Universität Ulm

Fakultät für Mathematik und Wirtschaftswissenschaften

Betweenness Relations

Dissertation

zur Erlangung des Doktorgrades Dr. rer. nat. der Fakultät für Mathematik und Wirtschaftswissenschaften der Universität Ulm

vorgelegt von

Philipp Matthias Schäfer

aus Darmstadt

2012

Amtierender Dekan:

Dr. Paul Wentges

Erstgutachter:

Dr. Dieter Rautenbach

Abstract

This thesis primarily gives an overview of research on so called betweenness relations with a focus on own contributions to that subject.

Betweenness relations represent a generalisation of the geometric notion that one point can lie between two others. Formally, a betweenness relation is a set of triples such that it contains the triple (a, b, c) if and only if it contains the triple (c, b, a).

An overview over results on characterisations of betweenness relations that are induced by various mathematical structures is given.

This is followed by algorithmic considerations on recognising special subsets of betweenness relations.

Finally, results related to abstract convexity problems–path convexity and conversion processes–are presented. Though they do not fit perfectly to the previous results, they are products of research conducted with my colleagues during the last three years.

Contents

Chapter 1

Introduction

1.1 Introduction

The Euclidean plane $\mathbb{R}^2$ is made up of points. For any two points a and b we have the set of all points between them, called the line segment between a and b. We call a set of points C convex, if for any two points in it, the line between them is contained in C. These notions have been around for more than two millennia. In this work, we consider abstract notions derived from them.

Following in this chapter, we begin by introducing notation used in this text as well as providing necessary definitions and, in the case of structures of particular interest, some context about the usage of terminology in the literature. This comprises the rest of this chapter.

The second chapter deals with axiomatic characterisations of so called betweenness relations, which capture the notion, that some elements of a set lie *between* two other elements of that set, just like all points of a line in the Euclidean plane lie between the endpoints of that line.

For different types of relations, induced by various discrete structures, theorems are presented that characterise these relations in terms of axioms about their elements. The goal is always to provide a set of axioms which is as simple as possible in terms of quantity as well as the complexity of the logic required to phrase those axioms.

Proofs of the presented theorems are only provided for our own results. Those can be found in the sections on tree betweenness, on forest betweenness, on P_3 betweenness and intersection betweenness (Section 2.1.2, Section 2.1.3, Section 2.2.2, and Section 2.2.3, respectively). Most of these results are already published in [9, 13, 44, 45].

In the third chapter, we look at results answering algorithmic questions regarding the recognition of structures related to betweenness relations. The first section of this chapter deals with the recognition of subsets of tree betweenness relations. The second section then looks at an 'induced' variant of that problem, which turns out to be easier, because more structure is required to be present in such a subset. In the last section we try, given an intersection betweenness, to fit a tree betweenness with the same ground set into that intersection betweenness.

All results in this chapter are our own results and hence all proofs are provided.

In the final chapter, two sections present results loosely connected to the above chapters as both deal with some sort of convexity notions in graphs. They are included since they are also part of my work of the past three years.

The first section looks at convexity spaces induced by graphs. A characterisation of convexity spaces which can be induced by a set of paths of a graph is given, followed by a proof that deciding if a convexity space can be induced by in such a way is in P.

In the second section, we look at results about algorithmic problems regarding conversion processes in graphs with deadlines. Two results show that finding a minimum conversion set for cliques and trees with given deadline and threshold functions is solvable in polynomial time.

Again, all results in this chapter are our own results and hence all proofs are provided.

1.2 Definitions and Notation

We begin this section by introducing some notation. After that we give definitions used in this text which are already established in the literature.

Exemplary references are given in each case.

Some Notation: We denote the set of the first n natural numbers (starting with 1) by $[n]$ and the union of $[n]$ and $\{0\}$ by $[n]_0$. Given a set V and a natural number k, we denote the set of all subsets of cardinality k of V by $\binom{V}{k}$ and the power set $\{U \mid U \subseteq V\}$ of V by 2^V.

Set Systems: A *set system* is a pair $(V, \mathcal{V})$ where V is some set and $\mathcal{V}$ is a subset of the power set of V, i.e. $\mathcal{V} \subseteq 2^V$.

Graphs: With a few exceptions the terminology and notation regarding graphs follows Diestel's book "Graph Theory" [12].

A *graph* G is a pair (V, E) consisting of a vertex set V, containing its *vertices*, and an edge set E, which is a subset of $\binom{V}{2}$, the two element subsets of V, and contains its *edges*[1]. Given a graph G, we may denote its vertex set by $V(G)$ and its edge set by $E(G)$.

Given a set $\mathcal{G}$ of graphs, we define their union $\bigcup_{G \in \mathcal{G}} G$ to be the graph G with vertex set $\bigcup_{G \in \mathcal{G}} V(G)$ and edge set $\bigcup_{G \in \mathcal{G}} E(G)$.

Given a vertex $v \in V(G)$ of a graph G, we call the set

$$\{u \in V(G) \mid \{u, v\} \in E(G)\}$$

the *neighbourhood* of v in G and denote it by $N_G(v)$. Its elements are called the *neighbours* of v in G.

A vertex is called *adjacent* to another vertex, if they are neighbours. A vertex and an edge are called *incident*, if the vertex is an element of that edge.

A graph H is called a *subgraph* of G, if $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$.

We say that a graph H arises from a graph G by deleting a vertex $v \in V(G)$ from G, if $V(H) = V(G) \setminus \{v\}$ and $E(H) = E(G) \cap \binom{V(H)}{2}$ hold. We say that a graph H arises from a graph G by contracting an edge $ab \in$

[1] So we only consider simple graphs.

$E(G)$, if $V(H) = (V(G) \setminus \{a,b\}) \cup \{r_{ab}\}$ with $r_{ab} \notin V(G)$ and $E(H) = \left(E(G) \cap \binom{V(H)}{2}\right) \cup \{r_{ab}c \mid ac \in E(G) \vee bc \in E(G)\}$ hold. r_{ab} is called the contraction of the edge ab.

A graph G with $l+1$-element vertex set $\{a_0, \ldots, a_l\}$ and edge set

$$\{\{a_{i-1}, a_i\} \mid i \in [l]\}$$

is called a *path* of length l between a_0 and a_l. We write $P : a_0 \ldots a_l$ to denote a path P with above mentioned vertex and edge set. The vertices $a_1, \ldots, a_{l-1}$ are called *inner vertices* of that path, while the vertices a_0 and a_l are called the *end vertices* of that path. A subgraph P of another graph G, that is a path, is called a *path in G*. If P is a path in a graph G, such that there is no path of smaller length in G between the end vertices of P, then P is called a *shortest* path in G. A path P in a graph G is called *induced*, if there are no edges in G between two non-consecutive vertices of that path.

We call a path $a_0 \ldots a_l$ with internal vertex b an a_0-b-a_l-path.

A path of length l is called a P_{l+1}. Given a graph $P : a_0 \ldots a_l$ and a vertex b not contained in $\{a_0, \ldots, a_l\}$ we denote a path $a_0 \ldots a_l b$ by Pb and a path $ba_0 \ldots a_l$ by bP.[2]

A graph is called *connected* if there is a path between any two its vertices.

A graph with vertex set $\{a_1, \ldots, a_l\}$ and edge set

$$\{\{a_i, a_{i+1}\} \mid i \in [l-1]\} \cup \{\{a_l, a_1\}\}$$

is called a *cycle* of length l. We write $C : a_1 \ldots a_l a_1$ to denote a cycle C with above mentioned vertex and edge set. The notions of a *cycle in a graph* and an *induced cycle* in a graph are defined analogously to the respective notions regarding paths. A cycle in a graph G is called a *shortest cycle* in G if there is no cycle of smaller length in G.

A graph is called *chordal*, if it contains no induced cycles of length 4 or more.

A graph is called *complete* if every of its vertices is adjacent to every other

[2] The definitions of this paragraph differ from the ones given in [12].

of its vertices. A complete subgraph of a graph G is called a *clique* in G.

A graph is called a *tree* if it contains no cycles and is connected.

A graph is called a *forest* if it contains no cycles.

Given a set system $(V, \mathcal{V})$, we define its *intersection graph* to be the graph with vertex set $\mathcal{V}$ and edge set $\{\{A, B\} \mid A \cap B \neq \emptyset\}$.

A graph G is called a *cactus* if every edge of G is contained in at most one cycle of G (see [3]).

Given a chordal graph G, we call the intersection graph of the maximal cliques of G, which is a tree (see Theorem 4.8 in [17]), the *clique tree* of G.

The *independence number* of a graph G is the maximum cardinality of a set I of vertices of G, such that no two elements of I are neighbours. A set with these properties is called an *independent set* of vertices.

The *domination number* of a graph G is the minimum cardinality of a set D of vertices of G, such that each element of $V(G) \setminus D$ has a neighbour in D (see [3]).

The *k-domination number* of a graph G is the minimum cardinality of a set D of vertices of G, such that each element of $V(G) \setminus D$ has k neighbours in D (see [15]).

The *distance-k-domination number* of a graph G is the minimum cardinality of a set D of vertices of G, such that for each element a of $V(G) \setminus D$ there is a path $a_0 \dots a_l$ in G with l at most k, $a = a_0$, and $a_l \in D$ (see [21]).

Metric Spaces: A *metric space* is a pair (V, δ), where δ is a function $\delta : V \times V \to \mathbb{R}$ with

- $\delta(a, b) = 0 \Leftrightarrow a = b$,
- $\delta(a, b) = \delta(b, a)$, and
- $\delta(a, c) \leq \delta(a, b) + \delta(b, c)$

for all $a, b, c \in V$. The function δ is called a metric on V.

For reference, see for example [49].

Posets: A *partially ordered set*, or short *poset*, is a pair $(V, \preceq)$ consisting of a ground set V and a *binary relation* $\preceq$ over V, i.e. a subset of V^2, which is reflexive ($a \preceq b \Rightarrow b \preceq a$), antisymmetric ($a \preceq b \wedge b \preceq a \Rightarrow a = b$), and transitive ($a \preceq b \wedge b \preceq c \Rightarrow a \preceq c$). The binary relation $\preceq$ of a partially ordered set is called a partial order. For two elements $a, b \in V$ of a partially ordered set we denote the fact that $a \preceq b \wedge a \neq b$ holds by $a \prec b$, in case we use $\preceq$ to denote the binary relation.

A *strict partially ordered set*, or short *strict poset*, is a tuple $(V, \prec)$ consisting of a ground set V and a *binary relation* $\prec$ ovper V, i.e. a subset of V^2, which is irreflexive ($a \prec b \Rightarrow a \neq b$), antisymmetric, and transitive. The binary relation $\prec$ of a strict partially ordered set is called strict partial order.

Note that, given a poset $(V, \preceq)$, the pair $(V, \preceq \setminus \{(a, a) \mid a \in V\})$ is a strict poset and, given a strict poset $(v, \prec)$, the pair $(V, \prec \cup \{(a, a) \mid a \in V\})$ is a poset.

For reference, see for example [19].

Ternary Relation Given a set V, the set of all ordered triples with elements in V is denoted by V^3 while V_s^3 denotes the subset of V^3 of triples with pairwise distinct elements. The elements of the latter set are called *strict triples*. We call a subset of V^3 a *ternary relation* on V and a subset of V_s^3 a *strict ternary relation* on V.

Given a ternary relation $\mathcal{R}$ we denote the fact that $(a, b, c) \in \mathcal{R}$ holds for some $a, b, c \in V$ by $[a, b, c]_{\mathcal{R}}$ and $(a, b, c) \notin \mathcal{R}$ for some $a, b, c \in V$ by $\neg[a, b, c]_{\mathcal{R}}$, or just $[a, b, c]$ and $\neg[a, b, c]$, respectively, if the context makes it unambiguous.

For reference, see for example [6]: n-ary relation.

Chapter 2

Axiomatic Characterisation of Betweenness Relations

Betweenness relations, which are ternary relations, aim to capture whether a certain element of some ground set is between two others. This is a natural question to ask about three points in Euclidean geometry, as stated in the introduction. According to Chvátal [8] this notion – not necessarily called betweenness though – was first used in efforts to axiomatise geometries by Pasch [41] in 1882 and later by Peano [42] in 1889, Hilbert [22] in 1899, Veblen [50] in 1904 and Coxeter [10] in 1961.

This chapter gives an overview over the results on the axiomatisation of betweenness relations induced by various structures, inspired by the case of Euclidean geometry. We almost exclusively deal with finite structures. The beginning of the section on metric betweenness (Section 2.1) being one exception. There we deal in part with betweennesses induced by metric spaces in general, i.e. non necessarily finite. The other exception is the section on shortest paths betweenness (Section 2.1.1), where we have a theorem on infinite graphs as well.

The first section of this chapter deals with metric betweenness in general and a few special cases. The final section of this chapter hosts all results on betweenness that do not fall in the category of metric betweennesses.

Before we start we give a formal definition of betweenness and an overview over the terminology used in the literature regarding such or similar structures.

Betweenness: The following paragraphs define the term betweenness relation as it is used in this text.

As stated above, betweenness relations aim to capture the notion of one thing being between two others, which we represent by a ternary relation $\mathcal{B}$ on a ground set V, that is, if some element b of V is between elements a and c, with $a, c \in V$, we want that $[a, b, c]$ holds, but then certainly b should also be considered being between c and a, so we want $[c, b, a]$ to hold as well. Therefore, we call a ternary relation $\mathcal{B} \subseteq V^3$ a *betweenness relation* if and only if $[a, b, c]$ implies $[c, b, a]$ for all $a, b, c \in V$ and we let it inherit the prefix *strict* if $\mathcal{B} \subseteq V_s^3$.

Though all betweenness relations presented here have, by definition, the property that $[a, b, c]$ implies $[c, b, a]$, there is an axiomatic characterisation of tree betweenness by Sholander [48] which does not include the axiom $[a, b, c] \Rightarrow [c, b, a]$, while all others do.

Given a betweenness relation $\mathcal{B}$, for all $a, b, c \in V$ we denote the fact that a, b, and c are pairwise distinct and $\neg[a, b, c]$, $\neg[b, a, c]$, and $\neg[a, c, b]$ hold by $N(a, b, c)$.

Given two betweenness relations $\mathcal{B}$ and $\mathcal{B}'$ on ground sets V and V', respectively, we say that they are *isomorphic*, if there is a bijection ϕ from V to V', such that

$$[a, b, c]_{\mathcal{B}} \Leftrightarrow [\phi(a), \phi(b), \phi(c)]_{\mathcal{B}'}$$

holds for all $a, b, c \in V$.

In the literature the term betweenness is sometimes used for different notions. In the introduction of this chapter we already mentioned its use in texts on axiomatisations of geometries. Morgana and Mulder also used the term *betweenness* to refer to a ternary relation $\mathcal{B}$ on a ground set V that satisfies

- $[a, a, b]$,

- $[a, b, c] \Rightarrow [a, b, c]$,
- $[a, b, c] \wedge b \notin \{a, c\} \Rightarrow \neg[a, c, b]$, and
- $[a, b, c] \wedge [a, d, b] \Rightarrow [a, d, c]$

for all $a, b, c, d \in V$ [31].

Interval Functions and Other Related Concepts: The following paragraphs give, for reference, an overview of notions related to the concept of a betweenness relation, as found in the literature.

An equivalent way of representing the same information as a betweenness relation, which satisfies $[a, b, c] \Rightarrow [c, b, a]$, is a function from V^2 to 2^V where each pair of vertices is mapped to all the vertices that lie between them, i.e. a function $I : V^2 \to 2^V$ which satisfies $\forall a, b, \in V : I(a, b) = I(b, a)$. Given a betweenness relation $\mathcal{B} \subseteq V^3$ we define $I_{\mathcal{B}}(a, c) = \{b \in V \mid [a, b, c]\}$ to get such a function and, conversely, given a function I of that type we can obtain a betweenness relation $\mathcal{B}_I = \{(a, b, c) \mid \exists a, b, c \in V : b \in I(a, c)\}$. We call a function $I : V^2 \to 2^V$ with $I(a, b) = I(b, a)$ for all $a, b \in V$ an *interval function*.

Mulder and Nebeský (see [33, 34, 35, 36, 38, 39]) have used the term 'interval function' to denote interval functions in our sense defined by shortest paths in graphs (see Section 2.1.1), but by our train of thought, which lead us to the definition of betweenness relations, interval function is probably a fitting name for this more general concept. Interval functions are not only an equivalent way of representing the betweenness notion, but in some cases also a more convenient way. So we use whichever notion makes it easier to express ourselves at a given point.

In the literature further related concepts have been used. These are being presented in the rest of this introduction for reference and completeness of presentation.

Mulder introduced the notion of a transit function [32]: Given a ground set V a function $T : V^2 \to 2^V$ is a transit function if it satisfies the axioms

(TF_1) $a \in T(a, b)$,

(TF_2) $T(a,b) = T(b,a)$, and

(TF_3) $T(a,a) = \{a\}$

for all $a, b \in V$. These axioms are called ($t1$), ($t2$), and ($t3$) in Mulder's paper. The second of these axioms corresponds to what we have in the definition of a interval functions. As for the other two axioms, they can not be satisfied by an interval function $I_{\mathcal{B}}$ corresponding to a strict betweenness $\mathcal{B}$.

Mulder and Nebeský also introduced the notion of a geometric function [33]. According to their definition $I : V^2 \to 2^V$ is a geometric function if it satisfies the axioms

(G_1) $a, b \in I(a,b)$,

(G_2) $I(a,b) = I(b,a)$,

(G_3) $b \in I(a,c), d \in I(a,b) \Rightarrow d \in I(a,c)$,

(G_4) $b \in I(a,c) \Rightarrow I(a,b) \cap I(b,c) = \{b\}$, and

(G_5) $b \in I(a,c), \in I(a,b) \Rightarrow b \in I(d,c)$,

for all $a, b, c, d \in V$. These axioms are called ($c1$) through ($c5$) in their paper. In Section 2.1.1, we see that these axioms, which correspond to the axioms (S_1) through (S_4) in that section, are a subset of those characterising betweennesses induced by shortest paths of graphs.

Hedlíková defines what she calls ternary spaces [20]. A ternary relation $\mathcal{T}$ on a ground set V is a ternary space, if it satisfies the axioms

(TS_1) $(a,b,c) \in \mathcal{T} \Rightarrow (c,b,a) \in \mathcal{T}$,

(TS_2) $(a,b,c), (a,c,b) \in \mathcal{T} \Rightarrow b = c$,

(TS_3) $(a,b,c), (a,c,d) \in \mathcal{T} \Rightarrow (b,c,d) \in \mathcal{T}$, and

(TS_4) $(a,b,c), (a,c,d) \in \mathcal{T} \Rightarrow (a,b,d) \in \mathcal{T}$

for all $a, b, c, d \in V$. These axioms are called (T_1) through (T_4) in Hedlíková's paper. Ternary spaces are (non-strict) betweennesses in our sense satisfying additional axioms.

2.1 Metric Betweenness

Every metric space (V, δ) induces a betweenness relation

$$\mathcal{B}(\delta) = \{(a, b, c) \in V^3 \mid \delta(a, c) = \delta(a, b) + \delta(b, c)\},$$

and a strict betweenness relation

$$\mathcal{B}_s(\delta) = \mathcal{B}(\delta) \cap V_s^3.$$

We call every betweenness relation induced in this way by a metric space *metric betweenness* and every strict betweenness relation induced in this way by a metric space *strict metric betweenness*.

Metric betweenness in this general form was first and briefly studied by Menger [30], who showed that metric betweennesses have the following properties:

Proposition 1. *(*Menger 1928, [30]*)*

Let $\mathcal{B}$ be a strict metric betweennesses on a ground set V (not necessarily finite), then

(M_1) $[a, b, c] \Rightarrow [c, b, a]$,

(M_2) $[a, b, c] \Rightarrow \neg[b, a, c], \neg[a, c, b]$,

(M_3) $[a, b, c], [a, c, d] \Rightarrow [a, b, d], [b, c, d]$, *and*

(M_4) $I_{\mathcal{B}}(a, b)$ *is closed*[1]

hold for all $a, b, c, d \in V$.

