Universität Ulm
Fakultät für Mathematik und Wirtschaftswissenschaften

Consumption-Investment problems with state-dependent discounting

Dissertation

zur Erlangung des Doktorgrades Dr. rer. nat.
der Fakultät für Mathematik und Wirtschaftswissenschaften
der Universität Ulm

vorgelegt von
Stefan Ehrenfried
aus Karlsruhe

7. November 2012

Amtierender Dekan:	Prof. Dr. Paul Wentges
1. Gutachter:	Prof. Dr. Ulrich Rieder
2. Gutachter:	Prof. Dr. Robert Stelzer
Tag der Promotion:	16. Januar 2013

Abstract

In this thesis we analyze consumption-investment problems with state-dependent discounting. We assume that the investor discounts his future utility by a non-exponential discount function which depends on his personal wealth and the state of the environment. This assumption makes the problem a so-called non-standard problem or time inconsistent problem, which has the consequence that the Bellman Optimality Principle does not hold. In order to find good time consistent strategies, we apply the concept of equilibrium strategies. We first analyze this problem in a discrete-time model, where we apply **Markov Decision Process (MDP)** methods to derive a characterization and calculation method for equilibrium strategies over the so-called extended Bellman equation.

In continuous-time we analyze the problem on a diffusion market with deterministic coefficients. We derive a PIDE, the so-called extended Hamilton-Jacobi-Bellman equation, by taking the limit from discrete to continuous-time via the extended Bellman equation. In a verification theorem we prove that the equilibrium strategies in continuous-time can be characterized and computed over the extended Hamilton-Jacobi-Bellman equation.

We compute closed-form solutions for equilibrium strategies in discrete and continuous-time for the three popular utility functions, logarithmic-, power- and exponential-utility in the case that the discount function is independent of the personal wealth of the investor.

Zusammenfassung

In dieser Arbeit analysieren wir Konsum-Investment Probleme mit zustandsabhängiger Diskontierung. Wir nehmen an, dass der Investor seinen zukünftigen Nutzen mit einer nicht exponentiellen Diskontierungsfunktion diskontiert, welche von seinem persönlichen Vermögen und dem Zustand der Umwelt abhängt. Diese Annahme macht das Problem zu einem sogenannten nicht-Standardproblem oder zeitinkonsistenten Problem, was zur Konsequenz hat, dass das Bellman-Optimalitätsprinzip nicht anwendbar ist. Um eine gute zeitkonsistente Strategie zu finden, wenden wir das Konzept der Gleichgewichtsstrategien an. Wir analysieren das Problem zuerst in diskreter Zeit, wobei wir Markov-Entscheidungsprozess-Methoden anwenden, um eine Charakterisierungs- und Berechnungsmethode für Gleichgewichtsstrategien über die sogenannte erweiterte Bellmangleichung herzuleiten. In stetiger Zeit analysieren wir das Problem auf einem Diffusionsmarkt mit deterministischen Koeffizienten. Wir leiten eine PIDE her, die sogenannte erweiterte Hamilton-Jacobi-Bellman Gleichung, indem wir den Grenzübergang von diskreter zu stetiger Zeit über die Bellmangleichung bilden. In einem Verifikationstheorem beweisen wir, dass man die Gleichgewichtsstrategien über die erweiterte Hamilton-Jacobi-Bellman Gleichung charakterisieren und berechnen kann. Wir berechnen geschlossene Lösungen, in diskreter und stetiger Zeit, für Gleichgewichtsstrategien für die drei beliebten Nutzenfunktionen Logarithmische-, Potenz- und Exponentielle Nutzenfunktion für den Fall, dass die Diskontierungsfunktion nicht vom Vermögen des Investors abhängt.

Contents

1 Introduction

1.1 Motivation and problem formulation

In this thesis we analyze consumption-investment problems with state-dependent discounting.

Historically, Merton first analyzed continuous-time consumption-investment problems in his seminal works Merton (1969) and Merton (1971). Since that time the Merton problem is one of the most analyzed problems in the area of portfolio optimization. We consider the Merton problem with general state-dependent discounting.

In classical portfolio optimization the time preferences of an investor are modeled by an exponential discount function, i.e. $\exp(-\rho t)$, $\rho > 0$, where the individual time preferences of an investor can only be described by the choice of the parameter ρ.

Many empirical studies have shown that exponential discounting is not a good way to model the individual time preference, see for instance Ainslie (1992) or Loewenstein and Prelec (1992). They have shown that the time preferences of individuals follow more a hyperbolic curve than an exponential curve. Individuals discount the near future more heavily than the long-term future. For example, when you offer someone the choice between 50 Euro now and 100 next year, many people will choose 50 Euro. But if you offer the choice between 50 Euro in five years and 100 Euro in six years almost everyone will choose 100 Euro in six years. Hyperbolic discount functions were introduced in Phleps and Pollak (1968) and extensively studied in the field of behavioral economics.

In the last years during the Financial Crisis and the Euro Crisis it was very visible that the state of the economy influences people's consumption behavior. To model this behavior we use the concept of endogenous time preferences, i.e. we let the discount function depend on an endogenous environment process, like the state of the economy. The concept of endogenous time preferences was first used in Koopmans (1960).

In this work, we also let the discount function depend on the wealth of the investor. It seems natural that the wealth situation of the investor affects his future perspective and thus his time preferences. To the best of our knowledge Björk et al. (2012) was the first work which considered a dependency of individual parameters of an investor on his personal wealth. They considered mean variance problems where the risk aversion parameter depends on the current wealth of the investor. As a result they showed that this gives a much more realistic description of the behavior of an investor, than in the case of a constant risk aversion.

To cover all three influences we model the time preferences with general state-dependent discount functions, i.e. we consider discount functions $\beta(t, x, i)$ which depend on the time t, the wealth of the investor x_t and on the state of the environment i_t. The environment process is

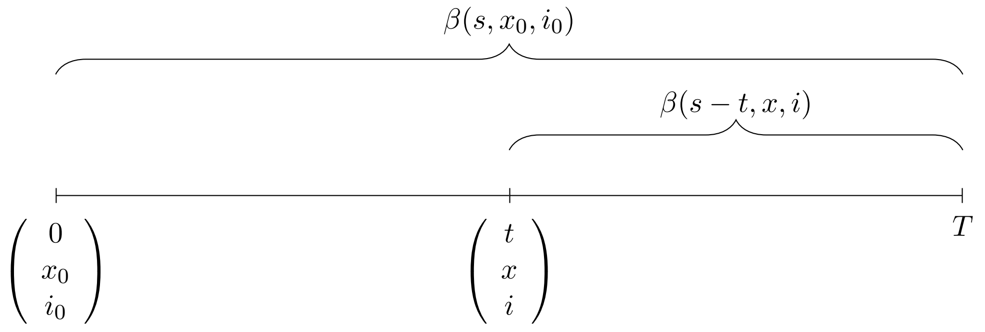

Figure 1.1:

described as a Markov chain with a finite state space.

We consider the Merton problem with these general state-dependent discount functions. The criterion of our investor at time t, wealth x_t and state of the environment i_t under a consumption-investment strategy $\pi = (c_t, a_t)$ is given by

$$V_\pi(t, x_t, i_t) := E_{t,x,i}[\int_t^T \beta_c(s-t, x_t, i_t)U_c(c_s)ds + \beta_p(T-t, x_t, i_t)U_p(X_T^\pi)],$$

where $E_{t,x,i}[\cdot] := E[\cdot|X_t = x_t, Y_{t-} = i_t]$, with (Y_t) being the environment process with discrete state space.

The horizon is T. The consumption of the investor is evaluated by the utility function U_c and discounted by the discount functions β_c. His terminal wealth X_T^π is evaluated by the utility function U_p and discounted by the discount functions β_p.

The choice of a non-exponential discount function turns the problem into a so-called non-standard problem or time inconsistent problem, in the sense that the Bellman Optimality Principle does not hold.
A strategy which is optimal on the interval $[0, T]$ is not optimal in general on the interval $[t, T]$, $t \in (0, T]$, since the optimal strategy on the interval $[0, T]$ depends in general on the initial state $(0, x_0, i_0)$. Since the discount function depends on the state (t, x, i) which is fixed during $[t, T]$, therefore the optimal strategy on $[t, T]$ in general depends on the initial state $(t, \tilde{x}, \tilde{i})$ (see figure 1.1.). There is no chance to find in general a strong solution, i.e. a strategy which is optimal for the whole family of optimization problems $(\mathcal{P}_{t,x,i})$, where

$$\mathcal{P}_{t,x,i} : \sup_{\pi \in \mathcal{A}} V_\pi(t, x, i).$$

In the case of standard discounting we have $\beta_c(t, x, i) = \beta_p(t, x, i) = \exp(-\rho t)$, $\rho > 0$. Here we obtain

$$V_\pi(t, x, i) = E_{t,x,i}[\int_t^T \exp(-\rho(s-t))U_c(c_s(X_s^\pi, Y_{s-})ds + \exp(-\rho(T-t))U_p(X_T^\pi)].$$

In this case the discount function only depends on the initial time t, but the optimal strategy only depends on the current time, since

$$\sup_\pi E_{t,x,i}[\int_t^T \exp(-\rho(s-t))U_c(c_s(X_s^\pi, Y_{s-})ds + \exp(-\rho(T-t))U_p(X_T^\pi)]$$

$$= \exp(\rho t) \sup_\pi E_{t,x,i}[\int_t^T \exp(-\rho s)U_c(c_s(X_s^\pi, Y_{s-})ds + \exp(-\rho T)U_p(X_T^\pi)],$$

which implies that every optimal solution is a strong solution.

In the literature there are two approaches to handle time inconsistency:

1) Using the so-called pre-committed problem, where optimal means optimal on the interval $[0, T]$.

2) Using the optimality concept of equilibrium strategies.

A drawback of the first approach is that in practice it is difficult to implement a pre-commitment strategy. If an investor reconsiders his pre-commitment strategy at a later point in time, his preferences change and the pre-commitment strategy is not longer optimal and therefore he will change it.

A good example to illustrate the problem of the implementation of pre-commitment solutions is a discrete stopping problem which is called the smoker problem. A smoker wants to stop smoking in the next N days. If he decides to stop at day n he gets the reward $r_n = -1$ (withdrawal) and at day $n+1$ the reward $r_{n+1} = 2$ (health) and for all other days his reward his zero. He discounts his future rewards by $\beta(n) = \frac{1}{2}\beta^n$, $n \geq 1$ and $\beta(0) = 1$, where $\beta \in (\frac{1}{2}, 1)$.

We model this problem as a finite time stopping problem with horizon N. The set of admissible strategies at time n is given by $A_n := \{\tau_n, \tau_{n+1}, \ldots, \tau_N, \tau_\infty\}$, where $\tau_k \widehat{=}$ stop smoking at day k and $\tau_\infty \widehat{=}$ never stop smoking. His reward function at stage n and strategy τ_k, $k \geq n$ is given by

$$r_n(\tau_k) = \begin{cases} -1, & \text{if } k = n, \\ 2, & \text{if } k = n+1, \\ 0, & \text{else} \end{cases}$$

The value function of the smoker is given by

$$V_n(\tau_k) = \begin{cases} (-1+\beta), & \text{if } k = n \\ -\beta^{k-n}(-\frac{1}{2}+\beta), & \text{if } k = n+1, \ldots, N-1 \\ 0, & \text{if } k = \infty \end{cases}$$

It holds that $V_0 = -1 + \beta < 0$ and if he stops tomorrow ($n = 1$), $V_1 = \frac{-\beta+2\beta^2}{2} > 0$ and $V_0(\tau_k) < V_0(\tau_2)$, $k = 3, \ldots, N, \infty$. Hence the optimal strategy at day 0 is to stop tomorrow (n=1), so the pre-commitment solution is to stop smoking tomorrow (at time $n = 1$). But if the smoker reconsider the problem again tomorrow $n = 1$ he has the value function V_1 and for that value function holds

$$V_1(\tau_2) > V_1(\tau_k),\ k = 1, 3, \ldots, N, \infty,$$

so the optimal strategy is τ_2 and he decides again to stop tomorrow ($n = 2$).

The concept of equilibrium strategies is motivated by the idea to view time inconsistency as a cooperative game we play with our future incarnation of our preferences. In this game we are interested in so-called subgame perfect Nash equilibrium strategies, which have the property that no future incarnation has the incentive to change the equilibrium strategy. For a precise game theoretic formulation we refer to Björk and Murgoci (2010).

This optimality concept has the advantage that equilibrium strategies are strong solutions, in the sense that an equilibrium strategy on the time interval $[0, T]$ is also an equilibrium strategy on every interval $[t, T]$, $t \in [0, T]$. It also coincides with the classical optimality concept in the case of time consistent problems.

The concept of equilibrium strategies in the context of discrete-time inconsistent problems is well-known and was introduced in Strotz (1955) and then used in many works, see for instance Laibson (1997), Barro (1999), Krusell and Smith (2003) or Vieille and Weibull (2009).

The first precise definition of equilibrium strategies in continuous time was given in Ekeland and Lazrak (2006) and first used in our context in Ekeland and Pirvu (2008b).

In the last years consumption-investment problems with non-exponential discounting got more attention and have been studied in several papers:

a) In Ekeland and Pirvu (2008a) they consider consumption-investment problems for two special types of discount functions, pseudo-exponential discounting type 1 $\beta_1(t) = \lambda \exp(-\rho_1 t) + (1 - \lambda) \exp(-\rho_2 t)$ and type 2 $\beta_2(t) = (1 + \lambda t) \exp(-\rho t)$ and found characterizations for the equilibrium value functions and the equilibrium strategies in these two cases.

b) In Ekeland and Pirvu (2008b) they consider a terminal wealth and pure consumption problem with a general time-dependent discount function $\beta(t)$ and found a characterization of the equilibrium value function via a PDE and PIDE and a solution of the problem in the case of CRRA utility.

c) In the working paper Pirvu and Zhang (2011) they consider exponential discount functions where the discount parameter ρ_i depends on the state of the environment.

In a),b) and c) stochastic maximum principle methods are used to characterize and compute equilibrium strategies. The idea to apply dynamic programming methods to non-standard problems has first been used in Björk and Murgoci (2010).
We will use dynamic programming methods to characterize equilibrium strategies and the equilibrium value function by a PIDE, the so-called extended Hamilton- Jacobi-Bellman equation (extended HJB). This approach provides a characterization of equilibrium strategies and of the equilibrium value function for a very general class of discount functions. Our discount function depends on the time, the wealth of the investor and the state of the economy and includes a),b) and c) as special cases.

First we will analyze the problem in a discrete-time Markov Decision Process (MDP) setup like in Bäuerle and Rieder (2011). We will characterize equilibrium strategies and the equilibrium value function over a difference equation, which we call the extended Bellman equation.
By taking the limit in time we derive the extended HJB heuristically and verify this result in a verification theorem. We present closed solutions for the three popular utility functions logarithmic-, power- and exponential-utility, in the case that the discount function depends on the time and on the environment process

1.2 Outline and contributions

The remaining parts of this thesis are structured as follows:

In **Chapter 2** we introduce the discrete version of our consumption-investment problems with state-dependent discounting. We derive a characterization and calculation method of equilibrium strategies in discrete time via a non-local difference equation, the so-called extended Bellman equation.

We start this chapter by introducing the discrete market model which is considered in the discrete part of the thesis. As a next step we introduce the environment process which we model as a discrete Markov chain with a finite state space and assume that it is independent of the financial market. In the following we formulate a precise definition of state depending discount functions. We formulate our problem as a classical Markov Decision Problem (MDP) with regime switching and introduce the value function of our investor. There is a lot of literature about MPD theory, see for instance Bertsekas and Shreve (1978) or Bäuerle and Rieder (2011). We mainly follow Bäuerle and Rieder (2011).

After discussing the issue of time inconsistency arising from the choice of our discount function we formulate the value iteration Theorem 2, which is fundamental for this chapter. Afterwards, we give a precise definition of equilibrium strategies with the help of the value iteration and derive the extended Bellman equation Corollary 1. We show in Corollary 2 that the extended Bellman equation reduces to the classical Bellman equation under constant discounting and that the equilibrium strategies coincide with classical optimal strategy in this standard case. Applying these tools we formulate a method for calculating equilibrium strategies via a sequence of one-dimensional optimization problems.
In the last section of this chapter we consider the special case of discount functions which only depend on the time and the environment process. We present closed-form solutions for the three popular utility functions log-, power- and exp-utility. This is far more difficult than in standard case, because we have to deal with a non-local difference equation which depends at stage n not only on the wealth at the point of time $n+2$, but also on the future wealth at the point of time $n+1,\dots,N$. The key to solve this difference equation is to find a good description of the dependency of the future wealth on the consumption-investment decision today.

In **Chapter 3** we present a characterization of continuous equilibrium strategies and the equilibrium value functions via a PIDE, the so-called extended Hamilton Jacobi Bellman equation

(HJB-equation). We derive the extended HJB-equation heuristically by taking the limit in time of the extended Bellman equation and verify this result by a verification theorem. There is a rich literature about stochastic control and investment-consumption problems in continuous-time, see for instance Yong.J and Zhou.X.Y. (1999), Korn and Korn (2001) or Pham (2009) and Karatzas et al. (1986), Wopperer (2010) or Rieder and Wopperer (2012).

Again we start the chapter by introducing the financial market driven by a d-dimensional-Brownian motion and the environment process which is independent of the market. Hereafter the wealth process associated with a trading strategy is introduced. We present a precise definition of state-dependent discount functions and move on to the definition of admissible strategies. We follow up with the introduction of the value function of the investor and the discussion about the issues of time inconsistency arising by the state-dependent discount function. Then we present a precise definition of continuous equilibrium strategies. In the next section we derive heuristically the so-called extended HJB by taking the limit in time of the Bellman equation. The idea to take the limit in time to derive the HJB heuristically was already used in the classical case, see for example Korn and Korn (2001). In the classical case the general idea is to consider a discrete model with step size h, express the value function at time $t+h$ with the help of Itô's formula and taking the limit $h \to 0$. In this case it is more delicate, since in the extended Bellman equation we have an additional non-local part that we have to deal with. We need a series of stochastic limit results which we present and prove in the Appendix. After the derivation we can define the extended HJB-equation and a solution of the extended HJB-equation.

The next major step is to verify this result in a verification theorem (Theorem 7). This verification theorem gives us a characterization of equilibrium strategies and the equilibrium value function via solutions of the extended HJB equation.

Afterwards we can show in Theorem 8 that the extended HJB-equation reduces to the classical HJB-equation in the case of exponential discounting and that in this case the equilibrium strategies coincide with optimal strategies.

In the last section of this chapter we consider again the special case where the discount function is independent of the wealth of the investor. We formulate a theorem for calculating equilibrium strategies and apply it to compute closed-form solutions in the case of log-, power and exp-utility.

In **Chapter 4** we present the main tools we use in chapter 3. We apply well-known results like Dynkin's formula for the process (t, X_t, Y_t), where (X_t) is a Markov-modeled diffusion and (Y_t) the corresponding continuous Markov chain. Further we present a series of limit results of functionals of this process. These results are used in the derivation of the HJB-equation and also in the proof of the Verification Theorem. We start with Itô's formula which is the major tool in this chapter. To derive the HJB-equation and to prove the Verification Theorem we need to analyze the following class of limits

$$\lim_{h \downarrow 0} \frac{1}{h} E[\big(\beta(t+h, X_{t+h}, Y_{(t+h)-}) - \beta(t,x,i)\big) E[U(X_T)|X_{t+h}, Y_{(t+h)-})]|X_t =, Y_{t-} = i]].$$

We call this limit above generalized infinitesimal operator. If we set $U \equiv 1$ we get the standard

infinitesimal operator of (t, X_t, Y_t).
In a series of Lemmata we prove a characterization of the generalized infinitesimal operator. After presenting Kolmogorov's backward equation Theorem we prove Dynkin's formula for our class of processes and formulate a Dynkin formula for the generalized infinitesimal operator.

In the next section we apply weak convergence results of SDE's from Kurtz and Protter (1996). We analyze the convergences of Markov modeled diffusion processes where we change the diffusion coefficients on an interval with length ϵ and take the weak limit as ϵ tends 0.

2 Discrete-time consumption-investment problems with state-dependent discounting

2.1 Problem formulation

We consider an investor with initial wealth $x_0 > 0$. At the beginning of each of N periods, he can decide how much of the wealth he consumes and how much he invests into a financial market.

The amount c_n which is consumed by the investor at time n is evaluated by a utility function U_c. The remaining wealth is invested in risky assets and a riskless bond. The terminal wealth of the investor X_N is valued by another utility function U_p. At stage n and current wealth x the investor discounts his future consumption at time $k > n$ according to a discount function $\beta_c(k-n, x, i)$, and the utility of his terminal wealth according to the discount function $\beta_p(N-n, x, i)$, where i is the value of an environment process $(Y_n)_{n\in\mathbb{N}}$ at time n which will be modeled as a Markov chain with finite state space.

We assume that all sources of randomness are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and $\mathcal{F}_0 := \{\emptyset, \Omega\}$.
One source of randomness are the price processes of assets on a discrete financial market and the other source is an environment process which is independent of the price processes. The discrete financial market consists of d risky assets and one riskless bond (compare Bäuerle and Rieder (2011)):

- The price process of the riskless bond is given by $S_0^0 \equiv 1$ and

$$S_{n+1}^0 := (1 + r_{n+1})S_n^0, \quad n = 0, 1, \dots N-1,$$

  where r_{n+1} denotes the deterministic interest rate for the time period $[n, n+1)$.

- The price process of the d risky assets is given by $S_0^k = s_0^k > 0$ and

$$S_{n+1}^k := S_n^k \cdot \tilde{R}_{n+1}^k, \quad n = 0, 1, \dots N-1,$$

  where $\tilde{R}_{n+1}^k \in (0, \infty)$. We assume that the random vectors $\tilde{R}_1, \dots, \tilde{R}_N$ are independent, where $\tilde{R}_n = (\tilde{R}_n^1, \dots, \tilde{R}_n^d)$. The processes (S_n^k), $k = 1, \dots, d$, are assumed to be adapted with respect to the filtration $(\mathcal{F}_n)$.

Definition 1. *A* ***consumption-investment strategy*** *is an $(\mathcal{F}_n)$-adapted stochastic process $\pi = (c_n, a_n^0, a_n)$, where $c_n \in \mathbb{R}_+$, $a_n^0 \in \mathbb{R}$ and $a_n = (a_n^1, \dots, a_n^d) \in \mathbb{R}^d$ for $n = 0, 1, \dots, N-1$. The quantity c_n is the amount of money which is consumed at time n, the quantity a_n^0 is the amount of money which is invested in the riskless bond during the time interval $[n, n+1)$ and the quantity a_n^k is the amount of money which is invested in the k-th asset during the time interval $[n, n+1)$.*

Let π be a consumption-investment strategy and denote by X^π_{n-} the value of the portfolio at time n before trading. Then

$$X^\pi_n := X^\pi_{n-} := a^0_{n-1}(1+r_n) + \sum_{k=1}^{d} a^k_{n-1}\tilde{R}^k_n - c_{n-1} = a^0_{n-1}(1+r_n) + a_{n-1}\cdot\tilde{R}_{n+1} - c_{n-1},$$

where $x\cdot y = \sum_{k=1}^d x_k y_k$ denotes the inner product of the vectors $x, y \in \mathbb{R}^d$.
The value of the portfolio π at time n after trading is given by

$$X_{n+} := \sum_{k=0}^{d} a^k_n - c_{n-1} = a^0_n + a_n\cdot e - c_{n-1},$$

where $e := (1,1,\dots,1) \in \mathbb{R}^d$.

We also consider pure portfolio strategies, i.e. consumption-investment strategies with $c_n \equiv 0$, for $n = 0,\dots,N-1$. The wealth process under portfolio strategy $a = (a^0_n, a_n)$ will be denote by X^a_n.

Definition 2. *A portfolio strategy $a = (a^0_n, a_n)$ is called **self-financing** if*

$$X^a_{n-} = X^a_{n+}, \quad \mathbb{P}-a.s.$$

for all $n = 1,\dots,N-1$, i.e. the current wealth is just reassigned to the assets.

In the case of self-financing portfolio strategies the amount a^0_n is implicit given by a_n, since the self-financing condition implies that

$$a^0_n = X^a_{n-1} - a_n\cdot e$$

We denote also a self-financing portfolio strategy by $a = (a_n)$.

Definition 3. *An arbitrage opportunity is a self-financing portfolio strategy $a = (a_n)$ with the following properties: $X^a_0 = 0$ and*

$$\mathbb{P}(X^a_N \geq 0) = 1 \quad \textit{and} \quad \mathbb{P}(X^a_N > 0) > 0.$$

Definition 4. *We call a consumption-investment strategy $\pi = (c_n, a^0_n, a_n)$ self-financing, if the portfolio strategy (a^0_n, a_n) is self-financing.*

We will only consider self-financing consumption-investment strategies which we will also denote by $\pi = (c_n, a_n)$.

We denote the wealth process under a self-financing consumption-investment strategy π by (X^π_n). The wealth process (X^π_n) with initial wealth $X^\pi_0 = x_0$ can be computed recursively by

$$X^\pi_{n+1} = a_n\cdot\tilde{R}_{n+1} + (1+r_{n+1})(X^\pi_n - c_n - e\cdot a_n)$$

To express the wealth process in a simpler way, we introduce the so-called relative risk process (R_n), where $R_n := (R^1_n,\dots,R^d_n)$ and

$$R^k_n := \frac{\tilde{R}^k_n}{1+r_n} - 1, \quad k = 1,\dots,d.$$

Note that $R_n^k \in (-1, \infty)$.

With the help of the relative risk process we can rewrite the recursion formula of the wealth process as

$$X_{n+1}^{\pi} = (1 + r_{n+1})(X_n^{\pi} - c_n + a_n \cdot R_{n+1}).$$

Is $X_n^{\pi} = x$, given we write

$$X_{n+1}^{c,a} = (1 + r_{n+1})(x - c_n + a_n \cdot R_{n+1}).$$

Later we will evaluate our wealth and our consumption by utility functions.

Definition 5 (Utility function)**.** *A function $U : dom(U) \to \mathbb{R}$ is called utility function, if U is strictly increasing, strictly concave and continuous on $dom(U)$. It is assumed that $dom(U) \subset \mathbb{R}$.*

The following utility functions are standard:

- *Logarithmic utility.* Here we have $U_c(x) = \log(x)$ and $dom(U) = (0, \infty)$.
- *Power utility.* Here we have $U_c(x) = \frac{1}{\gamma}x^{\gamma}$ and $dom(U) = [0, \infty]$ when $0 < \gamma < 1$. If $\gamma < 0$ we have $dom(U) = (0, \infty)$.
- *Exponential utility.* Here we have $U_c(x) = -\frac{1}{\gamma}\exp(-\gamma x)$ with $\gamma > 0$ and $dom(U) = \mathbb{R}$.

We want to characterize no arbitrage (NA) over the existence of an optimal solution for problem. We make the assumption that

$$E||R_n|| < \infty, \quad n = 1, \dots, N.$$

Let U be a utility function. We introduce the following one-period utility maximization problems:

$$\sup_{a \in D_n(x)} E[U((1 + r_{n+1})(x + a \cdot R_{n+1})], \quad n = 1, \dots, N, \tag{2.1}$$

where x is the wealth of the investor at time n and $D_n(x)$ is the set of admissible investment decisions, where $D_n(x)$ is given by

$$D_n(x) = \{a \in \mathbb{R}^d | (1 + r_{n+1})(x + a \cdot R_{n+1}) \in dom(U) \quad \mathbb{P} - a.s.\}.$$

If we define

$$u_n(x, a) := E[U((1 + r_{n+1})(x + a \cdot R_{n+1})], \quad n = 1, \dots, N,$$

an equivalent formulation of (2.1) is

$$v_n(x) := \sup_{a \in D_n(x)} u_n(x, a), \quad n = 1, \dots, N.$$

The following result shows that the absence of arbitrage opportunities is equivalent to the existence of an optimal solution for problem (2.1).In the following we set $K := dom(U)$.

Theorem 1. *Let $K = [0, \infty)$ or $K = (0, \infty)$. Then it holds:*

a) There are no arbitrage opportunities if and only if there exists a measurable $a^* : E \to D_n(x)$ such that

$$u_n(x, a^*(x)) = v_n(x), \quad x \in K.$$

b) The function $v(x)$ is strictly increasing, strictly concave and continuous on K.

Proof.
See Bäuerle and Rieder (2011), Theorem 4.1.1..

□

Remark 1. *If $dom(U) = \mathbb{R}$ and U is bounded from above, e.g. exp-utility, then Theorem 1 is also true. This is shown e.g. in Föllmer and Schied (2004).*

Environment process
The assets are not the only source of randomness. We also consider an environment process (Y_n) which influences the discount function of the investor (see below).
We model (Y_n) as a one-dimensional stationary Markov chain with finite state space E_Y. Further we denote the transition probabilities with p_{ij}, i.e. $p_{ij} := \mathbb{P}(Y_{n+1} = j | Y_n = i)$. We assume that the environment process (Y_n) and the relative risk process (R_n) are independent. For definition and properties of Markov chains we refer to Lawler (2006).

Definition 6 (Discount function)**.** *A discount function $\beta(n, x, i)$ is a mapping $\beta : \mathbb{N}_0 \times \mathbb{R} \times E_Y \to (0, 1]$, with the following properties:*

(i) $\beta(0, x, i) = 1$, $\forall x \in \mathbb{R}, \forall i \in E_Y$.

(ii) The mapping $x \mapsto \beta(n, x, i)$ is continuous for $n = 0, 1, \ldots, N$ and $\forall i \in E_Y$.

The assets, the wealth process and the environment process have a Markovian structure, that implies that we can without loss of generality consider only Markovian policies. That means, the consumption and investment decision at time n only depends on the state of the wealth process at time n and on the state of the environment process at time n, and not on the whole history.

We model our wealth process as a **Markov Decision Process** with **regime switching**. For further information we refer to Bäuerle and Rieder (2011).

As **state space** we choose $E = K \times E_Y$ with $K = dom(U_c) = dom(U_p) \subset \mathbb{R}$ where we restrict ourselves to utility functions with $dom(U_c) = dom(U_p)$, endowed with the σ-field $\mathcal{E} := \mathcal{B} \times \mathcal{P}(E_Y)$, where $\mathcal{B}$ is the Borel algebra of K and $\mathcal{P}(E_Y)$ is the power set of E_Y. Let $(x, i) \in E$, then x is the current wealth of the investor and i is the current state of the environment process. The **action space** A endowed with the Borel σ-field $\mathcal{A}$ is given by $A = \mathbb{R}_+ \times \mathbb{R}^d$, where $(c, a) \in A$ are the amounts of wealth we consume and invest in the financial market. We denote the set of all admissible actions at stage n with current wealth x by $D_n(x,i) = D_n(x)$. It is independent of the environment process and is given by

$$D_n(x) := \{(c,a) \in ([0,x] \cap K) \times \mathbb{R}^d | (1 + r_{n+1})(x - c + a \cdot R_{n+1}) \in K, \quad \mathbb{P} - a.s.\}.$$

This restriction guarantees that $X_N \in K$-$\mathbb{P}$-a.s, see Bäuerle and Rieder (2011) section 4.2..

The stochastic transition kernel $Q(X_{n+1} \in B, Y_{n+1} = j|x,i,c,a) = p_{ij}Q_n(B)$, where $Q_n(B)$ depends only on the distribution of R_{n+1}.

Our one stage reward function is given by $U_c(c)$ and our terminal reward function is given by $U_p(x)$. Both are independent of the environment process.

The state process is modeled as in the classical case. The only difference is that the relative risk process does not depend on the environment process. The fundamental difference is that here we have two discount functions: the investor discounts at time n with current wealth x and state of the environment i his consumptions at times $k > n$ by $\beta_c(k-n,x,i)$ and his terminal wealth by $\beta_p(N-n,x,i)$.

Before we can define our value function we need to define (Markovian) consumption-investment strategies.

Definition 7. *A **consumption-investment-decision rule** on stage n is a measurable mapping $f_n := (c_n, a_n) : E \to \mathbb{R}_+ \times \mathbb{R}^d$ with $f_n(x,i) = (c_n(x,i), a_n(x,i)) \in D_n(x)$, $\forall (x,i) \in E$. We denote the set of all decision rules on stage n by F_n.*

Definition 8 (Markovian Consumption-Investment Strategies)**.** *A **(Markovian) consumption-investment strategy** π is a sequence of consumption-investment decision rules $\pi = (f_0, f_1, \ldots, f_{N-1})$. We will denote the set of all consumption-investment strategies by F.*

In the following we will only consider Markovian consumption-investment strategies. Thus we will also use the short term consumption-investment strategies or strategies.

The Criterion of the Investor
We make the following assumption:
Assumption (A):

(i) no arbitrage (NA)

(ii) $E||R_n|| < \infty$, $n \in \mathbb{N}$.

For a fixed starting time n, initial wealth x, initial state of environment i and a consumption-investment strategy π the value function of the investor is given by

$$V_{n,\pi}(x,i) = E_{n,x,i}[\sum_{k=n}^{N-1} \beta_c(k-n,x,i)U_c(c_k(X_k^\pi, Y_k)) + \beta_p(N-n,x,i)U_p(X_N^\pi)], \quad (x,i) \in E. \quad (2.2)$$

The horizon of the investor is $N \in \mathbb{N}$. The consumption of the investor is evaluated by the utility function U_c and discounted by the discount functions β_c. His terminal wealth X_N^π is s evaluated by the utility function U_p and discounted by the discount functions β_p.

The discount functions depend on the starting time n and the initial state (x,i). So for every (n,x,i) we get a different functional.

The value function satisfies the following integrability condition:

$$E_{n,x,i}[\sum_{k=n}^{N-1} \beta_c(k-n,x,i)U_c^+(c_k(X_k^\pi, Y_k)) + \beta_p(N-n,x,i)U_p^+(X_N^\pi)] < \infty,$$

since $\beta_c(k-n,x,i) \leq 1$ and $\beta_p(k-n,x,i) \leq 1$ it holds that

$$E_{n,x,i}[\sum_{k=n}^{N-1} \beta_c(k-n,x,i)U_c^+(c_k(X_k^\pi, Y_k)) + \beta_p(N-n,x,i)U_p^+(X_N^\pi)]$$

$$\leq E_{n,x}[\sum_{k=n}^{N-1} U_c^+(c_k(X_k^\pi, Y_k)) + U_p^+(X_N^\pi)]$$

Under Assumption (A) it holds that

$$E_{n,x}[\sum_{k=n}^{N-1} U_c^+(c_k(X_k^\pi, Y_k)) + U_p^+(X_N^\pi)] < \infty$$

see Bäuerle and Rieder (2011).

Our aim is to find a strategy which is optimal for every starting time n, initial capital x and initial state environment i, i.e. we are looking for a strategy π which is optimal for the family of optimization problems $(\mathcal{P}_{n,(x,i)})_{n=0,\dots N-1,(x,i)\in E}$, where

$$\mathcal{P}_{n,x,i} : \sup_{\pi \in F} V_{n,\pi}(x,i).$$

In general there exists no strategy π that is optimal for the whole problem family $(\mathcal{P}_{n,x,i})$, because the optimal strategy for each optimization problem depends on the initial time n, the initial wealth x and initial state environment i. To make this fact clear, we fix (n, x, i) and look at the value function at a later time point $m \geq n$ with current wealth x_m and current state of the environment j. Then the value function is given by

$$V_{m,\pi}(n, x, i, x_m, j) = E_{m,x_m,j}[\sum_{k=m}^{N-1} \beta_c(k-n, x, i) U_c(c_k(X_k^\pi, Y_k)) + \beta_p(N-n, x, i) U_p(X_N^\pi)].$$

We see that the functional does not only depend on the current state, it depends also on the initial time n, initial wealth x and initial state of environment i. For a fixed starting time, the initial wealth x and initial state environment i we can compute an optimal strategy with the classical theory. This strategy is denoted by

$$\pi^{(n,x,i)} = \left(f_n^{(n,x,i)}, f_{n+1}^{(n,x,i)}, \ldots, f_{N-1}^{(n,x,i)} \right).$$

We can also determine an optimal strategy at the starting point $(n+1, \tilde{x}, j)$, which we denote by

$$\pi^{(n+1,\tilde{x},j)} = \left((f_{n+1}^{(n+1,\tilde{x},j)}, \ldots, f_{N-1}^{(n+1,\tilde{x},j)} \right).$$

If an optimal strategy for the family $(\mathcal{P}_{n,x,i})$ exists, then $\pi^{(n+1,\tilde{x},j)}$ has to be equal $\pi^{(n,x,i)}$ on $[n+1, N-1]$, i.e

$$f_k^{n,x,i} = f_k^{n+1,\tilde{x},j}, \quad k = n+1, \ldots, N-1$$

which in general is not the case.

An optimal strategy for the whole family exists if and only if:

1. an optimal strategy $\pi^{(0,x,i)}$ exists for $(0, x, i)$;
2. the strategy $\pi^{(0,x,i)}$ on $[k, N-1]$ is also optimal for all $(k, \tilde{x}, j)$.

In the case of standard problems we only have to check condition 1., because in the standard case (Bellman principle) condition 1. implies condition 2.. In the case of non standard discounting there is no hope to fulfill condition 2. The dependency of the functional from the initial state arises, because

- the discount function $\beta_c(k-n, x, i)$ depends on the initial state (n, x, i);
- the terminal discount function $\beta_p(N-n, x, i)$ depends on the the initial state (n, x, i).

We have to use a new optimality concept, which includes the optimality in the standard case. We use the concept of equilibrium strategies which is used in many papers in the field of non-standard stochastic optimization see, e.g. Björk and Murgoci (2010) for discrete time equilibrium strategies. In continuous time see, e.g. Björk and Murgoci (2010) and Björk et al. (2012) and especially in the context of non-standard discounting, see e.g. Ekeland and Pirvu (2008a), Ekeland and Pirvu (2008b), Ekeland and Lazarek (2010), Ekeland et al. (2012) and (Pirvu and Zhang, 2011).

2.2 Equilibrium Strategies and their computation

We define

$$\mathbb{M}(E) := \{v : E \to (-\infty, \infty) | \ v \text{ is measurable}\}.$$

Definition 9. *We define the following operators for $n = 0, 1, \ldots, N-1$: Let $v \in \mathbb{M}(E)$, $\pi = (f_0 \ldots, f_{N-1}) \in F$ and $(x, i) \in E$. Then we define for $(c, a) \in D_n(x)$:*

$$(L_n^\pi v)(x, i, c, a) := U_c(c) + E_{n,x,i}[v(X_{n+1}^{c,a}, Y_{n+1})]$$

$$-E_{n,x,i}[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, X_{n+1}^{c,a}, Y_{n+1}) - \beta_c(k-n, x, i)\big) U_c(c_k(X_k^\pi, Y_k))]$$

$$-E_{n,x,i}[\big(\beta_p(N-n-1, X_{n+1}^{c,a}, Y_{n+1}) - \beta_p(N-n, x, i)\big) U_p(X_N^\pi)],$$

whenever the right-hand side exists.

We restrict ourselves to strategies $\pi \in F$ such that

$$E_{n,x,i}[V_{n+1,\pi}(X_{n+1}^\pi, Y_{n+1})] > -\infty, \quad n = 0, \ldots N-1, \quad (x, i) \in E.$$

Theorem 2. *The reward iteration for a strategy $\pi \in F$ is given by*

$$V_{n,\pi}(x, i) = (L_n^\pi V_{n+1,\pi})(x, i, f_n(x, i)), \quad (x, i) \in E \quad \text{for } n = 0, \ldots, N-1$$

and

$$V_{N,\pi}(x, i) = U_p(x), \quad (x, i) \in E.$$

Proof.
We show that $V_{n,\pi}(x, i) = (L_n^\pi V_{n+1,\pi})(x, i, f_n(x, i))$. The expected value of $V_{n+1,\pi}(X_{n+1}^{c,a}, Y_{n+1})$ is given by

$$E_{n,x,i}[V_{n+1,\pi}(X_{n+1}^{c,a}, Y_{n+1})] = E_{n,x,i}[\sum_{k=n+1}^{N-1} \beta_c(k-n-1, X_{n+1}^{c,a}, Y_{n+1}) U_c(c_k(X_k^\pi, Y_k))] \tag{2.3}$$

$$+E_{n,x,i}[\beta_p(N-n-1, X_{n+1}^{c,a}, Y_{n+1}) U_p(X_N^\pi)].$$

We express $V_{n,\pi}(x, i)$ as

$$V_{n,\pi}(x, i) = E_{n,x,i}[V_{n+1,\pi}(X_{n+1}^{c,a}, Y_{n+1})] + V_{n,\pi}(x, i) - E_{n,x,i}[V_{n+1,\pi}(X_{n+1}^{c,a}, Y_{n+1})] \tag{2.4}$$

with the definition of $V_{n,\pi}(x, i)$ and (2.3) we get

$$V_{n,\pi}(x, i) - E_{n,x,i}[V_{n+1,\pi}(X_{n+1}^{c,a}, Y_{n+1})]$$

$$= U_c(c_n(x,i)) + E_{n,x,i}[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n,x,i) - \beta_c(k-n-1, X_{n+1}^{c,a}, Y_{n+1})\big) U_c(c_k(X_k^\pi, Y_k))]$$

$$+E_{n,x,i}[\big(\beta_p(N-n,x,i) - \beta_p(N-n-1, X_{n+1}^{c,a}, Y_{n+1})\big) U_p(X_N^\pi)].$$

Plugging that into (2.4) we get

$$V_{n,\pi}(x,i) = E_{n,x,i}[V_{n+1}(X_{n+1}^{c,a}, Y_{n+1})] + U_c(c_n(x,i))$$

$$-E_{n,x,i}[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, X_{n+1}^{c,a}, Y_{n+1}), Y_{n+1}) - \beta_c(k-n,x,i)\big) U_c(c_k(X_k^\pi, Y_k))$$

$$-E_{n,x,i}[\big(\beta_p(N-n-1, X_{n+1}^{c,a}, Y_{n+1}) - \beta_p(N-n,x,i)\big) U_p(X_N^\pi)]$$

$$= (L_n^\pi V_{n+1,\pi})(x,i,f_n(x,i))).$$

□

Definition 10. *A consumption-investment strategy π^* is called an* ***equilibrium strategy (eq-strategy)****, if for every (n,x,i) the following condition holds*

$$\sup_{(c,a)\in D_n(x)} (L_n^{\pi^*} V_{n+1,\pi^*})(x,i,c,a) = V_{n,\pi^*}(x,i), \quad n = 0,\dots,N-1.$$

If π^* is an equilibrium strategy, then we define the equilibrium value function V_n by

$$V_n(x,i) := V_{n,\pi^*}(x,i).$$

Definition 11. *Let $\pi = (f_0,\dots,f_{N-1})$. Then we call a measurable mapping $f_n^* = (c_n^*, a_n^*) : E \mapsto D_n(x)$ a $(f_{n+1},\dots,f_{N-1})$-maximizer of $V_{n+1,\pi}$, if*

$$\sup_{(c,a)\in D_n(x)} (L_n^\pi V_{n+1,\pi})(x,i,c,a) = (L_n^\pi V_{n+1,\pi})(x,i,c_n^*(x,i),a_n^*(x,i))$$

$$= (L_n^\pi V_{n+1,\pi})(x,i,f_n^*(x,i)), \forall (x,i) \in E.$$

Using the iteration in Theorem 2 we are able to compute an eq-strategy $\pi^* = (f_0^*,\dots,f_{N-1}^*)$ recursively in the following way:

- f_{N-1}^* as maximizer of U_p.
- f_{N-2}^* as f_{N-1}^*-maximizer of V_{N-1,π^*}.
- f_{N-3}^* as (f_{N-2}^*, f_{N-1}^*)-maximizer of V_{N-2,π^*}

  ⋮

Corollary 1 (Extended Bellman equation)**.** *Let $\pi^* = (f_0^*, \dots, f_{N-1}^*)$ be an equilibrium strategy. Then the equilibrium value functions V_n satisfy the extended Bellman equation:*

$$V_n(x,i) = \sup_{(c,a)\in D_n(x)} \Big\{ (L_n^{\pi^*} V_{n+1})(x,i,c,a) \Big\}$$

and

$$V_N(\cdot) = U_p(\cdot).$$

Corollary 2. *In the case of standard discounting(i.e.$\beta_c(n,x,i) = \beta_p(n,x,i) = \beta^n$ with $\beta \in (0,1]$) it holds:*

a) Every equilibrium strategy π^* is an optimal strategy, i.e.

$$V_{n,\pi^*}(x) = \sup_{\pi \in F} E_{n,x}[\sum_{k=n}^{N-1} \beta^{k-n} U_c(c_k(X_k^\pi)) + \beta^{N-n} U_p(X_N^\pi)].$$

b) The extended Bellman equation reduces to the standard Bellman equation

$$V_n(x) = \sup_{(c,a)\in D_n(x)} \Big\{ U_c(c) + \beta E_{n,x}[V_{n+1}(X_{n+1}^{c,a})] \Big\}, \quad \forall x \in K$$

and

$$V_N(\cdot) = U_p(\cdot).$$

Proof.
We prove a) and b) together.

$$(L_n^{\pi^*} V_{n+1})(x,c,a) = U_c(c) + E[V_{n+1,\pi^*}(X_{n+1}^{c,a})]$$

$$-E_{n,x}[\sum_{k=n+1}^{N-1} \big(\beta^{k-n-1} - \beta^{k-n}\big) U_c(c_k^*(X_k^{\pi^*})) - E_{n,x}[\big(\beta^{N-n-1} - \beta^{N-n}\big) U_p(X_N^{\pi^*})].$$

Furthermore, $E_{n,x}[V_{n+1}(X_{n+1}^{c,a})]$ is given by

$$E_{n,x}[V_{n+1,\pi^*}(X_{n+1}^{c,a})] = E_{n,x}[\sum_{k=n+1}^{N-1} \beta^{k-n-1} U_c(c_k^*(X_k^{\pi^*})) + \beta^{N-n-1} U_p(X_N^{\pi^*})].$$

Using that fact we get

$$\begin{aligned}(L_n^{\pi^*} V_{n+1})(x,c,a) &= U_c(c) + E_{n,x}[V_{n+1,\pi^*}(X_{n+1}^{c,a})] - E_{n,x}[V_{n+1}(X_{n+1}^{c,a})] + \beta E_{n,x}[V_{n+1}(X_{n+1}^{c,a})] \\ &= U_c(c) + \beta E_{n,x}[(X_{n+1}^{c,a})].\end{aligned}$$

□

Now we show that our definition of eq-strategies is equivalent to the definition given in Björk and Murgoci (2010).

For a fix time point n and a given strategy $\pi^* = (f_0^*, \dots, f_{N-1}^*)$ the corresponding hat strategy is defined by $\hat{\pi}_n^* = (f_{n-1}^*, \hat{f}_n, f_{n+1}^*, \dots, f_{N-1}^*)$, where $\hat{f}_n$ is a arbitrary decision rule.

Definition 12 (Equilibrium strategies)**.** *A strategy $\pi^* = (f_0^*, \ldots, f_{N-1}^*)$ is called eq-strategy if for all (n, x, i) holds that*

$$\sup_{\hat{f}_n \in F_n} V_{n,\hat{\pi}_n^*}(x, i) = V_{n,\pi^*}(x, i).$$

You can view $V_{n,\hat{\pi}^*}(x, i)$ as a value function where all future decisions are fixed and the only choices being made are the consumption and the investment decisions at time n.

Let $\pi_{n+1}^* := (f_{n+1}^*, \ldots, f_{N-1}^*)$ then it holds that

$$V_{n,\hat{\pi}_n}(x, i) = V_{n,\pi_{n+1}^*}(x, i, \hat{f}_n(x, i)).$$

The following theorem shows that the definition in Björk and Murgoci (2010) is equivalent to our definition.

Theorem 3. *Let $\pi^* = (f_0^*, \ldots, f_{N-1}^*)$ be a strategy. Then π^* is an equilibrium strategy if and only if for all (n, i, x) holds that*

$$\sup_{(c,a) \in D_n(x)} V_{n,\hat{\pi}_n}(x, i, c, a) = V_{n,\pi_{n+1}^*}(x, i, f_n^*(x, i)) = V_{n,\pi^*}(x, i). \tag{2.5}$$

Proof.
Let $\pi^* = (f_0^*, \ldots, f_{N-1}^*)$ be an eq-strategy. Then f_n^* is an $\pi_{n+1}^* = (f_{n+1}^*, \ldots, f_{N-1}^*)$-maximizer, i.e.

$$\sup_{(c,a) \in D_n(x)} (L_n^{\pi^*} V_{n+1,\pi^*})(x, i, c, a) = (L_n^{\pi^*} V_{n+1,\pi^*})(x, f_n^*(x, i)). \tag{2.6}$$

From the reward iteration follows directly that (2.6) is equivalent to

$$\sup_{(c,a) \in D_n(x)} V_{n,\pi_{n+1}^*}(x, i, c, a) = V_{n,\pi_{n+1}^*}(x, i, f_n^*(x, i)) = V_{n,\pi^*}(x, i).$$

Now let equation (2.5) be true for all (n, x, i). This is equivalent to

$$\sup_{(c,a) \in D_n(x)} V_{n,\pi^*}(x, i, c, a) = V_{n,\pi_{n+1}^*}(x, i, f_n^*(x, i)) = V_{n,\pi^*}(x, i).$$

This implies f_{N-1}^* is a maximizer, which implies that f_{N-1}^* is an f_{N-1}^*-maximizer. From Corollary 2, it follows that π^* is an equilibrium strategy.

