Wild quotient singularities of arithmetic surfaces and their regular models

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Chapter 1

Introduction

1.1 Introduction

Resolution of singularities is one of the oldest and yet still one of the most intriguing topics in mathematics. Ever since Isaac Newton's calculations on Puiseux series expansions of curves, it has been of major interest to describe or remove singularities on varieties. The first rigorous proof for desingularization of surfaces over the complex numbers was given 1934 by Walker [Wal34]. A few years later, Zariski gave proofs for desingularization of surfaces [Zar39] and three-folds [Zar44] over arbitrary fields of characteristic zero. In 1964, Hironaka gave his famous proof for reduced schemes of finite type over fields of characteristic zero [Hir64], Abhyankar presented his result for resolution over fields of characteristic seven or greater in 1966 [Abh66]. In 1978, Lipman proved desingularization for quasi-excellent surfaces which also include arithmetic surfaces [Lip78]. For the remaining cases, desingularization is still an open problem despite many efforts; however, Grothendieck has proved that quasi-excellency is a necessary condition for a locally Noetherian scheme to have a desingularization [Gro65, 7.9]. By Lipman's result, we know that it is also a sufficient condition for dimension two and less.

This thesis is motivated by the study of quotient singularities, occurring on the scheme of orbits by a finite group action on a regular scheme, the so-called quotient scheme. Quotient singularities are an important class of singularities in any characteristic which canonically arise in context of actions on schemes. The problem of quotient singularities was first studied by Jung [Jun08] and Hirzebruch [Hir53] for a cyclic group acting on a regular complex surface. Here the quotient surface exhibits the famous Hirzebruch-Jung singularities; their resolution combinatorics can be related directly to the action of the group G. General groups acting on complex varieties were later examined by Cartan [Car57] and Brieskorn [Bri68].

When one generalizes the problem to arbitrary (locally Noetherian quasi-excellent) schemes, one has to note that the problem of quotient singularities is essentially of local nature, so it is merely a question in commutative algebra. One can reduce to the following situation:

Let B a Noetherian local regular ring, and $G \hookrightarrow \operatorname{Aut} B$ a finite group action on B. Denote by k the residue field of B, by $p = \operatorname{char} k$ its characteristic. The ring of invariants $A = B^G$ is the subring of B consisting exactly of the elements invariant under G and it can happen that A is not regular. Here B plays the role of a local germ of the scheme in consideration, taken at a closed point; and A

is the corresponding germ on the quotient scheme. The main goal in the theory of quotient singularities is to relate the structure of A, e.g. the combinatorics of its minimal desingularization, to the group action of G on B.

The case where p is coprime to $\#G$, the so-called *tame* case, has been extensively studied in literature. In this context, the Hirzebruch-Jung singularities over the complex numbers can be generalized with minor efforts to the case of a cyclic group acting on varieties over arbitrary fields of characteristic zero. The same can be done for arbitrary group actions. There have been also some ad-hoc adaptations of these results to more general B mostly in dimension 2, e.g. the case of arithmetic surfaces in [Vie77]. Moreover, in the tame case, a celebrated theorem of Serre [Ser68] classifies exactly when the ring A is regular in terms of G, namely if and only if G is generated by so-called pseudo-reflections. There seems to be consensus that the tame case is relatively well understood, and that those results can be generalized to arbitrary rings in the tame case. However, as of 2010, there is no standard reference in literature which goes beyond particular applications.

The case where p divides $\#G$ is called the *wild* case, since the classical methods from the tame case fail by elementary means. To the author's knowledge, the only results concern the simplest non-trivial case where $G = \mathbb{Z}/p\mathbb{Z}$, and B is regular of very specific form: M. Artin [Art75] has obtained a few results for $p = 2$ and $B = k[[X_1, X_2]]$. In Peskin's thesis, [Pes83] several basic results about the wild case are collected and the results of Artin are generalized for specific group actions with $p \geq 3$.

There seems to have been little progress after that until Lorenzini's unpublished paper [Lor06], in which some structural results on quotient singularities with focus on quotients of stable models of curves by prime cyclic actions are obtained. The biggest part of those results uses the global machinery of Néron models and many ad-hoc combinatorial constructions which unfortunately gives no insight on how the G-action relates to the structure of the singularities. However, in the context of the local results of Artin these results suggest that it might be possible to obtain similar structural results only with local methods.

The goal of this thesis is to understand the relation between B and A in terms of the group action G on B, in the simplest non-trivial case where $G = \langle\sigma\rangle$ is prime cyclic.

In chapter 2, we collect and extend classical results on tame quotient singularities and discuss them in the context of toric geometry.

In chapter 3, we prove an algebraic result about the invariant morphism. We prove that B is a monogenous A-algebra if and only if the augmentation ideal

$$I_G = \{(\sigma - \mathrm{id})b \; ; \; b \in B\}$$

of B is principal. If in particular B is regular, this implies that A is also regular.

In chapter 4, we assume that B is a local regular normal crossings germ of an arithmetic surface over the spectrum of a complete discrete valuation ring R. Using birational geometry, combinatorial methods and the results from chapter 3, we relate the structure of the minimal normal crossings desingularization of A to the group action of G on B. Furthermore, we examine the behavior of the singularity of A with respect to tame base extension.

A more detailed overview on the content and the utilized methods can be found at the beginning of each chapter.

1.2 About notation

In this thesis we will try to use common symbols and notations. However, we want to point out several things which might be sources of confusion since the corresponding notation is not uniform worldwide. At most occasions, this will be also said in the text.

For a ring C, we will denote by $Q(C)$ its field of fractions. If C is local, then $\widehat{C}$ will denote the completion of C by the topology given by its maximal ideal.

For a scheme S and a closed point s on S, we will denote by $K(S)$ the function field of S, and by $k(s)$ the residue field of S at s. The structure sheaf of S will be denoted by $\mathcal{O}_S$.

Isomorphies will be denoted by $\cong$, congruences by $\equiv$.

The empty set will be denoted by $\varnothing$.

The symbol ∂ will always be used to denote boundaries. Partial derivatives do not occur in this thesis.

The general linear group of dimension n over a field k will be denoted by $\mathrm{GL}_n(k)$. Similarly, its special subgroup will be denoted $\mathrm{SL}_n(k)$.

The natural numbers will be denoted by $\mathbb{N}$ and contain zero. By $\mathbb{Q}_{\geq 0}$ we will denote the nonnegative rational numbers.

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Chapter 2

Tame quotient singularities

2.1 Overview

In this chapter we will summarize some classical facts on invariant rings. We will consider a Noetherian local normal ring B with finite group $G \hookrightarrow \operatorname{Aut} B$ acting on it, and try to understand the structure of the invariant ring $A = B^G$.

In section 2.2, *Basic facts on invariant rings*, we introduce the setting for this chapter in detail and make some basic definitions. In section 2.3, *Excellent rings*, we state some classical results about excellent rings, which will allow us under certain conditions to reduce to the case where B and A are complete.

In section 2.4, *Tame local actions*, we will review classical results in the case where the action of G on B is *tame*, i.e. when $\#G$ is coprime to the residue characteristic of B. We begin with classical linearization results for general tame group actions, utlizing the original argument of Cartan [Car57]. Then we concentrate on cyclic actions and explicitly describe the structure of the ring of invariants A, cf. Propositions 2.4.12 and 2.4.10. We will also relate those results to Serre's theorem on pseudo-reflections and regular rings, cf. Corollary 2.4.17.

The next objective will be to obtain the desingularization in the case where B is regular. If B is of dimension 2, this will lead us for example to the classical Hirzebruch-Jung singularities as in [Jun08] and [Hir53]. Instead of doing the calculations explicitly, we will introduce the notion of toric schemes and rings to avoid lengthy calculations obscuring the structural intuition, since it turns out that tame cyclic quotient singularities are toric. In section 2.5, *Toric schemes*, we will briefly introduce toric geometry using the book of Fulton [Ful93] and formulate our problem in this setting. As done already frequently by several authors, we will work over an arbitrary base field instead of $\mathbb{C}$.

In section 2.6, *Extension to mixed characteristic*, we extend the classic toric geometry over fields to arbitrary characteristic in the vein of Mumford. However, we have to broaden Mumford's setting in [KKMSD70, Chapter IV, §3] a bit; this allows us for example to also treat rings like $R[[X_1, X_2]]/(X_1X_2 - \pi)$; for this we will define the concept of a *locally toric ring* 2.6.2, which is the local equivalent of Kato's concept of log-regularity, cf. [Kat94, §3]. We refer the reader to the beginning of this section for a more detailed discussion.

In section 2.7, *Desingularization*, we will use the toric theory to desingularize the tame cyclic quotient singularities. We will do this by describing a method to desingularize toric rings in general. We will calculate the Hirzebruch-Jung resolution of the tame cyclic quotient singularities explicitly to illustrate the method. The example is divided in several steps which utilize toric geometry and establish the connection to the classical resolution by Hirzebruch-Jung fraction expansion.

In section 2.8, *A characterization of toric surface singularities*, we will refine several known results to characterize the simplest kind of surface singularities, the toric surface singularities, in terms of their desingularization. This will justify our definition of locally toric rings, since it will turn out that those are exactly the local complete rings whose resolution is chain-like, cf. Theorem 2.8.1.

2.2 Basic facts on invariant rings

In this section, we fix the setting for the next sections, introduce some fundamental definitions and prove some facts about invariant rings.

Let B be a local Noetherian normal ring with maximal ideal $\mathfrak{m}_B$ and residue field k_B. Let $G \hookrightarrow \operatorname{Aut} B$ be a finite group acting on B; we call this a local group action on B. If the cardinality of G is coprime to $p = \operatorname{char} k_B$, then the action of G on B is called *tame*, otherwise *wild*.

As usual, we will denote by

$$B^G = \{b \in B \mid \sigma(b) = b \text{ for all } \sigma \in G\}$$

the *ring of invariants* of B. We will denote it by $A = B^G$. We will make the assumption that A is also Noetherian - by [MS81], this is a non-trivial assumption which excludes some pathological cases. We will denote by $\mathfrak{m}_A$ the maximal ideal of A and by k_A its residue field.

The following fact is classic:

Proposition 2.2.1. *B is finite over A and A is again a normal ring.*

Proof. The finiteness of A follows directly from the fact that B and A are Noetherian, see e.g. [AM69, Proposition 6.2].

We will now demonstrate how to prove that A is again normal. It suffices to prove that A contains any $a \in Q(A)$ which is integral over A. Now since B is normal, we have $a \in B$, since a fulfills an integral equation over A and thus over B. On the other hand, a is invariant under G since $a \in Q(A)$, so we have in fact $a \in A$. □

Definition 2.2.2. If B is regular and A is not, then A is called a *quotient singularity.* If A is regular and B is not, then B is called a *normalization singularity.* The singularity is called *tame* resp. *wild* if G acts tamely resp. wildly.

2.3 Excellent rings

In this section, we cite some partially well-known results about excellent rings which we can apply to descend results about complete rings to local rings. Those can be found in greater detail e.g. in [Liu02, 8.2] or Grothendieck's original works. We keep the notations of section 2.2.

Definition 2.3.1. Let C be a local ring with maximal ideal $\mathfrak{m}_C$. We denote the *completion* of C at its maximal ideal by $\widehat{C}$.

Definition 2.3.2. A local Noetherian ring C with residue field k is called *excellent* if C is universally catenary and $\widehat{C} \otimes_C k'$ is regular for any finitely generated extension k' of k.

On of the main good properties of excellent rings is that completion of excellent rings commutes with normalization, see [Liu02, 8.2.41(a)]:

Proposition 2.3.3. *An excellent local Noetherian ring C is normal (resp. reduced) if and only $\widehat{C}$ is normal (resp. reduced).*

We will exhibit further good properties of excellent rings with respect to invariant group actions; they have already been pointed out by Lorenzini in his recent unpublished paper [Lor06, 2.1,2.2] for the particular case of regular rings in dimension 2 using the same techniques:

Lemma 2.3.4. *$(\widehat{B})^G$ is integral over $\widehat{A}$. Moreover, if $\#G$ is coprime to* $\operatorname{char} B$, *then $(\widehat{B})^G$ and $\widehat{A}$ have same field of fractions.*

Proof. Clearly $\widehat{A} \subseteq \widehat{B}^G$. Now let $x \in \widehat{B}^G$. Then the G-Norm $N(x) = x^{\#G}$ belongs to $\widehat{A}$, since its $\mathfrak{m}_B$-filtration has coefficients in $\mathfrak{m}_A$. Thus the polynomial

$$(T^{\#G} - N(x))$$

is a monic polynomial in T over $\widehat{A}$ which has $T = x$ as a zero. This proves that $(\widehat{B})^G$ is integral over $\widehat{A}$.

It remains to prove that $\widehat{A}$ and $\widehat{B}^G$ have same field of fractions if $\operatorname{char} B$ is coprime to $\#G$. Now note that to for $x_1, x_2 \in \widehat{B}^G$, the G-traces $\operatorname{Tr}(x_i) = \#G \cdot x_i$ lie in $\widehat{A}$ by the same reasoning, and are nonzero. Thus $\operatorname{Tr}(x_1)/\operatorname{Tr}(x_2) = x_1/x_2$ lies in $Q(\widehat{A})$. This proves the claim. □

Proposition 2.3.5. *If B and A are excellent rings with $\operatorname{char} B$ coprime to $\#G$, then completion and formation of invariants commute, i.e. $(\widehat{B})^G = \widehat{A}$.*

Proof. The rings $\widehat{B}$ and $\widehat{A}$ are normal by Proposition 2.3.3, because B and A are normal by assumption resp. by Proposition 2.2.1, and excellent by assumption. And $\widehat{B}^G$ is normal by Proposition 2.2.1 because $\widehat{B}$ is normal and G is finite. Now both rings coincide by uniqueness of the normalization since $\widehat{B}^G$ has same field of fractions as $\widehat{A}$ by Lemma 2.3.4. □

Finally, if G acts tamely, then completion always commutes with taking the invariants, even if the underlying rings are not excellent; this is probably also due to Lorenzini:

Proposition 2.3.6. *If G acts tamely, then $(\widehat{B})^G = \widehat{A}$.*

Proof. As in the proof of 2.3.4, we see that for $x \in \widehat{B}^G$, we have that $\operatorname{Tr}(x) = \#G \cdot x \in \widehat{A}$. But since G acts tamely, $\#G$ is invertible in $\widehat{A}$ and thus $x \in \widehat{A}$ which implies that both rings coincide. □

2.4 Tame local actions

In this section, we generalize a classical result on rings of invariants in characteristic zero to arbitrary characteristic. However, this is probably common knowledge. Also, we will be mainly interested in invariants by cyclic Galois groups. We keep the notations of the previous section and furthermore assume that B and A are *complete*. We can assume this without loss of generality by Proposition 2.3.6; in fact we can descend all structural results on the Galois morphism to the non complete case, since the G-action is tame.

Since A is complete, we can use Cohen's structure theorem [Coh46] to describe the structure of A. The theorem states that there exists a complete local ring R_A embedding into A having the same residue field as A, minimal with this property. This ring R_A is called the coefficient ring of A. If A is of equal characteristic, then $R_A = k_A$. If A is of mixed characteristic, then R_A is a complete discrete valuation ring. By the universal property of Witt vectors, it is isomorphic to the Witt ring $R_A = W(k_A)$ over k_A, if k_A is perfect. Analogously, we will denote the coefficient ring of B by R_B; similarly, we have $R_B = W(k_B)$, if k_B is perfect. For the convenience of the reader, we re-state the accumulated assumptions and notations for this chapter:

Notations 2.4.1.

B	complete local Noetherian normal ring
G	finite group acting on B
$A = B^G$	invariant ring, assumed Noetherian
$\mathfrak{m}_B, \mathfrak{m}_A$	maximal ideals of B, A
k_B, k_A	residue fields of B, A
$p = \operatorname{char} k_B = \operatorname{char} k_A$	residue characteristic of B, A
R_B, R_A	coefficient rings of B, A

We will now analyze the tame actions on B. We will see that they can be described by matrix actions over the basis R_A. The following lemma relates those actions in the equicharacteristic case to the ones of mixed characteristic:

Lemma 2.4.2. *For $n \in \mathbb{N}$, every finite subgroup of $\mathrm{GL}_n(k_A)$ of order prime to p admits a lift to a subgroup of $\mathrm{GL}_n(R_A)$.*

Proof. One has to only prove this if R_A is of mixed characteristic, since in the equal characteristic case, the isomorphy $R_A \cong k_A$ induces an embedding of $\mathrm{GL}_n(k_A)$ into $\mathrm{GL}_n(R_A)$. The existence of a lift follows from noncommutative group cohomology, see the first part of the proof of [Edi92, Lemma 3.3] where the vanishing of the $H^i(G, K_{m+1})$ is proved. □

The following statement is a generalization of classical theorems about quotient singularities, probably dating back to Cartan [Car57] and Brieskorn [Bri68]. The proof in this general setting is very similar to the arguments in [Edi92, Lemma 3.3].

Proposition 2.4.3. *Assume that G acts tamely. Let $T = \mathfrak{m}_B/\mathfrak{m}_B^2$ be the tangent space, which can be regarded as vector space over k_B or k_A; let $n = \dim_{k_B} T$, and respectively let $m = \dim_{k_A} T = n \cdot [k_B : k_A]$. Then:*

(i) *If $n \geq 1$, then the map*

$$\varepsilon : G \longrightarrow \mathrm{GL}_m(k_A) = \mathrm{GL}_{k_A}(T) \; ; \; \sigma \mapsto \sigma\,|_T,$$

where we consider T *as* k_A *-vector space, is injective.*

(ii) *If* $k_B = k_A$, *the group* G *is commutative, and a lift*

$$\gamma : G \longrightarrow \mathrm{GL}_n(R_A)$$

of ε *is fixed, then there exists a system* $(y_1, \dots, y_n)$ *of generators of* $\mathfrak{m}_B$ *on which* G *acts linearly in the following way:*

$$\sigma(y) = \gamma(\sigma) \cdot (y_1, \dots, y_n)^t \quad \textit{for any} \quad \sigma \in G$$

Proof. Note that the above defined group homomorphism

$$\varepsilon : G \longrightarrow \mathrm{GL}_m(k_A) \; ; \; \sigma \mapsto \sigma \mid_T$$

is canonically induced by restricting σ to the k_A -vector space $T = \mathfrak{m}_B/\mathfrak{m}_B^2$. Since $n \geq 1$, the tangent space T is nonzero.

First we prove (i), i.e. that ε is injective. Let $\tau \in G$ with $\varepsilon(\tau) = \mathrm{id}$. For injectivity, it suffices to prove that $\tau = \mathrm{id}$. First we show that there exists a system of generators of $\mathfrak{m}_B$ (as B -module) on which τ acts trivially. So let $(x_1, \dots, x_n)$ be an arbitrary system of generators of $\mathfrak{m}_B$. Due to the fact that $\varepsilon(\tau) = \mathrm{id}$, we have that

$$\tau(x_i) = x_i \mod \mathfrak{m}_B^2.$$

Let U be the cyclic subgroup of G generated by τ. Note that U acts tamely on B, since G does. Define $u := \#U$; note that u is a unit in the complete local ring R_B. Now consider the τ -traces of the x_i defined by

$$z_i := \frac{1}{u} \sum_{j=0}^{u-1} \tau^j(x_i).$$

Moreover, we have that $z_i \equiv x_i \mod \mathfrak{m}_B^2$ by our assumptions on τ. Thus the system $(z_1, \dots, z_n)$ is also a regular system of generators of $\mathfrak{m}_B$, and the images of the $z_1, \dots, z_n$ form a k_B -basis of T . One now calculates that $\tau(z_i) = z_i$. If $k_B = k_A$, we are done showing the injectivity of ε , since B is then generated over A by the z_i as power series algebra, thus $\tau = \mathrm{id}$, which we wanted to prove.

If this is not the case, i.e. $k_B \neq k_A$, we still have to prove that τ acts trivially on k_B. Then we are also done, since B is generated as power series algebra over R_B by the z_i , and k_B is separable over k_A . But τ acts indeed trivially on k_B , since

$$T = \bigoplus_{i=1}^{n} z_i \cdot k_B;$$

if now τ acted non-trivially on k_B, it would also act non-trivially on any of the direct factors of T and thus on T itself, since $n \geq 1$.

We now prove (ii). For this, we will construct the generating system $(y_1, \dots, y_n)$ of $\mathfrak{m}_B$. We start again with an arbitrary generating system $(x_1, \dots, x_n)$ of $\mathfrak{m}_B$. Since there exists a lift

$$\gamma : G \longrightarrow \mathrm{GL}_n(R_A)$$

of ε , we can define

$$y := (y_1, \dots, y_n)^t := \frac{1}{\#G} \sum_{\sigma \in G} \gamma(\sigma)^{-1} \cdot (\sigma(x_1), \dots, \sigma(x_n))^t.$$

Note that this is indeed a generating system, since $y \equiv x \mod \mathfrak{m}_B^2$. It is also an easy calculation to verify that G acts on the y_i in the claimed way. We will verify this explicitly now, writing $x = (x_1, \ldots, x_n)^t$. Let $\tau \in G$. Then

$$\begin{aligned}\tau(y) = \tau\left(\frac{1}{\#G}\sum_{\sigma\in G}\gamma(\sigma)^{-1}\cdot\sigma(x)\right) &= \frac{1}{\#G}\sum_{\sigma\in G}\gamma(\sigma)^{-1}\cdot\tau(\sigma(x)) \\ = \left(\frac{1}{\#G}\sum_{\sigma\in G}\gamma(\tau^{-1}\sigma)^{-1}\cdot\sigma(x)\right) &= \frac{1}{\#G}\sum_{\sigma\in G}\gamma(\sigma)^{-1}\cdot\gamma(\tau)\cdot\sigma(x) = \gamma(\tau)\cdot y\end{aligned}$$

where the last equality follows since G is commutative. □

Remark 2.4.4. If G is not commutative, the last equation is not true anymore. However, if one writes

$$r := \frac{1}{\#G}\sum_{\sigma\in G}\gamma(\sigma)^{-1}\cdot\sigma$$

for the so-called Reynolds operator which we have encountered implicitly, then one still has the equation

$$\tau \circ r = r \circ \gamma(\tau).$$

It would be interesting to know if the proposition still holds for all noncommutative G (obviously, it holds for actions which are already given by multiplication with $\tau(x) = \gamma(\tau)\cdot y$).

Proposition 2.4.3 can be stated also as a statement about graded algebrae:

Definition 2.4.5. Let C be a local ring with maximal ideal $\mathfrak{m}_C$. Then the *graded associated algebra* to C is the $C/\mathfrak{m}_C$-algebra

$$\operatorname{gr} C = \bigoplus_{n=0}^{\infty}\mathfrak{m}_C^i/\mathfrak{m}_C^{i+1}$$

where multiplication is canonically given by the $\mathfrak{m}_C$-grading.

Remark 2.4.6. Note that we have a canonical morphism of associated graded algebrae

$$\operatorname{gr} A \longrightarrow (\operatorname{gr} B)^G.$$

This is seen as follows: The mapping $A \hookrightarrow B$ is injective, so we have an injective morphism of graded algebrae $\operatorname{gr} A \hookrightarrow \operatorname{gr} B$. Moreover, G acts on $\operatorname{gr} B$ since G acts on $\mathfrak{m}_B^i$ and thus on the direct factor $\mathfrak{m}_B^i/\mathfrak{m}_B^{i+1}$ for any $i \in \mathbb{N}$. Since the elements of $\operatorname{gr} A$ are G-invariant, the morphism factors canonically as

$$\operatorname{gr} A \hookrightarrow (\operatorname{gr} B)^G \hookrightarrow \operatorname{gr} B.$$

The following Proposition can be proved similarly as Proposition 2.4.3 and can be found in [Bou68, Chap. V, Ex.7] in the case $k_B = k_A$:

Proposition 2.4.7. *If G acts tamely, then the canonical morphism of associated graded algebrae*

$$\operatorname{gr} A \longrightarrow (\operatorname{gr} B)^G$$

is an isomorphism.

Proof. The statement is trivial if $n = 0$, so we can assume $n \geq 1$. Since the morphism $\operatorname{gr} A \to (\operatorname{gr} B)^G$ is injective, as we have discussed in Remark 2.4.6, it suffices to prove that it is also surjective. So suppose that $\bar{b}$ is a G-invariant element in $\mathfrak{m}_B^i/\mathfrak{m}_B^{i+1}$. Now there exists a

$$b \in \mathfrak{m}_B^i \quad \text{such that} \quad \bar{b} = b \mod \mathfrak{m}_B^{i+1}.$$

We define an invariant element lifting $\bar{b}$ by

$$a = \frac{1}{\#G} \sum_{\sigma \in G} \sigma(b).$$

Since G acts tamely, we have that $\#G$ is invertible in B, and since G acts trivially on $\bar{b}$, the so-constructed a is an element of B in $\mathfrak{m}_B^i$ lifting $\bar{b}$. Also by construction, we have that $a \in A$. Thus there exists a $j \in \mathbb{N}$ such that the image $\bar{a}$ of a in $\mathfrak{m}_A^j/\mathfrak{m}_A^{j+1}$ is nonzero. But since $\operatorname{gr} A \to (\operatorname{gr} B)^G$ is injective, we must have that it maps $\bar{a}$ to $\bar{b}$. Since $\bar{b}$ was arbitrary, this completes the proof. □

The Propositions 2.4.3 and 2.4.7 are not true anymore if one omits the hypothesis that G acts tamely. For a counterexample, see Remark 2.7.13.

Furthermore, Proposition 2.4.3 implies that if G acts tamely, it injects canonically into the k_A-automorphism group of $T = \mathfrak{m}_B/\mathfrak{m}_B^2$. In particular, it has always to be isomorphic to a finite subgroup of $\mathrm{GL}_m(k_A)$ of tame order, where $m = \dim_{k_A} T$. One can say even a bit more: Consider the determinant map

$$\det : \mathrm{GL}_m(k_A) \longrightarrow \mathrm{GL}_1(k_A) = k_A^\times.$$

The group of unity roots μ_A is a subgroup of $k_A^\times$, its preimage in $\mathrm{GL}_m(k_A)$ is normal as well. Since G is of finite order, it has to inject into this preimage.

For fixed n, one can classify all finite subgroups of $\mathrm{GL}_n(k_A)$ by representation theory; for $n = 2$ and $k_B = k_A = \mathbb{C}$ see e.g. [Bri68, §2]; one can also consider the injective homomorphism

$$\mathrm{GL}_n(k_A) \longrightarrow \mathrm{SL}_{n+1}(k_A)\ ;\ M \mapsto \mathrm{Diag}(M, \det M^{-1})$$

and examine $\mathrm{SL}_n(k_A)$. Unfortunately, for any finite group G there exists an N such that G is a subgroup of SL_N resp. GL_N. There are also some properties of the invariant ring A which correspond to properties of the subgroup of GL_N considered, see for example the introduction chapter of [Blu07]. However, in the following, we will only concentrate on the case where G is cyclic, since we will later focus on the wild cyclic case (which is difficult enough).

In the following, we will examine in more detail the case where

$$G = \langle \sigma \rangle \cong \mathbb{Z}/q\mathbb{Z}$$

is cyclic.

In the cyclic case, the previous proposition admits a slight refinement if R_A contains a primitive q-th root of unity:

Proposition 2.4.8. *If $G \cong \mathbb{Z}/q\mathbb{Z}$ acts tamely and R_A contains a primitive q-th root of unity ζ_q, there exists a system $(y_1, \dots, y_n)$ of generators of $\mathfrak{m}_B$ on which G acts in the following way:*

$$\sigma(y_i) = \zeta_q^{a_i} y_i \quad \textit{with } a_i \in \mathbb{Z}.$$

Proof. Let σ be a generator of G ; note that σ acts by restriction on the m-dimensional k_A-vector space $T = \mathfrak{m}_B/\mathfrak{m}_B^2$, where $m = n \cdot [k_B : k_A]$. We will again construct the system $(y_1, \dots, y_n)$. We first show that one can diagonalize the induced action $\overline{\sigma}$ on T . Since $\overline{\sigma}^q = \mathrm{id}$, the minimal polynomial $m(\overline{\sigma}, X)$ divides $X^q - 1$. Since G acts tamely, k_A contains ζ_q and the polynomial

$$m(\overline{\sigma}^q, X^q) = X^q - 1 = \prod_{j=1}^{q}(X - \zeta_q^j),$$

splits. Hence $\overline{\sigma}$ is also diagonalizable as k_A-linear map.

So for a suitable choice of a k_A-basis $(b_1, \dots, b_m)$ of T , the action of $\overline{\sigma}$ is given by

$$\overline{\sigma}(b_i) = \zeta_q^{m_i} b_i \quad \text{with } m_i \in \mathbb{N}.$$

In particular, by taking some lifts z_i of b_i modulo $\mathfrak{m}_B^2$, one has

$$\sigma(z_i) \equiv \zeta_q^{a_i} z_i \mod \mathfrak{m}_B^2.$$

Setting

$$x_i := \frac{1}{q}\sum_{j=1}^{q} \zeta_q^{-ja_j}\sigma^j(z_i),$$

the x_i fulfill the equations

$$\sigma(x_i) = \zeta_q^{a_i} x_i \quad \text{and} \quad x_i \equiv z_i \mod \mathfrak{m}_B^2$$

and hence, $(x_1, \dots, x_m)$ give rise to a k_A-basis of T . Then one can select a k_B-basis $(y_1, \dots, y_n) = (x_{i_1}, \dots, x_{i_n})$ of T . The latter is the generating system of the B-ideal $\mathfrak{m}_B$ we were looking for. □

Remark 2.4.9. In particular, Proposition 2.4.8 implies that there exists a system $(b_1, \dots, b_\ell)$ generating B as A-algebra on which G acts as

$$\sigma(b_i) = \zeta_q^{a_i} b_i \quad \text{with } a_i \in \mathbb{Z}.$$

Namely since k_B is cyclic over k_A , the extension of coefficient rings is of the form $R_B = R_A[\alpha]$ with $\alpha^q \in R_A$, and we can then take $(b_1, \dots, b_\ell)$ as the set $(\alpha, y_1, \dots, y_n)$.

The following propositions tells what happens if the extension B/A is purely residual:

Proposition 2.4.10. *Assume that* $G \cong \mathbb{Z}/q\mathbb{Z}$ *acts tamely and that* R_A *contains a primitive* q*-th root of unity* ζ_q *. Assume that* $[k_B : k_A] = q$*. Then the extension* B/A *is induced by extension of coefficients, i.e. in the equicharacteristic case, we have* $B = k_B \otimes_{k_A} A$*; else we have* $B = R_B \otimes_{R_A} A$*.*

Proof. Let $G = \langle \sigma \rangle$. Due to Proposition 2.4.8, there exists a system $(y_1, \dots, y_n)$ of generators of $\mathfrak{m}_B$ such that

$$\sigma(y_i) = \zeta_q^{a_i} y_i \quad \text{with } a_i \in \mathbb{Z}.$$

Since the action of G is tame, there exists a Kummer element α with $R_B = R_A[\alpha]$ and $\alpha^q \in R$ such that $\sigma(\alpha) = \zeta_q \alpha$.

In particular, $(\alpha^{-a_1} y_1, \dots, \alpha^{-a_n} y_n)$ is again a generating system of $\mathfrak{m}_B$, which is also stable under G . Thus the extension B/A is induced by extension of coefficients. □

Now we will give an explicit result on the structure of quotient singularities by tame cyclic group actions. For this, we introduce an abbreviating multi-index notation:

Notations 2.4.11. Let $y = (y_1, \dots, y_n)^t$ be a vector of variables and $m = (m_1, \dots, m_n)^t \in \mathbb{Z}^n$ a vector of integers. Then we will write in multi-index notation

$$y^m := \prod_{i=1}^{n} y_i^{m_i}.$$

For two vectors $m, \tilde{m} \in \mathbb{Z}^n$ we will write $m \le \tilde{m}$ if $m_i \le \tilde{m}_i$ for all $1 \le i \le n$; we will denote by $m \cdot \tilde{m}$ the ordinary scalar product of m and $\tilde{m}$. Also, we will write $1 := (1, \dots, 1)^t$ and $0 := (0, \dots, 0)^t$.

The following proposition gives an exact structure statement about the invariant ring. Note that despite the fact that this Proposition also holds in the wild case, Proposition 2.4.8 does not; see e.g. example 3.1.4. This marks one of the main difficulties in the wild case.

Proposition 2.4.12. *Assume that $G \cong \mathbb{Z}/q\mathbb{Z}$ and that R_A contains a primitive q-th root of unity ζ_q. Assume that $k_B = k_A$. Let $(y_1, \dots, y_n)$ be a system of generators of $\mathfrak{m}_B$ on which G acts as*

$$\sigma(y_i) = \zeta_q^{a_i} y_i \quad \textit{with } a_i \in \mathbb{Z},$$

write $y = (y_1, \dots, y_n)^t, a = (a_1, \dots, a_n)^t$ for the corresponding vectors. Then A is generated over R_A as power series ring in the monomials

$$y^m \quad \textit{for vectors} \quad m \ge 0 \quad \textit{where} \quad m \cdot a \equiv 0 \mod q \quad \textit{and} \quad 1 \cdot m \le p.$$

Proof. Consider the Reynolds operator

$$r : B \longrightarrow A \; ; \; b \mapsto \frac{1}{q} \operatorname{Tr} b = \frac{1}{q} \sum_{i=1}^{q} \sigma^i(b).$$

First we assume that G acts tamely. Then q is invertible in R_A and thus the Reynolds operator is well-defined. Since $r(a) = a$ for any $a \in A$, the Reynolds operator is also surjective. We will now apply the Reynolds operator to an arbitrary element of B.

By definition, every element b of B can be written as

$$b = \sum_{j \ge 0}^{\infty} c_j y^j$$

where the c_j are taken from a fixed system of representatives of k_B in $R_B = R_A$. Since G acts trivially on k_B, we will assume without loss of generality that this system is G-stable. Now one directly calculates that

$$r(b) = r\left(\sum_{j \ge 0}^{\infty} c_j y^j\right) = \sum_{j \ge 0}^{\infty} c_j r\left(y^j\right)$$

since r operates trivially on the coefficients c_j. Now by writing out for arbitrary $m \in \mathbb{N}^n$ gives

$$r\left(y^m\right) = \frac{1}{q} \sum_{i=1}^{q} \sigma^i(y^m) = \frac{1}{q} \sum_{i=1}^{q} \prod_{k=1}^{n} \sigma^i(y_k^{m_k}) = \frac{1}{q} \sum_{i=1}^{q} \prod_{k=1}^{n} \zeta_q^{i a_k m_k} y_k^{m_k} = \frac{y^m}{q} \sum_{i=1}^{q} \zeta_q^{i m \cdot a},$$

we can deduce from the fact that for a fixed $c \in \mathbb{Z}$, one has

$$\sum_{i=1}^{q} \zeta_q^{i \cdot c} \neq 0 \quad \text{if and only if} \quad c \equiv 0 \mod q,$$

that the Reynolds operator acts on monomials as

$$r\left(y^m\right) \neq 0 \quad \text{if and only if} \quad m \cdot a \equiv 0 \mod q.$$

Since r is surjective and b was arbitrary, we see now that A consists precisely of all power series in the monomials y^m fulfilling

$$m \cdot a \equiv 0 \mod q.$$

Namely, on one hand it is now clear that every element of A is a power series in invariant monomials. On the other hand, any power series in invariant monomials is contained in B and thus in A since being invariant by definition.

If G does not act tamely, we can assume that $\operatorname{char} B = 0$, since else the only p -th root of unity is unity itself. We claim that in is this case, the Reynolds operator is still well-defined. A priori, the Reynolds operator is a map $r : B \to \{a/q \,;\; a \in A\}$ into the set of fractions in A with denominator dividing q. We first prove that the image is again A. This can be seen by considering the action on the monomials. If q is not coprime to p, we still have that as in our above considerations that $r(a) = a$ for any $a \in A$, and

$$r\left(y^m\right) \neq 0 \quad \text{if and only if} \quad m \cdot a \equiv 0 \mod q$$

for any monomial $y^m, m \in \mathbb{N}^n$. Since B is a power series ring over R_B in all monomials y^m , we can infer that the Reynolds operator maps indeed into A, and onto A.

