Quantum-noise quenching in the correlated spontaneous-emission laser as a multiplicative noise process. II. Rigorous analysis including amplitude noise

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An analytical steady-state distribution for the phase difference ψ in a correlated spontaneous-emission laser (CEL) is derived based on the amplitude and phase equations of a CEL. This distribution is shown to be an excellent approximation to that obtained from a numerical simulation of the complete set of CEL equations. In particular, the effects of amplitude noise on CEL operation are considered and it is shown that fluctuations in the relative amplitude are also noise quenched.

I. INTRODUCTION AND OVERVIEW

In paper I of this series[1] we have shown, via simple geometrical arguments, that the quantum-noise quenching in a correlated (spontaneous) emission laser[2–4] (CEL) can be understood as a result of a multiplicative noise process.[5] In particular, we have demonstrated that in contrast to the (two-mode) phase-locked laser[6] (PLL) the phase difference ψ between the two electric fields in a CEL obeys the Langevin equation

$$\dot{\psi}=a-b\sin\psi+\sin(\psi/2)\mathcal{F}(t)\ , \tag{1.1}$$

where a denotes the detuning of the cavity eigenfrequencies and b is the gain. The Gaussian white noise[7] $\mathcal{F}$ of strength

$$\langle \mathcal{F}(t)\mathcal{F}(s)\rangle=2(2\mathcal{D})\delta(t-s)$$

is assumed to have mean zero,

$$\langle \mathcal{F}(t)\rangle=0\ .$$

Moreover, the similarities and differences between CEL noise quenching and PLL noise quieting have been emphasized by an analysis of the corresponding Fokker-Planck equation,

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\psi}\{[a-(b-\tfrac{1}{2}\mathcal{D})\sin\psi]P\}+2\mathcal{D}\frac{\partial^2}{\partial\psi^2}[\sin^2(\psi/2)P]\ . \tag{1.2}$$

However, the derivation of Eqs. (1.1) and (1.2) given in Ref. 1 relies on general, geometrical arguments. The objective of the present article is therefore to present a mathematically rigorous derivation of this equation based on the complete set of equations governing[2] the CEL. In particular, we include noise in the average phase angle Ψ as well as amplitude fluctuations.

The paper is organized as follows: Starting from the quantum Langevin equations for the two modes of the CEL, we derive in Sec. II equations of motion for the phases and amplitudes of the two electric fields. The noise-free situation is discussed in Sec. III and special emphasis is made on the locking of the two fields to a constant relative phase angle ψ_0. In Sec. IV we then proceed to study the influence of the spontaneous emission noise on ψ_0. We here pursue two strategies. (a) In one approach we simulate the coupled system of equations for the phase difference ψ, phase sum Ψ, and the amplitude difference r, on a computer to obtain numerically a steady-state distribution $P_0=P_0(\psi)$. (b) In the second approach we utilize standard techniques from noise theory[7,8] to derive an approximate, effective Fokker-Planck equation which (up to a small noise-induced correction to the coupling coefficient b) is identical to Eq. (1.2) and thus to the one discussed in Ref. 1. Its solution reproduces the characteristic features of the steady-state distributions obtained from the numerical simulation for the range of parameters of interest here; that is, small frequency detunings between the waves, $|a/b|\ll 1$ and weak noise, $\mathcal{D}/b\ll 1$. Therefore, Eq. (1.1) with a very slightly modified coupling coefficient is an excellent description of the dynamics of the CEL. This confirms the notion of noise quenching as a result of multiplicative noise put forward in Paper I. Section V is a summary and discussion. In order to focus on the essentials we have banished all lengthy calculations to the Appendixes.

II. EQUATIONS OF MOTION

In this section we derive, starting from the quantum Langevin equations for the CEL modes, equations of motion for the relative phase ψ, for the average phase angle Ψ, and for the difference r and sum R of the electric field amplitudes.

In first-order laser theory[9,10] and at zero-cavity temperature, that is in the absence of blackbody photons, the evolution of the two fields described by the annihilation operators $\hat{a}_j$ ($j=1,2$) is given[2] by

$$\dot{\hat{a}}_1=[\tfrac{1}{2}(\alpha-\gamma)-i(\Omega_1-\nu_1)]\hat{a}_1+\tfrac{1}{2}\alpha\hat{a}_2e^{i\Phi(t)}+\hat{F}_1(t)\ , \tag{2.1a}$$

$$\dot{\hat{a}}_2=[\tfrac{1}{2}(\alpha-\gamma)-i(\Omega_2-\nu_2)]\hat{a}_2+\tfrac{1}{2}\alpha\hat{a}_1e^{-i\Phi(t)}+\hat{F}_2(t)\ , \tag{2.1b}$$

where for the sake of simplicity we have chosen the gain coefficients in the two modes, α_{jj}, to be equal to the cross-coupling coefficient α_{12} and to be real,[3] that is, $\alpha_{11}=\alpha_{22}=\alpha_{12}=\alpha_{21}\equiv\alpha$. Moreover, the two decay rates γ_j are assumed to be equal, $\gamma_1=\gamma_2\equiv\gamma$. The symbols Ω_j and ν_j denote the empty cavity frequencies and the frequencies of the transitions, respectively. The Gaussian noise operators $\hat{F}_j$ have mean zero