□

An eq-strategy $\pi^* = (f_0^*, \ldots, f_{N-1}^*)$ has the following property: At any time n, current wealth x and current environment state i the investor has no incentive to change his decision rule f_n^*. The definition of equilibrium strategies is motivated by game theory. We can view our problem as a game of N players $(M_0, \ldots, M_{N-1})$. Player M_n can choose an admissible decision rule f_n at time n. His pay-off depends on his decision and on the decisions of the player $M_{n+1}, \ldots, M_{N-1}$ and is given by $V_{n,\pi_{n+1}^*}(x, i, f_n)$. In this game theoretic context an eq-strategy is a subgame perfect Nash equilibrium. For more information about subgame perfect Nash equilibrium we refer to Bernighaus et al. (2005).

2.3 Special Cases

We will restrict ourselves now to discount functions without wealth dependency, i.e. $\beta_c(n,x,i) = \beta_c(n,i)$ and $\beta_p(n,x,i) = \beta_p(n,i)$. Our aim is to find eq-strategies for the three popular utility functions log-utility, power-utility and exp-utility. Therefore, we have to analyze the dependency of the wealth process X_k^π, $k = n+1,\dots,N-1$ on the decision at time n. In the case of linear decision rules f_k, $k = n+1,\dots N-1$, we get the following dependency.

Lemma 1. *Let $\pi = (c_k(x,i), a_k(x,i))$ be a strategy on $[n+1, N-1]$, where $(c_k(x,i), a_k(x,i)) := (\tilde{c}_k(i)\cdot x, \tilde{a}_k(i)\cdot x)$ with $(\tilde{c}_k(i), \tilde{a}_k(i)) \in (0,1]\times\mathbb{R}^d$ for $k = n+1,\dots,N-1$. Further let $(n,(x,i))$ be the current time, the current wealth, the current state of the environment. The strategy $\bar{\pi}$ on $[n, N-1]$ is given by $\bar{\pi} = \big((\bar{c}_n,\bar{a}_n),\pi\big) = \big((\bar{c}_n,\bar{a}_n),(c_{n+1},a_{n+1}),\dots,(c_{N-1},a_{N-1})\big)$, where $(\bar{c}_n,\bar{a}_n)$ is an arbitrary decision rule. Then we can write the wealth process $X_k^{\bar{\pi}}$ for $k = n+1,\dots,N$, as*

$$X_k^{\bar{\pi}} = Z_{n,k}^\pi (x - \bar{c}_n(x,i) + \bar{a}_n(x,i)\cdot R_{n+1}),$$

where, $Z_{n,k}^\pi$, $k = n+1,\dots,N$, are random variables with

$$Z_{n,n+1}^\pi := 1 + r_{n+1}$$

$$Z_{n,k+1}^\pi = (1+r_{k+1})(1-\tilde{c}_k(Y_k) + \tilde{a}_k(Y_k)R_{k+1})Z_{n,k}^\pi,\ k = n+2,\dots N-1.$$

Proof. We prove the statement by induction. For $k = n+2$ we get

$$X_{n+2}^{\bar{\pi}} = (1+r_{n+2})(X_{n+1}^{\bar{\pi}} - \tilde{c}_{n+1}(Y_{n+1})X_{n+1}^{\bar{\pi}} + X_{n+1}^{\bar{\pi}}\tilde{a}_{n+1}(Y_{n+1})\cdot R_{n+2})$$

Now we use that

$$X_{n+1}^{\bar{\pi}} = (1+r_{n+1})(x - \bar{c}_n(x,i) + \bar{a}_n(x,i)\cdot R_{n+1})$$

and get

$$\begin{aligned}
X_{n+2}^{\bar{\pi}} = (1+r_{n+2})\Big(&(1+r_{n+1})(x-\bar{c}_n(x,i)+\bar{a}_n(x,i)\cdot R_{n+1})\\
&-(1+r_{n+1})(x-\bar{c}_n(x,i)+\bar{a}_n(x,i)\cdot R_{n+1})\tilde{c}_{n+1}(Y_{n+1})\\
&+(1+r_{n+1})(x-\bar{c}_n(x,i)+\bar{a}_n(x,i)R_{n+1})\tilde{a}_{n+1}(Y_{n+1})\cdot R_{n+2})\Big)\\
&= \underbrace{(1+r_{n+1})(1+r_{n+2})(1-\tilde{c}_{n+1}(Y_{n+1})+\tilde{a}_{n+1}(Y_{n+1})\cdot R_{n+2})}_{=:Z_{n,n+2}^\pi} x\\
&+\bar{a}_n(x,i)\cdot R_{n+1}\underbrace{(1+r_{n+1})(1+r_{n+2})(1-\tilde{c}_{n+1}(Y_{n+1})+\tilde{a}_{n+1}(Y_{n+1})\cdot R_{n+2})}_{=Z_{n,n+2}^\pi}\\
&-\bar{c}_n(x,i)\underbrace{(1+r_{n+1})(1+r_{n+2})(1+\tilde{a}_{n+1}(Y_{n+1})R_{n+2}-\tilde{c}_{n+1}(Y_{n+2})}_{=Z_{n,n+2}^\pi}\\
&= Z_{n,n+2}^\pi(x+\bar{a}_n(x,i)\cdot R_{n+1}-\bar{c}_n(x,i)).
\end{aligned}$$

Now we make the step from $k \to k+1$.

$$X_{k+1}^{\bar{\pi}} \quad = \quad (1+r_{k+1})(X_k^{\bar{\pi}} - \tilde{c}_k(Y_k)X_k^{\bar{\pi}} + X_k^{\bar{\pi}}\tilde{a}_k(Y_k)\cdot R_{k+1})$$

$$
\begin{aligned}
&\overset{IH}{=} (1+r_{k+1})Z^{\pi}_{n,k}(x+\bar{a}_n(x,i)\cdot R_{n+1}-\bar{c}_n(x,i)) \\
&-(1+r_{k+1})\tilde{c}_k(Y_k)Z^{\pi}_{n,k}(x+\bar{a}_n(x,i)\cdot R_{n+1}-\bar{c}_n(x,i)) \\
&+(1+r_{k+1})\tilde{a}_k(Y_k)\cdot R_{k+1}Z^{\pi}_{n,k}(x+\bar{a}_n(x,i)\cdot R_{n+1}-\bar{c}_n(x,i)) \\
&= \underbrace{(1+r_{k+1})Z^{\pi}_{n,k}(1-\tilde{c}_k(Y_k)+\tilde{a}_k(Y_k)\cdot R_{k+1})}_{=:Z^{\pi}_{n,k+1}}x \\
&+\underbrace{(1+r_{k+1})Z^{\pi}_{n,k}(1-\tilde{c}_k(Y_k)+\tilde{a}_k(Y_k)\tilde{R}_{k+1})}_{Z^{\pi}_{n,k+1}}\bar{a}_n(x,i)R_{n+1} \\
&-\underbrace{(1+r_{k+1})Z^{\pi}_{n,k}(1-\tilde{c}_k(Y_k)+\tilde{a}_k(Y_k)\cdot R_{k+1})}_{=Z^{\pi}_{n,k+1}}\bar{c}_n(x,i) \\
&= Z^{\pi}_{n,k+1}(x-\bar{c}_n(x,i)+\bar{a}_n(x,i)\cdot R_{n+2}).
\end{aligned}
$$

□

2.3.1 Logarithmic-utility

We consider the case $U_c(x) = U_p(x) = \log(x)$.

We introduce the following one-dimensional optimization problems

$$\sup_{\rho_n \in A_n} \{E[\log(1 + \rho_n R_{n+1})]\}, \; n = 0, \dots, N-1, \tag{2.7}$$

where

$$A_n = \{\rho_n \in \mathbb{R}^d | 1 + \rho_n \cdot R_{n+1} > 0, \quad \mathbb{P} - a.s.\}. \tag{2.8}$$

From Theorem 1 follows directly that (2.7) has a solution ρ_n^* and we denote the optimal value by v_n.

Theorem 4. *a) The equilibrium value functions are given by*

$$V_N(x,i) = \log(x)$$

$$V_n(x,i) = \left(1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,Y_k)\right)\log(x) + d_n(i), \quad n = 0, \dots, N-1,$$

with $d_n(i) \in \mathbb{R}$, $\forall i \in E_Y$.

b) The equilibrium consumption-investment strategy π^* is given by $\pi^* = \left(f_0^*, \dots, f_{N-1}^*\right)$ with $f_n^*(x,i) = (c_n^*(x,i), a_n^*(x,i))$ and

$$c_n^*(x,i) := \frac{x}{1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)}, \quad x > 0,$$

$$a_n^*(x,i) := \frac{\beta_p(N-n,i) + \sum_{i=n+1}^{N-1} \beta_c(N-i)}{1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)} \rho_n^* x, \quad x > 0$$

where ρ_n^* is the optimal solution of (2.7).

Proof.
We will show the statement by induction.

$$V_{N-1}(x,i) = \sup_{(c,a) \in D_{N-1}(x)} (L_{N-1} V_N)(x,i,c,a)$$

$$= \sup_{(c,a) \in D_{N-1}} \Big\{\log(c) + E_{N-1,x,i}[V_N(X_N, Y_N)] - E_{N-1,x}[(1 - \beta_p(1,i))\log(X_N))]\Big\}$$

$$= \sup_{(c,a) \in D_{N-1}(x)} \Big\{\log(c) + \beta_p(1,i) E[\log((1 + r_N)(x - c + a \cdot R_N))]\Big\}.$$

We substitute $\tilde{c} = \frac{c}{x}$ and $\tilde{a} = \frac{a}{x}$ which are the fractions of wealth which will be consumed and invested in the financial market. The set of admissible fractions is given by

$$\tilde{D}_{N-1}(x) := (0,1] \times \tilde{A}_{N-1}^c(x),$$

where

$$\tilde{A}_{N-1}^c(x) = \{\tilde{a}_n \in \mathbb{R}^d | (1 + r_{n+1})(1 - \tilde{c}_n + \tilde{a}_n \cdot R_{n+1}) > 0, \quad \mathbb{P} - a.s.\}.$$

We get

$$V_{N-1}(x,i) = \log(x)+\beta_p(1,i)\log(x)+ \sup_{(c,a)\in\tilde{D}_{N-1}(x)} \Big\{\log(\tilde{c})+\beta_p(1,i)E[\log((1+r_N)(1-\tilde{c}+\tilde{a}\cdot R_N))]\Big\}$$

We make the transformation $\tilde{a} = \rho(1-\tilde{c})$, where $\rho \in A_{N-1}$ and get

$$\begin{aligned} V_{N-1}(x,i) &= \log(x) + \beta_p(1,i)\log(x) + \beta_p(1,i)\log(1+r_N) \\ &+ \sup_{(c,\rho)\in(0,1]\times A_{N-1}} \Big\{\log(\tilde{c}) + \beta_p(1,i)E[\log(1-\tilde{c}+\rho(1-\tilde{c})\cdot R_N)]\Big\} \\ &= \log(x) + \beta_p(1,i)\log(x) + \beta_p(1,i)\log(1+r_N) \\ &+ \sup_{(c,\rho)\in(0,1]\times A_{N-1}} \Big\{\log(\tilde{c}) + \beta_p(1,i)\log(1-\tilde{c}) + \beta_p(1,i)E[\log(1+\rho\cdot R_N)]\Big\}. \end{aligned}$$

We split the supremum

$$\begin{aligned} V_{N-1}(x,i) &= \log(x) + \beta_p(1,i)\log(x) + \beta_p(1,i)\log(1+r_N) \\ &+ \sup_{c\in(0,1]} \Big\{\log(\tilde{c}) + \beta_p(1,i))\log(1-\tilde{c}) + \beta_p(1,i) \sup_{\rho\in A_{N-1}} \big\{E[\log(1+\rho R_N)]\big\}\Big\}. \end{aligned}$$

The optimal ρ is given as the solution ρ^*_{N-1} of (2.7) with value v_{N-1}. We get

$$\begin{aligned} V_{N-1}(x,i) &= \log(x) + \beta_p(1,i)\log(x) + \beta_p(1,i)\log(1+r_N) + \beta_p(1,i)v_{N-1} \\ &+ \sup_{c\in(0,1]} \Big\{\log(\tilde{c}) + \beta_p(1,i))\log(1-\tilde{c})\Big\}. \end{aligned}$$

To get $\tilde{c}^*_{N-1}$ we have to solve the optimization problem

$$\sup_{c\in(0,1]} \Big\{\log(\tilde{c}) + \beta_p(1,i)\log(1-\tilde{c})\Big\}.$$

The solution c^*_{N-1} is given by

$$\tilde{c}^*_{N-1} = \frac{1}{\beta_p(1,i)+1}.$$

So we get that the optimal amount which is consumed is

$$c^*_{N-1}(x,i) = x\tilde{c}^*_{N-1} = \frac{x}{\beta_p(1,i)+1}.$$

and the optimal amounts which are invested in the stocks are

$$a^*_{N-1}(x,i) = x\tilde{a}^*_{N-1} = x\rho^*_N(1-\tilde{c}^*_{N-1}) = \frac{\beta_p(1,i)}{\beta_p(1,i)+1}\rho^*_N x.$$

Moreover we get

$$\begin{aligned} V_{N-1}(x,i) &= \log(x) + \beta_p(1,i)\log(x) + \beta_p(1,i)\log(1+r_N) \\ &+\beta_p(1,i)v_{N-1} + \log(\tilde{c}^*_{N-1}) + \beta_p(1,i)\log(1-\tilde{c}^*_{N-1}) \end{aligned}$$

$$
\begin{aligned}
&= \log(x) + \beta_p(1,i)\log(x) \\
&+ \underbrace{\beta_p(1,i)\log(1+r_N) + \beta_p(1,i)v_{N-1} - \log(\beta_p(1,i)+1) + \beta_p(1,i)\log(\frac{\beta_p(1,i)}{\beta_p(1,i)+1})}_{=:d_{N-1}(i)} \\
&= (1+\beta_p(1,i))\log(x) + d_{N-1}(i).
\end{aligned}
$$

Now we make the step from $n+1 \to n$

$$
\begin{aligned}
\sup_{(c,a)\in D_n(x)} (L_n V_{n+1})(x,i,c,a) = \sup_{(c,a)\in D_n(x)} \Big\{ & E_{n,x,i}[V_{n+1}(X^{c,a}_{n+1}, Y_{n+1})] + \log(c) \\
& - E_{n,x,i}[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1,Y_{n+1}) - \beta_c(k-n,i)\big) U_c(c^*_k(X^{\pi^*}_k, Y_k))] \\
& - E_{n,x,i}[\big(\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)\big)\log(X^{\pi^*}_N)] \Big\}
\end{aligned}
$$

From the induction hypothesis we know that

$$
V_{n+1}(X^{c,a}_{n+1}, Y_{n+1}) = \left(\beta_p(N-n-1,Y_{n+1}) + 1 + \sum_{k=n+2}^{N-1} \beta_c(N-k,Y_{n+1})\right) \log(X^{c,a}_{n+1}) + d_{n+1}(Y_{n+1})
$$

and $c^*_k(X^{\pi^*}_K, Y_k) = c^*_k(Y_k)X^{\pi^*}_k$, where $c^*_k(Y_k) = \Big(\beta_p(N-k,Y_k) + 1 + \sum_{j=k+1}^{N-1} \beta_c(N-k,Y_k)\Big)$.
We get

$$
\begin{aligned}
\sup_{(c,a)\in D_n(x)} (L_n V_{n-1})(x,i,c,a) = \sup_{(c,a)\in D_n(x)} \Big\{ & \log(c) + \\
& E_{n,x,i}[\left(\beta_p(N-n-1,Y_{n+1}) + 1 + \sum_{k=n+2}^{N-1} \beta_c(N-k,Y_{n+1})\right) \log(X^{c,a}_{n+1}) + d_{n+1}(Y_{n+1}) \\
& - E_{n,x,i}[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1,Y_{n+1}) - \beta_c(k-n,i)\big) \log(c^*_k(Y_k)X^{\pi^*}_k)] \\
& - E_{n,x,i}[\big(\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)\big)\log(X^{\pi^*}_N)] \Big\}.
\end{aligned}
$$

Now we apply Lemma 1 and use that $X^{c,a}_{n+1} = (1+r_{n+1})(x-c+a\cdot R_{n+1})$. We get

$$
\begin{aligned}
\sup_{(c,a)\in D_n(x)} (L_n V_{n+1})(x,i,c,a) = E_i[d_{n+1}(Y_{n+1})] + \sup_{(c,a)\in D_n(x)} \Big\{ & \log(c) \\
& + E_i[\left(\beta_p(N-n-1,Y_{n+1}) + 1 + \sum_{k=n+2}^{N-1} \beta_c(N-k,Y_{n+1})\right) \log\big((1+r_{n+1})(x-c+a\cdot R_{n+1})\big)] \\
& - E_i[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1,Y_{n+1}) - \beta_c(k-n,i)\big) \log(\tilde{c}^*_k(Y_k) Z^{\pi^*}_{n,k}\big(x-c+a\cdot R_{n+1}\big))]
\end{aligned}
$$

$$-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)\log(Z^{\pi^*}_{n,N}(x-c+a\cdot R_{n+1})]\Big\}$$

$$= E_i[d_{n+1}(Y_{n+1})\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\log(1+r_{n+1})]$$

$$+\sup_{(c,a)\in D_n(x)}\Big\{\log(c)$$

$$+E_i[\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\log\big(x-c+a\cdot R_{n+1}\big)]$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\log(\tilde{c}^*_k(Y_k)Z^{\pi^*}_{n,k}\big(x-c+a\cdot R_{n+1}\big))]$$

$$-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)\log(Z^{\pi^*}_{n,N}\big(x-c+a\cdot R_{n+1}\big))]\Big\}.$$

Now we substitute $\tilde{c}=\frac{c}{x}$ and $\tilde{a}_n=\frac{a}{x}$ which are the fractions of wealth which will be consumed and invested in the financial market. The set of admissible fractions is given by

$$\tilde{D}(x):=(0,1]\times\tilde{A}^c_n(x),$$

where

$$\tilde{A}^c_n(x)=\{\tilde{a}_n\in\mathbb{R}^d|(1+r_{n+1})(1-\tilde{c}_n+\tilde{a}_n\cdot R_{n+1})>0,\quad \mathbb{P}-a.s.\}.$$

We get

$$\sup_{(c,a)\in D_n(x)}(L_nV_{n+1})(x,i,c,a)=E_i[d_{n+1}(Y_{n+1})]$$

$$+E_i[\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\log(1+r_{n+1})]$$

$$+\sup_{(\tilde{c},\tilde{a})\in\tilde{D}_n(x)}\Big\{E_i[\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)$$

$$\cdot\big(\log(x)+\log\big(1-\tilde{c}+\tilde{a}\cdot R_{n+1}\big)\big)]$$

$$+\log(x)+\log(\tilde{c})-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)$$

$$\cdot\big(\log(\tilde{c}^*_k(Y_k))+\log(Z^{\pi^*}_{n,k})+\log(x)+\log\big(1-\tilde{c}+\tilde{a}\cdot R_{n+1}\big)\big)]$$

$$-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)\big(\log(Z^{\pi^*}_{n,N})+\log(x)+\log\big(1-\tilde{c}+\tilde{a}\cdot R_{n+1}\big)\big)]\Big\}$$

$$= E_i[d_{n+1}(Y_{n+1})+\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\log(1+r_{n+1})]$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\log(\tilde{c}^*_k(Y_k))]$$

$$+E_i[\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})]\log(x)$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)+\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)]\log(x)$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)Z^{\pi^*}_{n,k}]-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)Z^{\pi^*}_{n,N}]$$

$$+\sup_{(\tilde{c},\tilde{a})\in\tilde{D}_n(x)}\Big\{\log(\tilde{c})$$

$$+E_i[\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\log\big(1-\tilde{c}+\tilde{a}\cdot R_{n+1}\big)]$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\log\big(1-\tilde{c}+\tilde{a}\cdot R_{n+1}\big)]$$

$$-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)\log\big(1-\tilde{c}+\tilde{a}\cdot R_{n+1}\big)]\Big\}.$$

We substitute $\tilde{a}=\rho_n(1-\tilde{a})$, where $\rho_n\in A_n$. We get

$$V_n(x,i)=\log(x)$$

$$+E_i[d_{n+1}(Y_{n+1})+\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\log(1+r_{n+1})]$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\log(\tilde{c}^*_k(Y_k))]$$

$$+E_i[\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)]\log(x)$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)+\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)]\log(x)$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)Z^{\pi^*}_{n,k}]-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)Z^{\pi^*}_{n,N}]$$

$$+\sup_{(\tilde{c},\rho_n)\in(0,1]\times A_n}\Big\{\log(\tilde{c})$$

$$+E_i[\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\big(\log(1-\tilde{c})+\log(1+\rho_n\cdot R_{n+1})\big)$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\big(\log(1-\tilde{c})+\log(1+\rho_n\cdot R_{n+1})\big)]$$

$$-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)\big(\log(1-\tilde{c})+\log(1+\rho_n\cdot R_{n+1})\big)]\Big\}.$$

We simplify the summand in front of the supremum

$$\log(x)+E_i[d_{n+1}(Y_{n+1})+\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\log(1+r_{n+1})]$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\log(\tilde{c}_k^*(Y_k))]$$

$$+E_i[\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)]\log(x)$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)+\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)]\log(x)$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)Z_{n,k}^{\pi^*}]-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)Z_{n,N}^{\pi^*}]$$

$$=f(i)+E_i[\Big(1+\beta_p(N-n,i)+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})-\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\Big)]\log(x),$$

where

$$f(i):=E_i[d_{n+1}(Y_{n+1})+\left(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\right)\log(1+r_{n+1})]$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\log(\tilde{c}_k^*(Y_k))]$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)Z_{n,k}^{\pi^*}]-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)Z_{n,N}^{\pi^*}].$$

First we simplify the sums

$$1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})-\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)$$

$$=1+\sum_{k=n+1}^{N-1}\beta_c(k-n,i)+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})-\sum_{k=n+1}^{N-1}\beta_c(k-n-1,Y_{n+1})$$

$$=1+\sum_{k=n+1}^{N-1}\beta_c(k-n,i)+\sum_{k=n+1}^{N-2}\beta_c(k-n,Y_{n+1})-\sum_{k=n}^{N-2}\beta_c(k-n,Y_{n+1})$$

$$=1+\sum_{k=n+2}^{N-1}\beta_c(k-n,i)-1=\sum_{k=n+1}^{N-1}\beta_c(N-k,i).$$

So we get

$$f(i) + \Big(1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)\Big) \log(x).$$

Summarizing we get

$$\sup_{(c,a)\in D_n(x)} (L_n V_{n+1})(x,i,c,a) = f(i) + \Big(1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)\Big) \log(x)$$

$$+ \sup_{(\tilde{c},\rho_n(i))\in(0,1]\times A_n} \Big\{ \log(\tilde{c})$$

$$+E_i\Big[\left(\beta_p(N-n-1,Y_{n+1}) + 1 + \sum_{k=n+2}^{N-1} \beta_c(N-k,Y_{n+1})\right) \big(\log(1-\tilde{c}) + \log(1+\rho_n\cdot R_{n+1})\big)$$

$$-E_i\Big[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1) - \beta_c(k-n)\big)\big(\log(1-\tilde{c}) + \log(1+\rho_n\cdot R_{n+1})\big)\Big]$$

$$-E_i\Big[\big(\beta_p(N-n-1) - \beta_p(N-n)\big)\big(\log(1-\tilde{c}) + \log(1+\rho_n\cdot R_{n+1})\big)\Big]\Big\}.$$

The supremum exists and it is independent of x. The value of the supremum we will denote by $m(i)\in\mathbb{R}$, so we get

$$\sup_{(c,a)\in D_n(x)} (L_n V_{n+1})(x,i,c,a) = f(i) + m(i) + \Big(1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)\Big) \log(x)$$

$$= \Big(1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)\Big) \log(x) + d_n(i).$$

Now we have to compute $f_n^*(x,i) = (c_n^*(x,i), a_n^*(x,i))$, for that we have to compute the supremum

$$\sup_{(\tilde{c},\rho_n)\in(0,1]\times A_n} \Big\{ \log(\tilde{c})$$

$$+E_i\Big[\left(\beta_p(N-n-1,Y_{n+1}) + 1 + \sum_{k=n+2}^{N-1} \beta_c(N-k,Y_{n+1})\right) \big(\log(1-\tilde{c}) + \log(1+\rho_n\cdot R_{n+1})\big)\Big]$$

$$-E_i\Big[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1,Y_{n+1}) - \beta_c(k-n,i)\big)\big(\log(1-\tilde{c}) + \log(1+\rho_n\cdot R_{n+1})\big)$$

$$-E_i\Big[\big(\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)\big)\big(\log(1-\tilde{c}) + \log(1+\rho_n\cdot R_{n+1})\big)\Big]\Big\}$$

$$= \sup_{(\tilde{c},\rho_n)\in(0,1]\times A_n} \Big\{ E_i\Big[\left(\beta_p(N-n-1,Y_{n+1}) + 1 + \sum_{k=n+2}^{N-1} \beta_c(N-k,Y_{n+1})\right) \log(1-\tilde{c})\Big] + \log(\tilde{c})$$

$$-E_i\Big[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1,Y_{n+1}) - \beta_c(k-n,i)\big)\Big] \log(1-\tilde{c}) - E_i[\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)] \log(1-\tilde{c})$$

$$+E_i[\underbrace{\Big(\beta_p(N-n,i)+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})-\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\Big)}_{=\beta_p(N-n,i)+\sum_{k=n+1}^{N-1}\beta_c(N-k,i)}$$

$$\cdot\log(1+\rho_n\cdot R_{n+1})]\Big\}.$$

First we split the supremum and we get

$$\sup_{(\tilde{c}_n\in(0,1]}\Big\{E_i[\Bigg(\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})\Bigg)\log(1-\tilde{c})+\log(\tilde{c})]$$

$$-E_i[\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)]\log(1-\tilde{c})-E_i[\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)]\log(1-\tilde{c})$$

$$+\Big(\beta_p(N-n,i)+\sum_{k=n+1}^{N-1}\beta_c(N-k,i)\Big)\sup_{\rho_n\in A_n}\big\{E_{n,x}[\log(1+\rho_n\cdot R_{n+1})]\big\}\Big\}$$

The solution of the problem

$$\sup_{\rho_n\in A_n}\big\{E[\log(1+\rho_n\cdot R_{n+1})]\big\}$$

is given by ρ_n^* and the value of the problem is v_n. In order to compute $\tilde{c}_n^*$ we have to compute

$$\sup_{\tilde{c}_n\in(0,1]}\Big\{E_i[\beta_p(N-n-1,Y_{n+1})+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{k+1})]\log(1-\tilde{c})+\log(\tilde{c})]$$

$$-E_i[\sum_{k=n+1}^{N-1}\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)]\log(1-\tilde{c})$$

$$-E_i[\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)]\log(1-\tilde{c}))+(\beta_p(N-n,i)+\sum_{k=n+1}^{N-1}\beta_c(N-k,i))v_n]\Big\}$$

$$=\Big(\beta_p(N-n,i)+\sum_{k=n+1}^{N-1}\beta_c(N-k,i)\Big)v_n+\sup_{\tilde{c}\in(0,1]}\Big\{\log(\tilde{c})+$$

$$E_i[\Bigg(\beta_p(N-n,i)+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})-\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)\Bigg)]$$

$$\cdot\log(1-\tilde{c}_n)\Big\}.$$

From above we know that

$$\beta_p(N-n,i)+1+\sum_{k=n+2}^{N-1}\beta_c(N-k,Y_{n+1})-\sum_{k=n+1}^{N-1}\big(\beta_c(k-n-1,Y_{n+1})-\beta_c(k-n,i)\big)$$

$$=\beta_p(N-n,i)+\sum_{k=n+1}^{N-1}\beta_c(N-k,i).$$

So we get

$$= \Big(\beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)\Big) v_n$$

$$+ \sup_{\tilde{c}\in(0,1]} \underbrace{\left\{ \log(\tilde{c}) + \left(\beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)\right) \log(1-\tilde{c}) \right\}}_{=:l(\tilde{c})} \Bigg\}$$

$$l'(\tilde{c}) = \frac{1}{\tilde{c}} - \frac{\beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)}{1-\tilde{c}}$$

$$l''(\tilde{c}) = -\frac{1}{(\tilde{c})^2} - \frac{\beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)}{(1-\tilde{c})^2} < 0$$

and

$$l'(\tilde{c}) = 0 \quad \Leftrightarrow \quad \tilde{c} = \frac{1}{1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)}.$$

So we get

$$c_n^*(x,i) := \frac{x}{1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)}$$

$$a_n^*(x,i) := \frac{\beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)}{1 + \beta_p(N-n,i) + \sum_{k=n+1}^{N-1} \beta_c(N-k,i)} \rho_n^* x.$$

□

Corollary 3. *Let $\beta_c(n,i) = \beta_p(n,i) = \beta^n$, $\beta \in (0,1]$. Then it holds:*

a) The optimal value functions are given by

$$V_n(x,i) = \begin{cases} \frac{1-\beta^{N-n+1}}{1-\beta}\log(x) + d_n, & \text{if } \beta \in (0,1) \\ (N-n+1)\log(x) + d_n, & \text{if } \beta = 1 \end{cases}$$

with $d_n \in \mathbb{R}$.

b) The optimal consumption-investment strategy π^* is given by $\pi^* = \left(f_0^*, \dots, f_{N-1}^*\right)$ with $f_n^*(x,i) = (c_n^*(x,i), a_n^*(x,i))$ and

$$c_n^*(x,i) = \begin{cases} \frac{1-\beta}{1-\beta^{N-n+1}}x, & \text{if } \beta \in (0,1) \\ \frac{1}{N-n+1}x, & \text{if } \beta = 1 \end{cases} \quad x > 0$$

$$a_n^*(x,i) = \begin{cases} \frac{\beta(1-\beta^{N-n})}{1-\beta^{N-n+1}}\rho_n^* x, & \text{if } \beta \in (0,1) \\ \frac{N-n}{N-n+1}\rho_n^* x, & \text{if } \beta = 1 \end{cases} \quad x > 0$$

where ρ_n^* is the optimal solution of (2.7).

Proof.
It holds that

$$\beta_p(N-n)+1+\sum_{k=n+1}^{N-1}\beta_c(N-k) = \beta^{N-n}+1+\sum_{k=n+1}^{N-1}\beta^{N-k} = \sum_{k,=0}^{N-n}\beta^k = \begin{cases} \frac{1-\beta^{N-n+1}}{1-\beta}, & \text{if } \beta \in (0,1) \\ N-n+1, & \text{if } \beta = 1 \end{cases}.$$

We get

$$V_n(x) = \begin{cases} \frac{1-\beta^{N-n+1}}{1-\beta}\log(x) + d_n, & \text{if } \beta \in (0,1) \\ (N-n+1)\log(x) + d_n, & \text{if } \beta = 1 \end{cases}$$

and

$$c_n^*(x) = \begin{cases} \frac{1-\beta}{1-\beta^{N-n+1}}x, & \text{if } \beta \in (0,1) \\ \frac{1}{N-n+1}x, & \text{if } \beta = 1 \end{cases} \quad x > 0$$

$$a_n^*(x) = \begin{cases} \frac{\beta(1-\beta^{N-n})}{1-\beta^{N-n+1}}\rho_n^* x, & \text{if } \beta \in (0,1), \\ \frac{N-n}{N-n+1}\rho_n^* x, & \text{if } \beta = 1, \end{cases} \quad x > 0.$$

□

2.3.2 Power-utility

We consider the case $U_c(x) = U_p(x) = \frac{1}{\gamma}x^\gamma$, where $0 < \gamma < 1$.

We introduce the following one-dimensional optimization problems

$$\sup_{\rho_n \in A_n} \{E[(1+\rho_n R_{n+1})^\gamma]\}, \; n = 0, \dots, N-1, \tag{2.9}$$

where

$$A_n = \{\rho_n \in \mathbb{R}^d | 1 + \rho_n \cdot R_{n+1} \geq 0, \quad \mathbb{P} - a.s.\}. \tag{2.10}$$

From Theorem 1 follows directly that (2.9) has a solution ρ_n^* and we denote the optimal value by v_n.

Theorem 5. *a) The equilibrium value functions are given by*

$$V_n(x,i) = d_n(i)\frac{x^\gamma}{\gamma}, \quad x \geq 0, i \in E_Y,$$

where $d_n(i)$ can be computed recursively by $d_N \equiv 1$

$$d_n(i)^{-\delta} = E_i[1 + v_n^{-\delta} \cdot \Big((1+r_{n+1})^\gamma d_{n+1}(Y_{n+1})$$

$$- \sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n, i)\big)(d_k(Y_k))^{\gamma\delta}(1+r_{n+1})^\gamma$$

$$\cdot \Big(\prod_{j=n+1}^{k-1} (1+r_{j+1})(1+\rho_j^* \cdot R_{j+1})(1-d_j(Y_j))^\delta\Big)^\gamma$$

$$-\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n, i)\big)(1+r_{n+1})^\gamma \Big(\prod_{j=n+1}^{N-1} (1+r_{j+1})(1+\rho_j^* \cdot R_{j+1})(1-d_j(Y_j))^\delta\Big)^\gamma\Big)^{-\delta}],$$

with $\delta := (\gamma-1)^{-1}$ and ρ_n^* is the optimal solution of (2.9).

b) The equilibrium consumption-investment strategy π^* is given by $\pi^* = \Big(f_0^*, \dots, f_{N-1}^*\Big)$ with $f_n^*(x,i) = (c_n^*(x,i), a_n^*(x,i))$ and

$$c_n^*(x,i) = d_n^\delta x, \quad x \geq 0, i \in E_Y$$

$$a_n^*(x,i) = (1-d_n^\delta)\rho_n^* x, \quad x \geq 0, i \in E_Y.$$

Proof.
We prove the statement by induction.
We start with the case $n = N-1$:

$$V_{N-1}(x,i) = \sup_{(c,a) \in D_{N-1}(x)} (L_{N-1}V_N)(x,i,c,a)$$

$$= \sup_{(c,a) \in D_{N-1}} \Big\{\frac{1}{\gamma}c^\gamma + E_{N-1,x,i}[V_N(X_N^{c,a}, Y_N)] - E_{N-1,x}[(1-\beta_p(1,i))\frac{1}{\gamma}(X_N^{c,a})^\gamma)]\Big\}$$

$$= \sup_{(c,a)\in D_{N-1}(x)} \Big\{ \frac{1}{\gamma} c^\gamma + \beta_p(1,i) E[\frac{1}{\gamma}((1+r_N)(x-c+a\cdot R_N))^\gamma] \Big\}$$

Now we substitute $\tilde{c} = \frac{c}{x}$ and $\tilde{a}_n = \frac{a}{x}$ which are the fractions of wealth which will be consumed and invested in the financial market. The set of admissible fractions is given by

$$\tilde{D}_{N-1}(x) := [0,1] \times \tilde{A}^c_{N-1}(x),$$

where

$$\tilde{A}^c_{N-1}(x) = \{\tilde{a}_n \in \mathbb{R}^d | (1+r_N)(1-\tilde{c}_n + \tilde{a}_n \cdot R_N) \geq 0, \quad \mathbb{P}-a.s.)\}.$$

We get

$$V_{N-1}(x,i) = \frac{1}{\gamma} x^\gamma \sup_{(c,a)\in \tilde{D}_{N-1}(x)} \Big\{ \tilde{c}^\gamma + \beta_p(1,i) E[\frac{1}{\gamma}(1+r_N)^\gamma (1-\tilde{c}+\tilde{a}\cdot R_N)^\gamma] \Big\}.$$

We make the transformation $\tilde{a} = \rho(1-\tilde{c})$ where $\rho \in A_{N-1}$ and get

$$V_{N-1}(x,i) = \frac{1}{\gamma} x^\gamma \sup_{(c,\rho)\in[0,1]\times A_{N-1}} \Big\{ \tilde{c}^\gamma + \beta_p(1,i) E[(1+r_N)^\gamma (1-\tilde{c}+(1-\tilde{c})\rho\cdot R_N)^\gamma] \Big\}$$

$$= \frac{1}{\gamma} x^\gamma \sup_{(c,\rho)\in[0,1]\times A_{N-1}} \Big\{ \tilde{c}^\gamma + \beta_p(1,i)(1-\tilde{c})^\gamma (1+r_N)^\gamma E[(1+\rho\cdot R_N)^\gamma] \Big\}.$$

We split the supremum

$$V_{N-1}(x,i) = \frac{1}{\gamma} x^\gamma \sup_{\tilde{c}\in[0,1]} \Big\{ \tilde{c}^\gamma + \beta_p(1,i)(1-\tilde{c})^\gamma (1+r_N)^\gamma \sup_{\rho\in A_{N-1}} E[(1+\rho\cdot R_N)^\gamma] \Big\}.$$

The optimal ρ is given as solution ρ^*_{N-1} of 2.9 with value v_{N-1}. We get

$$V_{N-1}(x,i) = \frac{1}{\gamma} x^\gamma \sup_{c\in[0,1)} \Big\{ \tilde{c}^\gamma + \beta_p(1,i)(1+r_N)^\gamma v_{N-1}(1-\tilde{c})^\gamma \Big\}.$$

Note that a function $g(y) := by^\gamma + d(1-y)^\gamma$ with $b > 0, d > 0$ has its maximum point at

$$y^* = \frac{b^{-\delta}}{b^{\tilde{\delta}} + d^{-\delta}}$$

with value

$$g(y^*) = (b^{-\delta} + d^{-\delta})^{\frac{1}{-\delta}}.$$

This implies that the optimal fractions are given by

$$\tilde{c}^*_{N-1} = \frac{1}{1 + \beta_p^{-\delta}(1,i)(1+r_N)^{-\gamma\delta} v_{n-1}^{-\delta}}$$

and

$$\tilde{a}^*_{N-1} = (1-c^*_{N-1})\rho^*_{N-1} = \frac{\beta_p^{-\delta}(1,i)(1+r_N)^{-\gamma\delta} v_{n-1}^{-\delta}}{1 + \beta_p^{-\delta}(1,i)(1+r_N)^{-\delta} v_{n-1}^{-\delta}} \rho^*_{N-1}.$$

We get

$$V_{N-1}(x,i) = \underbrace{\left(1 + \beta_p^{-\delta}(1,i)(1+r_N)^{-\gamma\delta} v_{N-1}^{-\delta}\right)^{\frac{1}{-\delta}}}_{=d_{N-1}(i)} \frac{1}{\gamma} x^\gamma.$$

Since $d_N(i) = 1$ and get the following recursion for $d_{N-1}(i)$

$$d_{N-1}^{-\delta} = \big(1 + \beta_p^{-\delta}(1,i)(1+r_N)^{-\gamma\delta} v_{N-1}^{-\delta}\big) d_N^{-\delta}.$$

We can rewrite $c_{N-1}^*(x,i)$ and $a_{N-1}^*(x,i)$ as

$$c_{N-1}^*(x,i) = d_{N-1}(i)^\delta x$$

$$a_{N-1}^*(x,i) = (1 - d_{N-1}(i)^\delta)\rho_{N-1}^* x.$$

Now we make the induction step from $n+1 \to n$.

$$\begin{aligned} V_n(x,i) = \sup_{(c,a)\in D_n(x)} (L_n V_{n+1})(x,i,c,a) = \sup_{(c,a)\in D_n(x)} \Big\{ E_{n,x,i}[V_{n+1}(X_{n+1}^{c,a}, Y_{n+1})] + \frac{1}{\gamma} c^\gamma \\ - E_{n,x,i}[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big)\frac{1}{\gamma}(c_k^*(X_k^{\pi^*}, Y_k))^\gamma \\ - E_{n,x,i}[\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n,i)\big)\frac{1}{\gamma}(X_N^{\pi^*})^\gamma]\Big\}. \end{aligned}$$

From the induction hypothesis we know that $V_{n+1}(X_{n+1}^{c,a}, Y_{n+1}) = d_{n+1}(Y_{n+1})\frac{1}{\gamma}(X_{n+1}^{c,a})^\gamma$ and $c_k^*(X_k^{\pi^*}, Y_k) = c_k^*(Y_k) X_k^{\pi^*}$, where $c_k^*(Y_k) = d_k(Y_k)^\delta$.
We get

$$\begin{aligned} V_n(x,i) = \sup_{(c,a)\in D_n(x)} \Big\{ E_{n,x,i}[d_{n+1}(Y_{n+1})\frac{1}{\gamma}(X_{n+1}^{c,a})^\gamma] + \frac{1}{\gamma} c^\gamma \\ - E_{n,x,i}[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big)\frac{1}{\gamma}(c_k^*(Y_k)\cdot X_k^{\pi^*})^\gamma] \\ - E_{n,x,i}[\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n,i)\big)\frac{1}{\gamma}(X_N^{\pi^*})^\gamma]\Big\}. \end{aligned}$$

By applying Lemma 1 we get

$$\begin{aligned} V_n(x,i) = \sup_{(c,a)\in D_n(x)} \Big\{ E_i[d_{n+1}(Y_{n+1})\frac{1}{\gamma}(1+r_{n+1})^\gamma (x - c + a\cdot R_{n+1})^\gamma] + \frac{1}{\gamma} c^\gamma \\ - E_i[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big)\frac{1}{\gamma}(c_k^*(Y_k) Z_{n,k}^{\pi^*})^\gamma \cdot (x - c + a\cdot R_{n+1})^\gamma] \\ - E_i[\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n,i)\big)\frac{1}{\gamma}(Z_{n,N}^{\pi^*})^\gamma \cdot (x - c + a\cdot R_{n+1})^\gamma]\Big\}. \end{aligned}$$

Since $Z^{\pi^*}_{n,k}$ and R_{n+1} are independent we get

$$V_n(x,i) = \frac{1}{\gamma} \sup_{(c,a)\in D_n(x)} \Big\{ c^\gamma + E[(x-c+a\cdot R_{n+1})^\gamma]$$

$$\cdot E_i[(1+r_{n+1})^\gamma d_{n+1}(Y_{n+1}) - \sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big)(c^*_k(Y_k)\cdot Z^{\pi^*}_{n,k})^\gamma$$

$$-\big(\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)\big)(Z^{\pi^*}_{n,N})^\gamma]$$

$$= \frac{1}{\gamma} \sup_{(c,a)\in D_n(x)} \Big\{ c^\gamma + w(i) E[(x-c+a\cdot R_{n+1})^\gamma \Big\},$$

where

$$w(i) := E_i[(1+r_{n+1})^\gamma d_{n+1}(Y_{n+1}) - \sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big)(c^*_k(Y_k)\cdot Z^{\pi^*}_{n,k})^\gamma$$

$$-\big(\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)\big)(Z^{\pi^*}_{n,N})^\gamma].$$

Now we substitute $\tilde{c} = \frac{c}{x}$ and $\tilde{a} = \frac{a}{x}$ which are the fractions of wealth which will be consumed and invested in the financial market. The set of admissible fractions is given by

$$\tilde{D}_n(x) := [0,1] \times \tilde{A}^c_n(x),$$

where

$$\tilde{A}^c_n(x) = \{\tilde{a}_n \in \mathbb{R}^d | (1+r_{n+1})(1-\tilde{c}+\tilde{a}\cdot R_{n+1}) \geq 0 \quad \mathbb{P}-a.s.\}.$$

We get

$$V_n(x,i) = \frac{1}{\gamma} x^\gamma \sup_{(\tilde{c},\tilde{a})\in \tilde{D}_n(x)} \Big\{ \tilde{c}^\gamma + w_n(i) E_i[(1-\tilde{c}+\tilde{a}\cdot R_{n+1})^\gamma].$$

Now make a second substitution $\tilde{a} = \rho_n(1-\tilde{c}))$, where $\rho_n \in A_n$. We get

$$V_n(x,i) = \frac{1}{\gamma} x^\gamma \sup_{(\tilde{c},\tilde{a})\in \tilde{D}_n(x)} \Big\{ \tilde{c}^\gamma + w_n(i) E[(1-\tilde{c}+(1-\tilde{c})\rho_n\cdot R_{n+1})^\gamma] \Big\}.$$

$$= \sup_{(\tilde{c},\rho_n)\in[0,1]\times A_n} \Big\{ \tilde{c}^\gamma + w_n(i)(1-\tilde{c})^\gamma E[(1+\rho_n\cdot R_{n+1})^\gamma] \Big\}.$$

We split the supremum and get

$$V_n(x,i) = \frac{1}{\gamma} x^\gamma \sup_{\tilde{c}_n\in[0,1]} \Big\{ \tilde{c}^\gamma + w_n(i)(1-\tilde{c})^\gamma \sup_{\rho_n\in A_n} \{E[(1+\rho_n\cdot R_{n+1})^\gamma]\} \Big\}.$$

The solution of

$$\sup_{\rho_n\in A_n} \{E[(1+\rho_n\cdot R_{n+1})^\gamma]\}$$

is ρ^*_n and the value of the problem is v_n. We get

$$V_n(x,i) = \frac{1}{\gamma} x^\gamma \sup_{\tilde{c}_n\in[0,1]} \Big\{ \tilde{c}^\gamma + w_n(i)(1-\tilde{c})^\gamma v_n \Big\}.$$

Again we use that the maximum point of $g(y) := by^\gamma + d(1-y)^\gamma$ with $b > 0, d > 0$ is given by

$$y^* = \frac{b^{-\delta}}{b^{-\delta} + d^{-\delta}}$$
$$g(y) = (b^{-\delta} + d^{-\delta})^{\frac{1}{-\delta}}.$$

We get

$$\tilde{c}_n^*(i) = (1 + (w_n(i)v_n)^{-\delta},$$
$$\tilde{a}_n^*(i) = \rho_n^*(1 - \tilde{c}_n^*(i)) = (1 - (1 + w_n(i)v_n)^{-\delta})\rho_n^*$$
$$c_n^*(x,i) = \tilde{c}x = (1 + (w_n(i)v_n)^{-\delta}x$$
$$a_n^*(x,i) = \rho_n^*(1 - \tilde{c}_n^*(i))x = (1 - (1 + w_n(i)v_n)^{-\delta})\rho_n^* x$$

and

$$V_n(x,i) = \underbrace{(1 + (w_n(i)v_n)^{\delta})^{\frac{1}{-\delta}}}_{=:d_n(i)} \frac{1}{\gamma} x^\gamma = d_n(i) \frac{1}{\gamma} x^\gamma.$$

Now we show that the recursion for $d_n(i)$ is true

$$d_n(i)^{-\delta} = (1 + (w_n(i)v_n)^{-\delta})^{-\delta} =$$

$$E_i[1 + v_n^{-\delta} \cdot \Big((1 + r_{n+1})^\gamma d_{n+1}(Y_{n+1}) - \sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n, i)\big)(c_k^*(Y_k))^\gamma (Z_{n,k}^{\pi^*})^\gamma$$
$$-\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n, i)\big)(Z_{n,N}^{\pi^*})^\gamma\Big)^{-\delta}].$$

We can rewrite (c_n^*, a_n^*) as

$$c_n^*(x,i) = \tilde{c}_n^*(i)x = d_n(i)^\delta x$$
$$a_n(x,i) = \rho_n^*(1 - \tilde{c}_n^*(i))x = (1 - d_n(i)^\delta)\rho_n^* x.$$

We can plug $c_k^*(Y_k) = d_k(Y_k))\delta$ in the recursion for $d_n(i)$ and get

$$d_n(i)^{-\delta} = E_i[1 + v_n^{-\delta} \cdot \Big((1 + r_{n+1})^\gamma d_{n+1}(Y_{n+1})$$
$$- \sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n, i)\big) d_k(Y_k)^{\gamma\delta} (Z_{n,k}^{\pi^*})^\gamma$$
$$-\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n, i)\big)(Z_{n,N}^{\pi^*})^\gamma\Big)^{-\delta}].$$

The $Z_{n,k}^{\pi^*}$ can be computed with Lemma 1

$$Z_{n,n+2}^{\pi^*} = (1 + r_{n+1})(1 + r_{n+2})(1 - \tilde{c}_{n+1}(Y_{n+1}) + \tilde{a}_{n+1}(Y_{n+1}) \cdot R_{n+2})$$
$$= (1 + r_{n+1})(1 + r_{n+2})(1 - d_{n+1}(Y_{n+1}))^\delta + (1 - d_{n+1}(Y_{n+1}))^\delta)\rho_{n+1}^* \cdot R_{n+2})$$
$$= (1 + r_{n+1})(1 + r_{n+2})(1 + \rho_{n+1}^* \cdot R_{n+2})(1 - d_{n+1}(Y_{n+1}))^\delta).$$