Now we want to prove: As in the tame case, the ring A is generated over R_A by the invariant monomials y^m with $m \cdot a \equiv 0 \bmod q$. Again, every element b of B can be written as

$$b = \sum_{j \geq 0}^{\infty} c_j y^j$$

where the c_j are taken from a fixed system of representatives of k_B in $R_B = R_A$. Thus, as in the tame case, $r(b)$ can be written power series in the monomials y^m fulfilling

$$m \cdot a \equiv 0 \mod q.$$

Thus, A is again generated over R_B as power series ring in those invariant monomials.

It now remains to see in both the tame and wild cases, that it suffices to chose as generators the monomials with

$$1 \cdot m \leq p,$$

one has to see that if the sum is greater than p , one can factor the monomial in two invariant monomials. Namely, let w be a monomial which factors as

$$w = \prod_{i=1}^{p+1} f_i$$

with some monomials f_i . Let $0 \leq \alpha_j < p$ be the unique numbers such that

$$\sigma\left(\prod_{i=1}^{j} f_i\right) = \zeta_q^{\alpha_j} \prod_{i=1}^{j} f_i.$$

By the pigeonhole principle, at least two of the α_i have to coincide, say $\alpha_k = \alpha_\ell$ with $k < \ell$. Then

$$\prod_{i=k+1}^{\ell} f_i$$

is invariant under σ and a non-trivial factor of w. This finishes the proof of our claims. □

Remark 2.4.13. Proposition 2.4.12 still holds when B is a polynomial ring over R. Then A is generated over R_A as polynomial ring by the same system of monomials.

Corollary 2.4.14. *Keep the notations and assumptions of Proposition 2.4.12. Furthermore assume that B is regular of dimension n. Then A is regular if and only if $a_i \equiv 0 \mod q$ for at least $n-1$ of the generators a_i.*

Proof. The maximal ideal of A is generated by the monomials as described in Proposition 2.4.12. It can be seen that the number of generators needed is bigger than n if and only if $a_i \not\equiv 0 \mod q$ for at least two generators a_i. We will come back to this in more detail in the next section. □

This corollary is an alternative formulation of a classical theorem of Serre [Ser68] in the special case of cyclic groups. The most general statements of this kind in the tame case including arbitrary groups can be found in a recent work of Avramov and Borek [AB96]. We will prove it in the cyclic case. The theorem is formulated in terms of pseudo-reflections:

Definition 2.4.15. A linear map σ acting on a vector space is called *pseudo-reflection* if $\mathrm{rk}(\sigma - \mathrm{id}) \leq 1$. An element of $\mathrm{Aut}\, B$ is called *pseudo-reflection* if the induced action on the k_B -vector space $\mathfrak{m}_B/\mathfrak{m}_B^2$ is a pseudo-reflection.

Remark 2.4.16. Assume that $G \cong \mathbb{Z}/q\mathbb{Z}$ acts tamely. Then σ acts on the tangent space $\mathfrak{m}_B/\mathfrak{m}_B^2$ as a pseudo-reflection if and only if there exists a regular system of parameters $(y_1, \ldots, y_n)$ of B such that $y_1, \ldots, y_{n-1}$ are invariant under G.

Proof. The only if direction is clear. The if-direction works as in the proof of Proposition 2.4.8. Namely, one can linearize the action on B and then diagonalize it to find a regular system of parameters $(y_1, \ldots, y_n)$ on which G acts linearly and diagonally, i.e. $\sigma(y_i) = \zeta_q^{a_i} y_i$. Since $\mathrm{rk}(\sigma - \mathrm{id}) \leq 1$, at least $n-1$ of the a_i are zero, proving our claim. □

Corollary 2.4.17. *Assume that $G \cong \mathbb{Z}/q\mathbb{Z}$ acts tamely, $k_B = k_A$ and B is regular. Then A is regular if and only if G is generated by a pseudo-reflection.*

Proof. This follows directly from Corollary 2.4.14 and Remark 2.4.16. □

The corollary can also be reformulated in a statement about normalization singularities:

Corollary 2.4.18. *Assume that $G \cong \mathbb{Z}/q\mathbb{Z}$ acts tamely, $k_B = k_A$ and A is regular. Then B is regular if and only if $Q(B) = Q(A)(\alpha)$ for some $\alpha \in B$ with $\alpha^q \in \mathfrak{m}_A - \mathfrak{m}_A^2$.*

Proof. As in Remark 2.4.16, G is generated by a pseudo-reflection on B if and only if there exists a regular system of parameters $(y_1, \dots, y_n)$ of B such that $y_1, \dots, y_{n-1}$ are invariant under G. If B is regular, then G must act on B as a pseudo-reflection by Corollary 2.4.17, and so a regular system of $\mathfrak{m}_A$ is given by $(y_1, \dots, y_{n-1}, y_n^q)$. Thus $Q(B) = Q(A)(y_n)$. Conversely, if $Q(B) = Q(A)(\alpha)$ for some α with $\alpha^q \in \mathfrak{m}_A - \mathfrak{m}_A^2$, then one can find a regular system $(y_1, \dots, y_{n-1}, \alpha^q)$ of $\mathfrak{m}_A$. Now $(y_1, \dots, y_{n-1}, \alpha)$ is a regular system of parameters of $\mathfrak{m}_B$, since then $B = A[\alpha]$. Thus G acts on B as pseudo-reflection, and by 2.4.17, B is regular. □

Now that we have explicit structural results on tame cyclic quotient singularities, the next step is to calculate their desingularization. Rather than doing this explicitly, as in the celebrated papers of Hirzebruch [Hir53] and Jung [Jun08], we will use toric geometry to do so, since this approach will reveal a bit more of the structure behind the singularities. In fact, desingularizing tame cyclic quotient singularities over the complex numbers is a classical example to illustrate the power of the basic idea behind toric geometry.

2.5 Toric schemes

We will now give a brief introduction to the theory of toric schemes. We will use this later to desingularize tame cyclic quotient singularities, which are toric singularities, as we will see. We will hereby closely follow the exposition in the book of Fulton [Ful93], which treats toric geometry over the complex numbers.

One key point of this chapter is, that one can perform the whole theory over an arbitrary field instead of the complex numbers. For fields of characteristic zero, this is almost clear; however, one could think that problems occur when considering fields of positive characteristics. The key point that this does not matter is the following: Toric geometry builds on polynomial rings over k associated to certain submonoids of $\mathbb{Z}^n$; those correspond to the multiplicative structure of the variables over k. If we go from $\mathbb{C}$ to a field of positive characteristic, then the coefficient arithmetic changes considerably, but not the exponent resp. variable arithmetic on which toric geometry relies - this reflects also the fact that when calculating modulo p, one does this for the coefficients, but not for the exponents, e.g. $X^{p+1} \not\equiv X \mod p$ for an X transcendent over k. Since the multiplicative monoid of variables remains unchanged in arbitrary characteristic, most relevant results on toric geometry directly transfer to the new situation. We also want to mention that doing toric geometry over an arbitrary base field is not a new idea, as it has been already performed by several authors working with toric varieties over arbitrary ground fields, e.g. by Wilke [Wil09] or by Faltings [FC90, Chapter IV.2] over even more general base schemes.

So let k be a field fixed throughout this section. We will fix a commutative group M isomorphic to $\mathbb{Z}^n$. The real hulls will be denoted by $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. the group M embeds into $M_\mathbb{R}$ canonically and give rise to a lattice.

Let S be a finitely generated submonoid of M. We will denote by $S_\mathbb{R} = S \cdot \mathbb{R}_{\geq 0}$ the so-called *cone* of S in $M_\mathbb{R}$.

Definition 2.5.1. Let S be a finitely generated submonoid of M. We will denote by

$$k[S] = k[\chi^s \ ; \ s \in S]$$

the monoid algebra of S over k. Such a ring $k[S]$ is called *toric ring* over k or *toric.*

Proposition 2.5.2. *Let $A = k[S]$ for a finitely generated monoid $S \subseteq M$. Then A is normal if and only if $S = (S \otimes_{\mathbb{N}} \mathbb{Z}) \cap S_{\mathbb{R}}$.*

Proof. The monoid $(S \otimes_{\mathbb{N}} \mathbb{Z})$ is the subgroup of M generated by S . Without loss of generality, we can assume that $(S \otimes_{\mathbb{N}} \mathbb{Z}) = M$. So the if-part follows from [Ful93, Section 2.1, Proposition 1]. On the other hand, if S is strictly smaller than $S_{\mathbb{R}}$, say for example missing some $s \in M \cap S_{\mathbb{R}}$, we see that χ^s is integral over A , because then there exists an $n \in \mathbb{N}$ such that $(\chi^s)^n \in S$, see e.g.[KKMSD70, I.§1, Lemma 1]. So the ring A is not normal and we arrive at a contradiction. □

Definition 2.5.3. For a set $\{v_i \ ; \ i \in I\}$ in $M_{\mathbb{R}}$, the set of points

$$\sum_{i \in I} v_i \cdot \mathbb{R}_{\geq 0}$$

is called *cone* in $M_{\mathbb{R}}$, the v_i are called its generators. If I is finite, the cone is called *finitely generated.*

A point of $M_{\mathbb{R}}$ is called rational if it is contained in $M \otimes_{\mathbb{Z}} \mathbb{Q} \subset M_{\mathbb{R}}$. A cone in $M_{\mathbb{R}}$ is called a *finitely generated rational cone* in $M_{\mathbb{R}}$ if for a suitable choice all of its generators are rational.

We now introduce the dual space $N := M^{\vee} = \mathrm{Hom}(M, \mathbb{Z})$. It has also a real hull $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, where we can define rational points and cones in analogy. We also have a canonical pairing

$$M \times N \longrightarrow \mathbb{Z} \ ; \ (m, n) \mapsto n(m).$$

We will denote finitely generated rational cones in $N_{\mathbb{R}}$ usually by σ; we emphasize that this σ has nothing to do with the generator of G in the previous section.

For a finitely generated rational cone $\sigma \in \mathrm{N}_{\mathbb{R}}$, we define its dual $\sigma^{\vee} \in M_{\mathbb{R}}$ as

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} \ ; \ v(u) \geq 0 \text{ for all } v \in \sigma\}.$$

Definition 2.5.4. Let $\sigma \subseteq N_{\mathbb{R}}$ be a finitely generated rational cone. We call a rational cone $\tau \subseteq \sigma$ a face of σ if $\tau \subseteq \partial\sigma$ and $\tau = H \cap \sigma$ for some $\mathbb{R}$-subvector space H of $N_{\mathbb{R}}$, where $\partial\sigma$ denotes the boundary of σ with respect to the relative topology on $\sigma \otimes_{\mathbb{R}} \mathbb{R}$.

We cite some results on cones, which can be found e.g. in [Ful93, §1.2]:

Proposition 2.5.5. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a finitely generated rational cone. Then*

(i) $\sigma^{\vee}$ *is a finitely generated rational cone in* $M_{\mathbb{R}}$.
(ii) $(\sigma^{\vee})^{\vee} = \sigma$.
(iii) $\sigma \cap N$ *is finitely generated.*
(iv) *Any face of* σ *is a finitely generated rational cone.*

Notations 2.5.6. Let $\sigma \subseteq N_{\mathbb{R}}$ be a finitely generated rational cone. We will denote

$$S_{\sigma} = M \cap \sigma^{\vee} \text{ and } U_{\sigma} = \mathrm{Spec}\ k[S_{\sigma}].$$

We cite some results on toric geometry which easily generalize to our situation over k :

Proposition 2.5.7. (i) *Let* $\sigma \subseteq N_{\mathbb{R}}$ *be a finitely generated rational cone, let* τ *be a face of* σ*. Then the inclusion of sets* $\tau \subseteq \sigma$ *induces canonically an open embedding*

$$U_\tau \hookrightarrow U_\sigma$$

(ii) *Let* $\sigma_1, \sigma_2 \in N_{\mathbb{R}}$ *be finitely generated rational cones such that* $\tau = \sigma_1 \cap \sigma_2$ *is a common face of both. Then one has a canonical closed immersion*

$$U_\tau \longrightarrow U_{\sigma_1} \times U_{\sigma_2}$$

giving rise to an isomorphism

$$U_\tau \longrightarrow U_{\sigma_1} \cap U_{\sigma_2}.$$

In particular, let Z *be the scheme obtained by gluing* U_{σ_1} *and* U_{σ_2} *along* U_τ*. Then the morphism*

$$Z \longrightarrow U_{\sigma_1 \cup \sigma_2}$$

is birational.

Proof. The proofs can be found in [Ful93, Chapter 1] where it suffices to replace $\mathbb{C}$ by k . (i) is [Ful93, Section 1.3]; (ii) is [Ful93, Section 1.4]. □

Definition 2.5.8. A *fan of cones* or fan in $N_{\mathbb{R}}$ is a finite set F of cones in $N_{\mathbb{R}}$ such that for any two cones $\sigma_1, \sigma_2 \in F$, we have $\tau := \sigma_1 \cap \sigma_2 \in F$ and τ is a face of σ_1 and of σ_2.

A fan F_1 is called *refinement* of a fan F_2 if for every $\sigma \in F_2$ there exist finitely many $\sigma_1, \dots, \sigma_m \in F_1$ such that $\sigma = \bigcup_{i=1}^m \sigma_i$.

A fan F is called rational resp. finitely generated if every $\sigma \in F$ is rational resp. finitely generated. In the following, every fan is assumed to be rational and finitely generated, so we will omit those qualifiers.

Definition 2.5.9. Proposition 2.5.7 allows to associate a toric scheme to any fan F in $N_{\mathbb{R}}$, called the scheme of F. Namely for any σ_1, σ_2, one glues U_{σ_1} and U_{σ_2} along $U_{\sigma_1 \cap \sigma_2}$. This uniquely defines a scheme which we call a *toric scheme* and denote $U(F)$. Moreover, for any refinement F_1 of a fan F_2 , we have a canonically induced map $U(F_1) \to U(F_2)$ which is birational by Proposition 2.5.7 (ii).

Proposition 2.5.10. *Let* $\sigma \in N_{\mathbb{R}}$ *be a finitely generated rational cone. Then* U_σ *is regular except possibly at the maximal ideal* $(\chi^s, s \in S_\sigma)$*. Moreover,* U_σ *is regular if and only if* $\sigma \cap N$ *can be generated by part of a basis of* N*.*

Proof. Replace $\mathbb{C}$ by k in [Ful93, Section 2.1]. □

Definition 2.5.11. A fan of cones in $N_{\mathbb{R}}$ is called regular if the corresponding toric scheme is regular.

2.6 Extension to mixed characteristic

We will now explain how the situation over a field k extends to complete rings of arbitrary characteristic. In [KKMSD70, Chapter IV, §3], Mumford has already proposed how to extend toric rings to the situation of mixed characteristic. In the complete case, by Cohen's structure theorem, this is equivalent to working over a complete discrete valuation ring R. However, the class of schemes Mumford defines is not big enough for the application we have in mind. In Mumford's definition, the semigroup of monomials contains always an uniformizer π of the base ring R. This prohibits certain rings from being toric which nonetheless show similar features. For example, not every regular ring is toric by Mumford's definition, since the uniformizer of the coefficient field resp. one of its roots is not always a regular parameter; one of the simplest examples is (any regular compactification of) the geometric double point $R[[X_1, X_2]]/(X_1X_2 - \pi)$. On the other hand, every regular ring should be considered toric by any definition which intends to characterize toric rings as rings with the simplest kind of singularities. In this section, we will try to remedy that by adopting a slightly broader definition for toric rings in mixed characteristic; we will work locally to circumvent possible problems with the structure of the compactification. Our modified definition will then later be justified by the structural result of Theorem 2.8.1 which will expose the dimension 2 toric rings of our definition exactly as the type of singularities with chain-like type of minimal desingularization, the simplest possible ones which contain also the class of all regular rings.

The suitable definition for our means will be related to Kato's definition of log-regular schemes, which generalize the concept of toric schemes in the equicharacteristic case, see [Kat94]. Since we work locally, we will only be interested in their complete local germs, which we will call locally toric rings. They are exactly the complete local rings described in [Kat94, §3], and most arguments in the equicharacteristic case generalize well. We will now describe this in greater detail.

Notations 2.6.1. Consider a normal complete local Noetherian ring A with maximal ideal $\mathfrak{m}_A$, let R be the coefficient ring of A, let k be its residue field. We will fix the ring A for the rest of this section. As in section 2.5, we let M be isomorphic to the additive group $\mathbb{Z}^n$.

Definition 2.6.2. If A is an equicharacteristic ring, then A is called *locally toric ring* if it is a complete local germ of a toric ring over some field. If A is of mixed characteristic, then A is called a *locally toric ring* if there is an isomorphism

$$A \cong R[[S]]/\mathcal{I} = R[[\chi^s \ ; \ s \in S]]/\mathcal{I},$$

where S is a finitely generated submonoid of $M = \mathbb{Z}^n$, where $\mathcal{I}$ is generated by a relation of type $\pi - \varphi$, where R is the coefficient ring of A with uniformizer π , and

$$\varphi \equiv 0 \mod (\chi^s \ ; \ s \in S, s \neq 0).$$

We will say that S is the monoid associated to A. If A' is a normal local Noetherian ring which is not necessarily complete, we will say that A' is locally toric if its completion $\widehat{A'}$ is. Also, we will call an isolated singularity on some scheme toric if the corresponding local germ is locally toric.

If A is equicharacteristic, there is nothing new compared with the classical setting. Since completing a local ring is a flat local morphism, the results from the previous section transfer to the complete local setting in the following way by localization and completion. If A is of mixed characteristic, there are some non-trivial things to remark, so for the rest of the section, we will assume that A is of mixed characteristic $(0, p)$, and explain how the classical theory generalizes.

Locally toric rings are Noetherian by construction, so we can apply Cohen's structure theorem. One can see that any locally toric ring is isomorphic to a ring

$$R[[X_1, \ldots, X_n]]/\mathcal{I},$$

over the coefficient ring R, where $\mathcal{I}$ is generated by some relation $\pi - \varphi(\dots, \chi^s, \dots)$ with the same condition on φ as above, and relations of the type

$$\prod_{i=1}^{n} X_i^{\alpha_i} = \prod_{j=1}^{n} X_j^{\beta_j},$$

where $\alpha_i, \beta_j \in \mathbb{N}$ and both sides of the equation have same degree. We will call relations as the latter *of toric type* or *toric*.

Similarly as in the classic case, one can now also define $\sigma^\vee = S \otimes_{\mathbb{N}} \mathbb{R}_{\geq 0}$, then $\sigma = (\sigma^\vee)^\vee$ and $N, M, N_{\mathbb{R}}, M_{\mathbb{R}}$ analogously; for the rest of this section, we will use those notations in an analogous manner. However, given k , the ring A does not uniquely depend on S or σ in any way, since the choices of the relation $\pi - f(X_1, \dots, X_n)$ and the coefficient ring are not unique. For example, any regular ring has the same monoidal structure of S, but there are infinitely many isomorphy classes of complete regular local rings over R, unlike in the case over a fixed field. On the other hand, the graded algebra $\operatorname{gr} A$ associated to A carries the whole monoidal structure. One can verify that we have $\operatorname{gr} A = (R/\pi)[S] = k[S]$. This gives an analogy to the classic case.

Since we have assumed that A is normal, the semigroup S has similar properties as an equicharacteristic normal toric ring:

Lemma 2.6.3. *Let A be a locally toric ring, let S be the associated submonoid. We have that $S = (S \otimes_{\mathbb{N}} \mathbb{Z}) \cap S_{\mathbb{R}}$.*

Proof. This follows in complete analogy to the first direction in the classical case as in Proposition 2.5.2. □

We now want to apply the previous results in the case that A is toric. The idea is that since A is toric, only the structure of the monoid S counts in the desingularization, i.e. the toric relations on $\operatorname{gr} A$ are in one-to-one-correspondence to those on A. So it should be enough to desingularize A by desingularizing $\operatorname{gr} A$ instead and then performing some analogous operations on A.

Notations 2.6.4. Let $\tau \subseteq \sigma$ a finitely generated rational cone. Then we define $A_\tau := A[S_\tau]$.

We note that $A[S_\tau]$ is not necessarily local anymore. Going from a locally toric ring A to A_τ is analogous to the procedure in Proposition 2.5.7:

Proposition 2.6.5. *Let A be a toric ring, and S the associated submonoid. Let $\sigma_1, \sigma_2 \subseteq \sigma$ be finitely generated rational cones such that $\tau = \sigma_1 \cap \sigma_2$ is a common face of both. Then one has a canonical closed immersion*

$$A_\tau \longrightarrow A_{\sigma_1} \times A_{\sigma_2}$$

giving rise to an isomorphism

$$A_\tau \longrightarrow A_{\sigma_1} \cap A_{\sigma_2}.$$

In particular, let Z be the scheme obtained by gluing A_{σ_1} and A_{σ_2} along A_τ. Then the morphism

$$Z \longrightarrow A_{\sigma_1 \cup \sigma_2}$$

is birational.

Proof. This is a direct analogue to Proposition 2.5.7 (ii). $\square$

Lemma 2.6.6. *Let $\tau \subseteq \sigma$ be a finitely generated rational cone. Denote by $\mathfrak{m}_0$ the A_τ -ideal generated by the $\chi^s, s \in S_\tau$. Then $\widehat{(A_\tau)}_{\mathfrak{m}_0}$ is a locally toric ring, and one has that*

$$\operatorname{gr}\widehat{(A_\tau)}_{\mathfrak{m}_0} \cong U_\tau,$$

with $U_\tau = \operatorname{Spec} k[S_\tau]$ as in Notations 2.5.6.

Proof. By construction, the relations at $\widehat{(A_\tau)}_{\mathfrak{m}_0}$ are all of toric type. Also, the monoid of monomials is the same as of U_τ . Since U_τ is toric over k and $\operatorname{gr}\widehat{(A_\tau)}$ is also with same group of monomials, they have to coincide. $\square$

Definition 2.6.7. So in analogy to the classic case, given A ; one can associate to any fan F giving a refinement of σ a scheme $A(F)$ which admits a birational morphism to $\operatorname{Spec} A$. Also, any refinement F' of F defines a birational morphism $A(F') \to A(F)$.

The following analogy to the classic case are also true:

Proposition 2.6.8. *Let $\tau \subseteq \sigma$ be a finitely generated rational cone. Then A_τ is regular if and only if $\tau \cap N$ can be generated by part of a basis of N.*

Proof. Let $\mathfrak{m}_0$ denote the A_τ -ideal generated by the $\chi^s, s \in S_\tau$. One can now verify as in the classic case, that by construction, A_τ is regular, if and only if A_τ is regular at $\mathfrak{m}_0$. Now since the Krull dimensions of A_τ and U_τ coincide, and we have $\operatorname{gr}\widehat{(A_\tau)}_{\mathfrak{m}_0} \cong U_\tau$ by Lemma 2.6.6, one has that the embedding dimension of A_τ at $\mathfrak{m}_0$ equals that of U_τ. Also, A_τ is regular if and only if U_τ is. The rest follows from Proposition 2.5.10. $\square$

2.7 Desingularization

In this section, we will review the classic theory of desingularization of toric schemes. As before, the theory over $\mathbb{C}$ will extend to arbitrary fields of positive characteristic, and as we will see, also to mixed characteristic. We will in particular examine the example of tame cyclic quotient singularities which are locally toric, leading to the well-known Hirzebruch-Jung-desingularization.

A central theorem in the desingularization of toric singularities is the following:

Theorem 2.7.1. *Let F be a fan of cones in $N_\mathbb{R}$. Then F has a regular refinement.*

In the case of $\dim_\mathbb{R} N_\mathbb{R} = 2$, the fan F has a regular refinement $F_{\min}$ which is minimal, i.e. for any regular refinement F' of F , the fan F' is a refinement of $F_{\min}$.

Proof. This is done for example in [Ful93, Section 2.6]. It is easily seen that one can take k instead of $\mathbb{C}$, since the proofs rely only on the multiplicative structure of the underlying monoids. $\square$

Remark 2.7.2. By Proposition 2.5.10, the induced resolution of $U(F)$ is an isomorphism except over the ideal generated by the $\chi^s, s \in S$.

Remark 2.7.3. As explained in section 2.6, the desingularization process directly extends to the case of mixed characteristic. Namely, let A be a complete locally toric ring with monoid of monomials S. Let σ be the cone associated to S as in section 2.6. Then any regular refinement F of σ will define as in Definition 2.6.7 a scheme $A(F)$ birational to $\operatorname{Spec} A$. The scheme $A(F)$ is regular by Proposition 2.6.8 and is thus a desingularization of A. If the Krull dimension of A is 2, then by Theorem 2.7.1, there exists also a minimal desingularization which can be obtained from the minimal singular refinement of σ .

Remark 2.7.4. We recall a well-known fact: As blow-up commutes with flat base change [Liu02, 8.1.12(c)], and completion of a Noetherian ring is flat over the given ring [Eis99, 7.2.b], one can obtain the desingularization structure of any Noetherian local ring by desingularizing its completion.

We now define a well-known class of toric singularities in Krull dimension 2:

Definition 2.7.5. Let k be a field as in the previous section. For $1 \leq a < q$, we call the algebra

$$A_{(q,a)} = k[X^iY^j \; ; \; i,j \geq 0 \text{ with } i + aj \equiv 0 \mod q]$$

tame cyclic quotient singularity of type (q, a) over k .

Note that we have made no assumption on the characteristic of the base field. The associated monoid M to $A_{(q,a)}$ is isomorphic to the multiplicative monoid of generating monomials in $Q(A_{(q,a)})$, which we can embed into $\mathbb{N}^2$ by mapping a monomial X^iY^j to the point (i, j). Thus M can be generated by $(q, 0)$ and $(-a, 1)$, or equivalently by $(q, 0)$ and $(q - a, 1)$. Since $A_{(q,a)}$ is also normal, it corresponds to the rational cone

$$\sigma^\vee = (1, 0)\mathbb{R}_{\geq 0} + (0, 1)\mathbb{R}_{\geq 0} \subset M_\mathbb{R} = \mathbb{R}^2.$$

Now dualizing via the usual scalar product on $\mathbb{R}^2$, we see that N can be generated by $(0, 1)$ and $\frac{1}{q}(1, a)$. The cone associated to $A_{(q,a)}$ in $N_\mathbb{R}$ is then again

$$\sigma = (\sigma^\vee)^\vee = (1, 0)\mathbb{R}_{\geq 0} + (0, 1)\mathbb{R}_{\geq 0} \subset N_\mathbb{R} = \mathbb{R}^2.$$

We now can infer an important structural result:

Proposition 2.7.6. *Let A be a normal toric ring over k of Krull dimension 2 . Then A is a tame cyclic quotient singularity, i.e. there exist $q, a \in \mathbb{N}$ such that $A \cong A_{(q,a)}$.*

Proof. We have seen that $A_{(q,a)}$ is the unique toric ring over k associated to the rational cone

$$\sigma^\vee = (1, 0)\mathbb{R}_{\geq 0} + (0, 1)\mathbb{R}_{\geq 0}$$

and the lattice

$$M = (q, 0)\mathbb{Z} + (-a, 1)\mathbb{Z}.$$

On the other hand, since A is normal, by Proposition 2.5.2, there is also a rational cone $\tau^\vee$ and a lattice M' uniquely defining A. It suffices to prove that there is a linear transformation inducing an isomorphy between $\sigma^\vee$ and $\tau^\vee$ respectively M and M'. Now A is of Krull dimension 2, so the cone $\tau^\vee$ will be generated by exactly two vectors $e_1, e_2 \in \mathbb{N}$, unique up to length. Since $\tau^\vee$ is a rational cone,

one can choose their lengths such that the lattice generated by e_1, e_2 contains M. In particular, one can choose their lengths such that

$$M' = q'e_1\mathbb{Z} + (-a'e_1 + e_2)\mathbb{Z}$$

for some $q', a' \in \mathbb{N}$. But this implies that by taking $q = q', a = a'$ the desired linear transformation is exactly the one sending $(1,0)$ to e_1 and $(0,1)$ to e_2. $\square$

We will now calculate the minimal resolution of $A_{(q,a)}$; by Proposition 2.7.6 and Remark 2.7.3, this will yield the minimal desingularization for any locally toric singularity of Krull dimension 2. The desingularization process consists in finding the minimal regular subdivision of the fan F consisting of σ and its faces. For this, we will prove a lemma, which relates the generators of $\sigma \cap N$ in $N = (0,1)\mathbb{Z} + \frac{1}{q}(1,a)\mathbb{Z}$ and to the Hirzebruch-Jung fraction expansion of q/a - a sequence of integers occurring in many fields of mathematics, see [Hir53] and [Jun08].

Definition 2.7.7. Let $x, y \in \mathbb{N}$ be integers, set $s_0 = y$. Inductively define

$$x = r_1 y - s_1 \text{ with } 0 \leq s_1 < y$$

and for $n \in \mathbb{N}$, set

$$s_{n-1} = r_{n+1}s_n - s_{n+1} \text{ with } 0 \leq s_{n+1} < s_n.$$

Let N be the smallest number $N \in \mathbb{N}$ such that $s_N = 0$. Then the sequence $(r_1, \ldots, r_N)$ is called *Hirzebruch-Jung continued fraction expansion* or simply *Hirzebruch-Jung expansion* of the fraction x/y.

Definition 2.7.8. Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational finitely generated cone, without loss of generality of maximal dimension. We define the *Newton polygon* $N(\sigma)$ of σ to be the boundary of the convex hull of $\sigma \cap N$.

Remark 2.7.9. Two consecutive edges of $N(\sigma)$ form a basis of N, since the triangle formed by the origin and both edges can contain no other lattice point of N than the triangle's edges. Thus arbitrary two sides of the triangle span a fundamental domain of N and thus are a basis of N .

We will now explicitly calculate the desingularization of $A_{q,a}$, where we will follow partly the expositions in [Ful93, Chapter 2.2].

Lemma 2.7.10. *Consider* $N = (0,1)\mathbb{Z} + \frac{1}{q}(1,a)\mathbb{Z}$ *and the rational cone*

$$\sigma = (1,0)\mathbb{R}_{\geq 0} + (0,1)\mathbb{R}_{\geq 0} \subset N_{\mathbb{R}} = \mathbb{R}^2.$$

Label the consecutive edges of $N(\sigma)$ *by*

$$e_0 = (0,1), e_1 = \frac{1}{q}(1,a), \ldots, e_{k+1} = (1,0).$$

Then for $1 \leq i \leq k$, *there exist* $t_i \in \mathbb{N}$ *such that*

$$e_{i-1} + e_{i+1} = t_i e_i.$$

Moreover, the integers $(t_1, \ldots, t_k)$ *are the Hirzebruch-Jung expansion of* q/a.

Proof. By the previous remark, e_i and e_{i+1} are a basis of N for arbitrary i, and so are e_{i-1} and e_i. So we have

$$e_{i+1} = a_1 e_{i-1} + b_1 e_i \text{ and } e_{i-1} = a_2 e_{i+1} + b_2 e_i$$

for some integers $a_j, b_j \in \mathbb{N}$. Substituting e_{i-1} into the first equation, one obtains

$$e_{i+1}(1 - a_1 a_2) = (b_1 + a_1 b_2) e_i.$$

Since e_{i+1} and e_i are linearly independent, the coefficients have to be zero, thus $a_1 a_2 = 1$ resp. $a_j = \pm 1$. Since we have numbered the e_i consecutively, this must be $a_1 = a_2 = -1$. This proves the first claim for $t_i = b_1 = b_2$.

For the second claim, write $e_i = (x_i, y_i)$, and keep the notation for the s_i, r_i from Definition 2.7.7 where $x = q$ and $y = a$. We will prove by induction that we have $r_i = t_{i+1}$ for $i \geq 1$ and $q y_i = s_i$ for $i \geq 0$. One checks that the induction hypothesis is correct for $i = 0, 1$. So assume the claim is fulfilled for all $i \leq n$ for some $n \geq 1$. All we have to prove is that it is fulfilled for $n + 1 \geq 2$.

By construction of the Newton polygon, e_{n+1} is the unique linear combination $-e_{n-1} + \alpha_n e_n$ where α is minimal with the property such that e_{n+1} has positive second component, i.e. $-y_{n-1} + \alpha_n y_n \geq 0$. Thus

$$y_{n+1} = \alpha_n y_n - y_{n-1},$$

or reformulated,

$$y_{n-1} = \alpha_n y_n - y_{n+1},$$

where $0 \leq \alpha < y_n$. Comparing α_n with the above calculations, we see that $\alpha_n = t_n$. Moreover, by induction we have $q y_{n-1} = s_{n-1}$ and $q y_n = s_n$, thus remembering the equation

$$s_{n-1} = r_{n+1} s_i - s_{n+1}$$

from the definition of the Hirzebruch-Jung expansion 2.7.7 which fulfills equivalent conditions on s_{n+1}, we see by comparing coefficients and dividing by q, that indeed we have also $q y_{n+1} = s_{n+1}$, and $r_n = t_{n+1} = \alpha_{n+1}$. This completes the induction step and thus proves the second claim. □

Corollary 2.7.11. *Let as above F be the fan defined by σ and its faces. The refinement $F_{(q,a)}$ of F defined by the subdivision in the $k+1$ rational cones*

$$\tau_i = e_i \cdot \mathbb{R}_{\geq 0} + e_{i+1} \cdot \mathbb{R}_{\geq 0}, 0 \leq i \leq k$$

is the minimal resolution of F.

Proof. By the previous lemma, the τ_i are regular, since they are generated by a basis of N. Thus their faces are also regular, and so is the fan $F_{(q,a)}$. It remains to see that this refinement is also minimal: Since N is two-dimensional, any refinement is induced by dividing σ along finitely many rational straights. It can be seen that straights not going through one of the e_i can be omitted in any regular subdivision, and the straights through e_i are necessary, thus $F_{(q,a)}$ is indeed minimal. □

Corollary 2.7.12. *$U(F_{(q,a)})$ is the minimal resolution of $A_{(q,a)}$. The exceptional fiber of $U(F_{(q,a)})$ consists of a chain of k copies of $\mathbb{P}^1_k$.*

Remark 2.7.13. Assume that q is coprime to $\operatorname{char} k$. The $A_{(q,a)}$ occur naturally as quotients of $k[X,Y]$ by the q-cyclic action

$$\sigma : X \mapsto \zeta_q X, Y \mapsto \zeta_q^a Y$$

with trivial action on k, where ζ_q is a primitive root of unity.

In particular, Proposition 2.3.6 and Remark 2.7.4 show that we can desingularize any quotient singularity by a tame cyclic group this way.

By the previous considerations about mixed characteristics, the calculations over k also directly carry over to the following situation: Let R be a complete discrete valuation ring with uniformizer π, let $B = R[[X]]$. Consider the action of $G = \langle \sigma \rangle \cong \mathbb{Z}/q\mathbb{Z}$ given on B by

$$\sigma : \pi \mapsto \zeta_q \pi \ , \ X \mapsto \zeta_q^a X$$

and trivial action on the residue field of R. The associated graded algebra $\operatorname{gr} A$ of the invariant ring $A = B^G$ is a toric ring over the residue field of R, even if the action is wild. If the action of G is tame, we have by Proposition 2.4.7 that $(\operatorname{gr} B)^G = \operatorname{gr} A$; if G is wild, this is not true anymore, since the induced G-action on $\operatorname{gr} B$ is then trivial. Nonetheless, $\operatorname{gr} A$ is still toric. So as described above, in both the tame and wild cases the minimal resolution of A consists of a chain of rational components, as calculated explicitly in the nonclassical settings for example in [Vie77], [Edi92] and [Hal07].

Another example of toric singularities consist of a class of certain normalization singularities:

Example 2.7.14. Let q be coprime to $\operatorname{char} k$. Let A be the normalization of $k[X,Y]$ in the field $Q(k[X,Y])(\sqrt[q]{X^iY^j})$ for some $i,j \in \mathbb{N}$. We can assume without loss of generality that q is coprime to i and j. It is immediately seen that A is regular if $ij \equiv 0 \bmod q$; since then $A = k[\sqrt[q]{X}, Y]$ resp. $A = k[X, \sqrt[q]{Y}]$. On the other hand, if $ij \not\equiv 0 \bmod q$, then A is not regular. In particular, A is a subring of the regular ring $k[\tilde{X}, \tilde{Y}]$ where $\tilde{X} = \sqrt[q]{X}$ and $\tilde{Y} = \sqrt[q]{Y}$. It can now be seen that

$$\begin{aligned} A &= k[\tilde{X}^k\tilde{Y}^\ell \ ; \ k,\ell \geq 0, \ kj - \ell i \equiv 0 \mod q\,] \\ &= k[\tilde{X}^k\tilde{Y}^\ell \ ; \ k,\ell \geq 0, \ -kji^{-1} + \ell \equiv 0 \mod q\,] \ \cong A_{(q,q-ji^{-1})} \end{aligned}$$

where by i^{-1} we denote some modular inverse of i modulo q. Thus A is in particular a tame cyclic quotient singularity.

