# Totally Degenerated Formal Schemes

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"It is a mistake to think you can solve any major problems just with potatoes."

*Douglas Adams: Life, The Universe and Everything*

# Contents

# Introduction

In this thesis, we introduce a new class of rigid analytic varieties over a complete non-archimedean field $K$; namely those which have a *totally degenerated formal model*. These are natural generalizations of the well known Mumford curves to arbitrary dimension. Similar to the one-dimensional case, we will show that the Picard variety of these varieties is given by a quotient $\mathbb{G}_{m,K}^g/M$, where $M$ is a lattice in $\mathbb{G}_{m,K}^g$, not necessarily of full rank.

To any smooth projective curve $X$ over $\mathbb{C}$ (or, equivalently: a compact Riemann surface) of genus $g$, one can associate its Jacobian variety $\mathrm{Jac}(X)$; an abelian variety which parametrizes the equivalence classes of divisors on $X$ of degree $0$. The well-known Torelli theorem states that $X$ is uniquely determined by its (principal polarized) Jacobian. This makes the Jacobian a very important object for the study of Riemann surfaces. If $X$ is a Riemann surface of genus $1$, i.e. an analytic torus $\mathbb{C}/\Lambda$, the Jacobian $\mathrm{Jac}(X)$ is canonically isomorphic to $X$ itself. In general, the Jacobian is analytically isomorphic to a $g$-dimensional analytic torus $\mathbb{C}^g/M$, where $M$ is a lattice in $\mathbb{C}^g$ of rank $g$, the so-called *period lattice*.

Over a complete non-archimedean valued field $K$, such as the $p$-adic numbers $\mathbb{Q}_p$, the above situation does not extend without modification. In general, it is not true that the Jacobian of a curve is given by an analytic torus $\mathbb{G}_{m,K}^g/M$. This is related to the fact that only a certain class of $p$-adic curves has a complex analog; namely, the so-called *Mumford curves*. These curves $X_K$ have a formal model $X$ over the valuation ring $R$ such that every irreducible component of the special fibre $X_0$ is isomorphic to $\mathbb{P}^1$ and $X_0$ has only ordinary double points as singularities. Mumford proved in [28] that these are precisely the curves which have a *Schottky uniformization* $\Omega_K/\Gamma$, where $\Gamma \subset \mathrm{PGL}(2, K)$ is a Schottky group, and $\Omega_K \subset \mathbb{P}_K^1$ is the set of points where $\Gamma$ acts discontinuously; this is a direct analog of the classical Schottky uniformization over the complex numbers. In [26], Manin and Drinfeld proved that, as in the complex case, the Jacobian variety of a Mumford curve of genus $g$ is again isomorphic to an analytic torus $\mathbb{G}_{m,K}^g/M$, where $M$ is a multiplicative lattice in $\mathbb{G}_{m,K}^g$ of rank $g$.

If $\dim X > 1$, the analog of the Jacobian variety $\mathrm{Jac}(X)$ is the Picard variety $\mathrm{Pic}^0(X)$, which represents certain isomorphy classes of line bundles. The existence of the Picard variety of proper algebraic schemes over a field has been proven in the 1960s. An analogous result in the category of proper rigid analytic varieties over a complete discretely-valued field $K$ was established much later, in 2000, by Hartl and Lütkebohmert [21].

In this thesis, we will deal with the question when the Picard variety $\mathrm{Pic}^0(X_K)$ of a proper rigid-analytic variety $X_K$ over $K$ is again an analytic torus $\mathbb{G}_{m,K}^g/M$. The example of Mumford curves already shows that one can expect this to be true only in very special cases. In the work of Hartl and Lütkebohmert [21], it becomes apparent that the special fibre of a suitable formal model plays a key role in determining the structure of the Picard variety. This motivates the following generalization of Mumford curves:

We say a proper rigid-analytic variety $X_K$ over $K$ has a *totally degenerated model* $X$ over $R$ if the special fibre $X_0$ of $X$ consists of smooth rational components with normal crossings; i.e. locally, $X_0$ looks like the intersection of some coordinate hyperplanes in the affine space $\mathbb{A}^r$ (see Definition 4.1.1 for the precise conditions).

**Theorem 4.3.5.** *Let $X_K$ be the generic fibre of a totally degenerated formal scheme which is proper. On the category of smooth and connected rigid spaces, the Picard functor $\mathrm{Pic}^0_{X_K/K}$ is represented by a quotient $T_K/M$, where $T_K$ is a split torus, and $M$ is a lattice in $T_K$ such that $M \cap \bar{T}_K = \{1\}$.*

If $X_K$ is algebraizable, it is well-known that $\mathrm{Pic}^0(X_K)$ is always proper; i.e. $M$ has full rank $g$. If $X_K$ is not algebraizable, however, this need not be true. A standard example is the Hopf surface, introduced in the rigid analytic framework by Mustafin [29], which also has a totally degenerate model.

Generalizing the techniques for Mumford curves, we construct a suitable uniformization $X_K \cong \Omega_K/\Gamma$. As in the case of analytic tori, we show that any line bundle on $X_K$ which corresponds to a point of $\mathrm{Pic}^0(X_K)$ pulls back to the trivial line bundle on $\Omega_K$. Hence, line bundles on $X_K$ can be described by $\Gamma$-linearizations of constant type of the trivial line bundle on $\Omega_K$. This allows us to describe $\mathrm{Pic}^0(X_K)$ in terms of automorphic functions:

**Theorem 4.4.12.** *Let $\widehat{J} := \mathrm{Hom}(\widetilde{\Gamma}, \mathbb{G}_{m,K}) \cong \mathbb{G}_{m,K}^g$, where $\widetilde{\Gamma}$ is the free part of the abelianization $\Gamma/[\Gamma,\Gamma]$ of $\Gamma$, and let*

$$M := \{c \in \widehat{J}\,;\ c \text{ is the factor of automorphy for an invertible function } f \text{ on } \Omega_K\}$$

*Then $M$ is a lattice in $\widehat{J}$, and the quotient $J := \widehat{J}/M$ represents $\mathrm{Pic}^0(X_K)$.*

In arbitrary dimension, a large class of examples for rigid-analytic varieties with totally degenerated models is given by the so-called *general polytopal domains*. A general polytopal domains carries a rich combinatorial structure; the irreducible components of its special fibre are toric varieties. In fact, a great number of results from the theory of toric varieties carries over to the theory of polytopal domains. An affinoid piece of such a general polytopal domain is the pre-image of a polytope $\sigma \subset \mathbb{R}^n$ under the valuation map

$$\mathrm{val}: \mathbb{G}_{m,K}^n \to \mathbb{R}^n, \quad (x_1, \ldots, x_n) \mapsto (-\log|x_1|, \ldots, -\log|x_n|).$$

The fact that these general polytopal domains have indeed a totally degenerated model is proved using a combinatorial result of Kempf, Knudson, Mumford and Saint-Donat [24].

In the framework of algebraic geometry, polytopal domains have already been used by Mumford in [27]; the rigid-analytic version has been introduced by Gubler [19].

For such an affinoid polytopal domain, we prove the following cohomological result:

**Theorem 6.0.1.** *For an affinoid polytopal domain $X$ in $\mathbb{G}_m^n$, one has*

$$H^i(X, \mathcal{O}^\times) = 0 \textit{ for all } i \geqslant 1.$$

This implies that any line bundle on an affinoid polytopal domain is trivial; i.e. the Picard variety is trivial. The proof is done using techniques of van der Put [32], most notably the Base Change Theorem.

As an application of the theory of general polytopal domains, we then investigate totally degenerated varieties $X_K$ with universal covering $\Omega_K = \mathbb{G}_{m,K}^n$; i.e. $X_K \cong \mathbb{G}_{m,K}^n/\Gamma$ for a suitable subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{G}_{m,K}^n)$ (for the precise conditions on $\Gamma$, see Assumptions 5.6.2 and 5.6.7). A special case occurs when $\Gamma$ is a lattice in $\mathbb{G}_{m,K}^n$, i.e. $X_K$ is an analytic torus. This situation is already well-understood.

For general $\Gamma$, we can use elementary calculations in order to characterize the Picard variety of $X_K$. A pivotal role in the study of the Picard variety of $X_K$ is played by the translation subgroup $\Gamma_1 \subset \Gamma$, which is a lattice in $\mathbb{G}_{m,K}^n$, not necessarily of rank $n$. In fact, we prove the following central result:

**Theorem 5.6.13.** $\mathrm{Pic}^0(X_K)$ *is proper if and only if* $\mathrm{rk}\,\Gamma_1 = n$.

If $\mathrm{Pic}^0(X_K)$ is proper, one can ask when $X_K = \mathbb{G}_{m,K}^n/\Gamma$ is algebraizable. Similar to the complex case, the situation of an analytic torus $X_K = \mathbb{G}_{m,K}^n/M$, with $M$ a lattice, is well known. Namely, such a torus is algebraizable if and only if an analog of the classical Riemann period relations holds. Using this result, we prove the following:

**Theorem 5.6.6.** *If* $\mathrm{rk}\,\Gamma_1 = n$ *, then* $X_K$ *is algebraizable if and only if there exists a group morphism* $\lambda : \Gamma_1 \to M' := \mathrm{Hom}(\mathbb{G}_{m,K}^n, \mathbb{G}_{m,K})$ *such that the quadratic form* $\langle \lambda(m), m \rangle$ *is positive definite on* $\Gamma_1$ *; i.e.* $|\langle \lambda(m), m \rangle| < 1$ *for every* $m \in \Gamma_1$ *with* $m \neq 0$ *.*

Theorem 5.6.13 is illustrated by two new examples, which we present in Chapter 5. In §5.4, motivated by the classical Klein bottle, we construct a *Klein surface* over $K$ , which turns out to be algebraizable. In §5.5, we construct a *sheared torus*. This is the easiest example for a group $\Gamma$ with $\mathrm{rk}\,\Gamma_1 < n$ ; and we easily see that $\mathrm{Pic}^0(X_K)$ is not proper.

## Outline

In the first chapter, we will recall the basic facts about formal and rigid geometry. In the second chapter, we will gather the combinatorial facts we need in the following; most notably simplicial complexes and simplicial homology and cohomology. We will also recall basic facts about toric varieties, which we need later.

In Chapter 3, we present the theory of polytopal domains. Most results are rigid-analytic versions of similar results in [24] from the algebraic-geometric framework. §3.1 contains basic results which can mostly be found directly in [19]. In §3.2, we establish the connection between admissible formal blowing ups and subdivisions of the polytopal complex associated to a polytopal domain. In §3.3, following Gubler [19], we describe Cartier divisors in terms of polyhedral functions. The results of §3.2 and §3.3 are then used in §3.4 to establish the existence of a totally degenerated formal model of a polytopal domain, which is proved using a combinatorial result of Kempf, Knudson, Mumford and Saint-Donat [24]. As an application, we then show how to obtain two desingularizations results in [21] combinatorially. In §3.5, we recall that ampleness of a line bundle is equivalent to the strict convexity of its associated polyhedral function. We then generalize this in §3.6 and give a similar criterion when a line bundle is ample on the boundary of a certain subvariety.

In Chapter 4, we introduce the notion of a totally degenerated formal scheme and construct the quotient $\Omega_K/\Gamma$ (§4.2). We investigate the Picard variety of the special fibre $X_0$ (§4.3) and prove that it is a torus. Following [21], we then prove that $\mathrm{Pic}^0$ is a quotient of a torus by a lattice (Theorem 4.3.5). In §4.3, we show how to interpret this result in terms of $\Gamma$ -linearizations (Theorem 4.4.12). We then characterize general polytopal domains (§4.5).

These give a very restrictive subclass; namely, the universal covering $\Omega_K$ does not contain a subvariety isomorphic to $\mathbb{A}^1$ (Proposition 4.5.25).

In Chapter 5, we discuss examples of rigid analytic varieties with a totally degenerated formal model. Examples §§5.1 – 5.3 recall the well-known examples of Mumford curves, analytic tori, and the Hopf surface. We show how our framework reproduces the well-known results about the Picard varieties of these objects. Examples §5.4 and §5.5 are new; they give explicit examples of analytic quotients $\mathbb{G}_{m,K}^g/\Gamma$, where $\Gamma$ is not a lattice. The general case of these quotients is then treated in §5.6.

In Chapter 6, we recall van der Put's Base Change Theorem and use it to prove that an affinoid polytopal domain has trivial Picard group (Theorem 6.0.1).

# Chapter 1

# Formal and Rigid Geometry

In this chapter, we will give a short introduction into formal and rigid geometry. We will list the most important definitions and results, mostly without proof.

In the following, let $K$ be a field, endowed with a complete non-archimedean absolute value $|\cdot|$. Depending on the situation, $K$ will be either algebraically closed, or a discrete valued field. We denote with $R = \{z \in K : |z| \leqslant 1\}$ the corresponding valuation ring (of height 1), $\mathfrak{m}$ its maximal ideal, and $k = R/\mathfrak{m}$ its residue field.

## 1.1 Rigid Geometry

The *Tate algebra* $T_n = K\langle\zeta_1, \dots, \zeta_n\rangle$ is the $K$-algebra of strictly convergent power series

$$T_n = K\langle\zeta_1, \dots, \zeta_n\rangle = \left\{ \sum_{m \in \mathbb{N}^n} a_m \zeta_1^{m_1} \cdot \ldots \cdot \zeta_n^{m_n} \ ; \ \lim_{|m| \to \infty} |a_m| = 0 \right\}$$

It is the completion of the polynomial ring $K[\zeta_1, \dots, \zeta_n]$ with respect to the *Gauss norm*

$$\left| \sum_{m \in \mathbb{N}^n} a_m \zeta_1^{m_1} \cdot \ldots \cdot \zeta_n^{m_n} \right| := \max |a_m|.$$

An *affinoid* $K$*-algebra* is a quotient $T_n/I$ for some ideal $I \subset T_n$. An *affinoid variety* is a pair $\operatorname{Sp} A = (\operatorname{Max} A, A)$, where $\operatorname{Max} A$ is the set of maximal ideals of $A$. For $f \in A$, the *supremum semi-norm* is defined via

$$|f|_{\sup} := \sup\{|f(x)| \ ; \ x \in \operatorname{Max} A\}.$$

The affinoid algebra $A$ is *distinguished* if $|\cdot|_{\sup}$ agrees with the residue norm

$$|\bar{f}|_\alpha := \inf\{|f|_{\sup}\ ;\ \alpha(f) = \bar{f}\}$$

for a suitable epimorphism $\alpha : T_n \to A$. If $K$ is algebraically closed or discretely valued, $A$ is distinguished if and only if $A$ is reduced and $|\cdot|_{\sup}$ takes values in $|K|$, see [5, §6.4.3].

The *Berkovich spectrum* $\mathcal{M}(A)$ of $A$ is the set of multiplicative semi-norms

$$|\cdot|_p : A \to \mathbb{R}^{\geqslant 0}$$

satisfying $|\lambda|_p = |\lambda|$ for $\lambda \in K$ and $|f|_p \leqslant |f|_{\sup}$. Any point $x \in \operatorname{Max} A$ gives rise to such a semi-norm via $|f|_x := |f(x)|$. Thus, we have an injection $\operatorname{Sp} A \hookrightarrow \mathcal{M}(A)$. The elements of $\mathcal{M}(A)$ are also called *analytic points*. The *Berkovich topology* on $\mathcal{M}(A)$ is the weakest topology such that for all $f \in A$, the map $p \mapsto |f|_p$ is continuous. This makes $\mathcal{M}(A)$ into a compact Hausdorff space such that the topology on $\mathcal{M}(A)$ restricts to the canonical topology on $\operatorname{Sp} A$, which lies dense in $\mathcal{M}(A)$. Further details about the Berkovich topology will be given in Chapter 6.

The *residue algebra* $\tilde{A}$ of $A$ is given by $\tilde{A} := A^\circ / A^{\circ\circ}$, where

$$A^\circ := \{f \in A\ ;\ |f|_{\sup} \leqslant 1\}, \quad A^{\circ\circ} := \{f \in A\ ;\ |f|_{\sup} < 1\}$$

We have a functorial *reduction map*

$$\pi : \operatorname{Sp}(A) \to \operatorname{Spec}(\tilde{A}), \quad x \mapsto \tilde{x} := \operatorname{Ker}(\tilde{A} \mapsto (A/x)^{\sim})$$

which is surjective onto the set of closed points of $\operatorname{Spec}(\tilde{A})$. This extends to a map

$$\pi : \mathcal{M}(A) \to \operatorname{Spec}(\tilde{A}),$$

which is surjective by [3, Prop. 2.4.4].

A *rational domain* in $X = \operatorname{Sp}(A)$ is a subset

$$X(f_1/g, \ldots, f_r/g) := \{x \in X\ ;\ |f_j(x)| \leqslant |g(x)|, j = 1, \ldots, r\},$$

where $g, f_1, \ldots, f_r \in A$ generate the unit ideal. It is again an affinoid variety with corresponding affinoid algebra

$$A\langle f_1/g,\ldots,f_r/g\rangle := A\langle\xi\rangle/(g\xi - f_1,\ldots,g\xi - f_r).$$

If $g = 1$, we call $X(f_1,\ldots,f_r)$ a *Weierstrass domain*.

An *affinoid subdomain* of $X = \operatorname{Sp} A$ is a subset $U \subset X$ together with an affinoid morphism $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ mapping $\operatorname{Sp} B$ into $U$ such that every affinoid morphism $\varphi' : \operatorname{Sp} B' \to \operatorname{Sp} A$ with $\varphi'(\operatorname{Sp} B') \subset U$ factors uniquely through $\varphi'$. Any rational domain is an affinoid subdomain. By a theorem of Gerritzen and Grauert [5, 7.3.5.], every affinoid subdomain is a finite union of rational domains.

An affinoid space $X = \operatorname{Sp} A$ carries a *weak $G$-topology* $\mathfrak{T}$, defined as follows: The admissible open sets are the affinoid subdomains, and the admissible coverings are the finite unions of affinoid subdomains. The *strong $G$-topology* is the unique finest topology which is slightly finer than the weak $G$-topology; i.e. which satisfies the following conditions:

(i) $\mathfrak{T}'$ is finer than $\mathfrak{T}$,

(ii) The $\mathfrak{T}$-admissible open sets form a basis for $\mathfrak{T}'$,

(iii) For each $\mathfrak{T}'$-admissible covering $\mathfrak{U}$ of a $\mathfrak{T}$-admissible open subset $U \subset X$, there exists a $\mathfrak{T}$-admissible covering which refines it.

A subset $U$ of $X$ is called *formal open* if there exists an open subset $V \subset \operatorname{Spec} \tilde{A}$ with $U = \pi^{-1}(V)$. The resulting topology on $X$ is called the *formal topology*.

A *rigid-analytic variety* over $K$ is a locally $G$-ringed space $(X, \mathcal{O}_X)$ with an atlas $\mathfrak{U} = \{U_i\}$ of affinoid varieties $U_i = \operatorname{Sp} A_i$ such that the $G$-topology on $X$ restricts to the strong $G$-topology on $U_i$.

A *formal covering* of $X$ is an admissible open covering $\mathfrak{U} = \{U_i\}$ of affinoid subdomains $U_i$ of $X$ such that for every $i, j$, $U_i \cap U_j$ is a finite union of formal subdomains in $U_i$. Let $\pi_i : U_i \to \tilde{U}_i$ denote the reduction map. These reductions can be pasted together, which yields a scheme $\tilde{X}$ of locally finite type over $k$ and a reduction morphism $\pi : X \to \tilde{X}$ which is surjective onto the set of closed points of $\tilde{X}$. Moreover, the formal topologies on $U_i$ are compatible, so we can endow $X$ with a formal topology which restricts to the formal topology on each $U_i$. The resulting space $X_{\mathfrak{U}}$ is called a *formal analytic variety*. We call $X_{\mathfrak{U}}$ *distinguished* if $\mathcal{O}(U_i)$ is distinguished for every $i$.

The *analytification* $(\mathbb{A}^n_K)^{\mathrm{an}}$ of affine $n$-space $\mathbb{A}^n_K$ can be constructed by glueing the sequence of $n$-dimensional polydiscs $\mathbb{D}^n(|c_i|)$ with radii $|c_i|$, where $|c_i| \to \infty$ for $i \to \infty$.

This yields an analytic variety, which coincides pointwise with the closed points of $\mathbb{A}^n_K$. For any affine scheme $X = \operatorname{Spec} B \subset \mathbb{A}^n_K$ of finite type over $K$, one glues the corresponding closed subvarieties $X \cap \mathbb{D}^n(|c_i|)$ accordingly. Again, we call this analytic variety the *analytification* of $X$, and denote it again by $X^{\text{an}}$. Finally, for a scheme $X$ over $K$ locally of finite type, one constructs the analytification by glueing the analytifications of its affine parts. A rigid-analytic space $X$ over $K$ is *algebraizable* if it is the analytification of a scheme locally of finite type over $K$.

## 1.2 Admissible Formal Schemes

Let $S$ be any ring, commutative with $1$, and let $\mathfrak{a}$ be an ideal in $S$. The $\mathfrak{a}$*-adic topology* on $S$ is given as follows: A subset $U \subset S$ is open, if for each $x \in U$, there exists an $n \in \mathbb{N}$ such that $x + \mathfrak{a}^n \subset U$. Endowed with this topology, we call $S$ an *adic* ring.

For any ring $A$ which is complete and hausdorff with respect to some $\mathfrak{a}$-adic topology, let $\operatorname{Spf} A$ denote the set of all open prime ideals $\mathfrak{p} \subset A$. This set carries the structure of a locally ringed space $X = (\operatorname{Spf} A, A)$. We call this an *affine formal scheme*. A *formal scheme* is a locally topologically ringed space $(X, \mathcal{O}_X)$ with an atlas $\mathfrak{U} = \{U_i\}$ of affine formal schemes $U_i = \operatorname{Spf} A_i$.

Now, let $R$ be the valuation ring of $K$ corresponding to a non-archimedean valuation. As in the previous section, one defines the $R$-algebra $R\langle\zeta_1, \ldots, \zeta_n\rangle$ of strictly convergent power series with coefficients in $R$. An $R$-algebra $A$ is *topologically of finite presentation* if it is isomorphic to $R\langle\zeta_1, \ldots, \zeta_n\rangle/\mathfrak{a}$, where $\mathfrak{a}$ is a finitely generated ideal in $R\langle\zeta_1, \ldots, \zeta_n\rangle$. $A$ is called *admissible* if it has no $\mathfrak{m}$-torsion, where $\mathfrak{m}$ is the maximal ideal of $R$. A formal $R$-scheme $X$ is called *admissible* if it has an atlas of formal schemes $U_i = \operatorname{Spf} A_i$ such that the $A_i$ are admissible.

The *special fibre* of $X$ is a scheme $\tilde{X}$ of locally finite type over $k$ with the same underlying topological space as $X$ and structure sheaf $\mathcal{O}_{\tilde{X}} := \mathcal{O}_X \otimes_R k = \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X$. Note that $\tilde{X}$ is not necessarily reduced.

To any admissible formal scheme $X$, one can associate a formal analytic variety $X^{\text{f-an}}$ as follows: Locally, $X$ is given by $\operatorname{Spf} A$. Then $A_K := A \otimes_R K$ is a $K$-affinoid algebra. The formal analytic variety $X^{\text{f-an}}$ is given locally by $\operatorname{Sp} A_K$ with its formal topology. The corresponding rigid-analytic variety $X_K$ is called the *generic fibre* of $X$. In general, the special fibre $\tilde{X}$ of $X$ does not agree with the reduction of $X^{\text{f-an}}$; however, there is a finite surjective morphism $(X^{\text{f-an}})^{\sim} \to \tilde{X}$, given locally by

$$A \otimes_R k \to (A \otimes_R K)^{\sim}.$$

For the converse, if $X_{\mathfrak{U}}$ is a formal analytic variety given by a formal covering $\mathfrak{U}$, we can associate to $X_{\mathfrak{U}}$ an admissible formal scheme $X^{\text{f-sch}}$ as follows: If $U_i = \operatorname{Sp} A_i$ for an affinoid $K$-algebra $A_i$, then $X^{\text{f-sch}}$ is given locally by $\operatorname{Spf}(A_i^{\circ})$.

By a result of Bosch and Lütkebohmert [7, Lemma 1.1], the functors $X \mapsto X^{\text{f-an}}$ and $X \mapsto X^{\text{f-sch}}$ give an equivalence between

(i) the category of distinguished formal analytic varieties over $K$, and

(ii) the category of admissible formal schemes over $R$ with reduced special fibre.

Especially, if $X$ is a distinguished formal analytic variety over $K$, then its reduction $\tilde{X}$ is naturally isomorphic to the special fibre $(X^{\text{f-sch}})^{\sim}$ of $X^{\text{f-sch}}$.

For $\tilde{p} \in \tilde{X}$, we call $X_+(\tilde{p}) := \pi^{-1}(\tilde{p})$ the *formal fibre* over $\tilde{p}$. It is an open analytic subspace of $X_K$.

Now, let $\mathcal{J}$ be an open sheaf of ideals in $\mathcal{O}_X$. The *admissible formal blowing up* of $X$ in $\mathcal{J}$ is given by the morphism

$$X' := \varinjlim \operatorname{Proj} \bigoplus_{\nu \geqslant 0} (\mathcal{J}^{\nu} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}) \to X.$$

Due to a theorem of Raynaud [30], the functor

$$\text{rig} : X \to X_K$$

sending an admissible formal scheme $X$ to its generic fibre $X_K$, induces an equivalence between

(i) the category of all quasi-compact, quasi-separated admissible formal schemes over $R$, localized by admissible formal blowings-up, and

(ii) the category of all quasi-compact, quasi-separated rigid-analytic $K$-varieties.

Now, we assume that the valuation on $K$ is discrete, and that $\pi$ is a uniformizing parameter. Let $X$ be an admissible formal $R$-scheme, and let $X_0^{(1)}, \ldots, X_0^{(s)}$ be the irreducible components of the special fibre $X_0$ of $X$. For $M \subset \{1, \dots, s\}$, we define

$$X_0^M := \bigcap_{i \in M} X_0^{(s)}$$

as the scheme-theoretic intersection. We call $X$ *strictly semi-stable* if the following conditions hold:

(i) The generic fibre $X_K$ is smooth over $K$.

(ii) The special fibre $X_0$ is geometrically reduced.

(iii) $X_0^{(i)}$ is a Cartier divisor on $X$ for all $i = 1, \dots, s$.

(iv) $X_0^M$ is smooth over $k$ for all $M \subset \{1, \dots, s\}$, and $\dim X_0^M = \dim X - \#M$.

Note that conditions (ii) – (iv) already imply (i). This follows from the following equivalent characterization:

**Lemma 1.2.1.** *An admissible formal $R$-scheme is strictly semi-stable if and only if every closed point $x \in X_0$ of the special fibre $X_0$ admits an open neighbourhood which, for some $r \in \mathbb{N}$, is formally smooth over the formal scheme*

$$\mathrm{Spf}\, R\langle \zeta_1, \dots, \zeta_r \rangle / (\zeta_1 \cdot \ldots \cdot \zeta_r - \pi),$$

*Proof.* See [21, Prop. 1.3]. □

## 1.3 Formal Cartier and Weil Divisors

Let $X$ be an admissible formal scheme over $R$ with irreducible generic fibre $X_K$ and reduced special fibre $X_0$.

On $X$, let $\mathcal{S}$ denote the subsheaf of $\mathcal{O}_X$ consisting of elements which are not zero divisors. The *sheaf of meromorphic functions* on $X$ is given by the localization $\mathcal{M}_X := \mathcal{O}_X(\mathcal{S}^{-1})$. A *Cartier divisor* on $X$ is a global section of $\mathcal{M}_X^\times / \mathcal{O}_X^\times$, where $\mathcal{M}_X^\times$ resp. $\mathcal{O}_X^\times$ is the sheaf of invertible elements in $\mathcal{M}_X$ resp. $\mathcal{O}_X$. An *invertible meromorphic function* is a global section of $\mathcal{M}_X^\times$.

Let $\mathcal{L}$ be an invertible sheaf on $X$, and let $s$ be an invertible meromorphic section of $\mathcal{L}$; i.e. locally, under a trivialization, $s$ corresponds to a section of $\mathcal{M}_X^\times$. This section is independent of the trivialization up to $\mathcal{O}_X^\times$. Thus, $s$ induces a well defined Cartier divisor $\operatorname{div}(s)$ on $X$.

Let $X_K$ be the generic fibre of $X$. A *cycle* on $X$ is a locally finite formal sum

$$\sum n_Y Y_K$$

where $n_Y \in \mathbb{Z}$ and $Y_K$ ranges over all irreducible analytic subsets of $X_K$. A *Weil divisor* is a cycle on $X_K$ such that all $Y_K$ with $n_Y \neq 0$ have codimension $1$ in $X_K$.

A *horizontal cycle* on $X$ is a cycle on the generic fibre $X_K$. A *vertical cycle* on $X$ is a locally finite formal sum

$$\sum \lambda_W \tilde{W},$$

where $\lambda_W$ is in the valuation group of $K$ and $\tilde{W}$ ranges over all irreducible closed subsets of the special fibre $X_0$.

A *cycle* on $X$ is a sum of a horizontal and a vertical cycle on $X$.

Now, let $D$ be a Cartier divisor on $X$. We may associate to $D$ a Weil divisor on $X$. For the horizontal part, $D$ restricts to a Cartier divisor $D_K$ on the generic fibre $X_K$. We may associate to $D_K$ a Weil divisor on $X_K$ as follows:

Locally, $X_K$ is isomorphic to $\operatorname{Sp} A$ for a $K$-affinoid algebra $A$. We may assume that $D_K$ is given on $\operatorname{Sp} A$ by a single equation $s$. Then $s$ can be thought of as a rational function on the affine scheme $\operatorname{Spec} A$. As $\operatorname{Spec} A$ is noetherian, $s$ induces a Weil divisor on $\operatorname{Spec} A$. As there is a one-to-one correspondence between analytic subsets of $\operatorname{Sp} A$ and closed subsets of $\operatorname{Spec} A$, this Weil divisor can be thought of as a Weil divisor on $\operatorname{Sp} A$. One can show that these locally defined Weil divisors agree on overlaps. Thus, they give rise to a horizontal Weil divisor $\operatorname{cyc}_h(D)$ on $X$.

For the vertical part, let $\tilde{W}$ be an irreducible component of the special fibre $\tilde{X}$. Let $U$ be a formal affine open subset of $X$, which contains the generic point of $\tilde{W}$. We assume that $D$ is given on $U$ by $s = a/b$, where $a, b \in \mathcal{O}_X(U)$ are not zero-divisors. Then $U_K = \operatorname{Sp} A$ and $\mathcal{O}_X(U) \cong A^\circ$ for a $K$-affinoid algebra $A$, and let $\pi : U_K \to \tilde{U} = \operatorname{Spec} \tilde{A}$ be the reduction map. Then $\tilde{W} \cap U$ is an irreducible component of $\tilde{U}$. Let $\tilde{W}'$ be a non-empty open affine subset of $\tilde{W} \cap U$ which does not meet any other irreducible component of $\tilde{W} \cap U$. For

$a \in A$, we define

$$|a(W)| := \sup\{|a(x)| \; ; \; x \in X_K, \pi(x) \in \tilde{W}'\}.$$

This equals the supremum semi-norm on the formal open affinoid subspace $\pi^{-1}(\tilde{W}')$. Moroever, if $a$ is not a zero divisor, $|a(W)| > 0$. This allows us to define the order of $D$ in $\tilde{W}$ by

$$\mathrm{ord}(D, \tilde{W}) := \log|b(\tilde{W} \cap U)| - \log|a(\tilde{W} \cap U)|.$$

Then we define the vertical part of the Weil divisor associated to $D$ by

$$\mathrm{cyc}_v(D) := \sum_{\tilde{W}} \mathrm{ord}(D, \tilde{W}) \cdot \tilde{W},$$

where $\tilde{W}$ runs through the irreducible components of $X_0$.

# Chapter 2

# Convex Geometry and Toric Varieties

In the following, let $\Gamma = 1/m \cdot \mathbb{Z}$ for some $m \in \mathbb{N}$ denote a discrete subgroup of $\mathbb{R}$. For instance, $\Gamma$ may be the valuation group of a discrete valuation ring.

## 2.1 Preliminaries

Let $\langle \cdot, \cdot \rangle$ denote the standard pairing

$$\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}; \quad \langle m, x \rangle := m_1 x_1 + \cdots + m_n x_n.$$

A *polyhedron* $\sigma$ in $\mathbb{R}^n$ is the intersection of finitely many closed half-spaces

$$\{x \in \mathbb{R}^n : \langle m_i, x \rangle + c_i \geqslant 0 \ ; \ i = 1, \ldots, r\}.$$

$\sigma$ is called $\Gamma$*-rational* if we can choose $m_i \in \mathbb{Z}^n$, $c_i \in \Gamma$. A *closed face* of $\sigma$ is the intersection of $\sigma$ with a closed half-space $H$ which contains $\sigma$. An *open face* of $\sigma$ is the relative interior of a closed face $\tau$, which we will denote by $\operatorname{relint}(\tau)$. This is the same as taking $\tau$ minus all its properly contained closed faces. A face of dimension zero is a *vertex*. A bounded face of dimension one is an *edge*; if it is unbounded, it is called a *ray*. A face of codimension one is a *facet*.

A *polytope* is a bounded polyhedron. It is the convex hull of a finite set of points. An $r$-*simplex* is the convex hull of $r+1$ points which do not lie in a common $r$-dimensional hyperplane.

For $v_1, \ldots, v_r \in \mathbb{R}^n$, the set

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s \ ; \ r_i \geqslant 0\}$$

is called a *convex polyhedral cone*. The cone $\sigma$ is called *strongly convex* if $\sigma$ does not contain any nonzero linear subspace.

A *polyhedral complex* $\Delta$ is a topological space $X$, plus a family of subsets $\sigma$ of $X$ which are homeomorphic to polyhedra as above, such that the following conditions hold:

(i) $X = \bigcup_{\sigma \in \Delta} \sigma$

(ii) If $\sigma \in \Delta$, and $\tau$ is a face of $\sigma$, then also $\tau \in \Delta$

(iii) For $\sigma, \sigma' \in \Delta$, $\sigma \cap \sigma'$ is a face of both $\sigma$ and $\sigma'$

We call $|\Delta| := X$ its *support*. If $X$ is a subset of $\mathbb{R}^n$ and every $\sigma$ is $\Gamma$-rational, we call $\Delta$ a $\Gamma$-rational polyhedral complex in $\mathbb{R}^n$.

If every $\sigma$ is a polytope (resp. a simplex), we call $\Delta$ a *polytopal* (resp. *simplicial*) complex. If $\Delta$ is a $\Gamma$-rational polyhedral complex in $\mathbb{R}^n$ such that every $\sigma$ is a $\Gamma$-rational cone, then we call $\Delta$ a *fan*.

A polytope $\tau \in \Delta$ which is not contained in a larger polytope $\sigma \in \Delta$ is called a *maximal polytope*. A polytopal complex $\Delta$ is *of pure dimension* $d$ if every maximal polytope $\sigma \in \Delta$ has dimension dimension $d$.

If $\Delta$, $\Delta'$ are two polytopal complexes, we define their *intersection* as follows:

$$\Delta \cap \Delta' := \{\sigma \cap \tau; \ \sigma \in \Delta, \tau \in \Delta'\}.$$

It is again a polytopal complex with $|\Delta \cap \Delta'| = |\Delta| \cap |\Delta'|$. If $\tau_0$ is a polytope, we call $\Delta \cap \{\tau_0\}$ the *restriction* of $\Delta$ to $\tau_0$, or the *induced subdivision* on $\tau_0$.

A *polyhedral decomposition* of a set $S \subset \mathbb{R}^n$ is a polyhedral complex $\Delta$ such that $|\Delta| = S$. A *polyhedral subdivision* of a polyhedral complex $\Delta$ is a polyhedral complex $\Delta'$ such that every polyhedron $\sigma \in \Delta$ has a polyhedral decomposition in $\Delta'$. If $\tau \in \Delta$, then $\mathrm{star}(\tau)$ is the subcomplex of $\Delta$ defined by

$$\mathrm{star}(\tau) = \{\sigma \in \Delta : \tau \subset \sigma\}.$$

A function $f$ on a subset of $\mathbb{R}^n$ is called *affine* or *affine linear* if it can be written as $f(x) = \langle m, x\rangle + c$, with $m \in \mathbb{Z}^n$, $c \in \mathbb{R}$. It is called $\Gamma$*-rational* if $c \in \Gamma$. If $\Delta$ is a polyhedral complex in $\mathbb{R}^n$, a *polyhedral function* on $\Delta$ is a continuous function $f : |\Delta| \to \mathbb{R}^n$ which is affine linear on every $\sigma \in \Delta$.