$$\langle\hat{F}_j\rangle=0 \tag{2.2}$$

and

$$\langle\hat{F}_j^\dagger(t)\hat{F}_k(s)\rangle=2D_{jk}\delta(t-s)\ , \tag{2.3}$$

where

$$D_{11}=D_{22}=\tfrac{1}{4}\alpha\equiv D \tag{2.4a}$$

and

$$D_{12}=D_{21}^*=\tfrac{1}{4}\alpha e^{-i\Phi(t)}\ , \tag{2.4b}$$

with

$$\Phi(t)=(\nu_1-\nu_2-\omega_0)t-\tilde{\Delta}\ . \tag{2.5}$$

Here ω_0 and $\tilde{\Delta}$ are the frequency and the phase of the microwave which in the present treatment is considered to be noise free. The Hanle-effect laser[2] configuration is obtained for $\omega_0=0$ and $\tilde{\Delta}=0$.

The system, Eq. (2.1), can be solved exactly.[11] However, it is extremely difficult to extract from these solutions the relative phase ψ. We therefore utilize an ansatz

$$\hat{a}_j=\rho_j(t)e^{-i\theta_j(t)} \tag{2.6}$$

to obtain directly equations for the slowly varying phases θ_j and amplitudes ρ_j.

The ansatz (2.6), however, neglects[9,10] the operator character of $\hat{a}_j$. The following calculation is thus a semiclassical treatment as is adequate for a laser above threshold. Substituting the ansatz (2.6) into Eq. (2.1) and taking real and imaginary parts, we arrive at

$$\dot{\theta}_1=(\Omega_1-\nu_1)-\frac{\alpha}{2}\frac{\rho_2}{\rho_1}\sin(\Phi+\theta_1-\theta_2)$$
$$-\frac{1}{\rho_1}\frac{1}{2i}(F_1e^{i\theta_1}-F_1^*e^{-i\theta_1})\ , \tag{2.7a}$$

$$\dot{\theta}_2=(\Omega_2-\nu_2)+\frac{\alpha}{2}\frac{\rho_1}{\rho_2}\sin(\Phi+\theta_1-\theta_2)$$
$$-\frac{1}{\rho_2}\frac{1}{2i}(F_2e^{i\theta_2}-F_2^*e^{-i\theta_2})\ , \tag{2.7b}$$

and

$$\dot{\rho}_1=\tfrac{1}{2}(\alpha-\gamma)\rho_1+\tfrac{1}{2}\alpha\rho_2\cos(\Phi+\theta_1-\theta_2)$$
$$+\tfrac{1}{2}(F_1e^{i\theta_1}+F_1^*e^{-i\theta_1})\ , \tag{2.8a}$$

$$\dot{\rho}_2=\tfrac{1}{2}(\alpha-\gamma)\rho_2+\tfrac{1}{2}\alpha\rho_1\cos(\Phi+\theta_1-\theta_2)$$
$$+\tfrac{1}{2}(F_2e^{i\theta_2}+F_2^*e^{-i\theta_2})\ . \tag{2.8b}$$

Here we have ignored the operator nature of the noise operators. For a detailed discussion on this intricate point we refer to Refs. 8–10. Making use of Eqs. (2.7) and (2.8), the equations of motion for the new variables

$$\psi=\Phi+\theta_1-\theta_2\ , \tag{2.9a}$$

$$\Psi=\tfrac{1}{2}(\theta_1+\theta_2)\ , \tag{2.9b}$$

$$r\equiv\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\ , \tag{2.9c}$$

$$R\equiv\rho_1+\rho_2\ , \tag{2.9d}$$

read

$$\dot{\psi}=a-b\frac{1+r^2}{1-r^2}\sin\psi+\frac{1}{1-r^2}\frac{i}{R}(F-rG)\ , \tag{2.10a}$$

$$\dot{\Psi}=A+b\frac{r}{1-r^2}\sin\psi+\frac{1}{1-r^2}\frac{i}{2R}(G-rF)\ , \tag{2.10b}$$

$$\dot{r}=-\left[b\cos\psi+\frac{1}{2R}F_R\right]r+\frac{1}{2R}F_r\ , \tag{2.10c}$$

$$\dot{R}=\tfrac{1}{2}(\alpha-\gamma+\alpha\cos\psi)R+\tfrac{1}{2}F_R(t)\ , \tag{2.10d}$$

where

$$b=\alpha,\quad a\equiv\Omega_1-\Omega_2-\omega_0,\quad A\equiv\tfrac{1}{2}(\Omega_1-\nu_1+\Omega_2-\nu_2)\ ,$$