2.8 A characterization of toric surface singularities

We will now cite a result on toric surface singularities which shows that they are exactly the singularities with simplest minimal resolution. All parts of the proof can be already found in literature to a great extent.

Theorem 2.8.1. *Let A be a complete Noetherian normal local ring of Krull dimension 2 with algebraically closed residue field. Then the following are equivalent:*

(i) *A is locally toric.*

(ii) *A is locally toric, and $\operatorname{gr} A = A_{(n,q)}$ for some $n, q \in \mathbb{N}$.*

(iii) *The exceptional divisor of the minimal resolution of A is a normal crossings* *chain of rational components.*

In particular, the n, q in (ii) depend only on the combinatorics of the exceptional divisor in (iii) in terms of component multiplicities and self-intersections.

Proof. (ii) $\to$ (i) is obvious. (i) $\to$ (ii) follows from Proposition 2.7.6 for equicharacteristic and from Proposition 2.7.6 and Lemma 2.6.6 in mixed characteristic. (ii) $\to$ (iii) follows from our explicit calculations in from Corollary 2.7.12 in equicharacteristic and from Remark 2.7.3 and Corollary 2.7.12 in mixed characteristic. Note that for these directions we have not needed that the residue field is algebraically closed.

It remains to prove the crucial direction (iii) $\to$ (i):

Let $\mathfrak{m}_A$ be the maximal ideal of A, let X be the minimal desingularization of $\operatorname{Spec} A$ and E its exceptional divisor. Since E is by assumption a chain of normal crossing components, it is a rational divisor, thus A is a rational singularity.

Now as in the proof of [Liu02, 9.4.14], one can infer that

$$\mathfrak{m}_A = H^0(X, \mathcal{O}_X(-Z)),$$

where Z is the fundamental cycle of E. We have already chosen our symbols to match those of [Liu02, 9.4.14], for the rest one takes $U = X$ and $Y = \operatorname{Spec} A$ and $\mathcal{I} = \mathcal{O}_X(-Z)$. Note that it is not necessary to make the assumption that E is a divisor on an arithmetic surface; as in [Art66], the only necessary assumptions to carry out the proof correctly are that A is normal of Krull dimension 2, and that $H^1(X, \mathcal{O}_X(-Z)) = 0$, i.e. that the A is a rational singularity.

Now $\mathfrak{m}_A$ can be explicitly calculated as in [BHPVdV04, III.5] - the calculations over the complex numbers directly carry over to the general situation, since the module structure of $\mathfrak{m}_A$ depends only on the intersection combinatorics of E. It follows in particular that $\mathfrak{m}_A$ has generators which satisfy only toric relations among each other; since A is normal, it is a locally toric ring by definition. □

Remark 2.8.2. From Theorem 2.8.1, one can derive the following statement, using that an equicharacteristic complete Noetherian normal local ring is identical to the completion of its associated graded algebra at the origin:

Let A be an equicharacteristic complete Noetherian normal local ring of Krull dimension 2. Assume A has a rational singularity with chain-like resolution. Then the isomorphy type of A depends only on its resolution combinatorics in terms of component multiplicities and self-intersections.

In particular, there exist n, q such that A is the ring $A_{(n,q)}$ completed at some maximal ideal.

Note that this statement becomes false if we omit the equicharacteristic condition; since in mixed characteristic, not even all complete Noetherian regular local rings of Krull dimension 2 are isomorphic, for example

$$R[[X]] \quad \text{and} \quad R[[X_1, X_2]]/(X_1X_2 - \pi),$$

where R is a complete discrete valuation ring of mixed characteristic with uniformizer π, as discussed at the beginning of section 2.6.

Chapter 3

p-cyclic group actions on local normal rings

In this chapter, we will examine invariant rings of local normal rings by prime cyclic group actions. In section 3.1, *Motivating examples in dimension 2*, we will start with some elementary examples exhibiting the main problems we have to deal with. Section 3.2, *A criterion for monogeneity*, will contain the main result of this chapter: We will prove an algebraic result characterizing monogenous p-cyclic Galois extensions of Noetherian local normal rings. Section 3.3, *Application to models of curves*, will contain a more detailed analysis of p-cyclic actions on local complete germs of arithmetic surfaces. Among others, we will apply the main theorem of section 3.3, and derive some obstructions on possible p-cyclic actions in terms of the explicit action.

3.1 Motivating examples in dimension 2

We will begin our exposition with two important motivating examples in dimension 2 showing the main features we have to deal with quotient singularities.

Notations 3.1.1. Let R be a complete discrete valuation ring with uniformizer π and algebraically closed residue field k of characteristic p. Let v denote the π-valuation on R. Let $G = \langle\sigma\rangle \cong \mathbb{Z}/p\mathbb{Z} \hookrightarrow \operatorname{Aut} R$ be a Galois action on R with trivial induced action on k. We denote by $K = Q(R)$ the field of fractions of R. We will also denote by

$$N = \prod_{i=0}^{p-1} \sigma^i \text{ and } \operatorname{Tr} = \sum_{i=0}^{p-1} \sigma^i$$

the norm and trace induced by G, respectively.

In the first example, we will revisit the toric singularities of section 2.7 from a different viewpoint. Instead of starting immediately with a singular ring of invariants, we will expose some toric singularities as an invariant ring of a regular ring by a cyclic action. The main point of this example is that one can relate the group action on a ring to possibly occurring singularities.

Example 3.1.2. For simplicity, we assume that $\sigma(\pi) = \zeta \cdot \pi$, where ζ is a primitive p-th root of unity. We will consider an action arising from the following situation: Consider the projective line $\mathbb{P}^1_K$ over K and a parameter ξ of $\mathbb{P}^1_K$ with $\sigma(\xi) = \xi$; such a parameter arises by extending the G-action on R trivially to $\mathbb{P}^1_K$. Now we can consider a smooth model Y of $\mathbb{P}^1_K$ over R defined by a parameter $\eta = \xi \cdot \pi^a$ for some $a \in \mathbb{Z}$. We see that G acts on the model Y canonically by monodromy. If $a \equiv 0 \mod p$, the quotient model $X = Y/G$ is again a smooth model of $\mathbb{P}^1_{K^G}$ over R^G. If not, we will prove later that X is singular, and the special fiber of X is not reduced and of multiplicity p.

We will now explicitly calculate what happens in this example. Without loss of generality, we restrict ourselves to one of the affines containing the zero of η - namely the one defined by the ring $D = R[\eta]$. On it, the monodromy action of G can be described as

$$\sigma : R[\eta] \longrightarrow R[\eta] \; ; \; \begin{cases} \pi \longmapsto \zeta \cdot \pi \\ \eta \longmapsto \zeta^a \cdot \eta \end{cases}$$

with trivial induced action on the residue field k. We are now interested in the ring of invariants $C = D^G$ which corresponds to an open dense subset of the model X. We claim that any local germ of C is locally toric in the sense of 2.6, and the unique singularity of C lies under the closed point (η, π) of D. We will also see, that its desingularization is of Hirzebruch-Jung-type, despite the fact that the action of G is wild; compare Remark 2.7.13.

First we are interested in the structure of C. For this, remark that the ring D is generated over $R^G[N(\eta)] = R^G[\eta^p]$ by monomials $\eta^i\pi^j$ with $0 \le i, j \le p-1$, and such a monomial is G-invariant if and only if $j + ia \equiv 0 \mod p$. By Proposition 2.4.12 resp. Remark 2.4.13, we have that C is generated over $R^G[\eta^p]$ by the G-invariant monomials, so we can describe C explicitly as

$$C = R^G[\eta^p][\eta^i\pi^j \; ; \; i, j \in \mathbb{N}, \; j + ia \equiv 0 \bmod p, \; 0 \le i + j \le p - 1].$$

The above discussed monomials also generate all the maximal ideals of C in a certain sense:

Let P be a closed point of D. Denote by $B = D_P$ the localization of D at P. This gives rise to a local ring of invariants $A = B^G$. For this ring, we then have that

$$A = R^G[\eta^p][\eta^i\pi^j \; ; \; i, j \in \mathbb{N}, \; j + ia \equiv 0 \bmod p, \; 0 \le i + j \le p - 1]_P,$$

since localization commutes with formation of invariants. We will now distinguish two cases, based on the location of P. Since k is algebraically closed, We have that $P = (\pi, \eta - c)$ for some $c \in k$.

Case 1: One has $c \in k^\times$, i.e. P is not a zero of η. Then we claim that the ring A is always regular. For this note, that for any monomial $\eta^i\pi^j$ with $j + ia \equiv 0 \bmod p$ we can write

$$\eta^i\pi^j = \left(\frac{\pi}{\eta^{a^{-1}}}\right)^j (\eta^p)^b,$$

where a^{-1} is some inverse of a modulo p, and $b \in \mathbb{N}$ is chosen appropriately. Since η^p is a unit in A, its maximal ideal $\mathfrak{m}_A$ can then be generated by the two elements

$$\pi\eta^{-a^{-1}} \text{ and } \eta^p - c^p,$$

i.e. for the invariant ring we then have

$$A = R^G[\eta^p]\left[\pi\eta^{-a^{-1}}, \eta^p - c^p\right]_P = R^G\left[\pi\eta^{-a^{-1}}, \eta^p - c^p\right]_P$$

which is a regular ring. Note that $(\pi\eta^{-a^{-1}}, \eta - c)$ is a regular system of parameters of B such that at least one of its elements is G-invariant.

Case 2: One has $c = 0$. Then we distinguish two subcases.
Case 2a: One has $a \equiv 0 \mod p$. Then A is regular, one has

$$A = R^G[\eta]_P,$$

and the maximal ideal of A is generated by η and π^p. Note that (π, η) is a regular system of parameters of B such that one of its elements is G-invariant.
Case 2b: One has $a \not\equiv 0 \mod p$. Since A is a locally toric ring, we can examine its lattice of monomials. We see that the monomials $\pi^p, \eta^p, \pi\eta^{p-a^{-1}}$ are independent generators of the lattice, and so Proposition 2.6.8 states that A is singular.

This proves our claims from the beginning. There are several important things to note: In cases 1 and 2a, the invariant rings were regular. In both cases, we could exhibit a regular system of parameters of B with at least one G-invariant parameter. Later, we will say that in these cases, G acts on B as pseudo-reflection. In 2b, the invariant rings were singular, and it can be seen that is not possible to find such a system, else A would be regular. As it will turn out in Theorem 3.2.2, this can be reformulated to a more general condition, i.e. the invariant ring A is regular if G acts as a pseudo-reflection. Also, one should remark, that case 2b is the only case where one of the local parameters is sent to a non-trivial multiple of itself, and the other cannot be chosen G-invariant.

Similarly as in this example, one can prove that the minimal resolution of this singularity is a chain of projective lines, i.e. the minimal resolution of the singularity at $\eta = 0$ contains no component intersecting more than three others. For this, one has to note that a local parameter on every possible exceptional component will be given by some $\eta^i\pi^j$ for $i, j \in \mathbb{Z}$. Now the same arguments as above apply, and thus the exceptional components can be singular only at the zero or infinity of this parameter. And since the blow-up process terminates by the well-known fact that a desingularization exists, the desingularization of X is in fact a chain of components. We will not prove this in detail, this will follow later from more general considerations, see Theorem 4.3.12 and corollaries.

The following proposition will be important in our next example.

Proposition 3.1.3. *Let R, K and G be as in 3.1.1. Let $I = (\sigma - \mathrm{id})$ denote the augmentation map. Denote by $\mathrm{Tr} : Q(R) \to Q(R)$ the trace map on R, and set $\ell = v(I(\pi)) - 1$. Then for $n \in \mathbb{Z}$, we have*

$$I(\pi^n R) = I(\pi)\pi^{n-1}R \cap \operatorname{Ker}\operatorname{Tr} = \pi^{n+\ell}R \cap \operatorname{Ker}\operatorname{Tr}.$$

Proof. We fix an arbitrary $n \in \mathbb{Z}$ and denote $M := \pi^{n+\ell}R \cap \operatorname{Ker}\operatorname{Tr}$ First we will derive an important equation. We have by definition of ℓ that

$$I(\pi) = \varepsilon\pi^{\ell+1} \text{ for some } \varepsilon \in R^\times.$$

From this and the definition of I, it follows that

$$I(\pi^n) \equiv \varepsilon n\pi^{n+\ell} \equiv nI(\pi)\pi^{n-1} \mod \pi^{n+\ell+1} \text{ for } n \in \mathbb{N}$$

Since R is generated over R^G by π, so I maps π^n into $I(\pi)\pi^n R$. Since $\mathrm{Tr} \circ I = 0$, we have $I(\pi^n R) \subseteq \operatorname{Ker}\operatorname{Tr}$; thus we have in fact $I(\pi^n R) \subseteq M$.

To see that I is in fact also surjective, one has to apply the additive version of Hilbert's theorem 90: It states that $I(K)$ is a $(p-1)$-dimensional K^G-vector space. By our above equation, we see that the $I(\pi^n), 1 \leq n \leq p-1$ are linearly independent over K^G by the equation above. Thus $I(K)$ is generated over K^G by the π^i with $\ell+1 \leq i \leq \ell+p-1$.

Now we want to prove the inclusion $M \subseteq I(\pi^n R)$. So let

$$s = \sum_{i=\ell+1}^{\ell+p-1} r_i \pi^i \text{ with } r_i \in K^G$$

be an element of K with $\mathrm{Tr}(s) = 0$. It can be seen that $s \in \pi^{\ell+n}R$ if and only if $r_i\pi^i \in \pi^{\ell+n}R$ for all $\ell+1 \leq i \leq \ell+p-1$, since the summands have pairwise distinct valuation. These are also equivalent to stating that $r_i \in \pi^{n-i}R$ for all $1 \leq i \leq p-1$. The first equation then shows that if

$$s = \sum_{i=\ell+1}^{\ell+p-1} r_i \pi^i \in \pi^{\ell+n}R,$$

then we have

$$s = \sum_{i=1}^{p-1} r_i I(\pi^i)(i\varepsilon)^{-1},$$

and thus we have $I(\tilde{s}) = s$ for

$$\tilde{s} = \sum_{i=1}^{p-1} r_i \pi^i (i\varepsilon)^{-1} \in R.$$

We even have $\tilde{s} \in \pi^n R$, since by our assumption we have that $r_i \in \pi^{n-i}R$ for $1 \leq i \leq p-1$, as we have inferred above.

Since we have now shown the existence of a $\tilde{s} \in \pi^n R$ with $I(\tilde{s}) = s$, and $s \in M$ was arbitrary, we have in fact proven that $M \subseteq I(\pi^n R)$. Together with the other inclusion this implies equality. □

The next example will also show an important feature of wild quotient singularities: In this example, the regularity of the invariant ring depends on the augmentation of certain elements. Later in Theorem 3.2.2, we will see that the regularity of the ring of invariants can be indeed determined this way in general.

Example 3.1.4. This example will be similar to the first one; we keep the notations of 3.1.1 for R, K, k, v, G. Again we will consider a badly chosen smooth model Y of $\mathbb{P}^1_K$ over R. First consider the model Y with a G-invariant parameter ξ, i.e. $\sigma(\xi) = \xi$. Then the quotient $X = Y/G$ is a smooth model again. If we chose for the model Y some parameter $\tilde{\xi} = \xi + c$ instead with $c \in R$, this gives rise to an action

$$\sigma : \tilde{\xi} \mapsto \tilde{\xi} + I(c).$$

This action defines the same model as before with smooth quotient, since we can invert this substitution. Now we can perform the substitution also another way: If we take a parameter

$$\eta = (\xi + c)/N(\pi)^n \text{ with } n \in \mathbb{N}, c \in R$$

instead, the model defined by the parameter η needs not to be the same as before in general. We explain why this is the case: The induced action on the model Y is now given by

$$\sigma : \eta \mapsto \eta + I(c)/N(\pi)^n.$$

By Proposition 3.1.3, we make the above substitution to replace η by ξ if and only if $I(c)/N(\pi)^n \in I(\pi)R$, i.e. if $I(c) \in I(\pi)N(\pi)^n R$. In particular, if we cannot make this substitution, then the so-defined model is different to our first choice.

We will now fix the situation for our example and do explicit calculations to identify the main cases in our problem. We will deal locally with the problem by examining the points where Proposition 3.1.3 is relevant and take B as the local complete germ there. So consider the action of G on $B := R[[\eta]]$ given by

$$\sigma : B \longrightarrow B \; ; \; \begin{cases} \sigma : \pi \longmapsto \pi \cdot (1+\lambda) \; , \\ \sigma : \eta \longmapsto \eta + \varepsilon ; \; , \end{cases}$$

with constants $\varepsilon, \lambda \in R$ where $\mathrm{Tr}(\varepsilon) = \mathrm{Tr}(\pi\lambda) = 0$ follows from the fact that G is cyclic. We define $\ell = v(\lambda)$ and $\delta = v(\varepsilon)$, where we extend v to B canonically by taking the Gauss-valuation. We are interested in the rings of invariants $A = B^G$; we have to distinguish two main cases:

Case 1: $\ell < \delta$. Then by Proposition 3.1.3, there exists a $y \in R$ such that $I(y) = \varepsilon$. We thus can write $B = R[[\xi]]$ where $\xi := \eta - y$. We also have

$$\sigma(\xi) = \xi + I(\eta) - I(y) = \xi + \varepsilon - \varepsilon = \xi;$$

thus ξ is G-invariant and $A = R^G[[\xi]]$. We note that the extension B/A is totally ramified with respect to the Gauss-valuation v.

Case 2: $\ell \geq \delta$. We remark that in this case we have in fact $\ell > \delta$, since if we had $\ell = \delta$, then by Proposition 3.1.3 there would exist a $y \in R$ with $v(y) = 0$ such that $I(y) = \varepsilon$. But now we can write $y = y_1 + y_2$ with $y_1 \in R$ and $y_2 \in \pi R$, and thus $I(y) = I(y_2)$. But by Proposition 3.1.3, one has

$$v(\varepsilon) = v(I(y_2)) \geq \ell + 1 = \delta,$$

which is a contradiction to the assumption $\ell = \delta$.

Therefore we can assume that $\ell > \delta$. By Proposition 3.1.3, we cannot find a $y \in R$ with $I(y) = \varepsilon$. But we can do something very similar - as in case 1 for the parameter η which we have replaced by the G-invariant parameter ξ , we can construct a parameter $g \in A$ with $v(g) = v(\pi)$ which we can substitute for π. Then we can write $B = R^G[[g, \eta]]$, and thus the invariants will be $A = R^G[[g, N(\eta)]]$. This implies that the extension B/A is unramified with respect to v.

So all what remains to be proven is the existence of such a g . The fact that we can replace π by g as parameter of B is equivalent to saying that $g/\pi \in B^\times$. So we want to find a $g \in A$ with $v(g) = v(\pi)$ and $g/\pi \in B^\times$. We will construct this g by induction. We will prove, that for all $n \in \mathbb{N}$, there exist $z_n \in R$ fulfilling

(1.n) $z_n \equiv \pi \mod \pi^2$
(2.n) If $n > 1$, then $z_n \equiv z_{n-1} \mod \pi^{n-1}$
(3.n) $I(z_n) = \pi^{n+\ell} f_n$ with $f_n \in B$

We can start the induction with $z_1 = \pi$, fulfilling (1.1) to (3.1). For the induction step, we assume that there exists a z_n fulfilling (1.n) to (3.n), and we now want to construct a z_{n+1} fulfilling (1.n+1) to (3.n+1).

First we claim that

$$f_n \equiv 0 \bmod \pi \quad \text{or} \quad v(I(f_n)) > v(f_n) + \delta.$$

This is true, since f_n is a power series in monomials $\eta^i \pi^j$, $0 \le i, j \le p-1$, with coefficients in $R^G[[N(\eta)]]$. One can now write out $I(\eta^i\pi^j)$ explicitly in terms of $I(\eta)$ and $I(\pi)$, for example by using the rules in Remark 3.2.4; an elementary calculation then shows that

$$I(\eta^i\pi^j) = \eta^i I(\pi^j) + \sigma(\pi)^j I(\eta^i) \equiv j\eta^i\pi^{j+\ell} + i\pi^{j+\delta}\eta^{i-1} \mod \pi^{j+\min(\delta,\ell)+1}.$$

Since $\ell > \delta$, the formula implies in particular that

$$v(I(\eta^i\pi^j)) \ge v(\eta^i\pi^j) + \delta.$$

Since f_n is a power series in those monomials with G -invariant coefficients, this implies that $v(I(f_n)) \ge v(f_n) + \delta.$ If $f_n \not\equiv 0 \bmod \pi,$ then strict inequality holds by Proposition 3.1.3, since

$$I(R) = I(\pi R) = \pi^{1+\ell} R \cap \operatorname{Ker} \operatorname{Tr}.$$

This proves our above claim.

Now to complete our induction step, we use that $I(\eta) = \pi^\delta u$ with $u \in R^\times$. Thus we can write $I(z_n) = I(\eta)\pi^{n+\ell-\delta}u^{-1}f_n$. Set $y_n = \eta \cdot \pi^{n+\ell-\delta} \cdot u^{-1} \cdot f_n$. Then for $z_{n+1} = z_n - y_n,$ we have by elementary calculations that

$$\begin{aligned} I(z_{n+1}) &= I(z_n) - I(y_n) = -\eta \cdot I(\pi^{n+\ell-\delta}u^{-1}f_n) \\ &= -\eta \cdot I(\pi^{n+\ell-\delta}) \cdot u^{-1} \cdot f_n - \eta \cdot \sigma(\pi^{n+\ell-\delta}) \left(I(f_n) \cdot u^{-1} + I(u^{-1}) \cdot \sigma(f_n) \right). \end{aligned}$$

By assumption of $\ell \ge \delta$, we have $\ell - \delta > 0,$ as discussed above. Now we also have $v(I(\pi^{n+\ell-\delta})) \ge n + \ell + 1$ and $v(u^{-1}) \ge \ell + 1$ by Proposition 3.1.3, and by our previous considerations we also have $v(I(f_n)) \ge \delta + 1.$ Thus $v(I(z_{n+1})) \ge n + \ell + 1,$ i.e. z_{n+1} fulfills 3.n+1. Now by construction, $y_n \equiv 0 \mod \pi^n,$ thus z_{n+1} fulfills 2.n and 1.n.

Also, since B is complete and the z_n fulfill 2.n, the z_n converge with respect to the π -adic topology, denote their limit by g . Since the z_n fulfill 3.n, the element g is G -invariant. Since the z_n fulfill 1.n, we have $v(g) = v(\pi)$ and $g/\pi \in B^\times.$ This is the desired element.

Summarizing the two cases, we can make the following statements: The invariant ring A is always regular. In case 1, i.e. $\ell < \delta$, the invariant ring A and thus the component of the special fiber of $X = Y/G$ has multiplicity 1 over R , the ring extension B/A is totally ramified with respect to v . In case 2, i.e. $\ell \ge \delta$, the special fiber of X has multiplicity p over $R^G,$ the ring extension B/A is unramified with respect to $v.$

There are several points to observe: In both cases, the invariant ring is regular, and in both cases, there exists a regular system of $\mathfrak{m}_B$ such that at least one of its parameters is invariant. Namely, in case 1, the desired parameter system is $(\pi, \xi),$ and in case 2, it is $(g, \eta).$ This shows also the main difference in the behaviour of the invariant morphism. Moreover, there is also an important thing to remark: In case 1, $I(\pi)$ divides $I(\eta)$ due to the condition $\ell < \delta,$ in case 2 we have contrarily that $I(\eta)$ divides $I(\pi)$ due to $\delta < \ell.$ Similar divisibility conditions are present in case 1 and 2a of Example 3.1.2, where the invariant ring is also regular, but not in case 2b of Example 3.1.4 where the invariant ring is singular. It will later turn out in Theorem 3.2.2 that under certain conditions, such a divisibility condition is equivalent to the fact that G acts as a pseudo-reflection.

Finally, we want to remark that despite the model X we have defined in the beginning is regular under every point where the action is similar as on B , the scheme X is not regular in general. As we have remarked at the beginning, X is regular and again a smooth model of $\mathbb{P}^1_{K^G}$ if $\ell < \delta$; however, if $\ell \geq \delta$, then X has a singularity under the point at infinity where $\eta = \infty$. This can be also calculated by elementary methods as in Example 3.1.2; however, we will not do that in this example, as it works similarly.

One can now try and generalize the idea behind this iteration process to prove a more general result for arbitrary complete normal local rings, where one has to keep some strong conditions on the underlying rings; this approach has been followed in the paper of Wewers [Wew10]. Instead of doing this, we will develop in the next section a more elegant and direct criterion to see where and when invariant rings are singular, stemming from the insight gained from examples of the preceding type.

3.2 A criterion for monogeneity

In this section we will study local actions of a cyclic group G of prime order p on a normal Noetherian local ring B . We fix a generator σ of G and obtain the operator

$$I := I_\sigma := \sigma - \mathrm{id} : B \longrightarrow B \ ; \ b \longmapsto \sigma(b) - b \ ,$$

called *augmentation map.* We introduce the B-ideal

$$I_G := (I(b) \ ; \ b \in B) \subset B$$

which is generated by the image $I(B)$. This ideal is called *augmentation ideal.* If this ideal is generated by an element $I(y)$, we call y an *augmentation generator.* Note that this ideal does not depend on the chosen generator σ of G . Moreover, if y is an augmentation generator with respect to a generator σ , then y is also an augmentation generator for any other generator of G . Since B is local, the ideal I_G is generated by an augmentation generator if I_G is principal.

We introduce a notion of pseudo-reflection which is stronger than the classical one in [Bou68, Chapter 5, ex. 7], but coincides when the G-action is tame:

Definition 3.2.1. An action of a group G on a regular local ring B by local automorphisms is called a *pseudo-reflection* if there exists a system, of parameters $(y_1, \dots, y_d)$ of B such that $y_2, \dots, y_d$ are invariant under G .

This section will be dedicated to the proof of the following Theorem 3.2.2, which generalizes a result commonly attributed to Serre, see [Ser68] or [CES03, Theorem 2.3.9]; compare also Corollary 2.4.17 and Remark 2.4.16. Parts of the following proof have been facilitated by W. Lütkebohmert.

Theorem 3.2.2. *Let B be a Noetherian normal local ring with residue field $k_B := B/\mathfrak{m}_B$. Let p be a prime number and G a p-cyclic group of local automorphisms of B . Let I_G be the augmentation ideal of B . Let A be the ring of G-invariants of B and assume that A is Noetherian. Consider the following statements:*

(a) *The augmentation ideal I_G is principal.*

(b) *B is a monogenous A-algebra.*

(c) G *acts on* B *as pseudo-reflection.*

(d) B *is a free* A *-module.*

(e) A *is regular.*

Then we have the implications

$$(a) \Leftrightarrow (b) \Rightarrow (c) \wedge (d).$$

If the canonical map $k_A \longrightarrow k_B$ *is an isomorphism, then one has*

$$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (d).$$

In particular, if additionally B *is regular, then we have implications:*

$$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (d) \Leftrightarrow (e).$$

We first give examples showing that the implications given in the theorem are indeed strict:

Example 3.2.3. (i) First, we give an example showing that if $k_A \to k_B$ is not an isomorphism, the implication (c) $\Rightarrow$ (a) needs not to hold:

Let k be a field of positive characteristic p and set

$$B := k(Z)[Y, X_1, X_2]_{(Y,X_1,X_2)} \quad .$$

We define a p -cyclic action of $G = \langle \sigma \rangle$ on B by

$$\sigma|k := \mathrm{id}_k \, , \, \sigma(Z) = Z + X_1 \, , \, \sigma(Y) = Y + X_2 \, , \, \sigma(X_i) = X_i \text{ for } i = 1, 2 \quad .$$

This is a well-defined action on the polynomial ring $k[Z, Y, X_1, X_2]$ of order p , since $p \cdot X_i = 0$ for $i = 1, 2$ and it leaves the ideal (Y, X_1, X_2) invariant. Thus it gives rise to an action on the localization

$$C = k[Z, Y, X_1, X_2]_{(Y,X_1,X_2)} \quad .$$

For any $g \in k[Z] - \{0\}$ the image is given by $\sigma(g) = g + I(g)$ with $I(g) \in X_1 \cdot k[Z] \subset \mathfrak{m}_C$. In particular, we have $\sigma(g) \in B^{\times}$. So the automorphism σ of C extends to an automorphism of the localization B of C . We note that $A = B^G$ is Noetherian, since A is finite over the ring $k(N(Z))[N(Y), X_1, X_2]_{(N(Y),X_1,X_2)}$ which is a Noetherian ring. The augmentation ideal is $I_G = B \cdot X_1 + B \cdot X_2$, cf. Proposition 3.2.8, and is not principal although G acts through a pseudo-reflection.

(ii) Now we give an example that if B is not regular, and $k_A \to k_B$ is an isomorphism, the implication (d) $\Rightarrow$ (a) needs not to hold:

Consider the ring $C = \mathbb{C}[[X, Y]]$ and the action of $H = (\mathbb{Z}/3\mathbb{Z})^2 = \langle \sigma_1, \sigma_2$ on C via

$$\sigma_1 : X \mapsto \zeta_3 X, \; Y \mapsto Y \; ; \; \sigma_2 : X \mapsto X, \; Y \mapsto \zeta_3 Y$$

with trivial action on $\mathbb{C}$, where ζ_3 is a primitive third root of unity. Denote $H' = \langle \sigma_1 \circ \sigma_2 \rangle$.

Let $A = C^H = \mathbb{C}[[X^3, Y^3]]$, and $B = C^{H'} = \mathbb{C}[[X^3, X^2Y, XY^2, Y^3]]$. Then $k_A = k_B = \mathbb{C}$, the ring A is regular, and B is Cohen-Macaulay. Thus B is a free A -module. But then $G = H/H' = \langle \sigma \rangle$ with

$$\sigma : X \mapsto \zeta_3 X \; ; \; Y \mapsto \zeta_3^2 Y.$$

In particular, we have

$$I_G(X^3) = I_G(Y^3) = 0, \text{ and } I_G(X^2Y) = (\zeta_3 - 1)X^2Y, I_G(XY^2) = (\zeta_3^2 - 1)XY^2.$$

Thus the augmentation ideal is $I_G = B \cdot X^2Y + B \cdot XY^2$, cf. Proposition 3.2.8, and thus not principal, although B is a free A-module.

We start the proof of the theorem by several preparations.

Remark 3.2.4. For $b_1, b_2, b \in B$, the following relations are true:

(i) $I(b_1 \cdot b_2) = I(b_1) \cdot \sigma(b_2) + b_1 \cdot I(b_2)$

(ii) $I\left(\frac{b_1}{b_2}\right) = \frac{I(b_1)}{\sigma(b_2)} - b_1 \frac{I(b_2)}{b_2 \sigma(b_2)}$

(iii) $I(b^n) = \left(\sum_{i=1}^{n} \sigma(b)^{i-1} b^{n-i}\right) \cdot I(b)$

Proof. (i) and (ii) follow by a direct calculation and (iii) by induction from (i). □

For the implication (a) → (b) we need a technical lemma.

Lemma 3.2.5. *Let $y \in B$ be an augmentation generator. Then set, inductively,*

$$\begin{aligned} y_i^{(0)} &:= y^i && \text{for } i = 0, \ldots, p-1 \\ y_i^{(1)} &:= \frac{I\left(y_i^{(0)}\right)}{I\left(y_1^{(0)}\right)} && \text{for } i = 1, \ldots, p-1 \\ y_i^{(n+1)} &:= \frac{I\left(y_i^{(n)}\right)}{I\left(y_{n+1}^{(n)}\right)} && \text{for } i = n+1, \ldots, p-1 \end{aligned}$$

for $n + 1 = 1, \ldots, p-1$. Then

$$y_i^{(n)} = \sum_{0 \le k_1 \le \ldots \le k_{i-n} \le n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \text{ for } i = n, \ldots, p-1$$

and, in particular,

$$\begin{aligned} y_n^{(n)} &= 1 \\ y_{n+1}^{(n)} &= \sum_{j=1}^{n+1} \sigma^{j-1}(y) \\ I\left(y_{n+1}^{(n)}\right) &= \sigma^{n+1}(y) - y \end{aligned}$$

Furthermore, $y_{n+1}^{(n)}$ is again an augmentation generator for $n = 0, \ldots, p-2$.

Proof. We proceed by induction on n. For $n = 0$ the formulae are obviously correct. For the convenience of the reader we also display the formulae for $n = 1$. Due to Remark 3.2.4 one has

$$\begin{aligned} y_i^{(1)} &= \frac{I\left(y_i^{(0)}\right)}{I\left(y_1^{(0)}\right)} = \frac{I(y^i)}{I(y)} = \sum_{j=1}^{i} \sigma(y)^{j-1} y^{i-j} \\ &= \sum_{0 \le k_1 \le \ldots \le k_{i-1} \le 1} \prod_{\nu=1}^{i-1} \sigma^{k_\nu}(y) \end{aligned}$$

since the last sum can be viewed as a sum over an index j where $i-j$ is the number of the $k_\nu = 0$. In particular, the formulae are correct for $y_1^{(1)}$ and $y_2^{(1)}$. Moreover

$$I\left(y_2^{(1)}\right) = I(\sigma(y) - y) = \sigma^2(y) - y \ .$$

Since σ^2 is generator of G for $2 < p$, the element $y_2^{(1)}$ is an augmentation generator as well.
Now assume that the formulae are correct for n . Since $y_{n+1}^{(n)}$ is an augmentation generator, $I\left(y_{n+1}^{(n)}\right)$ divides $I\left(y_i^{(n)}\right)$ for $i = n+1, \ldots, p-1$. Then it remains to show

$$I\left(y_i^{(n)}\right) = \left(\sigma^{n+1}(y) - y\right) \cdot y_i^{(n+1)} \text{ for } i = n+1, \ldots, p-1 \ .$$

For the left hand side one computes

$$\begin{aligned} LHS &= I\left(\sum_{0 \le k_1 \le \ldots \le k_{i-n} \le n} \prod_{j=1}^{i-n} \sigma^{k_j}(y)\right) = \sum_{0 \le k_1 \le \ldots \le k_{i-n} \le n} I\left(\prod_{j=1}^{i-n} \sigma^{k_j}(y)\right) \\ &= \sum_{0 \le k_1 \le \ldots \le k_{i-n} \le n} \cdot \left(\prod_{j=1}^{i-n} \sigma^{k_j+1}(y) - \prod_{j=1}^{i-n} \sigma^{k_j}(y)\right) \\ &= \sum_{1 \le k_1 \le \ldots \le k_{i-n} \le n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) - \sum_{0 \le k_1 \le \ldots \le k_{i-n} \le n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \ . \end{aligned}$$

Now all terms occurring in both sums cancel. These are the terms with $k_{i-n} \le n$ in the first sum and $1 \le k_1$ in the second sum.
For the right hand side one computes

$$\begin{aligned} RHS &= \left(\sigma^{n+1}(y) - y\right) \cdot \sum_{0 \le k_1 \le \ldots \le k_{i-n-1} \le n+1} \prod_{j=1}^{i-n-1} \sigma^{k_j}(y) \\ &= \sum_{0 \le k_1 \le \ldots \le k_{i-n} = n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) - \sum_{0 = k_1 \le \ldots \le k_{i-n} \le n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \ . \end{aligned}$$

Comparing both sides one obtains $LHS = RHS$. In particular we have

$$\begin{aligned} y_{n+1}^{(n+1)} &= 1 \\ y_{n+2}^{(n+1)} &= \sum_{0 \le k_1 \le n+1} \prod_{j=1}^{1} \sigma^{k_1}(y) = \sum_{j=1}^{n+2} \sigma^{j-1}(y) \\ I\left(y_{n+2}^{(n+1)}\right) &= \sigma^{n+2}(y) - y \ . \end{aligned}$$

So $y_{n+2}^{(n+1)}$ is an augmentation generator for $n+2 < p$, since σ^{n+2} generates G . This concludes the technical part. □

Proposition 3.2.6. *Assume that the augmentation ideal I_G is principal and let $y \in B$ be an augmentation generator. Then B decomposes into the direct sum*

$$B = A \cdot y^0 \oplus A \cdot y^1 \oplus \ldots \oplus A \cdot y^{p-1} \ .$$

Proof. Since $I(y) \neq 0$, the element y generates the field of fractions $Q(B)$ over $Q(A)$. Therefore

$$Q(B) = Q(A) \cdot y^0 \oplus Q(A) \cdot y^1 \oplus \ldots \oplus Q(A) \cdot y^{p-1} \ .$$

Then it suffices to show the following claim:

Let $a, a_0, \ldots, a_{p-1} \in A$. Assume that a divides

$$b = a_0 \cdot y^0 + a_1 \cdot y^1 + \ldots + a_{p-1} \cdot y^{p-1} \quad .$$

Then a divides $a_0, a_1, \ldots, a_{p-1}$.