A polyhedral function $f$ is called *convex* if for every $\sigma$ in $\Delta$, there exist $m_\sigma \in \mathbb{Z}^n$, $c_\sigma$, such that

$$f(x) = \langle m_\sigma, x\rangle + c_\sigma \text{ for all } x \in \sigma,$$
$$f(x) \leqslant \langle m_\sigma, x\rangle + c_\sigma \text{ for all } x \in |\Delta|.$$

We say $f$ is $\Gamma$*-rational*, if we can choose $c_\sigma \in \Gamma$. This is equivalent to

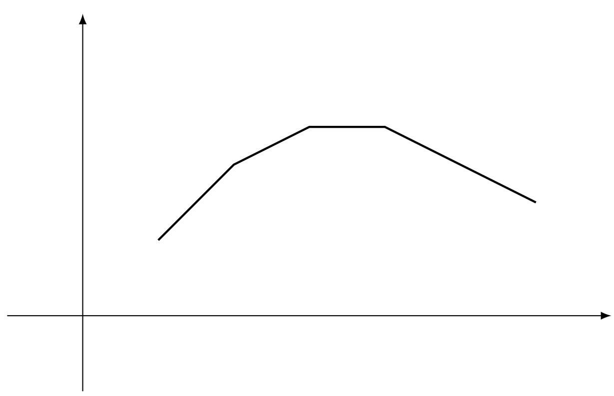
$$f(x) = \min_{\sigma\in\Delta} \langle m_\sigma, x\rangle + c_\sigma. \tag{2.1}$$

$f$ is called *strictly convex* if $m_\sigma$, $c_\sigma$ can be chosen such that

$$f(x) = \langle m_\sigma, x\rangle + c_\sigma \text{ for all } x \in \sigma,$$
$$f(x) < \langle m_\sigma, x\rangle + c_\sigma \text{ for all } x \in |\Delta| \setminus \sigma.$$

This is the case if and only if $\Delta$ is the maximal polytopal complex such that (2.1) holds.

**Remark 2.1.1.** Note that we have defined the notion of *convexity* as in [24]; this is exactly the opposite way as in calculus. Namely, a typical convex function on the real line looks as follows:



## 2.2 Toric Varieties

In this section, we will give a brief overview of the theory of toric varieties. For proofs, see [15].

In the following, let $N \cong \mathbb{Z}^n$ be a lattice, and let $M = \mathrm{Hom}(N, \mathbb{Z})$ denote the dual lattice, with $\langle \cdot, \cdot \rangle$ the canonical pairing on $M \times N$. Let $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ denote the real vector space with basis the generators of $N$. Similarly, let $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$ denote the real vector space corresponding to $M$; it is the dual vector space of $N_\mathbb{R}$.

**Proposition 2.2.1** (Gordon's Lemma)**.** *Let $\sigma$ be a rational convex polyhedral cone in $N_\mathbb{R}$. Let*

$$\sigma^\vee = \{m \in M_\mathbb{R} \ ; \ \langle m, v \rangle \geqslant 0 \, \textit{for all } v \in N_\mathbb{R}\} \subset M_\mathbb{R}$$

*denote the* dual cone. *Then $S_\sigma := \sigma^\vee \cap M$ is a finitely generated semigroup.*

Now, let $k$ be a field, and let $k[S_\sigma]$ denote the $k$-algebra generated by the characters

$$\chi^m := \langle m, \cdot \rangle : \mathbb{R}^n \to \mathbb{R}$$

with $m \in S_\sigma$. The multiplicative structure is given by

$$\chi^m \chi^{m'} := \chi^{m+m'}$$

Generators of $S_\sigma$ as a semigroup yield generators of $k[S_\sigma]$ as a $k$-algebra.

**Definition 2.2.2.** For a rational convex polyhedral cone $\sigma$ in $N_\mathbb{R}$, we call

$$U_\sigma := \mathrm{Spec}(k[S_\sigma])$$

an *affine toric variety*.

**Lemma 2.2.3.** *For $\tau \subset \sigma$, the morphism $U_\tau \to U_\sigma$ is an open immersion if and only if $\tau$ is a face of $\sigma$.*

**Remark 2.2.4.** For $\tau = \{0\}$, we get the algebraic torus $T := \mathrm{Spec}(k[M]) \cong (k^\times)^n$. It is a dense open subset of $U_\sigma$. Moreover, the action of $T$ on itself extends to an action of $T$ on $U_\sigma$. Namely, this action can be given by the algebra morphism

$$k[S_\sigma] \to k[S_\sigma] \otimes k[M], \quad \chi^m \mapsto \chi^m \otimes \chi^m$$

**Proposition 2.2.5.** *An affine toric variety $U_\sigma$ is nonsingular if and only if $\sigma$ is generated by part of a basis for the lattice $N$. In that case, for $r = \dim \sigma$, we have*

$$U_\sigma \cong k^r \times (k^\times)^{n-r}.$$

In the following, let $\Delta$ be a rational fan in $\mathbb{R}^n$. We can construct a *toric variety* $X_\Delta$ by glueing any two affine toric varieties $U_\sigma$, $U_{\sigma'}$ for $\sigma, \sigma' \in \Delta$ along the intersection $U_{\sigma\cap\sigma'}$ if $\sigma \cap \sigma' \neq \emptyset$. Especially, as every $U_\sigma$ contains the torus $T = U_{\{0\}}$, we have an action of $T$ on $X_\Delta$.

**Proposition 2.2.6.** *There is a one-to-one correspondence between torus orbits of $X_\Delta$ and cones $\tau \in \Delta$. For any $\tau \in \Delta$ with $k = \dim \tau$, the corresponding orbit $O_\tau$ is isomorphic to $(k^\times)^{n-k}$. Its closure $V(\tau)$ is a closed subvariety of $X_\Delta$.*

Note that $V(\tau)$ is again a toric variety. The torus orbit $O_\tau$ and its closure $V(\tau)$ can be constructed as follows:

Let $N_\tau$ denote the sublattice of $N$ generated as a group by $\tau \cap N$, and let $N(\tau)$ denote the quotient lattice $N/N_\tau$. Its dual lattice is $M(\tau) := \tau^\perp \cap M$. Then $O_\tau$ is the $(n-k)$-dimensional torus corresponding to the lattice $N(\tau)$. For any $\sigma \in \Delta$ with $\tau \subset \sigma$, let $\overline{\sigma}$ denote its image in $N(\tau)_\mathbb{R}$. The cones $\overline{\sigma}$ yield a fan in $N(\tau)$, which we denote by $\overline{\mathrm{star}(\tau)}$. Then $V(\tau)$ is the toric variety given by $\overline{\mathrm{star}(\tau)}$. For any affine toric variety $U_{\overline{\sigma}} \subset V(\tau)$, we have an embedding $U_{\overline{\sigma}} \hookrightarrow X(\Delta)$, which is given by the projection morphism

$$\begin{array}{ccc} k[\sigma^\vee \cap M] & \twoheadrightarrow & k[\sigma^\vee \cap \tau^\perp \cap M] \\ \chi^m & \mapsto & \begin{cases} \chi^m, & m \in \sigma^\vee \cap \tau^\perp \cap M \\ 0 & \text{otherwise} \end{cases} \end{array}$$

These embeddings glue to a closed embedding $V(\tau) \hookrightarrow X_\Delta$.

**Definition 2.2.7.** Let $\varphi : N \to N'$ be a homomorphism of lattices, and $\Delta, \Delta'$ fans in $N, N'$ respectively, such that, for each cone $\sigma' \in \Delta'$, its image $\varphi(\sigma')$ is contained in some $\sigma \in \Delta$. The morphism $S_\sigma \to S_{\sigma'}$ determines a morphism $U_{\sigma'} \to U_\sigma$ of affine toric varieties. These morphisms glue to a morphism $\varphi_* : X_{\Delta'} \to X_\Delta$ of toric varieties.

**Proposition 2.2.8.** *The morphism $\varphi_* : X_{\Delta'} \to X_\Delta$ constructed above is proper if and only if $\varphi^{-1}(|\Delta|) = |\Delta'|$. As a special case, a toric variety $X_\Delta$ is proper over $k$ if and only if $|\Delta| = N_\mathbb{R}$.*

## 2.3 Simplicial Homology and Cohomology

In the following, let $\Delta$ denote a simplicial complex, not necessarily in $\mathbb{R}^n$, with support $X := |\Delta|$. Let $C_n$ denote the free abelian group with basis consisting of the $n$-dimensional simplices of $\Delta$. We define a *boundary homomorphism* $\partial_n : C_n \to C_{n-1}$ via

$$\partial_n[p_0, \ldots, p_n] = \sum_{i=0}^{n} (-1)^i [p_0, \ldots, p_{i-1}, p_i, \ldots, p_n]$$

One has $\partial_n \circ \partial_{n-1} = 0$ for $n > 0$. This defines a *chain complex* of abelian groups

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$

We call $H_n(X) := \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$ the $n$-th *simplicial homology group* of $X$; it depends only on $X$ and not on the complex $\Delta$.

Now, let $G$ be an arbitrary abelian group, and let $C^n := \operatorname{Hom}(C_n, G)$. This yields homomorphisms $\delta_n := \partial_{n+1}^* : C^n \to C^{n+1}$. Dualizing this way, we get a *cochain complex*

$$0 \to C^0 \xrightarrow{\delta_0} C_1 \to \cdots \to C^n \xrightarrow{\delta_n} C^{n+1} \to \cdots$$

We call $H^n(X, G) := \operatorname{Ker} \delta_n / \operatorname{Im} \delta_{n-1}$ the $n$-th *cohomology group*.

Homology groups and cohomology groups are connected via the following result:

**Proposition 2.3.1** (Universal Coefficient Theorem for Cohomology)**.**

$$H^n(X, G) \cong \operatorname{Hom}(H_n(X), G) \oplus \operatorname{Ext}(H_{n-1}(X), G).$$

*Especially, for $n = 1$, as $H_0(X)$ is free, we have*

$$H^1(X, G) \cong \operatorname{Hom}(H_1(X), G).$$

**Definition 2.3.2.** An *edge-path* in $\Delta$ is a finite sequence of vertices $p_0, p_1, \ldots, p_s$ of $\Delta$ such that, for each $n = 0, \ldots, s-1$, $[p_n, p_{n+1}]$ is an edge of $\Delta$. It is called an *edge-loop* if $p_s = p_0$. For each edge-path $p_0 p_1 \cdots p_s$, we get an *equivalent* edge path by the following operations:

(i) If $[p_{i-1}, p_i, p_{i+1}]$ is a simplex of $\Delta$, we can replace $p_{i-1} p_i p_{i+1}$ by $p_{i-1} p_{i+1}$.

(ii) If $p_i = p_{i+1}$, we can replace $p_{i-1} p_i p_{i+1} p_{i+2}$ by $p_{i-1} p_{i+2}$.

Let $\pi_1(\Delta, p_0)$ denote the group of equivalence classes of edge-loops with starting point $p_0$, where the group operation is just the concatenation of edge-loops. It is called the *edge-path group* of $\Delta$ in $p_0$.

**Proposition 2.3.3.** *Assume that $X$ is connected.*

(i) *The edge-path group $\pi_1(\Delta, p_0)$ is isomorphic to the topological fundamental group $\pi_1(X, p_0)$ of $X$ in $p_0$.*

(ii) *The canonical morphism $\pi_1(\Delta, p_0) \to H_1(X)$, sending every edge-loop to the formal sum of the occurring edges, is an epimorphism, with kernel the commutator subgroup of $\pi_1(\Delta, p_0)$. In other words, $H_1(X)$ is isomorphic to the abelianization of $\pi_1(\Delta, p_0)$.*

## 2.4 Polytopal Complexes with Integral Structure

In this section, we will deal with a generalization of polytopal complexes in $\mathbb{R}^n$. Namely, we will consider polytopal complexes which have the same nice combinatorial structure as polytopal complexes in $\mathbb{R}^n$, but can only locally be embedded in $\mathbb{R}^n$.

An *integral structure over $\mu$* on a polytopal complex $\Delta$ is a set of finitely generated abelian groups $L_i$ of real-valued functions on $\sigma_i$ with values in $1/\mu \cdot \mathbb{Z}$ for every $\sigma_i \in \Delta$, such that the following holds:

(i) $L_i \supset n\mathbb{Z}$ for some $n \in \mathbb{N}$

(ii) If $n, f_1, \ldots, f_{n_i}$ are generators of $L_i$, then this yields an embedding

$$\varphi_i = (f_1, \ldots, f_{n_i}) : \sigma_i \hookrightarrow \mathbb{R}^{n_i}$$

which gives rise to a homeomorphism of $\sigma_i$ to a polytope in $\mathbb{R}^{n_i}$ which is not contained in a hyperplane.

(iii) If $\sigma_j$ is a face of $\sigma_i$, then $L_i|_{\sigma_j} = L_j$.

Let $V_i := L_i \otimes \mathbb{R}$. A subdivision $\Delta'$ of $\Delta$ is a *rational* subdivision, if, for all $\sigma_i, \sigma_j \in \Delta'$ with $\sigma_i \subset \sigma_j$, we have $V_j|_{\sigma_i} = V_i'$, and any function in $L_j$ takes rational values at the vertices of $\Delta'$. In particular, the integral structure of $\Delta$ restricts to a integral structure on $\Delta'$.

A subdivision $\Delta'$ of $\Delta$ is called *projective*, if there exists a continuous function $f : |\Delta'| \to \mathbb{R}$ such that $f$ is strictly convex on $\Delta' \cap \sigma$ for every $\sigma \in \Delta$; i.e. if the following two conditions hold:

(i) $f|_{\sigma_i} = \min_{j=1,\ldots,r} l_j$ for certain $l_1, \ldots, l_r \in V_i$

(ii) If $\sigma_i \in \Delta$ and $l \in V_i$ with $l \geqslant f|_{\sigma_i}$, then the set

$$\tau := \{x \in \sigma \ ; \ f(x) = l(x)\}$$

is either empty or a polyhedron of $X'$.

A function $f$ which satisfies the above conditions is called a *good* function for the subdivision $\Delta'$.

Let $\sigma_i \subset \Delta$ be a polytope of dimension $n_i$, and let $c \in \mathbb{Z}$ such that $c \in L_i$. Then we define the *multiplicity* of $\sigma_i$ with respect to $c$ as

$$m(\sigma_i, c) := c^{n_i} \cdot (n_i)! \cdot \operatorname{vol} \sigma_i,$$

where $\operatorname{vol}$ denotes the volume of $\varphi_i(\sigma_i)$ in $\mathbb{R}^{n_i}$.

For the following result, see [24, Th. 4.1].

**Proposition 2.4.1.** *Let $\Delta$ be a polytopal complex with $1/\mu$-rational structure. Then there exists an integer $\nu$ and a rational projective subdivision $\Delta'$ of $\Delta$ such that $\Delta'$ is $1/(\mu\nu)$-rational and $m(\sigma, \mu\nu) = 1$ for every $\sigma \in \Delta'$.*

# Chapter 3

# Polytopal Domains in $\mathbb{G}_m^n$

For simplicity, we will assume in the following that the complete non-archimedean field $K$ is algebraically closed. All results of this chapter hold for a discrete valued field as well, if we allow suitable finite field extensions. In the following, let $\Gamma$ be the value group of the (additive) valuation $v := -\log|\cdot| : K^\times \to \mathbb{R}$ on $K$.

## 3.1 Definitions and First Properties

Let $\mathbb{G}_m^n := (K^\times)^n$ denote the $n$-dimensional torus over $K$. The valuation

$$v := -\log|\cdot| : K^\times \to \mathbb{R}$$

induces a continuous mapping

$$\mathrm{val} : \mathbb{G}_m^n \to \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (v(x_1), \dots, v(x_n)).$$

As in Chapter 2, let $\langle \cdot, \cdot \rangle$ denote the standard pairing

$$\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}; \quad \langle m, x \rangle := m_1 x_1 + \cdots + m_n x_n.$$

Let $\chi = a\zeta^m$ with $a \in K^\times$, $m \in \mathbb{Z}^m$. Although $\chi$ is not necessarily monic, we call $\chi$ a *monomial*. We can describe the values $|\chi(u)|$ for $u \in \mathbb{G}_m^n$ by an affine linear function $f_\chi$ as follows:

We set $f_\chi(x) := \langle m, x \rangle + v(a)$ for $x \in \mathbb{R}^n$. For a point $u = (u_1, \dots, u_n) \in \mathbb{G}_m^n$ with

$\mathrm{val}(u) = x$, we have

$$-\log|\chi(u)| = -\log|a| - m_1 \log|u_1| - \cdots - m_n \log|u_n|$$
$$= v(a) + \langle m, \mathrm{val}(u)\rangle = f_\chi(x).$$

Let $\sigma \subset \mathbb{R}^n$ be a $\Gamma$-rational polytope which is given by a finite number of inequalities $\langle m_i, x\rangle + c_i \geqslant 0$, with $m_i \in \mathbb{Z}^n$, $c_i \in \Gamma$. We set $X_{\sigma,K} := \mathrm{val}^{-1}(\sigma)$; it is a Weierstrass domain in $\mathbb{G}_m^n$ given by

$$X_{\sigma,K} = \{x \in \mathbb{G}_m^n \; ; \; |a_i \zeta^{m_i}(x)| \leqslant 1 \text{ for } i = 1, \ldots, r\},$$

where $a_i \in K^\times$ with $v(a_i) = c_i$ for $i = 1, \ldots, r$. We define

$$\mathcal{O}(X_{\sigma,K}) := \left\{ \sum_{m \in \mathbb{Z}^n} a_m \zeta^m : \quad \lim_{|m| \to \infty} v(a_m) + \langle m, u\rangle = \infty \text{ for all } u \in \sigma \right\}.$$

It is a subring of the ring of formal Laurent series in $\zeta_1, \ldots, \zeta_n$. Endowing $\mathcal{O}(X_{\sigma,K})$ with the supremum norm

$$\left|\sum a_m \zeta^m\right|_{\sup} := \sup_{m \in \mathbb{Z}^n, x \in \sigma} e^{-\langle m,x\rangle - v(a_m)} = \max_{\substack{m \in \mathbb{Z}^n, \\ u \text{ vertex of } \sigma}} e^{-\langle m,u\rangle - v(a_m)} \tag{3.1}$$

on $X_{\sigma,K}$ makes $\mathcal{O}(X_{\sigma,K})$ into a $K$-Banach algebra. As $\sigma$ is $\Gamma$-rational, the supremum norm takes values in $K$.

**Lemma 3.1.1.** *$X_{\sigma,K}$ is an affinoid subdomain of $\mathbb{G}_m^n$ with $K$-affinoid algebra $\mathcal{O}(X_{\sigma,K})$.*

*Proof.* This has already been proven in [13, 3.1] or [19, Prop. 4.1]. We will however give a slightly different proof here.

We want to show that $\mathcal{O}(X_{\sigma,K})$ is $K$-affinoid. Let $u$ be a vertex of $\sigma$. For simplicity, we assume $u = 0$. Let $C_{\sigma,u} = \mathbb{R}^+ \cdot \sigma$ denote the cone over $\sigma$. By Gordon's Lemma (Proposition 2.2.1),

$$C_{\sigma,u}^\vee \cap \mathbb{Z}^n := \{m \in \mathbb{Z}^n : \langle m, x\rangle \geqslant 0 \text{ for all } x \in C_{\sigma,u}\}$$

is a finitely generated semigroup. Let $S_{\sigma,u}$ be a generating set of $C_{\sigma,u}^\vee \cap \mathbb{Z}^n$, and let $S_\sigma$ be the union of all $S_{\sigma,u}$, where $u$ runs through all the vertices of $\sigma$. For $m \in S_{\sigma,u}$, let $c_m := -\langle m, u\rangle$. Then $c_m \in \Gamma$, as $u$ has coordinates in $\Gamma$. We choose $a_m \in K^\times$ with $v(a_m) = c_m$

and set $\chi_m := a_m\zeta^m$. By construction, $|\chi_m|_{\sup} = 1$ on $X_{\sigma,K}$, and the maximum is assumed at all points of $\mathrm{val}^{-1}(u)$. Moreover, it is clear that any $m \in \mathbb{Z}^n$ lies in $C^\vee_{\sigma,u}$ for at least one vertex $u$. From the definition of $\mathcal{O}(X_{\sigma,K})$, one concludes

$$\mathcal{O}(X_{\sigma,K}) = K\langle\chi_m\,;\, m \in S_\sigma\rangle.$$

Thus, $\mathcal{O}(X_{\sigma,K})$ is a $K$-affinoid algebra, and $X_{\sigma,K}$ is an affinoid space. □

**Definition 3.1.2.** We call $X_{\sigma,K}$ the *affinoid polytopal domain associated to* $\sigma$.

**Remark 3.1.3.** Obviously, $X_{\sigma,K}$ is regular as an affinoid subdomain in $\mathbb{G}_{m,K}$.

The following lemma is a generalization of [5, Lemma 9.7.1/1].

**Lemma 3.1.4.** *Let* $g = \sum a_m\zeta^m \in \mathcal{O}(X_{\sigma,K})$. *Then* $g$ *is a unit on* $X_{\sigma,K}$ *if and only if there exists* $m_0 \in \mathbb{Z}^n$ *such that* $|a_{m_0}z^{m_0}| > |a_m z^m|$ *for all* $z \in X_{\sigma,K}, m \neq m_0$.

*Proof.* See [19, §6]. □

**Theorem 3.1.5.** $\mathcal{O}(X_{\sigma,K})$ *is factorial, hence normal.*

*Proof.* In Chapter 6, we will show that $\mathrm{Pic}(X_{\sigma,K}) = H^1(X_{\sigma,K}, \mathcal{O}^\times)$ is trivial. As $X_{\sigma,K}$ is regular, this proves that $\mathcal{O}(X_{\sigma,K})$ is factorial; cf. [14, Prop. 4.7.2.]. □

In the following, we want to associate a formal model to an affinoid polytopal domain. We need the following result:

**Lemma 3.1.6.** *Let* $\mathcal{O}(X_{\sigma,K})^\circ := \{f \in \mathcal{O}(X_{\sigma,K})\,;\, |f|_{\sup} \leqslant 1\}$. *Then* $\mathcal{O}(X_{\sigma,K})^\circ$ *is an admissible* $R$*-algebra; i.e. a flat* $R$*-algebra of topologically finite type.*

*Proof.* We choose $S_\sigma$ as in the proof of Lemma 3.1.1. We claim

$$\mathcal{O}(X_{\sigma,K})^\circ = R\langle\chi_m\,;\, m \in S_\sigma\rangle.$$

The "$\supset$" part is clear, since $|\chi_m| \leqslant 1$ on $X_{\sigma,K}$ for $m \in S_\sigma$. For the "$\subset$" part, let $\chi = a_m\zeta^m$ with $m \in \mathbb{Z}^n$, $a \in K^\times$ be a monomial with $|\chi|_{\sup} \leqslant 1$ on $X_{\sigma,K}$. Let $f_\chi$ be the corresponding affine linear function, and let $u$ be a vertex of $\sigma$ such that $f_\chi$ is minimal.

Then $f_\chi(x) \geqslant f_\chi(u)$ for all $x \in \sigma$. Thus $m \in C_{\sigma,u}^\vee$, and we can write $m = b_1 m_1 + \cdots + b_r m_r$ with $b_i \geqslant 0$ and $m_i \in S_{\sigma,u}$. Then

$$a_m \zeta^m = a'_m \cdot \chi_{m_1}^{b_1} \cdots \chi_{m_r}^{b_r}$$

for a unique $a'_m \in K$. As $|\chi_i| = 1$ on $\mathrm{val}^{-1}(u)$ for every $i = 1, \ldots, r$, we have $|\chi| = |a'_m|$ on $\mathrm{val}^{-1}(u)$. But $|\chi| \leqslant 1$ on $\mathrm{val}^{-1}(u)$, so $a'_m$ lies in $R$. Moreover, $\mathcal{O}(X_{\sigma,K})^\circ$ has no $R$-torsion. This proves the claim. □

**Remark 3.1.7.** Note that $\mathcal{O}(X_{\sigma,K})$ and $\mathcal{O}(X_\sigma) = \mathcal{O}(X_{\sigma,K})^\circ$ are integral domains. For $g, h \in \mathcal{O}(X_{\sigma,K})$, there exists $\alpha \in K^\times$ with $|\alpha g|, |\alpha h| \leqslant 1$ on $X_{\sigma,K}$. But then $g/h = (\alpha g)/(\alpha h)$; i.e. $\mathcal{O}(X_\sigma)$ and $\mathcal{O}(X_{\sigma,K})$ have the same field of fractions, which we denote by $\mathcal{M}(X_\sigma)$. This is the algebra of meromorphic functions on $X_{\sigma,K}$.

**Lemma 3.1.8.** *$\mathcal{O}(X_{\sigma,K})^\circ$ is normal.*

*Proof.* From Theorem 3.1.5, we see that $\mathcal{O}(X_{\sigma,K})$ is normal. Let $f \in \mathcal{M}(X_\sigma)$ satisfy an integral relation

$$f^n + a_{n-1} f^{n-1} + \cdots + a_0 = 0$$

with $a_i \in \mathcal{O}(X_{\sigma,K})^\circ$. As $\mathcal{O}(X_{\sigma,K})$ is normal, $f \in \mathcal{O}(X_{\sigma,K})$. By the ultrametric inequality, we get

$$|f|_{\sup}^n \leqslant \max_i |a_i| |f|_{\sup}^i.$$

Thus, $|f|_{\sup}^{n-i} \leqslant |a_i| \leqslant 1$ for a certain $i$. Hence, $|f|_{\sup} \leqslant 1$, and $f \in \mathcal{O}(X_{\sigma,K})^\circ$. This proves that $\mathcal{O}(X_{\sigma,K})^\circ$ is normal. □

This allows us to associate to $X_{\sigma,K}$ a canonical model $X_\sigma = \mathrm{Spf}(\mathcal{O}(X_{\sigma,K})^\circ$, which we will call an (affine) *formal polytopal domain*. Let $\tilde{X}_\sigma$ denote the special fibre of $X_\sigma$. This coincides with the reduction of $X_{\sigma,K}$, as the affinoid algebra $\mathcal{O}(X_{\sigma,K})$ is reduced, and hence distinguished; see Chapter 1. Let $\pi : X_{\sigma,K} \to \tilde{X}_\sigma$ denote the reduction map.

The affinoid torus $T = \{x \in (\mathbb{G}_m^n)_K \ ; \ |x_i| = 1\}$ acts on $X_{\sigma,K}$. Passing to reductions, we get an action of the algebraic torus $\tilde{T} = (k^\times)^n$ on $\tilde{X}_\sigma$.

Recall that the Berkovich spectrum of an affinoid algebra $A$ is the set of all multiplicative semi-norms on $A$ which are bounded by the supremum semi-norm, see Chapter 1.

**Definition 3.1.9.** The *Shilov boundary* of an affinoid algebra $A$ is the unique minimal subset $\Theta$ of the Berkovich spectrum such that every $f \in A$ assumes its minimum in $\Theta$.

From (3.1), it follows directly that the Shilov boundary of $\mathcal{O}(X_{\sigma,K})$ is given by

$$\Theta = \{|\cdot|_u \;;\; u \text{ vertex of } \sigma\}.$$

For the following result, see also [19, Prop. 4.4].

**Lemma 3.1.10.** *Let $X_{\sigma,K}$ be an affinoid polytopal domain. For the reduction $\tilde{X}_\sigma$, the following assertions hold:*

(i) *The irreducible components of $\tilde{X}_\sigma$ are in one-to-one correspondence with the vertices $u$ of $\sigma$. For a vertex $u \in \sigma$, the corresponding component $\tilde{X}_{\sigma,u}$ is the affine toric variety induced by the polyhedral cone $C_{\sigma,u} = \mathbb{R}^+(\sigma - u)$.*

(ii) *There is a one-to-one correspondence between torus orbits $Z$ of $\tilde{X}_\sigma$ and faces $\tau$ of $\sigma$, given by*

$$\tau \mapsto O_\tau := \pi(\mathrm{val}^{-1}(\mathrm{relint}(\tau))),$$

*where $\mathrm{relint}(\tau)$ denotes the relative interior of $\tau$.*
*Moreover, $\dim(O_\tau) = n - \dim(\tau)$.*

(iii) *If $\sigma' \subset \sigma$ is a $\Gamma$-rational polytope, then the canonical morphism $X_{\sigma,K} \to X_{\sigma',K}$ induces an open immersion of the reductions if and only if $\sigma'$ is a face of $\sigma$.*

*Proof.* By [3, Prop. 2.4.4], the Shilov boundary of $\mathcal{O}(X_{\sigma,K})$ consists of all analytic points $\xi_{\tilde{Y}}$ which reduce to the generic points of the irreducible components of $\tilde{X}_\sigma$. Thus, we have a one-to-one correspondence between irreducible components of $\tilde{X}_\sigma$ and vertices $u$ of $\sigma$.

Let $\tilde{X}_{\sigma,u}$ be the irreducible component corresponding to $u$. Then

$$\mathcal{O}(\tilde{X}_{\sigma,u}) = \mathcal{O}(X^\circ_{\sigma,K})/\{|\cdot|_u < 1\}.$$

Let $f = \sum_m a_m \zeta^m \in \mathcal{O}(X_{\sigma,K})$ with $|f|_{\sup} \leqslant 1$. This implies $|f|_u \leqslant 1$, so $|a_m\zeta^m|_u \leqslant 1$ for every $m \in \mathbb{Z}^n$. As in the proof of Lemma 3.1.6, we can write

$$a_m\zeta^m = a'_m \cdot \chi_{m_1}^{b_1} \cdots \chi_{m_r}^{b_r}$$

where $m_i \in S_{\sigma,u}$, $b_i \geqslant 0$ such that $m = b_1 m_1 + \cdots + b_r m_r$ and $|c_m| \leqslant 1$. Thus, we have $f \in R\langle \chi_{m_1}, \ldots, \chi_{m_r} \rangle$, and the reduction $\tilde{f}$ of $f$ modulo $\{|\cdot|_u < 1\}$ is a polynomial in $\tilde{\chi}_{m_1}, \ldots, \tilde{\chi}_{m_r}$ with coefficients in $k$. If $\tilde{f} = 0$, then $|a'_m| < 1$ for every $m \in \mathbb{Z}^n$. This proves

$$\mathcal{O}(\tilde{X}_{\sigma,u}) = k[\tilde{\chi}_{m_i},\ m_i \in S_{\sigma,u}].$$

By change of coordinates, we may assume $u = 0$. In this situation, every leading coefficient of $\chi_{m_i}$ has (multiplicative) valuation $1$; so we may replace $\tilde{\chi}_{m_i}$ by $\zeta^{m_i}$. As $S_{\sigma,u}$ generates the subgroup $C_{\sigma,u}^{\vee} \cap \mathbb{Z}^n$, and $C_{\sigma,u}$ is the cone over $\sigma$, claim (i) follows. Claim (iii) follows directly from the corresponding result for affine toric varieties, see Lemma 2.2.3.

Now, let $\tau$ be a face of $\sigma$. For $m \in \mathbb{Z}^n$, choose $a_m \in K^{\times}$ such that $\chi_m := a_m \zeta^m$ satisfies $|\chi_m|_{\sup} = 1$ on $X_{\sigma,K}$. Let $c_m := v(a_m)$. Then $\operatorname{relint}(\tau)$ is given by linear equations resp. inequalities

$$\langle m, x\rangle + c_m \begin{cases} = 0 & \text{if } m \in I, \\ > 0 & \text{if } m \in \mathbb{Z}^n \setminus I \end{cases}$$

for some index set $I \subset \mathbb{Z}^n$. Let $O_\tau := \pi(\operatorname{val}^{-1}(\tau))$. Then $\tilde{x} \in O_\tau$ if and only if

$$\tilde{\chi}_m(\tilde{x}) \begin{cases} \neq 0 & \text{for } m \in I \\ = 0 & \text{for } m \notin I. \end{cases}$$

Now, let $u$ be a face of $\tau$, and let $\tilde{X}_{\sigma,u}$ be the corresponding irreducible component of $\tilde{X}_\sigma$. From the proof of (i), we see that $\tilde{X}_{\sigma,u}$ is the vanishing locus of $\tilde{\chi}_m$ for $m \in \mathbb{Z}^n \setminus C_{\sigma,u}^{\vee}$. By definition of $I$, we have $I \subset C_{\sigma,u}^{\vee}$, so $O_\tau$ is contained in $\tilde{X}_{\sigma,u}$. Moreover, $O_\tau$ is given in $\tilde{X}_{\sigma,u}$ by $\tilde{\chi}_m = 0$ for $m \in C_{\sigma,u}^{\vee} \setminus I$ and $\tilde{\chi}_m \neq 0$ for $m \in I$. From the theory of toric varieties, we see that $O_\tau$ is a torus orbit. By definition, $\tau \subset \operatorname{val}(\pi^{-1}(O_\tau))$. But $\tilde{X}_\sigma$ is a disjoint union of its torus orbits, and $\sigma$ is a disjoint union of its open faces. So we have in fact equality. This sets up a bijective correspondence between open faces and torus orbits as claimed. □

To go from affine formal polytopal domains to global formal polytopal domains, we take a $\Gamma$-rational polytopal complex $\Delta$ in $\mathbb{R}^n$. Let $X_{\Delta,K} = \bigcup_{\sigma\in\Delta} X_{\sigma,K}$. By part (iii) of Lemma 3.1.10, $(X_{\sigma,K})_{\sigma\in\Delta}$ is a formal analytic atlas of $X_{\Delta,K}$. This gives rise to an admissible formal scheme $X_\Delta$ over $R$ with formal open affine atlas $(X_\sigma)_{\sigma\in\Delta}$. We call $X_\Delta$

a *formal polytopal domain* and $X_{\Delta,K}$ a *rigid polytopal domain*. Note that, once again, the algebraic torus $\tilde{T} = (k^{\times})^n$ acts on the special fibre $\tilde{X}_{\Delta}$. From Lemma 3.1.10, we derive the following global version:

**Proposition 3.1.11.** *Let $X_{\Delta}$ be a formal polytopal domain, $X_{\Delta,K}$ its generic fibre, $\tilde{X}_{\Delta}$ its special fibre. Then the following assertions hold:*

(i) *The irreducible components of $\tilde{X}_{\Delta}$ are in one-to-one correspondence with the vertices $u$ of $\Delta$. For a vertex $u \in \Delta$, the corresponding component $\tilde{X}_{\Delta,u}$ is a toric variety. Its fan is given by the cones $C_{\sigma,u} = \mathbb{R}^{+}(\sigma - u)$ for $\sigma \in \operatorname{star}(u)$.*

(ii) *There is a one-to-one correspondence between torus orbits $Z$ of $\tilde{X}_{\Delta}$ and polytopes $\tau \in \Delta$, given as in Lemma 3.1.10.*

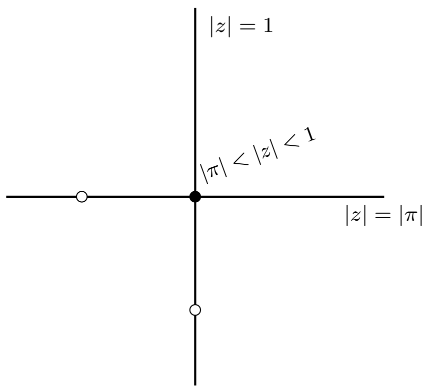
(iii) *Let $\sigma, \sigma' \in \Delta$. Then $X_{\sigma}$ is an open subset of $X_{\sigma'}$ if and only if $\sigma'$ is a face of $\sigma$.*

**Example 3.1.12.** We start with a simple example in dimension $1$:

(i) Take $\pi \in K$ with $|\pi| < 1$; set $c := -\log|\pi|$. As polytope, consider the line segment $[0, 2c] \subset \mathbb{R}^1$. The associated affinoid polytopal domain is given by the annulus $\{z : |\pi^2| \leqslant |z| \leqslant 1\}$ with the corresponding affinoid algebra

$$K\langle\zeta_1, \zeta_2\rangle/(\zeta_1\zeta_2 - \pi^2) = K\langle\zeta_1, \pi^2/\zeta_1\rangle.$$

The reduction is $k[\tilde{\zeta}_1, \tilde{\zeta}_2]/(\tilde{\zeta}_1\tilde{\zeta}_2)$ and thus consists of two copies of $\mathbb{A}^1$ intersecting in an ordinary double point. This gives the following picture:



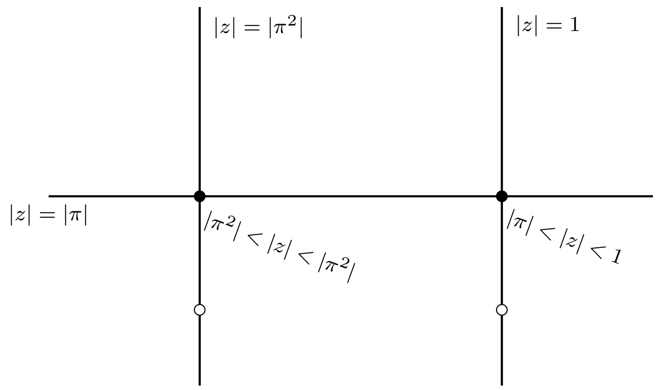

The white circle $\circ$ denotes the missing points at infinity. The reduction consists of the following 3 torus orbits:

$$\begin{array}{ccccc} \{\tilde{\zeta}_1 \neq 0\} & \leftrightarrow & |z| = 1 & \leftrightarrow & x = 0 \\ \{\tilde{\zeta}_2 \neq 0\} & \leftrightarrow & |z| = |\pi^2| & \leftrightarrow & x = 2c \\ \{\tilde{\zeta}_1 = \tilde{\zeta}_2 = 0\} & \leftrightarrow & |\pi^2| < |z| < 1 & \leftrightarrow & x \in (0, 2c) \end{array}$$

(ii) Now, consider the polytopal complex consisting of the two line segments $[0, c]$ and $[c, 2c]$. The affinoid polytopal domain is the union of the two annuli

$$\{z : |\pi^2| \leqslant |z| \leqslant |\pi|\} \cup \{z : |\pi| \leqslant |z| \leqslant 1\} = \{z : |\pi^2| \leqslant |z| \leqslant 1\},$$

which is the same annulus as in (i). Note, however, that each of the two annuli reduces to two affine lines as in (i); glueing them together gives the following picture:




We conclude this section with the following result:

**Proposition 3.1.13.** *Let $\Delta$ be a finite $\Gamma$-rational polytopal complex. $X_{\Delta,K}$ is affinoid if and only if the support $|\Delta|$ of $\Delta$ is a polytope.*

*Proof.* Let $\sigma := \text{conv}(|\Delta|)$ be the convex hull of $|\Delta|$; this is a polytope as $\Delta$ is finite. Let $U_{\sigma,K}$ be the affinoid polytopal domain corresponding to $\sigma$. If $\sigma = |\Delta|$, then $U_{\sigma,K} = X_{\Delta,K}$, so $X_{\Delta,K}$ is obviously affinoid. For the converse assertion, we claim

$$\mathcal{O}(X_{\Delta,K}) = \mathcal{O}(U_{\sigma,K}). \tag{3.2}$$

Obviously, $\mathcal{O}(U_{\sigma,K}) \subseteq \mathcal{O}(X_{\Delta,K})$. Now, let $g = \sum a_m \zeta^m \in \mathcal{O}(X_{\Delta,K})$; then $g \in \mathcal{O}(U_{\tau,K})$ for all $\tau \in \Delta$. Then

$$\lim_{|m| \to \infty} v(a_m) + \langle m, u \rangle = \infty \text{ for all } u \in |\Delta|.$$

Now, let $x \in \sigma$, i.e. $x = \lambda u_1 + (1-\lambda)u_2$ for a $\lambda \in [0,1]$, $u_1$, $u_2 \in |\Delta|$. Then

$$v(a_m) + \langle m, x\rangle = \lambda(v(a_m) + \langle m, u_1\rangle) + (1-\lambda)(v(a_m) + \langle m, u_2\rangle) \to \infty \text{ as } |m| \to \infty.$$

Hence $g \in \mathcal{O}(U_{\sigma,K})$. So we have proven (3.2). Thus, the restriction map

$$\mathcal{O}(U_{\sigma,K}) \to \mathcal{O}(X_{\Delta,K})$$

is an isomorphism and induces an isomorphism of affinoid spaces

$$\mathrm{Sp}(\mathcal{O}(X_{\Delta,K})) \cong \mathrm{Sp}(\mathcal{O}(U_{\sigma,K})) = U_{\sigma,K}.$$

If $X_{\Delta,K}$ is affinoid, then $\mathrm{Sp}(\mathcal{O}(X_{\Delta,K})) = X_{\Delta,K}$. Thus $X_{\Delta,K} \cong U_{\sigma,K}$. But then $|\Delta| = \sigma$, and $|\Delta|$ is convex. □

## 3.2 Subdivisions and Admissible Formal Blowing Ups

Now, let $\Delta$ be a polytopal complex, and let $\Delta'$ be a polytopal complex which subdivides $\Delta$. Then the atlas $(X_{\sigma,K})_{\sigma\in\Delta'}$ also yields a formal analytic structure on $\bigcup_{\sigma\in\Delta'} X_{\sigma,K}$ which is finer than the one given by $\Delta$. Let $X_{\Delta'}$ be the formal scheme associated to $\Delta'$, then we get a canonical morphism $X_{\Delta'} \to X_\Delta$ which acts as the identity on the generic fibre.