In Menger's paper these properties were called 1., 2., 3. and 4.. Property (M_4) is clearly not applicable when it comes to characterising metric betweennesses among ternary relations, because you need the metric to state that property. Chvátal [8] picked up Menger's work and proved that (M_1)

[1]Here, *closed* means with respect to the topology defined by the metric space on V, see for example in [49].

through (M_3) (called $(M2)$, $(M3)$ and $(M4)$ in his paper) are sufficient to characterise the strict metric betweennesses on ground sets V with $|V| = 5$ among all subsets of V_s^3:

Theorem 2. *(*Chvátal 2002, [8]*)*

Let V be a set with five elements and $\mathcal{B} \subseteq V_s^3$. $\mathcal{B}$ is a strict metric betweenness if and only if (M_1), (M_2) and (M_3) are satisfied.

Now given $(a, b, c) \in V_s^3$ let $v(a, b, c) \in \{0, 1, -1\}^{\binom{V}{2}}$ be defined by

$$v(a,b,c)_S = \begin{cases} 1 & , S = \{a, b\} \vee S = \{b, c\} \\ -1 & , S = \{a, c\} \\ 0 & , \text{otherwise.} \end{cases}$$

He goes on to observe that the following holds.

Let V be a set and $\mathcal{B} \subseteq V_s^3$. $\mathcal{B}$ is a strict metric betweenness if and only if it satisfies (M_1), (M_2) and there is a solution δ to the problem

$$\begin{aligned} v(a,b,c)^T \rho = 0, &\qquad \forall (a,b,c) \in \mathcal{B} \\ v(a,b,c)^T \rho > 0, &\qquad \forall (a,b,c) \in V_s^3 \setminus \mathcal{B}. \end{aligned}$$

Clearly a metric δ on V inducing $\mathcal{B}$ as a strict metric betweenness viewed as a vector with index set $\binom{V}{2}$ would solve the problem, and any solution ρ yields a metric δ_ρ on V by (M_2):

$$\delta_\rho(a,b) = \begin{cases} 0 & , a = b \\ \rho_{\{a,b\}} & , \text{otherwise.} \end{cases}$$

Now, Chvátal mentions that we can determine whether the above problem has a solution in polynomial time using Khachiyan's algorithm [23][2]. The existence of such a polynomial time algorithm implies the following theorem.

Theorem 3. *(*Chvátal 2002, [8]*) Metric betweenness can be recognised in polynomial time.*

[2] An English paper on the algorithm and proof of validity was published by Aspvall and Stone [2]. It is also commonly found in the literature under the name Ellipsoid Method

Using the notation introduced above we define $\mathcal{F}(V)$ to be the set of pairs $(\mathcal{B}^+, \mathcal{B}^-)$, such that there is a solution $\lambda \in \mathbb{R}^{V_s^3}$ of the equation

$$\sum_{(a,b,c)\in V_s^3} \lambda(a,b,c)v(a,b,c) = 0$$

with

$$\begin{aligned} \mathcal{B}^+ &= \{(a,b,c) \in V_s^3 \mid \lambda(a,b,c) > 0\}\text{and} \\ \mathcal{B}^- &= \{(a,b,c) \in V_s^3 \mid \lambda(a,b,c) < 0\}. \end{aligned}$$

Using Theorem 3 Chvátal then proved:

Theorem 4. *(*Chvátal 2002, [8]*)*
Let V be a set and $\mathcal{B} \subseteq V_s^3$. $\mathcal{B}$ is a strict metric betweenness if and only if (M_1) is satisfied and for every $(\mathcal{B}^+, \mathcal{B}^-) \in \mathcal{F}(V)$ we have that $\mathcal{B}^- \subseteq \mathcal{B}$ implies $\mathcal{B}^+ \subseteq \mathcal{B}$.

This concludes the part on general metric betweennesses. The following subsections each deal with a special case of metric betweenness.

We start with shortest path betweenness followed by tree betweenness, which is a special case of shortest path betweenness. Forest betweenness is dealt with in the subsequent section, a more general case than tree betweennesses. In the final two subsections on metric betweennesses we look at (strict) order betweennesses, which are those induced by posets, and ℓ_1- and ℓ_2-betweennesses. For order betweenness it is not immediately clear why it belongs into this section, but we will look at a result that shows that it does.

Remark 1. *Proofs of characterisations of betweennesses induced in some way by graphs often use a structure defined using the triples of the ternary relation similar to paths in graphs.*

You define a sequence $a_1, \ldots, a_l$ of elements of the ground set to be such a $\mathcal{B}$-path in a betweenness $\mathcal{B}$ on V, if

- $\forall i \in [l-1] \nexists c \in V : [a_i, c, a_{i+1}]$ *and*

- $\forall i \in [l-2] : [a_i, a_{i+1}, a_{i+2}]$

hold.

These or similarly defined structures were called chains [20] by Hedlíková, paths by Nebeský [36], processes by Mulder and Nebesky [38], [33], and $\mathcal{B}$-paths and chains by Düntsch and Urquhart [14].

The characterisation proofs usually use the fact that the axioms imply that $\mathcal{B}$-path correspond to paths in a graph G, where G has the ground set of $\mathcal{B}$ as the vertex set and an edge between two vertices a, c if and only if there is no b, such that $(a, b, c) \in \mathcal{B}$.

2.1.1 Shortest Path Betweenness

Every graph G induces a metric δ_G on its vertex set $V(G)$: For all $a, b \in V(G)$, let $\delta_G(a, b)$ be the length of a shortest path between vertices a and b of G. By way of this observation we see that G induces a metric betweenness

$$\mathcal{B}(G) = \{(a, b, c) \in V(G)^3 \mid \delta_G(a, c) = \delta_G(a, b) + \delta_G(b, c)\},$$

and a strict metric betweenness

$$\mathcal{B}_s(G) = \mathcal{B}(G) \cap V_s^3.$$

We call every betweenness relation induced in this way by a graph *shortest path betweenness* and every strict betweenness relation induced in this way by a graph *strict shortest path betweenness.*

Shortest path betweenness was primarily studied by Nebeský, first alone ([34, 35, 36, 38, 39]) and then together with Mulder ([33]), under the names *interval function* and *B-structure*. The study of this subject began with [34], where he gave a proof of the following theorem:

Theorem 5. *(*Nebeský 1994, [34]*)*

Let $\mathcal{B}$ be a ternary relation on a ground set V. $\mathcal{B}$ is a shortest path betweenness induced by a connected graph if and only if it satisfies the axioms

(S_1) $[a, b, c] \Rightarrow [c, b, a]$,

(S_2) $[a, a, b]$,

(S_3) $[a, b, c] \Rightarrow |I_\mathcal{B}(a, b) \cap I_\mathcal{B}(b, c)| = 1$,

(S_4) $[a, b, c], [b, d, c] \Rightarrow [a, d, c], [a, b, d]$,

(S_5) $|I_\mathcal{B}(a, b)| = 2, |I_\mathcal{B}(c, d)| = 2, [a, b, c], [a, d, c], [b, a, d] \Rightarrow [b, c, d]$, *and*

(S_6) $|I_\mathcal{B}(a, b)| = 2, |I_\mathcal{B}(c, d)| = 2, [a, b, c], \neg[a, d, c], \neg[a, b, d] \Rightarrow [b, c, d]$

for all $a, b, c, d \in V$.

Remark 2. *Given a ternary relation* $\mathcal{B}$*, we can define a graph* $G(\mathcal{B})$*, with* $V(G)$ *being the ground set of* $\mathcal{B}$ *and* $E(G)$ *being the set* $\{\{a, b\} \mid \nexists c \in V(G) : [a, b, c]\}$*. If* $\mathcal{B}$ *satisfies* (S_1) *through* (S_4)*, then* G *is connected, which is why we have to have the condition of connectedness in the above theorem (see also Corollary 7 in [38]).*

In [35] he published a slightly altered form of that theorem, where the main improvement lay in replacing axiom (S_5) by the slightly more general axiom (S_5a):

Theorem 6. *(*Nebeský 1998, [35]*)*

Let $\mathcal{B}$ *be a ternary relation on a ground set* V*.* $\mathcal{B}$ *is a shortest path betweenness if and only if it satisfies the axioms* (S_1), (S_2), (S_4), (S_6),

(S_3a) $|I(a, a)| = 1$, *and*

(S_5a) $|I_\mathcal{B}(a, b)| = 2, |I_\mathcal{B}(c, d)| = 2, [a, b, c], [b, a, d] \Rightarrow [b, c, d]$

for all $a, b, c, d \in V$.

This line of work was continued by a proof that also characterised shortest path betweenness induced by infinite graphs ([36]).

Theorem 7. *(*Nebeský 2001, [36]*)*

Let $\mathcal{B}$ *be a ternary relation on a (not necessarily finite) ground set* V*.* $\mathcal{B}$ *is a shortest path betweenness if and only if it satisfies the axioms* (S_1), (S_2), (S_3a), (S_4), (S_5), (S_6), *and*

(S_7) $a \neq c \Rightarrow (\exists e \in V : [a, e, c] \wedge (\nexists f : [a, f, e]))$

for all $a, b, c, d \in V$.

The other papers by Nebeský and Mulder publish simpler proofs or slight improvements of the above mentioned theorems.

2.1.2 Tree Betweenness

As every tree T is also a graph it induces a metric δ_G on its vertex set $V(T)$ Thus T induces a metric betweenness in the same way a graph does via its shortest paths. We call every shortest path betweenness induced by a tree a *tree betweenness* and every strict shortest path betweenness induced by a tree a *strict tree betweenness*.

What distinguishes trees from general graphs is that there is always only one induced and thus shortest path between any two vertices of the graph.[3]

In 1952 Sholander considered collections of so called segments, subsets of a ground set V, one for each ordered pair of elements of V, and called those collections trees, if they satisfied certain axioms [48]. He stated without proof that these are also trees in the sense defined by Kőnig [25][4], which is the definition given in Chapter 1 and which provides a definition for what is commonly understood if referred to by a tree today.

Let V be some set, then we call a function $t : V^2 \to 2^V$ a *Sholander tree* if and only if it satisfies the following three axioms:

(ST_1) $\forall a, b, c \in V\ \exists d \in V : t(a, b) \cap t(b, c) = t(b, d)$

(ST_2) $\forall a, b, c : t(a, b) \subseteq t(a, c) \Rightarrow t(a, b) \cap t(b, c) = \{b\}$

(ST_3) $\forall a, b, c : t(a, b) \cap t(b, c) = \{b\} \Rightarrow t(a, b) \cup t(b, c) = t(a, c)$

The above axioms were called (S), (T), and (U_1) respectively in [48].

Sholander trees have the structure of an interval function, but, given a Sholander tree t on a ground set V, what is missing in their definition is

[3]Note that there a graphs which have only one induced path between any two vertices but are not trees, for example any chordal cactus that is not a tree.

[4]This definition is the same used commonly today to define trees, see for example [12].

the requirement of satisfying $t(a,b) = t(b,a)$ for all $a,b \in V$. Sholander, however, proved in his paper, that (ST_1) and (ST_2) imply this property. Thus Sholander trees are an interval function and thus are equivalent to a betweenness relation.

The following lemma states this and some other deductions from the premises (ST_1), (ST_2), and (ST_3), which we use later on.

Lemma 8. *(*Sholander 1952, [48]*)*

Let t be a Sholander tree on the ground set V. Then it satisfies the following statements:

(SH_4) $\forall a,b : b \in t(a,b)$

(SH_5) $\forall a,b : t(a,b) = t(b,a)$

(SH_6) $\forall a,b,c \in V : b \in t(a,b) \Leftrightarrow t(a,b) \subseteq t(a,c)$

(SH_7) $\forall a,b,c \in V : (b \in t(a,c) \wedge c \in t(a,b)) \Rightarrow b = c$

(SH_8) $\forall a,b,c,d \in V : t(a,b) \cap t(b,c) = t(b,d) \Rightarrow t(a,d) \cap t(d,c) = \{d\}$

(SH_9) $\forall a,b,c \in V : b \in t(a,c) \Leftrightarrow t(a,b) \cap t(b,c) = \{b\} \Leftrightarrow t(a,b) \cup t(b,c) = t(a,c)$

(SH_{10}) $\forall a,b,c,d \in V : c,d \in t(a,b) \Rightarrow (c \in t(a,d) \wedge d \in t(c,b)) \vee (d \in t(a,c) \wedge c \in t(d,b))$

The above statements were called (1.2), (1.4), (1.5), (1.7), (1.10), (2.1), and (5.2) respectively in [48].

Since Sholander stated that his trees are the same as trees according to Kőnig, we proved that every Sholander tree, viewed as the equivalent betweenness relation, is in fact a tree betweenness [9].

Theorem 9. *(*Chvátal, Rautenbach, Schäfer 2011, [9]*)*

Given a set V, a function $t : V^2 \to 2^V$ is a Sholander tree if and only if there is a tree T with vertex set V such that $I_{\mathcal{B}(T)}(a,b) = t(a,b)$ for all $a,b \in V$.

The following theorem gives an equivalent statement using the definitions and terminology of betweenness relations.

Theorem 10. *(*Chvátal, Rautenbach, Schäfer 2011, [9]*)*

Let $\mathcal{B}$ be a ternary relation on a ground set V. $\mathcal{B}$ is a tree betweenness if and only if it satisfies the axioms

(T_1) $\forall a, b, c \in V\ \exists d \in V\ \forall e \in V : [a, e, b], [b, e, c] \Leftrightarrow [b, e, d]$

(T_2) $\forall a, b, c, d \in V : (\forall e \in V : [a, e, b] \Rightarrow [a, e, c]) \Rightarrow ([a, d, b] \wedge [b, d, c] \Rightarrow b = d)$

(T_3) $\forall a, b, c, d \in V : ([a, d, b] \wedge [b, d, c] \Rightarrow b = d) \Rightarrow (\forall e \in V : [a, e, b] \vee [b, e, c] \Leftrightarrow [a, e, c])$

For the proof of this theorem we use the properties Sholander already proved as presented in Lemma 8. Furthermore, we use the following well known fact.[5]

Lemma 11. *If $(V, \preceq)$ is a partially ordered set and $r \in V$ such that*

(i) $\forall a \in V : r \preceq a$ *and*

(ii) $\forall a, b, c \in V : (a \preceq c \wedge b \preceq c) \Rightarrow (a \preceq b \vee b \preceq a)$,

then there is a tree T with vertex set V such that $\forall a, b \in V : a \preceq b \Leftrightarrow a \in I_{\mathcal{B}(T)}(r, b)$ holds.

Proof of Theorem 9. First, suppose that there is a tree T with vertex set V such that $I_{\mathcal{B}(T)}(a, b) = t(a, b)$ holds for all $a, b \in V$ and a function $t : V^2 \to 2^V$, then clearly t is a Sholander tree.

Finally, suppose that $t : V^2 \to 2^V$ is a Sholander tree. Let $r \in V$ and define a relation $\preceq$ on V by

$$a \preceq b \Leftrightarrow a \in t(r, b).$$

[5] Statement and proof of this lemma can be found for example as Exercise 12 in Section 2.3 on page 314 of [24] and the corresponding answer on page 558.

By (SH_4), this relation is reflexive, by (SH_7) it is antisymmetric, and by (SH_6) it is transitive. So overall $\preceq$ is a partial order.

Now by (SH_4) and (SH_5), the relation $\preceq$ has property (i) of Lemma 11 and by (SH_{10}) it has property (ii) of Lemma 11. So there is a tree T with vertex set V such that

$$\forall a \in V : t(r,a) = I_{\mathcal{B}(T)}(r,a) \tag{2.1}$$

holds.

We continue to prove a generalisation of statement (2.1):

$$\forall a, b \in V : b \in I_{\mathcal{B}(T)}(r,a) \Rightarrow t(b,a) = I_{\mathcal{B}(T)}(b,a). \tag{2.2}$$

(SH_9) implies for arbitrary $a, b \in V$ that $t(r,b) \cap t(b,a) = \{b\}$ and $t(r,b) \cup t(b,a) = t(r,a)$ hold and hence that

$$t(b,a) = (t(r,a) \setminus t(r,b)) \cup \{b\} = (I_{\mathcal{B}(T)}(r,a) \setminus I_{\mathcal{B}(T)}(r,b)) \cup \{b\} = I_{\mathcal{B}(T)}(b,a)$$

holds.

Finally we proof the conclusion of the theorem, which is a generalisation of statement (2.2).

Let $a, b \in V$. Since T contains no cycles, there is a vertex $c \in V$ such that $I_{\mathcal{B}(T)}(r,a) \cap I_{\mathcal{B}(T)}(r,b) = I_{\mathcal{B}(T)}(r,c)$ and $I_{\mathcal{B}(T)}(a,b) = I_{\mathcal{B}(T)}(a,c) \cup I_{\mathcal{B}(T)}(c,b)$ hold. By (SH_5), we have $t(a,r) \cap t(r,b) = t(r,a) \cap t(r,b) = I_{\mathcal{B}(T)}(r,a) \cap I_{\mathcal{B}(T)}(r,b) = I_{\mathcal{B}(T)}(r,c) = t(r,c)$ and so (SH_8) implies $t(a,c) \cap t(c,b) = \{c\}$. Finally, (SH_9) implies $t(a,c) \cup t(c,b) = t(a,b)$ and thus, using (2.2) and (SH_5), we get

$$\begin{aligned} I_{\mathcal{B}(T)}(a,b) &= I_{\mathcal{B}(T)}(a,c) \cup I_{\mathcal{B}(T)}(c,b) \\ &= I_{\mathcal{B}(T)}(c,a) \cup I_{\mathcal{B}(T)}(c,b) \\ &= t(c,a) \cup t(c,b) \\ &= t(a,c) \cup t(c,b) \\ &= t(a,b). \end{aligned}$$

□

A characterisation of tree betweenness more in line with the other characterisations presented here, i.e. with the axiom

$$\forall a, b, c \in V : [a, b, c] \Leftrightarrow [c, b, a],$$

was published by Defays [11]:

Theorem 12. *(*Defays 1979, [11]*)*
Let $\mathcal{B}$ be a ternary relation on a ground set V. $\mathcal{B}$ is a tree betweenness if and only if it satisfies the axioms

(T_4) $[a, b, c], [b, a, c] \Leftrightarrow a = b,$

(T_5) $[a, b, c] \Rightarrow [c, b, a],$

(T_6) $b \neq c, [a, b, c], [b, c, d] \Rightarrow [a, c, d],$

(T_7) $[a, b, c], [a, d, b] \Rightarrow [a, d, c], [d, b, c],$ *and*

(T_8) $(\exists e \in V \setminus \{a, b\} : [a, e, b]) \vee (\forall e \in V : [a, b, e] \vee [b, a, e])$

for all $a, b, c, d \in V$.

For strict tree betweenness, Burigana published the first characterisation in 2009 [5]:

Theorem 13. *(*Burigana 2009, [5]*)*
Let $\mathcal{B}_s$ be a strict ternary relation on a ground set V. $\mathcal{B}_s$ is a strict tree betweenness if and only if it satisfies the axioms

(ST_1) $[a, b, c] \Rightarrow [c, b, a],$

(ST_2) $[a, b, c] \Rightarrow \neg[b, a, c],$

(ST_3) $[a, b, c], [b, c, d] \Rightarrow [a, c, d],$

(ST_4) $[a, b, c], [a, c, d] \Rightarrow [b, c, d],$ *and*

(ST_5) $N(a, b, c) \Rightarrow \exists d \in V : [a, d, b], [b, d, c], [c, d, a]$

for all $a, b, c, d \in V$.

The proof Burigana gave in his paper almost seven pages long. In a paper by Chvátal, Rautenbach, and Schäfer [9] one can find a shorter proof of this theorem, which uses even fewer axioms:

Theorem 14. *(*Chvátal, Rautenbach, Schäfer 2011, [9]*)*

Let $\mathcal{B}_s$ *be a strict ternary relation on a ground set* V. $\mathcal{B}_s$ *is a strict tree betweenness if and only if it satisfies the axioms* (ST_1), (ST_3), (ST_4), *and*

$$(ST_5a)\ \ N(a, b, c) \Rightarrow \exists e \in V : [a, e, b], [a, e, c]$$

for all $a, b, c, d \in V$.

Moreover, these axioms are independent.

Proof. A strict tree betweenness clearly satisfies the axioms (ST_1), (ST_3), (ST_4), and (ST_5a).

For the converse, suppose that a strict ternary relation $\mathcal{B}_s$ satisfies these axioms.