Since $Z_{n,k+1}^{\pi^*} = (1 + r_{k+1})(1 - \tilde{c}_k(Y_k) + \tilde{a}_k(Y_k) \cdot R_{k+1}) Z_{n,k}^{\pi^*}$ we get by induction

$$Z_{n,k}^{\pi^*} = (1 + r_{n+1}) \prod_{j=n+1}^{k-1} \Big((1 + r_{j+1})(1 + \rho_j^* \cdot R_{j+1})(1 - d_j(Y_j))^\delta)\Big).$$

Hence we get

$$d_n(i)^{-\delta} = E_i[1 + v_n^{-\delta} \cdot \Big((1 + r_{n+1})^\gamma d_{n+1}(Y_{n+1})$$

$$- \sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n, i)\big)(d_k(Y_k))^{\gamma\delta}(1 + r_{n+1})^\gamma$$

$$\cdot\Big(\prod_{j=n+1}^{k-1} (1 + r_{j+1})(1 + \rho_j^* \cdot R_{j+1})(1 - d_j(Y_j))^\delta\Big)^\gamma$$

$$-\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n, i)\big)(1+r_{n+1})^\gamma \Big(\prod_{j=n+1}^{N-1} (1+r_{j+1})(1+\rho_j^* \cdot R_{j+1})(1-d_j(Y_j))^\delta\Big)^\gamma\Big)^{-\delta}].$$

□

Corollary 4. *Let* $\beta_c(n,i) = \beta_p(n,i) = \beta^n$, $\beta \in (0,1]$. *Then it holds:*

a) The optimal value functions are given by

$$V_n(x) = d_n \frac{x^\gamma}{\gamma}, \quad x \geq 0,$$

where d_n *can be computed recursively by*

$$d_N := 1$$

$$d_n^\delta = 1 + (1 + r_{n+1})^{-\delta\gamma} \beta^{-\delta} v_n^{-\delta} d_{n+1}^{-\delta}.$$

b) The optimal consumption-investment strategy π^* *is given by* $\pi^* = \Big(f_0^*, \dots, f_{N-1}^*\Big)$ *with* $f_n^*(x,i) = (c_n^*(x,i), a_n^*(x,i))$ *and*

$$c_n^*(x) = d_n^\delta x, \quad x \geq 0$$

$$a_n^*(x) = (1 - d_n^\delta)\rho_n^* x, \quad x \geq 0,$$

where ρ_n^* *is the optimal solution of (2.9).*

Proof.
We have to show that

$$d_n^{-\delta} = 1 + (1 + r_{n+1})^{-\delta\gamma} \beta^{-\delta} v_n^{-\delta} d_{n+1}^{-\delta}.$$

We show the statement by induction.

$$d_{N-1}^{-\delta} = (1 + v_{N-1}^{-\delta}\Big((1 + r_N)^\gamma - (1-\beta)(1 + r_N)^\gamma\Big)^{-\delta} = (1 + v_{N-1}^{-\delta} \beta^{-\delta} (1 + r_N)^{-\delta\gamma}).$$

We make the induction step from $n+1 \to n$.
Since the R_i are independent we get

$$d_n^{-\delta} = \Big[1 + v_n^{-\delta} \cdot \Big((1+r_{n+1})^\gamma d_{n+1}$$

$$- \sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1) - \beta_c(k-n)\big)(d_k)^{\gamma\delta}(1+r_{n+1})^\gamma \Big(\prod_{j=n+1}^{k-1} (1+r_{j+1})^\gamma E[(1+\rho_j^* \cdot R_{j+1})^\gamma](1-d_j^\delta)^\gamma$$

$$-\big(\beta_p(N-n-1) - \beta_p(N-n)\big)(1+r_{n+1})^\gamma \Big(\prod_{j=n+1}^{N-1} (1+r_{j+1})^\gamma E[(1+\rho_j^* \cdot R_{j+1})^\gamma](1-d_j^\delta)^\gamma \Big)^{-\delta} \Big].$$

Moreover, it holds that $v_j = E[(1+\rho_j^* \cdot R_{j+1})^\gamma]$ so we get

$$d_n^{-\delta} = 1 + v_n^{-\delta} \cdot \Big[(1+r_{n+1})^\gamma d_{n+1}$$

$$- \sum_{k=n+1}^{N-1} \big(\beta^{k-n-1} - \beta^{k-n}\big)(d_k)^{-\gamma\delta}(1+r_{n+1})^\gamma \prod_{j=n+1}^{k-1} (1+r_{j+1})^\gamma v_j (1-d_j^\delta)^\gamma$$

$$-\big(\beta^{N-n-1} - \beta^{N-n}\big)(1+r_{n+1})^\gamma \prod_{j=n+1}^{N-1} (1+r_{j+1})^\gamma v_j (1-d_j^\delta)^\gamma \Big]^{-\delta}$$

$$= 1 + v_n^{-\delta} \cdot \Big[(1+r_{n+1})^\gamma d_{n+1}$$

$$-(1+r_{n+1})^\gamma \sum_{k=n+1}^{N} \big(\beta^{k-n-1} - \beta^{k-n}\big)(d_k)^{\gamma\delta} \prod_{j=n+1}^{k-1} (1+r_{j+1})^\gamma v_j (1-d_j^\delta)^\gamma \Big]^{-\delta}$$

$$= 1 + v_n^\delta (1+r_{n+1})^{-\delta\gamma} \cdot \Big[d_{n+1}$$

$$-(d_{n+1})^{\gamma\delta} + (1+r_{n+1})^\gamma \beta^{N-n} (d_N)^{\gamma\delta} \prod_{j=n+1}^{N-1} (1+r_{j+1})^\gamma v_j (1-d_j^\delta)^\gamma$$

$$- \sum_{k=n+1}^{N-1} \beta^{k-n} \Big(\prod_{j=n+1}^{k-1} (1+r_{j+1})^\gamma v_j (1-d_j^\delta)^\gamma \Big) \Big((d_{k+1})^{\gamma\delta}(1+r_{k+1})^\gamma v_k (1-d_j^\delta)^\gamma - (d_k)^{\gamma\delta} \Big) \Big]^{-\delta}.$$

Furthermore it holds that

$$(d_{k+1})^{\gamma\delta}(1+r_{k+1})^\gamma v_k (1-d_j^\delta)^\gamma - (d_k)^{\gamma\delta}$$

$$= (d_k)^{\gamma\delta} \cdot \Big((d_{k+1})^{\gamma\delta}(1+r_{k+1})^\gamma v_k ((d_k)^{-\delta} - 1)^\gamma - 1 \Big).$$

Now we use that $(d_k)^{-\delta} = 1 + (1+r_{k+1})^{-\delta\gamma} v_k^{-\delta} \beta^{-\delta} d_{k+1}^{-\delta}$ we get

$$= (d_k)^{\gamma\delta} \cdot \Big((d_{k+1})^{\gamma\delta}(1+r_{k+1})^\gamma v_k (1+r_{k+1})^{-\delta\gamma^2} v_k^{-\delta\gamma} \beta^{-\delta\gamma} (d_{k+1})^{-\delta\gamma} - 1 \Big)$$

$$= (d_k)^{\gamma\delta} \Big((1+r_{k+1})^{\gamma(-\delta\gamma+1)} v_k^{-\delta\gamma+1} \beta^{-\delta\gamma} - 1 \Big),$$

since $-\delta = (1-\gamma)^{-1}$ it holds that $-\delta\gamma + 1 = -\delta$ so we get

$$= (d_k)^{\gamma\delta}\Big((1+r_{k+1})^{-\gamma\delta} v_k^{\delta} \beta^{-\delta\gamma} - 1\Big).$$

So we get

$$d_n^{-\delta} = 1 + v_n^{-\delta}(1+r_{n+1})^{\gamma} \cdot \Big[d_{n+1}\gamma - (\gamma d_{n+1})^{\gamma\delta} + \beta^{N-n} \prod_{j=n+1}^{N-1} (1+r_{j+1})^{\gamma} v_j (1-d_j^{\delta})^{\gamma}$$

$$- \sum_{k=n+1}^{N} \beta^{k-n} \Big(\prod_{j=n+1}^{k-1} (1+r_{j+1})^{\gamma} v_j (1-d_j^{\delta})^{\gamma}\Big)(d_k)^{\gamma\delta}\Big((1+r_{k+1})^{-\gamma\delta} v_k^{-\delta} \beta^{-\delta\gamma} - 1\Big)\Big]^{-\delta}.$$

Now we use that $(d_j)^{-\delta} - 1 = (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta} d_{j+1}^{-\delta}$ and get

$$\prod_{j=n+1}^{k-1} (1+r_{j+1})^{\gamma} v_j (1-d_j^{\delta})^{\gamma} = \prod_{j=n+1}^{k-1} (1+r_{j+1})^{\gamma} v_j \Big(\frac{(1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta} (d_{j+1})^{-\delta}}{(d_j)^{-\delta}}\Big)^{\gamma}$$

$$= \frac{(d_k)^{-\gamma\delta}}{(d_{n+1})^{-\gamma\delta}} \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma(\gamma\delta+1)} v_j^{-\delta\gamma+1} \beta^{-\delta\gamma} = \frac{(d_k)^{-\gamma\delta}}{(d_{n+1})^{-\gamma\delta}} \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma}.$$

Hence we get

$$d_n^{-\delta} = 1 + v_n^{-\delta}(1+r_{n+1})^{\gamma} \cdot \Big[d_{n+1}(d_{n+1})^{\gamma\delta} + \beta^{N-n} \frac{(d_N)^{-\gamma\delta}}{(d_{n+1})^{-\gamma\delta}} \prod_{j=n+1}^{N-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma}$$

$$- \sum_{k=n+1}^{N-1} \beta^{k-n} \Big(\frac{(d_k)^{-\gamma\delta}}{(d_{n+1})^{-\gamma\delta}} \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma}\Big)(\gamma d_k)^{\gamma\delta}\Big((1+r_{k+1})^{-\gamma\delta} v_k^{-\delta} \beta^{-\delta\gamma} - 1\Big)\Big]^{-\delta}$$

$$= 1 + v_n - \delta(1+r_{n+1})^{\gamma} \cdot \Big[d_{n+1} - (d_{n+1})^{\gamma\delta} + \beta^{N-n}(d_{n+1})^{\gamma\delta} \prod_{j=n+1}^{N-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma}$$

$$- \sum_{k=n+1}^{N-1} \beta^{k-n}(d_{n+1})^{\gamma\delta} \Big(\prod_{j=n+1}^{k} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma} - \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma}\Big)]^{-\delta}$$

$$= 1 + v_n^{-\delta}(1+r_{n+1})^{\gamma} d_{n+1}^{-\delta} \cdot \Big[1 - (d_{n+1})^{\gamma\delta-1} + \beta^{N-n}(d_{n+1})^{\gamma\delta-1} \prod_{j=n+1}^{N-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma}$$

$$- \sum_{k=n+1}^{N-1} \beta^{k-n}(d_{n+1})^{\gamma\delta-1} \Big(\prod_{j=n+1}^{k} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma} - \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma}\Big)]^{-\delta}$$

$$= 1 + v_n^{\delta}(1+r_{n+1})^{\gamma} d_{n+1}^{-\delta} \cdot \Big[1 - (d_{n+1})^{\delta} + (d_{n+1})^{\delta}\beta^{N-n} \prod_{j=n+1}^{N-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma}$$

$$-(d_{n+1})^{\delta} \sum_{k=n+1}^{N-1} \beta^{k-n} \Big(\prod_{j=n+1}^{k} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} \beta^{-\delta\gamma} - \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma\delta} v_j^{\delta} \beta^{-\delta\gamma}\Big)]^{-\delta}.$$

Now we have to show that

$$\beta = 1 - (d_{n+1})^{\delta} + d_{n+1}^{\delta}\beta^{N-n}\prod_{j=n+1}^{N-1}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}\beta^{-\delta\gamma}$$

$$-d_{n+1}^{\delta}\sum_{k=n+1}^{N-1}\beta^{k-n}\Big(\prod_{j=n+1}^{k}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}\beta^{-\delta\gamma} - \prod_{j=n+1}^{k-1}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}\beta^{-\delta\gamma}\Big)$$

$$\Leftrightarrow (\beta-1)(d_{n+1})^{-\delta} = \Big(-1 + \beta^{N-n}\prod_{j=n+1}^{N-1}(1+r_{j+1})^{-\gamma\delta}v_j^{\delta}\beta^{-\delta\gamma}$$

$$-\sum_{k=n+1}^{N-1}\beta^{k-n}\Big(\prod_{j=n+1}^{k}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}\beta^{-\delta\gamma} - \prod_{j=n+1}^{k-1}(1+r_{j+1})^{-\gamma\delta}v_j^{\delta}\beta^{-\delta\gamma}\Big).$$

By induction it can be shown that a explicit formula for d_{n+1}^{δ} is given by

$$d_{n+1}^{-\delta} = 1 + \sum_{k=n+1}^{N-1}\beta^{-(k-n)\delta}\prod_{m=n+1}^{k}(1+r_{m+1})^{-\gamma\delta}v{-}\delta_m$$

$$\Leftrightarrow d_{n+1}^{-\delta} = 1 + \sum_{k=n+1}^{N-1}\beta^{-\delta(k+1-n)}\prod_{m=n+1}^{k}(1+r_{m+1})^{-\gamma\delta}v_m^{-\delta}.$$

Hence we get

$$(\beta-1)\Big(1+\sum_{k=n+1}^{N-1}\beta^{-(k-n)\delta}\prod_{m=n+1}^{k}(1+r_{m+1})^{-\gamma\delta}v_m^{-\delta}\Big) = -1 + \beta^{N-n}\prod_{j=n+1}^{N-1}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}\beta^{-\delta\gamma}$$

$$-\sum_{k=n+1}^{N-1}\beta^{k-n}\Big(\prod_{j=n+1}^{k}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}\beta^{-\delta\gamma} - \prod_{j=n+1}^{k-1}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}\beta^{-\delta\gamma}\Big)$$

$$\Leftrightarrow \beta + (\beta-1)\sum_{k=n+1}^{N-1}\beta^{-(k-n)\delta}\prod_{m=n+1}^{k}(1+r_{m+1})^{-\gamma\delta}v_m^{-\delta} = \beta^{N-n+(N-n-1)\gamma\delta}\prod_{j=n+1}^{N-1}(1+r_{j+1})^{\gamma\delta}v_j^{\delta}$$

$$-\sum_{k=n+1}^{N-1}\beta^{k-n-(k-n)\delta\gamma}\prod_{j=n+1}^{k}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}\beta^{-\delta\gamma} + \sum_{k=n+1}^{N-1}\beta^{k-n-(k-n-1)\delta\gamma}\prod_{j=n+1}^{k-1}(1+r_{j+1})^{\gamma\delta}v_j^{\delta}\beta^{\delta\gamma}$$

$$\Leftrightarrow \beta + \sum_{k=n+1}^{N-1}\beta^{-(k-n)\delta+1}\prod_{m=n+1}^{k}(1+r_{m+1})^{-\gamma\delta}v_m^{-\delta} - \sum_{k=n+1}^{N-1}\beta^{-(k-n)\delta}\prod_{m=n+1}^{k}(1+r_{m+1})^{\gamma\delta}v_m^{\delta}$$

$$= \beta^{N-n-(N-n-1)\gamma\delta}\prod_{j=n+1}^{N-1}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}$$

$$-\sum_{k=n+1}^{N-1}\beta^{k-n-(k-n)\delta\gamma}\prod_{j=n+1}^{k}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta} + \sum_{k=n+1}^{N-1}\beta^{k-n-(k-n-1)\delta\gamma}\prod_{j=n+1}^{k-1}(1+r_{j+1})^{-\gamma\delta}v_j^{-\delta}.$$

It holds thats

$$k - n - (k-n)\delta\gamma = -(k-n)\delta$$

$$k - n - (k-n-1)\delta\gamma = -(k-n-1)\delta + 1 \quad \text{and} \quad N - n - (N-n-1)\delta\gamma = -(N-n-1)\delta + 1.$$

So we get

$$\Leftrightarrow \beta + \sum_{k=n+1}^{N-1} \beta^{-(k-n)\delta+1} \prod_{m=n+1}^{k} (1+r_{m+1})^{-\gamma\delta} v_m^{-\delta} - \sum_{k=n+1}^{N-1} \beta^{-(k-n)\delta} \prod_{m=n+1}^{k} (1+r_{m+1})^{-\gamma\delta} v_m^{-\delta}$$

$$= \beta^{-(N-n-1)\delta+1} \prod_{j=n+1}^{N-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta}$$

$$- \sum_{k=n+1}^{N-1} \beta^{-(k-n)\delta} \prod_{j=n+1}^{k} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} + \sum_{k=n+1}^{N-1} \beta^{-(k-n-1)\delta+1} \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta}$$

$$\Leftrightarrow \beta + \sum_{k=n+1}^{N-1} \beta^{(k-n)\delta+1} \prod_{m=n+1}^{k} (1+r_{m+1})^{\gamma\delta} v_m^{\delta}$$

$$= \beta^{-(N-n-1)\delta+1} \prod_{j=n+1}^{N-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta} + \sum_{k=n+1}^{N-1} \beta^{-(k-n-1)\delta+1} \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta}$$

$$\Leftrightarrow \beta + \sum_{k=n+1}^{N-1} \beta^{-(k-n)\delta+1} \prod_{m=n+1}^{k} (1+r_{m+1})^{-\gamma\delta} v_m^{-\delta} = \beta + \sum_{k=n+2}^{N} \beta^{-(k-n-1)\delta+1} \prod_{j=n+1}^{k-1} (1+r_{j+1})^{-\gamma\delta} v_j^{-\delta}.$$

Now we only need to shift the index on the right-hand side and get the statement.

□

2.3.3 Exponential-utility

Now we consider the case $U_c(x) = U_p(x) = -\exp(-\gamma x)$, $\gamma > 0$ and $dom(U) = \mathbb{R}$. In this case we allow negative consumption, i.e. we choose $D_n(x) := \mathbb{R} \times \mathbb{R}^d$.

In this section we set $r_n \equiv 0$, $n = 1, \ldots, N$. We introduce the following one-dimensional optimization problems

$$\inf_{a_n \in \mathbb{R}} \left\{ E[-\exp(-\gamma \frac{1}{N-n} a_n \cdot R_n \cdot) \right\}, \; n = 0, \ldots, N-1. \tag{2.11}$$

From Theorem 1 and Remark 1 it follows directly that (2.11) has a solution a_n^* and we denote the optimal value by v_n.

Theorem 6. *a) The equilibrium value functions are given by*

$$V_n(x,i) = -d_n(i) \exp(-\frac{\gamma}{N-n+1} x),$$

where $d_N(i) = 1$ *and*

$$d_n(i) := \left(\frac{G_n(i)}{N-n} \right)^{\frac{N-n}{N-n+1}} + G_n(i) \left(\frac{G_n(i)}{N-n} \right)^{-\frac{1}{N-n+1}}.$$

The $G_n(i)$ *can be computed recursively by*

$$G_n(i) = v_n \Big(d_{n+1}(Y_{n+1})$$

$$-E_i[\sum_{k=n+1}^{N-1} \Big(\big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n, i)\big) \left(\frac{G_k(Y_k)}{N-k} \right)^{\frac{N-k}{N-k+1}} \exp(-\gamma W_{n,k}) \Big)$$

$$-E_i[\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n, i)\big) \exp(-\gamma W_{n,N})] \Big),$$

where $W_{n,n+1} = 0$ *and*

$$W_{n,k} = \sum_{j=n+1}^{k-1} \frac{N-k+1}{(N-j+1)\gamma} \log(\frac{G_j(Y_j)}{N-j}) + \frac{N-k+1}{N-j} a_j^* \cdot R_{j+1}.$$

b) The equilibrium consumption-investment strategy π^* *is given by* $\pi^* = \Big(f_0^*, \ldots, f_{N-1}^* \Big)$ *with* $f_n^*(x,i) = (c_n^*(x,i), a_n^*(x,i))$ *and*

$$c_n^*(x,i) = \frac{x}{N-n+1} - \frac{N-n}{(N-n+1)\gamma} \log(\frac{G_n(i)}{N-n})$$
$$a_n^*(x,i) := a_n^*,$$

where a_n^* *is the solution and* v_n *optimal value of (2.11).*

Lemma 2. *Let* $(c_k(x,i), a_k(x,i)) = (\frac{x}{N-k+1} - \frac{N-k}{(N-k+1)\gamma}\log\left(\frac{G_k(Y_k)}{N-k}\right), a_k^*)$ *for* $k = n+1, \ldots, N-1$. *Further let* (n, x, i) *be the current time, the current wealth, the current state of the and* $\bar{\pi} = \big((\bar{c}_n, \bar{a}_n), (c_{n+1}, a_{n+1}), \ldots, (c_{N-1}, a_{N-1})\big)$, *where* (c_n, a_n) *is an arbitrary decision rule. Then we can write the wealth process* $X_k^{\bar{\pi}}$, $k = n+1, \ldots, N$, *as*

$$X_k^{\bar{\pi}} = \frac{N-k+1}{N-n}(x - \bar{c}_n(x,i) + \bar{a}_n(x,i)\cdot R_{n+1}) + W_{n,k},$$

where

$$W_{n,n+1} := 0$$

$$W_{n,k} = \sum_{j=n+1}^{k-1} \frac{N-k+1}{(N-j+1)\gamma}\log(\frac{G_j(Y_j)}{N-j}) + \frac{N-k+1}{N-j}a_j^* \cdot R_{j+1}.$$

Proof.
To prove the statement we show by induction that $W_{n,k}$ can recursively computed by

$$W_{n,n+2} := \frac{N-n-1}{(N-n)\gamma}\log(\frac{G_{n+1}(Y_{n+1})}{N-n-1} + a_{n+1}^* \cdot R_{n+2}$$

$$W_{n,k+1} = \frac{N-k}{N-k+1}W_{n,k} + \frac{N-k}{(N-k+1)\gamma}\log\left(\frac{G_k(Y_k)}{N-k}\right) + a_k^* \cdot R_{k+1}.$$

Applying this recursion formula it follows that

$$W_{n,k} = \sum_{j=n+1}^{k-1} \frac{N-k+1}{(N-j+1)\gamma}\log(\frac{G_j(Y_j)}{N-j}) + \frac{N-k+1}{N-j}a_j^* \cdot R_{j+1}.$$

For $k = n+2$ we get

$$X_{n+2}^{\bar{\pi}} = X_{n+1}^{c,a} - c_{n+1}(x,i) + a_{n+1}(x,i)\cdot R_{n+2}$$

$$= X_{n+1}^{c,a} - \frac{X_{n+1}^{\bar{\pi}}}{N-n} + \frac{N-n-1}{(N-n)\gamma}\log\left(\frac{G_{n+1}(Y_{n+1})}{N-n-1}\right) + a_{n+1}^* \cdot R_{n+2}$$

$$= \frac{N-n-1}{N-n}X_{n+1}^{\bar{\pi}} + \frac{N-n-1}{(N-n)\gamma}\log\left(\frac{G_{n+1}(Y_{n+1})}{N-n-1}\right) + a_{n+1}^* \cdot R_{n+2}$$

$$= \frac{N-n-1}{N-n}(x - \bar{c}_n(x,i) + \bar{a}_n(x,i)\cdot R_{n+1}) + \frac{N-n-1}{(N-n)\gamma}\log\left(\frac{G_{n+1}(Y_{n+1})}{N-n-1}\right) + a_{n+1}^* \cdot R_{n+2}$$

$$= \frac{N-n-1}{N-n}(x - \bar{c}_n(x,i) + \bar{a}_n(x,i)\cdot R_{n+1}) + W_{n,n+2}.$$

Now we make the step from $k \to k+1$.

$$X_{k+1}^{\bar{\pi}} = X_k^{\bar{\pi}} - c_k(x,i) + a_k(x,i)\cdot R_{k+1} = X_k^{\bar{\pi}} - \frac{X_k^{\bar{\pi}}}{N-k+1} + \frac{N-k}{(N-k+1)\gamma}\log\left(\frac{G_k(Y_k)}{N-k}\right) + a_k^* \cdot R_{k+1}$$

$$= \frac{N-k}{N-k+1}X_k^{\bar{\pi}} + \frac{N-k}{(N-k+1)\gamma}\log\left(\frac{G_k(Y_k)}{N-k}\right) + a_k^* \cdot R_{k+1}$$

$$\begin{aligned}
&\overset{IH}{=} \frac{N-k}{N-k+1}\left(\frac{N-k+1}{N-n}(x-\bar{c}_n(x,i)+\bar{a}_n(x,i)\cdot R_{n+1})+W_{n,k}\right)\\
&\qquad +\frac{N-k}{(N-k+1)\gamma}\log\left(\frac{G_k(Y_k)}{N-k}\right)+a_k^*\cdot R_{k+1}\\
&= \frac{N-k}{N-n}(x-\bar{c}_n(x,i)+\bar{a}_n(x,i)\cdot R_{n+1})+\frac{N-k}{N-k+1}W_{n,k}\\
&\qquad +\frac{N-k}{(N-k+1)\gamma}\log\left(\frac{G_k(Y_k)}{N-k}\right)+a_k^*\cdot R_{k+1}\\
&= \frac{N-k}{N-n}(x-\bar{c}_n(x,i)+\bar{a}_n(x,i)\cdot R_{n+1})+W_{n,k+1},
\end{aligned}$$

where

$$W_{n,k+1}=\frac{N-k}{N-k+1}W_{n,k}+\frac{N-k}{(N-k+1)\gamma}\log\left(\frac{G_k(Y_k)}{N-k}\right)+a_k^*\cdot R_{k+1}.$$

□

Proof Theorem 6.
We will show the statement by induction.

$$V_{N-1}(x,i)=\sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d}(L_{N-1}V_N)(x,i,c,a)$$

$$=\sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d}\Big\{-\exp(-\gamma c)+E_{N-1,x,i}[V_N(X_N^{c,a},Y_N)]-E_{N-1,x}[(1-\beta_p(1,i))(-\exp(-\gamma X_N^{c,a}))]\Big\}$$

$$=\sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d}\Big\{-\exp(-\gamma c)-\beta_p(1,i)E[\exp(-\gamma(x-c+a\cdot R_N))]\Big\}$$

$$=\sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d}\Big\{-\exp(-\gamma c)-\beta_p(1,i)\exp(-\gamma(x-c))E[\exp(-\gamma a\cdot R_N))]\Big\}.$$

We split the supremum and get

$$V_n(x,i)=\sup_{c\in\mathbb{R}}\Big\{-\exp(-\gamma c)-\beta_p(1,i)\exp(-\gamma(x-c))\inf_{a\in\mathbb{R}^d}\{E[\exp(-\gamma a\cdot R_N))]\}\Big\}.$$

The optimal a is given as solution a_{N-1}^* of (2.11) with value v_{N-1}. We get

$$V_{N-1}(x,i)=\sup_{c\in\mathbb{R}}\Big\{-\exp(-\gamma c)-\beta_p(1,i)\exp(-\gamma(x-c))v_{N-1}\Big\}.$$

To get c_{N-1}^* we have to solve the optimization problem

$$\sup_{c\in\mathbb{R}}\Big\{-\exp(-\gamma c)-\beta_p(1,i)\exp(-\gamma(x-c))v_{N-1}\Big\}.$$

The solution c_{N-1}^* is given by

$$c_{N-1}^*=\frac{x}{2}-\frac{\log(v_{N-1}\beta_p(1,i))}{2\gamma}.$$

Moreover we get

$$V_{N-1}(x,i) = -\exp\left(-\gamma(\frac{x}{2} - \frac{\log(v_{n-1}\beta_p(1,i))}{2\gamma})\right)$$
$$-\beta_p(1,i)\exp\left(-\gamma x + \gamma(\frac{x}{2} - \frac{\log(v_{N-1}\beta_p(1,i))}{2\gamma})\right) v_{N-1}$$
$$= -\exp\left(-\gamma\frac{x}{2}\right)\left((v_{N-1}\beta_p(1,i))^{\frac{1}{2}} + v_{N-1}\beta_p(1,i)(v_{N-1}\beta_p(1,i))^{-\frac{1}{2}}\right)$$
$$= -\left(G_{N-1}(i)^{\frac{1}{2}} - G_{N-1}(i)G_{N-1}(i)^{-\frac{1}{2}}\right)\exp\left(-\gamma\frac{x}{2}\right) = -d_{N-1}\exp\left(-\gamma\frac{x}{2}\right).$$

Now we make the induction step from $n+1 \to n$:

$$V_n(x,i) = \sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d}(L_nV_{n+1})(x,i,c,a) = \sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d}\Big\{E_{n,x,i}[V_{n+1}(X_{n+1}^{c,a},Y_{n+1})] - \exp(-\gamma c)$$
$$-E_{n,x,i}[\sum_{k=n+1}^{N-1}\left(\beta_c(k-n-1,Y_{n+1}) - \beta_c(k-n,i)\right)\left(-\exp\left(-\gamma c_k^*(X_k^{\pi^*},Y_k)\right)\right)$$
$$-E_{n,x,i}[\left(\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)\right)\left(-\exp(\left(-\gamma X_N^{\pi^*}\right))\right]\Big\}.$$

From the induction hypothesis we know that $V_{n+1}(X_{n+1}^{c,a},Y_{n+1}) = -d_{n+1}(Y_{n+1})\exp(-\frac{\gamma}{N-n}X_{n+1}^{c,a})$, where $d_{n+1}(Y_{n+1}) = \left(\frac{G_{n+1}(Y_{n+1})}{N-n-1}\right)^{\frac{N-n-1}{N-n}} + G_{n+1}(Y_{n+1})\left(\frac{G_{n+1}(Y_{n+1})}{N-n-1}\right)^{-\frac{N-n-1}{N-n}}$ and $c_k^*(X_k^{\pi^*},Y_k) = \frac{X_k^{\pi^*}}{N-k+1} + \alpha_k(Y_k)$, where $\alpha_k(Y_k) = -\frac{N-k}{(N-k+1)\gamma}\log(\frac{G_k(Y_k)}{N-k})$.
We get

$$V_n(x,i) = \sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d}\Big\{E_{n,x,i}[-d_{n+1}(Y_{n+1})\exp\left(-\frac{\gamma}{N-n}(x-c+a\cdot R_{n+1})\right)] - \exp(-\gamma c)$$
$$-E_{n,x,i}[\sum_{k=n+1}^{N-1}\left(\beta_c(k-n-1,Y_{n+1}) - \beta_c(k-n,i)\right)\left(-\exp\left(-\gamma\frac{X_k^{\pi^*}}{N-k+1} - \gamma\alpha_k(Y_k)\right)\right)]$$
$$-E_{n,x,i}[\left(\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)\right)\left(-\exp\left(-\gamma X_N^{\pi^*}\right)\right)]\Big\}.$$

Applying Lemma 2 we get

$$V_n(x,i) = \sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d}\Big\{E[-d_{n+1}(Y_{n+1})\exp\left(-\frac{\gamma}{N-n}(x-c+a\cdot R_{n+1})\right)] - \exp(-\gamma c)$$
$$-E_i[\sum_{k=n+1}^{N-1}\left(\beta_c(k-n-1,Y_{n+1}) - \beta_c(k-n,i)\right)$$
$$\left(-\exp\left(-\frac{\gamma}{N-k+1}\frac{N-k+1}{N-n}(x-c+a\cdot R_{n+1}) - \gamma W_{n,k} - \gamma\alpha_k(Y_k)\right)\right)]$$
$$-E_i[\left(\beta_p(N-n-1,Y_{n+1}) - \beta_p(N-n,i)\right)\left(-\exp\left(-\gamma\frac{1}{N-n}(x-c+a\cdot R_{n+1}) - \gamma W_{n,N}\right)\right)]\Big\}$$

$$= \sup_{(c,a)\in\mathbb{R}\times\mathbb{R}^d} \Big\{ -\exp(-\gamma c) - E[\exp\left(-\gamma a \frac{1}{N-n}\cdot R_{n+1}\right)] \Big[d_{n+1}(Y_{n+1}) \exp(-\frac{\gamma}{N-n}(x-c))$$

$$-E_i[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big) \exp\left(-\frac{\gamma}{N-n}(x-c) - \gamma W_{n,k} - \gamma\alpha_k(Y_k)k\right)]$$

$$-E_i[\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n,i)\big) \exp\left(-\gamma\frac{1}{N-n}(x-c) - \gamma W_{n,N}\right)]\Big]\Big\}.$$

We split the supremum and get

$$V_n(x,i) = \sup_{c\in\mathbb{R}} \Big\{ -\exp(-\gamma c) - \inf_{a\in\mathbb{R}^d} \Big\{ E[-\exp(-\gamma \frac{1}{N-n} a \cdot R_{n+1})] \Big\}$$

$$\Big[d_{n+1}(Y_{n+1}) \exp\left(-\frac{\gamma}{N-n}(x-c)\right)$$

$$-E_i[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big) \exp\left(-\frac{\gamma}{N-n}(x-c) - \gamma W_{n,k} - \gamma\alpha_k(Y_k)\right)]$$

$$-E_i[\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n,i)\big) \exp\left(-\gamma\frac{1}{N-n}(x-c) - \gamma W_{n,N}\right)]\Big]\Big\}.$$

The the solution of the inner supremum is given by a_n^* and the value by v_n, so we get

$$V_n(x,i) = \sup_{c\in\mathbb{R}} \Big\{ -\exp(-\gamma c) - v_n \Big[d_{n+1}(Y_{n+1}) \exp\left(-\frac{\gamma}{N-n}(x-c)\right)$$

$$-E_i[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big) \exp\left(-\frac{\gamma}{N-n}(x-c) - \gamma W_{n,k} - \gamma\alpha_k(Y_k)\right)$$

$$-E_i[\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n,i)\big) \exp\left(-\gamma\frac{1}{N-n}(x-c) - \gamma W_{k,N}\right)]\Big]\Big\}$$

$$= \sup_{c\in\mathbb{R}} \Big\{ -\exp\left(-\gamma c\right) - G_n(i) \exp\left(-\frac{\gamma}{N-n}(x-c)\right)\Big\},$$

where

$$G_n(i) = v_n \Big[d_{n+1}(Y_{n+1}) - E_i[\sum_{k=n+1}^{N-1} \beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big) \exp(-\gamma W_{n,k} - \gamma\alpha_k(Y_k))]$$

$$-E_i[\big(\beta_p(N-n-1, Y_{n+1}) - \beta_p(N-n,i)\big)(\exp(-\gamma W_{k,N})]\Big].$$

We plug in $\alpha_k(Y_k) = -\frac{N-k}{(N-k+1)\gamma}\log(\frac{G_k(Y_k)}{N-k})$ and get

$$G_n(i) = v_n \Big[d_{n+1}(Y_{n+1} - E_i[\sum_{k=n+1}^{N-1} \big(\beta_c(k-n-1, Y_{n+1}) - \beta_c(k-n,i)\big) \left(\frac{G_k(Y_k)}{N-k}\right)^{\frac{N-k}{N-k+1}} \exp\left(-\gamma W_{n,k}\right)$$

$$-E_i[\big(\beta_p(N-n-1,Y_{n+1})-\beta_p(N-n,i)\big)\exp\left(-\gamma W_{n,N}\right)]\Big].$$

The solution of this one-dimensional optimization problem is given by

$$c_n^*(x,i)=\frac{x}{N-n+1}-\frac{N-n}{(N-n+1)\gamma}\log(\frac{G_n(i)}{N-n}).$$

Plugging that in we get

$$V_n(x,i)=-\exp\left(-\gamma\Big(\frac{x}{N-n+1}-\frac{N-n}{(N-n+1)\gamma}\log(\frac{G_n(i)}{N-n})\Big)\right)$$

$$-G_n(i)\exp\left(-\frac{\gamma}{N-n}(x-\frac{x}{N-n+1}+\frac{N-n}{(N-n+1)\gamma}\log(\frac{G_n(i)}{N-n}))\right)$$

$$=-\exp\left(-\frac{\gamma}{N-n+1}x\right)\left(\left(\frac{G_n(i)}{N-n}\right)^{\frac{N-n}{N-n+1}}+G_n(i)\left(\frac{G_n(i)}{N-n}\right)^{-\frac{1}{N-n+1}}\right).$$

$$=-d_n(i)\exp\left(-\frac{\gamma}{N-n+1}x\right).$$

From Lemma 2 follows that $W_{n,k}$ is given by

$$W_{n,k}=\sum_{j=n+1}^{k-1}\frac{N-k+1}{(N-j+1)\gamma}\log(\frac{G_j(Y_j)}{N-j})+\frac{N-k+1}{N-j}a_j^*\cdot R_{j+1}.$$

We proved the statement.

□

Corollary 5. *Let $\beta_c(n,i)=\beta_p(n,i)=\beta^n$, $\beta\in(0,1]$, then it holds:*

a) The optimal value functions are given by

$$V_n(x)=-d_n\exp(-\frac{\gamma}{N-n+1}x),\quad x\in\mathbb{R},$$

where d_n can be computed recursively by

$$d_N=1$$

$$d_n=\left(\frac{\beta v_n d_{n+1}}{N-n}\right)^{\frac{N-n}{N-n+1}}+\beta v_n d_{n+1}\left(\frac{\beta v_n d_{n+1}}{N-n}\right)^{-\frac{1}{N-n+1}}.$$

b) The optimal consumption-investment strategy π^* is given by $\pi^*=\left(f_0^*,\dots,f_{N-1}^*\right)$ with $f_n^*(x,i)=(c_n^*(x,i),a_n^*(x,i))$ and

$$c_n^*(x)=\frac{x}{N-n+1}-\frac{N-n}{(N-n+1)\gamma}\log(\frac{\beta v_n d_{n+1}}{N-n}),\quad x\in\mathbb{R}$$

$$a_n^*(x)=a_n^*,\quad x\in\mathbb{R},$$

where a_n^* is the solution and v_n optimal value of (2.11).

Proof.

In this case the extended Bellman equation reduces to the classical Bellman equation, see Corollary 2.

We prove the statement by induction.

$n = N-1$

$$\begin{aligned} V_{N-1}(x) &= \sup_{(c,a)\in D_{N-1}(x)} \{-\exp(-\gamma c) + \beta E[V_N(x-c+a\cdot R_N)]\} \\ &= \sup_{(c,a)\in D_{N-1}(x)} \{-\exp(-\gamma c) - \beta E[\exp(-\gamma(x-c+a\cdot R_N))]\} \\ &= \sup_{(c,a)\in D_{N-1}(x)} \{-\exp(-\gamma c) - \beta \exp(-\gamma(x-c))E[\exp(-\gamma a\cdot R_N))]\}. \end{aligned}$$

We split the supremum and get

$$\begin{aligned} V_{N-1}(x) &= \sup_{c\in[0,x]} \{-\exp(-\gamma c) - \beta\exp(-\gamma(x-c)) \underbrace{\sup_{a_{N-1}\in\mathbb{R}^d} E[\exp(-\gamma a\cdot R_N))]}_{v_{N-1}}\} \\ &= \sup_{c\in[0,x]} \{-\exp(-\gamma c) - \beta\exp(-\gamma(x-c))v_{N-1}\}. \end{aligned}$$

The optimal investment is given by a^*_{N-1} and the optimal c is given by

$$c^*_{N-1} = \frac{x}{2} - \frac{\log(\beta v_{N-1})}{2\gamma}.$$

We get

$$\begin{aligned} V_{N-1}(x) &= -\exp(-\gamma(\frac{x}{2} - \frac{\log(\beta v_{N-1})}{2\gamma})) - \beta\exp(-\gamma(x - \frac{x}{2} + \frac{\log(\beta v_{N-1})}{2\gamma}))v_{N-1} \\ &= -\underbrace{\left((\beta v_{N-1})^{\frac{1}{2}} + (\beta v_{N-1})^{-\frac{1}{2}}\beta v_{N-1}\right)}_{=d_{N-1}} \exp(-\gamma\frac{x}{2}) = -d_{N-1}\exp(-\gamma\frac{x}{2}). \end{aligned}$$

$n+1 \to n :$

$$\begin{aligned} V_{n+1}(x) &= \sup_{(c,a)\in D_n(x)} \{-\exp(-\gamma c) + \beta E[V_{n+1}(x-c+a\cdot R_{n+1})]\} \\ &= \sup_{(c,a)\in D_n(x)} \left\{-\exp(-\gamma c) - \beta E[d_{n+1}\exp(-\frac{\gamma}{N-n}(x-c+a\cdot R_{n+1}))]\right\} \\ &= \sup_{(c,a)\in D_n(x)} \left\{-\exp(-\gamma c) - \beta d_{n+1}\exp(-\frac{\gamma}{N-n}(x-c))E[\exp(-\frac{\gamma}{N-n}a\cdot R_{n+1})]\right\}. \end{aligned}$$

We split the supremum and get

$$V_{n+1}(x) = \sup_{c\in[0,x]} \{-\exp(-\gamma c) - \beta d_{n+1}\exp(-\frac{\gamma}{N-n}(x-c)) \underbrace{\sup_{a_n\in\mathbb{R}} \{E[\exp(-\frac{\gamma}{N-n}a\cdot R_{n+1})]\}}_{v_n}\}$$

$$= \sup_{c\in[0,x]} \left\{ - \exp(-\gamma c) - \beta d_{n+1} v_n \exp(-\frac{\gamma}{N-n}(x-c)) \right\}.$$

The optimal investment is given by a_n^* and the optimal c is given by

$$c_n^* = \frac{x}{N-n+1} - \frac{N-n}{(N-n+1)\gamma} \log(\frac{1}{N-n}\beta v_n d_{n+1}).$$

We get

$$V_n(x) = -\exp(-\gamma(\frac{x}{N-n+1} - \frac{N-n}{(N-n+1)\gamma}\log(\frac{1}{N-n}\beta v_n d_{n+1}))$$

$$-d_{n+1}\beta v_n \exp\big(-\frac{\gamma}{N-n}\big(x - \frac{x}{N-n+1} + \frac{N-n}{(N-n+1)\gamma}\log(\frac{1}{N-n}\beta v_n d_{n+1})\big)\big)$$

$$= -\underbrace{\Big((\frac{\beta v_n d_{n+1}}{N-n})^{-\frac{N-n}{N-n+1}} + \beta v_n d_{n+1}(\frac{\beta v_n d_{n+1}}{N-n})^{-\frac{1}{N-n+1}}\Big)}_{d_n} \exp(-\frac{\gamma}{N-n+1}x)$$

$$V_n(x) = -d_n \exp(-\frac{\gamma}{N-n+1}x).$$

□

3 Continuous-time consumption-investment problems with state-dependent discounting

Now we consider the continuous time non-standard investment-consumption problem. We analyze this problem in a financial market driven by a d-dimensional Brownian motion.

3.1 Problem formulation

We consider a fixed time horizon $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. On this space we have a d-dimensional Brownian motion (W_t) with Brownian filtration (F_t^W) and an environment process (Y_t) with natural filtration (F_t^Y), where (F_t^Y) is the filtration generated by (Y_t). In the following we consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $\mathcal{F}_t := \mathcal{F}_t^W \cup \mathcal{F}_t^Y$.

Financial market
We consider a complete financial market consisting of one riskless bond and d risky stocks. The bond has a deterministic interest rate (r_t), where (r_t) is an $\mathbb{R}$-valued, positive, bounded and continuous function, and the dynamic of the bond is given by

$$dS_t^0 = r_t S_t^0 dt, \quad r_t \geq 0 \quad S_0^0 = 1.$$

The d stocks have a deterministic drift α_t and a deterministic volatility matrix σ_t, where (α_t) and (σ_t) are $\mathbb{R}^d$-valued and $\mathbb{R}^{d\times d}$-valued, continuous and uniformly bounded functions and the dynamic of the stocks are

$$dS_t = diag(S_t)\Big(\alpha_t dt + \sigma_t dW_t\Big), \quad S_0 = (s_0^1, \dots, s_0^d)'.$$

The price process of the i-th stock satisfies

$$dS_t^i = S_t^i\Big(\alpha_t^i dt + \sigma_t^{i\cdot} dW_t\Big), \quad S_0^i = s_0^i,$$

where α_t^i is the i-th entry of α_t and $\sigma^{i\cdot}$ is the i-th row of σ_t. We assume that σ_t has full rank d, that $\sigma_t \sigma_t'$ is uniformly positive definite. Further we denote the excess return by $\mu_t := \alpha_t - r_t e_d$ for all $t \in [0, T]$, where $e_d = (1, ..., 1) \in \mathbb{R}^d$.

Definition 13. *A (self financing)* ***consumption-investment strategy*** *is an $(\mathcal{F}_t)$-predictable stochastic process $\pi = (c_t, a_t)$, where $c_t \in \mathbb{R}_+$ is the consumption rate at time t and $a_t = (a_t^1, \dots, a_t^d) \in \mathbb{R}^d$ are the amounts of money which are invested in the stocks at time t.*

Note that the self-financing condition implies that the amount which is invested in the bond at time t is given by $x_t - a_t' e$, where x_t is the wealth of the investor at time t and $e = (1, \dots, 1)' \in \mathbb{R}^d$.

The wealth equation
At any time t, the investor decides which amount of his wealth is invested in the stock and at which rate he consumes. The wealth of the investor under the consumption-investment strategy $\pi = (c_t, a_t)$ evolves according to

$$\begin{aligned} dX_t^\pi &= \sum_{i=1}^d a_t^i \frac{dS_t^i}{S_t^i} + (X_t^\pi - e'a_t)\frac{dS_t^0}{S_t^0} - c_t dt \\ &= \Big(\sum_{i=1}^d (\alpha_t^i a_t^i - a_t^i r_t^i) + r_t X_t^\pi - c_t\Big)dt + \sum_{i=1}^d \sigma_t^{i\cdot} a_t^i dW_t \\ &= \Big(\sum_{i=1}^d \mu_t^i a_t^i + r_t X_t^\pi - c_t\Big)dt + \sum_{i=1}^d \sigma_t^{i\cdot} a_t^i dW_t \end{aligned}$$

The wealth equation

$$X_0^\pi = x_0.$$

Environment Process
Our environment process (Y_t) is modeled as a continuous (time-homogeneous) Markov chain with finite state space E_Y. The Markov chain has the generator $\Lambda = (\lambda_{ij})_{E_Y \times E_Y}$ with $\lambda_{ij} \geq 0$ for $i \neq j$, and $\sum_{j \in E_Y} \lambda_{ij} = 0$ for all $i \in E_Y$. Further we assume that (Y_t) and (W_t) are independent.

Definition 14. *A* ***discount function*** $\beta(t, x, i)$ *is a mapping* $\beta : [0, \infty) \times \mathbb{R} \times E_Y \to (0, 1]$, *with* $\beta(\cdot, \cdot, i) \in C^{1,2}([0, \infty) \times \mathbb{R})$ *for all* $i \in E_Y$ *with the following property:*

$$\beta(0, x, i) = 1, \quad \forall x \in \mathbb{R}, i \in E_Y.$$

Markovian investment-consumption strategies

Definition 15. *An admissible (Markovian) consumption-investment strategy is a stochastic process* $\pi = (c_t, a_t)$, *where* $c_t := c_t(X_t, Y_{t-})$ *and* $a_t := a_t(X_t, Y_{t-})$ *are measurable mappings* $(c_t, a_t) : \mathbb{R} \times E_Y \to \mathbb{R}_+ \times \mathbb{R}^d$ *for* $t \in [0, T]$, *which have the following properties:*

(i) The process $\pi = (c_t, a_t)$ *take values in the sets* $D \subset \mathbb{R}^d \times \mathbb{R}_+$, *i.e* $(c_t(x, i), a_t(x, i)) \in D$. $X_t^\pi \in dom(U)$ *for all* $t \in [0, T]$, *where* X_t^π *is given by (3.1).*

(ii) The wealth equation given by (3.1) has for all $x > 0$, $i \in E_Y$ *and all* $\pi \in \mathcal{A}$ *a unique positive solution* (X_t^π). *This solution satisfies*

$$\sup_\tau E[|X_\tau^\pi|^k] < \infty, \quad \forall k \in \mathbb{N}.$$

Here the supremum is taken over all stopping times $0 < \tau \leq T$.

(iii) The investment strategy is square-integrable in the following sense:

$$E\left[\int_0^T ||a_s(X_s^\pi, Y_{s-})||^2 ds\right] < \infty.$$

(iv) The consumption strategy has moments of all order, i.e.

$$E\left[\int_0^T c_s^k(X_s^\pi, Y_{s-}) ds\right] < \infty, \quad k \in \mathbb{N}.$$

We denote the set of admissible (Markovian) strategies by $\mathcal{A}$.

