Conductors of Superelliptic Curves

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vorgelegt von
Roman Kohls
aus Serenda im Jahr 2019

Amtierender Dekan:
Prof. Dr. Martin Müller

Gutachter:
Prof. Dr. Stefan Wewers
Prof. Dr. Irene Bouw
Prof. Dr. Tim Dokchitser

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Introduction

The main object of this thesis are conductors associated to algebraic curves. Given a curve Y over a global field K the conductor of Y over K, is the ideal

$$\mathfrak{N}(Y/K) = \mathfrak{D}_K^{2g} \prod_{\mathfrak{p}} \mathfrak{p}^{f(Y/K_{\mathfrak{p}})}$$

where the product runs through the prime ideals of K, g is the genus of Y, $\mathfrak{D}_K$ is the different of K and $f(Y/K_{\mathfrak{p}})$ is the conductor exponent of Y over the local field $K_{\mathfrak{p}}$. The conductor exponent of a curve Y over a local field $K_{\mathfrak{p}}$ is a measure for the ramification of the representation associated to $Y/K_{\mathfrak{p}}$. With representation of $Y/K_{\mathfrak{p}}$ we refer to the first cohomology $H^1(Y_{\bar{K}_{\mathfrak{p}}}, \mathbb{Q}_\ell)$ with its action of the absolute Galois group of $K_{\mathfrak{p}}$ where ℓ is a prime different from the residue characteristic of $K_{\mathfrak{p}}$. As an example that $f(Y/K_{\mathfrak{p}})$ contains information about ramification, we state the Neron-Ogg-Shafarevich criterion. The representation of $Y/K_{\mathfrak{p}}$ is unramified, i.e. has trivial inertia image, if and only if $f(Y/K_{\mathfrak{p}}) = 0$, or in more geometric terms, the Jacobian of Y has good reduction over $K_{\mathfrak{p}}$ if and only if $f(Y/K_{\mathfrak{p}}) = 0$. In particular, if Y has good reduction over $K_{\mathfrak{p}}$, the conductor exponent of $Y/K_{\mathfrak{p}}$ is trivial. Conductors are closely related to the L-function $L(Y/K, s)$ associated to the curve Y/K. The relation being provided by a conjectural functional equation,

$$\Lambda(Y/K, s) = w(Y/K)\Lambda(Y/K, s-2)$$

where $w(Y/K) \in \{-1, 1\}$ and

$$\Lambda(Y/K, s) = \left(\frac{\mathrm{N}(\mathfrak{N}(Y/K))}{(2\pi)^{2g}}\right)^{\frac{s}{2}} \Gamma(s)^g \, L(Y/K, s).$$

This conjecture is known only in a few cases, for instance when Y is an elliptic curve over $\mathbb{Q}$. There are many conjectures concerning these L-functions such as conjectures on special values, e.g. the Birch and Swinnerton-Dyer conjecture, or a Riemann hypothesis.

To compute the conductor for a curve over a field is by definition a local problem, i.e. one has to compute the conductor exponents for all primes of the global field. In general the computation of the conductor exponent is difficult. However, there are results for some families of curves based on different methods.

The first is using the Semistable Reduction Theorem. Knowing an extension over which the curve has semistable reduction and a semistable model of the curve, there is an explicit formula to compute the conductor exponent (cf. [BW17, Theorem 2.9]). It is even possible to read off from the semistable model the representation associated to the curve (cf. [DDM18, Corollary 1.6]). Thus, the problem is to find an extension over which the curve has semistable reduction and a semistable model. For superelliptic curves over local fields of residue characteristic not dividing the degree it is known how to construct a semistable model [BW17, §§ 3-4]. More generally this result applies to covers of a projective line which are potentially Galois, and for which the residue characteristic does not divide the order of the Galois group.

For hyperelliptic curves given by $y^2 = f$ with a polynomial f defined over a local field of odd residue characteristic, the notion of cluster pictures has been recently introduced [DDMM18]. A cluster picture contains information about the configuration of the roots of f. It is possible to read off from a cluster picture invariants of the curve, such as the conductor [DDMM18, § 11].

A third approach is the construction of regular models. In [Dok18] a method for the construction of a regular model for general curves is discussed, yielding a description of the tame part of the Galois representation of a curve. In [Liu94] the minimal regular model is used for hyperelliptic curves of genus 2 in odd residue characteristic to prove an equality which implies the inequality

$$-A(Y/K) \leqslant v_K(\Delta_0)$$

where $A(Y/K)$ is the Artin conductor of a curve Y and Δ_0 is a minimal discriminant among integral equations for Y. The conductor exponent and the Artin conductor are related through the equality

$$-A(Y/K) = f(Y/K) + m - 1$$

where m is the number of irreducible components of the minimal regular model. Under the additional hypothesis that the curve has rational Weierstraß points, this inequality was extended in [Sri15] to all hyperelliptic curves in odd residue characteristic. The rationality hypothesis implies that the wild part of the conductor exponent vanishes and the Artin conductor is then the difference between the Euler-Poincaré characteristic of the generic and special fiber of the minimal regular model. The method used in [Sri15] is to explicitly construct a regular model of the curve and write the difference between the Euler-Poincaré characteristic of the generic and special fiber of the constructed model as a sum of local terms over the vertices of the dual graph of the special fiber. Writing the minimal discriminant also as a sum over the dual graph, the inequality is obtained by comparing local terms of the two sums.

The goal of this thesis is to find bounds for the conductor exponent of a curve over a local field. Such inequalities are useful, provided the bounds can be easily computed from an equation for the curve, for creating databases of curves of fixed genus ordered by the conductor such as [LMFDB]. The first main result (Theorem 4.1.4) of this thesis is the decomposition

$$f(Y/K) = \sum_{\chi \neq \mathbf{1}} f_\chi(Y/K)$$

of the conductor exponent of a superelliptic curve Y given by the equation $y^n = f$, where for each irreducible character $\chi \neq \mathbf{1}$ of a cyclic group of order n,

$$f_\chi(Y/K) = \sum_{v \in V_{\bar{Z}}} f_\chi(v) \; - \; f_\chi(\infty) \; + \sum_{a \in B_\chi} \delta_a.$$

In this formula a runs through Galois orbits of roots of the polynomial f, and the term δ_a is a valuation of a different and only depends on f but not on n, whilst the terms $f_\chi(v)$ and $f_\chi(\infty)$ are sums of conductor exponents of at most tamely ramified characters and v runs through vertices of a dual graph of a tree of projective lines. The sum over the Galois orbits a should be considered as the main contribution to the conductor exponent. For the tame part, the sum of $f_\chi(v)$, we can give sharp bounds. This gives our second main result (Theorem 4.2.7), the bound

$$f(Y/K) \leqslant (n-1)(r-2) + \sum_{\chi \neq \mathbf{1}} \sum_{a \in B_\chi} \delta_a.$$

Here r is the number of branch point orbits under the action of the absolute Galois group. We will provide examples showing that this bound is sharp. As a third result and an application to the proven inequalities, we prove that the conductor exponent of a Picard curve over a local field of residue characteristic not 2 or 3 is bounded by the valuation of a minimal discriminant of the curve, answering a question raised by Sutherland in these cases (cf. [BKSW19, § 5.2]).

Our general method is in essence (although not in terminology) very similar to the approach via cluster pictures by [DDMM18]. We will use the semistable method discussed above to obtain the upper bound. Therefore we can not determine the number m of components of the minimal regular model. Thus, our method does not provide bounds for the Artin conductor as in [Sri15] and [Liu94]. On the other hand, since $f(Y/K) \leqslant -A(Y/K)$, the inequality for the Artin conductor shows

$$f(Y/K) \leqslant v_K(\Delta(g))$$

for some monic polynomial g defined over the ring of integers of K such that Y is isomorphic over K to the curve given by $y^2 = g$. In Chapter 5 we will use our results to show

$$f(Y/K) \leqslant (n-1)v_K(\Delta(g))$$

for some monic polynomial g defined over the ring of integers of K and Y given by $y^n = g$. Furthermore, we will see that the discriminant of a defining polynomial g is often too large to give a useful bound.

We give a brief exposition of the proof of the main result. Let Y be a superelliptic curve over K. We can assume that K is maximally unramified and hence Y is a G-cover of a projective line. Let Γ be the Galois group of a Galois extension over which Y has semistable reduction. The G- and Γ-action commute. To obtain our main result we use that the conductor exponent of a curve can be computed as a sum of the conductor exponent of the semisimplification of the representation of the curve and the dimension of the inertia invariant space of the homology of the dual graph. The semisimplification of the representation can be expressed in terms of the special fiber. By [BW17, § 3] the special fiber $\bar{Y}$ is a cyclic G-cover of a tree $\bar{X}$ of projective lines. We will exploit this G action by considering the cover $\bar{Y} \to \bar{Y}/\tilde{\Gamma}$ with $\tilde{\Gamma}$ the direct product of G and Γ. In the case that $\bar{X}$ consist of exactly one projective line, a formula of Grothendieck-Ogg-Shafarevich type yields the Euler-Poincaré characteristic of the special fiber as a virtual representation of $\tilde{\Gamma}$. We than take the G-isotypical decomposition of this representation. The general case can be reduced to the treated case by considering the normalization of the special fiber. We obtain the Euler-Poincaré characteristic considered as a virtual representation of Γ as a sum over irreducible representations of G where each summand is a sum over the dual graph of $\bar{Y}/\tilde{\Gamma}$. By the Artin formalism this decomposition passes to the conductor exponent. For each irreducible representation of G we then show local inequalities for the conductor exponents of the representations appearing in the decomposition. Here local means that we consider one vertex of the dual graph of $\bar{Y}/\tilde{\Gamma}$ which corresponds to an irreducible component and therefore to components of $\bar{Y}$ that are a G-cover of a projective line. This is used to obtain the inequalities. Summing this local inequalities yields an upper bound for the conductor exponent of a superelliptic curve.

In Chapter 1 we introduce ℓ-adic Galois representations and conductor exponents associated to ℓ-adic Galois representations of local fields. We prove that conductor exponents only depend on the image of the inertia group and that they satisfy an Artin formalism, i.e. we investigate their behaviour under taking sums and induction from subgroups. From an object in arithmetic geometry one obtains a representation for each prime number. We discuss how to compare these through the concept of Weil-Deligne representations and compatible systems. To ℓ-adic Galois representations of global fields one defines L-functions as an Euler product over invariants of local ℓ-adic representations. We give an overview of the global theory.

The main part of Chapter 2 is a collection of three result becoming important in Chapters 3 and 4. The first result we explain is that for a curve that is a Galois cover of a projective line there exists a finite Galois extension and a semistable model after a base change such that the action of the Galois group extends to an action on the model. Furthermore, the special fiber of the semistable model admits a cyclic Galois cover of a tree of projective lines. The second result is a short exact sequence enabling us to express the semisimplification of the representation associated to a curve through the action on the special fiber and the dual graph of the special fiber of a quasistable model. The last result is a Grothendieck-Ogg-Shafarevich type formula expressing the Euler-Poincaré characteristic of a Galois cover of curves in terms of Artin characters for each point of the covered curve.

The semisimplification of the representation we are interested in can be given in terms of the cohomology of the special fiber which is an abelian Galois cover of a tree of projective lines. This is the situation we investigate in Chapter 3. We first analyze the situation that the tree of projective lines consists of only one irreducible component, i.e. the special fiber $\bar{Y}$ is an abelian G-cover of a projective line with an action of a finite quotient Γ of the absolute Galois group. Then we reduce the general case of a tree of projective lines to the case of one projective line by considering the normalization of the special fiber which is a disjoint union of covers of one projective line. The result of this chapter is a G-isotypical decomposition of the Euler-Poincaré characteristic considered as a virtual representation of Γ.

In Chapter 4 we obtain the main result of this thesis. We use the second fact of Chapter 2 and the result of Chapter 3 to obtain a G-isotypical decomposition of the semisimplification of the representation associated to the superelliptic curve where the G-isotypical parts are sums of monomial Γ-representations over the dual graph of $\bar{Y}/\tilde{\Gamma}$. We obtain a similar decomposition of the conductor exponent. For each irreducible representation of G we then bound the corresponding summand. This yields an upper bound for the conductor exponent of a superelliptic curve.

In the last chapter we apply the upper bound we have derived to give an upper bound for the conductor exponent of Picard curves. First, we compare the conductor exponent of a curve Y given by the equation $y^n = f$ for some polynomial $f \in K[x]$ with the discriminant of that polynomial. It turns out that the naive conjecture

$$f(Y/K) \leqslant (n-1)v_K(\mathrm{disc}(f))$$

is true under the assumption that f is monic and all roots of f are algebraic integers, but does not hold in general. To prove this, we write the discriminant of f as sum over the dual graph of $\bar{Y}/\tilde{\Gamma}$ and compare this sum with the sum for the conductor exponent obtained in Chapter 4. After that, we consider Picard curves. As plane quartics they have a notion of minimal discriminant. We show by a case by case analysis that away from residue characteristic 2 and 3 the conductor exponent of a Picard curve is bounded by the minimal discriminant of that curve.

Notation

$\mathbb{F}_K$	residue field of the local field K
q_K	number of elements in $\mathbb{F}_K$
$f_{L/K}$	residue degree of the finite extension L/K of local fields
$\mathcal{O}_K$	ring of integers of K
$\mathfrak{p}_K$	maximal ideal in $\mathcal{O}_K$
v_K	valuation of a local field K normalized such that $v_K(\pi) = 1$ for an element $\pi \in K$ with $(\pi) = \mathfrak{p}_K$
$\mathfrak{D}_{L/K}$	different of L/K
K^{nr}	maximal unramified extension of K
$R(G)$	Grothendieck group of the category of $k[G]$-modules that are finite dimensional over a field k
V_{he}	semisimplification of the representation V
r_G	regular representation of the group G
$\mathbf{1}_G$	trivial representation of the group G, denoted $\mathbf{1}$ if the group is clear from context
Γ_K	absolute Galois group of K
$\mathrm{ind}_L^K(\rho)$	Γ_K-representation induced from the Γ_L-representation ρ
V^I	subspace of the $k[G]$-module V invariant under the subgroup I of G, i.e. $V^I = \{v \in V : \sigma v = v \text{ for all } \sigma \in I\}$
$V(\Delta)$	vertex set of a graph Δ
$E(\Delta)$	edge set of a graph Δ
Y_L	the curve $Y \otimes_K L$ where Y is a curve over a field K and L/K is an extension

Chapter 1

Galois Representations

In this chapter we discuss the definition of the conductor exponent associated to Galois representations and prove that these conductors satisfy the same formalism as the L-functions introduced by Artin. After recalling basic facts about the semisimplification of representations and the definition of conductor exponents for representations on finite Galois groups in the first two sections, we will give in § 1.3 a proof of a theorem of Grothendieck on ℓ-adic representations stating that every ℓ-adic representation of a local field is potentially unipotent on the inertia group [ST68, Appendix]. This theorem makes it possible to define the conductor of a general ℓ-adic representation defining the wild part of the conductor as the wild conductor of its semisimplification which has finite inertia image by Grothendieck. This was first done by Serre in [Ser70, § 2]. In § 1.4 we take another approach to define a conductor generalizing a formula in the finite image case. This definition appears in [Wie12, 3.1.27] and [Ulm15]. We then show that our definition yields the same as the one given by Serre and prove that the conductor of an ℓ-adic representation satisfies the Artin formalism. In the next section we explain how to associate to an ℓ-adic representation a Weil-Deligne representation. These were introduced by Deligne [Del73, § 8] and give the possibility to compare ℓ-adic representations for different primes ℓ. Such an comparison becomes important, when considering the first cohomology group of a smooth projective curve. In the last section we consider the global setting. Adapting Artin's definition [Art24], [Art31] of the L-functions for a non-abelian extension of global fields, we define L-functions and conductors of compatible systems of Galois representations of global fields. As this definition is given as an Euler product of conductors (or L-functions) of local fields, we will focus in the rest of this thesis on the local case. At the end of this chapter, we state the functional equation conjectured for the global L-function of a curve as well as the conjectured Riemann Hypothesis for this L-function.

§ 1.1 Semisimple Representations and the Grothendieck Group

In this section we introduce the basic representation theoretic notions which will be used throughout this thesis.

Every representation $\varrho : G \to \mathrm{GL}(V)$ of a group G on a vector space V over a field k induces on V the structure of a $k[G]$-module. On the other hand, every $k[G]$-module V defines a representation $\varrho : G \to \mathrm{GL}(V)$. Therefore, we use these notions interchangeably and take in this section the more general situation of (left) modules over (non-commutative) rings.

Definition 1.1.1 Let A be a ring and V a A-module.

(1) The module V is called irreducible (or simple) if (0) and V are the only A-submodules of V.

(2) The module V is called semisimple if it is a direct sum of irreducible modules.

Theorem 1.1.2 ([CR62, 13.4, 13.7]) *Let A be an algebra over a field k and V an A-module that is finite dimensional as a k-module. Then there exists a descending chain*

$$V = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_n \supsetneq (0)$$

of A-submodules, called a composition series, such that the quotients V_i/V_{i+1}, called composition factors, are irreducible A-modules. Moreover, all composition series of V have the same length and (after renumbering if necessary) isomorphic composition factors.

Definition 1.1.3 Let A be an algebra over a field k and V a A-module that is finite dimensional as a k-module. A semisimplification of V is the direct sum of the composition factors in a composition series of V.

By the Theorem of Jordan-Hölder (Theorem 1.1.2) all semisimplifications of a $k[G]$-module V are isomorphic. We denote the isomorphism class of all semisimplifications of V by V_{he} and call it the semisimplification of V. The semisimplification of V defines an element of the Grothendieck group which is defined as follows.

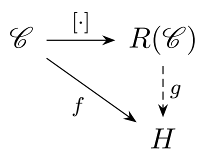
Definition 1.1.4 Let A be a ring and $\mathscr{C}$ a category of A-modules.

(1) Let G be an abelian group. A map $f : \mathscr{C} \to G$ is called additive if for all short exact sequences

$$0 \to V_1 \to V \to V_2 \to 0$$

in $\mathscr{C}$, $f(V) = f(V_1) + f(V_2)$ holds.

(2) A Grothendieck group of $\mathscr{C}$ is a pair consisting of an abelian group $R(\mathscr{C})$ and a map $[\cdot] : \mathscr{C} \to R(\mathscr{C})$ having the following universal property. For every abelian group H and every additive map $f : \mathscr{C} \to H$ there is a unique group homomorphism $g : R(\mathscr{C}) \to H$ such that $f = g \circ [\cdot]$, i.e. such that the diagram



commutes.

Remark 1.1.5 (1) Let k be a field, G a group and $\mathscr{C}$ the category of $k[G]$-modules that are finite dimensional over k. Then the Grothendieck group of $\mathscr{C}$ exists and is by definition unique up to isomorphism. It can be constructed as the free abelian group generated by isomorphism classes of irreducible objects of $\mathscr{C}$ such that for a short exact sequence

$$0 \to V_1 \to V \to V_2 \to 0$$

of objects of $\mathscr{C}$, $[V_1] + [V_2] = [V]$ holds where $[M]$ denotes the isomorphism class of M. In this case we denote the Grothendieck group of $\mathscr{C}$ by $R_k(G)$ or $R(G)$ if k is clear from the context.

(2) It follows from the definition that for an exact sequence

$$0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$$

of Objects of $\mathscr{C}$ we have

$$\sum_{i=1}^{n} (-1)^i [V_i] = 0$$

in the Grothendieck group.

(3) In the Grothendieck group $[V] = V_{\mathrm{he}}$ holds. Indeed, for a composition series

$$V = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_n \supsetneq (0)$$

we have exact sequences

$$0 \to V_{i+1} \to V_i \to V_i/V_{i+1} \to 0$$

and therefore

$$[V] = \sum_{i=1}^{n} [V_i/V_{i+1}] = V_{\mathrm{he}}.$$

In the rest of this chapter we collect results about semisimple modules which we use throughout this thesis. Let A be an algebra over a field k and V an A-module finite dimensional as a k-module. For $a \in A$ we have an endomorphism of V given by $l_a : v \mapsto av$. The map $A \to k, a \mapsto \mathrm{tr}(l_a)$ is called the character of V. In the case $A = k[G]$ for a group G the character is determined on its restriction to G and therefore we call this restriction also the character of V. In this case the character is $g \mapsto \mathrm{tr}(\varrho(g))$ for the representation ϱ corresponding to the $k[G]$-module V. In characteristic 0 the isomorphism classes of semisimple modules are determined by their character.

Theorem 1.1.6 ([Lan02, XVII.3.8]) *Let G be a group, k be a field of characteristic 0 and let V and W be semisimple $k[G]$-modules of finite dimension over k. The characters of V and W coincide if and only if V and W are isomorphic $k[G]$-modules.*

The following theorem is due to Clifford [Cli37].

Theorem 1.1.7 *Let G be a group, N a normal subgroup of G and let V be semisimple $k[G]$-modules of finite dimension over k. Then V is semisimple as a $k[N]$-module.*

Definition 1.1.8 Let V be a semisimple A-module. We call V isotypical if all irreducible submodules of V are isomorphic.

Let V be a semisimple A-module. Then V is direct sum of isotypical A-modules. If

$$V = \bigoplus_{i=1}^{n} V_i$$

is a decomposition into isotypical submodules such that for every pair V_i, V_j of the modules $V_1, \dots, V_n$ with $i \neq j$ an irreducible submodule of V_i is not isomorphic to an irreducible submodule of V_j, this decomposition is called the isotypical decomposition of V. The decomposition into isotypical components is canonical (cf. [Ser77, § 2.6]).

§ 1.2 Conductor Exponents of Galois Representations with finite image

The definition of conductor exponents of representations of the absolute Galois group of a local field is based on the case of Galois representations with finite image. These factor through a finite Galois group. We give a short summary of the results. For more details see [Ser79, VI].

Let L/K be a finite Galois extension of local fields. Then the Galois group Γ of L/K has a filtration $(\Gamma_i)_{i \geqslant -1}$ by the normal subgroups

$$\Gamma_i = \{\sigma \in \Gamma : v_L(\alpha - \sigma(\alpha)) \geqslant i + 1\},$$

the so called ramification groups. Here, $\alpha \in \mathcal{O}_L$ is an element such that $\mathcal{O}_K[\alpha] = \mathcal{O}_L$. We define functions i_Γ and a_Γ on Γ by setting $i_\Gamma(\sigma) = v_L(\alpha - \sigma(\alpha))$ and

$$a_\Gamma(\sigma) = \begin{cases} -f_{L/K}\, i_\Gamma(\sigma) & \sigma \neq 1 \\ f_{L/K} \displaystyle\sum_{\tau \in \Gamma \smallsetminus \{1\}} i_\Gamma(\sigma) & \sigma = 1 \end{cases} \cdot$$

The function a_Γ is called Artin character of L/K. Both functions i_Γ and a_Γ take values in $\mathbb{Z}$ and are class functions, i.e. are constant on conjugacy classes of Γ. In the following all representations are defined over a field E of characteristic 0. Then by Maschke's Theorem [CR62, 15.6] and the finiteness of Γ, all representations of Γ are semisimple. Therefore, all representations are determined by their character by Theorem 1.1.6.

Lemma 1.2.1 *Let L/K be a finite Galois extension of local fields with Galois group Γ. Then*

$$v_L(\mathfrak{D}_{L/K}) = \sum_{\sigma \neq 1} i_\Gamma(\sigma) = \sum_{i \geqslant 0} (|\Gamma_i| - 1).$$

Proof The first equality follows from $\mathfrak{D}_{L/K} = (f'(\alpha))$ where f' is the derivative of the minimal polynomial f of α over K and α is any element such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. The second equality holds since $i_\Gamma(\sigma) = i$ for $\sigma \in \Gamma_{i-1} \setminus \Gamma_i$. More details can be found in [Ser79, IV.1 Proposition 4]. □

Lemma 1.2.2 *Let Γ be the Galois group of a finite extension of local fields. Then*

$$a_\Gamma = \sum_{i \geqslant 0} \frac{1}{(\Gamma_0 : \Gamma_i)} \operatorname{ind}_{\Gamma_i}^{\Gamma}(r_{\Gamma_i} - \mathbf{1}).$$

Proof For $i \geqslant 0$ set $a_i = \operatorname{ind}_{\Gamma_i}^{\Gamma}(r_{\Gamma_i} - \mathbf{1})$. For $i \geqslant 0$, the subgroup Γ_i is normal in Γ. Therefore, by the formula for the induced character [Ser77, § 7.2] we have $a_i(\sigma) = 0$ if $\sigma \notin \Gamma_i$, and $a_i(\sigma) = -(\Gamma : \Gamma_i)$ if $\sigma \in \Gamma_i \setminus \{1\}$. We obtain

$$\sum_{i \geqslant 0} \frac{1}{(\Gamma_0 : \Gamma_i)} a_i(\sigma) = -(\Gamma : \Gamma_0) v_L(\alpha - \sigma(\alpha)) = a_\Gamma(\sigma)$$

for all $\sigma \neq 1$. For $\sigma = 1$, we have $a_i(1) = (\Gamma : \Gamma_i)(|\Gamma_i| - 1)$ and

$$\sum_{i \geqslant 0} \frac{1}{(\Gamma_0 : \Gamma_1)} a_i(1) = f_{L/K} \sum_{i \geqslant 0} (|\Gamma_i| - 1) = f_{L/K} \sum_{\sigma \neq 1} v_L(\alpha - \sigma(\alpha)) = a_\Gamma(1)$$

by Lemma 1.2.1. □

Theorem 1.2.3 ([Ser79, VI.2 Theorem 1]) *Let Γ be the Galois group of a finite extension of local fields. The Artin character a_Γ is a character of a representation of Γ.*

The proof amounts to showing that $(\chi \,|\, a_\Gamma)$ is a non-negative integer for every character χ of Γ defined over an algebraic closure $\bar{E}$ of E, since over $\bar{E}$ the space of class functions is spanned by the characters. Here, $(f \,|\, g)$ is defined as

$$(f \,|\, g) = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} f(\sigma) g(\sigma^{-1})$$

for class functions f and g.

Definition 1.2.4 Let Γ be the Galois group of a finite extension of local fields and let ϱ be a representation of Γ with character χ. We call $f(\varrho) = (\chi \,|\, a_\Gamma)$ the conductor exponent of ϱ.

We summarize properties of the conductor exponent.

Theorem 1.2.5 *Let Γ be the Galois group of a finite extension of local fields L/K.*

(a) Let ϱ be a representation of Γ. Then $f(\varrho)$ is a non-negative integer.

(b) *The assignment $\varrho \mapsto f(\varrho)$ is additive and induces therefore a group homomorphism $R(G) \to \mathbb{Z}$. In particular, if*

$$0 \to \varrho_1 \to \varrho \to \varrho_2 \to 0$$

is an exact sequence of representations of Γ, then $f(\varrho) = f(\varrho_1) + f(\varrho_2)$.

(c) *Let M be a intermediate field of the extension L/K, $\Gamma' = \mathrm{Gal}(L/M)$ and ϱ a representation of Γ'. Then*

$$f(\mathrm{ind}_{\Gamma'}^{\Gamma}(\varrho)) = f_{M/K} f(\varrho) + \dim(\varrho) v_M(\mathfrak{D}_{M/K}).$$

(d) *Let V be a representation of Γ. Then*

$$f(V) = \sum_{i \geqslant 0} \frac{1}{(\Gamma_0 : \Gamma_i)} \dim\bigl(V/V^{\Gamma_i}\bigr).$$

Proof We give the idea of the proof. More details can be found in [Ser79, VI.2]. Part (b) is a consequence of the definition of the conductor exponent. Part (c) follows from Frobenius reciprocity and the definition of the conductor exponent. Lemma 1.2.2 and Frobenius reciprocity imply (d). Using the other parts of this theorem and the Brauer induction theorem [Ser77, § 10.5] and the Hasse-Arf theorem [Ser79, V.7], yields (a).□

§ 1.3 ℓ-adic Galois Representations

In this section we prove that ℓ-adic Galois representations of local fields are potentially unipotent which makes it possible to define conductor exponents for these representations. The result is due to Grothendieck [ST68, Appendix]. We begin by giving the definition for Galois representations.

Definition 1.3.1 Let K be a field and Γ_K the absolute Galois group of K. Let E be a topological field and V an E-vector space. A homomorphism of topological groups

$$\varrho : \Gamma_K \to \mathrm{GL}(V)$$

is called a Galois representation of K over E. If $K/\mathbb{Q}_p$ is a finite extension for a prime $p \in \mathbb{Z}$ and $E/\mathbb{Q}_\ell$ is an extension for a prime number $\ell \neq p$, we call ϱ an ℓ-adic Galois representation.

We will interchangeably use ϱ and V to denote a representation $\varrho : G \to \mathrm{GL}(V)$ of a group G. In this thesis all ℓ-adic representations are finite dimensional ℓ-adic Galois representations. We will always assume this without mentioning it.

We recall basic facts about Galois groups of local fields. Let $K/\mathbb{Q}_p$ be a finite extension and denote by $\mathbb{F}_K$ the residue field of K. We choose once and for all an algebraic closure $\bar{K}$ of K and consider any algebraic extension L/K to be an intermediate extension of

$\bar{K}/K$. The inertia group I_K is defined as the absolute Galois group of the maximal unramified extension K^{nr}. There is a short exact sequence

$$1 \to I_K \to \Gamma_K \to \Gamma_{\mathbb{F}_K} \to 1.$$

An element of Γ_K mapping under the reduction map $\Gamma_K \to \Gamma_{\mathbb{F}_K}$ to the inverse of the automorphism $x \mapsto x^{|\mathbb{F}_K|}$ is called a geometric Frobenius element. Any finite tamely ramified extension of K^{nr} is cyclic and given by adjoining an mth root of a prime element of K where $p \nmid m$. Denoting by K^{tr} the maximal tamely ramified extension of K and by P_K its absolute Galois group we have a canonical isomorphism

$$I_K/P_K \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell(1).$$

The group P_K is called wild inertia group and is a pro-p group since every finite extension of K^{tr} is a p-group. For a prime $\ell \neq p$ let $K^{\mathrm{tr},\ell}$ be the maximal pro-ℓ extension of K^{nr}. It is a tamely ramified extension and therefore given by adjoining all ℓ-power roots of a prime element of K. Denoting by $I_{K,\ell}$ its absolute Galois group, there is a canonical isomorphism

$$I_K/I_{K,\ell} \simeq \mathbb{Z}_\ell(1).$$

Lemma 1.3.2 *Let ϱ be an ℓ-adic Galois representation of K. The image of $I_{K,\ell}$ under ϱ is finite.*

Proof Let $\varrho : \Gamma_K \to \mathrm{GL}(V)$ be defined over E. The stabilizer H of a lattice L in V is open in Γ_K and therefore Γ_K/H is finite. It follows that the sum Λ over the lattices gL where g runs through a representation system of Γ_K/H is a Γ_K-stable lattice in V. Choosing an $\mathcal{O}_E$-basis of Λ we have $\varrho : \Gamma_K \to \mathrm{GL}(\Lambda) \simeq \mathrm{GL}_n(\mathcal{O}_E)$. The kernel N of the reduction map $\mathrm{GL}_n(\mathcal{O}_E) \to \mathrm{GL}_n(\mathbb{F}_E)$ is a pro-ℓ group, for it is the limit of its quotients by the group of matrices congruent to the identity modulo ℓ^n. Since by definition $I_{K,\ell}$ is the absolute Galois group of the maximal pro-ℓ extension of K^{nr}, $\varrho(I_{K,\ell}) \cap N$ is trivial. Therefore, we have an injection of $\varrho(I_{K,\ell})$ into the finite group $\mathrm{GL}_n(\mathbb{F}_E)$. □

Theorem 1.3.3 *Let ϱ be a finite dimensional ℓ-adic representation of K. Then there exists a finite extension L/K such that $\varrho(\sigma)$ is unipotent for every $\sigma \in I_L$.*

Proof By Lemma 1.3.2 we can assume after a finite extension of K that $\varrho(I_{K,\ell})$ is trivial. Thus $\varrho|_{I_K}$ factors through the Galois group of $K^{\mathrm{nr}}(\pi^{1/\ell^n} : n \in \mathbb{N})/K^{\mathrm{nr}}$. For σ in this Galois group and a geometric Frobenius φ_K we get $\varphi_K^{-1}\sigma\varphi_K = \sigma^q$ where $q = |\mathbb{F}_K|$. Let $\sigma \in I_K/I_{K,\ell} \simeq \mathbb{Z}_\ell(1)$ be a topological generator. Since $\varrho(\sigma)$ and $\varrho(\sigma^q)$ have the same eigenvalues, the eigenvalues of $\varrho(\sigma)$ are roots of unity and thus there exists n such that $\varrho(\sigma^n)$ is unipotent. The closure of $\langle\sigma^n\rangle$ is of finite index in $I_K/I_{K,\ell}$, and for all τ in this closure, $\varrho(\tau)$ is unipotent. □

§ 1.4 Conductor associated to ℓ-adic Representations

The goal of this section is to extend the definition of the conductor exponent to all ℓ-adic representations of local fields and show that this conductor exponent satisfies an Artin formalism (Theorem 1.2.5 $(b), (c)$).