If $b = a \cdot \beta$, then $I(b) = a \cdot I(\beta)$. Since $I(\beta) = \beta_1 \cdot I(y)$, we get $I(b) = a\beta_1 \cdot I(y)$. So we see that a divides $I(b)/I(y) \in B$. Using the notations of Lemma 3.2.5, set

$$\begin{array}{lllll} b^{(0)} & := & b & = & a_0 \cdot y^0 + a_1 \cdot y^1 + \ldots a_{p-1} \cdot y^{p-1} \\ b^{(1)} & := & \dfrac{I\left(b^{(0)}\right)}{I(y)} & = & a_1 + a_2 \dfrac{I\left(y^2\right)}{I(y)} + \ldots + a_{p-1} \dfrac{I\left(y^{p-1}\right)}{I(y)} \\ & & & = & a_1 \cdot y_1^{(1)} + a_2 \cdot y_2^{(1)} + \ldots + a_{p-1} \cdot y_{p-1}^{(1)} \\ b^{(n)} & := & \dfrac{I\left(b^{(n-1)}\right)}{I\left(y_n^{(n-1)}\right)} & = & a_n \cdot y_n^{(n)} + a_{n+1} \cdot y_{n+1}^{(n)} + \ldots + a_{p-1} \cdot y_{p-1}^{(n)} \quad . \end{array}$$

Due to the observation above, we see by induction that a divides $b^{(0)}, b^{(1)}, \ldots, b^{(p-1)}$, since $y_{n+1}^{(n)}$ is an augmentation generator for $n = 1, \ldots, p-2$. So we obtain

$$a \,\big|\, b^{(p-1)} = a_{p-1} \cdot y_{p-1}^{(p-1)} = a_{p-1} \quad .$$

Now proceeding downwards, one obtains

$$a \,\big|\, b^{(p-2)} = a_{p-2} + a_{p-1} \cdot y_{p-1}^{(p-2)} \text{ and, hence, } a \,\big|\, a_{p-2}$$
$$a \,\big|\, b^{(n)} = a_n + a_{n+1} \cdot y_{n+1}^{(n)} + \ldots + a_{p-1} \cdot y_{p-1}^{(n)} \text{ and, hence, } a \,\big|\, a_n$$

for $n = p-1, p-2, \ldots, 0$. □

Lemma 3.2.7. *Let $R \subset B$ be a local subring of B which is invariant under G such that the canonical map $R/\mathfrak{m}_R \xrightarrow{\sim} B/\mathfrak{m}_B$ is an isomorphism. Let $(y_1, \ldots, y_d)$ be a generating system of the maximal ideal $\mathfrak{m}_B$. Then I_G is generated by $I(y_1), \ldots, I(y_d)$.*

Proof. Due to the assumption, we have $B = R + \mathfrak{m}_B$ and, hence, $I(B) = I(\mathfrak{m}_B)$. Furthermore, we have

$$\mathfrak{m}_B = \mathfrak{m}_B^2 + \sum_{i=1}^{d} R \cdot y_i \quad .$$

Since I is R-linear, we get

$$I(\mathfrak{m}_B) = I\left(\mathfrak{m}_B^2\right) + \sum_{i=1}^{d} R \cdot I(y_i) \quad .$$

Due to Remark 3.2.4, one knows $I\left(\mathfrak{m}_B^2\right) \subset \mathfrak{m}_B \cdot I(\mathfrak{m}_B)$. So one obtains

$$I(\mathfrak{m}_B) \subset \mathfrak{m}_B \cdot I(\mathfrak{m}_B) + \sum_{i=1}^{d} R \cdot I(y_i) \quad .$$

Since B is local, Nakayama's Lemma yields

$$I_G = B \cdot I(B) = B \cdot I(\mathfrak{m}_B) = \sum_{i=1}^{d} B \cdot I(y_i) \quad .$$

Thus the assertion is proved. □

Proposition 3.2.8. *Keep the assumption of the Corollary;* $k_A \xrightarrow{\sim} k_B$. *Let* $(y_1, \ldots, y_d)$ *be a generating system of the maximal ideal* $\mathfrak{m}_B$. *Then the following assertions are true:*

(i) $I_G = B \cdot I(y_1) + \ldots + B \cdot I(y_d)$

(ii) *If the ideal* $I_G = B \cdot I(B)$ *is principal, then there exists an index* $i \in \{1, \ldots, d\}$ *with* $I_G = B \cdot I(y_i)$.

Proof. (i) Let $\widehat{B}$ be the $\mathfrak{m}_B$-adic completion of B . Since $B \to \widehat{B}$ is faithfully flat and $\widehat{B} \cdot \mathfrak{m}_B = \mathfrak{m}_{\widehat{B}}$, it suffices to prove the assertion for the completion $\widehat{B}$. For complete local rings there exists a G-invariant lift R of the residue field k . Namely, in the mixed characteristic case $(0, p)$, by Cohen's structure theorem, there always exists a coefficient ring of k inside A which we can choose as R, and in the equal characteristic case, the residue field k lifts into $\widehat{A}$. Now we can apply Lemma 3.2.7 and obtain the assertion.
(ii) Since I_G is principal, $I_G/\mathfrak{m}_B I_G$ is generated by one of the $I(y_i)$ and, hence, again by Nakayama's Lemma $I_G = B \cdot I(y_i)$ for a suitable $i \in \{1, \ldots, d\}$. □

Proof of the Theorem

(a) → (b): This follows from Proposition 3.2.6.
(b) → (a): If $B = A[y]$, then the minimal polynomial of y over the field of fraction is of degree p and the coefficients of this polynomial belong to A . Then B has $y^0, y^1, , \ldots, y^{p-1}$ as an A-basis. Due to Remark 3.2.4, the ideal I_G is generated by $I(y)$ and, hence, principal.
(b) → (d) is clear: Any monogenous algebra is free.
(d) → (e) follows from [Mat80, Theorem 23.7]under the condition that B is regular.
(e) → (d): Since B is finite over A and B is Cohen-Macaulay, B is free over A .
(c) → (a): If G is a pseudo-reflection, I_G is generated by $I(y)$ due to Proposition 3.2.8 where $y, x_2, \ldots, x_p$ is a system of parameters with $x_i \in \mathfrak{m}_A$ for $i = 2, \ldots, p$ if $k_A = k_B$.
(a) → (c): First assume that the canonical map $k_A \to k_B$ of the residue fields is an isomorphism. If I_G is principal, one can choose an augmentation parameter $y \in \mathfrak{m}_B$ which is part of a system of parameters $(y, y_2, \ldots, y_d)$ due to Proposition 3.2.8. Due to Proposition 3.2.6, we know that B decomposes into the direct sum

$$B = A \cdot y^0 \oplus A \cdot y^1 \oplus \ldots \oplus A \cdot y^{p-1} \quad .$$

Now we can represent

$$y_j = \sum_{i=0}^{p-1} a_{i,j} \cdot y^i \ \text{ for } \ j = 2, \ldots, d \quad .$$

Then set

$$\tilde{y}_j := y_j - \sum_{i=1}^{p-1} a_{i,j} y^i = a_{0,j} \in A \cap \mathfrak{m}_B = \mathfrak{m}_A \ \text{ for } \ j = 2, \ldots, d \quad .$$

So $(y, \tilde{y}_2, \ldots, \tilde{y}_d)$ is a system of parameters of B as well. Thus G acts by a pseudo-reflection.
Now assume that the canonical map $k_A \to k_B$ of the residue fields is not an isomorphism. Then we show the direction (c) → (e). We know that B is a free A-module of rank p . Therefore $B \otimes_A k_A$ is k_A-vector space of rank p and

the residue map $B \otimes_A k_A \to k_B$ is surjective and, hence, the rank $[k_B : k_A] \leq p$. Actually, the rank is p . Namely, if G does not act trivially on k_B , then $[k_B : k_A] \geq \mathrm{card}(G)$, and if G acts trivially on k_B then for any $z \in B$ one has $z^p \equiv N(z) \mod \mathfrak{m}_B$ where $N(z) \in A$ is norm of z and so $k_B^p \subset k_A$ and, hence, $[k_B : k_A] = p$. Thus the rank of k_B over k_A is also p , we see $B \otimes_A k_A \to k_B$ is an isomorphism. This means $\mathfrak{m}_B$ is generated by $\mathfrak{m}_A$. Then G acts through a pseudo-reflection. □

3.3 Application to models of curves

In this section we will apply the regularity criterion from Theorem 3.2.2 to monodromy actions on regular models of curves. We will examine local rings arising in the following situation: Let R be a complete discrete valuation ring with uniformizer π , algebraically closed residue field k and quotient field K. Let C be an irreducible curve over K, and let Y be a normal model of C over R . We will consider a prime cyclic group action of $G = \langle\sigma\rangle \cong \mathbb{Z}/p\mathbb{Z}$ on Y , where we assume that G acts on R, i.e. $\sigma(R) \subseteq R$, and that the induced action on k is trivial.

Given a normal crossings closed point P of Y , several things can happen:

Definition 3.3.1. If Y is regular normal crossings at P , then two cases may occur: If P lies on exactly one component of the special fiber of Y , we will call it a *geometrically smooth* point, if P is a normal crossing point between two components, we call P a *geometric double points* of Y .

One has to note that the property to be normal crossing at a point is local, i.e. depends only on the localization at P. So it makes sense to use the properties normal crossing, smooth or double point to qualify local germs of fibered surfaces.

In the following, we will consider a complete local germ

$$B = \widehat{\mathcal{O}}_{Y,P}$$

for a closed point P on Y . We cite an impotant result on rings of that type:

Proposition 3.3.2. *If B is geometrically smooth, then there exists an isomorphism*

$$B \cong R[[u, v]]/(u^m - \pi a)$$

for some $a \in R^\times + (u, v) \subseteq B^\times$ and $m \in \mathbb{N}$. If B is a geometrical double point, then there exists an isomorphism

$$B \cong R[[u_1, u_2]]/(u_1^{m_1} u_2^{m_2} - \pi a)$$

for some $a \in R^\times + (u_1, u_2) \subseteq B^\times$ and $m_1, m_2 \in \mathbb{N}$.

Proof. See [Liu02, 9.2.34f] □

We will now mainly be interested in applying Proposition 3.3.4 to geometrically smooth points which are fixed by G . We will consider the following situation:

Notations 3.3.3.

R	complete discrete valuation ring
π	uniformizer of R
B	geometrically smooth local ring (in particular regular of Krull dimension 2)
$G=\mathbb{Z}/p\mathbb{Z}$	group acting nontrivially on B
$A=B^G$	invariant ring
$\mathfrak{m}_B,\mathfrak{m}_A$	maximal ideals of B, A
$k_B=k_A$	residue field of B, A and R

First we prove another characterization in this setting:

Proposition 3.3.4. *The following statements are equivalent:*

(a) *The augmentation ideal* I_G *is principal.*

(b) B *is a monogenous* A*-algebra.*

(c) G *acts on* B *as pseudo-reflection.*

(d) A *is geometrically smooth.*

Proof. The equivalence of (a)-(c) follows from Theorem 3.2.2 and our assumption that $k_B=k_A$. We will now prove (c) $\Leftrightarrow$ (d), i.e. that A is geometrically smooth if and only if G acts as pseudo-reflection.

(c) $\Rightarrow$ (d): By Proposition 3.3.2, we can fix an isomorphism

$$B\cong R[[u_B,v_B]]/(u_B^m-\pi a_B)\cong R^G[[u_B,v_B]]/(u_B^{m'}-\pi' a'_B)$$

with $m,m'\in\mathbb{N}, a_B,a'_B\in B^\times$ and π' the uniformizer of R^G. Since G acts as pseudo-reflection, there exists $\alpha\in(\mathfrak{m}_B-\mathfrak{m}_B^2)\cap A$. In particular, we have $\mathfrak{m}_B=(v_B,\alpha)$ or $\mathfrak{m}_B=(u_B,\alpha)$. We will now treat the two cases separately:

Case 1: $\mathfrak{m}_B=(u_B,\alpha)$. First we claim that $\mathfrak{m}_A=(N(u_B),\alpha)$, where by $N(u_B)$ we denote the G-norm of u_B. For this we note: Fix a G-invariant system S of representatives of k_B in R. Now any element in the complete ring B can be uniquely written as

$$b=\sum_{i=0}^{\infty}\alpha^i\sum_{j=0}^{\infty}s_{ij}u_B^j \quad \text{with } s_{ij}\in S.$$

Alternatively, we can write b as

$$b=\sum_{i=0}^{\infty}\alpha^i\sum_{j=0}^{\infty}N(u_B)^j\sum_{k=0}^{p-1}s_{ijk}u_B^k=\sum_{k=0}^{p-1}u_B^k\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}s_{ijk}\alpha^iN(u_B)^j\right) \quad \text{with } s_{ijk}\in S,$$

e.g. by successively replacing u_B^j by $u_B^kN(u_B)^m$ where $j=mp+k$ with $0\le k<p$ and then adding the difference terms. Since the B-valuations of the difference terms increases, one can go to the limes.

Now for an element b to be invariant under G , it is a sufficient condition that $s_{ijk}=0$ for $k\neq 0$. This condition is also necessary, since the extension of quotient fields $Q(B)/Q(A)$ is generated by u_B. So the invariant ring A is generated as power series ring in by the elements fulfilling these conditions, i.e. elements of the form

$$\sum_{j=0}^{\infty} s_{ij} N(u_B)^j \quad \text{with } s_{ij} \in S,$$

showing that the maximal ideal $\mathfrak{m}_A$ is indeed generated by $N(u_B)$ and α.

Now we proceed to prove that A is geometrically smooth. From the above it follows that $A/(\pi')$ is generated by $N(u_B)$ and α. Since α and u_B generate $\mathfrak{m}_A$, we must have that $\alpha = u_B f + v_B g$ with $f \in B$ and $g \in B^\times$. Also note that $N(u_B)$ is nilpotent in $A/(\pi')$, since u_B is. So the maximal ideal of $(A/(\pi'))_{red}$ is generated by the single element v_B , and thus again A is geometrically smooth by definition.

Case 2: $\mathfrak{m}_B = (v_B, \alpha)$. It follows in complete analogy to case 1 that $\mathfrak{m}_B = (N(v_B), \alpha)$, and $\alpha = u_B f + g$ with $f \in B^\times$ and $g \in B v_B$. We will now make several simplifications on the generators v_B and α.

If g is a unit times v_B , then we are in case 1, which has already been proved. So we can assume that $g \in v_B \mathfrak{m}_B$, say $g = u_B v_B h_1 + h_2$ where h_2 is an power series only in v_B over our chosen system of representatives S of k_B in A, and $m \geq 2$. By subtracting multiples of $N(v_B)$ from α , we can assume that

$$h_2 = \sum_{k=0}^{\infty} s_k v_b^k \quad \text{with } s_k \in S \text{ and } s_k = 0 \text{ if } p \text{ divides } k.$$

In particular, we can write $h_2 = v_B^a t$ with $a \geq 2$ and coprime to p , and $t \in B^\times$. Now it suffices to prove that $h_2 = 0$. Since then there exists a big integer $\mu \in \mathbb{N}$ such that

$$g^{p\mu} \equiv u_B^{p\mu} h_1^{p\mu} \equiv 0 \mod (\pi')$$

(because such an μ exists for u_B). Thus α is nilpotent modulo π' , and the maximal ideal $(A/(\pi'))_{red}$ is generated by $N(v_B)$.

Now let us show that $h_2 = 0$. For this, we assume the contrary and derive a contradiction; then we are done. As we have argued, we can write

$$\alpha = u_B(f + v_B h_1) + h_2 = u_B(f + v_B h_1) + v_B^a t,$$

where $(f + v_B h_1)$ and t are both units in B , and $a \in \mathbb{N}$ is coprime to p . Thus, without changing the isomorphy type of B, it is possible to replace u_B by $u_B(f + v_B h_1)$ and v_B by $v_B \sqrt[a]{-t}$. For the latter note that $\sqrt[a]{-t}$ is in B since a is coprime to p and B is complete (for more details cf. Prop 4.4.4). So we arrive at the situation where B is as in the beginning, and

$$\alpha = u_B - v_B^a \ \in A$$

with $a \geq 2$ coprime to p.

We want to remark that if p does not divide m , we can use [CES03, Lemma 2.3.2] and the structure of α to obtain a contradiction already at this point: The Lemma implies that one can choose u_B such that $I(u_B)$ lies in $R[[u_B]]$. But from above and Remark 3.2.4, we may infer that

$$I(u_B) = I(v_B) \left(\sum_{i=1}^{a} v_B^{a-i} v_B^{i-1} \right)$$

which is a contradiction, since the right side contains nonzero terms in v_B, and the left side does not.

We now derive a contradiction in the general case by showing that it is impossible for this α to be G-invariant. Hilbert's theorem 90 implies that $I(Q(B))$ is a $Q(A)$-vector space of dimension $p-1$. Consider the ring C which is generated over R^G by the norms $N(u_B)$ and $N(v_B)$. Note that we have

$$C := R^G[[N(u_B), N(v_B)]]/(N(u_B) - N(\pi)N(a_B)),$$

which is again a geometrically smooth ring. Since $Q(B)$ is a $Q(C)$-vector space of dimension p, the augmentation image $I(Q(B))$ is a $p(p-1)$-dimensional $Q(C)$-vector space.

We calculate that

$$I(v_B^n) = \left(\sum_{i=1}^{n} \sigma(v_B)^{i-1} v_B^{n-i} \right) \cdot I(v_B) =: \vartheta_n \cdot I(v_B)$$

(cf. 3.2.4), where ϑ_n is zero for $n = 0$, one for $n = 1$, and else a nonunit element in B containing the term v_B^{n-1}. Since $\alpha \in A$, we have $I(\alpha) = 0$ and thus in particular

$$I(u_B) = \vartheta_a I(v_B).$$

Also, we want to remark that u_B must divide $I(u_B)$, since $\sigma(u_B)$ is nilpotent modulo π. Thus we can write

$$\sigma(u_B) = \zeta u_B \quad \text{for some } \zeta \in B^{\times}.$$

So se can write

$$I(u_B^n) = \left(\sum_{i=1}^{n} \sigma(u_B)^{i-1} u_B^{n-i} \right) \cdot I(u_B) =: \psi_n \cdot u_B^{n-1} I(u_B),$$

where ψ_n is a unit for n coprime to p and zero for $n = 0$.

Now remark that the elements $v_B^i u_B^j, 0 \leq i, j \leq p-1$ form a $Q(C)$-basis of $Q(B)$. Thus their images under I generate the $Q(C)$-vector space $I(Q(B))$. If we can prove that at least $p(p-1)+1$ of the $I(v_B^i u_B^j)$ are linearly independent over $Q(C)$, we have the desired contradiction, since by our previous argumentation with Hilbert's theorem 90, this cannot be.

So using Remark 3.2.4, we now explicitly calculate

$$\begin{aligned} I(v_B^i u_B^j) &= I(v_B^i)\sigma(u_B^j) + v_B^i I(u_B^j) = \zeta u_B^j \cdot \vartheta_i I(v_B) + v_B^i u_B^{j-1} \cdot \psi_j I(u_B) \\ &= I(v_B)(\zeta^j u_B^j \cdot \vartheta_i + v_B^i u_B^{j-1} \cdot \psi_j \vartheta_a) \end{aligned}$$

It now can be seen that the number of linearly independent elements $I(v_B^i u_B^j)$ over $Q(C)$ stays the same when we divide all of them by $I(v_B)$, so it remains to determine how many of the elements

$$\zeta^j u_B^j \cdot \vartheta_i + v_B^i u_B^{j-1} \cdot \psi_j \vartheta_a, \quad 0 \leq i, j \leq p-1$$

are linearly independent over $Q(C)$. Since the right side contains the term $v_B^{i+a-1} u_B^{j-1}$ for $j > 0$ with nonzero coefficient in $Q(C)$ and the left side does not, the $p(p-1)$ elements for $0 \leq i, j \leq p-1, j \geq 1$ are linearly independent. Considering the element for $i = 1, j = 0$ yields

$$I(v_B^1 u_B^0)/I(v_B) = 1,$$

which is also linearly independent, since $a \geq 2$ and the chosen monomials in the previous elements were nonzero, linearly independent, and nonconstant.

This is the desired contradiction proving case 2.

(d) $\Rightarrow$ (c): Assume that A is geometrically smooth. By Proposition 3.3.2, we have isomorphisms

$$A \cong R^G[[u_A, v_A]]/(u_A^{m'} - \pi' a_A) \text{ and } B \cong R[[u_B, v_B]]/(u_B^m - \pi a_B)$$

for $a_A \in (R^G)^\times + u_A A + v_A A \subseteq A^\times \subseteq B^\times$ and $a_B \in R^\times + u_B B + v_B B \subseteq B^\times$, and $m, m' \in \mathbb{N}$. Note that one must have $m' = fm$ with $f = 1$ or $f = p$, since the localization of B at the prime ideal (u_B) induces an extension of discrete valuation rings.

In the following, for $a, b \in B$, we will write abbreviating $a \sim b$ if $a/b \in B^\times$. Note that $\sim$ is an equivalence relation. For example, we have $u_B^m \sim \pi$ and $u_A^{fm} \sim \pi'$ and $\pi'^p \sim \pi$.

It now suffices to show: There is an $\alpha \in (\mathfrak{m}_B - \mathfrak{m}_B^2) \cap A$.

Case 1: We have $f = p$. In this case, note that $u_A^m \sim \pi$. We also have noted that $u_B^m \sim \pi$. Thus $(u_B/u_A)^m \sim 1$. Since u_B/u_A lies in $Q(B)$, we have that $u_B \sim u_A$, i.e. $u_B/u_A \in B^\times$. Thus $u_A \in (\mathfrak{m}_B - \mathfrak{m}_B^2) \cap A$, which is what we wanted to prove.

Case 2: We have $f = 1$. Since A is regular, we have by Theorem 3.2.2 that B is free over A of rank p. Also, $B/u_A B$ is a free $A/u_A A$-module of rank p. We now have a canonical injection

$$\varphi : A/u_A A = k[[v_A]] \hookrightarrow B/u_B B = k[[v_B]].$$

We will show that φ is an isomorphism. From this it will follow that $v_A \in (\mathfrak{m}_B - \mathfrak{m}_B^2) \cap A$, proving our claim. Now φ is of finite degree and a morphism of normal rings. So it suffices to prove that the degree of φ is one. First note that the length of

$$Q(A/u_A A) \otimes_{A/u_A A} B/u_A B$$

as $Q(A/u_A A)$-module is p. Since $B/u_A B$ is a free $A/u_A A$-module of rank p, this must be the same as p times the length of

$$Q(A/u_A A) \otimes_{A/u_A A} B/u_B B.$$

which thus must have had length one, proving our claim. □

Now we want to show an interesting relation between this statement and some known invariants associated to the G-action on B in the case where the special fiber of B is reduced. So in the following we will assume that

$$B = R[[\eta]].$$

Denote by v_π the Gauss-valuation induced by the π-valuation on B. Define ramification numbers $\ell = v_\pi(I(\pi)) - 1$ and $\delta = v_\pi(I(\eta))$. Denote by ord the $\overline{\eta}$-valuation on $k[[\overline{\eta}]] = B/(\pi)$. Then define $a = \operatorname{ord}(I(\eta)/\pi^\delta \mod \pi)$, which is the vanishing order of $I(\eta)/\pi^\delta \mod \pi$ at $\overline{\eta} = 0$.

The above defined invariants occur in a natural way when dealing with p-cyclic R-automorphisms of B in the case of Hurwitz trees, where δ is called depth, and a occurs as pole degree of the Artin differential, see e.g. [Hen99] or [BW09]. ℓ can be interpreted as the different of the extension R/R^G.

Summarizing, our new setting is as follows:

Notations 3.3.5.

R	complete discrete valuation ring
π	uniformizer of R
B	$= R[[\eta]]$
$G = \mathbb{Z}/p\mathbb{Z}$	group acting nontrivially on B
$A = B^G$	invariant ring
$\mathfrak{m}_B, \mathfrak{m}_A$	maximal ideals of B, A
$k_B = k_A$	residue fields of B, A and R
v_π	Gauss-valuation on B induced by π
ℓ	$:= v_\pi(I(\pi)) - 1$, different of R/R^G
δ	$:= v_\pi(I(\eta))$
ord	the $\overline{\eta}$-valuation on $k[[\overline{\eta}]] = B/(\pi)$
a	$:= \mathrm{ord}(I(\eta)/\pi^\delta \mod \pi)$,

We can now apply proposition 3.3.4 to this situation:

Corollary 3.3.6. *The invariant ring* $A = B^G$ *is geometrically smooth if and only if* $\delta > \ell$ *, or* $\delta \leq \ell$ *and* $a = 0$*. We have in particular:*
(i) *If* $\delta > \ell$*, the extension* B/A *is totally ramified with respect to* v_π*.*
(ii) *If* $\delta \leq \ell$ *and* $a = 0$*, the extension* B/A *is unramified with respect to* v_π*.*

Proof. This is a direct application of Proposition 3.3.4 $(a) \Leftrightarrow (d)$ to the regular system (π, η) of $\mathfrak{m}_B$. By Proposition 3.2.8, the augmentation ideal is $I_G = (I(\eta), I(\pi))$. So in our case, (a) is equivalent to the fact that $I(\pi)$ divides $I(\eta)$, or $I(\eta)$ divides $I(\pi)$. In summary, we can conclude in our situation that A is geometrically smooth if and only if $I(\pi)$ divides $I(\eta)$, or $I(\eta)$ divides $I(\pi)$.

We will now reformulate this statement in terms of ℓ, δ, a : Since $I(\pi) \in R$ by our assumption that $\sigma R \subseteq R$, we have that $I(\pi)$ divides $I(\eta)$ if and only if $\delta > \ell$. On the other hand, $I(\eta)$ divides $I(\pi)$ if and only if $\delta \leq \ell$ and $I(\eta)/\pi^\delta \in B^\times$. But the latter is equivalent to $a = 0$, this proves the first part of our claim.

For the second part, we will exhibit regular systems of A . In the case of (i), i.e. when $I(\pi)$ divides $I(\eta)$, the element π is an augmentation generators of B. Since G is also a pseudo-reflection, we can then find a $\xi \in A$ such that (π, ξ) is a regular system of $\mathfrak{m}_B$. Thus $(N(\pi), \xi)$ is a regular system of $\mathfrak{m}_A$. Since it contains no element with valuation of B, the extension B/A is totally ramified.

Similarly one can treat the case of (ii), where $I(\eta)$ divides $I(\pi)$ and $a = 0$. Here η is an augmentation generator, and G is again a pseudo-reflection. So there has to exist an element $g \in A$ with $v(g) = v(\pi)$ such that (η, g) is a regular system of $\mathfrak{m}_B$. This implies that $(N(\eta), g)$ is a regular system of $\mathfrak{m}_A$; since it contains an element with same valuation as B (namely g), the extension B/A is unramified. This was the last remaining claim to prove. □

Remark 3.3.7. In Example 3.1.2, when localizing at the point $\eta = 0$ one is in the case $a = 1, \delta = \ell$. In every other point, one has $a = 0$ and $\delta = \ell$.

In Example 3.1.4, one had $a = 0$. The δ and ℓ defined in the example coincide with those defined above.

We will now further characterize certain actions on $B = R[[\eta]]$ and find obstructions on the G -action in terms of the invariants defined earlier.

Definition 3.3.8. A parameter ξ such that (π, ξ) is a regular system of parameters of $\mathfrak{m}_B$ is called *geometric parameter* of B. If there exists a geometric parameter ξ such that ξ divides $\sigma(\xi)$ we say that the G-action has a *geometric fixed point.*

Lemma 3.3.9. *If $a = 1$, then the G-action has a geometric fixed point.*

Proof. We can assume that $\delta \leq \ell$, since else by Corollary 3.3.6 we can take η such that $I(\eta) = 0$ and are done. Consider the power series expansion resp. the Weierstraß decomposition of $I(\eta)$ in η. We write

$$I(\eta) = \sum_{i=0}^{\infty} e_i \eta^i = \pi^\delta (\eta - \alpha) u \text{ where } e_i, \alpha \in R, u \in B^\times.$$

We will show the claim by induction: We will show that there exists a geometric parameter η' such that

$$I(\eta') = \sum_{i=0}^{\infty} e'_i \eta'^i,$$

with $v_\pi(e'_0) > v_\pi(e_0)$ and $\eta' \equiv \eta \mod e_0 \pi^{-\delta}$. Then ξ can be obtained as limit of a sequence of repeating the process, since B is complete.

We claim that $\eta' = \eta - \alpha$ is a valid choice. Consider

$$I(\eta') = I(\eta) - I(\alpha) = \pi^\delta \eta' u - I(\alpha) \text{ for some } u \in B^\times.$$

Now we have ist $v_\pi(e_0) = v_\pi(\pi^\delta \alpha)$ and $e'_0 = I(\alpha)$. By Proposition 3.1.3, one has that

$$v_\pi(I(\alpha)) > v_\pi(\alpha) + \delta v_\pi(\pi),$$

thus we have $v_\pi(e'_0) > v_\pi(e_0)$. This proves our claim and finishes the induction. □

Lemma 3.3.10. $\delta = \ell, a \neq 0$ *implies* $a = 1$.

Proof. Hilbert's theorem 90 implies that $I(Q(B))$ is a $Q(A)$-vector space of dimension $p-1$. Let $C := R^G[[N(\eta)]]$. Since $Q(B)$ is a $Q(C)$-vector space of dimension p, the augmentation image $I(Q(B))$ is a $p(p-1)$-dimensional $Q(C)$-vector space. We will assume that $\delta = \ell$ and $a \neq 1$ and derive a contradiction to this fact. For the convenience of the reader, since $\delta = \ell$, we will only use δ in the proof instead of ℓ.

We start by observing that the elements $\eta^i \pi^j, 0 \leq i, j \leq p-1$ form a $Q(C)$-basis of $Q(B)$. Thus their images under I generate the $Q(C)$-vector space $I(Q(B))$. We calculate, using Remark 3.2.4, that

$$I(\eta^i \pi^j) = I(\eta^i)\pi^j + \eta^i I(\pi^j) \equiv i\eta^{i-1}\pi^j I(\eta) + j\eta^i \pi^{j-1} I(\pi) \mod \pi^{j+\delta+1}.$$

Also, we see that both terms on the right hand side have the same valuation, namely $j + \delta$. By definition, we have that

$$I(\eta) \equiv \pi^\delta \eta^a u \mod \pi^{\delta+1} \text{ and } I(\pi) = \pi^{\delta+1} t \text{ for some } t, u \in k^\times$$

Thus, substituting the previous equation, we have that

$$I(\eta^i \pi^j) \equiv i\eta^{a+i-1}\pi^{j+\delta} u + j\eta^i \pi^{j+\delta} t = \pi^{j+\delta}\eta^i (iu\eta^{a-1} - jt) \mod \pi^{j+\delta+1}.$$

This shows that the $I(\eta^i\pi^j), 0 \leq i,j \leq p-1$ where i,j are not both zero, are linearly independent over $Q(C)$, since we have assumed $a \neq 1$ and thus the last bracket is never zero modulo $\pi^{j+\delta+1}$. Thus the $Q(C)$ -vector space $Q(I(B))$ would be at least (p^2-1) -dimensional under our assumption which contradicts the above consequence of Hilbert 90. $\square$

Chapter 4

Wild quotient singularities of surfaces

4.1 Brief overview on chapter 4

In this chapter, we will apply the results of chapter 3 to quotient singularities of normal fibered surfaces. The main motivation arises from the following problem: Let R be a complete discrete valuation ring with algebraically closed residue field k. We start with a regular fibered surface $\mathcal{Y}$ over R, and with a prime cyclic group action $G = \mathbb{Z}/p\mathbb{Z}$ on $\mathcal{Y}$ with trivial induced action on k and non-trivial action on R. Our goal is to characterize the minimal desingularization of the quotient singularities on $\mathcal{X} = \mathcal{Y}/G$ in terms of the group action on $\mathcal{Y}$. Since quotient singularities can occur only under G-stable points, we will consider a G-stable point y on $\mathcal{Y}$ and try to understand the singular point x of $\mathcal{X} = \mathcal{Y}^G$ in the image of y. Of particular interest is the case where the G-action is wild. The tame case can be treated with the methods of chapter 2; for the wild case, only few results are known. To the author's knowledge, the only results dealing with the desingularization structure of x can be found in a recent unpublished work of Lorenzini [Lor06], where rather special cases of quotient singularities are treated using global methods (see Theorem 6.1 or 8.1), in the paper of Artin where wild actions in the case $p = 2$ have been studied [Art75], and in their following generalization by Peskin [Pes83] under some probably unnatural assumptions on the G-action. Moreover, those results seem to give no or little insight on how the group action on y relates to the type of singularity at x.

The idea we will follow is that one should treat the problem only using local methods, since the singularities of $\mathcal{X}$ are isolated points in codimension 2. For the technical framework, one uses birational geometry to control the possible blowing-ups of $\mathcal{Y}$ with support in y resp. of $\mathcal{X}$ with support in x. One can now follow two approaches: On one hand, one can consider birational models of $\mathcal{X}$ and $\mathcal{Y}$ and restrict only to those models which are isomorphic to $\mathcal{X}$ resp. $\mathcal{Y}$ outside of x resp. y. Or, on the other hand, one can localize and consider the local rings $B = \mathcal{O}_{\mathcal{Y},y}$ and $A = B^G = \mathcal{O}_{\mathcal{X},x}$. As we will later do in section 4.2.1, one can show that the blowing-ups of B resp. A correspond one-to-one to those of $\mathcal{Y}$ and $\mathcal{X}$ with support only in y resp. x, so both formulations are equivalent, cf. Corollary 4.2.5. Since the classical theory of birational geometry is of global nature, in both cases, one has to prove or derive local versions of the classical results - either by restricting to those blowing-ups having support only in y resp. x, or by considering blowing-ups of B resp. A.

We have decided to follow the second approach involving local rings B and A, since many of the results seem not to depend on the fact that B and A are local germs of a global structure, but only on the local geometry, as are the results of chapter 3, or Lorenzini' local results in [Lor06, §2]. Even if the technical framework might seem a bit more complicated at the beginning than the alternative, the approach might become justified in the context of a birational geometry of more general local rings.

In section 4.2, *Models of local rings*, we construct the necessary tools to follow this approach, and prove some structural results on the Galois action in this context. In the first subsection 4.2.1, *Definition of models*, we make a central definition: For a local ring as B or A, we will define the concept of a model of B resp. A. The idea is that a model of B resp. A should be a scheme which is obtained from $\operatorname{Spec} B$ resp. $\operatorname{Spec} A$ by concatenation of finitely many blowing-ups in closed points with subsequent normalization, and blow-downs of entire components on the special fiber. Much of the birational geometry is similar as in the global case, however, there are two main differences: First, there are one or two special components, the so-called original components, which appear in any model of B resp. A. Those correspond exactly to the components which are already there at the beginning, i.e. in $\operatorname{Spec} B$ resp. $\operatorname{Spec} A$, and which cannot be blown down since the localization prohibits this. Second, only a certain set of components can appear in models of B or A, since we have restrict ourselves by localization to the neighborhood of a single point.

