom}(S, X), \psi \in \text{Hom}(S, Y)$ if it adheres to the universal property.

If $\varphi$ and $\psi$ are clear from the context we write $X \times_S Y$ for the pullback and $X \coprod_S Y$ for the pushout. The pullback relative to the terminal object is called the product of $X$ and $Y$ and the pushout relative to the initial object is called the coproduct of $X$ and $Y$. If $t_1$ and $t_2$ are the morphisms of the diagram we write $t_1 \times t_2: T \to X \times_S Y$ or $(t_1, t_2): X \coprod_S Y \to T$ for the respective unique morphism of the universal property.

If $x: T \to X$ and $y: T \to Y$ as well as $f: X \times Y \to Z$ are morphisms we write $f(x, y)$ for the morphism $f \circ (x \times y)$.

**Definition 3.1.6.** A locally small category $\mathcal{C}$ is called *additive* if it has a zero object, all finite products exist and $\text{Hom}(X, Y)$ has the structure of an abelian group, compatible with concatenation of arrows.

**Proposition 3.1.7.** An additive category has all finite coproducts and the coproduct $X \coprod Y$ is canonically isomorphic to $X \times Y$ for all objects $X$ and $Y$.

**Proof.** We only need to show that $X \times Y$ is the coproduct of $X$ and $Y$. We set $i_1 = \text{id}_X \times 0$ and $i_2 = 0 \times \text{id}_Y$. For any $t_1: X \to T, t_2: Y \to T$ we set

$$u = p_1 \circ t_1 + p_2 \circ t_2: X \times Y \to T.$$
3.2. Group objects

We then have

\[ i_j \circ u = i_j \circ (p_1 \circ t_1 + p_2 \circ t_2) = i_j \circ p_1 \circ t_1 + i_j \circ p_2 \circ t_2 = t_j \]

for \( j = 1, 2 \). Note that one needs the commutativity for \( u \) to be well defined and unique.

**Definition 3.1.8.** An additive category \( C \) is called pre-abelian if it contains every kernel and cokernel.

**Definition 3.1.9.** A pre-abelian category is called abelian if every monomorphism is the kernel and every epimorphism is the cokernel of some morphism.

### 3.2. Group objects

Let \( C \) be a locally small category with a terminal object 0 and all pullbacks. Let us also write 0 for the terminal morphism and \( p_i \) for the \( i \)th projection of the product. Likewise, we will omit \( \circ \) if no confusion is possible.

**Definition 3.2.1.** Let \( C \) be a category as above. We write \( P_C \) for the category with the objects \((X, e: 0 \to X)\) where \( X \) is an object in \( C \) and \( e \) is a section of 0 and the morphisms of \( C \) which respect \( e \), i.e.

\[ \text{Hom}_{P_C}(X,Y) = \{ \varphi \in \text{Hom}_C(X,Y) ; e_Y = \varphi \circ e_X \} . \]

**Remark.** The category \( P_C \) has 0 as the zero object and every morphism \( f: X \to Y \) has the kernel \( p_1: X \times Y 0 \to X \) as the universal property of the pullback is just the universal property of this kernel.

**Definition 3.2.2.** A group object \( G \) in \( C \) is an object \( G \) together with three morphisms \( e: 0 \to G, m: G \times G \to G, i: G \to G \) such that

1. \( m \circ ((m \circ p_{12}) \times p_3) = (m \circ (p_1 \times (m \circ p_{23}))) \circ \alpha \) with respect to the canonical isomorphism \( \alpha: (G \times G) \times G \to G \times (G \times G) \),
2. \( m \circ (i \circ (e \circ 0)) = m \circ ((e \circ 0) \times i) = i \circ (e \circ 0) \),
3. \( m \circ (i \times i) = m \circ (i \times i) = e \circ 0 \).

A group object is called commutative if moreover

4. \( m = m \circ \sigma \) holds for \( \sigma := p_2 \times p_1: G \times G \to G \times G \).

**Note.** To clarify notation, we will use \( f + g: T \to G \) or \( f \cdot g \) for \( m(f, g) \) where \( f, g: T \to G \) are morphisms. Likewise, we use \( -f \) or \( f^{-1} \) for \( i \circ f \) and 0 or 1 for the morphism \( e \circ 0 \). When dealing with two different group objects, we will use additive notation for one and multiplicative notation for the other to further clarify where the operation takes place. If the group object is commutative we use fractions for a compact notation of the inverse of the multiplicative notation.
3. Group objects and Jacobians

Note that groups are just the group objects of the category of sets, which we call abstract groups. Many elementary properties of abstract groups carry over to group objects. Since $e$ is a section of $0$, we know that $0$ is epic and can use the usual proofs to show that $e$ and $i$ are uniquely determined by the morphism $m$.

Group objects of a category together with the morphisms compatible with $m$, $i$ and $e$ form a category themselves which we denote by $\mathcal{G}_C$. We call these morphisms (group)homomorphisms and will specify the category if confusion is possible. As in the case of abstract groups the compatibility with $m$ induces those for $e$ and $i$ and we get

$$\text{Hom}_{\mathcal{G}_C}(X,Y) = \{ \varphi \in \text{Hom}_C(X,Y) : m_Y(\varphi, \varphi) = \varphi \circ m_X \}$$

One checks easily that this category has $0$ as zero object and the pullbacks $A \times_C B$ has canonical morphisms $m := m_A(p_{11}, p_{21}) \times m_B(p_{12}, p_{22})$, inducing $i$ and $e$ making it the pullback in $\mathcal{G}_C$.

Definition 3.2.3. We have a forgetful functor $\mathcal{G}_C \to \mathcal{G}_{rp}$ which associates the group of abstract points $\text{Hom}(0, G)$ to a group object $G$. For any $S \in \mathcal{C}$ we can equivalently define a forgetful functor which associates the group $\text{Hom}(S, G)$ to the $S$-valued points $G(S)$ of $G$.

The commutative group objects form a subcategory of $\mathcal{G}_C$ which we write as $\mathcal{Z}_C$. In this subcategory we see that $A \times B$ forms a coproduct of the group objects $A$ and $B$. For any object $T$ and a commutative group object $G$ in $\mathcal{C}$ we can induce an abelian group structure on $\text{Hom}(T, G)$, so $\mathcal{Z}_C$ is an additive category. Both $\mathcal{G}_C$ and $\mathcal{Z}_C$ are a subcategory of $\mathcal{P}_C$.

Lemma 3.2.4. Every morphism $\varphi : X \to Y$ in $\mathcal{G}_C$ has a kernel and this kernel is equal to the kernel of $\varphi$ in $\mathcal{P}_C$.

Proof. As said above $X \times Y 0$ has a group structure.

In many cases the category $\mathcal{Z}_C$ will have cokernels. Thereby $\mathcal{Z}_C$ is a pre-abelian category.

Definition 3.2.5. Assume that all cokernels in $\mathcal{Z}_C$ exist. For a homomorphism $\varphi$ we define the image of a morphism as $\text{im} \varphi := \ker \text{coker} \varphi$ and dually the coimage as $\text{coim} \varphi := \text{coker} \ker \varphi$.

We say that a series of morphisms $\varphi_i : G_i \to G_{i+1}$ forms an exact sequence if $\text{im} \varphi_i = \ker \varphi_{i+1}$. We say that an exact sequence is strict exact if the canonical morphism $\text{coim} \varphi_i \to \text{im} \varphi_i$ is an isomorphism for any $i$.

Note that the assertion of $\text{coim} \varphi$ being isomorphic to $\text{im} \varphi$ is automatically fulfilled in an abelian category. However, in the cases we are interested in, $\mathcal{Z}_C$ is not abelian.

Proposition 3.2.6. Let $\mathcal{Z}_C$ have all cokernels. For any morphism $\varphi$ we have $
ker \text{coker} \ker \varphi = \ker \varphi$ and $\text{coker} \ker \text{coker} \varphi = \text{coker} \varphi$. 


3.3. Central extensions of group objects

Proof. Look at the diagram

\[
\begin{array}{ccc}
K & \to & G \\
\downarrow x & & \downarrow ϕ \\
X & \to & T
\end{array}
\]

where \( k \) is the kernel of \( ϕ \) and \( c \) is the cokernel of \( k \), while \( x \) is arbitrary such that \( c \circ x = 0 \). Since \( c \) is the cokernel of \( k \) and \( ϕ \circ k = 0 \) there is a unique arrow \( u: C \to T \) such that \( ϕ = u \circ c \). So we get \( 0 = u \circ 0 = u \circ c \circ x = ϕ \circ x \). But \( k \) is the kernel of \( ϕ \), so there is a unique arrow \( v: X \to K \) making the diagram commutative which just means that \( k \) is the kernel of \( c \), as required.

The other part of the assumption follows by the dual diagram.

Therefore a short sequence is strict exact if the first morphism is the kernel of the second and the second morphism is the cokernel of the first.

Definition 3.2.7. Let \( 0 \to G \to E \to B \to 0 \) be a strict exact sequence in \( ZC \). We call \( E \) an extension of \( B \) by \( G \). Two extensions \( E \) and \( E' \) are isomorphic if there is an isomorphism \( f \) such that the diagram

\[
\begin{array}{ccc}
0 & \to & G \\
\downarrow \text{id} & & \downarrow f \\
0 & \to & G
\end{array}
\]

commutes. We denote by \( \text{Ext}(B, G) \) the class of extensions of \( B \) by \( G \) up to isomorphism.

3.3. Central extensions of group objects

We want to classify these group extensions. For abstract commutative groups this is done by using the notion of the central extension and group cohomology. The same process can, with some restrictions, be applied to commutative group objects which we will sketch in the next theorem.

For a strict exact sequence \( 0 \to G \to E \to B \to 0 \), we use multiplicative notation on \( G \) and additive notation on \( E \) and \( B \). Since all group objects are commutative, we use fractions for the inverse in multiplicative notation.

Definition 3.3.1. Let \( B \) and \( G \) be commutative group objects. We call the set of morphisms \( f \in \text{Hom}(B^2, G) \) satisfying

\[
f(y,z) \cdot f(x,y+z) = 1
\]

for the projections \( x, y, z \) of \( B \times B \times B \) the cocycles \( Z^2(B,G) \) of \( B \) with coefficients in \( G \). The coboundaries \( B^2(B,G) \) are morphisms given by

\[
f(x,y) = \frac{g(x) \cdot g(y)}{g(x+y)}
\]
where \( g: B \to G \) is an arbitrary morphism and \( x, y \) are the projections of \( B \times B \).

The second cohomology group of \( B \) with coefficients in \( G \) is given by

\[
H^2(B,G) = Z^2(B,G)/B^2(B,G)
\]

The restriction of \( Z^2(B,G) \) to symmetric morphisms \( f: B^2 \to G \) where \( f(x,y) = f(y,x) \) is denoted by \( Z^2(B,G)_s \). We set

\[
H^2(B,G)_s = Z^2(B,G)_s/B^2(B,G)
\]

**Theorem 3.3.2.** Let \( C \) be a category as in Section 3.2 such that \( Z_C \) has all cokernels. Let \( B \) and \( G \) be objects in \( Z_C \). The class of commutative extensions

\[
0 \to G \to E \to B \to 0
\]

of \( B \) by \( G \) up to isomorphism which admit a section of \( \rho \) in the category \( C \) is in one-to-one correspondence to the group \( H^2(B,G)_s \).

**Proof.** Take a symmetric cocycle \( f \in H^2(B,G)_s \). We can assume without restriction, that \( f(e_B,e_B) = e_G \) because we can otherwise modify \( f \) with the coboundary induced by \( g = f \circ \Delta \), where \( \Delta \) is the diagonal morphism. We endow \( E := G \times B \) with a group law by setting

\[
e := e_G \times e_B
\]

\[
m := (g_1 \cdot g_2 \cdot f(b_1,b_2)) \times (b_1 + b_2)
\]

\[
i := \left( \frac{1}{g \cdot f(b,-b)} \right) \times (-b)
\]

\[
\rho := p_2
\]

\[
\psi := \text{id}_G \times e_B
\]

\[
s := e_G \times \text{id}_B
\]

where \( g_1, g_2, b_1, b_2, g, b \) are the projections of \( (G \times B) \times (G \times B) \) and \( G \times B \) respectively.

Using the cocycle condition \([3.2]\) for \( \text{id}_B \times e_B \times e_B \) gives

\[
f(e_B,e_B) = f(e_B,b) = e_G
\]

so \( e \) is neutral and \( i \) is an inverse. The associativity follows directly by the cocycle condition \([3.2]\) and the commutativity is implied by \( f \) being symmetric.

One checks directly that \( \psi \) and \( \rho \) are indeed homomorphisms in \( Z_C \). Let \( X \in Z_C \) be any object and \( \varphi: E \to X \) a group homomorphism with \( \varphi \circ \psi = 0 \) so we get the following diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\psi} & E \\
\downarrow & & \downarrow \rho \\
0 & & B \\
\end{array}
\]

where \( \varphi \) is the identity on \( X \).
3.3. Central extensions of group objects

We set \( u = \varphi \circ s \). Let \( b_1 \) and \( b_2 \) be the projections of \( B \times B \) then

\[
\begin{align*}
 u \circ b_1 + u \circ b_2 &= \varphi \circ s \circ b_1 + \varphi \circ s \circ b_2 \\
 &= \varphi \circ (s \circ b_1 + s \circ b_2) \\
 &= \varphi \circ (1 \times b_1 + 1 \times b_2) \\
 &= \varphi \circ (f(b_1, b_2) \times (b_1 + b_2)) .
\end{align*}
\]

We know that \( \varphi \circ \psi \circ (f(b_1, b_2))^{-1} = 0 \) so we get

\[
\varphi \circ (f(b_1, b_2) \times (b_1 + b_2)) = \varphi \circ (1 \times (b_1 + b_2)) = \varphi \circ s \circ (b_1 + b_2) = u(b_1 + b_2) ,
\]

which means that \( u \) is a homomorphism, which is automatically unique since \( \rho \) is epic. By the same argument we get

\[
\varphi = \varphi(p_1, \rho) = \varphi(1, \rho) = \varphi \circ s \circ \rho = u \circ \rho .
\]

We also have

\[
\ker \rho = E \times_B 0 = (G \times B) \times_B 0 ,
\]

which is isomorphic to \( G \) via \( \psi \) by the universal properties of product and pullback in \( C \). Since the kernels in \( P_\mathcal{C} \) and \( Z_\mathcal{C} \) agree, we know that \( \psi = \ker \rho \). So \( \psi \) and \( \rho \) form a strict exact sequence.

On the other hand, for a given strict exact sequence

\[
\xymatrix{ 0 \ar[r] & G \ar[r]^{\psi} & E \ar[r]_{\rho}^{\circ} & B \ar[r]_{s} & 0 }
\]

with a section \( s \) of \( \rho \) we can assume without restriction that \( s \circ e_B = e_E \), since otherwise we can replace \( s \) by \( -s \circ e_B \).