Let L/K be a finite Galois extension of local fields. Denote the Galois group of L/K by Γ and the i-th ramification group by Γ_i. For a normal subgroup N of Γ Herbrand's Theorem [Ser79, IV.3 Proposition 14] can be stated as $\Gamma_i N/N = (\Gamma/N)_u$ with $u = \phi(i)$ where ϕ is the Herbrand function of L/K defined as

$$\phi(x) = \int_0^x \frac{\mathrm{d}t}{(\Gamma_0 : \Gamma_t)}.$$

As a piecewise linear, continuous and increasing function ϕ is invertible. Denote its inverse by ψ. Setting $\Gamma^u = \Gamma_{\psi(u)}$ for $u \geqslant -1$ the above stated theorem of Herbrand becomes $(\Gamma/N)^u = \Gamma^u N/N$ for a normal subgroup N of Γ. This allows us to define ramification group also for the case that L/K is an infinite Galois extension.

Definition 1.4.1 Let L/K be an (infinite) Galois extension of local fields with Galois group G and $u \geqslant -1$. We call

$$G^u = \varprojlim_M \mathrm{Gal}(M/K)^u,$$

where M runs through the finite Galois extensions of K inside L, the uth ramification group of L/K.

With this definition each Galois extension L/K of local fields comes with a filtration $(\Gamma^u)_{u \geqslant -1}$ of its Galois group Γ where $I = \Gamma^0$ is the inertia group of L/K, the quotient of the inertia groups of K and L, and $P = \Gamma^{>0}$, the union of Γ^u for $u > 0$, is the wild inertia group of L/K which is a pro-p group and the quotient of the wild inertia groups of K and L. Herbrand's Theorem also holds in the setting of infinite Galois extensions for normal subgroups with finite index in the Galois group.

Let ϱ be an ℓ-adic representation of K. If the image of I_K under ϱ is finite, let $G = \Gamma_K/\ker(\varrho)$. Then the indices $(G_0 : G_i)$ are finite and the conductor exponent of ϱ can be defined as

$$f(\varrho) = \sum_{i \geqslant 0} \frac{1}{(G_0 : G_i)} \dim\big(\varrho/\varrho^{G_i}\big).$$

Using that $G_i = G_{[i]+1}$ for $i \notin \mathbb{Z}$, where $[i]$ is the greatest integer smaller than i, the sum definig the conductor can be written as an integral and after substituting $i = \phi(u)$ we obtain

$$f(\varrho) = \int_{-1}^{\infty} \frac{1}{(G_0 : G_i)} \dim\big(\varrho/\varrho^{G_i}\big)\,\mathrm{d}i = \int_{-1}^{\infty} \dim\big(\varrho/\varrho^{G^u}\big)\,\mathrm{d}u. \tag{1.1}$$

The right hand side makes also sense for representations with infinite inertia image.

Definition 1.4.2 Let V be a finite dimensional ℓ-adic representation of a local field K. The number

$$f(V) = \int_{-1}^{\infty} \dim\bigl(V/V^{\Gamma_K^u}\bigr)\, \mathrm{d}u$$

is called the conductor exponent of V. We call

$$\delta(V) = \int_{0}^{\infty} \dim\bigl(V/V^{\Gamma_K^u}\bigr)\, \mathrm{d}u$$

the wild part and $\epsilon(V) = \dim(V) - \dim\bigl(V^{I_K}\bigr)$ the tame part of the conductor exponent of V.

In the rest of this section we justify this definition by proving that a version of Theorem 1.2.5 holds. The following observation is important in the rest of this section as well as in Chapters 3 and 4. It allows for the computation of the conductor exponent to pass to the inertia group.

Lemma 1.4.3 *Let ϱ be an ℓ-adic representation of K and let ϱ_0 be the restriction of ϱ to the inertia group I_K. Then $f(\varrho) = f(\varrho_0)$.*

Proof Note that $\Gamma_K^u \subset I_K$ for $u > -1$. This implies $\dim\bigl(\varrho^{\Gamma_K^u}\bigr) = \dim\Bigl(\varrho_0^{\Gamma_K^u}\Bigr)$ for all $u > -1$. By the definition of the conductor exponent the lemma follows. □

By Theorem 1.3.3 every ℓ-adic representation is unipotent on a finite index subgroup of I_K. For semisimple representations this amounts to

Lemma 1.4.4 *Let ϱ be a semisimple ℓ-adic representation of K. Then there exists a finite Galois extension L/K with Galois group Γ such that ϱ is trivial on I_L and*

$$f(\varrho) = \sum_{i \geqslant 0} \frac{1}{(\Gamma_0 : \Gamma_i)} \dim\bigl(\varrho/\varrho^{\Gamma_i}\bigr).$$

Proof By Theorem 1.3.3 there exists a finite extension L/K such that $\varrho(\sigma)$ is unipotent for all $\sigma \in I_L$. By extending L we can assume that L/K is a finite Galois extension. The restriction ϱ_1 of ϱ to the normal subgroup I_L is semisimple by Theorem 1.1.7 and its character is given by $\chi(\sigma) = \dim(\varrho)$ for all $\sigma \in I_L$ since $\varrho(\sigma)$ is unipotent for $\sigma \in I_L$. By Theorem 1.1.6, ϱ is trivial on I_L. Let Γ be the Galois group of the finite extension L/K. Herbrand's Theorem yields $\Gamma^u = \Gamma_K^u/\Gamma_K^u \cap \Gamma_L$ for all $u > -1$. Since $\Gamma_K^u \cap \Gamma_L \subset I_L$ for all $u > -1$ and ϱ is trivial on I_L, we obtain $\dim\bigl(\varrho^{\Gamma_K^u}\bigr) = \dim\bigl(\varrho^{\Gamma^u}\bigr)$ and therefore

$$f(\varrho) = \int_{-1}^{\infty} \dim\bigl(\varrho/\varrho^{\Gamma^u}\bigr)\, \mathrm{d}u.$$

Now the same computation as in (1.1) yields the lemma. □

A semisimple ℓ-adic representation has finite inertia image. Therefore, for conductor exponents of semisimple representations we obtain an analog of Theorem 1.2.5.

Theorem 1.4.5 *Let ϱ, ϱ_1 and ϱ_2 be semisimple ℓ-adic representations of L.*

(i) If there is a short exact sequence $0 \to \varrho_1 \to \varrho \to \varrho_2 \to 0$, the conductor exponent of ϱ is $f(\varrho) = f(\varrho_1) + f(\varrho_2)$.

(ii) If L is a finite extension of K, $f(\mathrm{ind}_L^K(\varrho)) = f_{L/K} f(\varrho) + \dim(\varrho) v_L(\mathfrak{D}_{L/K})$.

(iii) If ϱ is one dimensional, $f(\varrho) = 1 + \inf\{u > -1 : \varrho|_{\Gamma_L^u} = \mathbf{1}\}$.

Proof Part (*iii*) follows by Definition 1.4.2. By Lemma 1.4.4 there exists a finite Galois extension M/K such that the restrictions of ϱ, ϱ_1 and ϱ_2 to I_L factor through the finite inertia group $\Gamma_0 = I_L/I_M$. We denote the corresponding representations of $\Gamma_0 = \mathrm{Gal}(M^{\mathrm{nr}}/L^{\mathrm{nr}})$ by ϱ', ϱ_1' and ϱ_2'. The same lemma and Theorem 1.2.5 (*d*) show $f(\varrho) = f(\varrho')$, $f(\varrho_1) = f(\varrho_1')$ and $f(\varrho_2) = f(\varrho_2')$ where the right hand side is defined by Definition 1.2.4. Therefore, (*i*) follows by Theorem 1.2.5 (*b*). For (*ii*), by Mackey's formula [Ser77, 7.3 Proposition 22] we obtain

$$\mathrm{ind}_L^K(\varrho)|_{I_K} \simeq \bigoplus_{\sigma \in \Gamma_K/\Gamma_L I_K} \mathrm{ind}_{I_{\sigma^{-1}(L)}}^{I_K}((\varrho|_{I_L})^\sigma)$$

where $(\varrho|_{I_L})^\sigma(h) = \varrho(\sigma h \sigma^{-1})$ for $h \in I_{\sigma^{-1}(L)}$, since I_K is normal in Γ_K. We have

$$\dim\Big(\mathrm{ind}_{I_L}^{I_K}(\varrho|_{I_L})^{\Gamma_K^u}\Big) = \dim\Big(\mathrm{ind}_{I_{\sigma^{-1}(L)}}^{I_K}((\varrho|_{I_L})^\sigma)^{\Gamma_K^u}\Big)$$

for $\sigma \in \Gamma_K$ and $u > -1$. This follows from the fact that higher ramification groups of L and of $\sigma^{-1}(L)$ are conjugate and from the formula

$$\mathrm{ind}_H^G(V)^N = \mathrm{ind}_{\bar{H}}^{\bar{G}}\big(V^{N\cap H}\big)$$

with $\bar{G} = G/N$, $\bar{H} = H/H \cap N$ for a group G, a finite index subgroup H, a normal subgroup N of G and a representation V of H. Therefore, by Definition 1.4.2

$$f\Big(\mathrm{ind}_{I_L}^{I_K}(\varrho|_{I_L})\Big) = f\Big(\mathrm{ind}_{I_{\sigma^{-1}(L)}}^{I_K}((\varrho|_{I_L})^\sigma)\Big)$$

for $\sigma \in \Gamma_K$. Herbrand's Theorem yields $|\Gamma_K/\Gamma_L I_K| = f_{L/K}$. Putting everything together and using Lemma 1.4.3, we obtain

$$f(\mathrm{ind}_L^K(\varrho)) = f_{L/K} f(\mathrm{ind}_{I_L}^{I_K}(\varrho|_{I_L})) = f_{L/K} f(\mathrm{ind}_{\Gamma_0}^{\Gamma_0'}(\varrho'))$$

where $\Gamma_0' = \mathrm{Gal}(M^{\mathrm{nr}}/K^{\mathrm{nr}})$. Theorem 1.2.5 (*c*) applies to ϱ' which shows (*ii*). □

Theorem 1.4.6 *The conductor exponent of a semisimple ℓ-adic representation is a non-negative integer.*

Proof Using Lemma 1.4.4 as in the previous proof, there exists a finite extension L/K such that $\varrho|_{I_K}$ factors through a representation of $\mathrm{Gal}(L^{\mathrm{nr}}/K^{\mathrm{nr}})$. Thus, Theorem 1.2.5 (*a*) applies to $\varrho|_{I_K}$ and with Lemma 1.4.3 the theorem follows. □

Remark 1.4.7 Theorem 1.4.5 (*ii*) is referred to as inductivity in dimension 0, since for a semisimple element in the Grothendieck group of dimension 0, induction does not change the conductor exponent of that element. The assignment $\varrho \mapsto f(\varrho)$ for semisimple ϱ is additive by Theorem 1.4.5 (*i*). A version of the Brauer induction theorem for dimension 0 representations given by Deligne [Del73, Proposition 1.5] shows that such an assignment that is also inductive in dimension 0 is uniquely determined by the values it takes for one-dimensional representations. Identifying one-dimensional representations of Γ_K with one-dimensional representations of K^* through Artin reciprocity of local class field theory, Theorem 1.4.5 (*iii*) shows that the conductor exponent of ϱ with $\dim(\varrho) = 1$ is the smallest u such that ϱ is trivial on $1 + \mathfrak{p}_K^u$ or – in other words – the valuation of the conductor defined in class field theory (cf. [Neu92, § 5.1], [Ser79, XV.2]). In this sense f is the unique assignment from pairs consisting of a local field and a semisimple ℓ-adic representation of that field that is additive, inductive in dimension 0 and extends the conductor exponent of class field theory to higher dimensional representations.

The case of a general ℓ-adic representation can be reduced to the semisimple case.

Theorem 1.4.8 *Let ϱ be an ℓ-adic representation of K. Then $\delta(\varrho) = \delta(\varrho_{\mathrm{he}})$. In particular,*

$$f(\varrho) = \dim\left(\varrho_{\mathrm{he}}^{I_K}\right) - \dim\left(\varrho^{I_K}\right) + f(\varrho_{\mathrm{he}}).$$

Proof The image of the pro-p subgroup P_K under ϱ is finite by Lemma 1.3.2 and thus $\varrho|_{P_K}$ is semisimple. Since restrictions of semisimple representations to normal subgroups are semisimple by Theorem 1.1.7, $\varrho_{\mathrm{he}}|_{P_K}$ is semisimple. The semisimple representations $\varrho|_{P_K}$ and $\varrho_{\mathrm{he}}|_{P_K}$ are isomorphic because they have the same character. This implies $\delta(\varrho) = \delta(\varrho_{\mathrm{he}})$. The second statement is a direct consequence of the first.□

Now we can state a version of Theorem 1.2.5 for general ℓ-adic representations.

Theorem 1.4.9 *Let ϱ and ϱ' be ℓ-adic representations of L.*

- *(i) The conductor exponent of the sum of ϱ and ϱ' is $f(\varrho \oplus \varrho') = f(\varrho) + f(\varrho')$.*
- *(ii) If L/K is a finite extension, $f(\mathrm{ind}_L^K(\varrho)) = f_{L/K} f(\varrho) + \dim(\varrho) v_L(\mathfrak{D}_{L/K})$.*
- *(iii) If ϱ is one dimensional, $f(\varrho) = 1 + \inf\{u > -1 : \varrho|_{\Gamma_L^u} = \mathbf{1}\}$.*
- *(iv) The conductor exponent of ϱ is a non-negative integer.*

Proof Theorem 1.4.8 and Theorem 1.4.5 (*i*) yield (*i*). Part (*iv*) follows by Theorem 1.4.8 and Theorem 1.4.6. The definition of the conductor exponent implies (*iii*). For (*ii*), we use that

$$\mathrm{ind}_H^G(V)^N = \mathrm{ind}_{\bar{H}}^{\bar{G}}\left(V^{N \cap H}\right)$$

with $\bar{G} = G/N$, $\bar{H} = H/H \cap N$ holds for a group G, a finite index subgroup H, a normal subgroup N of G and a representation V of H. This implies

$$\dim\left(\mathrm{ind}_L^K(V)^{I_K}\right) = f_{L/K} \dim\left(V^{I_L}\right)$$

for $V \in \{\varrho_{\mathrm{he}}, \varrho\}$. Note that the restrictions of $\mathrm{ind}_L^K(\varrho_{\mathrm{he}})$ and $\mathrm{ind}_L^K(\varrho)_{\mathrm{he}}$ to I_K have finite inertia image and that their characters coincide. Thus, they are isomorphic and (*ii*) is a consequence of Theorem 1.4.8 and Theorem 1.4.5 (*ii*). □

Remark 1.4.10 The conductor exponent of an ℓ-adic representation is in general not additive since taking inertia invariants does not define an exact functor.

§ 1.5 Weil-Deligne Representations

In this section we introduce Weil-Deligne representations and prove that the categories of ℓ-adic representations and Weil-Deligne representations are equivalent. This makes it possible to compare ℓ-adic representations defined over different fields. For more on Weil groups and Weil-Deligne representations we refer to [Tat79, §§ 1, 4], [BH06, §§ 28, 32-33] and [Roh94, Part I].

Consider the discrete and dense subgroup Z of $\Gamma_{\mathbb{F}_K}$ generated by the automorphism $x \mapsto x^{|\mathbb{F}_K|}$. Let W_K be the subgroup of Γ_K such that the sequence

$$1 \to I_K \to W_K \to Z \to 1$$

is exact. We make W_K into a topological group by requiring that the topology on W_K is the coarsest topology such that the maps in the above exact sequence are continuous where we view I_K as topological group with the relative topology of Γ_K on it.

Definition 1.5.1 Let K be a local field. The topological group W_K defined above is called the Weil group of K.

The inclusion $W_K \hookrightarrow \Gamma_K$ is continuous and therefore every continuous representation of Γ_K gives by restriction to W_K a continuous representation of W_K. Since W_K is dense in Γ_K, an continuous representation of Γ_K is determined by its restriction to W_K. For a finite extension L/K the Weil group of L can be identified with an open subgroup of W_K and there is an bijection $W_K/W_L \to \Gamma_K/\Gamma_L$ which is a group isomorphism if L/K is a Galois extension.

Definition 1.5.2 Let K be a local field and V a finite dimensional, discrete $\mathbb{C}$-vector space. A Weil representation of K is a topological group homomorphism $W_K \to \mathrm{GL}(V)$.

Lemma 1.5.3 *Let K be a local field and V a finite dimensional, discrete $\mathbb{C}$-vector space. A group homomorphism $\varrho : W_K \to \mathrm{GL}(V)$ is a Weil representation if and only if the image of an open subset under ϱ is trivial.*

Proof This is a direct consequence of the compactness of I_K. □

Theorem 1.5.4 *Let $\varrho_\ell : W_K \to \mathrm{GL}(V_\ell)$ be an ℓ-adic representation and $t_\ell : I_K \to \mathbb{Z}_\ell$ a surjective, topological group homomorphism and φ_K a geometric Frobenius of a local field K.*

(a) There is a unique nilpotent endomorphism N_ℓ *of* V_ℓ *such that*

$$\varrho_\ell(\sigma) = e^{t_\ell(\sigma)N_\ell}$$

for all σ *in an open subgroup of* I_K*.*

(b) The map $\rho : W_K \to \mathrm{GL}(V_\ell)$ *given by*

$$\rho(\varphi_K^m \sigma) = \varrho_\ell(\varphi_K^m \sigma) e^{-t_\ell(\sigma)N_\ell}$$

is a group homomorphism which is trivial on an open subgroup of I_K*.*

(c) We have $\rho(\sigma)N_\ell\rho(\sigma)^{-1} = \omega_K(\sigma)N_\ell$ *for all* $\sigma \in W_K$*. Here* ω_K *is the unramified Weil of* K *representation determined by* $\omega_K(\varphi_K) = q_K^{-1}$*.*

(d) Let $t'_\ell : I_K \to \mathbb{Z}_\ell$ *be a surjective topological group homomorphism and* φ'_K *a geometric Frobenius of* K*. Further let* $N'_\ell \in \mathrm{End}(V)$ *and* $\rho' : W_K \to \mathrm{GL}(V_\ell)$ *be the maps given by* (a) *and* (b) *with respect to* t'_ℓ *and* φ'_K*. Then there exists an isomorphism* $\phi \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell \otimes V_\ell)$ *of* $\bar{\mathbb{Q}}_\ell$*-vector spaces such that* $\phi \circ \rho(\sigma) = \rho'(\sigma) \circ \phi$ *for all* $\sigma \in W_K$ *and* $\phi \circ N_\ell = N'_\ell \circ \phi$*.*

Proof Recall that we have a canonical isomorphism of topological groups

$$I_K/I_{K,\ell} \simeq \mathbb{Z}_\ell(1).$$

The map t_ℓ factors through the natural projection $I_K \to I_K/I_{K,\ell}$ and is therefore unique up to an element of $\mathbb{Z}_\ell^*$. Together with the relation $\varphi_K^{-1}\sigma\varphi_K = \sigma^{q_K}$ for $\sigma \in I_K/I_{K,\ell}$ this implies

$$t_\ell(\tau\sigma\tau^{-1}) = \omega_K(\tau)t_\ell(\sigma) \tag{1.2}$$

for all $\sigma \in I_K$ and all $\tau \in W_K$. By Theorem 1.3.3 there exists a finite extension L/K such that $\varrho_\ell(\sigma)$ is unipotent for all $\sigma \in I_L$. Choose $\tau \in I_L$ such that $t_\ell(\tau) \neq 0$. This is possible since t_ℓ is non-trivial on an open subgroups of I_K. We define

$$N_\ell = t_\ell(\tau)^{-1}\log(\varrho_\ell(\tau)).$$

Since $\varrho(\tau)$ is unipotent, N_ℓ is a nilpotent endomorphism of V_ℓ. By the proof of Theorem 1.3.3 $\varrho_\ell|_{I_L}$ factors through $I_L/I_{L,\ell}$ and therefore after identifying $I_K/I_{K,\ell} \simeq \mathbb{Z}_\ell$ via t_ℓ, we obtain a topological group homomorphism $f : \mathbb{Z}_\ell \to \mathrm{GL}(V_\ell)$ with $\varrho_\ell|_{I_L} = f \circ t_\ell|_{I_L}$. Let $g : \mathbb{Z}_\ell \to \mathrm{GL}(V_\ell)$, $x \mapsto e^{xN_\ell}$. Then g is also a topological group homomorphism and $f(t_\ell(\tau)) = g(t_\ell(\tau))$ implies that f and g coincide on the open subgroup $H = t_\ell(\tau)\mathbb{Z}_\ell$ of $\mathbb{Z}_\ell$. Therefore, $\varrho_\ell(\sigma) = e^{t_\ell(\sigma)N_\ell}$ for all σ in the open subgroup $t_\ell^{-1}(H)$ of I_K. This yields the existence statement of (a). The uniqueness of N_ℓ is clear. To prove (c) we compute for $\sigma \in W_K$

$$\begin{aligned}\varrho_\ell(\sigma)N_\ell\varrho_\ell(\sigma)^{-1} &= t_\ell(\tau)^{-1}\log(\varrho_\ell(\sigma\tau\sigma^{-1})) \\ &= t_\ell(\sigma\tau\sigma^{-1})t_\ell(\tau)^{-1}t_\ell(\sigma\tau\sigma^{-1})^{-1}\log(\varrho_\ell(\sigma\tau\sigma^{-1})) = \omega_K(\sigma)N_\ell\end{aligned} \tag{1.3}$$

where we used (1.2) and the uniqueness of N_ℓ in the last step. Part (c) follows directly from this computation and (b) can be easily verified using (c). We show (d) in two steps, starting with proving that N_ℓ and ρ are independent of the chosen Frobenius element up to an automorphism of V_ℓ. There exists $\xi \in I_K$ with $\varphi'_K = \varphi_K \xi$. Write $\gamma = (q-1)^{-1}$ and choose $\phi = e^{\gamma t_\ell(\xi) N_\ell} \in \mathrm{GL}(V_\ell)$. Clearly, N_ℓ does not depend on the choice of the geometric Frobenius element and ϕ commutes with N_ℓ. For $\sigma \in I_K$ we get $\phi \circ \rho(\sigma) \circ \phi^{-1} = \rho'(\sigma)$ by using the defining expression for ρ and ρ' and (1.3). With (1.3) we obtain

$$\begin{aligned}\phi \circ \rho(\varphi_K) \circ \phi^{-1} &= \varrho_\ell(\varphi_K) e^{q_K \gamma t_\ell(\xi) N_\ell} e^{-\gamma t_\ell(\xi) N_\ell} = \varrho_\ell(\varphi_K) e^{t_\ell(\xi) N_\ell} \\ &= \varrho_\ell(\varphi'_K \xi^{-1}) e^{-t_\ell(\xi^{-1}) N_\ell} = \rho'(\varphi_K).\end{aligned}$$

Thus, $\phi \circ \rho(\sigma) = \rho'(\sigma) \circ \phi$ for all $\sigma \in W_K$. Now, for the second step to show (d), let $t_\ell, t'_\ell : I_K \to \mathbb{Z}_\ell$ be surjective, continuous homomorphisms. We have already seen that there exists $a \in \mathbb{Z}_\ell^*$ such that $t_\ell = a t'_\ell$. Then $N'_\ell = a N_\ell$ and $\rho = \rho'$. There is n such that $\rho(\varphi_K^n)$ commutes with $\rho(W_K)$, for $\rho(I_K)$ is finite by (b). Let $\bar{V} = \bar{\mathbb{Q}}_\ell \otimes V_\ell$ and

$$\bar{V} = \bigoplus_\lambda V(\lambda)$$

be the decomposition into generalized eigenspaces of $\rho(\varphi_K^n)$. These eigenspaces are $\rho(W_K)$-stable and $N_\ell(V(\lambda)) \subset V(q_K^{-n}\lambda)$ by the commutativity relation of (c). Choosing a function $f : \bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_\ell$ with $f(q_K^{-n}\lambda) = a f(\lambda)$ for all $\lambda \in \bar{\mathbb{Q}}_\ell$ we set $\phi(v) = f(\lambda) v$ for $v \in V(\lambda)$. Then ϕ is an automorphism of $\bar{V}$ commuting with $\rho(W_K)$ and $\phi \circ N_\ell = N'_\ell \circ \phi$. This shows (d). □

Definition 1.5.5 Let K be a local field.

(1) A Weil-Deligne representation of K is a pair (ρ, N) consisting of a Weil representation $\rho : W_K \to \mathrm{GL}(V)$ and a nilpotent endomorphism N of V such that

$$\rho(\sigma) N \rho(\sigma)^{-1} = \omega_K(\sigma) N$$

for all $\sigma \in W_K$.

(2) Let (ρ, N) and (ρ', N') be Weil-Deligne representations of K. A homomorphism of Weil-Deligne representations is a linear map $\phi : \rho \to \rho'$ such that $\rho'(\sigma) \circ \phi = \phi \circ \rho(\sigma)$ for all $\sigma \in W_K$ and $N' \circ \phi = \phi \circ N$.

By Theorem 1.5.4 after choosing a surjective topological homomorphism $t_\ell : I_K \to \mathbb{Z}_\ell$, a geometric Frobenius φ_K and an isomorphism $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$, to every ℓ-adic representation ϱ_ℓ there is an associated Weil-Deligne representation $(\varrho_\ell)_{\mathrm{WD},\iota}$. Furthermore, the isomorphism class of $(\varrho_\ell)_{\mathrm{WD},\iota}$ is independent of the choices made for t_ℓ and φ_K.

Theorem 1.5.6 ([BH06, 32.6]) *Let $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism. The assignment $(\cdot)_{\mathrm{WD},\iota}$ is functorial and induces an equivalence between the category of finite dimensional ℓ-adic representations of W_K and the category of Weil-Deligne representations of K.*

Definition 1.5.7 Let $\mathcal{L}$ be a set of primes different from p. A system $(\varrho_\ell)_{\ell\in\mathcal{L}}$ of ℓ-adic representations of K is called compatible if for any two elements $\ell, l \in \mathcal{L}$ and any isomorphisms $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ and $\jmath : \bar{\mathbb{Q}}_l \to \mathbb{C}$ the Weil-Deligne representations $(\varrho_\ell)_{\mathrm{WD},\iota}$ and $(\varrho_l)_{\mathrm{WD},\jmath}$ are isomorphic.

Before considering the global situation in the next section, we want to define the conductor exponent of a Weil-Deligne representation such that the conductor exponents of an ℓ-adic representation and its corresponding Weil-Deligne representation coincide. Let $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism, let ϱ_ℓ be an ℓ-adic representation and $(\rho, N) = (\varrho_\ell)_{\mathrm{WD},\iota}$ the Weil-Deligne representation associated to ϱ_ℓ and a choice of t_ℓ and φ_K as in Theorem 1.5.4. Since t_ℓ is trivial on $I_{K,\ell}$ and therefore on the subgroup P_K, by Theorem 1.5.4 (*b*), $\varrho_\ell|_{P_K} = \rho|_{P_K}$. Hence, $\delta(\varrho_\ell) = \delta(\rho)$. Theorem 1.5.4 (*b*) also implies that $\varrho_\ell^{I_K} = \ker(N)^{\rho(I_K)}$. Thus, setting $V_N = \ker(N)$ we obtain

$$f(\varrho_\ell) = \epsilon(\varrho_\ell) + \delta(\varrho_\ell) = \dim(\rho) - \dim\big(V_N^{I_K}\big) + \delta(\rho) = f(\rho) + \dim\big(\rho^{I_K}\big) - \dim\big(V_N^{I_K}\big).$$

Definition 1.5.8 Let (V, N) be a Weil-Deligne representation and $V_N = \ker(N)$. The conductor exponent of (V, N) is defined as

$$f(V, N) = f(V) + \dim\big(V^{I_K}\big) - \dim\big(V_N^{I_K}\big).$$

The above discussion shows

Lemma 1.5.9 *Let ϱ be an ℓ-adic representation and $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Then $f(\varrho) = f((\varrho)_{\mathrm{WD},\iota})$.*

§ 1.6 L-Functions associated to Galois Representations

In the preceding sections we discussed Galois representation of local fields and their associated conductors. The goal of this section is to give a brief overview of the global theory. We associate to compatible systems of Galois representations an L-function and a conductor. In the case that the system comes from the first étale cohomology group of a curve, we also state open conjectures concerning an functional equation and the zeros of the L-function. More on L-functions and conductors associated to Galois representations can be found in [Del73, § 8] and [Tat79, § 4].

Let K and E be global fields. Let $\mathcal{V} = (\varrho_\ell)_{\ell\in\mathcal{L}}$ be a system of finite dimensional ℓ-adic representations of a global field K over E_ℓ where $\mathcal{L}$ consists of all finite places of E. For each finite place $\mathfrak{p}$ of K, fix an embedding $\bar{K} \to \bar{K}_\mathfrak{p}$ which induces an embedding $\Gamma_{K_\mathfrak{p}} \to \Gamma_K$ through restriction of elements. We obtain for each finite place $\mathfrak{p}$ a system $\mathcal{V}_\mathfrak{p} = (\varrho_\ell|_{\Gamma_{K_\mathfrak{p}}})_{\ell\in\mathcal{L}_\mathfrak{p}}$ of ℓ-adic representations of $K_\mathfrak{p}$ where $\mathcal{L}_\mathfrak{p}$ is the set of places of E not dividing the rational prime over which $\mathfrak{p}$ lies. The definition of Artin of the local L-polynomial at $\mathfrak{p}$ in this situation involves the non-natural choice of an $\ell \in \mathcal{L}_\mathfrak{p}$. However, if we assume that $\mathcal{V}_\mathfrak{p}$ is a compatible system, the local L-polynomial is independent of this choice. Thus, if $\mathcal{V}_\mathfrak{p}$ is a compatible system, we define the local L-polynomial of $\mathcal{V}$ at $\mathfrak{p}$ as

$$L(\mathcal{V}_\mathfrak{p}, T) = \det\Big(1 - \varrho_\ell(\varphi_\mathfrak{p})T\big|_{\varrho_\ell^{I_\mathfrak{p}}}\Big)$$

for some $\ell \in \mathcal{L}_\mathfrak{p}$ where $\varphi_\mathfrak{p}$ is a geometric Frobenius of $K_\mathfrak{p}$. In the same way, we can associate to $\mathcal{V}$ a local conductor exponent $f(\mathcal{V}_\mathfrak{p})$ at $\mathfrak{p}$ if $\mathcal{V}_\mathfrak{p}$ is compatible. Therefore, we call the system $\mathcal{V}$ compatible if $\mathcal{V}_\mathfrak{p}$ is compatible for all finite places $\mathfrak{p}$ of K and there exists a finite set S of places such that ϱ_ℓ is unramified at all finite places $\mathfrak{p} \notin S$ such that $\ell \notin \mathcal{L}_\mathfrak{p}$.