and

$$F\equiv\mathcal{F}_1-\mathcal{F}_2-\text{c.c.}\ , \tag{2.11a}$$

$$G\equiv\mathcal{F}_1+\mathcal{F}_2-\text{c.c.}\ , \tag{2.11b}$$

$$F_r\equiv\mathcal{F}_1-\mathcal{F}_2+\text{c.c.}\ , \tag{2.11c}$$

$$F_R\equiv\mathcal{F}_1+\mathcal{F}_2+\text{c.c.}\ , \tag{2.11d}$$

with

$$\mathcal{F}_1\equiv F_1e^{(i/2)(2\Psi+\psi-\Phi)}\ , \tag{2.11e}$$

$$\mathcal{F}_2\equiv F_2e^{(i/2)(2\Psi-\psi+\Phi)}\ . \tag{2.11f}$$

III. NOISE-FREE SOLUTIONS

The physics of the CEL hidden behind the rather complex system of Eqs. (2.10) stands out most clearly when we first discuss the noise-free Eqs. (2.10), that is,

$$\dot{\psi}=a-b\frac{1+r^2}{1-r^2}\sin\psi \ , \tag{3.1a}$$

$$\dot{\Psi}=A+b\frac{r}{1-r^2}\sin\psi \ , \tag{3.1b}$$

$$\dot{r}=-b\cos(\psi)r \ , \tag{3.1c}$$

$$\dot{R}=\tfrac{1}{2}(\alpha-\gamma+\alpha\cos\psi)R \ , \tag{3.1d}$$

and then include the spontaneous emission fluctuations. In particular we in this section analyze the steady-state solutions ($\partial/\partial t\equiv 0$) of Eq. (3.1). We start with Eq. (3.1c) which has the obvious steady-state solution

$$r_0=0 \ . \tag{3.2}$$

Substituting Eq. (3.2) into Eq. (3.1a) we arrive at

$$0=a-b\sin\psi_0 \ ,$$

which for $|a|<b$ allows the two steady-state solutions

$$\psi_0=\arcsin\left[\frac{a}{b}\right] \tag{3.3}$$

and

$$\psi_0'=\pi+\arcsin\left[\frac{a}{b}\right] ,$$

whereas no steady-state solution is possible for $|a|>b$. It is straightforward to show that $\psi_0=\arcsin(a/b)$ and $r_0=0$ are stable steady-state solutions whereas ψ_0' is unstable. In the remainder of this article we confine ourselves to detunings $|a|<b$ such that the stable solution (3.3) applies.

Note, however, that Eq. (3.1c) also has nonphysical steady-state solutions

$$\psi_0^{(n)}=(2n+1)\frac{\pi}{2} \ .$$

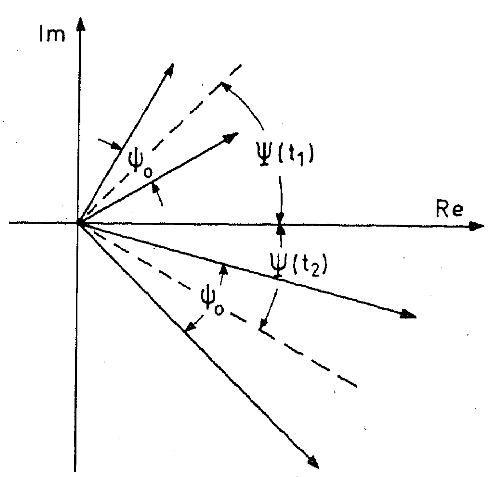

FIG. 1. Two electric fields in a correlated spontaneous-emission laser are locked to a constant relative phase angle ψ_0 while they rotate with a common rate A in the complex plane as indicated by the averaged phase angle Ψ. Their amplitudes are equal and grow exponentially as predicted by first-order laser theory.

This can be recognized by substituting $\psi_0^{(n)}$ into Eq. (3.1a) and solving for r_0^2,

$$r_0^2=\frac{a-(-1)^n b}{a+(-1)^n b} \ .$$

Since $|a|<b$ this condition is equivalent to $r_0^2<0$ in obvious contradiction to $r_0^2\geq 0$. In conclusion we note that according to Eq. (3.3) the two electric fields in a CEL are locked to the constant relative phase angle $\psi_0=\arcsin(a/b)$ as shown in Fig. 1.

Substituting Eq. (3.2) into Eq. (3.1b) yields after a trivial integration for the average phase angle Ψ,

$$\Psi(t)=\Psi_0+At \ . \tag{3.4}$$

While the two electric fields keep a constant relative phase angle ψ_0 they both rotate with a constant rate A in the complex plane as depicted in Fig. 1.

The amplitudes ρ_j of the two fields follow from Eqs. (2.9c) and (2.9d) as

$$\rho_1=\tfrac{1}{2}(1+r)R$$

and

$$\rho_2=\tfrac{1}{2}(1-r)R \ .$$

With the help of Eq. (3.2) we thus find that in steady state the two amplitudes are equal and given by

$$\rho_1=\rho_2=\tfrac{1}{2}R(t)=\tilde{R}_0 e^{(1/2)(\alpha+\Gamma-\gamma)t} \ , \tag{3.5}$$

where

$$\Gamma\equiv[(\alpha)^2-a^2]^{1/2} \ .$$

Here we have made use of Eqs. (3.1d) and (3.3). For $\alpha+\Gamma-\gamma>0$ the amplitudes ρ_1 and ρ_2 of the electric fields grow exponentially as expected from first-order laser theory.