In subsection 4.2.2, *Zariski valuations*, we introduce Zariski's view on birational geometry via so-called valuations of the first kind. The main point of the Zariski approach is that one can view the components on the special fiber of models of B and A inside a birational equivalence class, independently of the choice of an actual model. These concepts also transfer smoothly to our local setting, again with the difference that fewer valuations correspond to components which can potentially appear in a model due to the fact that B is local. This justifies to call a component which appears on some model of B (resp. A) simply a component of B (resp. a component of A), since it lives not only on a single model, but on a whole birational class of models.

In subsection 4.2.3, *Introducing the G-action*, we will use this framework to analyze the correspondence between models of B and A which is induced by the G-action. In particular, we will relate the components of B and A to each other. In this context, it is a corollary to Lüroth's theorem that any component on a model of A, in particular any exceptional component on the minimal desingularization of A, is rational. This theorem has been already proved by Lorenzini in his recent work [Lor06, 2.5] under more general conditions for the ring B; however we want to note that our proof should also work under more general conditions on B.

Corollary 4.2.41 *Any component of A which is not an original component is rational.*

We also want to emphasize that this result does not need to imply a-priori that A has a rational singularity, since rationality of the exceptional components is only a necessary condition for a singularity to be rational, see [Art66, Proposition 1]. An example for a non-rational singularity with only rational components can be found in [Art75].

Subsection 4.2.4, *The graph structure of models and the G-action*, pushes the combinatorial correspondence between models of B and A a bit further. The main result of this section is the following proposition which roughly states that the neighborship relation between components and A is unaltered with respect to the neighborship relations in B . Here, dual graphs of models of local rings are defined in complete analogy to those of classical models; the main difference to the classical case is that we consider by localization only a specific part of the combinatorics on the special fiber, and that we have special vertices corresponding to the original components.

Proposition 4.2.49 *Let Y be a model of B, let X be a model of A. Then the dual graph of Y resp. X is a tree (in particular a rooted tree with the original components as labelled roots).*

Also, the path combinatorics of the dual graphs transfers through the group action in the following way: Assume that G acts on Y. Let D be a component on Y , write $G \cdot D$ for the orbit of D under G . Assume that D is realized in X by a component C.

Let $D_1, D_2 \notin G \cdot D$ be two components of B realized in Y and in X, let C_1, C_2 be their images in X. Let $\mathcal{D}$ be the connected component of D_1 in $Y \times \operatorname{Spec} k - G \cdot D$. Let $\mathcal{C}$ be the connected component of C_1 in $X \times \operatorname{Spec} k - C$.

Then $G \cdot D_2 \cap \mathcal{D} \neq \varnothing$ if and only if $C_2 \in \mathcal{C}$.

Among others, this implies that the configuration of components on the special fiber of the minimal desingularization of A is tree-like, see 4.2.50. This fact has been also proven by Lorenzini [Lor06, Theorem 2.5] in a more general setting; however, Proposition 4.2.49 is not equivalent to the corollary, since it states in greater detail the structural interrelations between models of B and A on the level of their components. The proof of this proposition should be also generalizable for arbitrary local B.

In subsection 4.2.5, *Minimal regular realization*, we continue with some technical results in the birational geometry of B to obtain a better structural view on models of B and A. We show that for any component D of B resp. A, we can construct a model Y of B resp. A minimal with the property that D occurs in Y . We call this Y the minimal realization of D . One can additionally ask the model to be regular or regular normal crossings, or to dominate another model, and a minimal realization with this properties will still exist. We will also introduce some terminology for components of B resp. A to distinguish several combinatorial possibilities for their relative positions to the original components and relate those to the structure of the minimal regular realization.

In subsection 4.2.6, *Rings of components and parameters*, we examine the underlying algebraic structure of components of B and A. It turns out, that to each component we can associate a unique ring. This will become particularly useful later when we want to keep track on the structure of the desingularization in dependence of some regularity conditions on those rings.

In section 4.2.7, we illustrate the previous results by applying them to components of B which contain a $Q(R)$ -rational point. In this case, we can perform explicit algebraic calculations.

Section 4.3, *Resolution of wild quotient singularities*, is devoted to examining how the minimal desingularization of A is related to the group action of G on B. The main obstacle in the problem is the following: Take a model Y of B on which G acts. In general, the image of a double point of Y in Y/G will be singular. In particular, as soon as $Y \neq \operatorname{Spec} B$, the scheme Y/G will be singular. More general, in subsection 4.3.1, *Examining the naive approach*, we show that one cannot even find a model Y of B and a model X of A such that $X = Y^G$ and both Y and X are regular. This means two things: On one hand, no combinatorial description of the correspondence between B and A can descend combinatorics easily from B to A by considering models with corresponding components. On the other hand, a method which considers models Y of B and then the models Y/G of A without saying something about the singularities of Y/G also cannot work.

The main idea to resolve this situation is that one has to find the components at which the dual graph of the minimal normal crossings desingularization of A branches, and somehow identify them on the level of B. We will call them the critical components of B. Then the other components occurring in the minimal normal crossings desingularization of A can contribute only to the paths between the critical components. It will turn out that the critical components are exactly the components which have too many singularities on the level of A, and there are only finitely many of them (since e.g. the minimal desingularization of A exists). Moreover, the main result in subsection 4.3.2, *Critical components and correspondence of models*, shows that the preceding idea can be used to define a model of B solely in terms of the G-action, the so-called minimal model of the G-action on B, whose dual graph has - up to path lengths - the same branching structure as the minimal normal crossings desingularization of A. One can take as definition of the minimal model of the G action the following, see Corollary 4.3.15: The minimal model of the G-action on B is the model Y birationally minimal with the property that Y is regular and the singularities of $X = Y/G$ are toric (i.e. the simplest kind of singularities with chain-like resolution, see 2.8). The main theorem of subsection 4.3.2 relates this model and the minimal desingularization of A:

Theorem 4.3.12 *There exists a minimal model of the G-action on B.*

Let Y be a minimal model of the G-action on B, let X be the minimal normal crossings desingularization of A. Then the path reduction of the dual graph of Y is isomorphic to the path reduction of the dual graph of X. Moreover, the branch vertices in the path reductions and the intersection points correspond to each other under the quotient map.

In the proof, several results of section 4.2 are needed. The main ingredient is the combinatorics correspondence in Proposition 4.2.49. We also need most of the technical results in subsection 4.2.3 and 4.2.6. Additionally, we use some combinatorial techniques which are developed in the same subsection 4.3.2.

In subsection 4.3.3, *The augmentation ideal on critical components*, it then turns out that the minimal model of the G-action of B can be described completely in terms of the G-action on B, i.e. without informations on the level of A. The reason for this is the algebraic regularity criterion from Proposition 3.3.4 which we can apply locally at any point of the model of B. This fact has inherent importance since it is the last piece which makes the Theorem 4.3.12 really into a statement about Galois correspondence. For example, one can now characterize all quotient singularities with chain-like resolution in terms of the augmentation sheaf and Zariski valuations:

Corollary 4.3.20 *If there is exactly one original component, then the minimal resolution of A is a chain if and only if for any component D of B, the following holds: The augmentation ideal $\mathcal{I}(D)$ is invertible if D is not potentially terminal, and $\mathcal{I}(D)$ is invertible at all but at most one point if D is potentially terminal.*

If there are two original components, then the minimal resolution of A is a chain if and only if the augmentation ideal on any connecting component of B is invertible.

Also, this observation will be used in subsection 4.3.4 to continue to analyze the situation which we have started to examine in 4.2.7.

In section 4.4, *Tame descent*, we will analyze the Galois correspondence for group action with factor in a p-cyclic and a tame part. The situation we are interested in is as follows: We will assume that B has certain good properties, e.g. when $\widehat{B} = R[[\eta]]$. For q coprime to p, we will denote by B_q, the normalization of B in the unique extension of R with degree q. In section 4.4.1, *Galois extension and tame descent*, we will prove some local Galois theoretical statements from which most of the results in the chapter will follow. In particular, one can let G act on B_q in a canonical way, and then ask how the minimal desingularization of B_q^G or the minimal model of the G-action on B_q look like in dependence of the corresponding models of B resp. A. This question is answered by the following proposition, whose proof is performed in section 4.4.2, *Tame invariance of critical components*:

Proposition 4.4.12 *Let q be coprime to p. let D' be a component of B_q, let D be the corresponding component of B. Then D' is critical if and only if D is critical.*

Last but not least, in section 4.4.3, *Quotients of stable models*, we will prove a result on quotient singularities which is a local analog to an unpublished result of Lorenzini in a similar global situation. The proof will use most techniques developed before which justifies its concluding position.

4.2 Models of local rings

In this section, we will develop several tools which we will then use later to prove the main theorem in this chapter. For a more detailed overview, we refer to section 4.1.

4.2.1 Definition of models

First we will define what a model of a local ring should be. We will do this as in classical birational geometry; the main difference in our actual setting is that we have to deal with local rings of fibered surfaces instead of the fibered surfaces themselves. Since for our main application, all our local rings will be obtained from fibered surfaces by taking local germs, we will do this simply by restriction. For the convenience of the reader, we repeat some important definitions and theorems in the theory of fibered surfaces and their models. Along the way, we will then state the necessary corollaries which follow in our situation.

Definition 4.2.1. Let S be a Noetherian integral scheme of Krull dimension 1. An integral, projective, flat S-scheme X of Krull dimension 2 is called fibered surface over S.

If not stated otherwise, we will assume that all fibered surfaces in this chapter are normal.

Notations 4.2.2. For the rest of this subsection, we will fix a normal fibered surface $\mathcal{X}$ over a scheme S which is the spectrum of a Henselian discrete valuation ring R, a closed point x on $\mathcal{X}$, and the local germ $\mathcal{C} = \mathcal{O}_{\mathcal{X},x}$ of $\mathcal{X}$ at x.

We will assume that R is of mixed characteristic $(0, p)$, with uniformizer π. We will denote the induced valuation by $v_\pi : R \to \mathbb{N}$.

Models of $\mathcal{C}$ will defined by restriction of models of $\mathcal{X}$:

Definition 4.2.3. Let $\varphi : \mathcal{X}' \to \mathcal{X}$ be a birational morphism of normal fibered surfaces over S. Then $\mathcal{X}'$ is called *model* of $\mathcal{X}$.

Let as above $x \in \mathcal{X}$ be a closed point. We will call $\mathcal{X}' \times_{\mathcal{X}} \mathcal{O}_{\mathcal{X},x}$ a model of the local ring $\mathcal{C}$.

Since φ is defined at x, the morphism $\mathcal{X}' \times_{\mathcal{X}} \mathcal{O}_{\mathcal{X},x} \to \mathcal{O}_{\mathcal{X},x} = \mathcal{C}$ is well-defined, and so any model X of $\mathcal{C}$ comes with a morphism $X \to \operatorname{Spec} \mathcal{C}$. We will also admit $\operatorname{Spec} \mathcal{C}$ as a model of $\mathcal{C}$.

The base change $\operatorname{Spec} \mathcal{O}_{\mathcal{X},x} \to \mathcal{X}$ is flat, since $\mathcal{X}$ is Noetherian. (see [Liu02, 1.2.11]). In particular, the blowing-ups of $\mathcal{C}$ correspond one-to-one to the blowing-ups of $\mathcal{X}$ with center x. We will prove this using the following theorem, found e.g. in [Liu02, 8.1.12(c)].:

Proposition 4.2.4. *Let X be a locally Noetherian scheme, let $\mathcal{I}$ be a quasi-coherent ideal of sheafs of ideals on X. Let $X' \to X$ be the blowing-up of X with center $\mathcal{I}$, let $Y \to X$ be a flat morphism with Y locally Noetherian. Let $Y' \to Y$ be the blowing-up of Y with center $\mathcal{I}\mathcal{O}_Y$. Then $Y' = X' \times_X Y$.*

We will write out explicitly what this means in our situation:

Corollary 4.2.5. *The blowing-ups of $\mathcal{C}$ correspond one-to-one to the blowing-ups of $\mathcal{X}$ with support x in the following way:*

Let $\mathcal{I}$ be sheaf of ideals on $\mathcal{X}$ with locus $x = V(\mathcal{I})$. Let $\mathcal{X}' \to \mathcal{X}$ be the blowing-up of $\mathcal{X}$ with center $\mathcal{I}$. Let $\mathcal{C}'$ be the corresponding model to $\mathcal{X}'$. Then the canonical morphism $\mathcal{C}' \to \mathcal{C}$ is the blowing-up of $\mathcal{C}$ with center $\mathcal{I}\mathcal{O}_{\mathcal{X},x}$.

Conversely, let $\mathcal{C}' \to \mathcal{C}$ be a blowing-up of $\mathcal{C}$. Then $\mathcal{C}'$ is a model of $\mathcal{C}$, i.e. there exists be a sheaf of ideals $\mathcal{I}$ on $\mathcal{X}$ with locus $V(\mathcal{I}) = x$, such that the morphism $\mathcal{C}' \to \mathcal{C}$ is obtained from the blowing-up $\mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}$ in $\mathcal{I}$ by base change to $\mathcal{O}_{\mathcal{X},x}$.

Proof. $\mathcal{X}$ is Noetherian, and $\mathcal{I}$ is a sheaf of ideals on $\mathcal{X}$. After localizing at x and making the flat base change $\operatorname{Spec} \mathcal{O}_{\mathcal{X},x} \to \mathcal{X}$, the corollary follows from Proposition 4.2.4.

For the second assertion, note that $\mathcal{C}' \to \mathcal{C}$ is the blowing-up of an ideal $\mathcal{I}'$ generated by elements in $\mathcal{O}_{\mathcal{X},x}$. This induces a sheaf of ideals $\mathcal{I}$ on $\mathcal{X}$ with $V(\mathcal{I}) = x$ which is locally $\mathcal{I}'$. □

Definition 4.2.6. Let X be a model of $\mathcal{C}$. Let s be the closed point of S. Then we call $X \times_S \operatorname{Spec} k(s)$ the *special fiber* of X. We call $X \times_S \operatorname{Spec} K(S)$ the *generic fiber* of X at s.

Definition 4.2.7. Since $\mathcal{X}$ is projective, its special fiber consists of a configuration of irreducible projective curves. Some of them pass through x and thus appear on the special fiber of $\mathcal{C}$ and on any model of $\mathcal{C}$. We call these components *original components*.

Note that if $\mathcal{X}$ is normal crossings, there are at least one and at most two original components.

Since we obtain models of $\mathcal{C}$ only by localization, virtually all good properties of birational geometry over fibered surfaces transfer directly to our local situation. We will cite some classical results in the geometry of fibered surfaces and state the consequences in our setting.

Definition 4.2.8. We call a morphism of schemes which is a concatenation of finitely many blowing-ups in closed points and subsequent normalization a *σ-process.*

In the classical birational geometry of surfaces, it turns out that any birational morphism is a σ-process. This is also the case in this slightly modified setting. σ-processes are an important class of morphisms since their structure is simple and they have many good properties, some of which follow from Theorem [Liu02, 8.1.19]:

Theorem 4.2.9. *Let X be a regular locally Noetherian scheme, and $\pi : \tilde{X} \to X$ the blowing-up of X along a regular closed subscheme $Y = V(\mathcal{I})$. Then the following properties are true:*

(i) *The scheme $\tilde{X}$ is regular.*

(ii) *For any $x \in Y$, the fiber $\tilde{X}_x$ is isomorphic to $\mathbb{P}^{r-1}_{k(x)}$, where $r = \dim_x X - \dim_x Y$*

(iii) *Let $Y' = V(\mathcal{I}\mathcal{O}_{\tilde{X}})$ be the inverse image of Y under π. Then $Y' \to \tilde{X}$ is a regular immersion. Moreover, if X is affine and $\mathcal{I}/\mathcal{I}^2$ is free of rank r on Y, then we have $Y' \cong \mathbb{P}^{r-1}_Y$.*

In particular, blowing up a closed point produces only new components of genus zero.

Corollary 4.2.10. *Let $\pi : X' \to X$ be a morphism of models of $\mathcal{C}$ which is a blowing-up of a regular closed point $x \in X$. Then $\pi^{-1}(x) \cong \mathbb{P}^1_{k(x)}$, and X' is regular at $\pi^{-1}(x)$.*

Proof. This follows from Theorem 4.2.9, where we are in the special case that the maximal ideal $\mathfrak{m}_x$ at x is blown up. Note that since x is a regular point, $\mathfrak{m}_x/\mathfrak{m}_x^2$ is free of rank 2. $\square$

Also very important is a deep result about desingularization of fibered surfaces by Lipman which can be found in [Liu02, 8.3.44] in modern language:

Theorem 4.2.11. *Let X be an excellent, reduced, Noetherian scheme of Krull dimension 2. Then X admits a desingularization in the strong sense, i.e. there exists a regular scheme Z together with a proper birational morphism $Z \to X$ which is an isomorphism over every regular point of X (cf. [Liu02, 8.3.39]).*

In particular, there exists a birationally minimal desingularization when considering fibered surfaces, see [Liu02, 9.3.32]:

Theorem 4.2.12. *Let $X \to S$ be a normal fibered surface. If X admits a desingularization, then it admits a minimal desingularization. If X has only a finite number of singular fibers, then X admits a minimal normal crossings desingularization.*

Corollary 4.2.13. *Any model of $\mathcal{C}$ admits a desingularization in the strong sense. Furthermore, $\mathcal{C}$ admits a minimal desingularization, and a minimal normal crossings desingularization.*

Proof. This follows directly from [Liu02, 8.3.44] and [Liu02, 8.3.50]. Let X be a model of $\mathcal{C}$. By definition, this model comes from a model X' of $\mathcal{X}$. let $Z \to X'$ the desingularization in the strong sense which exists by Lipman's desingularization theorem 4.2.11. Then by localization and base change, one can infer by Proposition 4.2.4 that $Z \times \operatorname{Spec} \mathcal{C} \to X' \times \operatorname{Spec} \mathcal{C} = X$ is also a desingularization in the strong sense.

The existence of the minimal desingularizations follows similarly from Proposition 4.2.12. □

Another theorem which will be important is the elimination of indeterminacies, which is classical and can be found for example in [Liu02, 9.2.7]:

Theorem 4.2.14. *Let $X \to S$ a regular fibered surface. Let $\varphi : X \dashrightarrow Y$ be a rational map from X to a projective S-scheme Y. Then there exists a σ-process $f : X' \to X$ and a morphism $g : X' \to Y$ such that $\varphi \circ f = g$.*

Corollary 4.2.15. *Let X_1, X_2 be models of $\mathcal{C}$. Then there exists a model X of $\mathcal{C}$ such that there exist σ-processes $X \to X_1$ and $X \to X_2$.*

Proof. We use the theorem of elimination of indeterminacies 4.2.14. Again, X_1 and X_2 both correspond to models X_1' and X_2' of $\mathcal{X}$. Thus there exists a birational map $\varphi : X_1' \dashrightarrow X_2'$. We can apply the Theorem 4.2.14 to this map: It guarantees the existence of a scheme Y' and a σ-process $f' : X' \to X_1'$ such that $g' : X' \to X_2'$ is $g' = \varphi \circ f'$. But g is now birational, since φ and f are, and one can apply the theorem 4.2.14 again to g^{-1}, showing that g is also a σ-process. The morphisms g' and f' induce by Proposition 4.2.4 a model Y together with morphisms $f : X \to X_1$ and $g : X \to X_2$ which are the desired ones. □

Corollary 4.2.16. *Let X be a model of $\mathcal{C}$. Then there exists a scheme Z and σ-processes $Z \to X$ and $Z \to \operatorname{Spec} \mathcal{C}$. In other words, X can be obtained from $\operatorname{Spec} \mathcal{C}$ by performing finitely many blowing-ups in closed points and subsequent normalizations followed by finitely many contractions of irreducible components (in the sense of [Liu02, 8.3.27]).*

Proof. It suffices to give a finite sequence of blowing-ups resp. blow-downs of resp. to closed points between X and $\operatorname{Spec} \mathcal{C}$. Now by virtue of Lipman's desingularization theorem 4.2.13, the scheme X admits a desingularization in the strong sense, i.e. there exists a regular scheme X' such that $X' \to X$ is a σ-process. The same is true for $\mathcal{C}$, i.e. there exists a regular scheme Z' such that $Z' \to \operatorname{Spec} \mathcal{C}$ is a σ-process. So Z' and X' are birational to each other, and by the theorem of elimination of indeterminacies 4.2.15, one can find a scheme Z such that there exist σ-processes $Z \to X'$ and $Z \to Z'$. Composing $Z \to X' \to X$ and $Z \to Z' \to \operatorname{Spec} \mathcal{C}$ we see that this is what we wanted to prove. □

If we are in particular dealing with morphisms of regular models, we can obtain one model from the other without having to blow down again:

Lemma 4.2.17. *Let $X' \to X$ be a birational morphism of regular models of $\mathcal{C}$. Then $X' \to X$ is a σ-process.*

Proof. Let $\mathcal{X}$ be the fibered normal crossings surface from which we obtain by taking a local germ $B = \mathcal{O}_{\mathcal{X},x}$ at a point x. By definition of model, there exist fibered surfaces $\mathcal{Y}' \to \mathcal{X}$ and $\mathcal{Y} \to \mathcal{X}$ corresponding to X' resp. X. We also have a birational map $\mathcal{Y}' \dashrightarrow \mathcal{Y}$ which is in the preimage of x. Without loss of generality we can assume that it is an isomorphism away from x by Corollary 4.2.15, and that $\mathcal{Y}'$ and $\mathcal{Y}$ are regular by Corollary 4.2.13. Then we can apply [Liu02, 9.2.2] to infer that $\mathcal{Y}' \to \mathcal{Y}$ is a σ-process, and thus Corollary 4.2.5 implies that $X' \to X$ is also. □

Corollary 4.2.18. *Let Y be a regular model of $\mathcal{C}$. If $\mathcal{C}$ is regular, then Y can be obtained by a σ-process $Y \to \operatorname{Spec} \mathcal{C}$. In particular, Y is normal crossings if $\mathcal{C}$ is normal crossings.*

Proof. Lemma 4.2.17 states that any regular model Y of $\mathcal{C}$ can be obtained by a σ-process $Y \to \operatorname{Spec} \mathcal{C}$. Thus if $\mathcal{C}$ is normal crossings, Y is also. □

Another important tool is the process inverse to blowing-up, the so-called contraction [Liu02, 8.3.27]:

Definition 4.2.19. Let $X \to S$ be a normal fibered surface. Let $\mathcal{E}$ be a proper subset of the integral projective vertical curves on X. A normal fibered surface $Y \to S$ together with a projective birational morphism $f : X \to Y$ such that for every integral vertical curve on E on X, the set $f(E)$ is a point if and only if $E \in \mathcal{E}$, is called a *contraction* of the $E \in \mathcal{E}$ resp. of $\mathcal{E}$.

In our setting, we will again only consider the curves over x:

Definition 4.2.20. Let X be a model of $\mathcal{C}$, corresponding to a normal model X' of $\mathcal{X}$. Let $\mathcal{E}$ be a set of integral projective vertical curves on X, denote by $\mathcal{E}'$ the corresponding set of curves on X'. Let $X' \to Y'$ be a contraction morphism of $\mathcal{E}'$. Then we will call $X = X' \times \operatorname{Spec} \mathcal{C} \to Y' \times \operatorname{Spec} \mathcal{C}$ a *contraction* of the $E \in \mathcal{E}$ resp. of $\mathcal{E}$.

It is a classical result that the contraction of arbitrary vertical divisors exists if the base scheme is Henselian, see [BLR90, 6.7,4] or [Liu02, 8.3.36]:

Theorem 4.2.21. *Let X be a normal fibered surface over the spectrum of a Henselian discrete valuation ring. Let $\mathcal{E}$ be a finite proper subset of the components on the special fiber of X. Then the contraction morphism of all $E \in \mathcal{E}$ exists.*

Corollary 4.2.22. *Let X be a model of $\mathcal{C}$, let $\mathcal{E}$ be a finite subset of the components on the special fiber of $\mathcal{C}$, not containing any original components. Then the contraction morphism of all $E \in \mathcal{E}$ exists, i.e. there is a model Y of $\mathcal{C}$ and a morphism $\varphi : X \to Y$ such that $\varphi(E)$ is a closed point for any $E \in \mathcal{E}$, and φ is the blowing-up of the codimension two set $\varphi(\mathcal{E})$.*

Proof. This is a direct corollary to Theorem [Liu02, 8.3.36] and a consequence of the fact that $\mathcal{C}$ is a local germ of a fibered surface. Namely the components in $\mathcal{E}$ correspond one-to-one to components on some model X' of $\mathcal{X}$. By Theorem 4.2.21, there exists a contraction morphism $X' \to Y'$ of those. Since $\mathcal{E}$ contains only components on the special fiber of $\mathcal{C}$ which are not original components, the base change $X = X' \times \operatorname{Spec} \mathcal{C} \to Y' \times \operatorname{Spec} \mathcal{C} =: Y$ gives a model Y of $\mathcal{C}$ which is the desired contraction by Proposition 4.2.4. □

Models of $\mathcal{C}$ are endowed with some structure and local information associated to points and components. We have already seen certain good types of closed points: The geometrically smooth points and geometrical double points, the only point occurring on normal crossings models. Instead of repeating our earlier definition 3.3.1 of those types of points, we will give a more explicit definition which is equivalent by [Liu02, 9.2.34f], but a little bit more algebraic:

Definition 4.2.23. Let P be a closed point on a model X of $\mathcal{C}$. Denote by $Z = (X \times_S \operatorname{Spec} k)_{red}$ the reduced special fiber of X. We call P *geometrically smooth* if P is smooth on Z. We call P a *geometrical double point*, if there is an isomorphism

$$\widehat{\mathcal{O}}_{Z,P} \cong k[[X_1, X_2]]/(X_1X_2)$$

As said before, normal crossings models contain only closed points of this type. Also, the components on the special fiber of models come with algebraic invariants which will gain importance in the following.

Definition 4.2.24. Let X be a model of $\mathcal{C}$, denote by $Z = X \times_S \operatorname{Spec} k$ the special fiber of X.

Let D be an irreducible component on Z with generic point ξ. We define the *multiplicity* of D to be the length of the module $\mathcal{O}_{Z,\xi}$ (cf. [Liu02, 7.5.6]).

Let D_1, D_2 be two different irreducible components on Z , and assume that X regular in the support of D_1, D_2. Since $\mathcal{C}$ is a local germ of an arithmetic surface, the intersection number $D_1 \cdot D_2$ can be defined as in [Liu02, 9.1.15]. Similarly, the *self-intersection* D^2 of D can be defined as in [Liu02, 9.1.15].

Note that the multiplicity of any component D can be read off any geometrically smooth point P lying on it. Comparing with Proposition 3.3.2, one can see that it is equal to the number m associated to the local germ at P.

4.2.2 Zariski valuations

Keep the notations of the previous subsection. We now introduce Zariski's viewpoint on models. From this viewpoint, models will be viewed in terms of certain valuations of the field of fractions $Q(\mathcal{C})$.

Remark 4.2.25. Let D be a component on the special fiber of a model X of $\mathcal{C}$, let ξ be its generic point. Then the ring $\mathcal{O}_{X,\xi}$ is a valuation ring and induces a valuation $v_D : Q(\mathcal{C}) \to \mathbb{Q}$. Since D corresponds to a vertical divisor, we can choose the valuation v_D extending the v_π -valuation on R (i.e. $v_D(\pi) = 1$), which we will assume in the following.

So to a component on the special fiber of a model of $\mathcal{C}$, one can associate a valuation on $Q(\mathcal{C})$. On the other hand, by definition, given such a valuation with certain properties, one can always find a model and a component inducing it.

This notion is also independent from the chosen model X in the following sense: Let X' be a model of $\mathcal{C}$ and $\varphi : X \dashrightarrow X'$ is a birational map such that φ is defined in ξ and does not contract D. Let D' be the closure of the image of D under φ, and $\xi' = \varphi(\xi)$. Then one has $\mathcal{O}_{X,\xi} = \mathcal{O}_{X',\xi'}$, and hence $v_D = v_{D'}$.

We will use Zariski's terminology for valuations (see e.g. [Liu02, 8.3.18]):

Definition 4.2.26. For a component D on the special fiber of a model of $\mathcal{C}$, we will call the valuation v_D as in Remark 4.2.25 the D -valuation. We will also say that D induces the v_D -valuation.

If X' is another model of $\mathcal{C}$, and D' is a component on the special fiber of X with $v_{D'} = v_D$, we say that D resp. v_D is *realized* in X' by D'. Also, we will call any valuation $v : Q(\mathcal{C}) \to \mathbb{Q}$ of this type a valuation *of the first kind* (in $\mathcal{C}$).

More general, we will call any valuation v such that $v = v_D$ for some component D on some model of $\mathcal{C}$ a *valuation of the first kind* (on $\mathcal{C}$).

Remark 4.2.27. By Proposition [Liu02, 8.3.22], if a component D on the special fiber of a model X of $\mathcal{C}$ is realized by D' in a model X' of $\mathcal{C}$, then D' and D will be always mapped onto each other in the canonical birational map $X' \dashrightarrow X$, which is always defined at D'. In particular, by the Theorem of elimination of indeterminacy 4.2.15, there exists a regular scheme Z together with σ -processes $\varphi : Z \to X$ and $\varphi' : Z \to X'$ such that there exists a component $\tilde{D}$ realizing D and D' in Z. This means $\tilde{D}$ is mapped to D under φ and to D' under φ'.

Similarly, for any model Y and any valuation v of the first kind in $\mathcal{C}$, there exists a σ -process $Y' \to Y$ in such that v is realized in Y'.

This relative independence of components of the chosen model motivates the following definition:

Definition 4.2.28. Let D be a component on the special fiber of some model X of $\mathcal{C}$. We will call D a *component* of $\mathcal{C}$.

Remark 4.2.29. Let D be a component of $\mathcal{C}$. Then the multiplicity of D does not depend on the model realizing D . So the multiplicity of D is well-defined without referring to a model.

On the other hand, self-intersection and intersection numbers concerning D depend on the model on which D is realized.

4.2.3 Introducing the G -action

In this subsection, we will examine what happens if a group acts on a local ring and its models. This is the basic technical step introducing the Galois correspondence problem. We will restrict ourselves to prime cyclic group actions and impose some practical conditions on the underlying rings. We now state the exact situation which we will adopt for the rest of the thesis:

Notations 4.2.30. Let R be a complete discrete valuation ring of mixed characteristic $(0, p)$ with algebraically closed residue field k, quotient field K and uniformizer π . By v_π we denote the π -valuation on R. Let $S = \operatorname{Spec} R$.

Let B be a local germ at a closed point x of a regular normal crossings fibered surface $\mathcal{X}$ over S, and consider the group

$$G \cong \mathbb{Z}/p\mathbb{Z} = \langle \sigma \rangle$$

acting on $\mathcal{X}$. Assume that $\sigma(x) = x$, so G induces an action on B. Denote by $A = B^G$ the ring of invariants. By $\mathfrak{m}_B$ resp. $\mathfrak{m}_A$, we will denote the maximal ideals of B resp. A.

We will assume that the induced action of G on k is trivial.

Note that A is automatically local, and also Noetherian, since B is excellent.

Since B is a local germ of a regular normal crossings surface, it has only two or one original component. Namely, if the unique point of B is geometrically smooth, then there is only one, if it is a geometric double point, then there are two. Also, the number of original components of B must be equal to the number of original components of A, or a p-multiple. So if $p \neq 2$, then A and B have the same number of original components.

Example 4.2.31. Let $\mathcal{X}$ be a normal crossings model of some K-curve over R, let P be a closed point on the special fiber of X such that any component passing through P is reduced. In this case it is known that the completion of any local germ of $\mathcal{X}$ at P is either isomorphic to the ring $R[[\eta]]$, namely if P is geometrically smooth, or to $R[[\eta_1, \eta_2]]/(\eta_1\eta_2 - \pi)$, namely if P is a geometric double point.

We want to mention that Raynaud has studied the situation of Example 4.2.31 in the case where G acts trivially on R in [Ray90, Appendice] resp. [Ray99, 2.3].

In the following, we will examine how some properties of models of B transfer to the models of A via the group action of G.

Remark 4.2.32. Let Y a model of B on which G acts. Then the quotient scheme $X = Y/G$ exists (see [Liu02, Ex. 3.3.23]). X is a model of A, since the quotient scheme $\mathcal{X}/G$ of the fibered surface $\mathcal{X}$ inducing X also exists, and A is the local germ of $\mathcal{X}$ at the image of x. On the other hand, if X' is a model of A, then the normalization Y' of $X' \times_{S/G} S$ is a model of B, since it is birational to $\operatorname{Spec} B$. In both cases, we have induced morphisms of special fibers $Y \times \operatorname{Spec} k \to X \times \operatorname{Spec} k$ resp. $Y' \times \operatorname{Spec} k \to X' \times \operatorname{Spec} k$ and thus induced morphisms of irreducible components.

We now remark how the action on G on the components can be reflected in an action of G on the corresponding Zariski valuations:

Remark 4.2.33. Let Y be a model of B on which G acts, let D be a component on Y. Denote as before by v_D the valuation induced by D. If the orbit of D is trivial, then one sees that $v_D = \sigma \circ v_D$. If conversely D has non-trivial orbit, then $v_D \neq \sigma \circ v_D$.

Proof. This is immediate from the fact that different valuations of the first kind define different components. □

This allows us to characterize the models on which one can define a G-action.

Lemma 4.2.34. *Let Y be a model of B. Then the following are equivalent:*

(i) *G acts on Y.*

(ii) *Y is obtained from $\operatorname{Spec} B$ as a concatenation of finitely many blowing-ups in G-stable closed sets of codimension 2, followed by finitely many blow-downs of G-stable sets of components.*

(iii) *For any valuation v of the first kind realized in Y, the valuations $\sigma^i \circ v$ are also realized in Y for all $1 \leq i \leq p-1$.*

Proof. (i) ⇒ (ii) can be derived from Theorem 4.2.16 and the properties of blowing-up. First we consider the following situation: Let Z be a model on which G acts, let $\varphi : Z' \to Z$ be the blowing-up of some closed point z on Z followed by normalization such that there are morphisms

$$Y \to Z' \to Z \to \operatorname{Spec} B.$$

If z is fixed by G, then Z' is also a model on which G acts. If not, then $Y \to Z$ also factors through the blowing-up Z'' of the orbit $G(z)$ of z under G. This is true, since the morphism $Y \to Z'$ induces by twist with σ a morphism

$$Y = Y^\sigma \to (Z')^{\sigma^i} \to Z = Z^\sigma,$$

where $(Z')^{\sigma^i}$ is the blow-up of $\sigma^i(z)$ on Z. Now one can glue all those morphisms to obtain a morphism

$$Y \to Z'' \to Z.$$

In particular, we can now apply this process to express any σ-process as a concatenation of blowing-ups in G-stable closed subsets. The claim then follows from Theorem 4.2.16.

(ii) $\Rightarrow$ (iii) follows from the fact that the blow-ups are stable under the G actions, and thus the set of valuations of the first kind in Y is also.

(iii) $\Rightarrow$ (i) can be seen as follows: Since Y is obtained by σ-processes, it is the blowing-up of an ideal $\mathcal{I}$ on $\operatorname{Spec} B$. Then we can consider the model Y^σ which is the blowing-up of $\sigma(\mathcal{I})$ on $\operatorname{Spec} B$. Thus there is a birational map $Y^\sigma \dashrightarrow Y$. Note that G acts on Y if and only this map is a morphism, since this birational map induces by twist with σ maps

$$Y \dashrightarrow Y^{\sigma^{p-1}} \dashrightarrow \cdots \dashrightarrow Y^\sigma \dashrightarrow Y.$$

But the condition that for any v of the first kind, $\sigma^i \circ v$ is also of the first kind implies in particular that this map is a morphism, see [Liu02, 8.3.20] and thus that G acts on Y. □

Now we introduce a notion of components descending resp. ascending over the G-action:

Definition 4.2.35. Let D be a component of B. Let X be a model of A. We say that D is *realized* in X if the restriction of v_D to $Q(A)$ is realized in X.

Let C be a component of A. Let Y be a model of B. We say that D is *realized* in X if some extension of v_C to $Q(B)$ is realized in Y.

The following lemma can be considered as an analogue of the theorem of elimination of indeterminacy for descending or ascending the group action:

Lemma 4.2.36. *Let v' be a valuation of the first kind in B. Then the restriction of v' to $Q(A)$ is of the first kind in A. Conversely, if v is a valuation of the first kind in A, then any extension of v to $Q(B)$ is of the first kind in B.*

I.e. any component of A can be realized on a model of B, and any component of B can be realized on a model of A.