Definition 1.6.1 Let $\mathcal{V}$ be a compatible system of ℓ-adic representation of a global field K. The L-function associated to $\mathcal{V}$ is

$$L(\mathcal{V}, s) = \prod_\mathfrak{p} \frac{1}{L(\mathcal{V}_\mathfrak{p}, \mathrm{N}(\mathfrak{p})^{-s})}$$

where $\mathrm{N}(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ is the ideal norm, and the conductor associated to $\mathcal{V}$ is

$$\mathfrak{N}(\mathcal{V}) = \prod_\mathfrak{p} \mathfrak{D}_\mathfrak{p}^{\dim(\mathcal{V}_\mathfrak{p})} \mathfrak{p}^{f(\mathcal{V}_\mathfrak{p})}$$

where the products run through the prime ideals of K and $\mathfrak{D}_\mathfrak{p}$ is the different of $K_\mathfrak{p}$.

The additivity and inductivity of the local L-polynomial and the conductor exponent (Theorem 1.4.9) also hold in the global setting. More precisely, for compatible systems $\mathcal{V}$ and $\mathcal{W}$ of a finite extension M of a global field K, we have

$$L(\mathcal{V} \oplus \mathcal{W}, s) = L(\mathcal{V}, s)L(\mathcal{W}, s) \;\text{ and }\; \mathfrak{N}(\mathcal{V} \oplus \mathcal{W}) = \mathfrak{N}(\mathcal{V})\mathfrak{N}(\mathcal{W}),$$

$$L(\mathrm{ind}_M^K(\mathcal{V}), s) = L(\mathcal{V}, s) \;\text{ and }\; \mathfrak{N}(\mathrm{ind}_M^K(\mathcal{V})) = \mathfrak{N}(\mathcal{V}).$$

A source for systems of ℓ-adic representations is étale cohomology. Let Y be a smooth and proper variety over a global field K. Then the absolute Galois group of K acts on the base change $Y_{\bar{K}}$. By functoriality we obtain a Γ_K-action on the cohomology groups $H^i(Y_{\bar{K}}, \mathbb{Q}_\ell)$ for all $i \geqslant 0$ and all rational primes ℓ. Hence, for all $i \geqslant 0$ we have a system of ℓ-adic representations $H^i(Y) = (H^i(Y_{\bar{K}}, \mathbb{Q}_\ell))_\ell$ of K over $\mathbb{Q}$. For a prime ideal $\mathfrak{p}$ of K, the system $H^i(Y)_\mathfrak{p}$ is given by the cohomology spaces $H^i(Y_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell)$ for rational primes ℓ with $\mathfrak{p} \nmid \ell$. The system $H^0(Y)_\mathfrak{p}$ is compatible, since $H^0(Y_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell)$ is the permutation representation on the connected components of $Y_{\bar{K}_\mathfrak{p}}$ for each ℓ. For $d = \dim(Y)$, $H^{2d}(Y_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell)$ is the permutation representation on the irreducible components of $Y_{\bar{K}_\mathfrak{p}}$ twisted by ω_ℓ^d which also defines a compatible system. Here ω_ℓ is the ℓ-adic cyclotomic character. In particular, if Y is geometrically connected, we obtain

$$L(H^0(Y), s) = \zeta(s) \;\text{ and }\; \mathfrak{N}(H^0(Y)) = \mathfrak{d}_K,$$

$$L(H^{2d}(Y), s) = \zeta(s - d) \;\text{ and }\; \mathfrak{N}(H^{2d}(Y)) = \mathfrak{d}_K$$

where $\zeta(s)$ is the Riemann ζ-function. For curves we have

Theorem 1.6.2 ([Fon94, 2.4.6 iii)]) *Let Y be a smooth projective curve over a local field K of residue characteristic p. The system $(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell))_{\ell \neq p}$ of ℓ-adic representations is compatible.*

For varieties Y of dimension $d > 1$ it is conjectured but remains an open problem if $H^i(Y)_\mathfrak{p}$ for $0 < i < 2d$ is compatible. Therefore, we focus on the case of curves. In this case we define

$$L(Y/K, s) = L(H^1(Y), s) \quad \text{and} \quad \mathfrak{N}(Y/K) = \mathfrak{N}(H^1(Y)),$$

and call this the L-function respectively conductor of Y over K. As an Euler product $L(Y/K, s)$ converges in some right half-plane. It is expected for this L-function to define a holomorphic function on $\mathbb{C}$ and satisfy a functional equation.

Conjecture 1.6.3 *Let Y be a smooth projective curve over a global field K. Then $L(Y/K, s)$ has an analytic continuation to the complex plane and the completed L-function*

$$\Lambda(Y/K, s) = \left(\frac{\mathrm{N}(\mathfrak{N}(Y/K))}{(2\pi)^{2g_Y}} \right)^{\frac{s}{2}} \Gamma(s)^{g_Y} \, L(Y/K, s)$$

satisfies the functional equation

$$\Lambda(Y/K, s) = w(Y/K)\Lambda(Y/K, s - 2)$$

for a $w(Y/K) \in \{-1, 1\}$.

Conjecture 1.6.3 is known for elliptic curves over $\mathbb{Q}$ and curves with complex multiplication [BCDT01]. There is also a Riemann Hypothesis for the L-function of a curve. By the functional equation, $\Lambda(Y/K, s)$ has zeros in $s \in \{0, -1, -2, \dots\}$. These are called the trivial zeros of $\Lambda(Y/K, s)$.

Conjecture 1.6.4 *Let Y be a smooth projective curve over a global field K. Then all non-trivial zeros of the completed L-function $\Lambda(Y/K, s)$ have real part* 1.

Chapter 2

Étale Cohomology and Lefschetz Trace Formula

In this section we gather the results we use in the upcoming chapters. The first section gives an overview over the ℓ-adic representations coming from geometry. In § 2.2 we define the conductor exponent associated to a curve over a local field and discuss the computation of quasistable models for curves admitting covers of the projective line that are potentially Galois. In the next section we introduce (co)homology groups of graphs and derive exact sequences connecting the cohomology of a curve, the special fiber of a quasistable model and the dual graph of the special fiber. Sections 2.4 and 2.5 contain a proof for a Grothendieck-Ogg-Shafarevich formula expressing the Euler-Poincaré characteristic of a Galois cover of curves as a sum over local Artin characters using the Lefschetz trace formula.

§ 2.1 ℓ-adic Sheaves and Representations of the Fundamental Group

We briefly discuss the étale fundamental group and ℓ-adic representations arising from monodromy. The étale cohomology groups are such representations and we determine the 0-th and $2\dim(Y)$-th étale cohomology representation for a smooth proper variety Y from this description.

We assume throughout this section that all schemes are locally Noetherian.

Let X be a connected scheme and $\bar{x}$ a geometric point of X. Finite étale maps over X are classified by the étale fundamental group. More precisely, the étale fundamental group is defined as $\pi_1(X,\bar{x}) = \mathrm{Aut}(\mathrm{Fib}_{\bar{x}})$ where $\mathrm{Fib}_{\bar{x}}$ is the functor from the category of finite étale maps over X to the category of finite sets given by $Z \mapsto \mathrm{Hom}(\bar{x}, Z)$. By a theorem of Grothendieck [Gro71, V.4.1, V.7] this functor induces an equivalence between the finite étale maps over X and finite sets with a continuous $\pi_1(X,\bar{x})$-action. The étale fundamental group $\pi_1(X,\bar{x})$ is a pro-finite group [Sza09, 5.4.7]. More precisely, the set of automorphism groups of connected étale Galois covers of X forms an inverse system and

$$\pi_1(X,\bar{x}) = \varprojlim_j \mathrm{Aut}(Z_j/X)^{\mathrm{opp}}$$

where G^{opp} denotes the opposite group of a group G. Here a cover of X is a finite surjective morphism $Y \to X$ and a cover $Y \to X$ is called étale Galois if it is étale and the degree of the cover is equal to the order of $\operatorname{Aut}(Y/X)$.

Example 2.1.1 The fundamental group of $X = \operatorname{Spec}(K)$ for a field K is an absolute Galois group of K. The choice of a geometric point $\bar{x} : \operatorname{Spec}(\bar{K}) \to X$ in this case corresponds to the choice of an algebraic closure $\bar{K}$ of K. All finite étale covers of $\operatorname{Spec}(K)$ are spectra of a product of finite separable field extensions of K. The connected Galois covers of X therefore are of the form $\operatorname{Spec}(L)$ for a finite Galois extensions L/K. Hence, we have

$$\pi_1(\operatorname{Spec}(K), \bar{x}) = \varprojlim_L \operatorname{Aut}(\operatorname{Spec}(L)/\operatorname{Spec}(K))^{\text{opp}} = \varprojlim_L \operatorname{Gal}(L/K) = \operatorname{Gal}(\bar{K}/K).$$

The above equivalence of categories can be reformulated as an equivalence of the category of ℓ-adic $\pi_1(X,\bar{x})$-representations and the category of smooth ℓ-adic sheaves (cf. [Mil80, V.1 p. 165]). An ℓ-adic sheaf is given by a projective system $(\mathcal{F}_n)_{n\in\mathbb{N}}$ where $\mathcal{F}_n$ is a sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules and each map $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induces an isomorphism $\mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \to \mathcal{F}_n$. Morphisms in the category of ℓ-adic sheaves are given by

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = \varprojlim_n \operatorname{Hom}(\mathcal{F}_n, \mathcal{G}_n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

and stalks by

$$\mathcal{F}_{\bar{x}} = \varprojlim_n \mathcal{F}_{n,\bar{x}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

An ℓ-adic sheaf is called smooth if it can be given by a projective system such that every sheaf in the system is locally constant and constructible. For more on ℓ-adic sheaves and constructibility we refer to [Mil80, V.1] and [Gro77, VI].

Theorem 2.1.2 *Let X be a connected scheme, $\bar{x}$ a geometric point of X and ℓ a prime which is relatively prime to all residue characteristics of X. Then there exists a natural equivalence between the category of smooth ℓ-adic sheaves on X and finite dimensional ℓ-adic representations of $\pi_1(X,\bar{x})$ over $\mathbb{Q}_\ell$ given by $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$.*

Let X, $\bar{x}$ and ℓ be as in the theorem. We give the construction of the $\pi_1(X,\bar{x})$-representation $\mathcal{F}_{\bar{x}}$ for a given smooth ℓ-adic sheaf $\mathcal{F}$ given by $(\mathcal{F}_n)$. Let

$$\pi_1(X,\bar{x}) = \varprojlim_j \operatorname{Aut}(Z_j/X)$$

for a projective system $(Z_j)_j$ of connected étale Galois covers of X. Let $U \to X$ be an étale neighborhood of $\bar{x}$ and take $U_j = U \times_X Z_j$ for each j. Then $(U_j)_j$ is a projective system of étale neighborhoods of $\bar{x}$ and an element $\sigma \in \pi_1(X,\bar{x})$ induces a projective system of automorphisms of $(U_j)_j$ over X. Thus, $\sigma \in \pi_1(X,\bar{x})$ induces automorphisms $\mathcal{F}_n(U_j) \to \mathcal{F}_n(U_j)$, since $\mathcal{F}_n$ – as a locally constant sheaf – is represented by a finite étale cover of X. Composing with $\mathcal{F}_n(U) \to \mathcal{F}_n(U_j)$ and $\mathcal{F}_n(U_j) \to \mathcal{F}_{n,\bar{x}}$, we have a map $\mathcal{F}_n(U) \to \mathcal{F}_{n,\bar{x}}$ induced by σ. Taking the limit over étale neighborhoods U of $\bar{x}$, every $\sigma \in \pi_1(X, \bar{x})$ induces an automorphism $\mathcal{F}_{n,\bar{x}} \to \mathcal{F}_{n,\bar{x}}$. Taking the limit over n and tensoring with $\mathbb{Q}_\ell$ we get a morphism

$$\sigma^* : \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$$

for each $\sigma \in \pi_1(X, \bar{x})$ and thus an ℓ-adic $\pi_1(X, \bar{x})$-representation of $\mathcal{F}_{\bar{x}}$.

Example 2.1.3 (1) Let $\mathcal{F} = \mathbb{Q}_\ell$ be the constant ℓ-adic sheaf on a connected scheme X given by the projective system $(\mathbb{Z}/\ell^n\mathbb{Z})_n$ of constant torsion sheaves. Let $\bar{x}$ be a geometric point on X. It follows from the above description that the $\pi_1(X, \bar{x})$-representation $\mathbb{Q}_{\ell,\bar{x}}$ is the trivial representation of $\pi_1(X, \bar{x})$.

(2) Let X, $\bar{x} : \mathrm{Spec}(\bar{K}) \to X$ and ℓ be as in Theorem 2.1.2. We define the Tate twist of a sheaf and determine the corresponding representation of the fundamental group of X. Let $n \geqslant 1$ and μ_n be a sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on $X_{\text{ét}}$ such that

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m \to 1$$

is exact. Here $\mathbb{G}_m$ is the sheaf on $X_{\text{ét}}$ given by $U \mapsto \Gamma(U, \mathcal{O}_U)^*$ and $[n]$ is multiplication by n. Let $i \geqslant 0$. The sheaf obtained by tensoring μ_n i-times with itself is denoted $\mathbb{Z}/\ell^n\mathbb{Z}(i)$. The sheaf $\mathbb{Z}/\ell^n\mathbb{Z}(-i)$ is the dual of $\mathbb{Z}/\ell^n\mathbb{Z}(i)$, i.e. given by

$$U \mapsto \mathrm{Hom}_U(\mathbb{Z}/\ell^n\mathbb{Z}(i)|_U, \mathbb{Z}/\ell^n\mathbb{Z}|_U).$$

For $i \in \mathbb{Z}$ and a smooth ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n) \otimes \mathbb{Q}_\ell$ we define the i-th Tate twist of $\mathcal{F}$, denoted $\mathcal{F}(i)$, as the ℓ-adic sheaf of the projective system $(\mathcal{F}_n(i))$ where $\mathcal{F}_n(i) = \mathcal{F}_n \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z}(i)$. The i-th Tate twist of a smooth ℓ-adic sheaf is smooth. Let U be a connected étale Galois neighborhood of $\bar{x}$. The action of the Galois group on U over X induces an action on $\Gamma(U, \mathcal{O}_U)^*$ and therefore an action on the ℓ^n-th roots of unity in $\Gamma(U, \mathcal{O}_U)^*$, this means on $\mu_{\ell^n}(U)$. Taking limits and tensoring gives us the representation $\mathbb{Q}_\ell(1)_{\bar{x}}$. More concretely, the action of $\mathrm{Aut}(U/X)$ on $\mu_{\ell^n}(U)$ can be given by $\psi_{\ell^n} : \mathrm{Aut}(U/X) \to (\mathbb{Z}/\ell^n\mathbb{Z})^*$ defined by

$$\sigma(\zeta) = \zeta^{\psi_{\ell^n}(\sigma)}$$

for $\sigma \in \mathrm{Aut}(U/X)$, $\zeta \in \Gamma(U, \mathcal{O}_U)$ a root of unity and n maximal such that $\Gamma(U, \mathcal{O}_U)$ contains an ℓ^n-th root of unity. This shows that for $X = \mathrm{Spec}(K)$ for a finite extension $K/\mathbb{Q}_p$ with $p \neq \ell$ the representation associated by Theorem 2.1.2 with $\mathbb{Q}_\ell(1)$ is given by the cyclotomic ℓ-adic character ω_ℓ. Since $\mathbb{Q}_\ell(i)(j) = \mathbb{Q}_\ell(i+j)$ holds, the representation $\mathbb{Q}_\ell(m)_{\bar{x}}$ is the m-th tensor power of $\mathbb{Q}_\ell(1)_{\bar{x}}$. For a smooth ℓ-adic sheaf $\mathcal{F}$ we have $\mathcal{F}(i)_{\bar{x}} = \mathcal{F}_{\bar{x}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell(i)_{\bar{x}}$.

Let Y be a smooth and proper variety over a field K and $\ell \neq \mathrm{char}(K)$ a prime. We have a smooth and proper morphism $\phi : Y \to \mathrm{Spec}(K)$. By [Mil80, VI.2.2] the higher direct images $R^i\phi_*$ of the constant ℓ-adic sheaf $\mathbb{Q}_\ell$ are smooth for $i \geqslant 0$. Furthermore, by the proper base change theorem [Mil80, VI.2.3] we have a canonical isomorphism

$$(R^i\phi_*\mathbb{Q}_\ell)_{\bar{x}} \to H^i(Y_{\bar{K}}, \mathbb{Q}_\ell)$$

for $i \geqslant 0$ and $\bar{x} : \mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(K)$ a geometric point. By Theorem 2.1.2 the stalk $(R^i\phi_*\mathbb{Q}_\ell)_{\bar{x}}$ and therefore $H^i(Y_{\bar{K}}, \mathbb{Q}_\ell)$ is an ℓ-adic representation of $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$. In the cases $i = 0$ and $i = \dim(Y)$ these representations are easy to determine.

Let $i = 0$. Then $R^0\phi_* = \phi_*$. For an étale neighborhood $U \to \mathrm{Spec}(K)$ of $\bar{x}$ we have

$$\phi_*(\mathbb{Z}/\ell^n\mathbb{Z})(U) = (\mathbb{Z}/\ell^n\mathbb{Z})(Y \times_K U) = (\mathbb{Z}/\ell^n\mathbb{Z})^{\pi_0(Y\times_K U)}$$

where $\pi_0(Z)$ denotes the set of connected components of Z. Note that an étale morphism $U \to \mathrm{Spec}(K)$ is given by a product of separabel extensions of K. Thus, Γ_K acts naturally on $Y \times_K U$ and this action induces an action on the connected components of $Y \times_K U$. The group Γ_K acts on $\phi_*(\mathbb{Z}/\ell^n\mathbb{Z})(U)$ through this action and the stalk of $\phi_*(\mathbb{Z}/\ell^n\mathbb{Z})$ at $\bar{x}$ is $(\mathbb{Z}/\ell^n\mathbb{Z})^{\pi_0(Y_{\bar{K}})}$. Taking limits and tensoring with $\mathbb{Q}_\ell$ we see that the ℓ-adic Γ_K-representation $H^0(Y_{\bar{K}}, \mathbb{Q}_\ell)$ is the permutation representation of Γ_K on the connected components of $Y_{\bar{K}}$.

By Poincaré duality we have a canonical isomorphism

$$H^{2d}(Y_{\bar{K}}, \mathbb{Q}_\ell) \simeq H^0(Y_{\bar{K}}, \mathbb{Q}_\ell)(d)$$

for $d = \dim(Y)$. By Example 2.1.3 (2) we obtain $H^{2d}(Y_{\bar{K}}, \mathbb{Q}_\ell) = H^0(Y_{\bar{K}}, \mathbb{Q}_\ell) \otimes \omega_\ell^d$ as Γ_K-representations.

Remark 2.1.4 We have verified that the systems $H^0(Y) = (H^0(Y_{\bar{K}}, \mathbb{Q}_\ell))_{\ell \neq p}$ and $H^{2d}(Y) = (H^{2d}(Y_{\bar{K}}, \mathbb{Q}_\ell))_{\ell \neq p}$ are compatible. If Y is geometrically connected, these representations are unramified and hence their conductor exponent is 0.

§ 2.2 Conductors of Curves and Quasistable Models

In this section we define conductor exponents of smooth projective curves over local fields. Then we discuss how to compute these using semistable models and how to compute these models for covers which are potentially Galois.

To each cohomology group of a smooth proper variety Y over a local field K we obtain an ℓ-adic representation of K for ℓ different from the residue characteristic p of K. From now on we concentrate on the case that Y is a curve. In the previous section we already determined the cohomology groups $H^0(Y_{\bar{K}}, \mathbb{Q}_\ell)$ and $H^2(Y_{\bar{K}}, \mathbb{Q}_\ell)$ as representations of Γ_K. It is clear from this that for varying $\ell \neq p$ we have compatible systems. We recall that for curves it is known that the system $(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell))_{\ell \neq p}$ is compatible (Theorem 1.6.2). In particular, taking two representations of this system their conductor exponents coincide.

Definition 2.2.1 Let Y be a smooth projective curve over a local field K of residue characteristic p. The conductor exponent of Y over K is defined as $f(Y/K) = f(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell))$ for some prime $\ell \neq p$.

Let Y be a smooth projective curve over a local field K. By a theorem of Deligne and Mumford there exists a finite extension L/K such that there exists a semistable $\mathcal{O}_L$-model of Y_L. Under the assumption that the genus of Y is at least 2 there exists a minimal semistable model $\mathcal{Y}$ of Y_L by [DM69, Corollary 2.7]. We call the minimal semistable model $\mathcal{Y}$ the stable model of Y_L, its special fiber $\bar{Y}$ the stable reduction of Y_L and say that Y_L has stable reduction. For a finite extension M/L the stable reduction of Y_M is $\bar{Y} \otimes_{\mathbb{F}_L} \mathbb{F}_M$. This implies that the action of Γ_K on $Y_{\bar{K}}$ extends to an action on $\mathcal{Y}$.

Definition 2.2.2 Let Y be a smooth projective curve over a local field K and let L/K be a finite Galois extension with Galois group Γ such that Y_L has semistable reduction. We call an $\mathcal{O}_L$-model $\mathcal{Y}$ of Y_L quasistable if $\mathcal{Y}$ is semistable and the action of Γ on Y extends to an action on $\mathcal{Y}$.

The conductor exponent can be expressed in terms of quotients of the special fiber of a quasistable model.

Theorem 2.2.3 [BW17, Theorem 2.9, Corollary 2.6] *Let Y be a smooth projective curve over a local field K of residue characteristic p and L/K a finite Galois extension with Galois group Γ such that Y_L has a quasistable model $\mathcal{Y}$. Let $\bar{Y}$ be the special fiber of $\mathcal{Y}$ and $\bar{Y}^u = \bar{Y}/\Gamma^u$ for $u > -1$. Then $f(Y/K) = \epsilon(Y/K) + \delta(Y/K)$ with*

$$\epsilon(Y/K) = 2g_Y - \dim(H^1(\bar{Y}^0, \mathbb{Q}_\ell))$$

for a prime $\ell \neq p$, and

$$\delta(Y/K) = \int_0^\infty 2g_Y - 2g_{\bar{Y}^u} \,\mathrm{d}u.$$

Remark 2.2.4 (1) To prove Theorem 2.2.3 one shows $\dim\big(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)^{\Gamma^u}\big) = 2g_{\bar{Y}^u}$ for $u > 0$. For $u = 0$ this is false in general. However, one can show that $\dim(H^1(\bar{Y}^0, \mathbb{Q}_\ell)) = 2g_{\bar{Y}^0} - (i - j + 1)$ for i the number of singular points of $\bar{Y}^0$ and j the number of irreducible components of $\bar{Y}^0$ (cf. [BW17, Lemma 2.7]).

(2) Theorem 2.2.3 yields a proof that the conductor exponent of a curve is independent of the chosen prime ℓ in Definition 2.2.1 without using Theorem 1.6.2, since the stable reduction and the minimal Galois extension over which Y has semistable reduction are independent of ℓ. More generally, the Deligne-Mumford Theorem and the existence of the stable model is the reason for the compatibility of $(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell))_{\ell \neq p}$ (Theorem 1.6.2).

For covers of the projective line that are potentially Galois such that the residue characteristic does not divide the order of the Galois group of the cover it is known how to compute a quasistable model. This applies to superelliptic curves over a local field K of degree not divisible by the residue characteristic, since a superelliptic curve is a cover of $\mathbb{P}^1_K$ which is potentially Galois with cyclic Galois group. Our method to prove an upper bound for conductor exponents of superelliptic curves relies on the structure of the special fiber of this quasistable models. In the rest of this section we collect the results of [BW17, § 3] which we will use in Chapter 4. For proofs and more details, we refer to [BW17, § 3].

Definition 2.2.5 (1) Let S be a scheme and $\mathcal{X}$ a curve over S. A closed subscheme $\mathcal{D}$ of the smooth locus of $\mathcal{X}$ such that $\mathcal{D} \to S$ is finite étale of degree d, is called a marking of $\mathcal{X}/S$ of degree d. The pair $(\mathcal{X}, \mathcal{D})$ is called a marked semistable curve if $\mathcal{D}$ is a marking of $\mathcal{X}/S$ and $\mathcal{X}$ is semistable.

(2) Let X be a smooth projective curve over a local field K and D a smooth relative divisor of X of degree d. The divisor D is called split if D consist of d distinct K-rational points. The marked curve (X, D) has semistable reduction if D is split and there exists a marked semistable curve $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_K$ such that $\mathcal{X} \otimes_{\mathcal{O}_K} K \simeq X$ and this isomorphism induces $\mathcal{D} \otimes_{\mathcal{O}_K} K \simeq D$. In this case $(\mathcal{X}, \mathcal{D})$ is called a semistable model of (X, D).

Let Y and X be smooth projective curves over a local field K of residue characteristic p, let $\phi : Y \to X$ be a finite cover and D the branch divisor of ϕ. We assume the following.

Assumption 2.2.6
- The curve X is a projective line over K and the genus of Y is at least 2.
- There exists a finite extension L_0/K such that $\phi_{L_0} : Y_{L_0} \to X_{L_0}$ is a Galois cover with Galois group G and D_{L_0} is split.
- The order of the group G is not divisible by p.

One then can find a quasistable model of Y in two steps. First, compute a semistable model of the marked curve (X, D) where D is the branch divisor of ϕ. The existence of such a semistable model after a finite base change and the structure of its special fiber is the content of

Theorem 2.2.7 *Let X be a smooth projective curve over a local field K and D a smooth relative divisor of X of degree d.*

(a) There exists a finite extension L/K such that (X_L, D_L) has semistable reduction.

(b) Let $2g_X - 2 + d > 0$. Then there exists a unique minimal semistable model $(\mathcal{X}, \mathcal{D})$ of (X, D).

(c) Let the genus of X be 0 and let D be split. Then (X, D) is semistable.

(d) Let the genus of X be 0, $d \geqslant 3$ and let D be split. Let $(\bar{X}, \bar{D})$ be the special fiber of the minimal semistable model of (X, D). Then $\bar{X}$ is a tree of projective lines such that every irreducible component of $\bar{X}$ has at least three points which are either singular or in the support of $\bar{D}$.

Definition 2.2.8 Let X be a smooth projective curve over a local field K and D a smooth relative divisor of X of degree d such that $2g_X - 2 + d > 0$. The unique minimal semistable model of (X, D) is called the stable model of (X, D).

As the second step in the determination of a quasistable model, having already a finite extension L/K and a stable model of (X_L, D_L), one obtains a quasistable model of Y as a normalization.

Theorem 2.2.9 *Let Y and X be smooth projective curves over a local field K, let $\phi : Y \to X$ be a finite cover and D the branch divisor of ϕ such that Assumption 2.2.6 is satisfied. Then there exists a finite, tamely ramified extension L/L_0 such that the normalization of the stable model of (X_L, D_L) in the function field of Y_L is a quasistable model of Y_L.*

For superelliptic curves of degree different from the residue characteristic this construction of a quasistable model can be made explicit. We refer to [BW17, § 4].

§ 2.3 Dual Graph and Galois Representation associated to Curves

The main result of this section are two exact sequences determining the semisimplification of $H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)$ for a smooth projective curve Y over a local field K in terms of the cohomology of the dual graph of the special fiber of a quasistable model and the cohomology of the normalization of the special fiber. Since this results will be important in the upcoming chapters, we give a proof closely following [BW13, §§ 2.5-2.7].

As a starting point we define the homology and cohomology groups of a graph. Let $\Delta = (V, E)$ be a finite graph and Λ a commutative ring. For each $e = \{v, w\} \in E$ choose a vertex $t(e)$ in $\{v, w\}$ and denote by $s(e)$ the vertex in $\{v, w\}$ not equal to $t(e)$. Define $C_1(\Delta, \Lambda)$ (respectively $C_0(\Delta, \Lambda)$) to be the free Λ-module generated by E (respectively by V). Let $d : C_1(\Delta, \Lambda) \to C_0(\Delta, \Lambda)$ be the Λ-linear map defined by $d(e) = t(e) - s(e)$ for $e \in E$. With the definition $C_i(\Delta, \Lambda) = 0$ for $i \notin \{0, 1\}$, $(C_\bullet(\Delta, \Lambda), d)$ is a chain complex. Let $(C^\bullet(\Delta, \Lambda), \partial)$ be the dual cochain complex. This means that $C^0(\Delta, \Lambda)$ and $C^1(\Delta, \Lambda)$ are the dual modules to $C_0(\Delta, \Lambda)$ and $C_1(\Delta, \Lambda)$. The coboundary map $\partial : C^0(\Delta, \Lambda) \to C^1(\Delta, \Lambda)$ is given by $\partial(\phi) = -\phi \circ d$ for $\phi \in C^0(\Delta, \Lambda)$. Let $H_\bullet(\Delta, \Lambda)$ be the homology of $(C_\bullet(\Delta, \Lambda), d)$ and $H^\bullet(\Delta, \Lambda)$ be the cohomology of $(C^\bullet(\Delta, \Lambda), \partial)$. Note that (co)homology groups for different choices of orientation on the graph are canonically isomorphic. For $i \geqslant 0$ define

$$H_i(\Delta, \mathbb{Q}_\ell) = \varprojlim_m H_i(\Delta, \mathbb{Z}/\ell^m\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

and

$$H^i(\Delta, \mathbb{Q}_\ell) = \varprojlim_m H^i(\Delta, \mathbb{Z}/\ell^m\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Let G be a finite group acting on Δ. This action induces an action on the (co)homology groups provided we keep track of the orientation by setting $\sigma(e) = e'$ if $\sigma(t(e)) = t(e')$ and $\sigma(e) = -e'$ otherwise as the action on $C_1(\Delta, \Lambda)$ if, under the induced action of σ

on the edges, the edge e is mapped to the edge e'. For the induced G-representation ρ_E on $C_1(\Delta, \mathbb{Q}_\ell)$ we obtain

$$\rho_E = \bigoplus_e \mathrm{ind}_{G_e}^G(\psi_e) \tag{2.1}$$

where e runs through a system of representatives of $E(\Delta)/G$, G_e is the stabilizer of e, and ψ_e is the trivial representation if the vertices incident to e are fixed by G_e and otherwise ψ_e is the unique representation of G_e order 2. Note that the induced G-representation on $C_0(\Delta, \mathbb{Q}_\ell)$ is the permutation representation on V. The boundary map d is clearly G-equivariant.

Theorem 2.3.1 *Let $\Delta = (V, E)$ be a connected graph and G a group acting on Δ such that the quotient graph Δ/G is a tree. In $R(G)$ we have*

$$H_1(\Delta, \mathbb{Q}_\ell) = \rho_E - \rho_V + \mathbf{1}$$

where ρ_E is the permutation representation on the set E and ρ_V is the permutation representations on V.