IV. STEADY-STATE DISTRIBUTION $P_0(\psi)$

In Sec. III we have shown that in the absence of spontaneous emission noise the two electric fields in a CEL lock to a constant relative phase angle ψ_0. We now study the influence of the noise sources, Eq. (2.11), on ψ_0 as expressed by the system (2.10). However, we immediately face a problem: according to Eq. (3.5) the sum of the amplitudes R grows to infinity. Since the noise sources F, G, F_r, and F_R enter Eq. (2.10) always multiplied by the inverse of R, the spontaneous emission noise seems to die away. However, this is only an artifact of first-order laser theory. Due to the saturation of the laser,[9,10] however, R reaches a steady-state value $R_0=2\rho_0$ given by the nonlinear CEL theory.[12] This saturation effect can be incorporated into the system (2.10) in a well-known fashion[9,10] by assuming a damping constant γ such that no growth in R is possible, that is[13]

$$2\alpha\cos^2(\psi/2)-\gamma=0 \ .$$

Neglecting also the fluctuations δR around R_0 ($R=R_0+\delta R\cong R_0$), the equations of motion governing the CEL read

$$\dot{\psi}=a-b\frac{1+r^2}{1-r^2}\sin\psi+\frac{1}{1-r^2}\frac{i}{2\rho_0}(F-rG)\ , \tag{4.1a}$$

$$\dot{\Psi}=A+b\frac{r}{1-r^2}\sin\psi+\frac{1}{1-r^2}\frac{i}{4\rho_0}(G-rF)\ , \tag{4.1b}$$

$$\dot{r}=-\left[b\cos\psi+\frac{1}{4\rho_0}F_R\right]r+\frac{1}{4\rho_0}F_r\ , \tag{4.1c}$$

where the noise sources F, G, F_r, and F_R are defined by Eqs. (2.11a)–(2.11f). For a treatment of the nonlinear theory we refer to Ref. 12.

The influence of the spontaneous emission noise on the phase difference ψ is best illustrated by a probability distribution P_0 for ψ. Such a distribution requires the knowledge of all moments $\langle\psi^n\rangle$. However, since Eqs. (4.1) are highly coupled, and thus have to be solved simultaneously, it is not straightforward to evaluate these moments. A more promising approach consists of simulating[7] the Langevin forces F, G, F_r, and F_R by means of a random-number generator on a computer and integrating the system Eq. (4.1) numerically to obtain the probability distribution $P_0=P_0(\psi)$. The result of such a simulation is shown in Fig. 2 for $a/b=0.1$ and $2\mathcal{D}/b=0.1$ ($\mathcal{D}\equiv D/\rho_0^2$) by the solid "jagged" curve.

Although this numerical simulation is very useful it makes it difficult to gain some insight into the form of the distribution P_0 for various parameter values. For this reason we derive in this section an approximate expression valid in the physically relevant limit of small detunings $|a|/b\ll 1$ and weak noise $\mathcal{D}/b\ll 1$. Since ψ is the relative phase between the two CEL modes, periodic boundary conditions for ψ are adequate. We therefore expand the probability distribution $P=P(t,\psi)$ into

$$P(t,\psi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}c_n(t)e^{in\psi}\ , \tag{4.2}$$

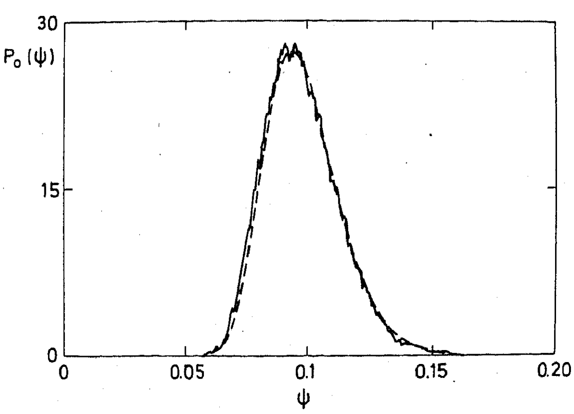

FIG. 2. Comparison between the steady-state distribution $P_0=P_0(\psi)$ (ψ in rad) obtained from a computer simulation of Eq. (4.1) (solid "jagged" line) and the approximate one found from the recurrence relation Eq. (4.8) (dashed line). The parameters chosen here are $a/b=0.1$, $2\mathcal{D}/b=0.1$, and $A=1$.

where the expansion coefficients c_n are given by

$$c_n(t)=\int_{-\pi}^{\pi}d\psi\, e^{-in\psi}P(t,\psi)=\langle e^{-in\psi}\rangle\ . \tag{4.3}$$

We now derive a recurrence relation for the coefficients c_n. Multiplying Eq. (4.1a) by $e^{-in\psi}$ and noting that

$$e^{-in\psi}\dot{\psi}=\frac{i}{n}\frac{d}{dt}(e^{-in\psi})\ ,$$

we arrive at

$$\frac{i}{n}\dot{c}_n=ac_n-\frac{b}{2i}\left\langle\frac{1+r^2}{1-r^2}(e^{-i(n-1)\psi}-e^{-i(n+1)\psi})\right\rangle+\left\langle\frac{1}{1-r^2}\frac{i}{2\rho_0}(F-rG)e^{-in\psi}\right\rangle\ . \tag{4.4}$$