Proof. Assume $v' = v_D$ for some component D of B. Let Y be a model of B realizing D. By virtue of Lemma 4.2.34, we can assume that G acts on Y. Indeed, if Y is some model obtained from B by blowing up an ideal $\mathcal{I}$, then we may replace Y by the model where we blow up the orbit ideal

$$\mathcal{I}' = \{\sigma^i(b)\ ;\ b \in \mathcal{I}, i \in \mathbb{N}\}.$$

Then Y/G will be a model of A realizing D. In particular, the component on Y/G realizing D will give the valuation v' restricted to $Q(A)$.

Conversely, if $v = v_C$ for some component C of A, let X be a model of A realizing C. The normalization X' of X in $Q(B)$ will be a model realizing D; and any extension of v_C to $Q(B)$ will be realized in X'. □

From the viewpoint of valuations, one can directly derive how the multiplicities of components behave with respect to the quotient map:

Remark 4.2.37. Let D be a component of B with multiplicity m, let f be the order of the inertia group of D resp. the D-valuation with respect to the G-action. Let C be the component of A under D. Then the multiplicity of C is fm. Since G is p-cyclic, f can be only 1 or p; thus the multiplicity of C can be only m or pm.

We can refine Lemma 4.2.36 also in a statement about the relation of arbitrary models of B and A, which can be seen as an analog to the theorem of elimination of indeterminacy:

Corollary 4.2.38. *Let X be a model of A, let Y be a model of B. Then there exist a σ-process $Y' \to Y$ and a birational morphism $X' \to X$ such that $Y'/G = X'$.*

In particular, for any model Y of B, there exists a σ-process $Y' \to Y$ such that G acts on Y'.

Proof. Combine Lemma 4.2.36 and Corollary 4.2.15. By Corollary 4.2.15, we can blow up Y in closed points to Y' in order to realize all possible extensions of any first kind valuation realized in X, and all possible images of the first kind valuations under G realized in Y. Thus G acts on Y', as we have seen in Remark 4.2.33.

Now any valuation realized in X is realized in Y' by definition and thus also on $X' := Y'/G$. Thus there exists a birational morphism $X' \to X$. □

We will now make some important statements on the structure of B and A. In particular, we will see that any component of B resp. A is rational.

Proposition 4.2.39. *Any component of B which is not an original component is rational (i.e. is of genus zero).*

Proof. This follows directly from the definition of models, that B is regular, k algebraically closed, and the fact that blowing up a regular closed point yields rational components as exceptional fiber, see Corollary 4.2.10. Thus the proposition follows, since an irreducible curve is rational if and only its underlying reduced structure is rational. □

For models of A, the same is true the general version of Lüroth's theorem.

Theorem 4.2.40. *Any unirational curve is rational. I.e. given an arbitrary projective k-curve C and a dominant map $\mathbb{P}^1_k \dashrightarrow C$, one automatically has $C \cong \mathbb{P}^1_k$.*

Proof. It suffices to prove that the function field of C is isomorphic to $k(X)$. But this follows from [Bou81, V. §3 Ex.11] or [vdW66, 63]. □

Corollary 4.2.41. *Any component of A which is not an original component is rational.*

Proof. Let C be a component of A. By Lemma 4.2.36, we can realize C in a model Y of B, e.g. as a component D giving a dominant map $D \dashrightarrow C$. This induces a morphism of reduced k-curves $D_{red} \to C_{red}$. By Proposition 4.2.39, one has $D_{red} \cong \mathbb{P}^1_k$ so by the Theorem of Lüroth 4.2.40, C is also rational. □

This corollary also has an interesting implication when applying Theorem 2.8.1:

Corollary 4.2.42. *A is a toric ring if and only if the exceptional divisor of its minimal resolution is a chain.*

Proof. By Proposition 4.2.41, any component of A rational, in particular any component occurring in the exceptional fiber of the minimal resolution of A. Then the assertion follows from Theorem 2.8.1 (i) $\Leftrightarrow$ (iii). □

4.2.4 The graph structure of models and the G-action

To each model of A or B, more precisely to their special fiber, we can associate a combinatorial structure, its so-called dual graph, which will be defined in our local situation analogously as in [Liu02, 10.1.4] for projective schemes. For this, we first review some graph theory which allows to easier formulate also later results. All graphs in the following will be simple and undirected.

Definition 4.2.43. Let v be a vertex of a graph. We denote the adjacency degree of v by $\deg v$. We will call v differently depending on its degree:

$\deg v$	how v is called
0	isolated
1	terminal or leaf (vertex)
2	path vertex
≥ 3	branch vertex

Definition 4.2.44. A *tree* is a connected acyclic graph.
A *chain* is a tree without branch vertex.
A *path* in a graph $\mathcal{G}$ is a subgraph of $\mathcal{G}$ which is a chain. If v_1, v_2 are the unique leaf vertices in a path $\mathcal{P}$, we will say that $\mathcal{P}$ is a path between v_1 and v_2.

Unfortunately, branch vertices are sporadically called nodes in mathematical literature. On the other hand, in mainstream graph theory, node is synonymous to vertex. Therefore we will avoid using the word node.

We will now define the concept of path reduction to simplify our statements. The idea is elementary: Consider a graph; given one of its path vertices, we can remove this vertex and connect the two neighbors by an edge. In the definition, we will do this for all path vertices contained in a proper subset of vertices.

Definition 4.2.45. Let $\mathcal{G}$ be a graph. Let $\tilde{V} \subseteq V(\mathcal{G})$ be a subset of its vertices. The *path reduction* of $\mathcal{G}$ with respect to $\tilde{V}$ is the graph $\mathcal{H}$ defined in the following way:

$$V(\mathcal{H}) = \{v \in \mathcal{G}\ ;\ \deg(v) \neq 2 \text{ or } v \notin \tilde{V}\}$$
$$E(\mathcal{H}) = \{(v_1, v_2)\ ;\ \text{there is a path between } v_1, v_2 \text{ with every path vertex in } \tilde{V}\}$$

We will call the path reduction of $\mathcal{G}$ with respect to $V(\mathcal{G})$ just the path reduction of $\mathcal{G}$.

Now we will associate to any model a graph with additional combinatorial structure:

Definition 4.2.46. Let Y be a model of B resp. A. The *dual graph* $\Gamma(Y)$ associated to Y is defined as follows.

First we will define the *unreduced intersection graph* $\tilde{\Gamma}(Y)$ associated to Y.

Denote by V_c the set of components on the special fiber of Y, denote by V_x the set of closed points of Y lying at which a component intersects an other component in V_c or itself.

The vertex set of $\tilde{\Gamma}(Y)$ is then defined as

$$V(\tilde{\Gamma}(Y)) = V_c \cup V_x.$$

The edges of $\tilde{\Gamma}(Y)$ is the set of pairs

$$E(\tilde{\Gamma}(Y)) = \{(C, x)\ ;\ C \in V_c, x \in V_x, x \text{ lies on } C\}.$$

Furthermore, we label the original components Y specifically. To every vertex in V_c , we will associate the multiplicity of the irreducible components.

The dual graph associated to Y is then the path reduction of $\tilde{\Gamma}(Y)$ with respect to V_x.

By convention, we will use qualifiers for components of models of B or A when they are true for the respective vertex in the corresponding dual graph. I.e. if the vertex V in the dual graph of a model Y corresponds to a component D on Y or a valuation v on $Q(B)$ and V satisfies a property P, we will also say that D and v satisfy property P.

Note that if Y is regular normal crossings, then V_x is empty in the dual graph $\Gamma(Y)$, and the definition above coincides with the classical definition of dual graph.

Remark 4.2.47. Blowing up a model Y in a closed point y changes the unreduced intersection graph of Y in the following way: Assume that n components corresponding to vertices $V_1, \dots, V_n$ run through y. By performing the blowing-up, a new vertex E in V_c corresponding to the single exceptional component is introduced (see Corollary 4.2.10). The vertex y is replaced by vertices $y_1, \dots, y_m$ in V_x, and every edge (y, V_i) is replaced by a set of edges (y_j, V_i) , where the j depend only on i. As we will see in a moment in Proposition 4.2.49, this is in fact only a single edge per i .

The dual graphs change analogously. In particular, if Y was regular normal crossings, then y had only one neighbor V_1 or two neighbors V_1, V_2. In any case, a new vertex E is introduced into V_c in the dual graph, and connected to every V_i with an edge (E, V_i)

Blowing down an irreducible component E on the special fiber of a model Y is easier to describe: Let $V_1, \dots, V_n$ the neighbors of E in the dual graph. If $n \leq 2$, then E is removed from the vertex set together with adjacent edges, and if $n = 2$, then the edge (V_1, V_2) is introduced into the vertex set. If $n \geq 3$, then E is moved from V_c into V_x and identified with all its neighbors in V_x.

We want to expose certain properties of the dual graphs of models of B resp. A which can be derived by the regularity of B and the quotient map mapping B to A.

Remark 4.2.48. Let Y be a model of B on which G acts. Then G acts on the set of vertical divisors, or equivalently, the set of components on the special fiber.

Also, G acts naturally on the set of closed points of Y. If D is a component on Y, then one can distinguish two cases:

Either D is G-stable, or it is not. One can also relate this to the σ-process by which Y is obtained: At one point of the blowing-up process, there had to be a point d such that blowing up d resulted in the emergence of D. Now either d was G-stable, then any point in the blowing-up D of d has to be mapped into D by G, since blowing up is an isomorphism away from its center, and we have blown up a G-invariant ideal. Or, d had non-trivial orbit, then the blowing-up of $d \cup \sigma(d) \cup \dots \sigma^{p-1}(d)$ is p copies of D, and the Galois action is permutation of isomorphic points resp. components.

Proposition 4.2.49. *Let Y be a model of B, let X be a model of A. Then the dual graph of Y resp. X is a tree (in particular a rooted tree with the original components as labelled roots).*

Also, the path combinatorics of the dual graphs transfers through the group action in the following way: Assume that G acts on Y. Let D be a component on Y, write $G \cdot D$ for the orbit of D under G. Assume that D is realized in X by a component C.

Let $D_1, D_2 \notin G \cdot D$ be two components of B realized in Y and in X, let C_1, C_2 be their images in X. Let $\mathcal{D}$ be the connected component of D_1 in $Y \times \operatorname{Spec} k - G \cdot D$. Let $\mathcal{C}$ be the connected component of C_1 in $X \times \operatorname{Spec} k - C$.

Then $G \cdot D_2 \cap \mathcal{D} \neq \varnothing$ if and only if $C_2 \in \mathcal{C}$.

Proof. We prove the first claims: The dual graph of a model Y of B is always a tree, since B is regular and Y is obtained from B by definition in successive blowing-ups and blowing-downs in closed points. By 4.2.16, we can exhibit Y by concatenating two σ-processes $Z \to \operatorname{Spec} B$ and $Z \to Y$. Since B was regular, and executing single blowing-ups $Z = Z_n \to \dots \to Z_0 = \operatorname{Spec} B$ in closed points yield regular schemes Z_i which also can contain no cylces, see [Liu02, 8.1.19]. By the second part of Remark 4.2.47, performing the blowing-down $Z \to Y$ also cannot introduce cycles.

We will now prove that the dual graph of the model X of A is always a tree. So let C, D, Y be as above. By Corollary 4.2.38, there exist models Y', X' and blowing-up morphisms $Y' \to Y$ and $X' \to X$ such that $Y'/G = X'$. We can also choose X' such that no component on its special fiber intersects itself. Note that the dual graph of X' is a tree only if the dual graph of X is a tree. So it suffices to prove that the dual graph of X' is a tree. For this, it suffices to prove that for a given original component on X', there exists only one path from this component to C.

Now under the quotient map, points and components of Y' are mapped surjectively to those of X', since this map is open and proper. Also, intersection points between two components are mapped to intersection points between their images. Points lying on a single component can only be mapped to a point lying on a single component, and by our assumption above, we have excluded the possibility that this might be a self-intersection point of the component. Thus for the dual graph of X' to be a tree, since D is arbitrary, it suffices to prove: Let O be an original component, assume that D is not G-stable. Denote by $P_i, 0 \leq i \leq p-1$ paths from O to $\sigma^i(D)$. Those are unique, since by our above reasoning, the dual graph of Y' is a tree. Then there cannot exist distinct components D_i on the path P_i such that D_i is G-stable and lies on P_j if and only if $i = j$.

Assume the contrary. Now the model Y' can be blown down to a model $Y' \to Z$

such that the components D_i are all terminal on Z , since the dual graph of Y' is a tree. On the model Z , then any points on the components D_i would be mapped onto D_i by the G -action, with exception of the images of the $\sigma^i(D)$ in Z on the D_i which would be mapped to a $D_j, j \neq i$. But this cannot be by Remark 4.2.48. This proves that the dual graph of X' is a tree, on which the original components - which can be labelled or not - are terminal.

The second claim follows in analogy. Without loss of generality, we can again consider $Y'/G = X'$ instead X and Y. Since no cycles can occur in any model of A, the connected components of $Y \times \operatorname{Spec} k - G \cdot D$ must be mapped surjectively onto the connected components of $X \times \operatorname{Spec} k - C$. That implies the statement about neighborship of the D_1, D_2 and C_1, C_2. □

Corollary 4.2.50. *Keep the notations of Proposition 4.2.49. We have seen in the proof: If $X = Y/G$, then the quotient map maps the connected components $\mathcal{D}_i$ surjectively onto the connected components $\mathcal{C}_i$.*

Also, if D_3 is a component on the unique path between D_1 and D_2, and D_3 is realized in X by the component C_3, then C_3 lies on the unique path between C_1 and C_2.

Proof. This follows directly from Proposition 4.2.49. The dual graph of any model of A is a tree since the $\mathcal{D}_i$ map surjectively. Were the dual graph of some model of A not a tree, one could remove a component of A obtaining only one connected component $\mathcal{C}_1$. □

This allows us to state the following definition which we will use later to describe the relative position of components with respect to the original components:

Definition 4.2.51. Let D be a component of B. Assume there are two original components in B. If D lies on the path between those two original components, then we call D a *connecting component*.

4.2.5 Minimal regular realization of a valuation

In this section we will examine how one can realize a component of B with a minimal number of blowing-ups. We will need these technical details later on. In this subsection, we will again assume that $\mathcal{C}$ is a local germ of an arithmetic surface as in 4.2.2.

Definition 4.2.52. Let D be a component of $\mathcal{C}$. A regular model X of $\mathcal{C}$ realizing D is called *minimal regular realization* of D if for any regular model X' realizing D, the canonical birational map $X' \dashrightarrow X$ is a morphism.

A regular normal crossings model X of $\mathcal{C}$ realizing D is called *minimal regular normal crossings realization* of D if for any regular normal crossings model X' realizing D, the canonical birational map $X' \dashrightarrow X$ is a morphism.

We will prove that for any component D of $\mathcal{C}$ a minimal regular realization resp. a minimal regular normal crossings realization exists.

Lemma 4.2.53. *Let D be a component of $\mathcal{C}$. There exists a regular and a normal crossings model realizing D.*

Proof. Since D is a component of B resp. A , there exists a model Y realizing D. By Lipman's desingularization theorem 4.2.13, there exists a desingularization $Y' \to Y$ of Y also realizing D , which one can choose normal crossings. □

Proposition 4.2.54. *Let D be a component of $\mathcal{C}$. There exists a minimal regular realization and a minimal normal crossings realization of D.*

Proof. By Lemma 4.2.53, there exists a regular model Y realizing D. Due to Corollary 4.2.22, there exists a morphism $Y \to Y'$ which contracts all components on the special fiber of Y except D and the original components. Due to Proposition 4.2.12, there exists a minimal desingularization $Z \to Y'$ of Y'. We claim: Z is the minimal regular realization of D.

Now the definition of minimal regular realization Z is that Z contains D, is regular , and birationally minimal with that property. But this can be taken also as a defining property for the minimal desingularization of Y', since D is the unique occurring non-original component. This proves that D has a minimal regular realization. To see that D has a minimal regular normal crossings desingularization, one can proceed in complete analogy by replacing the word regular with regular normal crossings. □

Proposition 4.2.55. *Let D be a component of $\mathcal{C}$ which is not realized in the minimal regular normal crossings model of $\mathcal{C}$. Then D is a terminal or a path component in its minimal regular realization, and in particular also in its minimal regular normal crossings realization.*

Proof. Let Y be the minimal regular realization of D which exists by Proposition 4.2.54. Then by Lemma 4.2.17, we can find a sequence

$$Y = Y_n \to Y_{n-1} \to \cdots \to Y_0$$

of n blowing-ups in closed points, where Y_0 is the minimal regular normal crossings model of $\mathcal{C}$. Since Y_0 was normal crossings, the Y_i are also normal crossings. Furthermore, the last blowing-up $Y_n \to Y_{n-1}$ is the unique blowing-up where D occurs in the exceptional fiber by definition of minimal regular realization. Since Y_{n-1} was normal crossing, the blowing-up was performed either in a geometrically smooth point or geometric double point y of Y_{n-1}. As explained in Remark 4.2.47, the component D thus has only two or less neighbors in the dual graph of Y_n. □

Definition 4.2.56. Let D be a component on B. If D is terminal in its minimal regular realization, we call D *potentially terminal*.

To get more intuition on what it means for a component to be potentially terminal, one has to note some combinatorial conditions about multiplicities of blowing-ups:

Remark 4.2.57. Let Y be a model of $\mathcal{C}$, let y be a regular point on Y.

If y is geometrically smooth and lies on a single component D of multiplicity m, then the blowing-up of y yields a rational component of multiplicity m intersecting D transversally in a single point.

If y is a geometric double point and lies at the intersection of two components D_1, D_2 with multiplicities m_1, m_2, then the blowing-up of y yields a rational component of multiplicity $m_1 + m_2$ intersecting D_1 and D_2 transversally at exactly one point each.

In particular, if y is a geometrically smooth point on a component of multiplicity m, then successive blowing-ups of regular points over y yield only components whose multiplicity is an integer multiple of m.

Proof. Theorem [Liu02, 9.1.23] tells us that for any not original component Γ in the special fiber Y_s of Y, we have that the fiber product $Y_s \cdot \Gamma = 0$. This is in particular true if Γ is an exceptional component of Y, we will further examine this case:

Let $D_1, \dots, D_n$ be the neighboring components of Γ in Y', with respective multiplicities $m_1, \dots, m_n$. Let μ be the multiplicity of Γ. Then we have

$$Y_s \cdot \Gamma = \left(\mu\Gamma + \sum_{i=1}^{n} m_i D_i\right) \cdot \Gamma = \mu\Gamma^2 + \sum_{i=1}^{n} m_i (D_i \cdot \Gamma) = 0.$$

Now $\Gamma^2 = -1$ by Castelnuovo's criterion (see [Liu02, 9.3.8]), since Γ^2 was exceptional resp. y was regular. So the above equation amounts to

$$\mu = \sum_{i=1}^{n} m_i (D_i \cdot \Gamma).$$

If y is geometrically smooth, then $n = 1$ and Γ and $D_1 = D$ intersect transversally, so we have $(\Gamma \cdot D_1) = 1$ and thus $\mu = m_1 = m$, as claimed. If y is a geometric double point, then $n = 2$ and γ and D_1, D_2 intersect transversally, so we have analogously $\mu = m_1 + m_2$, as claimed. □

Remark 4.2.58. Using the fact that any component arises via a σ-process (cf. 4.2.15) and Remark 4.2.57, we can now derive some implications on certain properties of components of B :

Assume that the original components of B are reduced. Then:

Connecting components (cf. 4.2.51) are not potentially terminal. But not every not potentially terminal component is connecting.

Any reduced and not connecting component is potentially terminal, but not all potentially terminal components are reduced. In fact, given B, the number of different multiplicities which can occur for a potentially terminal component of B is infinite.

We will also need a relative notion of minimal regular desingularization:

Definition 4.2.59. Let D be a component of $\mathcal{C}$, let X be a model of $\mathcal{C}$. A regular model $Y \to X$ of $\mathcal{C}$ dominating X and realizing D is called *relatively minimal regular realization* of D (with respect to X) if for any regular model $Y' \to X$ dominating X realizing D, the canonical birational map $Y' \dashrightarrow Y$ is a morphism.

Let D be a component of $\mathcal{C}$, let X be a model of $\mathcal{C}$. A regular normal crossings model $Y \to X$ of $\mathcal{C}$ dominating X and realizing D is called *relatively minimal regular normal crossings realization* of D (with respect to X) if for any regular normal crossings model $Y' \to X$ dominating X and realizing D, the canonical birational map $Y' \dashrightarrow Y$ is a morphism.

The results for the minimal regular desingularization transfer to the relative setting, the proofs are analogous to those of Propositions 4.2.54 and 4.2.55:

Proposition 4.2.60. *Let D be a component of $\mathcal{C}$, let X be a model of $\mathcal{C}$. Then there exists a relatively minimal regular realization and a relatively minimal regular normal crossings realization of D with respect to X.*

Proposition 4.2.61. *Let D be a component of $\mathcal{C}$, let X be a regular normal crossings model of $\mathcal{C}$ not realizing D. Then D is a terminal or a path component in its relatively minimal regular realization with respect to X.*

4.2.6 Rings of components and parameters

We will now describe the inherent algebraic structure of the models' components in some greater detail.

Remark 4.2.62. Let D be a component of B, let Y be the minimal regular realization of D , which exists by Proposition 4.2.54. By Proposition 4.2.49, the graph of Y is a tree. So for each original component E , there exists a unique point at which D intersects the connected component of $Y \times \operatorname{Spec} k - D$ which contains E. We denote those points by $P_1, \dots, P_n$; note that some of those points can coincide. Also, as we have remarked earlier, one has $1 \leq n \leq 2$, since B is normal crossing.

We can now take the formal completion of Y at D and then the formal open subset U reducing to $D - \{P_1, \dots, P_n\}$. We can also take the formal open subset U_{reg} reducing to the regular locus of $D - \{P_1, \dots, P_n\}$. It is now crucial to note that both U and U_{reg} both depend only on D, and are formal affine, since $D - \{P_1, \dots, P_n\}$ is affine.

Definition 4.2.63. Let D be a component of B. Then we define $B(D)$ to be the ring associated to the formal affine U from Remark 4.2.62. We define $B(D^\times)$ to be the ring associated to the formal affine U_{reg}.

Lemma 4.2.64. *Let D be a component of B. Then $B(D)$ has at most one singularity. D is potentially terminal or a connecting component if and only if $B(D)$ is regular.*

Proof. Let Y be the minimal regular realization of D. Since B is already regular normal crossings by assumption, it is its own minimal regular normal crossings model, we can use Proposition 4.2.55, which states that D has at most two neighbors in Y. So the formal completion of Y at D can have at most two singularities which correspond to the intersection points of D with the neighbors. Then we have to omit the points corresponding to the original components, the P_i from Remark 4.2.62. The exact number of singularities is thus the number of neighbors of D on Y minus the cardinality of the P_i. Since the first is at most two and the latter at least one, the first assertion follows. For the rest note that if D is potentially terminal, it has only one neighbor in Y. Moreover, if D is connecting, then there are two distinct P_i. And connecting components are never potentially terminal. The assertion follows direct from these facts. □

We will now define parameters associated to components of B giving local coordinates on the $B(D)$:

Definition 4.2.65. Let D be a component of B .

We will call any $f \in B(D)$ such that f is a uniformizer of v_D an *arithmetic parameter* of D.

The component $D \bmod f$ is reduced, and by Proposition 4.2.39 rational. Thus there exists a coordinate $\bar{g}$ on the projective line $D \bmod f$ which lifts to an element g in $B(D)$. We will call any such lift *geometric parameter* of D.

Remark 4.2.66. Fix an arithmetic parameter f and a geometric parameter g defined on an affine over D. On an open dense subset of D, the maximal ideal of any closed point will be of the form $(g - c, f)$, where $c \in R^G$.

Proof. By Proposition 4.2.39 and the definition of f, the factor ring $B(D)/(f)$ is isomorphic to the projective line over k with finitely many points removed. By definition, $g \mod f$ is a coordinate on this line. So the points on this line are of form $(g-c)$ for $c \in k$, and we can lift those to closed points of the form $(g-c, f)$ for an open dense subset of D. □

Now we want to look at the quotient $X = Y/G$ or the ring of $A = B^G$:

Definition 4.2.67. Let C be a component of A. Let then D be some corresponding component of B realizing C which exists by Lemma 4.2.36. Let U be the formal open set from Remark 4.2.62 associated to D, by Corollary 4.2.38 we can assume that G acts on the model Y' on which U lives. Let V be the orbit of U under the group action given by G. Then we define

$$A(C) = A(D) = \mathcal{O}(V)^G.$$

Similarly, if instead of the component C of A, we start with a component D on B, we can still make the same definition, since there exists by Lemma 4.2.36 a component C of A realizing B.

4.2.7 Application to components with K-rational center

In general, it seems to be difficult to obtain well-behaved arithmetic and geometric parameters for arbitrary components. In [Gra68, 5], it is stated that parameters can be obtained in terms of polynomials in $R[\eta]$; however, it seems that the polynomials associated to the components are difficult to control.

In the following, we will assume that $\widehat{B} = R[[\eta]]$, and $\eta \in B$. The case where $\widehat{B} = R[[\eta_1, \eta_2]]/(\eta_1\eta_2 - \pi)$ can be easily derived from these considerations by blowing up B at its maximal ideal and localizing at the double point. In this subsection, we will consider a specific type of components of B; namely those which lie on the path between a reduced component and the original component. In terms of rigid geometry, those correspond to concentric circles containing a K-rational point, therefore the title of this section.

I.e. we start with blowing up only reduced points. Then we arrive at a component with geometric parameter η/π^n for a suitable choice of η and arithmetic parameter π. Then we continue blowing up only geometric double points and arrive at a component with geometric parameter η^i/π^j for some $i, j \in \mathbb{N}$ coprime. Associated to this component is a valuation v of the first kind. It turns out that one has

$$v = \log |.|_\rho, \quad \text{where } \rho = j/i$$

and $|.|_\rho$ is the ρ-value from rigid geometry, i.e. the sup-norm of functions on the circle with radius ρ. On the other hand, one can prove that for each i, j coprime, one can find such a component as above. So the following definition makes sense:

Definition 4.2.68. Let $D(\rho)$ be the component associated to the geometric parameter η^i/π^j where $\rho = j/i \in \mathbb{Q}$ with $i, j \in \mathbb{N}$ coprime. We define

$$\mathcal{O}_B(\rho)$$

to be the formal ring associated to $\operatorname{Spf} B(D)$ minus the unique zero of η on $D(\rho)$.

For $\rho_1 < \rho_2 \in \mathbb{Q}_{\geq 0}$, let Y be the model of B where only the components $D(\rho_1)$ and $D(\rho_2)$ are realized. Then consider the formal affine whose support on the special fiber is $D_1 \cup D_2$ minus the original component and the unique zero of η on $D(\rho_2)$. Then we define

$$\mathcal{O}_B(\rho_1, \rho_2)$$

to be the associated ring.

Note that the definition of $\mathcal{O}_B(\rho_1, \rho_2)$ is independent of Y and depends only on the ρ_i and B, compare 4.2.62. Also note that the definition of $\mathcal{O}_B(\rho)$ and $\mathcal{O}_B(\rho_1, \rho_2)$ depend on the choice of η, so we may use them only when η is clear from the context or fixed.

The next lemma tells that in this particular case, one can always find a geometric parameter which has a simple normal form:

Lemma 4.2.69. *Let $\rho \in \mathbb{Q}_{\geq 0}$, with $\rho = j/i$ with i, j coprime. Then as geometric parameter for $\mathcal{O}_B(\rho)$, one can take η^i/π^j. As arithmetic parameter for $\mathcal{O}_B(\rho)$, one can take any π^m/η^n where*

$$\det \begin{pmatrix} i & m \\ j & n \end{pmatrix} = 1.$$

Proof. This follow directly from the fact that the ring $\mathcal{O}_B(\rho)$ is obtained by performing successive blowing-ups in geometrically smooth points starting with $\operatorname{Spec} B$. So one will arrive at a geometric parameter of the desired form, and the arithmetic parameter will satisfy the determinant condition. To see that one can take the arithmetic parameter arbitrarily fulfilling this condition, one has to note the geometric parameter is a unit on $\mathcal{O}_B(\rho)$, and thus can multiply any power of it to the arithmetic parameter to obtain another arithmetic parameter. □

We will now analyze when the above defined rings are regular:

Remark 4.2.70. By construction, $\mathcal{O}_B(\rho)$ is always regular.

Lemma 4.2.71. *For $\rho_i = m_i/n_i \in \mathbb{Q}, i = 1, 2$, and m_1, n_1 resp. m_2, n_2 coprime, the ring $\mathcal{O}_B(\rho_1, \rho_2)$ is regular if and only if*

$$\det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = 1.$$

Proof. Since, as a union of sets,

$$\operatorname{Spec} \mathcal{O}_B(\rho_1, \rho_2) = \operatorname{Spec} \mathcal{O}_B(\rho_1) \cup \operatorname{Spec} \mathcal{O}_B(\rho_2) \cup P$$

where P is the single double point, we see by Remark 4.2.70 that $\operatorname{Spec} \mathcal{O}_B(\rho_1, \rho_2)$ is regular if and only if P is regular. Since the local germ at P is excellent, this is true if and only if $\widehat{\mathcal{O}}_P$ is regular. But the latter is isomorphic to the normalization of

$$R[[\eta^{m_1}/\pi^{n_1}, \pi^{n_2}/\eta^{m_2}, \eta]]$$

since $\eta^{m_1}/\pi^{n_1}, \pi^{n_2}/\eta^{m_2}$ are the geometric parameters from the neighboring components. So the ring $\widehat{\mathcal{O}}_P$ is toric, and by the regularity criterion of toric rings 2.6.8, one sees that $\widehat{\mathcal{O}}_P$ is regular if and only if $\eta^{m_1}/\pi^{n_1}, \pi^{n_2}/\eta^{m_2}$ generate the multiplicative monoid of monomials in $\widehat{\mathcal{O}}_P$. By considering the monoid of monomials and by identifying $\eta^i\pi^j$ with $(i, j) \in \mathbb{Z}^2$, the latter is equivalent to the fact that $(m_1, -n_1)$ and $(-m_2, n_2)$ generate the group of monoids. This is in turn equivalent to the fact that

$$\{(1, 0), (0, 1)\} \subseteq \mathbb{N} \cdot (m_1, -n_1) + \mathbb{N} \cdot (-m_2, n_2),$$

and this is true if and only if the determinant condition is fulfilled. □

Remark 4.2.72. The multiplicity of a component with geometric parameter η^{m_1}/π^{n_1} and arithmetic parameter π^{n_2}/η^{m_2} is m_1 , since one has

$$\left(\frac{\eta^{m_1}}{\pi^{n_1}}\right)^{m_2}\left(\frac{\pi^{n_2}}{\eta^{m_2}}\right)^{m_1} = \pi^{m_1 n_2 - m_2 n_1} = \pi,$$

which means that the reduction of the arithmetic parameter modulo π has nilpotent order m_1. One has the analogous equation

$$\left(\frac{\eta^{m_1}}{\pi^{n_1}}\right)^{n_2}\left(\frac{\pi^{n_2}}{\eta^{m_2}}\right)^{n_1} = \eta^{m_1 n_2 - m_2 n_1} = \eta.$$

For both equations, on has to remark that the arithmetic parameter is not unique since a unit on the component, so only the exponents m_1 and n_1 of the geometric parameter are independent of the choice of the parameter.

Definition 4.2.73. Let $\rho \in \mathbb{Q}$. Then we will denote the valuation on $\mathcal{O}(\rho)$ associated to $D(\rho)$, by

$$v_\rho : \mathcal{O}_B(\rho) \to \mathbb{Q},$$

which we normalize by $v_\rho(\pi) = 1$. Since $B \subseteq \mathcal{O}_B(\rho)$ for any $\rho > 0$, we can define a function

$$v_.(.) : \mathbb{Q}_{\geq 0} \times B \to \mathbb{Q}, (\rho, b) \mapsto v_\rho(b).$$

Remark 4.2.74. For $r \in R$, we have $v_\rho(r) = v_\pi(r)$ for all ρ. Also, we have $v_\rho(\eta) = \rho$.

Proof. The first assertion follows by multiplicativity of the valuation since we have normalized the valuation such that $v_\rho(\pi) = v_\pi(\pi) = 1$. The second assertion follows from the equations in Remark 4.2.72:

Let η^{m_1}/π^{n_1} be the geometric parameter with $n_1/m_1 = \rho$; similarly, let π^{m_2}/η^{n_2} be the arithmetic parameter of $\mathcal{O}_B(\rho)$. By Remark 4.2.72, we have that

$$\left(\frac{\eta^{m_1}}{\pi^{n_1}}\right)^{n_2}\left(\frac{\pi^{n_2}}{\eta^{m_2}}\right)^{n_1} = \eta,$$

and using the multiplicativity of v_ρ, we see that

$$v_\rho(\eta) = n_2 v_\rho\left(\frac{\eta^{m_1}}{\pi^{n_1}}\right) + n_1 v_\rho\left(\frac{\pi^{n_2}}{\eta^{m_2}}\right).$$

Since η^{m_1}/π^{n_1} is a unit on $\mathcal{O}_B(\rho)$, and $v_\rho(.)$ is by definition the $\pi^{n_2}\eta^{m_2}$ valuation normalized by $v_\rho(\pi) = 1$, we see by the second equation in Remark 4.2.72

$$\left(\frac{\eta^{m_1}}{\pi^{n_1}}\right)^{m_2}\left(\frac{\pi^{n_2}}{\eta^{m_2}}\right)^{m_1} = \pi,$$

that we have

$$v_\rho(\eta) = n_1 v_\rho\left(\frac{\pi^{n_2}}{\eta^{m_2}}\right) = \frac{n_1}{m_1} v_\rho(\pi) = \frac{n_1}{m_1} = \rho,$$

which we wanted to prove. □

We now want to show by an elementary calculation how the function v_ρ behaves with respect to ρ; this will become useful later.

Lemma 4.2.75. *Let* $f \in B$. *Then the function* $v_{(.)}(f) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}$ *is continuous, piecewise linear; the breaks are precisely at the valuations of the zeroes of* f *with respect to* η . *Both* $v_{(.)}(f)$ *and its negative derivative are monotonously increasing. In particular, if*

$$f = \sum_{i=-\infty}^{\infty} f_i \eta^i \text{ with } f_i \in S,$$

then for $\rho \in \mathbb{Q}$ *one has*

$$v_\rho(f) = \min_i (\rho \cdot i + v_\pi(f_i))$$

Proof. Clearly proving the last formula suffices since all other statements are implied. Since $f \in S$ implies that $i + v_\pi(f_i) \geq 0$ and so the minimum is taken over straights with nonnegative coefficient.

The main thing one has to remark is that the non-Archimedic inequality is strict for monomials, i.e. we have

$$v_\rho(g_1\eta^i + g_2\eta^j) = \min\left(v_\rho(g_1\eta^i), v_\rho(g_2\eta^j)\right) \text{ for any } \rho \in \mathbb{Q}, g_i \in S \text{ and } i \neq j \in \mathbb{Z}.$$

Thus, we have

$$\begin{aligned} v_\rho(f) = v_\rho\left(\sum_{i=-\infty}^{\infty} f_i\eta^i\right) &= \min_i \left(v_\rho(f_i\eta^i)\right) \\ = \min_i \left(v_\rho(f_i) + v_\rho(\eta^i)\right) &= \min_i(\rho \cdot i + v_\pi(f_i)). \end{aligned}$$

The last equality is true by Remark 4.2.74. This proves the lemma. □

We will continue the analysis of this particular situation in the next section, namely in subsection 4.3.4.

4.3 Resolution of wild quotient singularities

In this section we will show how the group action on B and the minimal desingularization of A are related.

4.3.1 Examining the naive approach

In this subsection we will show that the naive approach to the problem of wild quotient singularities does not work. We will briefly explain this approach: It has been suggested that one can always construct a regular model Y of B on which G acts such that Y/G is regular. Then all what one would have to do is to give a sort of algorithm to find Y, examine its combinatorial structure, transfer it to Y/G, which then is a model of A, and relate it to the minimal regular model of A.