We define \( r' : E \to E ; \ r' = \text{id}_E - s \circ \rho \), so \( \rho \circ r' = e_B \) which implies that there is a morphism \( r : E \to G ; \ \psi \circ r = r' \) since \( \psi = \ker \rho \) in \( P_\mathcal{C} \) as well. But \( r' \circ \psi = \psi \), so \( r \) is a retraction of \( \psi \).

Thus we get an isomorphism \( \varphi : G \times B \to E \) by \( \varphi = \psi \circ g + s \circ b \) and \( \varphi^{-1} = r \times \rho \). We set \( f' := s \circ b_1 + s \circ b_2 - s \circ (b_1 + b_2) \). Then \( \rho \circ f' = e_B \) so there is \( f \) such that \( f' = \psi \circ f \). We can define a group law on \( G \times B \) via \( \varphi \) as

\[
m : (G \times B) \times (G \times B) \to G \times B
\]

\[
m = \varphi^{-1} \circ m_E \circ (\varphi \circ p_1 \times \varphi \circ p_2)
\]

\[
= \varphi^{-1} \circ (\psi \circ g_1 + s \circ b_1 + \psi \circ g_2 + s \circ b_2)
\]

\[
= (g_1 \cdot g_2 \cdot f(b_1, b_2)) \times (b_1 + b_2) .
\]

For another section \( s' \) of \( \rho \) we can define \( g : B \to G \) such that \( \psi \circ g = s' - s \) since \( \rho \circ (s' - s) = e_B \). The cocycles corresponding to \( s \) and \( s' \) just differ by the coboundry induced by \( g \). On the other hand any morphism \( g : B \to G \) can be induced this way by setting \( s' = s + \psi \circ g \). By the definition of isomorphy of extensions, this shows that two extensions are isomorphic if and only if they induce the same cohomology class in \( H^2(B, G)_s \).
Definition 3.3.3. Let $\alpha: G \to G'$ be a homomorphism in $Z_C$ and let

\[
1 \to G \xrightarrow{\psi} E \xrightarrow{\rho} B \to 1
\]

be a strict exact sequence admitting a section. Then the extension $\alpha_*E \in \text{Ext}(B, G')$ corresponding to $\alpha_*f := \alpha \circ f$, where $f$ is the cocycle describing $E$, is called the pushout of $\alpha$.

Definition 3.3.5. Let $\gamma: B' \to B$ be a homomorphism in $Z_C$ and

\[
1 \to G \xrightarrow{\psi} E \xrightarrow{\rho} B \to 1
\]

be a strict exact sequence admitting a section. The extension $\gamma_*E$ of $B'$ by $G$ given through the cocycle $\gamma^*f := f \circ (\gamma \circ p_1) \times (\gamma \circ p_2)$ where $f$ is the cocycle corresponding to $E$ is called the pullback of $\gamma$.

Proposition 3.3.4. The pushout of a homomorphism $\alpha$ is the categorical theoretic pushout of $\alpha$ and $\psi$. It is the unique extension of $B$ by $G'$ such that a homomorphism $A$ exists such that the diagram

\[
1 \to G \xrightarrow{\psi} E \xrightarrow{id} B \to 1
\]

commutes.

Proof. The pushout of $\alpha$ and $\psi$ is given by the cokernel of $-\alpha \times \psi$. This cokernel is just $p_1 + \alpha \circ r \times \rho \circ p_2$. We set $A := \alpha \circ p_1 \times p_2$ which makes the diagram commutative. The uniqueness of $E'$ follows by the uniqueness of the categorical pushout.

Definition 3.3.6. Let $\gamma: B' \to B$ be a homomorphism in $Z_C$ and

\[
1 \to G \xrightarrow{\psi} E \xrightarrow{\rho} B \to 1
\]

as underlying object for the pullback in $Z_C$. Using this canonical isomorphism to induce a group law produces the given central extension.

Proposition 3.3.5. The pullback of a homomorphism $\gamma$ is the categorical theoretic pullback of $\gamma$ and $\rho$. It is the unique extension of $B'$ by $G$ such that a homomorphism $\Gamma$ exists such that the diagram

\[
1 \to G \xrightarrow{id} E' \xrightarrow{\gamma} B' \to 1
\]

commutes.

Proof. Since the categorical pullbacks of $Z_C$ and $P_C$ coincide we get

\[E' = (G \times B) \times_B B' \to G \times B'\]

as underlying object for the pullback in $Z_C$. Using this canonical isomorphism to induce a group law produces the given central extension.
3.4. Algebraic and formal analytic groups

We are interested in group objects of the category of algebraic varieties, i.e. separated, connected and integral schemes of finite type over an algebraically closed field which we call algebraic groups and the group objects of the category of connected, quasi-separated, quasi-paracompact rigid $K$-spaces, which we will call analytic groups. We are especially interested in analytic groups with a fixed reduction $\pi: X \to \tilde{X}$, where $\tilde{X}$ is an algebraic group, on which the group laws are given as the reduction of the group laws on $X$, in other words a formal group scheme over $R$.

In the rest of this section we will collect some statements about algebraic groups and look to generalize them to their analytic counterparts.

Proposition 3.4.1. Algebraic and analytic groups are non-singular.

Proof. Let $G$ be the group in question and $\tau_x: G \to G$ be the left translation by a closed point $x \in G$. This induces an isomorphism of the tangent space of $G$ at 0 and at $x$. Since we defined algebraic and analytic groups as reduced schemes this proves the assumption. \qed

Proposition 3.4.2. Let $G$ and $H$ be algebraic groups and $U \subset G$ a non empty open subvariety. Let $\varphi: U \to H$ be a morphism compatible with the group law. Then there is a unique group homomorphism $\hat{\varphi}: G \to H$ which restricts to $\varphi$.

Proof. Let $x \in G$ be any closed point. Then $U - x$ is isomorphic to $U$ and open. $G$ is integral and therefore irreducible, so $U \cap (x - U)$ is non empty. Take $a = x - b$ from that intersection and set $\hat{\varphi}(x) = \varphi(a) + \varphi(b)$. Since $\varphi$ is compatible with the group law the choice of $a$ does not matter and $\hat{\varphi}$ restricts to $\varphi$. Furthermore for any fixed $a \in U$ we can set $\varphi_a: U + a \to H$ by $\varphi_a(x) = \varphi(x - a) + \varphi(a)$ and obtain a morphism that coincides with $\varphi$ on the common intersection. So $\hat{\varphi}$ is a morphism of algebraic varieties and respects the group law. It is unique since rational functions are uniquely determined by an open subvariety. \qed

Corollary 3.4.3. Let $G$ be a formal group scheme and $H$ be any analytic group. Let $U \subset G$ be a non empty formal open subset of $G$ and $\varphi: U \to H$ be a morphism which respects the group law, i.e. $\varphi(u_1 + u_2) = \varphi(u_1) + \varphi(u_2)$ for $u_1, u_2, u_1 + u_2 \in U$. Then there is a unique group homomorphism $\hat{\varphi}: G \to H$ which restricts to $\varphi$.

Proof. Take any point $x \in G$. We’ll get again an isomorphy between $U$ and $x - U$ and both are formal open subsets. Since $G$ has an irreducible reduction, we know that $U \cap (x - U)$ is not empty, so we can define $\hat{\varphi}(x)$ as above. We can again prove locally that $\hat{\varphi}$ is a morphism and obtain uniqueness since a morphism is defined by its values on any non empty formal open subset. \qed

The last corollary suggests the definition of a formal rational function as follows.
3. Group objects and Jacobians

Definition 3.4.4. A formal rational function on a smooth formal connected scheme $X$ over $R$ is a pair $(f, U)$ where $U$ is a non-empty admissible formal open subset of $X$ and $f \in \mathcal{O}_X(U)$. Two rational functions $(f, U)$ and $(g, V)$ are called equal if the restrictions of $f$ and $g$ to $U \cap V$ are equal.

Proposition 3.4.5. Let $(f, U)$ be a formal rational function on $X$ and let $V \supset U$ be another admissible formal open subset of $X$. If there is an extension of $f$ on $V$, it is unique.

Proof. We can assume that $U$ is a rational subdomain of an affinoid set $V$. Then the proposition follows according to the identity theorem of power series.

Remark. One should note that meromorphic functions are not equivalent to rational functions as one might expect. Just take $\mathbb{P}^1_K$ as $U_1 = \text{Sp} K(\zeta)$ and $U_2 = \text{Sp} K(\eta)$ glued on the subset $\text{Sp} K(\zeta, \eta)/(\zeta \eta - 1)$. A meromorphic function without poles can be written as a power series on both $U_1$ and $U_2$, which implies that it must be constant. So a meromorphic function on $\mathbb{P}^1_K$ is always the quotient of two polynomials in $\zeta$ as one might expect. But $K(\zeta)$ contains power series that are not the quotient of two polynomials, for example if $|a| > 1$ then $1/(\zeta - a)$ has a square root in $K(\zeta)$ as simple calculations show.

Proposition 3.4.6. A short exact sequence of commutative algebraic or formal analytic groups

\[ 0 \to G \xrightarrow{\psi} E \xrightarrow{\rho} B \to 0 \]

is strict exact if and only if the induced sequence on the tangent spaces

\[ 0 \to T_G \xrightarrow{d\psi} T_E \xrightarrow{d\rho} T_B \to 0 \]

is also exact.

Proof. As in [Sect. III, §3 Cor. 2 and 3] the condition on the tangent spaces implies that $G$ is a closed subvariety of $E$ and that $B$ is the geometric quotient $E/G$. For any object $X$ in $\mathcal{Z}_C$ we get the exact sequences

\[ 0 \to \text{Hom}(B, X) \to \text{Hom}(E, X) \to \text{Hom}(G, X) \]
\[ 0 \to \text{Hom}(X, G) \to \text{Hom}(X, E) \to \text{Hom}(X, B) \]

which implies that $\rho$ is the cokernel of $\psi$ and $\psi$ is the kernel of $\rho$. So the sequence is strict exact.

Starting with a strict exact sequence we also get

\[ 0 \to \text{Hom}(X, G) \to \text{Hom}(X, E) \to \text{Hom}(X, B) \]

for any $X$ in $\mathcal{Z}_C$. But since kernels in $\mathcal{Z}_C$ and $\mathcal{P}_C$ are equal, the same sequence is exact for $\mathcal{P}_C$. We can expand $C$ to include non-reduced varieties. Using $X = \text{Spec} k[\varepsilon]$ gives the desired sequence on the tangent spaces, with the surjectivity of the last morphism implied by dimension.

Remark. The condition on the tangent spaces make $\psi$ a reduced closed immersion and $\rho$ a smooth morphism.
3.4. Algebraic and formal analytic groups

We know how to describe the extensions of algebraic/analytic groups if the strict exact sequence $0 \to G \xrightarrow{\psi} E \xrightarrow{\rho} B \to 0$ admits a regular section. We put $H^2_{\text{reg}}(B,G)_s$ for these extensions.

If $\rho$ only admits a rational section, we end up with a “birational” group $E$ with a group law only defined on a non-empty open subset. There is always a unique algebraic group birationally equivalent to a given birational group as seen in [Ser88, VII, §1 Prop. 4]. So we still end up with a unique extension this way. We put $H^2_{\text{rat}}(B,G)_s$ for this subgroup of $\text{Ext}(B,G)$. Since the extension is constructed by gluing techniques the same works for the formal analytic case. Note that a rational section of a morphism of formal analytic groups is a morphism defined on a formal open subset and not necessarily meromorphic. We will construct this group explicitly for the case that $G$ is a linear group in Proposition 3.5.3.

**Proposition 3.4.7.** Let

$$0 \to G \to E \xrightarrow{\rho} B \to 0$$

be a strict exact sequence of formal groups schemes over $R$ with an algebraically closed field of fractions with the reduction

$$0 \to \tilde{G} \to \tilde{E} \to \tilde{B} \to 0$$

of algebraic groups. Then the sequence has a rational section if and only if the reduced sequence has a rational section.

**Proof.** As any section of the analytic sequence reduces to a section of the reduction, this part of the proof is trivial.

On the other hand, since we are talking about rational sections, we can assume $E$ and $B$ to be affinoid. Then the proof is given by Proposition 1.9.7. □

**Proposition 3.4.8.** Let $1 \to G \to E \xrightarrow{\rho} B \to 1$ be a strict exact sequence of connected commutative algebraic groups, where $G$ is linear. Then the sequence has a rational section.

**Proof.** We will sketch the proof as it is done in [Ser88 VII, §1 Prop. 6].

Let $x$ be any point of $B$. Then $E_x := \rho^{-1}(x) = E \times_B \text{Spec} \ k(x)$ is a principal homogeneous space for $G$ over $k(x)$, i.e. $G$ acts transitively on $E_x$. The action and the induced division map are both regular.

A rational section of $\rho$ is the same as a $k(x)$ rational point in $E_x$, where $x$ is the generic point of $B$, in other words $E_x$ is isomorphic to $G$ and the trivial homogeneous space.

By a result of Lang and Tate in [LT58 Prop. 4] the isomorphism classes of principal homogeneous spaces over an arbitrary field $k$ are in one-to-one correspondence to $H^1(g_s, G)$, where $g_s$ is the Galois group of $k_s/k$ with the topology induced by étale extensions, where $k_s$ is the separable closure of $k$ acting on $G$ by action on the coordinates of the points.
3. Group objects and Jacobians

The group $H^1(\mathfrak{g}_s, G)$ is trivial for a commutative linear group $G$. We will only show this for the case $G = \mathbb{G}_{m,K}$ since this is the case we are most interested in. The general case can be proven by showing the result for $\mathbb{G}_{a,K}$ and combining these two cases to form the arbitrary linear group $G$ as seen in [Ser88, VII, §1 Prop. 6].

So for the case $G = \mathbb{G}_{m,K}$ take a finite extension $K/k$ with Galois group $A$. A cocycle in $H^1(A, \mathbb{G}_{m,K})$ is a map $f : A \to \mathbb{G}_{m,K}$ satisfying $f(\sigma \circ \tau) = f(\sigma) \cdot \sigma(f(\tau))$ and the coboundary are the maps $b_\alpha : A \to \mathbb{G}_{m,K}$ $\sigma \mapsto \sigma(\alpha)/\alpha$.

Now for a fixed automorphism $\sigma$ we can calculate the norm of $f(\sigma)$ in the extension induced by $\langle \sigma \rangle$ by the cocycle condition to be

$$N^K_{K(\sigma)}(f(\sigma)) = \prod_{i=0}^{r-1} \sigma^i(f(\sigma)) = \prod_{i=0}^{r-1} \frac{f(\sigma^{i+1})}{f(\sigma^i)} = 1 \ .$$

So according to Hilbert 90 there is an $\alpha \in K$ with $f(\sigma) = \sigma(\alpha)/\alpha$ i.e. $f$ is a coboundary.

So since there is only one principal homogeneous space of $G$ over $k(x)$ we conclude that $E_x$ is isomorphic to $G$ and therefore has a $k(x)$-rational point leading to the needed section.