Proof Note that vertices incident to an edge e have to be fixed by the stabilizer G_e of e, since Δ/G is a tree. Thus, by (2.1) the representation on $C_1(\Delta, \mathbb{Q}_\ell)$ defined by the action of G is the permutation representation on E. Set $\Lambda = \mathbb{Z}/\ell^m\mathbb{Z}$. We have $H_0(\Delta, \Lambda) = \mathrm{coker}(d)$, $H_1(\Delta, \Lambda) = \ker(d)$, $H_i(\Delta, \Lambda) = 0$ for $i \geqslant 2$. Thus, the sequence

$$0 \to H_1(\Delta, \Lambda) \to C_1(\Delta, \Lambda) \xrightarrow{d} C_0(\Delta, \Lambda) \to H_0(\Delta, \Lambda) \to 0$$

is exact and G-equivariant. In $R(G)$ we obtain

$$H_1(\Delta, \mathbb{Q}_\ell) - H_0(\Delta, \mathbb{Q}_\ell) = \rho_E - \rho_V$$

and $H_0(\Delta, \mathbb{Q}_\ell) = \mathbf{1}$ since Δ is connected. □

Let Y be a smooth projective curve over a local field K and let L/K be a finite extension such that there exists a quasistable model $\mathcal{Y}$ of Y_L. We denote by $\bar{Y}$ the special fiber of $\mathcal{Y}$ and by $\pi : \tilde{Y} \to \bar{Y}$ the normalization of $\bar{Y}$. Let $\Delta_{\bar{Y}} = (V_{\bar{Y}}, E_{\bar{Y}})$ denote the dual graph of $\bar{Y}$, i.e. the graph whose vertex set is in bijection with the set of irreducible components of $\bar{Y}$ such that two vertices are adjacent if the corresponding irreducible components meet in a singular point. The Γ_K-action on $\bar{Y}$ induces an action on $\Delta_{\bar{Y}}$ and therefore an action on the homology and cohomology groups of $\Delta_{\bar{Y}}$. The first exact sequence we will derive gives an Γ_K-equivariant embedding of the cohomology of $\Delta_{\bar{Y}}$ in the cohomology of $\bar{Y}$ with the cohomology of the normalization of $\bar{Y}$ as the quotient.

Let $\Lambda = \mathbb{Z}/\ell^m\mathbb{Z}$ where ℓ is a prime different from the residue characteristic of K. Define a sheaf $\mathcal{E}$ on the étale site of $\bar{Y}$ such that

$$0 \to \Lambda \to \pi_*\Lambda \to \mathcal{E} \to 0$$

is an exact sequence. By [Mil80, II.3.5(c)] the stalk at a closed point y of $\bar{Y}$ is

$$(\pi_*\Lambda)_y = \Lambda^{\pi^{-1}(y)}.$$

The normalization map π is an isomorphism on the smooth locus of $\bar{Y}$ and, since $\bar{Y}$ is semistable, the inverse image of a singular point consists of exactly two points of $\tilde{Y}$. By the exact sequence defining $\mathcal{E}$ we therefore have $\mathcal{E}_y = 0$ for all non-singular closed points $y \in \bar{Y}$ and $\mathcal{E}_y \simeq \Lambda$ for singular points $y \in \bar{Y}$. The exact sequence defining $\mathcal{E}$ induces a long exact sequence,

$$0 \to H^0(\bar{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \to H^0(\bar{Y}_{\bar{\mathbb{F}}_K}, \pi_*\Lambda) \to H^0(\bar{Y}_{\bar{\mathbb{F}}_K}, \mathcal{E}) \to H^1(\bar{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \to H^1(\bar{Y}_{\bar{\mathbb{F}}_K}, \pi_*\Lambda) \to 0.$$

We recall that $C^0(\Delta_{\bar{Y}}, \Lambda)$ and $C^1(\Delta_{\bar{Y}}, \Lambda)$ are the dual modules of the Λ-modules generated by $V_{\bar{Y}}$ and $E_{\bar{Y}}$. The projection formula yields an isomorphism

$$H^0(\bar{Y}_{\bar{\mathbb{F}}_K}, \pi_*\Lambda) \simeq H^0(\tilde{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \to C^0(\Delta_{\bar{Y}}, \Lambda),$$

since the normalization $\tilde{Y}$ is the disjoint union of the irreducible components of the special fiber. By the above description of $\mathcal{E}$ we also have an isomorphism

$$H^0(\bar{Y}_{\bar{\mathbb{F}}_K}, \mathcal{E}) \to C^1(\Delta_{\bar{Y}}, \Lambda)$$

which can be chosen canonically if we consider the orientation on $\Delta_{\bar{Y}}$ chosen to define the graph cohomology. We obtain a commutative diagram

$$\begin{array}{ccc} H^0(\tilde{Y}_{\bar{\mathbb{F}}_K}, \Lambda) & \longrightarrow & H^0(\bar{Y}_{\bar{\mathbb{F}}_K}, \mathcal{E}) \\ \downarrow & & \downarrow \\ C^0(\Delta_{\bar{Y}}, \Lambda) & \longrightarrow & C^1(\Delta_{\bar{Y}}, \Lambda) \end{array}$$

where the map $C^0(\Delta_{\bar{Y}}, \Lambda) \to C^1(\Delta_{\bar{Y}}, \Lambda)$ is the coboundary map and the vertical maps are the above isomorphisms. The above long exact sequence therefore yields the short exact Γ_K-equivariant sequence

$$0 \to H^1(\Delta_{\bar{Y}}, \Lambda) \to H^1(\bar{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \to H^1(\tilde{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \to 0. \tag{2.2}$$

We have a cartesian and Γ_K-equivariant diagram

$$\begin{array}{ccccc} \bar{Y}_{\bar{\mathbb{F}}_K} & \xrightarrow{\ i\ } & \mathcal{Y} & \xleftarrow{\ j\ } & Y_{\bar{K}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\bar{\mathbb{F}}_K) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) & \longleftarrow & \mathrm{Spec}(\bar{K}) \end{array}$$

Following [Gro72, I.2], we set $\Psi^m(\Lambda) = i^* R^m j_* \Lambda$ as sheaf on $\bar{Y}_{\text{ét}}$ for $m \geqslant 0$, the so called sheaves of vanishing cycles. These are constructible sheaves of Λ-modules with a natural Γ_K-action. For $m \geqslant 2$, $\Psi^m(\Lambda) = 0$ and for $m = 0$ the sheaf $\Psi^0(\Lambda)$ is the constant sheaf Λ. We have $\Psi^1(\Lambda)_y = 0$ for all smooth points $y \in \bar{Y}$ and a canonical isomorphism (up to sign) $\Psi^1(\Lambda)_y = \Lambda(-1)$ for all singular points $y \in \bar{Y}$ by [DK73, XV.2.2.5]. The Leray spectral sequence [Gro72, I.2.2.3]

$$H^p(\bar{Y}_{\bar{\mathbb{F}}_K}, \Psi^q(\Lambda)) \Rightarrow H^{p+q}(Y_{\bar{K}}, \Lambda)$$

yields a Γ_K-equivariant exact sequence

$$0 \to H^1(\bar{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \to H^1(Y_{\bar{K}}, \Lambda) \to \bigoplus_{e \in E_{\bar{Y}}} \Lambda(-1) \to H^2(\bar{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \tag{2.3}$$

where the map

$$H^1(Y_{\bar{K}}, \Lambda) \to \bigoplus_{e \in E_{\bar{Y}}} \Lambda(-1)$$

is given by $a \mapsto (a \cup \delta_e)_{e \in E_{\bar{Y}}}$ by [DK73, XV]. Here $\cup$ is the cup product

$$\cup : H^1(Y_{\bar{K}}, \Lambda) \times H^1(Y_{\bar{K}}, \Lambda) \to \Lambda(-1)$$

and $\delta_e \in H^1(Y_{\bar{K}}, \Lambda)$ are the so called vanishing cycles which are defined as follows. For $e \in E_{\bar{Y}}$ let $d_e \in H^1(\Delta_{\bar{Y}})$ be the cycle corresponding to $\phi_e \in C^1(\Delta_{\bar{Y}}, \Lambda)$ given by

$$\phi_e(e') = \begin{cases} 0 & e' \neq e \\ 1 & e' = e \end{cases}$$

for $e' \in E_{\bar{Y}}$. Then δ_e is the image of d_e under the injection $H^1(\Delta_{\bar{Y}}, \Lambda) \to H^1(Y_{\bar{K}}, \Lambda)$ in (2.2). From the exact sequence (2.3) we obtain a Γ_K-equivariant short exact sequence

$$0 \to H^1(\bar{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \to H^1(Y_{\bar{K}}, \Lambda) \to N \to 0 \tag{2.4}$$

by defining

$$N = \ker\Big(\bigoplus_{e \in E_{\bar{Y}}} \Lambda(-1) \to H^2(\bar{Y}_{\bar{\mathbb{F}}_K}, \Lambda) \Big).$$

We note that by the exactness of the sequence (2.3) and the expilicit description of the middle map of (2.3), $a \cup \delta_e$ only depends on the image of $a \in H^1(Y_{\bar{K}}, \Lambda)$ under the map $H^1(Y_{\bar{K}}, \Lambda) \to N$ for all $e \in E_{\bar{Y}}$ and thus the cup product induces a pairing

$$H^1(\Delta_{\bar{Y}}, \Lambda) \times N \to \Lambda(-1). \tag{2.5}$$

Lemma 2.3.2 *The pairing $H^1(\Delta_{\bar{Y}}, \Lambda) \times N \to \Lambda(-1)$ is perfect.*

Proof Write $\langle\cdot,\cdot\rangle$ for the pairing (2.5). Since all involved Λ-modules have finite rank, it suffices to show that $\langle\cdot,\cdot\rangle$ is non-degenerate. Let $\langle a,b\rangle = 0$ for all $a \in H^1(\Delta_{\bar{Y}},\Lambda)$. By exactness of the sequence (2.3), N is the image of

$$H^1(Y_{\bar{K}},\Lambda) \to \bigoplus_{e\in E_{\bar{Y}}} \Lambda(-1),\ a \mapsto (a \cup \delta_e)_{e\in E_{\bar{Y}}}. \tag{2.6}$$

Thus, taking $a = d_e \in H^1(\Delta_{\bar{Y}},\Lambda)$ and $b' \in H^1(Y_{\bar{K}},\Lambda)$ an element in the preimage of b under $H^1(Y_{\bar{K}},\Lambda) \to N$, we obtain $0 = \langle d_e, b\rangle = \delta_e \cup b'$ for all $e \in E_{\bar{Y}}$. Since N is the image of the map (2.6), $b = 0$ follows. Now let $\langle a,b\rangle = 0$ for all $b \in N$. Since the cup product is non-degenerate and it only depends on the image under the map $H^1(Y_{\bar{K}},\Lambda) \to N$, and $\langle\cdot,\cdot\rangle$ is induced by the cup product, $a = 0$ follows. □

By Lemma 2.3.2 we have a canonical isomorphism $N \simeq H_1(\Delta_{\bar{Y}},\Lambda)(-1)$ and thus combined with (2.4) a Γ_K-equivariant short exact sequence

$$0 \to H^1(\bar{Y}_{\bar{\mathbb{F}}_K},\Lambda) \to H^1(Y_{\bar{K}},\Lambda) \to H_1(\Delta_{\bar{Y}},\Lambda)(-1) \to 0. \tag{2.7}$$

After taking $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ and inverse limits over n, and tensoring with $\mathbb{Q}_\ell$ in (2.2) and (2.7), the above discussion can be summarized as

Theorem 2.3.3 *Let Y be a smooth projective curve over a local field K, $\bar{Y}$ the special fiber of a quasistable model of Y_L with L/K a finite extension, $\Delta_{\bar{Y}}$ the dual graph of $\bar{Y}$ and $\tilde{Y}$ the normalization of $\bar{Y}$. Then there exist Γ_K-equivariant short exact sequences*

$$0 \to H^1(\Delta_{\bar{Y}},\mathbb{Q}_\ell) \to H^1(\bar{Y}_{\bar{\mathbb{F}}_K},\mathbb{Q}_\ell) \to H^1(\tilde{Y}_{\bar{\mathbb{F}}_K},\mathbb{Q}_\ell) \to 0$$

and

$$0 \to H^1(\bar{Y}_{\bar{\mathbb{F}}_K},\mathbb{Q}_\ell) \to H^1(Y_{\bar{K}},\mathbb{Q}_\ell) \to H_1(\Delta_{\bar{Y}},\mathbb{Q}_\ell)(-1) \to 0.$$

Remark 2.3.4 In [DDM18, Corollary 1.6] it is shown that

$$H^1(Y_{\bar{K}},\mathbb{Q}_\ell) \simeq H^1(\tilde{Y}_{\bar{\mathbb{F}}_K},\mathbb{Q}_\ell) \oplus (H^1(\Delta_{\bar{Y}},\mathbb{Q}_\ell) \otimes \mathrm{sp}(2))$$

as Γ_K-representations where sp(2) is the special representation in dimension 2. This can be deduced from the exact sequences in Theorem 2.3.3 using the Picard-Lefschetz formula [BW13, Proposition 2.8].

A consequence of Theorem 2.3.3 is the following formula for the conductor exponent of a curve.

Lemma 2.3.5 *Let Y be a smooth projective curve over a local field K, $\bar{Y}$ the special fiber of a quasistable model of Y_L with L/K a finite extension and $\Delta_{\bar{Y}}$ the dual graph of $\bar{Y}$. Then*

$$f(Y/K) = \dim\big(H_1(\Delta_{\bar{Y}},\mathbb{Q}_\ell)^{I_K}\big) + f(H^1(Y_{\bar{K}},\mathbb{Q}_\ell)_{\mathrm{he}}).$$

Proof By Theorem 1.4.8 the conductor exponent of Y/K is

$$f(Y/K) = \dim\Big(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\mathrm{he}}^{I_K}\Big) - \dim\big(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)^{I_K}\big) + f(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\mathrm{he}}).$$

By the second exact sequence in Theorem 2.3.3, we have in $R(\Gamma_K)$

$$\dim\Big(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\mathrm{he}}^{I_K}\Big) = \dim\big(H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)^{I_K}\big) + \dim\big(H^1(\bar{Y}_{\bar{\mathbb{F}}_K}, \mathbb{Q}_\ell)^{I_K}\big).$$

By [BW17, Proposition 2.8] we have an isomorphism $H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)^{I_K} \simeq H^1(\bar{Y}_{\bar{\mathbb{F}}_K}, \mathbb{Q}_\ell)^{I_K}$. We obtain

$$f(Y/K) = \dim\big(H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)^{I_K}\big) + f(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\mathrm{he}}). \qquad \square$$

§ 2.4 The Lefschetz Trace Formula

In this section we state the Lefschetz Trace Formula which is the main ingredient for the formula we prove in the next section. We first discuss the intersection number appearing in the trace formula.

Let Y be a smooth projective curve over an algebraically closed field. Let $\phi : Y \to Y$ be a morphism, Γ_ϕ its graph and Δ the diagonal in $Y \times Y$. The intersection number $(\Gamma_\phi.\Delta)$ is the number of fixed points of ϕ counted with multiplicity. More precisely, if Γ_ϕ and Δ intersect properly, we set

$$(\Gamma_\phi.\Delta) = \sum_{x\in\Gamma_\phi\cap\Delta} \mathrm{length}(\mathcal{O}_{Y,y(x)}/I_{y(x)})$$

where $y(x)$ is the image of $x \in \Gamma_\phi \cap \Delta$ under the projection $Y \times Y \to Y$ on the first coordinate and I_y is the ideal generated by elements of the form $t - \phi(t)$ for $t \in \mathcal{O}_{Y,y}$. Here that Γ_ϕ and Δ intersect properly means that the intersection scheme $\Gamma_\phi \cap \Delta$ is empty or has dimension 0. Since $\mathcal{O}_{Y,y}$ is a discrete valuation ring, after taking a uniformising element t_y of $\mathcal{O}_{Y,y}$, the ideal I_y is generated by $t_y - \phi(t_y)$ and therefore

$$\mathrm{length}(\mathcal{O}_{Y,y}/I_y) = v_y(t_y - \phi(t_y)).$$

We obtain

$$(\Gamma_\phi.\Delta) = \sum_{y\in Y} v_y(t_y - \phi(t_y)).$$

We can now state the Lefschetz trace formula. For a proof see [Mil80, VI.12.3].

Theorem 2.4.1 *Let Y be a smooth projective curve over an algebraically closed field. Let $\phi : Y \to Y$ be a morphism, Γ_ϕ its graph and Δ the diagonal in $Y \times Y$. If Γ_ϕ and Δ intersect properly,*

$$(\Gamma_\phi.\Delta) = \sum_{i\geqslant 0} (-1)^i \,\mathrm{tr}\big(\phi|_{H^i(Y,\mathbb{Q}_\ell)}\big).$$

§ 2.5 A Grothendieck-Ogg-Shafarevich Formula

The Grothendieck-Ogg-Shafarevich formula [Mil80, V.2.12] gives an expression of the Euler-Poincaré characteristic as a sum of local terms. In this section we prove a formula in the Grothendieck group of a group $G \subset \mathrm{Aut}(Y)$ that expresses the Euler-Poincaré characteristic of a smooth projective curve Y as a sum over Artin characters associated to points of the quotient Y/G. This formula is well known (see e.g. [Mil80, V.2.9]) and goes back to Weil [Wei48, V].

Let $Y \xrightarrow{\phi} Z$ be a Galois cover with Galois group G of smooth projective curves over an algebraically closed field k, i.e. ϕ is a finite and surjective morphism that is an étale Galois cover with Galois group G on the étale locus of Y over Z. By purity of the branch locus [AK70, VI.6.8] the complement of the étale locus is a divisor, called ramification divisor.

Remark 2.5.1 By the definition of Galois cover which we use throughout this thesis the curve Y is not necessarily connected if $Y \to Z$ is a Galois cover.

Let $y \in Y$ be a point. Since Y and Z are smooth, the rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,\phi(y)}$ are discrete valuation rings. Denote by $\hat{\mathcal{O}}_{Y,y}$ and $\hat{\mathcal{O}}_{Z,\phi(y)}$ the completions of $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,\phi(y)}$. Furthermore, denote by L/K the extension of fraction fields of $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,\phi(y)}$, and by L_y/K_y the extension of fraction fields of $\hat{\mathcal{O}}_{Y,y}$ and $\hat{\mathcal{O}}_{Z,\phi(y)}$. Then L/K and L_y/K_y are Galois extensions, since ϕ is a Galois cover, and restricting elements of $\mathrm{Gal}(L_y/K_y)$ to L defines an embedding $\mathrm{Gal}(L_y/K_y) \to \mathrm{Gal}(L/K)$. Composing with the embedding of $\mathrm{Gal}(L/K)$ into G, we obtain an injection $\mathrm{Gal}(L_y/K_y) \to G$ with image $G_y = \{\sigma \in G : \sigma(y) = y\}$. Thus, L_y/K_y is a finite Galois extension of local fields with Galois group G_y. We are in the situation of § 1.2.

Definition 2.5.2 Let y be a closed point of Y and t_y a uniformising element at y.

(1) We define the decomposition group of y by $G_y = \{\sigma \in G : \sigma(y) = y\}$.

(2) The different $\mathfrak{D}_y$ of y is the different of the extension L_y/K_y.

(3) The Artin character of y is defined by

$$a_y(\sigma) = \begin{cases} -v_y(t_y - \sigma(t_y)) & \sigma \neq 1 \\ \sum\limits_{\tau \in G_y \setminus \{1\}} v_y(t_y - \tau(t_y)) & \sigma = 1 \end{cases}.$$

Remark 2.5.3 Let y be a closed point of Y. Since Y and Z are curves over an algebraically closed field, the extension L_y/K_y is totally ramified. Hence, the inertia group of this extension is $G_{y,0} = G_y$ and the Artin character of y is the Artin character a_{G_y} of the extension L_y/K_y defined as in § 1.2.

By Theorem 1.2.3 the Artin character of a closed point $y \in Y$ is the character of a representation of G_y. This character induces a character of G.

Definition 2.5.4 Let z be a closed point of Z and choose a point y of Y over z. The character $a_z = \operatorname{ind}_{G_y}^G(a_y)$ is called Artin character of z.

For a closed point $z \in Z$ the Artin character a_z is independent of the choice of a point $y \in Y$ over z. Indeed, the formula for the induced character shows

$$a_z(\sigma) = \sum_{y \in \phi^{-1}(z)} a_y(\sigma)$$

for $\sigma \neq 1$ and

$$a_z(1) = -\sum_{\sigma \neq 1} a_y(\sigma)$$

where we set $a_y(\sigma) = 0$ if $\sigma \notin G_y$. The following direct consequence of Lemma 1.2.2 will prove usefull in the next chapter.

Lemma 2.5.5 *Let z be a closed point of Z and choose a point y of Y over z. Then*

$$a_z = \sum_{i \geqslant 0} \frac{1}{(G_y : G_{y,i})} \operatorname{ind}_{G_{y,i}}^G(r_{G_{y,i}} - \mathbf{1}).$$

The ℓ-adic Euler-Poincaré characteristic of a curve Y is

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell) = \sum_{i \geqslant 0} (-1)^{i-1} H^i(Y, \mathbb{Q}_\ell),$$

considered as an element of $R(G)$ for a group G acting on Y. The Lefschetz Trace formula can be used to express the Euler-Poincaré characteristic through the Artin characters of closed points of Z.

Theorem 2.5.6 *Let $Y \xrightarrow{\phi} Z$ be a Galois cover with Galois group G of smooth projective curves over an algebraically closed field k, let Z be irreducible and $\ell \neq \operatorname{char}(k)$. Then in the Grothendieck group $R(G)$ we have*

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell) = (2g_Z - 2) r_G + \sum_{z \in Z} a_z.$$

Proof Let $\sigma \in G$ and let σ^* be the induced map on

$$\bigoplus_{i \geqslant 0} H^i(Y, \mathbb{Q}_\ell).$$

By the Lefschetz trace formula (Theorem 2.4.1),

$$\sum_{i \geqslant 0} (-1)^i \operatorname{tr}\big(\sigma^*|_{H^i(Y,\mathbb{Q}_\ell)}\big) = (\Gamma_\sigma.\Delta)$$

for $\sigma \neq 1$. We have

$$(\Gamma_\sigma.\Delta) = \sum_{y \in Y} v_y(t_y - \sigma(t_y)) = -\sum_{z \in Z} a_z(\sigma).$$

This shows

$$-\mathrm{tr}(\chi_{\mathrm{EP}}(Y,\mathbb{Q}_\ell)(\sigma)) = (2-2g_Z)r_G(\sigma) - \sum_{z\in Z} a_z(\sigma)$$

for all non-trivial $\sigma \in G$. The Riemann-Hurwitz formula [Har77, IV, Corollary 2.4] yields

$$-\dim(\chi_{\mathrm{EP}}(Y,\mathbb{Q}_\ell)) = |G|(2-2g_Z) - \deg(R)$$

where

$$R = \sum_{y\in Y} \mathrm{length}(\Omega_{Y/Z,y})y$$

is the ramification divisor of ϕ and $\Omega_{Y/Z}$ is the sheaf of relative differentials. By [Ser79, IV.1 Proposition 4] the different $\mathfrak{D}_y$ is the annihilator of $\Omega_{Y/Z,y}$ and therefore

$$\mathrm{length}(\Omega_{Y/Z,y}) = v_y(\mathfrak{D}_y).$$

By Lemma 1.2.1 we have

$$\deg(R) = \sum_{y\in Y} v_y(\mathfrak{D}_y) = \sum_{y\in Y}\sum_{\sigma\neq 1} v_y(t_y - \sigma(t_y)) = \sum_{z\in Z} a_z(1).$$

This concludes the proof. □

Chapter 3

G-isotypical Decomposition of the Euler-Poincaré Characteristic

A superelliptic curve over K is a cover of a projective line over K which is potentially Galois with cyclic Galois group. Since we are interested in the conductor exponent, we can assume that the curve is defined over K^{nr} whose residue field is algebraically closed. Assuming that the order of the Galois group is relatively prime to the residue characteristic, we can find a Galois extension L/K^{nr} such that the special fiber of a quasistable $\mathcal{O}_L$-model is a Galois cover of a tree of projective lines by Theorem 2.2.9, say with Galois group G. In this chapter we determine the Euler-Poincaré characteristic of the special fiber as an element in the Grothendieck group of $\Gamma = \mathrm{Gal}(L/K^{\mathrm{nr}})$. In § 3.1 we consider the case that the tree of projective lines consist of exactly one projective line. That means, we have a G-cover of smooth projective curves on which Γ acts. Since the G-cover is defined over K^{nr}, the G- and the Γ-action commute. We consider the cover of the quotient curve by the direct product of G and Γ and apply Theorem 2.5.6 to this cover. We then decompose the occuring Artin characters in their G-isotypical components yielding a formula for the Euler-Poincaré characteristic as a sum over the irreducible representations of G where each summand is a sum over the branch points of the cover. In §§ 3.2 and 3.3 we consider the general case. The Euler-Poincaré characteristic of the special fiber is the sum of the Euler-Poincaré characteristic of the normalization and a permutation character on the singular points of the special fiber. Since the normalization is the disjoint union of the covers of the projective lines, the determination of the Euler-Poincaré characteristic of the normalization can be reduced to the case in § 3.1. We obtain a decomposition of the Euler-Poincaré characteristic as a sum over irreducible representations of G where each summand is a sum over the vertices and edges of the dual graph of the quotient.

§ 3.1 G-isotypical Decomposition for Smooth Curves

Let $Y \to X$ be a Galois cover with abelian Galois group G of smooth projective curves over an algebraically closed field k of characteristic $p > 0$. Let Γ be a group acting on Y. We assume the following.

Assumption 3.1.1
- The curve X is isomorphic to a projective line over k.
- The order of the abelian group G is not divisible by p.

- The group Γ is a subgroup of a semidirect product $H \ltimes \Gamma_1$ where H is a cyclic group of order not divisible by p and Γ_1 is a p-group.
- The group $G \times \Gamma$ acts on Y. In particular, Γ acts on X.
- There exist a point on X, denoted ∞, which is fixed by Γ.

As noted in Remark 2.5.1, Y is not assumed to be irreducible. We define $\tilde{\Gamma} = G \times \Gamma$ and $Z = Y/\tilde{\Gamma}$, and we denote the point of Z below $\infty \in X$ also by ∞ if no confusion can arise.

With Theorem 2.5.6 it is possible to determine $\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell)$ where $\ell \neq p$ in $R(\tilde{\Gamma})$. To determine the G-isotypical decomposition of this representation we will use

Lemma 3.1.2 *Let G be an abelian group, Γ a group and $\tilde{\Gamma} = G \times \Gamma$. Let $U \subset \tilde{\Gamma}$ be a subgroup. Then as representations of Γ*

$$\mathrm{ind}_U^{\tilde{\Gamma}}(\mathbf{1}) = \bigoplus_\chi \rho_\chi$$

where

$$\rho_\chi = \begin{cases} 0 & \chi|_{U\cap G} \neq \mathbf{1} \\ \mathrm{ind}_{GU\cap\Gamma}^{\Gamma}(\eta_{U,\chi}) & \chi|_{U\cap G} = \mathbf{1} \end{cases}$$

and $\eta_{U,\chi}$ is a character of $GU \cap \Gamma$ defined by $\eta_{U,\chi}(\gamma) = \chi^{-1}(g)$ for some $g \in G$ with $(g, \gamma) \in U$ if $\chi\big|_{U\cap G} = \mathbf{1}$ and the sum runs over the irreducible characters of G.

Proof Since $\tilde{\Gamma}$ is the direct product of G and Γ, the G-isotypical decomposition

$$\mathrm{ind}_U^{\tilde{\Gamma}}(\mathbf{1}) = \bigoplus_\chi \rho_\chi$$

is also a decomposition of Γ-representations. Let χ be an irreducible character of G. The projection p_χ of $\mathrm{ind}_U^{\tilde{\Gamma}}(\mathbf{1})$ onto ρ_χ is given by

$$p_\chi(x) = \frac{1}{|G|} \sum_{g\in G} \chi^{-1}(g) g x.$$

We have $gx = x$ and therefore $p_\chi(x) = \chi^{-1}(g) p_\chi(x)$ for every $g \in U \cap G$ and every $x \in \mathrm{ind}_U^{\tilde{\Gamma}}(\mathbf{1})$. This implies $\rho_\chi = 0$ if χ is not trivial on $U \cap G$. Assume now that $U \cap G \subset \ker(\chi)$. Then the map $\eta_{U,\chi}$ given by $\eta_{U,\chi}(\gamma) = \chi^{-1}(g)$ for some $g \in G$ with $(g, \gamma) \in U$ is a well-defined character of $GU \cap \Gamma$. Let $\gamma \in GU \cap \Gamma$. We identify $\mathrm{ind}_U^{\tilde{\Gamma}}(\mathbf{1}) = \mathbb{Q}_\ell[\tilde{\Gamma}] \otimes_{\mathbb{Q}_\ell[U]} \mathbf{1}$ and set $x = p_\chi(1_{\tilde{\Gamma}} \otimes v)$ for some non-zero $v \in \mathbf{1}$. Clearly, $x \neq 0$ and $g\gamma(1_{\tilde{\Gamma}} \otimes v) = 1_{\tilde{\Gamma}} \otimes v$ for every $g \in G$ with $(g, \gamma) \in U$. Thus, $\gamma x = \chi^{-1}(g) x$ for every $g \in G$ with $(g, \gamma) \in U$. This shows that $\eta_{U,\chi}$ is a $\mathbb{Q}_\ell[GU \cap \Gamma]$-submodule of ρ_χ. The inclusion induces an injective $\mathbb{Q}_\ell[\Gamma]$-homomorphism

$$\mathbb{Q}_\ell[\Gamma] \otimes_{\mathbb{Q}_\ell[GU\cap\Gamma]} \eta_{U,\chi} \to \rho_\chi.$$

By [Ser77, § 7.3 Proposition 22] the restriction of $\mathrm{ind}_U^{\tilde{\Gamma}}(\mathbf{1})$ to G is the direct sum of $(\Gamma : GU \cap \Gamma)$ copies of $\mathrm{ind}_{U\cap G}^{G}(\mathbf{1})$ and therefore $\dim(\rho_\chi) = (\Gamma : GU \cap \Gamma)$. Hence,

$$\mathbb{Q}_\ell[\Gamma] \otimes_{\mathbb{Q}_\ell[GU\cap\Gamma]} \eta_{U,\chi} \to \rho_\chi$$

is a $\mathbb{Q}_\ell[\Gamma]$-isomorphism. □

For the rest of this section we fix the following notations and conventions.

Notation 3.1.3 • $\tilde{N} = \ker(\tilde{\Gamma} \to \mathrm{Aut}(Y))$, $N = G\tilde{N} \cap \Gamma = \ker(\Gamma \to \mathrm{Aut}(X))$,

- for each point z of Z we choose a point y of Y (respectively x of X) above z and denote by $\tilde{\Gamma}_z$ (respectively Γ_z) the stabilizer of y (respectively x) with respect to the $\tilde{\Gamma}$- (respectively Γ-)action,
- write a_z for the Artin character of a closed point z of Z associated to the cover $Y \to Z$,
- for an irreducible character χ of G and a point z of Z with $\chi|_{\tilde{\Gamma}_z\cap G} = \mathbf{1}$ write $\eta_\chi = \eta_{\tilde{N},\chi}$ and $\eta_{z,\chi} = \eta_{\tilde{\Gamma}_z,\chi}$ where $\eta_{\tilde{N},\chi}$ and $\eta_{\tilde{\Gamma}_z,\chi}$ are defined as in Lemma 3.1.2,
- sums over χ run over the irreducible characters of G.

We give a description of the characters $\eta_{z,\chi}$. Let z be a point of Z and χ an irreducible character of G. We have projection maps

$$\begin{array}{ccc} & \tilde{\Gamma}_z & \\ \swarrow & & \searrow \\ \Gamma_z & & G \end{array}.$$

Take $\eta_{z,\chi}$ as the composition of a section $\Gamma_z \to \tilde{\Gamma}_z$, the projection $\tilde{\Gamma}_z \to G$ and χ^{-1}. If $\chi|_{\tilde{\Gamma}_z\cap G} = \mathbf{1}$, this definition is independent of the chosen section and defines a character of Γ_z. The character η_χ is also the composition of the maps $N \to \mathrm{Aut}(Y/X) \simeq G$ and χ.