Since in the absence of noise we have $r_0=0$ [Eq. (3.2)], a nonvanishing contribution to r can only be due to the noise and is thus at least proportional to $\mathcal{D}$. In particular, we show in Appendix A that

$$\langle r^2\rangle\cong\frac{1}{8}\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}\ll 1\ . \tag{4.5}$$

We therefore neglect the contributions r^2 in Eq. (4.4) compared to unity, and the recurrence relation Eq. (4.4) reduces to

$$\frac{i}{n}\dot{c}_n=ac_n-\frac{b}{2i}c_{n-1}+\frac{b}{2i}c_{n+1}+\frac{i}{2\rho_0}(\langle Fe^{-in\psi}\rangle-\langle rGe^{-in\psi}\rangle)\ . \tag{4.6}$$

Since according to Eq. (4.1a) the phase ψ is driven by the Langevin forces F and rG, the Fourier component $e^{-in\psi}$ contains these noise sources as well. We therefore expect the averages $\langle Fe^{-in\psi}\rangle$ and $\langle rGe^{-in\psi}\rangle$ to be nonvanishing. The detailed calculations of Appendixes B and C yield

$$\langle rGe^{-in\psi}\rangle=\frac{\rho_0}{2}\mathcal{D}(c_{n+1}-c_{n-1})\ ,$$

$$\langle Fe^{-in\psi}\rangle=\rho_0\mathcal{D}(-2nc_n+nc_{n+1}+nc_{n-1})\ .$$

These results simplify Eq. (4.6) to the three-term recurrence relation

$$\dot{c}_n=-n\left\{(ia+n\mathcal{D})c_n-\left[\frac{b}{2}+\frac{\mathcal{D}}{4}+n\frac{\mathcal{D}}{2}\right]c_{n-1}+\left[\frac{b}{2}+\frac{\mathcal{D}}{4}-n\frac{\mathcal{D}}{2}\right]c_{n+1}\right\}\ . \tag{4.7}$$

In steady state ($\dot{c}_n\equiv 0$) Eq. (4.7) can be solved by the iteration

$$c_n=S_nc_{n-1}\ , \tag{4.8}$$

where S_n is the scalar continued fraction

$$S_n=\frac{b+\frac{\mathcal{D}}{2}+n\mathcal{D}}{2(ia+n\mathcal{D})+\left[b+\frac{\mathcal{D}}{2}-n\mathcal{D}\right]S_{n+1}} \tag{4.9}$$

and $c_0=1$ and $c_{-n}=c_n^*$ from Eq. (4.3).

Thus the approximate steady-state distribution $P_0=P_0(\psi)$ is given by Eqs. (4.2), (4.8), and (4.9) and is shown in Fig. 2 by the dashed line for $a/b=0.1$ and $2\mathcal{D}/b=0.1$. We emphasize the excellent agreement between the numerical simulation of the complete CEL equations and this approximate, analytical treatment.

V. DISCUSSION AND SUMMARY

In Sec. IV we have derived an approximate steady-state distribution for the phase difference ψ reproducing the characteristic features of the numerical simulation. Comparing this approximate solution and, in particular, the recurrence relation Eq. (4.7) to the corresponding one [Eq. (3.13)] derived in Ref. 1 by geometrical considerations, we recognize that the coupling coefficient b between the two CEL modes is replaced by

$$\tilde{b}\equiv b+\mathcal{D} . \tag{5.1}$$

This small noise-induced correction is due to the coupling of Eq. (4.1a) to Eqs. (4.1b) and (4.1c). However, aside from this modification, the equations have the same structure as those of Ref. 1. In particular, according to Eqs. (4.2) and (4.7) the distribution for ψ satisfies a Fokker-Planck type of equation

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\psi}\{[a-(\tilde{b}-\tfrac{1}{2}\mathcal{D})\sin\psi]P\}+2\mathcal{D}\frac{\partial^2}{\partial\psi^2}[\sin^2(\psi/2)P] , \tag{5.2}$$

which is identical to Eq. (1.2) [and (3.10) of Ref. 1] when b is replaced by $\tilde{b}$. Moreover, the Langevin equation for ψ corresponding to Eq. (5.2) is indeed Eq. (1.1), however, with the modified coupling coefficient $\tilde{b}$. Since in paper I (Ref. 1) we have already shown that Eq. (1.1) shows noise quenching, we thus see that the CEL noise quenching can be described by Eq. (1.1) and therefore by multiplicative noise. We conclude by noting that according to Eq. (4.5) noise quenching also occurs in the difference r of the amplitudes.

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APPENDIX A: QUENCHING OF FLUCTUATIONS IN DIFFERENCE OF AMPLITUDES

In this appendix we calculate the average $\langle r^2\rangle$ in the limit of small detunings, $|a/b|\ll 1$, and weak noise, $\mathcal{D}/b\ll 1$, and thus show that noise quenching also occurs in the difference r of the amplitudes of the two electric fields.