We begin by showing that $\mathcal{B}_s$ has the following properties:

$$\forall a, b, c, d \in V : [a, b, c], [a, c, d] \Rightarrow [a, b, d] \tag{2.3}$$

$$\forall a, b, c, d \in V : [a, b, c], [b, c, d] \Rightarrow [a, c, d] \tag{2.4}$$

$$\forall a, b, c \in V : [a, b, c] \Rightarrow \neg[b, a, c] \tag{2.5}$$

$$\forall a, b, c, d \in V : [a, b, d], [a, c, d] \Rightarrow b = c \vee [a, b, c] \vee [a, c, b] \tag{2.6}$$

$$\forall a, b, c, d \in V : [a, b, d], [a, c, d] \Rightarrow b = c \vee [c, b, a] \vee [c, b, d] \tag{2.7}$$

$$\forall r, a, b, c, d \in V : [r, a, b], [r, a, d], [b, c, d] \Rightarrow c = a \vee [r, a, c] \tag{2.8}$$

(2.3) follows directly from (ST_4) and (ST_3).

(2.4) follows directly from (ST_1), (ST_4) and (ST_3).

For (2.5), assume the contrary, i.e. that $[a, b, c]$ and $[b, a, c]$ hold. Then (ST_1) and (ST_3) imply $[c, b, c]$, which is a contradiction to the strictness of $\mathcal{B}_s$.

The proofs for properties (2.6) and (2.7) can be found in Burigana's proof of Lemma 1 in [5].

For (2.6), assume the contrary, i.e. that $[a, b, d]$, $[a, c, d]$, $b \neq c$, $\neg[a, b, c]$, and $\neg[a, c, b]$ hold.

From $[a, b, d]$, we get $\neg[b, a, d]$ by (2.5); in turn, from $[a, c, d]$ and $\neg[b, a, d]$, we get $\neg[c, a, b]$ by (ST_1) and (ST_3). Now $N(a, b, c)$ holds and so two applications of (ST_5a) give points e and f such that $[c, e, a]$, $[c, e, b]$, $[b, f, a]$, and $[b, f, c]$ hold. From $[c, e, a]$ and $[a, c, d]$, we get $[e, c, d]$ by (ST_1) and (ST_4); in turn, from $[c, e, b]$ and $[e, c, d]$, we get $[b, c, d]$ by (ST_1) and (ST_3). Similarly, from $[b, f, a]$ and $[a, b, d]$, we get $[f, b, d]$ by (ST_1) and (ST_4); in turn, from $[b, f, c]$ and $[f, b, d]$, we get $[c, b, d]$ by (ST_1) and (ST_3). But then (2.5) is contradicted by the fact that $[b, c, d]$ and $[c, b, d]$ hold.

Property (2.7) follows from (2.6) and (ST_4).

To derive (2.8), assume that $[r, a, b]$, $[r, a, d]$, and $[b, c, d]$ hold. We define a relation $\prec$ on V:

$$x \prec y \Leftrightarrow [r, x, y].$$

This binary relation is a strict partial order: It is irreflexive and antisymmetric since $\mathcal{B}_s$ is strict and it is transitive by (2.3). By assumption, the set $\{v \in V \mid v \prec b, v \prec d\}$ is not empty; consider any of its maximal elements and call it m. By (2.6) and by maximality of m, we have $m = a$ or $a \prec m$, and so (2.3) reduces proving $c = a \vee [r, a, c]$ to proving $c = m \vee [r, m, c]$.

By maximality of m, no element v of V with $m \prec v$ satisfies $c \prec b \wedge c \prec d$; from (ST_1) and (2.3), it follows that no element v of V satisfies $[m, v, b] \wedge [m, v, d]$; since $\mathcal{B}_s$ is strict, m, b, and d are pairwise distinct; now (ST_5a) implies that at least one of $[m, b, d]$, $[b, d, m]$, and $[d, m, b]$ holds. Symmetry allows us to assume that at least one of $[m, b, d]$ and $[d, m, b]$ holds. In case $[m, b, d]$ holds, we get first $[m, b, c]$ by (2.4) and then $[r, m, c]$ by (ST_1) and (ST_3). In case $[d, m, b]$ holds, (2.7) guarantees that $c = m \vee [m, c, b] \vee [m, c, d]$ holds; if $[m, c, b] \vee [m, c, d]$ holds, then $[r, m, c]$ holds by (ST_1) and (ST_4).

Now that (2.3) through (2.8) are established, we proof the remaining implication of the assertion of the theorem by induction on $|V|$.

If $|V| = 1$ holds, then the statement is trivial.

If $|V| > 1$ holds, we choose an arbitrary element $r \in V$ and define the relation $\prec$ as above. And by the same arguments as above, we get that $\prec$ is a strict partial order.

Let $\{r_1, \ldots, r_k\}$ be the minimal elements of the strict partially ordered set $(V \setminus \{r\}, \prec)$ and let $V_i = \{r_i\} \cup \{v \in V \mid r_i \prec v\}$ for $i \in [k]$. Property (2.6) and strictness of $\mathcal{B}_s$ guarantee that the sets $V_1, \ldots, V_k$ form a partition of $V \setminus \{r\}$.

By the induction hypothesis, there are trees $T_1, \ldots, T_k$ such that the strict tree betweenness $\mathcal{B}_s(T_i)$ induced by T_i is the same as $\mathcal{B}_s \cap V_i^3$. Let T be the tree with vertex set

$$V(T) = \{r\} \cup V(T_1) \cup \ldots \cup V(T_k)$$

and edge set

$$E(T) = \{rr_1, \ldots, rr_k\} \cup E(T_1) \cup \ldots \cup E(T_k).$$

We go on to show that $\mathcal{B}_s(T) = \mathcal{B}_s$ and thus proof the theorem. To prove that, we need to show for all $a, b, c \in V$ that $[a, b, c]$ holds if and only if b is an internal vertex of the path between a and c in T. By symmetry, we only have to consider the following three cases:

Case 1: Suppose $a, c \in V_i$ holds for some $i \in [k]$. In this case the path in T joining a and c lies in T_i and the assertion follows from the induction hypothesis.

Case 2: Suppose $a = r, c \in V_i$ holds for some $i \in [k]$. If $c = r_i$, then the path in T joining a and c has no internal vertices and the minimality of r_i with respect to $\prec$ guarantees that $[a, b, c]$ can not hold.

So we assume $c \neq r_i$, which implies that if b is an internal vertex of a path in T between a an c if and only if either $r_i = b$ holds or b is an internal vertex of a path in T between r_i and c. The latter implies by Case 1 that this is true if and only if $[r_i, b, c]$ holds; (ST_1) and (2.3) then imply that $[a, b, c]$ holds. Conversely, (2.7) together with the minimality of r_i with respect to $\prec$ say that $[a, b, c]$ implies $b = r_i \vee [r_i, b, c]$, which completes the proof of this case.

Case 3: Suppose $a \in V_i, c \in V_j$ holds for some $i, j \in [k]$ with $i \neq j$. We begin by showing that $[a, r, c]$ holds. For contradiction, assume that $[a, r, c]$ does not hold. Since $a \in V_i$ and $\neg[r, r_i, c]$ holds, property (2.3) implies $\neg[r, a, c]$; similarly, $z \in V_j$ and $\neg[r, r_j, a]$ imply by (2.3) that $\neg[r, c, a]$ holds. Now (ST_5a) implies that there is a d in V such that $[r, d, a]$ and $[r, d, c]$ hold. Since $z \notin V_i$ holds, we have that $\neg[r, r_i, c]$ holds and so $d \neq r_i$. Property (2.3) together with $[r, d, c]$ and $\neg[r, r_i, c]$ implies $\neg[r, r_i, d]$. And now we know that $[r, d, a]$, $[r, r_i, a]$, $c \neq r_i$ and $\neg[r, r_i, d]$ hold, so (2.6) implies that $[r, d, r_i]$ holds, which contradicts the minimality of r_i with respect to $\prec$. So $[a, r, c]$ holds.

The vertex b is an internal vertex of the path in T between a and c if and only if $b = r$ or b is an internal vertex of the path in T between a and r or b is an internal vertex of the path in T between c and r. By Case 2 we know that this is the case if and only if $b = r$ or $[a, b, r]$ holds or $[r, b, c]$ holds. (ST_1) and property (2.3) guarantee that $b = r \vee [a, b, r] \vee [r, b, c]$ implies $[a, b, c]$; Conversely, property (2.7) guarantees that $[a, b, c]$ implies $b = r \vee [a, b, r] \vee [r, b, c]$, which completes the proof of this case.

To verify the independence of the axioms consider $V = \{a, b, c, d\}$ and the following four ternary relations on V:

$$\begin{aligned}
\mathcal{B}_{s1} &= \{(a,b,c),(a,b,d),(a,c,d),(b,c,d)\},\\
\mathcal{B}_{s3} &= \{(a,b,c),(b,c,d),(c,d,a),(d,a,b),\\
&\qquad (c,b,a),(d,c,b),(a,d,c),(b,a,d)\},\\
\mathcal{B}_{s4} &= \{(a,b,c),(a,b,d),(a,c,d),(c,b,d),\\
&\qquad (c,b,a),(d,b,a),(d,c,a),(d,b,c)\},\\
\mathcal{B}_{s5a} &= \{(a,b,c),(a,b,d),(c,b,a),(d,b,a)\}.
\end{aligned}$$

For each $i \in \{1, 3, 4, 5a\}$ the ternary relation $\mathcal{B}_i$ satisfies all axioms but (ST_i).[6] □

[6] Note, that $\mathcal{B}_{5a}$ differs from the set given in [9], which is wrong.

Using this theorem it is easy to give a characterisation of tree betweenness using similar axioms.

Corollary 15. *(*Chvátal, Rautenbach, Schäfer 2011, [9]*)*

Let $\mathcal{B}$ be a ternary relation on a ground set V. $\mathcal{B}$ is a tree betweenness if and only if it satisfies the axioms (T_4), (T_5), (T_6),

(T_7a) $[a,b,c],[a,c,d] \Rightarrow [b,c,d]$, *and*

(T_9) $N(a,b,c) \Rightarrow \exists e \in V : e \neq a, [a,e,b],[a,e,c]$

for all $a,b,c,d \in V$.

Moreover, these axioms are independent.

Proof. It is easy to verify that every tree betweenness satisfies theses axioms. So assume that $\mathcal{B}$ is a ternary relation satisfying the axioms (T_4), (T_5), (T_6), (T_7a), and (T_9).

Let $\mathcal{B}_s = \mathcal{B} \cap V_s^3$, the set of strict triples in $\mathcal{B}$. By (T_5), (T_6), (T_7a), and (T_9), $\mathcal{B}_s$ satisfies (ST_1), (ST_3), (ST_4), and (ST_5a), and so it is a strict tree betweenness. So there is a tree T such that $\mathcal{B}_s(T) = \mathcal{B}_s$.

We already know that $\mathcal{B}(T) \cap V_s^3 = \mathcal{B} \cap V_s^3$. Let $V_=$ be the set $\{(a,b,c) \in V^3 \mid a = b \vee b = c\}$. By (T_4) and (T_5) we have that $\mathcal{B} \cap (V^3 \setminus V_s^3) = V_=$ holds and it is easy to see that $\mathcal{B}(T) \cap (V^3 \setminus V_s^3) = V_=$ holds. So overall we get $\mathcal{B}(T) = \mathcal{B}$.

To verify the independence of the axioms consider $V = \{a,b,c,d\}$ and the following five ternary relations on V (for $\mathcal{B}_1$, $\mathcal{B}_3$, $\mathcal{B}_4$, and $\mathcal{B}_{5a}$ see proof of Theorem 14):

$$\mathcal{B}_4 = V^3$$
$$\mathcal{B}_5 = \mathcal{B}_{s1} \cup V_=$$
$$\mathcal{B}_6 = \mathcal{B}_{s3} \cup V_=$$
$$\mathcal{B}_{7a} = \mathcal{B}_{s4} \cup V_=$$
$$\mathcal{B}_9 = \mathcal{B}_{s5a} \cup V_=$$

For each $i \in \{4, 5, 6, 7a, 9\}$ the ternary relation $\mathcal{B}_i$ satisfies all axioms but (T_i).[7] □

Compared to Theorem 12 by Defays, the above corollary uses a weakened form of (T_7) and replaces axiom (T_8) with the axiom (T_9). Theorem 12 and Corollary 15 also imply the following proposition.

Proposition 16. *Under the assumption that (T_4), (T_5), (T_6) and (T_7a) hold for a ternary betweenness $\mathcal{B}$, the axioms (T_8) and (T_9) are equivalent.*

Induced Subbetweenness of a Tree

In this section we do not look at a characterisation of a type of betweenness relation, but of specific subsets of betweenness relations. In particular, we are going to look at induces subbetweennesses of trees as introduced by Rautenbach, Santos, Schäfer and Szwarcfiter in [45].

We call a ternary relation $\mathcal{S}$ an *(strict) induced subbetweenness of a tree* if there is a (strict) tree betweenness $\mathcal{B}$ on a ground set V and a subset U of V such that $\mathcal{S} = \mathcal{B} \cap U^3$.

The following theorem provides a characterisation of strict induced subbetweenness of a tree. The characterisation uses similar axioms as Burigana's characterisation of tree betweenness (see Theorem 13). Axiom (ST_5) is replaced by axioms (ST_5b) and (ST_5c), which handles the fact that an induced subbetweenness does not carry as much information about the original structure in comparison to a tree betweenness.

In Chapter 3 on algorithmic betweenness problems, we consider the problem of recognising induced subbetweennesses of trees comparing its complexity to a similar problem.

Theorem 17. *(Rautenbach, Santos, Schäfer and Szwarcfiter 2011, [45])*

Let $\mathcal{S}_s$ be a strict ternary relation on a ground set V. $\mathcal{S}_s$ is a strict induced subbetweenness of a tree if and only if it satisfies the axioms (ST_1), (ST_3), (ST_4),

[7] The sets given here to prove independence differ from the ones given in [9], which are wrong.

(ST_5b) $(N(a,b,c) \wedge [a,d,b]) \Rightarrow ([a,d,c] \vee [b,d,c])$, *and*

(ST_5c) $(N(a,b,c) \wedge [d,b,a]) \Rightarrow [d,b,c]$.

for all $a,b,c,d \in V$.

Proof. If $\mathcal{S}_s$ is a strict induced subbetweenness of a tree it clearly satisfies axioms (ST_1), (ST_3), (ST_4), (ST_5b), and (ST_5c).

Now assume that $\mathcal{S}_s$ satisfies axioms (ST_1), (ST_3), (ST_4), (ST_5b), and (ST_5c).

First note that axiom (ST_2), which is equivalent to (2.5) in the proof of Theorem 14, follows from (ST_1) and (ST_3) as can be seen in the proof of Theorem 14.[8]

For this proof we introduce another short hand and write $N^*(a,b,c)$ if

$$N(a,b,c) \wedge \nexists d \in V : [a,d,b] \vee [a,d,c] \vee [b,d,c]$$

holds for some $a,b,c \in V$.

Now let $\mathcal{T} = \left\{a,b,c \in \binom{V}{3} \,\middle|\, N^*(a,b,c)\right\}$ and let G be the graph with vertex set $\mathcal{T}$ and edge set $\left\{\{t,s\} \in \binom{\mathcal{T}}{2} \,\middle|\, |t \cap s| = 2\right\}$. Let $\{C_1,\dots,C_k\}$ be the set of components of G. And for a subgraph H of G let

$$N(H) = \bigcup_{t \in V(C)} t.$$

As the next step of the proof, we show that the following holds:

$$\forall i \in [k]\ \forall a,b,c \in N(C_i) : a \neq b, b \neq c, c \neq a \Rightarrow N^*(a,b,c). \tag{2.9}$$

To accomplish that, we show that the assertion

$$\forall a,b,c \in N(C) : a \neq b, b \neq c, c \neq a \Rightarrow N^*(a,b,c)$$

holds for every non-empty connected subgraph C of C_i for some $i \in [k]$ by induction over $|V(C)|$.

[8]For this reason, the theorem differs from the one published in [45].

For $|V(C)| = 1$, the desired statement holds by definition.

Now suppose that $|V(C)| \geq 2$ holds. Choose $t' = \{a', b', c'\} \in V(C)$, such that $C' = G[V(C) \setminus \{t'\}]$ is connected. Since t' has a neighbour in $V(C')$, it holds by construction that $|N(C) \setminus N(C')| \leq 1$. Let a, b, and c be three distinct elements of $N(C)$. If $a, b, c \in N(C')$ holds, the desired statement follows by induction. Hence, we may assume that $a, b \in N(C')$, $\{c\} = N(C) \setminus N(C')$, and $c = c'$ hold.

For contradiction, we assume that $N^*(a, b, c)$ does not hold.

First, we assume that $a = a'$ and $b \neq b'$ hold, which implies that $N^*(a, b', c)$ holds. By induction $N^*(a, b, b')$ holds as well. If $N(a, b, c)$ does not hold, then $N^*(a, b', c)$ and $N^*(a, b, b')$ imply $[b, a, c]$. By (ST_5c), $N^*(a, b', c)$ and $[b, a, c]$ imply $[b, a, b']$, which is a contradiction to $N^*(a, b, b')$. Hence $N(a, b, c)$ does hold.

Since $N^*(a, b, c)$ does not hold, $N^*(a, b', c)$ and $N^*(a, b, b')$ imply that there is some $d \in V$ such that $[b, d, c]$ holds. Now, by (ST_5b), $N(a, b, c)$ and $[b, d, c]$ imply $[a, d, b] \vee [a, d, c]$, which contradicts $N^*(a, b, b') \wedge N^*(a, b', c)$.

Next, we assume that $a \neq a'$ and $b \neq b'$ hold. By similar arguments as in the previous case, we obtain that $N^*(a', b', c)$ implies $N^*(a', b, c)$, and that $N^*(a', b, c)$ implies $N^*(a, b, c)$, which completes the proof of (2.9).

Let T be the graph with vertex set $V(T) = V \cup \{m_1, \ldots, m_k\}$, where $m_1, \ldots, m_k$ are k distinct elements not contained in V, and edge set

$$\begin{aligned} E(T) = \{&m_i a \mid i \in [k], a \in N(C_i)\} \cup \\ &\{ab \mid a, b \in V, \nexists d \in V : (N(a, b, d) \vee [a, d, b])\}. \end{aligned}$$

By (2.9), if $a, c \in V$ and $m \in V(T) \setminus V$ are such that $a \neq c$ and $ma, mc \in E(T)$ hold, then there is some $b \in V$ such that $N^*(a, b, c)$ and $mb \in E(T)$.

We continue by proving four claims, which imply that T is a tree.

Claim 1: If $a, b \in V$ are two distinct non adjacent vertices of T and there is no $d \in V$ such that $[a, d, b]$ holds, then there is some $c \in V$ such that $N^*(a, b, c)$ holds.

Proof of Claim 1: Let a and b be as in the statement of the claim. Since $ab \notin E(T)$ holds, there is, by definition of $E(T)$, some $c \in V$ such that $N(a,b,c)$ holds. We assume that c is chosen such that $Z(c)$ defined to be the set

$$\{d \in V \mid [a,d,c] \vee [b,d,c]\}$$

has minimum possible cardinality.

For contradiction, we assume that $N^*(a,b,c)$ does not hold, i.e. that $Z(c)$ is not empty. Since there is no $d \in V$ such that $[a,d,b]$ holds, (ST_1) and (ST_5b) imply the existence of some $d \in V$ such that $[a,d,c]$ and $[b,d,c]$ hold. If $[a,b,d]$ holds, then (ST_1), (ST_3), and $[a,d,c]$ imply $[a,b,c]$, which is a contradiction. Hence $\neg[a,b,d]$ holds and, by symmetry, $[d,a,b]$ as well, which implies $N(a,b,d)$. Let $e \in Z(d)$. If $[a,e,d]$ holds, then (ST_4) and $[a,d,c]$ imply $[e,d,c]$. Now, (ST_1), (ST_3), $[a,e,d]$, and $[e,d,c]$ imply $[a,e,c]$, i.e. $e \in Z(c)$. Similarly, if $[b,e,d]$ holds, then $[b,e,c]$ holds, i.e. $e \in Z(c)$. Altogether, $Z(e)$ is a subset of $Z(c)$. Since $e \in Z(c) \setminus Z(e)$, we obtain a contradiction to the choice of c, which completes the proof of Claim 1.

Claim 2: If $P : a_0 \dots a_l$ is a path in T such that $a_0, a_i, a_l \in V$ for some $i \in [l-1]$, then $[a_0, a_i, a_l]$ holds.

Proof of Claim 2: We proof the assertion by induction on the cardinality $v(P)$ of $V(P) \cap V$.

First, suppose that $v(P) = 3$ holds. If $l = 2$, then, by construction, $N(a_0, a_1, a_2)$ does not hold. Since $a_0a_1, a_1a_2 \in E(T)$ holds, this implies, again by construction, that $[a_0, a_1, a_2]$ holds.