The wealth process X_t^π under an admissible strategy is given by the following SDE

$$dX_t^\pi = \Big(r_t X_t^\pi + a_t'(X_t^\pi, Y_{t-})\mu_t - c_t(X_t^\pi, Y_{t-})\Big)dt + a_t'(X_t^\pi, Y_{t-})\sigma_t dW_t, \tag{3.1}$$

where $X_0 = x \in (0, \infty)$ is the initial wealth and $Y_0 = i \in E_Y$ is the initial state of the environment.

The wealth process X_t^π has the following representation

$$\begin{aligned} X_t^\pi = X_0^\pi \exp\Big(\int_0^t \big(r_s + \frac{a_s'(X_s^\pi, Y_{s-})\mu_s}{X_s^\pi} - \frac{c_s(X_s^\pi, Y_{s-})}{X_s^\pi} - \frac{1}{2}\frac{a_s'\left(X_s, Y_{s-}\right)\sigma_s\sigma_s' a_s\left(X_s, Y_{s-}\right)}{(X_s^\pi)^2}\big)ds\Big) \\ \cdot \exp\Big(\int_0^t \frac{a_s'(X_s^\pi, Y_{s-})\sigma_s}{X_s^\pi} dW_s\Big). \end{aligned} \tag{3.2}$$

Criterion of the investor
We only consider utility functions which satisfy a polynomial growth condition, we assume $|U(x)| \leq K(1 + |x|^k)$ for some $k \in \mathbb{N}$ and $K \in \mathbb{R}_+$.

The criterion of our investor for initial time t, initial wealth x, initial state of the environment i and $\pi \in \mathcal{A}$ is given by

$$V_\pi(t, x, i) := E_{t,x,i}\left[\int_t^T \beta_c(s - t, x, i)U_c(c_s(X_s^\pi, Y_{s-}))ds + \beta_p(T - t, x, i)U_p(X_T^\pi)\right],$$

where $E_{t,x,i}[\cdot] := E[\cdot | X_t = x, Y_{t-} = i]$, $(t, x, i) \in E := [0, T] \times dom(U) \times E_Y$.

The horizon of the investor is T. The consumption of the investor is evaluated by the utility function U_c and discounted by the discount functions β_c. His terminal wealth X_T^π is evaluated by the utility function U_p and discounted by the discount functions β_p.

The discount functions depends on the initial state $(t, x, i) \in E$. So for every (t, x, i) we get a different functional.

The value function $V_\pi(t,x,i)$ is well-defined for every $(t,x,i) \in E$ and $\pi \in \mathcal{A}$, since

$$\begin{aligned}
|V_\pi(t,x,i)| &\leq E_{t,x,i}[\int_t^T \beta_c(s-t,x,i)|U_c(c_s(X_s^\pi,Y_{s-})|ds + \beta_p(T-t,x,i)|U_p(X_T^\pi)|] \\
&\leq E_{t,x,i}[\int_t^T |U_c(c_s(X_s^\pi,Y_{s-})|ds + |U_p(X_T^\pi)|] \\
&\leq E_{t,x,i}[\int_t^T K(1+|c_s(X_s^\pi,Y_{s-})|^k)ds + \tilde{K}(1+|X_T^\pi|^{\tilde{k}})] \\
&< \infty.
\end{aligned}$$

In the case of standard discounting we have $\beta_c(t,x,i) = \beta_p(t,x,i) = \exp(-\rho t)$, $\rho > 0$, we obtain the value function of the Merton problem

$$V_\pi(t,x,i) = E_{t,x,i}[\int_t^T \exp(-\rho(s-t))U_c(c_s(X_s^\pi,Y_{s-})ds + \exp(-\rho(T-t))U_p(X_T^\pi)].$$

In this case the discount function only depends on the initial time t, but the optimal strategy only depends on the current time, since

$$\begin{aligned}
&\sup_{\pi\in\mathcal{A}} E_{t,x}[\int_t^T \exp(-\rho(s-t))U_c(c_s(X_s^\pi,Y_{s-})ds + \exp(-\rho(T-t))U_p(X_T^\pi)] \\
&= \exp(\rho t)\sup_{\pi\in\mathcal{A}} E_{t,x}[\int_t^T \exp(-\rho s)U_c(c_s(X_s^\pi,Y_{s-})ds + \exp(-\rho T)U_p(X_T^\pi)].
\end{aligned}$$

In general an optimal strategy will depend on the initial state (t,x,i). Like in the discrete case we are interested to find a strategy which is optimal for every initial state $(t,x,i) \in E$, so we are looking for a strategy π which is optimal for the family of optimization problems $(\mathcal{P}_{t,x,i})_{(t,x,i)\in E}$, where

$$\mathcal{P}_{t,x,i} : \sup_{\pi\in\mathcal{A}} V_\pi(t,x,i).$$

In general there exists no optimal strategy for the whole family $(\mathcal{P}_{t,x,i})$, so we use again the concept of equilibrium strategies which are widely used in the field of non-standard optimization, especially in the case of non-standard discounting. If an optimal strategy exists it coincides with the equilibrium strategy.

3.2 Equilibrium strategies

We use the definition for equilibrium strategies, which was first given in (Ekeland and Lazrak, 2006).

Let $\pi = (c_t, a_t)$ be a consumption-investment strategy. Then we define the corresponding ϵ-strategy $\pi^\epsilon = (c_t^\epsilon, a_t^\epsilon)$ for the current state (t, x, i) for $\epsilon > 0$:

$$c_s^\epsilon(x,i) = \begin{cases} c_s(x,i), & s \in (t+\epsilon, T] \\ \hat{c}_s(x,i), & s \in [t, t+\epsilon] \end{cases},$$

$$a_s^\epsilon(x,i) = \begin{cases} a_s(x,i), & s \in (t+\epsilon, T] \\ \hat{a}_s(x,i), & s \in [t, t+\epsilon] \end{cases},$$

where $(\hat{c}_s(x,i), \hat{a}_s(x,i))_{s\in[t,t+\epsilon]}$ is an arbitrary strategy, such that π^ϵ is an admissible strategy.

Lemma 17 in the Appendix implies that $X_t^{\pi^\epsilon}$ convergences weakly against X_t^π for $\epsilon \downarrow 0$, i.e.

$$\lim_{\epsilon\downarrow 0} X_t^{\pi^\epsilon} \stackrel{D}{=} X_t^\pi.$$

The definition of equilibrium strategies in continuous time is motivated by the characterization of equilibrium strategies in discrete time of Theorem 3. Unlike the discrete case we have to allow the decision maker to change his decision on a small interval $[t, t+\epsilon]$, since a change of a decision at one time point would be only a change on a Lebesgue zero set. A decision maker has no incentive to change his decision if

$$\lim_{\epsilon\downarrow 0} \frac{V_{\pi^*}(t,x,i) - V_{\pi^\epsilon}(t,x,i)}{\epsilon} \geq 0. \tag{3.3}$$

Like in the discrete case we fix the state and the decisions of the decision maker on the interval $(t+\epsilon, T]$ and view the value function as a function in the decision of the decision maker on the interval $[t, t+\epsilon]$. The inequality (3.3) guarantee that the equilibrium strategy at time t is a stationary point of this function, since the left hand side of inequality (3.3) can be viewed as the derivative of V with respect to the decision at time t. If the value function is concave we also have a maximum point like in the discrete case. Note that it is necessary to divide by ϵ, since

$$\lim_{\epsilon\downarrow 0} V_{\pi^*}(t,x,i) - V_{\pi^\epsilon}(t,x,i) = 0.$$

3.3 Extended Hamilton-Jacobi-Bellman equation approach

We want to derive an extended **Hamilton-Jacobi-Bellman equation (HJB-equation)** as the limit of the discrete time extended Bellman equation (Corollary 1). This approach is often used in the standard case and also was used in the non standard case, see Björk and Murgoci (2010). This derivation will be done heuristically. We will confirm our result in a verification theorem later.

We discretize the time scale by dividing the interval $[t, T]$ into N periods of constant length $h = \frac{T-t}{N}$. Our investor only make decision at the time points $t, t+h, \ldots, T-h$, that means that he invests and consumes a fix amount of his wealth on the intervals $[t+kh, t+(k+1)h)$, $k = 0, \ldots N-1$. Thus the decision maker only choose a discrete strategy.

Now let $\pi_h^* = (c^*, a^*)$ be a discrete equilibrium strategy for the discrete problem with step size h, then the wealth process under π^* is given for $k = 0, \ldots, N-1$ and $s \in [t+kh, t+(k+1)h)$ by

$$dX_s^{\pi_h^*} = \Big(r_s X_s^{\pi_h^*} + a_{t+kh}^{*'}(X_{t+kh}^{\pi_h^*}, Y_{(t+kh)-})\mu_s - c_{t+kh}^*(X_{t+kh}^{\pi_h^*}, Y_{(t+kh)-})\Big)ds$$

$$+a_{t+kh}^{*'}(X_{t+kh}^{\pi_h^*}, Y_{(t+kh)-})\sigma_s dW_s.$$

This implies that the discretized value function under π_h^* is given by

$$\begin{aligned} V_{\pi_h^*}^h(t,x,i) &= E_{t,x,i}[\sum_{k=0}^{N-1} \beta_c(kh,x,i) \int_{kh}^{(k+1)h} U_c(c_{t+kh}^*(X_{t+kh}^{\pi_h^*}, Y_{(t+kh)-})ds + \beta_p(T-t,x,i)U_p(X_T^{\pi_h^*})] \\ &= E_{t,x,i}[\sum_{k=0}^{N-1} \beta_c(kh,x,i)U_c(c_{t+kh}(X_{t+kh}^{\pi_h^*}, Y_{(t+kh)-})h + \beta_p(T-t,x,i)U_p(X_T^{\pi_h^*})]. \end{aligned}$$

We make the following Assumptions for the heuristic approach:

(i) The equilibrium strategy π_h^* converges weakly against an admissible continuous strategy π, i.e. $\lim_{h\downarrow 0} \pi_h^* = \pi \in \mathcal{A}$

(ii) The wealth process under π_h^* converges weakly against the wealth process under π , i.e. $\lim_{h\downarrow 0} X^{\pi_h^*} \overset{D}{=} X^\pi$

(iii) $\lim_{h\downarrow 0} V_{\pi_h^*}^h(t,x,i) = V_\pi(t,x,i)$

To ease the notation we set $X_t^h := X_t^{\pi_h^*}$, $V^h(t,x,i) := V_{\pi_h^*}(t,x,i)$ and $V(t,x,i) := V_\pi^h(t,x,i)$. Further we write $c_{t+kh} := c_{t+kh}^*(X_{t+kh}^{\pi_h^*})$ and $a_{t+kh} := a^*(X_{t+kh}^{\pi_h^*})$.

We can now apply the extended Bellman equation (Corollary 1) to express $V^h(t,x,i)$ by

$$\left.\begin{array}{c} V^h(t,x,i) = \sup_{(c,a)\in D_t(x)}\Big\{U_c(c)h + E_{t,x,i}[V^h(t+h, X_{t+h}^{h,c,a}, Y_{(t+h)-})] \\ -E_{t,x,i}[\sum_{k=1}^{N-1}\Big(\beta_c((k-1)h, X_{t+h}^{h,c,a}, Y_{(t+h)-}) - \beta_c(kh,x,i)\Big)U_c\left(c_{t+kh}\right)h] \\ -E_{t,x,i}[\Big(\beta_p(T-t-h, X_{t+h}^{h,c,a}, Y_{(t+h)-}) - \beta_p(T-t,x,i)\Big)U_p(X_T^h)]\Big\} \end{array}\right\} \quad (3.4)$$

Our aim is to compute the limit of the extended Bellman equation for $h \downarrow 0$. This will be done in three steps:

i) We apply Itô's formula to $V^h(t+h, X^{h,c,a}_{t+h}, Y_{(t+h)-})$ and get with the help of Appendix Proposition 4

$$E_{t,x,i}[V^h\left(t+h, X^{h,c,a}_{t+h}, Y_{(t+h)-}\right)] = V^h(t,x,i) + E_{t,x,i}[\int_0^h \frac{\partial}{\partial t} V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)$$

$$+\left(r_{t+s}X^{h,c,a}_{t+s} + a'\mu_{t+s} - c\right)\frac{\partial}{\partial x}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)$$

$$+\frac{1}{2}a'\sigma_{t+s}\sigma'_{t+s}a\frac{\partial}{\partial x^2}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)ds]$$

$$+E_{t,x,i}[\int_0^h a'\sigma_{t+s}\left(X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)\frac{\partial}{\partial x}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)dW_s]$$

$$+E_{t,x,i}[\sum_{0<s\leq h}\left(V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)}\right) - V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)\right)].$$

We make the assumption that

$$E_{t,x,i}[\int_0^h \left(\frac{\partial}{\partial x}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)\right)^2 ds] < \infty,$$

which implies that

$$E_{t,x,i}[\int_0^h a'\sigma_{t+s}\frac{\partial}{\partial x}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)dW_s] = 0.$$

Moreover we get

$$E_{t,x,i}[V^h\left(t+h, X^{h,c,a}_{t+h}, Y_{(t+h)-}\right)] = V^h(t,x,i) + E_{t,x,i}[\int_0^h \frac{\partial}{\partial t} V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)$$

$$+\left(r_{t+s}X^{h,c,a}_{t+s} + a'\mu_{t+s} - c\right)\frac{\partial}{\partial x}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)$$

$$+\frac{1}{2}a'\sigma_{t+s}\sigma'_{t+s}a\frac{\partial}{\partial x^2}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)ds]$$

$$+E_{t,x,i}[\sum_{0<s\leq h}\left(V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{t+s}\right) - V\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)\right)].$$

(II) Now we plug this expression in (3.4) and divide the equation by h and get

$$0 = \sup_{(c,a)\in D_t(x)}\Big\{U_c(c)h + \frac{1}{h}E_{t,x,i}[\int_0^h \frac{\partial}{\partial t}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)$$

$$+\left(r_{t+s}X^h_{t+s} + a'\mu_{t+s} - c\right)\frac{\partial}{\partial x}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)$$

$$+\frac{1}{2}a\sigma_{t+s}\sigma'_{t+s}a\left(X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)\frac{\partial}{\partial x^2}V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)ds]$$

$$+E_{t,x,i}[\sum_{0<s\leq h}\left(V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right) - V^h\left(t+s, X^{h,c,a}_{t+s}, Y_{(t+s)-}\right)\right)]$$

$$
\begin{aligned}
&-\frac{1}{h}E_{t,x,i}[\sum_{k=1}^{N-1}\Big(\beta_c((k-1)h,X_{t+h}^{h,c,a},Y_{(t+h)-})-\beta_c(kh,x,i)\Big)U_c\left(c_{t+kh}\right)h\\
&-\frac{1}{h}E_{t,x,i}[\Big(\beta_p(T-t-h,X_{t+h}^{h,c,a},Y_{(t+h)-})-\beta_p(T-t,x,i)\Big)U_p(X_T)]\Big\}\\
\Leftrightarrow 0=&\sup_{(c,a)\in D_t(x)}\Big\{U_c(c)h+G(t,x,i,h,c,a)-H(t,x,i,h,c,a)-K(t,x,i,h,c,a)\Big\}, \quad (3.5)
\end{aligned}
$$

where

$$
\begin{aligned}
G(t,x,i,h,c,a):=&\frac{1}{h}E_{t,x,i}[\int_0^h\frac{\partial}{\partial t}V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)\\
&+\Big(r_{t+s}X_{t+s}^{h,c,a}+a'\mu_{t+s}-c\Big)\frac{\partial}{\partial x}V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)\\
&+\frac{1}{2}a'\sigma_{t+s}\sigma_{t+s}'a\frac{\partial}{\partial x^2}V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)ds]\\
&+\frac{1}{h}E_{t,x,i}[\sum_{0<s\le h}\left(V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)-V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)\right)]\\
H(t,x,i,h,c,a):=&\frac{1}{h}E_{t,x,i}[\sum_{k=1}^{N-1}\Big(\beta_c((k-1)h,X_{t+h}^{h,c,a},Y_{(t+h)-})-\beta_c(kh,x,i)\Big)U_c\left(c_{t+kh}\right)h]\\
K(t,x,i,h,c,a):=&\frac{1}{h}E_{t,x,i}[\Big(\beta_p(T-t-h,X_{t+h}^{h,c,a},Y_{(t+h)-})-\beta_p(T-t,x,i)\Big)U_p(X_T)].
\end{aligned}
$$

(III) We derive the extended HJB-equation by taking the limit $h\downarrow 0$.

First we compute $G(t,x,i,c,a):=\lim_{h\downarrow 0}G(t,x,i,h,c,a)$. First note that from Lemma 5 in the Appendix follows that

$$
\lim_{h\downarrow 0}\frac{1}{h}E_{t,x,i}[\sum_{0<s\le h}\left(V(t+s,X_{t+s}^{h,c,a},Y_{(t+s)})-V(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-})\right)]=\sum_{j\in E_Y}\lambda_{ij}V(t,x,j).
$$

With the help of the Dominated Convergence Theorem we get

$$
\begin{aligned}
G(t,x,i,c,a)=&\lim_{h\downarrow 0}\frac{1}{h}E_{t,x,i}[\int_0^h\frac{\partial}{\partial t}V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)\\
&+\Big(r_{t+s}X_{t+s}^{h,c,a}+a'\mu_{t+s}-c\Big)\frac{\partial}{\partial x}V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)\\
&+\frac{1}{2}a'\sigma_{t+s}\sigma_{t+s}'a\frac{\partial}{\partial x^2}V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)ds]\\
&+E_{t,x,i}[\sum_{0<s\le h}\left(V\left(t+s,X_{t+s}^{h,c,a},Y_{t+s}\right)-V^h\left(t+s,X_{t+s}^{h,c,a},Y_{(t+s)-}\right)\right)]\\
=&\frac{\partial}{\partial t}V(t,x,i)+\frac{\partial}{\partial x}V(t,x,i)\cdot\Big(r_tx+a\mu_t-c\Big)\\
&+\frac{1}{2}a'\sigma_t\sigma_t'a\frac{\partial}{\partial x^2}V(t,x,i)+\sum_{j\in E_Y}\lambda_{ij}V(t,x,j).
\end{aligned}
$$

Now we compute the limit

$$H(t,x,i,c,a) = \lim_{h\downarrow 0} \frac{1}{h} E_{t,x,i}[\sum_{k=1}^{N-1} \Big(\beta_c((k-1)h, X^{h,c,a}_{t+h}, Y_{(t+h)-}) - \beta_c(kh,x,i)\Big) U_c\left(c_{t+kh}\right) h]$$

$$= \lim_{h\downarrow 0} \sum_{k=1}^{N-1} E_{t,x,i}[\Big(\beta_c((k-1)h, X^{h,c,a}_{t+h}, Y_{(t+h)-}) - \beta_c(kh,x,i)\Big) U_c\left(c_{t+kh}\right)]h = \lim_{h\downarrow 0} \sum_{k=1}^{N-1} f_h(kh)h,$$

where

$$f_h(hk) := E_{t,x,i}[\Big(\beta_c((k-1)h, X^{h,c,a}_{t+h}, Y_{(t+h)-}) - \beta_c(kh,x,i)\Big) U_c\left(c_{t+kh}\right)].$$

Appendix Lemma 8 implies that $f_h(s)$ converges uniformly against $f(s)$, where

$$f(s) := E_{t,x,i}[U_c(c_s(X_s,Y_{s-})\Big(-\frac{\partial}{\partial t}\beta_c(s-t,x,i)$$
$$+\Big(r_t x + a'\mu_t - c\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i) + \frac{1}{2}a'\sigma_t\sigma_t' a\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)du$$
$$+a'\sigma_t\sigma_t' a E_{t,x,i}[\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^{\pi}_s(t,x,i)ds$$
$$+\sum_{j\in E_Y}\lambda_{ij}E_{t,x,j}[\int_t^T \Big(\beta_c(s-t,x,j) - \beta_c(t,x,i)\Big)U_c(c_s(X_s,Y_{s-}))ds].$$

Appendix Lemma 18 implies that $\lim_{h\downarrow 0}\sum_{k=1}^{N-1} f_h(kh)h = \int_t^T f(s)ds$, so we get

$$H(t,x,i,c,a) = E_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-})\Big(-\frac{\partial}{\partial t}\beta_c(s-t,x,i)$$
$$+\Big(r_t x + a'\mu_t - c\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i) + \frac{1}{2}a'\sigma_t\sigma_t' a\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)du$$
$$+a'\sigma_t\sigma_t' a E_{t,x,i}[\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}E_{t,x,i}[U_c(c_s(X_s,Y_{s-}))]ds$$
$$+\sum_{j\in E_Y}\lambda_{ij}E_{t,x,j}[\int_t^T \Big(\beta_c(s-t,x,j) - \beta_c(t,x,i)\Big)U_c(c_s(X_s,Y_{s-}))ds].$$

Next we compute $K(t,x,i,c,a) := \lim_{h\downarrow 0}\frac{1}{h}K(t,x,i,h,c,a)$

$$= \lim_{h\downarrow 0}\frac{1}{h}E_{t,x,i}[\Big(\beta_p(T-t-h, X^{h,c,a}_{t+h}, Y_{(t+h)-}) - \beta_p(T-t,x,i)\Big)U_p(X^h_T)].$$

We apply Appendix Lemma 10 and get

$$K(t,x,i,c,a) = E_{t,x,i}[U_p(X_T)]\Big(-\frac{\partial}{\partial t}\beta_p(T-t,x,i) + \Big(r_t x + a'\mu_t - c\Big)\frac{\partial}{\partial x}\beta_p(s-t,x,i)$$
$$+\frac{1}{2}a'\sigma_t\sigma_t' a\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)\Big) + a'\sigma_t\sigma_t' a\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}E_{t,x,i}[U_p(X_T)]$$
$$+\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(T-t,x,j)-\beta_p(T-t,x,i)\big)E_{t,x,j}[U_p(X_T)].$$

(IV) Plugging the limits in (3.5) we get

$$0 = \sup_{(c,a)\in D}\Big\{U_c(c) + \frac{\partial}{\partial t}V(t,x,i)$$
$$+\frac{\partial}{\partial x}V(t,x,i)\cdot\Big(r_t x + a'\mu_t - c\Big) + \frac{1}{2}a'\sigma_t\sigma_t' a\frac{\partial}{\partial x^2}V(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}V(t,x,j)$$
$$+E_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-}))\frac{\partial}{\partial t}\beta_c(s-t,x,i)ds]$$
$$-\Big(r_t x + a'\mu_t - c\Big)E_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-}))\frac{\partial}{\partial x}\beta_c(s-t,x,i)ds]$$
$$-\frac{1}{2}a'\sigma_t\sigma_t' aE_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-}))\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)ds]$$
$$-a'\sigma_t\sigma_t' a\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}E_{t,x,i}[U_c(c_s(X_s,Y_{s-}))]ds$$
$$-\sum_{j\in E_Y}\lambda_{ij}\int_t^T\big(\beta_c(s-t,x,j)-\beta_c(t,x,i)\big)E_{t,x,j}[U_c(c_s(X_s,Y_{s-}))]ds$$
$$+E_{t,x,i}[U_p(X_T)\frac{\partial}{\partial t}\beta_p(T-t,x,i)] - \Big(r_t x + a'\mu_t - c\Big)E_{t,x,i}[U_p(X_T)\frac{\partial}{\partial x}\beta_p(T-t,x,i)]$$
$$-\frac{1}{2}a'\sigma_t\sigma_t' aE_{t,x,i}[U_p(X_T)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)] - a'\sigma_t\sigma_t' a\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}E_{t,x,i}[U_p(X_T)]$$
$$-\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(T-t,x,j)-\beta_p(T-t,x,i)\big)E_{t,x,j}[U_p(X_T)].$$

Now we re-sort and get

$$\Leftrightarrow 0 = \frac{\partial}{\partial t}V(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}V(t,x,j)$$
$$+\underbrace{E_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-}))\frac{\partial}{\partial t}\beta_c(s-t,x,i)ds + U_p(X_T)\frac{\partial}{\partial t}\beta_p(T-t,x,i)]}_{=:L_\pi(t,x,i)}$$
$$\left.\begin{matrix}-\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T\big(\beta_c(s-t,x,j)-\beta_c(s-t,x,i)\big)E_{t,x,j}[U_c(c_s(X_s,Y_{s-}))]ds \\ +\big(\beta_p(T-t,x,j)-\beta_p(T-t,x,i)\big)E_{t,x,j}[U_p(X_T)]\Big)\end{matrix}\right\} =: -J_\pi(t,x,i)$$

$$+\sup_{(c,a)\in D}\Big\{U_c(c)+\frac{\partial}{\partial x}V(t,x,i)\cdot\Big(r_tx+a'\mu_t-c\Big)+\frac{1}{2}a'\sigma_t\sigma_t\sigma_t'2a\frac{\partial}{\partial x^2}V(t,x,i)$$

$$-\Big(r_tx+a'\mu_t-c\Big)$$

$$\cdot\underbrace{E_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-}))\frac{\partial}{\partial x}\beta_c(s-t,x,i)ds+U_p(X_T)\frac{\partial}{\partial x}\beta_p(T-t,x,i)]}_{=:M_\pi(t,x,i)}$$

$$-\frac{1}{2}a'\sigma_t\sigma_t'a\underbrace{E_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-}))\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)ds+U_p(X_T)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)]}_{=:N_\pi(t,x,i)}$$

$$-a'\sigma_t\sigma_t'a\underbrace{\Big(\int_t^T\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}E_{t,x,i}[U_c(c_s(X_s,Y_{s-}))]ds+\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}E_{t,x,i}[U_p(X_T)]\Big)}_{:=W_\pi(t,x,i)}\Big\}$$

$$\Leftrightarrow 0=\frac{\partial}{\partial t}V(t,x,i)+L_\pi(t,x,i)+\sum_{j\in E_Y}\lambda_{ij}v(t,x,j)-J_\pi(t,x,i)$$

$$+\sup_{(c,a)\in D}\Big\{U_c(c)+\big(r_tx+a'\mu_t-c\big)\Big(\frac{\partial}{\partial x}V(t,x,i)-M_\pi(t,x,i)\Big)$$

$$+\frac{1}{2}a'\sigma_t\sigma_t'a\Big(\frac{\partial}{\partial x^2}V(t,x,i)-N_\pi(t,x,i)-2W_\pi(t,x,i)\Big)\Big\}.$$

We call the equation above **extended Hamilton-Jacobi-Bellman equation (HJB)-equation.**

Definition 16 (Extended HJB-equation)**.** *A pair $(v,\pi)\in C^{1,2}\times\mathcal{A}$ is called solution of the extended HJB-equation if it solves*

$$0=\frac{\partial}{\partial t}v(t,x,i)+L_\pi(t,x,i)+\sum_{j\in E_Y}\lambda_{ij}v(t,x,j)-J_\pi(t,x,i)$$

$$+\sup_{(c,a)\in D}\Big\{U_c(c)+\Big(r_tx+a'\mu_t-c\Big)\left(\frac{\partial}{\partial x}v(t,x,i)-M_\pi(t,x,i)\right)$$

$$+\frac{1}{2}a'\sigma_t\sigma_t'a\left(\frac{\partial}{\partial x^2}v(t,x,i)-N_\pi(t,x,i)-2W_\pi(t,x,i)\right)\Big\},$$

with

$$L_\pi(t,x,i):=E_{t,x,i}[\int_t^T U_c(c_s(X_s^\pi,Y_{s-}))\frac{\partial}{\partial t}\beta_c(s-t,x,i)ds+U_p(X_T^\pi)\frac{\partial}{\partial t}\beta_p(T-t,x,i)]$$

$$M_\pi(t,x,i):=E_{t,x,i}[\int_t^T U_c(c_s(X_s^\pi,Y_{s-})\frac{\partial}{\partial x}\beta_c(s-t,x,i)ds+U_p(X_T^\pi)\frac{\partial}{\partial x}\beta_p(T-t,x,i)]$$

$$N_\pi(t,x,i) := E_{t,x,i}[\int_t^T U_c(c_s(X_s^\pi, Y_{s-})\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)ds + U_p(X_T^\pi)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)]$$

$$J_\pi(t,x,i) := \sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T \big(\beta_c(s-t,x,j)-\beta_c(s-t,x,i)\big)E_{t,x,j}[U_c(c_s(X_s^\pi,Y_{s-}))]ds$$
$$+\big(\beta_p(T-t,x,j)-\beta_p(T-t,x,i)\big)E_{t,x,j}[U_p(X_T^\pi)]\Big).$$

$$W_\pi(t,x,i) := \int_t^T \frac{\partial}{\partial x}E_{t,x,i}[U_c(s,X_s^\pi,Y_{s-})]\frac{\partial}{\partial x}\beta_c(s-t,x,i)ds + \frac{\partial}{\partial x}E_{t,x,i}[U_p(X_t^\pi)]\frac{\partial}{\partial x}\beta_p(T-t,x,i)$$

and with terminal condition

$$v(T,x,i) = U_p(x).$$

Proposition 1. *In case the discount functions are given by* $\beta_c(t,x,i) = \beta_p(t,x,i) = \exp(-\rho_i t)$, $\rho_i \in \mathbb{R}_+$, $\forall i \in E_Y$, *the extended HJB-equation reduces to*

$$0 = \frac{\partial}{\partial t}v(t,x,i) - \rho_i V_\pi(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}v(t,x,j) - J_\pi(t,x,i)$$
$$+\sup_{(c,a)\in D}\Big\{U_c(c) + \Big(r_t x + a'\mu_t - c\Big)\frac{\partial}{\partial x}v(t,x,i) + \frac{1}{2}a'\sigma_t\sigma_t' a\frac{\partial}{\partial x^2}v(t,x,i)\Big\},$$

where

$$J_\pi(t,x,i) := \sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T\big(\exp(-\rho_j(s-t)) - \exp(-\rho_i(s-t))\big)E_{t,x,j}[U_c(c_s(X_s^\pi,Y_{s-}))]ds$$
$$+\big(\exp(-\rho_j(T-t)) - \exp(-\rho_i(T-t))\big)E_{t,x,j}[U_p(X_T^\pi)]\Big)$$

and with terminal condition

$$v(T,x,i) = U_p(x).$$

Proof.
Since β_c and β_p are independent of x it holds that $M_\pi = N_\pi = W_\pi = 0$. Further it holds that

$$L_\pi(t,x,i) = E_{t,x,i}[\int_t^T U_c(c_s(X_s^\pi,Y_{s-}))(-\rho_i\exp(-\rho_i(s-t)))ds + U_p(X_T^\pi)(-\rho_i\exp(-\rho_i(T-t)))]$$
$$= -\rho_i V_\pi(t,x,i).$$

□

3.4 Verification Theorem

The next step will be a verification theorem for the computation of eq-strategies.

We restrict ourselves to pairs (v, π) of solutions of the extended HJB-equation, which satisfy the following **Assumption (T)**:

$$(T)\begin{cases} E[\int_0^T a_s'(X_s^\pi, Y_{s-})a_s(X_s^\pi, Y_{s-})\left(\frac{\partial}{\partial x}v(s, X_s^\pi, Y_s)\right)^2 ds] < \infty \\ E[\int_0^T a_s'(X_s^\pi, Y_{s-})a_s(X_s^\pi, Y_{s-})\left(\frac{\partial}{\partial x}\beta_c(s, X_s, Y_s)\right)^2 ds] < \infty \\ E[\int_0^T a_s'(X_s^\pi, Y_{s-})a_s(X_s^\pi, Y_{s-})\left(\frac{\partial}{\partial x^2}\beta_p(s, X_s, Y_s)\right)^2 ds] < \infty \\ E_{t,x,i}[\int_0^T a_s'(X_s, Y_{s-})a_s(X_s, Y_{s-})\left(\frac{\partial}{\partial x}E_{s,X_s,Y_{s-}}[U_c(c(X_u, Y_u))]\right)^2 ds] < \infty, \ \forall u \in [0,T] \\ E_{t,x,i}[\int_0^T a_s'(X_s, Y_{s-})a_s(X_s, Y_{s-})\left(\frac{\partial}{\partial x}E_{t,x,i}[U_p(X_T)]\right)^2 ds] < \infty \end{cases}$$

Theorem 7 (Verification)**.** *Let* $\pi^* = (c_t^*, a_t^*) \in \mathcal{A}$ *,*$v(\cdot, \cdot, i) \in C^{1,2}$*,* $\forall i \in E_Y$ *be a solution of the extended HJB-equation which satisfy Assumption (T) and* $(c_t^*(x,i), a_t^*(x,i))$ *be a maximum point of*

$$\begin{aligned}(c,a) \quad \to \quad & U_c(c) + \Big(r_t x + a'\mu_t - c\Big)\left(\frac{\partial}{\partial x}v(t,x,i) - M_{\pi^*}(t,x,i)\right) \\ & + \frac{1}{2}a'\sigma_t\sigma_t'a\left(\frac{\partial}{\partial x^2}v(t,x,i) - N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i)\right), \ (c,a) \in D.\end{aligned}$$

Then it holds: $v = V_{\pi^*}$ *is the equilibrium value function and* $\pi^* = (c_t^*, a_t^*)$ *is an equilibrium strategy.*

Proof.
We prove the Theorem in two steps:

i) First we show that $v(t,x,i) = V_{\pi^*}(t,x,i)$.

ii) Then we show that $\pi^* = (c_t^*, a_t^*)$ is indeed an equilibrium strategy.

The idea to prove the statement in this two steps can also be found in Björk and Murgoci (2010).

To ease the notation we write $X_t := X_t^{\pi^*}$, $\varphi_s^{\pi^*}(t,x,i) := E_{t,x,i}[U_c(c_s(X_s^{\pi^*}, Y_{s-})]$, for $s \geq t$, and $\psi^{\pi^*}(t,x,i) := E_{t,x,i}[U_p(X_T)]$.

i) Let (v, π^*) be a solution of the HJB-equation and let the supremum be realized by $(c_t^*(x,i), a_t^*(x,i))$, i.e.

$$0 = \frac{\partial}{\partial t}v(t,x,i) + U_c(c_t^*(x,i)) + L_{\pi^*}(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}v(t,x,j) - J_{\pi^*}(t,x,i)$$

$$-\Big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\Big)M_{\pi^*}(t,x,i) - \frac{1}{2}a^{*'}(x,i)\sigma_t\sigma_t' a^*(x,i)(N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i))$$

$$+\frac{\partial}{\partial x}v(t,x,i)\cdot\Big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\Big) + \frac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)\frac{\partial}{\partial x^2}v(t,x,i)$$

$$\Leftrightarrow \frac{\partial}{\partial t}v(t,x,i) + \Big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\Big)\frac{\partial}{\partial x}v(t,x,i)$$

$$+\frac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)\frac{\partial}{\partial x^2}v(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}v(t,x,j)$$

$$= -U_c(c_t^*(x,i)) - L_{\pi^*}(t,x,i) + J_{\pi^*}(t,x,i) \tag{3.6}$$

$$+\Big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\Big)M_{\pi^*}(t,x,i) + \frac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)(N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i)).$$

Appendix Lemma 11 implies that the infinitesimal operator A^{π^*} (see Appendix Definition 17) of $v(t,x,i)$ is given by

$$A^{\pi^*}v(t,x,i) = \frac{\partial}{\partial t}v(t,x,i) + \Big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\Big)\frac{\partial}{\partial x}v(t,x,i)$$

$$+\frac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)\frac{\partial}{\partial x^2}(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}v(t,x,j).$$

Hence we can write equation (3.6) as

$$\left.\begin{aligned} &A^{\pi^*}v(t,x,i) = -U_c(c_t^*(x,i)) - L_{\pi^*}(t,x,i) + J_{\pi^*}(t,x,i) \\ &+\Big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\Big)M_{\pi^*}(t,x,i) \\ &+\tfrac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)(N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i)) \end{aligned}\right\} \tag{3.7}$$

With the help of Dynkin's Formula (Appendix Theorem 15) we can write v as

$$v(t,x,i) = E_{t,x,i}[v(T,X_T,Y_{T-})] - E_{t,x,i}[\int_t^T A^{\pi^*}v(s,X_s,Y_{s-})ds].$$

Using (3.7) we get

$$v(t,x,i) = E_{t,x,i}[v(T,X_T,Y_{T-})] - E_{t,x,i}[\int_t^T \Big(-U_c(c_s^*(X_s,Y_{s-})) - L_{\pi^*}(s,X_s,Y_{s-})$$

$$+J_{\pi^*}(s,X_s,Y_{s-}) + \Big(r_s X_s + a_s^{*'}(X_s,Y_{s-})\mu_s - c_s^*(X_s,Y_{s-})\Big)M_{\pi^*}(s,X_s,Y_{s-})$$

$$+\frac{1}{2}a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})N_{\pi^*}(s,X_s,Y_{s-})\Big)ds]$$

$$\left.\begin{aligned}
&= E_{t,x,i}[v(T,X_T,Y_{T-}) + \textstyle\int_t^T U_c(c_s^*(X_s,Y_{s-}))ds] \\
&+E_{t,x,i}[\textstyle\int_t^T L_{\pi^*}(s,X_s,Y_{s-}) - J_{\pi^*}(s,X_s,Y_{s-})ds] \\
&-E_{t,x,i}[\textstyle\int_t^T \Big(r_sX_s + a_s^{*'}(X_s,Y_{s-})\mu_t - c_s^*(X_s,Y_{s-})\Big)M_{\pi^*}(s,X_s,Y_{s-}) \\
&+\tfrac{1}{2}a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})(N_{\pi^*}(s,X_s,Y_{s-}) - 2W_{\pi^*}(s,X_s,Y_{s-}))\Big)ds]
\end{aligned}\right\} \quad (3.8)$$

In order to compute

$$\left.\begin{aligned}
&E_{t,x,i}[\textstyle\int_t^T L_{\pi^*}(s,X_s,Y_{s-}) - J_{\pi^*}(s,X_s,Y_{s-})ds] \\
&-E_{t,x,i}[\textstyle\int_t^T \Big(r_sX_s + a_s^{*'}(X_s,Y_{s-})\mu_s - c_s^*(X_s,Y_{s-})\Big)M_{\pi^*}(s,X_s,Y_{s-}) \\
&+\tfrac{1}{2}a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})(N_{\pi^*}(s,X_s,Y_{s-}) - 2W_{\pi^*}(s,X_s,Y_{s-}))\Big)ds]
\end{aligned}\right\} \quad (3.9)$$

we first plug in the definitions of $L_{\pi^*}(t,x,i)$, $M_{\pi^*}(t,x,i)$, $N_{\pi^*}(t,x,i)$, $J_{\pi^*}(t,x,i)$ and $W_{\pi^*}(t,x,i)$ and get

$$(3.9) = E_{t,x,i}[\int_t^T E_{s,X_s,Y_{s-}}[\int_s^T U_c(c_u^*(X_u,Y_{u-}))\frac{\partial}{\partial t}\beta_c(u-s,X_s,Y_{s-})duds$$

$$+\int_t^T U_p(X_T)\frac{\partial}{\partial t}\beta_p(T-s,X_s,Y_{s-})ds]]$$

$$-E_{t,x,i}[\int_t^T \sum_{j\in E_Y}\int_s^T [\lambda_{Y_{s-}j}\big(\beta_c(u-s,X_s,j)-\beta_c(u-s,X_s,Y_{s-})\big)E_{s,X_s,j}[U_c(c_u^*(X_u,Y_{u-}))]duds]$$

$$-E_{t,x,i}[\int_t^T \sum_{j\in E_Y}\lambda_{Y_{s-}j}\big(\beta_p(u-s,X_s,j)-\beta_p(u-s,X_s,Y_{s-})\big)E_{s,X_s,j}[U_p(X_T)]ds]$$

$$-E_{t,x,i}[\int_t^T E_{s,X_s,Y_{s-}}[\Big(r_sX_s + a_s^{*'}(X_s,Y_{s-})\mu_s - c_s^*(X_s,Y_{s-})\Big)$$

$$\cdot\int_s^T U_c(c_u^*(X_u,Y_{u-})\frac{\partial}{\partial x}\beta_c(u-s,X_s,Y_{s-})duds]]$$

$$-E_{t,x,i}[\int_t^T E_{s,X_s,Y_{s-}}[\Big(r_sX_s+a_s^{*'}(X_s,Y_{s-})\mu_s-c_s^*(X_s,Y_{s-})\Big)U_p(X_T)\frac{\partial}{\partial x}\beta_p(u-s,X_s,Y_{s-})ds]]$$

$$-E_{t,x,i}[\int_t^T E_{s,X_s,Y_{s-}}[\frac{1}{2}a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})$$

$$\cdot\int_s^T U_c(c_u^*(X_u,Y_{u-})\frac{\partial}{\partial x^2}\beta_c(u-s,X_u,Y_{u-})duds]]$$

$$-E_{t,x,i}[\int_t^T E_{s,X_s,Y_{s-}}[\frac{1}{2}a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})U_p(X_T)\frac{\partial}{\partial x^2}\beta_p(T-s,X_s,Y_{s-})ds]]$$

$$-E_{t,x,i}[\int_t^T a^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a^*(X_s,Y_{s-})\int_s^T \frac{\partial}{\partial x}\varphi_u^{\pi^*}(s,X_s,Y_{s-})\frac{\partial}{\partial x}\beta_c(u-s,X_s,Y_{s-})du\,ds]$$

$$-E_{t,x,i}[\int_t^T a^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a^*(X_s,Y_{s-})\frac{\partial}{\partial x}\psi^{\pi^*}(s,X_s,Y_{s-})\frac{\partial}{\partial x}\beta_p(T-s,X_s,Y_{s-})ds]$$

Now we re-sort the terms and get

$$\left.\begin{aligned}
&E_{t,x,i}[\textstyle\int_t^T E_{s,X_s,Y_{s-}}[\int_s^T \Big(\frac{\partial}{\partial t}\beta_c(u-s,X_s,Y_{s-})\\
&-\Big(r_sX_s + a_s^{*'}(X_s,Y_{s-})\mu_s - c_s^*(X_s,Y_{s-})\Big)\tfrac{\partial}{\partial x}\beta_c(u-s,X_s,Y_{s-})\\
&-\tfrac{1}{2}a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})\tfrac{\partial}{\partial x^2}\beta_c(u-s,X_s,Y_{s-})\Big)U_c(c_u^*(X_u,Y_{u-}))duds]\\
&-E_{t,x,i}[\textstyle\int_t^T a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})\\
&\cdot\textstyle\int_s^T \frac{\partial}{\partial x}\varphi_u^{\pi^*}(s,X_s,Y_{s-})\frac{\partial}{\partial x}\beta_c(u-s,X_s,Y_{s-})du\,ds]\\
&-E_{t,x,i}[\textstyle\sum_{j\in E_Y}\lambda_{Y_{s-}j}\int_s^t\big(\beta_c(u-s,X_s,j)-\beta_c(u-s,X_s,Y_{s-})\big)\\
&\cdot E_{s,X_s,j}[U_c(c_u^*(X_u,Y_{u-}))\tfrac{\partial}{\partial x}\beta_p(T-s,X_s,Y_{s-})]]\\
&+E_{t,x,i}[\textstyle\int_t^T E_{s,X_s,Y_{s-}}[\Big(\frac{\partial}{\partial t}\beta_p(T-s,X_s,Y_{s-})\\
&-\Big(r_sX_s + a_s^{*'}(X_s,Y_{s-})\mu_s - c_s^*(X_s,Y_{s-})\Big)\\
&-\tfrac{1}{2}a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})\tfrac{\partial}{\partial x^2}\beta_p(T-s,X_s,Y_{s-})\Big)U_p(X_T)ds]]\\
&-E_{t,x,i}[\textstyle\int_t^T a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})\frac{\partial}{\partial x}\psi^{\pi^*}(s,X_s,Y_{s-})\frac{\partial}{\partial x}\beta_p(T-s,X_s,Y_{s-})ds]\\
&-E_{t,x,i}[\textstyle\int_t^T \sum_{j\in E_Y}\lambda_{Y_{s-}j}\big(\beta_p(u-s,X_s,j)-\beta_p(u-s,X_s,Y_{s-})\big)E_{s,X_s,j}[U_p(X_T)]ds]
\end{aligned}\right\} \tag{3.10}$$

Appendix Lemma12 implies that

$$\begin{aligned}
-A^\pi(\beta_p(T-t,x,i)\psi^{\pi^*}(t,x,i)) &= -\psi^{\pi^*}(t,x,i)\frac{\partial}{\partial t}\beta_p(T-t,x,i)\\
&-\Big(r_tx + a_t'(x,i)\mu_t - c_t(x,i)\Big)\psi^{\pi^*}(t,x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\\
&-\frac{1}{2}a_t'(x,i)\sigma_t\sigma_t' a_t(x,i)\psi^{\pi^*}(t,x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)\\
&-a_t'(x,i)\sigma_t\sigma_t' a_t(x,i)\Big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i)\\
&-\sum_{j\in E_Y}\lambda_{ij}\psi(t,x,j)\left(\beta_p(T-t,x,j)-\beta_p(T-t,x,i)\right)
\end{aligned}$$

and

$$-A^\pi(\beta_c(s-t,x,i)\varphi^{\pi^*}(t,x,i)) = -\varphi^{\pi^*}(t,x,i)\frac{\partial}{\partial t}\beta_c(s-t,x,i)$$

$$-\Big(r_t x + a_t'(x,i)\mu_t - c_t(x,i)\Big)\varphi^{\pi^*}(t,x,i)\frac{\partial}{\partial x}\beta_c(s-t,x,i)$$
$$-\frac{1}{2}a_t'(x,i)\sigma_t\sigma_t' a_t(x,i)\varphi^{\pi^*}(t,x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)$$
$$-a_t'(x,i)\sigma_t\sigma_t' a_t(x,i)\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^{\pi^*}(t,x,i)$$
$$-\sum_{j\in E_Y}\lambda_{ij}\varphi(t,x,j)\left(\beta_c(s-t,x,j)-\beta_c(s-t,x,i)\right).$$

Therefore we get

$$(3.10) = E_{t,x,i}[\int_t^T E_{u,X_u,Y_{u-}}[\int_s^T \Big(-A^\pi(\beta_p(u-s,X_u,Y_{u-})\psi_s^\pi(u,X_u,Y_{u-}))\Big)du]ds]$$
$$+\int_t^T E_{s,X_s,Y_{s-}}[-A^\pi(\beta_p(T-s,X_s,Y_{s-})\varphi^{\pi^*}(s,X_s,Y_{s-}))]ds].$$

We apply Appendix Lemma 16 and get

$$E_{t,x,i}[\int_t^T E_{s,X_s,Y_{s-}}[-A^\pi(\beta_p(T-s,X_s,Y_{s-})\varphi^{\pi^*}(s,X_s,Y_{s-}))]ds]$$
$$= -E_{t,x,i}[U_p(X_T)\Big(1-\beta_p(T-t,x,i)\Big)]$$

and from Appendix Lemma 14 follows that

$$E_{t,x,i}[\int_t^T E_{u,X_u,Y_{u-}}[\int_s^T \Big(-A^\pi(\beta_p(u-s,X_u,Y_{u-})\psi_s^\pi(u,X_u,Y_{u-}))\Big)du]ds]$$
$$= -E_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-})\Big(1-\beta_c(s-t,x,i)\Big)ds].$$

This finally yields

$$E_{t,x,i}[\int_t^T L_{\pi^*}(s,X_s,Y_{s-}) - J_{\pi^*}(s,X_s,Y_{s-})ds]$$
$$-E_{t,x,i}[\int_t^T \Big(r_s X_s + a_s^{*'}(X_s,Y_{s-})\mu_s - c_s^*(X_s,Y_{s-})\Big)M_{\pi^*}(s,X_s,Y_{s-})$$
$$+\frac{1}{2}a_s^{*'}(X_s,Y_{s-})\sigma_s\sigma_s' a_s^*(X_s,Y_{s-})(N_{\pi^*}(s,X_s,Y_{s-}) - 2W_{\pi^*}(s,X_s,Y_{s-})\Big)ds]$$
$$= -E_{t,x,i}[\int_t^T U_c(c_s(X_s,Y_{s-})\Big(1-\beta_c(s-t,x,i)\Big)ds - U_p(X_T)\Big(1-\beta_p(T-t,x,i)\Big)].$$

Plugging this result in (3.8) we get

$$v(t,x,i) = E_{t,x,i}[v(T,X_T,Y_{T-}) + \int_t^T U_c(c_s^*(X_s,Y_{s-}))ds$$
$$-\int_t^T (1-\beta_c(s-t,x,i))U_c(c_s(X_s,Y_{s-})ds + (1-\beta_p(T-t,x,i))U_p(X_T)]$$

$$= E_{t,x,i}[v(T, X_T, Y_{T-}) + \int_t^T \beta(s-t,x,i)U_c(c_s^*(X_s, Y_{s-})ds - (1-\beta_p(T-t,x,i))U_p(X_T)].$$

Finally we use the Terminal Condition $v(T, X_T, Y_{T-}) = U_p(X_T)$ and get

$$v(t,x,i) = E_{t,x,i}[\int_t^T \beta_c(s-t,x,i)U_c(c_s^*(X_s, Y_{s-})ds + \beta_p(T-t,x,i)U_p(X_T)] = V_{\pi^*}(t,x,i).$$

So we finished the first part of the proof.

ii) To prove the second part we have to show that $\pi^* = (c_t^*, a_t^*)$ is indeed an equilibrium strategy, i.e. we have to show that π^* satisfies for every (t,x,i)

$$\lim_{\epsilon \downarrow 0} \frac{V_{\pi^*}(t,x,i) - V\pi^\epsilon(t,x,i)}{\epsilon} \geq 0.$$

The wealth process under the equilibrium strategy π^* is given as a solution of the following SDE

$$X_t^{\pi^*} = x,\ dX_s^{\pi^*} = \Big(rX_s^{\pi^*} + a_s^{*'}(X_s^{\pi^*}, Y_{s-})\mu_s - c_s^*(X_s^{\pi^*}, Y_{s-})\Big)ds + a_s^{*'}(X_s^{\pi^*}, Y_{s-})\sigma_s dW_s,\ s \in [t,T]$$

and the wealth process under the strategy π^ϵ which we denote by X_t^ϵ is given as a solution of the following SDE

$$X_t^\epsilon = x,\ dX_s^\epsilon = \Big(rX_s^\epsilon + a_s^{\epsilon'}(X_s^\epsilon, Y_{s-})\mu_s - c_s^\epsilon(X_s^\epsilon, Y_{s-})\Big)ds + a_s^{\epsilon'}(X_s^\epsilon, Y_{s-})\sigma_s dW_s,\ s \in [t,T].$$

Appendix Lemma 15 implies that (X_t^ϵ) converges weakly against $(X_t^{\pi^*})$.