**Proposition 3.2.1.** *Let $U_\sigma$ be the affine formal subdomain corresponding to a polytope $\sigma$, and let $\Delta$ be a subdivision of $\sigma$. Then the morphism $X_\Delta \to U_\sigma$ is proper.*

*Proof.* This result follows directly from the general result that a morphism of admissible formal schemes is proper if and only if the induced rigid-analytic map on the generic fibre is proper; cf. [25]. As the map on the generic fibre is the identity, and hence proper, the assertion is trivial. However, the result can also be verified easily by applying the properness criterion for maps of toric varieties.

We need to show that the induced morphism $\tilde{X}_\Delta \to \tilde{U}_\sigma$ on the special fibres is proper. By [17, Cor. 5.4.5], it is enough to find a family of closed subsets $\{\tilde{Y}_u\}$ of $\tilde{U}_\sigma$ such that, for any vertex $u$ of $\Delta$, the morphism $\tilde{X}_\Delta \to \tilde{U}_\sigma$ restricts to a proper morphism $\tilde{X}_{\Delta,u} \to \tilde{Y}_u$.

Fix a vertex $u$ of $\Delta$, and let $\tilde{X}_{\Delta,u}$ be the corresponding irreducible component. If $u$ is also a vertex of $\sigma$, then $u$ corresponds to an irreducible component $\tilde{U}_{\sigma,u}$ of $\tilde{U}_\sigma$. In that case, the morphism $\tilde{X}_{\Delta,u} \to \tilde{U}_{\sigma,u}$ is proper by Proposition 2.2.8, as the fans at $u$ have the same support.

However, if $u$ is not a vertex of $\sigma$, we can not apply Proposition 2.2.8 directly. Thus, we need to find a suitable closed toric subvariety $\tilde{Y}_u$ in $\tilde{U}_\sigma$ such that $\tilde{X}_{\Delta,u}$ is mapped into $\tilde{Y}_u$.

Let $\tau$ be the unique face of $\sigma$ such that $u$ is contained in $\mathrm{relint}(\tau)$. Then the image of $\tilde{X}_{\Delta,u}$ is contained in the orbit closure $V(\tau) := \overline{O}_\tau$, which is closed in $\tilde{U}_\sigma$.

Let $u'$ be a vertex of $\tau$, then $V(\tau)$ is a closed toric subvariety of $\tilde{U}_{\sigma,u'}$. Let $N$ denote the underlying lattice of $\tilde{U}_{\sigma,u'}$, and let $N_\tau$ denote the sublattice of $N$ which is generated by $\tau \cap N$ as a group, and let $N(\tau) = N/N_\tau$ denote the quotient lattice. For every $\tau' \in \mathrm{star}(\tau)$, let $C_{\tau',u'}$ be the cone over $\tau' - u'$ in $N$. Let $\overline{C}_{\tau',u'}$ denote the image of $C_{\tau',u'}$ in $N(\tau) \otimes_{\mathbb{Z}} \mathbb{R}$. Let

$$\overline{\mathrm{star}(\tau)} := \{\overline{C}_{\tau',u'}\ ;\ \tau' \in \mathrm{star}(\tau)\};$$

this defines a fan of cones in $N(\tau)$. As detailed in Section 2.2, this is the fan that gives the toric variety $V(\tau)$. Note that this does not depend on the actual choice of the vertex $u'$ of $\tau$. Now, let $N'$ be the underlying lattice of $\tilde{X}_{\Delta,u}$. Then there is a natural isomorphism of lattices $N' \cong N$, which induces a natural epimorphism $\varphi : N' \twoheadrightarrow N(\tau)$. Now, let $\tau' \in \mathrm{star}(u)$ with associated cone $C_{\tau',u}$ in $N'$. Let $\tau''$ be a face of $\sigma$ such that $\tau' \subset \tau''$. Then $\varphi$ maps $C_{\tau',u}$ into $\overline{C}_{\tau'',u'}$. So $\varphi$ maps the fan defining $\tilde{X}_{\Delta,u}$ into the fan $\overline{\mathrm{star}(\tau)}$. This is exactly the map of fans defining the morphism $\tilde{X}_{\Delta,u} \to V(\tau)$. On the other hand, if $\tau'' \in \mathrm{star}(\tau)$, we see easily that $\varphi^{-1}(\overline{C}_{\tau'',u'})$ consists of exactly those $C_{\tau',u}$ with $\tau' \in \mathrm{star}(u) \subset \Delta$ such that $\tau' \subset \tau''$. But this is exactly the properness criterion for maps of toric varieties; cf. Proposition 2.2.8. This proves the claim. □

In the following, we will determine under which conditions the above proper morphism is in fact a blowing up. For any monomial $\chi = a\zeta^m$, let $f_\chi(x) := \langle m, x\rangle + v(a)$ be the associated affine linear function.

**Definition 3.2.2.** Let $X = X_\sigma$ be an affine formal polytopal domain with generic fibre $X_K$ and special fibre $\tilde{X}$. A *monomial ideal* of $\mathcal{O}(X_K)^\circ$ is an $\mathcal{O}(X_K)^\circ$-submodule $I$ of $\mathcal{O}(X_K)^\circ$ which is generated by a finite number of monomials $\chi_i = a_i\zeta^{m_i} \in \mathcal{O}(X_K)^\circ$ with $a_i \in K$, $m_i \in \mathbb{Z}$.

Similarly, a *fractional monomial ideal* of $\mathcal{O}(X_K)^\circ$ is a $\mathcal{O}(X_K)^\circ$-submodule $I$ of the field of fractions $\mathcal{M}(X_K)$ of $\mathcal{O}(X_K)^\circ$ generated by finitely many monomials $\chi_i$.

If $X$ is a formal polytopal domain associated to a polytopal complex $\Delta$, then a *(fractional) monomial ideal* on $X$ is a sheaf of modules $\mathcal{I}$ on $X$ whose restriction $\mathcal{I}_\sigma := \mathcal{I}|_{X_\sigma}$ to any $X_\sigma$ for $\sigma \in \Delta$ is of the form $\mathcal{I}_\sigma = I_\sigma^\Delta$ for a (fractional) monomial ideal $I_\sigma$ of $\mathcal{O}(X_K)^\circ$.

A fractional monomial ideal $I$ of $\mathcal{O}(X_K)^\circ$ is *complete*, if it is integrally closed in $\mathcal{M}(X_K)$; i.e. if any $f \in \mathcal{M}(X_K)$ satisfying a relation

$$f^r + a_1 f^{r-1} + \cdots + a_r = 0, \quad a_i \in \mathcal{I}^i$$

satisfies $f \in \mathcal{I}$. The *completion* of $I$ is the integral closure of $I$ in $\mathcal{M}(X_K)$.

If $f_1, \ldots, f_r$ are affine linear functions on $\sigma$, then we can define a fractional monomial ideal $I$ of $\mathcal{O}(X_K)^\circ$ by setting

$$I_{(f_1,\ldots,f_r)} := (\chi_1, \ldots, \chi_r)\mathcal{O}(X_K)^\circ,$$

where $\chi_i := \chi_{f_i}$ is the monomial corresponding to $f_i$ as in Section 3.1. Recall that $\chi_i$ is only unique up to multiplication by a unit in $\mathcal{O}(X_K)^\circ$; however, this does not change the ideal defined above. By abuse of notation, we call $\{f_1, \ldots, f_r\}$ a *generating set* of $I$.

Note that any fractional monomial ideal $I$ is automatically open. Namely, as $\sigma$ is compact and every $f_i$ is continuous, there exists a $c \in \Gamma$, $c > 0$ such that $c \geqslant f_i$ on $\sigma$. Choosing $t \in R$ with $v(t) = c$, we have $t/\chi_{f_i} \in \mathcal{O}(X_K)^\circ$, and hence, $t \in I$.

**Lemma 3.2.3.** *Let $X = X_\sigma$ be an affine formal polytopal domain, and let $I$ be a fractional monomial ideal generated by $f_1, \ldots, f_r$ as above, and let $I'$ denote the completion of $I$. Then*

$$I' = \widehat{\bigoplus} R \cdot \chi \subset \mathcal{M}(X),$$

*where the topological sum runs through all $\chi$ such that $f_\chi(x) \geqslant \min_i f_i(x)$ for all $x \in \sigma$*

*Proof.* Let $\chi$ be a monomial with $f_\chi(x) \geqslant \min_i f_i(x)$ on $\sigma$. Let $\tau$ denote the $\Gamma$-rational cone in $\mathbb{R}^r$ generated by the elements

$$(f_\chi(x) - f_1(x), \ldots, f_\chi(x) - f_r(x)), \quad x \in \sigma.$$

This cone contains no vector $y \leqslant 0$, $y \neq 0$. Hence, the dual cone $\tau^\vee$ contains a vector

$\lambda \geqslant 0$ , $\lambda \neq 0$ . We may choose $\lambda \in \mathbb{Z}^n$ . Thus, the function

$$g := \sum_i \lambda_i (f_\chi - f_i)$$

satisfies $g \geqslant 0$ on $\sigma$ , which yields a relation

$$\lambda f_\chi = g + \sum_i \lambda_i f_i,$$

with $\lambda := \sum_i \lambda_i$ . Defining $\chi_g$ accordingly, we have $\chi_g \in \mathcal{O}(X_K)^\circ$ . This in turn yields an integral relation over $I$ :

$$\chi^\lambda = \chi_g \cdot \prod_i \chi_{f_i}^{\lambda_i}$$

Thus, $\chi \in I'$ .

For the converse, let $\chi \in I'$ . Then $\chi$ satisfies some integral relation

$$\chi^t + g_1 \chi^{t-1} + \cdots + g_t = 0, \quad g_i \in I^i.$$

For $x \in X_K$ , we have the inequality

$$|\chi(x)| \leqslant \max_i |g_i(x)|^{1/i}.$$

Thus, there exists an $1 \leqslant N \leqslant t$ such that $|\chi(x)|^N \leqslant |g_N(x)|$ . We write

$$g_N = \sum_{\substack{i, m_1, \ldots, m_r, m_j \geqslant 0 \\ m_1 + \cdots + m_r = N}} \chi_{i,m} \cdot \chi_1^{m_1} \cdot \ldots \cdot \chi_r^{m_r}$$

with $\chi_{i,m} \in \mathcal{O}(X_K)^\circ$ . Hence, there exist $m_1, \ldots, m_r$ with $m_1 + \cdots + m_r = N$ , such that

$$|g_N(x)| \leqslant |\chi_1(x)|^{m_1} \cdot \ldots \cdot |\chi_r(x)|^{m_r}.$$

Setting $f := f_\chi$ , and $u := \mathrm{val}(x) \in \sigma$ , we have

$$N \cdot f(u) \geqslant m_1 f_1(u) + \cdots + m_r f_r(u) \geqslant N \cdot \min_i f_i(u).$$

Dividing by $N$ yields $f(u) \geqslant \min_i f_i(u)$ , which proves the claim. □

**Example 3.2.4.** For a non-complete monomial ideal $I$ , consider the following example: Let

$X_K := \{|\pi| \leqslant |z| \leqslant 1\}$, and let $I := (z^3, \pi^3/z^3)$. Then $\pi^2 \notin I$; but $\pi^2$ is integral over $I$; namely, we have $(\pi^2)^2 = \pi \cdot z^3 \cdot (\pi^3/z^3)$. For any $z \in X_K$, we have either $|z^3| \leqslant |\pi^2|$ or $|\pi^3/z^3| \leqslant |\pi^2|$. Thus, $\pi^2 \in I'$, where $I'$ is given as above.

As above, let $I$ be generated by $f_1, \ldots, f_r$. Consider a polytopal subdivision $\Delta$ of $\sigma$, with maximal polytopes $\sigma_1, \ldots, \sigma_r$ given as follows:

$$\sigma_i := \sigma \cap \{x : f_j(x) \geqslant f_i(x) \text{ for all } j \neq i\}.$$

This is the unique minimal subdivision of $\sigma$ such that $f(x) := \min_i f_i(x)$ is a strictly convex polyhedral function on $\Delta$. Conversely, any such subdivision $\Delta$ of $\sigma$, together with a strictly convex polyhedral function $f$ defines a complete fractional monomial ideal $I_f$; it is the completion of the ideal $I_{f_1,\ldots,f_r}$. This yields the following result:

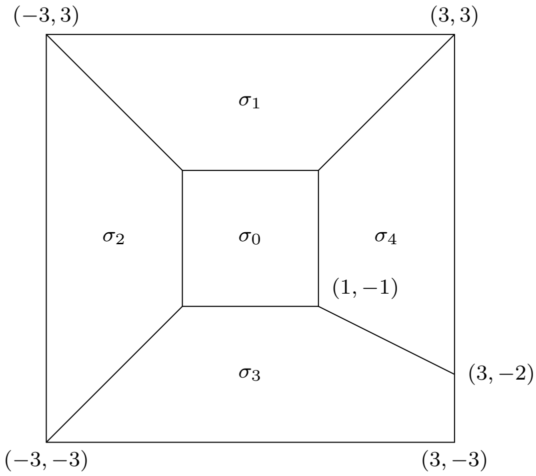
**Lemma 3.2.5.** *Let $X = X_\sigma$ be an affine formal polytopal domain. There is a one-to-one correspondence between*

(i) *complete fractional monomial ideals of $\mathcal{O}(X_K)^\circ$,*

(ii) *pairs $(\Delta, f)$ where $\Delta$ is a polytopal subdivision of $\sigma$ and $f$ is a strictly convex polyhedral function on $\Delta$.*

Recall that a polytopal subdivision $\Delta$ of $\sigma$ is *projective* if there exists a strictly convex piecewise linear function $f$ on $\Delta$. The reason for the term *projective* will become clear in the following section; we will see that the reduction $\tilde{X}_\Delta$ for a polytopal complex $\Delta$ can be embedded into projective space if and only if $\Delta$ is projective in the above sense.

**Example 3.2.6.** For a polytopal complex $\Delta$ which is not projective, consider the complex $\Delta$ given by the following picture:




There exists no strictly convex polyhedral function $f$ on $\Delta$. Namely, assume that $f$ is a convex polyhedral function on $\Delta$. We may assume without loss of generality that $f = 0$ on $\sigma_0$. Then there exist constants $a_1, \ldots, a_4 \geqslant 0$ such that

$$f|_{\sigma_1} = a_1 \cdot (1 - x_2), \quad f|_{\sigma_2} = a_2 \cdot (x_1 + 1),$$
$$f|_{\sigma_3} = a_3 \cdot (x_2 + 1), \quad f|_{\sigma_4} = a_4 \cdot (1 - x_1).$$

The coefficients $a_i$ have to be chosen such that these settings agree on overlaps. This yields the equation

$$a_4 = a_1 = a_2 = a_3 = 2a_4,$$

which has only the trivial solution. Thus, $f = 0$ on $|\Delta|$, so $f$ is not strictly convex.

From Lemma 3.2.5, we derive the following global version:

**Proposition 3.2.7.** *Let $\Delta$ be a polytopal complex, $X_\Delta$ the corresponding formal polytopal domain. Then there is a one-to-one correspondence between*

(i) *complete fractional monomial ideal sheaves on $X_\Delta$,*

(ii) *pairs $(\Delta', f)$, where $\Delta'$ is a polytopal subdivision of $\Delta$ and $f$ is a polyhedral function on $\Delta'$ which is strictly convex on $\Delta' \cap \sigma$ for every $\sigma \in \Delta$, where $\Delta' \cap \sigma$ is the subdivision of $\sigma$ induced by $\Delta'$.*

*Proof.* It is clear from Lemma 3.2.5 that any pair $(\Delta', f)$ as in (ii) defines a complete fractional monomial sheaf of ideals $\mathcal{I}_\sigma$ on every $\sigma \in \Delta$. From Lemma 3.2.3, we see that $\mathcal{I}_\sigma$

agrees with $\mathcal{I}_\tau$ on $\sigma \cap \tau$, as $f$ is continuous. Thus, all $\mathcal{I}_\sigma$ can be glued together to a fractional monomial sheaf of ideals $\mathcal{I}$.

Conversely, let $\mathcal{I}$ be a complete fractional monomial sheaf of ideals. Locally on $\sigma \in \Delta$, the ideal $\mathcal{I}$ induces a polytopal subdivision $\Delta'_\sigma$ of $\sigma \in \Delta$ together with a strictly convex polyhedral function $f_\sigma$ on $\Delta'_\sigma$. We only have to check that these $\Delta'_\sigma$ give rise to a polytopal subdivision $\Delta'$ of $\Delta$. This means that $\Delta'_\sigma$ and $\Delta'_\tau$ induce the same subdivision of $\sigma \cap \tau$. But $\sigma \cap \tau$ is again a polytope in $\Delta$ with a subdivision $\Delta'_{\sigma\cap\tau}$, and it is obvious that this subdivision is the restriction both of $\Delta'_\sigma$ and $\Delta'_\tau$, as $\sigma \cap \tau$ is a face of both $\sigma$ and $\tau$. Again, by Lemma 3.2.3, we see that $f_\sigma$ agrees with $f_\tau$ on $\sigma \cap \tau$, thus there is a polyhedral function $f$ on $\Delta'$ such that $f|_\sigma = f_\sigma$. □

## 3.3 Cartier Divisors, Line Bundles and Polyhedral Functions

In the following, we will describe the relationship between Cartier divisors and polyhedral functions. Let $\Delta$ be a polytopal complex, and let $X_\Delta$ be the corresponding formal polytopal domain. For simplicity, we assume that $\Delta$ is of pure dimension $n$. For a maximal polytope $\sigma_i$ of $\Delta$, let $X_i := X_{\sigma_i}$ be the corresponding affine formal polytopal domain.

Now, let $f$ be a polyhedral function on $\Delta$, then $f$ is associated to a complete fractional monomial principal ideal on $X_\Delta$ which we denote by $\mathcal{L}_f$. On the other hand, $f$ defines a formal Cartier divisor on $X$ by choosing $\chi_{-f_i}$ as local equation on $X_i$, where $f$ is given on $\sigma_i$ by the affine linear function $f_i$. Let $D_f$ denote this Cartier divisor. By construction, $\mathcal{L}_f = \mathcal{O}(D_f)$.

**Lemma 3.3.1.** *Let $D$ be a Cartier divisor on $X$ given by a polyhedral function $f$ as above. Then $D$ is trivial on $X_K$, and the vertical part of the Weil divisor associated to $D$ is given by*

$$\mathrm{cyc}_v(D) = \sum -f(u) \cdot \tilde{X}_u,$$

*where $u$ runs through the vertices of $\Delta$ and $\tilde{X}_u$ is the corresponding irreducible component of $\tilde{X}$.*

*Proof.* It is easy to see that $D_K = 0$, as every monomial $\chi_i$ is a unit in $\mathcal{O}(X_{\sigma_i,K})$. For the vertical cycle, we fix a vertex $u$. Let $\tilde{X}_u$ be the corresponding irreducible component, and let $O_u$ denote the torus orbit corresponding to $u$. Then $O_u$ is open and affine in $\tilde{X}_u$ and does not meet any other irreducible component. Moreover, it is the image of $\mathrm{val}^{-1}(u)$

under the reduction map $\pi$ , and $D$ is given on $O_u$ by $\chi_{(-f_\sigma)}$ , where $\sigma$ is any polytope in $\mathrm{star}(u)$ , and $f_\sigma = f|_\sigma$ . Then the order of $D$ in $\tilde{X}_u$ is given by

$$\mathrm{ord}(D, \tilde{X}_u) = -\log\left|\chi_{(-f_\sigma)}(\tilde{X}_u \cap O_u)\right| = -f(u).$$

This proves the claim. □

Moreover, we will see in the following proposition that all Cartier divisors with trivial horizontal part and trivialization $(X_\sigma)_{\sigma\in\Delta}$ arise this way.

**Lemma 3.3.2.** *Let $X$ be a formal polytopal domain corresponding to a polytopal complex $\Delta$ with generic fibre $X_K$ . Then there is a one-to-one correspondence between polyhedral functions $f$ on $\Delta$ and formal Cartier divisors $D$ on $X$ with trivialization $(X_\sigma)_{\sigma\in\Delta}$ and $D_K = 0$ on $X_K$ . If $f$ is a polyhedral function given by $f_\sigma$ on $\sigma$ , then $D_f$ is given by the equation $\chi_{(-f_\sigma)}$ on $X_\sigma$ .*

*Proof.* We have already seen above that every polyhedral function $f$ induces a Cartier divisor $D_f$ as claimed. For the converse, let $D$ be a formal Cartier divisor which is given by a rational function (not necessarily a polynomial) $g_\sigma$ on $X_\sigma$ . As $D$ is trivial on the generic fibre, $g_\sigma$ is in fact a unit in $\mathcal{O}(X_{\sigma,K})$ . But by Lemma 3.1.1, there is a monomial $\chi_\sigma$ such that $|g_\sigma(x)| = |\chi_\sigma(x)|$ for all $x \in X_{\sigma,K}$ . We may thus assume that $D$ is given by monomials $\chi_\sigma$ ; hence $\mathcal{O}(-D)$ is a fractional monomial sheaf of ideals generated by $\chi_\sigma$ on $X_\sigma$ , which gives rise to a polyhedral function $f_D$ as claimed. Obviously, $f \mapsto D_f$ and $D \mapsto f_D$ are inverse to each other. This proves the claim. □

If $\mathcal{I}$ is a fractional monomial sheaf of ideals given by a polytopal subdivision $\Delta'$ and $f$ where $f \geqslant 0$ , then $\mathcal{I}$ is in fact an ordinary sheaf of ideals; by Lemma 3.2.3, its sections over each $X_\sigma$ form a submodule of $\mathcal{O}(X_\sigma)$ . In this situation, the polytopal subdivision $\Delta'$ has an interpretation in terms of admissible formal blowing ups as follows:

**Proposition 3.3.3.** *Let $\Delta$ be a polytopal complex with associated formal polytopal domain $X_\Delta$ . Let $\Delta'$ be a subdivision of $\Delta$ induced by a monomial sheaf of ideals $\mathcal{I}$ . Then the canonical morphism $X_{\Delta'} \to X_\Delta$ is the normalization of the admissible formal blowing up of $\mathcal{I}$ on $X$ .*

*Proof.* It suffices to check this locally for $X = X_\sigma$ , $\sigma \in \Delta$ . Let $\sigma_1, \ldots, \sigma_r$ be the polytopes in $\Delta'$ subdividing $\sigma$ . By [9, Lemma 2.2], the $i$ -th patch of the admissible formal blowing up is given by $\mathrm{Spf}(A_i)$ , where

$$A_i = \mathcal{O}(X_\sigma)\left\langle \chi_j/\chi_i, j \neq i\right\rangle/(\chi_i\text{-torsion}).$$

At first, we claim $A_i = B_i$, where

$$B_i := R\langle\{\chi_m;\ m \in S_\sigma\} \cup \{\chi_j/\chi_i;\ j \neq i\}\rangle.$$

The generating system of $B_i$ satisfies relations of the form

$$a\chi_1^{b_1} \cdots \chi_r^{b_r} = a'\chi_{r+1}^{b_{r+1}} \cdots \chi_s^{b_s},$$

where $a = 1$ or $a' = 1$, and either $\chi_k = \chi_j/\chi_i$ for a certain $i$, or $\chi_k = \chi_m$ for a certain $m \in S_\sigma$. By multiplying with $\chi_i^N$ for $N$ large enough, we get a relation which holds in $\mathcal{O}(X_\sigma)$. After dividing out the $\chi_i$-torsion, we see that the original relation comes from a relation in $A_i$. Conversely, every relation in $A_i$ comes from a relation in $B_i$. Moreover, as $|\chi_j/\chi_i| \leqslant 1$ on $\sigma_i$ by definition, we see that $A_i \subset \mathcal{O}(X_{\sigma_i})$. By Lemma 3.1.8, $\mathcal{O}(X_{\sigma_i})$ is normal.

It remains to show that $\mathcal{O}(X_{\sigma_i})$ is in fact the normalization of $A_i$. Let $\chi \in \mathcal{O}(X_{\sigma_i})$ be a monomial, and let $f := f_\chi$. For any point $x \in \sigma$; we have either $f(x) \geqslant 0$ if $x \in \sigma_i$, or $f_i(x) - f_j(x) \geqslant 0$, if $x \in \sigma_j$. Thus, as in the proof of Lemma 3.2.3 there exist integers $d > 0$, $\lambda_1, \ldots, \lambda_r \geqslant 0$, such that

$$g := df + \sum_j \lambda_j(f_i - f_j) \geqslant 0 \text{ on } \sigma.$$

Hence,

$$\chi^d = \chi_g \prod_j (\chi_j/\chi_j)^{\lambda_j} \in A_i$$

for a suitably chosen $\chi_g$ with $|\chi_g| \leqslant 1$ on $\sigma$, and $\chi$ is integral over $A_i$. □

Note that the converse of Proposition 3.3.3 is not necessarily true: Not every polytopal subdivision of $\Delta$ comes from a blowing up, as not every polytopal complex $\Delta'$ allows a strictly convex polyhedral function $f$; see Example 3.2.6 However, this is true after possibly refining the subdivision $\Delta'$.

**Proposition 3.3.4.** *Let $\Delta$ be a polytopal complex with associated formal polytopal domain $X_\Delta$. Let $\Delta'$ be a subdivision of $\Delta$, and let $X_{\Delta'}$ be the corresponding formal polytopal domain. Then there exists a subdivision $\Delta''$ of $\Delta'$ such that both morphisms $X_{\Delta''} \to X_\Delta$ and $X_{\Delta''} \to X_{\Delta'}$ are normalization of admissible formal blowing ups of monomial ideal sheaves.*

*Proof.* At first, consider the case where $X := X_\Delta$ is affine; i.e. $\Delta$ consists only of a polytope $\sigma$ plus its faces. Let $\Delta'$ be a decomposition of $\sigma$, and let $X' := X_{\Delta'}$ be the corresponding formal polytopal domain. Fix a polytope $\tau \in \Delta'$, and let $U' := U'_\tau$ be the corresponding affine formal subscheme of $X'$; let $U'_K$ be its generic fibre. In the first step, we want to construct a projective subdivision $\Delta'$ of $\Delta$ which contains $\tau$.

The polytope $\tau$ is given as a subset of $\sigma$ by linear inequalities $f_i \geqslant 0$, $i = 1, \ldots, r$, where $f_i$ is an affine linear function. Then $U'_K$ is the rational subdomain of $X_K$ which is given by $|\chi_{f_i}| \leqslant 1$. We choose an affine linear function $f_0$ such that $f_0 \geqslant 0$ on $\sigma$ and $f_0 > 0$ on $\tau$. For $n > 0$ large enough, we will get $f_i + nf_0 \geqslant 0$ on $\sigma$ for all $i$. We set $\chi_0 := \chi_{nf_0}$, $\chi_i := \chi_{f_i} \cdot \chi_0$ for $i = 1, \ldots, r$. Then $\chi_0, \ldots, \chi_r \in \mathcal{O}(X)$, and $U'_K$ is a rational subdomain of $X_K$ given by $U'_K = X_K(\chi_1/\chi_0, \ldots, \chi_r/\chi_0)$. Let $\mathcal{I}_\tau \subset \mathcal{O}_X$ denote the completion of the monomial ideal generated by $\chi_0, \ldots, \chi_r$, and let $X_\tau \to X$ be the corresponding blowing up of $\mathcal{I}$. By [4, §2.6, Prop. 7], $X_\tau$ has an open affine covering $U_{\tau,j} := \mathrm{Spf}(A_{\tau,j})$, where

$$A_{\tau,j} := A\langle \chi_i/\chi_j, i \neq j\rangle/(\chi_j - \text{torsion})$$

and $A := \mathcal{O}(X)$. For $j \neq 0$, $U_{\tau,j}$ is an affine formal polytopal domain, corresponding to a polytope $\tau'_j$ which is given as a subset of $\sigma$ by the inequalities $f_i(x) \geqslant f_j(x)$, $i = 1, \ldots, r$. By construction, $\tau'_0 = \tau$. Thus, $\Delta''_\tau = \{\tau'_0 = \tau, \tau'_1, \ldots, \tau'_n\}$ defines a projective subdivision of $\Delta$ containing $\tau$.

Repeating this for every polytope $\tau \in \Delta'$, we can construct complete monomial ideals $\mathcal{I}_{\tau_1}, \ldots, \mathcal{I}_{\tau_r}$ with corresponding projective subdivisions $\Delta''_{\tau_1}, \ldots, \Delta''_{\tau_r}$. The intersection

$$\Delta'' := \Delta''_{\tau_1} \cap \cdots \cap \Delta''_{\tau_r} = \{\rho_1 \cap \cdots \cap \rho_r \; ; \; \rho_i \in \Delta''_{\tau_i}\}$$

is then again a projective subdivision corresponding to blowing up the complete monomial ideal

$$\mathcal{J} := \mathcal{I}_{\tau_1} \cdot \ldots \cdot \mathcal{I}_{\tau_r}.$$

Thus, we have proven the claim in the case that $X_\Delta$ is affine.

Now, let $X = X_\Delta$ be a formal polytopal domain associated to an arbitrary polytopal complex $\Delta$. On every $\sigma \in \Delta$, we have a complete monomial ideal $\mathcal{J}_\sigma$ determining a blowing up $X'_\sigma \to X_\sigma$ and a projective subdivision $\Delta''_\sigma$ of $\Delta' \cap \sigma$. By Lemma 3.3.5, we can enlarge $\mathcal{J}_\sigma$ to a complete monomial sheaf of ideals $\bar{\mathcal{J}}_\sigma$ on $X$. Then the sheaf of ideals $\prod_{\sigma\in\Delta} \bar{\mathcal{J}}_\sigma$ induces a subdivision $\Delta''$ of $\Delta$ and a blow up $X_{\Delta''} \to X_\Delta$. Obviously, the sheaf of ideals

$\prod \bar{\mathcal{J}}_\sigma$ pulls back to an sheaf of ideals on $X_{\Delta'}$. As $\Delta''$ subdivides $\Delta'$ by construction, the pull back induces exactly the subdivision $\Delta''$ of $\Delta'$ and thus $X_{\Delta''} \to X_{\Delta'}$ is also a blow up. □

**Lemma 3.3.5.** *Let $\mathcal{I}_\sigma$ be a monomial sheaf of complete ideals on $X_\sigma$ for $\sigma \in \Delta$. There exists a monomial sheaf of complete ideals $\bar{\mathcal{I}}$ on $X_\Delta$ such that $\bar{\mathcal{I}}|_{X_\sigma} = \mathcal{I}_\sigma$.*

*Proof.* Let $\mathcal{I}_\sigma$ be generated by $f_1, \ldots, f_r$ on $X_\sigma$. For $\tau \in \Delta$, we take $\mathcal{I}_\tau$ to be the completion of the monomial ideal on $X_\tau$ generated by all affine linear $f$ with $f \geqslant 0$ such that $f(x) \geqslant \min_i f_i(x)$ for all $x \in \tau \cap \sigma$. If $\tau \cap \sigma = \emptyset$, then $\mathcal{I}_\tau = \mathcal{O}(X_\tau)$. Obviously, these monomial ideals agree on intersections and thus can be glued together. □

## 3.4 Strictly Semi-Stable Formal Models

In this section, assume that $R$ is a discrete valuation ring with uniformizing parameter $\pi$, and set $v(\pi) := 1$.

Recall that an $n$-*simplex* is the convex hull of $n+1$ affinely independent vertices $u_0, \ldots, u_n$ with coordinates in $\Gamma$. Assume that $\sigma$ is a $1/e$-rational simplex; i.e. $u_i \in 1/e \cdot \mathbb{Z}^n$ for all vertices $u_i$, $i = 1, \ldots, n$. We define the *multiplicity of $\sigma$ (with respect to $c$)* as

$$m(\sigma, e) := e^n n! \operatorname{vol}(\sigma) = e^n \cdot |\det(u_1 - u_0, \ldots, u_n - u_0)|.$$

Note that the multiplicity does not depend on the choice of the vertex $u_0$.

**Lemma 3.4.1.** *Let $\sigma$ be an $n$-simplex with vertices in $\mathbb{Z}^n$ such that $m(\sigma, 1) = 1$. Then*

$$X_\sigma = \operatorname{Spf} R\langle \zeta_0, \ldots, \zeta_n \rangle / (\zeta_0 \cdot \ldots \cdot \zeta_n - \pi).$$

*Especially, $X_\sigma$ is strictly semi-stable.*

*Proof.* If $m(\sigma, 1) = 1$, then $\{u_i - u_0 \ ; \ i = 1, \ldots, n\}$ is a basis of $\mathbb{Z}^n$. After a change of coordinates, we may assume $u_0 = 0$ and $u_i = e_i$. This simplex is given by the inequalities $x_i \geqslant 0$ for $i = 1, \ldots, n$, $x_1 + \cdots + x_n \leqslant 1$. Thus, we have

$$\mathcal{O}(X_\sigma) = R\langle \zeta_1, \ldots, \zeta_n, \pi \cdot (\zeta_1 \cdots \zeta_n)^{-1} \rangle.$$

Setting $\zeta_{n+1} := \pi \cdot (\zeta_1 \cdots \zeta_n)^{-1}$ proves the claim. □

Now, for the converse:

**Lemma 3.4.2.** *Let* $\sigma$ *be a polytope such that* $X = X_\sigma$ *is strictly semi-stable. Assume that* $\dim \sigma = \dim X_\sigma$ *. Then* $\sigma$ *is a simplex with vertices in* $\mathbb{Z}^n$ *and* $m(\sigma, 1) = 1$ *.*

*Proof.* At first, assume that $\sigma$ is not a simplex. Then we choose two distinct vertices $u$ , $v$ of $\sigma$ which are not connected by an edge. Let $\tau$ be the unique smallest face containing both $u$ and $v$ . Then $\dim \tau > 1$ . Let $\tilde{X}_u$ , $\tilde{X}_v$ be the irreducible components of $\tilde{X}$ corresponding to $u$ and $v$ respectively. Then $\tilde{X}_u \cap \tilde{X}_v$ has dimension $n - \dim \tau$ , which is strictly smaller than $n-1$ . This contradicts the strict semi-stability.