Lemma 3.1.4 *There exists a cyclic group C of order m, $p \nmid m$, and a group $T \simeq \mathbb{F}_p^r$ such that $\Gamma/N \simeq C \ltimes T$.*

Proof The group $\bar{\Gamma} = \Gamma/N$ acts faithfully and as affine transformations on $X = \mathbb{P}_k^1$ since the point $\infty \in X$ is fixed by Γ. Let T be the subgroup of translations. Then $T \simeq \mathbb{F}_p^r$ as a subgroup of k and $\bar{\Gamma}/T$ is cyclic of order not divisible by p as a subgroup of k^*. In particular, the short exact sequence

$$1 \to T \to \bar{\Gamma} \to \bar{\Gamma}/T \to 1$$

splits. □

In the rest of this section we denote by C' and T' the subgroups of Γ corresponding to the subgroups C and T of Γ/N of Lemma 3.1.4, i.e. C' respectively T' is the preimage of C respectively T under the natural projection $\Gamma \to \Gamma/N$.

We begin with determining the G-isotypical decomposition of the Artin characters of points of Z different from ∞.

Lemma 3.1.5 *(a) There exists a closed point $z_0 \neq \infty$ of Z with Γ_{z_0} conjugate to the subgroup C' of Γ. If $C' \neq N$, the point z_0 is unique.*

(b) For all closed points $z \notin \{z_0, \infty\}$ of Z we have $\Gamma_z = N$. For $z \neq \infty$

$$a_z = \sum_{\chi} a_{z,\chi}$$

holds as Γ-representations where for $z \notin \{z_0, \infty\}$

$$a_{z,\chi} = \begin{cases} \operatorname{ind}_{\Gamma_z}^{\Gamma}(\eta_\chi) & \chi|_{\tilde{\Gamma}_z \cap G} \neq \mathbf{1} \\ 0 & \chi|_{\tilde{\Gamma}_z \cap G} = \mathbf{1} \end{cases}$$

and

$$a_{z_0,\chi} = \begin{cases} \operatorname{ind}_N^{\Gamma}(\eta_\chi) & \chi|_{\tilde{\Gamma}_{z_0} \cap G} \neq \mathbf{1} \\ \operatorname{ind}_N^{\Gamma}(\eta_\chi) - \operatorname{ind}_{\Gamma_{z_0}}^{\Gamma}(\eta_{\tilde{\Gamma}_{z_0},\chi}) & \chi|_{\tilde{\Gamma}_{z_0} \cap G} = \mathbf{1} \end{cases}.$$

Proof By Lemma 3.1.4, $\bar{\Gamma} = \Gamma/N = C \ltimes T$ acts as an affine group on X. Hence, all $\gamma \in \bar{\Gamma} \setminus T$ have exactly one fixed point different from ∞. Let $x_0 \neq \infty$ be a fixed point of a generator of the cyclic group C. Then the stabilizer of x_0 is C and the orbit of x_0 has length $|T|$. Since elements in the same orbit have conjugate stabilizers, the orbit of x_0 contains the fixed points different from ∞ of $|T|(|C|-1)$ elements of $\bar{\Gamma}$. Because there are $|T|(|C|-1)$ elements of $\bar{\Gamma}$ with fixed points different from ∞, the orbit of x_0 contains exactly all the fixed points different from ∞ of non-trivial elements of $\bar{\Gamma}$. Thus, Γ_{z_0} is a subgroup conjugate to the subgroup C' of Γ for z_0 the point of Z below x_0 and for all other points $z \neq \infty$ of Z we have $\Gamma_z = N$. In particular, the cover $X \to Z$ is tame at all points $z \neq \infty$ of Z. Since $Y \to X$ is tame at all points, the cover $Y \to Z$ is tame at all points $z \neq \infty$ of Z and Lemma 2.5.5 yields

$$a_z = \operatorname{ind}_{\tilde{\Gamma}_z}^{\tilde{\Gamma}}(r_{\tilde{\Gamma}'_z} - \mathbf{1}) = \operatorname{ind}_{\tilde{N}}^{\tilde{\Gamma}}(\mathbf{1}) - \operatorname{ind}_{\tilde{\Gamma}_z}^{\tilde{\Gamma}}(\mathbf{1})$$

where $\tilde{\Gamma}'_z = \tilde{\Gamma}_z/\tilde{N}$. We conclude by applying Lemma 3.1.2. □

Since ∞ is wildly ramified, we first determine the jumps in the ramification filtration of ∞.

Lemma 3.1.6 *Let e_∞ be the ramification index of ∞ in $Y \to X$. We have*

$$a_\infty = \operatorname{ind}_{\tilde{\Gamma}_\infty}^{\tilde{\Gamma}}(r_{\tilde{\Gamma}_\infty} - \mathbf{1}) + \frac{e_\infty}{(\tilde{\Gamma}_\infty : \tilde{\Gamma}_{\infty,1})} \operatorname{ind}_{\tilde{\Gamma}_{\infty,1}}^{\tilde{\Gamma}}(r_{\tilde{\Gamma}_{\infty,1}} - \mathbf{1}).$$

Proof First consider the cover $X \to Z$. A local uniformizer at ∞ is $t_\infty = 1/x$. For the affine transformation $\gamma : x \mapsto ax + b$ on X we have

$$v_\infty(t_\infty - \gamma(t_\infty)) = v_\infty\left(\frac{(a-1)x + b}{x(ax+b)}\right)$$

which is 1 if γ is not a translation and 2 if γ is a non-trivial translation. Since $\Gamma/N = C \ltimes T$ acts as an affine group, we have $\Gamma_\infty = \Gamma$, the first ramification group is the group T of translations in Γ and all other ramification groups are trivial. The cover $Y \to X$ is tame of ramification degree e_∞ at ∞. Therefore, the $(e_\infty + i)$-th ramification group at ∞ of $Y \to Z$ is trivial for $i \geqslant 1$ and $\tilde{\Gamma}_{\infty,1} = \tilde{\Gamma}_{\infty,2} = \ldots = \tilde{\Gamma}_{\infty,e_\infty}$. Applying Lemma 2.5.5 gives the lemma. □

To decompose the wild part of the Artin character of ∞ into its G-isotypical components, we will use

Lemma 3.1.7 *Let $\Gamma \subset H \ltimes \Gamma_1$ with H a cyclic group, N a subgroup of Γ and η a character of N with $\eta|_{N\cap\Gamma_1} = \mathbf{1}$.*

(a) Then there exists a character η' of Γ with $\eta'\big|_N = \eta$.

(b) Furthermore, let N be normal in Γ such that $\Gamma/N \simeq C \ltimes T$ acts faithfully on $\mathbb{A}^1_k$ with T the subgroup of translations. Write C' and T' for the subgroups of Γ corresponding to C and T. Then

$$\operatorname{ind}_N^\Gamma(\eta) = \operatorname{ind}_{T'}^\Gamma(\eta'|_{T'}) + (\Gamma : T')\big(\operatorname{ind}_{C'}^\Gamma(\eta'|_{C'}) - \eta'\big)$$

holds for every character η' of Γ with $\eta'\big|_N = \eta$.

Proof Since $\eta|_{\Gamma_1\cap N} = \mathbf{1}$, η factors through $N/\Gamma_1 \cap N = N\Gamma_1/\Gamma_1$ and is therefore a character of $N\Gamma_1$. Choose a character η'' of H with $\eta''|_{H\cap N\Gamma_1} = \eta|_{H\cap N\Gamma_1}$ and define η''' as the composition of the natural projection $H \ltimes \Gamma_1 \to H$ and η''. Then η''' is a character of $H \ltimes \Gamma_1$ with $\eta'''\big|_N = \eta$. Take η' to be the restriction of η''' to Γ.

For a representation ρ of a group G and for a representation σ of a subgroup $H \subset G$ one has

$$\rho \otimes \operatorname{ind}_H^G(\sigma) = \operatorname{ind}_H^G(\rho|_H \otimes \sigma).$$

Therefore, part (b) of the lemma follows from

$$\operatorname{ind}_N^\Gamma(\mathbf{1}) = \operatorname{ind}_{T'}^\Gamma(\mathbf{1}) + (\Gamma : T')\big(\operatorname{ind}_{C'}^\Gamma(\mathbf{1}) - \mathbf{1}\big)$$

which is easily verified. □

Remark 3.1.8 Since Γ_1 is a p-group and $p \nmid |G|$, $(g,\gamma) \in \tilde{\Gamma}_{\infty,1}$ only if $g = 1$ for all $\gamma \in \Gamma_1$. All characters $\eta_{U,\chi}$ defined by Lemma 3.1.2 for an irreducible character χ of G and a subgroup U of $\tilde{\Gamma}$ with $U \cap G \subset \ker(\chi)$ are trivial on $GU \cap \Gamma \cap \Gamma_1$. By the proof of Lemma 3.1.7 extensions of these characters to Γ are trivial on $\Gamma \cap \Gamma_1$.

Lemma 3.1.9 *For each irreducible character χ of G let η'_χ be a character of Γ such that $\eta'_\chi|_N = \eta_\chi$. Then there is a decomposition*

$$a_\infty = \sum_\chi a_{\infty,\chi}$$

with

$$a_{\infty,\chi} = \begin{cases} \mathrm{ind}_N^\Gamma(\eta_\chi) + \mathrm{ind}_{C'}^\Gamma(\eta'_\chi|_{C'}) - \eta'_\chi - \eta_{\infty,\chi} & \chi|_{\tilde{\Gamma}_\infty \cap G} = \mathbf{1} \\ \mathrm{ind}_N^\Gamma(\eta_\chi) + \mathrm{ind}_{C'}^\Gamma(\eta'_\chi|_{C'}) - \eta'_\chi & \chi|_{\tilde{\Gamma}_\infty \cap G} \neq \mathbf{1} \end{cases}.$$

Proof Since $Y \to X$ is tame at ∞, we have $(\tilde{\Gamma}_\infty : \tilde{\Gamma}_{\infty,1}) = e_\infty(\Gamma_\infty : \Gamma_{\infty,1}) = e_\infty(\Gamma : T')$ where e_∞ is the ramification index of ∞ in $Y \to X$. Using first Lemma 3.1.6 and then Lemma 3.1.2 we get a decomposition

$$a_\infty = \sum_\chi a_{\infty,\chi}$$

with

$$a_{\infty,\chi} = \begin{cases} \mathrm{ind}_N^\Gamma(\eta_\chi) + \frac{1}{(\Gamma : T')}\Big(\mathrm{ind}_N^\Gamma(\eta_\chi) - \mathrm{ind}_{T'}^\Gamma\big(\eta_{\tilde{\Gamma}_{\infty,1,\chi}}\big)\Big) & \chi|_{\tilde{\Gamma}_\infty \cap G} \neq \mathbf{1} \\ \mathrm{ind}_N^\Gamma(\eta_\chi) - \eta_{\infty,\chi} + \frac{1}{(\Gamma : T')}\Big(\mathrm{ind}_N^\Gamma(\eta_\chi) - \mathrm{ind}_{T'}^\Gamma\big(\eta_{\tilde{\Gamma}_{\infty,1,\chi}}\big)\Big) & \chi|_{\tilde{\Gamma}_\infty \cap G} = \mathbf{1} \end{cases}.$$

Let χ be an irreducible character of G. Since Γ_1 is a p-group and $p \nmid |G|$, $(g, \gamma) \in \tilde{\Gamma}_{\infty,1}$ only if $g = 1$ for all $\gamma \in \Gamma_1$ and thus the character $\eta_{\tilde{\Gamma}_{\infty,1,\chi}}$ is trivial on $T' \cap \Gamma_1$ as is the character η'_χ. The order of an element $\gamma \in T' \cap H$ is prime to p and $T'/N \simeq T$ is a p-group implying that $\gamma \in N$. We have $\tilde{N} \subset \tilde{\Gamma}_{\infty,1}$ and therefore $\eta_{\tilde{\Gamma}_{\infty,1}}$ coincides with the extension η'_χ of η_χ on $T' \cap H \subset N$. Since T' is a subgroup of $H \ltimes \Gamma_1$, $\eta'_\chi|_{T'} = \eta_{\tilde{\Gamma}_{\infty,1,\chi}}$ and we can conclude with Lemma 3.1.7. □

Let χ be an irreducible character of G and $z \neq \infty$ a point of Z. If $\chi|_{\tilde{\Gamma}_z \cap G} = \mathbf{1}$, the character $\eta_{\tilde{\Gamma}_z,\chi}$, defined as in Lemma 3.1.2, is well-defined. We note that this character is trivial on $\tilde{\Gamma}_z \cap \Gamma_1$, since Γ_1 is a p-group and $p \nmid |G|$, and therefore has an extension to Γ by Lemma 3.1.7. By Lemma 3.1.5 there exists a point $z_0 \neq \infty$ on Z such that $\Gamma_{z_0} = C'$.

Definition 3.1.10 Let $z_0 \neq \infty$ be a point on Z such that $\Gamma_{z_0} = C'$ and let χ be an irreducible character of G. If χ is trivial on $\tilde{\Gamma}_{z_0} \cap G$, a character of Γ which is an extension of $\eta_{\tilde{\Gamma}_{z_0},\chi}$ is called a χ-character associated to the cover $Y \to Z$. If $\chi|_{\tilde{\Gamma}_{z_0} \cap G} \neq \mathbf{1}$, a character of Γ which is an extension of η_χ is called χ-character associated to the cover $Y \to Z$.

Theorem 3.1.11 *For each irreducible character of G let η'_χ be a χ-character associated to the cover $Y \to Z$. Then there is decomposition in $R(\Gamma)$,*

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell) = \sum_\chi \chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell)_\chi$$

where

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell)_\chi = \begin{cases} \sum\limits_{\substack{z \neq \infty \\ \chi|_{\tilde{\Gamma}_z \cap G} \neq \mathbf{1}}} \mathrm{ind}_{\Gamma_z}^\Gamma(\eta'_\chi|_{\Gamma_z}) - \eta'_\chi & \chi|_{\tilde{\Gamma}_\infty \cap G} \neq \mathbf{1} \\ \sum\limits_{\substack{z \neq \infty \\ \chi|_{\tilde{\Gamma}_z \cap G} \neq \mathbf{1}}} \mathrm{ind}_{\Gamma_z}^\Gamma(\eta'_\chi|_{\Gamma_z}) - \eta'_\chi - \eta_{\infty,\chi} & \chi|_{\tilde{\Gamma}_\infty \cap G} = \mathbf{1} \end{cases}.$$

Proof By Theorem 2.5.6 the character of $H^1(Y)$ as $\tilde{\Gamma}$-representation is

$$\sum_{z \in Z} a_z - 2r_{\tilde{\Gamma}'}$$

with $\tilde{\Gamma}' = \tilde{\Gamma}/\tilde{N}$. By Lemmata 3.1.5 and 3.1.9 we have a decomposition

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell) = \sum_{\chi} \chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell)_\chi$$

where for $\chi|_{\tilde{\Gamma}_\infty \cap G} = \mathbf{1}$

$$\begin{aligned}
\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell)_\chi &= \sum_{z \neq \infty, z_0} a_{z,\chi} + a_{z_0,\chi} + a_{\infty,\chi} - 2\,\mathrm{ind}_N^\Gamma(\eta_\chi) \\
&= \sum_{\substack{z \neq \infty, z_0 \\ \chi|_{\tilde{\Gamma}_z \cap G} \neq \mathbf{1}}} \mathrm{ind}_{\Gamma_z}^\Gamma(\eta_\chi) + a_{z_0,\chi} - \mathrm{ind}_N^\Gamma(\eta_\chi) + \mathrm{ind}_{\Gamma_{z_0}}^\Gamma(\eta'_\chi|_{\Gamma_{z_0}}) - \eta'_\chi - \eta_{\infty,\chi} \\
&= \sum_{\substack{z \neq \infty \\ \chi|_{\tilde{\Gamma}_z \cap G} \neq \mathbf{1}}} \mathrm{ind}_{\Gamma_z}^\Gamma(\eta'_\chi|_{\Gamma_z}) - \eta'_\chi - \eta_{\infty,\chi}
\end{aligned}$$

and for $\chi|_{\tilde{\Gamma}_\infty \cap G} \neq \mathbf{1}$

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell)_\chi = \sum_{\substack{z \neq \infty \\ \chi|_{\tilde{\Gamma}_z \cap G} \neq \mathbf{1}}} \mathrm{ind}_{\Gamma_z}^\Gamma(\eta'_\chi|_{\Gamma_z}) - \eta'_\chi$$

by the same calculation as above. □

§ 3.2 Orientation on a Tree of Projective Lines

Let $Y \to X$ be a Galois cover with abelian Galois group G of semistable projective curves over an algebraically closed field k of characteristic $p > 0$. Let Γ be a group acting on Y. We make the following assumption.

Assumption 3.2.1 • The curve X is a tree of projective lines over k.

- The order of the abelian group G is not divisible by p.
- The group Γ is a semidirect product $H \ltimes \Gamma_1$ of a p-group Γ_1 and a cyclic group H such that $p \nmid |H|$.
- The group $G \times \Gamma$ acts on Y.
- There exist a point on X, denoted x_r, which is fixed by Γ.

Define $\tilde{\Gamma} = G \times \Gamma$ and $Z = Y/\tilde{\Gamma}$.

Lemma 3.2.2 *Let X_w be an irreducible component of X such that $x_r \notin X_w$, let X_v be the irreducible component of X such that removing X_v from X the curve X_w is in another connected component than x_r, and let x_e be the point in which X_v and X_w meet. Let Γ_v, Γ_w respectively Γ_e be the stabilizer of X_v, X_w respectively x_e under the action of Γ on X. Then $\Gamma_e = \Gamma_w \subset \Gamma_v$.*

Proof Let X_r be the irreducible component of X with $x_r \in X_r$. Removing the point x_e from the tree X we obtain two connected components denoted X_1 and X_2. Let X_1 be the connected component containing x_r. By the hypothesis X_v is an irreducible component of X_1 and X_w is an irreducible component of X_2. For $\sigma \in \Gamma_e$ we have $\sigma(X_w) \in \{X_v, X_w\}$ and, since x_r is fixed by Γ, the connected component X_1 is fixed by σ. In particular, σ fixes X_v. It follows that every element of $\sigma \in \Gamma_e$ fixes X_w and X_v. This shows $\Gamma_e \subset \Gamma_v$ and $\Gamma_e \subset \Gamma_w$. Let $\sigma \in \Gamma_w$. If σ would not fix x_e, it would not fix the connected component X_1 and thus it would not fix x_r. This implies $\Gamma_w \subset \Gamma_e$. □

Denote by Δ_Y, $\Delta_X = (V_X, E_X)$ respectively $\Delta_Z = (V_Z, E_Z)$ the dual graphs of Y, X respectively Z. We declare the vertex corresponding to the irreducible component X_r of X with $x_r \in X_r$ as the root vertex of the tree Δ_X. Choose the orientation on the rooted tree Δ_X in which every edge points away from the root vertex. This induces an orientation on $\Delta_Z = \Delta_X/\Gamma$ and throughout the rest of the chapter we understand Δ_Z as this directed tree. The edge $e \in E_Z$ incident to the vertices $v, w \in V_Z$ and pointing towards w will be denoted (v, w).

Notation 3.2.3
- The root vertex of Δ_Z is denoted by r and the component of Z corresponding to $v \in V_Z$ is denoted Z_v.
- For each $v \in V_Z$ we choose a component, denoted X_v, of X over the component Z_v, define Y_v to be the preimage of X_v under $Y \to X$ and write Γ_v respectively $\tilde{\Gamma}_v$ for the stabilizer of X_v respectively Y_v. Note that $\tilde{\Gamma}_v = G \times \Gamma_v$ for each $v \in V_Z$.
- By T_Z denote the directed tree having the same vertices and directed edges as Δ_Z and a vertex adjacent to $v \in V_Z$ added for each non-singular point of Z_v over which a branch point of $Y_v \to X_v$ different from x_r lies. Every added edge of T_Z points towards the added vertex incident to it.
- By B we denote the set of leafs of the tree T_Z. Note that we do not view the vertex corresponding to a point below x_r as a leaf.
- Let e be an edge of T_Z. We denote the stabilizer of a point of Y respectively X over the point of Z corresponding to the edge e by $\tilde{\Gamma}_e$ respectively Γ_e. Note that $\Gamma_e = G\tilde{\Gamma}_e \cap \Gamma$.

For each vertex $v \in V_Z$, the point x_e of X corresponding to the unique edge ending in v is fixed by Γ_v by Lemma 3.2.2. Thus, for each vertex $v \in V_Z$, Assumption 3.1.1 is satisfied for the Galois cover $Y_v \to X_v$ with abelian Galois group G of smooth projective curves over k and the group Γ_v acting on Y_v. Therefore, for every irreducible character χ and every edge $v \in V_Z$ there is an character attached, a χ-character associated to the cover $Y_v \to Z_v$ (Definition 3.1.10), denoted $\eta_{v,\chi}$. We continue with introducing notations.

Notation 3.2.4 • For each irreducible character χ of G and each vertex v of Δ_Z, we define a representation of Γ by

$$\rho_{v,\chi} = \mathrm{ind}_{\Gamma_v}^{\Gamma}(\eta_{v,\chi}).$$

- We attach to every irreducible character χ of G and every edge $e = (v, w)$ of T_Z the character $\eta_{e,\chi}$ defined by $\eta_{e,\chi} = \eta_{v,\chi}|_{\Gamma_e}$ and set

$$\rho_{e,\chi} = \mathrm{ind}_{\Gamma_e}^{\Gamma}(\eta_{e,\chi}).$$

  Note that $\eta_{e,\chi} = \eta_{\tilde{\Gamma}_e,\chi}$ if χ is trivial on $\tilde{\Gamma}_e \cap G$.
- For each leaf $a \in B$ and each irreducible character of G define $\rho_{a,\chi} = \rho_{e,\chi}$, $\Gamma_a = \Gamma_e$ and $\tilde{\Gamma}_a = \tilde{\Gamma}_e$ for the edge e incident to a.
- Let $\tilde{\Gamma}_\infty$ be the stabilizer of the point of Z below x_r under the action of $\tilde{\Gamma}$ on Y. We define $\eta_{\infty,\chi} = \eta_{\tilde{\Gamma}_\infty,\chi}$ if $\tilde{\Gamma}_\infty \cap G \subset \ker(\chi)$ and $\eta_{\infty,\chi} = 0$ otherwise.

Using Theorem 3.1.11 for each local cover $Y_v \to X_v$, enables us to determine the G-isotypical decomposition of the Euler-Poincaré characteristic of the normalization of Y.

Theorem 3.2.5 *Let $\tilde{Y}$ be the normalization of Y. In $R(\Gamma)$ we have*

$$\chi_{\mathrm{EP}}(\tilde{Y}, \mathbb{Q}_\ell) = \sum_{\chi} \chi_{\mathrm{EP}}(\tilde{Y}, \mathbb{Q}_\ell)_\chi$$

where

$$\chi_{\mathrm{EP}}(\tilde{Y}_{\bar{k}}, \mathbb{Q}_\ell)_\chi = \sum_{\substack{e \in E(T_Z) \\ \chi|_{\tilde{\Gamma}_e \cap G} \neq \mathbf{1}}} \rho_{e,\chi} - \sum_{\substack{e \in E_Z \\ \chi|_{\tilde{\Gamma}_e \cap G} = \mathbf{1}}} \rho_{e,\chi} - \sum_{v \in V_Z} \rho_{v,\chi} - \eta_{\infty,\chi}.$$

Proof The curve $\tilde{Y}$ is smooth and the disjoint union of the curves Y_j which are the preimages of the irreducible components X_j of X under $Y \to X$ for $j \in V_X$. Thus,

$$H^i(\tilde{Y}, \mathbb{Q}_\ell) = \bigoplus_{j \in V_X} H^i(Y_j, \mathbb{Q}_\ell)$$

as $\mathbb{Q}_\ell$-vector spaces for $i \geqslant 0$. This implies

$$\chi_{\mathrm{EP}}(\tilde{Y}, \mathbb{Q}_\ell) = \sum_{v \in V_Z} \mathrm{ind}_{\Gamma_v}^{\Gamma}(\chi_{\mathrm{EP}}(Y_v, \mathbb{Q}_\ell))$$

in $R(\Gamma)$. We conclude by applying Theorem 3.1.11 to each curve Y_v and noting that the possible branch points of $Y_v \to X_v$ correspond to edges $(v, w) \in E(T_Z)$ and the unique edge e pointing towards v corresponds to the point on X_v fixed by Γ_v. □

§ 3.3 G-isotypical Decomposition for Semistable Curves

We keep the notation and assumptions of § 3.2. Let $\pi : \tilde{Y} \to Y$ be the normalization of Y. Let $\mathcal{E}$ be a sheaf on $Y_{\text{ét}}$ such that the sequence

$$0 \to \mathbb{Q}_\ell \to \pi_* \mathbb{Q}_\ell \to \mathcal{E} \to 0$$

is exact. We recall from § 2.3 that the stalk of $\pi_* \mathbb{Q}_\ell$ is 1-dimensional at smooth points and 2-dimensional at singular points of Y, and hence the sheaf $\mathcal{E}$ is concentrated with 1-dimensional stalks in the singular points. Thus, the above exact sequence induces the long exact sequence

$$0 \to H^0(Y, \mathbb{Q}_\ell) \to H^0(Y, \pi_* \mathbb{Q}_\ell) \to H^0(Y, \mathcal{E}) \to H^1(Y, \mathbb{Q}_\ell) \to H^1(Y, \pi_* \mathbb{Q}_\ell) \to 0.$$

By Remark 1.1.5 (2) and the projection formula we have

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell) = \chi_{\mathrm{EP}}(\tilde{Y}, \mathbb{Q}_\ell) - \chi_{\mathrm{EP}}(Y, \mathcal{E}) = \chi_{\mathrm{EP}}(\tilde{Y}, \mathbb{Q}_\ell) + H^0(Y, \mathcal{E})$$

in the Grothendieck group $R(\tilde{\Gamma})$. Combining this with Theorem 3.2.5 – in order to determine the G-isotypical decomposition of the Euler-Poincaré characteristic of Y – we are left with computing the G-isotypical decomposition of $H^0(Y, \mathcal{E})$.

Lemma 3.3.1 *As Γ-representations we have*

$$H^0(Y, \mathcal{E}) = \bigoplus_{\chi} H^0(Y, \mathcal{E})_\chi$$

where

$$H^0(Y, \mathcal{E})_\chi = \sum_{\substack{e \in E_Z \\ \chi|_{\tilde{\Gamma}_e \cap G} = \mathbf{1}}} \rho_{e,\chi}.$$

Proof Since $\mathcal{E}$ is concentrated in the singular points of Y with 1-dimensional stalks, we have

$$H^0(Y, \mathcal{E}) = \bigoplus_{e \in E_Z} \mathrm{ind}_{\tilde{\Gamma}_e}^{\tilde{\Gamma}}(\mathbf{1}).$$

We conclude with Lemma 3.1.2. □

Combining Lemma 3.3.1 and Theorem 3.2.5 we obtain

Theorem 3.3.2 *We have the following decomposition $R(\Gamma)$,*

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell) = \sum_{\chi} \chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell)_\chi$$

where

$$\chi_{\mathrm{EP}}(Y, \mathbb{Q}_\ell)_\chi = \sum_{\substack{a \in B \\ \chi|_{\tilde{\Gamma}_a \cap G} \neq \mathbf{1}}} \rho_{a,\chi} + \sum_{\substack{e \in E_Z \\ \chi|_{\tilde{\Gamma}_e \cap G} \neq \mathbf{1}}} \rho_{e,\chi} - \sum_{v \in V_Z} \rho_{v,\chi} - \eta_{\infty,\chi}.$$

Chapter 4

Conductor Exponents of Superelliptic Curves

This chapter contains the main result of this thesis, a formula for the conductor exponent of a superelliptic curve which immediately gives an upper bound. Having the G-isotypical decomposition of the Euler-Poincaré characteristic of the special fiber of the previous chapter and the exact sequences in Theorem 2.3.3 at hand, we obtain in § 4.1 a G-isotypical decomposition of the semisimplification of the Galois representation associated to a superelliptic curve. As a consequence the conductor exponent of a superelliptic curve is a sum over irreducible representations of G. For an irreducible representation, the summand in this decomposition has two parts. The first part is a sum of valuations of the differents at the branch points. The second part is a sum of local terms over the dual graph of a quotient curve. These local terms are sums of conductor exponents of at most tamely ramified one-dimensional representations associated to points in the special fiber. We use this in § 4.2 to obtain upper bounds for the local terms yielding an upper bound for the conductor exponent.

The situation throughout this chapter is as follows. Let $Y \xrightarrow{\phi} X$ be a cyclic cover of smooth projective curves over a finite extension $K/\mathbb{Q}_p^{\mathrm{nr}}$ with Galois group $G \simeq \mathbb{Z}/n\mathbb{Z}$ such that $p \nmid n$, Y is absolutely irreducible of genus $g_Y \geqslant 2$ and the genus of X is 0. Let D denote the branch divisor of ϕ and let L_0 be a splitting field of D. Then D_L has at least three points for any finite extension L/L_0 and therefore the marked curve (X_L, D_L) has a unique minimal semistable $\mathcal{O}_L$-model $(\mathcal{X}, \mathcal{D})$ by Theorem 2.2.7 (b). By Theorem 2.2.7 (d) the special fiber $(\bar{X}, \bar{D})$ of $(\mathcal{X}, \mathcal{D})$ is a marked tree of projective lines. By Theorem 2.2.9 there exists a tame extension L/L_0 such that L/K is a Galois extension and the normalization $\mathcal{Y}$ of $\mathcal{X}$ in Y_L is a quasistable model of Y. Thus, the action of the Galois group $\Gamma = \mathrm{Gal}(L/K)$ extends to an action on $\mathcal{Y}$. In particular, Γ acts on the special fiber $\bar{Y}$ and, since ϕ is defined over K, this action commutes with the action of G on $\bar{Y}$. In other words, the group $\tilde{\Gamma} = G \times \Gamma$ acts on the special fiber of $\mathcal{Y}$. Define $\bar{Z} = \bar{Y}/\tilde{\Gamma}$. Since K is an extension of the maximal unramified extension of $\mathbb{Q}_p$, the residue field of K is algebraically closed and the Galois group Γ is a quotient of the absolute inertia group of K. We choose a point x_r on $\bar{X}$ which is fixed by Γ. This defines an orientation on the trees $\Delta_{\bar{X}}$ and $\Delta_{\bar{Z}}$ by § 3.2. Assumption 3.2.1 is satisfied for the cover $\bar{Y} \to \bar{X}$. Throughout the chapter we use Notations 3.2.3 and 3.2.4 adapted to the cover $\bar{Y} \to \bar{X}$.

§ 4.1 G-isotypical Decomposition of the Conductor Exponent

As in Chapter 1 we denote by $H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\text{he}}$ the semisimplification of the Γ-representation $H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)$. By Lemma 2.3.5,

$$f(Y/K) = \dim\big(H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)^\Gamma\big) + f(H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\text{he}}).$$

In this section we determine the G-isotypical decompositions of $H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\text{he}}$ and of $H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)^\Gamma$. We obtain a corresponding decomposition of $f(Y/K)$.