Unfortunately, this does not work. The point of failure is that an Y as claimed above does not exist in general. In this section, we will show: In general, there does not exists a regular model Y of B on which G acts such that Y/G is regular - neither in the tame nor in the wild case.

Example 4.3.1. We will exhibit a simple class of examples where we will prove our claim. One can also see that in the tame case, a similar argument can be found for all possible examples. Note that we have already seen similar examples in section 2.7 and Example 3.1.2.

Consider the local ring $B = R[\eta]_{(\eta,\pi)}$, where R is a complete discrete valuation ring with uniformizer π and residue field k. Assume that $G = \mathbb{Z}/p\mathbb{Z} = \langle\sigma\rangle$ acts on R by

$$\sigma : \eta \mapsto \zeta_p^a \eta,\ \pi \mapsto \zeta_p \pi$$

and trivial action on the residue field k, where ζ_p is a primitive p-th root of unity and $a \not\equiv 0 \mod p$. Note that we do not make any assumption on the characteristic of R.

We claim: If $p \geq 5$ and $a \not\equiv 1 \mod p$ then there exists no regular model Y of B on which G acts such that Y/G is regular (we will see later that for $p \leq 3$ or $a \equiv 1$, the claim is false).

We will prove the claim in the case where $2a \not\equiv -1 \mod p$. We will make a short remark at the end of the proof how the case $2a \equiv -1 \mod p$ can be treated - it can be done by a minor modification. However, this modification is not a-priori clear and might confuse the reader, and the case $2a \not\equiv -1 \mod p$ suffices to expose the main problems; so we will assume the latter for the exposition without losing critical features.

First we remark that any non-trivial regular model of B has to contain the component with geometric parameter η/π and arithmetic parameter π by Corollary 4.2.18. In particular, any non-trivial regular model must contain a finite number of components $D(\rho)$ with $0 < \rho \leq 1$; compare section 4.2.7.

We will now prove our claim by contradiction. Assume that there was a regular model Y of B such that Y/G is also regular. Then by Corollary 4.2.18, we can construct Y by a finite sequence of blowing-ups in closed points. Now as in Example 3.1.4, we can see that the invariant ring of $\mathcal{O}_B(\rho)$ is regular. However, not that a singularity may or may not occur under the point $\eta = 0$ which was removed from $B(D(\rho))$ to obtain $\mathcal{O}_B(\rho)$. This means that there exists a model Y with the claimed properties if and only if there exists a model Y with the claimed properties realizing only components of type $D(\rho)$. We will now focus on the finitely many components realized in Y with $0 < \rho < 1$.

Also, as we have seen, the image of a $D(\rho)$ under the quotient map can contain a singular point only under some double point, i.e. the intersection point with the next component. One now calculates that the maximal ideal in any regular double point P on $D(\rho, \rho')$ for $0 < \rho < \rho' < 1$ is generated by two elements of the form η^i/π^j and π^m/η^n where $\rho = j/i$ and $\rho' = m/n$ with $i, j, m, n \in \mathbb{N}$ where i, j resp. m, n are coprime satisfying the regularity condition in Lemma 4.2.71 - the geometric parameters of the neighboring components. I.e. the group action on the two elements is given by

$$\sigma : \frac{\eta^i}{\pi^j} \mapsto \zeta_p^{ai-j} \frac{\eta^i}{\pi^j}\ ,\ \frac{\pi^m}{\eta^n} \mapsto \zeta_p^{m-an} \frac{\pi^m}{\eta^n}.$$

Since those two elements form a regular system of the local germ $\mathcal{O}_{Y,P}$, we can apply Corollary 2.4.14 to the G-action on $\mathcal{O}_{Y,P}$ and conclude that the image of P under the quotient map is regular if and only if one of the two elements η^i/π^j and π^m/η^n is invariant under σ. In particular, this is equivalent to the fact that $ai - j \equiv 0 \mod p$ or $an - m \equiv 0 \mod p$.

We will now develop a notation to formalize this for a sequence of arbitrary ρ_k

which correspond to a chosen regular model of Y. Above we have reasoned that we need to consider only the models realizing components of the type $D(\rho_k)$. We will consider only those lying between the original component and $D(1)$, as this will suffice to derive a contradiction. So assume that we have chosen a model of B realizing exactly the components $D(\rho_k)$ for

$$0 = \rho_0 < \rho_1 < \dots \rho_r = 1,$$

between the original component and $D(1)$. To it, we will associate the ordered tuple

$$(\theta_0 = a, \theta_1, \dots, \theta_r = a - 1),$$

where $\theta_k := ia - j \mod p$. This means that for the geometric parameter g of $D(\rho_k)$, we have $\sigma(g) = \zeta_p^{\theta_k} g$.

With our above considerations on the double points and their invariants, we now see that there exists a regular model Y of B such that Y/G is regular only if there exist $0 < \rho_1 < \dots \rho_r = 1$ such that $D(\rho_k, \rho_{k+1})$ is regular for all $1 \leq k \leq r-1$. The latter is equivalent to the fact that for any ρ_k with odd k, we have $\theta_k \equiv 0 \mod p$; since no two subsequent θ_k can be zero in a regular model, else π would be invariant. The condition on the ρ_k corresponds to the fact that at each double point of Y, we can chose a G -invariant parameter coming from one of the intersecting components.

Now we prove by an elementary calculation that such a model cannot exist. For this, we prove that a sequence of ρ_i with the above properties cannot exist. Since Y is regular, we must be able to construct such a sequence of ρ_k by a σ -process. A blowing-up in the double point of $D(\rho_k, \rho_{k+1})$ adds a component between $D(\rho_k)$ and $D(\rho_{k+1})$, and thus transforms the tuple

$$(\theta_0, \dots, \theta_k, \theta_{k+1}, \dots, \theta_r)$$

into the tuple

$$(\theta_0, \dots, \theta_k, \theta_k + \theta_{k+1}, \theta_{k+1}, \dots, \theta_r),$$

where a θ and a ρ for the new component have been added between the k -th and the $(k+1)$ -th number. So beginning at

$$(a, a-1),$$

we must reach the conditions above by finitely many blow-ups, since $Y \to \operatorname{Spec} B$ is a σ -process as we have seen. Blowing up once, we get

$$(a, a-1) \mapsto (a, 2a-1, a-1).$$

By our assumptions that $a \not\equiv 0, 1 \mod p$ and $2a \not\equiv -1 \mod p$, none of these three numbers is zero modulo p, and they are pairwise distinct. We will now derive a contradiction by showing that any blow-up in a double point changes this tuple in a way that three subsequent numbers are nonzero modulo p. Then we are done since this means that the quotient of the corresponding model will be singular, and by our previous considerations, this happens for every model of B.

We now proceed to derive the contradiction by induction. Assume that we have a tuple

$$(\dots, \theta_{k-1}, \theta_k, \theta_{k+1}, \dots)$$

where $\theta_{k-1}, \theta_k, \theta_{k+1} \neq 0 \mod p$. In order to obtain a regular quotient, we have to blow up both double points at the component $D(\rho_k)$, obtaining

$$\begin{aligned}(\dots, \theta_{k-1}, \theta_k, \theta_{k+1}, \dots) \mapsto &(\dots, \theta_{k-1}, \theta_{k-1} + \theta_k, \theta_k, \theta_k + \theta_{k+1}, \theta_{k+1}, \dots) \\ &=: (\dots, \theta'_{k'-2}, \theta'_{k'-1}, \theta'_{k'}, \theta'_{k'+1}, \theta'_{k'+2}, \dots).\end{aligned}$$

Now an elementary calculation shows that

$$\theta'_{k'+1} - \theta'_{k'-1} \equiv \theta_{k+1} - \theta_{k-1} \mod p,$$

so $\theta'_{k'+1}$ and $\theta'_{k'-1}$ are equal if and only if θ_{k+1} and θ_{k-1} are. In particular, if the latter were not, then $\theta'_{k'+1}$ and $\theta'_{k'-1}$ cannot both be zero modulo p. Moreover, if $\theta_{k-1}, \theta_k, \theta_{k+1}$ were pairwise distinct, then either $\theta'_{k'}, \theta'_{k'+1}, \theta'_{k'+2}$ are pairwise distinct and nonzero, or $\theta'_{k'-2}, \theta'_{k'-1}, \theta'_{k'}$ are.

In particular, starting at

$$(a, 2a-1, a-1),$$

where these three numbers are nonzero and pairwise distinct, performing blowing-ups in closed points, we can never arrive at a sequence where every second number is zero modulo p.

By our above considerations, this implies that in this example and under our conditions on a and p, we can never find a regular model Y of B such that Y/G is regular.

We now shortly discuss our assumption $2a \not\equiv -1 \mod p$: Without this assumption, i.e. in the case when $2a \equiv -1 \mod p$, the above proof fails, since one can choose a partition of $D(0,1)$ such that the quotient model becomes regular. However, the arguments in the proof carry over when considering $D(1,2)$ instead of $D(0,1)$. We omit the proof since this works in complete analogy.

Also, we show now why the claim is false when $p \leq 3$ or $a = 1$: If $a = 1$, as it is always the case in this example when $p = 2$, one can take

$$(\dots, \rho_k, \dots) = (0,1) \; ; \; (\dots, \theta_k, \dots) = (1,0).$$

If $p = 3, a = 2$ then one arrives at

$$(\dots, \rho_k, \dots) = \left(0, \frac{1}{2}, 1, 2\right) \; ; \; (\dots, \theta_k, \dots) = (2,0,1,0).$$

The main point which is to be observed in this subsection is the following: It is hopeless to try to achieve regularity on both the niveaus of B and A except in very special cases (compare [Art75] for $p = 2$). Instead, one must describe the combinatorial structure of well-chosen models of B resp. A and interrelate them.

This is what we try in the next subsections: We will find a finite set of components on which the main part of information is located - namely the branch components of the minimal regular model of A - and try to describe them on both levels. The space in-between those components can then be filled with path components whose combinatorics seems only to depend on the respective base ring, and not on the Galois action. This also explains the result of this subsection in a certain way. We will re-examine these observations in more details later.

4.3.2 Critical components and correspondence of models

We keep the assumptions and notations 4.2.30 of the previous section. In this section, we will study the correspondence of the group action on a ring to the minimal resolution of the ring of its ring of invariants. We give first some observations how one can relate models of B to models of A. We will define a model of B corresponding to the minimal desingularization of A.

Definition 4.3.2. Let D be a component of B. Let n be the number of distinct points on D leading to the original components (compare 4.2.62). The component D is called *critical* if $A(D)$ is not geometrically smooth at $3-n$ points or more. All other components are called *non-critical*.

Note that $n=1$ or $n=2$ depending on the number of original components and the position of the component D. If B has only one original component, then we have always that $n=1$. If there are two original components in B, then $n=2$ if D is a connecting component, else $n=1$.

Remark 4.3.3. If there is only one original component, then D is critical if $A(D)$ is geometrically smooth at 2 points or more.

If there are two original components, then there are two cases: Let Y be the minimal regular normal crossings realization of D.
Case 1: If D is a connecting component, then D is critical if $A(D)$ contains a not geometrically smooth point.
Case 2: If D is not a connecting components, then D is critical if $A(D)$ is not geometrically smooth at 2 points or more.

The central idea of this section is that critical components correspond exactly to the branch components in the minimal regular model of A. The main observation here is: Since $A(D)$ is lifting almost all of D, any model realizing D will have geometrical singular points under geometrically smooth points corresponding to the geometrical singularities of $A(D)$. We will make this more precise in the following.

We first state that critical components are all G-stable:

Lemma 4.3.4. *Let D be a component which has non-trivial orbit under G. Then D is not critical.*

Proof. By assumption, the G-action permutes p copies of $B(D)$. Thus the component lying under D is also regular, since $A(D)=B(D)$, thus non-critical by definition. □

Now we examine further the behaviour of critical components under the quotient map:

Lemma 4.3.5. *Let D be a critical component of B. Then D is realized in any regular normal crossings model of A by a branching component.*

Proof. Define n and $P_1,\dots P_n$ as in Definition 4.3.2. Assume there would exist a regular normal crossings model Z of A not realizing D. Then by Proposition 4.2.60, the relative minimal regular realization $Z'\to Z$ of D with respect to Z exists. Let C be the component on Z' realizing D. By Proposition 4.2.61, C has two neighbors or less. Also, since Z' is regular, all points on C except at most two are geometrically smooth. Now the ring $A(C)$ is then not geometrically smooth at $3-n$ points or more, since D was critical. So there have to be at least three points on C which are not geometrically smooth. But this is a contradiction to what we have derived above. This proves that D is realized on any regular normal crossings model of A.

It remains to prove that D is realized by a branching component. So let X be a regular normal crossings model of A. Let C' be the component realizing D in X. Since X is normal crossings, the special fiber of X can have only geometrically smooth points and geometric double points. So C has to have geometric double points on X where the geometrical singularities of $A(D)$ on C' are, resulting from distinct neighboring components by Proposition 4.2.49. This proves the second claim. □

A converse is given by the following Lemma:

Lemma 4.3.6. *Let C be a component realized in the minimal regular normal crossings model of A which is a branching component there. Then there is a unique G-stable component D of B lying over C, and D is critical.*

Proof. Define n and $P_1, \dots P_n$ as in Definition 4.3.2. Let Z be the minimal regular normal crossings model of A. Since C has three or more neighbors, there exists at least one connected component in $Z \otimes k - C$ not containing an original component and intersecting C. Contracting all connected components not containing an original component yields a model $Z \to X$ of A such that C is terminal or connecting and $3 - n$ or more geometrical singularities lie on C. Thus every component D over C must be critical by definition, and thus G-stable by Lemma 4.3.4. □

Together with Lemma 4.3.5, this gives the following statement:

Corollary 4.3.7. *Let D be a component on B, let C be the component on A obtained from the restriction of v_D. Then D is critical if and only if C is realized in the minimal regular normal crossings model of A as a branching component.*

Proof. Combine Lemma 4.3.5 and Lemma 4.3.6. □

Proposition 4.3.8. *Let D be a component of B such that $B(D)$ is singular, i.e. a not potentially terminal component which is not connecting, cf. Lemma 4.2.64. Denote by y the by Lemma 4.2.64 unique singular point on $B(D)$. Then the image x of y in $A(D)$ is singular.*

Proof. Let Y be the minimal regular realization of D, which exists by 4.2.54, and is normal crossings, since B is regular. Since D is not potentially terminal, it has exactly two neighboring components in Y by Proposition 4.2.55. Also, since the dual graph of Y is a tree by 4.2.49, there exists a unique terminal T component on Y such that D lies on the path from T to the original component. Let m be the multiplicity of T. Then the multiplicity of D is nm with $n \in \mathbb{N}, n \geq 2$, as one can see from Remark 4.2.57.

We distinguish the two cases of Remark 4.2.48: If D has trivial orbit under G, consider a model Y' of B where D is realized as a terminal component (thus Y' is singular). The point y corresponds to a point on Y'. Also, G acts by permuting p copies of y. Thus x is locally isomorphic to y and singular (we even have $A(D) = B(D)$).

If D has trivial G-orbit, we will prove that x is singular by contradiction. Assume that x is regular. Then there exists a model X of A such that D is realized in X by a component C and the point x can be identified with a single regular point of C. Let X' be the relatively minimal regular realization of T with respect to X. This exists by Proposition 4.2.60, assume T' is the component realizing T in X'. As described in Remark 4.2.57, the multiplicity of C will divide the multiplicity of T'. By the ramification formula (see Remark 4.2.37), we can distinguish two cases:

Case 1: The multiplicity of C is pnm. The multiplicity of T' is m or pm. But the multiplicity of C has to divide the multiplicity of T', as stated earlier, which cannot be, since $n \geq 2$. This gives the desired contradiction.

Case 2: The multiplicity of C is nm. If n is not p, then one can infer a contradiction as in case 1, so we can assume that $n = p$. In this case, T' can be obtained as successive blowing-up only in closed geometrically smooth points, see Remark 4.2.57. Since the multiplicity of C is the same as the multiplicity of D, it is mandatory that on an open dense subset of D one can choose a geometric parameter which is G-invariant, say f. By adding elements in R^G, we can assume that f has a zero at y and at T'. The points on this subset will have maximal ideal $(f - c, g)$, where g is the arithmetic parameter on D, and $c \in R^G$ signifies the position of the point on D, see Remark 4.2.66 So the points on C will have maximal ideal $(f - c, N(g))$ on an open dense subset. Since T' is now obtained by blowing up in closed geometrically smooth points, on an open dense subset of T', the maximal ideals will be of form $(f/N(q)^d - c, N(g))$ for some $d \in \mathbb{N}$ which is the number of performed blowing-ups to reach T' from C. But normalizing T' shows that points on T will be of form $(f/N(q)^d - c, g)$ on an open dense subset, which means that the multiplicity of T' equals that of T, i.e. equals m. But by assumption, it also equals pm, which cannot be, giving the desired contradiction.

This finishes the proof of the proposition. □

Corollary 4.3.9. *The minimal desingularization of A contains a terminal component D of multiplicity 1 or p. As a component of B, the multiplicity of D is 1.*

Proof. We claim that by Proposition 4.3.8 and Theorem 4.2.16, there has to be a potentially terminal component T of A whose multiplicity on the level of B is one. By Proposition 4.3.8, any component of B which is not potentially terminal cannot be terminal in a regular model of A. And because every regular model of B contains by Theorem 4.2.16 a component of multiplicity 1, the claim follows by the ramification formula, see Remark 4.2.37. □

These results motivate the following definition:

Definition 4.3.10. A model Y of B is called *model of the G-action* (on B) if the following condition is fulfilled: Y is regular, every critical component is realized in Y, and for any component D and any geometrically smooth point x on D, the image of x in $A(D)$ is regular.

Moreover, Y called *minimal model of the G-action* if it is birationally minimal with that property.

Remark 4.3.11. The minimal model of the G-action is unique if it exists. In particular, Y can be algorithmically constructed from B as blowing-ups in regular points. By Lemma 4.3.4, all components on the special fiber of a model of the G-action are G-stable.

Theorem 4.3.12. *There exists a minimal model of the G-action on B.*

Let Y be a minimal model of the G-action on B, let X be the minimal normal crossings desingularization of A. Then the path reduction of the dual graph of Y is isomorphic to the path reduction of the dual graph of X. Moreover, the branch vertices in the path reductions and the intersection points correspond to each other under the quotient map.

Proof. First we prove the existence claim. Assume that A has a minimal normal crossings desingularization X. Then the normalization Y' of X in $Q(B)$ is a G-stable model of B. There exists a minimal desingularization Z of Y', since B is regular. The model Z is a model of the G-action on B, since any exceptional component will be non-critical by Lemma 4.3.5. Contracting finitely many divisors (which is possible by Corollary 4.2.22) will then give a minimal model Y of the G-action.

Now we continue with the structure statement. By Corollary 4.3.7 and Proposition 4.2.49, there is a one-to-one-correspondence between the critical components of B and the branch components on the minimal regular normal crossings model of A. So the branch vertices of the dual graphs of the minimal normal crossings desingularization of A and the minimal model of the G-actions are the same.

To prove that the path reductions of X and Y coincide, we still need to prove two things: First, that any branching component on Y is G-stable, and second, that the intersection points on the branching components of Y correspond to those on the branching components of X. Since then Proposition 4.2.49 shows that the combinatorial structures coincide.

The first fact follows from Lemma 4.3.4. There we have proved that the critical components of B are G-stable. And every branching component of B is critical by the definition of critical. The second fact follows almost from the definition of the model of the G-action, since for an arbitrary component D, any not geometrically smooth point on $A(D)$ induces an intersection point on any model of the G-action above it. It remains to prove that any intersection point on a critical component of B lies over a not geometrically smooth point of $A(D)$. For the regular points of $B(D)$ this follows again by definition, for the possible singular point of $B(D)$ this follows from Prop 4.3.8. □

Remark 4.3.13. It might be interesting to ask if one can prove the existence of minimal G-model without using Lipman's desingularization theorem, and then use this to prove an alternative desingularization theorem. For example, one could prove that the singularities of the minimal G-model are always toric, and then use simpler results on desingularization of toric singularities.

For future characterizations of chain-like desingularizations, we want to note that the minimal normal crossings desingularization of A is a chain if and only if the minimal desingularization of A is a chain.

Corollary 4.3.14. *The minimal desingularization of A is a chain if and only B has no critical components.*

Proof. First assume that B has no critical components. In this case, the minimal model of the G-action is B itself by definition. So Theorem 4.3.12 implies that the path reduction of the minimal normal crossings desingularization X of A consists only of the original components linked by an edge. But this can only occur if the dual graph of X is a chain. In particular, the dual graph of the minimal desingularization of A is also a chain.

On the other hand, if B has a critical component, then the minimal desingularization of A cannot be a chain by Theorem 4.3.12, since it must then contain a branch component. □

The next corollary uses Theorem 4.3.12 and is thus not suitable to fulfill Remark 4.3.13:

Corollary 4.3.15. *Let Y be a model of the G-action on B. Then the singularities of $X = Y/G$ are toric.*

Moreover, the minimal model of the G-action on B is the model Y birationally minimal with the property that Y is regular and the singularities of X are toric.

Proof. By the definition of Y, singularities occur only under the double points of Y. Now one can apply Corollary 4.3.14 to the local germs at the double points to see that the minimal desingularizations of the singularities of X are chains. By 2.8.1, this is equivalent for the singularities to be locally toric.

The second assertion follows by applying Corollary 4.3.14 to the local germs at the double points. □

4.3.3 The augmentation ideal on critical components

The results of chapter 3 give us a tool to describe when a component of B is critical. In Lemma 4.3.4, we have already seen that critical components are always G-stable. We now want to examine in terms of augmentation ideals when the G-stable components are critical.

Definition 4.3.16. Let D be a G-stable component of B. We will denote by $\mathcal{I}(D)$ the augmentation ideal on $B(D^\times)$. If $D = D(\rho)$, we will also write $\mathcal{I}(\rho) = \mathcal{I}(D(\rho))$.

Corollary 4.3.17. *Let D be a G-stable component of B.*

If D is not potentially terminal, then D is critical if and only if $\mathcal{I}(D)$ is not invertible. If D is potentially terminal, then D is critical if and only the locus of non-invertibility of $\mathcal{I}(D)$ is at most a single point.

Proof. This follows directly from the regularity criterion 3.3.4, Remark 4.3.3 and Proposition 4.3.8. □

Proposition 4.3.18. *Let D, D' be two G-stable components of B, assume that D lies on the path from any original component to D'. Let y be the point on $B(D)$ where the connected component of D' in the complement of D meets D. If $\mathcal{I}(D)$ is invertible at y, then D' is not critical.*

Proof. Since $\mathcal{I}(D)$ is invertible at y, the ring $A(D)$ is geometrically smooth at the image of y by Prop 3.2.2. So D' cannot be realized in the minimal normal crossings desingularization X of A. Since else the connected component meeting in the preimage of y in X would contain no original component by Proposition 4.2.49 and could be blown down to a geometrically smooth point by Proposition 4.2.12. Thus Theorem 4.3.12 implies that D' is not critical. □

Remark 4.3.19. A model Y of B is a model of the G-action if the following condition is fulfilled: Y is regular, every critical component is realized in Y, and the augmentation sheaf of Y is principal on any geometrically smooth point.

Corollary 4.3.20. *If there is exactly one original component, then the minimal resolution of A is a chain if and only if for any component D of B, the following holds: The augmentation ideal $\mathcal{I}(D)$ is invertible if D is not potentially terminal, and $\mathcal{I}(D)$ is invertible at all but at most one point if D is potentially terminal.*

If there are two original components, then the minimal resolution of A is a chain if and only if the augmentation ideal on any connecting component of B is invertible.

Proof. Corollary 4.3.17 states that the critical components can be characterized in terms of the points of non-invertibility of $\mathcal{I}(D)$. Using the corollary, the statement of the corollary is equivalent to:

If there is exactly one original component, then the minimal resolution of A is a chain if and only if no component of B is critical.

If there are two original components, then then the minimal resolution of A is a chain if and only if no connecting component of B is critical.

The first assertion follows directly from Corollary 4.3.14; the second follows from Corollary 4.3.14 and the following: No connecting component of B is critical if and only no component of B is critical by Proposition 4.3.18. □

Corollary 4.3.21. *If $\widehat{B} = R[[\eta]]$, then the minimal resolution of A is a chain if and only if for an appropriate choice of η and for any $\rho \geq 0$, the augmentation ideal is invertible on $\mathcal{O}_B(\rho)$.*

Proof. We are here in the case where B has only one original component. We have to prove two directions. First we prove the if-direction. If $\mathcal{I}(\rho)$ is invertible on $\mathcal{O}_B(\rho)$, then since the dual graph of A is a tree (see Proposition 4.2.49), by Proposition 4.3.18, no component of B is critical. By Corollary 4.3.20, the minimal resolution of A is a chain.

We now prove the only if-direction. If $\mathcal{I}(\rho)$ was not invertible on some $\mathcal{O}_B(\rho)$, the component $D(\rho)$ was critical, and thus the minimal resolution of A would contain a branching component by Theorem 4.3.12. □

4.3.4 Application to components with K-rational center

Now we will apply the results from the previous sections to control the minimal desingularization in the example of section 4.2.7 by the augmentation sheaf. We fix a geometrically smooth ring B and a parameter η of B, such that $\widehat{B} = R[[\eta]]$. For each $\rho \in \mathbb{Q}$, we will describe the augmentation ideal $\mathcal{I}(\rho)$ on $\mathcal{O}_B(\rho)$. We will assume that η is chosen such that if $\mathcal{I}(\rho)$ is not principal on a reduced component D , then D is critical. The latter is no loss of generality in our setting by Proposition 4.3.18 and is postulated only to ensure that we have chosen η as corresponding to a somewhat maximal blowing-up in loci of non-invertibility.

We will mainly use Proposition 3.3.4 on the explicit choice of geometric and arithmetic parameters which we can make in this case, as we have seen in Lemma 4.2.69.

So fix the regular system (η, π) of $\mathfrak{m}_B$ as above. We define

$$\sigma(\pi) =: \pi\lambda \ , \ \ell := v_\pi(\lambda - 1)$$

for the augmentation of π and

$$I(\eta) =: \pi^\delta f(\eta) u \quad \text{where } v_\pi(f) = 0, f \in R[\eta] \text{ and } u \in B^\times$$

is the Weierstraß decomposition of $I(\eta)$ in $\widehat{B}$. Now it follows from Corollary 3.3.6 that A is geometrically smooth if and only if $\ell < \delta$ or $\deg f = 0$. We now want to calculate the augmentation ideal $\mathcal{I}(\rho)$ for arbitrary $\rho \in \mathbb{Q}$.

First we introduce some abbreviating notation:

Definition 4.3.22. Let $f_1, f_2 \in \mathcal{O}_B(\rho)$. If

$$f_1 = f_2 u \text{ with } u \in \mathcal{O}_B(\rho)^\times,$$

we will write this by

$$f_1 \sim_\rho f_2.$$

If ρ is fixed resp. clear by the context, we will just write $f_1 \sim f_2$.

Remark 4.3.23. Note that for fixed $\rho \in \mathbb{Q}$, the relation $\sim_\rho$ is an equivalence relation on $\mathcal{O}_B(\rho)$.

Lemma 4.3.24. *One has* $\sigma(\eta) \sim_\rho \eta$ *for* $\rho > 0$.

Proof. The geometric parameter on $\mathcal{O}_B(\rho)$ can be taken as η^i/π^j for $j/i = \rho$ coprime, according to Lemma 4.2.69. Since $\mathcal{O}_B(\rho)$ does not contain the unique zero of the geometric parameter by definition, η^i/π^j is invertible. Thus

$$\sigma\left(\frac{\eta^i}{\pi^j}\right) = \lambda^{-j}\frac{\sigma(\eta^i)}{\pi^j}$$

is also invertible. Since λ is a unit, we have that

$$\frac{\eta^i}{\pi^j} \cdot \frac{\pi^j}{\sigma(\eta^i)} = \left(\frac{\eta}{\sigma(\eta)}\right)^i$$

is an element of $\mathcal{O}_B(\rho)$. Analogously, we see that

$$\left(\frac{\sigma(\eta)}{\eta}\right)^i$$

lies in $\mathcal{O}_B(\rho)$. Since η is also contained in $\mathcal{O}_B(\rho)$, it follows that $\sigma(\eta)/\eta \in \mathcal{O}_B(\rho)^\times$, which we wanted to prove. □

Remark 4.3.25. Also, one has $\pi \sim_\rho \sigma(\pi)$ for all $\rho \geq 0$, since $\sigma(\pi)/\pi$ is a already unit in R and R is contained in $\mathcal{O}_B(\rho)$.

Notations 4.3.26. We will denote the incomplete σ -Norm by

$$N_{(j)}^{(k)} := \prod_{i=j}^{k-1} \sigma^i \text{ and } N^{(k)} := N_{(0)}^{(k)} = \prod_{i=0}^{k-1} \sigma^i$$

Furthermore, we will write

$$J_k := \frac{\sigma^k}{\mathrm{id}} - 1 \text{ and } J := J_1.$$

Remark 4.3.27. One directly calculates that

$$I(N_{(j)}^{(k)}) = (\sigma^k - \sigma^j)N_{(j+1)}^{(k)}.$$

Remark 4.3.28. Note that by Lemma 4.3.24 and Remark 4.3.25, the element $N^{(i)}(\eta)/N^{(j)}(\pi)$ is a geometric parameter on $\mathcal{O}_B(\rho)$ for $\rho = j/i \in \mathbb{Q}_{\geq 0}$ with i, j coprime. Similarly, any element $N^{(m)}(\pi)/N^{(n)}(\eta)$ with $im - jn = 1$ is an arithmetic parameter on $\mathcal{O}_B(\rho)$. This is true since

$$N^{(i)}(\eta) = \eta^i \prod_{k=1}^{i} \sigma^{i-k}(\xi^k) \text{ with } \xi = \frac{\sigma(\eta)}{\eta},$$

where ξ and thus $\sigma^{i-k}(\xi)$ is a unit in $\mathcal{O}_B(\rho)$ by Lemma 4.3.24. Thus $N^{(i)}(\eta) \sim \eta^i$. Analogous calculations hold for $N^{(j)}(\eta)$, so we have $N^{(i)}(\eta)/N^{(j)}(\pi) \sim \eta^i/\pi^j$ and is thus itself a geometric parameter; in analogy, $N^{(m)}(\pi)/N^{(n)}(\eta) \sim \pi^m/\eta^n$ is an arithmetic parameter.

Under our conditions on the chosen parameter η we can now prove a criterion on the invertibility of the augmentation ideal $\mathcal{I}(\rho)$ only in terms of the augmentation elements of the chosen generators η, π of $\mathfrak{m}_B$:

Lemma 4.3.29. *Let $\rho \in \mathbb{Q}$ with $\rho = j/i$ with $i, j \in \mathbb{N}$ coprime. Then $\mathcal{I}(\rho)$ is a principal $\mathcal{O}_B(\rho)$ -ideal if and only if one of the following conditions is fulfilled:*

(i) $v_\rho(J(\eta)) < \ell$ *or* p *divides* ij, *and* $I(\eta)$ *is invertible on* $\mathcal{O}_B(\rho)$.

(ii) $v_\rho(J(\eta)) = \ell = v_\rho(I(g))$ *and* $I(\eta) + \eta(1 - N^{(\rho)}(\lambda))$ *has no zeroes on* $\mathcal{O}_B(\rho)$.

(iii) $v_\rho(J(\eta)) = \ell < v_\rho(I(g))$.

(iv) $v_\rho(J(\eta)) > \ell$.

(v) p *divides* i *and* $v_\rho(J(\eta)) + 2i^{-1} > \ell$

If $\mathcal{I}(\rho)$ is not invertible, then the locus of non-invertibility on $\mathcal{O}_B(\rho)$ is exactly

(a) *The zero locus of* $I(\eta)$ *on* $\mathcal{O}_B(\rho)$, *if* $v_\rho(I(\eta)/\eta) < \ell$ *or* p *divides* ij.

(b) *The zero locus of* $I(\eta) + \eta(1 - N^{(\rho)}(\lambda))$ *on* $\mathcal{O}_B(\rho)$, *interpreting* ρ *as integer modulo* p, *if* $v_\rho(J(\eta)) = \ell = v_\rho(I(g))$, *and* p *does not divide* ij.

Moreover, if $v_\rho(J(\eta)) = \ell < v_\rho(I(g))$, *then* $I(\eta)$ *is invertible on* $\mathcal{O}_B(\rho)$.

Proof. Fix a $\rho = j/i$ and the corresponding component $\mathcal{O}_B(\rho)$. As in Remark , we can take for the component $\mathcal{O}_B(\rho)$ as geometric parameter $g := N^{(i)}(\eta)/N^{(j)}(\pi)$ with $\rho = j/i$. Similarly, the arithmetic parameter is of form $h := N^{(m)}(\pi)/N^{(n)}(\eta)$. Any point on $\mathcal{O}_B(\rho)$ is regular with maximal ideal $(g - c, h)$ for some $c \in R^G$. We will first calculate

$$I(g - c) = I(g).$$

By Remark 3.2.4, we calculate that

$$\begin{aligned} I(g) = I(N^{(i)}(\eta)/N^{(j)}(\pi)) &= \frac{I(N^{(i)}(\eta))}{\sigma(N^{(j)}(\pi))} + \frac{N^{(j)}(\pi) - \sigma(N^{(j)}(\pi))}{N^{(j)}(\pi)\sigma(N^{(j)}(\pi))} N^{(i)}(\eta) \\ &\sim \frac{I(N^{(i)}(\eta))}{N^{(j)}(\pi)} - \frac{I(N^{(j)}(\pi))}{N^{(j)}(\pi)} \cdot g. \end{aligned}$$

Since g is invertible on $\mathcal{O}_B(\rho)$, one obtains

$$\begin{aligned} I(g) &\sim \frac{I(N^{(i)}(\eta))}{N^{(i)}(\eta)} - \frac{I(N^{(j)}(\pi))}{N^{(j)}(\pi)} \sim \frac{(\sigma^i - \mathrm{id})\eta}{\eta} - \frac{(\sigma^j - \mathrm{id})\pi}{\pi} \\ &= \frac{(\sigma^i - \mathrm{id})\eta}{\eta} + 1 - N^{(j)}(\lambda) = J_i(\eta) - J_j(\pi), \end{aligned}$$

where the last $\sim$ follows by Remark 4.3.27.Analogously, one calculates

$$\begin{aligned} I(h) = I(N^{(m)}(\pi)/N^{(n)}(\eta)) &\sim h\left(\frac{(\sigma^n - \mathrm{id})\eta}{\eta} - \frac{(\sigma^m - \mathrm{id})\pi}{\pi}\right) \\ &= h\left(\frac{(\sigma^n - \mathrm{id})\eta}{\eta} + 1 - N^{(m)}(\lambda)\right) = h\left(J_n(\eta) - J_m(\pi)\right) \end{aligned}$$

We now must describe both augmentations in terms of the h -valuation and divisibility.

This can be directly done by the above presentation and depends on which of the i, j, m, n are divisible by p . We remark: J_k is the zero map if and only if p divides k. If k is coprime to p , then we have

$$v_\rho(J_k(\eta)) = v_\rho(I(\eta)) - \rho \text{ and } v_\rho(J_k(\pi)) = v_\pi(J_k(\pi)) = \ell.$$

We also note that we may multiply any power of the geometric parameter g to the arithmetic parameter h , yielding another arithmetic parameter, since the geometric parameter is a unit. Thus we may assume that p divides n if p does not divide i. Else we may assume that p divides m. This is true by the following: If i is divisible by p, then j and n are not. If j is divisible by p , then i and m are not - see Lemma 4.2.71. Thus we may without loss of generality replace the arithmetic parameter h by hg or h/g.

We will now distinguish some cases:

Case 1: p is coprime to i, j. Then we can assume that p divides n, and thus m is also coprime to p. We then have

$$I(g) \sim J_i(\eta) - J_j(\pi) \text{ and } I(h) \sim hJ_m(\pi).$$

Since $h^i \sim \pi$, and m is coprime to p, we have that

$$I(h) \sim \pi^\ell h \text{ and thus } v_\pi(I(h)) = \ell + i^{-1}.$$

Note that $I(h)$ has no zeroes on $\mathcal{O}_B(\rho)$. Thus $\mathcal{I}(\rho)$ is invertible at a point P if and only if

$$v_\rho(I(g)) \geq v_\rho(I(h)) = \ell + i^{-1} \text{ or } I(g) \text{ has no zero at } P.$$

Since the h -valuation maps to $\mathbb{N}/i$, this is equivalent to

$$v_\rho(I(g)) > \ell \text{ or } I(g) \text{ has no zero at } P.$$

We will now further distinguish cases depending whether $v_\rho(I(g)) \geq v_\rho(I(h))$ or not.