The solution of Eq. (4.1c) reads

$$r(t)=\frac{1}{4\rho_0}\int_0^t dt' F_r(t')\exp\left[-\int_{t'}^t dt''[b\cos\psi(t'')+\frac{1}{4\rho_0}F_R(t'')]\right] . \tag{A1}$$

For $|a/b|\ll 1$, that is, $\psi_0\cong a/b$ and $\mathcal{D}\ll b$ we find $\cos\psi(t'')\cong\cos(a/b)\cong 1$. When we neglect the noise source F_R in the exponent of Eq. (A1), we thus arrive at

$$r(t)\cong\frac{1}{4\rho_0}\int_0^t dt' F_r(t')e^{-b(t-t')} . \tag{A2}$$

Hence

$$\begin{aligned}\langle r^2(t)\rangle&=\frac{1}{16\rho_0^2}\int_0^t dt'\int_0^t dt''\langle F_r(t')F_r(t'')\rangle e^{-b(2t-t'-t'')}\\&=\frac{1}{4\rho_0^2}\int_0^t dt'(D_{11}-D_{21}e^{i(\psi-\Phi)}-D_{12}e^{-i(\psi-\Phi)}+D_{22})e^{-2b(t-t')}\\&=\frac{D}{\rho_0^2}\int_0^t dt'\sin^2\left[\frac{\psi(t')}{2}\right]e^{-2b(t-t')} ,\end{aligned}$$

where we have made use of Eqs. (2.4). Since $\sin^2[\psi(t')/2]\cong(a/2b)^2$, we can perform the integration in the limit $t\to\infty$ (steady state) to find

$$\langle r^2(t)\rangle\cong\tfrac{1}{8}\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b} .$$

APPENDIX B: THE AVERAGE $\langle rGe^{-in\psi}\rangle$

In this appendix we calculate the average $\langle rGe^{-in\psi}\rangle$. When we make use of Eqs. (2.11b), (2.11e), and (2.11f), we find

$$\langle rGe^{-in\psi}\rangle = \langle rF_1 e^{(i/2)(2\Psi+\psi-\Phi)}e^{-in\psi}\rangle - \langle rF_1^* e^{-(i/2)(2\Psi+\psi-\Phi)}e^{-in\psi}\rangle + \langle rF_2 e^{(i/2)(2\Psi-\psi+\Phi)}e^{-in\psi}\rangle - \langle rF_2^* e^{-(i/2)(2\Psi-\psi+\Phi)}e^{-in\psi}\rangle \ . \tag{B1}$$

Averages such as $\langle rF_1 e^{(i/2)(2\Psi+\psi-\Phi)}e^{-in\psi}\rangle$ can be obtained with the help of Eqs. (A2) and (2.11c), which yield

$$\begin{aligned}\langle rF_1 e^{(i/2)(2\Psi+\psi-\Phi)}e^{-in\psi}\rangle &= \frac{1}{4\rho_0}\int_0^t dt' e^{-b(t-t')}\langle F_r(t')F_1(t)e^{(i/2)[2\Psi(t)+\psi(t)-\Phi(t)]}e^{-in\psi(t)}\rangle \\ &= \frac{1}{4\rho_0}\int_0^t dt' e^{-b(t-t')}\langle F_1^*(t')F_1(t)\rangle \\ &\quad\times\langle \exp(-(i/2)\{2[\Psi(t')-\Psi(t)]+\psi(t')-\psi(t)-\Phi(t')+\Phi(t)\})e^{-in\psi(t)}\rangle \\ &\quad -\frac{1}{4\rho_0}\int_0^t dt' e^{-b(t-t')}\langle F_2^*(t')F_1(t)\rangle \\ &\quad\times\langle \exp(-(i/2)\{2[\Psi(t')-\Psi(t)]-\psi(t')-\psi(t)+\Phi(t')+\Phi(t)\})e^{-in\psi(t)}\rangle \\ &= \frac{D_{11}}{4\rho_0}c_n - \frac{D_{21}}{4\rho_0}e^{-i\Phi(t)}c_{n-1} = \frac{\rho_0}{4}\mathcal{D}(c_n - c_{n-1}) \ .\end{aligned} \tag{B2}$$

Here we have made use of Eqs. (2.3) and (2.4) and followed the convention

$$\int_0^t dt'\delta(t') = \tfrac{1}{2} \ . \tag{B3}$$

Since

$$\langle rF_1^* e^{-(i/2)(2\Psi+\psi-\Phi)}e^{-in\psi}\rangle = \langle rF_1 e^{(i/2)(2\Psi+\psi-\Phi)}e^{-i(-n)\psi}\rangle^* \ ,$$

we find from Eq. (B2)

$$\begin{aligned}\langle rF_1^* e^{-(i/2)(2\Psi+\psi-\Phi)}e^{-in\psi}\rangle &= \frac{\rho_0}{4}\mathcal{D}(c_{-n} - c_{-n-1})^* \\ &= \frac{\rho_0}{4}\mathcal{D}(c_n - c_{n+1}) \ .\end{aligned} \tag{B4}$$