Theorem 4.1.1 *The semisimplification of $H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)$ decomposes in $R(\Gamma)$ as*

$$H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\text{he}} = \sum_{\chi \neq \mathbf{1}} H^1(Y)_\chi$$

where

$$H^1(Y)_\chi = \sum_{\substack{a \in B \\ \chi|_{\tilde{\Gamma}_a \cap G} \neq \mathbf{1}}} \rho_{a,\chi} + \sum_{e \in E_{\bar{Z}}} \rho_{e,\chi} - \sum_{v \in V_{\bar{Z}}} \rho_{v,\chi} - \eta_{\infty,\chi}.$$

Proof By Theorem 2.3.3 there exists a Γ-equivariant exact sequence

$$0 \to H^1(\bar{Y}, \mathbb{Q}_\ell) \to H^1(Y_{\bar{K}}, \mathbb{Q}_\ell) \to H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)(-1) \to 0.$$

Thus, in $R(\Gamma)$ we obtain

$$H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)_{\text{he}} = H^1(\bar{Y}, \mathbb{Q}_\ell) + H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)(-1).$$

Since L/K is a totally ramified extension, $H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)(-1) = H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)$ as Γ-representations. By Theorem 2.3.1 it holds that $H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell) = \rho_E - \rho_V + \mathbf{1}$, where ρ_E is the permutation representation of $\tilde{\Gamma}$ on the edges of $\Delta_{\bar{Y}}$ and ρ_V is the permutation representation on vertices. Since $\bar{Y}$ is connected, we have $H^0(\bar{Y}, \mathbb{Q}_\ell) = \mathbf{1}$. It holds that $H^2(\bar{Y}, \mathbb{Q}_\ell) \simeq H^2(\tilde{Y}, \mathbb{Q}_\ell)$ for the normalization $\tilde{Y}$ of $\bar{Y}$. As $\mathbb{Q}_\ell$-vector space

$$H^2(\tilde{Y}, \mathbb{Q}_\ell) = \bigoplus_{i \in V(\Delta_{\bar{Y}})} H^2(\bar{Y}_i, \mathbb{Q}_\ell)$$

where $\bar{Y}_i$ is the irreducible component of $\bar{Y}$ corresponding to the vertex $i \in V(\Delta_{\bar{Y}})$. The stabilizer Γ_i of $\bar{Y}_i$ acts trivially on the one dimensional space $H^2(\bar{Y}_i, \mathbb{Q}_\ell)$ since L/K is totally ramified. Hence, $H^2(\bar{Y}, \mathbb{Q}_\ell) = \rho_V$ as Γ-representation. To sum up, in $R(\Gamma)$ we have

$$\begin{aligned} H^1(Y, \mathbb{Q}_\ell)_{\text{he}} &= H^1(\bar{Y}, \mathbb{Q}_\ell) + H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)(-1) \\ &= H^0(\bar{Y}, \mathbb{Q}_\ell) + H^2(\bar{Y}, \mathbb{Q}_\ell) + \chi_{\text{EP}}(\bar{Y}, \mathbb{Q}_\ell) + H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)(-1) \\ &= 2 \cdot \mathbf{1} + \rho_E + \chi_{\text{EP}}(\bar{Y}, \mathbb{Q}_\ell). \end{aligned}$$

By Lemma 3.1.2, ρ_E is, as an element of $R(\Gamma)$, the sum of the Γ-representations

$$E_\chi = \sum_{\substack{e \in E_{\bar{Z}} \\ \chi|_{\tilde{\Gamma}_e \cap G} = \mathbf{1}}} \rho_{e,\chi}.$$

We use Theorem 3.3.2 to obtain a decomposition

$$H^1(Y, \mathbb{Q}_\ell)_{\mathrm{he}} = \sum_\chi H^1(Y)_\chi$$

where

$$H^1(Y)_{\mathbf{1}} = 2 \cdot \mathbf{1} + \sum_{(v,w) \in E_{\bar{Z}}} \mathrm{ind}_{\Gamma_w}^{\Gamma}(\mathbf{1}) + \sum_{(v,w) \in E_{\bar{Z}}} \mathrm{ind}_{\Gamma_w}^{\Gamma}(\mathbf{1}) - \sum_{v \in V_{\bar{Z}}} \mathrm{ind}_{\Gamma_v}^{\Gamma}(\mathbf{1}) - \sum_{v \in V_{\bar{Z}}} \mathrm{ind}_{\Gamma_v}^{\Gamma}(\mathbf{1}) = 0$$

and

$$\begin{aligned} H^1(Y)_\chi &= E_\chi + \sum_{\substack{a \in B \\ \chi|_{\tilde{\Gamma}_a \cap G} \neq \mathbf{1}}} \rho_{a,\chi} + \sum_{\substack{e \in E_{\bar{Z}} \\ \chi|_{\tilde{\Gamma}_e \cap G} \neq \mathbf{1}}} \rho_{e,\chi} - \sum_{v \in V_{\bar{Z}}} \rho_{v,\chi} - \eta_{\infty,\chi} \\ &= \sum_{\substack{a \in B \\ \chi|_{\tilde{\Gamma}_a \cap G} \neq \mathbf{1}}} \rho_{a,\chi} + \sum_{e \in E_{\bar{Z}}} \rho_{e,\chi} - \sum_{v \in V_{\bar{Z}}} \rho_{v,\chi} - \eta_{\infty,\chi} \end{aligned}$$

for $\chi \neq \mathbf{1}$. □

Lemma 4.1.2 *We have a decomposition in* $R(\Gamma)$*,*

$$H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell) = \sum_{\chi \neq \mathbf{1}} H_1(\Delta_{\bar{Y}})_\chi$$

with

$$H_1(\Delta_{\bar{Y}})_\chi = \sum_{\substack{e \in E_{\bar{Z}} \\ \chi|_{\tilde{\Gamma}_e \cap G} = \mathbf{1}}} \rho_{e,\chi} - \sum_{\substack{v \in V_{\bar{Z}} \\ \chi|_{\tilde{\Gamma}'_v \cap G} = \mathbf{1}}} \rho'_{v,\chi}$$

where $\rho'_{v,\chi} = \mathrm{ind}_{\Gamma_v}^{\Gamma}(\eta'_{v,\chi})$ *and* $\eta'_{v,\chi}$ *is defined as in Lemma 3.1.2 for the stabilizer* $\tilde{\Gamma}'_v$ *of an irreducible component of* $\bar{Y}_v$*.*

Proof By Lemma 2.3.1,

$$H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell) = \sum_{e \in E_{\bar{Z}}} \mathrm{ind}_{\tilde{\Gamma}_e}^{\tilde{\Gamma}}(\mathbf{1}) - \sum_{v \in V_{\bar{Z}}} \mathrm{ind}_{\tilde{\Gamma}'_v}^{\tilde{\Gamma}}(\mathbf{1}) + \mathbf{1}$$

in $R(\tilde{\Gamma})$. Applying Lemma 3.1.2 yields the lemma. □

Before stating the main theorem of this section, we introduce notations that we will use throughout the rest of this thesis.

Notation 4.1.3 • For $v \in V_{\bar{Z}}$ and an irreducible character $\chi \neq \mathbf{1}$ of G, denote by $\tilde{\Gamma}'_v$ the stabilizer of an irreducible component of $\bar{Y}_v$, and by $\eta'_{v,\chi}$ the character $\eta_{\tilde{\Gamma}'_v,\chi}$ defined in Lemma 3.1.2 for $U = \tilde{\Gamma}'_v$.

- Let $E(v)$ be the set of edges e of $E(T_{\bar{Z}})$ such that $v \in V_{\bar{Z}}$ is the source vertex of e, and $E'(v) = E(v) \cap E_{\bar{Z}}$.
- For an irreducible character $\chi \neq \mathbf{1}$ of G and $v \in V_{\bar{Z}}$ define

$$f_\chi(v) = \sum_{\substack{a \in B \cap E(v) \\ \chi|_{\tilde{\Gamma}_a \cap G} \neq \mathbf{1}}} f(\eta_{a,\chi}) + \sum_{e \in E'(v)} f(\eta_{e,\chi}) - f(\eta_{v,\chi}) + \sum_{\substack{e \in E'(v) \\ \chi|_{\tilde{\Gamma}_e \cap G} = \mathbf{1}}} (1 - f(\eta_{e,\chi})) - (1 - f(\eta'_{v,\chi}))$$

  if $\chi|_{\tilde{\Gamma}'_v \cap G} = \mathbf{1}$, and

$$f_\chi(v) = \sum_{\substack{a \in B \cap E(v) \\ \chi|_{\tilde{\Gamma}_a \cap G} \neq \mathbf{1}}} f(\eta_{a,\chi}) + \sum_{e \in E'(v)} f(\eta_{e,\chi}) - f(\eta_{v,\chi}) + \sum_{\substack{e \in E'(v) \\ \chi|_{\tilde{\Gamma}_e \cap G} = \mathbf{1}}} (1 - f(\eta_{e,\chi}))$$

  if $\chi|_{\tilde{\Gamma}'_v \cap G} \neq \mathbf{1}$.
- We set $f_\chi(\infty) = f(\eta_{\infty,\chi})$.
- For a leaf $a \in B$ we define $\delta_a = v_{L^{\Gamma_a}}(\mathfrak{D}_{L^{\Gamma_a}/K})$.
- For an irreducible character χ of G set $B_\chi = \{a \in B : \chi|_{\tilde{\Gamma}_a \cap G} \neq \mathbf{1}\}$.

Theorem 4.1.4 *For the conductor exponent of Y/K we have*

$$f(Y/K) = \sum_{\chi \neq \mathbf{1}} f_\chi(Y/K)$$

with

$$f_\chi(Y/K) = \sum_{v \in V_{\bar{Z}}} f_\chi(v) \; - \; f_\chi(\infty) \; + \; \sum_{a \in B_\chi} \delta_a.$$

Proof By Theorem 4.1.1 and Lemma 4.1.2,

$$f(Y/K) = f(H^1(Y, \mathbb{Q}_\ell)_{\mathrm{he}}) + \dim\big(H_1(\Delta_{\bar{Y}}, \mathbb{Q}_\ell)^\Gamma\big) = \sum_{\chi \neq \mathbf{1}} \big(f\big(H^1(Y)_\chi\big) + \dim\big(H_1(\Delta_{\bar{Y}})^\Gamma_\chi\big)\big)$$

where $H^1(Y)_\chi$ and $H_1(\Delta_{\bar{Y}})^\Gamma_\chi$ are defined as in Theorem 4.1.1 and Lemma 4.1.2. Recall that $\Gamma_e = \Gamma_w$ for $e = (v, w) \in E_{\bar{Z}}$ by Lemma 3.2.2. Therefore,

$$\rho_{e,\chi} - \rho_{w,\chi} = \mathrm{ind}_{\Gamma_w}^{\Gamma}(\eta_{e,\chi} - \eta_{w,\chi}).$$

Using that $T_{\bar{Z}}$ is a tree and the inductivity in dimension 0 of the conductor exponent (Theorem 1.4.5 (ii)) we get

$$f\big(H^1(Y)_\chi\big) = \sum_{a \in B_\chi} (v_{L^{\Gamma_a}}(\mathfrak{D}_{L^{\Gamma_a}/K}) + f(\eta_{a,\chi})) + \sum_{e \in E_{\bar{Z}}} f(\eta_{e,\chi}) - \sum_{v \in V_{\bar{Z}}} f(\eta_{v,\chi}) - f(\eta_{\infty,\chi}).$$

For a monomial representation $\rho = \mathrm{ind}_U^H(\eta)$ of a group H we have

$$\dim\left(\rho^H\right) = \begin{cases} 1 & \eta = \mathbf{1} \\ 0 & \eta \neq \mathbf{1} \end{cases}.$$

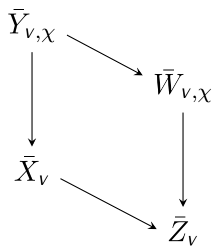
Hence,

$$\dim\left(H_1(\Delta_{\bar{Y}})^\Gamma_\chi\right) = \sum_{\substack{e \in E_{\bar{Z}} \\ \chi|_{\tilde{\Gamma}_e \cap G} = \mathbf{1}}} (1 - f(\eta_{e,\chi})) - \sum_{\substack{v \in V_{\bar{Z}} \\ \chi|_{\tilde{\Gamma}'_v \cap G} = \mathbf{1}}} (1 - f(\eta'_{v,\chi}))$$

since the one-dimensional representations $\eta_{e,\chi}$ and $\eta'_{v,\chi}$ are tamely ramified if and only if they are non-trivial and unramified otherwise (cf. Remark 3.1.8). □

§ 4.2 An Upper Bound for the Conductor Exponent

In order to prove an upper bound, we have to give upper bounds for the local summands $f_\chi(v)$ in Theorem 4.1.4. Let $v \in V_{\bar{Z}}$ and χ be an irreducible and non-trivial character of G. We recall the situation and the definition of the characters $\eta_{e,\chi}$ (cf. § 3.1). We have a cyclic Galois cover $\bar{Y}_v \to \bar{X}_v$ with Galois group G of order n and a cover $\bar{X}_v \to \bar{Z}_v$ on which Γ_v acts. Define $\tilde{N}_v = \ker(\tilde{\Gamma}_v \to \mathrm{Aut}(\bar{Y}_v))$ and $N_v = G\tilde{N}_v \cap \Gamma_v$. By Lemma 3.1.4 the quotient Γ_v/N_v is isomorphic to the semidirect product of a p-group T and a cyclic group C of order not divided by p. We denote by T' and C' the preimage of T and C under $\Gamma_v \to \Gamma_v/N_v$. Then there exists a point $z_0 \in \bar{Z}_v$ such that a point $x_0 \in \bar{X}_v$ above z_0 has stabilizer $\Gamma_{z_0} = C'$ and all other points of $\bar{Z}_v$ have stabilizer N_v by Lemma 3.1.5. For a subgroup $U \subset \tilde{\Gamma}_v$ with $U \cap G \subset \ker(\chi)$ we have defined in Lemma 3.1.2 the character $\eta_{U,\chi}$ which we denote by $\eta_{z,\chi}$ if U is the stabilizer $\tilde{\Gamma}_z$ of a point of $\bar{Y}_v$ above $z \in \bar{Z}_v$. The map $\eta_{U,\chi}$ is the composition of a section $GU \cap \Gamma_v \to \tilde{\Gamma}_v$ of the projection $\tilde{\Gamma}_v \to \Gamma_v$, the projection $\tilde{\Gamma}_v \to G$ and the inverse of χ. We introduce some terminology. Setting $\bar{W}_v = \bar{Y}_v/\Gamma_v$, we obtain a commutative diagram



where $\bar{Y}_{v,\chi} = \bar{Y}_v/\ker(\chi)$ and $\bar{W}_{v,\chi} = \bar{W}_v/\ker(\chi)$. For a point $z \in \bar{Z}_v$, $\chi|_{\Gamma\tilde{\Gamma}_z \cap G} = \mathbf{1}$ is equivalent to z is unramified in the cover $\bar{W}_{v,\chi} \to \bar{Z}_v$, and $\chi|_{\tilde{\Gamma}_z \cap G} = \mathbf{1}$ is equivalent to points of $\bar{X}_v$ above z are unramified in $\bar{Y}_{v,\chi} \to \bar{X}_v$.

Definition 4.2.1 Let $v \in V_{\bar{Z}}$, $z \in \bar{Z}_v$ and χ an irreducible character of G. The point z is called χ-unramified in $\bar{Y}_v \to \bar{X}_v$ if $\chi|_{\tilde{\Gamma}_z \cap G} = \mathbf{1}$. If $\chi|_{\Gamma\tilde{\Gamma}_z \cap G} = \mathbf{1}$, we call z χ-unramified in $\bar{W}_v \to \bar{Z}_v$.

The character $\eta_{v,\chi}$ is defined as an extension $\eta_{z_0,\chi}$ to Γ_v if z_0 is χ-unramified in $\bar{Y}_v \to \bar{X}_v$, and otherwise as an extension of $\eta_{\tilde{N}_v,\chi}$ to Γ_v. If $z \in \bar{Z}_v$ is χ-ramified, we define $\eta_{z,\chi}$ to be the restriction of $\eta_{v,\chi}$ to the stabilizer of z. For each Γ_v-orbit of branch points of the cover $\bar{Y}_v \to \bar{X}_v$ there is an edge $e \in E(v)$ and $\eta_{e,\chi}$ denotes the character $\eta_{z_e,\chi}$ where z_e is the point corresponding to e. We will say that $e \in E(v)$ is χ-unramified in $\bar{Y}_v \to \bar{X}_v$ (respectively in $\bar{W}_v \to \bar{Z}_v$) if points above the point z_e (respectively the point z_e) are χ-unramified in $\bar{Y}_v \to \bar{X}_v$ (respectively in $\bar{W}_v \to \bar{Z}_v$).

Remark 4.2.2 (1) The character $\eta'_{v,\chi}$ appearing in the definition of $f_\chi(v)$ is equal to $\eta_{\xi,\chi}$ for ξ the generic point of $\bar{Z}_v$ if ξ is χ-unramified. Indeed, under the surjective morphism $\bar{Y}_v \to \bar{Z}_v$ generic points are mapped to generic points, and the subgroup $\tilde{\Gamma}'_v$ is the stabilizer of an irreducible component of $\bar{Y}_v$ under the $\tilde{\Gamma}_v$-action and is thus the stabilizer of the generic point corresponding to this irreducible component.

(2) In case that z_0 is χ-ramified, it is also possible and amounts to the same if one first defines the character η_{z_0} as an extension of $\eta_{\tilde{N},\chi}$ to C' and then defines $\eta_{v,\chi}$ as an extension of $\eta_{z_0,\chi}$ to Γ_v. Note that the characters $\eta_{v,\chi}$ and $\eta_{e,\chi}$ are determined by $\eta_{z_0,\chi}$. Since the character $\eta_{v,\chi}$ is trivial on $\Gamma_v \cap \Gamma_1$ (cf. Remark 3.1.8) and since $\eta_{z_0,\chi}$ is defined on C', the extension $\eta_{v,\chi}$ is unique. The characters $\eta_{e,\chi}$ are the restrictions of $\eta_{z_0,\chi}$ to Γ_e. Indeed, if e is χ-ramified, this holds by definition, and if e is χ-unramified and does not correspond to z_0, we have $\Gamma_e = N_v$ and since for a χ-unramified e the character $\eta_{e,\chi}$ is independent of the chosen section $\Gamma_e \to \tilde{\Gamma}_e$, we have $\eta_{e,\chi} = \eta_{\tilde{N}_v,\chi}$ which is the restriction of $\eta_{z_0,\chi}$ to N_v.

By Remark 3.1.8 all the characters discussed above are tamely ramified or unramified. This implies that their conductor exponents are 0 or 1, and it is 1 if and only if the character is non-trivial, since Γ is an inertia group as the Galois group of a totally ramified extension. Hence, determining or bounding $f_\chi(v)$ amounts to determining whether each of the characters $\eta_{v,\chi}$ and $\eta_{e,\chi}$ for $e \in E(v)$ are trivial or not. Note that for a point $z \in \bar{Z}_v$ the character $\eta_{z,\chi}$ is trivial if and only if z is χ-unramified in $\bar{W}_v \to \bar{Z}_v$ since $\Gamma\tilde{\Gamma}_z \cap G$ is the stabilizer under the G-action of a point of $\bar{W}_v$ above z, and that $\eta_{v,\chi} = \mathbf{1}$ is equivalent to $\chi|_{\Gamma\tilde{N}_v \cap G} = \mathbf{1}$.

Abusing notation, for a subset $E \subset E(v)$ we write $z_0 \in E$ if there exists an edge in E that corresponds to the point z_0, and for a leaf $a \in B$ we identify a with the unique edge incident to a and pointing towards a.

Lemma 4.2.3 *(a) Let χ be non-trivial on $\Gamma\tilde{N}_v \cap G$. Then*

$$f_\chi(v) = |\{a \in E(v) \cap B : \chi|_{\tilde{\Gamma}_a \cap G} \neq \mathbf{1}\}| + |E(v) \setminus B| - 1.$$

(b) Let χ be trivial on $\Gamma\tilde{N}_v \cap G$ and $E_1(v) = E(v) \setminus \{z_0\}$. Then

$$f_\chi(v) = |\{e \in E_1(v) \setminus B : \chi|_{\tilde{\Gamma}_e \cap G} = \mathbf{1}\}| + j_0 - j'$$

with

$$j_0 = \begin{cases} 1 & \chi|_{\Gamma\tilde{\Gamma}_{z_0}\cap G} = \mathbf{1},\ z_0 \in E(v) \smallsetminus B \\ -1 & \chi|_{\Gamma\tilde{\Gamma}_{z_0}\cap G} \neq \mathbf{1},\ z_0 \notin E(v) \\ 0 & \textit{otherwise} \end{cases}$$

and

$$j' = \begin{cases} 0 & \chi|_{\Gamma\tilde{\Gamma}'_v\cap G} \neq \mathbf{1} \\ 1 & \chi|_{\Gamma\tilde{\Gamma}'_v\cap G} = \mathbf{1} \end{cases}.$$

Proof First we consider the case $\Gamma\tilde{N}_v \cap G \not\subset \ker(\chi)$. Clearly, $\tilde{N}_v$ is a subset of $\tilde{\Gamma}_z$ for every $z \in \bar{Z}_v$. This implies $\eta_{e,\chi} \neq \mathbf{1}$ for all $e \in E(v)$ which yields part (a) of the lemma by the definition of $f_\chi(v)$.

Now let χ be trivial on $\Gamma\tilde{N}_v \cap G$. Then Remark 4.2.2 (2) shows that $\eta_{e,\chi} = \eta_{\tilde{N}_v,\chi} = \mathbf{1}$ for all $e \in E(v)$ not corresponding to the point z_0. It follows that

$$\sum_{\substack{a\in B\cap E_1(v)\\ \chi|_{\tilde{\Gamma}_a\cap G}\neq\mathbf{1}}} f(\eta_{a,\chi}) + \sum_{e\in E_1'(v)} f(\eta_{e,\chi}) + \sum_{\substack{e\in E_1'(v)\\ \chi|_{\tilde{\Gamma}_e\cap G}=\mathbf{1}}} (1 - f(\eta_{e,\chi})) = |\{e \in E_1(v) \smallsetminus B : \chi|_{\tilde{\Gamma}_e\cap G} = \mathbf{1}\}| \quad (4.1)$$

with $E_1'(v) = E'(v) \smallsetminus \{z_0\}$. We first consider the case that $z_0 \in E(v)$. Remark 4.2.2 (2) shows $f(\eta_{z_0,\chi}) = f(\eta_{v,\chi})$. Hence, by (4.1) and the definition of $f_\chi(v)$, we have

$$f_\chi(v) = |\{e \in E_1(v) \smallsetminus B : \chi|_{\tilde{\Gamma}_e\cap G} = \mathbf{1}\}| + j_0 - j'$$

with j' as in the statement and $j_0 \in \{0, 1\}$, and $j_0 = 1$ if and only if z_0 is χ-unramified in $\bar{W}_v \to \bar{Z}_v$ and $z_0 \notin B$. This proves (b) if $z_0 \in E(v)$. Now assume $z_0 \notin E(v)$. In this case we have

$$f_\chi(v) = \sum_{\substack{a\in B\cap E_1(v)\\ \chi|_{\tilde{\Gamma}_a\cap G}\neq\mathbf{1}}} f(\eta_{a,\chi}) + \sum_{e\in E_1'(v)} f(\eta_{e,\chi}) + \sum_{\substack{e\in E_1'(v)\\ \chi|_{\tilde{\Gamma}_e\cap G}=\mathbf{1}}} (1 - f(\eta_{e,\chi})) - f(\eta_{v,\chi}) - j'.$$

Setting $j_0 = -f(\eta_{v,\chi}) = -f(\eta_{z_0,\chi}) \in \{0, -1\}$, part (b) follows by (4.1), since $f(\eta_{z_0,\chi}) = 1$ if and only if z_0 is χ-ramified in $\bar{W}_v \to \bar{Z}_v$. □

We use Lemma 4.2.3 to get an upper bound for the local terms in the decomposition of the conductor exponent.

Lemma 4.2.4 *Let $\chi \neq \mathbf{1}$ be an irreducible character of G and $v \in V_{\bar{Z}}$. Then*

$$f_\chi(v) \leqslant |E(v)| - 1.$$

Proof If χ is trivial on $\Gamma\tilde{N}_v \cap G$, the lemma follows immediately from Lemma 4.2.3 (a). Assume that χ is non-trivial on $\Gamma\tilde{N}_v \cap G$. By part (b) of Lemma 4.2.3 the lemma holds if $B \cap E(v) \neq \emptyset$ or if there exists an in $\bar{Y}_v \to \bar{X}_v$ χ-ramified point corresponding to an edge $e \in E(v)$ or if z_0 is χ-ramified in $\bar{W}_v \to \bar{Z}_v$. Thus, we can assume that non of these hold. This assumption implies that all points of $\bar{Z}_v$ are χ-unramified in $\bar{W}_v \to \bar{Z}_v$ making $\bar{W}_{v,\chi} \to \bar{Z}_v$ into an unramified cover. Therefore, $\Gamma\tilde{\Gamma}'_v \cap G \subset \ker(\chi)$ and the lemma follows by 4.2.3 (b) since the there defined j' equals 1. □

Lemma 4.2.5 *Let $\chi \neq \mathbf{1}$ be an irreducible character of G and $\mathsf{v} \in V_{\bar{Z}}$. Let e' be the edge incident to $\mathsf{v} \neq \mathsf{r}$ and pointing towards v or, if $\mathsf{v} = \mathsf{r}$, let e' be the point of $\bar{Z}$ below x_r.*

(a) Let $\chi|_{\Gamma\tilde{N}_{\mathsf{v}} \cap G} = \mathbf{1}$ for $\tilde{N}_{\mathsf{v}} = \ker(\tilde{\Gamma}_{\mathsf{v}} \to \mathrm{Aut}(\bar{Y}_{\mathsf{v}}))$. Then $f_\chi(\mathsf{v}) \leqslant |E(\mathsf{v}) \smallsetminus B|$.

(b) Let $\chi|_{\Gamma\tilde{\Gamma}_{e'} \cap G} = \mathbf{1}$. Then

$$f_\chi(\mathsf{v}) \leqslant \min\{|E(\mathsf{v})| - 2, |E(\mathsf{v}) \smallsetminus B|\}$$

or $\chi|_{\Gamma\tilde{\Gamma}_e \cap G} = \mathbf{1}$ for all $e \in E(\mathsf{v})$.

Proof Part (a) is a direct consequence of Lemma 4.2.3 (a). Let e' be χ-unramified in $\bar{W}_{\mathsf{v}} \to \bar{Z}_{\mathsf{v}}$. Since $\Gamma\tilde{N}_{\mathsf{v}} \cap G \subset \Gamma\tilde{\Gamma}_{e'} \cap G$, the inequality $f_\chi(\mathsf{v}) \leqslant |E(\mathsf{v}) \smallsetminus B|$ follows by Lemma 4.2.3 (b). We show that $f_\chi(\mathsf{v}) \leqslant |E(\mathsf{v})| - 2$ or that every edge $e \in E(\mathsf{v})$ is χ-unramified in $\bar{W}_{\mathsf{v}} \to \bar{Z}_{\mathsf{v}}$. We can assume that there exists at most one edge $e \in E(\mathsf{v})$ such that e is χ-ramified in $\bar{Y}_{\mathsf{v}} \to \bar{X}_{\mathsf{v}}$, for otherwise the inequality $f_\chi(\mathsf{v}) \leqslant |E(\mathsf{v})| - 2$ follows by Lemma 4.2.3 (b). The cover $\bar{W}_{\mathsf{v},\chi} \to \bar{Z}_{\mathsf{v}}$ has at most two branch points. Indeed, the possible branch points of $\bar{Y}_{\mathsf{v}} \to \bar{Z}_{\mathsf{v}}$ are the points below branch points of $\bar{Y}_{\mathsf{v}} \to \bar{X}_{\mathsf{v}}$ and z_0 (cf. Lemma 3.1.5), and therefore $\bar{W}_{\mathsf{v},\chi} \to \bar{Z}_{\mathsf{v}}$ can only have the points z_e and z_0 as branch points by the above assumption. This implies that, as a cyclic cover, $\bar{W}_{\mathsf{v},\chi} \to \bar{Z}_{\mathsf{v}}$ is either unramified, in which case all edges in $E(\mathsf{v})$ are χ-unramified in $\bar{W}_{\mathsf{v}} \to \bar{Z}_{\mathsf{v}}$, or has exactly two branch points, in which case e and z_0 are χ-ramified in $\bar{W}_{\mathsf{v}} \to \bar{Z}_{\mathsf{v}}$. In the latter case, points over z_e must be branched in $\bar{Y}_{\mathsf{v},\chi} \to \bar{X}_{\mathsf{v}}$, since z_0 is the unique branch point of $\bar{W}_{\mathsf{v},\chi} \to \bar{Z}_{\mathsf{v}}$ below points not ramified in $\bar{Y}_{\mathsf{v},\chi} \to \bar{X}_{\mathsf{v}}$. Thus, χ is non-trivial on $\tilde{\Gamma}_e \cap G$, trivial on $\tilde{\Gamma}_{z_0} \cap G$, and non-trivial on $\Gamma\tilde{\Gamma}_{z_0} \cap G$ and $\Gamma\tilde{\Gamma}_e \cap G$. This implies

$$f_\chi(\mathsf{v}) \leqslant |E(\mathsf{v})| - 2$$

by Lemma 4.2.3 (b). □

Using Lemmata 4.2.4 and 4.2.5 to bound the local terms, and summing over the tree $T_{\bar{Z}}$ we obtain an upper bound for the conductor exponent.

Theorem 4.2.6 *Let r be the number of Γ-orbits of D_L. We have a decomposition*

$$f(Y/K) = \sum_{\chi \neq \mathbf{1}} f_\chi(Y/K)$$

with

$$f_\chi(Y/K) \leqslant r - 2 + \sum_{a \in B_\chi} \delta_a$$

for all $\chi \neq \mathbf{1}$.

Proof By Theorem 4.1.4 we have the decomposition

$$f(Y/K) = \sum_{\chi \neq \mathbf{1}} f_\chi(Y/K)$$

where for irreducible characters $\chi \neq \mathbf{1}$ of G,

$$f_\chi(Y/K) = \sum_{v \in V_{\bar{Z}}} f_\chi(v) - f_\chi(\infty) + \sum_{a \in B_\chi} \delta_a.$$

Let $\chi \neq \mathbf{1}$ be an irreducible character of G. First assume that the point x_r which defines the orientation on $\bar{X}$ is a branch point of $\bar{Y} \to \bar{X}$. Then $|B| = r - 1$ (cf. the definition of B in Notation 3.2.3) and by Lemma 4.2.4

$$f_\chi(Y/K) \leqslant \sum_{v \in V_{\bar{Z}}} (|E(v)| - 1) + \sum_{a \in B_\chi} \delta_a = |B| - 1 + \sum_{a \in B_\chi} \delta_a = r - 2 + \sum_{a \in B_\chi} \delta_a.$$

Now assume that x_r is not a branch point of $\bar{Y} \to \bar{X}$. Then $|B| = r$. If $f_\chi(\infty) = 1$, the theorem follows as above if we use in the first inequality that $f_\chi(\infty) = 1$. Thus, we can assume that $f_\chi(\infty) = 0$. Then $\eta_{\infty,\chi} = \mathbf{1}$ and we can use Lemma 4.2.5 (b) with $v = r$. Using Lemma 4.2.5 (b) with induction over the tree $\Delta_{\bar{Z}}$ shows the existence of a vertex $v_0 \in V_{\bar{Z}}$ with $f_\chi(v_0) \leqslant |E(v_0)| - 2$. Combining this with the inequality of Lemma 4.2.4 for all vertices of $\Delta_{\bar{Z}}$ different from v_0 implies the theorem. □

As a direct corollary to Theorem 4.2.6 we have

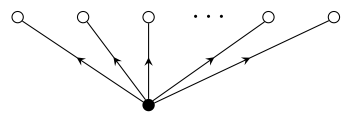
Theorem 4.2.7 *Let r be the number of Γ-orbits of D_L. Then*

$$f(Y/K) \leqslant (n-1)\left(r - 2 + \sum_{a \in B} \delta_a\right).$$

Remark 4.2.8 It should be possible to give a lower bound with the same methods. We plan to work this out in a later publication.

We discuss the upper bound of Theorem 4.2.7 in a few examples. In particular, we will see that the inequality is sharp.

Example 4.2.9 (1) Let $K/\mathbb{Q}_p^{\mathrm{nr}}$ be a finite extension, $f \in K[x]$ be a monic separable polynomial of degree d with $v_K(\mathrm{disc}(f)) = 0$ and roots in $\mathcal{O}_L$ where L is the splitting field of f. Let $c \in K^*$, $n \geqslant 2$, $p \nmid n$ and consider the curve Y over K given by $y^n = cf$. We assume Y to be absolutely irreducible and each root of Y to be branched in $Y \to X$. Since $v_K(\mathrm{disc}(f)) = 0$, the set of roots of f is a subset of $\mathcal{O}_L$ and $\mathbb{F}_K$ is algebraically closed, the reduction of f in $\mathbb{F}_K[x]$ splits into linear factors. By Hensel's Lemma all roots of f are elements of $\mathcal{O}_K$. The tree $T_{\bar{Z}}$ consists of one vertex r and a vertex adjacent to r for each root of f.