Now, let $\sigma$ be generated by affine independent vectors $u_0, \dots, u_n$ . Without loss of generality, we may assume $u_0 = 0$ . As $X$ is strictly semi-stable, all irreducible components $\tilde{X}_i := \tilde{X}_{u_i}$ are smooth. We start with $\tilde{X}_0$ . Let $u_i'$ be the first lattice point in $\mathbb{Z}^n$ along the ray generated by $u_i$ . As $\tilde{X}_0$ is smooth, $u_1', \dots, u_n'$ is a basis of $\mathbb{Z}^n$ due to Proposition 2.2.5. We may therefore perform a change of coordinates such that $u_i = \lambda_i e_i$ for some $\lambda_i \in \mathbb{N}$ . It remains to show that $\lambda_1 = \cdots = \lambda_n = 1$ .

Now, let $j > 0$ . The cone belonging to the lattice point $u_j$ is generated over $\mathbb{R}$ by

$$-\lambda_j e_j, \lambda_1 e_1 - \lambda_j e_j, \dots, \lambda_n e_n - \lambda_j e_j.$$

As above, $\tilde{X}_{u_j}$ is smooth, so the first lattice points along the rays generating the cone form a basis of $\mathbb{Z}^n$ . These lattice points are given by

$$-e_j, (\lambda_1/d_{1j})e_1 - (\lambda_j/d_{1j})e_j, \dots, (\lambda_n/d_{nj})e_n - (\lambda_j/d_{nj})e_j,$$

where $d_{ij} = \gcd(\lambda_i, \lambda_j)$ . The determinant of the matrix whose columns are given by these vectors is calculated as $\prod_{i \neq j} \lambda_i/d_{ij}$ . This equals $1$ if and only if $\lambda_i = d_{ij}$ for all $i, j$ . But then $\lambda := \lambda_1 = \cdots = \lambda_n \in \mathbb{N}$ . Thus, $\sigma$ is given by the inequalities $x_i \geqslant 0$ for $i = 1, \dots, n$ , and $x_1 + \cdots + x_n \leqslant \lambda$ . As in the proof of the above Lemma, $X$ is given by

$$X = \operatorname{Spf} R\langle \zeta_0, \dots, \zeta_n \rangle / (\zeta_0 \cdot \ldots \cdot \zeta_n - \pi^\lambda).$$

However, for $\lambda > 1$ , the ideal corresponding to the $i$ -th irreducible component $\tilde{X}_i$ of the special fibre is generated by the two elements $\pi, \zeta_i$ , and hence, $\tilde{X}_i$ is no Cartier divisor. Thus, we have $\lambda = 1$ , which proves the claim. $\square$

**Definition 3.4.3.** We call

$$\operatorname{Sp} K\langle \zeta_0, \dots, \zeta_r \rangle (\zeta_0 \cdot \ldots \cdot \zeta_r - \pi)$$

the *affinoid standard* $r$*-simplex.*

Now, let $\Delta$ be a polytopal complex in $\mathbb{R}^n$. From the above, we see that a formal polytopal domain $X_\Delta$ is strictly semi-stable if and only if every maximal polytope $\sigma \in \Delta$ is isomorphic to the standard simplex. In general, by Proposition 2.4.1, there exists a projective subdivision $\Delta'$ of $\Delta$ which is $1/e$-rational, such that every simplex $\tau \in \Delta'$ is a simplex with multiplicity $m(\tau, e) = 1$ with respect to the lattice $1/e \cdot \mathbb{Z}^n$. Thus, the formal polytopal domain $X'_\Delta$ is defined over $R' := R[\sqrt[e]{e}]$, and strictly-semistable over $R'$. This yields the following result:

**Proposition 3.4.4.** *Let $R$ be a discrete valuation ring, and let $X_\Delta$ be a formal polytopal domain over $R$, where $\Delta$ is a polytopal complex in $\mathbb{R}^n$. Then there exists a finite extension $R'$ of $R$ and a subdivision $\Delta'$ of $\Delta$ such that $X'_\Delta$ is strictly semi-stable over $R'$ and $X'_\Delta \to X_\Delta \times_R R'$ is the normalization of an admissible formal blowing up.*

This yields a combinatorial interpretation for the two desingularization results in [21, §1.3]. We will illustrate this in the following:

**Proposition 3.4.5.** *For $n \geqslant 1$, $e \geqslant 1$, let*

$$A := R\langle \zeta_1, \dots, \zeta_{n+1} \rangle / (\zeta_1 \cdot \ldots \cdot \zeta_{n+1} - \pi^e)$$

*Then there exists a strictly semi-stable formal scheme $X$ over $R$ such that $X \to \operatorname{Spf} A$ is an admissible formal blowing up.*

*Proof.* The formal scheme $\operatorname{Spf} A$ corresponds to the affine formal polytopal domain given by the simplex $\sigma$ in $\mathbb{R}^n$ with vertices $0$ and $e \cdot e_i$, where $e_i$ is the $i$-th unit vector. Let $x_1, \dots, x_n$ denote the standard coordinates on $\mathbb{R}^n$. From these, we define *cumulative coordinates* $y_0, \dots, y_n$ via

$$y_0 := 0, \quad y_1 = x_1, \quad \dots \quad y_k = x_1 + \cdots + x_k, \quad \dots \quad y_n = x_1 + \cdots + x_n.$$

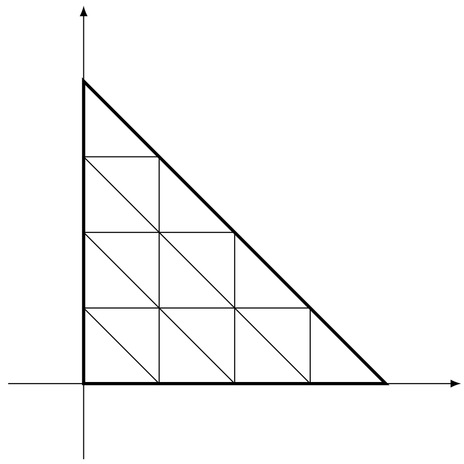


Figure 3.1: The regular subdivision for $n = 2$, $e = 4$

With respect to these coordinates, the simplex $\sigma$ is given by the inequalities

$$0 \leqslant y_1 \leqslant \cdots \leqslant y_n \leqslant e.$$

For $0 \leqslant j < i \leqslant n$ and $0 \leqslant k \leqslant e$, let $H_k^{i,j}$ denote the hyperplane given by the equation $y_i - y_j = k$. These hyperplanes define a subdivision $\Delta$ of $\sigma$, where every maximal polytope $\tau \in \Delta$ is isomorphic to the standard simplex. Thus, $X_\Delta$ is a strictly semi-stable formal scheme. Note that each hyperplane $H_k^{i,j}$ corresponds to blowing up the ideal

$$\mathcal{I}_k^{i,j} := (\zeta_{j+1} \cdot \ldots \cdot \zeta_i, \pi^k)$$

Thus, the morphism $X_\Delta \to \operatorname{Spf} A$ corresponds to blowing up the product of all these $\mathcal{I}_k^{i,j}$. □

**Remark 3.4.6.** The subdivision constructed above is called the *regular subdivision* of $\sigma$. Figure 3.1 shows the regular subdivision for $n = 2$, $e = 4$.

**Proposition 3.4.7.** *For* $r, s \geqslant 1$, *let*

$$A := R\langle \zeta_0, \ldots, \zeta_r \rangle / (\zeta_0 \cdot \ldots \cdot \zeta_r - \pi)$$
$$B := R\langle \xi_0, \ldots, \xi_s \rangle / (\xi_0 \cdot \ldots \cdot \xi_s - \pi),$$

*and let* $C := A \hat{\otimes}_R B$. *Then there is a strictly semi-stable formal scheme* $X$ *over* $R$ *such that* $X \to \operatorname{Spf} C$ *is an admissible formal blowing up.*

*Proof.* Let $\sigma_r$ denote the standard $r$-simplex in $\mathbb{R}^r$, $\sigma_s$ denote the standard $s$-simplex in $\mathbb{R}^s$. Then consider $\sigma := \sigma_r \times \sigma_s \subset \mathbb{R}^r \times \mathbb{R}^s$. We construct a suitable subdivision of $\sigma$. Let $x_1, \ldots, x_r$ resp. $y_1, \ldots, y_s$ denote the coordinates on $\mathbb{R}^r$, resp. $\mathbb{R}^s$. Again, define cumulative coordinates

$$X_1 := x_1, \quad \ldots \quad X_k := x_1 + \cdots + x_k, \quad \ldots \quad X_r := x_1 + \cdots + x_r$$
$$Y_1 := y_1, \quad \ldots \quad Y_l := y_1 + \cdots + y_l, \quad \ldots \quad Y_s := y_1 + \cdots + y_s$$

Then $\sigma$ is given by the inequalities

$$0 \leqslant X_1 \leqslant X_2 \leqslant \cdots \leqslant X_r \leqslant 1 \tag{$*$}$$
$$0 \leqslant Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_s \leqslant 1. \tag{$**$}$$

For $1 \leqslant k \leqslant r$, $1 \leqslant l \leqslant s$, consider the hyperplane $H^{k,l}$ given by the equation $X_i = Y_j$. These hyperplanes define a subdivision $\Delta$ of $\sigma$ such that every maximal polytope $\tau$ in $\Delta$ is isomorphic to the standard $r+s$-simplex. Namely, any maximal polytope is described by a total ordering $\leqslant$ on the variables $X_i, Y_j$, such that the induced ordering on the sets $\{X_i\}$, $\{Y_j\}$ is given by ($*$), ($**$). Any such ordering describes a $r+s$-simplex. Moreover, there are exactly $\binom{r+s}{s}$ such orderings. As $\sigma$ has volume $1/r! \cdot 1/s!$, any maximal simplex $\tau$ has volume $1/(r+s)!$, and hence, multiplicity $1$. Thus, $X_\Delta$ is a strictly semi-stable formal polytopal domain. Note that the hyperplane $H^{k,l}$ corresponds to blowing up the ideal

$$\mathcal{I}^{k,l} := (\zeta_1 \cdot \ldots \cdot \zeta_k, \xi_1 \cdot \ldots \cdot \xi_l).$$

The morphism $X_\Delta \to \operatorname{Spf} C$ is the blowing up of the product of all these $\mathcal{I}^{k,l}$. □

## 3.5 Ampleness

In the following, we assume that $\Delta$ is a subdivision of a polytope $\sigma$ in $\mathbb{R}^n$. Let $X_\Delta$ be the corresponding formal polytopal domain. Recall that a polyhedral function $f$ defines a formal line bundle on $X_\Delta$ which is locally generated by $\chi_{f_{\sigma_i}}$ on $X_{\sigma_i}$ for $\sigma_i \in \Delta$. It is easy to check the following result:

**Lemma 3.5.1.** *$f$ is convex if and only if $\mathcal{L}_f$ is generated by the global sections $\chi_i := \chi_{f_i}$, where $f$ is given by the affine linear function $f_i$ on $\sigma_i$, and $\sigma_i$ runs through the maximal polytopes of $\Delta$.*

*Proof.* If $f$ is convex, then we have $f_i \geqslant f_j$ on $\sigma_j$ for all $i, j$. This is equivalent to $|\chi_i| \leqslant |\chi_j|$ on $X_{\sigma_j,K}$. This means that $\chi_i$ is a section on every $X_{\sigma_j}$ and thus a global section which generates $\mathcal{L}_f$ on $X_{\sigma_i}$. The converse statement follows in the same way. □

Now, let $f$ be convex, and let $\chi_0, \ldots, \chi_r$ be the generators of $\mathcal{L}_f$ as in the above lemma. Let $\chi_{r+1}, \ldots, \chi_s$ be other global sections of $\mathcal{L}_f$. Then the system $\chi_0, \ldots, \chi_s$ induces a morphism $\psi : X_\Delta \to \mathbb{P}_R^s$. This morphism restricts to a morphism on the special fibre $\tilde{\psi} : \tilde{X}_\Delta \to \mathbb{P}_k^s$ as follows:

Let $\sigma_i$ be a maximal polytope of $\Delta$. Then $|\chi_j/\chi_i| \leqslant 1$ on $X_{\sigma_i,K}$ for every $j = 0, \ldots, s$. Thus, $\chi_j/\chi_i$ reduces to a well-defined function $(\chi_j/\chi_i)^\sim$ on $\tilde{X}_{\sigma_i}$. Let $\{T_i\}$ be the homogeneous coordinates on $\mathbb{P}_k^s$, and let $V_i := \{T_i \neq 0\} \subset \mathbb{P}_k^s$. Then $\tilde{\psi}$ is given on $\tilde{X}_{\sigma_i}$ by

$$\tilde{\psi} : \tilde{X}_{\sigma_i} \longrightarrow V_i \subset \mathbb{P}_k^s, \quad x \mapsto ((\chi_0/\chi_i)^\sim(x) : \ldots : (\chi_s/\chi_i)^\sim(x)).$$

We recall the standard definitions for ampleness in the algebraic situation over a field:

**Definition 3.5.2.** Let $X$ be an algebraic scheme of finite type over a field $k$.

(i) A line bundle $\mathcal{L}$ on $X$ is called *very ample* if there exists a finite set of global sections $s_0, \ldots, s_r$ generating $\mathcal{L}$ such that the corresponding morphism

$$i : X \longrightarrow \mathbb{P}_k^r, \quad x \mapsto (s_0 : \ldots : s_r)$$

is an immersion.

(ii) $\mathcal{L}$ is called *ample*, if there exists $n > 0$ such that $\mathcal{L}^{\otimes n}$ is *very ample*.

As for toric varieties (see [24, Ch. 1, Th. 13]), one can characterize the ampleness of the canonical reduction $\tilde{\mathcal{L}}_f$ directly as follows:

**Lemma 3.5.3.** *$\tilde{\mathcal{L}}_f$ is ample if and only if $f$ is strictly convex.*

*Proof.* Let $f$ be strictly convex. For a maximal polytope $\sigma_i$, $i = 0, \ldots, r$, let $\chi_i$ be the corresponding generator of $\mathcal{L}_f$ on $X_{\sigma_i}$ as above. Due to the strict convexity, $\chi_i$ generates $\mathcal{L}_f$ exactly on $X_{\sigma_i}$. On every $\tilde{X}_{\sigma_i}$, we choose generators $\tilde{\chi}_{ij}$, $j = 1, \ldots, l_i$ of the coordinate ring $\mathcal{O}(\tilde{X}_{\sigma_i})$. These correspond to affine linear functions $f_{ij}$ which satisfy $f_{ij} \geqslant 0$ on $\sigma_i$ and $f_{ij}(u) = 0$ for some vertex $u$ of $\sigma_i$. Again due to the strict convexity of $f$, we have $f_i > f_k$ on $\sigma_k \setminus \sigma_i$ and $f_i = f_k$ on $\sigma_i \cap \sigma_k$. Thus, we find an $n > 0$ such that

$f_{ij} + n(f_i - f_k) \geqslant 0$ on $\sigma_k$ for every $k$. We set $\chi'_{ij} := \chi_{ij} \cdot \chi_i^n$, then $|\chi'_{ij}| \leqslant |\chi_k^n|$ for every $i, j, k$ by construction. Thus, the system

$$\{\chi_i^n,\, i = 0, \dots, r\} \cup \{\chi'_{ij},\, i = 0, \dots, r,\, j = 1, \dots, l_i\}$$

is a system of global generators of the line bundle $\mathcal{L}_{nf}$. We have to show that the corresponding morphism is an immersion on the special fibre:

Let $\mathbb{P}^s$ be the projective space with homogeneous coordinates $\{T_i\}$ and $\{T_{ij}\}$ corresponding to $\{\chi_i^n\}$ and $\{\chi_{ij}\}$ as above. Let $V_i = \{T_i \neq 0\}$ for $i = 0, \dots, r$. Then $\tilde{\psi}^{-1}(V_i) = \tilde{X}_{\sigma_i}$, again by the strict convexity. The corresponding map of coordinate rings is given by

$$T_j/T_i \mapsto (\chi_j^n/\chi_i^n)^{\sim}, \quad T_{kj}/T_i \mapsto (\chi_{kj} \cdot \chi_k^n/\chi_i^n)^{\sim}.$$

Especially, for $k = i$, $T_{ij}/T_i$ maps to the generator $\tilde{\chi}_{ij}$ of $\mathcal{O}(\tilde{X}_{\sigma_i})$. Thus, the corresponding map of rings is surjective; and hence, $\tilde{\psi}$ is an immersion. This proves that $\tilde{\mathcal{L}}_{nf}$ is very ample. □

By Proposition 3.2.7, this has an interpretation in terms of admissible formal blowing ups.

**Proposition 3.5.4.** *Let $\sigma$ be a polytope, and let $U_\sigma$ be the affine formal polytopal domain associated to $\sigma$. Let $\Delta$ be a polytopal subdivision of $\sigma$, and let $\mathcal{L}$ be an invertible monomial sheaf of ideals on $X_\Delta$. Then $\tilde{\mathcal{L}}$ is ample on $\tilde{X}_\Delta$ if and only if there exists a monomial ideal $\mathcal{I}$ on $U_\sigma$ such that the canonical morphism $\varphi : X_\Delta \to U_\sigma$ is the normalization of the admissible formal blowing up of $\mathcal{I}$ on $U_\sigma$ and $\mathcal{L} = \varphi^*\mathcal{I}$.*

*Proof.* Let $\mathcal{L}$ be an ample line bundle given by a strictly convex polyhedral function $f$ on $\Delta$ with $f \geqslant 0$. Let $\mathcal{I}$ be the ideal on $U_\sigma$ generated by $\chi_{f_\sigma}$, where $\sigma$ is a maximal polytope of $\Delta$ and $f$ agrees where $\sigma$ is a maximal polytope of $\Delta$ and $f$ agrees with $f_\sigma$ on $\sigma$. By construction and Proposition 3.3.3, $\mathcal{I}$ induces the subdivision $\Delta$ of $\sigma$ and $\varphi : X_\Delta \to U_\sigma$ is the normalization of the blowing up of $\mathcal{I}$. It is clear that $\mathcal{L} = \varphi^*\mathcal{I}$.

Now, let $\mathcal{I}$ be a monomial sheaf of ideals inducing the subdivision $\Delta$ as in Lemma 3.2.5 and a strictly convex polyhedral function $f$ on $\Delta$. But then, the monomial sheaf of ideals $\varphi^*\mathcal{I}$ is invertible and given locally by $\chi_{f_\sigma}$, where $f|_\sigma = f_\sigma$. As $f$ is strictly convex, the claim follows from Lemma 3.2.5. □

By Proposition 3.3.4, we also have the following:

**Corollary 3.5.5.** *In the situation of the previous theorem, there exists a subdivision* $\Delta'$ *of* $\Delta$ *and a formal line bundle* $\mathcal{L}'_f$ *of* $X_{\Delta'}$ *which is ample on* $\tilde{X}_{\Delta'}$ *.*

Combining Lemma 3.5.3 and Proposition 3.5.4, we conclude:

**Proposition 3.5.6.** *Let* $\sigma$ *be a polytope. For a polytopal complex* $\Delta$ *with* $|\Delta| = \sigma$ *, the following assertions are equivalent:*

(i) $\tilde{X}_\Delta$ *is quasi-projective.*

(ii) *There exists a piecewise affine linear function* $f$ *which is strictly convex on* $\Delta$ *(i.e.* $\Delta$ *is projective).*

(iii) *There is a monomial ideal* $\mathcal{I}$ *on* $U_\sigma$ *such that* $X_\Delta \to U_\sigma$ *is the normalization of the admissible formal blowing up of* $\mathcal{I}$ *.*

## 3.6 Ampleness on the Boundary

If $f$ is convex, but not necessarily strictly convex, then the corresponding line bundle $\mathcal{L}_f$ is not necessarily ample on $\tilde{X}_\Delta$ , so it does not necessarily yield an embedding. The question is: How far is the corresponding morphism from being an embedding? This question can be answered by the following proposition:

**Proposition 3.6.1.** *Let* $f$ *be a convex function on* $\Delta$ *, not necessarily strictly convex. Let* $\Delta_f$ *be the unique polytopal subdivision of* $\sigma$ *such that* $f$ *is strictly convex on* $\Delta_f$ *. Then there exists* $n > 0$ *and monomials* $\chi_i$ *,* $i = 0, \ldots, r$ *generating* $\mathcal{L}_{nf}$ *on* $X_\Delta$ *, such that the morphism*

$$\tilde{\psi} : \tilde{X}_\Delta \longrightarrow \mathbb{P}^r_k, \quad x \mapsto (\tilde{\chi}_0 : \ldots : \tilde{\chi}_r)$$

*factorizes as follows:*

$$\begin{array}{ccc} \tilde{X}_\Delta & \longrightarrow & \tilde{X}_{\Delta_f} \\ & \searrow^{\tilde{\psi}} & \downarrow i \\ & & \mathbb{P}^r_k \end{array}$$

*where* $\tilde{X}_\Delta \to \tilde{X}_{\Delta_f}$ *is the natural morphism and* $i$ *is an immersion.*

We can think of the resulting morphism $\tilde{\psi}$ as a *blow-down* of $\tilde{X}_\Delta$ to $\tilde{X}_{\Delta_f}$ .

*Proof.* Let $\mathcal{L}'_f$ denote the line bundle on $X_{\Delta_f}$ given by $f$. Thus, by Lemma 3.5.3, there is an $n > 0$ such that $\tilde{\mathcal{L}}'_{nf}$ is very ample on $\tilde{X}_{\Delta_f}$. As in the proof of Lemma 3.5.3, we take generators $\chi_i$, $\chi'_{ij}$ of $\mathcal{O}(\tilde{X}_{\sigma_i})$ for every maximal polytope $\sigma_i \in \Delta_f$. Let $i$ be the corresponding immersion. Let $\{T_i\}$ resp. $\{T_{ij}\}$ be the homogeneous coordinates on $\mathbb{P}^l$ corresponding to $\chi_i$ resp. $\chi'_{ij}$. Then $i$ maps $\tilde{X}_{\sigma_i}$ to $V_i = \{T_i \neq 0\}$. The corresponding map of rings is given by

$$k[\{T_j/T_i\}, \{T_{kj}/T_i\}] \twoheadrightarrow \mathcal{O}(\tilde{X}_{\sigma_i})$$
$$T_j/T_i \mapsto (\chi_j/\chi_i)^{\sim}, \quad T_{kj}/T_i \mapsto (\chi_{kj}/\chi_i)^{\sim},$$

where the reduction is taken in $\mathcal{O}(\tilde{X}_{\sigma_i})$. Now, let $\sigma'_{is} \in \Delta$ be a maximal polytope with $\sigma'_{is} \subset \sigma_i$, $\sigma_i \in \Delta_f$. The natural morphism $\mathcal{O}(\tilde{X}_{\sigma_i}) \to \mathcal{O}(\tilde{X}_{\sigma'_{is}})$ sends each $\tilde{\chi} \in \mathcal{O}(\tilde{X}_{\sigma_i})$ to the corresponding reduction on $\tilde{X}_{\sigma'_{is}}$. This gives a chain of morphisms

$$k[\{T_j/T_i\}, \{T_{kj}/T_i\}] \to \mathcal{O}(\tilde{X}_{\sigma_i}) \to \mathcal{O}(\tilde{X}_{\sigma'_{is}})$$

This corresponds to a morphism $\tilde{X}_{\sigma'_{is}} \to \tilde{V}_i$ given by $\{(\chi_j/\chi_i)^{\sim}\}$ and $\{(\chi'_{kj} \cdot \chi_k/\chi_i)^{\sim}\}$ which factors through $\tilde{X}_{\sigma_i}$. But then the morphism $\tilde{\psi} : \tilde{X}_\Delta \to \mathbb{P}^l_k$ which is given by the corresponding global sections $\{\chi_i\}$ and $\{\chi'_{ij}\}$ of $\mathcal{L}_{nf}$ factors through $\tilde{X}_{\Delta_f}$ as claimed. □

For the rest of the section, we will fix the following situation:

**Notation 3.6.2.** let $\sigma_0$ be a polytope, and $\Delta_0$ be a polytopal decomposition of $\sigma_0$; i.e. $|\Delta_0| = \sigma_0$. Let $\Delta$ be a polytopal complex with support $|\Delta| = \sigma$ such that $\sigma_0$ lies in the relative interior of $\sigma$ and that $\Delta$ agrees with $\Delta_0$ on $\sigma_0$. Let $\tilde{Y}_\Delta$ denote the closed subscheme of $\tilde{X}_\Delta$ which consists only of those components $\tilde{X}_{\Delta,u}$ of $\tilde{X}_\Delta$ where $u \in \Delta_0$. Then $\tilde{Y}_\Delta$ is the schematic closure of $\tilde{X}_{\Delta_0}$ in $\tilde{X}_\Delta$. As every vertex of $\Delta_0$ lies in the interior of $\sigma$, the corresponding fan of cones has support $\mathbb{R}^n$. Thus, $\tilde{Y}_\Delta$ is proper over $k$ by Proposition 2.2.8; i.e. a $k$*-compactification* of $\tilde{X}_{\Delta_0}$.

In the above situation, we will call $\Delta$ a *polyhedral extension* of $\Delta_0$. We say, $\Delta$ is *minimal with respect to* $\sigma_0$, if every maximal polytope $\tau \in \Delta$ has non-trivial intersection with $\sigma_0$. This means that every component of $\tilde{X}_\Delta$ meets $\tilde{X}_{\Delta_0}$. Note that $\tilde{Y}_\Delta$ depends only on the vertices $u$ of $\Delta$ and the corresponding fans of cones. These fans are determined only by the maximal polytopes in $\Delta$ which meet $\sigma_0$. Therefore, to any extension $\Delta$ of $\Delta_0$, we can construct a *minimal extension* $\Delta'$ such that $\tilde{Y}_\Delta = \tilde{Y}_{\Delta'}$.

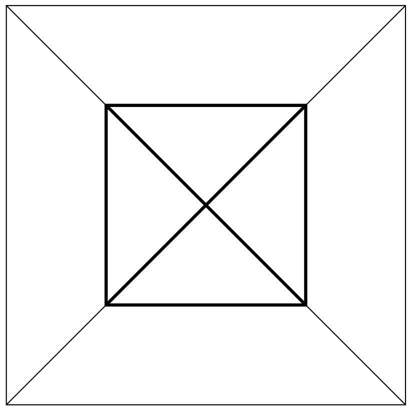


Figure 3.2: A minimal extension of $\Delta_0$ by $\Delta$

Namely, if $\tau \in \Delta$ is a maximal polytope which does not meet $\sigma_0$, we find an affine linear $f$ with $f < 0$ on $\tau$ and $f > 0$ on $\sigma_0$. Replacing $\sigma$ with $\sigma' := \sigma \cap \{f \geqslant 0\}$ and $\Delta$ with its restriction to $\sigma'$ does not change $\tilde{Y}_\Delta$. Thus, $\Delta'$ is a minimal extension of $\Delta$.

Figure 3.2 illustrates Notation 3.6.2. The thick lines denote the edges of the complex $\Delta_0$, the whole picture denotes the complex $\Delta$.

In the following, we want to discuss the notion of *ampleness on the boundary* in the above situation. This notion has been introduced by Lütkebohmert [25] for Cartier divisors; we will use the language of line bundles instead for our situation.

**Definition 3.6.3.** Let $X$ be a proper, separated scheme of finite type over $k$. Let $U$ be an open dense subscheme of $X$. A line bundle $\mathcal{L}$ on $X$ is called *ample on the boundary of $U$ in $X$*, if there exists a finite set of global sections $s_0, \ldots, s_r$ generating $\mathcal{L}$ such that the induced morphism $p : X \to \mathbb{P}_k^r$ satisfies the following two conditions:

(i) $p$ is finite on $X \setminus U$.

(ii) $p^{-1}(\mathbb{A}_k^r) = U$ for a suitable $\mathbb{A}_k^r \subset \mathbb{P}_k^r$.

**Remark 3.6.4.** If the above conditions hold, then $p|_U : U \to \mathbb{A}_k^r$ is proper, and the *centre*

$$B = \{y \in \mathbb{A}_k^r : \dim p^{-1}(\{y\}) \geqslant 1\}$$

of $p|_U$ is a finite set. Moreover, the Stein factorization of $p|_U$ shows that in fact $U$ is the modification of an affine scheme of finite type. For details, see [25].

To characterize ampleness on the boundary for our situation, we will need the following conditions on $f$:

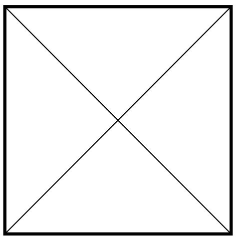
**Definition 3.6.5.** Let $\Delta$, $\Delta_0$ as in Notation 3.6.2, and let $f$ be a convex polyhedral function on $\Delta$. Then $f$ is called *strictly convex on the boundary of* $\Delta_0$ *in* $\Delta$, if the following conditions hold:

(i) There exists an affine linear function $f_0$ such that $f|_{\sigma_0} = f_0$ and $f < f_0$ on $\sigma \setminus \sigma_0$.

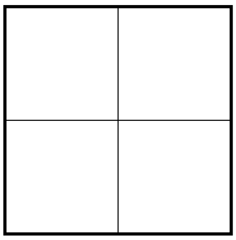
(ii) For every maximal polytope $\tau_i \in \Delta$ which does not lie in $\sigma_0$, there exists an affine linear function $f_i$ such that $f|_{\tau_i} = f_i$ and $f < f_i$ on $\sigma \setminus \tau_i$.

**Definition 3.6.6.** Let $\sigma$ be a polytope, and let $\Delta$ be a polytopal subvidision of $\sigma$. We say $\Delta$ is *strictly convex*, if every proper face of $\sigma$ is an element of $\Delta$; i.e. if $\Delta$ induces the trivial decomposition on every proper face of $\sigma$.

**Example 3.6.7.** Let $\sigma$ be the unit square. The following two decompositions of $\sigma$ are strictly convex, resp. not strictly convex:



strictly convex



not strictly convex

**Remark 3.6.8.** If $\Delta_0$ is strictly convex, and $f$ is strictly convex on the boundary of $\Delta_0$ in $\Delta$, then the polytopal complex $\Delta_f$ where $f$ is strictly convex is given by

$$\Delta_f = \{\tau \in \Delta : \tau \not\subset \sigma_0\} \cup \{\tau \ ; \ \tau \text{ is a face of } \sigma_0\}.$$

Note that the strict convexity of $\Delta_0$ guarantees that this is indeed a polytopal complex.

On the other hand, we have the following result:

**Lemma 3.6.9.** *If there exists a polyhedral function* $f$ *which is strictly convex on the boundary of* $\Delta_0$ *in* $\Delta$, *then* $\Delta_0$ *is strictly convex.*

*Proof.* Assume that $\Delta_0$ is not strictly convex. Thus, there is a face $\tau$ of $\sigma_0$ which is subdivided into polytopes $\tau_1, \ldots, \tau_r$ with $r \geqslant 2$. We may assume that $\tau$ has codimension $1$ in $\sigma_0$. Counting only those $\tau_i$ of maximal dimension, we may assume as well that $\tau, \tau_1, \ldots, \tau_r$ have codimension 1 in $\sigma_0$. Let $\tau_1', \ldots, \tau_r'$ be the polytopes in $\Delta$ such that $\tau_i' \cap \sigma_0 = \tau_i$. Now, let $g$ be an affine linear function with $g = 0$ on $\tau$ and $g \geqslant 0$ on $\sigma_0$. As $\tau$ has codimension $1$ in $\sigma_0$, note that $g$ is unique up to a positive multiplicative constant. As $f$ is continuous on $\tau_i'$, there exists $c_i > 0$ such that $f = c_i \cdot g$ on $\tau_i'$. Without loss of generality, we may assume that $\tau_1' \cap \tau_2' \neq \emptyset$. From the continuity of $f$ on $\tau_1' \cap \tau_2'$, we see that $c_1 = c_2$. But this contradicts the strict convexity of $f$. □

**Proposition 3.6.10.** *Let $f$ be strictly convex on the boundary of $\Delta_0$ in $\Delta$. Then $\tilde{\mathcal{L}}_f$ is ample on the boundary of $\tilde{X}_{\Delta_0}$ in $\tilde{X}_\Delta$.*

*Proof.* Let $\tau_1, \ldots, \tau_r$ be the maximal polytopes in $\Delta$ which do not lie in $\sigma_0$. Let $f = f_i$ on $\tau_i$ and $f = f_0$ on $\sigma_0$. For $i = 0, \ldots, r$, we choose corresponding monomials $\chi_i = a_i \zeta^{m_i}$ with $f_{\chi_i} = f_i$. Due to Lemma 3.5.1, the monomials $\chi_0, \ldots, \chi_r$ are global sections generating $\mathcal{L}_f$. We consider the corresponding morphism

$$\psi : X_\Delta \longrightarrow \mathbb{P}_R^r, \quad x \mapsto (\chi_0 : \ldots : \chi_r).$$

Now, let $\tilde{\psi}$ be the restriction of $\psi$ to $\tilde{Y}_\Delta$. We want to show that $\tilde{\psi}$ satisfies the conditions for ampleness on the boundary. At first, let $V_0 := \{T_0 \neq 0\} \cong \mathbb{A}_k^r \subset \mathbb{P}_k^r$. As $f_0 > f$ outside of $\sigma_0$, we see directly that $\tilde{\psi}^{-1}(V_0) = \tilde{X}_{\Delta_0}$.

Now, fix $i \in \{1, \ldots, r\}$, and let $V_i := \{T_i \neq 0\}$. Again, as $f_i > f$ outside of $\tau_i$, we have $\tilde{\psi}^{-1}(V_i) = \tilde{X}_{\tau_i} \cap \tilde{Y}_\Delta =: \tilde{Y}_{\tau_i}$. As $\tilde{X}_{\tau_i}$ is affine, so is $\tilde{Y}_{\tau_i}$. As a $k$-module, $\mathcal{O}(\tilde{Y}_{\tau_i})$ is generated by the set of monomials

$$\{\tilde{\chi}_m \ ; \ m \in C_{\tau_i,u}^\vee,\ u \text{ vertex of } \tau_i \cap \sigma_0\}.$$

The corresponding morphism of rings is given by

$$A_i := k[T_0/T_i, \ldots, T_r/T_i] \longrightarrow \mathcal{O}(\tilde{Y}_{\tau_i}), \quad T_j/T_i \mapsto (\chi_j/\chi_i)^\sim.$$

Thus we have to show that $\mathcal{O}(\tilde{Y}_{\tau_i})$ is a finite module over $A_i$. Fix a common vertex $u$ of $\tau_i$ and $\sigma_0$. Then the set

$$I_u := \{m_j - m_i \ ; \ j = 0 \text{ or } u \text{ vertex of } \tau_i \cap \tau_j\}$$

generates $C^\vee_{\tau_i,u}$ as a cone, and thus a sub-semigroup of $C^\vee_{\tau_i,u} \cap \mathbb{Z}^n$ of finite index. Then there exists a finite set $\{m_{j,u}\}$ such that $\{\tilde{\chi}_{m_{j,u}}\}$ generates $k[\tilde{\chi}_m;\ m \in C^\vee_{\tau_i,u}]$ as a module over $k[\tilde{\chi}_m;\ m \in I_u]$. Taking $I := \bigcup I_u$, the set $\{\tilde{\chi}_m;\ m \in I\}$ generates $\mathcal{O}(\tilde{Y}_{\tau_i})$ as a module over $k[\{(\chi_i/\chi_k)^\sim\}]$. This proves that the restriction of $\tilde{\psi}$ to $\bigcup \tilde{Y}_{\tau_i}$ is a finite morphism. As $\tilde{Y}_\Delta \setminus \tilde{X}_{\Delta_0}$ is a closed subset of $\bigcup \tilde{Y}_{\tau_i}$, the claim follows. □

For the existence of a polyhedral function $f$ which is strictly convex on the boundary of $\Delta_0$, we need the following result, which is a stronger version of Proposition 3.3.4:

**Proposition 3.6.11.** *Let $\Delta_0$ be strictly convex, and let $\Delta$ be a polyhedral extension of $\Delta_0$. Then there exists*

(i) *a polytopal subdivision $\Delta'$ of $\Delta$ which coincides with $\Delta_0$ on $\sigma_0$,*

(ii) *a subcomplex $\Delta''$ of $\Delta'$ which is a minimal extension of $\Delta_0$, and*

(iii) *a polyhedral function $f$ on $\Delta''$ which is strictly convex on the boundary of $\Delta_0$ in $\Delta''$.*

*Proof.* Assume first that $\Delta$ is a minimal extension of $\Delta_0$. We have to show that there exists a monomial ideal $\mathcal{I}$ inducing a subdivision $\Delta'$ of $\Delta$ which does not subdivide $\sigma_0$. Let $\sigma_0$ be given by $f_1 \geqslant 0, \dots, f_r \geqslant 0$. We fix a maximal polytope $\tau \in \Delta$ not contained in $\sigma_0$ which is given by $g_1 \geqslant 0, \dots, g_s \geqslant 0$. By the strict convexity of $\Delta_0$, $\tau$ meets $\sigma_0$ in a common face $\tau'$. Then there exists an affine linear function $f_0$ with $f_0 = 0$ on $\tau'$ such that $f_0 > 0$ on $\sigma_0 \setminus \tau'$ and $f_0 < 0$ on $\tau \setminus \tau'$. For $c \in \mathbb{N}$ large enough, we have $g_j + c \cdot f_0 \geqslant 0$ on $\sigma_0$ for all $j$, and $c \cdot f_0 \leqslant f_i$ on $\tau$ for all $i$. Now, define $\mathcal{I}_\tau$ by

$$\mathcal{I}_\tau := (0, f_1, \dots, f_r, c \cdot f_0, g_1 + c \cdot f_0, \dots, g_s + c \cdot f_0).$$

This induces a subdivision $\Delta_\tau$ of $\sigma$. One checks that $\tau, \sigma_0 \in \Delta_\tau$. Repeating this process for every maximal polytope $\tau_k \in \Delta$ which is not contained in $\sigma_0$, we find monomial ideals $\mathcal{I}_1, \dots, \mathcal{I}_t$ with corresponding subdivisions $\Delta_1, \dots, \Delta_t$ such that $\tau_k, \sigma_0 \in \Delta_k$ for all $k$. Taking $\mathcal{J}$ as the product of the $\mathcal{I}_k$, the associated subdivision $\Delta'$ is the intersection of all $\Delta_k$. Thus, $\sigma_0 \in \Delta'$, and $\Delta'$ is a subdivision of $\Delta$. By subdiving further, we may assume that $\Delta'$ contains a minimal extension $\Delta''$ of $\Delta_0$. By construction, $\Delta''$ allows a polyhedral function which is strictly convex on the boundary of $\Delta_0$ in $\Delta''$.