If $l = 3$, then, by (ST_1), we may assume that $a_2 \in V(T) \setminus V$. By construction, there is some component C of G with $a_1, a_3 \in N(C)$. Therefore, by (2.9), there is some $c \in V$ with $N^*(a_1, a_3, c)$, which implies that $\neg[a_1, a_0, a_3]$ holds. Since $a_0a_1 \in E(T)$ holds, we get that $\neg[a_0, a_3, a_1]$ holds and that $N(a_0, a_1, a_3)$ does not hold. This implies that $[a_0, a_1, a_3]$ holds.

If $l = 4$, we have $a_1, a_3 \in V(T) \setminus V$. Since a_0 and a_2 have a common neighbour in $V(T) \setminus V$, (2.9) implies that $\neg[a_0, a_4, a_2]$ holds. Similarly, $\neg[a_2, a_0, a_4]$ holds. For contradiction, we assume that $\neg[a_0, a_2, a_4]$ holds, i.e. $N(a_0, a_2, a_4)$. If $[a_0, d, a_4]$ holds for some $d \in V \setminus \{a_2\}$, then (ST_5b) implies that either $[a_0, d, a_2]$ or $[a_2, d, a_4]$ holds, which is a contradiction. Hence $N^*(a_0, a_2, a_4)$ holds. Now the construction of G implies the existence of some component C of G with $a_0, a_2, a_4 \in N(C)$, which implies the contradiction $a_1 = a_3$. Since $V(T) \setminus T$ is by construction an independent set in T, this concludes the case $v(P) = 3$.

Now let $v(P) \geq 4$. We may assume that there is some j with $j \in [i-1]$ such that $a_j \in V$ holds. By induction applied to the subpath of P between a_0 and a_i, we get that $[a_0, a_j, a_i]$ holds. Similarly, by induction applied to the subpath of P between a_j and a_l, we get that $[a_j, a_i, a_l]$ holds. Now (ST_3) implies $[a_0, a_i, a_l]$, which completes the proof of Claim 2.

Claim 3: T has no cycle.

Proof of Claim 3: For contradiction, we assume that G contains a cycle D. Since $V(T) \setminus V$ is an independent set in T, the cycle D contains at least two elements of V. We first assume that D contains exactly two elements b and c of V. If $D : bcmb$ for some $m \in V(T) \setminus V$, then $bc \in E(T)$ implies that there is no $a \in V$ with $N(a, b, c)$, which is a contradiction. If $D : bm_1cm_2b$ for some $m_1, m_2 \in V(T) \setminus V$, then (2.9) implies that there is exactly one component C of G with $\{b, c\} \subseteq N(C)$, which implies in turn the contradiction $m_1 = m_2$. Hence D contains at least three elements a, b, and c of V. Applying Claim 2 to the path in D between a and b with c as an internal vertex implies $[a, c, b]$, which contradicts (ST_2). This completes the proof of Claim 3.

Claim 4: T is connected.

Proof of Claim 4: Since every vertex in $V(T) \setminus V$ has a neighbour in V, it suffices to show that all elements of V belong to the same component of T. For contradiction, we assume that there are two elements a and b of V that belong to different components of T. We assume that a and b are chosen such that $Z(a,b)$, defined to be the set

$$\{d \in V \mid [a,d,b]\},$$

has minimum possible cardinality. If $Z(a,b)$ is not empty, then let $d \in Z(a,b)$. By symmetry, we may assume that a and d belong to different components of T. If $e \in Z(a,d)$ holds, then (ST_1), (ST_3), (ST_4) imply $[a,e,b]$. Therefore $Z(a,d) \subseteq Z(a,e) \setminus \{d\}$ holds, which contradicts the choice of a and b. Hence $Z(a,b)$ is empty. Since $ab \notin E(T)$ holds, Claim 1 implies that there is some $c \in V$ with $N^*(a,b,c)$. Hence, by construction, a and b have a common neighbour in $V(T) \setminus V$, which contradicts our assumption. This completes the proof of Claim 4.

Altogether, we have shown that T is a tree, which implies that $\mathcal{B}_s(T) \cap V^3$ satisfy the axioms (ST_1), (ST_3), (ST_4), (ST_5b), (ST_5c). To finish the proof we show that $\mathcal{B}_s = \mathcal{B}_s(T) \cap V^3$.

By Claim 2, $\mathcal{B}_s(T) \cap V^3 \subseteq \mathcal{B}_s$ holds. Therefore, it remains to proof that $\mathcal{B}_s \subseteq \mathcal{B}_s(T) \cap V^3$ holds as well.

Let $\mathcal{B}'_s$ be the set $\mathcal{B}_s(T) \cap V^3$. For contradiction, we assume that $\mathcal{B}_s \setminus \mathcal{B}'_s$ is not empty. We choose $(a,b,c) \in \mathcal{B}_s \setminus \mathcal{B}'_s$ such that the cardinality of $Z(a,b,c)$, defined to be the set

$$\{d \in V \mid [a,d,b] \vee [b,c,d]\},$$

is minimum possible.

If $Z(a,b,c)$ is not empty, then, by symmetry, we may assume, that there is some $d \in V$ such that $[a,d,b]$ holds. This, together with (ST_3) and (ST_4), implies that $[d,b,c]$ holds. As before, we obtain that $Z(a,d,b)$ as well as $Z(d,b,c)$ are proper subsets of $Z(a,b,c)$. By the minimality of $|Z(a,b,c)|$, this implies $(a,d,b), (d,b,c) \in \mathcal{B}'_s$. By (ST_4) for $\mathcal{B}'_s$, we get that $(a,b,c) \in \mathcal{B}'_s$ holds, which is a contradiction. Hence $Z(a,b,c)$ is empty.

By Claim 2, we may assume $ab \notin E(T)$. By Claim 1, there is some $e_1 \in V$ with $N^*(a,b,e_1)$ and hence there is some $m_1 \in V(T) \setminus V$ with $a,b \in N_T(m_1)$. Since $(a,b,c) \notin \mathcal{B}'_s$ holds, the existence of m_1 implies $bc \notin E(T)$. Applying the above argument again, we obtain that there is some $m_2 \in V(T) \setminus V$ with $b, c \in N_T(m_2)$. $[a, b, c]$ and (2.9) imply that m_1 and m_2 are distinct. Hence am_1bm_2c is a path in T and $(a, b, c) \in \mathcal{B}'_s$ holds, which is a contradiction.

This completes the proof that $\mathcal{B}_s \subseteq \mathcal{B}_s(T) \cap V^3$ holds, and hence the proof of the whole theorem. □

2.1.3 Forest Betweenness

For the graph class of forests we can not use the metric that we defined for graphs in the chapter on shortest path betweenness, since a forest is not necessarily connected. However, given a forest F, we can take the metrics induced by the components and combine them to get a function δ_F that is defined on all pairs of vertices of F that belong to the same component.

In this way a forest F induces a betweenness relation

$$\begin{aligned}\mathcal{B}(F) = \ & \{(a, b, c) \in V(F)^3 \mid \delta_F(a, c) = \delta_F(a, b) + \delta_F(b, c) \\ & \text{and } a,\ b, \text{ and } c \text{ belong to the same component}\},\end{aligned}$$

and a strict betweenness relation

$$\mathcal{B}_s(F) = \mathcal{B}(F) \cap V_s^3.$$

We call every betweenness relation induced in this way by a forest *forest betweenness* and every strict betweenness relation induced in this way by a forest *strict forest betweenness.*

In [45] Rautenbach, Santos, Schäfer, and Szwarcfiter gave a characterisation of strict forest betweenness. Since any component, i.e. tree T, of a forest F only contributes triples to the betweenness induced by F, if it has at least 3 elements, we only consider forests in which every component has at least 3 elements.

To capture the concept of a component based just on knowledge of the triples of a betweenness relation we introduce the binary relation $\sim_{\mathcal{B}}$ defined on a ground set V of a ternary relation $\mathcal{B}$:

$$\forall a, b \in V: \ a \sim_{\mathcal{B}} b \Leftrightarrow (a = b) \vee (\exists c \in V : [c, a, b] \vee [a, c, b] \vee [a, b, c]).$$

Theorem 18. *(*Rautenbach, Santos, Schäfer, Szwarcfiter 2011, [45]*)*[9]

Let $\mathcal{B}_s$ be a strict ternary relation on a ground set V. $\mathcal{B}_s$ is a strict forest betweenness if and only if it satisfies for all $a, b, c, d \in V$ the axioms (ST_1), (ST_3), (ST_4), and

$$(SF_5) \ \ N(a, b, c) \wedge (a \sim_{\mathcal{B}_s} b) \wedge (a \sim_{\mathcal{B}_s} c) \Rightarrow \exists d \in V : [a, d, b], [a, d, c], [b, d, c].$$

Proof. A strict tree betweenness clearly satisfies the axioms (ST_1), (ST_3), (ST_4), and (SF_5).

Suppose $\mathcal{B}_s$ satisfies the axioms (ST_1), (ST_3), (ST_4), and (SF_5). By definition and (ST_1) the relation $\sim_{\mathcal{B}_s}$ is reflexive and symmetric. We continue with proving that it is transitive as well.

For contradiction, assume that for some $a, b, c \in V$ we have that $a \sim_{\mathcal{B}_s} b$, $b \sim_{\mathcal{B}_s} c$, and $a \not\sim_{\mathcal{B}_s} c$ hold. This implies that a, b, and c are distinct elements of V and $N(a, b, c)$. By (SF_5), there is an element d of V with $[a, d, c]$, a contradiction.

Thus, $\sim_{\mathcal{B}_s}$ is an equivalence relation. Let $V = V_1 \cup \ldots V_k$ be the partition of V into equivalence classes of $\sim_{\mathcal{B}_s}$. For $i \in [k]$, let $\mathcal{B}_{si} = \mathcal{B}_s \cap V_i^3$. Since, by definition, two elements of V appear in an element of $\mathcal{B}_s$ if and only if they belong to the same equivalence class, we have that $\mathcal{B}_s = \mathcal{B}_{s1} \cup \ldots \cup \mathcal{B}_{sk}$. For a ternary relation, who has only one equivalence class according to $\sim_{\mathcal{B}_s}$, axiom (SF_5) reduces to axiom (ST_5) and thus every $\mathcal{B}_{si}$ for $i \in [k]$ satisfies (ST_1), (ST_3), (ST_4) and (ST_5) and by Theorem 14 there is a tree T_i for every $i \in [k]$ with $\mathcal{B}_s(T_i) = \mathcal{B}_s$. Finally, let F be the disjoint union of $T_1, \ldots, T_k$ and note that $\mathcal{B}_s(F) = \mathcal{B}_{s1} \cup \ldots \cup \mathcal{B}_{sk} = \mathcal{B}_s$ holds, which completes the proof. □

Using the same arguments as in the proof of Theorem 14 one can see that the Axioms (ST_1), (ST_3), (ST_4), and (SF_5) are independent.

[9] Note that in [45] the theorem was stated using (ST_2) as well, but from the proof of Theorem 14 we can see, that (ST_1), (ST_3), and strictness already imply (ST_2).

2.1.4 ℓ_1- and ℓ_2-Betweenness

We say a metric δ on a ground set V is an ℓ_p-metric if and only if there is a bijection $\phi : V \to \mathbb{R}^m$, such that $\delta(a,b) = ||\phi(a) - \phi(b)||_p$ holds for all $a, b \in V$ where $||\cdot||_p$ is the p-norm on $\mathbb{R}^m$, i.e. given $(x_1, \ldots, x_m) \in \mathbb{R}^m$, the value of $||(x_1, \ldots, x_m)||_p$ is defined as

$$\sqrt[p]{\sum_{i \in [m]} x_i^p}.$$

Based on this definition, we call a (strict) metric betweenness a *(strict) ℓ_p-betweenness*, if it is induced by an ℓ_p-metric.

In [8], Chvátal gave a characterisation of ℓ_1-betweenness. In order to state the corresponding theorem we need a small definition.

Given a ternary relation $\mathcal{B}$ on a finite ground set V, we say that a subset S of V is convex, if and only if it satisfies

$$[a,b,c] \wedge a,c \in S \Rightarrow b \in S.$$

Theorem 19. *(*Chvátal 2004, [8]*)*

Let $\mathcal{B}_s$ be a strict ternary relation on a finite ground set V. $\mathcal{B}_s$ is a strict ℓ_1-betweenness if and only if it satisfies the axioms

(SL_1) $[a,b,c] \Rightarrow \neg[c,a,b]$ *and*

(SL_2) $\neg[a,b,c] \Rightarrow \exists S,T \subseteq V : S,T$ *convex* $\wedge S \cap T = \emptyset \wedge S \cup T = V \wedge a,c \in S \wedge b \in T$

for all $a,b,c \in V$.

From this Chvátal deduced the following corollary.

Corollary 20. *(Chvátal 2004) Every ℓ_2-betweenness on a finite set is an ℓ_1-betweenness.*

2.1.5 Order Betweenness

Every partially ordered set $P = (V, \leq)$ induces a betweenness relation

$$\mathcal{B}(P) = \{(a, b, c) \in V^3 \mid a \leq b \leq c \vee c \leq b \leq a\},$$

and a strict betweenness relation

$$\mathcal{B}_s(P) = \mathcal{B}(P) \cap V_s^3.$$

We call every betweenness relation induced in this way by a partially ordered set *order betweenness* and every strict betweenness relation induced in this way by a graph *strict order betweenness.*

Altwegg published a axiomatic characterisation of order betweenness in 1950 [1]:

Theorem 21. *(*Altwegg 1950, [1]*)*
Let $\mathcal{B}$ be a ternary relation on a ground set V. $\mathcal{B}$ is a order betweenness if and only if it satisfies the axioms

(O_1) $[a, b, c] \Rightarrow [c, b, a]$,

(O_2) $[a, a, a]$,

(O_3) $[a, b, c] \Rightarrow [a, a, b]$,

(O_4) $[a, b, a] \Rightarrow a = b$,

(O_5) $[a, b, c], [b, c, d], b \neq c \Rightarrow [a, b, d]$, *and*

(O_6) $(\forall i \in [2k+1] : [a_{i-1}, a_{i-1}, a_i]), (\forall i \in [2k] : \neg[a_{i-1}, a_i, a_{i+1}]) \Rightarrow [a_{2n}, a_0, a_1]$

for all $a, b, c, d, a_i \in V$ with $i \in [2k]_0$ and for some $k \in \mathbb{N}$.
Moreover, these axioms are independent.

Sholander improved Altwegg's result in 1952 [48]:

Theorem 22. *(*Sholander 1952, [48]*)*

Let $\mathcal{B}$ be a ternary relation on a ground set V. $\mathcal{B}$ is a order betweenness if and only if it satisfies the axioms (O_2), (O_4), (O_6), and

(O_7) $[a,b,c],[b,d,e] \Rightarrow [c,b,d] \vee [e,b,a]$

for all $a,b,c,d,e,a_i \in V$ with $i \in [2k]_0$ and for some $k \in \mathbb{N}$.

In 2000 Lihová published a characterisation of strict order betweenness completing the picture [27]. To state the theorem we define the following axioms.

(SO_1) $[a,b,c] \Rightarrow [c,b,a]$

(SO_2) $[a,b,c],[d,b,e] \Rightarrow [a,b,e] \vee [a,b,d]$

(SO_3) $[a,b,c],[b,c,d] \Rightarrow [a,b,d]$

(SO_4) $[a,b,c],[b,d,e] \Rightarrow [a,b,d] \vee [c,b,d]$

(SO_5) $[a,b,c],[b,d,e] \Rightarrow [e,b,a] \vee [c,b,d]$

(SO_6) $(\forall i \in [k-1]_0 : (a_i, y_{i+1}, a_{i+1}]), (\forall i \in [k-1] : \neg[a_{i-1}, a_i, a_{i+1}]),$
$(a_0, y_1, a_1) = (a_{k-1}, y_k, a_k) \Rightarrow k \equiv 1 \mod 2$

Theorem 23. *(*Lihová 2000, [27]*)*

Let $\mathcal{B}_s$ be a strict ternary relation on a ground set V. The following conditions are equivalent:

(i) *$\mathcal{B}_s$ is a strict order betweenness.*

(ii) *$\mathcal{B}_s$ satisfies the axioms (SO_1), (SO_2), (SO_3), (SO_4), (SO_6) for all $a,b,c,d,e,a_i,y_j \in V$ with $i \in [k]_0, j \in [k]$ and for some $k \in \mathbb{N}$.*

(iii) *$\mathcal{B}_s$ satisfies the axioms (SO_1), (SO_2), (SO_5) (SO_6) for all a, b, c, d, e, a_i, $y_j \in V$ with $i \in [k]_0, j \in [k]$ and for some $k \in \mathbb{N}$.*

Lihová's proof makes use of Altwegg's theorem (Theorem 21).

In his paper [8] Chvátal shows that every strict order betweenness is a ℓ_1-betweenness. This implies that every strict order betweenness is a metric betweenness. This is the reason for why this subsection is found in the section on metric betweennesses.

In 2006 Düntsch and Urquhart picked up on Altwegg's paper and reproved his result, Theorem 21. What they added to the theory is the following theorem [14]:

Theorem 24. *(*Düntsch and Urquhart 2006, [14]*)*

There exists no finite set S of first-order logic sentences such that, given a ternary relation $\mathcal{B}_s$, $\mathcal{B}_s$ is an order betweenness if and only if it satisfies each sentence in S.

One can find a similar result in Section 2.2.1 on induced path betweenness.

In [46] Rautenbach and Schäfer characterised the pairs (G, P) of graphs and partially ordered sets on the same ground set, whose induced strict shortest path betweenness and strict order betweenness, respectively, coincide.

2.2 Other Betweennesses

All the betweenness relations we looked at so far were metric betweennesses. The betweenness relations we consider in this section differ from them by not satisfying (M_2), that is, if we have that b is between a and c for some a, b and c in a betweenness relation we can have that a is between b and c as well, without a equalling b. For that, imagine a circle in the plane and three distinct points on it. If you just consider shortest connections for determining if one of the points lies between the other two, we have (M_2)-like behaviour again, but if we consider any connection between two points, the axiom would have to be dropped.

2.2.1 Induced Path Betweenness

Every graph G, given its set of induced paths $\mathcal{P}(G)$, induces a betweenness relation

$$\mathcal{B}(G) = \{(a_0, b, a_l) \in V(G)^3 \mid \exists P : a_0, \ldots, a_l \in \mathcal{P}(G) \text{ and } b \in V(P)\},$$

and a strict betweenness relation

$$\mathcal{B}_s(G) = \mathcal{B}(G) \cap V_s^3.$$

We call every betweenness relation induced in this way by a graph *induced path betweenness* and every strict betweenness relation induced in this way by a graph *strict induced path betweenness*.

Nebeský used the term Γ-*structure* in [37] to denote the interval functions equivalent to induced path betweennesses.

For many of the betweenness relations we looked at so far, there was always a finite set of first-order logic axioms which characterised the betweenness relations, with Order betweenness being a exception as shown by Theorem 24. Nebeský proved that there is no finite set of first-order logic axioms characterising induced path betweenness [37]:

Theorem 25. *(*Nebeský 2002, [37]*)* *There exists no finite set S of first-order logic sentences, such that, given a ternary relation $\mathcal{B}$, $\mathcal{B}$ is an induced path betweenness if and only if it satisfies each sentence in S.*

2.2.2 P_3 Betweenness

We call a ternary relation $\mathcal{B}_s \subseteq V_s^3$ a *strict P_3-betweenness* if there is a graph G, such that $[a, b, c]$ if and only if abc is a path, i.e. a P_3 subgraph, of G. A strict P_3-betweenness which has this property with respect to some graph G is denoted by $\mathcal{B}_s(G)$, the strict P_3-betweenness induced by G.

Theorem 26. *Let $\mathcal{B}_s \subseteq V_s^3$. $\mathcal{B}_s$ is a strict P_3-betweenness if and only if it satisfies*

(P_1) $[a, b, c] \Rightarrow [c, b, a]$,

(P_2) $[a, b, c], [c, d, e], b \neq d \Rightarrow [b, c, d]$,

(P_3) $[a, b, c], [b, d, e], d \neq a \Rightarrow [a, b, d]$, *and*

(P_4) $[a, b, c], [d, b, e], d \neq a \Rightarrow [d, b, a]$

for all $a, b, c, d, e \in V$. Moreover, these axioms are independent.