3.5. Extensions by tori

There is a different way to interpret $H^2_{rat}(B, G)_s$. A rational section of $B$ to $E$ can be translated by the multiplication on $B$ and yields a family of (admissible) open sets $U_i$ on $B$, each with a section $s_i$. Together with the morphism $\psi : G \to E$ of the exact sequence this gives open embeddings $s_i \circ p_1 \times \psi \circ p_2 : U_i \times G \to E$, making $E$ a fiber space of $B$ with the fiber $G$. So if $G_B$ is the sheaf of germs of regular functions from $B$ to $G$, we see that there is a map

$$\pi : H^2_{rat}(B, G)_s \to H^1(B, G_B) \ .$$

To make things more explicit let $f$ correspond to the extension $E$ with the rational section $s : U \to E$ where $U$ is an open subset of $B$. Recall that then

$$\psi \circ f(b_1, b_2) = s(b_1) + s(b_2) - s(b_1 + b_2)$$

for any $b_1, b_2 \in U$ with $b_1 + b_2 \in U$.

The section $s$ gives an isomorphism $\varphi_0$ between a (formal) open subset of $E$ and $U \times G$. Covering $B$ with the sets $b + U$ for $b \in U$ let us calculate

$$s_b : b + U \to B ; \ b + u \mapsto s(b) + s(u)$$

as the translated sections with $\varphi_b$ as corresponding isomorphisms.

So for $b_1, b_2 \in U$ we get for

$$\varphi^{-1}_{b_1} \circ \varphi_{b_2} : [(b_2 + U) \cap (b_1 + U)] \times G \to [(b_1 + U) \cap (b_2 + U)] \times G$$

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the map defined by
\[ \varphi_{b_1}^{-1} \circ \varphi_{b_2}(u, g) = \varphi_{b_1}^{-1}(\psi(g) + s(b_2) + s(u - b_2)) \]
\[ = \varphi_{b_1}^{-1}(\psi(g) + s(b_1) + s(u - b_1)) \]
\[ - s(b_1) - s(u - b_1) + s(u) - s(u) + s(b_2) + s(u - b_2)) \]
\[ = \varphi_{b_1}^{-1}(\psi(g) + s_{b_1}(u) - \psi \circ f(b_1, u - b_1) + \psi \circ f(b_2, u - b_2)) \]
\[ = \left( u, g \cdot \frac{f(u - b_2, b_2)}{f(u - b_1, b_1)} \right) \]
So we see that
\[ \pi(f) = \left( \frac{f(u - b_2, b_2)}{f(u - b_1, b_1)} \right)_{b_i \in U} \]
is the image of a cocycle.

**Proposition 3.5.1.** Let \( 0 \to G \to E \to B \to 0 \) be a strict exact sequence of formal analytic or algebraic groups. The map \( \pi: H^2_{rat}(B, G)_s \to H^1(C, \mathcal{G}_B) \) is a group homomorphism which commutes with pullbacks and pushouts.

**Proof.** As already noted we get \( \pi(f) = (f(u - b_2, b_2)/f(u - b_1, b_1))_{b_i \in U} \) as the image of a cocycle. So we see directly that \( \pi(f \cdot g) = \pi(f) \cdot \pi(g) \) after choosing a common open set \( U \) so \( \pi \) is a group homomorphism.

With that we can directly calculate for any group homomorphism \( \alpha: G \to G' \) that
\[ \pi(\alpha \circ f)_{b_1, b_2} = \alpha \circ f(u - b_2, b_2) \]
\[ = \alpha \circ (f(u - b_2, b_2)/f(u - b_1, b_1)) \]
\[ = \alpha \circ \pi(f)_{b_1, b_2} \]
holds and that for any group homomorphism \( \gamma: B' \to B \)
\[ \pi(\gamma \circ f)_{\gamma(b_1), \gamma(b_2)} = f(\gamma(u - b_2), \gamma(b_2)) \]
\[ = f(\gamma(u - b_1), \gamma(b_1)) \]
\[ = f(\gamma(u) - \gamma(b_1), \gamma(b_1)) \]
\[ = \gamma \ast \pi(f)_{\gamma(b_1), \gamma(b_2)} \]
holds as well, which concludes the proof. \( \square \)

**Proposition 3.5.2.** Let \( G \) be a commutative linear and \( B \) be a commutative projective algebraic or formal analytic group. Then the map \( \pi: H^2_{rat}(B, G)_s \to H^1(B, \mathcal{G}_B) \) is injective.

**Proof.** Let again be \( E \) the extension corresponding to \( f \in H^2_{rat}(B, G)_s \) and \( s: U \to E \) the rational section of \( \rho \), defined on the open subset \( U \) of \( B \). If \( f \) is in \( \ker \pi \) then \( f(x - b, b) = s(x - b) + s(b) - s(x) \) is regular on all of \( U \times U \) after the change by a coboundary. But then \( s \) is a regular section which in turn implies that \( f \) is regular on \( B \times B \) and since \( B \) is projective and \( G \) is affine, \( f \) is necessarily constant and thus induced by the constant coboundary. \( \square \)
3. Group objects and Jacobians

**Proposition 3.5.3.** Under the assumptions of Proposition 3.5.2, the image of the map \( \pi : H^2_{\text{rat}}(B,G)_s \to H^1(B,\mathcal{G}_B) \) consists of all elements \( x \in H^1(B,\mathcal{G}_B) \) for which \( m^*_B x = p_1^* x \cdot p_2^* x \) holds on \( B \times B \).

**Proof.** First we note that for any \( f \in H^2_{\text{rat}}(B,G)_s \) we get

\[ m^*_B \pi(f) = \pi(m^*_B f) = \pi(p_1^* f \cdot p_2^* f) = p_1^* \pi(f) \cdot p_2^* \pi(f) \]

according to Lemma 3.5.1 and the coboundaries of \( H^2(B \times B, G) \).

On the other hand, if \( (U_i)_{i \in I} \) is an open covering for which \( (g_{ij}) \) represents an element in \( H^1(B,\mathcal{G}_B) \) we can reformulate the stated property, as

\[ \frac{g_{ij}(x+y)}{g_{ij}(x)g_{ij}(y)} = \frac{f_j(x,y)}{f_i(x,y)} \]

where \( (f_i) \) is in \( H^0(B \times B, \mathcal{G}_B^{\times B}) \). We want to show that any \( f_i \) represents a preimage of \( (g_{ij}) \).

By the definition of \( H^2_{\text{rat}}(B,G)_s \), all of the \( (f_i) \) differ only by a coboundary. For any \( i \) the \( f_i \) also fulfills the cocycle condition, since

\[ h(x,y,z) := \frac{f_i(y,z)f_i(x,y+z)}{f_i(x+y,z)f_i(x,y)} \]

defines the same function for all \( i \in I \) and is thus constant as a regular function from the projective variety \( B \times B \times B \) to the affine variety \( G \) and \( h(0,0,0) = 1 \).

This map is obviously a group homomorphism. It is also injective. Suppose we have \( (g_{ij}) \) in the kernel i.e.

\[ \frac{g_{ij}(x+y)}{g_{ij}(x)g_{ij}(y)} = \frac{h_j(x+y)h_i(x)h_i(y)}{h_i(x+y)h_j(x)h_j(y)} \]

Changing \( (g_{ij}) \) by the coboundary \( (h_j/h_i) \) gives

\[ g_{ij}(x+y) = g_{ij}(x) \cdot g_{ij}(y) \]

So \( g_{ij} \) is a group homomorphism from \( B \) to \( G \) by Proposition 3.4.2. Therefore \( g_{ij} \) is constant as a map from a projective to an affine space and \( (g_{ij}) \) is the trivial bundle.

For \( f \in H^2_{\text{rat}}(B,G)_s \) this trivialization for \( \pi(f) \) is given by

\[ \frac{\pi(f)_{b_i,b_j}(x+y)}{\pi(f)_{b_i,b_j}(x) \cdot \pi(f)_{b_i,b_j}(y)} = \frac{f(x+y-b_j,b_j) \cdot f(x-b_i,b_i) \cdot f(y-b_i,b_i)}{f(x+y-b_i,b_i) \cdot f(x-b_j,b_j) \cdot f(y-b_j,b_j)} \]

\[ = \frac{f(x,y) \cdot f(x-b_i,y) \cdot f(y-b_i,b_i)}{f(x-b_j,y) \cdot f(y-b_j,b_j) \cdot f(x,y)} \]

\[ = \frac{f(x-b_i,y-b_i) \cdot f(x+y-2b_i,b_i)}{f(x-b_j,y-b_j) \cdot f(x+y-2b_j,b_j)} \]

which for \( b_i = 0 \) recovers the cocycle \( f \). \( \square \)
3.5. Extensions by tori

Remark. One can even construct an explicit extension of $B$ by $G$ this way. Take a principal fiber space $L$ of $x$. Then $m_B^t x = p_1^t x + p_2^t x$ gives a regular function $g: L \times L \to L$ compatible with the multiplication on $B$. One can use this function $g$ to define a group law on $L$, see [Ser88, pg. 182 et seq.] for details. Since the construction works as well for analytic groups, this shows that all rational cocycles indeed come from an extension.

Proposition 3.5.4. Let $B$ be an abelian variety or the generic fiber of a proper formal group scheme. Then an extension of $B$ by $\mathbb{G}_{m,K}$ is equal to a point of $\operatorname{Pic}^r(B)$, the dual abelian variety of $B$.

Proof. We have already seen in Proposition 3.5.2 that the morphism
$$\pi: H^2_{\text{rat}}(B, \mathbb{G}_{m,K}) \to H^1(B, \mathcal{O}_B^*)$$
is injective and that its image consists of the line bundles $\mathcal{L} \in H^1(B, \mathcal{O}_B^*)$ which fulfill $m_B^t \mathcal{L} = p_1^t \mathcal{L} \otimes p_2^t \mathcal{L}$ which by the Theorem of the square as in [Mum08, II. 8./III. 13.] happens if and only if $\mathcal{L} \in \operatorname{Pic}^r(B)$.

Proposition 3.5.5. Let $B$ be an abelian variety or the generic fiber of a proper formal group scheme and $t \in \mathbb{N}$. Then an extension of $B$ by $\mathbb{G}_{m,K}^t$ is equal to a point of $\operatorname{Pic}^r(B)^t$.

Proof. This follows from the previous proposition and the fact that the cohomology group $H^1(B, \mathcal{O}_B^*)^t$ equals $H^1(B, \mathcal{O}_B^*)^t$ by definition.

We can formulate the last proposition without introducing coordinates.

Corollary 3.5.6. Let $B$ be a proper formal group scheme. Then an extension of the generic fiber of $B$ by a torus $T$ is equal to a group homomorphism $\operatorname{Hom}(T, \mathbb{G}_{m,K}) \to \operatorname{Pic}^r(B)$ of the character group of $T$ to the translation invariant line bundles over $B$.

Proof. We can choose coordinates on $T$ so that we get $T \to \mathbb{G}_{m,K}^t$. An extension $1 \to \mathbb{G}_{m,K}^t \to E \to B \to 1$ gives rise to the map
$$\varphi: \operatorname{Hom}(\mathbb{G}_{m,K}^t, \mathbb{G}_{m,K}) \to \operatorname{Pic}^r(B)$$via $\alpha \mapsto \alpha_* E$ which is a total fiber space of a line bundle in $\operatorname{Pic}^r(B)$. By the definition of the pushout, this is a group homomorphism. On the other hand any such group homomorphism is determined by the images of the projections which give the point of $\operatorname{Pic}^r(B)^t$ of Proposition 3.5.5.

Corollary 3.5.7. Let $0 \to T \to E \to B \to 0$ be an extension of the generic fiber of a proper formal group scheme by a torus $T$. An $S$-valued point $\sigma: S \to E$ is equivalent to a family of $S$-valued points $\sigma_\chi: S \to \mathcal{P}_{B \times \phi(\chi)}$ with $\sigma_{\chi_1 + \chi_2} = \sigma_{\chi_1} \otimes \sigma_{\chi_2}$ and where $\phi$ is the homomorphism of the last corollary and $\mathcal{P}_{B \times B'}$ is the Poincaré bundle over $B$.

In other words we can write $E = \prod_{i=1}^n \mathcal{P}_{B \times \phi(c_i')}$ where $c_i'$ is a basis of the character group and a point $t = (t_1, \ldots, t_n) \in \tilde{E}$ corresponds to the family $t_\chi = t_1^m_1 c_1' \otimes \cdots \otimes t_n^m_n c_n' \in \mathcal{P}_{B \times \phi(\chi)}$ where $\chi = m_1 c_1' + \cdots + m_n c_n'$ with $m_i \in \mathbb{Z}$.
3. Group objects and Jacobians

Note. If one does not care about the embedding $\psi: \mathbb{G}_{m,K} \to E$ of $\mathbb{G}_{m,K}$ in the extension $E$, one needs to divide by the automorphism group of $\mathbb{G}_{m,K}$. Therefore two line bundles $L$ and $L^{-1}$ yield the same extension up to $\psi$.

**Definition 3.5.8.** Let $1 \to \bar{\mathbb{G}}_{m,K} \to \bar{E} \to B \to 1$ be an extension of formal group schemes. Denote by $1 \to \mathbb{G}_{m,K} \to E \to B \to 1$ the pushout given by the map $\bar{\mathbb{G}}_{m,K} \to \mathbb{G}_{m,K}$ on the generic fibers. We define $|·|: E \to \mathbb{R}^*_0$ by $x \mapsto |a|$ with $a \in \mathbb{G}_{m,K}$ and $x \cdot a^{-1} \in \bar{E}$. A lattice $M$ in $E$ is an analytic subgroup of $E$ so that $|·|_M$ is injective and $-\log|M|$ is a lattice in $\mathbb{R}^t$. The rank of $M$ is just the rank of $-\log|M|$.

Remark. There is a more formal approach to define a value on $E$. We know that $E$ is a line bundle over the formal scheme $B$. So the defining cocylces $(g_{ij})$ have absolute value 1 as they are functions of a formal scheme to $\mathbb{G}_{m,K}$. This induces the absolute value on $E$ with the absolute value of $\mathbb{G}_{m,K}$ and leads to the same value that we defined.

**Proposition 3.5.9.** Let $E$ be an extension of the generic fiber of a proper formal group scheme $B$ by a Torus $T$ as before. Let $M$ be a lattice of full rank in $E$. Then $E/M$ is again a proper analytic group variety.

Proof. We write

$$
\begin{array}{c}
M \\
0 \longrightarrow T \xrightarrow{\psi} E \xrightarrow{\rho} B \longrightarrow 0 \\
A
\end{array}
$$

with $A = E/M$. As an abstract group $A$ obviously has a group structure. $E$ is a total fiber space of a line bundle, so we can find an affinoid subset $U \subset B$ such that $E$ can be covered by a finite amount of translations of $\rho^{-1}(U)$ and $\rho^{-1}(U) = \mathbb{G}_{m,K} \times U$. We write $T_c$ for the annulus $T\langle c^{-1}\zeta, c\zeta^{-1} \rangle$ where $|c| < 1$ and $\zeta$ is a coordinate on $T$. Since the rank of the lattice is full we find $c \in K^\times$ such that the translates of $T_c \times U$ already cover $A$. Since $B$ is assumed to be proper and is formal so its generic fiber is proper in the rigid analytic sense and we can cover $U$ by relatively compact subsets. This means that $A$ is proper.