Then we get

$$V_{\pi^*}(t,x,i) - V_{\pi^\epsilon}(t,x,i) =$$

$$E_{t,x,i}[\int_t^T \beta_c(s-t,x,i)U_c(c_s^*(X_s^{\pi^*}, Y_{s-}))du + \beta_p(T-t,x,i)U_p(X_T^{\pi^*})]$$

$$-E[\int_t^T \beta_c(s-t,x,i)U_c(c_s^\epsilon(X_s^\epsilon, Y_{s-})ds + \beta_p(T-t,x,i)U_p(X_T^\epsilon)]$$

$$= E_{t,x,i}[\int_t^{t+\epsilon} \beta_c(s-t,x,i)U_c(c_s^*(X_s^{\pi^*}, Y_{s-})ds]$$

$$+ \underbrace{E_{t,x,i}[\int_{t+\epsilon}^T \beta_c(s-t, X_{t+\epsilon}^{\pi^*}, Y_{(t+\epsilon)-})U_c(c_s^*(X_s^{\pi^*}, Y_{s-})ds + \beta_p(T-t, X_{t+\epsilon}^{\pi^*}, Y_{(t+\epsilon)-})U_p(X_T^{\pi^*}))]}_{=V_{\pi^*}(t+\epsilon, X_{t+\epsilon}^{\pi^*}, Y_{(t+\epsilon)-})}$$

$$+E_{t,x,i}[\int_{t+\epsilon}^T \Big(\beta_c(s-t,x,i) - \beta_c(s-t-\epsilon, X_{t+\epsilon}^{\pi^*}, Y_{(t+\epsilon)-})\Big) U_c(c_s^*(X_s^{\pi^*}, Y_{s-}))ds]$$

$$+E_{t,x,i}[\big(\beta_p(T-t,x,i) - \beta_p(T-t-\epsilon, X_{t+\epsilon}^{\pi^*}, Y_{(t+\epsilon)-})\big)U_p(X_T^{\pi^*})]$$

$$-E_{t,x,i}[\int_t^{t+\epsilon} \beta_c(s-t,x,i)U_c(c_s^\epsilon(X_s^\epsilon,Y_{s-}))ds]$$

$$-\underbrace{E_{t,x,i}[\int_{t+\epsilon}^T \beta_c(s-t,X_{t+\epsilon}^\epsilon,Y_{(t+\epsilon)-})U_c(c_s^\epsilon(X_s^\epsilon,Y_{s-}))ds+\beta_p(T-t,X_{t+\epsilon}^\epsilon,Y_{(t+\epsilon)-})U_p(X_T^\epsilon)]}_{=V_{\pi^\epsilon}(t+\epsilon,X_{t+\epsilon}^\epsilon,Y_{(t+\epsilon)-})}$$

$$-E_{t,x,i}[\int_{t+\epsilon}^T \big(\beta_c(s-t,x,i)-\beta_c(s-t-\epsilon,X_{t+\epsilon}^\epsilon,Y_{(t+\epsilon)-})\big)\,cU\big((c_s^\epsilon(X_s^\epsilon,Y_{s-})\big)ds]$$

$$-E_{t,x,i}[\big(\beta_p(T-t,x,i)-\beta_p(T-t-\epsilon,X_{t+\epsilon}^\epsilon,Y_{(t+\epsilon)-}\big)U_p(X_T^\epsilon)].$$

We use the fact that $c_t^\epsilon(x,i)=c_t^*(x,i)$ on $[t+\epsilon,T]$ and $a^\epsilon(t,x,i)=a_t^*(x,i)$ on $[t+\epsilon,T]$ which implies that

$$V_{\pi^\epsilon}(t+\epsilon,X_{t+\epsilon}^\epsilon,Y_{(t+\epsilon)-})=V_{\pi^*}(t+\epsilon,X_{t+\epsilon}^{\pi^*},Y_{(t+\epsilon)-}).$$

After re-sorting, dividing the equation by ϵ and taking the limit $\epsilon\downarrow 0$ we get

$$\lim_{\epsilon\downarrow 0}\frac{V_{\pi^*}(t,x,i)-V_{\pi^\epsilon}(t,x,i)}{\epsilon}$$

$$=\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}E_{t,x,i}[\int_t^{t+\epsilon}\beta_c(s-t,x,i)\big(U_c(c_s^*(X_s^{\pi^*},Y_{s-})-U_c(c_s^\epsilon(X_s^\epsilon,Y_{s-}))\big)ds]$$

$$+\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}E_{t,x,i}[V_{\pi^*}(t+\epsilon,X_{t+\epsilon}^{\pi^*},Y_{(t+\epsilon)-})-V_{\pi^*}(t+\epsilon,X_{(t+\epsilon)-}^\epsilon,Y_{(t+\epsilon)-})]$$

$$+\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}E_{t,x,i}[\int_{t+\epsilon}^T\Big(\beta_c(s-t,x,i)-\beta_c(s-t-\epsilon,X_{t+\epsilon}^{\pi^*},Y_{(t+\epsilon)-})\Big)\,U\big(c_s^*(X_s^{\pi^*},Y_{s-})\big)ds]$$

$$-\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}E_{t,x,i}[\int_{t+\epsilon}^T\big(\beta_c(s-t,x,i)-\beta_c(s-t-\epsilon,X_{t+\epsilon}^\epsilon,Y_{(t+\epsilon)-})\big)\,U\big(c_s^*(X_s^\epsilon,Y_{s-})\big)ds]$$

$$+\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}E_{t,x,i}[\big(\beta_p(T-t,x,i)-\beta_p(T-t-\epsilon,X_{t+\epsilon}^{\pi^*},Y_{(t+\epsilon)-})\big)U_p(X_T^{\pi^*})]$$

$$-\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}E_{t,x,i}[\big(\beta_p(T-t,x,i)-\beta_p(T-t-\epsilon,X_{t+\epsilon}^\epsilon,Y_{(t+e)-})\big)U_p(X_T^\epsilon)].$$

We will treat each of the six terms separately. The idea to compute the limit by separating the expression into several therms is also used in Ekeland et al. (2012), where the discount function only depends on the time t.

1)

$$\lim_{\epsilon\downarrow 0}\frac{E_{t,x,i}[\int_t^{t+\epsilon}\beta_c(s-t,x,i)\left(U_c(c_s^*(X_s^{\pi^*},Y_{s-})-U_c(c_s^\epsilon(X_s^\epsilon,Y_{s-}))\right)ds]}{\epsilon}$$

$$=U_c(c_t^*(x,i))-U_c(c_t^\epsilon(x,i)).$$

2) It holds that

$$E_{t,x,i}[V_{\pi^*}(t+\epsilon, X^{\pi^*}_{(t+\epsilon)-}, Y_{(t+\epsilon)-}) - V_{\pi^*}(t+\epsilon, X^{\epsilon}_{t+\epsilon}, Y_{(t+\epsilon)-})]$$

$$= E_{t,x,i}[V_{\pi^*}(t+\epsilon, X^{\pi^*}_{t+\epsilon}, Y_{(t+\epsilon)-}) - V_{\pi^*}(t,x,i)] - E[V_{\pi^*}(t+\epsilon, X^{\epsilon}_{t+\epsilon}, Y_{(t+\epsilon)-}) - V_{\pi^*}(t,x,i)].$$

Moreover

$$E_{t,x,i}[V_{\pi^*}(t+\epsilon, X^{\pi^*}_{t+\epsilon}, Y_{(t+\epsilon)-}) - V_{\pi^*}(t,x,i)] = E_{t,x,i}[\int_t^{t+\epsilon} dV_{\pi^*}(s, X^{\pi^*}_s, Y_{s-})]$$

$$E_{t,x,i}[V_{\pi^*}(t+\epsilon, X^{\epsilon}_{t+\epsilon}, Y_{(t+\epsilon)-}) - V_{\pi^*}(t,x,i)] = E_{t,x,i}[\int_t^{t+\epsilon} dV_{\pi^*}(s, X^{\epsilon}_s, Y_{s-}))].$$

Ito's formula yields

$$\lim_{\epsilon\downarrow 0} \frac{E_{t,x,i}[\int_t^{t+\epsilon} dV_{\pi^*}(s, X^{\pi^*}_s, Y_{s-})]}{\epsilon} = \frac{\partial}{\partial t} V_{\pi^*}(t,x,i) + \Big(r_t x + a^{*'}_t(x,i)\mu_t - c^*_t(x,i)\Big)\frac{\partial}{\partial x} V_{\pi^*}(t,x,i)$$
$$+\frac{1}{2} a^{*'}_t(x,i)\sigma_t\sigma'_t a^*_t(x,i)\frac{\partial}{\partial x^2} V_{\pi^*}(t,x,i) + \sum_{j\in E_Y} \lambda_{ij} V_{\pi^*}(t,x,j)$$

$$\lim_{\epsilon\downarrow 0} \frac{E_{t,x,i}[\int_t^{t+\epsilon} dV_{\pi^*}(s, X^{\epsilon}_s, Y_{s-})]}{\epsilon} = \frac{\partial}{\partial t} V_{\pi^*}(t,x,i) + \Big(r_t x + a^{\epsilon'}_t(x,i)\mu_t - c^{\epsilon}_t(x,i)\Big)\frac{\partial}{\partial x} V_{\pi^*}(t,x,i)$$
$$+\frac{1}{2} a^{\epsilon'}_t(x,i)\sigma_t\sigma'_t a^{\epsilon}_t(x,i)\frac{\partial}{\partial x^2} V_{\pi^*}(t,x,i) + \sum_{j\in E_Y} \lambda_{ij} V_{\pi^*}(t,x,j).$$

Summarizing we get

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} E_{t,x,i}[V_{\pi^*}(t+\epsilon, X^{\pi^*}_{(t+\epsilon)-}, Y_{(t+\epsilon)-}) - V_{\pi^*}(t+\epsilon, X^{\epsilon}_{t+\epsilon}, Y_{(t+\epsilon)-})]$$
$$= \Big(r_t x + a^{*'}_t(x,i)\mu_t - c^*_t(x,i)\Big)\frac{\partial}{\partial x} V_{\pi^*}(t,x,i) + \frac{1}{2} a^{*'}_t(x,i)\sigma_t\sigma'_t a^*_t(x,i)\frac{\partial}{\partial x^2} V(t,x,i)$$
$$-\Big(r_t x + a^{\epsilon'}_t(x,i)\mu_t - c^{\epsilon}_t(x,i)\Big)\frac{\partial}{\partial x} V_{\pi^*}(t,x,i) - \frac{1}{2} a^{\epsilon'}_t(x,i)\sigma_t\sigma'_t a^{\epsilon}_t(x,i)\frac{\partial}{\partial x^2} V(t,x,i).$$

3) By using Appendix Lemma 9 we get

$$\lim_{\epsilon\downarrow 0} \frac{E_{t,x,i}[\int_{t+\epsilon}^T \big(\beta_c(s-t,x,i) - \beta_c(s-t-\epsilon, X^{\pi^*}_{t+\epsilon}, Y_{(t+\epsilon)-})\big) U_c(c^*_s(X^{\pi^*}_s, Y_{s-})\big)ds]}{\epsilon}$$

$$= -\lim_{\epsilon\downarrow 0} \frac{E_{t,x,i}[\int_{t+\epsilon}^T \big(\beta_c(s-t-\epsilon, X^{\pi^*}_{t+\epsilon}, Y_{(t+\epsilon)-}) - \beta_c(s-t,x,i)\big) U_c(c^*_s(X^{\pi^*}_s, Y_{s-})\big)ds]}{\epsilon}$$

$$= E_{t,x,i}[\int_t^T \Big(\frac{\partial}{\partial t}\beta_c(s-t,x,i) - \Big(r_t x + a^{*'}_t(x,i)\mu_t - c^*_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i)$$
$$-\frac{1}{2} a^{*'}_t(x,i)\sigma_t\sigma'_t a^*_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)\Big) U_c(c^*_s(X^{\pi^*}_s, Y_{s-}))ds]$$
$$-a^{*'}_t(x,i)\sigma_t\sigma'_t a^*_t(x,i)\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^{\pi^*}_s(t,x,i)ds$$
$$-E_{t,x,j}[\int_t^T \sum_{j\in E_Y} \lambda_{ij}\big(\beta_c(s-t,x,j) - \beta_c(s-t,x,i)\big) U_c(c^*_s(X^{\pi^*}_s, Y_{s-}))ds].$$

4) Now we use Appendix Lemma 17 and get

$$
\begin{aligned}
&-\lim_{\epsilon\downarrow 0}\frac{E_{t,x,i}[\int_{t+\epsilon}^{T}\big(\beta_c(s-t,x,i)-\beta_c(s-t-\epsilon,X^{\epsilon}_{t+\epsilon},Y_{(t+\epsilon)-})\big)U_c(c^*_s(X^{\epsilon}_s,Y_{s-})\big)ds]}{\epsilon}\\
&=\lim_{\epsilon\downarrow 0}\frac{E_{t,x,i}[\int_{t+\epsilon}^{T}\big(\beta_c(s-t-\epsilon,X^{\epsilon}_{t+\epsilon},Y_{(t+\epsilon)-})-\beta_c(s-t,x,i)\big)U_c(c^*_s(X^{\epsilon}_s,Y_{s-})\big)ds]}{\epsilon}\\
&=E_{t,x,i}[\int_t^T\Big(-\frac{\partial}{\partial t}\beta_c(s-t,x,i)+\Big(r_tx+a^{\epsilon'}_t(x,i)\mu_t-c^{\epsilon}_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i)\\
&\qquad+\frac{1}{2}a^{\epsilon'}_t(x,i)\sigma_t\sigma'_ta^{\epsilon}_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)\Big)U_c(c^*_s(X^{\pi^*}_s,Y_{s-}))ds]\\
&\qquad+a^{\epsilon'}_t(x,i)\sigma_t\sigma'_ta^{\epsilon}_t(x,i)\int_t^T\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^{\pi^*}_s(t,x,i)ds\\
&\quad+E_{t,x,j}[\int_t^T\sum_{j\in E_Y}\lambda_{ij}\big(\beta_c(s-t,x,j)-\beta_c(s-t,x,i)\big)U_c(c^*_s(X^{\pi^*}_s,Y_{s-}))ds].
\end{aligned}
$$

3+4 is given by

$$
\begin{aligned}
&=-E_{t,x,i}[\int_t^T\Big(\Big(r_tx+a^{*'}_t(x,i)\mu_t-c^*_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i)\\
&-\frac{1}{2}a^{*'}_t(x,i)\sigma_t\sigma'_ta^*_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)\Big)U_c(c^*_s(X^{\pi^*}_s,Y_{s-}))ds]\\
&\quad-a^{*'}_t(x,i)\sigma_t\sigma'_ta^*_t(x,i)\int_t^T\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^{\pi^*}_s(t,x,i)ds\\
&\quad+E_{t,x,i}[\int_t^T\Big(\Big(r_tx+a^{\epsilon'}_t(x,i)\mu_t-c^{\epsilon}_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i)\\
&+\frac{1}{2}a^{\epsilon'}_t(x,i)\sigma_t\sigma'_ta^{\epsilon}_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)\Big)U_c(c^*_s(X^{\pi^*}_s,Y_{s-}))ds]\\
&\quad+a^{\epsilon'}_t(x,i)\sigma_t\sigma'_ta^{\epsilon}_t(x,i)\int_t^T\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^{\pi^*}_s(t,x,i)ds.
\end{aligned}
$$

5) Using Appendix Lemma 10 we get

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}\frac{\Big(\beta_p(T-t,x,i)-\beta_p(T-t-\epsilon,X^{\pi^*}_{t+\epsilon},Y_{(t+\epsilon)-})\Big)U\big(X^{\pi^*}_T)]}{\epsilon}\\
&=-\lim_{\epsilon\downarrow 0}\frac{E_{t,x,i}[\Big(\beta_p(T-t-\epsilon,X^{\pi^*}_{t+\epsilon},Y_{(t+\epsilon)-})-\beta_p(T-t,x,i)\Big)U\big(X^{\pi^*}_T)]}{\epsilon}\\
&=\Big(\frac{\partial}{\partial t}\beta_p(T-t,x,i)-\Big(\Big(r_tx+a^{*'}_t(x,i)\mu_t-c^*_t(x,i)\Big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\\
&\qquad-\frac{1}{2}a^{*'}_t(x,i)\sigma_t\sigma'_ta^*_t(x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)\Big)E_{t,x,i}[U_p(X^{\pi^*}_T)]\\
&\qquad-a^{*'}_t(x,i)\sigma_t\sigma'_ta^*_t(x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i)\\
&\qquad-\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(s-t,x,j)-\beta_p(s-t,x,i)\big)E_{t,x,j}[U_p(X^{\pi^*}_T)].
\end{aligned}
$$

6) Using Appendix Lemma 17 we get

$$
\begin{aligned}
&-\lim_{\epsilon\downarrow 0}\frac{E_{t,x,i}[\Big(\beta_p(T-t,x,i)-\beta_p(T-t-\epsilon,X^{\epsilon}_{t+\epsilon},Y_{(t+\epsilon)-})\Big)U\big(X^{\epsilon}_T\big)]}{\epsilon}\\
&=\lim_{\epsilon\downarrow 0}\frac{E_{t,x,i}[\Big(\beta_p(T-t-\epsilon,X^{\epsilon}_{t+\epsilon},Y_{(t+\epsilon)-})-\beta_p(T-t,x,i)\Big)U\big(X^{\epsilon}_T\big)]}{\epsilon}\\
&=E_{t,x,i}[\Big(-\frac{\partial}{\partial t}\beta_p(T-t,x,i)+\big(r_tx+a^{\epsilon'}_t(x,i)\mu_t-c^{\epsilon}_t(x,i)\big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\\
&\quad+\frac{1}{2}a^{\epsilon'}_t(x,i)\sigma_t\sigma_t'a^{\epsilon}_t(x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)\Big)E_{t,x,i}[U_p(X^{\pi^*}_T)]\\
&\quad+a^{\epsilon'}_t(x,i)\sigma_t\sigma_t'a^{\epsilon}_t(x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i)\\
&\quad+\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(s-t,x,j)-\beta_p(s-t,x,i)\big)E_{t,x,j}[U_p(X^{\pi^*}_T)].
\end{aligned}
$$

5+6 is given by

$$
\begin{aligned}
&=-\Big(\big(r_tx+a^{*'}_t(x,i)\mu_t-c^{*}_t(x,i)\big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\\
&\quad+\frac{1}{2}a^{*'}_t(x,i)\sigma_t\sigma_t'a^{*}_t(x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)\Big)E_{t,x,i}[U_p(X^{\pi^*}_T)]\\
&\quad-a^{*'}_t(x,i)\sigma_t\sigma_t'a^{*}_t(x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i)\\
&\quad+\Big(\big(r_tx+a^{\epsilon'}_t(x,i)\mu_t-c^{\epsilon}_t(x,i)\big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\\
&\quad+\frac{1}{2}a^{\epsilon'}_t(x,i)\sigma_t\sigma_t'a^{\epsilon}_t(x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)\Big)E_{t,x,i}[U_p(X^{\pi^*}_T)]\\
&\quad+a^{\epsilon'}_t(x,i)\sigma_t\sigma_t'a^{\epsilon}_t(x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i).
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}\frac{V_{\pi^*}(t,x,i)-V_{\pi^\epsilon}(t,x,i)}{\epsilon}\\
&=U_c(c^*_t(x,i))-U_c(c^{\epsilon}_t(x,i))\\
&+\Big(r_tx+a^{*'}_t(x,i)\mu_t-c^*_t(x,i)\Big)\frac{\partial}{\partial x}V_{\pi^*}(t,x,i)+\frac{1}{2}a^{*'}_t(x,i)\sigma_t\sigma_t'a^*_t(x,i)\frac{\partial}{\partial x^2}V_{\pi^*}(t,x,i)\\
&-\Big(r_tx+a^{\epsilon'}_t(x,i)\mu_t-c^{\epsilon}_t(x,i)\Big)\frac{\partial}{\partial x}V_{\pi^*}(t,x,i)-\frac{1}{2}a^{\epsilon'}_t(x,i)\sigma_t\sigma_t'a^{\epsilon}_t(x,i)\frac{\partial}{\partial x^2}V_{\pi^*}(t,x,i)\\
&-E_{t,x,i}[\int_t^T\Big(\big(r_tx+a^{*'}_t(x,i)\mu_t-c^*_t(x,i)\big)\frac{\partial}{\partial x}\beta_c(s-t,x,i)\\
&-\frac{1}{2}a^{*'}_t(x,i)\sigma_t\sigma_t'a^*_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)\Big)U_c(c^*_s(X^{\pi^*}_s,Y_{s-}))ds]\\
&-a^{*'}_t(x,i)\sigma_t\sigma_t'a^*_t(x,i)\int_t^T\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^{\pi^*}_s(t,x,i)ds
\end{aligned}
$$

$$+E_{t,x,i}[\int_t^T \Big(\big(r_t x + a_t^{\epsilon'}(x,i)\mu_t - c_t^{\epsilon}(x,i)\big)\frac{\partial}{\partial x}\beta_c(s-t,x,i)$$
$$+\frac{1}{2}a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^{\epsilon}(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)\Big)U_c(c_s^*(X_s^{\pi^*},Y_{s-}))ds]$$
$$+a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^{\epsilon}(x,i)\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi_s^{\pi^*}(t,x,i)ds$$
$$-\Big(\big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^{*}(x,i)\big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)$$
$$+\frac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^{*}(x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)\Big)E_{t,x,i}[U_p(X_T^{\pi^*})]$$
$$-a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^{*}(x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i)$$
$$+\Big(\big(r_t x + a_t^{\epsilon'}(x,i)\mu_t - c_t^{\epsilon}(x,i)\big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)$$
$$+\frac{1}{2}a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^{\epsilon}(x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)\Big)E_{t,x,i}[U_p(X_T^{\pi^*})]$$
$$+a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^{\epsilon}(x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i)$$

We re-sort and get

$$= U_c(c_t^*(x,i)) + \big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\big)\frac{\partial}{\partial x}V_{\pi^*}(t,x,i) + \frac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)\frac{\partial}{\partial x^2}V_{\pi^*}(t,x,i)$$
$$-\big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\big)$$
$$\cdot \underbrace{E_{t,x,i}[\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)U_c(c_s^*(X_s^{\pi^*},Y_{s-}))ds + \frac{\partial}{\partial x}\beta_p(T-t,x,i)U_p(X_T^{\pi^*})]}_{=M_{\pi^*}(t,x,i)}$$
$$-\frac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)$$
$$\cdot \underbrace{E_{t,x,i}[\int_t^T \frac{\partial}{\partial x^2}\beta_c(s-t,x,i)U_c(c_s^*(X_s^{\pi^*},Y_{s-}))ds + \frac{\partial}{\partial x^2}\beta_p(T-t,x,i)U_p(X_T^{\pi^*})]}_{=N_{\pi^*}(t,x,i)}$$
$$-a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)\Big(\underbrace{\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi_s^{\pi^*}(t,x,i)ds + \frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i)}_{=W_{\pi^*}(t,x,i)}\Big)$$
$$-U_c(c_t^{\epsilon}(x,i)) - \big(r_t x + a_t^{\epsilon'}(x,i)\mu_t - c_t^{\epsilon}(x,i)\big)\frac{\partial}{\partial x}V_{\pi^*}(t,x,i) - \frac{1}{2}a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^{\epsilon}(x,i)\frac{\partial}{\partial x^2}V_{\pi^*}(t,x,i)$$
$$+\big(r_t x + a_t^{\epsilon'}(x,i)\mu_t - c_t^{\epsilon}(x,i)\big)$$
$$\cdot \underbrace{E_{t,x,i}[\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)U_c(c_s^*(X_s^{\pi^*},Y_{s-}))ds + \frac{\partial}{\partial x}\beta_p(T-t,x,i)U_p(X_T^{\pi^*})]}_{=M_{\pi^*}(t,x,i)}$$

$$+\frac{1}{2}a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^{\epsilon}(x,i)$$

$$\cdot \underbrace{E_{t,x,i}[\int_t^T \frac{\partial}{\partial x^2}\beta_c(s-t,x,i)U_c(c_s^*(X_s^{\pi^*},Y_{s-}))ds + \frac{\partial}{\partial x^2}\beta_p(T-t,x,i)U_p(X_T^{\pi^*})]}_{=N_{\pi^*}(t,x,i)}$$

$$+a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^{\epsilon}(x,i)\underbrace{\Big(\int_t^T \frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi_s^{\pi^*}(t,x,i)ds + \frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^{\pi^*}(t,x,i)\Big)}_{=W_{\pi^*}(t,x,i)}.$$

So we can re-write the equation

$$\lim_{\epsilon\downarrow 0}\frac{V_{\pi^*}(t,x,i)-V_{\pi^\epsilon}(t,x,i)}{\epsilon}$$

$$= U_c(c_t^*(x,i)) + \big(r_t x + a_t^{*'}(x,i)\mu_t - c_t^*(x,i)\big)\left(\frac{\partial}{\partial x}V_{\pi^*}(t,x,i) - M_{\pi^*}(t,x,i)\right)$$
$$+\frac{1}{2}a_t^{*'}(x,i)\sigma_t\sigma_t' a_t^*(x,i)\left(\frac{\partial}{\partial x^2}V(t,x,i) - N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i)\right)$$
$$-\Big[U_c(c_t^\epsilon(x,i)) + \big(r_t x + a_t^{\epsilon'}(x,i)\mu_t - c_t^\epsilon(x,i)\big)\left(\frac{\partial}{\partial x}V_{\pi^*}(t,x,i) - M_{\pi^*}(t,x,i)\right)$$
$$+\frac{1}{2}a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^\epsilon(x,i)\left(\frac{\partial}{\partial x^2}V_{\pi^*}(t,x,i) - N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i)\right)\Big].$$

Now we use that (c_t^*, a_t^*) are maximizer of

$$\sup_{(c,a)\in D}\Big\{U_c(c) + \big(r_t x + a'\mu_t - c(x,i)\big)\left(\frac{\partial}{\partial x}V_{\pi^*}(t,x,i) - M_{\pi^*}(t,x,i)\right)$$
$$+\frac{1}{2}a'\sigma_t\sigma_t' a\left(\frac{\partial}{\partial x^2}V_{\pi^*}(t,x,i) - N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i)\right)\Big\}$$

and re-write the equation as

$$\lim_{\epsilon\downarrow 0}\frac{V_{\pi^*}(t,x,i)-V_{\pi^\epsilon}(t,x,i)}{\epsilon}$$

$$= \sup_{(c,a)\in D}\Big\{U_c(c) + \big(r_t x + a'\mu_t - c\big)\left(\frac{\partial}{\partial x}V_{\pi^*}(t,x,i) - M_{\pi^*}(t,x,i)\right)$$
$$+\frac{1}{2}a'\sigma_t\sigma_t' a\left(\frac{\partial}{\partial x^2}V_{\pi^*}(t,x,i) - N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i)\right)\Big\}$$
$$-\Big[U_c(c_t^\epsilon(x,i)) + \big(r_t x + a_t^{\epsilon'}(x,i)\mu_t - c_t^\epsilon(x,i)\big)\left(\frac{\partial}{\partial x}V_{\pi^*}(t,x,i) - M_{\pi^*}(t,x,i)\right)$$
$$+\frac{1}{2}a_t^{\epsilon'}(x,i)\sigma_t\sigma_t' a_t^\epsilon(x,i)\left(\frac{\partial}{\partial x^2}V_{\pi^*}(t,x,i) - N_{\pi^*}(t,x,i) - 2W_{\pi^*}(t,x,i)\right)\Big] \geq 0.$$

□

Theorem 8. *In the case of standard discounting we have* $\beta_c(t,x,i) = \beta_p(t,x,i) = \exp(-\rho t)$, $\rho > 0$. *Then it holds:*

a) The extended HJB-equation reduces to

$$0 = \frac{\partial}{\partial t} v(x,t) - \rho V_\pi(t,x) + \sup_{(c,a)\in D} \Big\{ U_c(c) + \Big(rx + a'\mu_t - c \Big) \frac{\partial}{\partial x} v(t,x) + \frac{1}{2} a'\sigma_t\sigma_t' a \frac{\partial}{\partial x^2} v(t,x) \Big\},$$

with terminal condition

$$v(T,x) = U_p(X_T).$$

b) Let $\pi^* = (c_t^*, a_t^*) \in \mathcal{A}$,$v(\cdot,\cdot,i) \in C^{1,2}$, $\forall i \in E_Y$ *be a solution of the extended HJB-equation which satisfy Assumption (T) and* $(c_t^*(x,i), a_t^*(x,i))$ *is a maximum point of*

$$(c,a) \;\rightarrow\; U_c(c) + \Big(r_t x + a'\mu_t - c \Big) \frac{\partial}{\partial x} v(t,x) + \frac{1}{2} a'\sigma_t\sigma_t' a \frac{\partial}{\partial x^2} v(t,x), \; (c,a) \in D, \tag{3.11}$$

then it holds that the equilibrium strategy π^* *is an optimal strategy and* v *is the optimal value function, i.e.*

$$v(t,x) = V_{\pi^*}(t,x) = \sup_{\pi \in \mathcal{A}} E_{t,x,i}[\int_t^T \exp(-\rho(s-t)) U_c(c_s(X_s^\pi, Y_{s-}) ds + \exp(-\rho(T-t)) U_p(X_T^\pi)].$$

Proof. a) Since β_c and β_p are independent of x, it holds that $M_\pi(t,x,i) = 0$, $N_\pi(t,x,i) = 0$, $J_\pi(t,x,i) = 0$, $W_\pi(t,x,i) = 0$ and $\sum_{j\in E_Y} \lambda_{ij} v(t,x,j) = 0$. Moreover we get

$$L_\pi(t,x,i) = E_{t,x,i}[\int_t^T U_c(c_s(X_s^\pi, Y_{s-}))(-\rho) \exp(-\rho(s-t)) ds + U_p(X_T^\pi)(-\rho) \exp(-\rho(T-t))]$$

$$= -\rho E_{t,x}[\int_t^T U_c(c_s(X_s^\pi, Y_{s-})) \exp(-\rho(s-t)) ds + U_p(X_T^\pi) \exp(-\rho(T-t))] = -\rho V_\pi(t,x).$$

So we get that the extended HJB-equation reduces to

$$0 = \frac{\partial}{\partial t} v(x,t) - \rho V_\pi(t,x) + \sup_{(c,a)\in D} \Big\{ U_c(c) + \Big(rx + a'\mu_t - c \Big) \frac{\partial}{\partial x} v(t,x) + \frac{1}{2} a'\sigma_t\sigma_t' a \frac{\partial}{\partial x^2} v(t,x) \Big\}.$$

b) Let (v, π^*) be a solution of the extended HJB-equation, then the Verification Theorem implies that $v = V_\pi^*$ and satisfies

$$0 = \frac{\partial}{\partial t} v(x,t) - \rho V_{\pi^*}(t,x) + U_c(c_t^*(x)) + \Big(r_t x + a_t^{*'}(x)\mu_t - c_t^*(x) \Big) \frac{\partial}{\partial x} v(t,x)$$
$$+ \frac{1}{2} a_t^{*'}(x)\sigma_t\sigma_t' a_t^*(x) \frac{\partial}{\partial x^2} v(t,x)$$
$$\Leftrightarrow 0 = \frac{\partial}{\partial t} v(x,t) - \rho v(t,x) + U_c(c_t^*(x)) + \Big(r_t x + a_t^{*'}(x)\mu_t - c_t^*(x) \Big) \frac{\partial}{\partial x} v(t,x)$$
$$+ \frac{1}{2} a_t^{*'}(x)\sigma_t\sigma_t' a_t^*(x) \frac{\partial}{\partial x^2} v(t,x),$$

since $v = V_{\pi^*}$. Now we use (c_t^*, a_t^*) is the maximum point of (3.11)

$$0 = \frac{\partial}{\partial t} v(x,t) - \rho v(t,x) + \sup_{(c,a)\in D} \Big\{ U_c(c) + \Big(rx + a'\mu_t - c \Big) \frac{\partial}{\partial x} v(t,x) + \frac{1}{2} a'\sigma_t\sigma_t' a \frac{\partial}{\partial x^2} v(t,x) \Big\}. \tag{3.12}$$

So we get that (v, π^*), with $\pi^* = (c_t^*, a_t^*)$ is a solution of (3.12) which is the classical HJB-equation. The statement follows from the classical theory.

3.5 Special cases

Now we only consider discount functions, which are independent of the wealth x, i.e $\beta_c(t,x,i)=\beta_c(t,i)$ and $\beta_p(t,x,i)=\beta_p(t,i)$.

Proposition 2. *Let the discount functions $\beta_c(t,x,i)$ and $\beta_p(t,x,i)$ are independent of the wealth, i.e. $\beta_c(t,x,i)=\beta_c(t,i)$ and $\beta_p(t,x,i)=\beta_p(t,i)$, the extended HJB-equation reduces to*

$$0=\frac{\partial}{\partial t}v(t,x,i)+L_\pi(t,x,i)+\sum_{j\in E_Y}\lambda_{ij}v(t,x,j)-J_\pi(t,x,i)$$

$$+\sup_{(c,a)\in D}\Big\{U_c(c)+\Big(r_tx+a'\mu_t-c\Big)\frac{\partial}{\partial x}v(t,x,i)+\frac{1}{2}a'\sigma_t\sigma_t'a\frac{\partial}{\partial x^2}v(t,x,i)\Big\},$$

where

$$L_\pi(t,x,i):=E_{t,x,i}[\int_t^T U_c(c_s(X_s^\pi,Y_{s-}))\frac{\partial}{\partial t}\beta_c(s-t,x,i)ds+U_p(X_T^\pi)\frac{\partial}{\partial t}\beta_p(T-t,x,i)]$$

$$J_\pi(t,x,i)=:\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T\big(\beta_c(s-t,j)-\beta_c(s-t,i)\big)E_{t,x,j}[U_c(c_s(X_s^\pi,Y_{s-}))]ds$$

$$+\big(\beta_p(T-t,j)-\beta_p(T-t,i)\big)E_{t,x,j}[U_p(X_T^\pi)]\Big).$$

with terminal condition

$$v(T,x,i)=U_p(x).$$

Proof. It holds that $M_\pi(t,x,i)=0$ and $N_\pi(t,x,i)=0,$.

□

Proposition 3. *We assume that $v\in C^{1,2}$ and $\pi^*=(c_t^*,a_t^*)\in\mathcal{A}$ is a solution of the extended HJB which satisfy Assumption (T) and (c_t^*,a_t^*) are maximum points of*

$$(c,a)\to U_c(c)+\Big(r_tx+a'\mu_t-c\Big)\frac{\partial}{\partial x}v(t,x,i)+\frac{1}{2}a'\sigma_t\sigma_t'a\frac{\partial}{\partial x^2}v(t,x,i),\ (c,a)\in D.$$

Then v is given as the solution of the PIDE

$$\left.\begin{aligned}&0=\tfrac{\partial}{\partial t}v(t,x,i)+L_{\pi^*}(t,x,i)+\textstyle\sum_{j\in E_Y}\lambda_{ij}v(t,x,j)-J_{\pi^*}(t,x,i)+U_c(I(\tfrac{\partial}{\partial x}v(t,x,i)))\\&+r_tx\tfrac{\partial}{\partial x}v(t,x,i)-\tfrac{1}{2}\tfrac{\left(\frac{\partial}{\partial x}v(t,x,i)\right)^2}{\frac{\partial}{\partial x^2}v(t,x,i)}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t-I(\tfrac{\partial}{\partial x}v(t,x,i))\tfrac{\partial}{\partial x}v(t,x,i)\end{aligned}\right\},\tag{3.13}$$

where $I(x):=(U'(x))^{-1}$.

Proof.
First note that the maximizers of

$$(c,a) \to \Big\{U_c(c) + \Big(r_t x + a'\mu_t - c\Big)\frac{\partial}{\partial x}v(t,x,i) + \frac{1}{2}a'\sigma_t\sigma_t' a\frac{\partial}{\partial x^2}v(t,x,i)\Big\},\ (c,a)\in D$$

are given by

$$c_t^*(x,i) := I(\frac{\partial}{\partial x}v(t,x,i)).$$

$$a_t^*(x,i) := -\frac{\frac{\partial}{\partial x}v(t,x,i)}{\frac{\partial}{\partial x^2}V(t,x,i)}(\sigma_t\sigma_t')^{-1}\mu_t$$

Plugging that in the extended HJB-equation we get

$$0 = \frac{\partial}{\partial t}v(t,x,i) + L_{\pi^*}(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}v(t,x,j) - J_{\pi^*}(t,x,i)$$

$$+U_c(I(\frac{\partial}{\partial x}v(t,x,i))) + \Big(r_t x - \frac{\frac{\partial}{\partial x}v(t,x,i)}{\frac{\partial}{\partial x^2}v(t,x,i)}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\frac{\partial}{\partial x}v(t,x,i) - I(\frac{\partial}{\partial x}v(t,x,i))\Big)\frac{\partial}{\partial x}v(t,x,i)$$

$$+\frac{1}{2}\Big(\frac{\frac{\partial}{\partial x}v(t,x,i)}{\frac{\partial}{\partial x^2}v(t,x,i)}\Big)^2\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\frac{\partial}{\partial x^2}v(t,x,i)$$

$$\Leftrightarrow 0 = \frac{\partial}{\partial t}V(t,x,i) + L_{\pi^*}(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}V(t,x,j) - J_{\pi^*}(t,x,i) + U_c(I(\frac{\partial}{\partial x}v(t,x,i)))$$

$$+r_t x\frac{\partial}{\partial x}v(t,x,i) - \frac{1}{2}\frac{\big(\frac{\partial}{\partial x}v(t,x,i)\big)^2}{\frac{\partial}{\partial x^2}v(t,x,i)}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t - I(\frac{\partial}{\partial x}v(t,x,i))\frac{\partial}{\partial x}v(t,x,i).$$

□

Theorem 9 (Construction Theorem)**.** *Let $v \in C^{1,2}$ and let v satisfies the following conditions:*

1) The maximum point of

$$(c,a) \to U_c(c) + \Big(r_t x + a'\mu_t - c\Big)\frac{\partial}{\partial x}v(t,x,i) + \frac{1}{2}a'\sigma_t\sigma_t' a\frac{\partial}{\partial x^2}v(t,x,i), \quad (c,a)\in D$$

exists and is denoted by $c_t^*(x,i)$ and $a_t^*(x,i)$. Further it holds that $\pi^* := (c_t^*, a_t^*) \in \mathcal{A}$.

2) (v,π^*) is a solution of (3.13) and fulfills Assumption (T).

Then it holds that $v = V_{\pi^*}$ is the equilibrium value function and the equilibrium strategy is given by

$$c_t^*(x,i) = I(\frac{\partial}{\partial x}v(t,x,i))$$

$$a_t^*(x,i) = -\frac{\frac{\partial}{\partial x}v(t,x,i)}{\frac{\partial}{\partial x^2}v(t,x,i)}(\sigma_t'\sigma_t)^{-1}\mu_t.$$

Proof.
Follows directly from the Verification theorem and Proposition 3.

Remark 2. *The dependency of the equilibrium strategy on the value function is like in the standard case. In the case that the value has a structure where we can separate the wealth (for example $v(t,x,i) = g(t,i)h(x)$), the equilibrium investment strategy a_t^* is independent of i.*

In this section we present also the solution in the case of standard discounting, i.e. $\beta_c(t,i) = \beta_p(t,i) = \exp(-\rho t)$, $\rho \geq 0$. Therefore the value function is given by

$$V_\pi(t,x) = E_{t,x}[\int_t^T \exp(-\rho(s-t))U_c(c_s(X_s^\pi, Y_{s-})ds + \exp(-\rho(T-t))U_p(X_T^\pi)].$$

In the literature they often discuss the following value function (classical problem)

$$V_\pi^1(t,x) := E_{t,x}[\int_t^T \exp(-\rho s)U_c(c_s(X_s^\pi, Y_{s-})ds + \exp(-\rho T)U_p(X_T^\pi)].$$

$\exp(-\rho t)$ has no influence on the optimal strategy, since

$$\sup_{\pi\in\mathcal{A}} E_{t,x}[\int_t^T \exp(-\rho(s-t))U_c(c_s(X_s^\pi, Y_{s-})ds + \exp(-\rho(T-t))U_p(X_T^\pi)]$$

$$= \exp(\rho t) \sup_{\pi\in\mathcal{A}} E_{t,x}[\int_t^T \exp(-\rho s)U_c(c_s(X_s^\pi, Y_{s-})ds + \exp(-\rho T)U_p(X_T^\pi)].$$

Hence for the optimal value functions it holds

$$V(t,x) = \exp(-\rho t)V_1(t,x).$$

3.5.1 Logarithmic-utility

We consider the case $U_c(x) = U_p(x) = \log(x)$ with $dom(U) = (0, \infty)$. Then the solution is given by the following Theorem.

Theorem 10. *a) The equilibrium value function is given by*

$$V(t,x,i) = \Big(\int_t^T \beta_c(s-t,i)ds + \beta_p(T-t,i)\Big)\log(x) + d(t,i), \quad x \in (0,\infty), i \in E_Y,$$

where $d(t,i)$ is given as the unique solution of the first order linear differential equation

$$\begin{aligned} 0 = \frac{\partial}{\partial t} d(t,i) + \sum_{j \in E_Y} \lambda_{ij} d(t,j) + \alpha_1(t,i) - \alpha_2(t,i) - 1 \\ - \log\Big(\int_t^T \beta_c(s-t,i)ds + \beta_p(T-t,i)\Big) \\ + \big(r_t + \frac{1}{2}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\big)\Big(\int_t^T \beta_c(s-t,i)ds + \beta_p(T-t,i)\Big), \end{aligned}$$

with boundary condition $d(T,i) = 0$, where

$$\begin{aligned} \alpha_1(t,i) := E_i\Big[\int_t^T \Big(\kappa(t,s) - \int_t^s \frac{1}{d_1(u,Y_{u-})}du - \log(d_1(s,Y_{s-}))\Big)\frac{\partial}{\partial t}\beta_c(s-t,i)ds \\ + \big(\kappa(t,T) - \int_t^T \frac{1}{d_1(u,Y_{u-})}du\big)\frac{\partial}{\partial t}\beta_p(T-t,i)\Big] \\ \alpha_2(t,i) := \sum_{j \in E_Y} \lambda_{ij}\Big(\int_t^T E_j\big[\kappa(t,s) - \int_t^s \frac{1}{d_1(u,Y_{u-})}du - \log(d_1(s,Y_{s-}))\big)\big] \\ \cdot\big(\beta_c(s-t,j) - \beta_c(s-t,i)\big)ds \\ + E_j\big[\kappa(t,T) - \int_t^T \frac{1}{d_1(u,Y_{u-})}du\big]\big(\beta_p(T-t,j) - \beta_p(T-t,i)\big)\Big) \end{aligned}$$

and

$$\begin{aligned} d_1(t,i) := \int_t^T \beta_c(s-t,i)ds + \beta_p(T-t,i) \\ \kappa(t,s) := \int_t^s \big(r_u + \frac{1}{2}\mu_u'(\sigma_u\sigma_u')^{-1}\mu_u\big)du + \int_t^s \mu_s'(\sigma_u\sigma_u')^{-1}\mu_u dW_u. \end{aligned}$$

b) The equilibrium strategy $\pi^* = (c_t^*, a_t^*)$ is given by

$$\begin{aligned} c_t^*(x,i) := \frac{x}{\int_t^T \beta_c(s-t,i)ds + \beta_p(T-t,i)}, \quad x \in (0,\infty), i \in E_Y \\ a_t^*(x,i) := (\sigma_t\sigma_t')^{-1}\mu_t x, \quad x \in (0,\infty), i \in E_Y. \end{aligned}$$

Remark 3. *We invest the Merton ratio $(\sigma_t\sigma_t')^{-1}\mu_t$ of our wealth in the financial market like in the standard case. The consumption rate is linear in the wealth and it holds that $c_t^*(x,i) \geq \frac{x}{T-t+1}$.*

Proof of Theorem 10.
We solve the problem with a special Ansatz for the value function

$$V(t,x,i) = d_1(t,i)\log(x) + d_2(t,i).$$

Applying Theorem 9 we get

$$c_t^*(x,i) = \frac{x}{d_1(t,i)} \quad \text{and} \quad a_t^*(x,i) = (\sigma_t\sigma_t')^{-1}\mu_t x.$$

To ease the notation we write $X_t := X_t^{\pi^*}$.