Proof. First suppose that $\mathcal{B}_s$ is a strict P_3-betweenness, then (P_1) through (P_4) are certainly satisfied.

Suppose next that $\mathcal{B}_s$ satisfies (P_1) through (P_4). Let G be the graph with vertex set $V(G) = V$ and edge set $E(G) = \{ab \in \binom{V}{2} \mid \exists c \in V : [a, b, c]_{\mathcal{B}_s}\}$. We prove that $\mathcal{B}_s(G) = \mathcal{B}_s$ holds.

Suppose that $[a, b, c]_{\mathcal{B}_s}$ holds, then $[a, b, c]_{\mathcal{B}_s(G)}$ holds by definition and (P_1). Suppose now that $[a, b, c]_{\mathcal{B}_s(G)}$ holds, then there is a path $P : abc$ in G. Note that this implies that $a \neq c$ holds, and hence that $([a, b, d]_{\mathcal{B}_s} \vee [b, a, d]_{\mathcal{B}_s})$ holds for some $d \in V$ and that $([c, b, e]_{\mathcal{B}_s} \vee [b, c, e]_{\mathcal{B}_s})$ holds for some $e \in V$.

If either $d = e$ or $e = a$, we are done. Otherwise, for each of the four possible outcomes one of the axioms (P_2), (P_3), and (P_4), respectively, together with (P_1) imply that $[a, b, c]_{\mathcal{B}_s}$ holds, which concludes the proof.

To prove independence of the axioms we consider four example relations $\mathcal{B}_{si} \subseteq [5]^3, i \in [4]$, such that $\mathcal{B}_{si}$ satisfies all axioms but (P_i):

$$\begin{aligned}
\mathcal{B}_{s1} &= \{(1,2,3)\}, \\
\mathcal{B}_{s2} &= \{(1,2,3),(3,4,5),(3,2,1),(5,4,3)\}, \\
\mathcal{B}_{s3} &= \{(1,2,3),(2,4,5),(3,2,1),(5,4,2)\}, \\
\mathcal{B}_{s4} &= \{(1,2,3),(4,2,3),(3,2,1),(3,2,4)\}.
\end{aligned}$$

□

2.2.3 Intersection Betweenness

Given a finite set system $(V, \mathcal{V})$, we can define a betweenness relation

$$\mathcal{B}(\mathcal{V}) = \{(a,b,c) \in \mathcal{V}^3 \mid a \cap c \subseteq b\},$$

and a strict betweenness relation

$$\mathcal{B}_s(\mathcal{V}) = \mathcal{B}(T) \cap \mathcal{V}_s^3.$$

We call every betweenness relation isomorphic to a betweenness relation induced in this way by a set system *intersection betweenness* and every strict betweenness relation isomorphic to a strict betweenness relation induced in this way by a set system *strict intersection betweenness.*[10]

The above defined notion was introduced by Burigana in 2009 [5]. In his paper he claims that the following axioms hold for every strict intersection betweenness:

(I_1) $[a,b,c] \Rightarrow [c,b,a]$

(I_2) $[a,b,c],[a,d,b] \Rightarrow [a,d,c]$

(I_3) $[d,a,e],[d,c,e],[a,b,c] \Rightarrow [d,b,e]$

[10] A difference of this definition to all the other betweenness relations considered so far is that in the case of intersection betweenness we make use of the fact that we can form the intersection of two sets, i.e. we do not look at the elements of the ground set as abstract elements but use a property–their elements–of them.

These axioms clearly hold for every intersection betweenness. In [44], Rautenbach, Santos, Schäfer, and Szwarcfiter note that (I_1) also holds for strict intersection betweenness, but that (I_2) and (I_3) do not hold for every strict intersection betweenness, because they potentially imply the existence of non strict triples. In our paper we then go on and give axiomatic characterisations for both the strict and non-strict case.

An interesting property of intersection betweenness is, that given one, say $\mathcal{B}$, it can be defined by two different finite set systems, that are not isomorphic. This differs from most kinds of betweenness relations considered so far, at least if you do not consider pathological cases, like graphs with isolated vertices. The property is due to the fact that for intersection betweenness the ground set of the ternary relation does not coincide with the ground set of the inducing structure.

Algorithm 1: Algorithm for representing intersection and strict intersection betweennesses.

Input: A ternary relation $\mathcal{B}$ on a finite ground set V that is an intersection betweenness or a strict intersection betweenness.

Output: A set system $(V', \mathcal{V}')$ such that $|V'| \leq \binom{|V|}{2}$. Furthermore, if $\mathcal{B}$ is an intersection betweenness, then $\mathcal{B}$ is isomorphic to $\mathcal{B}(\mathcal{V}')$, and if $\mathcal{B}$ is a strict intersection betweenness, then $\mathcal{B}$ is isomorphic to $\mathcal{B}_s(\mathcal{V}')$.

for $\{a, c\} \in \binom{V}{2}$ **do**
    $\mathcal{B} \leftarrow \mathcal{B} \cup \{(a, a, c), (a, c, c), (a, a, a)\}$
for $b \in V$ **do**
    $M_b \leftarrow \emptyset$
for $(a, b, c) \in \mathcal{B}$ **do**
    $M_b \leftarrow M_b \cup \{\{a, c\}\}$
$V' \leftarrow \bigcup_{b \in V} M_b$
$\mathcal{V}' \leftarrow \{M_b \mid b \in V\}$
return $(V', \mathcal{V}')$

Theorem 27. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [44]*)*

Let $\mathcal{B}$ be a ternary relation on a ground set V. $\mathcal{B}$ is an intersection betweenness if and only if it satisfies (I_1), (I_3), and

(I_4) $[a, a, b]$

for all $a, b, c, d, e \in V$*.*

In addition, if $\mathcal{B}$ *is an intersection betweenness on a ground set* V*, then there is a set system* $(V', \mathcal{V}')$ *such that* $\mathcal{B}(\mathcal{V}')$ *is isomorphic to* $\mathcal{B}$*,* $|V'| \leq \binom{|V|}{2}$ *holds, and* $\mathcal{V}'$ *can be constructed from* $\mathcal{B}$ *in polynomial time.*

Proof. If $\mathcal{B}$ is an intersection betweenness, then it certainly satisfies (I_1), (I_3), and (I_4).

Now assume that $\mathcal{B}$ is a betweenness relation that satisfies (I_1), (I_3), and (I_4). For every $b \in V$, let M_b be the set $\{\{a, c\} \mid a, c \in V \wedge [a, b, c]\}$. We show that $\mathcal{B}$ is isomorphic to $\mathcal{B}(\mathcal{M})$ with $\mathcal{M} = \{M_b \mid b \in V\}$. In order to do that, we show that $\mathcal{B} = \{(a, b, c) \mid (M_a, M_b, M_c) \in \mathcal{B}(\mathcal{M})\}$ holds. Therefore, let $(a, b, c) \in V^3$. By (I_1) and (I_4), $[a, a, c]$ and $[a, c, c]$ hold, which implies $\{a, c\} \in M_a \cap M_c$. First, we assume that $\neg[a, b, c]$ holds. By definition, this implies that $\{a, c\} \in (M_a \cap M_c) \setminus M_b$, and hence that $\neg[a, b, c]_{\mathcal{B}(\mathcal{M})}$ holds. Conversely, we assume that $[a, b, c]$ holds. For contradiction, we assume that $\neg[a, b, c]_{\mathcal{B}(\mathcal{M})}$ holds. This implies that there is some $\{d, e\} \in (M_a \cap M_c) \setminus M_b$. By definition, this implies that $[d, a, e]$ and $[d, c, e]$ hold. Now (I_3) together with $[a, b, c]$ implies that $[d, b, e]$ holds, and hence, by definition, we have $\{d, e\} \in M_b$, a contradiction.

The second part of the theorem follows from our construction of $\mathcal{M}$ and Algorithm 1. □

Theorem 28. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [44]*)*

Let $\mathcal{B}_s$ *be a strict ternary relation on a ground set* V*.* $\mathcal{B}_s$ *is a strict intersection betweenness if and only if it satisfies*

(SI_1) $[a, b, c] \Rightarrow [c, b, a]$,

(SI_2) $[a, b, c], [a, d, b], c \neq d \Rightarrow [a, d, c]$, *and*

(SI_3) $[d, a, e], [d, c, e], [a, b, c], d \neq b, b \neq e \Rightarrow [d, b, e]$

for all $a, b, c, d, e \in V$*.*

In addition if $\mathcal{B}_s$ *is a strict intersection betweenness on a ground set* V*, then there is a set system* $(V', \mathcal{V}')$ *such that* $\mathcal{B}_s(\mathcal{V}')$ *is isomorphic to* $\mathcal{B}_s$*,* $|V'| \leq \binom{|V|}{2}$*, and* $\mathcal{V}'$ *can be constructed from* $\mathcal{B}_s$ *in polynomial time.*

To prove Theorem 28, we use the following lemma, which relates strict intersection betweennesses to special intersection betweennesses containing them.

Lemma 29. *If $\mathcal{B}_s$ is a strict intersection betweenness on a ground set V, then*

$$\mathcal{B} = \mathcal{B}_s \cup \{(a,a,c) \mid a,c \in V\} \cup \{(a,c,c) \mid a,c \in V\}$$

is an intersection betweenness.

Proof. Let $(V', \mathcal{V}')$ be a set system such that $\mathcal{B}_s(\mathcal{V}')$ is isomorphic to $\mathcal{B}_s$. Let $\{b_a \mid a \in \mathcal{V}'\}$ be a set of $|\mathcal{V}'|$ distinct elements that do not belong to V', let $a' = a \cup \{b_a\}$ for $a \in \mathcal{V}'$, and let $\mathcal{V}'' = \{a' \mid a \in \mathcal{V}'\}$. Since $a' \not\subseteq c'$ and $a' \cap c' = a \cap c$ hold for every two distinct elements $a, c \in \mathcal{V}'$, we get that $\mathcal{B}_s(\mathcal{V}'')$ is isomorphic to $\mathcal{B}_s(\mathcal{V}')$, which in turn is isomorphic to $\mathcal{B}_s$, and that $\mathcal{B}(\mathcal{V}'')$ is isomorphic to $\mathcal{B}$, which completes the proof.

Note that $\mathcal{B}(\mathcal{V}')$ might contain triples of the form (a,b,a) for some $a, b \in \mathcal{V}'$ with $a \neq b$. This implies that for some strict intersection betweennesses there may be two set systems inducing them up to isomorphism that induce different intersection betweennesses. □

The following proof of Theorem 28 also shows that Algorithm 1 constructs set systems inducing betweenness relations isomorphic to the ones given as the input in polynomial time regardless of whether the input was a strict relation or not.

Proof of Theorem 28: If $\mathcal{B}_s$ is a strict intersection betweenness, then it certainly satisfies (SI_1), (SI_2), and (SI_3).

Now assume that $\mathcal{B}_s$ satisfies (SI_1), (SI_2), and (SI_3). Let $\mathcal{B}$ be defined as in Lemma 29. Clearly, $\mathcal{B}$ satisfies (I_1) and (I_4). Note that it does not contain any triple of the form (a,b,a) for $a, b \in V$ with $a \neq b$.

To prove that $\mathcal{B}$ also satisfies (I_3), let $a, b, c, d, e \in V$ such that $[d,a,e]_{\mathcal{B}}$, $[d,c,e]_{\mathcal{B}}$, and $[a,b,c]_{\mathcal{B}}$ hold.

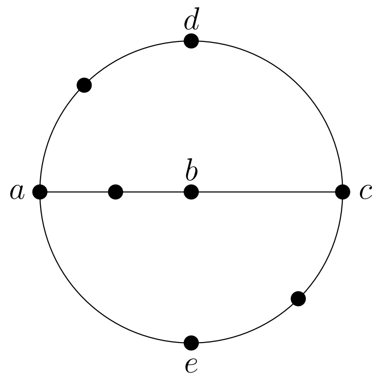

Figure 2.1: A distance theta

If $a = b$ or $c = b$ hold then trivially $[d, b, e]_{\mathcal{B}}$ holds. Hence, we may assume that neither $a \neq b$ nor $c \neq b$ holds. The definition of $\mathcal{B}$ and $[a, b, c]_{\mathcal{B}}$ imply that $a \neq c$ holds as well, i.e. (a, b, c) is a strict triple.

If $d = b$ or $e = b$ holds, then the definition of $\mathcal{B}$ implies $[d, b, e]_{\mathcal{B}}$. Hence, we may assume that $d \neq b$ and $e \neq b$ hold.

If $d = a$ holds, then $[a, c, e]_{\mathcal{B}}$ holds, which together with $a \neq c$ implies $a \neq e$. If $c = e$ holds, then $(d, b, e) = (a, b, c)$ and thus $[d, b, e]_{\mathcal{B}}$ would hold. Thus, we may assume that $c \neq e$ holds, which would make (a, c, e) a strict triple. Since $b \neq e$ holds, (SI_2) implies that $[a, b, e]_{\mathcal{B}}$ holds, which together with $d = a$ implies $[d, b, e]_{\mathcal{B}}$. Hence, we may assume that $d \neq a$ holds, and by symmetry, that $d \neq c$, $e \neq a$, and $e \neq c$ hold. It follows that the tree triples (d, a, e), (d, c, e), and (a, b, c) are strict. This together with (SI_3) implies that $[d, b, e]_{\mathcal{B}}$ holds. So $\mathcal{B}$ satisfies (I_3).

By Theorem 27, there is a set system $(V', \mathcal{V}')$ such that $\mathcal{B}(\mathcal{V}')$ is isomorphic to $\mathcal{B}$ and $|V'| \leq \binom{|V|}{2}$, which can be constructed in polynomial time by Algorithm 1. Since $\mathcal{B}_s = \mathcal{B} \cap V_s^3$, it follows that $\mathcal{B}_s$ is isomorphic to $\mathcal{B}_s(\mathcal{V}')$, which completes the proof. $\square$

Another result of [44] is a characterisations using forbidden induced subgraphs of those graphs, whose strict shortest path betweenness is also an intersection betweenness, similar to the result mentioned at the end of the section on order betweenness.

To state the theorem, we first have to introduce the notion of a distance theta in a graph, which are used as a forbidden substructure in the theorem. A *distance theta* (see Figure 2.1) in a graph G is a subgraph H of G that is a subdivision of the complete bipartite graph with partite sets $\{a, c\}$ and $\{d, b, e\}$, where a, b, c, d, and e are distinct vertices, such that the following four conditions are satisfied:

- The path in H between d and e containing a but not b has length $\delta_G(d, e)$.[11]
- The path in H between d and e containing c but not b has length $\delta_G(d, e)$.
- The path in H between a and c containing b has length $\delta_G(a, c)$.
- $\delta_G(d, e) < \delta_G(d, b) + \delta_G(b, e)$

Theorem 30. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [44]*)*
Let G be a graph. The strict shortest path betweenness induced by G is a strict intersection betweenness if and only if G does not contain a distance theta as an induced subgraph.

Before we proof the theorem. We mention the following immediate corollary.

Corollary 31. *Every strict tree betweenness is an intersection betweenness.*

The converse is not true as we see in Section 3.3 where we look at an algorithm that tries to embed a strict tree betweenness in a given strict intersection betweenness.

Proof of Theorem 30. Let $\mathcal{B}_s$ denote the strict shortest path betweenness induced by G. Clearly, $\mathcal{B}_s$ satisfies (SI_1). We continue to show that $\mathcal{B}_s$ also satisfies (SI_2). Suppose $[a, b, c]$ and $[a, e, b]$ hold such that $c \neq e$ holds. By

[11] δ_G is the metric induced by a graph G via its shortest paths, see Section 2.1.1.

definition,

$$
\begin{aligned}
\delta_G(a,c) &= \delta_G(a,b) + \delta_G(b,c) \\
&= \delta_G(a,e) + \delta_G(e,b) + \delta_G(b,c) \\
&\geq \delta_G(a,e) + \delta_G(e,c)
\end{aligned}
$$

and

$$\delta_G(a,c) \leq \delta_G(a,e) + \delta_G(e,c)$$

hold, which implies that $\delta_G(a,c) = \delta_G(a,e) + \delta_G(e,c)$ holds and hence $[a,e,c]$ holds. Thus, $\mathcal{B}_s$ satisfies (SI_2) and, by Theorem 28, $\mathcal{B}_s$ is a strict intersection betweenness if and only if $\mathcal{B}_s$ satisfies (SI_3). Therefore, it remains to be shown that $\mathcal{B}_s$ does not satisfy (SI_3) if and only if G contains a distance theta.

If G contains a distance theta, then $[a,b,c]$, $[d,a,e]$, $[d,c,e]$, and $\neg[d,b,e]$ hold for some $a,b,c,d,e \in V(G)$, i.e. $\mathcal{B}_s$ does not satisfy (SI_3).

Conversely, suppose that $\mathcal{B}_s$ does not satisfy (SI_3), then there are five vertices, a_0, b, c_0, d_0, and e_0, in G, such that

$$
\begin{aligned}
\delta_G(d_0,e_0) &= \delta_G(d_0,a_0) + \delta_G(a_0,e_0) \\
&= \delta_G(d_0,c_0) + \delta_G(c_0,e_0), \\
\delta_G(a0,c_0) &= \delta_G(a_0,b) + \delta_G(b,c_0), \text{ and} \\
\delta_G(d_0,e_0) &< \delta_G(d_0,b) + \delta_G(b,e_0)
\end{aligned}
$$

hold.

Let Q_0 be the shortest path in G between d_0 and e_0 that contains a_0, and let R_0 be the shortest path in G between d_0 and e_0 that contains c_0.

If c_0 lies on Q_0, then we may assume by symmetry that c_0 lies between

a_0 and e_0 on Q_0, and hence that

$$\begin{aligned}\delta_G(d_0, e_0) &= \delta_G(d_0, a_0) + \delta_G(a_0, c_0) + \delta_G(c_0, e_0) \\ &= \delta_G(d_0, a_0) + \delta_G(a_0, b) + \delta_G(b, c_0) + \delta_G(c_0, e_0) \\ &\geq \delta_G(d_0, b) + \delta_G(b, e_0)\end{aligned}$$

holds, which implies a contradiction to $\delta_G(d_0, e_0) < \delta_G(d_0, b) + \delta_G(b, e_0)$. Hence c_0 does not lie on Q_0, and, by symmetry, a_0 does not lie on R_0 either.

Let C be the shortest cycle in the graph $Q_0 \cup R_0$ that contains a_0 and c_0. Note that C contains exactly two vertices, d and e, that belong to $V(Q_0) \cap V(R_0)$. We may assume by symmetry that d lies between d_0 and u_0 on Q_0. This implies that d lies between d_0 and c_0 on R_0. Let Q denote the subpath of Q_0 between d and e and let R denote the subpath of R_0 between d and e.

Let S_0 be the shortest path in G between a_0 and c_0 that contains v. If d lies on S_0, then we may assume by symmetry that b lies between d and u_0 on S_0, and hence that

$$\begin{aligned}\delta_G(d_0, e_0) &= \delta_G(d_0, d) + \delta_G(d, a_0) + \delta_G(a_0, e_0) \\ &= \delta_G(d_0, d) + \delta_G(d, b) + \delta_G(b, a_0) + \delta_G(a_0, e_0) \\ &\geq \delta_G(d_0, b) + \delta_G(b, e_0)\end{aligned}$$

holds, which is a contradiction. Therefore, d and, by symmetry, e do not lie on S_0. Furthermore, it holds by a similar argument that b does not lie on Q or R.

Let S be the shortest subpath of S_0, that contains b as an internal vertex and contains two vertices from $V(Q) \cup V(R)$. Since $\delta_G(d_0, e_0) < \delta_G(d_0, b) + \delta_G(b, e_0)$, S is a shortest path between an internal vertex, say a, of Q and an internal vertex, say c, of R.

Now the five distinct vertices a, b, c, d, and e together with the three paths Q, R, and S comprise a distance theta in G, which completes the proof. □

Chapter 3

Algorithmic Betweenness Problems

In this chapter we look at three algorithmic problems related to betweenness relations.

Before we do this, note that recognising a ternary relation as one of the specific betweenness relations presented in Chapter 2, is possible in polynomial time for every specific betweenness relation that can be characterised by a finite set of first-order logic axioms:

Consider a betweenness relation $\mathcal{B}$ on a finite ground set V. Let k be the number of bound variables in one of the axioms. To check whether $\mathcal{B}$ satisfies that axiom, we have to check for $|V|^k$ many combinations of elements, whether the axiom is satisfied. This has to be done a fixed number of times (once for each axioms), so the overall process has a running time in $O(|V|^l)$ for some $l \in \mathbb{N}$.