**Proposition 3.5.10.** Let $A$ be proper analytic group and let $\bar{E}$ be an open formal analytic subgroup of $A$. Suppose there is a morphism $\psi: \mathbb{G}_{m,K} \to A$ that restricts to $\bar{\psi}: \mathbb{G}_{m,K} \to \bar{E}$ and gives $\bar{E}$ as extension of a proper formal group scheme $B$ by $\mathbb{G}_{m,K}$. Suppose that there is an $\varepsilon \in \mathbb{R}$ such that every element of $A$ can be written as $\psi(g) \cdot e$ with $e \in \bar{E}$ and $\varepsilon \leq |g| \leq \varepsilon^{-1}$. Then $A$ is isomorphic to $E/M$ where $E$ is the pushout as in Definition 3.5.8 and $M$ is a lattice of full rank.

Proof. We can write the elements of $E$ as tuples $(g,e) \in \mathbb{G}_{m,K} \times \bar{E}$ with the equivalence relation $(g,e) \sim (g',e')$ if and only if $\bar{\psi}(g/g') = e'/e$ according to Proposition 3.3.4.
3.5. Extensions by tori

Define the group homomorphism $\eta: E \to A$ by setting $\eta(g, e) = \psi(g) \cdot \iota(e)$ with $\iota: \bar{E} \to A$ as the open embedding of $\bar{E}$ in $A$. The kernel of $\eta$ is a lattice in $E$, since $\iota$ and $\psi$ are injective and the value of an element $(g, e)$ of $E$ is the value of $g$.

The assumption that every element of $A$ can be written as $\psi(g) \cdot e$ where $g$ is bounded and $e \in \bar{E}$ implies the surjectivity and that the lattice is of full rank.
4. The Jacobian of a formal analytic curve

4.1. The cohomology of graphs

In this section $G(V, E)$ will be an arbitrary connected graph with vertex set $V$ and edge set $E$.

**Definition 4.1.1.** A graph $G'(V', E')$ with $V' \subset V$ and $E' \subset E$, where every edge of $E'$ has source or target in $V'$ is called a subgraph of $G(V, E)$. A subgraph is called complete, if $E'$ contains all edges of $E$ with source and target in $V'$.

We want to discuss homology and cohomology of a connected graph. For this to be defined, we need to fix a topology on $G$. This is done by declaring all complete subgraphs of $G$ to be open. Since cycles in a homology group have an orientation, we will have to choose an arbitrary orientation for each edge $e \in E$.

In this section a *simple cycle* is a directed closed path in the graph which visits an edge only once. The general term *cycle* is used in the homological sense of the word, i.e. a chain of edges with zero boundary. Cycles can be obtained by taking formal sums of simple cycles modulo the standard concatenation of cycles. We say that an edge *ends* in a vertex if said vertex is either target or source of the edge. The graph $G$ may contain loops.

**Definition 4.1.2.** Denote by $n$ the number vertices of $G$ and by $e$ the number of edges. The number $t := e - n + 1$ is called the *cyclomatic number* of $G$.

The cyclomatic number of a graph is fundamental to describe its homology, as the following proposition shows.

**Proposition 4.1.3.** Let $t$ be the cyclomatic number of a connected graph $G$. Then one can fix $t$ edges $\epsilon_1, \ldots, \epsilon_t$ in $E$ such that $G'(V, E \setminus \{\epsilon_1, \ldots, \epsilon_t\})$ is a tree. Each of these edges $\epsilon_i$ corresponds uniquely to a simple cycle $\gamma_i$ on $G$ in a natural way.

**Proof.** A connected graph is a tree if and only if its cyclomatic number $t$ equals 0. If a connected graph $G$ contains cycles, i.e. $t \geq 1$ we remove an edge $\epsilon_1$ of one of the cycles, thereby getting a graph $\tilde{G}$ with cyclomatic number $t - 1$. Repeating that $t$ times yields a tree and $t$ edges $\epsilon_1, \ldots, \epsilon_t$. The graph $\tilde{G}(V, E \setminus \{\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon_{i+1}, \ldots, \epsilon_t\})$ has exactly one simple cycle matching orientation with $\epsilon_i$ which we denote by $\gamma_i$. In other words this is the unique cycle in $G$ which passes exactly one time through $\epsilon_i$ but none of the $\epsilon_j$ for $j \neq i$.\]

To compute the homology groups of $G$ we first note that the homology groups for the constant sheaf $\mathbb{Z}$ defined by our topology coincide with the homology...
groups of $G$ if we regard $G$ as a simplicial complex of dimension 1. This gives us the group $H_1(G, \mathbb{Z})$ as the group of formal sums of the closed paths of $G$ with coefficients in $\mathbb{Z}$, i.e. the cycles of $G$.

**Proposition 4.1.4.** The group $H_1(G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^t$ with $t$ the cyclomatic number of Definition 4.1.3. Higher homology groups vanish.

*Proof.* We define a group homomorphism from cycles in $G$ to $\mathbb{Z}^t$, by counting how often the cycle passes through the edges $\varepsilon_1, \ldots, \varepsilon_t$ of Proposition 4.1.3 where the sign encodes the direction of the cycle in this edge. For an element $(a_i)_{i=1}^t$ of $\mathbb{Z}^t$ we obtain a preimage as the cycle $\sum_{i=1}^t a_i \gamma_i$. Suppose two cycles $\gamma$ and $\gamma'$ have the same image. Then $\gamma - \gamma'$ is mapped to 0, i.e. it is equivalent to a cycle which passes through none of the $\varepsilon_i$. But $G'(V, E \setminus \{\varepsilon_1, \ldots, \varepsilon_t\})$ is a tree, thus $\gamma - \gamma'$ is trivial and $\gamma = \gamma'$. □

We can now compute the cohomology groups $H^0(G, \mathbb{Z})$ via Čech cohomology.

**Proposition 4.1.5.** Let $(A, \cdot)$ be any abelian group. Then the cohomology group $H^0(G, A)$ is isomorphic to $A$. The group $H^1(G, A)$ arises as $\text{Hom}(H_1(G, \mathbb{Z}), A)$ and is as such isomorphic to $A^t$. All higher cohomology groups vanish.

*Proof.* A graph is connected in the graph-theoretical sense if and only if it is connected topologically. Therefore $H^0(G, A)$, the group of global sections of the constant sheaf $A$, is trivially isomorphic to $A$.

To calculate the higher cohomology groups we define an open covering. Let $G_e$ be the subgraph of $G$ consisting of the edge $e$ and the two vertices it joins. Then $\mathfrak{S} = \{G_e : e \in E(G)\}$ is an open covering of $G$, as the $G_e$ are complete. With respect to this covering, we have

$$
\check{C}^q(\mathfrak{S}, A) = \prod_{(e_0, \ldots, e_q) \in E(G)^{q+1}} A(G_{e_0} \cap \cdots \cap G_{e_q})
$$

and

$$
d_q : \check{C}^q \to \check{C}^{q+1} : \alpha \mapsto \left( \prod_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, i_k-1, i_{k+1}, \ldots, i_{p+1}} \right)_{i_0, \ldots, i_{p+1}},
$$

as in [Har77] pg. 218] for standard Čech cohomology. The elements of $\check{C}^0(\mathfrak{S}, A)$ therefore assign an element of $A$ to each edge of $E$, whereas the elements of $\check{C}^1(\mathfrak{S}, A)$ assign an element of $A$ to each pair of edges which share a vertex of $G$.

To get the group $H^1(G, A) = \ker d_1 / \text{im} d_0$ we need to discuss the kernel of $d_1$ and the image of $d_0$. As always, the elements of $\ker d_1$ are the elements of $\check{C}^1$ which satisfy the cocycle relations $\alpha_{e_i, e_j} = \alpha_{e_i, e_k} \cdot \alpha_{e_k, e_j}$ for every three edges $e_i, e_j, e_k$ sharing a vertex $v$. In particular, we have $\alpha_{e,v} = 1$ and $\alpha_{e, e} = 1$.

To finish our description of $H^1(G, A)$, we need to look at the coboundaries $\text{im} d_0$. By definition, these are the elements defined by $\alpha_{e, e} = \beta_e \beta_e^{-1}$ for a 0-cocycle $(\beta_e)_{e \in E}$.

4. The Jacobian of a formal analytic curve
4.1. The cohomology of graphs

Given any cycle \( \gamma = \sum_{k=1}^{m} e_k \in H_1(G, \mathbb{Z}) \), we get
\[
\prod_{k=1}^{m} a_{e_k, e_{k+1}} = \prod_{k=1}^{m} (\beta_{e_k+1} \beta_{e_k}^{-1}) = 1 ,
\]
(4.1)

where \( e_1 = e_{m+1} \).

Conversely, suppose we have an element \( \alpha \) of \( \hat{\mathcal{C}}_1(G, \mathcal{A}) \) such that the product of (4.1) is trivial for every cycle of \( G \). We can define a 0-cocycle \((\beta_{e})_{e \in E}\) of \( G \) by setting \( \beta_{e_0} = 1 \) and recursively defining \( \beta_{e_i} = \beta_{e_j} a_{e_i, e_j}^{-1} \) if \( \beta_{e_j} \) is already defined and \( e_i \) and \( e_j \) share a vertex. Since the product of (4.1) is trivial this does not depend on the path we choose from \( e_i \) to \( e_0 \).

This means that the elements of \( \hat{H}^1(G, \mathcal{A}) \) are uniquely defined by the possible values of the left hand side of (4.1). Therefore we get a dualizing form
\[
\Phi: H_1(G, \mathbb{Z}) \times \hat{H}^1(G, \mathcal{A}) \rightarrow \mathcal{A}
\]

by sending \((\gamma, \alpha)\) to this left hand side. This gives us an isomorphism
\[
\varphi: \hat{H}^1(G, \mathcal{A}) \rightarrow \mathcal{A}^t
\]

by dualizing using the generators \((\gamma_i)\) of \( H_1(G, \mathbb{Z}) \).

By the same argument the higher cohomology groups must be dual to the corresponding homology groups. Since these vanish, the cohomology groups vanish too.

Since any intersection of elements of \( \mathcal{G} \) is connected or empty the sheaf \( \mathcal{A} \) is acyclic on each of these intersections. This implies that \( H^q(G, \mathcal{A}) = \hat{H}^q(G, \mathcal{A}) \) for all \( q \in \mathbb{N} \).

**Definition 4.1.6.** For an arbitrary edge \( e \) with target vertex \( v \) and an element \( a \in \mathcal{A} \) we define the weighted cocycle \( \alpha(e, a) = (\alpha_{e_i, e_j})_{(e_i, e_j) \in E^2} \) by setting
\[
\alpha_{e_i, e_j} = \begin{cases} 
a & \text{if } e_i = e, e_j \neq e \text{ and } e_j \text{ ends in } v \\
 a^{-1} & \text{if } e_j = e, e_i \neq e \text{ and } e_i \text{ ends in } v \\
 1 & \text{otherwise.}
\end{cases}
\]

One checks easily that every weighted cocycle obeys the cocycle conditions. Recall that an edge ends in a vertex if said vertex is either target or source.

**Lemma 4.1.7.** The image \( \varphi(\alpha(e, a)) \) of a weighted cocycle under the map \( \varphi: H^1(G, \mathcal{A}) \rightarrow \mathcal{A}^t \) can be computed as
\[
\varphi(\alpha(e, a))_i = \begin{cases} 
a & \text{if } e \in \gamma_i \text{ and } \gamma_i \text{ traverses } e \text{ in the same direction} \\
 a^{-1} & \text{if } e \in \gamma_i \text{ and } \gamma_i \text{ traverses } e \text{ in the other direction} \\
 1 & \text{otherwise.}
\end{cases}
\]

where \( \gamma_i \) are the simple cycles of Proposition 4.1.4.
4. The Jacobian of a formal analytic curve

Proof. If the cycle $\gamma_i$ does not contain $e$, every factor of (4.1) is trivial. For $\gamma_i = \sum_{k=1}^{m} e_k$ with $e_r = e$ and $\alpha(e,a) = (\alpha_{e_j,e_k})(e_j,e_k) \in E^2$, we have

$$ \varphi(\alpha(e,a))_i = \prod_{k=1}^{m} \alpha_{e_k,e_{k+1}} = \alpha_{e_{r-1},e_r} \alpha_{e_r,e_{r+1}} \cdot $$

If $\gamma_i$ traverses $e$ in the direction of $e$ we get $\alpha_{e_{r-1},e_r} = 1$ and $\alpha_{e_r,e_{r+1}} = a$. If on the other hand $\gamma_i$ traverses $e$ in the opposite direction we have $\alpha_{e_{r-1},e_r} = a^{-1}$ and $\alpha_{e_r,e_{r+1}} = 1$.

Corollary 4.1.8. Denote by $\hat{e}$ the edge $e$ with reversed direction. Then we get $\alpha(\hat{e},a) = \alpha(e,a^{-1})$.

Since the cycle $\gamma_i$ corresponding to the edge $\varepsilon_i$ was defined as the cycle containing $\varepsilon_i$ and none of the $\varepsilon_j$ for $i \neq j$, we see that

$$ \varphi(\alpha(\varepsilon_i,a)) = (1, \ldots, 1, a, 1, \ldots, 1), $$

the element of $A^t$ with $a$ at the $i$th component. This means that the weighted cocycles are dual to the simple cycles in the sense that

$$ \Phi(\alpha(\varepsilon_i,a), \gamma_j) = a^{\delta_{ij}}. $$

With this construction we can give an explicit inverse morphism of $\varphi$. Let $(a_i)_{i=1}^t$ be an element of $A^t$. Then the cocycle

$$ (\alpha_{e_j,e_k})(e_j,e_k) \in E^2 = \prod_{i=1}^{t} \alpha(\varepsilon_i,a_i) $$

is a preimage of $(a_i)$.

4.2. The cohomology of curves with semi-stable reduction

We recall the definition of a semi-stable curve of Chapter 2.

Definition 4.2.1. A projective, connected, geometrically reduced curve $\tilde{X}/k$ that has only ordinary double points as singularities is called semi-stable.

Let $X$ be a formal analytic curve with semi-stable reduction $\tilde{X}$. We write $\tilde{X}_i, i = 1, \ldots, n$ for the irreducible components of the reduction. The components $\tilde{X}_i$ can at most have nodes as singularities. If there is a node which lies only on one component, we know by Chapter 2 that it has a formal fiber which is isomorphic to an open annulus. We can use a formal blow-up to subdivide this open annulus in order to replace this node by two nodes lying on different components. Since all statements in this section are trivial for a curve with good reduction, this means that we can assume that the reduction $\tilde{X}$ has at least two irreducible components and that every node lies on two different components.
4.2. The cohomology of curves with semi-stable reduction

Example 4.2.2. To get an example for a curve with semi-stable reduction and non-regular components, we look at the Weierstraß-equation \( \eta^2 \zeta - \xi (\xi - \zeta) (\xi - \lambda \zeta) \) defining an elliptic curve \( X \) in \( \mathbb{P}^2_K \) for a \( \lambda \in K \) with \( |\lambda| < 1 \) and coordinates \( (\xi, \eta, \zeta) \). The standard affinoid cover of \( \mathbb{P}^2_K \) with three copies of \( \mathbb{B}^2_K \) gives the nodal curve defined by \( T_0^2 T_2 - T_1^2 (T_1 - T_2) \) in \( \mathbb{P}^2_k \) as the reduction of \( X \). Blowing up \( (\xi, \lambda) \) in this model and changing coordinates in a suitable manner gives us the curve of Example 1.11.4. The reduction of \( \tilde{X} \) now has two components \( \tilde{X}_1, \tilde{X}_2 \) of genus 0, which are regular.