Now, if $\Delta$ is not a minimal extension, we may assume after a suitable subdivision that $\Delta$ contains a minimal extension $\Delta'$ as a subcomplex. Applying the above construction to $\Delta'$ and extending the ideal $\mathcal{J}$ to an ideal $\overline{\mathcal{J}}$ on $\Delta$, we get the desired result. □

We conclude this section with a further result on the canonical morphism $\tilde{X}_\Delta \to \tilde{U}_\sigma$, where $\Delta$ is a decomposition of $\sigma$, and $\tilde{U}_\sigma$ is the reduction of the affinoid polytopal domain $U_{\sigma,K}$ corresponding to $\sigma$.

**Definition 3.6.12.** Let $X$, $Y$ be schemes of finite type over a field $k$. A morphism $f : X \to Y$ is called a *modification*, if it satisfies the following conditions:

(i) $f$ is proper

(ii) $f_*\mathcal{O}_X = \mathcal{O}_Y$

(iii) The *centre* $B = \{y \in Y \ ; \ \dim_{k(y)} X \times_Y k(y) \geqslant 1\}$ is finite.

Note that the centre $B$ is a closed subset of $Y$ by [18, 13.1.5], and that $f$ is an isomorphism outside of $B$.

**Proposition 3.6.13.** *Let $\Delta$ be a decomposition of $\sigma$. The canonical morphism $\tilde{\psi} : \tilde{X}_\Delta \to \tilde{U}_\sigma$ is a modification if and only if $\Delta$ is strictly convex.*

*Proof.* By [25, Prop. 5.4], $\tilde{X}_\Delta$ is a modification of an affine scheme if and only if there exists a compactification $\tilde{Y}$ of $\tilde{X}_\Delta$ and a line bundle which is ample on the boundary of $\tilde{X}_\Delta$ in $\tilde{Y}$. Due to Proposition 3.6.11, starting from any minimal extension of $\Delta$, we can construct a suitable extension. One only has to check that $\tilde{X}_\Delta$ is a modification of $\tilde{U}_\sigma$. We can however see the above result directly as follows:

Due to Proposition 3.2.1, we see that $\tilde{\psi}$ is proper. For the second condition, it is enough to check that $\Gamma(\tilde{X}_\Delta, \mathcal{O}_{\tilde{X}_\Delta}) = \Gamma(\tilde{U}_\sigma, \mathcal{O}_{\tilde{U}_\sigma})$. Note that $f \in \Gamma(X_\Delta, \mathcal{O}_{X_\Delta})$ if and only if $|f| \leqslant 1$ on $X_{\tau,K}$ for every $\tau \in \Delta$. But this is equivalent to $|f|_{\sup} \leqslant 1$ on $U_{\sigma,K}$. For such a function $f$, $\tilde{f} = 0$ in $\Gamma(\tilde{X}_\Delta, \mathcal{O}_{\tilde{X}_\Delta})$ holds if and only if $|f| < 1$ on $X_{\tau,K}$ for every $\tau \in \Delta$. Again, this is equivalent to $|f|_{\sup} < 1$ on $U_{\sigma,K}$. This proves the second claim.

For the third claim, let $\tau$ be a face of $\sigma$, and let $O_\tau$ denote the corresponding torus orbit in $\tilde{U}_\sigma$. Then the inverse image of $O_\tau$ under $\tilde{\psi}$ is given as follows:

$$\tilde{\psi}^{-1}(O_\tau) = \bigcup O_{\tau'},$$

where $\tau'$ runs through all polytopes in $\Delta$ which satisfy $\operatorname{relint}(\tau') \subset \operatorname{relint}(\tau)$. As $\tilde{\psi}$ is torus invariant, a point $y \in O_\tau$ lies in $B$ if and only if $O_\tau \subset B$. But $\tilde{\psi}^{-1}(y)$ is finite for $y \in O_\tau$ if and only if $\dim \tilde{\psi}^{-1}(O_\tau) = \dim O_\tau$. This is true if and only if every proper face $\tau$ of $\sigma$ is not subdivided by $\Delta$. But this is exactly the strict convexity of $\Delta_0$, so the claim follows. □

# Chapter 4

# Totally Degenerated Formal Schemes

In the following, let $R$ be a discrete valuation ring with uniformizing parameter $\pi$, let $K$ be its field of fractions, $k$ its residue field. We assume further that the residue field $k$ is separably closed. This condition is crucial for the construction of the Picard variety.

## 4.1 Definitions

**Definition 4.1.1.** Let $X$ over $R$ be a quasi-compact admissible formal scheme, and let $X_0^{(\nu)}, \nu \in N$ denote the irreducible components of the special fibre $X_0$ of $X$. We call $X$ *totally degenerated*, if the following conditions hold:

(i) The irreducible components of $X_0$ are rational varieties over $k$ with *normal crossings*; i.e. every point $x \in X_0$ has an open neighbourhood $U$ such that its special fibre $U_0$ is isomorphic to an open subset of

$$\operatorname{Spec} k[\tilde{\xi}_1, \ldots, \tilde{\xi}_s; \tilde{\zeta}_0, \ldots, \tilde{\zeta}_r]/(\tilde{\zeta}_0 \cdot \ldots \cdot \tilde{\zeta}_r)$$

(ii) $X_0^{(\nu)}$ is a Cartier divisor in $X$ for every $\nu \in N$.

**Remark 4.1.2.** Condition (i) implies that, for every $M \subset N$, the intersection

$$X_0^M := \bigcap_{\nu \in M} X_0^{(\nu)}$$

is *strictly rational* over $k$; i.e. every point $x \in X_0^M$ has an open neighbourhood which is isomorphic to an open subset of $\mathbb{A}_k^{\dim X - \# M}$. Moreover, $X_0$ is geometrically reduced. Assertion (ii) then implies directly that any totally degenerated formal scheme is strictly semi-stable.

**Lemma 4.1.3.** *Let $X$ be a totally degenerated formal scheme, and let $U = \operatorname{Spf} A$ be a neighbourhood of $x \in X_0$ as in condition (i) of Definition 4.1.1. Let $\zeta_i$ denote a lift of $\tilde{\zeta}_i$. Then there exists a unit $u \in A^\times$ such that*

$$\zeta_0 \cdot \ldots \cdot \zeta_r \cdot u = \pi.$$

*Especially, on the generic fibre $U_K$ of $U$, we have*

$$|\zeta_0(x)| \cdot \ldots \cdot |\zeta_r(x)| = |\pi|$$

*for every $x \in U_K$.*

*Proof.* Assume without loss of generality that $X = U$, and that $x$ is the point given by $\tilde{\xi}_j(x) = \tilde{\zeta}_k(x) = 0$ for all $j = 1, \ldots, s$, $k = 0, \ldots, r$. As $X$ is strictly semi-stable, every irreducible component $X_0^{(i)}$ of $X_0$ is a Cartier divisor. The corresponding ideal is generated by $\zeta_i$, where $\zeta_i$ is a lift of $\tilde{\zeta}_i$. Hence, we have $\pi \in (\zeta_0)$, and we may write $\pi = u_0 \cdot \zeta_0$ for some $u_0 \in A^\times$. Again, we have $\pi \in (\zeta_1)$. As $(\zeta_1)$ is a prime ideal and $\zeta_0 \notin (\zeta_1)$, we have $\pi = u_1 \cdot \zeta_0 \zeta_1$. Continuing this way, we get $\pi = u \cdot \zeta_0 \cdot \ldots \cdot \zeta_r$ with $u \in A^\times$. □

**Notation 4.1.4.** In the following, we will always assume that $X$ is a totally degenerated admissible formal scheme which is proper and connected. We fix a covering

$$\mathfrak{U} = \{U^{(1)}, \ldots, U^{(l)}\}$$

of $X$ such that, for each $i$, the special fibre $U_0^{(i)}$ is given as in condition (i) of Definition 4.1.1. Moreover, we will assume that each $U_0^{(i)}$ contains the point $x_0^{(i)}$ given by $\tilde{\xi}_j(x_0^{(i)}) = \tilde{\zeta}_k(x_0^{(i)}) = 0$ for all $j = 1, \ldots, s$, $k = 0, \ldots, r$. Let $\mathfrak{U}_0 = \{U_0^{(1)}, \ldots, U_0^{(l)}\}$ denote the corresponding covering of $X_0$. We will further assume that, for any subset $J \subset \{1, \ldots, l\}$, the intersection $\bigcap_{j \in J} U^{(j)}$ is connected.

**Remark 4.1.5.** The most important examples of totally degenerated formal schemes are those which have an atlas $\mathfrak{U}$, where each $U^{(i)}$ is isomorphic to an open subsets of

$$\operatorname{Spf} R\langle \xi_1, \ldots, \xi_s; \zeta_0, \ldots, \zeta_r \rangle / (\zeta_0 \cdot \ldots \cdot \zeta_r - \pi).$$

All examples which we will presents in Chapter 5, such as Mumford curves or analytic tori $\mathbb{G}_m^n / M$, are of this type. However, it is not clear whether the converse holds; i.e. if every totally degenerated formal scheme locally arises this way.

**Proposition 4.1.6.** *Let $X$ be a proper totally degenerated formal scheme. Let $X_0^{(0)}$ be an irreducible component of its special fibre $X_0$, which is not the whole of $X_0$, and let $Y_0^{(0)}$ be the intersection of the non-singular locus of $X_0$ with $X_0^{(0)}$. If $r$ is the rank of $\mathcal{O}^\times(Y_0^{(0)})/k^\times$, then $X_0^{(0)}$ meets at least $r+1$ other irreducible components.*

*Proof.* Let $X_0^{(1)},\ldots,X_0^{(s)}$ be the other irreducible components meeting $X_0^{(0)}$. As $X_0^{(0)}$ is not the whole of $X_0$, we have $s \geqslant 1$. Let $Z_0^{(i)} := X_0^{(i)} \cap X_0^{(0)}$. Due to the strict semi-stability, $Z_0^{(i)}$ is a Weil divisor on $X_0^{(0)}$. As $X_0^{(0)}$ is smooth, we have $Y_0^{(0)} = X_0^{(0)} \setminus \bigcup_{i=0}^s Z_0^{(i)}$. Now, let $\mathfrak{D}$ denote the group of Weil divisors on $X_0^{(0)}$, and let $\mathfrak{D}_Z$ denote the subgroup generated by $Z_0^{(1)},\ldots,Z_0^{(s)}$. Furthermore, let $\mathfrak{D}_H$ denote the group of principal divisors. Consider the group morphism

$$\varphi : \mathcal{O}^\times(Y_0^{(0)})/k^\times \to \mathfrak{D}_H, \quad f \mapsto \operatorname{div}(f).$$

Its image is contained in $\mathfrak{D}_H \cap \mathfrak{D}_Z$, as any meromorphic function which is invertible on $Y_0^{(0)}$ gives rise to a Weil divisor with support in $Z_0^{(1)} \cup \ldots \cup Z_0^{(s)}$. As $X_0^{(0)}$ is proper, the only meromorphic functions $f$ with $\operatorname{div}(f) = 0$ are constants, so $\varphi$ is injective. If $r = \operatorname{rk} \mathcal{O}^\times(Y_0^{(0)})/k^\times$, then $\operatorname{rk}(\mathfrak{D}_Z \cap \mathfrak{D}_H) \geqslant r$. Moreover, as $n \cdot Z_0^{(1)}$ is not a principal divisor for any $n \neq 0$, $Z_0^{(1)}$ yields a non-torsion element in $\mathfrak{D}_Z/(\mathfrak{D}_Z \cap \mathfrak{D}_H)$, so $\mathfrak{D}_Z$ has at least rank $r+1$. As $\mathfrak{D}_Z$ is generated by $Z_0^{(1)},\ldots,Z_0^{(s)}$, we have $s \geqslant r+1$. □

**Remark 4.1.7.** The assertion of Proposition 4.1.6 may also hold if $X$ is not proper. Namely, consider the formal scheme $X$ constructed by gluing the two affine formal schemes

$$U_1 := \operatorname{Spf} R\langle \zeta_1, \zeta_2, \pi/(\zeta_1\zeta_2)\rangle, \quad U_2 := \operatorname{Spf} R\langle \zeta_1, 1/\zeta_2, \pi\zeta_2/\zeta_1\rangle.$$

This is the formal scheme associated to the polytopal complex in $\mathbb{R}^2$ given by Figure 4.1. The special fibre $X_0$ consists of four components, one of which, say $X_0^{(0)}$, is isomorphic to $\mathbb{A}_K^1 \times \mathbb{P}_K^1$. The intersection with the other components $X_0^{(1)}$, $X_0^{(2)}$, and $X_0^{(3)}$ is given by $\{0\} \times \mathbb{P}_K^1$, $\mathbb{A}_K^1 \times \{0\}$ and $\mathbb{A}_K^1 \times \{\infty\}$. Hence, $X_0^{(0)} \setminus (X_0^{(1)} \cup X_0^{(2)} \cup X_0^{(2)}) = \mathbb{G}_{m,K}^2$. Hence, its unit group has rank $2$, $X_0^{(0)}$ meets $3$ components, but $X_0^{(0)}$ itself is not proper.

Refining the proof of Proposition 4.1.6 yields the following stronger result:

**Proposition 4.1.8.** *Let $X_0^{(0)}$, $Z_0^{(1)},\ldots,Z_0^{(s)}$ be as in the proof of Proposition 4.1.6. If the images of $Z_0^{(1)},\ldots,Z_0^{(s)}$ generate the Picard group $\operatorname{Pic} X_0^{(0)} = \mathfrak{D}/\mathfrak{D}_H$, then*

$$s \geqslant \operatorname{rk} \mathcal{O}^\times(Y_0^{(0)}) + \operatorname{rk} \operatorname{Pic} X_0^{(0)}.$$

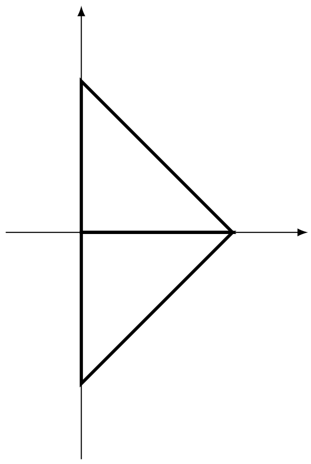


Figure 4.1: Polytopal complex for Remark 4.1.7

*Proof.* Due to the assumptions, $\operatorname{Pic} X_0^{(0)} = \mathfrak{D}_Z/(\mathfrak{D}_Z \cap \mathfrak{D}_H)$. But then, as in the proof of Proposition 4.1.6, we see that

$$\operatorname{rk}\operatorname{Pic} X_0^{(0)} = s - \operatorname{rk}(\mathfrak{D}_Z \cap \mathfrak{D}_H) \leqslant s - \operatorname{rk}\mathcal{O}^\times(Y_0^{(0)}).$$

This proves the claim. □

The following is a useful result for the cohomology of the special fibre $X_0$ of $X$, which will allow us to easily compute the cohomology by the combinatorial configuration of the sets $U^{(i)}$.

**Lemma 4.1.9.** *Let $X$ be an open subset of $\operatorname{Spec} k[x_1, \dots, x_n]/(x_1 \cdot \ldots \cdot x_r)$ for some $r \leqslant n$, and let $G$ be a constant sheaf on $X$. Then $H^i(X, G) = 0$ for all $i > 0$.*

*Proof.* The case is clear for $r = 0$. Namely, in that case $X$ is irreducible, and hence, $G$ is flasque on $X$. Now, we perform induction by $r$ and $n$. If $r > 1$, let $X_1$ be an irreducible component of $X$, and let $U := X \setminus X_1$. Let $j : X_1 \to X$ and $i : U \to X$ be the corresponding closed resp. open immersions, and let $G_{X_1} := j_*(G|_{X_1})$ resp. $G_U := i_!(G|_U)$ denote the extension of $G$ by zero outside of $X_1$ and $U$ respectively. This yields an exact sequence of sheaves

$$0 \to G_U \to G \to G_{X_1} \to 0.$$

This yields the following long exact sequence:

$$\begin{aligned} 0 \to H^0(X, G_U) \to H^0(X, G) \to H^0(X, G_{X_1}) \to \\ \to H^1(X, G_U) \to H^1(X, G) \to H^1(X, G_{X_1}) \to \cdots \\ \cdots \to H^{i-1}(X, G_{X_1}) \to H^i(X, G_U) \to H^i(X, G) \to H^i(X, G_{X_1}) \to \cdots \end{aligned}$$

Note that we have $H^i(X, G_{X_1}) = H^i(X_1, G)$ by [22, Lemma 2.10]. For $i = 0$, this yields

$$H^0(X, G) = H^0(X_1, G) = G,$$

as $X$ and $X_1$ are both connected. Thus, we get the following long exact sequence:

$$\begin{aligned} 0 \to H^1(X, G_U) \to H^1(X, G) \to H^1(X_1, G) \to H^2(X, G_U) \to \cdots \\ \cdots \to H^{i-1}(X, G_{X_1}) \to H^i(X, G_U) \to H^i(X, G) \to H^i(X_1, G) \to \cdots \end{aligned} \tag{4.1}$$

For $i \geqslant 1$, we have $H^i(X_1, G) = 0$ by the induction hypothesis. Hence, (4.1) yields

$$H^i(X, G_U) \cong H^i(X, G)$$

for any $i \geqslant 1$. Thus, it remains to show $H^i(X, G_U) = 0$ for $i > 0$. Let $\bar{U}$ denote the Zariski closure of $U$ in $X$. We can then identify $G_U$ with a sheaf on $\bar{U}$ with the same cohomology. Let $Y := \bar{U} \setminus U$. Then, again, we have an exact sequence of sheaves on $\bar{U}$ as follows:

$$o \to G_U \to G \to G_Y \to 0.$$

Note that $Y$ satisfies the conditions of the Lemma with $n' := n - 1$, $r' := r - 1$, and $\bar{U}$ satisfies the conditions of the Lemma with $n' := n$, $r' := r - 1$. Thus, the induction hypothesis yields

$$H^i(\bar{U}, G) = H^i(Y, G) = 0.$$

Using the long exact sequence of cohomology again, this proves $H^i(X, G_U) = 0$, and the claim follows. □

Using a standard Leray argument, Lemma 4.1.9 yields the following result:

**Proposition 4.1.10.** *Let $X$ be a totally degenerated formal scheme, and let $\mathfrak{U}_0$ denote the covering* *of $X_0$ as in Notation 4.1.4. Then, for any constant sheaf $G$ on $X_0$, we have $H^i(X_0, G) = \check{H}^i(\mathfrak{U}_0, G)$.*

## 4.2 The Universal Covering

**Definition 4.2.1.** Let $X$ be a totally degenerated formal scheme, and let $\mathfrak{U} = \{U^{(i)}\}$ be an affine covering of $X$ as in Notation 4.1.4. We associate to $X$ a simplicial complex $\Delta(X)$ as follows: The vertices $v_i$ of $\Delta(X)$ are the affine sets $U^{(i)}$. A set of vertices $v_{i_0}, \dots, v_{i_r}$ build an $r$-simplex if the intersection $U^{(i_0)} \cap \ldots \cap U^{(i_r)}$ is non-empty. We call $\Delta(X)$ the *nerve* of the covering $\{U^{(i)}\}$.

**Remark 4.2.2.** In Notation 4.1.4, we assumed that every intersection of the sets $U^{(i)}$ is connected. Thus, any simplex is uniquely determined by its vertices.

In the following, let $X_K$ be the generic fibre of a totally degenerated formal scheme. We want to construct a rigid analytic variety $\Omega_K$, which we will call the *universal covering* of $X_K$:

Let $u_\Delta : \Delta' \to \Delta(X)$ be the universal covering of $\Delta(X)$ in the category of simplicial complexes. For any vertex $v$ of $\Delta'$, let $\Omega(v)$ be the affine formal scheme $U^{(i)}$ corresponding to the vertex $u(v)$ in $\Delta(X)$. Two affine formal schemes $\Omega(v_1)$, $\Omega(v_2)$ are glued together if and only if $v_1$ and $v_2$ are connected by an edge of $\Delta'$. Locally on triangles, the universal covering map is an isomorphism, so this construction preserves triple intersections. Hence, this glueing yields an admissible formal scheme $\Omega$, which is locally isomorphic to $X$, and hence, also totally degenerated. The corresponding simplicial complex is exactly $\Delta'$; we will also write $\Delta(\Omega)$ instead. As always, let $\Omega_K$ denote the generic fibre, $\Omega_0$ the special fibre.

Let $\Gamma$ denote the group of deck transformations of $u_\Delta : \Delta' \to \Delta$; i.e. the set of automorphisms $\gamma$ of $\Delta'$ satisfying $u_\Delta \circ \gamma = u_\Delta$. We can interpret $\Gamma$ as the fundamental group of $\Delta(X)$; i.e. each element $\gamma \in \Gamma$ can be interpreted as a closed edge-path in $\Delta$. Let $\overline{\Gamma} := \Gamma/[\Gamma, \Gamma]$ be the abelianization of $\Gamma$, where $[\Gamma, \Gamma]$ is the commutator subgroup of $\Gamma$. By Proposition 2.3.3, $\overline{\Gamma}$ is isomorphic to the first simplicial homology group $H_1(\Delta(X))$ of $\Delta$.

Any $\gamma \in \Gamma$ induces an automorphism of $\Omega_K$, which we will denote again by $\gamma$, and which satisfies $u \circ \gamma = u$. We may consider $X_K$ as the rigid analytic quotient of $\Omega_K$ by the group $\Gamma$; we write $X_K = \Omega_K/\Gamma$.

We want to prove the following result:

**Proposition 4.2.3.** *Any bounded holomorphic function on* $\Omega_K$ *is constant.*

In order to prove Proposition 4.2.3, we will apply methods similar to those of [6, §3]. Assume for the moment that $K$ is algebraically closed.

**Definition 4.2.4.** Let $X$ be an admissible formal scheme with generic fibre $X_K$ and reduced special fibre $X_0$, and let $X_0^{(i)}$ be an irreducible component of $X_0$. We choose an open affine subset $Y_0^{(i)}$ of $X_0^{(i)}$ which does not meet any other irreducible component, and let $Y_K^{(i)} := \pi^{-1}(Y_0^{(i)})$ denote the affinoid formal open subset of $X_K$ corresponding to $Y_0^{(i)}$. By [5, Prop. 6.2.3./5], the supremum norm $\|\cdot\|_{Y_K^{(i)}}$ is multiplicative. If $f$ is a holomorphic function on $X_K$, we set $|f|_i := \|f\|_{Y_K^{(i)}}$ and call it the *norm of* $f$ *over the component* $X_0^{(i)}$. Note that this is independent of the choice of the open set $Y_K^{(i)}$. If $f \neq 0$, there exists a constant $c_i \in K^\times$ such that $|f|_i = |c_i|$. Then the holomorphic function $f_i := c_i^{-1} f$ has norm $1$ over $X_0^{(i)}$, so it reduces to a rational function $\tilde{f}_i$ on $X_0^{(i)}$ which is regular on $Y_0^{(i)}$.

Now, consider the special case where $U_K = U_K^{(i)}$ is an affine piece of a totally degenerated formal scheme as in Notation 4.1.4; i.e. the reduction $U_0$ is given by an open subset of

$$\operatorname{Spec} k[\tilde{\xi}_1, \ldots, \tilde{\xi}_s; \tilde{\zeta}_0, \ldots, \tilde{\zeta}_r]/(\tilde{\zeta}_0 \cdot \ldots \cdot \tilde{\zeta}_r)$$

Let $\pi : U_K \to U_0$ denote the reduction. Let $U_0^{(i)}$ denote the irreducible component of $U_0$ given by $\tilde{\zeta}_i = 0$, and let $Y_0^{(i)}$ denote the non-singular locus of $U_0^{(i)}$. Then $Y_0^{(i)}$ is given in $U_0$ by $\tilde{\zeta}_j \neq 0$ for $j \neq i$. Let $Y_K^{(i)} := \pi^{-1}(Y_0^{(i)})$, then $Y_K^{(i)}$ is the formal open subset of $U_K$ given by $|\zeta_j| = 1$ for $j \neq i$. By Lemma 4.1.3, we have

$$|\zeta_0(x)| \cdot \ldots \cdot |\zeta_r(x)| = |\pi|$$

for every $x \in U_K$, and hence $|\zeta_i| = |\pi|$ on $Y_K^{(i)}$.

**Lemma 4.2.5.** *Let* $f$ *be a holomorphic function on* $U_K$. *Then the following assertions hold:*

(i) *If* $|f|_j > |f|_i$, *then* $\tilde{f}_j$ *vanishes on* $U_0^{(j)}$ *along* $U_0^{(ij)} := U_0^{(i)} \cap U_0^{(j)}$.

(ii) *Let*

$$m_{ji} := \operatorname{ord}_{U_0^{(ij)}} \tilde{f}_j$$

*denote the order of* $\tilde{f}_j$ *on* $U_0^{(j)}$ *along* $U_0^{(ij)}$. *Then* $|f|_i \geqslant |f|_j \cdot |\pi|^{m_{ji}}$.

*Proof.* Without loss of generality, we may assume $i = 1$, $j = 0$. As we only need to consider the behaviour at $Y_K^{(0)}$ and $Y_K^{(1)}$, we may replace $U_K$ by its formal open subset given by the equation $|\zeta_2| = \ldots = |\zeta_s| = 1$, which we denote again by $U_K$. Thus, the reduction $U_0$ of $U_K$ has only two irreducible components $U_0^{(0)}$ and $U_0^{(1)}$. For assertion (i), we may assume $|f|_0 = 1$. Then $f$ reduces to a regular function $\tilde{f}$ on $U_0$ which does not vanish completely on $U_0^{(0)}$ and coincides with $\tilde{f}_0$ there. On the other hand, $\tilde{f} = 0$ on $U_0 \setminus U_0^{(0)}$, as $|f| < 1$ on $Y_K^{(1)}$. As the vanishing locus is closed, $\tilde{f}$ vanishes on the closure of $U_0 \setminus U_0^{(0)}$ in $U_0$, which is $U_0^{(1)}$. This proves (i).

On $U_0^{(0)}$, the ideal corresponding to $U_0^{(01)}$ is generated by $\tilde{\zeta}_1$. Thus, we may write $\tilde{f}_0 = \tilde{g}\tilde{\zeta}_1^{m_{01}}$, where $m_{01}$ is the order of $\tilde{f}_0$ along $U_0^{(1)}$, and $\tilde{g}$ is a regular function on $U_0^{(0)}$ which does not vanish completely on $U_0^{(01)}$. Now, consider the holomorphic function $g := \zeta_1^{-m_{01}} f$ on $U_K$. As $|\zeta_1| = 1$ on $Y_K^{(0)}$, we have

$$|g|_0 = |f|_0 \cdot |\zeta_1|_0^{-m_{01}} = |f|_0 = 1,$$

so $\tilde{g}_0$ coincides with $\tilde{g}$ on $U_0^{(0)}$. As $\tilde{g}$ does not vanish along $U_0^{(0)} \cap U_0^{(1)}$, assertion (i) yields $|g|_0 \leqslant |g|_1$. Thus, using $|\zeta_1| = |\pi|$ on $Y_K^{(1)}$, we get

$$|f|_1 = |\zeta_1|_1^{m_{01}} \cdot |g|_1 = |\pi|^{m_{01}} \cdot |g|_1 \geqslant |\pi|^{m_{01}} \cdot |g|_0 = |\pi|^{m_{01}} \cdot |f|_0.$$

This proves (ii). □

**Corollary 4.2.6.** *In the situation of Lemma 4.2.5, we have $m_{ji} + m_{ij} \geqslant 0$, where*

$$m_{ij} := \operatorname{ord}_{U_0^{(ij)}} \tilde{f}_i.$$

*Proof.* Reversing the roles of $i$ and $j$ yields $|f|_j \geqslant |f|_i \cdot |\pi|^{m_{ij}}$. Hence $1 \geqslant |\pi|^{m_{ij}+m_{ji}}$, from which the claim follows immediately. □

*Proof of Proposition 4.2.3.* If $f$ is not a constant, we may assume that $f(x_1) = 0$ for some $x_1 \in \Omega_K$. Otherwise, consider $f' = f - f(x_1)$; this is again a bounded holomorphic function on $\Omega_K$. Let $\Omega_0^{(1)}$ be a component of $\Omega_0$ containing the reduction $\tilde{x}_1$. Then $\tilde{f}_1$ vanishes at $\tilde{x}_1$. As $\Omega_0^{(1)}$ is proper, $\tilde{f}_1$ necessarily has a pole along some prime divisor of $\Omega_0^{(1)}$. But $\tilde{f}_1$ is regular over the non-singular locus of $\Omega_0^{(1)}$, so $\tilde{f}_1$ has a pole only along an intersection with some component $\Omega_0^{(2)}$. We choose an affinoid formal open subset $U_K$ of $\Omega_K$ as above, such that its special fibre $U_0$ has non-trivial intersection with $\Omega_0^{(1)} \cap \Omega_0^{(2)}$. Then we can apply Lemma 4.2.5 to see that $|f|_2/|f|_1 \geqslant |\pi|^{m_1} \geqslant 1/|\pi|$, where $m_1 < 0$ is the order of $\tilde{f}_1$ along the intersection. But then, by Corollary 4.2.6, $\tilde{f}_2$ has a zero along the intersection with $\Omega_0^{(1)}$ of order at least $-m_1$. By the same reasoning as above, $\tilde{f}_2$ has a pole somewhere, so we find a component $\Omega_0^{(3)}$ such that $|f|_3 \geqslant 1/|\pi| \cdot |f|_2 \geqslant 1/|\pi|^2 \cdot |f|_1$. Continuing this way, we find an infinite sequence of components $\Omega_0^{(1)}, \Omega_0^{(2)}, \dots$ such that $|f|_k \geqslant 1/|\pi|^{k-1}|f|_1$ for every $k \geqslant 1$. But then $f$ is unbounded. This proves the claim. □

## 4.3 The Picard Variety

In the following, let $X_K$ be a proper smooth rigid-analytic variety over $K$, together with a $K$-rational point $x_K$. Assume that $X_K$ has a strictly semi-stable formal model $X$ over the valuation ring $R$.

Let $\mathfrak{C}_K$ denote the category of pointed rigid-analytic varieties $(V_K, v_K)$, where $V_K$ is smooth and connected over $K$ and $v_k \in V_K(K)$ is a $K$-rational point. The morphisms in this category are the rigid morphisms respecting the points.

One defines the *Picard functor*

$$\mathrm{Pic}^0_{X_K/K} : \mathfrak{C}_K \to \text{(sets)}, \quad (V_K, v_K) \mapsto \mathrm{Pic}^0_{X_K/K}(V_K, v_K),$$

where

$$\mathrm{Pic}^0_{X_K/K}(V_K, v_K) := \left\{ \mathrm{Isoclass}(\mathcal{L}_K, \lambda) \;\middle|\; \begin{array}{c} \mathcal{L}_K \text{ line bundle on } X_K \times_K V_K, \\ \lambda : \mathcal{O}_{V_K} \xrightarrow{\sim} (x_K \times \mathrm{id}_{V_K})^* \mathcal{L}_K, \\ (\mathrm{id}_{X_K} \times v_K)^* \mathcal{L}_K \cong \mathcal{O}_{X_K} \end{array} \right\}$$

Hartl and Lütkebohmert proved in [21] that this functor is represented by a smooth connected group variety $(P_K, 1)$, which is an extension

$$1 \to T_K \to P_K \to Q_K \to 1,$$

of an abeloid rigid-analytic group $Q_K$ by an affine torus $T_K$; i.e. $Q_K$ is smooth, connected and proper. This means that there exists a line bundle $\mathcal{P}$ on $X_K \times P_K$ and an isomorphism $\lambda_{\mathcal{P}} : \mathcal{O}_{P_K} \xrightarrow{\sim} (x_K \times \mathrm{id}_{P_K})^* \mathcal{P}$ such that for any smooth rigid space $V_K$ and any pair $(\mathcal{L}, \lambda) \in \mathrm{Pic}^0_{X_K/K}(V_K)$, there exists a unique morphism $\varphi : V_K \to P_K$ and a unique isomorphism

$$(\mathcal{L}, \lambda) \xrightarrow{\sim} (\mathrm{id}_{X_K} \times \varphi)^* (\mathcal{P}, \lambda_{\mathcal{P}}).$$

The line bundle $\mathcal{P}$ is called the *Poincaré bundle.*

Moreover, the representing space $P_K$ is the 1-component of the general Picard functor, which is given by

$$\mathrm{Pic}_{X_K/K} : (\text{smooth rigid spaces}) \to (\text{sets}), \quad V_K \mapsto \mathrm{Pic}_{X_K/K}(V_K),$$

where

$$\mathrm{Pic}_{X_K/K}(V_K) := \left\{ \mathrm{Isoclass}(\mathcal{L}_K, \lambda) : \begin{array}{c} \mathcal{L}_K \text{ line bundle on } X_K \times_K V_K \\ \lambda : \mathcal{O}_{V_K} \xrightarrow{\sim} (x_K \times \mathrm{id}_{V_K}) * \mathcal{L}_K \end{array} \right\}$$

In this section, we will assume that $X_K$ has a totally degenerated formal model. We will use the construction of Hartl and Lütkebohmert in order to describe $\mathrm{Pic}^0_{X_K/K}$.

Let $\pi$ be a uniformizing parameter of $R$. We set $R_n := R/(\pi^{n+1})$. As $X_K$ is proper over $K$, we see that $X_n := X \times_R R_n$ is a proper flat $R_n$-scheme with geometrically reduced special fibre. Due to a theorem of Artin [1, Theorem 7.1], the functor $\mathrm{Pic}^0_{X_n/R_n}$ on the category of $R_n$-schemes locally of finite type is representable by an algebraic space $P'_n$ locally of finite type over $R_n$. This is a group scheme over $R_n$ due to [2, Theorem 3.5], as $R_n$ is artinian. Due to [12, I, Exposé $IV_A$, Proposition 2.4], it is of finite type. On every level $X_n \times_{R_n} P'_n$, we have a Poincaré bundle $\mathcal{P}'_n$. Note that $X_{n+1} \times_{R_{n+1}} R_n = X_n$; hence $P'_{n+1} \times_{R_{n+1}} R_n = P'_n$. Thus, we have a projection $P'_{n+1} \to P'_n$. We can now consider the direct limit

$$P' := \varinjlim P'_n.$$

then $P'$ is a formal scheme which is topologically of finite type over $R$. In the same way, setting $\mathcal{P}' := \varprojlim \mathcal{P}'_n$ yields the Poincaré bundle on $X \times P'$.