This does not answer, what complexity the problem of recognising a ternary relation as a specific betweenness relation has, in case the latter can not be characterised by a finite set of first-order logic axioms.

The first two problems we look at in this chapter are related in that both deal with subsets of strict tree betweennesses. The corresponding results were published by Rautenbach, Santos, Schäfer, and Szwarcfiter in 2011 [45]. The first problem considered is shown to be NP-complete, while the second can be solved in polynomial time. Though both problems are similar in nature, the second requires 'Yes'-instances of the problem to have considerably more structure, which can explain the difference in complexity.

The third problem looks at the problem of partially representing a strict intersection betweenness $\mathcal{B}$ by a strict tree betweenness of a tree with the same ground set as $\mathcal{B}$, if possible.

For definitions regarding algorithms and their complexity, we refer the reader to chapter 15 of [26].

3.1 Subbetweenness of a Tree

The first problem we consider is Subbetweenness of a tree:

Subbetweenness of a tree

Instance:	A strict ternary relation $\mathcal{B}_s$ on a ground set V.
Question:	Is there a tree T such that $\mathcal{B}_s \subseteq \mathcal{B}_s(T)$ holds?

Before we look at the NP-completeness proof, we consider the following lemma, which captures properties of trees that solve Subbetweenness of a tree.

Lemma 32. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [45]*)*
If T is a tree and $V \subseteq V(T)$ holds, then the following two assertions hold.

(i) If a has degree at most 2 for some $a \in V(T) \setminus V$ and T' is the tree we get by deleting a from T and joining all pairs of distinct neighbours of a in T by a new edge, then $\mathcal{B}_s(T) \cap V^3 = \mathcal{B}_s(T') \cap V^3$ holds.

(ii) If ab is some edge of T with $a, b \in V(T) \setminus V$ and T' is the tree we get by contracting the edge ab in T, then $\mathcal{B}_s(T) \cap V^3 = \mathcal{B}_s(T') \cap V^3$ holds.

Proof. (i) If $a \in V(T) \setminus V$ is a vertex of degree at most 1, then $\mathcal{B}_s(T') \cap V^3 = \mathcal{B}_s(T) \cap V^3$ holds because no path in T between two elements of V is influenced by the deletion.

If $a \in V(T) \setminus V$ is a vertex of degree 2, then every path in T between vertices in V contains either both or none of the two edges incident with a. Thus, $\mathcal{B}_s(T') \cap V^3 = \mathcal{B}_s(T) \cap V^3$ holds as well.

(ii) If r_{ab} denotes the contraction of the edge ab, then the paths in T that are between two vertices of V and contain a vertex in the set $\{a, b\}$ are in one to one correspondence with the paths in T' that are between two vertices of V and contain the vertex r_{ab}. Thus, $\mathcal{B}_s(T') \cap V^3 = \mathcal{B}_s(T) \cap V^3$ holds. □

To prove that Subbetweenness of a tree is NP-complete, we reduce Total ordering, which is a specialisation of our problem, to it:

Total ordering
Instance: A strict ternary relation $\mathcal{B}_s$ on a ground set V.
Question: Is there a path P such that $\mathcal{B}_s \subseteq \mathcal{B}_s(P)$ holds?

For an NP-completeness proof for Total ordering see [40].

Theorem 33. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [45]*)* Subbetweenness of a tree *is NP-complete.*

Proof. In order to prove that Subbetweenness of a tree is in NP, we assume that $\mathcal{B}_s$ is a strict ternary relation on a ground set V, such that there is some tree T with $\mathcal{B}_s \subseteq \mathcal{B}_s(T)$, i.e. that $\mathcal{B}_s$ is a 'Yes'-instance of the problem. We now want to show that T is a certificate for $\mathcal{B}_s$.

By Lemma 32, we may assume that $V(T) \setminus V$ is an independent set of vertices of degree at least 3 in T. Hence T contains at least $|V(T) \setminus V|$ end vertices, which all belong to V. This implies that the order of T is polynomially bounded in terms of $|V|$, and thus the coding length of T is polynomially bounded in $|V|$.

This also implies that we can check whether $\mathcal{B}_s \subseteq \mathcal{B}_s(T)$ in polynomial time in $|V|$. Thus T is a certificate for $\mathcal{B}_s$, and thus Subbetweenness of a tree is in NP.

We proceed with the proof by reducing the NP-complete problem Total ordering to Subbetweenness of a tree in order to show that also the latter is NP-complete.

Let $\mathcal{B}_s$ be a strict ternary relation on a ground set V, i.e an instance of TOTAL ORDERING. Let V' be the set $V \cup \{x, y\}$ for two distinct elements x, y that do not belong to V and let $\mathcal{B}'_s$ be the strict ternary relation $\mathcal{B}_s \cup \{(x, u, y) \mid u \in V\}$. It is easy to see that there is a path P such that $\mathcal{B}_s \subseteq \mathcal{B}_s(P)$ holds if and only if there is a tree T such that $\mathcal{B}'_s \subseteq \mathcal{B}_s(T)$ holds. Since we can construct $\mathcal{B}'_s$ from $\mathcal{B}_s$ in polynomial time in the coding length of $\mathcal{B}_s$, we found a polynomial reduction of TOTAL ORDERING to SUBBETWEENNESS OF A TREE, which completes the proof. □

For the TOTAL ORDERING problem there is a $\frac{1}{2}$-approximation algorithm published by Chor and Sudan in 1998 [7]. In 2009 Makarychev published a much simpler algorithm achieving the same [28]. We now consider an adaptation of his algorithm to the maximisation version of SUBBETWEENNESS OF A TREE yielding a $\frac{1}{2}$-approximation algorithm if restricted to 'Yes'-instances of SUBBETWEENNESS OF A TREE.

Maximisation version of SUBBETWEENNESS OF A TREE
Instance: A strict ternary relation $\mathcal{B}_s$ on a ground set V.
Task: Determine a tree T such that $\frac{|\mathcal{B}_s \cap \mathcal{B}_s(T)|}{|\mathcal{B}_s|}$ is maximum.

Theorem 34. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [45]*)*
Algorithm 2 works correctly and can be implemented to run in linear time, i.e. there is a linear time $\frac{1}{2}$-approximation algorithm for the maximisation version of SUBBETWEENNESS OF A TREE *restricted to 'Yes'-instances of* SUBBETWEENNESS OF A TREE.

Proof. First we show that a vertex as selected in line 4 of Algorithm 2 always exists. Let T be a tree and let U be a subset of $V(T)$. If we root T in an arbitrary vertex and select a vertex $b \in U$ of maximal depth, then there are no two elements $a, c \in U$ with $(a, b, c) \in \mathcal{B}_s(T)$.

Now to prove that the returned path has the claimed property, note that the path P in line 10 has vertex set $V(P) = \{b_1, \ldots, b_{i-1}\}$. Hence, by the choice of b_i in line 4, there are no two elements $a_i, c_i \in V(P)$ with $(a_i, b_i, c_i) \in \mathcal{B}_s$. Therefore, for one path Q among Pb_i and b_iP, $\mathcal{B}_s(Q)$ contains at least half the triples in $\{(a, b, c) \in \mathcal{B}_s \mid b_i \in \{a, b, c\}, \{a, b, c\} \subseteq \{b_1, \ldots, b_i\}\}$, which completes the proof of correctness.

The linear running time was already established by Makarychev in [28].

□

Algorithm 2: Algorithm solving the maximisation version of SUBBETWEENNESS OF A TREE.

Input: A strict ternary relation $\mathcal{B}_s$ on a finite ground set V, such that there is a tree T with $\mathcal{B}_s \subseteq \mathcal{B}_s(T)$.

Output: A path P with $\frac{|\mathcal{B}_s \cap \mathcal{B}_s(P)|}{|\mathcal{B}_s|} \geq \frac{1}{2}$.

$n \leftarrow |V|$
$i \leftarrow n$
while $i \geq 1$ **do**
    Select $b \in V$ such that $\nexists a, c \in V : (a, b, c) \in \mathcal{B}_s \cap V^3$ holds
    $b_i \leftarrow b$
    $V \leftarrow V \setminus \{b\}$
    $i \leftarrow (i - 1)$
$P \leftarrow b_1 b_2$
for i *from* 3 *to* n **do**
    if $|\mathcal{B}_s \cap |\mathcal{B}_s(Pb_i)| \geq |\mathcal{B}_s \cap \mathcal{B}_s(b_iP)|$ **then**
        $P \leftarrow Pb_i$
    else
        $P \leftarrow b_iP$
return P

3.2 Induced Subbetweenness of a Tree

The second problem we consider is INDUCED SUBBETWEENNESS OF A TREE:

INDUCED SUBBETWEENNESS OF A TREE

Instance: A strict ternary relation $\mathcal{B}_s$ on a ground set V.

Question: Is there a tree T such that $\mathcal{B}_s = \mathcal{B}_s(T) \cap V^3$ holds?

In Section 2.1.2 we already looked at a characterisation of 'Yes'-instances of the problem. Since this characterisation only used first-order logic axioms, we already know that INDUCED SUBBETWEENNESS OF A TREE is in P. However, we look at an algorithm that does not use a brute force approach and returns an adequate tree T in polynomial time, if there is one.

Before we look at the algorithm and its correctness proof we prove the following lemma.

Lemma 35. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [45]*)*

Let T be a tree and let V be a subset of $V(T)$ such that $V(T) \setminus V$ is an independent set of vertices of degree at least 3 in T. Let $\mathcal{B}_s$ be the set $\mathcal{B}_s(T) \cap V^3$. For the remainder of the assertion and its proof let $[a, b, c]$ denote $[a, b, c]_{\mathcal{B}_s}$ for all $a, b, c \in V$. Let $\mathcal{T}$ denote the set of all sets $\{a, b, c\} \in \binom{V}{3}$ such that $\neg[a, b, c]$, $\neg[a, c, b]$, and $\neg[b, a, c]$ hold and there is no $d \in V$ such that $[a, d, b] \vee [a, d, c] \vee [b, d, c]$ holds. Let G be the graph with vertex set $\mathcal{T}$ in which two vertices s and t are adjacent if and only if $|s \cap t| = 2$ holds. The following assertions are true:

(i) $V(T) \setminus V$ contains a vertex b with neighbourhood N if and only if there is some component C of G with $N = \bigcup_{t \in V(C)} t$.

(ii) Two distinct vertices a and b are adjacent in T if and only if there is no $t \in \mathcal{T}$ with $a, b \in t$ and there is no $d \in V$ such that $[a, d, b]$ holds.

Proof. (i) Let $v \in V(T) \setminus V$ and let $N = N_T(v)$. By construction, all elements of $\binom{N}{3}$ belong to one component C of G. If $C \neq \binom{N}{3}$ holds, then there is some $t \in C \setminus \binom{N}{3}$ and some $s \in \binom{N}{3}$ with $|t \cap s| = 2$. Let $t \cap s = \{x, y\}$. By construction, there is some vertex $v' \in V(T) \setminus T$ that is distinct from v such that $x, y \in N_T(v) \cap N_T(v')$ holds. This implies the existence of a cycle $xvyv'x$ in T, which is a contradiction. Hence $C = \binom{N}{3}$ is true, which shows that $N = \bigcup_{t \in V(C)} t$ holds.

Conversely, let C be a component of G and let $N = \bigcup_{t \in V(C)} t$. Let t be an element of $V(C)$ with $t = \{a, b, c\}$. Since $\neg[a, b, c]$, $\neg[a, c, b]$, and $\neg[b, a, c]$ hold, the minimal subtree T_t of T that contains a, b, and c is the subdivision of a claw $K_{1,3}$ with end vertices a, b, and c. Since there is no $d \in V$ such that $[a, d, b] \vee [a, d, c] \vee [b, d, c]$ holds, the tree T_t is isomorphic to $K_{1,3}$ and the vertex m_t of degree 3 in T_t belongs to $V(G) \setminus V$. If t and s are two elements of $\mathcal{T}$ with $|t \cap s| = 2$, then $m_t = m_s$ holds since T has no cycles. By the definition of G, this implies that all vertices in N are adjacent to the same vertex m in $V(T) \setminus V$, i.e. $N \subseteq N_T(m)$ holds. So let $d \in N_T(m)$ and let t be an element of $V(C)$ with $t = \{a, b, c\}$ and such that $d \notin \{a, b, c\}$ holds. By construction, $\{d, a, b\} \in V(C)$, which implies $d \in N$. Therefore, $N_T(m) \subseteq N$ and thus $N = N_T(m)$ holds, which completes the proof of (i).

(ii) Let a and b be two distinct vertices in V. If $ab \in E(T)$ holds, then there is no $t \in \mathcal{T}$ with $a, b \in t$ and there is no $d \in V$ such that $[a, d, b]$ holds. Conversely, if $ab \notin E(T)$ holds, then the path P in T between a and b contains at least one internal vertex. If some internal vertex d of P belongs to V, then $[a, d, b]$ holds. If no internal vertex of P belongs to V, then P contains exactly one internal vertex $d \in V(T) \setminus V$, since $V(T) \setminus V$ is an independent set. The vertex d has degree at least 3, so there is some $c \in N_T(d) \setminus \{a, b\}$, which implies $\{a, b, c\} \in \mathcal{T}$, which completes the proof. □

Algorithm 3: Algorithm for INDUCED SUBBETWEENNESS OF A TREE.

Input: A strict ternary relation $\mathcal{B}_s$ on a finite ground set V.
Output: A tree T with $\mathcal{B}_s = \mathcal{B}_s(T) \cap V^3$ or the answer "No" if there is no such tree.

$V(T) \leftarrow V$
$E(T) \leftarrow \emptyset$
Construct $\mathcal{T}$ and G as in Lemma 35
for *every component* C *of* G **do**
    $N \leftarrow \bigcup_{t \in V(C)} t$
    Let m be a new vertex not in $V(T)$
    $V(T) \leftarrow V(T) \cup \{m\}$
    $E(T) \leftarrow E(T) \cup \{ma \mid a \in N\}$
for *every* $\{a, b\} \in \binom{V}{2}$ **do**
    if $(\nexists t \in \mathcal{T} : a, b \in t) \wedge (\nexists d \in V : [a, d, b])$ **then**
        $E(T) \leftarrow E(T) \cup \{ab\}$
if T *is not a tree* **then**
    return *"No"*
Construct $\mathcal{B}_s(T)$
if $\mathcal{B}_s = \mathcal{B}_s(T) \cap V^3$ **then**
    return T
else
    return *"No"*

Theorem 36. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [45]*)* *Algorithm 3 works correctly and runs in $O(|V|^l)$ time for some $l \in \mathbb{N}$.*

Proof. The correctness follows from Lemma 32 and Lemma 35: The graphs $\mathcal{T}$ and $\mathcal{G}$ in Lemma 35 are defined using only V and $\mathcal{B}_s$. Therefore, by Lemma 35 (i) and (ii), the tree T is uniquely determined by V and $\mathcal{B}_s$ up to isomorphism. The steps executed by Algorithm 3 during the construction of T correspond exactly to Lemma 35: Lines 4 to 8 correspond to Lemma 35 (i), while lines 9 to 11 correspond to Lemma 35 (ii).

Furthermore, note that the task in line 3 can be executed in polynomial time, because it only requires knowledge of $\mathcal{B}$. The rest of the polynomiality proof is obvious. □

3.3 Partial Tree Representation

In the previous two sections we looked at problems, the goal of which was to find a tree such that the strict tree betweenness it induces contains a certain ternary relation. The following problem suggested by Burigana [5] reverses the inclusion direction in a certain sense:

PARTIAL TREE REPRESENTATION

Instance: A strict intersection betweenness $\mathcal{B}_s$ on a ground set V.

Task: Decide whether there is a tree T with vertex set V such that $\mathcal{B}_s(T) \subseteq \mathcal{B}_s$, and construct such a tree if possible.

In his paper [5] Burigana shows that the GYO algorithm of Graham [18], Yu, and Ozsoyolu [52] can be used to solve PARTIAL TREE REPRESENTATION. While doing that he assumes that the strict intersection betweenness $\mathcal{B}_s$ is given by an arbitrary set system $(V, \mathcal{V})$ inducing the betweenness. But then the running time of the algorithm can at best be polynomial in the encoding length of $(V, \mathcal{V})$, which with $(V, \mathcal{V})$ being an arbitrary set system is not necessarily polynomial in the cardinality of V, while the encoding length of $\mathcal{B}_s$ always is.

We consider an algorithm published by Rautenbach, Santos, Schäfer, and

Szwarcfiter [44] solving the problem Partial tree representation with a running time polynomially bounded in $|V|$ using the set $\mathcal{B}_s$ as the input.

Algorithm 4: Algorithm for Partial tree representation

Input: A strict intersection betweenness $\mathcal{B}_s$ on a ground set V.
Output: A tree T with vertex set V such that $\mathcal{B}_s(T) \subseteq \mathcal{B}_s$ holds or the answer "No" if there is no such tree.

Construct a set system $(X, \mathcal{V})$, such that $\mathcal{B}_s(\mathcal{V})$ is isomorphic to $\mathcal{B}_s$, $|X| = \left|\bigcup_{U \in \mathcal{V}} U\right| = O(|V|^2)$ holds, and

$$\forall U \in \mathcal{V} : U \setminus \bigcup_{W \in \mathcal{V} \setminus \{U\}} W \neq \emptyset$$

holds as well. Let $\phi : \mathcal{V} \to V$ be an isomorphism between $\mathcal{B}_s(\mathcal{V})$ and $\mathcal{B}_s$.
for $x \in X$ **do**
    $M_x \leftarrow \{U \in \mathcal{V} \mid x \in U\}$
Construct the intersection graph G of $\{M_x \mid x \in X\}$, representing each set M_x by the vertex x.
if *G is not chordal* **then**
    return "No"
if *$\mathcal{V}$ is not the set of exactly $|V|$ many maximal cliques of G* **then**
    return "No"
Construct a clique tree T_G of G using $\phi(U)$ to denote the maximal clique U for $U \in \mathcal{V}$
if $\mathcal{B}_s(T_G) \not\subseteq \mathcal{B}_s$ **then**
    return "No"
else
    return T_G

Theorem 37. *(*Rautenbach, Santos, Schäfer, and Szwarcfiter 2011, [44]*)*
Algorithm 4 correctly solves Partial tree representation *in polynomial time.*

Proof. Let $\mathcal{B}_s$ be a strict intersection betweenness on a ground set V. By Theorem 27, we can construct a set system $(X, \mathcal{V})$, such that $\mathcal{B}_s(\mathcal{V})$ is isomorphic to $\mathcal{B}_s$ and $\left|\bigcup_{U \in \mathcal{V}} U\right| = O(|V|^2)$ holds, in polynomial time. Since $\mathcal{B}_s$

is strict we may assume that

$$\forall U \in \mathcal{V} : U \setminus \bigcup_{W \in \mathcal{V} \setminus \{U\}} W \neq \emptyset \tag{3.1}$$

holds as well (see proof of Lemma 29). Let ϕ and M_x be defined as in Algorithm 4.

We assume now that there exists some tree T with vertex set V such that $\mathcal{B}_s(T) \subseteq \mathcal{B}_s$ holds. Consider a path in T between two vertices a and c that contains b as an internal vertex, and let $x \in \phi^{-1}(a) \cap \phi^{-1}(c)$. Since $[a,b,c]_{\mathcal{B}_s(T)}$ holds and since $\mathcal{B}_s(T)$ is a subset of $\mathcal{B}_s$, we obtain that $\phi^{-1}(a) \cap \phi^{-1}(c) \subseteq \phi^{-1}(b)$, and hence $x \in \phi^{-1}(b)$. Therefore, for every $x \in X$, the set $\phi(M_x)$ induces a subtree T_x of T. It follows from a well-known result [4, 16, 51] that the intersection graph G as defined in Algorithm 4 is chordal.

Let C be a maximal clique in G. By definition, for every two elements x and y in C, the sets M_x and M_y intersect, i.e. the two subtrees T_x and T_y share a vertex. Since subtrees of a tree have the Helly-property (see Proposition 4.7 in [17]), there is some set $U_C \in \mathcal{V}$ such that $C \subseteq U_C$. Since U_C clearly induces a clique in G, the maximality of C implies that $C = U_C$. This implies that $\mathcal{V}$ is the collection of all maximal cliques of G. Furthermore, by (3.1), no two of these maximal cliques are equal, and thus G has exactly $|V|$ many distinct maximal cliques that are in bijective correspondence with the elements of V.