Definition 4.2.3. For a semi-stable curve \( \tilde{X} \), we want to define the dual graph \( G(\tilde{X}) \). The vertex set \( V \) of \( G \) is the set of irreducible components of \( \tilde{X} \). The edge set \( E \) of \( G \) is the set of double points of \( \tilde{X} \), where an edge \( e \) connects the vertices corresponding to the components which intersect in the corresponding double point.

For a semi-stable curve \( \tilde{X} \) we will denote the double points corresponding to the edges \( e_i \) (as in Proposition 4.1.3) by \( \tilde{y}_i \).

Example 4.2.4. The dual graph \( G(\tilde{X}) \) of the elliptic curve of Example 1.11.4 has two vertices and two edges between these edges. Removing one of these edges yields a tree. We denote the double point corresponding to this edge as \( \tilde{y} \).

We now apply the results for graphs of Section 4.1 to a \( p \)-adic curve \( X \) with semi-stable reduction \( \tilde{X} \).

Proposition 4.2.5. Let \( X \) be a smooth projective formal analytic curve with semi-stable reduction \( \tilde{X} \). Let \( G(V,E) \) be the dual graph of \( \tilde{X} \). For any abelian group \( A \) we have 

\[
H^q(X, A) = H^q(\tilde{X}, A) = H^q(G, A) .
\]

Proof. Let \( e \in E \) be an edge corresponding to a double point \( \tilde{x} \) between the components \( \tilde{X}_i \) and \( \tilde{X}_j \). Define

\[
Z_e := \pi^{-1} \left( \left( \tilde{X}_i \cup \tilde{X}_j \right) \setminus \text{Sing} \tilde{X} \right) \cup \tilde{x} .
\]

By our assumption, \( \tilde{x} \) is not the only singularity of \( \tilde{X} \), so \( \mathfrak{Z} := \{ Z_e : e \in E \} \) is an open affinoid covering of \( X \). The part of the graph associated to the reductions \( \tilde{Z}_e \) are the \( G_e \) of the proof of Proposition 4.1.5. This implies that \( H^q(\mathfrak{Z}, A) = H^q(G, A) = H^q(G, A) \).

To prove that \( H^q(\mathfrak{Z}, A) = H^q(X, A) \) it suffices to show that \( A \) is acyclic on the intersections of elements of \( \mathfrak{Z} \). This can be done by a case-by-case analysis of the intersections as in [BL84, Prop. 2.2].

Corollary 4.2.6. The isomorphism \( H^1(G, A) \to A^1 \) gives rise to the isomorphism \( \psi : G_{m,K}^t \to H^1(X, K^\times) \), \( \tilde{\psi} : G_{m,K}^t \to H^1(X, R^\times) \) and its reduction \( \tilde{\psi} : G_{m,K}^t \to H^1(X, k^\times) \). This gives these cohomology groups are canonical structure as analytic or algebraic groups.
4. The Jacobian of a formal analytic curve

4.3. The Jacobian of a semi-stable curve

Let \( \tilde{X} \) be a semi-stable curve. As seen in Chapter 3 the Picard group \( \text{Pic} \tilde{X} \) of \( \tilde{X} \) is defined as the isomorphism classes of line bundles over \( \tilde{X} \). This group is canonically isomorphic to \( H^1(\tilde{X}, \mathcal{O}_\tilde{X}^\times) \). In this situation the group \( \text{CaCl}(\tilde{X}) \) of Cartier divisors modulo linear equivalence coincides with \( \text{Pic} \tilde{X} \).

Proposition 4.3.1. The group \( \text{Pic} \tilde{X} \) can be represented as \( \text{Div} \tilde{X}/\sim \) where \( \text{Div} \tilde{X} \) are the Weil divisors on \( \tilde{X} \) with support in the non-singular locus of \( \tilde{X} \) and \( \sim \) is the usual linear equivalence.

Proof. See [BL84, pg. 274]

Note that our definition of the Jacobian variety in Chapter 3 is only valid for non-singular curves. We can now generalize this concept.

The Weil divisors with support in the non-singular locus have a well-defined degree on each irreducible component of \( \tilde{X} \).

Definition 4.3.2. The Jacobian variety \( \text{Jac}(\tilde{X}) \) of \( \tilde{X} \) is the subgroup of \( \text{Pic}(\tilde{X}) \) composed of the classes of divisors with degree 0 on each component.

Theorem 4.3.3. The natural map \( H^1(\tilde{X}, k^\times) \to H^1(\tilde{X}, \mathcal{O}_\tilde{X}^\times) \) induces a short strict exact sequence

\[
1 \to G^m_{m,k} \xrightarrow{\text{\psi}} \text{Jac} \tilde{X} \xrightarrow{\tilde{\rho}} \tilde{B} := \prod \text{Jac} \tilde{X}_i \to 1 
\]  

(4.2)

Proof. The second morphism \( \tilde{\rho} \) is obtained by projecting a divisor pointwise onto the irreducible components of \( \tilde{X} \), i.e. the pull-back \( f_i^*\mathcal{L} \) of a line bundle \( \mathcal{L} \) by the canonical immersion \( f_i: \tilde{X}_i \to \tilde{X} \). This map is obviously surjective and compatible modulo principle divisors.

Let now \( [D] \in \ker \tilde{\rho} \). This means that \( [D]_{\tilde{X}_i} = 1 \), i.e. \( D|_{\tilde{X}_i} = (f_i) \) for a rational function \( f_i \) on \( \tilde{X}_i \). Since \( D \) has support in the non-singular locus, every \( f_i \) is defined and non-zero in every double point \( \tilde{x} \) of \( \tilde{X} \). Let \( \tilde{x}_1, \tilde{x}_2 \in \tilde{X}_i \) be two double points on the same component of \( \tilde{X} \) with corresponding edges \( e_1, e_2 \) with a common vertex \( v \) in the dual graph of \( \tilde{X} \). Setting \( \alpha_{e_1,e_2} = f_i(\tilde{x}_1)/f_i(\tilde{x}_2) \) yields an element of \( H^1(\tilde{X}, k^\times) \) which corresponds to the natural map \( H^1(\tilde{X}, k^\times) \to H^1(\tilde{X}, \mathcal{O}_\tilde{X}^\times) \).

On the other hand take an element \( \alpha \in H^1(\tilde{X}, k^\times) \). We know by a well-known corollary of the Riemann–Roch Theorem that there exists a function with given values in the finitely many double points of \( \tilde{X}_i \), so we can set the values at the given double points as we wish. Therefore every element of \( H^1(\tilde{X}, k^\times) \) yields appropriate functions as before. Since every component of \( \tilde{X} \) is a projective curve, the divisor corresponding to these function will have degree 0 on each component. This proves the exactness of the sequence.

To see that (4.2) is strict exact, we have to look at the geometric structure of \( \text{Jac} \tilde{X} \). Without restriction, we can assume that every marked double point \( \tilde{x}_i \) is blown up to a component isomorphic to \( \mathbb{P}^1_k \), intersecting the rest of \( \tilde{X} \) in the points at zero and infinity. A collection \( (f_i) \) of functions can be renormed at
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the unmarked double points and represented by \((c_iZ - 1)/(Z - 1)\) for a suitable \(c_i \in \mathbb{G}_{m,k}\) on the blown up \(\mathbb{P}^1_k\) with coordinate \(Z\) for the marked double point \(\tilde{x}_i\), yielding a rational function on all of \(\tilde{X}\).

This way, we can set

\[
\tilde{X}^{(*)} = \mathbb{G}_{m,k}^t \times \prod_{i=1}^n \tilde{X}_i^{(g_i)}
\]

where \(\tilde{X}_i^{(g_i)}\) is the \(g_i\)th symmetric product of the \(i\)th component with \(g_i\) as the genus of that component. Thereby we can construct the Jacobian variety with the usual Weil construction, see for example [Mil86, §7]. Since \(\prod \text{Jac } X_i\) is constructed the same way, we see that \(\tilde{\psi}\) embeds \(\mathbb{G}_{m,k}^t\) as a closed subvariety and that \(\tilde{\rho}\) is smooth.

\[
\square
\]

We can now directly describe the group law of the Jacobian by the methods of Chapter 3. We need to make the Weil construction of the Jacobian more explicit. We select a non-singular base point \(\tilde{x}_i\) for every component \(\tilde{X}_i\) of \(\tilde{X}\).

Then Jac \(\tilde{X}_i\) has an open subset \(U_i\) isomorphic to an open subset of \(\tilde{X}_i^{(g_i)}\) where \(g_i\) is the genus of \(\tilde{X}_i\) given via the morphism

\[
\tilde{X}_i^{(g_i)} \rightarrow \text{Jac } \tilde{X}_i ; D \mapsto [D - g_i\tilde{x}_i]
\]

We can without restriction assume that \(U_i\) has no support in the singular points of \(\tilde{X}\). On \(U_i\), the group law is given via \(D_1 + D_2 = \text{div } h_i + D_1 + D_2 - g\tilde{x}_i\) where \(h_i\) is the global section of \(\mathcal{L}(D_1 + D_2 - g\tilde{x}_i)\), unique up to a constant.

Let \(\tilde{X}^{(*)} = \prod \tilde{X}_i^{(g_i)}\) and let \(U\) be the product of the subsets \(U_i\). On \(U\) we can define a section of \(\rho\) by sending \(D \in \tilde{X}^{(*)}\) to \([D - \sum g_i\tilde{x}_i]\) on Jac \(\tilde{X}\). This map defines the open subset \(\mathbb{G}_{m,k}^t \times U\) of Jac \(\tilde{X}\) with the group law

\[
(a_1, D_1) + (a_2, D_2) = (a_1a_2f(D_1, D_2), D_1 + D_2)
\]

The cocycle \(f\) is defined by selecting the \(h_i\) so that they agree on the unmarked double points and then taking the quotient of the two values at the double point corresponding to \(\varepsilon_i\) as the \(i\)th component as described in the proof of Theorem 3.5.3.

**Proposition 4.3.4.** Let \(\tilde{X}'\) be a regular curve of genus \(g\) and \(\tilde{x}_1, \tilde{x}_2\) two points on \(\tilde{X}'\). Denote by \(\tilde{X}\) the semi-stable curve obtained by identifying \(\tilde{x}_1\) and \(\tilde{x}_2\) on \(\tilde{X}'\). If \(\tilde{x}_1 = \tilde{x}_2\) this gives a rational nodal curve glued with \(\tilde{X}'\) in \(\tilde{x}_1\). The line bundle of Proposition 3.5.4 corresponding to the extension (4.2) is given by the point \([\tilde{x}_2 - \tilde{x}_1]\) of Jac \(\tilde{X}'\) regarded as a line bundle on Jac \(\tilde{X}'\) via the autoduality Jac \(\tilde{X}' \rightarrow \text{Pic}^0(\text{Jac } \tilde{X}')\) given by any base point \(\tilde{x}_0\).

**Proof.** Proposition 3.5.4 states that (4.2) means that \(\tilde{E} := \text{Jac } \tilde{X}\) is a \(\mathbb{G}_{m,k}\)-torsor associated to a line bundle over \(\tilde{B} := \text{Jac } \tilde{X}'\).

Recall that this line bundle is given by a rational section \(s\) \(\tilde{U} \rightarrow E\) of \(\rho\) in

\[
0 \rightarrow \mathbb{G}_{m,k} \xrightarrow{\psi} \tilde{E} \xrightarrow{\rho} \tilde{B} \rightarrow 0
\]

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together with its translations \( s_\xi : \tilde{c} + \tilde{U} \to \tilde{E} \) as \( s_\xi(x) = s(x - \tilde{c}) + s(\tilde{c}) \) for any \( \tilde{c} \in \tilde{U} \). These give isomorphisms

\[
\varphi_\xi : \mathbb{G}_{m,k} \times (\tilde{c} + \tilde{U}) \to \rho^{-1}(\tilde{c} + \tilde{U})
\]

by \( \varphi_\xi(\tilde{a}, \tilde{u}) = \psi(\tilde{u}) + s_\xi(\tilde{u}) \).

A rational section \( t \) of \( \rho \) gives rise to a rational function \( t^\xi = p_1 \circ \varphi_\xi^{-1} \circ t \) for \( t^\xi : \tilde{c} + \tilde{U} \to \mathbb{P}^1 \). By the definition of \( \varphi_\xi \) we get \( t^\xi = p_1 \circ (t - s_\xi) \). Since poles and zeroes of \( t^\xi \) behave well when changing \( \tilde{c} \) this gives the Cartier divisor on \( C \) that belongs to the extension.

So to calculate the divisor associated to \( \text{Jac } X \) as a line bundle we only need to determine the divisor of \( s \).

With [Mil86, Lemma 6.9./Remark 6.10.] we know that

\[
l^* : \text{Pic}^r(\text{Jac } X') \to \text{Pic}^0 \tilde{X}'
\]

is an isomorphism, where

\[
l : \tilde{X}' \to \text{Jac } X' : \tilde{x} \mapsto [\tilde{x} - \tilde{x}_0]
\]

is the canonical embedding with respect to \( \tilde{x}_0 \).

So when we fix a point \( \tilde{x}_0 \) on \( \tilde{X}' \) any divisor class on \( \text{Jac } \tilde{X}' \) is defined by an element of \( \text{Jac } X' \). This leaves us with calculating the divisor of \( s \circ l \) as described above.

Let \( \tilde{U} \subset (\tilde{X}' \setminus \{\tilde{x}_1, \tilde{x}_2\})^g \) be the set of effective divisors with degree \( g \) such that \( D - g\tilde{x}_0 \) is non special for all \( D \in \tilde{U} \). Since divisors of the form \( x - x_0 \) are always non special we see that \( l|_{\tilde{X}' \setminus \{\tilde{x}_1, \tilde{x}_2\}} \) is restricted to \( \tilde{U} \).

We get a rational section \( s : \tilde{U} \to \text{Jac } \tilde{X} \) of (4.2) by sending \( [D - g\tilde{x}_0] \in \text{Jac } X' = \tilde{B} \) to \( [D - g\tilde{x}_0] \in \text{Jac } X = \tilde{E} \). Choose any \( D_\xi \in \tilde{U} \) such that \( D_\xi \neq g\tilde{x}_0 \) and that \( D_i \in \tilde{U} \) where \( D_i \) is defined by

\[
[\tilde{x}_i - \tilde{x}_0] + [D_\xi - g\tilde{x}_0] = [D_i - g\tilde{x}_0],
\]

for \( i = 1, 2 \).