We will now give an explicit description of the Picard scheme $P'_0$ of the special fibre $X_0$ of $X$.

Let $V$ be an irreducible $k$-scheme, and let $X_0^{(1)}, \ldots, X_0^{(r)}$ denote the irreducible components of $X_0$. Let $X'_0$ be the disjoint union $\coprod X_0^{(i)}$, and let $p : X'_0 \to X_0$ be the projection. Furthermore, let $X''_0$ be the disjoint union $\coprod_{i<j} X_0^{(i)} \cap X_0^{(j)}$. Let $p_i : X''_0 \to X_0$ for $i = 1, 2$ be the projection onto the first resp. second coordinate, and let $q = p \circ p_1 = p \circ p_2$.

Now, let $\mathcal{L}$ be a line bundle on $X_0 \times V$. The pull-back $(p, \mathrm{id})^*\mathcal{L}$ is given by line bundles $\mathcal{L}_i$ on $X_0^{(i)} \times V$. Moreover, we have isomorphisms $\varphi_{ij} : \mathcal{L}_i|_{(X_0^{(i)} \cap X_0^{(j)}) \times V} \xrightarrow{\sim} \mathcal{L}_j|_{(X_0^{(i)} \cap X_0^{(j)}) \times V}$

which satisfy the cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on triple overlaps $(X_0^{(i)} \cap X_0^{(j)} \cap X_0^{(k)}) \times V$. We call $(\varphi_{ij})$ a *descent datum*.

The morphism $X_0' \times V \to X_0 \times V$ is neither flat, nor does it have a section. However, we will see in the following that descent in this situation has nice enough properties.

For the following result, see [20, Satz 4.8]:

**Lemma 4.3.1.** *Let $A = B[\zeta_0, \dots, \zeta_s]/(\zeta_0, \dots, \zeta_s)$, where $B$ is a $k$-algebra. Let $A_i := A/(\zeta_i)$ and $A_{ij} := A/(\zeta_i, \zeta_j)$. Then the following sequence is exact:*

$$\begin{array}{ccccccc} 0 \longrightarrow & A & \xrightarrow{\alpha} & A_0 \times \cdots \times A_s & \xrightarrow{\beta} & \bigoplus_{i<j} A_{ij} \\ & f & \longmapsto & (\bar f, \dots, \bar f) & & \\ & & & (\bar f_0, \dots, \bar f_s) & \longmapsto & (\dots, \bar f_i - \bar f_j, \dots) \end{array}$$

*Proof.* If $f \in \mathrm{Ker}(\alpha)$, then $f \in (\zeta_i)$ for all $i$. Thus $f = \zeta_0 \cdot f_1$. As $\zeta_0 \notin (\zeta_1)$, we have $f_1 \in (\zeta_1)$, hence $f = \zeta_0 \zeta_1 \cdot f_2$. Iteratively, we find $f = \zeta_0 \cdots \zeta_s \cdot f_{s+1}$, which vanishes in $A$. Hence, $\alpha$ is injective.

Let $f_i = \sum_\nu a_\nu^{(i)} \zeta$ such that $(\bar f_i) \in \mathrm{Ker}(\beta)$. We construct an element $f$ with $\bar f = \bar f_i$ as follows: Let

$$b_\nu := \begin{cases} a_\nu^{(i)} & \text{if } \nu_i = 0 \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$$

This is well-defined, as $a_\nu^{(i)} = a_\nu^{(j)}$ if $\nu_i = \nu_j = 0$. By construction, $f$ reduces to $\bar f_i$ modulo $\zeta_i$. This proves the claim. □

**Lemma 4.3.2.** *The functor $\mathcal{L} \to (\mathcal{L}_i, \varphi_{ij})$ from line bundles on $X_0 \times V$ to line bundles on $X_0^{(i)} \times V$ together with descent data is fully faithful.*

*Proof.* See also [10, §6, Prop. 1]. Let $\mathcal{L}$, $\mathcal{M}$ be line bundles on $X_0 \times V$. We want to show that the following sequence of canonical maps is exact:

$$0 \to \mathrm{Hom}_{X_0 \times V}(\mathcal{L}, \mathcal{M}) \to \mathrm{Hom}_{X_0' \times V}(p^*\mathcal{L}, p^*\mathcal{M}) \rightrightarrows \mathrm{Hom}_{X_0'' \times V}(q^*\mathcal{L}, q^*\mathcal{M}).$$

The assertion is local on $X_0 \times V$, so we may assume that $V$ is affine, say $V = \mathrm{Spec}(B)$. Moreover, as $X$ totally degenerated, we may replace $X_0$ by an open affine subset $\mathrm{Spec}(A')$ of $\mathrm{Spec}\, k[\zeta_1, \dots, \zeta_r]/(\zeta_1 \cdots \zeta_r)$. Then $X_0 \times V$ is isomorphic to an open subset $\mathrm{Spec}(A')$ of

$\mathrm{Spec}(A)$, with $A$ given as above. Let $A'_i = A'/(\zeta_i) = A_i \otimes_A A'$. Note that $A'_i$ is non-zero if and only if there is an irreducible component $X_0^{(i)}$ of $X_0$ given by $\zeta_i = 0$. Furthermore, set $A'_{ij} = A'/(\zeta_i, \zeta_j)$. Due to Lemma 4.3.1, the following sequence is exact:

$$0 \to A \to \oplus_i A_i \rightrightarrows \oplus_{ij} A_{ij}$$

As $A'$ is just a localization of $A$, and hence flat, this sequence stays exact if we replace $A$ by $A'$, $A_i$ by $A'_i$ and $A_{ij}$ by $A'_{ij}$ respectively. As $\mathcal{L}$ resp. $\mathcal{M}$ are coherent, they are given by invertible modules $L$ resp. $M$ on $X_0 \times V$. As $L$ and $M$ are flat, tensoring yields the following exact sequences

$$0 \to L \to L \otimes_{A'} (\oplus_i A'_i) \rightrightarrows L \otimes_{A'} (\oplus_{ij} A'_{ij})$$
$$0 \to M \to M \otimes_{A'} (\oplus_i A'_i) \rightrightarrows M \otimes_{A'} (\oplus_{ij} A'_{ij})$$

From the injectivity of the map $L \to L \otimes_{A'} (\oplus_i A'_i)$, we conclude that the canonical map

$$\mathrm{Hom}_{X_0 \times V}(\mathcal{L}, \mathcal{M}) \to \mathrm{Hom}_{X'_0 \times V}(p^*\mathcal{L}, p^*\mathcal{M})$$

is injective. Similarly, every homomorphism $L \otimes_{A'} (\oplus_i A'_i) \to M \otimes_{A'} (\oplus_i A'_i)$ corresponding to an element of $\mathrm{Ker}(p_1^*, p_2^*)$ restricts to an $A$-homomorphism $L \to M$. Hence, $\mathrm{Im}\, p^* \supset \mathrm{Ker}(p_1^*, p_2^*)$. The opposite inclusion is clear. From this, the claim follows. □

In the following, we want to describe $\mathrm{Pic}^0_{X_0}(V)$ by studying the corresponding descent data. Let $\mathcal{L}$ be a line bundle on $X_0 \times V$ corresponding to an element of $\mathrm{Pic}^0_{X_0}(V)$. Let $\mathcal{L}_i$ be the pull back of $\mathcal{L}$ to $X_0^{(i)} \times V$. As $X_0^{(i)}$ is a rational variety over $k$, we know that $\mathrm{Pic}^0_{X_0^{(i)}}$ is trivial. As the whole problem is local on $V$, we may assume that $V$ is affine, say $V = \mathrm{Spec}(B)$ and hence, that $\mathcal{L}_i$ is trivial. Thus, the descent datum $(\varphi_{ij})$ is given by a cocycle $(c_{ij})$ with $c_{ij} \in \Gamma((X_0^{(i)} \cap X_0^{(j)}) \times V, \mathcal{O}^\times_{X_0 \times V})$. However, as $X_0^{(i)}$ is proper, so is $X_0^{(i)} \cap X_0^{(j)}$, and we have in fact $c_{ij} \in \Gamma(X_0^{(i)} \cap X_0^{(j)}, \mathcal{O}_V^\times)$, where we identify $\mathcal{O}_V^\times$ with the constant sheaf $\Gamma(V, \mathcal{O}_V^\times)$ on $X_0$.

We associate to $X$ the *dual simplicial complex* $\Delta^D$ as follows: The vertices of $\Delta^D$ are the irreducible components of $X_0$. An $l$-simplex $[v_0, \ldots, v_l]$ corresponds to the intersection of the corresponding irreducible components. Now, let $G$ be an arbitrary abelian group. The simplicial cochain complex with coefficients in $G$ is then given by

$$0 \to \bigoplus_i \Gamma(X_0^{(i)}, G) \xrightarrow{\delta_0} \bigoplus_{i<j} \Gamma(X_0^{(i)} \cap X_0^{(j)}, G) \xrightarrow{\delta_1} \bigoplus_{i<j<k} \Gamma(X_0^{(i)} \cap X_0^{(j)} \cap X_0^{(j)}, G) \to \cdots$$

Let $H^1(\Delta^D, G) = \operatorname{Ker}\delta_1/\operatorname{Im}\delta_0$ denote the first cohomology group with coefficients in $G$. We write $H^1(\Delta^D, \mathbb{G}_{m,k})$ for the group functor $V \mapsto H^1(\Delta^D, \mathcal{O}_V^\times)$.

**Lemma 4.3.3.** *The group functor $H^1(\Delta^D, \mathbb{G}_{m,k})$ is isomorphic to a finite product of copies of $\mathbb{G}_{m,k}$ and various $\boldsymbol{\mu}_m$, where $\boldsymbol{\mu}_m$ denotes the group of $m$-th roots of unity.*

*Proof.* The universal coefficient theorem of cohomology (Proposition 2.3.1) yields

$$H^1(\Delta^D, \mathbb{G}_{m,k}) \cong \operatorname{Hom}(H_1(\Delta^D), \mathbb{G}_{m,k}),$$

and this isomorphism is functorial. As the simplicial complex $\Delta^D$ is finite, $H_1(\Delta^D)$ is a finitely generated abelian group. After a suitable choice of generators, we may write

$$H_1(\Delta^D) = \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_i\mathbb{Z}.$$

This yields the desired decomposition

$$H^1(\Delta^D, \mathbb{G}_{m,k}) = \mathbb{G}_{m,k}^r \oplus \boldsymbol{\mu}_{m_1} \oplus \cdots \oplus \boldsymbol{\mu}_{m_s}.$$

□

**Lemma 4.3.4.** *Let $p = \operatorname{char} k$. Then*

$$\operatorname{Pic}^0_{X_0} \cong \mathbb{G}_{m,k}^r \oplus \boldsymbol{\mu}_{p^{l_1}} \oplus \cdots \oplus \boldsymbol{\mu}_{p^{l_s}}.$$

*Proof.* At first, we consider the morphism $\operatorname{Pic}^0_{X_0}(V) \to H^1(\Delta^D, \mathbb{G}_{m,k})$, sending a line bundle on $X_0 \times V$ to its descent datum $(c_{ij})$. Changing the isomorphism $\mathcal{L}_i \xrightarrow{\sim} \mathcal{O}_{X_0^{(i)} \times V}$ corresponds to changing the cocycle $(c_{ij})$ by a coboundary. Hence, by Lemma 4.3.2, this morphism is injective.

On the other hand, let $(c_{ij}) \in H^1(\Delta^D, \mathbb{G}_{m,k})$. We want to construct a line bundle $\mathcal{L} \in \operatorname{Pic}_{X_0}(V)$ which has exactly this descent datum. We may cover $X_0 \times V$ with open affine subsets $Y^{(k)} = \operatorname{Spec}(A'^{(k)})$ such that $Y^{(k)}$ is isomorphic to an open subset of $\operatorname{Spec}(A)$ as above. We assume that $Y^{(k)}$ meets the components $X_0^{(1)}, \ldots, X_0^{(r)}$, with $X_0^{(j)} \cap Y^{(k)} = V(\zeta_j)$. By applying the Chinese Remainder Theorem, we can construct a rational function $l_k$ on $Y^{(k)}$ which is $1$ on $V(\zeta_0)$ and $c_{i0}$ on $V(\zeta_i)$ for $i > 0$. Repeating this for every $Y^{(k)}$ defines a Cartier divisor $(Y^{(k)}, l_k)$ on $X_0 \times V$ which gives rise to a line bundle $\mathcal{L}'$ on $X_0 \times V$. By construction, $\mathcal{L}$ induces the descent datum $(c_{ij})$ we started with. If $(c_{ij})$ comes from a line bundle $\mathcal{L}' \in \operatorname{Pic}^0_{X_0}(V)$, we have $\mathcal{L}' \cong \mathcal{L}$.

Thus, the canonical inclusion $\mathrm{Pic}^0_{X_0} \hookrightarrow \mathrm{Pic}_{X_0}$ factors as follows:

$$\begin{array}{ccc} \mathrm{Pic}^0_{X_0} & \hookrightarrow & H^1(\Delta^D, \mathbb{G}_{m,k}) \\ & \searrow & \downarrow \\ & & \mathrm{Pic}_{X_0} \end{array}$$

Hence, the identity component $\mathrm{Pic}^0_{X_0}$ of $\mathrm{Pic}_{X_0}$ agrees with the identity component of $H^1(\Delta^D, \mathbb{G}_{m,k})$. However, if $\mathrm{char}\, k = p$, the identity component of the group scheme $\boldsymbol{\mu}_{m_i}$ is given by $\boldsymbol{\mu}_{p^{l_i}}$, where $l_i$ is maximal such that $p^{l_i}$ divides $m_i$. Thus, the claim follows with Lemma 4.3.3. □

Now we can continue to construct the rigid analytic Picard variety $\mathrm{Pic}^0_{X_K/K}$. Note, that in general, the formal scheme $P'$ does not need to be flat over $R$. This is related to the fact that a line bundle on $X_n$ does not necessarily lift to a line bundle on $X_{n+1}$. By dividing out the nilpotent structure of $P'$ and the $\pi$-torsion, we get a closed subscheme

$$\bar{P}' := P'/(\mathcal{N} : \pi) \hookrightarrow P',$$

where, for any open subset $U \subset P'$, we define

$$\mathcal{N}(U) := \{f \in \mathcal{O}_{P'}(U)\ ;\ f \text{ nilpotent}\},$$
$$(\mathcal{N} : \pi)(U) := \{f \in \mathcal{O}_{P'}(U)\ ;\ \pi^n f \in \mathcal{N}(U) \text{ for some } n \geqslant 0\}.$$

Thus, $\bar{P}'$ is reduced and flat over $R$ and has a group structure which is induced from the group structure of $P'$. The pointed formal scheme $(\bar{P}', 1)$ represents the functor $\mathrm{Pic}^0_{X/R}$ of trivialized and rigidified line bundles on the category of pointed admissible formal $R$-schemes which are reduced and connected.

The generic fibre $\bar{P}'_{\text{rig}}$ of $\bar{P}'$ is geometrically reduced, and hence smooth. Its $1$-component $\bar{P}_K := (\bar{P}'_{\text{rig}})^0$ has finite index in $\bar{P}'_{\text{rig}}$. It is quasi-compact and hence has a smooth formal model $\bar{P}$ over $R$. The pointed rigid space $(\bar{P}_K, 1)$ represents the functor $\mathrm{Pic}^0_{X_K/K}$ on the category $\bar{\mathfrak{C}}_K$ of pointed rigid space $(V_K, v_K)$ over $K$, where $V_K$ is smooth and connected over $K$ and has a smooth formal model $V$ over $K$. The canonical map $\bar{P} \to \bar{P}' \to P'$ induces a finite surjective map $\bar{P}_0 \to P'_0$ on special fibres. Pulling back the line bundle $\mathcal{P}'$, we obtain the Poincaré bundle $\bar{\mathcal{P}}$ on $X \times_R \bar{P}$.

In our case, as $P'_0$ is affine, $\bar{P}_0$ is also affine, and smooth, as $\bar{P}$ is smooth. Thus, using Lemma 4.3.4, $\bar{P}_0$ is a torus $T_0 \cong \mathbb{G}^r_{m,k}$. But then $\bar{P}$ is a formal torus $\bar{T} \cong \bar{\mathbb{G}}^r_{m,R}$, and its generic fibre is given by

$$\bar{T}_K := \{(x_1, \ldots, x_r) \; ; \; |x_i| = 1\}$$

It is in a natural way embedded into the affine torus $\hat{T}_K := \mathbb{G}_{m,K}^r$. The rigidified Poincaré bundle $(\bar{\mathcal{P}}, \bar{\rho})$ on $X_K \times_K \bar{T}_K$ extends to a unique rigidified line bundle $(\hat{\mathcal{P}}_K, \hat{\rho})$ on $X_K \times_K \hat{T}_K$.

Now, let $\mathbb{K}$ denote the topological algebraic closure of $K$. We define

$$M := \{p \in \hat{T}_K(\mathbb{K}) \; ; \; (\mathrm{id}_{X_k} \times p)^{\times} \hat{\mathcal{P}} \cong \mathcal{O}_{X_K}\}.$$

Due to [21, Lemma 3.10], $M$ is a $K$-rational lattice in $\hat{T}_K(\mathbb{K})$. Moreover, $M$ satisfies $M \cap \bar{T}_K = \{1\}$. Dividing out the lattice $M$ yields a rigid group variety $T_K = \hat{T}_K / M$. Due to [21, Theorem 3.14], the pair $(T_K, 1)$ represents the functor $\mathrm{Pic}^0_{X_K}$ on the category of pointed rigid spaces $(V_K, v_K)$ where $V_K$ is *smooth* and connected, and $v_K$ is a $K$-rational point. Thus, we have shown the following result:

**Theorem 4.3.5.** *Let $X_K$ be the generic fibre of a totally degenerated formal scheme which is proper. On the category of smooth and connected rigid spaces, the Picard functor $\mathrm{Pic}^0_{X_K/K}$ is represented by a quotient $T_K/M$, where $T_K$ is a split torus, and $M$ is a lattice in $T_K$ such that $M \cap \bar{T}_K = \{1\}$.*

**Remark 4.3.6.** The lattice $M$ does not necessarily have full rank; i.e. $\mathrm{Pic}^0_{X_K/K}$ is not necessarily proper. A well-known example is the Hopf surface, which is discussed in §4.5.26. A new example, the *sheared torus*, will be presentend in §5.5.

## 4.4 Automorphic Functions

In the last section, we saw that the Picard variety of $X_K$ is given by $\mathbb{G}_{m,K}^r/M$, if $X_K$ has a totally degenerated formal model. The rank $r$ of the torus was given by the rank of $H^1(\Delta^D, \mathbb{G}_{m,k})$, where $H^1(\Delta^D, \mathbb{G}_{m,k})$ parametrizes the glueing of trivial line bundles on each irreducible component of $X_0$ along their intersections. However, in order to find a more suitable description of line bundles on $X_K$, it is better to see this from a dual point of view.

Namely, by [20, Proposition 3.8], the torus $\mathbb{G}_{m,K}^r$ can be recovered as follows: We choose a basis $n^{(1)}, \ldots, n^{(r)}$ of $H^1(X_K, \mathbb{Z})$, which is canonically isomorphic to $H^1(X_0, \mathbb{Z})$. However, due to Lemma 4.1.9, we have $H^1(X_0, \mathbb{Z}) = \check{H}^1(\mathfrak{U}, \mathbb{Z})$, where $\mathfrak{U} = \{U^{(i)}\}$ denotes the covering of $X$ as in Notation 4.1.4. Thus, the cocycles $n^{(k)} = (n_{ij}^{(k)})$ can be chosen with respect to the formal covering $\{U_K^{(i)}\}$ of $X_K$. Then, for $(t_1, \ldots, t_r) \in \mathbb{G}_{m,K}$, setting

$$(t_{ij}) := (t_1^{n_{ij}^{(1)}} \cdot \ldots \cdot t_r^{n_{ij}^{(r)}})$$

yields an element of $\check{H}^1(\mathfrak{U}, \mathbb{G}_{m,K})$ which describes a line bundle in $\mathrm{Pic}^0_{X_K/K}$. Any line bundle in $\mathrm{Pic}^0_{X_K/K}$ arises this way. Note, however, that not every element of $\check{H}^1(\mathfrak{U}, \mathbb{G}_{m,K})$ describes an element of $\mathrm{Pic}^0_{X_K/K}$, see also Lemma 4.4.6.

We will explain in the following how to interpret the torus in terms of the universal covering. Let $u : \Omega_K \to X_K$ be the universal covering of $X_K$, $\Gamma$ the group of Deck transformations.

**Definition 4.4.1.** A $\Gamma$*-automorphic form* on $\Omega$ is a meromorphic function $u$ such that for all $\gamma \in \Gamma$ there exists a constant $c(\gamma) \in K^\times$ with

$$u(\gamma(z)) = c(\gamma) \cdot u(z)$$

for all $z \in \Omega$. The mapping $c : \Gamma \to \mathbb{G}_{m,K}$, $\gamma \mapsto c(\gamma)$ is called the *factor of automorphy* of $f$. We denote the group of $\Gamma$-automorphic forms by $\Theta$, and the subgroup of invertible automorphic forms by $\Theta^\times$.

**Remark 4.4.2.** The factor of automorphy $c$ is automatically a group homomorphism. Thus, $c$ factors through the commutator factor group $\overline{\Gamma} := \Gamma/[\Gamma, \Gamma]$.

**Definition 4.4.3.** Let $L = \Omega_K \times \mathbb{A}^1$ denote the trivial line bundle on $\Omega_K$. A $\Gamma$*-linearization* $\alpha$ of $L$ is a $\Gamma$-action on $L$ of the form

$$\begin{array}{rcll} \alpha_\gamma : \Omega_K \times \mathbb{A}^1 & \longrightarrow & \Omega_K \times \mathbb{A}^1, & \gamma \in \Gamma \\ (x, a) & \longmapsto & (\gamma(x), e_\gamma(x) \cdot a), & \end{array}$$

where $e_\gamma \in \mathcal{O}_{\Omega_K}^\times$ and $\gamma \to e_\gamma$ is a $1$-cocycle for $\Gamma$, i.e.

$$e_{\gamma' \cdot \gamma}(x) = e_{\gamma'}(\gamma(x)) \cdot e_\gamma(x)$$

for all $x \in \Omega_K$. Two $\Gamma$-linearizations $\alpha$, $\alpha'$ are *isomorphic* if there exists an invertible function $f \in \mathcal{O}_{\Omega_K}^\times$ such that

$$e'_\gamma(x) = e_\gamma(x) \cdot f(\gamma(x))/f(x).$$

We say the linearization $\alpha$ is *of constant type* if $e_\gamma(x)$ is constant on $\Omega_K$. In that case, $\gamma \to e_\gamma$ is a group homomorphism from $\Gamma$ to $K^\times$.

**Lemma 4.4.4.** *Let $\mathcal{L}$ be a line bundle on $X_K$ which is given by a cocycle $(t_{ij}) \in \check{H}^1(\mathfrak{U}, \mathbb{G}_{m,K})$, where $\mathfrak{U} := \{U_K^{(i)}\}$. Then the pull-back $u^*\mathcal{L}$ on $\Omega_K$ is trivial.*

*Proof.* The cocycle $(t_{ij})$ pulls back to a cocycle on $\Omega_K$ with respect to the covering

$$\{\gamma(U_K^{(i)})\}_{\gamma\in\Gamma, i\in I},$$

which defines the line bundle $u^*\mathcal{L}$ on $\Omega_K$. The cocycle $(t_{ij})$ then yields a cocycle on $\Delta(\Omega)$ for the simplicial cohomology. However, as $\Delta(\Omega)$ is the universal covering of $\Delta(X)$ and hence simply connected, this cocycle is trivial on $\Delta(\Omega)$. Hence, it is trivial on $\Omega_K$, and $u^*\mathcal{L}$ is the trivial line bundle. □

We will now show how to explicitly construct a line bundle to any $\Gamma$-linearization $\alpha$ of constant type. Let $\alpha$ be given by a group homomorphism $c : \Gamma \to \mathbb{G}_{m,K}$. At first, we will construct a cocycle $(t_{ij}) \in \check{H}^1(\mathfrak{U}, \mathbb{G}_{m,K})$ from $c$. For any $U_K^{(i)}$, choose a connected component $\Omega_K^{(i)}$ of $u^{-1}(U_K^{(i)})$. Let $u_i : \Omega_K^{(i)} \to U_K^{(i)}$ denote the restriction of $u$ to $\Omega_K^{(i)}$. Then, for every pair $i, j$ such that $U_K^{(i)} \cap U_K^{(j)} \neq \emptyset$, there exists $\gamma_{ij} \in \Gamma$ with

$$u_j^{-1}(U_K^{(i)} \cap U_K^{(j)}) = \gamma_{ij}(u_i^{-1}(U_K^{(i)} \cap U_K^{(j)})).$$

If $U_K^{(i)} \cap U_K^{(j)} \cap U_K^{(k)} \neq \emptyset$, we have $\gamma_{ij} \cdot \gamma_{jk} = \gamma_{ik}$.

Now, we can define a cocycle $(t_{ij}) \in \check{H}^1(\mathfrak{U}, \mathbb{G}_{m,K})$ via $t_{ij} := c(\gamma_{ij})$. Let $\mathcal{L}(c)$ denote the line bundle on $X_K$ given by the cocycle $(t_{ij})$. Then $\mathcal{L}(c)$ is given by patching the trivial line bundles $U_K^{(i)} \times \mathbb{A}_K^1$ via the cocycle $(t_{ij})$.

**Lemma 4.4.5.** *The above construction yields a one-to-one correspondence between the following sets:*

(i) *Group homomorphisms $c : \Gamma \to \mathbb{G}_{m,K}$,*

(ii) *$\Gamma$-linearizations of the trivial line bundle on $\Omega_K$ of constant type,*

(iii) *Cocycles $(t_{ij}) \in \check{H}^1(\mathfrak{U}, \mathbb{G}_{m,K})$ with respect to the covering $\mathfrak{U} = \{U_K^{(i)}\}$ of $X_K$.*

*Proof.* The correspondence between (i) and (ii) follows from Definition 4.4.3. For the rest, let $\mathcal{L}(c)$ be defined as above. As $U_K^{(i)} \times \mathbb{A}_K^1$ is isomorphic to $\Omega_K^{(i)} \times \mathbb{A}_K^1$ via $u_i$, the line bundle

$\mathcal{L}(c)$ can also be described by patching $\Omega_K^{(i)} \times \mathbb{A}_K^1$ with $\Omega_K^{(j)} \times \mathbb{A}_K^1$ via the $\Gamma$-linearization

$$\begin{array}{ccc} \Omega_K^{(i)} \times \mathbb{A}_K^1 & & \Omega_K^{(j)} \times \mathbb{A}_K^1 \\ \cup & & \cup \\ u_i^{-1}(U_K^{(i)} \cap U_K^{(j)}) \times \mathbb{A}_K^1 & \longrightarrow & u_j^{-1}(U_K^{(i)} \cap U_K^{(j)}) \times \mathbb{A}_K^1 \\ (x,a) & \longmapsto & (\gamma_{ij}(x), c(\gamma_{ij}) \cdot x) \end{array}$$

Thus, $\mathcal{L}(c)$ is just the quotient of the trivial line bundle $\Omega_K \times \mathbb{A}_K^1$ by the given action of $\Gamma$.

For the converse, let $\mathcal{L}$ be a line bundle given by a cocycle $(t_{ij})$. Due to Lemma 4.4.4, its pull back $u^*\mathcal{L}$ to $\Omega_K$ is trivial; i.e. there is an isomorphism $u^*\mathcal{L} \xrightarrow{\sim} \Omega_K \times \mathbb{A}_K^1$. Moreover, as seen in the proof, it can be trivialized by a coboundary $(t_i)$ with values in $\mathbb{G}_{m,K}$. Note that $\Gamma$ acts canonically on $u^*\mathcal{L}$ such that the quotient of $u^*\mathcal{L}$ modulo this action is just the line bundle $\mathcal{L}$. This $\Gamma$-action carries over to the trivial line bundle $\Omega_K \times \mathbb{A}_K^1$ as a $\Gamma$-linearization. As the cocycle $(t_{ij})$ on $\Omega_K$ can be trivialized in $\mathbb{G}_{m,K}$, this $\Gamma$-linearization is constant. □

As isomorphism classes of line bundles correspond bijectively to isomorphism classes of $\Gamma$-linearizations, the following holds:

Now, let $T(\overline{\Gamma})$ denote the torsion subgroup of $\overline{\Gamma}$, and let $\widetilde{\Gamma} = \overline{\Gamma}/T(\overline{\Gamma})$. Then $\widetilde{\Gamma}$ is a free abelian group.

**Lemma 4.4.6.** *Let $\mathcal{L}(c)$ be a line bundle on $X_K$ given by a group homomorphism $c : \Gamma \to \mathbb{G}_{m,K}$. Then $\mathcal{L}(c)$ gives rise to a point of $\mathrm{Pic}^0_{X_K/K}$ if and only if $c$ factorizes through $\widetilde{\Gamma}$.*

*Proof.* Let $n^{(1)}, \dots, n^{(r)}$ denote a basis of $H^1(\mathfrak{U}, \mathbb{Z})$. As above, this induces a morphism

$$\mathbb{G}_{m,K}^r \hookrightarrow \check{H}^1(\mathfrak{U}, \mathbb{G}_{m,K}) \cong \mathrm{Hom}(\Gamma, \mathbb{G}_{m,K}).$$

Again, due to the universal coefficient theorem of cohomology (Proposition 2.3.1), we have

$$\check{H}^1(\mathfrak{U}, \mathbb{Z}) = H^1(\Delta(X), \mathbb{Z}) \cong \mathrm{Hom}(\Gamma, \mathbb{Z}).$$

Thus, after a suitable choice of generators for $\overline{\Gamma}$, we can write $\overline{\Gamma} = \bigoplus_{i=1}^s \mathbb{Z}/m_i\mathbb{Z} \oplus \mathbb{Z}^r$, where $r = \mathrm{rk}\, H^1(X(\Delta), \mathbb{Z})$, and hence

$$\mathrm{Hom}(\Gamma, \mathbb{G}_{m,K}) \cong \bigoplus_{i=1}^s \boldsymbol{\mu}_{m_i} \oplus \mathbb{G}_{m,K}^r,$$

where $\boldsymbol{\mu}_{m_i}$ denotes the group of $m_i$-th roots of unity. As $\mathbb{G}_{m,K}^r$ is connected, it is mapped into a connected subgroup of $\mathrm{Hom}(\Gamma, \mathbb{G}_{m,K})$. Thus, the image is exactly the torus part $\mathbb{G}_{m,K}^r$, which corresponds to those $c : \overline{\Gamma} \to \mathbb{G}_{m,K}$ which are trivial on the torsion part $T(\overline{\Gamma})$. Hence, every $c$ coming from an element of $\mathbb{G}_{m,K}^r$ factorizes through $\widetilde{\Gamma}$, and vice versa. □

**Lemma 4.4.7.** *The line bundle $\mathcal{L}(c)$ is isomorphic to the trivial line bundle on $X_K$ if and only if $c$ is the factor of automorphy of an invertible function $f$ on $\Omega_K$.*

*Proof.* By Definition 4.4.3, the $\Gamma$-linearization of constant type corresponding to $c$ is isomorphic to the trivial linearization if and only if there exists an invertible function $f \in \mathcal{O}_{\Omega_K}^{\times}$ such that

$$c(\gamma) = f(\gamma(x))/f(x)$$

for all $x \in \Omega_K$; i.e. $f$ is an invertible automorphic function with factor of automorphy $c$. □

**Definition 4.4.8.** We define the following groups:

- $\widehat{J} := \mathrm{Hom}(\widetilde{\Gamma}, \mathbb{G}_{m,K})$,
- $\overline{J} := \mathrm{Hom}(\widetilde{\Gamma}, \overline{\mathbb{G}}_{m,K})$
- $M := \{c \in \widehat{J}\,;\ c$ is a factor of automorphy of an invertible function $\}$.

**Lemma 4.4.9.** *For the subgroup $M$, we have $M \cap \overline{J} = \{1\}$.*

*Proof.* Let $c$ be the factor of automorphy of an invertible function $f$, and let $|c(\gamma)| = 1$ for all $\gamma \in \Gamma$. Then $|f(z)| = |f(\gamma(z))|$ for all $\gamma \in \Gamma$. For any vertex $v$ of $\Delta(X)$, choose a lift $v' \in \Delta(\Omega)$ of $v$. Let $F := \bigcup_{v \in \Delta(X)} \Omega_K(v')$. Then $F$ is a finite union of affinoid subsets. Hence, $|f|$ is bounded on $F$. Moreover, as $\|f\|_F = \|f\|_{\gamma(F)}$ for all $\gamma \in \Gamma$, we see that $|f|$ is bounded on the whole of $\Omega_K$. But then $f$ is constant, due to Proposition 4.2.3. Thus, $c$ is trivial, and the claim follows. □

In the following, we fix a basis $\gamma_1, \ldots, \gamma_r$ of $\widetilde{\Gamma}$. This yields an isomorphism

$$\widehat{J} \longrightarrow \mathbb{G}_{m,K}^r, \quad c \longmapsto (c(\gamma_1), \ldots, c(\gamma_n)).$$

We will always identify $\widehat{J}$ with $\mathbb{G}_{m,K}^r$ via this map.

**Lemma 4.4.10.** *$M$ is a lattice in $\widehat{J}$.*

*Proof.* As above, identify $\widehat{J}$ with $\mathbb{G}_{m,K}^r$. On the quotient $\widehat{J}/\overline{J}$, we have a valuation map

$$\mathrm{val} : \widehat{J}/\overline{J} \to \mathbb{Z}^r, \quad c \mapsto (-\log_{|\pi|}|c_1|, \ldots, -\log_{|\pi|}|c_r|),$$

this is an injective group homomorphism. Then $M$ is mapped bijectively to a subgroup of $\mathbb{Z}^r$, hence it is a lattice. □

**Lemma 4.4.11.** *Let $\Theta^\times$ denote the group of invertible automorphic forms on $\Omega_K$. Identify $\widehat{J}$ with $\mathbb{G}_{m,K}^r$ as above. Then*

$$\psi : \Theta^\times \longrightarrow M, \quad f \longmapsto (\gamma_1^*(f)/f, \ldots, \gamma_r^*(f)/f)$$

*is a group epimorphism with $\operatorname{Ker}\psi = K^\times$.*

*Proof.* Note that $\psi$ sends every invertible automorphic form $f$ to its factor of automorphy, so $\psi$ is surjective by the definition of $M$. If $f$ is an invertible automorphic form with trivial factor of automorphy, then a similar argument as in the proof of Lemma 4.4.9 shows that $f$ is constant. This proves the claim. □

Combining Lemma 4.4.5 and Lemma 4.4.6, we get the following result:

**Theorem 4.4.12.** *The functorial mapping*

$$\widehat{J} \longrightarrow \mathrm{Pic}^0_{X_K/K}, \quad c \longmapsto \mathcal{L}(c)$$

*is a group epimorphism with kernel $M$; i.e. the quotient $J := \widehat{J}/M$ represents the functor $\mathrm{Pic}^0_{X_K/K}$ on the category of smooth and connected rigid spaces.*

Note that, as already mentioned in Theorem 4.3.5, the lattice $M$ does not necessarily have full rank. In our situation, Lemma 4.4.11 yields the following result.

**Theorem 4.4.13.** *The rigid analytic Picard variety $\mathrm{Pic}^0_{X_K/K}$ is proper if and only if*

$$\operatorname{rk}\Theta^\times/K^\times = \operatorname{rk}\Gamma/[\Gamma,\Gamma];$$

*i.e. if and only if there are "enough" invertible automorphic forms.*

## 4.5 General Polytopal Domains

In this section, we will discuss a special class of totally degenerated formal schemes, which are built from polytopal domains.

**Definition 4.5.1.** Let $U_K = \mathrm{Sp}(A)$ be an affinoid variety such that $U_K$ is isomorphic to some affinoid polytopal domain. An affinoid subdomain $V \subset U_K$ is called a *face* of $U_K$, if there exists a function $f \in A^\times$ such that

$$V = \{x \in U_K \; ; \; |f(x)| = \max_{u \in U_K} |f(u)|\}$$

If $\varphi : U_K \to \mathbb{G}_m^n$ exhibits $U_K$ as an affinoid polytopal domain in $\mathbb{G}_m^n$, the coordinates $\zeta_1, \ldots, \zeta_n$ of $\mathbb{G}_m^n$ give rise to coordinates on $U_K$. Let $\sigma := \mathrm{val}(U_K) \subset \mathbb{R}^n$. If $V$ is a face of $U_K$ as above, then $f$ can be written as $f = c\zeta_1^{a_1} \cdot \ldots \cdot \zeta_n^{a_n}(1+h)$. Then $\tau := \mathrm{val}(V)$ is the subset of $\sigma$ where $u \mapsto \langle a, u\rangle$ assumes its maximum. Hence, $\tau$ is a face of $\sigma$. Hence, the above definition is just a way to characterize subsets of $U_K$ corresponding to faces $\tau$ of $\sigma$, without actually choosing coordinates on $U_K$.