Case 1.1 $v_\rho(I(g)) \geq v_\rho(I(h))$. We remark that one has

$$v_\rho(I(g)) \geq v_\rho(I(h)) \text{ if and only if } v_\rho(J_i(\eta)) = v_\rho(J_j(\pi)) = \ell < v_\rho(I(g)).$$

Since else $v_\rho(I(g)) = \min(v_\rho(J_i(\eta)), v_\rho(J_j(\pi)))$ which is certainly smaller or equal than ℓ. So in this case, $\mathcal{I}(\rho)$ is always invertible. Also, the zero set of $J_i(\eta)$ is void, since we must have $I_i(\eta) \equiv J_j(\pi)\eta \mod \pi^{\ell+1}$.

Case 1.2 $v_\rho(I(g)) < v_\rho(I(h))$. From our previous considerations in case 1.1, it follows that $v_\rho(I(g)) = \min(v_\rho(J_j(\pi)), v_\rho(J_i(\eta))$. Comparing the conditions of this case with the regularity criterion, $\mathcal{I}(\rho)$ is invertible at a point P if and only $I(g)$ has no zero at P. We now analyze the behavior of $I(g)$ in this case, depending on whether we have $v_\rho(J_i(\eta)) < \ell = J_j(\pi))$, or not.

Case 1.2.1 If $v_\rho(J_i(\eta)) > \ell = v_\rho(J_j(\pi))$, then the term $J_j(\pi)$ dominates, and we have $v_\rho(I(g)) = \ell$. Then $I(g)$ has no zero at a point P if and only if $J_j(\pi)$ has no zero at P . But the latter is a constant, so $\mathcal{I}(\rho)$ is invertible.

Case 1.2.2 If $v_\rho(J_i(\eta)) < \ell = v_\rho(J_j(\pi))$, then the term $J_i(\eta)$ dominates, and we have $v_\rho(I(g)) < \ell$. Then $I(g)$ has no zero at a point P if and only if $J_i(\eta)$ has no zero at P if and only if $I(\eta)$ has no zero at P .

Case 1.2.3 If $v_\rho(J_i(\eta)) = \ell = v_\rho(J_j(\pi))$, then also $v_\rho(I(g)) = \ell$ by our precious considerations. Summarizing, we have that $\mathcal{I}(\rho)$ is invertible at a point P if and only if P is not in the zero locus of $J_i(\eta) - J_j(\pi)$.

We will now summarize all cases. For this, we first remark that since the choice of the generator σ of G was arbitrary, and i and j are both prime to p , one has $v_\rho(J_i(\eta)) = v_\rho(J(\eta))$, and $v_\rho(J_j(\pi)) = \ell$. Similarly, the zero set of $J_i(\eta) - J_j(\pi)$ is the same as the zero locus of $J(\eta) - J_{j/i}(\pi) = J(\eta) - J_\rho(\pi)$, where the rational ρ has to interpreted as integer modulo p.

One has now to observe that these are indeed all cases which can occur. Now in summary: If $v_\rho(J(\eta)) > \ell$, we are in case 1.2.1, and $\mathcal{I}(\rho)$ is invertible. If

$v_\rho(J(\eta)) < \ell$, we are in case 1.2.2, and $\mathcal{I}(\rho)$ is invertible exactly where $I(\eta)$ has no zeroes. If $v_\rho(J(\eta)) = \ell$, we are either in case 1.2.3 or in case 1.1. In both cases, $\mathcal{I}(\rho)$ is invertible exactly where $J(\eta) - J_\rho(\pi)$ has no zeroes.

Case 2: p divides j . Then p is coprime to m and i , and we can assume that p divides n. So we get

$$I(g) \sim J_i(\eta) \text{ and } I(h) \sim hJ_m(\pi).$$

As in case 1, $\mathcal{I}(\rho)$ is invertible at a point P if and only if

$$v_\rho(I(g)) > \ell \text{ or } I(g) \text{ has no zero at } P.$$

In this case, since $I(g) = J_i(\eta)$, this is equivalent to

$$v_\rho(J_i(\eta)) > \ell \text{ or } J_i(\eta) \text{ has no zero at } P,$$

and we have that

$$v_\rho(I(g)) = v_\rho(J_i(\eta)) = v_\rho(I(\eta)) - \rho.$$

So as in case 1, it follows that $\mathcal{I}(\rho)$ is invertible at P if and only if

$$v_\rho(J(\eta)) > \ell \text{ or } I(\eta) \text{ has no zero at } P.$$

Case 3: p divides i . Then p is coprime to n and j , and we can assume that p divides m. So we get

$$I(g) \sim J_j(\pi) \text{ and } I(h) \sim hJ_n(\eta).$$

Thus $I(g)$ has no zeroes and we have $v_\rho(I(g)) = v_\pi(I(g)) = \ell$. So $\mathcal{I}(\rho)$ is invertible at a point P if and only if

$$v_\rho(I(h)) \geq v_\rho(I(g)) = \ell \text{ or } I(h) \text{ has no zero at } P.$$

Since $I(h) \sim hJ_n(\eta)$, this is equivalent to

$$v_\rho(J_n(\eta)) > \ell - 2i^{-1} \text{ or } J_n(\eta) \text{ has no zero at } P.$$

As in case 1, if follows now that $\mathcal{I}(\rho)$ is invertible at P if and only if

$$v_\rho(J(\eta)) + 2i^{-1} > \ell \text{ or } I(\eta) \text{ has no zero at } P.$$

□

We will now derive some criteria for the minimal desingularization of A be a chain. The previous results allow us to make some simplificational assumptions:

Remark 4.3.30. Corollary 4.3.9 guarantees us that the minimal desingularization of A contains a terminal component which T is reduced on the level of B. So we may assume that the geometric parameter η of B is chosen such that it has a zero of one of the terminal reduced components of B , i.e. the geometric parameter of T is η^i/π^j for suitable $i, j \in \mathbb{Q}$.

We will assume this for η in the rest of our section.

Combining Lemma 4.3.29 and Corollary 4.3.20, we can infer the following:

Corollary 4.3.31. *Let $\rho_k \in \mathbb{Q}$ be the unique smallest radius such that $v_{\rho_k}(J(\eta)) = v_\pi(J(\pi))$ (a priori, this can also be ∞), let $a = \operatorname{ord} I(\eta)$. Then the normal crossings resolution of $\mathfrak{m}_A$ is a chain if and only if one of the following conditions holds:*

(i) *A is geometrically smooth.*

(ii) *$a = 1$.*

(iii) *$I(\eta)$ has no zero on $\mathcal{O}_\rho$ such that $\rho < \rho_k$ and $\rho \neq \rho_k - \frac{1}{np}$ for any $n \in \mathbb{N}$. If $v_\pi(\rho_k) = 0$, then the Weierstraß polynomial of $I(\eta)$ at ρ_k has degree 1.*

Proof. Since the minimal resolution of $\mathfrak{m}_A$ contains a component with multiplicity 1, we can assume without loss of generality that its parameter is π^i/η^j for some $i, j \in \mathbb{N}$. Also, it is equivalent to say that $\mathcal{I}(\rho)$ is invertible for all $\rho \in \mathbb{Q}_{\geq 0}$ and to say that the minimal resolution of $\mathfrak{m}_A$ is a chain by Corollary 4.3.20. But the first we can do exactly by Lemma 4.3.29.

We first prove the direction: One of the conditions (i)-(iii) is fulfilled, then the minimal resolution is a chain. If (i) is true, that is directly clear. If (ii) is true, then by Lemma 3.3.9, any possible locus of non-invertibility is located at $\eta = 0$, thus any $\mathcal{I}(\rho)$ is invertible.

Now to case (iii). We can assume that $a \geq 2$ since the rest is already treated in (i) and (ii). Corollary 4.3.20 states that the minimal resolution of A is a chain if and only if $\mathcal{I}(\rho)$ is invertible for all ρ. All we have to do now is to translate this by Lemma 4.3.29 in a statement about $I(\eta)$. By Lemma 4.2.75, the function $v_\rho(I(\eta))$ monotonously increases with ρ until reaching $\rho = \rho_k$. Lemma 4.3.29 now states: $\mathcal{I}(\rho)$ is invertible, if and only if: (iv) $\rho > \rho_k$, (i),(ii),(iii) the Weierstraß polynomial of $I(\eta)$ at ρ_k has degree 1, (i) $I(\eta)$ has no zero on $\mathcal{O}_\rho$ for $\rho < \rho_k$ where case (v) does not apply. But a short calculation shows that case (v) applies only for $\rho = \rho_k - \frac{1}{np}$ with $n \in \mathbb{N}$. Since the statement of Lemma 4.3.29 was if and only if, this amounts to the statement of (iii), which is also if and only if. This proves our claims. □

Corollary 4.3.32. *Let $\rho_k \in \mathbb{Q}$ be the unique smallest radius such that $v_{\rho_k}(J(\eta)) = v_\pi(J(\pi))$, let $a = \operatorname{ord} I(\eta)$, assume that the zeroes of $I(\eta)$ are all tame over R. Then the normal crossings resolution of $\mathfrak{m}_A$ is a chain if and only if one of the following conditions holds:*

(i) *A is geometrically smooth.*

(ii) *$a = 1$.*

(iii) *$v_\pi\left(\frac{\ell-\delta}{a-1}\right) < 0$, and the series $I(\eta)$ has no zero on $\mathcal{O}_\rho$ such that $\rho \leq \rho_k$.*

Proof. By postulating that the zeroes of $I(\eta)$ are all tame over R, we have excluded case (v) of Lemma 4.3.29. Again, as in Corollary 4.3.31, it will suffice to translate the statement $\mathcal{I}(\rho)$ is invertible by means of Lemma 4.3.29. In particular, any zero of $I(\eta)$ lying on $\mathcal{O}_B(\rho)$ with $\rho < \rho_k$ will cause $\mathcal{I}(\rho)$ not to be invertible. So all Weierstraß zeroes of $I(\eta)$ must coincide until ρ_k, i.e. lie on the point $\eta = 0$. So using Lemma 4.2.75, the function $v_\rho(I(\eta))$ consists of a single edge of slope $a - 1$. In particular, one has $\rho_k = \frac{\ell-\delta}{a-1}$. Lemma 4.4.5, which we will prove later to properly distinguish methods, implies that $v(\rho_k) \leq 0$. Note that the proof of Lemma 4.4.5 is independent of the proof of this corollary.

Now if $a \not\equiv 1 \mod p$, then $v(\rho_k) = 0$, and for $\mathcal{I}(\rho_k)$, we are in case (ii) of Lemma 4.3.29. Since $a \geq 2$, the function $I(\eta) + \eta(1 - N^{(\rho_k)}(\lambda))$ has always zeroes on $\mathcal{O}_B(\rho_k)$, and thus $\mathcal{I}(\rho_k)$ is not invertible. So we must have $a \equiv 1 \mod p$ if the minimal resolution is a chain. The same applies if $v_\pi\left(\frac{\ell-\delta}{a-1}\right) = 0$. On the other hand, if $v_\pi\left(\frac{\ell-\delta}{a-1}\right) < 0$, then for $\mathcal{I}(\rho_k)$, one has to consider case (i) of Lemma 4.3.29, which implies that then $\mathcal{I}(\rho_k)$ is invertible. On the other hand, if one of the conditions in (iii) is not met, then either $I(\eta)$ has zeroes which are not tame over R, or $\mathcal{I}(\rho)$ is not invertible for some ρ. Thus we have proved our claims. □

Remark 4.3.33. Note that (iii) of Corollary 4.3.32 in fact implies $a \equiv 1 \mod p$. On the other hand, since $a < p$ implies that $I(\eta)$ has only tame zeroes, one then has the following statement:

If $a < p$, then the resolution of A is a chain if and only if $a = 1$ or A is regular.

Using Corollary 3.3.6, one can then reformulate this in

If $a < p$, then the normal crossings resolution of A is a chain if and only if $a \leq 1$ or $\ell < \delta$.

However, it seems that the most interesting cases still are the situations where $I(\eta)$ can have zeroes which are wild over R. One may conjecture that the cases (iii) in Corollaries 4.3.31 and 4.3.32 never occur. So an example for the cases (iii) might also be extremely interesting.

4.4 Tame descent

In this section, we will analyze how the model of the G-action behaves under tame extension of the base ring R. Using those results, we will then develop a description of the minimal desingularization of A under certain conditions which can be obtained only in terms of the generic fiber of B.

4.4.1 Galois extension and tame descent

In this subsection, we will examine possible tame extensions of the base ring and the morphisms of corresponding models. We keep the notations of 4.2.30. First we say what a Galois extension of R is supposed to be:

Definition 4.4.1. Let L be a Galois extension of $Q(R)$, let R' be the normalization of R in L. Then we say that R' is a *Galois extension* of R. We also define $\mathrm{Gal}(R'/R) := \mathrm{Gal}(L/Q(R))$. This group acts canonically on R'.

Now we will explain how one can extend the Galois action from R to its tame closure.

Notations 4.4.2. We will denote by R^* the tame closure of R, i.e. the field consisting of algebraic elements of tame degree over R. By a well-known result from Galois theory, one can non-canonically extend the automorphism σ from $Q(R)$ to R^*. This induces, for any finite Galois extension R' of R, a finite cyclic extension of the G-action from B onto $B \otimes_R R'$ and its normalization which is generated by σ. Note that this cyclic group is in general not the complete Galois group $\mathrm{Gal}(R'/R)$.

Note that R^* is the pro-finite limes of every finite extension of $Q(R)$ of finite prime-to-p degree. From what we will prove in Proposition 4.4.4, it follows that G is a direct factor of the pro-finite Galois group $\mathrm{Gal}(R^*/R)$; in particular we can canonically chose an extension $\sigma : R^* \to R^*$ such that σ acts trivially on $(R^*)^G$. We will fix this choice for the rest of the chapter.

Also, the R-valuation canonically extends to a valuation on R^* which we will denote by

$$v_\pi : R^* \to \mathbb{Q}$$

and which we normalize by $v_\pi(\pi) = 1$. We will also keep our notation

$$\ell = v_\pi(J(\pi)))$$

Remark 4.4.3. This generalization of σ from R to R^* of course also applies to the

$$I, J_k, N_{(j)}^{(k)} \text{ and so on.}$$

Proposition 4.4.4. *Let R' be a finite Galois extension of R of order q coprime to p. Then*

$$\mathrm{Gal}(R'/R^G) = G \times \mathrm{Gal}(R'/R) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}.$$

Also, the element $\pi' = \sqrt[q]{\pi}$ is a uniformizer of R'; in particular, R' is unique. Moreover, we have that

$$v_\pi(J(\pi'^n)) = v_\pi(J(\pi^{n/q})) = \ell \textit{ if } p \textit{ is coprime to } n.$$

Proof. We will first prove the following claim: Let $v \in R'^\times$ be a unit. Then there exists a q-th root v in $R'^\times$, i.e. a unit $u \in R'$ such that $u^q = v$.

First remark that one can write $v = \alpha(1+t)$, where α is a unit in a multiplicative system of representatives of k and $t \in \mathfrak{m}_{R'}$. Then we have

$$\sqrt[q]{v} = \zeta_q^m \sqrt[q]{\alpha} \sum_{i=0}^{\infty} t^i \begin{pmatrix} 1/q \\ i \end{pmatrix},$$

where ζ_q is a primitive q-th root of unity. Since q is coprime to p, the binomial coefficients are in R'. Since k is algebraically closed, and R is complete the power series is also in R' for any m. The exponent m can be chosen arbitrarily in $\mathbb{Z}$.

Now we prove that $\sqrt[q]{\pi}$ is a uniformizer of R'. Since k is algebraically closed, and R' is a complete discrete valuation ring, there exists a uniformizer $\tilde{\pi}$ of R' with $q \cdot v_\pi(\tilde{\pi}) = v_\pi(\pi)$. Also, $\mathrm{Gal}(R'/R) = \mathbb{Z}/q\mathbb{Z}$, in particular, there exists a unit $v \in R'$ such that $N_R^{R'}(\tilde{\pi}) = v \cdot \tilde{\pi}^q = \pi$. By our previous calculation there exists a q-th root $u = \sqrt[q]{v}$ of v in $R'^\times$. Thus $\pi' = u\tilde{\pi}$ is a uniformizer of R', and we have $\pi'^q = \pi$ resp. $\pi' = \sqrt[q]{\pi}$ as claimed.

Now we continue with the remaining assertions: From the above reasoning it follows that there exists a unique extension $Q(S)$ of order q of $Q(R^G)$. Also we see that the composite $Q(S)Q(R)$ is an extension of $Q(R)$ of degree q, which must be isomorphic to $Q(R')$ by our above reasoning. Since R' is the corresponding normal extension of R^G, this implies that

$$\begin{aligned}\mathrm{Gal}(R'/R^G) &= \mathrm{Gal}(Q(R')/Q(R^G)) = \mathrm{Gal}(Q(R)Q(S)/Q(R^G)) \\ &= \mathrm{Gal}(Q(R)/Q(R^G)) \times \mathrm{Gal}(Q(S)/Q(R^G)) \\ &= G \times \mathrm{Gal}(R'/R) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}.\end{aligned}$$

For the last assertion, we note that we can express

$$\sigma(\pi) = \pi(1 + \pi^\ell u) \text{ for some } u \in S^\times.$$

So in summary, we have that

$$\sigma(\pi') = \sqrt[q]{\sigma(\pi)} = \pi' \sqrt[q]{1+\pi^{\ell} u} = \pi' \zeta_q^m \sum_{i=0}^{\infty} \pi^{i\ell} u^i \begin{pmatrix} 1/q \\ i \end{pmatrix}$$

where i is determined by the actual choice of the extension of σ to R'. Since we have postulated trivial action on $\mathrm{Gal}(R'/R)$, we have $m = 0$. Thus we see that

$$v_\pi(I(\pi')/\pi') = \ell.$$

Moreover, for n coprime to p, it directly follows by an elementary calculation that

$$v_\pi(I(\pi'^n)/\pi'^n) = \ell.$$

If p divides n, we have that $\pi'^n \in R'^G$ by our choice that σ acts trivially on the second factor of $G \times \mathrm{Gal}(R'/R)$. This was the last claim to prove. □

We illustrate an application of this to our original situation by deriving a new obstruction for certain local actions:

Lemma 4.4.5. *Assume* $\widehat{B} = R[[\eta]]$ *, and write* $\delta = v_\pi(I(\eta))$*. Assume* $\ell > \delta, a \neq 0$*, and that the zeroes of* $I(\eta)$ *are identic zero modulo* π^L *with* $L = \frac{\ell-\delta}{a-1}$*. Then one has* $v_\pi(a-1) \geq v_\pi(\ell-\delta)$*.*

Proof. By definition, we have that

$$I(\eta) \equiv \pi^\delta \eta^a u \mod (\pi^{\delta+1}, \eta^{a+1}) \text{ with } u \in k^\times.$$

Now let $q \in \mathbb{Q}$ with $v_\pi(q) > 0, q \leq L$. Then π^q lies in $(R^*)^G$ by Proposition 4.4.4. In particular, we have that

$$\begin{aligned} I\left(\frac{\eta}{N(\pi)^q}\right) &\equiv \pi^\delta \left(N(\pi)^q\right)^{a-1} \left(\frac{\eta}{N(\pi)^q}\right)^a u \\ &\equiv \pi^{\delta+(a-1)qp} \left(\frac{\eta}{N(\pi)^q}\right)^a u \mod \left(\pi^{\delta+1-pq}, \frac{\eta^{a+1}}{N(\pi)^q}\right), \end{aligned}$$

since $N(\pi) \equiv \pi^p \mod \pi^{p+1}$, and the zeroes of $I(\eta)$ were identic zero, so the term above is still the leading term after dividing with $N(\pi)^q$. One can now interpret the action on $\eta/N(\pi)^q$ as an action on a ring B' of same type, with possibly expanded R, and associated invariants $a' = a$ and $\delta' = \delta + (a-1)qp$ and $\ell' = \ell$ Since $a' > 1$, it is forbidden by Lemma 3.3.10 that $\delta' = \ell'$, which is equivalent to

$$\ell - \delta = (a-1)qp.$$

Now if one would have $v_\pi(a-1) < v_\pi(\ell-\delta)$, then it was possible to choose q such that both sides equate, namely by setting $q := \frac{\ell-\delta}{(a-1)p} = \frac{L}{p}$. Since by assumption, this q then would fulfill the imposed restrictions $v_\pi(q) > 0, q \leq L$. This proves the claim. □

Remark 4.4.6. In particular, if all the zeroes of $I(\eta)$ are identic zero modulo π^L, this implies that always $v_\pi(L) \leq 0$.

In Proposition 4.4.4, we have seen that tame extensions of R are uniquely determined by their degree. We introduce some abbreviating notation reflecting this:

Notations 4.4.7. For $q \in \mathbb{N}$, coprime to p, we will denote by R_q the unique Galois extension of R of degree n.

The main application of Proposition 4.4.4 lies in the following method of tame descent:

Proposition 4.4.8. *Let D be a complete local germ at a geometrically smooth point of a G-stable model of B. Let $q \in \mathbb{N}$ be coprime to p, Let D' be the normalization of $R_q \otimes_R D$. By Proposition 4.4.4, the group G acts on D' canonically as direct factor of the Galois group $\mathrm{Gal}(R_q/R^G)$. Denote by $C' = D'^G$ and $C = D^G$ the corresponding invariant rings. Then C is geometrically smooth if and only if C' is geometrically smooth.*

Proof. Denote the multiplicity of D over R by m. First we will assume that m is coprime to p. In this case, one can show that $D \cong R_m[[\eta]]$, see e.g. [CES03, Lemma 2.3.2]. The normalization D' of $D \otimes_R R_q$ then is explicitly calculated as

$$D' = \bigoplus_{i=1}^{\gcd(q,m)} R_{\mathrm{lcm}(q,m)}[[\eta]] = \left(\bigoplus_{i=1}^{\gcd(q,m)} R_q R_m[[\eta]] \right).$$

The Galois group $\mathrm{Gal}(R_q/R^G) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ is then generated by our chosen extension of σ, and an element τ of order q which acts on D' in the following way: τ independently permutes the $\gcd(q,m)$ factors of D' and acts on each factor $R_q R_m[[\eta]]$ trivially on η and on $R_q R_m = R_{\mathrm{lcm}(q,m)}$ as generator of the Galois group $\mathrm{Gal}(R_{\mathrm{lcm}(q,m)}/R_m)$. The extended σ on the other hand does not permute the factors of D' and acts on $R_q R_m[[\eta]]$ through the induced action on $R_m[[\eta]]$ with trivial action on R_q. So in summary, we have

$$D' = \left(\bigoplus_{i=1}^{\gcd(q,m)} R_q \right) R_m[[\eta]] = R_q^{\oplus \gcd(q,m)} D,$$

where σ acts only on the factor D and τ on the copies of R_q. Thus

$$C' = (D')^{\langle \sigma \rangle} = R_q^{\oplus \gcd(q,m)} D^G = R_q^{\oplus \gcd(q,m)} C.$$

The latter is geometrically smooth, if and only if $R_q C$ is geometrically smooth, since C' is just a finite number of copies of C. But by the regularity criterion 3.3.4, this is geometrically smooth if and only if σ acts as a pseudo-reflection on every factor $R_q D$ of D'. And considering that D is geometrically smooth, this is true if and only if σ acts as pseudo-reflection on D, because σ acts trivially on R_q. Using Proposition 3.3.4 again, this is true if and only if C is geometrically smooth. This completes the proof in the case where m is coprime to p.

If p divides m, the situation is similar. The ring D' again decomposes as

$$D' = \left(\bigoplus_{i=1}^{\gcd(q,m)} R_q \right) D,$$

since one can write D as a extension of degree $p^i, i \in \mathbb{N}$ of $R_t[[\eta]]$ for some $t \in \mathbb{N}$ coprime to p with $m = tp^i$, and non-split tame extensions of complete discrete valuations rings are unique up to degree. Since q and p are coprime, the Galois group again decomposes as $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} = \langle \sigma \rangle \times \langle \tau \rangle$, such that σ acts only on the second factor D, and τ only on the first factor given by the copies of R_q. The rest of our reasoning above also applies for this case, so the claims are proved. □

The situation directly transfers to the non-complete case:

Corollary 4.4.9. *Let D be a local germ at a geometrically smooth point of a G-stable model of B. Let $q \in \mathbb{N}$ be coprime to p, Let D' be the normalization of $R_q \otimes_R D$. By Proposition 4.4.4, the group G acts on D' canonically as direct factor of the Galois group $\mathrm{Gal}(R_q/R^G)$. Denote by $C' = D'^G$ and $C = D^G$ the corresponding invariant rings. Then C is geometrically smooth if and only if C' is geometrically smooth.*

Proof. The fact that D is a complete local germ implies that D is an excellent ring. Also, D' is the completion of an excellent ring. So by Proposition 2.3.5, the corollary follows from Proposition 4.4.8 □

4.4.2 Tame invariance of critical components

The results of the last subsection have consequences for the models of B. We will try to understand how the models of B transfer under tame base extensions:

Notations 4.4.10. For q coprime to p, we will denote by B_q the normalization of $B \otimes_R R_q$. We will assume that for all such q, the ring B_q is again a local normal Noetherian ring with the same number of original components as B, and the minimal desingularization of B_q is a chain. Note that $G = \langle\sigma\rangle \cong \mathbb{Z}/p\mathbb{Z}$ acts canonically on B_q by Proposition 4.4.4 resp. our choice of continuation of σ. We will denote the G-invariants of B_q by $A_q := B_q^G$.

Example 4.4.11. The conditions on B_q are fulfilled for example if B is a local germ at a geometrically smooth point, or if B is a local germ at a geometrical double points where both components passing through B have multiplicity coprime to p.

Proposition 4.4.12. *Let q be coprime to p. let D' be a component of B_q, let D be the corresponding component of B with $v_{D'} |_{Q(B)} = v_D$. Then D' is critical if and only if D is critical.*

Proof. First we will prove this in the case where q is prime. We distinguish several cases:
Case 1: q does not divide the multiplicity of D. Then as in the proof of Proposition 4.4.8, there is exactly one point y in D' over any geometrically smooth point x on D. Then corollary 4.4.1 shows that $A_q(D'^{\times})$ is not geometrically smooth at exactly as many points as $A(D^{\times})$. Since q is coprime to p, the component is D' is potentially terminal if and only if D is, and since B_q has the same number of original components as B, the component D' is a connecting component if and only if D is. Thus Proposition 4.3.8 implies that $A(D)$ has the same number of geometrical singularities as $A_q(D')$. The definition of a critical component 4.3.2 then implies that D' is critical if and only if D is.
Case 2: q divides the multiplicity of D, and the components D, D' are both potentially terminal. Then by the inherent structure of B, see 4.2.57, $\mathrm{Gal}(R_q/R)$ permutes q copies of D'. It is then easily seen by the definition of a critical component 4.3.2 that D' is critical if and only if D is.
Case 3: q divides the multiplicity of D, and the components D, D' are both not potentially terminal. Here, two subcases can happen. Either (i), $\mathrm{Gal}(R_q/R)$ permutes q copies of D', or (ii) $\mathrm{Gal}(R_q/R)$ leaves D stable, and the induced map $(D')_{red} \dashrightarrow D_{red}$ has no inertia. In case (i), one can reason as in case 2; in case (ii), one has to see that $\mathrm{Gal}(R_q/R)$ permutes all points of $B_q(D')$. Since D'

and D are both not potentially terminal, D' resp. D is critical by Proposition 4.3.8 if and only if $A(D^{\times})$ resp. $A_q(D'^{\times})$ has a not geometrically smooth point. And those latter two conditions are equivalent since $\mathrm{Gal}(R_q/R)$ acts on any point of $A_q(D'^{\times})$ without inertia.

Case 4: q divides the multiplicity of D, the component D is not potentially terminal, and D is potentially terminal. Then D is critical if and only if $A(D^{\times})$ has a geometrical singularity by Proposition 4.3.8, and D' is critical if and only if $A_q(D')$ has 2 geometrical singularities or more. Now the main point to remark is that the unique singularity of $B(D)$ gets regular in $B_q(D')$, since D' is potentially terminal. So the corresponding point on $A_q(D')$ is also regular by 4.3.8. Moreover, on any other point of $A(D')$, the group $\mathrm{Gal}(R_q/R)$ acts without inertia. So this implies that $A(D^{\times})$ has n geometrical singularities of and only if $A_q(D')$ has qn geometrical singularities for any $n \in \mathbb{N}$. In particular, by our considerations at the beginning, this implies that D is critical if and only if D' is critical, since $q \geq 2$.

Now all that remains to prove is the case where q is not prime. But we have seen that $\mathrm{Gal}(R_q/R) \cong \mathbb{Z}/q\mathbb{Z}$, and thus $\mathrm{Gal}(R_q/R)$ can be decomposed in prime cyclic subextensions to which we can apply the previous reasonings, this proves the proposition. □

Proposition 4.4.12 gives a tool to relate the minimal desingularizations of B resp. B_q : The path reduction of the dual graph of the minimal model of the G-action on B is almost the graph theoretical quotient of the path reduction of the dual graph of the minimal model of the G-action on B_q. However, there can occur parts on B which do not correspond to parts of the graph on B_q, as in case 4 of the proof of 4.4.12, when terminal components on B_q become nonterminal in B.

An interesting consequence of Proposition 4.4.12 is the following:

Corollary 4.4.13. *A_q is locally toric if and only if A is locally toric.*

Proof. By Theorem 2.8.1, a ring is locally toric if and only its minimal resolution is a chain. By Corollary 4.3.14, the latter is equivalent to the fact that it has no critical components. And by Proposition 4.4.12, the fact that A has no critical component is equivalent to the fact that A_q has no critical component. □

4.4.3 Quotients of stable models

A few years ago, Lorenzini has obtained a result which is probably one of the first non-trivial results on wild quotient singularities. We will state the result and explain several links to our work.

The situation is nearly identical with that considered earlier, but comes from a global setting rather than a local one. Let R be a complete discrete valuation ring of mixed characteristic $(0,p)$ with uniformizer π and algebraically closed residue field k containing a primitive p-th root of unity. We will denote the field of fractions by $K = Q(R)$. We assume that a group $G \cong \mathbb{Z}/q\mathbb{Z}$ for a prime number q acts non-trivially on G with trivial induced action on k. Let C be a smooth proper geometrically connected curve over K of genus g with good reduction over R. Assume that C does not have good reduction over R^G. Let Y be the smooth model of C over R. By universal property of the stable model, the G-action on R induces a canonical G-action on Y. Let $X = Y/G$. Since X is normal, it has finitely many quotient singularities in codimension 2. Lorenzini's theorem [Lor06, 6.1] states:

Assume that $g > 1$, let Z be the minimal normal crossings desingularization of X, let x be a singularity of X.
If $q \neq p$, then there exists no component on the special fiber of Z in the exceptional fiber of x which is a branch component in the special fiber of Z.

If $q = p$, then there exists a component on the special fiber Z in the exceptional fiber of x which is a branch component in the special fiber of Z.

The first part where $q \neq p$ is already clear by our consideration in section 2.7: If q is coprime to p, then the singularities are tame, and since the problem is local, the desingularization is just of Hirzebruch-Jung-type. This result also follows from the more general result in Theorem 4.3.12.

The second part is more complicated. Rephrased, it states that the exceptional fiber of any quotient singularity on X contains a branching component of the special fiber of X if the action is wild. Lorenzini's proof is global and uses the machinery of Néron models. We will prove a local version under certain conditions. From now on we assume that $q = p$.

Remark 4.4.14. By the assumption that C has no good reduction over R^G, it can be shown that the induced action of G on the special fiber of Y has generically no inertia, i.e. acts transitively on all but finitely many points of the special fiber. On the other hand, the ramification points are exactly the points lying over the quotient singularities on X. For details see the beginning of §6 in Lorenzini's unpublished paper [Lor06].

We will now prove a local version of Lorenzini's Theorem:

Theorem 4.4.15. *Let B be a local complete Notherian ring with $\widehat{B} = R[[\eta]]$, assume that $G = \mathbb{Z}/p\mathbb{Z}$ acts on B, and the induced action on the residue field of B is trivial. Let $A = B^G$, let a be the invariant from Corollary 3.3.6. Assume that $I(\eta)$ has only tame zeroes, that G acts without inertia on the special fiber of B , and that the invariant morphism of the special fiber is ramified at B.*

Then the dual graph of the minimal desingularization of A has a branch vertex.

Proof. We can also associate to the ring B the invariants δ, ℓ from Corollary 3.3.6. We see, that $\delta = 0$ and $a \geq 1$ by assumption, thus by Corollary 3.3.6 A is not geometrically smooth. We can apply Corollary 4.3.32.

First we will prove that $a \neq 1$. We will do this by contradiction, so assume that $a = 1$. By 3.3.9, we can without loss of generality assume that η divides $I(\eta)$ by choosing the geometric parameter appropriately. Write now

$$\sigma(\eta) = \eta + I(\eta) = \sum_{i=1}^{\infty} e_i \eta^i.$$

Since $a = 1$, we must have that $N(e_1) = 1$ and $v_\pi(e_i) \geq v_\pi(e_1 - 1)$ for all $i > 1$. Thus $e_1 \equiv 1 \mod \pi$, since $\operatorname{char} k = p$. This means that $v_\pi(e_1 - 1) > 0$ and thus $v_\pi(e_i) > 0$ and, by our above considerations, $\delta = v_\pi(I(\eta)) > 0$, which is a contradiction to our assumption $\delta = 0$. This proves that $a \geq 2$, and in particular $a \not\equiv 1 \mod p$.

So in summary, by Corollary 4.3.32, the minimal resolution of A is a chain if and only if B fulfills condition (iii) of the Corollary, since above we have excluded the possibility (i) and (ii). Also, by 4.3.9, the minimal resolution of A contains a terminal component of multiplicity p or 1. First assume it contains a terminal component of multiplicity p. Then the minimal resolution of A cannot be a chain, since then p would have to divide the multiplicity of any component of A, and the original component also has multiplicity p. Thus any component had multiplicity exactly p and self-intersection -1 by the intersection formula. But this would mean that any component except possibly the original could be contracted by Castelnuovo's criterion, which contradicts the construction of the minimal regular model. So in this case, the minimal resolution of A contains a branch component.

It remains to treat the case where the minimal resolution of A contains a terminal component of multiplicity 1. By Remark 4.3.33, condition (iii) implies that $a \equiv 1 \mod p$. This will be disproved shortly in Lemma 4.4.16. □

Lemma 4.4.16. *Keep the conditions of Theorem 4.4.15. Assume the minimal resolution of A contains a terminal component of multiplicity 1. Then one has $a \not\equiv 1 \mod p$.*

Proof. We will prove this by contradiction. Assume $a \equiv 1 \mod p$. To derive a contradiction, one has to calculate explicitly $\sigma^p(\eta)$. So write

$$\sigma(\eta) = \sum_{i=1}^{\infty} e_i \eta^i.$$

Note that we can take without loss of generality $e_0 = 0$ by the considerations in the previous chapter, since by Lemma 3.3.9, the G-action on B has a geometric fixed point. Explicitly writing out $\sigma^p(\eta)$ shows: $e_1 = \zeta_p$, where ζ_p is some primitive p-th root of unity. A lengthy but elementary calculation then shows that the leading term of $\sigma^p(\eta) - \eta$ is

$$\sum_{i=0}^{p-1} \zeta^{p-i} \sigma^i(e_a) \sigma^i(\eta^a) = \mathrm{Tr}(e_a)\eta^a.$$

But $v_\pi(e_a) = 0$ since $\delta = 0$, so by Proposition 3.1.3, the element $\mathrm{Tr}(e_a)$ is nonzero. Thus the leading term of $\sigma^p(\eta) - \eta$ is nonzero, which means that $\sigma^p(\eta) \neq \eta$. But this is clearly a contradiction to $G = \mathbb{Z}/p\mathbb{Z}$ resp. $\sigma^p = \mathrm{id}$. □

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