If \( D \) is any effective divisor in \( \tilde{U} \cap D_\xi + \tilde{U} \) we can define \( D' \) as the unique effective divisor that fulfills

\[
[D - g\tilde{x}_0]_{\text{Jac } X'} - [D_\xi - g\tilde{x}_0]_{\text{Jac } X'} = [D' - g\tilde{x}_0]_{\text{Jac } X'}, \tag{4.3}
\]

and we get

\[
s^\xi([D - g\tilde{x}_0]_{\text{Jac } X'}) = p_1 \circ \varphi_\xi^{-1}([D - g\tilde{x}_0]_{\text{Jac } X'})
= p_1 \left( s \left( [D - g\tilde{x}_0]_{\text{Jac } X'} \right) - s_\xi \left( [D - g\tilde{x}_0]_{\text{Jac } X'} \right) \right)
= p_1 \left( s \left( [D - g\tilde{x}_0]_{\text{Jac } X'} \right) - s \left( [D_\xi - g\tilde{x}_0]_{\text{Jac } X'} \right) \right)
= p_1 \left( s \left( [D_\xi - g\tilde{x}_0]_{\text{Jac } X'} \right) \right) - s \left( [D_\xi - g\tilde{x}_0]_{\text{Jac } X'} \right).
\]
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We can now use 4.3 to simplify the second summand to get

\[ p_t \left( s([D - g\tilde{x}_0] \text{Jac} X') - s([D' - g\tilde{x}_0] \text{Jac} X') \right) - s([D' - g\tilde{x}_0] \text{Jac} X') \]
\[ = p_t([D - g\tilde{x}_0] \text{Jac} X') - (D' - g\tilde{x}_0) \text{Jac} X' = h_D(\tilde{x}_1) \]

where \( h_D \in \mathcal{M}_X' \) is a rational function with the divisor

\[ \text{div} h_D = D' + D_c - D - g\tilde{x}_0 \]

and \( h_D(\tilde{x}_2) = 1 \) by the definition of \( s \) and \( \psi \).

We can interpret the \( h_D \) as a rational function \( h \) on \( \text{Jac} X' \times X' \). To find the divisor associated to the extension, we need to find the zeroes and poles of \( h(\cdot, \tilde{x}_1) \). By applying \( \iota \) we can interpret \( h \) as a rational function on \( X' \times X' \) and reduce the problem to finding the zeroes and poles of the first argument.

We can choose \( D_c \) in such a way that the support of \( D_c, D_1 \) and \( D_2 \) does not contain \( \tilde{x}_1 \) and \( \tilde{x}_2 \). Using the definition of \( h \) on \( X' \times X' \) we see that the diagonal is a simple pole. This is the only component of the divisor that meets the point \((\tilde{x}_1, \tilde{x}_1)\) by our assumption on \( D_c \).

Therefore \( h(\iota(\tilde{x}), \tilde{x}_1) \) has a single pole at \( \tilde{x}_1 \) and by the same argument a single zero at \( \tilde{x}_2 \).

This means that \( \text{Jac} X \) is a total fiber space of the divisor class \([\tilde{x}_2 - \tilde{x}_1]\) over \( \text{Jac} X' \) for any base point \( \tilde{x}_0 \in X' \).

If \( \tilde{x}_1 = \tilde{x}_2 \), we see that the rational functions on \( X \) are the same as on \( X' \), thus the extension corresponds to the trivial line bundle. \( \square \)

**Proposition 4.3.5.** Let \( \tilde{X} \) be a semi-stable curve. Denote by \( \tilde{x}_i,\gamma,1 \) and \( \tilde{x}_i,\gamma,2 \) the two double points of the component \( \tilde{X}_i \) which lie on the cycle \( \gamma \) with the corresponding orientation. The line bundles given by the extension \( 4.3 \) are given by \( \mathcal{L}_\gamma = \otimes [\tilde{x}_i,\gamma,2 - \tilde{x}_i,\gamma,1] \).

**Proof.** We can use the same proof as in 4.3.4 by replacing \( \tilde{X}' \) with the disjoint union of the irreducible components \( \tilde{X}_i \) of \( \tilde{X} \). We fix a base point \( \tilde{x}_i \) for every irreducible component and let \( s \) be the section induced by these base points, i.e.

\[ s(\otimes[D_i - g_i\tilde{x}_i] \text{Jac} \tilde{X}_i) = \otimes[D_i - g_i\tilde{x}_i] \text{Jac} \tilde{X} \]

for suitable effective divisors \( D_i \) on \( \tilde{X}_i \). We can choose a divisor \( D,\tilde{c} \) with support solely in \( \tilde{X}_i \) to translate \( s \) and thereby calculate the divisor of \( s \) on the component \( \tilde{X}_i \) by the same method as in 4.3.4 which gives \( \mathcal{L}_\gamma|_{\tilde{X}_i} = [\tilde{x}_i,\gamma,2 - \tilde{x}_i,\gamma,1] \). This proves the assumption. \( \square \)

**Proposition 4.3.6.** The sequence \( 4.2 \) of Theorem 4.3.3 will split if and only if the vertices belonging to the simple curves \( \gamma_i \) are rational curves.

**Proof.** By the virtue of Proposition 4.3.5, we only need to determine when the bundle \( [\tilde{x}_2 - \tilde{x}_1] \) on every component \( X_i \) is trivial. If \( \tilde{x}_1 \neq \tilde{x}_2 \) the triviality of the bundle gives an isomorphism of \( \tilde{X}_i \) to \( \mathbb{P}^1 \), so \( \tilde{X}_i \) is rational. \( \square \)
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To apply this result to the original curve \( X \), we lift the exact sequence of Theorem 4.3.3.

We want to describe the set of Weil divisors analogous to the Weil divisors used in Definition 4.3.2.

**Definition 4.4.1.** We say that a Weil Divisor \( D \) on \( X \) has support in the non-singular locus of the reduction if we have \( \text{supp} \ D \subset \pi^{-1}(\tilde{X} \setminus \text{Sing} \tilde{X}) \).

We say that \( D \) has degree 0 in the reduction if \( \tilde{D} \) has degree 0 on every irreducible component of \( \tilde{X} \).

The Divisors with support in the non-singular locus of the reduction and degree 0 in the reduction are denoted by \( \text{Div} X \).

With \([BL84, \text{Thm. 5.1}]\) we know that \( \text{Div} X/\sim \) yields an open analytic subgroup \( \bar{\text{J}} \) of \( \text{Jac} X \) with the canonical reduction \( \text{Jac} \tilde{X} \).

**Theorem 4.4.2.** There is a strict exact sequence

\[
1 \rightarrow \bar{G}_{m,K} \xrightarrow{\bar{\psi}} \bar{\text{J}} \xrightarrow{\bar{\rho}} B \rightarrow 1 \tag{4.4}
\]

which reduces to \( (4.2) \) of Theorem 4.3.3.

**Proof.** See \([BL84, \text{Theorem 6.6}]\). \( \square \)

**Note.** If the sequence \( (4.4) \) splits, then the exact sequence \( (4.2) \) of the reduction will also split, since the splitting morphism can be reduced. On the other hand knowing that the sequence of the reduction splits is not sufficient to deduce whether or not sequence \( (4.4) \) splits.

We want to extend this exact sequence to the whole \( G_{m,K} \). This can be done via the theory of group objects as we have seen in the previous chapter.

**Theorem 4.4.3.** There is a exact sequence

\[
1 \rightarrow G_{m,K} \xrightarrow{\psi} \hat{\text{J}} \rightarrow B \rightarrow 1 \tag{4.5}
\]

with

\[
\hat{J} := (G_{m,K} \times \bar{J}) / \{(g,j) \in G_{m,K} \times \bar{J} ; \bar{\psi}(g) = j^{-1}\} .
\]

**Proof.** There is a unique group homomorphism \( \varphi : \hat{G}_{m,K} \rightarrow G_{m,K} \), so \( \hat{J} \) is just the push forward \( \varphi_* \bar{J} \). \( \square \)

The projection on the first factor of the direct product gives a surjective morphism

\[
\varphi : \hat{J} \rightarrow |K^\times|^t \rightarrow \mathbb{R}^t ; \ (g,j) \mapsto |g|^{-\log|g|} . \tag{4.6}
\]

In the rest of this section we will discuss how we can use the analytic group \( \hat{J} \) to describe the Jacobian \( \text{Jac} X \).

**Definition 4.4.4.** Let \( \hat{\mathcal{O}}_X \) be the sheaf of functions with supremum norm 1 on every component. Let \( \hat{\mathcal{O}}_X^\times \) be the subsheaf of functions without zeros.
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Definition 4.4.5. With the natural morphisms, we can regard $H^1(X, K^\times)$ and $H^1(X, \mathcal{O}_X^\times)$ as part of $H^1(X, \mathcal{O}_X^\times)$. We call the bundles coming from $H^1(X, K^\times)$ toric and the bundles coming from $H^1(X, \mathcal{O}_X^\times)$ formal. Note that the bundles in $H^1(X, R^\times)$ are both toric and formal.

Lemma 4.4.6. Every element $f \in H^1(X, \mathcal{O}_X^\times)$ can be written as a product $f = \alpha \cdot g$ of a toric bundle $\alpha \in H^1(X, K^\times)$ with a formal bundle $g \in H^1(X, \mathcal{O}_X^\times)$.

Proof. Let $f = (f_{i,j}) \in H^1(X, \mathcal{O}_X^\times)$ be represented by a cocycle on an open covering $U = (U_i)_{i \in I}$ which is a refinement of the covering $\mathcal{U}$. With the appropriate blow-up and further refinement we can achieve that $U_i \cap U_j \supset \pi^{-1}(\tilde{X}_k \setminus \text{Sing} \tilde{X})$ for every $i \neq j$. Blowing up double points of the reduction only adds curves of genus 0, on which every divisor is trivial. Hence the cohomology group $H^1(X, \mathcal{O}_X^\times)$ remains unchanged.

We now calculate these factorizations explicitly for line bundles corresponding to divisors of the form $[x_1 - x_2], x_1, x_2 \in X$. For a line bundle $\mathcal{L}$ denote by $D_\mathcal{L}$ the corresponding divisor and by $f_\mathcal{L} \in H^1(X, \mathcal{O}_X^\times)$ the representing element of the cohomology group.

Lemma 4.4.7. An analytic Weil divisor with support in the non-singular locus of the reduction can be solved locally and thus be written as a Cartier divisor.

Proof. Let $x \in X$ be a point with the regular point $\tilde{x}$ as reduction. We can find a local parameter of $\tilde{x}$, i.e. a function $f \in \mathcal{O}_X(\tilde{U})$ which has only one zero on $\tilde{U}$ in $\tilde{x}$ and $\text{ord}_x f = 1$. A lift $\tilde{f}$ of $f$ can only have a single zero in $X_+(\tilde{x})$ and has $|f(y)| < 1$ for all $y \in X_+(\tilde{x})$ according to Corollary 2.2.7. Therefore $f - f(x) \in \mathcal{O}_X(U)$ solves the divisor $[x]$ locally on $U = \pi^{-1}(\tilde{U})$.

Proposition 4.4.8. Let $x_1, x_2 \in X_i := \pi^{-1}(\tilde{X}_i \setminus \text{Sing} \tilde{X})$ be two points on the same component of $X$. Then the line bundle $\mathcal{L}$ corresponding to $D_\mathcal{L} = [x_1 - x_2]$, is represented by an element $f_\mathcal{L} = (f_{i,j}) \in H^1(X, \mathcal{O}_X^\times)$ that is already in $H^1(X, \mathcal{O}_X^\times)$.

Proof. The set $U_1 = X \setminus \pi^{-1}(\{\tilde{x}_1, \tilde{x}_2\})$ is formal open on $X$ and since $\mathcal{L}$ is trivial on $U_1$ it can be represented by the constant function $m_1 = 1$. Since $\mathcal{L}$ has no support outside of $X_i$, we can choose a formal open covering $U_2, \ldots, U_r$ of $X_i$ and meromorphic functions $m_i$ on the $U_i$ which represent $D_\mathcal{L}$ by Lemma 4.4.7. Without restriction, we can set the absolute value of $m_i$ to 1, since the supremum norm is multiplicative on $X_i$. Therefore the line bundle $\mathcal{L}$ will be represented by $(m_i)$ and the transformation functions $f_{i,j} = m_j/m_i$ on $U_i \cap U_j$ have absolute value 1. Thus $f \in H^1(X, \mathcal{O}_X^\times)$.

Corollary 4.4.9. The analytic group $\tilde{J}$ is a subgroup of $H^1(X, \mathcal{O}_X^\times)$.
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**Proposition 4.4.10.** The map

\[ \eta: \tilde{J} \to \text{Jac} X; \quad (g, j) \mapsto g \cdot j \]

is well-defined and a surjective morphism of analytic groups.

**Proof.** In order to show that the morphism \( \eta \) is well defined, we need to show that every toric line bundle \( \mathcal{M} \) has degree zero. But the connected variety \( \mathbb{C}^*_m \times K \) parameterizes the toric line bundles, so their degree cannot change and the trivial bundle has degree zero. By Corollary 4.4.9 we know that \( \tilde{J} \) is a subgroup of \( H^1(X, \mathcal{O}_X^\times) \) and the canonical map \( H^1(X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times) \) coincides with the natural embedding \( \tilde{J} \to \text{Jac} X \) for bundles of degree zero. The equivalence relation \( \tilde{J} = \mathbb{C}^*_m \times K \) is corresponding to the fact that \( H^1(X, R^\times) \) is a subgroup of both \( H^1(X, K^\times) \) and \( H^1(X, \mathcal{O}_X^\times) \).

By Lemma 4.4.6 we can write any line bundle \( \mathcal{L} \) of degree zero as a product of a toric bundle \( \mathcal{M} \) and a formal bundle \( \mathcal{N} \). As we already noted, the toric bundle \( \mathcal{M} \) has degree zero which implies that \( \mathcal{N} \) has degree zero as well and is in \( \tilde{J} \). Hence \( \eta \) is surjective. \( \square \)

**Note.** The morphism \( \eta \) is not injective as there are line bundles that are both formal and toric and yet not induced by \( R^\times \) as we will see later in this chapter.

We can assume without restriction that the dual graph contains no loops, as we have discussed in the introduction of this section. This leaves only the next proposition as second case to evaluate.

**Proposition 4.4.11.** Let \( e \in E \) be an edge with corresponding double point \( \tilde{x} \), lying on components \( \tilde{X}_i \) and \( \tilde{X}_j \), where \( \tilde{X}_i \) corresponds to the source of \( e \) and \( \tilde{X}_j \) corresponds to the target of \( e \). Let \( x_1 \in \pi^{-1}(\tilde{X}_i \setminus \text{Sing} \tilde{X}) \) and \( x_2 \in \pi^{-1}(\tilde{X}_j \setminus \text{Sing} \tilde{X}) \). We denote by \( q \) the height of the annulus of the formal fiber of \( \tilde{x} \).