We plug V in (3.13) and get

$$0 = \frac{\partial}{\partial t}d_1(t,i)\log(x) + \frac{\partial}{\partial t}d_2(t,i) + L_{\pi^*}(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}\big(d_1(t,j)\log(x) + d_2(t,j)\big) - J_{\pi^*}(t,x,i)$$

$$+\log(\frac{x}{d_1(t,i)}) + r_t x\frac{d_1(t,i)}{x} + \frac{1}{2}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t x\frac{d_1(t,i)}{x} - \frac{x}{d_1(t,i)}\frac{d_1(t,i)}{x} + \frac{1}{2}\mu_t'(\sigma_t\sigma_t')^{-1}\sigma_t\sigma_t'$$

$$\Leftrightarrow 0 = \Big(\frac{\partial}{\partial t}d_1(t,i) + 1 + \sum_{j\in E_Y}\lambda_{ij}d_1(t,j)\Big)\log(x) + \frac{\partial}{\partial t}d_2(t,i) + L_{\pi^*}(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}d_2(t,j) - J_{\pi^*}(t,x,i)$$

$$-\log(d_1(t,i)) + r_t d_1(t,i) + \mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\frac{d_1(t,i)}{2} - 1.$$

Now we compute $L_{\pi^*}(t,x,i)$

$$L_{\pi^*}(t,x,i) = E_{t,x,i}[\int_t^T U_c(c_s^*(X_s,Y_{s-}))\frac{\partial}{\partial t}\beta_c(s-t,i)ds + U_p(X_T)\frac{\partial}{\partial t}\beta_p(T-t,i)]$$

$$= E_{t,x,i}[\int_t^T \log(\frac{X_s}{d_1(s,Y_{s-})})\frac{\partial}{\partial t}\beta_c(s-t,i)ds + \log(X_T)\frac{\partial}{\partial t}\beta_p(T-t,i)]$$

$$= E_{t,x,i}[\int_t^T \big(\log(X_s) - \log(d_1(s,Y_{s-}))\big)\frac{\partial}{\partial t}\beta_c(s-t,i)ds + \log(X_T)\frac{\partial}{\partial t}\beta_p(T-t,i)].$$

X_s is given by

$$X_s = X_t\exp\Big(\int_t^s\big(r_u + \frac{a_u^{*'}(X_u,Y_{u-})\mu_u}{X_u} - \frac{c_u^*(X_u,Y_{u-})}{X_u} - \frac{1}{2}\frac{a_u^{*'}(X_u,Y_{u-})\sigma_u'\sigma_u a_u^*(X_u,Y_{u-})}{X_u^2}\big)du\Big)$$

$$\cdot\exp\Big(\int_t^s\frac{a_u^{*'}(X_u,Y_{u-})\sigma_t}{X_u}dW_u\Big).$$

We plug $c_t^*(x,i)$ and $a_t^*(x,i)$ in and get

$$X_s = X_t\exp\Big(\int_t^s\big(r_u + \frac{1}{2}\mu_u'(\sigma_u\sigma_u')^{-1}\mu_u - \frac{1}{d_1(u,Y_{u-})}\big)du + \int_t^s\mu_s'(\sigma_u'\sigma_u)^{-1}\mu_u dW_u\Big)]$$

$$= X_t\exp\Big(\underbrace{\int_t^s\big(r_u + \frac{1}{2}\mu_u'(\sigma_u\sigma_u')^{-1}\mu_u\big)du + \int_t^s\mu_s'(\sigma_u'\sigma_u)^{-1}\mu_u dW_u}_{:=\kappa(s,t)}\Big)\exp\big(-\int_t^s\frac{1}{d_1(u,Y_{u-})}\big)du\big)$$

$$= X_t \exp(\kappa(t,s)) \exp\big(-\int_t^s \frac{1}{d_1(u,Y_{u-})}\big) du\big).$$

So we get

$$L_{\pi^*}(t,x,i) =$$

$$E_{t,x,i}[\int_t^T \Big(\log\Big(X_t \exp(\kappa(t,s)) \exp\big(-\int_t^s \frac{1}{d_1(u,Y_{u-})}\big) du\Big) - \log(d_1(s,Y_{s-}))\Big) \frac{\partial}{\partial t}\beta_c(s-t,i)ds$$

$$+ \log\Big(X_t \exp(\kappa(t,T) \exp\big(-\int_t^T \frac{1}{d_1(u,Y_{u-})}\Big) du\Big) \frac{\partial}{\partial t}\beta_p(T-t,i)]$$

$$= E_i[\int_t^T \Big(\log(x) + \kappa(t,s) - \int_t^s \frac{1}{d_1(u,Y_{u-})} du - \log(d_1(s,Y_{s-}))\Big) \frac{\partial}{\partial t}\beta_c(s-t,i)ds$$

$$+ \Big(\log(x) + \kappa(t,T) - \int_t^T \frac{1}{d_1(u,Y_{u-})} du\Big) \frac{\partial}{\partial t}\beta_p(T-t,i)]$$

$$= \Big(\int_t^T \frac{\partial}{\partial t}\beta_c(s-t,i)ds + \frac{\partial}{\partial t}\beta_p(T-t,i)\Big) \log(x)$$

$$+ E_i[\int_t^T \Big(\kappa(t,s) - \int_t^s \frac{1}{d_1(u,Y_{u-})} du - \log(d_1(s,Y_{s-}))\Big) \frac{\partial}{\partial t}\beta_c(s-t,i)ds$$

$$+ \Big(\kappa(t,T) - \int_t^T \frac{1}{d_1(u,Y_{u-})} du\Big) \frac{\partial}{\partial t}\beta_p(T-t,i)]$$

$$= \Big(\int_t^T \frac{\partial}{\partial t}\beta_c(s-t,i)ds + \frac{\partial}{\partial t}\beta_p(T-t,i)\Big) \log(x) + \alpha_1(t,i),$$

where

$$\alpha_1(t,i) := E_i[\int_t^T \Big(\kappa(t,s) - \int_t^s \frac{1}{d_1(u,Y_{u-})} du - \log(d_1(s,Y_{s-}))\Big) \frac{\partial}{\partial t}\beta_c(s-t,i)ds$$

$$+ \big(\kappa(t,T) - \int_t^T \frac{1}{d_1(u,Y_{u-})} du\big) \frac{\partial}{\partial t}\beta_p(T-t,i)]$$

Next we compute $J_{\pi^*}(t,x,i)$

$$J_{\pi^*}(t,x,i) = \sum_{j \in E_Y} \lambda_{ij} \Big(\int_t^T E_{t,x,j}[U_c(c_s(X_s,Y_{s-}))]\big(\beta_c(s-t,j) - \beta_c(s-t,i)\big)ds$$

$$+ E_{t,x,j}[U_p(X_T)]\big(\beta_p(T-t,j) - \beta_p(T-t,i)\big)\Big)]$$

$$= \sum_{j \in E_Y} \lambda_{ij} \Big(\int_t^T E_{t,x,j}[\big(\log(X_s) - \log(d_1(s,Y_{s-}))\big)]\big(\beta_c(s-t,j) - \beta_c(s-t,x,i)\big)ds$$

$$+ E_{t,x,j}[\log(X_T)]\big(\beta_p(T-t,j) - \beta_p(T-t,i)\big)\Big)].$$

Again we plug X_s in and get

$$J_{\pi^*}(t,x,i) =$$

$$\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T E_j[\Big(\log(x)+\kappa(t,s)-\int_t^s \frac{1}{d_1(u,Y_{u-})}du-\log(d_1(s,Y_{s-}))\Big)]\big(\beta_c(s-t,j)-\beta_c(s-t,i)\big)ds$$

$$+E_j[\big(\log(x)+\kappa(t,T)-\int_t^T \frac{1}{d_1(u,Y_{u-})}du\big)]\big(\beta_p(T-t,j)-\beta_p(T-t,i)\big)\Big)]$$

$$=\log(x)\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T \big(\beta_c(s-t,j)-\beta_c(s-t,i)\big)ds+\big(\beta_p(T-t,j)-\beta_p(T-t,i)\big)\Big)$$

$$+\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T E_j[\Big(\kappa(t,s)-\int_t^s \frac{1}{d_1(u,Y_{u-})}du-\log(d_1(s,Y_{s-}))\Big)]\big(\beta_c(s-t,j)-\beta_c(s-t,i)\big)ds$$

$$+E_j[\big(\kappa(t,T)-\int_t^T \frac{1}{d_1(u,Y_{u-})}du\big)]\big(\beta_p(T-t,j)-\beta_p(T-t,i)\big)\Big)]$$

$$=\log(x)\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T \beta_c(s-t,j)ds+\beta_p(T-t,j)\Big)$$

$$-\log(x)\underbrace{\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T \beta_c(s-t,i)ds+\beta_p(T-t,i)\Big)}_{=0}+\alpha_2(t,i)$$

$$=\log(x)\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T \beta_c(s-t,j)ds+\beta_p(T-t,j)\Big)+\alpha_2(t,i),$$

where

$$\alpha_2(t,i):=\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T E_j[\kappa(t,s)-\int_t^s \frac{1}{d_1(u,Y_{u-})}du-\log(d_1(s,Y_{s-})))]\big(\beta_c(s-t,j)-\beta_c(s-t,i)\big)ds$$

$$+E_j[\kappa(t,T)-\int_t^T \frac{1}{d_1(u,Y_{u-})}du]\big(\beta_p(T-t,j)-\beta_p(T-t,i)\big)\Big)$$

.

Now we plug $L_{\pi^*}(t,x,i)$ and $J_{\pi^*}(t,x,i)$ in and get

$$0=\Big(\frac{\partial}{\partial t}d_1(t,i)+1+\sum_{j\in E_Y}\lambda_{ij}d_1(t,j)\Big)\log(x)+\frac{\partial}{\partial t}d_2(t,i)+\sum_{j\in E_Y}\lambda_{ij}d_2(t,j)$$

$$+\Big(\int_t^T \frac{\partial}{\partial t}\beta_c(s-t,i)ds+\frac{\partial}{\partial t}\beta_p(T-t,x,i)\Big)\log(x)+\alpha_1(t,i)$$

$$-\log(x)\sum_{j\in E_Y}\lambda_{ij}\Big(\int_t^T \beta_c(s-t,j)ds+\beta_p(T-t,j)\Big)-\alpha_2(t,i)$$

$$-\log(d_1(t,i))+r_td_1(t,i)+\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\frac{d_1(t,i)}{2}-1$$

$$\Leftrightarrow 0 = \Big(\frac{\partial}{\partial t} d_1(t,i) + \sum_{j \in E_Y} \lambda_{ij} d_1(t,j) + \beta_c(T-t,i) + \frac{\partial}{\partial t}\beta_p(T-t,x,i)$$

$$- \sum_{j \in E_Y} \lambda_{ij} \Big(\int_t^T \beta_c(s-t,j)ds + \beta_p(T-t,j)\Big) \log(x)$$

$$+\frac{\partial}{\partial t} d_2(t,i) + \sum_{j \in E_Y} \lambda_{ij} d_2(t,j) + \alpha_1(t,i) - \alpha_2(t,i) - \log(d_1(t,i)) + r_t d_1(t,i) + \mu_t'(\sigma_t\sigma_t')^{-1}\mu_t \frac{d_1(t,i)}{2} - 1.$$

Since the equation above must be satisfied for every $x \in (0,\infty)$, it is necessary that $d_1(t,i)$ solves the differential equation

$$\frac{\partial}{\partial t} d_1(t,i) + \sum_{j \in E_Y} \lambda_{ij} d_1(t,j) + \beta_c(T-t,i) + \frac{\partial}{\partial t}\beta_p(T-t,x,i)$$

$$- \sum_{j \in E_Y} \lambda_{ij} \Big(\int_t^T \beta_c(s-t,j)ds + \beta_p(T-t,j)\Big) = 0,$$

with boundary condition $d_1(T,i) = 1$ for all $i \in E_Y$.

The unique solution is given by

$$d_1(t,i) = \int_t^T \beta_c(s-t,i)ds + \beta_p(T-t,i).$$

Plugging $d_1(t,i)$ in we get

$$\left.\begin{aligned} 0 = \tfrac{\partial}{\partial t} d_2(t,i) + \textstyle\sum_{j \in E_Y} \lambda_{ij} d_2(t,j) + \alpha_1(t,i) - \alpha_2(t,i) - 1 \\ - \log\big(\textstyle\int_t^T \beta_c(s-t,i)ds + \beta_p(T-t,i)\big) + \big(r_t + \tfrac{1}{2}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\big)\Big(\textstyle\int_t^T \beta_c(s-t,i)ds + \beta_p(T-t,i)\Big) \end{aligned}\right\} \quad (3.14)$$

Further it holds that $\alpha_1(t,i)$ and $\alpha_1(t,i)$ are independent of d_2

So $d_2(t,i)$ is the solution of the differential equation (3.14). This differential equation can be viewed as a system of $m = |E_Y|$ linear differential equations

$$0 = \frac{\partial}{\partial t} d_2(t,j_1) + \sum_{j \in E_Y} \lambda_{j_1 j} d_2(t,j) + \alpha_1(t,j_1) - \alpha_2(t,j_1) - 1$$

$$- \log\Big(\int_t^T \beta_c(s-t,j_1)ds + \beta_p(T-t,j_i)\Big) + \big(r_t + \frac{1}{2}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\big)\Big(\int_t^T \beta_c(s-t,j_1)ds + \beta_p(T-t,j_1)\Big)$$

$$\vdots$$

$$0 = \frac{\partial}{\partial t} d_2(t,j_m) + \sum_{j \in E_Y} \lambda_{j_m j} d_2(t,j) + \alpha_1(t,j_m) - \alpha_2(t,j_m) - 1$$

$$- \log\Big(\int_t^T \beta_c(s-t,j_m)ds + \beta_p(T-t,j_m)\Big) + \big(r_t + \frac{1}{2}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\big)\Big(\int_t^T \beta_c(s-t,j_1)ds + \beta_p(T-t,j_m)\Big),$$

with boundary condition $d_2(T,i) = 0$ for all $i \in E_Y$.

□

Corollary 6. *In the case of standard discounting, i.e.* $\beta_c(t,i) = \beta_p(t,i) = \exp(-\rho t)$, $\rho \geq 0$, *it holds:*

a1) *The optimal value function in the case* $\rho \neq 0$ *is given by*

$$V(t,x) = \left(\frac{1}{\rho} + (1-\frac{1}{\rho})\exp(-\rho(T-t))\right)\log(x) + d(t),$$

where $d(t)$ *is given by the unique solution of the first oder ODE*

$$0 = \frac{\partial}{\partial t}d(t) - \rho d(t) - 1 - \log\left(\frac{1}{\rho} + (1-\frac{1}{\rho})\exp(-\rho(T-t))\right)$$

$$+\big(r_t + \frac{1}{2}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\big)\left(\frac{1}{\rho} + (1-\frac{1}{\rho})\exp(-\rho(T-t))\right),$$

with boundary condition $d(T) = 0$.

a2) *The optimal value function in the case* $\rho = 0$ *is given by*

$$V(t,x) = (T-t+1)\log(x) + d(t), \quad x \in (0,\infty)$$

where $d(t)$ *is given by the unique solution of the first oder ODE*

$$0 = \frac{\partial}{\partial t}d(t) - \rho d(t) - 1 - \log\left(T-t+1\right)$$

$$+\big(r_t + \frac{1}{2}\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\big)\left(T-t+1\right),$$

with boundary condition $d(T) = 0$.

b) *The optimal strategy* $\pi^* = (c_t^*, a_t^*)$ *is given by*

$$c_t^*(x) := \begin{cases} \frac{x}{\frac{1}{\rho}+(1-\frac{1}{\rho})\exp(-\rho(T-t))}, & \text{, if } \rho \neq 0 \\ \frac{x}{T-t+1}, & \text{, if } \rho = 0 \end{cases}, \qquad x \in (0,\infty),$$

$$a_t^*(x) := (\sigma_t\sigma_t')^{-1}\mu_t x, \qquad x \in (0,\infty).$$

Proof. Follows directly form Theorem 8 and Theorem 10.

□

3.5.2 Power-utility

We consider the case $U_c(x) = U_p(x) = \frac{x^\gamma}{\gamma}$, $\gamma \in (0,1)$ and $dom(U) = [0,\infty)$. Let $\delta := \frac{1}{\gamma-1}$, then the solution is given by the following Theorem:

Theorem 11. *a) The equilibrium value function is given by*

$$V(t,x,i) = d(t,i)\frac{x^\gamma}{\gamma}, \quad x \geq 0, i \in E_Y$$

where $d(t,i) > 0$ is given as the unique solution of the integro-differential equation

$$0 = \frac{\partial}{\partial t}d(t,i) + \Big(r_t + \mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\frac{1}{2(1-\gamma)}\Big)\gamma d(t,i) - (\gamma-1)d^{\delta\gamma}(t,i) + \sum_{j\in E_Y}\lambda_{ij}d(t,j)$$
$$+E_i[\int_t^T d^{\delta\gamma}(s,Y_{s-})\exp(\gamma\kappa(t,s))\exp(-\int_t^s d^\delta(u,Y_{u-})du)\frac{\partial}{\partial t}\beta_c(s-t,i)ds]$$
$$+E_i[\exp(\gamma\kappa(t,T))$$
$$\cdot\exp(-\int_t^T d^\delta(u,Y_{u-})du)\frac{\partial}{\partial t}\beta_p(T-t,i)]$$
$$-\sum_{j\in E_Y}\lambda_{ij}E_j[\int_t^T d^{\delta\gamma}(s,Y_{s-})\exp(\gamma\kappa(t,s))\exp(-\int_t^s d^\delta(u,Y_{u-})du)\big(\beta_c(s-t,j)-\beta_c(s-t,i)\big)ds]$$
$$-\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(T-t,j)-\beta_p(T-t,i)\big)E_j[\exp(\gamma\kappa(t,T))\exp(-\int_t^T d^\delta(u,Y_{u-})du)],$$

with boundary condition $d(T,i) = 1$, where

$$\kappa(t,s) = \int_t^s r_u - \mu_u'(\sigma_u\sigma_u')^{-1}\mu_u\frac{1}{2(\gamma-1)})du - \int_t^s \mu_u'(\sigma_u')^{-1}\frac{1}{(\gamma-1)}dW_u.$$

b) The equilibrium strategy $\pi^* = (c_t^*, a_t^*)$ is given by

$$c_t^*(x,i) := d^\delta(t,i)x, \quad x \geq 0, i \in E_Y$$
$$a_t^*(x,i) := \frac{(\sigma_t\sigma_t')^{-1}\mu_t}{1-\gamma}x, \quad x \geq 0, i \in E_Y.$$

Remark 4. *We invest the Merton ratio $\frac{(\sigma_t\sigma_t')^{-1}\mu_t}{1-\gamma}$ of our wealth in the financial market and case and the consumption rate is linear in the wealth like in the standard.*

Proof of Theorem 11
We solve the problem with a special Ansatz for the value function

$$V(t,x,i) = d(t,i)\frac{x^\gamma}{\gamma}.$$

Applying Theorem 9 we get

$$c_t^*(x,i) = d^\delta(t,i)x \quad \text{and} \quad a_t^*(x,i) = (\sigma_t\sigma_t')^{-1}\mu_t\frac{x}{1-\gamma}.$$

To ease the notation we write $X_t := X_t^{\pi^*}$.

We plug V in (3.13) and get

$$0 = \frac{\partial}{\partial t}d(t,i)\frac{x^\gamma}{\gamma} + \sum_{j\in E_Y}\lambda_{ij}d(t,j)\frac{x^\gamma}{\gamma}$$

$$+d^{\delta\gamma}(t,i)\frac{x^\gamma}{\gamma} + r_t x d(t,i)x^{\gamma-1}\frac{1}{2}\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t\frac{d(t,i)x^\gamma}{1-\gamma} - d^\delta(t,i)xd(t,i)x^{\gamma-1}$$

$$\left.\begin{aligned} 0 &= L_{\pi^*}(t,x,i) - J_{\pi^*}(t,x,i) + \Big(r_t + \mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t\tfrac{1}{2(1-\gamma)}\Big)d(t,i)x^\gamma \\ &+\Big(\tfrac{\partial}{\partial t}d(t,i) - (\gamma-1)d^{\delta\gamma}(t,i) + \textstyle\sum_{j\in E_Y}\lambda_{ij}d(t,j)\Big)\tfrac{x^\gamma}{\gamma}\end{aligned}\right\} \tag{3.15}$$

note that $\delta + 1 = \delta\gamma$.

Now we compute $L_{\pi^*}(t,x,i)$

$$L_{\pi^*}(t,x,i) = E_{t,x,i}[\int_t^T U_c(c_s^*(X_s,Y_{s-}))\frac{\partial}{\partial t}\beta_c(s-t,i)ds + U_p(X_T)\frac{\partial}{\partial t}\beta_p(T-t,x,i)]$$

$$= E_{t,x,i}[\int_t^T d^{\delta\gamma}(s,Y_{s-})\frac{X_s^\gamma}{\gamma}\frac{\partial}{\partial t}\beta_c(s-t,i)ds + \frac{X_T^\gamma}{\gamma}\frac{\partial}{\partial t}\beta_p(T-t,i)].$$

X_s is given by

$$X_s = X_t\exp\Big(\int_t^s\big(r_u + \frac{a_u^{*'}(X_u,Y_{u-})\mu_u}{X_u} - \frac{c_u(X_u,Y_{u-})}{X_u} - \frac{1}{2}\frac{a_u^{*'}(X_u,Y_{u-})\sigma_u\sigma_u' a_u^*(X_u,Y_{u-})}{X_u^2}\big)du\Big)$$

$$\cdot\exp\Big(\int_t^s\frac{a_u^{*'}(X_u,Y_{u-})\sigma_u}{X_u}dW_u\Big).$$

We plug $c_t^*(x,i)$ and $a_t^*(x,i)$ in and get

$$X_s = X_t\exp\Big(\underbrace{\int_t^s(r_u + \mu_u'(\sigma_u\sigma_u')^{-1}\mu_u\frac{1}{2(1-\gamma)})du + \int_t^s\mu_u'(\sigma_u')^{-1}\frac{1}{(1-\gamma)}dW_u}_{:=\kappa(s,t)}\Big)$$

$$\cdot\exp(-\int_t^s d^\delta(u,Y_{u-})du)$$

$$= X_t \exp(\kappa(t,s)) \exp(-\int_t^s d^\delta(u, Y_{u-}) du).$$

So we get

$$L_{\pi^*}(t,x,i) = E_{t,x,i}[\int_t^T d^{\delta\gamma}(s, Y_{s-}) \frac{X_t^\gamma \exp(\gamma\kappa(t,s))}{\gamma} \exp(-\int_t^s d^\delta(u, Y_{u-}) du) \frac{\partial}{\partial t}\beta_c(s-t,i) ds$$

$$+ \frac{X_t \exp(\gamma\kappa(t,T))}{\gamma} \exp(-\int_t^T d^\delta(u, Y_{u-}) du) \frac{\partial}{\partial t}\beta_p(T-t,i)]$$

$$= \frac{x^\gamma}{\gamma}\Big(E_i[\int_t^T d^{\delta\gamma}(s, Y_{s-}) \exp(\gamma\kappa(t,s)) \exp(-\int_t^s d^\delta(u, Y_{u-}) du) \frac{\partial}{\partial t}\beta_c(s-t,i) ds]$$

$$+ E_i[\exp(\gamma\kappa(t,T)) \exp(-\int_t^T d^\delta(u, Y_{u-}) du) \frac{\partial}{\partial t}\beta_p(T-t,i)]\Big)$$

Next we compute $J_{\pi^*}(t,x,i)$

$$J_{\pi^*}(t,x,i) = \sum_{j \in E_Y} \lambda_{ij} E_{t,x,j}[\int_t^T U_c(c_s(X_s, Y_{s-}))\big(\beta_c(s-t,j) - \beta_c(s-t,i)\big) ds$$

$$+ U_p(X_T)\big(\beta_p(T-t,j) - \beta_p(T-t,i)\big)]$$

$$= \sum_{j \in E_Y} \lambda_{ij} E_{t,x,j}[\int_t^T d^{\delta\gamma}(s, Y_{s-}) \frac{X_s^\gamma}{\gamma}\big(\beta_c(s-t,j) - \beta_c(s-t,i)\big) ds + \frac{X_T^\gamma}{\gamma}\big(\beta_p(T-t,j) - \beta_p(T-t,i)\big)].$$

Again we plug X_s in and get

$$J_{\pi^*}(t,x,i) = \frac{x^\gamma}{\gamma}\Big(\sum_{j \in E_Y} \lambda_{ij} E_j[\int_t^T d^{\delta\gamma}(s, Y_{s-}) \exp(\gamma\kappa(t,s))$$

$$\cdot \exp(-\int_t^s d^\delta(u, Y_{u-}) du)\big(\beta_c(s-t,j) - \beta_c(s-t,i)\big) ds]$$

$$+ \sum_{j \in E_Y} \lambda_{ij}\big(\beta_p(T-t,j) - \beta_p(T-t,i)\big) E_i[\exp(\gamma\kappa(t,T)) \exp(-\int_t^T d^\delta(u, Y_{u-}) du)]\Big)$$

Plugging $L_{\pi^*}(t,x,i)$ and $J_{\pi^*}(t,x,i)$ in and get

$$0 = \Big(r_t + \mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{1}{2(1-\gamma)}\Big) d(t,i) x^\gamma + \Big(\frac{\partial}{\partial t} d(t,i) - (\gamma-1) d^{\delta\gamma}(t,i) + \sum_{j \in E_Y} \lambda_{ij} d(t,j)\Big)\frac{x^\gamma}{\gamma}$$

$$+ \frac{x^\gamma}{\gamma}\Big(E_i[\int_t^T d^{\delta\gamma}(s, Y_{s-}) \exp(\gamma\kappa(t,s)) \exp(-\int_t^s d^\delta(u, Y_{u-}) du) \frac{\partial}{\partial t}\beta_c(s-t,i) ds]$$

$$+ E_i[\exp(\gamma\kappa(t,T)) \exp(-\int_t^T d^\delta(u, Y_{u-}) du) \frac{\partial}{\partial t}\beta_p(T-t,i)]\Big)$$

$$-\frac{x^\gamma}{\gamma}\Big(\sum_{j\in E_Y}\lambda_{ij}E_j[\int_t^T d^{\delta\gamma}(s,Y_{s-})\exp(\gamma\kappa(t,s))\cdot\exp(-\int_t^s d^\delta(u,Y_{u-})du)\big(\beta_c(s-t,j)-\beta_c(s-t,i)\big)ds]$$

$$+\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(T-t,j)-\beta_p(T-t,i)\big)E_i[\exp(\gamma\kappa(t,T))\exp(-\int_t^T d^\delta(u,Y_{u-})du)]\Big)$$

$$\Leftrightarrow 0=\Big(r_t+\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t\frac{1}{2(1-\gamma)}\Big)\gamma d(t,i)+\frac{\partial}{\partial t}d(t,i)-(\gamma-1)d^{\delta\gamma}(t,i)+\sum_{j\in E_Y}\lambda_{ij}d(t,j)$$

$$+E_i[\int_t^T d^{\delta\gamma}(s,Y_{s-})\exp(\gamma\kappa(t,s))\exp(-\int_t^s d^\delta(u,Y_{u-})du)\frac{\partial}{\partial t}\beta_c(s-t,i)ds]$$

$$+E_i[\exp(\gamma\kappa(t,T))\exp(-\int_t^T d^\delta(u,Y_{u-})du)\frac{\partial}{\partial t}\beta_p(T-t,i)]$$

$$-\sum_{j\in E_Y}\lambda_{ij}E_j[\int_t^T d^{\delta\gamma}(s,Y_{s-})\exp(\gamma\kappa(t,s))\exp(-\int_t^s d^\delta(u,Y_{u-})du)\big(\beta_c(s-t,j)-\beta_c(s-t,i)\big)ds]$$

$$-\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(T-t,j)-\beta_p(T-t,i)\big)E_j[\exp(\gamma\kappa(t,T))\exp(-\int_t^T d^\delta(u,Y_{u-})du)].$$

$d(t,i)$ is the unique solution with $d(t,i)>0$ of the differential equation above with boundary condition $d(T,i)=1$, $\forall i\in E_Y$ (see Lemma 3).
This differential equation can be viewed as system of $m:=|E_Y|$ ordinary differential equations with boundary condition $d(T,i)=1$, $\forall i\in E_Y$.

□

Corollary 7. *In the case of standard discounting, i.e.* $\beta_c(t,i)=\beta_p(t,i)=\exp(-\rho t)$, $\rho\geq 0$, *it holds:*

a) The optimal value function is given by

$$V(t,x)=d(t)\frac{x^\gamma}{\gamma},\quad x\geq 0,$$

where $d(t)$ *is the unique solution of the linear first order ODE*

$$\Leftrightarrow 0=\frac{\partial}{\partial t}d(t)+\Big(r_t-\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\frac{1}{2(\gamma-1)}\Big)\gamma d(t)-(\gamma-1)d^{\delta\gamma}(t)-\rho d(t),$$

with boundary condition $d(T)=1$.

b) The optimal strategy $\pi^*=(c_t^*,a_t^*)$ *is given by*

$$c_t^*(x):=d^\delta(t)x,\quad x\geq 0,$$
$$a_t^*(x):=(\sigma_t\sigma_t')^{-1}\mu_t\frac{x}{1-\gamma},x\geq 0.$$

Proof. Follows directly from Theorem 8 and Theroem 11.

□

Let $(\mathcal{X}[K,\infty), ||\cdot||)$ be the Banach space of the functions $d : [0,T] \times E_Y \to [K,\infty)$ with $d(\cdot,i) \in C^b[0,T]$ for all i endowed with the supremums norm, which we denote by $||\cdot||$, i.e $||d(t,i)|| := \sup_{t\in[0,T]} |d(t,i)|$.

Lemma 3. *The integro-differential equation*

$$0 = \Big(r_t + \mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{1}{2(1-\gamma)}\Big)\gamma d(t,i) + \frac{\partial}{\partial t} d(t,i) - (\gamma-1)d^{\delta\gamma}(t,i) + \sum_{j\in E_Y} \lambda_{ij} d(t,j)$$
$$+E_i[\int_t^T d^{\delta\gamma}(s,Y_{s-}) \exp(\gamma\kappa(t,s)) \exp(-\int_t^s d^{\delta}(u,Y_{u-})du) \frac{\partial}{\partial t}\beta_c(s-t,i) ds]$$
$$+E_i[\exp(\gamma\kappa(t,T)) \exp(-\int_t^T d^{\delta}(u,Y_{u-})du) \frac{\partial}{\partial t}\beta_p(T-t,i)]$$
$$-\sum_{j\in E_Y} \lambda_{ij} E_j[\int_t^T d^{\delta\gamma}(s,Y_{s-}) \exp(\gamma\kappa(t,s)) \exp(-\int_t^s d^{\delta}(u,Y_{u-})du) \big(\beta_c(s-t,j) - \beta_c(s-t,i)\big) ds]$$
$$-\sum_{j\in E_Y} \lambda_{ij} \big(\beta_p(T-t,j) - \beta_p(T-t,i)\big) E_j[\exp(\gamma\kappa(t,T)) \exp(-\int_t^T d^{\delta}(u,Y_{u-})du)],$$

with boundary condition $d(T,i) = 1$, has a unique solution on $[0,T] \times E_Y \times (\mathcal{X}[K,\infty), ||\cdot||)$, $K \in (0,\infty)$.

Proof.
We define

$$\alpha_1(t) := \Big(r_t + \mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{1}{2(1-\gamma)}\Big)\gamma$$
$$\alpha_2(t,s,i) := \exp(\gamma\kappa(t,s)) \frac{\partial}{\partial t}\beta_c(s-t,i)$$
$$\alpha_3(t,s,i) := \exp(\gamma\kappa(t,s)) \big(\beta_c(s-t,j) - \beta_c(s-t,i)\big)$$
$$\alpha_4(t,T,i) := \exp(\gamma\kappa(T,s)) \frac{\partial}{\partial t}\beta_p(T-t,i)$$
$$\alpha_5(t,T,i) := \exp(\gamma\kappa(T,s)) \big(\beta_p(T-t,j) - \beta_p(T-t,i)\big)$$

We plug this definitions in and get

$$0 = \alpha_1(t)d(t,i) + \frac{\partial}{\partial t} d(t,i) - (\gamma-1)d^{\delta\gamma}(t,i) + \sum_{j\in E_Y} \lambda_{ij} d(t,j))$$
$$\underbrace{+E_i[\int_t^T \alpha_2(t,s) d^{\delta\gamma}(s,Y_{s-}) \exp(-\int_t^s d^{\delta}(u,Y_{u-})du) ds]}_{=:g_1(t,i,d)}$$
$$\underbrace{+E_i[\alpha_4(t,T) \exp(-\int_t^T d^{\delta}(u,Y_{u-})du)]}_{=:g_2(t,i,d)}$$

$$\underbrace{-\sum_{j\in E_Y}\lambda_{ij}E_j[\int_t^T \alpha_3(t,s)d^{\delta\gamma}(s,Y_{s-})\exp(-\int_t^s d^{\delta}(u,Y_{u-})du)ds]}_{=:g_3(t,i,d)}$$

$$\underbrace{-\sum_{j\in E_Y}\lambda_{ij}E_j[\alpha_5(t,T)\exp(-\int_t^T d^{\delta}(u,Y_{u-})du)]}_{=:g_4(t,i,d)}$$

So we can write

$$0=\alpha_1(t)d(t,i)+\frac{\partial}{\partial t}d(t,i)-(\gamma-1)d^{\delta\gamma}(t,i)+\sum_{j\in E_Y}\lambda_{ij}d(t,j))+g_1(t,d)+g_2(t,d)-g_3(t,d)-g_4(t,d)$$

$$\left.\begin{aligned}&\Leftrightarrow \tfrac{\partial}{\partial t}d(t,i)=-\alpha_1(t)d(t,i)+(\gamma-1)d^{\delta\gamma}(t,i)-\textstyle\sum_{j\in E_Y}\lambda_{ij}d(t,j))\\ &-g_1(t,d)-g_2(t,d)+g_3(t,d)+g_4(t,d)\end{aligned}\right\}\qquad(3.16)$$

To prove the existence of a unique solution of (3.16) on $(\mathcal{X}[K,\infty),||\cdot||)$ we use the existence and uniqueness Theorem for ODE's from Picard-Lindelöf, so we have to show that

$$f(t,i,d):=-\alpha_1(t)d(t,i)+(\gamma-1)d^{\delta\gamma}(t,i)-\sum_{j\in E_Y}\lambda_{ij}d(t,j))-g_1(t,d)-g_2(t,d)+g_3(t,d)+g_4(t,d)$$

is continuous on $[0,T]\times(\mathcal{X}[K,\infty),||\cdot||)$ for all i and Lipschitz continuous in d on $[0,T]\times(\mathcal{X}[K,\infty),||\cdot||)$ for all i.

$f(t,i,d)$ is obviously continuous on $[0,T]\times(\mathcal{X}[K,\infty),||\cdot||)$ for all $i\in E_Y$, so we only have to prove that $f(t,i,d)$ is Lipschitz continuous in d.

It holds $\alpha_1(t)d(t,i)$, $\sum_{j\in E_Y}\lambda_{ij}d(t,j))$ are Lipschitz and $d^{\delta\gamma}(t,i)$ is uniformly Lipschitz with $L=|\delta\gamma K^{\delta\gamma}|$ on $[0,T]\times(\mathcal{X}[K,\infty),||\cdot||)$ for all i , since

$$|(d^{\delta\gamma}(t,i))'|=|\delta\gamma d^{\delta\gamma-1}(t,i))|\overset{(1)}{\leq}|\delta\gamma K^{\delta\gamma}|$$

(1) $\delta\gamma-1=\delta\in(-\infty,-1]$ and $d(t,i)\geq K$.
(2) $d^{\delta\gamma}(t,i)$ is uniformly Lipschitz with $L=|\delta\gamma K^{\delta\gamma}|$ on $[0,T]\times(\mathcal{X}[K,\infty),||\cdot||)$ for all i.

It is left to show that g_1,g_2,g_3 and g_4 are Lipschitz in d.

$$||g_1(t,i,d)-g_1(t,i,\tilde{d})||$$

$$=||E_i[\int_t^T\alpha_2(t,s)\Big(d^{\delta\gamma}(s,Y_{s-})\exp(-\int_t^s d^{\delta}(u,Y_{u-})du)-\tilde{d}^{\delta\gamma}(s,Y_{s-})\exp(-\int_t^s\tilde{d}^{\delta}(u,Y_{u-})du)\Big)ds]||$$

$$\leq E_i[\int_t^T||\alpha_2(t,s)||\,||d^{\delta\gamma}(s,Y_{s-})\exp(-\int_t^s d^{\delta}(u,Y_{u-})du)-\tilde{d}^{\delta\gamma}(s,Y_{s-})\exp(-\int_t^s\tilde{d}^{\delta}(u,Y_{u-})du)||ds]$$

$$\leq E_i[\int_t^T ||\alpha_2(t,s)|| \underbrace{||\exp(-\int_t^s \tilde{d}^\delta(u,Y_{u-}))||}_{\leq 1}$$

$$\cdot ||d^{\delta\gamma}(s,Y_{s-})\exp(-\int_t^s \Big(d^\delta(u,Y_{u-}) - \tilde{d}^\delta(u,Y_{u-})\Big)\,du) - \tilde{d}^{\delta\gamma}(s,Y_{s-})||ds]$$

$$\leq E_i[\int_t^T ||\alpha_2(t,s)||\,||d^{\delta\gamma}(s,Y_{s-})\exp((T-t)||d^\delta(u,Y_{u-}) - \tilde{d}^\delta(u,Y_{u-})||) - \tilde{d}^{\delta\gamma}(s,Y_{s-})||ds]$$

$$\overset{(2)}{\leq} \exp((T-t)|\delta\gamma K^{\delta\gamma}|)E_i[\int_t^T ||\alpha_2(t,s)||\,||d^{\delta\gamma}(s,Y_{s-}) - \tilde{d}^{\delta\gamma}(s,Y_{s-})||ds]$$

$$\overset{(2)}{\leq} \exp((T-t)|\delta\gamma K^{\delta\gamma}|)E_i[\int_t^T ||\alpha_2(t,s)||\,|\delta\gamma K^{\delta\gamma}|\,||d(s,Y_{s-}) - \tilde{d}(s,Y_{s-})||ds]$$

$$\leq \exp((T-t)|\delta\gamma K^{\delta\gamma}|)(T-t)\,|\delta\gamma K^{\delta\gamma}|\,\max_{k\in E_Y}\{||d(t,k) - \tilde{d}(t,k)||\}|E_i[||\alpha_2(t,s)||]$$

We denote $M := \frac{\max_{k\in E_Y}\{||d(s,k)-\tilde{d}(s,k)||\}}{||d(s,i)-\tilde{d}(s,i)||}$ and get

$$\leq \exp((T-t)|\delta\gamma K^{\delta\gamma}|)(T-t)\,|\delta\gamma K^{\delta\gamma}|M\,E_i[||\alpha_2(t,s)||]\,||d(s,i) - \tilde{d}(s,i)|| = \bar{L}||d(s,i) - \tilde{d}(s,i)||,$$

where $\bar{L} = \exp((T-t)|\delta\gamma K^{\delta\gamma}|)(T-t)\,|\delta\gamma K^{\delta\gamma}|M\,E_i[||\alpha_2(t,s)||]$.
It is left to show that $E_i[||\alpha_2(t,s)||] < \infty$

$$E_i[||\alpha_2(t,s)||] < \infty \Leftrightarrow E_i[|\alpha_2(t,s)|] < \infty,\ \forall s \in [t,T]$$

$$E_i[|\alpha_2(t,s)|] = E_i[\gamma\exp(\gamma\kappa(t,s))|\frac{\partial}{\partial t}\beta_c(s-t,i)|] = \gamma|\frac{\partial}{\partial t}\beta_c(s-t,i)|E_i[|\exp(\gamma\kappa(t,s))|].$$

We plug $\kappa(t,s) = \int_t^s r_u + \mu_u'(\sigma_u'\sigma_u)^{-1}\mu_u\frac{1}{2(1-\gamma)})du + \int_t^s \sigma_u^{-1}\mu_u\frac{1}{(1-\gamma)}dW_u$ and get

$$E_i[|\alpha_1(t,s)|] = \gamma|\frac{\partial}{\partial t}\beta_c(s-t,i)|$$

$$\cdot E_i[|\exp(r_u + \mu_u'(\sigma_u\sigma_u')^{-1}\mu_u\frac{1}{2(1-\gamma)})du + \int_t^s \sigma_u^{-1}\mu_u\frac{1}{(1-\gamma)}dW_u)|] < \infty, \forall s \in [t,T]$$

since μ_t and σ_t are uniformly bounded.

Analogously for g_2, g_2 and g_4. Which implies that $f(t,i,d)$ is Lipschitz.

□

3.5.3 Exponential-utility

We consider the case $U_c(x) = U_p(x) = -\exp(-\gamma x)$, $\gamma > 0$, and $dom(U) = \mathbb{R}$. In this case we also allow negative consumption, i.e. $c_t(x, i) \in \mathbb{R}$ for all $(x, i) \in \mathbb{R} \times E_Y$.

Furthermore we set set $r_t \equiv 0$.

Theorem 12. *a) The equilibrium value function is given by*

$$V(t, x, i) = -d(t, i) \exp(-\frac{\gamma}{1 + T - t} x), \quad x \in \mathbb{R}, i \in E_Y,$$

where $d(t, i)$ is given as the unique solution of the differential equation

$$0 = \frac{\partial}{\partial t} d(t, i) + \sum_{j \in E_Y} \lambda_{ij} d(t, j) + \frac{d(t, i)}{1 + T - t}$$

$$-\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{d(t, i)}{2} - \log(d(t, i)) \frac{d(t, i)}{1 + T - t} + \log(1 + T - t) \frac{d(t, i)}{1 + T - t}$$

$$+ \sum_{j \in E_Y} \lambda_{ij} E_j \Big[\int_t^T \frac{d(s, Y_{s-})}{1 + T - s} \exp\left(-\frac{1}{1 + T - s} \int_t^s \log\left(\frac{d(u, Y_{u-})}{1 + T - u} \right) du \right)$$

$$\cdot \exp(-\frac{\gamma}{1 + T - s} \kappa(t, s)) \big(\beta_c(s - t, j) - \beta_c(s - t, x, i) \big) ds$$

$$+ \exp\left(-\int_t^T \log\left(\frac{d(u, Y_{u-})}{1 + T - u} \right) du \right) \exp(-\gamma\kappa(t, T)) \big(\beta_p(T - t, j) - \beta_p(T - t, x, i) \big) \Big]$$

$$-E_i \Big[\int_t^T \frac{d(s, Y_{s-})}{1 + T - s}$$

$$\exp\left(-\frac{1}{1 + T - s} \int_t^s \log\left(\frac{d(u, Y_{u-})}{1 + T - u} \right) du \right) \exp(-\frac{\gamma}{1 + T - s} \kappa(t, s)) \frac{\partial}{\partial t} \beta_c(s - t, i) ds$$

$$+ \exp(-\gamma\kappa(t, T)) \exp\left(-\int_t^T \log\left(\frac{d(u, Y_{u-})}{1 + T - u} \right) du \right) \frac{\partial}{\partial t} \beta_p(T - t, x, i) \Big],$$

with boundary condition $d(T, i) = 1$, where

$$\kappa(t, s) = \int_t^s \frac{1}{\gamma} \mu_u'(\sigma_u\sigma_u')^{-1}\mu_u du + \int_t^s \frac{1}{\gamma} \mu_u'(\sigma_u)^{-1} dW_u.$$

b) The equilibrium strategy $\pi^* = (c_t^*, a_t^*)$ is given by

$$c_t^*(x, i) := \frac{x}{1 + T - t} - \frac{1}{\gamma} \log(\frac{d(t, i)}{1 + T - t}), \quad x \in \mathbb{R}, i \in E_Y,$$

$$a_t^*(x, i) = a_t^* := \frac{1 + T - t}{\gamma} (\sigma_t\sigma_t')^{-1}\mu_t, \quad x \in \mathbb{R}, i \in E_Y.$$

Remark 5. *We invest for every time point t constants amounts $a_t^* = \frac{1+T-t}{\gamma}(\sigma_t\sigma_t')^{-1}\mu_t$ in the financial market and the consumption rate is linear in the wealth.*

Proof.

We solve the problem with a special Ansatz for the value function

$$V(t,x,i) = -d(t,i)\exp(-\frac{\gamma}{1+T-t}x),\ d(t,i)>0.$$

Applying Theorem 9 we get

$$c_t^*(x,i) = \frac{x}{1+T-t} - \frac{1}{\gamma}\log(\frac{d(t,i)}{1+T-t}) \quad \text{and} \quad a_t^*(x,i) = \frac{1+T-t}{\gamma}(\sigma_t\sigma_t')^{-1}\mu_t.$$

To ease the notation we write $X_t := X_t^{\pi^*}$.

We plug in (3.13) and get

$$0 = -\frac{\partial}{\partial t}d(t,i)\exp(-\frac{\gamma}{1+T-t}x) + \gamma\frac{x}{(1+T-t)^2}d(t,i)\exp(-\frac{\gamma}{1+T-t}x)$$

$$+L_{\pi^*}(t,x,i) - \sum_{j\in E_Y}\lambda_{ij}d(t,j)\exp(-\frac{\gamma}{1+T-t}x) - J_{\pi^*}(t,x,i) - \exp(-\frac{\gamma}{1+T-t}x)\frac{d(t,i)}{1+T-t}$$

$$+\mu_t'(\sigma_t\sigma_t')^{-1}\mu_t\frac{d(t,i)}{2}\exp(-\frac{\gamma}{1+T-t})$$

$$-(\frac{x}{1+T-t} - \frac{\log(d(t,i))}{\gamma} + \frac{\log(1+T-t)}{\gamma})\frac{\gamma}{1+T-t}d(t,i)\exp(-\frac{\gamma}{1+T-t}x)$$

$$\left.\begin{aligned}&\Leftrightarrow 0 = \tfrac{\partial}{\partial t}d(t,i) + \textstyle\sum_{j\in E_Y}\lambda_{ij}d(t,j) + \tfrac{d(t,i)}{1+T-t} - \mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t\tfrac{d(t,i)}{2} - \log(d(t,i))\tfrac{d(t,i)}{1+T-t}\\ &+\log(1+T-t)\tfrac{d(t,i)}{1+T-t} + \exp(\tfrac{\gamma}{1+T-t}x)(J_{\pi^*}(t,x,i) - L_{\pi^*}(t,x,i))\end{aligned}\right\} \tag{3.17}$$

We compute $L_{\pi^*}(t,x,i)$

$$L_{\pi^*}(t,x,i) = E_{t,x,i}[\int_t^T U_c(c_s^*(X_s,Y_{s-}))\frac{\partial}{\partial t}\beta_c(s-t,i)ds + U_p(X_T)\frac{\partial}{\partial t}\beta_p(T-t,x,i)]$$

$$= -E_{t,x,i}[\int_t^T \frac{d(s,Y_{s-})}{1+T-s}\exp(-\frac{\gamma}{1+T-s}X_s)\frac{\partial}{\partial t}\beta_c(s-t,i)ds + \exp(-\gamma X_T)\frac{\partial}{\partial t}\beta_p(T-t,x,i)].$$

X_s is given by

$$dX_s = \left(a_s^{*'}(X_s,Y_{s-})\mu_s - c_s^*(X_s,Y_{s-})\right)ds + a_s^{*'}(X_s,Y_{s-})\sigma_s dW_s$$

We plug $c^*(t,x,i)$ and $a^*(t,x,i)$ in and get

$$\left.\begin{aligned}&dX_s = \left(\tfrac{1+T-s}{\gamma}\mu_s'(\sigma_s'\sigma_s)^{-1}\mu_s + \tfrac{1}{\gamma}\log(\tfrac{d(s,Y_{s-})}{1+T-s}\right)ds - \tfrac{X_s}{1+T-s}ds\\ &+\tfrac{1+T-s}{\gamma}\mu_s'(\sigma_s)^{-1}dW_s\end{aligned}\right\} \tag{3.18}$$

The solution of an affine linear SDE of the type

$$dX_s = \eta_s^1 ds + \eta_s^2 X_s ds + \eta^3 s dW_s$$

is given by

$$X_s = \frac{\eta_t^2}{\eta_s^2} X_t + \frac{1}{\eta_s^2} \int_t^s \eta_s^1 \eta_s^2 ds + \frac{1}{\eta_s^2} \int_t^s \eta_s^2 \eta_s^3 dW_s$$

The solution of the SDE (3.18) above is given by

$$X_s = \frac{T-s+1}{T-t+1} X_t - \frac{1}{1+T-s} \int_t^s \log\left(\frac{d(u, Y_{u-})}{1+T-u}\right) du + \kappa(t,s),$$

$$\kappa(t,s) := \int_t^s \frac{1}{\gamma} \mu_u' (\sigma_u' \sigma_u)^{-1} \mu_u du + \int_t^s \frac{1}{\gamma} \mu_u' (\sigma_u)^{-1} dW_u.$$

So we get

$$\begin{aligned}
L_{\pi^*}(t,x,i) = -E_{t,x,i}[\int_t^T \frac{d(s,Y_{s-})}{1+T-s} \exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \\
\exp\left(-\frac{\gamma}{1+T-s}\frac{T-s+1}{T-t+1} X_t - \frac{\gamma}{1+T-s}\kappa(t,s)\right) \frac{\partial}{\partial t}\beta_c(s-t,i) ds \\
+ \exp\left(-\frac{\gamma}{1+T-t} X_t - \gamma\kappa(t,T)\right) \exp\left(-\int_t^T \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \frac{\partial}{\partial t}\beta_p(T-t,x,i)] \\
= -\exp(-\frac{\gamma}{1+T-t}x) E_i[\int_t^T \frac{d(s,Y_{s-})}{1+T-s} \\
\exp(-\frac{1}{T-s+1}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du) \exp(-\frac{\gamma}{1+T-s}\kappa(t,s)) \frac{\partial}{\partial t}\beta_c(s-t,i) ds \\
+ \exp(-\gamma\kappa(t,T)) \exp(-\int_t^T \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du) \frac{\partial}{\partial t}\beta_p(T-t,x,i)].
\end{aligned}$$

Further it holds that

$$\begin{aligned}
J_{\pi^*}(t,x,i) = -\sum_{j\in E_Y} \lambda_{ij} E_{t,x,j}[\int_t^T \frac{d(s,Y_{s-})}{1+T-s} \exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \\
\cdot \exp\left(-\frac{\gamma}{1+T-s}\frac{T-s+1}{T-t+1} X_t - \frac{\gamma}{1+T-s}\kappa(t,s)\right) \big(\beta_c(s-t,j) - \beta_c(s-t,x,i)\big) ds \\
+ \exp\left(-\int_t^T \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \exp\left(-\frac{\gamma}{1+T-t} X_t - \gamma\kappa(t,T)\right) \big(\beta_p(T-t,j) - \beta_p(T-t,x,i)\big)] \\
= -\exp(-\gamma x) \sum_{j\in E_Y} \lambda_{ij} E_j[\int_t^T \frac{d(s,Y_{s-})}{1+T-s} \exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \\
\cdot \exp(-\frac{\gamma}{1+T-s}\kappa(t,s)) \big(\beta_c(s-t,j) - \beta_c(s-t,x,i)\big) ds \\
+ \exp\left(-\int_t^T \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \exp(-\gamma\kappa(t,T)) \big(\beta_p(T-t,j) - \beta_p(T-t,x,i)\big)].
\end{aligned}$$

We plug $L_{\pi^*}(t,x,i)$ and $J_{\pi^*}(t,x,i)$ in the extended HJB-equation and we get

$$0 = \frac{\partial}{\partial t} d(t,i) + \sum_{j\in E_Y} \lambda_{ij} d(t,j) + \frac{d(t,i)}{1+T-t}$$

$$-\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{d(t,i)}{2} - \log(d(t,i))\frac{d(t,i)}{1+T-t} + \log(1+T-t)\frac{d(t,i)}{1+T-t}$$

$$+\sum_{j\in E_Y} \lambda_{ij} E_j[\int_t^T \frac{d(s,Y_{s-})}{1+T-s} \exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right)$$

$$\cdot \exp(-\frac{\gamma}{1+T-s}\kappa(t,s))\big(\beta_c(s-t,j) - \beta_c(s-t,x,i)\big)ds$$

$$+\exp\left(-\int_t^T \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \exp(-\gamma\kappa(t,T))\big(\beta_p(T-t,j) - \beta_p(T-t,x,i)\big)]$$

$$-E_i[\int_t^T \frac{d(s,Y_{s-})}{1+T-s}$$

$$\exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \exp(-\frac{\gamma}{1+T-s}\kappa(t,s))\frac{\partial}{\partial t}\beta_c(s-t,i)ds$$

$$+\exp(-\gamma\kappa(t,T))\exp\left(-\int_t^T \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \frac{\partial}{\partial t}\beta_p(T-t,x,i)].$$

The integro-differential equation above has a unique solution see Lemma 4.