**Definition 4.5.2.** A *general polytopal domain* is a separated rigid analytic space $X_K$ which has a covering by affinoid subsets $U_K^{(i)}$, where $U_K^{(i)}$ is isomorphic to an affinoid polytopal domain in $\mathbb{G}_m^n$ for some $n$, and $U_K^{(i)} \cap U_K^{(j)}$ is a collection of faces of $U_K^{(i)}$ resp. $U_K^{(j)}$.

**Remark 4.5.3.** Any covering by general polytopal domains $U_K^{(i)}$ as above is a formal covering, so it gives rise to an admissible formal scheme $X$ with generic fibre $X_K$. We call this a *general formal polytopal domain*.

**Example 4.5.4** (The Tate curve)**.** The *Tate curve* $\mathbb{G}_m/q^{\mathbb{Z}}$, where $q \in K$ with $0 < |q| < 1$ can be constructed as follows: Let

$$\begin{aligned}
U_K^{(1)} &:= \{z \in \mathbb{G}_m \; ; \; |q| \leqslant |z| \leqslant |q|^{1/2}\} \\
U_K^{(2)} &:= \{z \in \mathbb{G}_m \; ; \; |q|^{1/2} \leqslant |z| \leqslant 1\} \\
V_K^{(1)} &:= \{z \in \mathbb{G}_m \; ; \; |z| = |q|\} \subset U_K^{(1)} \\
V_K^{(2)} &:= \{z \in \mathbb{G}_m \; ; \; |z| = 1\} \subset U_K^{(2)}
\end{aligned}$$

We identify $V_K^{(2)}$ with $V_K^{(1)}$ via multiplication with $q$. Glueing along this identification and canonically along the intersection $U_K^{(1)} \cap U_K^{(2)}$ yields the analytic torus $\mathbb{G}_m/q^{\mathbb{Z}}$.

**Remark 4.5.5.** Being a general polytopal domain is a very restrictive condition. In dimension $1$, the Tate curve constructed above is the only example of a proper general polytopal domain.

Recall that, on $\mathbb{G}_m^n$, we have the valuation map

$$\mathrm{val} : \mathbb{G}_m^n \to \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (-\log|x_1|, \dots, -\log|x_n|).$$

Similarly, one can associate to a general polytopal domain $X_K$ a *valuation space*.

**Notation 4.5.6.** Let $\mathcal{O}^\times(1)$ be the subsheaf of $\mathcal{O}^\times$ on $X_K$ defined by

$$\mathcal{O}^\times(1)(U) := \{1 + h \; ; \; |h| < 1 \text{ on } U\}.$$

Let $S = \mathcal{O}^\times / \mathcal{O}^\times(1)$ denote the quotient sheaf.

**Lemma 4.5.7.** *Let $U_K$ be an affinoid polytopal domain of dimension $n$, then*

$$S(U_K) \cong (K')^\times \oplus \mathbb{Z}^n,$$

*where $(K')^\times = K^\times / \{1 + c \; ; \; |c| < 1\}$.*

*Proof.* As shown in Chapter 6, the sheaf $\mathcal{O}^\times(1)$ has trivial cohomology on $U_K$. Thus, we have $H^1(U_K, \mathcal{O}^\times(1)) = 0$, and hence $H^0(U_K, S) = H^0(U_K, \mathcal{O}^\times(1))/H^0(U_K, \mathcal{O}^\times)$. However, after a choice of coordinates $\zeta_1, \dots, \zeta_n$ on $U_K$, every element $f \in \mathcal{O}^\times$ can be written in the form $f = c \cdot \zeta_1^{m_1} \cdot \ldots \cdot \zeta_n^{m_n}(1+h)$, with $|h| < 1$ on $U_K$, $c \in K^\times$, where $c$ is unique up to multiplication by an element of the form $1+\varepsilon$, $|\varepsilon| < 1$. From this, the claim follows. □

In the following, let $\mathfrak{U} := \{U_K^{(i)}\}$ denote the covering of $X_K$ by affinoid polytopal domains. We assume that the covering is *fine*, i.e. that the intersection $U_K^{(i)} \cap U_K^{(j)}$ is connected for all $i, j$.

Any element $\xi \in S(U_K^{(i)})$ gives a unique mapping

$$f : U_K^{(i)} \to \mathbb{R}, \quad x \mapsto -\log|\xi(x)|.$$

Any choice of a basis $c, \xi^{(1)}, \dots, \xi^{(n)}$ of $S(U_K^{(i)})$ yields a mapping

$$(f_1, \dots, f_n) : U_K^{(i)} \to \mathbb{R}^n, \quad x \mapsto (-\log|\xi^{(1)}(x)|, \dots, -\log|\xi^{(n)}(x)|).$$

The image of $U_K^{(i)}$ is a $\Gamma$-rational polytope $\sigma^{(i)}$ in $\mathbb{R}^n$. A different choice of basis gives a different polytope, which can be obtained from $\sigma^{(i)}$ by an affine-linear transformation. Thus, we can associate to $U_K^{(i)}$ a topological polytope $\sigma^{(i)}$. A face of $U_K^{(i)}$ is identified with a face of $\sigma^{(i)}$. Thus, the set $\Delta := \{\sigma^{(i)}\}$ has the structure of a polytopal complex. Moreover, any $\xi \in S(U_K^{(i)})$ induces a real-valued polyhedral function $f$ on $\sigma^{(i)}$. Thus, we get a finitely generated abelian group $L_i$ of real-valued functions on every $\sigma^{(i)}$. We have $L_i \cong v(K^\times) \oplus \mathbb{Z}^r$, where $r = \dim \sigma^{(i)}$, and $v(K^\times)$ is the value group of $K^\times$. This yields an integral structure on $\Delta$. We call $\Delta$ the *valuation space* of $X_K$.

In contrast to polytopal domains in $\mathbb{G}_m^n$, a general polytopal domain $X_K$ is not uniquely determined by the associated polytopal complex $\Delta$, as $\Delta$ does not contain all necessary glueing data:

**Example 4.5.8.** Consider again the Tate curve. The sets $U_K^{(1)}$ and $U_K^{(2)}$ correspond to the line segments $[0, 1/2c]$ and $[1/2c, c]$ of $\mathbb{R}^1$, where $c = v(q)$. The polytopal complex is then given by these two line segments, glued via identifying the point $c$ with the point $0$. Note that this depends only on $c$, so if $|q_1| = |q_2|$, both give the same polytopal complex, although the corresponding Tate curves are not isomorphic.

From Proposition 2.4.1 and § 3.4, we get the following:

**Proposition 4.5.9.** *Let $X$ be a general formal polytopal domain over $R$. Then there exists a finite extension $R'$ of $R$ and a general polytopal domain $X'$ over $R'$ such that $X'$ is a strictly semi-stable formal model of $X_K \otimes_K K'$, where $K'$ is the field of fractions of $R'$. Especially, $X'$ is totally degenerated.*

Thus, in the following we will assume that $X_K$ has a strictly semi-stable formal model which is general polytopal. This means that $X_K$ has a formal covering $\mathfrak{U} := \{U_K^{(i)}\}$ where every $U_K^{(i)}$ is isomorphic to the affinoid standard simplex. This allows us to construct the universal covering $u : \Omega_K \to X_K$. In the situation of general polytopal domains, we can associate to $\Omega_K$ a *universal valuation map* $\mathrm{val} : \Omega_K \to \mathbb{R}^n$. We will do so in the following:

**Lemma 4.5.10.** *Let $\xi^{(i)}$ be a unit on $\Omega_K^{(i)}$. Then, for $j \neq i$ with $\Omega_K^{(i)} \cap \Omega_K^{(j)} \neq \emptyset$, there is a unit $\xi^{(j)}$ on $\Omega_K^{(j)}$ such that $\xi^{(i)}/\xi^{(j)} = 1 + h$, where $|h| < 1$ on $\Omega_K^{(i)} \cap \Omega_K^{(j)}$. The unit $\xi^{(j)}$ is unique up to multiplication by $1 + h'$, with $|h'| < 1$ on $\Omega_K^{(j)}$.*

*Proof.* As $\xi^{(i)}$ is a unit on $\Omega_K^{(i)} \cap \Omega_K^{(j)}$, we may write $\xi^{(i)} = c\zeta_1^{m_1} \cdot \ldots \zeta_n^{m_n}(1 + h)$, where $\zeta_1, \ldots, \zeta_n$ denote coordinates on $\Omega_K^{(j)}$. Thus, $\xi^{(j)} := c\zeta_1^{m_1} \cdot \ldots \cdot \zeta_n^{m_n}$ satisfies the conditions of the lemma. □

**Lemma 4.5.11.** *Under the conditions of the above lemma, if* $\Omega_K^{(i)} \cap \Omega_K^{(j)} \cap \Omega_K^{(k)} \neq \emptyset$ *, then* $\xi^{(j)}/\xi^{(k)} = 1 + h$ *holds on* $\Omega_K^{(j)} \cap \Omega_K^{(k)}$ *.*

*Proof.* On $\Omega_K^{(j)} \cap \Omega_K^{(k)}$ , we can write $\xi^{(j)} = c\zeta_1^{m_1} \cdot \ldots \cdot \zeta_n^{m_n}(1+h)$ with $|h| < 1$ on $\Omega_K^{(k)}$ , where $\zeta_1, \ldots, \zeta_n$ denote the coordinates on $\Omega_K^{(k)}$ . On the other hand, we have $\xi^{(i)} = \xi^{(j)}(1+g)$ on $\Omega_K^{(i)} \cap \Omega_K^{(j)}$ and $\xi^{(i)} = \xi^{(k)}(1+g')$ on $\Omega_K^{(i)} \cap \Omega_K^{(k)}$ . Without loss of generality, we may write $\xi^{(k)} = c'\zeta_1^{m'_1} \cdot \ldots \cdot \zeta_n^{m'_n}$ . Thus, on $\Omega_K^{(i)} \cap \Omega_K^{(j)} \cap \Omega_K^{(k)}$ , we have the identity

$$\begin{aligned}\xi^{(j)} &= c\zeta_1^{m_1} \cdot \ldots \cdot \zeta_n^{m_n}(1+h) \\ &= c'\zeta_1^{m'_1} \cdot \ldots \cdot \zeta_n^{m'_n}(1+g')/(1+g)\end{aligned}$$

Thus, we have $c = c'$ and $m_1 = m'_1, \ldots, m_n = m'_n$ . Hence, $\xi^{(j)} = \xi^{(k)}(1+h)$ on $\Omega_K^{(j)} \cap \Omega_K^{(k)}$ . This proves the claim. □

**Proposition 4.5.12.** *Let* $S = \mathcal{O}^\times/\mathcal{O}^\times(1)$ *as in Notation 4.5.6. Then* $S(\Omega_K) \cong K' \oplus \mathbb{Z}^n$ *, where* $K' := K^\times/\{1+c; |c| < 1\}$ *, and* $n = \dim \Omega_K$ *.*

*Proof.* By Lemma 4.5.7, it is enough to show that any element of $S(\Omega_K^{(i)})$ extends to a unique element of $S(\Omega_K)$ . Without loss of generality, we choose $i = 0$ . Let $\xi^{(0)} \in S(\Omega_K^{(0)})$ . For any $j$ , we choose an edge-path $\alpha := v_0, v_1, \ldots, v_j$ in $\Delta(\Omega)$ . Due to Lemma 4.5.10, we can iteratively choose elements $\xi^{(1)}, \ldots, \xi^{(j)}$ with $\xi^{(k)} \in S(\Omega_K^{(k)})$ such that $\xi^{(k)} = \xi^{(k+1)}$ in $S(\Omega_K^{(k)} \cap \Omega_K^{(k+1)})$ . Due to Lemma 4.5.11, the choice of $\xi^{(j)}$ does not depend on the equivalence class of $\alpha$ . As $\Delta(\Omega_K)$ is simply connected, all such edge-paths are equivalent. Hence, $\xi^{(j)}$ is well-defined. Repeating this construction for every $j$ , we can glue all $\xi^{(j)}$ together to a unique $\xi \in S(\Omega_K)$ which extends $\xi^{(i)}$ . On the other hand, every $\xi \in S(\Omega_K)$ restricts to a unique $\xi^{(i)} \in S(\Omega_K^{(i)})$ . This shows $S(\Omega_K^{(i)}) \cong S(\Omega_K)$ for every $i$ , which proves the claim. □

Again, fix coordinates $\zeta_1, \ldots, \zeta_n$ on $\Omega_K^{(i)}$ for some fixed $i$ . Let $\xi_1, \ldots, \xi_n$ denote the images of $\zeta_1, \ldots, \zeta_n$ in $S(\Omega_K)$ . Then

$$\mathrm{val} : \Omega_K \to \mathbb{R}^n, \quad x \mapsto (-\log_{|\pi|} |\xi_1(x)|, \ldots, -\log_{|\pi|} |\xi_n(x)|)$$

is a well-defined function on $\Omega_K$ .

**Lemma 4.5.13.** *For every* $j$ *,* $\mathrm{val}(\Omega_K^{(j)})$ *is an* $n$ *-simplex in* $\mathbb{R}^n$ *of multiplicity* $1$ *. Its vertices are given by* $u_0^{(j)}, \ldots, u_n^{(j)}$ *, where*

$$u_i^{(j)} = (-\log|\xi_1|_i, \ldots, -\log|\xi_n|_i),$$

*where* $|\cdot|_i$ *denotes the norm over the* $i$ *-th irreducible component of* $\Omega_0^{(j)}$ *, as in Definition 4.2.4.*

*Proof.* Choose a set of coordinates $\zeta_1, \ldots, \zeta_n$ on $\Omega_K^{(j)}$ . Set $\zeta_0 := \pi/\zeta_1 \cdot \ldots \cdot \zeta_n$ . Then $\Omega_K^{(j)}$ is given in $\mathbb{G}_{m,K}^n$ by $|\zeta_i| \leqslant 1$ for $i = 0, \ldots, n$ . Thus, under the valuation map corresponding to $\zeta_1, \ldots, \zeta_n$ , $\Omega_K^{(j)}$ is mapped to an $n$ -simplex $\sigma^{(j)}$ with multiplicity $1$ in $\mathbb{R}^n$ . As $\xi_i = \varphi(\zeta_i)$ for a suitable automorphism $\varphi$ of $S(\Omega_K^{(j)})$ , the image of $\Omega_K^{(j)}$ under $\mathrm{val}$ is given by the image of the simplex $\sigma^{(j)}$ under the corresponding linear transformation, which does not change the multiplicity. This proves the claim. □

Thus, the image of $\Omega_K$ under $\mathrm{val}$ is a polytopal complex in $\mathbb{R}^n$ , which we will again denote by $\Delta$ .

**Lemma 4.5.14.** *There is a bijective correspondence between irreducible components of* $\Omega_0$ *and vertices of* $\Delta$ *. Every irreducible component* $\Omega_{0,u}$ *of* $\Omega_0$ *is a proper toric variety. Its fan is given by the fan generated by* $\sigma_i - u$ *,* $i = 1, \ldots, r$ *, where* $\sigma_1, \ldots, \sigma_r$ *are the polytopes in* $\Delta$ *containing* $u$ *.*

*Proof.* The bijective correspondence is clear from Lemma 3.1.10. Now, let $u$ be a vertex of $\Delta$ , and let $\Omega_{0,u}$ be the corresponding irreducible component. We may assume without loss of generality that $u = 0$ . Let $\sigma_i$ be a polytope containing $u$ , and let $\Omega_K^{(i)}$ denote the corresponding affinoid part of $\Omega_K$ . Let $\Omega_{0,u}^{(i)} := \Omega_{0,u} \cap \Omega_0^{(i)}$ . As $\xi_1, \ldots, \xi_n$ differ only by elements of type $1 + h$ on $\Omega_0^{(i)} \cap \Omega_0^{(j)}$ , they reduce to well-defined regular functions $\tilde{\xi}_1, \ldots, \tilde{\xi}_n$ on $\Omega_{0,u}$ . Let $T_K := \mathrm{val}^{-1}(0)$ , this is a formal open subdomain of $\Omega_K$ . Its reduction $T_0$ is a torus, which is given by

$$T_0 = \operatorname{Spec} k[\tilde{\xi}_1^{\pm 1}, \ldots, \tilde{\xi}_n^{\pm 1}].$$

Thus, the action of $T_0$ on itself extends to an action on every $\Omega_{0,u}^{(i)}$ . As these actions agree on $\Omega_{0,u}^{(i)} \cap \Omega_{0,u}^{(j)}$ , we get an action of $T_0$ on $\Omega_{0,u}$ . This makes $\Omega_{0,u}$ into a toric variety, which is proper, as it is isomorphic to an irreducible component of $X_0$ . Again, it follows from Lemma 3.1.10 that the fan of cones is given as claimed. □

In the following, we choose a field extension $K'$ of $K$ such that $|K'| = \mathbb{R}$ . Then, for any affinoid part $\Omega_K^{(i)}$ of $\Omega_K$ , the valuation map $\mathrm{val} : \Omega_K^{(i)} \times_K K' \to \sigma \subset \mathbb{R}^n$ is surjective.

**Lemma 4.5.15.** *The image of* $\Omega_{K'} := \Omega_K \times_K K'$ *under* val *is open in* $\mathbb{R}^n$ .

*Proof.* Let $v = \mathrm{val}(x)$ for some $x \in \Omega_K \times K'$ . If $v$ is a vertex of $\Delta$ , then $v$ is an interior point of $\mathrm{val}(\Omega_{K'})$ by Lemma 4.5.14, as the corresponding irreducible component is proper, and the cone at $u$ has support $\mathbb{R}^n$ . Now, assume that $v$ is not a vertex of $\Delta$ . Let $\tau$ be the unique face of $\Delta$ such that $v$ lies in the relative interior of $\tau$ , and let $u$ be a vertex of $\tau$ . By the same reasoning, the fan of cones at $u$ has support $\mathbb{R}^n$ . As $\tau$ is contained in this cone, $\tau$ must hence be the intersection of polytopes $\sigma_1, \ldots, \sigma_r$ of $\Delta$ with $\dim \sigma_i = n$ . But then every point of $\tau$ is an interior point of $\mathrm{val}(\Omega_{K'})$ , which proves the claim. □

In the following, we will show that a general polytopal domain as above does not contain a copy of $\mathbb{A}^1$ . In the following, a morphism $\varphi : X \to Y$ between rigid analytic varieties will be called *affinoid*, if the inverse image of any affinoid subset $U \subset Y$ is again affinoid in $X$ . By [5, Prop. 9.4.4./1], any finite morphism (and hence, any closed immersion) is affinoid.

**Proposition 4.5.16.** *Let* $X$ *be an analytic variety, and let* $\varphi : \mathbb{A}^1 \to X$ *be an affinoid morphism. Then* $X$ *is not a polytopal domain.*

In the following, we will always assume that $X$ is an arbitrary polytopal domain.

**Lemma 4.5.17.** *Let* $\varphi : D := \mathbb{D}^1(r) \to X$ *be an affinoid morphism. Let* $U_\sigma \subset X$ *be an affinoid polytopal domain, and let* $D'$ *be a non-empty connected component of* $\varphi^{-1}(U_\sigma)$ *. If* $D'$ *is isomorphic to a disc in* $D$ *, then already* $D' = D$ *.*

*Proof.* For the contrary, we may assume $D' = \mathbb{D}^1(r')$ for some $r' < r$ . If $\varphi^{-1}(U_\sigma)$ is disconnected, let $D'_1, \ldots, D'_m$ denote the other connected components. Let $\zeta$ denote the coordinate on $D$ , then $\zeta$ has no zeros on $D'_1, \ldots, D'_m$ . Set $r_i := \min_{D'_i} |\zeta|$ . We have $r_i \geqslant r'$ . By the Maximum Modulus Principle, there exists $x_i \in D'_i$ with $|x_i| = r_i$ . As $D'_i$ is disjoint from $D'$ , we see that $r_i > r'$ for all $i$ . Choosing $r''$ with $r' < r'' < \min_i r_i$ , we see that $D'$ is the only connected component of $\varphi^{-1}(U_\sigma)$ meeting $\mathbb{D}^1(r'')$ . After replacing $r$ with $r''$ and restricting $\varphi$ to $\mathbb{D}^1(r'')$ , we may assume that $\varphi^{-1}(U_\sigma) = D'$ is connected.

Let $\zeta_i$ denote the $i$ -th coordinate on $\mathbb{G}_m^n \supset U_\sigma$ . Then $f_i := \varphi^*\zeta_i$ is a unit on $D'$ for $i = 1, \ldots, n$ . We can write $f_i = c_i(1 + h_i)$ with $|h_i| < 1$ on $D'$ . Set $s_i := |c_i| \cdot \|h_i\|_{D'} < |c_i|$ , where $\| \cdot \|_{D'}$ denotes the supremum norm on $D'$ . We have $\varphi(D') \subset \mathbb{D}^n(y, \underline{s})$ , where $\mathbb{D}^n(y, \underline{s})$ denotes the closed polydisc with radii $s_1, \ldots, s_n$ and centre $y = (y_1, \ldots, y_n) = \varphi(0)$ . Note that the *open* polydisc $\mathring{\mathbb{D}}^n(y, \underline{c})$ with radii $|c_1|, \ldots, |c_n|$ is still contained in $U_\sigma$ .

Thus, $\varphi(D \setminus D')$ is contained in $X \setminus \mathring{\mathbb{D}}^n(y, \underline{c})$, and we have $\varphi(D) \subset X' := (X \setminus \mathring{\mathbb{D}}^n(y, \underline{c})) \cup \mathbb{D}^n(y, \underline{s})$. Note that $X'$ is a disconnected admissible subset of $X$.

As $D \neq D'$, we have $\varphi^{-1}(X \setminus \mathring{\mathbb{D}}^n(y, \underline{c})) \neq \emptyset$, and we can write $D$ as a disconnected admissible subset

$$D = \varphi^{-1}(X \setminus \mathbb{D}^n(y, \underline{c})) \cup D',$$

which is absurd, since $D$ is connected. This proves the claim. □

In the following, we will need some results on reductions of standard domains in $\mathbb{P}^1_K$:

**Definition 4.5.18.** A *standard domain* in $\mathbb{P}^1_K$ is an affinoid subset $C = \mathbb{P}^1_K \setminus \bigcup_{i=0}^r B_i$, where $B_i$ is an open disc in $\mathbb{P}^1_K$.

If $\infty \notin C$, then we may assume $\infty \in B_0$, so that $\mathbb{P}^1_K \setminus B_0$ is a closed disc $D \subset \mathbb{A}^1_K$. Hence, in that case, we have $C = D \setminus \bigcup_{i=1}^r B_i \subset \mathbb{A}^1_K$.

For the reduction of a standard domain, we cite the following results; cf. [16, III.2 and V.2].

**Lemma 4.5.19.** *Let $C$ be a standard domain.*

(i) *The canonical reduction of $C$ consists of finitely many components $\tilde{C}_1, \dots, \tilde{C}_r$, where each $\tilde{C}_i$ is isomorphic to a Zariski-open subset of $\mathbb{A}^1_k$. Moreover, every intersection of components is* quasi-normal*; i.e. the local ring is isomorphic to*

$$k[[T_1, \dots, T_s]]/(T_iT_j)_{i \neq j}.$$

(ii) *$C$ has a unique stable reduction; i.e. every component is isomorphic to a Zariski-open subset of $\mathbb{P}^1_k$, every singularity is an ordinary double point, and every component isomorphic to $\mathbb{P}^1_k$ meets the other components in at least three points.*

(iii) *Every semi-stable reduction of $C$ can be derived from the stable reduction by blowing up points. The intersection graph of any semi-stable reduction is a tree.*

(iv) *Let $\tilde{C}$ be any reduction of $C$, and let $\overline{\tilde{C}}$ be its compactification. Then $\overline{\tilde{C}} \setminus \tilde{C}$ consists of exactly $r+1$ points, each* missing point *corresponding to one open disc $B_i$ as in Definition 4.5.18.*

For the next step of the proof, we will need a special version of the Maximum Modulus Principle for units.

**Definition 4.5.20.** Let $C$ be a standard domain in $\mathbb{A}^1$. Let $\pi : C \to \tilde{C}$ be the canonical reduction of $C$, and let $\tilde{C}_1, \dots, \tilde{C}_r$ denote the irreducible components of $\tilde{C}$. A *peripheral domain* of $C$ is a formal open subset $P \subset C$ such that

$$P = \pi^{-1}\left(\tilde{C} \setminus \bigcup_{i \in I} \tilde{C}_i\right)$$

for some index set $I \subset \{1, \dots, r\}$.

**Example 4.5.21.** (i) Let $C = \{|\pi| \leqslant |\zeta| \leqslant 1\}$ be an annulus with height $|\pi| < 1$. The reduction $\tilde{C}$ of $C$ consists of two copies of $\mathbb{A}^1_k$, intersecting in an ordinary double point; cf. Example 3.1.12. Thus, $C$ has three peripheral domains: $P_1 = \{|\zeta| = |\pi|\}$, $P_2 = \{|\zeta| = 1\}$, and $P_3 = C$.

(ii) Let $C = \{|\zeta| = 1\}$, then $\tilde{C} \cong \mathbb{A}^1_k \setminus \{0\}$, and the only peripheral domain of $C$ is $C$ itself.

(iii) Let $C = \mathbb{D}^1$, then $\tilde{C} = \mathbb{A}^1_k$, and the only peripheral domain of $C$ is $C$ itself.

**Lemma 4.5.22** (Maximum Modulus Principle for units)**.** *Let $f$ be a unit on a standard domain $C \subset \mathbb{A}^1$. Then the set $P := \{x \in C \; ; \; |f(x)| = |f|_{\sup}\}$ is a peripheral set of $C$. If $|f|_{\sup} = 1$, keeping the situation of Definition 4.5.20, we have $P = \pi^{-1}(\tilde{C} \setminus \bigcup_{i \in I} \tilde{C}_i)$, where $I = \{i \; ; \; \tilde{f}|_{\tilde{C}_i} = 0\}$.*

In the three cases of Example 4.5.21, this result is immediate. Namely, in cases (i) and (ii), we can write $f = \zeta^n(1 + h)$ with $n \in \mathbb{Z}$, and $|h| < 1$, whereas in (iii), we may write $f = c(1 + h)$ with $|h| < 1$, and the claim follows directly from this representation.

*Proof of Lemma 4.5.22.* We may assume $|f|_{\sup} = 1$, so $f$ reduces to a non-zero element $\tilde{f}$ on $\tilde{C}$. The set where $|f|$ assumes its maximum on $C$ is the formal open subset $P = \pi^{-1}(\tilde{P})$, where $\tilde{P} := \{\tilde{x} : \tilde{f}(\tilde{x}) \neq 0\} \subset \tilde{C}$. Let $\tilde{x} \in \tilde{C}$ with $\tilde{f}(\tilde{x}) = 0$. It remains to show that $\tilde{x} \in \tilde{C}_i$ for some $i \in I$, where $I$ is defined as in Definition 4.5.20.

At first, assume that $\tilde{x}$ is a smooth point of $\tilde{C}$ which lies on the component $\tilde{C}_i$. The formal fibre $C_+(\tilde{x})$ is isomorphic to the open unit disc $D = \{|\zeta| < 1\}$; see [6, Prop. 2.2]. Assume that $\tilde{x}$ is an isolated zero of $\tilde{f}$, say of order $m$. As in the proof of [6, Prop. 3.1], one can show that $f$ has $m$ zeros on $D$, which is a contradiction, since $f$ is a unit. This shows that $\tilde{f}$ vanishes everywhere on $\tilde{C}_i$.

Now, let $\tilde{x}$ be a singular point with $\tilde{f}(\tilde{x}) = 0$. For the contrary, assume that for every $\tilde{C}_i$ with $\tilde{x} \in \tilde{C}_i$, there exists a point on $\tilde{C}_i$ where $\tilde{f}$ does not vanish. By the first part of the proof, we see that $\tilde{f}$ has no zeros on $\mathrm{Reg}(\tilde{C}_i)$, where $\mathrm{Reg}(\tilde{C}_i)$ denotes the non-singular locus of $\tilde{C}_i$. Then the set

$$\tilde{E} := \bigcup_{\tilde{x}\in\tilde{C}_i} \mathrm{Reg}(\tilde{C}_i) \cup \{\tilde{x}\}$$

is open and connected in $\tilde{C}$. Let $E := \pi^{-1}(\tilde{E})$. By the above, we have $|f| < 1$ on $C_+(\tilde{x})$ and $|f| = 1$ on $E \setminus C_+(\tilde{x})$. As $f$ is a unit on $C$, we may consider $g := 1/f$ on $E$. Let $c := \|g\|_{\sup}$. Then $\widetilde{(g/c)}(\tilde{x}) \neq 0$ and $\widetilde{(g/c)} = 0$ on $\tilde{E} \setminus \tilde{x}$. This is a contradiction, as $\tilde{E} \setminus \{\tilde{x}\}$ is not closed in $\tilde{E}$. □

This allows us to show the following result:

**Lemma 4.5.23.** *Let $\varphi : \mathbb{A}^1 \to X$ be an affinoid morphism. Let $U_\sigma \subset X$ be an affinoid polytopal domain, and let $C$ be a connected component of $\varphi^{-1}(U_\sigma)$. If $\tau$ is a face of $\sigma$, then $\varphi^{-1}(U_\tau) \cap C$ is a peripheral domain of $C$.*

*Proof.* Let $\varphi|_C$ be given by units $f_1, \dots, f_n$ of $\mathcal{O}(C)$. If $\tau$ is a face of $\sigma$, there is a linear function $g := m_1x_1 + \dots + m_nx_n$ on $\sigma$ such that $g$ assumes its minimum exactly on $\tau$. Thus, the element $f = f_1^{m_1} \cdots f_n^{m_n}$ assumes its maximal value on $\varphi^{-1}(U_\tau) \cap C$, if the latter is not the empty set. As $f$ is a unit, the claim follows from the Maximum Modulus Principle 4.5.22. □

**Lemma 4.5.24.** *Let $\varphi : D := \mathbb{D}^1(r) \to X$ be an affinoid morphism. Then $\varphi(D) \subset U_\sigma$ for some affinoid polytopal domain $U_\sigma \subset X$.*

*Proof.* For any maximal polytope $\sigma$, let $V_\sigma := \varphi^{-1}(U_\sigma)$. As $D$ is quasi-compact, we may assume that the covering $\mathfrak{V} = \{V_\sigma\}$ of $D$ is finite. Without loss of generality, we will assume further that every $V_\sigma$ is connected; otherwise we split $V_\sigma$ into connected components $V_{\sigma,1}, \dots V_{\sigma,s}$. We choose a semi-stable reduction $\pi : D \to \tilde{D}$ such that the formal structure on $D$ given by $\tilde{D}$ is finer than the one given by the formal covering $\mathfrak{V}$; i.e. every $V_\sigma$ is formal open with respect to $\tilde{D}$.

Due to Lemma 4.5.19, the incidence graph of irreducible components of $\tilde{D}$ is a tree, with one component $\tilde{D}_0$ isomorphic to $\mathbb{P}^1$ minus one point, and all other components $\tilde{D}_i$, $i > 0$ isomorphic to $\mathbb{P}^1$. By fixing $\tilde{D}_0$ as its root, we can define an orientation on the tree.

Now, assume that the assertion is false. Let $\tilde{V}_{\sigma_0}$ meet $\tilde{D}_0$ for some $\sigma_0$. By Lemma 4.5.17, $V_{\sigma_0}$ is not a disc in $D$. As $V_{\sigma_0}$ is connected, it is a standard domain, and hence it is isomorphic to a closed disc minus $t$ open discs, where $t \geqslant 1$. Due to Lemma 4.5.19, the reduction $\tilde{V}_{\sigma_0}$ has $t+1 \geqslant 2$ missing points; one for each open disc in $\mathbb{P}^1_K \setminus V_{\sigma_0}$. One of these missing points, say $\tilde{x}_1$, is contained in $\tilde{D}$. Let $\tilde{D}_1$ be the irreducible component of $\tilde{D}$ such that $\tilde{x}_1 \in \tilde{D}_1$ and $\tilde{D}_1 \cap \tilde{V}_{\sigma_0}$ is open in $\tilde{V}_{\sigma_0}$. Choose $\tilde{V}_{\sigma_1}$ with $\tilde{x}_1 \in \tilde{V}_{\sigma_1}$. As $\tilde{V}_{\sigma_0} \cap \tilde{V}_{\sigma_1} = \tilde{V}_\tau$, where $\tau = \sigma_0 \cap \sigma_1$ is a common face of both $\sigma_1$ and $\sigma_2$, we see from Lemma 4.5.23 that $\tilde{V}_{\sigma_0} \cap \tilde{V}_{\sigma_1}$ contains the non-singular locus of $\tilde{D}_1$. Hence, $\tilde{x}_1$ is a double point of $\tilde{D}$ lying on an irreducible component of $\tilde{D}$ disjoint from $\tilde{V}_{\sigma_0}$.

Again, by Lemma 4.5.17, $V_{\sigma_1}$ is not a disc in $D$. By the same reasoning as above, we see that $\tilde{V}_{\sigma_1}$ has at least two points missing. We have to show that one of these is contained in $\tilde{D}$ and lies downwards from $\tilde{x}_1$. Let $\tilde{T}$ be the open subset of $\tilde{D}$ lying downwards from $\tilde{x}_1$. For the contrary, we assume that $\tilde{T}$ is contained in $\tilde{V}_{\sigma_1}$. Note that $\tilde{T}$ is again a tree consisting of projective lines and one affine line where $\tilde{x}_1$ is the missing point. Hence, $T := \pi^{-1}(\tilde{T})$ is a disc in $D$. As $\tilde{V}_{\sigma_0}$ is connected and contains no points lying downwards from $\tilde{x}_1$, we see that $\tilde{V}_{\sigma_0}$ is disjoint from $\tilde{T}$. Set $E := V_{\sigma_0} \cup V_{\sigma_1}$, then $\tilde{E}$ is an open subset of $\tilde{D}$ containing $\tilde{T}$. In the following, we will only look at $\tilde{E}$. We may change $\tilde{E}$ by blowing down $\tilde{T}$. As $T$ is a disc in $E$, this yields another semi-stable reduction $\tilde{E}'$ of $E$ such that $\tilde{x}_1$ is a non-singular point of $\tilde{E}'$. Moreover, $V_{\sigma_0}$ and $V_{\sigma_1}$ are still formal open in $E$ with respect to the corresponding formal topology, as $\tilde{T} \subset \tilde{V}_{\sigma_1}$ and $\tilde{T} \cap \tilde{V}_{\sigma_0} = \emptyset$. Applying Lemma 4.5.23 as before to $\tilde{V}'_{\sigma_0}$ and $\tilde{V}'_{\sigma_1}$, we see that $\tilde{x}$ still has to be a singular point of $\tilde{E}'$, which is a contradiction. This proves that one of the missing points of $\tilde{V}_{\sigma_1}$ lies downwards from $\tilde{x}_1$.

Continuing inductively, we construct an infinite sequence of points $(\tilde{x}_i)$ with $\tilde{x}_{i+1}$ lying downwards from $\tilde{x}_i$. This is obviously a contradiction, as $\mathfrak{V}$ is a finite covering. Hence $D = V_{\sigma_0}$ for some $\sigma_0$, which proves the claim. □

We can now complete the proof of Proposition 4.5.16.

*Proof of Proposition 4.5.16.* Let $\varphi : \mathbb{A}^1 \to X$ be an affinoid morphism. We choose an admissible covering $\mathbb{A}^1 = \bigcup_{n\in\mathbb{N}} \mathbb{D}^1(r_n)$ with $r_n \to \infty$. The restriction $\varphi_n : \mathbb{D}^1(r_n) \to X$ of $\varphi$ to $\mathbb{D}^1(r_n)$ is again affinoid. By Lemma 4.5.24, there exists $\sigma_n$ such that $\mathbb{D}^1(r_n)$ is mapped into $U_{\sigma_n}$. We will always choose the unique minimal $\sigma_n$ such that this holds. In that case, $\sigma_n \subset \sigma_{n+1}$ holds for all $n$; i.e. $\sigma_n$ is a face of $\sigma_{n+1}$. Since $\dim \sigma_n$ is bounded by $\dim X$, the sequence $\sigma_0 \subset \sigma_1 \subset \dots$ is stationary. Hence, there exists $\sigma_N$ such that $\varphi$ maps $\mathbb{A}^1$ into $U_{\sigma_N}$. Let $\zeta_i$ denote the $i$-th coordinate on $\mathbb{G}_m^n \supset U_{\sigma_N}$, then $f_i := \varphi^*\zeta_i$ is a unit on $\mathbb{A}^1$; especially on every $\mathbb{D}^1(r_n)$. Hence, we can write $f_i = c_i(1+h_i)$ with $|h_i| < 1$ everywhere on

$\mathbb{A}^1$. But then $h_i$ is a constant, so $\varphi$ is a constant morphism. In that case, $\varphi^{-1}(U_{\sigma_N}) = \mathbb{A}^1$ is not affinoid, which is a contradiction. This proves the claim. □

This yields the following result:

**Proposition 4.5.25.** *Let $\Omega_K$ be the universal covering of a general polytopal domain. Then $\Omega_K$ does not contain an analytic subvariety isomorphic to $\mathbb{A}^1$.*

**Example 4.5.26.** Let $q \in K^\times$ with $|q| < 1$. Let $\gamma$ denote the action on $\mathbb{A}^2_K \setminus \{0,0\}$ given by

$$\gamma(z_1, z_2) = (qz_1, qz_2).$$

The quotient $H_K := (\mathbb{A}^2_K \setminus \{0,0\})/\gamma^{\mathbb{Z}}$ is called the *rigid-analytic Hopf Surface*. We will show in § 5.3 that $H_K$ has a totally degenerated formal model with universal covering $\Omega_K := \mathbb{A}^2_K \setminus \{0,0\}$. As the vanishing locus of $\zeta_1 - 1$ in $\Omega_K$ is isomorphic to $\mathbb{A}^1$, we see that $H_K$ is not a general polytopal domain.