Then the line bundle \( \mathcal{L} \) with \( D_{\mathcal{L}} = [x_1 - x_2] \) can be factored into \( \alpha \cdot q \) with \( \alpha = \alpha(e, q) \in H^1(X, K^\times) \), the weighted cocycle of \( q \) in \( e \) as in Definition 4.1.6 and \( g \in H^1(X, \mathcal{O}_X^\times) \).

**Proof.** Let \( U_1 \) be a formal neighborhood of \( \pi^{-1}(\tilde{x}) \). We have seen in Chapter 2 that the formal fiber of \( \tilde{x} \) is an annulus with parameter \( \zeta \) such that \( |\zeta(x)| = 1 \) for \( x \in X_i \) and \( |\zeta(x)| = q \) for \( x \in X_j \). By proposition 1.4.8 we can adjust \( x_1 \) and \( x_2 \) on their respective component so that both \( x_1 \) and \( x_2 \) are in \( U_1 \). Then \( \mathcal{L} \) can be represented on the interior of \( U_1 \) by the meromorphic function

\[ m_1 = \frac{\zeta - x_1}{\zeta - x_2}, \]

which expands uniquely to \( U_1 \). On \( U_2 := X \setminus \pi^{-1}(\{\tilde{x}_1, \tilde{x}_2\}) \) we choose \( m_2 = 1 \). Then \( \mathcal{U} = \{U_1, U_2\} \) is a formal open covering of \( X \) and the \( m_i \) represent \( D_{\mathcal{L}} \).

\( m_1 \) has absolute value 1 on \( X_i \) and absolute value \( q^{-1} \) on \( X_j \). So \( m_2/m_1 \) has absolute value 1 on \( X_i \) and \( q \) on \( X_j \). Refining \( \mathcal{U} \) as in Lemma 4.4.6 therefore gives us the toric line bundle \( \alpha(e, q) \) in the factorization of \( f \). \( \square \)
Recall that we have defined
\[ \varphi : J \to \mathbb{R}^t \]
by
\[ \varphi(a, g) = -\log|a| . \]
To further describe \( \eta \) we need to look at its kernel. As it turns out this kernel is a lattice in \( J \) i.e. a discrete subgroup. Such a group is mapped to a lattice in \( \mathbb{R}^t \) under \( \varphi \), retaining most of the important information. The next theorem uses the previous two propositions to give a basis for that lattice. In the proof we will also construct the kernel of \( \eta \) without applying \( \varphi \) but this construction is rather abstract, since the formal factor is left implicit.

**Theorem 4.4.12.** The group \( \varphi(\ker(\eta)) \), the projection of first factor of the kernel of the morphism \( \eta \) of Corollary 4.4.10 is a lattice of rank \( t \) in \( \mathbb{R}^t \). If the simple cycles of Proposition 4.1.4 are denoted by \( \gamma_i \), then this lattice is generated by \( v_{ij} = (v_{ij})_{j=1}^t \) with
\[ v_{ij} = \sum_{e \in \gamma_i \cap \gamma_j} -d(e) \cdot \log|q(e)| \]
where \( q(e) \) is the height of the formal fiber of the double point corresponding to \( e \) and \( d(e) = 1 \) if \( e \) has the same orientation in \( \gamma_i \) as in \( \gamma_j \) and \( d(e) = -1 \) otherwise.

**Proof.** We factorize the trivial line bundle in a non trivial way for each cycle \( \gamma = (v_1, \ldots, v_m), v_{m+1} = v_1 \) in \( G(X) \). Select a point \( x_j \) on each component \( X_j := \pi^{-1}(\tilde{X}_j \setminus \text{Sing} \tilde{X}) \) corresponding to \( v_j \). We know that \( \bigotimes_{k=1}^m [x_k - x_{k+1}] \cong \mathcal{O} \). So by using Proposition 4.4.11 we get a toric factor \( M = (\alpha_{ij}) \). The corresponding element \( a = (a_i) \in \mathbb{G}_m^m \) of \( \alpha \) is easily computed with Lemma 4.1.7 to be
\[ a_i = \prod_{e \in \gamma_i \cap \gamma} q(e)^d(e) \]
where \( q(e) \) is the height of the formal fiber of the double point corresponding to \( e \) and \( d(e) \) is the multiplicity of \( e \) in \( \gamma \) counted positive if the direction of \( e \) in \( \gamma \) and \( \gamma_i \) coincide, negative if not. Note that the cycles \( \gamma_i \) are simple, so \( e \) is only found once in \( \gamma_i \).

Since every cycle can be written as a formal sum of the \( \gamma_i \), each factorization obtained by this procedure will have an image in the given lattice.

Furthermore we obviously get \( \varphi(J) = 0 \) and \( \eta^{-1}(J) \cap J = J \), since \( \varphi \) was defined this way and \( \eta \) is an isomorphism when restricted to \( J \). So \( \varphi(\ker \eta) \) is a lattice and we can concentrate on the first factor of every factorization of the trivial line bundle.

Suppose there is another factorization \( \mathcal{O} = \mathcal{M} \otimes \mathcal{N} \) of the trivial line bundle into a toric line bundle \( \mathcal{M} \) and a formal line bundle \( \mathcal{N} \). We represent these factors by the cocycles \( (\alpha_{ij}) \) and \( (g_{ij}) \) on a formal open covering \( \mathfrak{U} \) as in 4.4.6, i.e. \( \mathfrak{U} \) refines \( \mathfrak{J} \) and \( U_i \cap U_j \subset X_k := \pi^{-1}(\tilde{X}_k \setminus \text{Sing} \tilde{X}) \). This means that for each \( U_i \in \mathfrak{U} \), we have an \( f_i \in \mathcal{O}_X^\times(U_i) \) with \( \alpha_{ij}g_{ij} = f_j/f_i \) on \( U_i \cap U_j \).
4. The Jacobian of a formal analytic curve

We have two cases to consider. There are $U_i$ for which $U_i \subset X_k$ holds for some $k$. In this case we can set $|f_i| = 1$, changing $\alpha$ only by a coboundary. The other case we have to consider is $U_i \subset Z_e$ with non-empty intersection with both components of $Z_e$. For each $e \in E$ there is only one such $U_i$ since we have chosen the refinement of $\mathcal{U}$ accordingly. Denote by $\tilde{X}_k$ the component corresponding to the source of $e$ and by $\tilde{X}_j$ the component corresponding to the target of $e$. We can adjust $f_i$ such that $|f_i|_{X_k} = 1$. Then $\tilde{f}_i$, the reduction of $f_i$ on $\tilde{X}_k$, is well defined. Set $n_e := \text{ord}_\tilde{x} \tilde{f}_i$ the order of $\tilde{f}_i$ in $\tilde{x}$ the double point corresponding to $e$. We have $|f_i|_{U_j} = q^{ne}$.

Select an arbitrary point of non-singular reduction $x_k$ on $X_k$ and $x_j$ on $X_j$. By Proposition 4.4.11 the line bundle $L_e$ corresponding to the divisor class $D_e = [−n_e x_k + n_e x_j]$ will yield the same toric factor on $U_i$. This means that $L = \bigotimes_{e \in E} L_e$ can be factored to $L = M \otimes N'$.

We have $\mathcal{O}, \mathcal{N}, \mathcal{N}' \in \hat{J}$ by definition, hence $\mathcal{M}, \mathcal{L} \in J$. The divisor $D = \sum_{e \in E} D_e$ has support in the non-singular locus of the reduction. We can modify $D$ with the divisor of a rational function such that $\text{deg} D|_{X_k} = 0$ holds for each component $\tilde{X}_k$. Modifying $D$ by the divisor of a rational function does not change the cocycles, since a rational function corresponds to the unit cocycle in both factors. This means that the chain $\sum_{e \in E} -n_e e$ has zero boundary on $G(X)$ and is a cycle. Therefore $\varphi(M)$ is already an element of the lattice defined above.

**Proposition 4.4.13.** In the case where the sequence (4.4) splits, the lattice of Theorem 4.4.12 lies completely in the $\mathbb{G}^t_{m,K}$ factor of $\hat{J}$ i.e. we get

$$\text{Jac } X = \left( \mathbb{G}^t_{m,K}/M \right) \times B$$

with $B$ being the analytic group variety with good reduction of sequence (4.4).

**Proof.** We have the following diagram of exact sequences.

```
1 → $\mathbb{G}^t_{m,K}$ → $\hat{J}$ → $B$ → 1
```

Now the injective splitting morphism $\sigma$ of (4.4) implies an arrow to $\text{Jac } X$ which is necessarily injective. Furthermore, in the splitting case we have $\hat{J} = \mathbb{G}^t_{m,K} \times B$ and therefore $\hat{J} = \mathbb{G}^t_{m,K} \times B$ by its definition. Thus the kernel of $\eta$ is restricted to $\mathbb{G}^t_{m,K} \times \{0\}$, which gives us a way to see $M$ as a lattice purely in $\mathbb{G}^t_{m,K}$.

We want to calculate the abelian variety $B$ in a specific example. For this we need the following lemma.
4.4. The Jacobian of a curve with semi-stable reduction

Lemma 4.4.14. Let $X$ and $Y$ be two projective smooth curves and $\rho: X \to Y$ be an algebraic surjective map of degree $n$ coprime to $\text{char } K$. Then $\rho_* \circ \rho^* = n \cdot \text{id}_{\text{Jac} Y}$, where $\rho^*$ is the induced map and $\rho_* := (\rho^*)^*$ its dual.

Proof. The situation is as follows

\[
\begin{array}{ccc}
X & \xrightarrow{\rho} & \text{Jac} X \\
\downarrow & & \downarrow \rho^* \rho_* \\
Y & \xrightarrow{\rho} & \text{Jac} Y
\end{array}
\]

Let us look at a more explicit description of $\rho^*$ and $\rho_*$. If $D = \sum_{y \in Y} n_y y$ is a divisor, then $\rho^* D = \sum_{y \in Y} n_y \rho^{-1}(y)$. This definition coincides with the one of Chapter 3 as one can easily check by writing $D$ as a Cartier divisors. Similarly for Divisors $D = \sum_{x \in X} n_x x$ with $n_x = n_{x_2}$ for all $x_1, x_2 \in X$ with $\rho(x_1) = \rho(x_2)$ the map $\rho_*(D)$ is given by $\rho_*(D) = \sum_{x \in X} n_x \rho(x)$. On can check this once again by calculating $(\rho^*)^*$ using cocycles and the fact that according to Torelli’s theorem the map $\rho^*$ maps the theta structure of $\text{Jac} Y$ onto that of $\text{Jac} X$. Clearly $\rho_* \circ \rho^* = n \cdot \text{id}$. Another way to put this is that $\rho_* \rho^* \mathcal{L} = N(\rho^* (\mathcal{L})) = \mathcal{L}^\otimes n$ where $N$ is the norm induced by $\rho$ on the corresponding function fields. 

Proposition 4.4.15. Let $X$ be an analytic curve of genus $g$ with semi-stable reduction $\tilde{X}$ such that the dual graph of $\tilde{X}$ has cyclomatic number 1. Let furthermore $\rho: X \to Y$ be an algebraic surjective map of degree $n$ such that $Y$ has a reduction whose dual graph is a tree and genus $g - 1$. Then the analytic variety $B$ of (4.4) is isogenous to $\text{Jac} Y$ and $\tilde{J}$ is the total fiber space of an $n$-torsion point of $\text{Pic}^0(B)$.

Proof. The dual of the map $\rho^*$ is $\rho_*: \text{Jac} X \to \text{Jac} Y$ and has a kernel of dimension one. The reduced subscheme of this kernel is an elliptic curve $E$ with multiplicative reduction. We get $E = \mathbb{G}_{m,K}$ and a morphism $\check{\psi}: \mathbb{G}_{m,K} \to \tilde{J}$ from the embedding. We know there is only one way up to automorphism to embed $\mathbb{G}_{m,K}$ in $\tilde{J}$ so $\check{\psi}$ is the $\check{\psi}$ of Theorem 4.4.2.

Let $\check{\rho}: \tilde{J} \to B$ be the cokernel of $\check{\psi}$. Since $\rho_* \circ \check{\psi} = 0$ we get a homomorphism $\gamma: B \to \text{Jac} Y$. Since $\text{Jac} Y$ and $B$ have the same dimension $g - 1$ and since $\rho_* = \gamma \circ \check{\rho}$ we see that $\gamma$ is an isogeny.

We construct $\sigma: B \to \tilde{J}$ as $\sigma = \rho^* \circ \gamma$. With Lemma 4.4.14 we calculate

\[
\gamma \circ \check{\rho} \circ \sigma = \gamma \circ \check{\rho} \circ \rho^* \circ \gamma \\
= \rho_* \circ \rho^* \circ \gamma \\
= n \cdot \text{id}_{\text{Jac} Y} \circ \gamma .
\]

so $\check{\rho} \circ \sigma = n \cdot \text{id}_B$. This means that for a rational section $s: B \to \tilde{J}$ of $\check{\rho}$ we get $ns = \sigma$ up to coboundries. Therefore $\tilde{J}$ is the total fiber space of an $n$-torsion point of $\text{Pic}^0(B)$. 

Note. It should be noted that even if $B = \text{Jac} Y$ is the Jacobian of some curve, the existence of the map $\check{\rho}: \tilde{J} \to B$ does not imply a map $\rho: X \to Y$, since the
curve $X$ cannot be embedded in the variety $\bar{J}$. In fact it is easy to construct an example where $X$ has the same reduction properties as in Proposition 4.4.15 which does not come from an $n$-torsion point. Take any line bundle with good reduction over any elliptic curve $B$. Dividing by a lattice gives a two dimensional abelian variety which is the Jacobian of an genus two curve $X$ by Torelli’s Theorem. Since there are line bundles which are not $n$-torsion points there will be no elliptic curve on which $X$ can map onto.

4.5. Examples

Example 4.5.1. We will discuss Theorem 4.4.12 in the example of an elliptic curve $X$. If $X$ has good reduction the dual graph $G(X)$ contains only one vertex and has cyclomatic number $t = 0$. We therefore get $\bar{J} = \text{Jac} X = X$ as usual. In the case of bad reduction consider the model of Example 1.11.4. This means that $X$ has two components of genus 0 meeting in two double points with $|\sqrt{q}|$ as height of the formal fiber. We have exactly one simple cycle in $G(X)$, which means that our lattice is generated by the logarithm of $|\sqrt{q}| = |\sqrt{q_1}| \cdot |\sqrt{q_2}|$. Since $\bar{J} = \mathbb{G}_{m, K}$ the sequence (4.4) splits and thus we know that $\hat{J} = \mathbb{G}_{m, K}$ and that there is a point $q \in \mathbb{G}_{m, K}$ such that all trivial toric bundles are represented by $q^2$. And indeed $\mathbb{G}_{m, K}/q^2 \mathbb{Z}$ is equal to $\text{Jac} X = X$.

Example 4.5.2. Let us calculate the lattice for curves of genus 2. In case of good reduction we have $\text{Jac} X = \bar{J}$ as before. For bad reduction, we have three cases to consider.

First we can have $t = 0$ as cyclomatic number. In this case the reduction of $X$ consists of two elliptic curves meeting at one ordinary double point. Since $t = 0$ the toric part is trivial and $\text{Jac} X = \bar{J} = B$. Describing $B$ and its relation to the elliptic curves of the reduction in this case is rather difficult, some cases are described in [FK91].