Let T_G denote a clique tree of G, where we use $\phi(U)$ to denote the maximal clique U for $U \in \mathcal{V}$. By the definition of a clique tree, for every $x \in X$, the set M_x induces a subtree of T_G. This implies that, if $[a,b,c]_{\mathcal{B}_s(T_G)}$ holds for some $a,b,c \in V$ and $x \in \phi^{-1}(a) \cap \phi^{-1}(c)$, then $x \in \phi^{-1}(b)$ holds, i.e. $\phi^{-1}(a) \cap \phi^{-1}(c) \subseteq \phi^{-1}(b)$ holds, and hence $[a,b,c]_{\mathcal{B}_s}$ holds. Therefore, T_G satisfies $\mathcal{B}_s(T_G) \subseteq \mathcal{B}_s$ and solves Partial tree representation.

Overall this proves that Algorithm 4 works correctly.

The task in line 1 can be done in polynomial time using Algorithm 1. Furthermore, the tasks in line 4, 5, 7, and 9 can be performed in polynomial time using standard methods [17, 29]. Therefore, the overall running time is polynomially bounded in terms of $|V|$. □

Chapter 4

Two Graph Convexity Problems

In this chapter we look at two problems in the field of graph convexity in general. They are also part of my research conducted during the last three years but do not belong into the field of betweenness relations.

The first section looks at convexity spaces induced by sets of paths in graphs. A characterisation of convexity spaces which can be induced by a set of paths of a graph is given, followed by a proof that deciding if a convexity space can be induced by a set of paths of a graph is in P. The results presented in this section are published in [13].

In the second section algorithmic problems related to conversion processes in graphs with deadlines are considered. Two results show that finding a minimum conversion set for cliques and trees with given deadline and threshold functions can be accomplished in polynomial time. The results of this section are submitted for publication coauthored by Rautenbach, dos Santos, and Schäfer [43].

4.1 Convexity Spaces Induced by Paths in Graphs

Before we begin to look at the problem, we need to give definitions of the structures we are going to talk about.

A *finite convexity space* is a set system $(V, \mathcal{C})$ with finite ground set V which satisfies

- $\emptyset \in \mathcal{C}, V \in \mathcal{C}$ and
- $\forall C_1, C_2 \in \mathcal{C} : C_1 \cap C_2 \in \mathcal{C}$,

i.e. a set system closed under intersection. We call the elements of $\mathcal{C}$ *convex sets*. Given a subset S of V, we call the smallest convex set containing S its $\mathcal{C}$-convex hull and denote this hull by $H_{\mathcal{C}}(S)$.

All convexity spaces considered in this text are going to be finite, so we just call them *convexity space*.

Sets of paths of a graph induce convexity spaces. Given a graph G and a set $\mathcal{P}$ of paths of G we can define a convexity spaces $(V(G), \mathcal{C}(\mathcal{P}))$, where a set C of vertices of G belongs to $\mathcal{C}(\mathcal{P})$ if and only if the vertices of every path of $\mathcal{P}$ whose end vertices are contained in C are contained in C.

Theorem 38. *(*Dourado, Rautenbach, and Schäfer 2011, [13]*)*

Let $(V, \mathcal{C})$ be a convexity space and let G be a graph with vertex set V. There is a set $\mathcal{P}$ of paths in G with $\mathcal{C} = \mathcal{C}(\mathcal{P})$ if and only if

(P_1) $\forall S \subseteq V : S \notin \mathcal{C} : (\exists a, c \in S : (a \neq c) \wedge (H_{\mathcal{C}}(\{a, c\}) \nsubseteq S))$ *and*

(P_2) $\forall a, c \in V : \forall b \in H_{\mathcal{C}}(\{a, c\}) \setminus \{a, c\} : G[H_{\mathcal{C}}(\{a, c\})]$ *contains an a-b-c-path*

hold.

In order to proof this theorem we need the following two lemmas.

Lemma 39. *(*Dourado, Rautenbach, and Schäfer 2011, [13]*)*

Let G be a graph and let $P_0, \ldots, P_l$ be a sequence of paths of length at least 1 *in G such that* *(i) P_0 has the end vertices a and c and*

(ii) for all $i \in [l]$, the path P_i and the graph $G_{i_1} = P_0 \cup \ldots \cup P_{i-1}$ have exactly the end vertices of P_i in common.

For every two vertices u and v of G_l, the graph G_l contains two paths P and Q between $\{a, c\}$ and $\{u, v\}$ such that, if $u \neq v$ holds, then P and Q are vertex-disjoint and, if $u = v$ holds, then P and Q share exactly u.

Proof. We proof the Lemma by induction on l. For $l = 0$, the graph G equals the path P_0 and the assertion holds.

Now assume that $l \geq 1$. Denote the end vertices of P_l by a' and c'.

If u and v belong to G_{l-1}, the assertion holds by induction. If u and v belong to P_l, then, since a' and c' are distinct, G_{l-1} contains two vertex-disjoint paths P' and Q' between $\{a, c\}$ and $\{a', c'\}$ by induction. Furthermore, P_l contains two paths P'' and Q'' between $\{a', c'\}$ and $\{u, v\}$ such that, if $u \neq v$ holds, then P'' and Q'' are vertex-disjoint and, if $u = v$ holds, then P'' and Q'' share exactly u. Combining these four paths at common end vertices in $\{a', c'\}$, we get the desired two paths between $\{a, c\}$ and $\{u, v\}$.

This leaves the remaining case with one of $\{u, v\}$ belonging to G_{l-1} and the other being an inner vertex of P_l. Without loss of generality we may assume that u belongs to G_{l-1}, while v is an inner vertex of P_l. We may further assume that $u \neq a'$ holds. By induction, there are vertex-disjoint paths P' and Q' in G_{l-1} between $\{a, c\}$ and $\{a', u\}$. Furthermore, P_l contains a path P'' between a' and v. Combining the paths containing a' and taking the left over path in addition, we get the desired two paths between $\{a, c\}$ and $\{u, v\}$. □

Lemma 40. *(*Dourado, Rautenbach, and Schäfer 2011, [13]*)*

Let G be a graph and let $\mathcal{P}$ be a set of paths in G. For every two distinct vertices a and c of G such that $H_{\mathcal{C}(\mathcal{P})}(\{a, c\}) \neq \{a, c\}$ holds, there is a sequence $P_0, \ldots, P_l$ of paths in G such that (i) and (ii) from Lemma 39 are satisfied and $H_{\mathcal{C}(\mathcal{P})}(\{a, c\})$ is the vertex set of $G_l = P_0 \cup \ldots \cup P_l$.

Proof. By definition, the $\mathcal{C}(\mathcal{P})$-convex hull of $\{a, c\}$ can be constructed by the following procedure.

Algorithm 5: $\mathcal{C}(\mathcal{P})$-convex hull of $\{a, c\}$

$C \leftarrow \{a, c\}$
while *there is a path P in $\mathcal{P}$ whose end vertices belong to C but whose vertex set is not contained in C* **do**
  $C \leftarrow C \cup V(P)$
return C

The set C maintained by the above procedure is always a subset of the $\mathcal{C}(\mathcal{P})$-convex hull of $\{a, c\}$. The finiteness of V implies that the procedure terminates and hence produces the smallest $\mathcal{C}(\mathcal{P})$-convex set containing the set $\{a, c\}$.

Since $\mathcal{H}_{\mathcal{C}(\mathcal{P})}(\{a, c\}) \neq \{a, c\}$ holds, in the first iteration of the **while**-loop, the procedure adds the vertices of a path P_0 to C, whose end vertices are a and c.

By an inductive argument, we may assume that $P_0, \ldots, P_j$ is a sequence of paths in G such that (i) and (ii) from Lemma 39 are satisfied and the vertex set of $P_0 \cup \ldots \cup P_j$ is the set C between iterations $i-1$ and i of the **while**-loop. In the i-th iteration of the **while**-loop the procedure adds the vertices of a path P in $\mathcal{P}$ to C, whose end vertices belong to C but whose vertex set is not fully contained in C. The path P decomposes into maximal subpaths $P_{j+1}, \ldots, P_{j+k}$ for some $k \in \mathbb{N}$, whose end vertices belong to C and whose internal vertices do not belong to C. Now $P_0, \ldots, P_{j+k}$ is a sequence of paths in G such that (i) and (ii) from Lemma 39 are satisfied and the vertex set of $P_0 \cup \ldots \cup P_{j+k}$ equals the set C just after the execution of the i-th iteration of the **while**-loop.

This completes the proof. □

Proof of Theorem 38. First we assume the existence of a set $\mathcal{P}$ of paths in G with $\mathcal{C} = \mathcal{C}(\mathcal{P})$.

If a set S of vertices is not $\mathcal{C}$-convex, then the definition of $\mathcal{C}(\mathcal{P})$ implies the existence of some path P in $\mathcal{P}$ whose end vertices, say a and c, belong to S but which contains a vertex, say b, which does not belong to S. By definition, $b \in H_{\mathcal{C}}(\{a, c\})$ holds, which implies that $H_{\mathcal{C}}(\{a, c\}) \not\subseteq S$ holds. This implies that $(V, \mathcal{C})$ satisfies (P_1).

Now let a and c be in V and let b be in $H_{\mathcal{C}}(\{a,c\}) \setminus \{a,c\}$. By Lemma 39 and Lemma 40 there is a path between a and b and a path between b and c which share exactly b and whose vertex sets are contained in $H_{\mathcal{C}}(\{a,c\})$. Concatenating these two paths yields an a-b-c-path in $G[H_{\mathcal{C}}(\{a,c\})]$. This implies that $(V,\mathcal{C})$ satisfies (P_2).

Next we assume that $(V,\mathcal{C})$ satisfies (P_1) and (P_2). Let $\mathcal{P}$ be the set of all paths P in G such that for every $C \in \mathcal{C}$ that contains the end vertices of P, the set C contains all vertices of P.

If C belongs to $\mathcal{C}$, then the definition of P implies that $C \in \mathcal{C}(\mathcal{P})$ holds.

If S does not belong to $\mathcal{C}$, then (P_1) implies that there are two distinct elements a and c of S whose $\mathcal{C}$-convex hull $H_{\mathcal{C}}(\{a,c\})$ is not contained in S. Let $b \in H_{\mathcal{C}}(\{a,c\}) \setminus S$. (P_2) implies that the graph $G[H_{\mathcal{C}}(\{a,c\})]$ contains an a-b-c-path P. Since $H_{\mathcal{C}}(\{a,c\})$ is a subset of every $\mathcal{C}$-convex set containing a and c, the definition of $\mathcal{P}$ implies that P belongs to $\mathcal{P}$. This implies that S does not belong to $\mathcal{C}(\mathcal{P})$.

Overall, we get $\mathcal{C} = \mathcal{C}(\mathcal{P})$, which completes the proof. □

Corollary 41. *(*Dourado, Rautenbach, and Schäfer 2011, [13]*)*

Let $(V,\mathcal{C})$ be a convexity spaces. There is a graph G with vertex set V and a set $\mathcal{P}$ of paths in G with $\mathcal{C} = \mathcal{C}(\mathcal{P})$ if and only if (P_1) is satisfied.

Proof. Note that (P_1) implies that all sets of cardinality 1 are $\mathcal{C}$-convex. This implies that (P_2) is trivially satisfied for the complete graph with vertex set V. Now the assertion follows from Theorem 38. □

In order to look at the algorithmic side of the problem, we now define the corresponding decision problem.

Path Convexity of a Graph

Instance:	A convexity space $(V,\mathcal{C})$ and a graph G with vertex set V.
Task:	Decide whether there is a set $\mathcal{P}$ of paths of G with $\mathcal{C} = \mathcal{C}(\mathcal{P})$ and construct $\mathcal{P}$ if it exists.

The proof of Theorem 38 gives us a characterisation of the 'Yes'-instances of the problem. While the property (P_2) can be checked in polynomial time in a straight forward way using efficient algorithms for the 2-disjoint path problem [47], trying to do that for property (P_1) may take exponential time in $\mathcal{C}$ because we are looking at the complement of $\mathcal{C}$ with respect to 2^V.

However, the following theorem shows that Path Convexity of a Graph can be solved in polynomial time.

Theorem 42. *(*Dourado, Rautenbach, and Schäfer 2011, [13]*)*
Path Convexity of a Graph *can be solved in polynomial time.*

Proof. Let $(V, \mathcal{C})$ and G be an instance of Path Convexity of a Graph. The $\mathcal{C}$-convex hull of a set of vertices of G can be determined in time polynomially bounded in $|V|$ and $|\mathcal{C}|$, e.g. just check all elements of $\mathcal{C}$ for being the smallest set containing the set of vertices, which can be done in $O(|V||\mathcal{C}|)$ time.

For all triples (a, b, c) such that a and c are distinct vertices of G and b is in $H_{\mathcal{C}}(\{a, c\}) \setminus \{a, c\}$, we can efficiently determine an a-b-c-path P in $G[H_{\mathcal{C}}(\{a, c\})]$ using an algorithm for the 2-disjoint path problem [47]. If we fail to find such a path for some triple, then $(V, \mathcal{C})$ does not satisfy property (P_2) and, by Theorem 38, there is no set $\mathcal{P}$ of paths in G with $\mathcal{C} = \mathcal{C}(\mathcal{P})$. Hence we may assume that there are such paths. Let $\mathcal{P}$ denote the set of all these paths. Note that $|\mathcal{P}|$ is polynomially bounded in $|V|$.

Since the vertex set of every a-b-c-path P in $\mathcal{P}$ belongs to $H_{\mathcal{C}}(\{a, c\})$, every $\mathcal{C}$-convex set C that contains the end vertices a and c of the path P must contain all vertices of P. Therefore, $\mathcal{C} \subseteq \mathcal{C}(\mathcal{P})$ holds.

For every C in $\mathcal{C}$ and every b in $V \setminus C$, we can check whether the $\mathcal{C}(\mathcal{P})$-convex hull $H_{\mathcal{C}(\mathcal{P})}(C \cup \{b\})$ of $C \cup \{b\}$ belongs to $\mathcal{C}$. This can be done in polynomial time in $|V|$ and $|\mathcal{C}|$.

First we assume that there is some C in $\mathcal{C}$ and an element b of $V \setminus C$ such that $H_{\mathcal{C}(\mathcal{P})}(C \cup \{b\})$ is not $\mathcal{C}$-convex. If there is a pair a' and c' of distinct vertices in $H_{\mathcal{C}(\mathcal{P})}(C \cup \{b\})$ such that $H_{\mathcal{C}}(\{a', c'\}) \not\subseteq H_{\mathcal{C}(\mathcal{P})} C \cup \{b\}$ holds, then $\mathcal{P}$ contains by definition an a'-b'-c' path for some vertex $b' \in H_{\mathcal{C}}(\{a', c'\}) \setminus H_{\mathcal{C}(\mathcal{P})} C \cup \{b\}$, which implies the contradiction that $H_{\mathcal{C}(\mathcal{P})} C \cup \{b\}$ is not $\mathcal{C}(\mathcal{P})$-convex. Therefore, we have $H_{\mathcal{C}}(\{a', c'\}) \subseteq H_{\mathcal{C}(\mathcal{P})} C \cup \{b\}$ for every pair a' and c' of distinct vertices in $H_{\mathcal{C}(\mathcal{P})} C \cup \{b\}$. This implies that $(V, \mathcal{C})$ does not satisfy (P_1), which together with Theorem 38 implies that there is no set $\mathcal{P}$

of paths in G with $\mathcal{C} = \mathcal{C}(\mathcal{P})$. Hence, we may assume that the $\mathcal{C}(\mathcal{P})$-convex hull of $\mathcal{C} \cup \{b\}$ belongs to $\mathcal{C}$ for every C in $\mathcal{C}$ and every b in $V \setminus C$.

We are going to show that $\mathcal{C} = \mathcal{C}(\mathcal{P})$. We already established that $\mathcal{C} \subseteq \mathcal{C}(\mathcal{P})$ holds, so we assume for contradiction that there is a set $S \in \mathcal{C}(\mathcal{P}) \setminus \mathcal{C}$. Let C be a subset of S of maximum order that belongs to $\mathcal{C}$. Since the empty set is in $\mathcal{C}$, the set C is well defined. Since $S \notin \mathcal{C}$ holds, there is some $b \in S \setminus C$. Our observations above imply that the $\mathcal{C}(\mathcal{P})$-convex hull of $C \cup \{b\}$ is a subset of S, which is strictly larger than C and belongs to $\mathcal{C}$. This is a contradiction, which completes the proof. □

4.2 Conversion with Deadlines in Graphs

The problems we are going to deal with in this section are conversion processes in networks with individual deadlines for the nodes. In our case we will be looking at irreversable conversion processes, i.e. in case a node is 'converted' it will remain so.

This notion is a generalisation of hull operators on sets. Whereas for hulls, we only are interested in the final set, which can, as in the previous chapter, be constructed by iterating a certain fuction, here, in this case we also care about how long it takes vor an individual element to be added to the set.

An instance of our problems is given by a triple (G, t_d, f), where

- G is a graph,
- t_d is a function $t_d : V(G) \to \mathbb{N}_0$, called the *deadline function*, and
- f is a function $f : \{(u, t) : u \in V(G), t \in \mathbb{N}_0, 0 \leq t \leq t_d - 1\} \to \mathbb{N}_0$, called the *time-dependent threshold function*.

The graph represents the network in which we want to model the conversion process, the deadline function tells us for each vertex of the graph after at most how many steps we want that vertex to be converted, and the time-dependent threshold functions tells us for every vertex of the graph and every natural number less than the deadline, representing a conversion step, how many neighbours of the vertex need to be converted to convert the vertex in the current conversion step.

To formalise this notion we define what a conversion process on such a problem instance is. We call a sequence $\mathcal{C} = (c_t)_{t\in\mathbb{N}_0}$ a *valid conversion process on* (G, t_d, f) if and only if it satisfies

- for every $t \in \mathbb{N}_0$ the element c_t is a function $c_t : V(G) \to \{0, 1\}$ and
- for every $t \in \mathbb{N}_0$ and every $a \in V(G)$ the equality $c_t(a) = 1$ holds if and only if $c_{t-1}(a) = 1$ holds or $|\{b \in N_G(a) \mid c_{t-1}(b) = 1\}| \geq f(a, t-1)$ holds.

Note that the deadline functions in this structure could be modelled equivalently in a structure of just a graph and a time-dependent threshold function, by setting the threshold to a natural number greater than $d_G(u)$ for all $t \geq t_d(u)$. But including the deadline function made it easier to state and proof the following results.

Given a valid conversion process $\mathcal{C} = (c_t)_{t\in\mathbb{N}_0}$ on a problem instance (G, t_d, f), we say that a vertex $a \in V(G)$ is *converted by* $\mathcal{C}$ *at time* t if $c_t(a) = 1$ and $c_s(a) = 0$ hold for every $s \in [t]_0$. We say that a vertex $a \in V(G)$ is *converted on time* by $\mathcal{C}$ if $c_{t_d(a)}(a) = 1$ holds. We say the process $\mathcal{C}$ *converges to* 1 *on time*, if all vertices of G are converted on time by $\mathcal{C}$. If $\mathcal{C}$ converges to 1 on time, the set $c_0^{-1}(1)$ is called a *conversion set* of (G, t_d, f). The minimum cardinality of a conversion set of (G, t_d, f) is denoted by $m(G, t_d, f)$. Since $V(G)$ is always a conversion set of (G, t_d, f), the invariant $m(G, t_d, f)$ is well-defined.

Choosing special functions for t_d and f our invariant $m(G, t_d, f)$ corresponds to certain well-known graph parameters:

- If $t_d(\cdot) \equiv 1$ and $f(\cdot, 0) \equiv 1$ hold, then $m(G, t_d, f)$ corresponds to the domination number of G.
- If $t_d(\cdot) \equiv k$ and $f(\cdot, \cdot) \equiv 1$ hold, then $m(G, t_d, f)$ corresponds to the distance-k-domination number of G.
- If $t_d(\cdot) \equiv 1$ and $f(\cdot, 0) \equiv k$ hold, then $m(G, t_d, f)$ corresponds to the k-domination number of G.

- If $\forall a \in V(G): \ f(a) = d_G(a)$, then $|V(G)| - m(G, t_d, f)$ corresponds to the independence number of G.

These observations imply that calculating $m(G, t_d, f)$ is a problem that is algorithmically hard to solve. The results we look at in this section deal with the problem for special graph classes.