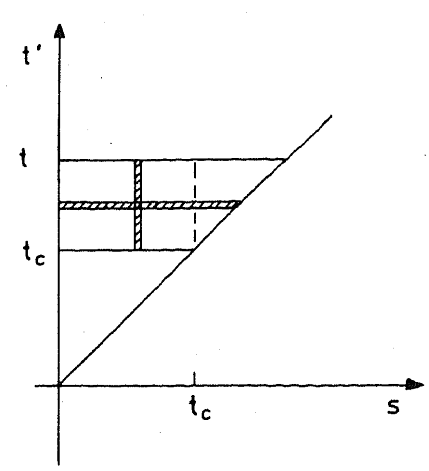

FIG. 3. Change of order of integration.

In the last step we have used $c_{-n}^* = c_n$ following from Eq. (4.3).

Similarly we arrive at

$$\langle rF_2 e^{(i/2)(2\Psi-\psi+\Phi)}e^{-in\psi}\rangle = \frac{\rho_0}{4}\mathcal{D}(c_{n+1} - c_n) \ , \tag{B5}$$

which yields

$$\begin{aligned}\langle rF_2^* e^{-(i/2)(2\Psi-\psi+\Phi)}e^{-in\psi}\rangle &= \langle rF_2 e^{(i/2)(2\Psi-\psi+\Phi)}e^{-i(-n)\psi}\rangle^* \\ &= \frac{\rho_0}{4}\mathcal{D}(c_{n-1} - c_n) \ .\end{aligned} \tag{B6}$$

Substituting Eqs. (B2), (B4), (B5), and (B6) into Eq. (B1) yields

$$\langle rGe^{-in\psi}\rangle = \frac{\rho_0}{2}\mathcal{D}(c_{n+1} - c_{n-1}) \ .$$

APPENDIX C: THE AVERAGE $\langle Fe^{-in\psi}\rangle$

In this appendix we calculate the average $\langle Fe^{-in\psi}\rangle$ neglecting terms quadratic in the diffusion constant. With the help of Eqs. (2.11a), (2.11e), and (2.11f) we find

$$\begin{aligned}\langle Fe^{-in\psi}\rangle &= \langle F_1 e^{(i/2)(2\Psi+\psi-\Phi)}e^{-in\psi}\rangle \\ &\quad - \langle F_1^* e^{-(i/2)(2\Psi+\psi-\Phi)}e^{-in\psi}\rangle \\ &\quad - \langle F_2 e^{(i/2)(2\Psi-\psi+\Phi)}e^{-in\psi}\rangle \\ &\quad + \langle F_2^* e^{-(i/2)(2\Psi-\psi+\Phi)}e^{-in\psi}\rangle \ .\end{aligned} \tag{C1}$$

The averages in Eq. (C1) can be performed using a method developed in Ref. 8. Introducing the earlier time $t_c \equiv t - \tau$ where $0 < \tau \to 0$, we find

$$\langle F_1 e^{(i/2)(2\Psi+\psi-\Phi)} e^{-in\psi}\rangle = \langle F_1(t) e^{(i/2)[2\Psi(t_c)+\psi(t_c)-\Phi(t_c)]} e^{-in\psi(t_c)}\rangle$$
$$+\left\langle F_1(t)\int_{t_c}^{t} dt' \frac{d}{dt'}\left(e^{(i/2)[2\Psi(t')+\psi(t')-\Phi(t')]} e^{-in\psi(t')}\right)\right\rangle$$
$$=\left\langle F_1(t)\int_{t_c}^{t} dt' e^{(i/2)[2\Psi(t')+\psi(t')-\Phi(t')]} e^{-in\psi(t')}\left[\frac{i}{2}(2\dot\Psi+\dot\psi-\dot\Phi)-in\dot\psi\right]\right\rangle . \tag{C2}$$

In the last step we have made use of the fact that the phases ψ, and Ψ at the earlier time t_c are uncorrelated to the noise F_1 at the later time t together with Eq. (2.2). When we neglect the terms r^2 we find from Eqs. (4.1a), (4.1b), and (2.5),

$$\frac{i}{2}(2\dot\Psi+\dot\psi-\dot\Phi)-in\dot\psi=\frac{i}{2}\Bigg|[2A+a(1-2n)-(\nu_1-\nu_2-\omega_0)-b(1-2n)\sin\psi]$$
$$+2rb\sin\psi+\frac{i}{2\rho_0}\{G+(1-2n)F-r[F+(1-2n)G]\}\Bigg| . \tag{C3}$$

After substituting this result back into Eq. (C2) we can perform the average. We start with the contributions in the first set of square brackets of Eq. (C3) consisting of terms independent of the stochastic variables and of phases $e^{\pm i\psi(t')}$ which can be combined with the phase factors already present in Eq. (C2). Due to the integration over t' all phases in Eq. (C2) are taken at a time prior to the time t of the noise F_1 and are thus uncorrelated. Since $\langle F_1\rangle=0$ [Eq. (2.2)] these times do not contribute and the terms in the bracket in Eq. (C3) do not contribute at all.