Second if the cyclomatic number $t$ equals 1, then $X$ consists of an elliptic curve intersecting a rational curve in two double points of height $q_1$ and $q_2$ respectively. There is only one simple cycle and we see that the lattice is generated by $\log|q_1 q_2|$. In the splitting case we furthermore get $\text{Jac} X = \mathbb{G}_{m, K}/(q_1 q_2)^2 \times E$ with an elliptic curve $E$ with good reduction.

The lattice is the most interesting in the total degenerate case of $t = 2$ where $X$ consists entirely of rational curves. Since $B = 0$ we know that the lattice lies completely in $\mathbb{G}_{m, K}^2$. There are two possibilities for $G(X)$ save for blow-ups.

In the first case $G(X)$ will look like

$$
\begin{array}{cccc}
\bullet & \vdots & \bullet & \vdots \\
& e_2 & & e_4 \\
& \bullet & \bullet & \bullet
\end{array}
$$

with corresponding heights $q_1, q_2, q_3$ and $q_4$. We select $\gamma_1 = e_1 + e_2$ and $\gamma_2 = e_3 + e_4$. These cycles have no intersection, so the lattice is generated by $(q_1 q_2, 1)$ and $(1, q_3 q_4)$.
4.5. Examples

In the second case we get the graph $G(X)$

$$
\begin{array}{c}
\bullet \\
\downarrow e_1 \\
\downarrow e_2 \\
\downarrow e_3 \\
\bullet
\end{array}
$$

with the height of the double points being $q_1, q_2$ and $q_3$. We select the simple cycles $\gamma_1 = e_1 + e_2$ and $\gamma_2 = e_1 + e_3$. $\gamma_1$ and $\gamma_2$ intersect at $e_1$ and $e_1$ has the same direction in both cycles. This means that the lattice is generated by $(q_1, q_2, q_3)$ and $(q_1, q_1, q_3)$.

**Example 4.5.3.** We want to describe $B$ in detail in the case of a special genus two curve with cyclomatic number 1.

For this let $\lambda, a, b \in K$ with $|\lambda| < 1$ and $|a| = |b| = 1$. We want to describe the Jacobian of the genus two curve $X$ given by

$$
U_1 := \text{Sp} K(\xi, \eta)/(\eta^2 - (\xi^2 - \lambda^2)(\xi^2 - a^2)(\xi^2 - b^2))
$$

$$
U_2 := \text{Sp} K(\sigma, \tau)/(\tau^2 - (1 - \lambda^2 \sigma^2)(1 - a^2 \sigma^2)(1 - b^2 \sigma^2))
$$

glued via $\sigma = 1/\xi$ and $\tau = \eta/\sigma^3$.

The reduction induced by $U_i$ is given by

$$
\tilde{U}_1 := \text{Spec} k[X, Y]/(Y^2 - X^2(X^2 - \tilde{a}^2)(X^2 - \tilde{b}^2))
$$

$$
\tilde{U}_2 := \text{Spec} k[S, T]/(T^2 - (1 - \tilde{a}^2 S^2)(1 - \tilde{b}^2 S^2))
$$

with $S = 1/X$ and $T = Y/S^3$. This curve has a self intersection at $X = 0, Y = 0$. To get rid of this self intersection we blow $U_1$ up by subdividing it in parts $|\xi| \leq \lambda$ and $|\xi| \geq \lambda$. We get

$$
V_1 := \text{Sp} K(\chi, \vartheta)/(\vartheta^2 - (\chi^2 - 1)(\lambda^2 \chi^2 - a^2)(\lambda^2 \chi^2 - b^2))
$$

$$
V_2 := \text{Sp} K(\xi, \eta, \zeta)/(\zeta \xi - \lambda, \xi^2 - (1 - \zeta^2)(\xi^2 - a^2)(\xi^2 - b^2))
$$

$$
V_3 := \text{Sp} K(\sigma, \tau)/(\tau^2 - (1 - \lambda^2 \sigma^2)(1 - a^2 \sigma^2)(1 - b^2 \sigma^2))
$$

where $\chi = \lambda^{-1} \xi$ and $\eta$ is modified accordingly.

This gives the reduction $\tilde{X}$

$$
\tilde{V}_1 := \text{Spec} k[V, W]/(W^2 - (V^2 - 1))
$$

$$
\tilde{V}_2 := \text{Spec} k[X, Y, Z]/(XZ, Y^2 - (1 - Z^2)(X^2 - \tilde{a}^2)(X^2 - \tilde{b}^2))
$$

$$
\tilde{V}_3 := \text{Spec} k[S, T]/(T^2 - (1 - \tilde{a}^2 S^2)(1 - \tilde{b}^2 S^2))
$$

which is the elliptic curve given by the ramification points $\pm \tilde{a}$ and $\pm \tilde{b}$ glued with a rational curve intersecting at $X = 0, Y = \pm \tilde{a} \tilde{b}$. Any divisor with support in the non singular locus of the reduction is a divisor with support in $V_1 \cup V_2$.

Since $\tilde{J}$ is part of the Jacobian which is birational to $X^{(2)}$ we can deduce that

$$
\iota: V_2(\lambda \xi^{-1}, \lambda^{-1} \xi) \times V_2(\xi^{-1}) \longrightarrow \tilde{J}
$$

$$
((\xi_p, \eta_p), (\xi_q, \eta_q)) \longmapsto [(\xi_p, \eta_p) + (\xi_q, \eta_q) - (\lambda, 0) - (a, 0)]
$$
4. The Jacobian of a formal analytic curve

is a formal birational map describing $\bar{J}$.

There are two elliptic curves $E_1$ and $E_2$ gained by

$U_1 := \text{Sp} K\langle \xi_1, \eta_1 \rangle / (\eta_1^2 - \xi_1(\xi_1 + \lambda^2)(\xi_1 + a^2)(\xi_1 + b^2))$

$U_2 := \text{Sp} K\langle \sigma_1, \tau_1 \rangle / (\tau_1^2 - (1 + \sigma_1 \lambda^2)(1 + \sigma_1 a^2)(1 + \sigma_1 b^2))$

and

$V_1 := \text{Sp} K\langle \xi_2, \eta_2 \rangle / (\eta_2^2 - (\xi_2 - \lambda^2)(\xi_2 - a^2)(\xi_2 - b^2))$

$V_2 := \text{Sp} K\langle \sigma_2, \tau_2 \rangle / (\tau_2^2 - \sigma_2(1 - \sigma_2 \lambda^2)(1 - \sigma_2 a^2)(1 - \sigma_2 b^2))$

with $\sigma_i = 1/\xi_i$ and $\tau_i = \eta_i/\xi_i^2$ as gluing relations. We also get maps of degree 2

$\psi: X \to E_1$

$\psi(\xi, \eta) = (-\xi^2, \eta \cdot \xi)$

$\rho: X \to E_2$

$\rho(\xi, \eta) = (\xi^2, \eta)$

These maps induce pullbacks on Pic which give the maps

$\psi^*: E_1 \to \text{Jac } X$

$[(\xi_p, \eta_p) - (a^2, 0)] \mapsto
[(\xi_p, \eta_p/\xi_p) + (-\xi_p, -\eta_p/\xi_p) - (a, 0) - (-a, 0)]$

$\rho^*: E_2 \to \text{Jac } X$

$[(\xi_p, \eta_p) - (a^2, 0)] \mapsto
[(\xi_p, \eta_p) + (-\xi_p, \eta_p) - (a, 0) - (-a, 0)]$

$\psi_*: \text{Jac } X \to E_1$

$[(\xi_p, \eta_p) + (\xi_q, \eta_q) - (a, 0) - (-a, 0)] \mapsto
[(-\xi_p^2, \xi_p \eta_p) + (-\xi_q^2, \xi_q \eta_q) - 2(-a^2, 0)]$

$\rho_*: \text{Jac } X \to E_2$

$[(\xi_p, \eta_p) + (\xi_q, \eta_q) - (a, 0) - (-a, 0)] \mapsto
[(\xi_p^2, \eta_p) + (\xi_q^2, \eta_q) - 2(a^2, 0)]$

where $\psi_*$ and $\rho_*$ are the dual maps of $\psi^*$ and $\rho^*$ with $\{(\xi_p, \eta_p), (\xi_q, \eta_q)\}$ as the preimage of $\psi^*(x)$ or $\rho^*(x)$, extended to $\text{Jac } X$ uniquely.

As noted in Lemma [4.4.14] we have

$\psi_*(\psi^*([(-\xi_p^2, \eta_p) - (a^2, 0)])) = \psi_*([(\xi_p, \eta_p/\xi_p) + (-\xi_p, -\eta_p/\xi_p) - (a, 0) - (-a, 0)])$

$= [2(-\xi_p^2, \eta_p) - 2(-a^2, 0)]$

$\rho_*(\rho^*([\xi_p^2, \eta_p) - (a^2, 0)]) = \rho_*([(\xi_p, \eta_p) + (-\xi_p, \eta_p) - (a, 0) - (-a, 0))]$

$= [2(\xi_p^2, \eta_p) - 2(a^2, 0)]$

One checks that

$\rho_*(\psi^*([(-\xi_p^2, \eta_p) - (a^2, 0)])) = [(\xi_p^2, \eta_p/\xi_p) + (\xi_p^2, -\eta_p/\xi_p) - 2(a^2, 0)] = 0$. 

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4.5. Examples

Furthermore $\psi^*$ is injective and $\rho_*$ is surjective so we get the exact but not strict exact sequence

$$
0 \longrightarrow E_1 \xrightarrow{\psi^*} \text{Jac } X \xrightarrow{\rho_*} E_2 \longrightarrow 0 .
$$

(4.7)

To see that (4.7) is not strict exact, look at the fibers of $\rho_*$. The points $[(\xi_p, \eta_p) + (\xi_0, \eta_0) - (a, 0) - (-a, 0)]$ that map to zero are given by $\xi_p^2 - \xi_0^2$ and $\eta_p + \eta_0$ on $X^2$. This is a singular variety in $\text{Jac } X$ so $\rho_*$ is not smooth according to [Har77 III. Lemma 10.5]. The morphism $\psi^*$ however is indeed a closed immersion.

The curve $E_1$ has multiplicative reduction. Blowing up at $|\xi| = |\lambda^2|$ again resolves the self intersection and leaves us with $\tilde{E}_1$ as the points of $E_1$ with a $\xi$ coordinate with absolute value 1 by mapping $(-\xi_p, \eta_p)$ to the line bundle $[(-\xi_p^2, \eta_p) - (-a^2, 0)]$. The map $\psi^*$ maps this line bundle to the line bundle $[\xi_p, \eta_p/\xi_p + (\xi_p, -\eta_p/\xi_p) - (a, 0) - (-a, 0)]$ which lies in $J$.

We know by [BGR84 9.7] that there is an element $q \in \mathbb{G}_{m,K}$ with $|q| < 1$ and an isomorphism $\varphi: \mathbb{G}_{m,K}/q \rightarrow E_1$ which restricts to $\tilde{\varphi}: \mathbb{G}_{m,K} \rightarrow \tilde{E}_1$. Then $\psi^* \circ \tilde{\varphi}: \mathbb{G}_{m,K} \rightarrow \text{Jac } X$ has its image in $J$ and is therefore a closed immersion $\tilde{\psi}: \mathbb{G}_{m,K} \rightarrow J$. Since there is up to automorphism of $\mathbb{G}_{m,K}$ only one injective map from $\mathbb{G}_{m,K}$ to $J$ we can without restriction assume that $\tilde{\psi}$ is the $\psi$ of Theorem 4.4.2.

Let $\bar{\rho}: J \rightarrow B$ be the cokernel of this morphism. Since $\rho_* \circ \psi^* = 0$ and $\tilde{\psi}$ is a restriction of $\psi^*$ we get $\rho_* \circ \tilde{\psi} = 0$ and therefore a morphism $\gamma: B \rightarrow E_2$. Both $B$ and $E_2$ have dimension 1 and since $\rho_* = \gamma \circ \bar{\rho}$ we see that $\gamma$ is surjective and is therefore an isogeny.

We can directly calculate that $B$ is $\text{Spec } k[X,Y]/(Y^2 - (X^2 - a^2)(X^2 - b^2))$ which is isogenous of order two to $E_2$.

Furthermore we know that $\rho_* \circ \rho^* = 2 \cdot \text{id}$ and that we get the diagram

$$
\begin{array}{ccc}
\mathbb{G}_{m,K} & \xrightarrow{\tilde{\psi}} & E_2 \xrightarrow{\rho^*} J \longrightarrow \text{Jac } X \\
\downarrow & & \downarrow \tilde{\rho} \\
B & & B
\end{array}
$$

One calculates that $\tilde{\psi}(\mathbb{G}_{m,K}) \cap \rho^*(E_2)$ consists of 0 and the two torsion point $[(b, 0) + (a, 0) - (-a, 0)]$. So $\gamma$ is an isogeny of order two and $B$ is obtained by completing $\text{Sp } K(\xi, \eta)/Q^2 = (\xi^2 - (a^2 - \lambda^2)(b^2 - \lambda^2))$ in the canonical way.

Let’s set $\sigma = \rho^* \circ \gamma: B \rightarrow \bar{J}$. Since

$$
\begin{array}{ccc}
\mathbb{G}_{m,K} & \xrightarrow{\tilde{\psi}} & \bar{J} \xrightarrow{\tilde{\rho}} B \xrightarrow{\gamma} E_2 \xrightarrow{\rho^*} \bar{J} \\
\downarrow & & \downarrow \rho_* & \downarrow \bar{\rho} & \downarrow \rho_* \\
\mathbb{G}_{m,K} & & B & & E_2
\end{array}
$$

we know that $\tilde{\rho} \circ \sigma = \bar{\rho} \circ \rho^* \circ \gamma: B \rightarrow B$ is an isogeny. And since $\rho_* \circ \rho^* = 2 \cdot \text{id}$

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4. The Jacobian of a formal analytic curve

we get

\[ \gamma \circ \bar{\rho} \circ \sigma = \gamma \circ \bar{\rho} \circ \rho^* \circ \gamma = \rho_* \circ \rho^* \circ \gamma = 2 \cdot \text{id}_{E^2} \circ \gamma. \]

thus \( \bar{\rho} \circ \sigma = 2 \cdot \text{id}_B. \)

Let \( s : B \to J \) be a rational section of \( \bar{\rho}. \) Then \( 2s - \sigma : U \to \bar{J} \) is in the image of \( \bar{\psi} \) by the exactness of (4.4) and therefore \( 2s \) and \( \sigma \) only differ by a cocycle. But \( \sigma \) is a group homomorphism and therefore defining a trivial bundle. This means that \( J \) is the total space of a two torsion point of \( \text{Pic}^0(B). \)
A. Bibliography


A. Bibliography

Ehrenwörtliche Erklärung

Hiermit erkläre ich, dass ich diese Arbeit selbstständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung der Universität Ulm zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

Ulm, der . Juni 2013

______________________________
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