□

Corollary 8. *In the case of standard discounting, i.e.* $\beta_c(t,i) = \beta_p(t,i) = \exp(-\rho t)$, $\rho \geq 0$, *it holds:*

a) The optimal value function is given by

$$V(t,x) = -d(t)\exp(-\frac{\gamma}{1+T-t}x), \quad x \in \mathbb{R},$$

where $d(t) > 0$ *is the unique solution of the ODE*

$$0 = \frac{\partial}{\partial t} d(t) + \frac{d(t)}{1+T-t} - \mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{d(t)}{2}$$

$$-\log(d(t))\frac{d(t)}{1+T-t} + \log(1+T-t)\frac{d(t)}{1+T-t} - \rho d(t)$$

with boundary condition $d(T) = 1$.

b) The optimal strategy $\pi^* = (c_t^*, a_t^*)$ *is given by*

$$c_t^*(x) := \frac{x}{1+T-t} - \frac{1}{\gamma}\log(\frac{d(t,i)}{1+T-t}), \quad x \in \mathbb{R}$$

$$a_t^*(x) := a_t^*(x,i) = \frac{1+T-t}{\gamma}(\sigma_t\sigma_t')^{-1}\mu_t, \quad x \in \mathbb{R}.$$

Proof. Follows directly from Theorem 8 and Theroem 12.

□

We need the following remark to prove the existence of a solution of the ODE for $d(t,i)$.

Remark 6. *It holds that $\nu \le d(t,i)$ for all $t \in [0,T]$ and $i \in E_Y$, where $\nu := \min_{t\in[0,T]} \bar{d}(t) > 0$ and $\bar{d}(t)$ is given as the solution of the ODE*

$$0 = \frac{\partial}{\partial t}\bar{d}(t) + \frac{\bar{d}(t)}{1+T-t} - \mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t\frac{\bar{d}(t)}{2}$$

$$-\log(\bar{d}(t))\frac{\bar{d}(t)}{1+T-t} + \log(1+T-t)\frac{\bar{d}(t)}{1+T-t}$$

with boundary condition $\bar{d}(T) = 1$.

Proof.
We denote the optimal value function under standard discounting with $\rho = 0$ by V^0. From Corollary 8 we know that $V^0(t,x) = -\bar{d}(t)\exp(-\frac{\gamma x}{1+T-t})$. Further we know that

$$V^0(t,x) \ge V(t,x,i) \;\Leftrightarrow -\bar{d}(t)\exp(-\frac{\gamma x}{1+T-t}) \ge -d(t,i)\exp(-\frac{\gamma x}{1+T-t})\,\forall i \in E_Y$$

$$\Leftrightarrow \bar{d}(t) \le d(t,i)\,\forall i \in E_Y$$

From the classical Theory it is well-known that $\bar{d}(t)$ exists. It holds that $V^0(t,x) < 0$ which implies that $\bar{d}(t) > 0$ for all $t \in [0,T]$. Since $\bar{d}(t)$ is continuous it follows that

$$\nu = \min_{t\in[0,T]} \bar{d}(t) > 0.$$

□

Let $(\mathcal{X}[\nu, K], ||\cdot||)$, where $K \in (\nu, \infty)$ be the Banach space of the functions $d : [0, T] \times E_Y \to [\nu, K]$ with $d(\cdot, i) \in C^b[0, T]$ for all i endowed with the supremums norm, which we denote by $||\cdot||$, i.e $||d(t, i)|| := \sup_{t\in[0,T]} |d(t, i)|$.

Lemma 4. *The integro-differential equation*

$$0 = \frac{\partial}{\partial t} d(t, i) + \sum_{j\in E_Y} \lambda_{ij} d(t, j) + \frac{d(t, i)}{1 + T - t}$$

$$-\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{d(t, i)}{2} - \log(d(t, i))\frac{d(t, i)}{1 + T - t} + \log(1 + T - t)\frac{d(t, i)}{1 + T - t}$$

$$+ \sum_{j\in E_Y} \lambda_{ij} E_j[\int_t^T \frac{d(s, Y_{s-})}{1 + T - s} \exp\left(-\frac{1}{1 + T - s}\int_t^s \log\left(\frac{d(u, Y_{u-})}{1 + T - u}\right) du\right)$$

$$\cdot \exp(-\frac{\gamma}{1 + T - s}\kappa(t, s))\big(\beta_c(s - t, j) - \beta_c(s - t, x, i)\big)ds$$

$$+ \exp\left(-\int_t^T \log\left(\frac{d(u, Y_{u-})}{1 + T - u}\right) du\right) \exp(-\gamma\kappa(t, T))\big(\beta_p(T - t, j) - \beta_p(T - t, x, i)\big)]$$

$$-E_i[\int_t^T \frac{d(s, Y_{s-})}{1 + T - s}$$

$$\exp\left(-\frac{1}{1 + T - s}\int_t^s \log\left(\frac{d(u, Y_{u-})}{1 + T - u}\right) du\right) \exp(-\frac{\gamma}{1 + T - s}\kappa(t, s))\frac{\partial}{\partial t}\beta_c(s - t, i)ds$$

$$+ \exp(-\gamma\kappa(t, T)) \exp\left(-\int_t^T \log\left(\frac{d(u, Y_{u-})}{1 + T - u}\right) du\right) \frac{\partial}{\partial t}\beta_p(T - t, x, i)],$$

with boundary condition $d(T, i) = 1$*, where*

$$\kappa(t, s) = \int_t^s \frac{1}{\gamma}\mu_u'(\sigma_u\sigma_u')^{-1}\mu_u du + \int_t^s \frac{1}{\gamma}\mu_u'(\sigma_u)^{-1}dW_u.$$

has a unique solution on $[0, T] \times E_Y \times (\mathcal{X}[\nu, K], ||\cdot||)$*,* $K \in (\nu, \infty)$ *.*

Proof.

$$\alpha_1(t, s) := \gamma \exp(-\frac{\gamma}{1 + T - s}\kappa(t, s))\frac{\partial}{\partial t}\beta_c(s - t, i)$$

$$\alpha_2(t, s) := \gamma \exp(-\frac{\gamma}{1 + T - s}\kappa(t, s))\big(\beta_c(s - t, j) - \beta_c(s - t, i)\big)$$

$$\alpha_3(t, T) := \gamma \exp(\gamma\kappa(t, T))\frac{\partial}{\partial t}\beta_p(T - t, i)$$

$$\alpha_4(t, s) := \gamma \exp(\gamma\kappa(t, T))\big(\beta_p(T - t, j) - \beta_p(T - t, i)\big)$$

Plugging that in the integro-differential equation we get

$$0 = \frac{\partial}{\partial t} d(t,i) + \sum_{j \in E_Y} \lambda_{ij} d(t,j) + \frac{d(t,i)}{1+T-t}$$

$$-\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{d(t,i)}{2} - \log(d(t,i))\frac{d(t,i)}{1+T-t} + \log(1+T-t)\frac{d(t,i)}{1+T-t}$$

$$\underbrace{-E_i[\int_t^T \frac{d(s,Y_{s-})}{1+T-s} \exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \alpha_1(t,s)) ds]}_{g_1(t,i,d)}$$

$$\underbrace{-E_i[\exp(-\int_t^T \log(d(u,Y_{u-}))du)\alpha_3(t,s)]}_{g_2(t,i,d)}$$

$$\underbrace{+\sum_{j \in E_Y} \lambda_{ij} E_j[\int_t^T \frac{d(s,Y_{s-})}{1+T-s} \exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \alpha_2(t,s)) ds]}_{g_3(t,i,d)}$$

$$\underbrace{+\sum_{j \in E_Y} \lambda_{ij} E_j[\exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right) du\right) \alpha_4(t,s)]}_{g_4(t,i,d)}$$

$$\left.\begin{array}{c} \Leftrightarrow 0 = \frac{\partial}{\partial t} d(t,i) + \sum_{j \in E_Y} \lambda_{ij} d(t,j) + \frac{d(t,i)}{1+T-t} \\ -\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{d(t,i)}{2} - \log(d(t,i))\frac{d(t,i)}{1+T-t} + \log(1+T-t)\frac{d(t,i)}{1+T-t} \\ +\log(\gamma d(t,i))d(t,i) + g_1(t,i,d) + g_2(t,i,d) - g_3(t,i,d) - g_4(t,i,d) \end{array}\right\} \quad (3.19)$$

To prove the existence of a unique solution of (3.16) on $(\mathcal{X}[\nu,K], ||\cdot||)$ we use the existence and uniqueness Theorem for ODEs from Picard-Lindelöf, so we have to show that

$$f(t,i,d) := \frac{\partial}{\partial t} d(t,i) + \sum_{j \in E_Y} \lambda_{ij} d(t,j) + \frac{d(t,i)}{1+T-t}$$

$$-\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{d(t,i)}{2} - \log(d(t,i))\frac{d(t,i)}{1+T-t} + \log(1+T-t)\frac{d(t,i)}{1+T-t}$$

$$+g_1(t,i,d) + g_2(t,i,d) - g_3(t,i,d) - g_4(t,i,d)$$

is continuous on $[0,T] \times (\mathcal{X}[\nu,K], ||\cdot||)$ for all i and Lipschitz continuous in d on f$[0,T] \times (\mathcal{X}[\nu,K], ||\cdot||)$ for all i.

$f(t,i,d)$ is obviously continuous on $[0,T] \times (\mathcal{X}[\nu,K], ||\cdot||)$ for all $i \in E_Y$, so we only have to prove that $f(t,i,d)$ is Lipschitz continuous in d.

It holds $\frac{d(t,i)}{1+T-t}$, $\sum_{j \in E_Y} \lambda_{ij} d(t,j))$, and $\mu_t'(\sigma_t'\sigma_t)^{-1}\mu_t \frac{d(t,i)}{2}$ are Lipschitz and $\log(d(t,i))\frac{d(t,i)}{1+T-t}$ is uniformly Lipschitz with $L = \frac{1}{1+T-t} \max\{|1+\log(\nu)|, |1+\log(K)|\}$ on $[0,T] \times \mathcal{X}$ for all i , since

$$|(\log(d(t,i))\frac{d(t,i)}{1+T-t})'| = \frac{1}{1+T-t}|1+\log(d(t,i))| \leq \frac{1}{1+T-t} \max\{|1+\log(\nu)|, |1+\log(K)|\}$$

It is left to show that g_1, g_2, g_3 and g_4 are Lipschitz in d.

$$||g_1(t,i,d) - g_1(t,i,\tilde{d})||$$

$$= ||E_i[\int_t^T \alpha_1(t,s)\Big(\frac{d(s,Y_{s-})}{1+T-s}\exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{d(u,Y_{u-})}{1+T-u}\right)du\right)$$

$$-\frac{\tilde{d}(s,Y_{s-})}{1+T-s}\exp\left(-\frac{1}{1+T-s}\int_t^s \log\left(\frac{\tilde{d}(u,Y_{u-})}{1+T-u}\right)du\right)ds]||$$

$$\leq E_i[\int_t^T \frac{1}{1+T}\exp(\int_t^s \log(1+T-u)du)$$

$$||d(s,Y_{s-})\exp(-\int_t^s \log(d(u,Y_{u-}))du) - \tilde{d}(s,Y_{s-})\exp(-\int_t^s \log(\tilde{d}(u,Y_{u-}))du)||ds]$$

$$\leq E_i[\int_t^T \frac{\exp(\log(T-t+1)(s-t))}{1+T}||\alpha_2(t,s)||$$

$$||\exp(-\int_t^s \log(\underbrace{\tilde{d}(u,Y_{u-})}_{\geq \nu})du)||\,||d(s,Y_{s-})\exp(-\int_t^s (\log(\frac{d(u,Y_{u-})}{\tilde{d}(u,Y_{u-})})du) - \tilde{d}(s,Y_{s-})||ds]$$

$$\leq \exp(-(T-t)\log(\nu))\frac{\exp(\log(T-t+1)(T-t))}{1+T}E_i[\int_t^T ||\alpha_1(t,s)||$$

$$\cdot||d(s,Y_{s-})\exp(-\int_t^s (\log(\underbrace{\frac{d(u,Y_{u-})}{\tilde{d}(u,Y_{u-})}}_{\geq \frac{\nu}{K}})du) - \tilde{d}(s,Y_{s-})||ds]$$

$$\leq \exp(-(T-t)\log(\nu))\frac{\exp(\log(T-t+1)(T-t))}{1+T}$$

$$\cdot E_i[\int_t^T ||\alpha_1(t,s)||\,||\exp(-(T-t)\log(\frac{\nu}{K}))d(s,Y_{s-}) - \tilde{d}(s,Y_{s-})||ds]$$

$$\leq \exp(-(T-t)\log(\nu))\frac{\exp(\log(T-t+1)(T-t))}{1+T}$$

$$\cdot E_i[\int_t^T ||\alpha_1(t,s)||\exp(-(T-t)\log(\frac{\nu}{K}))\max_{k\in E_Y}\{||d(s,k) - \tilde{d}(s,k)||\}ds]$$

$$= \exp(-(T-t)(2\log(\nu) - \log(K)))\frac{\exp(\log(T-t+1)(T-t))}{1+T}(T-t)$$

$$\cdot \max_{k\in E_Y}\{||d(t,k) - \tilde{d}(t,k)||\}|\frac{d(s,Y_{s-})}{1+T-s}E_i[||\alpha_1(t,s)||]$$

We denote $M := \frac{\max_{k\in E_Y}\{||d(s,k)-\tilde{d}(s,k)||\}}{||d(s,i)-\tilde{d}(s,i)||}$ and get

$$\leq \exp(-(T-t)(2\log(\nu) - \log(K)))\frac{\exp(\log(T-t+1)(T-t))}{1+T}(T-t)$$

$$\cdot M\, E_i[||\alpha_1(t,s)||]\,||d(s,i) - \tilde{d}(s,i)||$$

$$= \bar{L}||d(s,i) - \tilde{d}(s,i)||,$$

where $\bar{L} := \exp(-(T-t)(2\log(\nu) - \log(K)))\frac{\exp(\log(T-t+1)(T-t))}{1+T}1 + T(T-t)M\, E_i[||\alpha_2(t,s)||]$.

It is left to show that $E_i[||\alpha_1(t,s)||] < \infty$.

$$E_i[||\alpha_1(t,s)||] < \infty \Leftrightarrow E_i[|\alpha_1(t,s)|] < \infty,\ \forall s \in [t,T]$$

$$E_i[|\alpha_1(t,s)|] = E_i[\gamma \exp(\gamma\kappa(t,s))|\frac{\partial}{\partial t}\beta_c(s-t,i)|] = \gamma|\frac{\partial}{\partial t}\beta_c(s-t,i)|E_i[|\exp(\gamma\kappa(t,s))|]$$

We plug $\kappa(t,s) = \int_t^s \frac{1}{\gamma}\mu_u'(\sigma_u'\sigma_u)^{-1}\mu_u du + \int_t^s \frac{1}{\gamma}\mu_u'(\sigma_u)^{-1}dW_u$ and get

$$E_i[|\alpha_1(t,s)|] = \gamma|\frac{\partial}{\partial t}\beta_c(s-t,i)|E_i[|\exp(\int_t^s \mu_u'(\mu_u'(\sigma_u\sigma_u')^{-1})\mu_u du + \int_t^s \mu_u'(\sigma_u)^{-1}dW_u)|] < \infty, \forall s \in [t,T]$$

since μ_t and σ_t are uniformly bounded.

Analog for g_2, g_2 and g_4. Which implies that $f(t,i,d)$ is Lipschitz.

□

4 Appendix

4.1 Stochastic Analysis

4.2 Stochastic Analysis

Theorem 13 (Itô's Formula)**.** *Let J be a semimartingale and let a $f \in C^2$ be real function. Then $f(J)$ is again a semimartingale, and the following formula holds:*

$$f(J_t) - f(J_0) = \int_{0+}^t f'(J_{s-})dJ_s + \frac{1}{2}\int_{0+}^t f''(J_{s-})d[J,J]_s^c + \sum_{0<s\leq t} f(J_s) - f(J_{s-}) - f'(J_{s-})\Delta J_s.$$

Proof. See Protter (2005) Theorem 32, page 78.

Proposition 4. *Let $f(\cdot,\cdot,i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ for all $i \in E_Y$. Then it holds:*

$$f(t, X_t^\pi, Y_{t-}) - f(0, X_0^\pi, Y_0) = \int_{0+}^t \frac{\partial}{\partial t} f(s, X_s^\pi, Y_{s-})ds$$

$$+ \int_{0+}^t \Big(r_s X_s^\pi + a_s'(X_s^\pi, Y_{s-})\mu_s - c_s(X_s^\pi, Y_{s-})\Big) \frac{\partial}{\partial x} f(s, X_s^\pi, Y_{s-})ds$$

$$+ \int_0^t \frac{1}{2} a_s'(X_s^\pi, Y_{s-})\sigma_s\sigma_s' a_s(X_s^\pi, Y_{s-}) \frac{\partial}{\partial x^2} f(s, X_s^\pi, Y_{s-})ds + \int_0^t a_s'(X_s^\pi, Y_{s-})\sigma_s \frac{\partial}{\partial x} f(s, X_s^\pi, Y_{s-})dW_s$$

$$+ \sum_{0<s\leq t} f(s, X_s^\pi, Y_s) - f(s, X_s^\pi, Y_{s-}).$$

Proof.
We apply Itô's formula for $J = (t, X_t^\pi, Y_t)$. Further it holds that $[t,t]^c = 0$, $[t, X_t^\pi]^c = 0$, $[t, Y_t] = 0$, $[Y_t, X_t^\pi]^c = 0$ and $d[X_t^\pi, X_t^\pi]^c = a_t'(X_t^\pi, Y_{t-})\sigma_t\sigma_t' a_t(X_t^\pi, Y_{t-})dt$. We get

$$f(t, X_t^\pi, Y_{t-}) - f(0, X_0^\pi, Y_0) = \int_{0+}^t \frac{\partial}{\partial t} f(s, X_s^\pi, Y_{s-})ds + \int_{0+}^t \frac{\partial}{\partial x} f(s, X_s^\pi, Y_{s-})dX_s^\pi$$

$$+ \int_{0+}^t \frac{\partial}{\partial y} f(s, X_s^\pi, Y_{s-})dY_{s-} + \frac{1}{2}\int_{0+}^t a_s'(X_s^\pi, Y_{s-})\sigma_s\sigma_s' a_s(X_s^\pi, Y_{s-}) \frac{\partial}{\partial x^2} f(s, X_s^\pi, Y_{s-})ds$$

$$+ \sum_{0<s\leq t} \left(f(s, X_s^\pi, Y_s) - f(s, X_s^\pi, Y_{s-}) - \frac{\partial}{\partial y} f(s, X_s^\pi, Y_{s-})(Y_s - Y_{s-}) \right).$$

Note that $\int_0^t \frac{\partial}{\partial y} f(s, X_s^\pi, Y_{s-})dY_{s-} = \sum_{0<s\leq t} \frac{\partial}{\partial y} f(s, X_s^\pi, Y_{s-})(Y_s - Y_{s-})$ and
$dX_s^\pi = r_s X_s^\pi + a_s'(X_s^\pi, Y_s)\mu_s - c_s(X_s^\pi, Y_{s-})ds + a_s'(X_s^\pi, Y_{s-})\sigma_s' dW_s$ which implies the statement.

□

Lemma 5. *Let $f(\cdot,\cdot,i) \in C^{1,2}$ for all $i \in E_Y$ and f fulfill a polynomial growth condition in x. Then it holds that*

$$\lim_{h\downarrow 0} \frac{1}{h} E_{t,x,i}[\sum_{0<s\leq h} \big(f(t+s, X^\pi_{t+s}, Y_{t+s}) - f(t+s, X^\pi_{t+s}, Y_{(t+s)-})\big)] = \sum_{j\in E_Y} \lambda_{ij} f(t,x,j).$$

Proof. First note that the probability for more then one jump from Y_t on $[t+t\epsilon]$ is $o(\epsilon)$. Since we consider the limit $\epsilon \downarrow 0$ we only have to look for jumps at time t. So we get

$$\lim_{h\downarrow 0} \frac{1}{h} E_{t,x,i}[\sum_{0<s\leq h} \big(f(t+s, X^\pi_{t+s}, Y_{t+s}) - f(t+s, X^\pi_{t+s}, Y_{(t+s)-})\big)]$$

$$= \lim_{h\downarrow 0} \frac{1}{h} \sum_{j\in E_Y} p_{ij}(h) f(t,x,j) - f(t,x,i) = \sum_{j\in E_Y} \lim_{h\downarrow 0} \frac{p_{ij}(\epsilon) - \delta_{ij}}{\epsilon} f(t,x,j)$$

$$= \sum_{j\in E_Y} \lambda_{ij} f(t,x,j)$$

□

Lemma 6. *It holds that*

$$\lim_{h\downarrow 0} \frac{1}{h} E_{t,x,i}[\sum_{0<s\leq h} \Big(\big(\beta_p(T-t-s, X^\pi_{t+s}, Y_{t+s}) - \beta_p(T-t,x,i)\big) E_{t+s, X^\pi_{t+s}, Y_{t+s}}[U_p(X^\pi_T)]$$

$$- \big(\beta_p(T-t-s, X^\pi_{t+s}, Y_{(t+s)-}) - \beta_p(T-t,x,i)\big) E_{t+s, X^\pi_{t+s}, Y_{(t+s)-}}[U_p(X^\pi_T)]\Big)$$

$$= \sum_{j\in E_Y} \lambda_{ij} \big(\beta_p(T-t,x,j) - \beta_p(T-t,x,i)\big) E_{t,x,j}[U_p(X^\pi_T)].$$

Proof.
To ease notation we set $X_t := X^\pi_t$.

$$\lim_{h\downarrow 0} \frac{1}{h} E_{t,x,i}[\sum_{0<s\leq h} \Big(\big(\beta_p(T-t-s, X_{t+s}, Y_{t+s}) - \beta_p(T-t,x,i)\big) E_{t+s, X_{t+s}, Y_{t+s}}[U_p(X_T)]$$

$$- \big(\beta_p(T-t-s, X_{t+s}, Y_{(t+s)-}) - \beta_p(T-t,x,i)\big) E_{t+s, X_{t+s}, Y_{(t+s)-}}[U_p(X_T)]\Big)$$

$$= \lim_{h\downarrow 0} \frac{1}{h} E_{t,x,i}[U(X_T)\left(\beta_p(T-t-h, X_{t+h}, Y_{t+h}) - \beta_p\left(T-t,x,i\right)\right))] + o(1)$$

$$= \lim_{h\downarrow 0} \frac{1}{h} \sum_{j\in E_Y} E_{t,x,i}[U(X_T)\mathbb{1}(Y_{t+h}=j)\left(\beta_p(T-t-h, X_{t+h}, j)-) - \beta_p\left(T-t,x,i\right)\right))] + o(1)$$

$$= \lim_{h\downarrow 0} \frac{1}{h} \sum_{j\in E_Y} P_{t,x,i}(Y_{t+h}=j) \frac{E_{t,x,i}[U(X_T)I(Y_{t+e}=j)\left(\beta_p(T-t-h, X_{t+h},) - \beta_p\left(T-t,x,i\right)\right))]}{P_{t,x,i}(Y_{t+h}=j)} + o(1)$$

$$= \lim_{h\downarrow 0} \frac{1}{h} \sum_{j\in E_Y} p_{ij}(h) \frac{E_{t,x,i}[U(X_T)\mathbb{1}(Y_{t+h}=j)\left(\beta_p(T-t-h, X_{t+h}, j) - \beta_p\left(T-t,x,i\right)\right)]}{P_{t,x,i}(Y_{t+h}=j)} + o(1)$$

$$= \lim_{h\downarrow 0} \frac{1}{h} \sum_{j\in E_Y} (\lambda_{ij}h + o(1)) E_{t,x,i}[U(X_T)\left(\beta_p(T-t-h, X_{t+h}, j)-\right) - \beta_p\left(T-t,x,i\right)))\,|Y_{t+h}=j] + o(1)$$

$$= \sum_{j\in E_Y} \lambda_{ij}\big(\beta_p(T-t,x,j) - \beta_p(T-t,x,i)\big) E_{t,x,j}[U_p(X_T)].$$

□

Lemma 7. *It holds that*

$$\lim_{h\downarrow 0} \frac{1}{h} E_{t,x,i}[\sum_{0<u\le h} \Big(\big(\beta_c(s-t-u, X^\pi_{t+u}, Y_{t+u}) - \beta_c(s-t,x,i)\big) E_{t+u, X^\pi_{t+u}, Y_{t+u}}[U_c(c_s(X^\pi_s, Y_{s-})]$$

$$- \big(\beta_c(s-t-u, X^\pi_{t+u}, Y_{(t+u)-}) - \beta_c(s-t,x,i)\big) E_{t+u, X^\pi_{t+u}, Y_{(t+u)-}}[U_c(c_s(X^\pi_s, Y_{s-})]\Big)$$

$$= \sum_{j\in E_Y} \lambda_{ij}\left(\beta_c(s-t,x,j) - \beta_c(s-t,x,i)\right) E_{t,x,j}[U_c(c_s(X^\pi_s, Y_{s-})]$$

Proof. Analog to Lemma 7

Lemma 8. *Let the following conditions be satisfied*

$$E_{t,x,i}[\int_0^h a'_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) a_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) \left(\frac{\partial}{\partial x}\beta_c(s-t-u, X^\pi_{t+u}, Y_{(t+u)-})\right)^2 du] < \infty$$

$$E_{t,x,i}[\int_0^h a'_{t+u}(X^\pi_u, Y_{u-}) a_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) \left(\frac{\partial}{\partial x}\varphi^\pi_s(t+u, X^\pi_{t+u}, Y_{t+u})\right)^2 du] < \infty.$$

Then it holds that

$$\lim_{h\downarrow 0} E_{t,x,i}[\frac{\beta_c(s-t-h, X^\pi_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)}{h} U_c(c(s, X^\pi_s, Y_{s-})]$$

$$= E_{t,x,i}[U_c(c(s, X^\pi_s, Y_{s-})\Big(-\frac{\partial}{\partial t}\beta_c(s-t,x,i)$$

$$+\Big(r_t x + a'_t(x,i) - c_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i) + \frac{1}{2}a'_t(x,i)\sigma_t\sigma'_t a_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i))\Big)]$$

$$+a'_t(x,i)\sigma'_t\sigma_t a_t(x,i)\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^\pi_s(t,x,i)$$

$$+ \sum_{j\in E_Y} \lambda_{ij}\left(\beta_c(s-t,x,j) - \beta_c(s-t,x,i)\right) E_{t,x,j}[U_c(c_s(X^\pi_s, Y_{s-})],$$

where $\varphi^\pi_s(t,x,i) := E_{t,x,i}[U_c(c_s(X^\pi_s, Y_{s-})]$.

Proof.
To ease notation we set $X_t := X_t^\pi$.

$$E_{t,x,i}[\frac{\beta_c(s-t-h, X_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)}{h} U_c(c(s, X_s, Y_{s-})]$$

$$= E_{t,x,i}[\frac{\beta_c(s-t-h, X_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)}{h} \underbrace{E_{t+h, X_{t+h}, Y_{(t+h)-}} U_c(c(s, X_s, Y_{s-})]}_{:=\varphi_s^\pi(t+h, X_{t+h}, Y_{(t+h)-})}$$

We apply Itô's Formula to

$$\beta_c(s-t-h, X_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)\varphi_s^\pi(t+h, X_{t+h}, Y_{(t+h)-})$$

and get

$$\Big(\beta_c(s-t-h, X_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)\Big)\varphi_s^\pi(t+h, X_{t+h}, Y_{t+h})$$

$$= -\int_0^h \varphi_s^\pi(t+u, X_{t+u}, Y_{(t+u)-}) \frac{\partial}{\partial t}\beta_c(s-t-u, X_{t+u}, Y_{(t+u)-})$$

$$+\Big(\beta_c(s-t-u, X_{t+u}, Y_{(t+u)-}) - \beta_c(s-t,x,i)\Big)\frac{\partial}{\partial t}\varphi_s^\pi(t+u, X_{t+u}, Y_{(t+u)-})du$$

$$+\int_0^h \Big(r_{t+u}X_{t+u} + a'_{t+u}(X_{t+u}, Y_{(t+u)-})\mu_{t+u} - c_{t+u}(X_{t+u}, Y_{(t+u)-})\Big)$$

$$\cdot \varphi_s^\pi(t+u, X_{t+u}, Y_{t+u})\frac{\partial}{\partial x}\beta_c(s-t-u, X_{t+u}, Y_{(t+u)-})du$$

$$+\int_0^h \Big(r_{t+u}X_{t+u} + a'_{t+u}(X_{t+u}, Y_{(t+u)-})\mu_{t+u} - c_{t+u}(X_{t+u}, Y_{(t+u)-})\Big)$$

$$\cdot\Big(\beta_c(s-t-u, X_{t+u}, Y_{(t+u)-}) - \beta_c(s-t,x,i)\Big)\frac{\partial}{\partial x}\varphi_s^\pi(t+u, X_{t+u}, Y_{t+u})du$$

$$+\int_0^h \frac{1}{2}a'_{t+u}(X_{t+u}, Y_{(t+u)-})\sigma_{t+u}\sigma'_{t+u}a_{t+u}(X_{t+u}, Y_{(t+u)-})$$

$$\cdot\varphi_s^\pi(t+u, X_{t+u}, Y_{t+u})\frac{\partial}{\partial x^2}\beta_c(s-t-u, X_{t+u}, Y_{(t+u-)})du$$

$$+\int_0^h a'_{t+u}(X_{t+u}, Y_{(t+u)-})\sigma_{t+u}\sigma'_{t+u}a_{t+u}(X_{t+u}, Y_{(t+u)-})\frac{\partial}{\partial x}$$

$$\cdot\varphi_s^\pi(t+u, X_{t+u}, Y_{t+u})\frac{\partial}{\partial x}\beta_c(s-t-u, X_{t+u}, Y_{(t+u-)})du$$

$$+\int_0^h \frac{1}{2}a'_{t+u}(X_{t+u}, Y_{(t+u)-})\sigma_{t+u}\sigma'_{t+u}a_t(x,i)$$

$$\cdot\Big(\beta_c(s-t-u, X_{t+u}, Y_{(t+u)-}) - \beta_c(s-t,x,i)\Big)\frac{\partial}{\partial x^2}\varphi_s^\pi(t+u, X_{t+u}, Y_{t+u})du$$

$$+\int_0^h a'_{t+u}(X_{t+u}, Y_{(t+u)-})\sigma_{t+u}\frac{\partial}{\partial x}\beta_c(s-t-u, X_{t+u}, Y_{(t+u)-})$$

$$+a'_{t+u}(X_{t+u}, Y_{(t+u)-})\sigma_{t+u}\Big(\beta_c(s-t-u, X_{t+u}, Y_{(t+u)-}) - \beta_c(s-t,x,i)\Big)\frac{\partial}{\partial x}\varphi_s^\pi(t+u, X_{t+u}, Y_{t+u})dW_u$$

$$+\sum_{0<s\le h}\Big((\beta_c(s-t-u,X_{t+u},Y_{t+u})-\beta_c(s-t,x,i))\,\varphi_s(t+u,X_{t+u},Y_{t+u})$$

$$-\left(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\right)\varphi_s^{\pi}(t+u,X_{t+u},Y_{(t+u)-}\Big).$$

So we get

$$H(t,x,i):=\lim_{h\downarrow 0}\frac{1}{h}E_{t,x,i}[-\int_0^h \varphi_s^{\pi}(t+u,X_{t+u},Y_{(t+u)-})\frac{\partial}{\partial t}\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})$$

$$+\Big(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\Big)\frac{\partial}{\partial t}\varphi_s^{\pi}(t+u,X_{t+u},Y_{(t+u)-})du$$

$$+\int_0^h\Big(r_{t+u}X_{t+u}+a'_{t+u}(X_{t+u},Y_{(t+u)-})\mu_{t+u}-c_{t+u}(X_{t+u},Y_{(t+u)-})\Big)$$

$$\cdot\varphi_s^{\pi}(t+u,X_{t+u},Y_{t+u})\frac{\partial}{\partial x}\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})du$$

$$+\int_0^h\Big(r_{t+u}X_{t+u}+a'_{t+u}(X_{t+u},Y_{(t+u)-})\mu_{t+u}-c_{t+u}(X_{t+u},Y_{(t+u)-})\Big)$$

$$\cdot\Big(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\Big)\frac{\partial}{\partial x}\varphi_s^{\pi}(t+u,X_{t+u},Y_{t+u})du$$

$$+\int_0^h\frac{1}{2}a'_{t+u}(X_{t+u},Y_{(t+u)-}))\sigma_{t+u}\sigma'_{t+u}a_{t+u}(X_{t+u},Y_{(t+u)-})$$

$$\cdot\varphi_s^{\pi}(t+u,X_{t+u},Y_{t+u})\frac{\partial}{\partial x^2}\beta_c(s-t-u,X_{t+u},Y_{(t+u-)})du$$

$$+\int_0^h a'_{t+u}(X_{t+u},Y_{(t+u)-})\sigma_{t+u}\sigma'_{t+u}a_{t+u}(X_{t+u},Y_{(t+u)-})$$

$$\cdot\frac{\partial}{\partial x}\varphi_s^{\pi}(t+u,X_{t+u},Y_{t+u})\frac{\partial}{\partial x}\beta_c(s-t-u,X_{t+u},Y_{(t+u-)})du$$

$$+\int_0^h\frac{1}{2}a'_{t+u}(X_{t+u},Y_{(t+u)-})\sigma_{t+u}\sigma'_{t+u}a_{t+u}(X_{t+u},Y_{(t+u)-})$$

$$\cdot\Big(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\Big)\frac{\partial}{\partial x^2}\varphi_s^{\pi}(t+u,X_{t+u},Y_{t+u})du$$

$$+\int_0^h a'_{t+u}(X_{t+u},Y_{(t+u)-})\sigma_{t+u}\frac{\partial}{\partial x}\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})$$

$$+a'_{t+u}(X_{t+u},Y_{(t+u)-})\sigma_{t+u}\Big(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\Big)\frac{\partial}{\partial x}\varphi_s^{\pi}(t+u,X_{t+u},Y_{t+u})dW_u$$

$$+\sum_{0<u\le h}\Big((\beta_c(s-t-u,X_{t+u},Y_{t+u})-\beta_c(s-t,x,i))\,\varphi_s^{\pi}(t+u,X_{t+u},Y_{t+u})$$

$$-\left(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\right)\varphi_s^{\pi}(t+u,X_{t+u},Y_{(t+u)-}\Big).$$

It holds that

$$0=E_{t,x,i}[\int_0^h a'_{t+u}(X_{t+u},Y_{(t+u)-})\sigma_{t+u}\frac{\partial}{\partial x}\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})$$

$$+a'_{t+u}(X_{t+u},Y_{(t+u)-})\sigma_{t+u}$$

$$\cdot\Big(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\Big)\frac{\partial}{\partial x}\varphi_s^\pi(t+u,X_{t+u},Y_{t+u})dW_u],$$

since

$$E_{t,x,i}[\int_0^h a'_{t+u}(X_{t+u},Y_{(t+u)-})a_{t+u}(X_{t+u},Y_{(t+u)-})\left(\frac{\partial}{\partial x}\beta_p(s-t-u,X_{t+u},Y_{(t+u)-})\right)^2du]<\infty$$

$$E_{t,x,i}[\int_0^h a'_{t+u}(X_{t+u},Y_{(t+u)-})a_{t+u}(X_{t+u},Y_{(t+u)-})\left(\frac{\partial}{\partial x}\varphi_s^\pi(t+u,X_{t+u},Y_{t+u})\right)^2du]<\infty.$$

Using dominated convergence we get

$$H(t,x,i)=E_{t,x,i}[U_c(c(s,X_s,Y_{s-})\Big(-\frac{\partial}{\partial t}\beta_c(s-t,x,i)$$

$$+\Big(r_tx+a'_t(x,i)\mu_t-c_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i)+\frac{1}{2}a'_t(x,i)\sigma_t\sigma'_ta_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)du$$

$$+a'_t(x,i)\sigma_t\sigma'_ta_t(x,i)\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi_s^\pi(t,x,i)$$

$$+\frac{1}{h}E_{t,x,i}[\lim_{h\downarrow 0}\sum_{0<u\le h}\Big(\left(\beta_c(s-t-u,X_{t+u},Y_{t+u})-\beta_c(s-t,x,i)\right)\varphi_s(t+u,X_{t+u},Y_{t+u})$$

$$-\left(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\right)\varphi_s(t+u,X_{t+u},Y_{(t+u)-}\Big)$$

Applying Lemma 7 we get

$$\lim_{h\downarrow 0}\frac{1}{h}E_{t,x,i}[\sum_{0<u\le h}\Big(\left(\beta_c(s-t-u,X_{t+u},Y_{t+u})-\beta_c(s-t,x,i)\right)\varphi_s(t+u,X_{t+u},Y_{t+u})$$

$$-\left(\beta_c(s-t-u,X_{t+u},Y_{(t+u)-})-\beta_c(s-t,x,i)\right)\varphi_s(t+u,X_{t+u},Y_{(t+u)-}\Big)$$

$$=\sum_{j\in E_Y}\lambda_{ij}\left(\beta_c(s-t,x,j)-\beta_c(s-t,x,i)\right)E_{t,x,j}[U_c(c_s(X_s,Y_{s-})].$$

Summarizing we get

$$H(t,x,i)=E_{t,x,i}[U_c(c(s,X_s,Y_{s-})\Big(-\frac{\partial}{\partial t}\beta_c(s-t,x,i)$$

$$+\Big(r_tx+a'_t(x,i)-c_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i)+\frac{1}{2}a'_t(x,i)\sigma_t\sigma'_ta_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i)\Big)U_c(c(s,X_s,Y_{s-}))$$

$$+a'_t(x,i)\sigma_t\sigma'_ta_t(x,i)\frac{\partial}{\partial x}\beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi_s^\pi(t,x,i)$$

$$+\sum_{j\in E_Y}\lambda_{ij}\int_t^T\left(\beta_c(s-t,x,j)-\beta_c(s-t,x,i)\right)E_{t,x,j}[U_c(c_s(X_s,Y_{s-})]$$

□

Lemma 9. *Let the following conditions be satisfied*

$$E_{t,x,i}[\int_0^h a'_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) a_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) \left(\frac{\partial}{\partial x}\beta_p(s-t-u, X^\pi_{t+u}, Y_{(t+u)-})\right)^2 du] < \infty$$

$$E_{t,x,i}[\int_0^h a'_{t+u}(X^\pi_u, Y_{u-}) a_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) \left(\frac{\partial}{\partial x}\varphi^\pi_s(t+u, X^\pi_{t+u}, Y_{t+u})\right)^2 du] < \infty.$$

Then it holds that

$$\lim_{h\downarrow 0} E_{t,x,i}[\int_{t+h}^T \frac{\beta_c(s-t-h, X^\pi_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)}{h} U_c(c(s, X^\pi_s, Y_{s-})ds]$$

$$= E_{t,x,i}[\int_t^T U_c(c(s, X^\pi_s, Y_{s-})\Big(-\frac{\partial}{\partial t}\beta_c(s-t,x,i)$$

$$+\Big(r_t x + a'_t(x,i) - c_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i) + \frac{1}{2}a'_t(x,i)\sigma_t\sigma'_t a_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i))\Big)ds]$$

$$+a'_t(x,i)\sigma_t\sigma'_t a_t(x,i)\frac{\partial}{\partial x}\int_{t+h}^T \beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi^\pi_s(t,x,i)ds$$

$$+\sum_{j\in E_Y}\lambda_{ij}\int_t^T (\beta_c(s-t,x,j) - \beta_c(s-t,x,i))\, E_{t,x,j}[U_c(c_s(X^\pi_s, Y_{s-})]ds,$$

where $\varphi^\pi_s(t,x,i) := E_{t,x,i}[U_c(c_s(X^\pi_s, Y_{s-})]$.

Proof.

$$\lim_{h\downarrow 0} E_{t,x,i}[\int_{t+h}^T \frac{\beta_c(s-t-h, X^\pi_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)}{h} U_c(c(s, X^\pi_s, Y_{s-})ds]$$

$$= \lim_{h\downarrow 0} E_{t,x,i}[\int_t^T \frac{\beta_c(s-t-h, X^\pi_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)}{h} U_c(c(s, X^\pi_s, Y_{s-})\mathbb{1}(s \geq t+h)ds]$$

We use dominated convergence theorem twice and get

$$E_{t,x,i}[\int_t^T \lim_{h\downarrow 0} \frac{\beta_c(s-t-h, X^\pi_{t+h}, Y_{(t+h)-}) - \beta_c(s-t,x,i)}{h} U_c(c(s, X^\pi_s, Y_{s-})\mathbb{1}(s \geq t+h)ds].$$

Lemma 8 implies the statement

□

Lemma 10. *Let the following condition be satisfied*

$$E_{t,x,i}[\int_0^h a'_{t+s}(X^\pi_{t+s}, Y_{(t+s)-})a_{t+s}(X^\pi_{t+s}, Y_{(t+s)-}) \left(\frac{\partial}{\partial x}\beta_p(T-t-s, X^\pi_{t+s}, Y_{(t+s)-})\right)^2 ds < \infty]$$

$$E_{t,x,i}[\int_0^h a'_{t+s}(X^\pi_{t+s}, Y_{(t+s)-})a_{t+s}(X^\pi_{t+s}, Y_{(t+s)-}) \left(\frac{\partial}{\partial x}\psi^\pi(t+s, X^\pi_{t+s}, Y_{t+s})\right)^2 ds] < \infty.$$

Then it holds that

$$\lim_{h\downarrow 0}\frac{1}{h}E_{t,x,i}[\Big(\beta_p(T-t-h, X^\pi_{t+h}, Y_{(t+h)-}) - \beta_p(T-t,x,i)\Big)U_p(X^\pi_T)]$$

$$= E_{t,x,i}[U_p(X^\pi_T)\Big(-\frac{\partial}{\partial t}\beta_p(T-t,x,i) + \Big(r_t x + a'_t(x,i)\mu_t - c_t(x,i)\Big)\frac{\partial}{\partial x}\beta_p(s-t,x,i)$$

$$+\frac{1}{2}a'_t(x,i)\sigma_t\sigma'_t a_t(x,i))\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)]\Big)] + a'_t(x,i)\sigma_t\sigma'_t a_t(x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(T-t,x,j) - \beta_p(T-t,x,i)\big)E_{t,x,j}[U_p(X^\pi_T)],$$

where $\psi^\pi(t,x,i) := E_{t,x,i}[U_p(X^\pi_T)]$.

Proof. Analog to Lemma 8.

Definition 17. *Let* (J_t) *be a (time-homogeneous) Markov process in* $\mathbb{R}^n$. *The infinitesimal generator* A *of* (J_t) *is defined by*

$$Af(j) = \lim_{h\downarrow 0}\frac{E_j[f(J_h)] - f(j)}{h}; \quad x \in \mathbb{R}^n.$$

The set of functions $f: \mathbb{R}^n \to \mathbb{R}$ *such that the limit exists at* j *is denoted by* $D_A(j)$, *while* D_A *denotes the set of functions for which the limit exists for all* $j \in \mathbb{R}^n$.

Lemma 11. *Let* $(J_t) = (t, X^\pi_t, Y_t)$ *and* $f(\cdot,\cdot,i) \in C^{1,2}(\mathbb{R}^2)$ *for all* $i \in E_Y$, *further let*

$$E_{t,x,i}[\int_0^T a'_s(X^\pi_s, Y_{s-})a_s(X^\pi_s, Y_{s-})\left(\frac{\partial}{\partial x}f(s, X^\pi_s, Y_-)\right)^2 ds] < \infty,$$

then $f \in D_A$ *and*

$$A^\pi f(t,x,i) = \frac{\partial}{\partial t}f(t,x,i) + \big(rx + a'_t(x,i)\mu_t - c_t(x,i)\big)\frac{\partial}{\partial x}f(t,x,i)$$

$$+\frac{1}{2}a'_t(x,i)\sigma'_t\sigma_t a_s(x,i)\frac{\partial}{\partial x^2}f(t,x,i) + \sum_{j\in E_Y}\lambda_{ij}f(t,x,j).$$

Proof. Applying Itô's formula to $f(t+h, X^{\pi}_{t+h}, Y_{(t+h)-})$ we get

$$A^{\pi} f(t,x,i) = \lim_{h \downarrow 0} \frac{E_{t,x,i}[f(t+h, X^{\pi}_{t+h}, Y_{(t+h)-}] - f(t,x,i)]}{h}$$

$$= \lim_{\epsilon \downarrow 0} \frac{1}{h} E_{t,x,i}[\int_t^{t+h} \frac{\partial}{\partial t} f(s, X^{\pi}_s, Y_{s-}) ds + \int_t^{t+h} \left(rX^{\pi}_s + a'_s(X_s, Y_{s-})\mu_s - c_s(X^{\pi}_s, Y_{s-}) \right) \frac{\partial}{\partial x} f(s, X^{\pi}_s, Y_{s-}) ds$$

$$+ \int_t^{t+h} \frac{1}{2} a_s(X^{\pi}_s, Y_{s-}) \sigma'_s \sigma_s a_s(X^{\pi}_s, Y_{s-}) \frac{\partial}{\partial x^2} f(s, X^{\pi}_s, Y_{s-}) ds + \sum_{j \in E_Y} \lambda_{ij} f(t,x,j)]$$

$$+ \lim_{\epsilon \downarrow 0} \frac{1}{h} \underbrace{E_{t,x,i}[\int_t^{t+h} a'_s(X^{\pi}_s, Y_{s-}) \sigma_s f_x(s, X^{\pi}_s, Y_{s-}) dW_s]}_{=0}$$

$$= \frac{\partial}{\partial t} f(t,x,i) + \left(rx + a'_t(x,i)\mu_t - c_t(x,i) \right) \frac{\partial}{\partial x} f(t,x,i)$$

$$+ \frac{1}{2} a'_t(x,i) \sigma'_t \sigma_t a_s(x,i) \frac{\partial}{\partial x^2} f(t,x,i) + \sum_{j \in E_Y} \lambda_{ij} f(t,x,j).$$

Note that

$$\int_t^{t+h} a'_s(X^{\pi}_s, Y_{s-}) \sigma_s f_x(s, X^{\pi}_s, Y_{s-}) dW_s] = 0,$$

since

$$\int_t^{t+h} a'_s(X^{\pi}_s, Y_{s-}) \sigma_s f_x(s, X^{\pi}_s, Y_{s-}) ds] < \infty.$$

□

Theorem 14 (Kolmogorov's Backward equation)**.** *Let* $(J_t) = (t, X^{\pi}_t, Y_t)$ *and* $f(\cdot,\cdot,i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ *for all* $i \in E_Y$*,* f *fulfill a polynomial growth in* x *and* $u(t,x,i) := E_{t,x,i}[U_p(X^{\pi}_T)]$*,* $s > t$*, then it holds that*

$$Au(t,x,i) = 0.$$

Proof. We denote by A_t the time-homogeneous generator, i.e.

$$A_t u(t,x,i) := \lim_{h \downarrow 0} \frac{E_{t,x,i}[u(t, X^{\pi}_{t+h}, Y_{(t+h)-})] - u(t,x,i)}{h}.$$

It holds that

$$Au(t,x,i) = \frac{\partial}{\partial t} u(t,x,i) + A_t u(t,x,i)$$

It is sufficient to show that

$$A_t u(t,x,i) = -\frac{\partial}{\partial t} u(t,x,i)$$

First note that by the strong Markov property, it follows that

$$u(t-h,x,i) = E_{t-h,x,i}[U(X^{\pi}_T)|\mathcal{F}_t] = E_{t-h,x,i}[E_{t,X_t,Y_{t-}}[U(X^{\pi}_T)]] = E_{t-h,x,i}[u(t, X^{\pi}_t, Y_{t-})] \quad (4.1)$$

$$\begin{aligned} A_t u(t,x,i) &= \lim_{h\downarrow 0} \frac{E_{t,x,i}[u(t, X^\pi_{t+h}, Y_{(t+h)-})] - u(t,x,i)}{h} \\ &= \lim_{h\downarrow 0} \frac{E_{t-h,x,i}[u(t, X^\pi_{t}, Y_{t-})] - u(t,x,i)}{h} \\ &\overset{(4.1)}{=} \lim_{h\downarrow 0} \frac{u(t-h,x,i) - u(t,x,i)}{h} \\ &= -\frac{\partial}{\partial t} u(t,x,i). \end{aligned}$$

□

Lemma 12. *Let the following conditions be satisfied*

$$E_{t,x,i}[\int_0^h a'_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) a_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) a \left(\frac{\partial}{\partial x}\beta_p(T-t-u, X^\pi_{t+u}, Y_{(t+u)-})\right)^2 du] < \infty$$

$$E_{t,x,i}[\int_0^h a'_{t+u}(X^\pi_{u}, Y_{u-}) a_{t+u}(X^\pi_{t+u}, Y_{(t+u)-}) \left(\frac{\partial}{\partial x}\psi^\pi_s(t+u, X^\pi_{t+u}, Y_{t+u})\right)^2 du] < \infty.$$

Then it holds that

$$A^\pi\left(\psi^\pi(t,x,i)\beta_p(T-t,x,i)\right)$$

$$= -\psi^\pi(t,x,i)\frac{\partial}{\partial t}\beta_p(T-t,x,o) + \Big(r_t x + a'_t(x,i)\mu_t - c_t(x,i)\Big)\psi^\pi(t,x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)$$

$$+\frac{1}{2}a'_t(x,i)\sigma_t\sigma'_t a'_t(x,i)\psi^\pi(t,x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i) + a'_t(x,i)\sigma_t\sigma'_t a'_t(x,i)\Big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\sum_{j\in E_Y}\lambda_{ij}\psi^\pi(T-t,x,j)\left(\beta_p(T-t,x,j) - \beta_p(T-t,x,i)\right),$$

where $\psi^\pi(t,x,i) := E_{t,x,i}[U_p(X^\pi_T)]$.