Theorem 4.2.7 yields

$$f(Y/K) \leqslant \begin{cases} (n-1)(d-1) & n \nmid d \\ (n-1)(d-2) & n \mid d \end{cases}.$$

We will see that this bound is attained if $n \nmid v_K(c)$, but also $f(Y/K) = 0$ if $n \mid v_K(c)$. Since all roots of f are in $\mathcal{O}_K$, we have $\delta_a = 0$ for all $a \in B$ and for the conductor exponent of Y/K,

$$f(Y/K) = \sum_{\chi \neq \mathbf{1}} (f_\chi(r) - f_\chi(\infty)).$$

Taking $y_r = y/c^{1/n}$ we obtain a smooth model over $L = K(c^{1/n})$ with special fiber $\bar{y}_r = \bar{f}$. The action on the special fiber is given by $\sigma(\bar{x}, \bar{y}_r) = (\bar{x}, \zeta\bar{y}_r)$ where ζ is a primitive $n/\gcd(n, v_K(c))$-th root of unity and σ is a generator of the cyclic Galois group of L/K. Thus, the group $\Gamma\tilde{N}_r \cap G$ has order $n/\gcd(n, v_K(c))$ and therefore χ is trivial on $\Gamma\tilde{N}_r \cap G$ if and only if $\mathrm{ord}(\chi) \mid v_K(c)$. This implies together with Lemma 4.2.3 that $\mathrm{ord}(\chi) \mid v_K(c)$ if and only if $f_\chi(r) = 0$ and $f_\chi(\infty) = 0$. In the case $\mathrm{ord}(\chi) \nmid v_K(c)$,

$$f_\chi(\infty) = \begin{cases} 0 & \mathrm{ord}(\chi) \nmid d \\ 1 & \mathrm{ord}(\chi) \mid d \end{cases}$$

and

$$f_\chi(r) = \left| \left\{ a \in B : \mathrm{ord}(\chi) \nmid \tfrac{n}{e_a} \right\} \right| - 1$$

where e_a is the ramification index in $Y \to X$ of the root of f corresponding to $a \in B$. Taking for n a prime number,

$$f(Y/K) = \begin{cases} 0 & n \mid v_K(c) \\ (n-1)(d-1) & n \nmid v_K(c), n \nmid d \\ (n-1)(d-2) & n \nmid v_K(c), n \mid d \end{cases}.$$

(2) Let Y be the curve over a finite extension $K/\mathbb{Q}_p^{\mathrm{nr}}$ given by the equation $y^n = f$ with $f \in K[x]$ irreducible and $p \nmid n$. The tree $T_{\bar{Z}}$ for this curve is depicted below.

By Theorem 4.2.7

$$f(Y/K) \leqslant (n-1)\delta_a$$

if the point ∞ is ramified, and

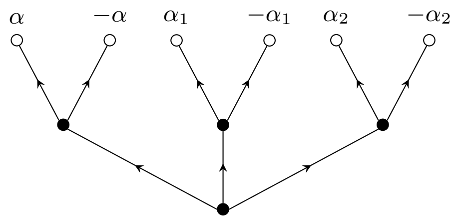
$$f(Y/K) \leqslant (n-1)(-1+\delta_a)$$

otherwise where δ_a is the valuation of the different of the extension $\bar{K}^{\Gamma_a}/K$ and Γ_a the stabilizer of a root of f under the Γ_K-action.

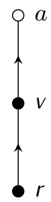
Consider more concretely the polynomial $f = x^6 - 2$ over $K = \mathbb{Q}_2^{\mathrm{nr}}$. Let $\alpha = 2^{1/6}$, $\alpha_1 = \zeta_6 2^{1/6}$ and $\alpha_2 = \zeta_6^2 2^{1/6}$ denote roots of f in $\bar{K}$. Then the roots of f are exactly $\pm\alpha$, $\pm\alpha_1$, $\pm\alpha_2$. To obtain a quasistable model, we split the roots of f. We can take

$$x_0 = \frac{x-\alpha}{\alpha_1 - \alpha}, \; x_v = \frac{x-\alpha}{-2\alpha} \quad \text{and} \quad x_i = \frac{x-\alpha_i}{-2\alpha_i}.$$

The trees $T_{\bar{X}}$ and $T_{\bar{Z}}$ are depicted below. Here $T_{\bar{X}}$ is the tree $\Delta_{\bar{X}}$ with a vertex added for each root of f adjacent to the vertex corresponding to the component to which the root specializes.



$T_{\bar{X}}$



$T_{\bar{Z}}$

The splitting field of f is a cyclic extension of degree 6. Writing f in the variable x_0 respectively x_v we see that taking $y_0 = y/2^{1/n}$ respectively $y_v = y/2^{3/n}$ gives a semistable model. Taking $n = 3$, we have $f_\chi(\infty) = 1$, since the point ∞ on $\bar{X}_r$ is unramified and points above ∞ are permuted by Γ_r. It holds that $\Gamma\tilde{N}_r \cap G = \{1\}$ since only the wild inertia subgroup of Γ acts trivially on $\bar{X}_r$. The edges of $T_{\bar{X}}$ incident to the root vertex are permuted by Γ_r. Thus, the unique point z_0 below the point of $\bar{X}_r$ fixed by the tame subgroup C_r' of Γ_r does not correspond to the edge (r, v). We obtain $\Gamma\Gamma_{z_0} \cap G = G$ and thus $f_\chi(r) = -1$ by Theorem 4.2.3 since we have a non-trivial action on $y_0 = y/2^{1/n}$ by elements of C_r'. The stabilizer Γ_v is the order 2 wild inertia subgroup of Γ. Thus, $\Gamma\tilde{N}_v \cap G = \{1\}$ and $f_\chi(v) = 0$ for all non-trivial irreducible χ by Lemma 4.2.3. Hence, by Theorem 4.1.4

$$f(Y/K) = 2(-2 + \delta_a).$$

In this example, we have $\delta_a = v_{K(\alpha)}(\mathfrak{D}_{K(\alpha)/K}) = v_{K(\alpha)}(f'(\alpha)) = v_{K(\alpha)}(6\alpha^5) = 11$, since f is an Eisenstein polynomial. If $3 \nmid n$, we obtain $f_\chi(\infty) = f_\chi(r) = f_\chi(v) = 0$ for all $\chi \neq \mathbf{1}$ by Lemma 4.2.3, since $\Gamma\tilde{N}_r \cap G = \Gamma\tilde{N}_v \cap G = G$, and thus

$$f(Y/K) = (n-1)\delta_a = 11(n-1).$$

Since the residue characteristic is 2, in the case $3 \nmid n$ the point ∞ is ramified and the bound provided by Theorem 4.2.7 is attained.

Chapter 5

Discriminants as Upper Bounds

In this chapter we apply the inequality we proved in the preceding chapter. First, in § 5.1 we assume that a superelliptic curve is given birationally by $y^n = f$ for a polynomial f. We compare the conductor exponent of this curve with the valuation of the discriminant of f by choosing an appropriate orientation on the tree of projective lines and writing the discriminant of f as a sum of local terms over this tree. We then compare this local terms with the local terms in the decomposition of the conductor exponent obtained in § 4.1. It turns out that the conductor exponent is bounded by $n-1$ times the valuation of $\mathrm{disc}(f)$ for monic polynomials f whose roots are algebraic integers. Here, $\mathrm{disc}(f)$ is the discriminant of the radical of f. In § 5.2 we introduce Picard curves. As plane quartic curves they have a notion of discriminant for an defining equation. Every Picard curve admits a superelliptic equation such that the discriminant of this equation is minimal among all defining equations. That makes the inequality of the previous chapter applicable to the question whether the conductor exponent of a Picard curve is bounded by the valuation of its minimal discriminant. We prove that this inequality holds if the residue characteristic of the field over which the Picard curve is defined is different from 2 and 3 (cf. [BKSW19, § 5.2]).

§ 5.1 Discriminants of Polynomials and Conductor Exponents of Superelliptic Curves

Let $n \geqslant 2$, p a prime with $p \nmid n$, $K/\mathbb{Q}_p^{\mathrm{nr}}$ a finite extension and $f \in K[x]$ a polynomial. Let Y be the curve over K given by the equation $y^n = f$. Since K contains all n-th roots of unity, this defines a Galois cover of curves $\phi : Y \to X = \mathbb{P}^1_K$ with Galois group $G \simeq \mathbb{Z}/n\mathbb{Z}$. Let L_0 be the splitting field of f over K and $R \subset L_0$ the set of roots of f. Then we have over L_0

$$f = c \prod_{\alpha \in R} (x - \alpha)^{m_\alpha}$$

with $c \in K^*$ and $m_\alpha \geqslant 0$ for all $\alpha \in R$. We can assume $0 < m_\alpha < n$ for all $\alpha \in R$ and $\gcd(n, m_\alpha : \alpha \in R) = 1$, i.e. all roots of f are branch points of ϕ and Y is absolutely irreducible. Let D be the branch divisor of ϕ. We have

$$D_L = \begin{cases} R & n \mid \deg(f) \\ R \cup \{\infty\} & \text{otherwise} \end{cases}$$

for every extension L/L_0. We assume that the genus of Y is at least 2. This implies $|D_L| \geqslant 3$. We also add the point $\infty \in X$ to the divisor D if $n \mid \deg(f)$. By Theorem 2.2.7 the marked curve (X_L, D_L) has a unique minimal semistable $\mathcal{O}_L$-model $(\mathcal{X}, \mathcal{D})$, the special fiber $(\bar{X}, \bar{D})$ of $(\mathcal{X}, \mathcal{D})$ is a marked tree of projective lines. By Theorem 2.2.9 there exists a tame extension L/L_0 such that the normalization $\mathcal{Y}$ of $\mathcal{X}$ in Y_L is quasistable and L/K is a Galois extension. Let $\Gamma = \mathrm{Gal}(L/K)$ and define $\bar{Z} = \bar{X}/\Gamma$. Assumption 3.2.1 is satisfied for the cover $\bar{Y} \to \bar{X}$. Throughout the chapter we use Notations 3.2.3 and 3.2.4 adapted to the cover $\bar{Y} \to \bar{X}$ and Notation 4.1.3.

We added the point $\infty \in X$ to D to ensure that the specialization $\overline{\infty}$ of ∞ is a smooth point of $\bar{X}$. Since ∞ is a K-rational point, it is fixed by Γ and can therefore be chosen as the point x_r on $\bar{X}$ defining the orientation of $\Delta_{\bar{X}}$ and $T_{\bar{Z}}$. In the following we justify this choice by introducing a metric d on $\Delta_{\bar{Z}}$ such that $d_v < d_w$ for every edge $(v, w) \in E_{\bar{Z}}$.

Let $D' \subset D_L^3$ be the set of triples of pairwise distinct elements of D_L. For such a triple $t = (\alpha, \beta, \gamma) \in D'$ let $\lambda_t : X_L \to \mathbb{P}^1_L$ be a L-linear isomorphism with $\lambda_t(\alpha) = 0$, $\lambda_t(\beta) = 1$, $\lambda_t(\gamma) = \infty$. Explicitly,

$$\lambda_t(x) = \frac{\beta - \gamma}{\beta - \alpha} \frac{x - \alpha}{x - \gamma}. \tag{5.1}$$

Write $t \sim t'$ if $\lambda_t \circ \lambda_{t'}^{-1}$ extends to an automorphism of $\mathbb{P}^1_{\mathcal{O}_L}$. This defines an equivalence relation on D'. We state Proposition 4.2 of [BW17].

Theorem 5.1.1 *(a) For $t \in D'$ the morphism λ_t extends to a proper $\mathcal{O}_L$-morphism $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_L}$. The reduction to the special fibre is a contraction morphism*

$$\bar{\lambda}_t : \bar{X} \to \mathbb{P}^1_{\mathbb{F}_L}$$

contracting all but one component to closed points.

(b) There is a bijection $V(\Delta_{\bar{X}}) \to D'/\sim$ under which a vertex v is mapped to the class of a triple t such that $\bar{\lambda}_t$ does not contract $\bar{X}_v$.

(c) For $\alpha \in D_L$ let $\bar{\alpha} \in \bar{X}$ denote its specialization and let $\bar{\cdot} : \mathbb{P}^1_L \to \mathbb{P}^1_{\mathbb{F}_L}$ be the specialization map. Then $\bar{\lambda}_t(\bar{\alpha}) = \overline{\lambda_t(\alpha)}$ for $t \in D'$ and $\alpha \in D_L$.

Notation 5.1.2
- Let $T_{\bar{X}}$ be the tree $\Delta_{\bar{X}}$ where we add for every root α of f a vertex that is adjacent to a vertex v if and only if $\bar{\alpha} \in \bar{X}_v$. Then $T_{\bar{X}}$ is again a directed tree with every edge pointing away from the root.
- For $e = (v, w) \in E(\Delta_{\bar{X}})$ we define the directed tree $T_{\bar{X}}(e)$ as the connected component of the graph $(V(T_{\bar{X}}), E(T_{\bar{X}}) \setminus \{e\})$ which does not contain the root vertex.

Take $v \in V(\Delta_{\bar{X}})$ and two different edges $e_1, e_2 \in E(T_{\bar{X}})$ and let $e' \in E(T_{\bar{X}})$ be the edge incident to v and pointing towards v. Let $t = (\alpha, \beta, \gamma)$ where α respectively $\beta \in R$ is a root of f corresponding to a leaf of $T_{\bar{X}}(e_1)$ respectively $T_{\bar{X}}(e_2)$ and $\gamma \in R$ corresponds to a leaf of $T_{\bar{X}}$ not in $T_{\bar{X}}(e')$. Then $\bar{\lambda}_t$ does not contract $\bar{X}_v$. The reason for that is that $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are mapped under $\bar{\lambda}_t$ to different points, since $\bar{\lambda}(\bar{\alpha}) = 0, \bar{\lambda}(\bar{\beta}) = 1$ and $\bar{\lambda}(\bar{\gamma}) = \infty$ by Theorem 5.1.1 (c) and (5.1).

Lemma 5.1.3 *Let $e = (v, w), e' = (v, w') \in E(\Delta_{\bar{X}})$ be different edges and let $\alpha, \alpha' \in R$ (respectively $\beta, \beta' \in R$) be roots of f that correspond to leafs of $T_{\bar{X}}(e)$ (respectively $T_{\bar{X}}(e')$). The following statements are true.*

(a) We have $v_K(\alpha - \beta) < v_K(\alpha' - \alpha)$.

(b) We have $v_K(\alpha - \beta) = v_K(\alpha' - \beta')$.

Proof We can assume that $\alpha \neq \alpha'$. First, we find a vertex $v' \in V(\Delta_{\bar{X}})$ such that there are different edges $e_1 = (v', w_1), e_2 = (v', w_2) \in E(\Delta_{\bar{X}})$ with the property that $T_{\bar{X}}(e_1)$ (respectively $T_{\bar{X}}(e_2)$) contains the vertex corresponding to α (respectively α'). Either w has this property or there is an edge $e'' = (w, w'')$ such that α and α' correspond to vertices in $T_{\bar{X}}(e'')$. By iterating this process we find an edge v' with the desired property. Let $t = (\alpha', \alpha, \beta) \in D'$. Since by the above there exists an edge $e_0 = (v_0, v')$ such that $T_{\bar{X}}(e_0)$ does not contain the vertex corresponding to β, $\bar{\lambda}_t$ does not contract $\bar{X}_{v'}$ and $\bar{\lambda}_t(\overline{\infty}) = \bar{\lambda}_t(\bar{\beta}) = \infty$ by Theorem 5.1.1 (c). Together with (5.1) this implies (a). Part (b) follows from (a) and the non-Archimedean triangle inequality. □

We define a map $d' : V(\Delta_{\bar{X}}) \to \mathbb{Q}$. For $v \in V(\Delta_{\bar{X}})$ choose two different edges $e = (v, w), e' = (v, w')$ and roots $\alpha, \beta \in R$ corresponding to vertices of $T_{\bar{X}}(e)$ and $T_{\bar{X}}(e')$. Then define $d'_v = v_K(\alpha - \beta)$. This definition is independent of the chosen roots α, α' and the chosen edges $(v, w), (v, w')$ by part (b) of Lemma 5.1.3. Clearly, $d'_{v'} = d'_{v''}$ for $v', v'' \in V(\Delta_{\bar{X}})$ above a vertex $v \in V_{\bar{Z}}$. Thus, d' induces a map $d : V_{\bar{Z}} \to \mathbb{Q}$. By part ($a$) of Lemma 5.1.3, $d_w > d_v$ for every edge $(v, w) \in E_{\bar{Z}}$.

With the metric d we are able to express the discriminant of the polynomial f as a sum over local terms disc(v) for $v \in V_{\bar{Z}}$. We first clarify what we understand under the term discriminant in our context and introduce some notations.

Definition 5.1.4 Let $f \in K[x]$ be a polynomial and let $R \subset L$ be the set of roots of f in a splitting field L of f. We call

$$\operatorname{disc}(f) = \prod_{\alpha \in R} \prod_{\substack{\beta \in R \\ \alpha \neq \beta}} (\alpha - \beta)$$

the discriminant of f.

Remark 5.1.5 Usually the discriminant of a polynomial f of degree d and with (not necessarily distinct) roots $\alpha_1, \ldots, \alpha_d$ and leading coefficient c is defined by

$$\Delta(f) = c^{2d-2}(-1)^{\frac{d(d-1)}{2}} \prod_{i \neq j} (\alpha_i - \alpha_j).$$

The relation with disc(f) is

$$(-1)^{\frac{d(d-1)}{2}} c^{2d-2} \operatorname{disc}(f) = \Delta(\operatorname{rad}(f))$$

where rad(f) is the radical of f, more precisely $\operatorname{rad}(f) = f/\gcd(f, f')$ where $\gcd(f, f')$ is the monic greatest common divisor of f and f' is the derivative of f.

Notation 5.1.6
- For $e = (\mathsf{v}, \mathsf{w}) \in E_{\bar{Z}}$ we define the directed tree $T(e)$ as the connected component of the graph $(V(T_{\bar{Z}}), E(T_{\bar{Z}}) \smallsetminus \{e\})$ which does not contain the root vertex r.
- Let B_v be the set of leafs of the tree $T(e)$ for the edge $e \in E_{\bar{Z}}$ incident and pointing towards the vertex $\mathsf{v} \in V_{\bar{Z}}$.
- For $\mathsf{v} \in V_{\bar{Z}}$ and $a \in B_\mathsf{v}$ let $e_\mathsf{v}(a) \in E(\mathsf{v})$ be the edge with $a \in V(T(e_\mathsf{v}(a)))$.
- Let f_a denote the irreducible factor of f over K corresponding to $a \in B$.

Lemma 5.1.7 *We have*

$$v_K(\operatorname{disc}(f)) = \sum_{\mathsf{v} \in V_{\bar{Z}}} \operatorname{disc}(\mathsf{v}) + \sum_{a \in B} v_K(\operatorname{disc}(f_a))$$

where for $\mathsf{v} \in V_{\bar{Z}}$

$$\operatorname{disc}(\mathsf{v}) = \sum_{a \in B_\mathsf{v}} \sum_{\substack{b \in B_\mathsf{v} \\ b \neq a}} (\Gamma : \Gamma_a) \nu_\mathsf{v}(a, b) d_\mathsf{v}$$

with

$$\nu_\mathsf{v}(a, b) = \begin{cases} (\Gamma_\mathsf{v} : \Gamma_b) - (\Gamma_{e_\mathsf{v}(a)} : \Gamma_b) & b \in V(T(e_\mathsf{v}(a))) \\ (\Gamma_\mathsf{v} : \Gamma_b) & b \notin V(T(e_\mathsf{v}(a))) \end{cases}$$

for $a, b \in B_\mathsf{v}$ *with* $a \neq b$.

Proof For $a \in B$ choose a root $\alpha_a \in L$ of f_a and write R_a for the set of roots of f_a in L. We have

$$v_K(\operatorname{disc}(f)) = \sum_{a \in B} \left(v_K(\operatorname{disc}(f_a)) + (\Gamma : \Gamma_a) \sum_{\substack{b \in B \\ b \neq a}} \sum_{\beta \in R_b} v_K(\alpha_a - \beta) \right). \tag{5.2}$$

Let $A(\mathsf{v}) \subset V_{\bar{Z}}$ be the set of ascendants of a vertex $\mathsf{v} \in V(T_{\bar{Z}})$. Let $a \in B$, $\mathsf{v} \in A(a)$ and $b \in B_\mathsf{v}$ with $a \neq b$. If there are different edges $e_a, e_b \in E(\mathsf{v})$ with $a \in V(T(e_a))$ and $b \in V(T(e_b))$, then there are $(\Gamma_\mathsf{v} : \Gamma_b)$ roots $\beta \in R_b$ with $v_K(\alpha_a - \beta) = d_\mathsf{v}$. Otherwise $b \in V(T(e_\mathsf{v}(a)))$ and there are $(\Gamma_\mathsf{v} : \Gamma_b) - (\Gamma_{e_\mathsf{v}(a)} : \Gamma_b)$ roots $\beta \in R_b$ with $v_K(\alpha_a - \beta) = d_\mathsf{v}$. Thus,

$$\sum_{\substack{b \in B \\ b \neq a}} \sum_{\beta \in R_b} v_K(\alpha_a - \beta) = \sum_{\mathsf{v} \in A(a)} \sum_{\substack{b \in B_\mathsf{v} \\ b \neq a}} \nu_\mathsf{v}(a, b) d_\mathsf{v}. \tag{5.3}$$

The lemma follows from (5.2) and (5.3). □

We recall that by Theorem 4.1.4 we have an expression for $f(Y/K)$ as a sum over local terms $f_\chi(v)$ and δ_a. In order to obtain an upper bound for the conductor exponent of Y in terms of the valuation of the discriminant, we compare these local terms with the local terms for the discriminant of f in Lemma 5.1.7. The following lemma gives a lower bound for $\mathrm{disc}(v)$.

Lemma 5.1.8 *Let $v_K(\alpha) \geqslant 0$ for all roots $\alpha \in R$ of f. For all $v \in V_{\bar{Z}} \setminus \{r\}$ we have*

$$\mathrm{disc}(v) \geqslant |B_v|(|E(v)|-1).$$

In particular, $f_\chi(v) \leqslant \mathrm{disc}(v)$ for all $v \in V_{\bar{Z}} \setminus \{r\}$ and all non-trivial irreducible characters χ of G.

Proof Let $v \in V_{\bar{Z}} \setminus \{r\}$ and $a, b \in B_v$ with $a \neq b$. Since all roots of f are integral, d_v is positive. We have $L^{\Gamma_v} \subset L^{\Gamma_a}, L^{\Gamma_b}$ and

$$(\Gamma : \Gamma_a)(\Gamma_v : \Gamma_b) = [L^{\Gamma_a} : K][L^{\Gamma_b} : L^{\Gamma_v}] \geqslant [L^{\Gamma_a} : K][L^{\Gamma_a}L^{\Gamma_b} : L^{\Gamma_a}] = [L^{\Gamma_a}L^{\Gamma_b} : K].$$

Therefore, $d_v(\Gamma : \Gamma_a)(\Gamma_v : \Gamma_b) \geqslant 1$. It is clear that $d_v(\Gamma : \Gamma_a)\nu_v(a,b) \geqslant 0$. For every edge $e \in E(v) \setminus \{e_v(a)\}$ the tree $T(e)$ has at least one leaf. Thus,

$$\mathrm{disc}(v) = \sum_{a \in B_v} \sum_{\substack{b \in B_v \\ b \neq a}} (\Gamma : \Gamma_a)\nu_v(a,b)d_v \geqslant \sum_{a \in B_v} |B_v \setminus V(T(e_v(a)))| \geqslant |B_v|(|E(v)|-1). \square$$

Before we state the main theorem of this section, we give a lemma which is useful for computing the local terms $f_\chi(v)$. For $v \in V_{\bar{Z}}$ the group Γ_v is a subgroup of an inertia group and therefore of the form $H_v \ltimes \Gamma_{v,1}$ with H_v a cyclic group of order prime to p and $\Gamma_{v,1}$ a p-group. Take a generator σ of H_v. By [BW17, Proposition 5.6] we can assume that the action of σ on $\bar{Y}_v$ is given by $\sigma(\bar{x}_v, \bar{y}_v) = (c\bar{x}_v, \gamma\bar{y}_v)$ with $c, \gamma \in \mathbb{F}_K^*$. Then we denote by m_v the order of c and by u_v the order of γ. Further, we set $\mu_v = \mathrm{lcm}(m_v, u_v)$ which is the order of σ as an automorphism of $\bar{Y}_v$. The point z_0 is the unique point of $\bar{Z}_v$ below a point of $\bar{X}_v$ fixed by σ. Let e_0 be the ramification index in $\bar{Y}_v \to \bar{X}_v$ of a point of $\bar{X}_v$ above z_0 and e_∞ the ramification index in $\bar{Y}_r \to \bar{X}_r$ of the point x_r which defines the orientation on $\bar{X}$.

Lemma 5.1.9 *Let $v \in V_{\bar{Z}}$ and χ an irreducible character of G.*

(a) *The character χ is trivial on $\Gamma\tilde{N}_v \cap G$ if and only if $\mathrm{ord}(\chi) \mid \frac{nm_v}{\mu_v}$.*

(b) *The order of χ divides $n/\mathrm{lcm}(u_v, e_0)$ if and only if $\chi|_{\Gamma\tilde{\Gamma}_{z_0} \cap G} = \mathbf{1}$.*

(c) *The order of χ divides $\deg(f)$ and $\mathrm{ord}(\chi) \nmid \frac{n}{\mathrm{lcm}(u_v, e_\infty)}$ if and only if $f_\chi(\infty) = 1$.*

(d) *Let υ_v be the number of irreducible components of $\bar{Y}_v$, $n'_v = n/\upsilon_v$. The order of χ divides $n/\mathrm{lcm}(n'_v, u_v)$ if and only if $\chi|_{\Gamma\tilde{\Gamma}'_v \cap G} = \mathbf{1}$.*

Proof Since G is cyclic, we have that χ is trivial on a subgroup of G of order r if and only if the order of χ divides n/r. Thus, for (a) it suffices to show $|\Gamma\tilde{N}_v \cap G| = \mu_v/m_v$. We have $(g, \sigma^{m_v}) \in \tilde{N}_v$ for some $g \in G$, and since $\Gamma_{v,1}$ is a p-group, this implies $|\Gamma\tilde{N}_v \cap G| = \mu_v/m_v$. We obtain $|\Gamma\tilde{\Gamma}_{z_0} \cap G| = \mathrm{lcm}(u_v, e_0)$ since a point of $\bar{X}_v$ over z_0 is fixed by σ. This implies (b). Parts (c) and (d) follow similarly. □

Remark 5.1.10 The local ring at the generic point of $\bar{X}_v$ is a discrete valuation ring. Let ϑ_v be the corresponding valuation on $L(x)$ normalized such that $\vartheta_v(\pi_K) = 1$. The curve $\bar{Y}_v$ can be given by $\bar{y}_v^n = \bar{f}_v$ where $\bar{f}_v$ is the reduction of $f_v = \pi_K^{-\vartheta_v(f)} f$ and $y_v = \pi_K^{-\vartheta_v(f)/n} y$ (cf. [BW17, § 4.3]). The valuation $\vartheta_v(f)$ can be computed from the metric d and the valuation of the leading coefficient of f. The integer u_v is determined by $\vartheta_v(f)$. The order m_v depends only on the action of Γ_K on the roots of f. Thus, by Lemmata 5.1.9 and 4.2.3 it is possible to determine the local terms $f_\chi(v)$ from the valuation of the leading coefficient of f and the action of Γ_K on the roots of f.

Theorem 5.1.11 *Let $v_K(\alpha) \geqslant 0$ for all roots $\alpha \in R$ of f, let c be the leading coefficient of f and r the number of irreducible factors of $f \in K[x]$.*

(a) If $n \mid v_K(c)$,

$$f(Y/K) \leqslant (n-1)v_K(\mathrm{disc}(f)).$$

(b) If $n \nmid v_K(c)$,

$$f(Y/K) \leqslant (n-1)(v_K(\mathrm{disc}(f)) + r - 1).$$

Proof We first show

$$v_{L^{\Gamma_a}}(\mathfrak{D}_{L^{\Gamma_a}/K}) \leqslant v_K(\mathrm{disc}(f_a)) \tag{5.4}$$

for $a \in B$. Let $a \in B$, $\alpha \in L$ be a root of f_a and $R_a \subset L$ be the set of roots of f_a. Then by [Ser79, III.6 Corollary 2]

$$v_{L^{\Gamma_a}}(\mathfrak{D}_{L^{\Gamma_a}/K}) \leqslant v_{L^{\Gamma_a}}(f_a'(\alpha)) = (\Gamma : \Gamma_a) \sum_{\substack{\beta \in R_a \\ \beta \neq \alpha}} v_K(\alpha - \beta) = v_K(\mathrm{disc}(f_a)).$$

Together with Theorem 4.2.7 this yields the upper bound in the case $n \nmid v_K(c)$. From now on suppose $n \mid v_K(c)$. Let $\chi \neq \mathbf{1}$ be an irreducible character of G. Since $n \mid v_K(c)$, the curve is K-isomorphic to the curve given by $y^n = c^{-1}f$ and we can assume that f is monic. Since all roots of f are integral and f is monic, we have $\vartheta_r(f) = 0$ and hence $u_r = 1$, $m_r = 1$. By Lemmata 5.1.9 and 4.2.3 (b) we have $f_\chi(r) = |\{e \in E'(r) : \chi|_{\Gamma\tilde{\Gamma}_e \cap G} = \mathbf{1}\}|$. Let $e \in E'(r)$ with $\chi|_{\Gamma\tilde{\Gamma}_e \cap G} = \mathbf{1}$. If $T(e)$ has exactly one leaf,

$$\sum_{v \in V(T(e))} f_\chi(v) = -1$$

by Lemma 4.2.5 (b). Assume $T(e)$ has at least two leafs. Then by Lemmata 4.2.4 and 5.1.8,

$$\sum_{v \in V(T(e))} f_\chi(v) \leqslant \sum_{v \in V(T(e))} \mathrm{disc}(v) - 1.$$

Therefore,

$$\sum_{v \in V_{\bar{Z}}} f_\chi(v) \leqslant \sum_{v \in V_{\bar{Z}}} \operatorname{disc}(v)$$

and together with (5.4) this yields the inequality in the case $n \mid v_K(c)$. □

Given a polynomial $f \in K[x]$ and the curve given by $y^n = f$, one easily finds an isomorphic curve given by $y^n = g$ for a polynomial g with integral roots. This yields

Theorem 5.1.12 *Let $d = \deg(f)$, c be the leading coefficient of f, r the number of irreducible factors of f over K and $m = \max\{l \in \mathbb{Z} : l \leqslant v_K(\alpha)$ for all $\alpha \in R\}$.*

(a) If $n \mid v_K(c) + md$,

$$f(Y/K) \leqslant (n-1)(v_K(\operatorname{disc}(f)) - md(d-1)).$$

(b) If $n \nmid v_K(c) + md$,

$$f(Y/K) \leqslant (n-1)(v_K(\operatorname{disc}(f)) - md(d-1) + r - 1).$$

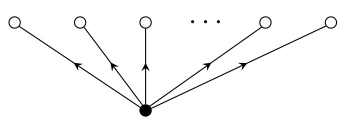
Proof Set $g = f(\pi^m x) \in K[x]$ for a $\pi \in K$ with $v_K(\pi) = 1$. The superelliptic curves defined by $y^n = f$ and $y^n = g$ are isomorphic over K. All roots of g are algebraic integers and the leading coefficient of g has K-valuation $v_K(c) + md$. Thus,

$$f(Y/K) \leqslant (n-1)(v_K(\operatorname{disc}(g)) + r')$$

where $r' = 0$ if $n \mid v_K(c) + md$ and $r' = r - 1$ if $n \nmid v_K(c) + md$ by Theorem 5.1.11. We conclude by noticing that $\operatorname{disc}(g) = \pi^{-md(d-1)} \operatorname{disc}(f)$. □

We give examples comparing the bounds given by Theorems 4.2.7 and 5.1.11. As indicated by Lemma 5.1.8 the bound provided by the discriminant does not have the right order of magnitude if $T_{\bar{Z}}$ has a vertex v such that v is not the root vertex and $|B_v|$ is large.