A somewhat similar argument applies to the term $rb\sin\psi$. According to Eq. (A2) r contains the noise sources F_1 and F_2 via F_r. Thus the average is of the form

$$\langle F_1(t)\int_{t_c}^{t} dt'\int_0^{t'} ds\, F_1^*(s)\rangle = 2D_{11}\int_{t_c}^{t} dt'\int_0^{t'} ds\,\delta(t-s)$$
$$=2D_{11}\left[(t-t_c)\int_0^{t_c} ds\,\delta(t-s)+\int_{t_c}^{t} ds(t-s)\delta(t-s)\right]$$
$$=2D_{11}\left[\int_0^{t_c} ds\int_{t_c}^{t} dt'\delta(t-s)+\int_{t_c}^{t} ds\int_s^{t} dt'\delta(t-s)\right] .$$

Here we have used Eq. (2.3) and changed the order of integration with the new limits of integration apparent from Fig. 3. Since $t_c\le t$ the first integral is nonvanishing and equal to $\frac{1}{2}$ only for $t_c=t$, however, in this case the prefactor vanishes. A similar argument holds for the second integral, and therefore

$$\left\langle F_1(t)\int_{t_c}^{t} dt'\int_0^{t'} ds\, F_1^*(s)\right\rangle=0 .$$

We now turn to the contributions from the curly brackets in Eq. (C3). Let us first consider the terms proportional to r. The resulting averages are typically of the form

$$\left\langle F_1(t)\int_{t_c}^{t} dt' F_1^*(t')r(t')e^{-in\psi(t')}\right\rangle = D_{11}\langle r(t)e^{-in\psi(t)}\rangle .$$

According to Eq. (4.1a) the phase ψ is driven by the noise r. Hence there is a correlation between r and $e^{-in\psi}$. Therefore, the average is nonzero and at least proportional to the diffusion constant D. Thus the averages due to the contributions rF and rG involve terms proportional to $(\mathcal{D}/b)^2\ll 1$, which we neglect.

We now turn to the last two terms in Eq. (C3) and Eq. (C2) reads

$$\langle F_1 e^{(i/2)(2\Psi+\psi-\Phi)} e^{-in\psi}\rangle = \frac{(-1)}{4\rho_0}\int_{t_c}^{t} dt'\langle F_1(t)[G(t')+F(t')-2nF(t')]e^{(i/2)[2\Psi(t')+\psi(t')-\Phi(t')]} e^{-in\psi(t')}\rangle . \tag{C4}$$

When we substitute Eqs. (2.11a) and (2.11b) into the above result and perform the averages with the help of Eq. (2.3) and the convention (B3), we arrive at

$$\langle F_1 e^{(i/2)(2\Psi+\psi-\Phi)} e^{-in\psi}\rangle = \frac{\rho_0}{2}\mathcal{D}(c_n-nc_n+nc_{n-1}) . \tag{C5}$$

From this result we immediately find

$$\langle F_1^* e^{-(i/2)(2\Psi+\psi-\Phi)} e^{-in\psi}\rangle = \frac{\rho_0}{2}\mathcal{D}(c_n+nc_n-nc_{n+1}) . \tag{C6}$$

The average $\langle F_2 e^{(i/2)(2\Psi-\psi+\Phi)} e^{-in\psi}\rangle$ can be calculated in an analogous way yielding

$$\langle F_2 e^{(i/2)(2\Psi-\psi+\Phi)} e^{-in\psi}\rangle = \frac{(-1)}{4\rho_0}\int_{t_c}^{t} dt'\langle F_2(t)[G(t')-F(t')-2nF(t')]e^{(i/2)[2\Psi(t')-\psi(t')+\Phi(t')]} e^{-in\psi(t')}\rangle ,$$

and thus

$$\langle F_2 e^{(i/2)(2\Psi-\psi+\Phi)} e^{-in\psi}\rangle = \frac{\rho_0}{2}\mathcal{D}(c_n + nc_n - nc_{n+1}) \tag{C7}$$

and

$$\langle F_2^* e^{-(i/2)(2\Psi-\psi+\Phi)} e^{-in\psi}\rangle = \frac{\rho_0}{2}\mathcal{D}(c_n - nc_n + nc_{n-1}) \ . \tag{C8}$$

When we substitute Eqs. (C5), (C6), (C7), and (C8) back into (C1) we arrive at

$$\langle F e^{-in\psi}\rangle = \rho_0 \mathcal{D}(-2nc_n + nc_{n+1} + nc_{n-1}) \ .$$

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[13]Oriented towards application in gravitational wave detection (Ref. 4) or ring laser gyroscopes (Ref. 2) we confine ourselves to small detunings $|a|/b \ll 1$, which yields $|\psi| \cong |\arcsin(a/b+\Delta)| \cong |a/b+\Delta| \ll 1$, where $|\Delta| \ll 1$ includes the correction due to the spontaneous-emission noise. Thus $0 = 2\alpha\cos^2(\psi/2) - \gamma \cong 2\alpha - \gamma$, that is, $\gamma \cong 2\alpha$.