Proof.
To ease notation we set $X_t := X^\pi_t$.

Apply Lemma 11 we get

$$A\psi^\pi(t,x,i)\beta_p(T-t,x,i)$$

$$= -\psi^\pi(t,x,i)\frac{\partial}{\partial t}\beta_p(T-t,x,i) + \beta_p(T-t,x,i)\frac{\partial}{\partial t}\psi^\pi(t,x,i)$$

$$\Big(r_t x + a'_t(x,i)\mu_t - c_t(x,i)\Big)\psi^\pi(t,x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)$$

$$+\Big(r_t x + a'_t(x,i)\mu_t - c_t(x,i)\Big)\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\frac{1}{2}a'_t(x,i)\sigma_t\sigma'_t a_t(x,i)\psi^\pi(t,x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)$$

$$+\frac{1}{2}a_t'(x,i)\sigma_t^2 a_t(x,i)\beta_p(T-t,x,i)\frac{\partial}{\partial x^2}\psi^\pi(t,x,i)$$

$$+a_t'(x,i)\sigma_t\sigma_t' a_t(x,i)\Big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\sum_{j\in E_Y}\lambda_{ij}\left(\beta_p(T-t,x,j)\psi(T-t,x,j)-\beta_p(T-t,x,i)\psi(T-t,x,i)\right).$$

We resort and get

$$A\psi^\pi(t,x,i)\beta_p(T-t,x,i)$$

$$=-\psi^\pi(t,x,i)\frac{\partial}{\partial t}\beta_p(T-t,x,i)+$$

$$\Big(r_t x+a_t'(x,i)\mu_t-c_t(x,i)\Big)\psi^\pi(t,x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)$$

$$+\frac{1}{2}a_t'(x,i)\sigma_t'\sigma_t a_t(x,i)\psi^\pi(t,x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)$$

$$+a_t'(x,i)\sigma_t'\sigma_t a_t(x,i)\Big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\sum_{j\in E_Y}\lambda_{ij}\psi(T-t,x,j)\left(\beta_p(T-t,x,j)-\beta_p(T-t,x,i)\right)$$

$$-\beta(T-t,x,i)\Big(\frac{\partial}{\partial t}\psi^\pi(t,x,i)+\Big(r_t x+a_t'(x,i)\mu_t'-c_t(x,i)\Big)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\frac{1}{2}a_t'(x,i)\sigma_t'\sigma_t a_t(x,i)\beta_p(T-t,x,i)\frac{\partial}{\partial x^2}\psi^\pi(t,x,i)+\sum_{j\in E_Y}\lambda_{ij}\psi(T-t,x,j))\Big).$$

Theorem 14 implies that

$$0=A\psi^\pi(t,x,i)=\frac{\partial}{\partial t}\psi^\pi(t,x,i)+\Big(r_t x+a_t'(x,i)\mu_t-c_t(x,i)\Big)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\frac{1}{2}a_t'(x,i)\sigma_t\sigma_t' a_t(x,i)\beta_p(T-t,x,i)\frac{\partial}{\partial x^2}\psi^\pi(t,x,i)+\sum_{j\in E_Y}\lambda_{ij}\psi(T-t,x,j),$$

which implies the statement.

□

Lemma 13. *Let the following conditions be satisfied*

$$E_{t,x,i}[\int_0^h a'_{t+u}(X^\pi_{t+u},Y_{(t+u)-})a_{t+u}(X^\pi_{t+u},Y_{(t+u)-})\left(\frac{\partial}{\partial x}\beta_p(s-t-u,X^\pi_{t+u},Y_{(t+u)-})\right)^2 du]<\infty$$

$$E_{t,x,i}[\int_0^h a'_{t+u}(X^\pi_u,Y_{u-})a_{t+u}(X^\pi_{t+u},Y_{(t+u)-})\left(\frac{\partial}{\partial x}\varphi^\pi_s(t+u,X^\pi_{t+u},Y_{t+u})\right)^2 du]<\infty.$$

Then it holds that It holds that

$$A^\pi\left(\psi^\pi(t,x,i)\beta_p(T-t,x,i)\right)$$

$$=-\psi^\pi(t,x,i)\frac{\partial}{\partial t}\beta_p(T-t,x,o)+\Big(r_tx+a'_t(x,i)\mu_t-c_t(x,i)\Big)\psi^\pi(t,x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)$$

$$+\frac{1}{2}a'_t(x,i)\sigma'_t\sigma_ta'_t(x,i)\psi^\pi(t,x,i)\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)+a'_t(x,i)\sigma'_t\sigma_ta'_t(x,i)\Big)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\sum_{j\in E_Y}\lambda_{ij}\psi(T-t,x,j)\left(\beta_p(T-t,x,j)-\beta_p(T-t,x,i)\right),$$

where $\varphi^\pi_s(t,x,i):=E_{t,x,i}[U_c(c_s(X^\pi_s,Y_{s-})]$.

Proof. Analog Lemma 12

Theorem 15 (Dynkin's formula)**.** *Let* $(J_t)=(t,X^\pi_t,Y_t)$ *and* $f(\cdot,\cdot,i)\in C^{1,2}(\mathbb{R}_+\times\mathbb{R})$ *for all* $i\in E_Y$ *and* f *fulfill a polynomial growth condition in* x*. Further let the following assumption be satisfied*

$$E_{t,x,i}[\int_t^s a'_s(X^\pi_s,Y_{s-})a_s(X^\pi_s,Y_{s-})\left(\frac{\partial}{\partial x}f(s,X^\pi_s,Y_{s-})\right)^2 ds]<\infty.$$

Then it holds that

$$E_{t,x,i}[f(u,X^\pi_u,Y_{u-})]=f(t,x,i)+E_{t,x,i}[\int_t^u Af(s,X^\pi_s,Y_{s-})ds].$$

Proof.
To ease notation we set $X_t:=X^\pi_t$.

We write $f(s,X_s,Y_{s-})$ with Itô and get

$$f(u,X_u,Y_{s-})=f(t,x,i)+\int_t^s\frac{\partial}{\partial t}f(s,X_s,Y_{s-})ds+\int_t^s\frac{\partial}{\partial x}a'_s(s,X_s,Y_{s-})\mu'_sf(s,X_s,Y_{s-})ds$$

$$+\int_t^s a'_s(X_s,Y_{s-})\sigma_s\frac{\partial}{\partial x}f(s,X_s,Y_{s-})dW_s+\int_t^s a'_s(X_s,Y_{s-})\sigma_s\sigma'_sa_s(X_s,Y_{s-})\frac{\partial}{\partial x^2}f(s,X_s,Y_{s-})ds$$

$$+\sum_{t<s\le u}f(s,X_s,Y_s)-f(s,X_s,Y_{s-}).$$

It holds that

$$E_{t,x,i}[\int_t^s a'_s(X_s,Y_{s-})\sigma_s\frac{\partial}{\partial x}f(s,X_s,Y_{s-})dW_s]=0,$$

since

$$E_{t,x,i}[\int_t^s a_s'(X_s,Y_{s-})a_s(X_s,Y_{s-})\left(\frac{\partial}{\partial x}f(s,X_s,Y_{s-})\right)^2 ds]<\infty.$$

So we get

$$E_{t,x,i}[f(u,X_u,Y_{u-})]=f(t,x,i)+E_{t,x,i}[\int_t^s \frac{\partial}{\partial t}f(s,X_s,Y_{s-})ds$$
$$+\int_t^s \frac{\partial}{\partial x}a_s'(s,X_s,Y_{s-})\mu_s' f(s,X_s,Y_{s-})ds$$
$$+\int_t^s a_s'(X_s,Y_{s-})\sigma_s\sigma_s' a_s(X_s,Y_{s-})\frac{\partial}{\partial x^2}f(s,X_s,Y_{s-})ds].$$
$$+\sum_{t<s\leq u} f(s,X_s,Y_s)-f(s,X_s,Y_{s-})].$$

On the other hand we get

$$f(t,x,i)+E_{t,x,i}[\int_t^u Af(s,X_s,Y_{s-})ds]=f(t,x,i)+E_{t,x,i}[\int_t^s \frac{\partial}{\partial t}f(s,X_s,Y_{s-})ds$$
$$+\int_t^s \frac{\partial}{\partial x}a_s'(s,X_s,Y_{s-})\mu_s' f(s,X_s,Y_{s-})ds+\int_t^s a_s'(X_s,Y_{s-})\sigma_s\sigma_s' a_s(X_s,Y_{s-})\frac{\partial}{\partial x^2}f(s,X_s,Y_{s-})ds$$
$$+\int_t^u \sum_{j\in E_Y}\lambda_{Y_{s-}j}f(s,X_s,j)ds].$$

It is left to show that

$$E_{t,x,i}[\sum_{t<s\leq u} f(s,X_s,Y_s)-f(s,X_s,Y_{s-})]=E_{t,x,i}[\int_t^u \sum_{j\in E_Y}\lambda_{Y_{s-}j}f(s,X_s,j)ds].$$

Therefore we make a partition of the interval $[t,u]$ in N parts with length h, and get

$$E_{t,x,i}[\sum_{t<s\leq u} f(s,X_s,Y_s)-f(s,X_s,Y_{s-})]$$

$$=E_{t,x,i}[\lim_{h\to 0}\sum_{k=1}^{N}\sum_{t+(k-1)h<u\leq t+kh}\frac{f(s,X_s,Y_s)-f(s,X_s,Y_{s-})}{h}h]$$

$$=E_{t,x,i}[\lim_{h\to 0}E_{t+h(k-1),X_{t+h(k-1)}Y_{(t+h(k-1))-}}[\sum_{k=1}^{N}\sum_{t+(k-1)h<u\leq t+kh}\frac{f(s,X_s,Y_s)-f(s,X_s,Y_{s-})}{h}]h].$$

We define $\iota_k := t+h(k-1), X_{t+h(k-1)}Y_{(t+h(k-1))-}$ and

$$l_h(\iota_k):=E_{\iota_k}\sum_{t+(k-1)h<u\leq t+kh}\frac{f(s,X_s,Y_s)-f(s,X_s,Y_{s-})}{h},$$

so we get

$$E_{t,x,i}[\lim_{h\to 0}\sum_{k=1}^{N}\sum_{t+(k-1)h<s\leq t+kh}\frac{f(s,X_s,Y_s)-f(s,X_s,Y_{s-})}{h}h=E_{t,x,i}[\lim_{h\to 0}\sum_{k=1}^{N}l_h(s,x,i,h)]h.$$

From Lemma 5 follows that

$$\lim_{h\to 0} l_h(t+h(k-1), X_{t+h(k-1)}Y_{(t+h(k-1))-}) = \sum_{j\in E_Y} \lambda_{Y_{s-}j} f(s, X_s, j).$$

Further Lemma 18 that $\lim_{h\to 0}\sum_{k=1}^{N} l_h(t+h(k-1), X_{t+h(k-1)}Y_{(t+h(k-1))-})h = \int_t^u l(s, X_s, Y_{s-})ds$, so we get

$$E_{t,x,i}[\sum_{t<s\le u} f(s, X_s, Y_s) - f(s, X_s, Y_{s-})] = E_{t,x,i}[\int_t^u l(s, X_s, Y_{s-})ds]$$

$$= E_{t,x,i}[\int_t^u \sum_{j\in E_Y} \lambda_{Y_{s-}j} f(s, X_s, j)ds].$$

□

Theorem 16 (Generalized Dykin's Formula)**.** *Let the following assumption be satisfied*

$$E_{t,x,i}[\int_t^s a'_u(X_u^\pi, Y_{u-})a_u(X_u^\pi, Y_{u-})\left(\frac{\partial}{\partial x}\beta_p(T-u, X_u^\pi, Y_{u-})\right)^2 du <]\infty$$

$$E_{t,x,i}[\int_t^s a'_u(X_u^\pi, Y_{u-})a_u(X_u^\pi, Y_{u-})\left(\frac{\partial}{\partial x}\psi^\pi(u, X_u^\pi, Y_{u-})\right)^2 du] < \infty.$$

Then it holds that

$$E_{t,x,i}[\Big(\beta_p(T-u, X_u^\pi, Y_u) - \beta_p(T-t, x, i)\Big)\psi^\pi(u, X_u^\pi, Y_u)]$$

$$= E_{t,x,i}[\int_t^s A\big(\psi^\pi(u, X_u^\pi, Y_{u-})\beta_p(T-u, X_u^\pi, Y_{u-})\big)du],$$

where $\psi^\pi(t,x,i) := E_{t,x,i}[U_p(X_T^\pi)]$.

Proof.
To ease notation we set $X_t := X_t^\pi$.

Using Lemma 12 we get

$$A\psi^\pi(t,x,i)\beta_p(T-t,x,i)$$

$$= E_{t,x,i}[\int_t^s (-\psi^\pi(u, X_u, Y_{u-}))\frac{\partial}{\partial t}\beta_p(T-u, x_u, Y_{u-})$$

$$+\Big(r_u X_u + a'_u(X_u, Y_{u-})\mu_u - c_u(X_u, Y_{u-})\Big)\psi^\pi(u, X_u, Y_{u-})\frac{\partial}{\partial x}\beta_p(T-t, x, i)$$

$$+\frac{1}{2}a'_u(X_u, Y_{u-})\sigma'_u\sigma_u a_u(X_u, Y_{u-})\psi^\pi(t,x,i)\frac{\partial}{\partial x^2}\beta_p(T-u, X_u, Y_{u-})$$

$$+a'_u(X_u, Y_{u-}))\sigma'_u\sigma_u a_u(X_u, Y_{u-}))\Big)\frac{\partial}{\partial x}\beta_p(T-u, X_u, Y_{u-})\frac{\partial}{\partial x}\psi^\pi(u, X_u, Y_{u-})$$

$$+\sum_{j\in E_Y}\lambda_{Y_{u-}j}\psi(T-u, X_u, j)\left(\beta_p(T-u, X_u, j) - \beta_p(T-u, X_u, Y_{u-})\right)du]$$

We apply Itô's Formula to $\Big(\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i)\Big)\psi^\pi(u,X_u^\pi,Y_{u-})$ and get

$$\Big(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i)\Big)\psi(u,X_u,Y_{u-})$$

$$=-E_{t,x,i}[\int_t^s \psi(u,X_u,Y_{u-})\frac{\partial}{\partial t}\beta_p(T-u,X_u,Y_{u-})$$

$$+\Big(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i)\Big)\frac{\partial}{\partial t}\psi(u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \Big(r_uX_u+a_u'(X_u,Y_{u-})\mu_u-c_u(X_u,Y_{u-})\Big)$$

$$\cdot\psi(u,X_u,Y_{u-})\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \Big(r_uX_u+a_u'(X_u,Y_{u-})\mu_u-c_u(X_u,Y_{u-})\Big)$$

$$\cdot\Big(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i)\Big)\frac{\partial}{\partial x}\psi(u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a_u'(X_u,Y_{u-})\sigma_u\sigma_u'a_u(X_u,Y_{u-})\psi^\pi(u,X_u,Y_{u-})\frac{\partial}{\partial x^2}\beta_p(T-u,X_u,Y_{(u-)})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a_u'(X_u,Y_{u-})\sigma_u\sigma_u'a_u(X_u,Y_{u-})\frac{\partial}{\partial x}\psi^\pi(u,X_u,Y_u)\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{(u-)})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a_u'(X_u,Y_{u-})\sigma_u\sigma_u'a_u(X_u,Y_{u-})\Big(\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i)\Big)\frac{\partial}{\partial x^2}\psi^\pi(u,X_u,Y_u)du]$$

$$+\int_t^s a_u(X_u,Y_{u-})\sigma_u\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{u-})$$

$$+a_u(X_u,Y_{u-})\sigma_u\Big(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i)\Big)\frac{\partial}{\partial x}\psi^\pi(u,X_u,Y_{u-})dW_s$$

$$+E_{t,x,i}[\sum_{t<u\le s}\Big(\left(\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i)\right)\psi^\pi(u,X_u,Y_u)$$

$$-\left(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i)\right)\psi^\pi(u,X_u,Y_{u-})\Big)].$$

It holds that

$$0=\int_t^s a_u(X_u,Y_{u-})\sigma_u\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{u-})$$

$$+a_u(X_u,Y_{u-})\sigma_u\Big(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i)\Big)\frac{\partial}{\partial x}\psi^\pi(u,X_u,Y_{u-})dW_s,$$

since

$$E_{t,x,i}[\int_t^s a_u'(X_u,Y_{u-})a_u(X_u,Y_{u-})\left(\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{u-})\right)^2 ds<\infty]$$

$$E_{t,x,i}[\int_t^s a_u'(X_u,Y_{u-})a_u(X_u,Y_{u-})\left(\frac{\partial}{\partial x}\psi^\pi(u,X_u,Y_{u-})\right)^2 ds]<\infty.$$

We resort and get

$$E_{t,x,i}[\Big(\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i)\Big)\psi(u,X_u,Y_{u-})]$$

$$= -E_{t,x,i}[\int_t^s \psi(u,X_u,Y_{u-})\frac{\partial}{\partial t}\beta_p(T-u,X_u,Y_{u-}) + \beta_p(T-u,X_u,Y_{u-})\frac{\partial}{\partial t}\psi^\pi(u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \Big(r_uX_u + a'_u(X_u,Y_{u-})\mu_u - c_u(X_u,Y_{u-})\Big)\psi^\pi(u,X_u,Y_{u-})\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \Big(r_uX_u + a'_u(X_u,Y_{u-})\mu_u - c_u(X_u,Y_{u-})\Big)\cdot\beta_p(T-u,X_u,Y_{u-})\frac{\partial}{\partial x}\psi(u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a'_u(x,i)\sigma_u\sigma'_u a_u(x,i)\psi(u,X_u,Y_{u-})\frac{\partial}{\partial x^2}\beta_p(T-u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a'_u(x,i)\sigma_u\sigma'_u a_t(x,i)\frac{\partial}{\partial x}\psi^\pi(u,X_u,Y_{u-})\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a'_u(x,i)\sigma_u\sigma'_u a_t(x,i)\beta_p(T-u,X_u,Y_{u-})\frac{\partial}{\partial x^2}\psi^\pi(u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\sum_{t<u\le s}\Big((\beta_p(T-u,X_u,Y_u) - \beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_u)$$

$$-(\beta_p(T-u,X_u,Y_{u-}) - \beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_{u-})\Big)]$$

$$E_{t,x,i}[\int_t^s (\beta(T-u,X_u,Y_{u-}) - \beta_p(T-t,x,i))\,\Big(\frac{\partial}{\partial t}\psi^\pi(u,X_u,Y_{u-})]$$

$$+\Big(r_uX_u + a'_u(X_u,Y_{u-})\mu_u - c_u(X_u,Y_u)\Big)\frac{\partial}{\partial x}\psi(u,X_u,Y_{u-})]$$

$$+\int_t^s \frac{1}{2}a'_u(X_u,Y_{u-})\sigma'_u\sigma_u a_t(X_u,Y_{u-})\frac{\partial}{\partial x^2}\psi(u,X_u,Y_{u-})du\Big)].$$

Theorem 14 implies that

$$E_{t,x,i}[\int_t^s (\beta(T-u,X_u,Y_{u-}) - \beta_p(T-t,x,i))\,\Big(\frac{\partial}{\partial t}^\pi\psi(u,X_u,Y_{u-})]$$

$$+\Big(r_uX_u + a'_u(X_u,Y_{u-})\mu_u - c_u(X_u,Y_{u-})\Big)\frac{\partial}{\partial x}\psi^\pi(u,X_u,Y_{u-})]$$

$$+\int_t^s \frac{1}{2}a'_u(X_u,Y_{u-})\sigma_u\sigma'_u a_t(X_u,Y_{u-})\frac{\partial}{\partial x^2}\psi(u,X_u,Y_u)du\Big)]$$

$$= E_{t,x,i}[\int_t^s (\beta(T-u,X_u,Y_{u-}) - \beta_p(T-t,x,i))\,\underbrace{A\psi(u,X_u,Y_{u-})}_{=0}du] = 0$$

So we get

$$E_{t,x,i}[\Big(\beta_p(T-u,X_u,Y_{u-}) - \beta_p(T-t,x,i)\Big)\psi^\pi(u,X_u,Y_{u-})]$$

$$= -E_{t,x,i}[\int_t^s \psi(u,X_u,Y_{u-})\frac{\partial}{\partial t}\beta_p(T-u,X_u,Y_{u-}) + \beta_p(T-u,X_u,Y_{u-})\frac{\partial}{\partial t}\psi^\pi(u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \Big(r_uX_u + a'_u(X_u,Y_{u-})\mu_u - c_u(X_u,Y_{u-})\Big)\psi^\pi(u,X_u,Y_{u-})\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \Big(r_uX_u + a'_u(X_u,Y_{u-})\mu_u - c_u(X_u,Y_{u-})\Big)\cdot\beta_p(T-u,X_u,Y_{u-})\frac{\partial}{\partial x}\psi^\pi(u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a_u'(X_u,Y_{u-})\sigma_u\sigma_u' a_u(X_u,Y_{u-})\psi^\pi(u,X_u,Y_{u-})\frac{\partial}{\partial x^2}\beta_p(T-u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a_u'(x,i)\sigma_u\sigma_u' a_t(x,i)\frac{\partial}{\partial x}\psi^\pi(u,X_u,Y_{u-})\frac{\partial}{\partial x}\beta_p(T-u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\int_t^s \frac{1}{2}a_u'(X_u,Y_{u-})\sigma_u\sigma_u' a_t(X_u,Y_{u-})\beta_p(T-u,X_u,Y_u)\frac{\partial}{\partial x^2}\psi(u,X_u,Y_{u-})du]$$

$$+E_{t,x,i}[\sum_{t<u\le s}\Big((\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_u)$$

$$-(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_{u-})\Big)]$$

Which implies that

$$E_{t,x,i}[\Big(\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i)\Big)\psi^\pi(u,X_u,Y_u)]$$

$$-E_{t,x,i}[\int_t^s A\psi^\pi(u,X_u^\pi,Y_{u-}\beta_p(T-u,X_u^\pi,Y_{u-})du]$$

$$=E_{t,x,i}[\sum_{t<u\le s}\Big((\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_u)$$

$$-(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_{u-})\Big)]$$

$$-E_{t,x,i}[\int_t^s \sum_{j\in E_Y}\lambda_{Y_{u-}j}\psi(T-u,X_u,j)\,(\beta_p(T-u,X_u,j)-\beta_p(T-u,X_u,Y_{u-}))\,du].$$

Next we compute

$$E_{t,x,i}[\sum_{t<u\le s}\Big((\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_u)$$

$$-(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_{u-})\Big)].$$

Therefore we make a partition of the interval $[t,s]$ in N parts with length h, so we get

$$E_{t,x,i}[\sum_{t<u\le s}\Big((\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_u)$$

$$-(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_{u-})\Big)]$$

$$=\lim_{h\to 0}E_{t,x,i}[\sum_{k=1}^N\sum_{t+(k-1)h<u\le t+kh}\Big((\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_u)$$

$$-(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_{u-})\Big)]$$

$$=\lim_{h\to 0}E_{t,x,i}[\sum_{k=1}^N\frac{1}{h}\sum_{t+(k-1)h<u\le t+kh}\Big((\beta_p(T-u,X_u,Y_u)-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_u)$$

$$-(\beta_p(T-u,X_u,Y_{u-})-\beta_p(T-t,x,i))\,\psi^\pi(u,X_u,Y_{u-})\Big)h]$$

From the strong Markov property and the tower property for conditional expectation we have that $E_{t,x,i}[\cdot] = E_{t,x,i}[E_{s,X_s,Y_{s-}}[\cdot]]$, so we get

$$\lim_{h\to 0} E_{t,x,i}[\sum_{k=1}^{N} \frac{1}{h} \sum_{t+(k-1)h<u\le t+kh} \Big((\beta_p(T-u, X_u, Y_u) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_u)$$
$$- (\beta_p(T-u, X_u, Y_{u-}) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_{u-})\Big) h]$$

$$\lim_{h\to 0} E_{t,x,i}[\sum_{k=1}^{N} \frac{1}{h} E_{\kappa}[\sum_{t+(k-1)h<u\le t+kh} \Big((\beta_p(T-u, X_u, Y_u) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_u)$$
$$- (\beta_p(T-u, X_u, Y_{u-}) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_{u-})\Big)]h]$$

where $\kappa := (t+(k-1)h, X_{t+(k-1)h}, Y_{(t+(k-1)h)-})$. From Lemma 6 we know that

$$E_{\kappa}[\sum_{t+(k-1)h<u\le t+kh} \Big((\beta_p(T-u, X_u, Y_u) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_u)$$
$$- (\beta_p(T-u, X_u, Y_{u-}) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_{u-})\Big)]$$
$$= \sum_{j\in E_Y} \lambda_{ij} \big(\beta_p(T-u,x,j) - \beta_p(T-u,x,i)\big) E_{t,x,j}[U_p(X_T)].$$

Moreover Lemma 18 implies that $\lim_{h\downarrow 0} \sum_{k=1}^{N} f_h(kh)h \to \int f(s)ds$, so we get

$$\lim_{h\to 0} E_{t,x,i}[\sum_{k=1}^{N} \frac{1}{h} \sum_{t+(k-1)h<u\le t+kh} \Big((\beta_p(T-u, X_u, Y_u) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_u)$$
$$- (\beta_p(T-u, X_u, Y_{u-}) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_{u-})\Big) h]$$

$$\lim_{h\to 0} E_{t,x,i}[\sum_{k=1}^{N} \frac{1}{h} E_{\kappa}[\sum_{t+(k-1)h<u\le t+kh} \Big((\beta_p(T-u, X_u, Y_u) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_u)$$
$$- (\beta_p(T-u, X_u, Y_{u-}) - \beta_p(T-t,x,i))\, \psi^{\pi}(u, X_u, Y_{u-})\Big)]h]$$
$$= E_{t,x,i} \int_t^s \sum_{j\in E_Y} \lambda_{ij} \big(\beta_p(T-s, X_s, j) - \beta_p(T-t, X_s, Y_{s-})\big) E_{s,X_s,j}[U_p(X_T)]$$

So plugging this in and get

$$E_{t,x,i}[\Big(\beta_p(T-u, X_u, Y_u) - \beta_p(T-t,x,i)\Big)\psi^{\pi}(u, X_u, Y_u)]$$
$$-E_{t,x,i}[\int_t^s A\psi^{\pi}(u, X_u^{\pi}, Y_{u-}\beta_p(T-u, X_u^{\pi}, Y_{u-})du]$$
$$= E_{t,x,i} \int_t^s \sum_{j\in E_Y} \lambda_{ij} \big(\beta_p(T-s, X_s, j) - \beta_p(T-t, X_s, Y_{s-})\big) E_{s,X_s,j}[U_p(X_T)]$$

$$-E_{t,x,i}[\int_t^s \sum_{j\in E_Y} \lambda_{Y_{u-}j}\psi(t,X_u,j)\left(\beta_p(T-u,X_u,j)-\beta_p(T-u,X_u,Y_{u-})\right)du]$$

$$= E_{t,x,i}\int_t^s \sum_{j\in E_Y} \lambda_{ij}\big(\beta_p(T-s,X_s,j)-\beta_p(T-t,X_s,Y_{s-})\big)E_{u,X_u,j}[U_p(X_T)]U_p(X_T)]du$$

$$-E_{t,x,i}[\int_t^s \sum_{j\in E_Y} \lambda_{Y_{u-}j}\left(\beta_p(T-u,X_u,j)-\beta_p(T-u,X_u,Y_{u-})\right)E_{u,X_u,j}[U_p(X_T)]du]=0$$

□

Lemma 14. *Let the following assumption be satisfied*

$$E_{t,x,i}[\int_t^T a_s'(X_s^\pi,Y_{s-})a_s(X_s^\pi,Y_{s-}))\left(\frac{\partial}{\partial x}\beta_c(s,X_s^\pi,Y_s)\right)^2 ds]<\infty$$

$$E_{t,x,i}[\int_t^T a_s'(X_s^\pi,Y_{s-})a_s(X_s^\pi,Y_{s-})\left(\frac{\partial}{\partial x}\varphi_s^\pi(s,X_s^\pi,Y_s)\right)^2 ds]<\infty.$$

Then it holds that

$$E_{t,x,i}[\int_w^T E_{u,X_u^\pi,Y_{u-}}[\int_s^T \Big(-A^\pi(\beta_p(u-s,X_u,Y_{u-})\varphi_s^\pi(u,X_u^\pi,Y_{u-}))\Big)du]ds]$$

$$=-E_{t,x,i}[\int_w^T U_c(c_s(X_s^\pi,Y_{s-})\Big(\beta_c(w-s,X_w^\pi,Y_{w-})-\beta_c(s-t,x,i)\Big)ds],$$

where $\varphi_s^\pi(t,x,i):=E_{t,x,i}[U_c(c_s(X_s^\pi,Y_{s-})]$.

Proof.Analog to Theorem 16.

4.3 Weak Convergence

In this section we prove some weak convergence results, which help us to prove the Verification Theorem. First we present some general results for weak convergence of stochastic differential equations. We follow Kurtz and Protter (1996) chapter 8..

Let $(X_n)_{n\geq 1}$ be a sequence of semimartingales, and $(J_n)_{n\geq 1}$ be a sequence of processes in $\mathbb{D}$, where $\mathbb{D}$ denotes the space of adapted processes with càdlàg paths. Let

$$F, F^n : D_{\mathbb{R}^k}[0,\infty) \to D_{\mathbb{M}^{mk}}[0,\infty)$$

have the property that for $t > 0$ we have $F^n(X)_t = F^n(X(t))_t$ and $F(X)_t = F(X(t))_t$, which is a non-anticipation requirement. We will study equations of the type

$$X^n(t) = J^n(t) + \int_0^t F^n(X^n(s-))dZ(s)^n \tag{4.2}$$

and give conditions that imply $X^n \Rightarrow X$, where X is a solution of the limiting equation

$$X(t) = J(t) + \int_0^T F(X(s-))dZ(s.) \tag{4.3}$$

Definition 18. *Let (X_n) be a sequence of $\mathbb{R}^d$-valued semimartingales on Θ^n, $n \geq 1$ and assume $(X)_n \Rightarrow X$. The sequence X_n is good if for any sequence $(H_n)_{n\geq 1}$ of $d \times k$ matrix processes in D defined on Θ^n such that $(H^n, X^n) \to (H, X)$, then X is semimartingale and $\int_0^t H(s-)dX^n(s) \Rightarrow \int_0^t H(s-)dX(s)$*

If we assume that the solution of 4.2 are relatively compact, we have the following result:

Theorem 17. *Suppose that (J^n, X^n, Z^n) satisfies equation 4.2, that $(J^n, X^n, Z^n)$is relatively compact in the Skorohod topology for $D_{\mathbb{R}^{2k+m}}[0,\infty)$, and that $(J^n, Z^n) \Rightarrow (J, Z)$ and that $(Z^n)_{n\geq 1}$ is good. Assume further that F^n, F satisfy:*

- *if $(x_n, y_n) \to (x, y)$ in the Skorohod topology, then $(x_n, y_F^n(x_n)) \to (x, y, F(x))$ in the Skorohod topology.*

Then any limit point of the sequence $(X^n)_{n\geq 1}$ satisfies 4.3.

Proof. See Kurtz and Protter (1996) Theorem 8.1..

Our aim is to show that the wealth process under the ϵ-strategy corresponding to π convergence weakly to the wealth process under the strategy π. Let $\pi = (c, a)$ be an admissible strategy and $\pi^\epsilon = (c^\epsilon, a^\epsilon)$ the corresponding ϵ-strategy, then the wealth process (X_t^π) under π is given by a solution of the SDE

$$X_t^\pi = x \quad dX_s^\pi = X_s^\pi r + a_s^{\epsilon'}(X_s^\pi, Y_{s-})\mu_s - c_s^\epsilon(X_s^\pi, Y_{s-})ds + a_s(X_s^\pi, Y_{s-})\sigma_s dW_s, \quad s \in [t,T] \tag{4.4}$$

and the wealth process under π^ϵ which we denote by (X_s^ϵ) is given as the solution of SDE

$$X_t^\epsilon = x \quad dX_s^\epsilon = X_s^\epsilon r + a_s^{\epsilon'}(X_s^\epsilon, Y_{s-})\mu_s - c_s^\epsilon(X_s^\epsilon, Y_{s-})ds + a_s^{\epsilon'}(X_s^\epsilon, Y_{s-})\sigma_s dW_s, \quad s \in [t,T] \tag{4.5}$$

The next Lemma shows that in our case we get the weak convergence from $\lim_{\epsilon\downarrow 0} X_s^\epsilon = X_s^\pi$, $s \in [t,T]$.

Lemma 15. *Let (X_s^π) be given by 4.4 and (X_s^ϵ) e given by 4.5. Then it holds that*

$$\lim_{\epsilon\downarrow 0} E[f(X_s^\epsilon)] = E[f(X_s^\pi)], \quad \forall f \in C_b[\mathbb{R}), \quad s \in [t,T].$$

Proof:
We want to use Theorem 17, so we have to check the assumptions first. $Y_n \equiv x$ and $Z_n \equiv (s, W(s))$. Therefore $(J^n, Z^n) \Rightarrow (J, Z)$ and $(Z^n)_{n\geq 1}$ is good.
$F^n = F^\epsilon = \Big(X_s^\epsilon r + a^{\epsilon'}(s, X_s^\epsilon, Y_{s-})\mu_s - c^\epsilon(s, X_s^\epsilon, Y_{s-}), a^{\epsilon'}(s, X_s^\epsilon, Y_{s-})\sigma_s)\Big)'$. We only need the right-hand limit so it is sufficient to show that

$$\lim_{\epsilon\downarrow 0} x^\epsilon = x \Rightarrow \lim_{\epsilon\downarrow 0} F^\epsilon(x^\epsilon) = F(x),$$

Since (c,a) are càdlàg processes the condition is fulfilled and the statement follows from Theorem 17.

□

Lemma 16. *Let (X_s^π) be given by (4.4) and (X_s^ϵ) is given by (4.5) and let the following condition be satisfied*

$$E_{t,x,i}[\int_0^h a'_{t+s}(X_{t+s}^\epsilon, Y_{(t+s)-})a_{t+s}(X_{t+s}^\epsilon, Y_{(t+s)-})\left(\frac{\partial}{\partial x}\beta_p(T-t-s, X_{t+s}^\epsilon, Y_{(t+s)-})\right)^2 ds] < \infty$$

$$E_{t,x,i}[\int_0^h a'_{t+s}(X_{t+s}^\epsilon, Y_{(t+s)-})a_{t+s}(X_{t+s}^\epsilon, Y_{(t+s)-})\left(\frac{\partial}{\partial x}\psi^\pi(t+s, X_{t+s}^\epsilon, Y_{t+s})\right)^2 ds] < \infty.$$

Then it holds that

$$\lim_{h\downarrow 0}\frac{1}{h}E_{t,x,i}[\Big(\beta_p(T-t-h, X_{t+h}^\epsilon, Y_{(t+h)-}) - \beta_p(T-t,x,i)\Big)U_p(X_T^\epsilon)]$$

$$= E_{t,x,i}[U_p(X_T^\pi)\Big(-\frac{\partial}{\partial t}\beta_p(T-t,x,i) + \Big(r_t x + a'_t(x,i)\mu_t - c_t(x,i)\Big)\frac{\partial}{\partial x}\beta_p(s-t,x,i)$$

$$+\frac{1}{2}a'_t(x,i)\sigma_t\sigma'_t a_t(x,i))\frac{\partial}{\partial x^2}\beta_p(T-t,x,i)]\Big)] + a'_t(x,i)\sigma_t\sigma'_t a_t(x,i)\frac{\partial}{\partial x}\beta_p(T-t,x,i)\frac{\partial}{\partial x}\psi^\pi(t,x,i)$$

$$+\sum_{j\in E_Y}\lambda_{ij}\big(\beta_p(T-t,x,j) - \beta_p(T-t,x,i)\big)E_{t,x,j}[U_p(X_T^\pi)],$$

where $\psi^\pi(t,x,i) = E_{t,x,i}[U_p(X_T^\pi)]$.

Proof.
Follows from Lemma 10 and the fact that $\lim_{\epsilon\downarrow 0} X_s^\epsilon \overset{D}{\rightarrow} X_s^\pi$ (Lemma 15).

Lemma 17. *Let* (X_s^π) *be given by (4.4) and* (X_s^ϵ) *is given by (4.5) and let the following condition be satisfied*

$$E_{t,x,i}[\int_0^h a'_{t+u}(X_{t+u}^\epsilon, Y_{(t+u)-})a_{t+u}(X_{t+u}^\epsilon, Y_{(t+u)-})\left(\frac{\partial}{\partial x}\beta_p(s-t-u, X_{t+u}^\epsilon, Y_{(t+u)-})\right)^2 du] < \infty$$

$$E_{t,x,i}[\int_0^h a'_{t+u}(X_u^\epsilon, Y_{u-})a_{t+u}(X_{t+u}^\epsilon, Y_{(t+u)-})\left(\frac{\partial}{\partial x}\varphi_s^\pi(t+u, X_{t+u}^\epsilon, Y_{t+u})\right)^2 du] < \infty.$$

Then it holds that

$$\lim_{h\downarrow 0} E_{t,x,i}[\int_{t+h}^T \frac{\beta_c(s-t-h, X_{t+h}^\epsilon, Y_{(t+h)-}) - \beta_c(s-t,x,i)}{h} U_c(c(s, X_s^\epsilon, Y_{s-})ds]$$

$$= E_{t,x,i}[\int_t^T U_c(c(s, X_s^\pi, Y_{s-})\Big(-\frac{\partial}{\partial t}\beta_c(s-t,x,i)$$

$$+\Big(r_t x + a'_t(x,i) - c_t(x,i)\Big)\frac{\partial}{\partial x}\beta_c(s-t,x,i) + \frac{1}{2}a'_t(x,i)\sigma_t\sigma'_t a_t(x,i)\frac{\partial}{\partial x^2}\beta_c(s-t,x,i))\Big)ds]$$

$$+a'_t(x,i)\sigma_t\sigma'_t a_t(x,i)\frac{\partial}{\partial x}\int_{t+h}^T \beta_c(s-t,x,i)\frac{\partial}{\partial x}\varphi_s^\pi(t,x,i)ds$$

$$+\sum_{j\in E_Y}\lambda_{ij}\int_t^T (\beta_c(s-t,x,j) - \beta_c(s-t,x,i))\, E_{t,x,j}[U_c(c_s(X_s^\pi, Y_{s-})]ds,$$

where $\varphi_s^\pi(t,x,i) := E_{t,x,i}[U_c(c_s(X_s^\pi, Y_{s-})]$.

Proof.
Follows from Lemma 9 and the fact that $\lim_{\epsilon\downarrow 0} X_s^\epsilon \overset{D}{\to} X_s$ (Lemma 15).

4.4 Analysis

Lemma 18. *Let* f_h *be a sequence of continuous functions which converges uniformly against* f*. Further let* $\{h, 2h, \dots Nh\}$ *a partition of the intverval* $[0,T]$ *with equidistant step size* $h > 0$*, then it holds*

$$\lim_{h\downarrow 0}\sum_{k=1}^N f_h(kh)h = \int_0^T f(s)ds.$$

Proof

$$|\sum_{k=1}^N f_h(kh)h - \int_0^T f(s)ds| = |\sum_{k=1}^N f_h(kh)h - \int_0^T f(s)ds + \sum_{k=1}^N f(kh)h - \sum_{k=1}^N f(kh)h|$$

$$\leq |\sum_{k=1}^N f_h(kh)h - \sum_{k=1}^N f(kh)h| + |\int_0^T f(s)ds - \sum_{k=1}^N f(kh)h|$$

$$\leq \sum_{k=1}^N |f_h(kh) - f(kh)|h + |\int_0^T f(s)ds - \sum_{k=1}^N f(kh)h|$$

It holds that

$$\lim_{h\downarrow 0}\sum_{k=1}^{N}|f_h(kh)-f(kh)|h=0,$$

since f_h converges uniformly against f and

$$\lim_{h\downarrow 0}|\int_0^T f(s)ds-\sum_{k=1}^{N}f(kh)h|=0.$$

□

Bibliography

Ainslie, G. (1992): "Picoeconomics," *Cambridge Univerity Press.*

Barro, R. (1999): "Ramseys meets laibson in the neoclassical growth model," *The Quarterly Journal of Economics*, 114, 1125–1152.

Bernighaus, S., K. Ehrhart, and W. Güth (2005): *Strategische Spiele*, Springer.

Bertsekas, P. and S. Shreve (1978): *Stochastic Optimal Control: The Discrete Time Case*, Academic Press.

Björk, T. and A. Murgoci (2010): "A general theory of markovian time inconsistent stochastic control problems," Technical report, `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1694759`.

Björk, T., A. Murgoci, and X. Y. Zhou (2012): "Mean–variance portfolio optimization with state dependent risk aversion," *Mathematical Finance*, 22, 1–24.

Bäuerle, N. and U. Rieder (2011): *Markov Decision Processes with Applications to Finance*, Springer.

Ekeland, I. and A. Lazarek (2010): "The golden rule when preferences are time inconsistent," *Mathematics and Financial Economics*, 4, 29–55.

Ekeland, I. and A. Lazrak (2006): "Being serious about non-commitment: subgame perfect equilibrium in continuous time," Technical report, `http://arxiv.org/abs/math/0604264`.

Ekeland, I., O. Mbodji, and T. Pirvu (2012): "Time consistent portfolio management," *SIAM Journal on Financial Mathematics*, 3, 57–86.

Ekeland, I. and T. Pirvu (2008a): "Investment and consumption without commitment," *Mathematics and Financial Economics*, 2, 57–86.

Ekeland, I. and T. Pirvu (2008b): "On a non-standard stochastic control problem," Technical report, `http://arxiv.org/abs/0806.4026`.

Fleming, W. and M. Soner (2006): *Controlled Markov Processes and Viscosity Solutions*, Sprin.

Föllmer, H. and A. Schied (2004): *Stochastic finance*, Walter de Gruyter & Co., Berlin.

Frederick, S., G. Loewenstein, and T. O'Donoghue (2002): "Time discounting and time preference: A critical review," *Journal of Economic Literature*, XL(june 2002), 351–401.

Goldman, S. (1980): "Consistent plans," *The Review of Financial Studies*, 47, 533–537.

Jacod, J. and A. Shiryaev (2002): *Limit Theorems for Stochastic Processes*, Springer.

Karatzas, I., J. Lehoczky, S. Sethi, and S. Shreve (1986): "Explicit solutions of a general consumption/ investment problem," *Mathematics of Operations Research*, 11, 261–294.

Koopmans, T. (1960): "Stationary utility and impatience,," *Econometrica*, 28, 287–309.

Korn, R. and E. Korn (2001): *Option Pricing and Portfolio Optimization*, Graduate Studies in Mathematics,.

Krusell, P. and A. Smith (2003): "Consumption and saving decisions with quasi-geometric discounting," *Econometrica*, 71, 365–375.

Kurtz, T. and P. Protter (1996): "Weak convergence of stochastic integrals and differential equations," *Lecture Notes in Mathematics*, 1627/1996, 14–41, `http://www.stat.purdue.edu/research/technical_reports/pdfs/1996/tr96-15.pdf`.

Laibson, D. (1997): "Golden eggs and hyperbolic discounting," *Quarterly Journal of Economics*, 112, 443–477.

Lawler, G. (2006): *Introduction to stochastic processes*, Chapman Hall, Boca Raton.

Loewenstein, G. and D. Prelec (1992): "Anomalies in intertemporal choice: Evidence and an interpretation," *Quarterly Journal of Economics*, 57, 573–598.

Marín-Solano, J. and J. Navas (2009): "Consumption and portfolio rules for time-inconsistent investors," *European Journal of Operational Research*, 201, 860–872.

Merton, R. (1969): "Lifetime portfolio selection under uncertainty: The continuous time case," *Review of Economics and Statistics,*, 51, 247–257.

Merton, R. (1971): "Optimal consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, 3, 373–413.

Oksendal, B. (1998): *Stochastic Differential Equations*, Springer.

Pham, H. (2009): *Continuous-Time Stochastic Control and Optimization with Financial Applications*, Springer.

Phleps, E. and R. Pollak (1968): "On second-best national saving and game-equilibrium growth," *The Review of Economic Studies*, 35, 185–199.

Pirvu, T. and H. Zhang (2011): "On investment-consumption with regime-switching," Technical report, `http://arxiv.org/abs/1107.1895`.

Pollak, R. (1968): "Consistent planning," *Review of Economics and Statistics*, 35, 185–199.

Protter, P. (2005): *Stochastic Integration and Differential Equations*, Springer.

Ramsey, F. (1928): "A mathematical theory of saving," *The Economic Journal*, 38, 543–559.

Rieder, U. and C. Wopperer (2012): "Robust consumption-investment problems with random market coefficients," *Mathematics and Financial Economics.*

Strotz, R. (1955): "Myopia and inconsistency in dynamic utility maximization," *Review of Economics and Statistics*, 23, 165–180.

Vieille, N. and J. Weibull (2009): "Multiple solutions under quasi-exponential discounting," *Economic Theory*, 39, 513–526.

Wopperer, C. (2010): *Robust consumption-investment problems with stochastic coefficients*, Ph.D. thesis, University of Ulm.

Yong.J and Zhou.X.Y. (1999): *Stochastic Controls*, Springer.

Acknowledgments

First and foremost, I would like to express my gratitude to Prof. Dr. Ulrich Rieder for all his support in research and teaching and for providing me with the opportunity to prepare this doctoral thesis. Professor Rieder gave me the idea to write this thesis about non-standard portfolio optimization, taught me how to question thoughts and express ideas, thus contributing a lot to the results given in this work. Also I would like to thank Prof. Dr. Robert Stelzer for being the co examiner of this thesis and for supporting me in many ways during my time in the institute for financial mathematics. I extend my gratitude to Jun. Prof. Dr. Mikhail Urusov for many comments and fruitful discussions. My special thanks also go to Roland Wehner for proofreading this thesis. Furthermore, I want to thank my colleagues and friends for the wonderful time in Ulm. Last but not least I would like to express my deepest gratitude to my family for supporting me in so many ways.

Declaration

I hereby declare that this thesis was performed and written on my own and that references and resources used within this work have been explicitly indicated.

I am aware that making a false declaration may have serious consequences.

Ulm, November, the 7th, 2012

(Signature)

Der Inhalt der Seiten 138 und 139 wurde aus Gründen des Datenschutzes entfernt.