Example 5.1.13 (1) Let $K/\mathbb{Q}_p^{\mathrm{nr}}$ be a finite extension, $f \in K[x]$ be a monic separable polynomial of degree d with $v_K(\operatorname{disc}(f)) = 0$ and roots of positive valuation. Let $c \in K^*$, $n \geqslant 2$, $p \nmid n$ and consider the curve Y over K given by $y^n = cf$. The tree $T_{\bar{Z}}$ consists of one vertex r and a vertex adjacent to r for each root of f.



Theorem 5.1.11 yields

$$f(Y/K) = \begin{cases} 0 & n \mid v_K(c) \\ (n-1)(d-1) & n \nmid v_K(c) \end{cases}.$$

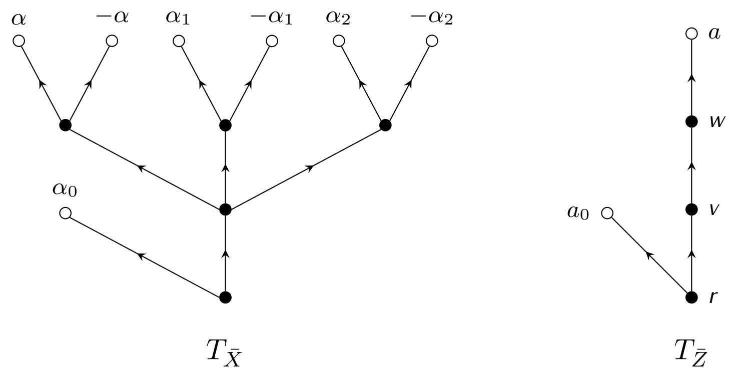
Recall from Example 4.2.9 (1) that for n a prime number,

$$f(Y/K) = \begin{cases} 0 & n \mid v_K(c) \\ (n-1)(d-1) & n \nmid v_K(c), n \nmid d. \\ (n-1)(d-2) & n \nmid v_K(c), n \mid d \end{cases}$$

(2) We take a slight modification of Example 4.2.9 (2). Let Y be given by

$$y^3 = (x-1)(x^6-2)$$

over $\mathbb{Q}_2^{\mathrm{nr}}$. The trees $T_{\bar{X}}$ and $T_{\bar{Z}}$ are depicted below.



Here Theorem 4.2.7 gives

$$f(Y/K) \leqslant 2(1+\delta_a)$$

with $\delta_a = 11$, while by Theorem 5.1.11 we have

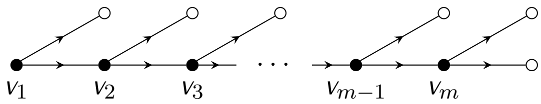
$$f(Y/K) \leqslant 2\delta_a = 22.$$

Note that ∞ is ramified in this case and for both non-trivial irreducible χ we have $f_\chi(\infty) = 0$. We use Lemma 5.1.9 to determine the conductor exponent. For the root vertex we have $m_r = 1$, since the edges of $T_{\bar{X}}$ starting in the root vertex are fixed by Γ_r, and $\vartheta_r(f) = 0$, since $d_r = 0$, implying $u_r = 1$ and $\mu_r = 1$. By Lemma 5.1.9, $\chi \neq \mathbf{1}$ is trivial on $\Gamma\tilde{N}_r \cap G$, and by Lemma 4.2.3 we have $f_\chi(r) = 1$ since (r, v) is unramified. The remaining computations are very similar to Example 4.2.9 (2). We obtain $m_v = 3$, $\vartheta_v(f) = 1$, $u_v = 3$ and $\mu_v = 3$. By Lemma 5.1.9 the characters $\chi \neq \mathbf{1}$ are trivial on $\Gamma\tilde{N}_v \cap G$. Since the edge (v, w) has stabilizer of order 2, the point z_0 below the point of $\bar{X}_v$ fixed by the cyclic prime to p subgroup of Γ_v is unramified in $\bar{Y}_v \to \bar{X}_v$, and by Lemma 5.1.9, χ is non-trivial on $\Gamma\Gamma_{z_0} \cap G$. This yields $f_\chi(v) = -1$ by Lemma 4.2.3. We have $m_w = 1$ and $\vartheta_w(f) = 3$, so $u_w = 1$. Lemmata 5.1.9 and 4.2.3 imply $f_\chi(w) = 0$. This gives us $f(Y/K) = 2\delta_a = 22$.

(3) Let $\pi \in K$ be an element with $v_K(\pi) = 1$ and let

$$f = x\prod_{i=1}^{m}(x - \pi^i) \in K[x]$$

for some $m \geqslant 1$. Consider the curve Y over K given by $y^n = f$ for some n which is not divisible by the residue characteristic of K. The tree $T_{\bar{Z}}$ then is depicted below. The tree $T_{\bar{X}}$ coincides with $T_{\bar{Z}}$ since the action of Γ_K on the roots is trivial.



An easy computation yields

$$v_K(\operatorname{disc}(f)) = 2\sum_{i=1}^{m} v_K(\pi^i) + 2\sum_{i=1}^{m}\sum_{j=1}^{i-1} v_K(\pi^i - \pi^j) = \frac{m(m+1)(m+2)}{3}.$$

By Theorem 4.2.7 the conductor exponent of Y is bounded by $(n-1)m$ if $n \nmid m+1$, since the roots of f are fixed under the action of Γ and we have $m+2$ branch points in $Y \to X$, and $(n-1)(m-1)$ if $n \mid m+1$, since in this case ∞ is not a branch point. This shows that the bound provided by the discriminant has the wrong order of magnitude. However, we show that the bound of Theorem 4.2.7 is attained for infinitely many n.

The action of Γ_K on the roots of f is trivial. In particular, we obtain $m_v = 1$ for all $v \in V_{\bar{Z}}$. We have

$$\vartheta_{v_i}(f) = i + \sum_{j=1}^{i-1} j + \sum_{j=i}^{m} i = im - \frac{1}{2}i(i+1)$$

for $i \in \{1, \ldots, m\}$ since $d_{v_i} = i$. For $n > m(m-1)/2$ a prime number we obtain $u_{v_i} = n$ and hence $f_\chi(v_i) = 1$ for all irreducible characters χ and all $i \in \{1, \ldots, m\}$ by Lemmata 5.1.9 and 4.2.3. Thus,

$$f(Y/K) = \sum_{\chi \neq \mathbf{1}} \Big(\sum_{v \in V_{\bar{Z}}} f_\chi(v) - f_\chi(\infty) \Big) = \sum_{\chi \neq \mathbf{1}} \sum_{v \in V_{\bar{Z}}} 1 = (n-1)m.$$

§ 5.2 An Upper Bound for the Conductor Exponent of Picard Curves

Let K be a local field.

Definition 5.2.1 A Picard curve Y over K is smooth projective curve of genus 3 over K such that $Y_{\bar{K}}$ admits a Galois cover of degree 3. A Picard curve Y over K is called special if $Y_{\bar{K}}$ admits more than one distinguished group of automorphisms.

It is shown in [BKSW19, Theorem 1.18] that every non-special Picard curve can birationally be given by $y^3 = f$ with $f \in K[x]$ a separable polynomial of degree 4.

Moreover, for ternary quartic forms $F \in K[y,x,z]$ it is possible to define a discriminant by

$$\Delta(F) = \mathrm{Res}(\mathrm{D}_y F, \mathrm{D}_x F, \mathrm{D}_z F)/2^{14}$$

where $\mathrm{D}_y F, \mathrm{D}_x F, \mathrm{D}_z F$ are the partial derivatives of F and Res is the resultant of 3 ternary quartics.

Definition 5.2.2 Let Y be a Picard curve over K. The minimal discriminant of Y, denoted $\Delta(Y/K)$, is defined to be the ideal $\mathfrak{p}_K^e$ of $\mathcal{O}_K$ where e is the minimum of the valuations $v_K(\Delta(F))$ with $F \in \mathcal{O}_K[y,x,z]$ running through the quartic forms that define a model of Y.

It is conjectured that $f(Y/K) \leqslant v_K(\Delta(Y/K))$ holds for every Picard curve Y. Theorem 2.2.12 in [BKSW19] states that for every non-special Picard curve Y over K of residue characteristic different from 3 there exist $b \in \mathcal{O}_K$ and a separable polynomial $f \in \mathcal{O}_K[x]$ of degree 4 such that a model for Y is birationally given by $by^3 = f$ and

$$v_K(\Delta(Y/K)) = v_K(\Delta(F))$$

for $F \in \mathcal{O}_K[y,x,z]$ the ternary quartic form given by homogenizing the polynomial $by^3 - f \in \mathcal{O}_K[y,x]$. In particular, to show

$$f(Y/K) \leqslant v_K(\Delta(Y/K))$$

for non-special Picard curves Y, it suffices to consider superelliptic curves given by $by^3 = f$ with a separable polynomial $f \in \mathcal{O}_K[x]$ of degree 4. Using Theorem 4.2.7 we prove

Theorem 5.2.3 *Let Y be a smooth projective curve over a local field K of residue characteristic different from 3 given by $by^3 = f$ with $b \in \mathcal{O}_K$ and $f \in \mathcal{O}_K[x]$ a separable polynomial of degree 4 with leading coefficient c. Then*

$$f(Y/K) \leqslant 2(v_K(\mathrm{disc}(f)) + 7v_K(c) + 3v_K(b)).$$

Since

$$\Delta(by^3z - z^4 f(x/z)) = -3^9 b^{12} c^{15}\,\mathrm{disc}(f)^2$$

holds by [BKSW19, Lemma 2.1.4] for every separable polynomial $f \in K[x]$ of degree 4 with leading coefficient c, by the above discussion Theorem 5.2.3 establishes

Theorem 5.2.4 *Let Y be a non-special Picard curve over a local field K of residue characteristic different from 3. Then*

$$f(Y/K) \leqslant v_K(\Delta(Y/K)).$$

For the proof of Theorem 5.2.3 let $b \in \mathcal{O}_K$ and $f \in \mathcal{O}_K[x]$ be a separable polynomial of degree 4 with leading coefficient c and consider the curve Y over K given by $by^3 = f$. The inequality in Theorem 4.2.7 depends on the number of irreducible factors of f over the maximal unramified extension of K. Therefore we will distinguish cases depending on how f splits over K^{nr}. In all cases the following lemma is useful.

Lemma 5.2.5 *Let f be an irreducible polynomial of degree n over a local field K with leading coefficient c and constant coefficient d. Then $\tilde{f} = x^n f(1/x) \in K[x]$ is irreducible and*

$$v_K(\mathrm{disc}(f)) = v_K(\mathrm{disc}(\tilde{f})) + 2(n-1)(v_K(d) - v_K(c)).$$

Proof Let $R_f \subset \bar{K}$ be the set of roots of f. Since the roots of $\tilde{f}$ are the reciprocals of the roots of f, we have

$$\begin{aligned} v_K(\mathrm{disc}(\tilde{f})) &= \sum_{\alpha \in R_f} \sum_{\substack{\beta \in R_f \\ \beta \neq \alpha}} v_K\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) = \sum_{\alpha \in R_f} \sum_{\substack{\beta \in R_f \\ \beta \neq \alpha}} v_K(\beta - \alpha) - \sum_{\alpha \in R_f} \sum_{\substack{\beta \in R_f \\ \beta \neq \alpha}} v_K(\alpha\beta) \\ &= v_K(\mathrm{disc}(f)) - \sum_{\alpha \in R_f} \sum_{\substack{\beta \in R_f \\ \beta \neq \alpha}} v_K(\alpha\beta). \end{aligned}$$

The polynomial f is irreducible over K and therefore all roots of f have valuation $-(v_K(c) - v_K(d))/n$. With $|\{(\alpha, \beta) \in R_f^2 : \alpha \neq \beta\}| = n(n-1)$, we obtain

$$v_K(\mathrm{disc}(\tilde{f})) = v_K(\mathrm{disc}(f)) + 2(n-1)(v_K(c) - v_K(d)). \qquad \square$$

Proof of Theorem 5.2.3 Theorem 5.2.3 follows under the additional assumption $v_K(c) = v_K(b) = 0$ from Theorem 5.1.11 (a). Thus, for the proof of Theorem 5.2.3 we assume that $v_K(c) \geqslant 1$ or $v_K(b) \geqslant 1$. We denote by $R_g \subset \bar{K}$ the set of roots of a polynomial $g \in K[x]$.

(1) As first case we consider that f is irreducible over K^{nr}. Note that, since f is irreducible, the number of orbits under the action of the inertia group I_K on the set of branch points of ϕ over $\bar{K}$ is 2. If the roots of f are algebraic integers, Theorem 4.2.7 yields

$$f(Y/K) \leqslant 2v_K(\mathrm{disc}(f)).$$

Otherwise set $g = x^4 f(1/x)$. Then Y is isomorphic to the curve given by $y^3 = x^2 g$. By Lemma 5.2.5, g is an irreducible polynomial over K. Thus, by Theorem 4.2.7 and Lemma 5.2.5 we have

$$f(Y/K) \leqslant 2v_K(\mathrm{disc}(g)) \leqslant 2(v_K(\mathrm{disc}(f)) + 6v_K(c))$$

proving Theorem 5.2.3 in this case.

(2) For the second case assume that $f = (c_1x - d_1)g$ with $c_1, d_1 \in \mathcal{O}_{K^{\mathrm{nr}}}$ and $g \in \mathcal{O}_{K^{\mathrm{nr}}}[x]$ irreducible over K^{nr} of degree 3. Write c_g for the leading and d_g for the constant coefficient of g and note that $c = c_1c_g$. Define $\tilde{g} = x^3g(1/x)$. By Theorem 4.2.7,

$$f(Y/K) \leqslant 2(\delta_a + 1)$$

where $a \in B$ is the leaf corresponding to g. The polynomial g has integral roots or $\tilde{g}$ has integral roots and thus $\delta_a \leqslant v_K(\mathrm{disc}(g))$ or $\delta_a \leqslant v_K(\mathrm{disc}(\tilde{g}))$. Together with Lemma 5.2.5 this yields the inequality

$$f(Y/K) \leqslant 2(v_K(\mathrm{disc}(g)) + 4v_K(c_g) + 1).$$

We have

$$\begin{aligned} v_K(\mathrm{disc}(f)) &= v_K(\mathrm{disc}(g)) + 2\sum_{\alpha \in R_g} v_K\Big(\alpha - \tfrac{d_1}{c_1}\Big) \\ &\geqslant v_K(\mathrm{disc}(g)) + 6\min\big\{v_K(d_1) - v_K(c_1), \tfrac{1}{3}(v_K(d_g) - v_K(c_g))\big\} \\ &\geqslant v_K(\mathrm{disc}(g)) - \min\{6v_K(c_1), 2v_K(c_g))\}. \end{aligned}$$

Combining the two inequalities and noting $v_K(c) = v_K(c_1) + v_K(c_g)$,

$$f(Y/K) \leqslant 2(v_K(\mathrm{disc}(f)) + 6v_K(c) + 1)$$

follows. Since we have reduced to the case that $v_K(c) \geqslant 1$ or $v_K(b) \geqslant 1$, this yields Theorem 5.2.3 in the considered case.

(3) In the third case let f be the product of irreducible polynomials $g, h \in \mathcal{O}_{K^{\mathrm{nr}}}[x]$ of degree 2. Denote by c_r and d_r the leading and constant coefficient of a polynomial $r \in K^{\mathrm{nr}}[x]$ and set $\tilde{g} = x^2g(1/x)$ and $\tilde{h} = x^2h(1/x)$. Then

$$\begin{aligned} v_K(\mathrm{disc}(f)) &= v_K(\mathrm{disc}(g)) + v_K(\mathrm{disc}(h)) + 2\sum_{\alpha \in R_g}\sum_{\beta \in R_h} v_K(\alpha - \beta) \\ &\geqslant v_K(\mathrm{disc}(g)) + v_K(\mathrm{disc}(h)) + 4\min\{v_K(d_g) - v_K(c_g), v_K(d_h) - v_K(c_h)\} \\ &\geqslant v_K(\mathrm{disc}(g)) + v_K(\mathrm{disc}(h)) - 4v_K(c). \end{aligned}$$

By Lemma 5.2.5,

$$v_K(\mathrm{disc}(\tilde{g})) \leqslant v_K(\mathrm{disc}(g)) + 2v_K(c_g) \text{ and } v_K(\mathrm{disc}(\tilde{h})) \leqslant v_K(\mathrm{disc}(h)) + 2v_K(c_h),$$

and Theorem 4.2.7 yields

$$f(Y/K) \leqslant 2(v_K(\mathrm{disc}(g)) + v_K(\mathrm{disc}(h)) + 2v_K(c) + 1) \leqslant 2(v_K(\mathrm{disc}(f)) + 6v_K(c) + 1).$$

Hence, Theorem 5.2.3 holds in this case, for we assumed that $v_K(c) \geqslant 1$, $v_K(b) \geqslant 1$.

(4) As next case we assume that $f = (c_1x - d_1)(c_2x - d_2)g$ with $c_1, c_2, d_1, d_2 \in \mathcal{O}_{K^{\mathrm{nr}}}$ and $g \in \mathcal{O}_{K^{\mathrm{nr}}}[x]$ an irreducible polynomial of degree 2 with leading coefficient c_g and constant coefficient d_g. Set $\tilde{g} = x^2 g(1/x)$. Without loss of generality we assume $v_K(c_1) \geqslant v_K(c_2)$. Then

$$\begin{aligned} v_K(\mathrm{disc}(f)) &= v_K(\mathrm{disc}(g)) + 2\sum_{\alpha\in R_g} v_K\Big(\alpha - \tfrac{d_1}{c_1}\Big) + 2\sum_{\alpha\in R_g} v_K\Big(\alpha - \tfrac{d_2}{c_2}\Big) + 2v_K\Big(\tfrac{d_2}{c_2} - \tfrac{d_1}{c_1}\Big) \\ &\geqslant v_K(\mathrm{disc}(g)) + 4\min\Big\{\tfrac{1}{2}v_K\Big(\tfrac{d_g}{c_g}\Big), v_K\Big(\tfrac{d_1}{c_1}\Big)\Big\} \\ &\qquad + 4\min\Big\{\tfrac{1}{2}v_K\Big(\tfrac{d_g}{c_g}\Big), v_K\Big(\tfrac{d_2}{c_2}\Big)\Big\} + 2\min\Big\{v_K\Big(\tfrac{d_2}{c_2}\Big), v_K\Big(\tfrac{d_1}{c_1}\Big)\Big\} \\ &\geqslant v_K(\mathrm{disc}(g)) - M \end{aligned}$$

with $M = \max\{4v_K(c_g) + 2v_K(c_1)\,,\, 6v_K(c_1) + \max\{2v_K(c_g), 4v_K(c_2)\}\}$. Using Lemma 5.2.5 and $v_K(c) = v_K(c_1) + v_K(c_2) + v_K(c_g)$,

$$v_K(\mathrm{disc}(\tilde{g})) \leqslant v_K(\mathrm{disc}(g)) + 2v_K(c_g) \leqslant v_K(\mathrm{disc}(f)) + 6v_K(c).$$

By Theorem 4.2.7

$$f(Y/K) \leqslant 2(v_K(\mathrm{disc}(f)) + 6v_K(c) + 2)$$

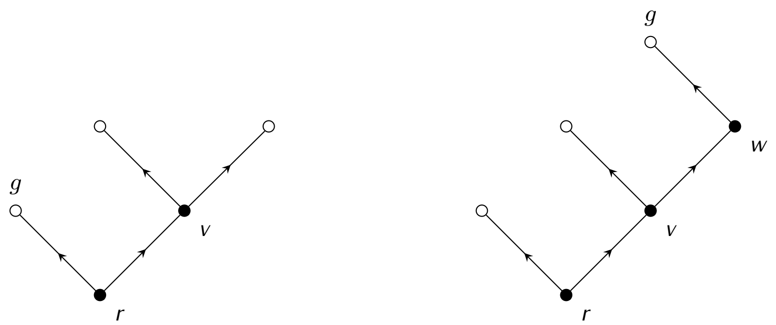
which yields Theorem 5.2.3 if $v_K(c) \geqslant 2$ or $v_K(b) \geqslant 1$. Now let $v_K(b) = 0$ and $v_K(c) = 1$. Then exactly one of $v_K(c_1), v_K(c_2), v_K(c_g)$ is 1 and the other two are 0. We can further reduce to the case that $v_K(\alpha - \beta), v_K(\alpha - \gamma), v_K(\beta - \gamma)$ are non-positive for all $\alpha \in R_g$ and $\beta = d_1/c_1, \gamma = d_2/c_2$ since otherwise we would obtain

$$\begin{aligned} v_K(\mathrm{disc}(f)) &= v_K(\mathrm{disc}(g)) + 2\sum_{\alpha\in R_g} v_K\Big(\alpha - \tfrac{d_1}{c_1}\Big) + 2\sum_{\alpha\in R_g} v_K\Big(\alpha - \tfrac{d_2}{c_2}\Big) + 2v_K\Big(\tfrac{d_2}{c_2} - \tfrac{d_1}{c_1}\Big) \\ &\geqslant v_K(\mathrm{disc}(g)) + 1 - \max\{4v_K(c_g), 6v_K(c_1) + \max\{v_K(c_g), 2v_K(c_2)\}\}. \end{aligned}$$

As above this would lead to

$$f(Y/K) \leqslant 2(v_K(\mathrm{disc}(f)) + 6v_K(c) + 1).$$

After this reduction, we are left with the following two possible configurations for the tree $T_{\bar{\mathcal{Z}}}$ if we take the specialization of the K-rational branch point $\infty \in X$ as x_r.



$v_K(c_g) = 1$ $\qquad$ $v_K(c_1) = 1$

Let $v, w \in V_{\bar{Z}}$ be as in the above trees. Then $d_v = 0$ and in the notations used in Lemma 5.1.9, $m_v = 1$, $\vartheta_v(f) = 0$ and hence $u_v = 1$, $\mu_v = 1$. Thus, Lemmata 5.1.9 and 4.2.3 imply $f_X(v) = 0$ in both cases. We obtain

$$f(Y/K) \leqslant 2(1 + \delta_a) \leqslant 2(1 + v_K(\mathrm{disc}(f)) + 6v_K(c))$$

which shows Theorem 5.2.3 in this case.

(5) In the last case f splits into linear factors over K^{nr}. Then $f(Y/K) \leqslant 2 \cdot 3$ by Theorem 4.2.7. For $v_K(c) \geqslant 1$ or $v_K(b) \geqslant 1$, Theorem 5.2.3 follows.

The proof of Theorem 5.2.3 is now complete. □

As discussed above this proves Theorem 5.2.4. Our next goal is to prove a similar inequality for special Picard curves. More precisely, we want to show

Theorem 5.2.6 *Let Y be a special Picard curve over a local field K of residue characteristic different from 2. Then*

$$f(Y/K) \leqslant v_K(\Delta(Y/K)).$$

[BKSW19, Theorem 4.5.15] states that every special Picard curve Y over K of residue characteristic different from 2 admits a superelliptic equation of the form $by^4 = f$ with $b \in \mathcal{O}_K$ and a separable polynomial $f \in \mathcal{O}_K[x]$ of degree 4 such that the corresponding homogeneous equation $F = 0$ has minimal discriminant. The discriminant of such a equation is given by

$$\Delta(F) = -2^{16}b^9c^{18}\,\mathrm{disc}(f)^3$$

by [BKSW19, Remark 4.5.3] if c denotes the leading coefficient of f. Therefore, Theorem 5.2.6 is a corollary of

Theorem 5.2.7 *Let Y be a smooth projective curve over a local field K of residue characteristic different from 2 given by $by^4 = f$ with $b \in \mathcal{O}_K$ and $f \in \mathcal{O}_K[x]$ a separable polynomial of degree 4 with leading coefficient c. Then*

$$f(Y/K) \leqslant 3(v_K(\mathrm{disc}(f)) + 6v_K(c) + 2v_K(b)).$$

Proof As in the proof of Theorem 5.2.3 we proceed by a case-by-case analysis with the same cases. We enumerate the cases as in the proof of Theorem 5.2.3 and use in each case the notation of the corresponding case in this proof. The proof is similar to the preceding proof and therefore we will just give the needed adjustments in each case.

If $v_K(c) = v_K(b) = 0$, we have $f(Y/K) \leqslant 3v_K(\mathrm{disc}(f))$ by Theorem 5.1.11 (a). Therefore, we assume in all of the following cases that $v_K(c) = 1$ or $v_K(b) = 1$.

(1) The first case can be copied word for word with the exception that we get

$$f(Y/K) \leqslant 3v_K(\mathrm{disc}(f))$$

if the roots of f have non-negative valuation, and otherwise

$$f(Y/K) \leqslant 3(v_K(\mathrm{disc}(f)) + 6v_K(c)).$$

(2) For the second case we note that ∞ is not branched in $Y \to X$ since $4 \mid \deg(f)$. Hence, there are 2 Γ-orbits of branch points and Theorem 4.2.7 yields

$$f(Y/K) \leqslant 3\delta_a.$$

Combining this with the steps in the corresponding case in the proof of Theorem 5.2.3, we obtain

$$f(Y/K) \leqslant 3(v_K(\mathrm{disc}(f)) + 6v_K(c)).$$

(3) As in case (2) the only adjustment needed is due to the fact that ∞ is not a branch point of $Y \to X$. This amounts to only changing the last inequality in case (3) of the proof of Theorem 5.2.3 to

$$f(Y/K) \leqslant 3(v_K(\mathrm{disc}(g)) + v_K(\mathrm{disc}(h)) + 2v_K(c)) \leqslant 3(v_K(\mathrm{disc}(f)) + 6v_K(c)).$$

(4) With the same changes as in cases (2) and (3) – that means changing the factor 2 to 3 and reducing the expression in the brackets by 1 in all appearing upper bounds for $f(Y/K)$ – we reduce this case to the two cases depicted in case (4) of the proof of Theorem 5.2.3. We obtain again $f_\chi(v) = 0$ as in case (4) of the proof of Theorem 5.2.3. However, this is not sufficient to prove Theorem 5.2.7. We have

$$\vartheta_r(f) = v_K(c) + 4d_r = \begin{cases} -3 & v_K(c_1) = 1 \\ -1 & v_K(c_g) = 1 \end{cases}$$

and hence $u_r = 4$. Since $\deg(f) = 4$, we obtain $f_\chi(\infty) = 1$ for every irreducible χ by Lemma 5.1.9. With $f_\chi(r) \leqslant 1$ and $f_\chi(w) \leqslant 0$ (Lemma 4.2.4), we get in both cases

$$f(Y/K) = \sum_{\chi \neq \mathbf{1}} \left(\sum_{v' \in V_{\bar{Z}}} f_\chi(v') - f_\chi(\infty) + \delta_a \right) \leqslant 3(v_K(\mathrm{disc}(f)) + 6v_K(c)).$$

(5) For this case we can copy the corresponding case of the proof of Theorem 5.2.3 to the letter.

Theorem 5.2.7 follows. □

As a consequence of Theorem 5.2.4 and Theorem 5.2.6 we have

Theorem 5.2.8 *Let Y be a Picard curve over a local field K of residue characteristic different from 2 and 3. Then*

$$f(Y/K) \leqslant v_K(\Delta(Y/K)).$$

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit Führern superelliptischer Kurven. Führer von Kurven über globalen Körpern hängen eng mit L-Funktionen zusammen und sind wie diese als Produkt

$$\mathfrak{N}(Y/K) = \mathfrak{D}_K^{2g} \prod_{\mathfrak{p}} \mathfrak{p}^{f(Y/K_{\mathfrak{p}})}$$

lokaler Faktoren definiert. Hierbei ist $\mathfrak{D}_K$ die Differente des Körpers K, g das Geschlecht der Kurve und das Produkt durchläuft die Primideale von K. Zunächst definieren wir Führerexponenten endlichdimensionaler Galoisdarstellungen zu lokalen Körpern und zeigen, dass diese den Artinschen Formalismus erfüllen und nur vom Bild der Trägheitsgruppe abhängen. Die lokalen Faktoren $f(Y/K_{\mathfrak{p}})$ des Führers einer Kurve sind definiert als die Führerexponenten einer Darstellung der absoluten Galoisgruppe des lokalen Körpers $K_{\mathfrak{p}}$, nämlich der ersten Kohomologiegruppe der Kurve.

Die Berechnung von Führerexponenten von Kurven erweist sich im Allgemeinen als schwierig. Für einige Familien von Kurven sind explizite Methoden bekannt, etwa für superelliptische Kurven, also Kurven, die durch eine Gleichung der Form $y^n = f$ mit $f \in K[x]$ gegeben sind, unter der Voraussetzung, dass die Restklassenkörpercharakteristik von K nicht n teilt. Im weiteren Teil dieser Arbeit sind wir an Abschätzungen für Führerexponenten von Kurven interessiert. Der Führerexponent ist durch die Wirkung der Trägheitsgruppe auf die ℓ-adische Kohomologie der speziellen Faser eines quasistabilen Modells festgelegt. Um letztere zu bestimmen, verwenden wir die explizite Beschreibung der speziellen Faser aus [BW17] als eine zyklische G-Überlagerung eines Baums projektiver Geraden. Wir nutzen dies, indem wir die Kohomologie der speziellen Faser als Darstellung des Produktes von G und der Trägheitsgruppe bestimmen. Dazu verwenden wir, dass der Quotient der speziellen Faser nach diesem Produkt ein Baum projektiver Geraden ist. Eine Grothendieck-Ogg-Shafarevich-Formel lässt sich für jede dieser Geraden auf die entsprechende Überlagerung anwenden. Dies ergibt die Kohomologie als Summe lokaler Terme. Wir zerlegen diese Darstellungen in ihre isotypischen Komponenten bezüglich der Gruppe G. Da die Trägheitsgruppenwirkung mit der G-Wirkung kommutiert, ist dies auch eine Zerlegung in invariante Unterräume bezüglich der Trägheitsgruppe. Durch den Artinschen Formalismus übertragen sich die Zerlegung in G-isotypische Komponenten und die Zerlegung in lokale Terme auf den Führerexponenten. Jede G-isotypische Komponente besteht aus zwei wesentlich unterschiedlichen Summanden. Einer dieser Summanden hängt nur von der Wirkung der Trägheitsgruppe auf den Nullstellen des Polynoms f aber nicht von n ab. Der zweite Summand lässt sich nach oben durch die Anzahl der Nullstellenorbits unter der Wirkung der Trägheitsgruppe abschätzen.

Als Anwendung dieses Resultats zeigen wir, dass der Führerexponent einer Picardkurve über einem Körper von Restklassenkörpercharakteristik ungleich 2 oder 3 durch eine minimale Diskriminante einer Gleichung für die Kurve beschränkt ist. Wir verwenden das Ergebnis aus [BKSW19], dass sich eine superelliptische Gleichung mit minimaler Diskriminante für jede Picardkurve finden lässt. Der Beweis basiert dann auf einer Fallunterscheidung nach der Anzahl der Nullstellenorbits einer superelliptischen Gleichung der Picardkurve und der bewiesenen oberen Schranke für Führerexponenten.

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Dissertation selbstständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe.

Ulm, den

Roman Kohls

Lebenslauf

Der Lebenslauf ist aus Gründen des Datenschutzes nicht enthalten.