Minimum Conversion Set

Instance:	A triple (G, t_d, f), where G is a graph, t_d is a deadline function, and f is a threshold function.
Task:	Find a conversion set of cardinality $m(G, t_d, f)$.

We begin by looking at the problem just for the class of forests. In this case, the problem is efficiently solvable by Algorithm 6.

Theorem 43. *(*Rautenbach, Santos, and Schäfer [43]*)*
Algorithm 6 is correct and runs in polynomial time.

Proof. Since in every execution of the **while**-loop starting in line 2, the order of G is reduced by at least 1 and all manipulations of and calculations based on t_d and f can be implemented to run in time polynomial in $|(G, t_d, f)|$, the overall running time of Algorithm 6 is polynomial in the coding length of the input.

Next we proof correctness. Every conversion set of (G, t_d, f) has to contain all vertices a such that $f(a, t) > d_G(a)$ holds for all t with $0 \leq t < t_d(a)$. This is taken into account by the **while**-loop starting in line 3. When we remove such vertices from G, we have to reduce the threshold values of their neighbours accordingly (see line 5). This proofs the correctness of the reduction applied in the **while**-loop starting in line 3. After its completion, every remaining vertex a of G satisfies

$$\exists t: \ 0 \leq t < t_d(a), f(a, t) \leq d_G(a).$$

If some remaining vertex a of G has degree 0, then this implies that $f(a, t) = 0$ holds for some $0 \leq t < t_d(a)$. Such vertices are always converted

Algorithm 6: Algorithm solving MINIMUM CONVERSION SET for forests.

Input: An instance (G, t_d, f) of MINIMUM CONVERSION SET, where G is a forest.

Output: A conversion set C of (G, t_d, f) of cardinality $m(G, t_d, f)$.

$C \leftarrow \emptyset$;
while $V(G) \neq \emptyset$ **do**
    while $\exists a \in V(G): \ \forall t \in \mathbb{N}_0: \ (0 \leq t < t_d(a)) \Rightarrow (f(a,t) > d_G(a))$ **do**
        for $b \in N_G(a)$ *and* $0 \leq t < t_d(b)$ **do**
            $f(b,t) \leftarrow f(b,t) - 1$
        $C \leftarrow C \cup \{a\}$
        $G \leftarrow G - a$
    $G \leftarrow G - \{a \in V(G): \ d_G(a) = 0\}$
    if $V(G) \neq \emptyset$ **then**
        Let $a \in V(G)$ be such that $d_G(a) = 1$ holds.
        Let $b \in V(G)$ be the unique neighbour of a.
        if $\exists t \in \mathbb{N}_0: \ 0 \leq t < t_d(a), f(a,t) = 0$ **then**
            $t_0 \leftarrow \min\{t \in \mathbb{N}_0 \mid 0 \leq t < t_d(a), f(a,t) = 0\}$
            for $t_0 + 1 \leq t < t_d(b)$ **do**
                $f(b,t) \leftarrow f(b,t) - 1$
        else
            $t_1 \leftarrow \max\{t \in \mathbb{N}_0 \mid 0 \leq t < t_d(a), f(a,t) = 1\}$
            $t_d(b) \leftarrow \min\{t_d(b), t_1\}$
        $G \leftarrow G - a$
return C

on time and can just be removed from G, which is done in line 8. After line 8 all remaining vertices have degree at least 1.

If G is not empty, then we can select a vertex a of degree 1, since G is a forest. Let b denote its neighbour. If there is some t with $0 \leq t < t_d(a)$ and $f(a,t) = 0$, then no minimum conversion set of (G, t_d, f) needs to contain a. Since a is guaranteed to be converted at the latest at time $t_0 + 1$ (cf. line 13), we can remove a from G and reduce the threshold value of its unique neighbour b for all t with $t \geq t_0 + 1$, which is done in line 15. In line 16, the vertex a satisfies $f(a,t) \geq 1$ for every $0 \leq t \leq t_d(a)$ and t_1 defined in line 17 is the largest t with $f(a,t) = 1$. This implies that there is a conversion set of (G, t_d, f) of order $m(G, t_d, f)$, which does not contain a, assuming b is converted by time t_1 (cf. line 17).

These observations imply the correctness of the reductions executed by Algorithm 6. □

The next class of graphs we consider is the class of cliques. We look at an efficient algorithm that, given an instance (G, t_d, f) with G being a clique, determines a conversion set of cardinality k if such a set exists. So, for starters, we look at a slightly different overall problem this time:

k-Conversion Set

Instance: A triple (G, t_d, f), where G is a graph, t_d is a deadline function, and f is a threshold function and a natural number k.

Task: Determine a conversion set of (G, t_d, f) of cardinality k if such a set exists.

Algorithm 7 simulates a conversion process $\mathcal{C}' = (c'_t)_{t \in \mathbb{N}_0}$ on a triple (G', t'_d, f') derived from the instance (G, t_d, f) by extending the complete graph G by k vertices to another complete graph G'. The added vertices compose the set of vertices with initial label 1, i.e. the set ${c'}_0^{-1}(1)$.

T is set to the maximum deadline occurring in the instance. The variables c_t for $0 \leq t \leq T$ count the vertices converted at time t' for some $t' \leq t$, which is all the algorithm needs to know to determine if a vertex with label 0 gets converted at time $t + 1$, since G' is a clique. Hence c_0 is set to k the number

Algorithm 7: Algorithm solving k-Conversion Set for cliques.

Input: An instance $(G, t_d, f), k$ of k-Conversion Set, where G is a clique.

Output: A conversion set C of (G, t_d, f) of cardinality k if such a set exists; otherwise No.

1 $c_0 \leftarrow k$
2 $t \leftarrow 0$
3 $T \leftarrow \max\{t_d(a) \mid a \in V(G)\}$
4 **while** $t \leq T$ **do**
5 $\quad S_t \leftarrow \{a \in V(G) \mid t \leq t_d(a), f(a, t-1) \leq c_{t-1}\}$
6 $\quad c_t \leftarrow c_{t-1} + |S_t|$
7 $\quad$ Denote the elements of S_t by $u_{c_{t-1}+1}, \ldots, u_{c_t}$
8 $\quad V(G) \leftarrow V(G) \setminus S_t$
9 $\quad t \leftarrow t + 1$
10 **if** $|V(G)| \leq k$ **then**
11 $\quad C \leftarrow V(G) \cup \{u_{c_T - i} \mid 0 \leq i \leq (k - |V(G)|) - 1\}$
12 $\quad$ **return** C
13 **else**
14 $\quad$ **return** No

of vertices with initial label 1.

For every t with $1 \leq t \leq T$ the set S_t consists of those vertices of G that are converted by $\mathcal{C}'$ at time t as long as they would be converted on time as well. The vertices in $S_1 \cup \ldots \cup S_T$ are ordered as $u_{k+1}, \ldots u_{c_T}$ representing a non-decreasing time of conversion.

After the **while**-loop starting in line 4 the set $V(G)$ contains exactly those vertices of G that have not been converted by the simulation, i.e. those that where not converted on time or not at all by $\mathcal{C}'$. If $V(G)$ contains at most k vertices, then $V(G)$ together with the first $k - |V(G)|$ vertices of $S_1 \cup \ldots \cup S_T$ of maximum possible time of conversion get returned as the set C in line 12. If $|V(G)|$ contains more than k vertices, then No is return.

Lemma 44. *(*Rautenbach, Santos, and Schäfer [43]*)*

If Algorithm 7 returns a set C in line 12, then C is a conversion set of (G, t_d, f) of order k.

Proof. Let $\mathcal{C}'' = (c''_t)_{t \in \mathbb{N}_0}$ be the conversion process on (G, t_d, f) with $C =$

${c''_0}^{-1}(1)$, i.e. C return by Algorithm 7 is the set of vertices with initial label 1. By an inductive argument it follows that, if $a \in S_t \setminus C$ holds for some $1 \leq t \leq T$, then $c''_t(a) = 1$ and $t \leq t_d(v)$ hold. This implies that every vertex of G is converted by $\mathcal{C}''$ on time, i.e. C is in fact a conversion set of (G, t_d, f) of order k. □

Lemma 45. *(*Rautenbach, Santos, and Schäfer [43]*)*

If Algorithm 7 returns No *in line 14, then there is no conversion set of (G, t_d, f) of order k.*

Proof. For contradiction, we assume that $\tilde{\mathcal{C}}$ is a conversion set of (G, t_d, f) of order k. Let $\tilde{\mathcal{C}} = (\tilde{c}_t)_{t \in \mathcal{N}_0}$ be the conversion process on (G, t_d, f) with $\tilde{C} = \tilde{c}_0^{-1}(1)$. All vertices of G get converted on time by $\tilde{\mathcal{C}}$.

We prove by induction on t with $1 \leq t \leq T$ that if some vertex a in $V(G) \setminus \tilde{C}$ gets converted by $\tilde{\mathcal{C}}$ at time t, then $a \in S_s$ for some $1 \leq s \leq t$.

Since $\tilde{C}$ has exactly k this is true for $t = 1$. Now let $t > 1$ and let a be converted by $\tilde{\mathcal{C}}$ at time t. We assume that $a \notin S_s$ for s with $1 \leq s \leq t-1$. The number of elements in $\tilde{C}$ plus the number x of vertices of G converted by $\tilde{\mathcal{C}}$ at some time s with $1 \leq s \leq t-1$ is at least $f(a, t-1)$. Furthermore, $t \leq t_d(a)$ holds. By induction, $S_1 \cup \ldots S_{t-1}$ contains at least x elements. This implies that c_{t-1} is at least $k + x$. Hence $f(a, t-1) \leq c_{t-1}$. Now $a \in S_t$ holds since $t \leq t_d(a)$ holds. This completes the proof of the induction step.

Since all vertices do get converted on time by $\tilde{\mathcal{C}}$, the set $V(G)$ in line 10 cannot have more than k elements of $\tilde{C}$ and the algorithm would not return No in line 14, which is a contradiction. □

Lemma 44 and Lemma 45 together imply the following theorem.

Theorem 46. *(*Rautenbach, Santos, and Schäfer [43]*)*

Algorithm 7 is correct and runs in polynomial time.

This theorem implies that sc Minimum Conversion Set is solvable in polynomial time for cliques as well:

Corollary 47. *(*Rautenbach, Santos, and Schäfer [43]*)*

Minimum Conversion Set *is solvable in polynomial time for instances (G, t_d, f) where G is a clique.*

Proof. Running Algorithm 7 for all natural numbers k with $0 \leq k \leq |V(G)|$ allows to calculate $m(G, t_d, f)$ in polynomial time. $\square$

Bibliography

[1] Martin Altwegg, *Zur Axiomatik der teilweise geordneten Mengen*, Commentarii Mathematici Helvetici, **24**, 149-155, 1950

[2] Bengt Aspvall and Richard Even Stone, *Khachiyan's Linear Programming Algorithm*, Journal of Algorithms, **1**, 1-13, 1980

[3] Claude Berge, *Graphs and Hypergraphs*, North-Holland, 1973

[4] P. Buneman, *A characterization of rigid circuit graphs*, Discrete Mathematics, **9**, 205-212, 1974

[5] Luigi Burigana, *Tree Representation of Betweenness Relations Defined by Intersection and Inclusion*, Mathematics and Social Sciences, **185**, 5-36, 2009

[6] Peter Jephson Cameron, *Sets, Logic and Categories*, Springer, 2000

[7] Benny Chor and Madhu Sudan, *A geometric approach to betweenness*, SIAM Journal on Discrete Mathematics, **11**, 511-523, 1998

[8] Vašek Chvátal, *Sylvester-Gallai Theorem and Metric Betweenness*, Discrete and Computational Geometry, **31**, 175-195, 2004

[9] Vašek Chvátal, Dieter Rautenbach, and Philipp Matthias Schäfer, *Finite Sholander trees, trees, and their betweenness*, Discrete Mathematics, **311**, 2143-2147, 2011

[10] Harold Scott MacDonald Coxeter, *Introduction to Geometry*, Wiley, New York, 1969

[11] D. Defays, *Tree Representations of Ternary Relations*, Journal of Mathematical Psychology, **19**, 208-218, 1979

[12] Reinhard Diestel, *Graph Theory*, 3rd revised edition, Springer, 2009

[13] Mitre Costa Dourado, Dieter Rautenbach, and Philipp Matthias Schäfer, *On finite convexity spaces induced by sets of paths in graphs*, Discrete Mathematics, **311**, 616–619, 2011

[14] Ivo Düntsch and Alasdair Urquhart, *Betweenness and Comparability Obtained from Binary Relations*, Lecture Notes in Computer Science, **4136**, 148-161, 2006

[15] John Frederick Fink and Michael S. Jacobson, *n-domination in graphs*, Graph Theory with Applications to Algorithms and Computer Science, John Wiley and Sons, 282-300, 1985

[16] F. Gavril, *The intersection graphs of subtrees in trees are exactly the chordal graphs*, Journal of Combinatorial Theory Series B, **16**, 47-56, 1974

[17] Martin C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Annals of Discrete Mathematics, **57**, 2004

[18] Marc Henry Graham, *On the universal relation*, Computer Systems Research Group Report, University of Toronto, 1979

[19] Paul Richard Halmos, *Naive Set Theory*, Van Nostrand, 1960

[20] Jarmila Hedlíková, *Ternary spaces, media, and Chebyshev Sets*, Czechoslovak Mathematical Journal, **33**, 373-389

[21] Michael Anthony Henning, *Distance domination in graphs*, Domination in Graphs; Advanced Topics, Marcel Dekker, Inc., 335-365, 1998

[22] David Hilbert, *Grundlagen der Geometrie*, 1899

[23] Leonid Genrikhovich Khachiyan, *A Polynomial Algorithm in Linear Programming*, Doklady Akademiia Nauk SSSR, **244**, 1979

[24] Donald Ervin Knuth, *The Art of Computer Programming. Vol. 1: Fundamental Algorithms*, second printing, Addison Wesley, 1969

[25] Dénes Kőnig, *Theorie der endlichen und unendlichen Graphen*, Leipzig: Akad. Verlagsges. mbH, 1936

[26] Bernhard Korte und Jens Vygen, *Combinatorial Optimization, Theory and Algorithms, Second Edition*, Springer, 2002

[27] Judita Lihová, *Strict Order-Betweenness*, Acta Univ. M. Belii ser. Mathematics, **8**, 27-33, 2000

[28] Yury Makarychev, *Simple linear time approximation algorithm for betweenness*, Microsoft Research Technical Report MSR-RT-2009-74, 2009

[29] T.A. McKee and F.R. McMorris, *Topics in intersection graph theory*, SIAM Monographs on Discrete Mathematics and Applications, **2**, 1999

[30] Karl Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, **100**, 75-162, 1928

[31] Maria Aurora Morgana and Henry Martyn Mulder, *The induced path convexity, betweenness, and svelte graphs*, Discrete Mathematics, **254**, 349-379, 2002

[32] Henry Martyn Mulder, *Transit Functions on Graphs (and Posets)*, Econometric Insitute Report EI 2007-13, Erasmus University Rotterdam, 2007

[33] Henry Martyn Mulder and Ladislav Nebeský, *Axiomatic characterization of the interval function of a graph*, European Journal of Combinatorics, **30**, 1172-1185, 2009

[34] Ladislav Nebeský, *A characterization of the interval function of a connected graph*, Czechsolovak Mathematical Journal, **44**, 173-178, 1994

[35] Ladislav Nebeský, *Characterizing the interval function of a connected graph*, Mathematica Bohemica, **2**, 137-144, 1998

[36] Ladislav Nebeský, *A Characterization of the Interval Funtion of a (Finite or Infinite) Connected Graph*, Czechoslovak Mathematical Journal, **51**, 635-642, 2001

[37] Ladislav Nebeský, *The induced paths in a connected graph an a ternary relation determined by them*, Mathematica Bohemia, **127**, 397-408, 2002

[38] Ladislav Nebeský, *Intervals and steps in a connected graph*, Discrete Mathematics, **286**, 151-156, 2004

[39] Ladislav Nebeský, *The interval function of a connected graph and road systems*, Discrete Mathematics, **307**, 2067-2073, 2007

[40] Jaroslav Opatrny, *Total ordering Problem*, SIAM Jornal on Computing, **8**, 111-114, 1979

[41] Moritz Pasch, *Vorlesungen über neuere Geometrie*, Teubner, Leipzig, 1882

[42] Giuseppe Peano, *I principii di Geometria logicamente esposti*, Fratelli Bocca, Torino, 1889

[43] Dieter Rautenbach, Vinícius Fernandes dos Santos, Philipp Matthias Schäfer, *Conversion Processes with Deadlines*, submitted manuscript

[44] Dieter Rautenbach, Vinícius Fernandes dos Santos, Philipp Matthias Schäfer, and Jayme Luiz Szwarcfiter, *Characterization and represenation problems for intersection betweenness*, Discrete Applied Mathematics, **159**, 389-395, 2011

[45] Dieter Rautenbach, Vinícius Fernandes dos Santos, Philipp Matthias Schäfer, and Jayme Luiz Szwarcfiter, *On subbetweenness of trees: Hardness, algorithms, and characterizations*, Computers and Mathematics with Applications, **62**, 4674-4681, 2011

[46] Dieter Rautenbach and Philipp Matthias Schäfer, *Strict Betweennesses Induced by Posets as well as by Graphs*, Order, **28**, 89-97, 2011

[47] Paul D. Seymour, *Disjoint Paths in Graphs*, Discrete Mathematics, **29**, 293-309, 1980

[48] Marlow Sholander, *Trees, Lattices, Order, and Betweenness*, Proceedings of the American Mathematical Society **3**, 369-381, 1952

[49] Wilson Alexander Sutherland, *Introduction to Metric and Topological Spaces*, Oxford University Press, 1975

[50] Oswald Veblen, *A System of Axioms for Geometry*, Transactions of the American Mathematical Society **5**, 344-384, 2004

[51] J.R. Walter, *Representation of chordal graphs as subtrees of a tree*, Journal of Graph Theory, **2**, 265-267, 1978

[52] Clement Tak Yu and M.Z. Ozsoyoglu, *An algorithm for tree-query membership of a distributed query*, Proceedings 1979 IEEE COMPSAC, Institute of Electrical and Electronic Engineers, New York, 306-312, 1979

Erklärung über Zusammenarbeit

Im Rahmen der Forschung, die zu den hier präsentierten Resultaten geführt hat, habe ich mit einigen anderen Mathematikern zusammengearbeitet.

Die Ergebnisse, die in Zusammenarbeit mit Vašek Chvátal, Mitre Costa Dourado, Dieter Rautenbach, Vinícius Fernandes dos Santos und Jayme Luiz Szwarcfiter entstanden sind, sind bereits publiziert [9, 13, 45, 44].

Eine Arbeit, die in Zusammenarbeit mit Dieter Rautenbach und Vinícius Fernandes dos Santos entstanden ist, ist zur Veröffentlichung eingereicht [43].

Danksagung

Es gibt einige Menschen, die mir meine Zeit als Doktorand erleichtert haben und denen ich dafür danken möchte:

Ich danke den oben genannten Kollegen für die angenehme Zusammenarbeit, insbesondere Vinícius der mir ein Freund geworden ist und Dieter Rautenbach, der durch seine Betreuungsarbeit zur Stressfreiheit der letzten drei Jahre beigetragen hat.

Außerdem danke ich meiner Familie. Details hierzu schreibe ich nicht in ein zu veröffentlichenden Text.

Curriculum Vitae

Philipp Matthias Schäfer

Personal Information

Date of Birth:	June 5, 1985 (Darmstadt, Germany)
Nationality:	German

Education

08.1991 – 06.1995	Elementary School in Klein-Gerau
08.1995 – 03.2004	Secondary School in Mainz
10.2004 – 10.2009	Studies of Mathematics, TU Ilmenau
10.2007 – 06.2008	Studies of Mathematics, University of Cambridge
06.2008	Certificate of Advanced Study, University of Cambridge
10.2009	Diploma in Mathematics, TU Ilmenau
11.2009 – 09.2010	PhD Student in Mathematics, TU Ilmenau PhD Scholarship by the Landesgraduiertenförderung Thüringen
03.2010 – 04.2010	Research Stay at Universidade Federal do Rio de Janeiro, Brazil
since 10.2010	PhD Student in Mathematics, University of Ulm
02.2011 – 03.2011	Research Stay at Universidade Federal do Rio de Janeiro, Brazil
03.2012	Research Stay at Universidade Ferderal do Rio de Janeiro, Brazil