# Chapter 5

# Examples

In this chapter, we will apply the methods of the previous chapter in order to find the Picard variety for some rigid-analytic varieties. For the first three examples, this is well known; we will see that our approach agrees with the classical results. Afterwards, we will discuss two new examples for rigid-analytical varieties with totally-degenerate models in dimension two, where we can easily calculate the Picard variety using our methods.

## 5.1 Mumford Curves

In [28], David Mumford describes the $p$-adic uniformization of curves of genus $g \geqslant 2$ with degenerate reduction. These degenerate curves can be described analytically by taking a copy of $\mathbb{P}^1$ minus $g$ pairs of open disks and identifying the boundaries of each pair. This leads to the study of Schottky groups.

In this section, we will first give a short review of the analytic construction of such a Mumford curve, based on [16]. As Manin and Drinfeld [26] have shown, the identity component of the Picard variety is an analytic torus $\mathbb{G}_{m,K}^g/M$ of dimension $g$, where $M$ is a lattice in $\mathbb{G}_{m,K}^g$ of full rank $g$. This construction can be made explicit by using automorphic forms, as described in the previous chapter.

**Remark 5.1.1.** The group of automorphisms of $\mathbb{P}^1(K)$ is $\mathrm{PGL}(2, K)$, where a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ operates on $\mathbb{P}^1$ via $z \mapsto \frac{az+b}{cz+d}$.

**Definition 5.1.2.** Let $\Gamma$ be a subgroup of $\mathrm{PGL}(2, K)$. An element $p \in \mathbb{P}^1$ is called a *limit point* of $\Gamma$ if there exists $q \in \mathbb{P}^1$ and an infinite sequence $\{\gamma_n\} \subset \Gamma$ with $\gamma_n \neq \gamma_m$ for $m \neq n$, such that $\lim \gamma_n(q) = p$. The group $\Gamma$ is called *discontinuous*, if the following two conditions hold:

(i) The set of limit points does not equal $\mathbb{P}^1$.

(ii) For all $p \in \mathbb{P}^1$, the closure of the orbit $\Gamma p$ in $\mathbb{P}^1$ is compact.

**Remark 5.1.3.** Condition (i) implies that $\Gamma$ is a discrete subgroup of $\mathrm{PGL}(2, K)$. Namely, if $\Gamma$ is not discrete, we have a sequence $\{\gamma_n\}$ with $\lim \gamma_n = \gamma$ for some $\gamma$. But then $\gamma_n' := \gamma_n \gamma^{-1}$ satisfies $\lim \gamma_n'(p) = p$ for all $p \in \mathbb{P}^1$, so every point $p \in \mathbb{P}^1$ is a limit point.

**Definition 5.1.4.** A subgroup $\Gamma$ of $\mathrm{PGL}(2, K)$ is called a *Schottky group*, if the following conditions hold:

(i) $\Gamma$ is finitely generated,

(ii) $\Gamma$ has no elements of finite order (other than $1$)

(iii) $\Gamma$ is discontinuous.

Now, take $2g$ disjoint open disks $B_1, \ldots, B_g$, $C_1, \ldots, C_g$ with radii in $|K|$ such that the corresponding closed disks, which we will denote by $B_i^+$ and $C_i^+$ respectively, are still disjoint. Set $F_K := \mathbb{P}^1 \setminus \bigcup B_i \setminus \bigcup C_i$. We assume in the following that $\infty \in F_K$.

For every $i = 1, \ldots, r$, there exists $\gamma_i \in \mathrm{PGL}(2, K)$ such that $\gamma_i(\mathbb{P}^1 \setminus B_i) = C_i^+$ and $\gamma_i(\mathbb{P}^1 \setminus B_i^+) = C_i$; i.e. $\gamma_i$ maps the boundary of $B_i$ to the boundary of $C_i$.

Set $\Gamma$ be the subgroup of $\mathrm{PGL}(2, K)$ generated by $\gamma_1, \ldots, \gamma_g$, then $\Gamma$ is a Schottky-group with $\gamma_1, \ldots, \gamma_g$ as free generators. Moreover, if we set $\Omega_K := \bigcup_{\gamma \in \Gamma} \gamma F$, then

$$\Omega_K = \mathbb{P}^1 \setminus \{ \text{ limit points of } \Gamma \ \}.$$

We call the analytic quotient $X_K := \Omega_K / \Gamma$ a *Mumford curve of genus* $g$.

The set $F_K$ defined above is a fundamental domain for $X_K$. The analytic structure on the quotient $\Omega_K/\Gamma$ is given as follows: We take a suitable formal covering $\{U_K^{(1)}, \ldots, U_K^{(r)}\}$ of $F_K$, such that every $U_K^{(j)}$ is a formal open subset of an annulus. After a suitable extension of $K$, we may assume that each corresponding annulus has height $\pi$. The $\{U_K^{(j)}\}$ are then glued together by identifying the boundary of $B_i$ with the boundary of $C_i$ via $\gamma_i$. We may further assume that the covering $\{U_K^{(j)}\}$ is fine enough so that the intersection of the sets $\{U_K^{(j)}\}$ in $X_K$ is always connected. Thus, $X_K$ is the generic fibre of a totally degenerated formal scheme, and $\Omega_K$ is the universal covering of $X_K$ with $\Gamma$ the group of deck transformations.

We can now make the theory of automorphic functions on $\Omega_K$ rather explicit. For $a, b \in \Omega$, we define

$$\theta_{a,b}(z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}$$

This defines a meromorphic function on $\Omega$. For $\gamma \in \Gamma$, we set $u_\gamma(z) := \theta_{a,\gamma(a)}(z)$. We cite the following results without proof:

**Lemma 5.1.5.** (i) $\theta_{a,b}$ *is an automorphic function with constant factor of automorphy.*

(ii) *The definition of* $u_\gamma$ *is independent of the choice of* $a$.

(iii) $u_\gamma \cdot u_{\gamma'} = u_{\gamma \circ \gamma'}$.

(iv) $u_\gamma$ *is an invertible automorphic function.*

(v) $u_\gamma = u_{\gamma'}$ *if and only if* $\gamma \equiv \gamma' \mod [\Gamma, \Gamma]$, *where* $[\Gamma, \Gamma]$ *denotes the commutator subgroup of* $\Gamma$.

(vi) $u_\gamma$ *is constant if and only if* $\gamma \in [\Gamma, \Gamma]$.

This shows the following:

**Proposition 5.1.6.** *Let* $X_K$ *be a Mumford curve of genus* $g$. *Then the rigid analytic Picard variety* $\mathrm{Pic}^0_{X_K/K}$ *is isomorphic to* $\mathbb{G}^g_{m,K}/M$, *where* $M$ *is a lattice of rank* $g$ *in* $\mathbb{G}^g_{m,K}$.

*Proof.* Due to (v) and (vi) in the above lemma, we have an embedding $\bar{\Gamma} \to \Theta^\times / K^\times$, where $\bar{\Gamma} := \Gamma/[\Gamma, \Gamma]$ is the commutator factor group of $\Gamma$, and $\Theta^\times$ is the group of invertible automorphic forms. Hence, $\mathrm{rk}\,\Gamma = \mathrm{rk}\,\Theta^\times / K^\times$. The claim now follows with Theorem 4.4.13. □

## 5.2 Analytic Tori

Let $T_K$ be a split torus of rank $g$, and let $M$ be a split lattice of full rank in $T_K$. Then the quotient $A_K := T_K/M$ is a rigid-analytic group variety. These analytic tori are studied in detail in [8, §2]. We will desribe these results in brief and show how to interpret the construction of $\mathrm{Pic}^0(A_K)$ in terms of automorphic functions.

It is well-known that the rigid-analytic Picard variety of $A_K$ is just the dual $A'_K$. We will make this explicit in the following section by applying again the theory of automorphic forms.

Let $M' = \mathrm{Hom}(T_K, \mathbb{G}_{m,K})$ denote the character group of $T_K$; it is a split lattice in $T_K$ of rank $g$. This yields a bilinear pairing

$$\langle\cdot,\cdot\rangle : M' \times T_K \to \mathbb{G}_{m,K}, \quad \langle m', x\rangle = m'(x).$$

After a choice of coordinates $\zeta_1, \ldots, \zeta_n$, we may identify $T_K$ with $\mathbb{G}_{m,K}^g$, and $M'$ with $\mathbb{Z}^n$. Let $\mathrm{val} : \mathbb{G}_{m,K}^g \to \mathbb{R}^n$ denote again the valuation map. Then $M$ is mapped bijectively to a lattice in $\mathbb{R}^n$, which we will denote again by $M$. We construct a $M$-invariant decomposition of $\mathbb{R}^n$ into $n$-simplices of volume $1$. This decomposition can be guaranteed by means of Proposition 2.4.1. Over a suitable finite extension of $K$, this yields a totally degenerated formal model $T$ for $T_K$, such that the quotient of $T$ by $M$ is a totally degenerated formal model for $A_K$. Hence, $T_K$ is the universal covering of $A_K$, and $M$ is the group of deck transformations.

The key ingredient for the construction of the Picard variety is the following:

**Lemma 5.2.1.** *Let $c : M \to \mathbb{G}_{m,K}$ be a group morphism. There exists an automorphic function $f$ with factor of automorphy $c$ if and only if there exists a character $m' \in M'$ such that $c(m) = \langle m', m\rangle$ for all $m \in M$.*

*Proof.* If $m' \in M'$ is a character, then we have

$$m'(mx) = m'(m) \cdot m'(x) = \langle m', m\rangle \cdot m'(x)$$

for all $m \in M$, so $m'$ is an invertible automorphic form with factor of automorphy $c(m) := \langle m', m\rangle$. On the other hand, every invertible function on $T_K$ is a character up to an element of $K^\times$. As the factor of automorphy ignores scaling by elements of $K^\times$, the claim follows. □

We may thus identify the lattice $M'$ with the group of all automorphy factors coming from invertible automorphic functions. This yields the following result:

**Theorem 5.2.2.** *Let $T'_K := \mathrm{Hom}(M, \mathbb{G}_{m,K})$ be the split torus with character group $M$. Then the Picard variety $\mathrm{Pic}^0_{X_K/K}$ of $A_K = T_K/M$ is represented by the quotient $A'_K = T'_K/M'$.*

Indeed, we have a description of all line bundles as follows:

**Proposition 5.2.3.** *There is a one-to-one correspondence between isomorphism classes of line bundles* $\mathcal{L}$ *on* $\mathbb{G}_m^n/M$ *, and* $M$ *-linearisations* $\alpha$ *of the trivial line bundle* $\mathbb{G}_m^n \times \mathbb{A}^1$ *on* $\mathbb{G}_m^n$ *. The* $M$ *-linearisations can be desribed by pairs* $(\lambda, r)$ *, where* $\lambda : M \to M'$ *is a group homomorphism and* $r : M \to \mathbb{G}_m$ *satisfies*

$$\langle \lambda(m_2), m_1 \rangle = r(m_1 + m_2) \cdot r(m_1)^{-1} \cdot r(m_2)^{-1}$$

*for all* $m_1, m_2 \in M$ *. The action* $\alpha$ *corresponding to* $(\lambda, r)$ *is given by*

$$\begin{aligned} \alpha_m : \mathbb{G}_m^n \times \mathbb{A}^1 \to \mathbb{G}_m^n \times \mathbb{A}^1, \quad m \in M \\ (x, a) \mapsto (x + m, r(m) \cdot \langle \lambda(m), x \rangle \cdot a) \end{aligned}$$

*Two pairs* $(\lambda_1, r_1)$ *and* $(\lambda_2, r_2)$ *define isomorphic line bundles on* $\mathbb{G}_m^n/M$ *if and only if* $\lambda_1 = \lambda_2$ *and there exists some* $m \in M'$ *such that* $r_2(m) = \langle m', m \rangle \cdot r_1(m)$ *for all* $m \in M$ *.*

By the above description, there exists an analogue of Riemann's period relations as follows:

**Theorem 5.2.4.** *A line bundle* $\mathcal{L}$ *on* $A_K$ *is ample if and only if the corresponding quadrativ form* $\langle m, \lambda(m) \rangle$ *is positive definite; i.e.* $|\langle \lambda(m), m \rangle| < 1$ *if* $m \neq 0$ *. Especially,* $A_K$ *is algebraizable if and only if there exists a group homomorphism* $\lambda : M \to M'$ *such that* $\langle \lambda(m), m \rangle$ *is positive definite.*

## 5.3 The Hopf Surface

The following example has been studied by Mustafin in [29] and gives an example of a proper smooth rigid-analytic variety whose Picard variety is not proper.

Let $q \in K^\times$ with $|q| < 1$ . Let $\gamma$ denote the action on $\mathbb{A}_K^2 \setminus \{(0,0)\}$ given by

$$\gamma(z_1, z_2) = (qz_1, qz_2).$$

The quotient $H_K := (\mathbb{A}_K^2 \setminus \{(0,0)\})/\gamma^{\mathbb{Z}}$ is called the *rigid-analytic Hopf Surface*.

**Lemma 5.3.1.** *The Hopf surface* $H_K$ *has a totally degenerated formal model.*

*Proof.* This has been constructed explicitly by H. Voskuil in his doctoral thesis; see [34]. We will explain the construction. Assume that $q = \pi^k$ for some $k \geqslant 3$ ; we can achieve this after a suitable finite extension of $K$ if necessary. For $i = 1, \ldots, k$, we choose affinoid subsets $F_i, G_i$ of $\mathbb{A}^2_K \setminus \{(0,0)\}$ as follows:

$$F_i := \{(z_1, z_2) \; ; \; |\pi|^i \leqslant |z_1| \leqslant |\pi|^{i-1}, |z_2| \leqslant |z_1|\},$$
$$G_i := \{(z_1, z_2) \; ; \; |\pi|^i \leqslant |z_2| \leqslant |\pi|^{i-1}, |z_1| \leqslant |z_2|\}.$$

Then $\{\gamma^r(F_i), \gamma^s(G_j) \; ; \; i, j = 1, \ldots, k; r, s \in \mathbb{Z}\}$ is a formal covering of $\mathbb{A}^2_K \setminus \{(0,0)\}$, which induces a formal covering of $H_K$ by copies of $F_i, G_i$. We claim that this covering gives a totally degenerated formal model of $\mathbb{A}^2_K \setminus \{(0,0)\}$. Namely, consider $F_1 = \operatorname{Sp} A_1$, where

$$A_1 = K\langle \zeta_1, \pi/\zeta_1, \zeta_2/\zeta_1 \rangle \cong K\langle \zeta_1, \pi/\zeta_1, \eta \rangle.$$

Hence, $F_1$ is isomorphic to a product of a rigid-analytic disc with an annulus of height $|\pi|$; hence, the canonical model of $F_1$ is totally degenerated. The same holds by analogy for all $F_i, G_i$; hence, the formal model given by this covering is totally degenerated. As $H_K$ is covered by copies of $F_i, G_i$, the same holds for $H_K$. □

**Lemma 5.3.2.** *In the context of the previous chapter, the universal covering of $H_K$ is $\mathbb{A}^2 \setminus \{(0,0)\}$, and the group of deck transformations is $\Gamma = \gamma^{\mathbb{Z}}$.*

*Proof.* This should be clear from the construction, but we will check explicitly that our notion of universal covering and deck transformation yields what we expect.

From the proof of the previous lemma, we see that $H_K$ is covered by the images of the sets $F_i$, $G_i$ under the projection map. We construct the nerve $\Delta(H)$ corresponding to this formal structure on $H_K$. For $i = 1, \ldots, k$, let $a_i$, $b_i$ denote the vertices corresponding to $F_i$ resp. $G_i$, and set $a_0 = a_k$, $b_0 = b_k$. Then the nerve $\Delta(H)$ consists of the tetrahedra $[a_i, b_i, a_{i+1}, b_{i+1}]$ for $i = 0, \ldots, k$, together with all its faces. Now, remove the tetrahedron $[a_0, b_0, a_k, b_k]$, leaving the edges $[a_0, b_0]$ and $[a_k, b_k]$ intact. One checks that the resulting simplicial complex is simply connected. Therefore, the universal covering of $\Delta(H)$ is given by joining copies of this situation as follows:

For $i = 1, \ldots, r$, $l \in \mathbb{Z}$, let $a_i^{(l)}, b_i^{(l)}$ be copies of $a_i, b_i$ respectively. Construct the tetrahedra $[a_i^{(l)}, b_i^{(l)}, a_{i+1}^{(l)}, b_{i+1}^{(l)}]$ for $i = 1, \ldots, k-1$, and $[a_k^{(l)}, b_k^{(l)}, a_1^{(l+1)}, b_1^{(l+1)}]$. This yields a simply connected complex $\Delta'$. The deck transformation group is generated by the automorphism $\gamma$ with $\gamma(a_i^{(l)}) = a_i^{(l+1)}$ and $\gamma(b_i^{(l)} = b_i^{(l+1)}$. According to this configuration, we construct the universal covering of $H_K$ by gluing a new copy of $F_1, G_1$ to $F_k$, $G_k$ respectively, and continuing from there. This, corresponds to multiplication by $q$ in both coordinates, so we get the desired covering of $\mathbb{A}^2_K \setminus \{(0,0)\}$. □

**Remark 5.3.3.** In Example 4.5.26, we showed that $H_K$ is not a general polytopal domain; i.e. we can not choose a covering $\mathfrak{U} = \{U_K^{(i)}\}$ such that every $U_K^{(i)}$ is isomorphic to an affinoid polytopal domain.

**Proposition 5.3.4.** *For the rigid analytic Picard variety of the Hopf surface, we have*

$$\mathrm{Pic}^0_{X_K/K} \cong \mathbb{G}_{m,K}.$$

*Proof.* As discussed above, $\mathbb{A}^2_K \setminus \{(0,0)\}$ is the universal covering of $H_K$, and $\Gamma$ is its group of deck automorphisms. As $\Gamma \cong \mathbb{Z}$, $\mathrm{Pic}^0_{X_K/K}$ is isomorphic to a quotient of $\mathbb{G}_{m,K}$ by a lattice $M$, where the rank of $M$ is given by the rank of invertible $\Gamma$-automorphic functions modulo constants. Let $g$ be an invertible analytic function on $\mathbb{A}^2_K \setminus \{(0,0)\}$. Then $g$ is invertible on $\mathbb{G}^2_{m,K}$; hence $g = c\zeta_1^{k_1}\zeta_2^{k_2}$ for some $c \in K^\times$, $k_1, k_2 \in \mathbb{Z}$. This is only invertible on $\mathbb{A}^2_K \setminus \{(0,0)\}$ if $k_1 = k_2 = 0$; i.e. if $g$ is a constant. This shows that there are no non-trivial invertible $\Gamma$-automorphic functions. Thus, the claim follows directly from Theorem 4.4.12. □

## 5.4 A Rigid Analytic Klein Surface

In the following, we will construct a new example of a rigid analytic variety with a totally degenerated formal model. The construction is analogous to the well-known construction of the Klein bottle.

Let $q_1, q_2 \in K^\times$ with $|q_1|, |q_2| < 1$. Consider two automorphisms $\gamma_1, \gamma_2$ of $\mathbb{G}^2_m$, given by

$$\begin{aligned}\gamma_1 :&(z_1, z_2) \mapsto (q_1 z_1, q_2/z_2)\\ \gamma_2 :&(z_1, z_2) \mapsto (z_1, q_2 z_2)\end{aligned}$$

Let $\Gamma := \langle \gamma_1, \gamma_2 \rangle$.

**Proposition 5.4.1.** *The quotient $X_K = \mathbb{G}^2_m/\Gamma$ exists as a rigid-analytic variety. We call $X_K$ a* rigid-analytic Klein surface.

*Proof.* We consider the following Weierstrass domain in $\mathbb{G}_m^n$:

$$F_K := \{(z_1, z_2)\ ;\ |q_1| \leqslant |z_1| \leqslant 1, |q_2| \leqslant |z_2| \leqslant 1\}.$$

Consider the valuation map $\mathrm{val} : \mathbb{G}_m^2 \to \mathbb{R}^2$. Under $\mathrm{val}$, the domain $F_K$ is mapped to the rectangle with vertices $(0,0)$, $(0,c_2)$, $(c_1,0)$ and $(c_1,c_2)$, where $c_i := -\log|q_i| > 0$. We may identify $\gamma_1, \gamma_2$ with the following affine-linear transformations of $\mathbb{R}^2$:

$$\begin{aligned}\gamma_1 :& (x_1, x_2) \mapsto (x_1 + c_1, c_2 - x_2)\\ \gamma_2 :& (x_1, x_2) \mapsto (x_1, c_2 + x_2).\end{aligned}$$

Thus, we see that $\gamma_1, \gamma_2$ each send one edge of $\mathrm{val}(F_K)$ to its opposite edge, where $\gamma_1$ reverses the direction. This is exactly analogous to the construction of the classical Klein bottle. We may thus view the Klein bottle as the *valuation space* of $X_K$. From the classical case, one knows that $\mathrm{val}(F_K)$ is a fundamental domain of the $\Gamma$-action on $\mathbb{R}^2$. Hence, $F_K$ is a fundamental domain for the $\Gamma$-action on $\mathbb{G}_m^2$, and the quotient is constructed by identifying the affinoid subsets

$$\{|z_1| = 1\}, \{|z_1| = |q_1|\}, \{|z_2| = 1\}, \{|z_2| = |q_2|\}$$

of $F_K$ via $\gamma_1$, $\gamma_2$. Hence, the quotient exists as a rigid-analytic variety. □

**Remark 5.4.2.** The automorphisms $\gamma_1, \gamma_2$ satisfy $\gamma_1 \circ \gamma_2 = \gamma_2^{-1} \circ \gamma_1$. As in the classical case, the group $\Gamma$ is not abelian; its commutator subgroup is given by $[\Gamma, \Gamma] = \langle \gamma_2^2 \rangle$. Hence, the abelianization $\overline{\Gamma} = \Gamma/[\Gamma, \Gamma]$ of $\Gamma$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where $\gamma_1$ generates the free part and $\gamma_2$ generates the torsion part.

From the construction, one sees that $X_K$ is a general polytopal domain. Thus, by Proposition 4.5.9, after a finite extension $R'$ of $R$, $X_K$ has a totally degenerate model $X$. Moreover, it is clear that $\mathbb{G}_m^2$ is the universal covering of $X_K$, and $\Gamma$ is its group of Deck transformations.

**Theorem 5.4.3.** *For the rigid analytic Picard variety of the Klein surface, we have*

$$\mathrm{Pic}^0_{X_K/K} \cong \mathbb{G}_{m,K}/q_1^{\mathbb{Z}}$$

*Proof.* Let $\widetilde{\Gamma}$ denote the torsion free part of $\overline{\Gamma}$. Then $\widetilde{\Gamma}$ is isomorphic to $\mathbb{Z}$ with generator $\gamma_1$. This shows that $\mathrm{Pic}^0_{X_K/K}$ is a quotient of $\mathbb{G}_{m,K}$ by a lattice $M$. To determine the lattice $M$, we have to look for invertible automorphic functions on the universal covering $\mathbb{G}_{m,K}^2$.

Consider the coordinate $\zeta_1$. Then $\gamma_1^* \zeta_1^k = q_1^k \zeta_1^k$, and $\gamma_2^* \zeta_1^k = \zeta_1^k$, so $\zeta_1^k$ is an automorphic function with factor of automorphy given by $c(\gamma_1) = q_1^k$ and $c(\gamma_2) = 1$. The claim follows now directly with Theorem 4.4.12. □

**Theorem 5.4.4.** *The rigid-analytic Klein surface $X_K$ is algebraizable.*

*Proof.* Note that $\gamma_1^2(z_1, z_2) = (q_1^2 z_1, z_2)$. Define $\Gamma_1 := \langle \gamma_1^2, \gamma_2 \rangle$. One can check easily that $\Gamma_1$ is a normal subgroup of $\Gamma$ of index $2$ with cosets $\Gamma_1$ and $\gamma_1 \Gamma_1$. Moreover,

$$X'_K := \mathbb{G}_m^2 / \Gamma_1 = \mathbb{G}_m / q_1^{2\mathbb{Z}} \times \mathbb{G}_m / q_2^{\mathbb{Z}}.$$

Hence, $X'_K$ is algebraizable as a product of two elliptic curves. But then $X_K$ is algebraizable as the quotient $\pi : X'_K \to X_K = X'_K / (\Gamma / \Gamma_1)$ of $X'_K$ by the finite group $\Gamma / \Gamma_1$. □

**Remark 5.4.5.** Contrary to what one might expect, the non-orientability of the Klein bottle does not prevent us from defining a corresponding object analytically. Moreover, we have seen that rigid-analytic varieties with non-orientable valuation space can still be algebraizable!

In the following, we will show how to derive the structure of $\mathrm{Pic}^0_{X_K}$ from $\mathrm{Pic}^0_{X'_K}$ directly and to give an interpretation of Theorem 5.4.3. Namely, $X'_K$ is the product of the two elliptic curves

$$E_K^{(1)} = \mathbb{G}_m / q_1^{2\mathbb{Z}}, \quad E_K^{(2)} = \mathbb{G}_m / q_2^{\mathbb{Z}}$$

Let $p : X'_K \to E_K^{(1)}$ denote the first projection. On the other hand, we have a closed immersion of $E_K^{(1)}$ into $X'_K$ via

$$i : E_K^{(1)} \to X'_K, \quad z \mapsto (z, \sqrt{q_2})$$

This is a section of $p$. Now, let $\pi : X'_K \to X_K$ be the natural projection. As seen above, $X_K$ is the quotient of $X'_K$ by the action of $\gamma_1$ on $X'_K$. Note that $iE_K^{(1)}$ is invariant under $\gamma_1$. The quotient of $iE_K^{(1)}$ under $\gamma_1$ is the elliptic curve $E_K^{(1)} / \bar{q}_1 = \mathbb{G}_m / q_1^{\mathbb{Z}}$, where $\bar{q}_1$ is the image of $q_1$ in $E_K^{(1)}$. The morphisms $i$, $p$ then restrict to morphisms $p_1 : X_K \to E_K^{(1)} / \bar{q}_1$ resp. $i_1 : E_K^{(1)} / \bar{q}_1 \to X_K$. Again, $i_1$ is a section of $p_1$. Let $\pi_1$ denote the projection morphism $E_K^{(1)} \to E_K^{(1)}/\bar{q}_1$ . Thus, we have the following commutative diagram:

$$\begin{array}{ccc} X'_K & \xrightarrow{\pi} & X_K \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i_1} \\ E_K^{(1)} & \xrightarrow{\pi_1} & E_K^{(1)}/\bar{q}_1 \end{array}$$

For the Picard varieties, this yields the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Pic}^0_{X'_K} & \xleftarrow[\pi^*]{} & \mathrm{Pic}^0_{X_K} \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i_1^*} \\ \mathrm{Pic}^0_{E_K^{(1)}} & \xleftarrow[\pi_1^*]{} & \mathrm{Pic}^0_{E_K^{(1)}/\bar{q}_1} \end{array}$$

Note that, as $X'_K$ is the product of two Tate curves, we have

$$\mathrm{Pic}^0_{X'_K} = \mathbb{G}_m/q_1^{2\mathbb{Z}} \times \mathbb{G}_m/q_2^{\mathbb{Z}}$$

**Proposition 5.4.6.** *In the above situation, the following assertions hold:*

(i) $\pi^* : \mathrm{Pic}^0_{X_K} \to \mathrm{Pic}^0_{X'_K}$ *is an isogeny onto the abelian subvariety* $\mathbb{G}_m/q_1^{2\mathbb{Z}} \times \{1\}$ *of* $\mathrm{Pic}^0_{X'_K}$ *with* $\mathrm{Ker}\,\pi^* \cong \mathbb{Z}/2\mathbb{Z}$ .

(ii) *The restriction* $i^* : \pi^* \mathrm{Pic}^0_{X_K} \to \mathrm{Pic}^0_{E_K^{(1)}}$ *is an isomorphism.*

(iii) $i_1^* : \mathrm{Pic}^0_{X_K} \to \mathrm{Pic}^0_{E_K^{(1)}/\bar{q}_1}$ *is an isomorphism.*

*Proof.* For simplicity, we will assume that $\mathrm{char}\, K \neq 2$ .

In the following, we will always identify $\mathrm{Pic}^0_{X_K}$ with certain classes of Weil divisors on $X_K$ . Let $C$ be a Weil divisor such that its class $[C]$ lies in $\mathrm{Pic}^0_{X_K}$ . Then $\pi^*C$ is a $\gamma_1$ - invariant Weil divisor on $X'_K$ . As $X'_K$ is the product of $E_K^{(1)}$ and $E_K^{(2)}$ , the divisor $\pi^*C$ is linearly equivalent to a unique divisor

$$D' := ((\alpha_1) - (1)) \otimes ((\alpha_2) - (1)) = ((\alpha_1) - (1)) \times E_K^{(2)} - E_K^{(1)} \times ((\alpha_2) - (1))$$

with $(\alpha_1, \alpha_2) \in \mathbb{G}_m/q_1^{2\mathbb{Z}} \times \mathbb{G}_m/q_2^{\mathbb{Z}}$ . As $\gamma_1^* D = D$ , necessarily $D'$ is linearly equivalent to $\gamma_1^* D'$ . Calculating $D' - \gamma_1^* D'$ yields

$$D' - \gamma_1^* D' = ((\alpha_1) - (q_1\alpha_1) + (q_1) - (1)) \times E_K^{(2)} + E_K^{(1)} \times ((\alpha_2) - (1/\alpha_2))$$

Using the group laws on $E_K^{(1)}$ and $E_K^{(2)}$, we see that $(\alpha_1)-(q_1\alpha_1)+(q_1)-(1)$ is a principal divisor on $E_K^{(1)}$, and $(\alpha_2)-(1/\alpha_2)$ is linearly equivalent to $(\alpha_2^2)-(1)$ on $E_K^{(2)}$. Hence, $D'-\gamma_1^*D'$ is linearly equivalent to

$$E_K^{(1)}\times((\alpha_2^2)-(1))$$

This divisor is principal if and only if $\alpha_2^2=1$ on $E_K^{(2)}$; i.e. $\alpha_2$ is a 2-torsion point of $E_K^{(2)}$. As $\mathrm{Pic}^0_{X_K}$ is connected, its image in $\mathrm{Pic}^0_{X'_K}$ is connected as well. But then $\pi^*\mathrm{Pic}^0_{X_K}$ is necessarily contained in $\mathbb{G}_m/q_1^{2\mathbb{Z}}\times\{1\}$. On the other hand, any divisor class in $\mathbb{G}_m/q_1^{2\mathbb{Z}}\times\{1\}$ contains a divisor $((\alpha)+(q_1\alpha)-(q_1)-(1))\times E_K^{(2)}$ which is $\gamma_1$-invariant and hence comes from a divisor on $X_K$. Thus, $\pi^*$ maps onto $\mathbb{G}_m/q_1^{2\mathbb{Z}}\times\{1\}$. By the following Lemma 5.4.7, we have $\operatorname{Ker}\pi^*\cong\mathbb{Z}/2\mathbb{Z}$. This proves claim (i).

As $\mathrm{Pic}^0_{E_K^{(1)}}=\mathbb{G}_m^n/q_1^{2\mathbb{Z}}$, we see directly that $i^*$ is an isomorphism on

$$\pi^*\mathrm{Pic}^0_{X_K}=\mathbb{G}_m/q_1^{2\mathbb{Z}}\times\{1\}.$$

This proves assertion (ii). Moreover, $\mathrm{Pic}^0_{X_K}$ has dimension at most 1.

It remains to show that $i_1^*$ is injective. Combining (i) and (ii), we see that $i^*\circ\pi^*$ is an isogeny of degree $2$. On the other hand, $\pi_1^*$ is obviously an isogeny of degree $2$. Thus, we have

$$2=\deg(i^*\circ\pi^*)=\deg\pi_1^*\cdot\deg i_1^*=2\deg i_1^*.$$

But then $i_1^*$ is an isomorphism. This proves (iii). □

Thus, in terms of line bundles on $X'_K$, one can describe $\pi^*\mathrm{Pic}^0_{X_K}$ as those line bundles which are pull-backs from $E^{(1)}$ under $p$. The line bundles on $X_K$ which are trivial on $X'_K$ are the pull-backs of the trivial line bundle and the line bundle $(-1)-(1)$ on $E_K^{(1)}/\bar{q}_1$, respectively.

**Lemma 5.4.7.** *Let $X$ be a proper scheme of finite type over a field $K$ which is smooth and integral. Let $p$ be a prime with $\operatorname{char}K\neq p$, and let $G\cong\mathbb{Z}/p\mathbb{Z}$ act on $X$, such that the quotient $X/G$ exists. Let $\pi:X\to X/G$ denote the projection, and let $\pi^*:\mathrm{Pic}^0(X/G)\to\mathrm{Pic}^0(X)$ denote the pull back of line bundles. Then $\operatorname{Ker}\pi^*$ is either trivial or isomorphic to $\mathbb{Z}/p\mathbb{Z}$.*

*Proof.* Let $\mathcal{M}(X)$ denote the field of meromorphic functions on $X$, $\mathcal{M}(X/G)$ the field of meromorphic functions on $X/G$, which is the field of $G$-invariant meromorphic functions on $X$. Note that $G$ is the Galois group of $\mathcal{M}(X)$ over $\mathcal{M}(X/G)$. Let $C$ be a Weil divisor on $X/G$ such that its class $[C]$ lies in $\operatorname{Ker}\pi^*$. Then $\pi^*C = \operatorname{div}(f)$ for some $f \in \mathcal{M}(X)$. Let $\sigma$ be a generator of $G$, then $\pi^*C$ is $\sigma$-invariant; hence $\operatorname{div}(f) = \operatorname{div}(\sigma^* f)$, and $\sigma^* f = \xi f$. As $\sigma^p = 1$, we have $f = \xi^p f$, so $\xi$ is a $p$-th root of unity. But then $\sigma^* f^p = \xi^p f^p = f^p$, so $f^p$ is $G$-invariant; hence $f^p \in \mathcal{M}(X/G)$, and $pC = \operatorname{div}(f^p)$ is a principal divisor on $X/G$. Assume further that $\operatorname{Ker}\pi^*$ is not trivial, so we can choose $C$ such that $C$ is not a principal divisor. Hence, the class $[C]$ has order $p$ in $\operatorname{Pic}^0(X/G)$, and $\xi \neq 1$. We can now apply Hilbert's Theorem 90 to find an element $y \in \mathcal{M}(X)$ with $\sigma^* y = \xi y$. But then $f/y$ is invariant under $\sigma^*$, so we can write $f = yg$ for some $g \in \mathcal{M}(X/G)$. Let $C'$ be another Weil divisor with $[C'] \in \operatorname{Ker}\pi^*$. Again, $\pi^* C' = \operatorname{div}(f')$ for some $f' \in \mathcal{M}(X)$ with $\sigma^* f' = \xi' f'$, where $\xi'$ is another $p$-th root of unity. Write $\xi' = \xi^k$ for some $k \in \mathbb{Z}$. In that case, $f'/y^k$ is $\sigma^*$-invariant, so that $f' = y^k g'$ for some $g' \in \mathcal{M}(X/G)$. But then

$$\pi^*(C' - kC) = \operatorname{div}(y^k g') - \operatorname{div}(y^k g^k) = \operatorname{div}(g'/g^k).$$

Note that $g'/g^k$ is $G$-invariant, so $C' - kC = \operatorname{div}(g'/g^k)$ is a principal divisor on $X/G$, and $C'$ is linearly equivalent to $kC$. But then $[C]$ is a generator of $\operatorname{Ker}\pi^*$. Hence, $\operatorname{Ker}\pi^*$ is cyclic of order $p$. □

## 5.5 The Sheared Torus

In the following, we will construct another two-dimensional example whose construction is very similar to the construction of the two-dimensional torus $\mathbb{G}_{m,K}^2/M$, where $M$ is a lattice in $\mathbb{G}_{m,K}^2$ of rank 2.

Again, let $q_1, q_2 \in K^\times$ with $|q_1|, |q_2| < 1$. Furthermore, let $r \in \mathbb{Z}$. Let $\Gamma := \langle \gamma_1, \gamma_2 \rangle$, where $\gamma_1, \gamma_2$ are automorphisms of $\mathbb{G}_m^2$, acting via

$$\begin{aligned} \gamma_1 &: (z_1, z_2) \mapsto (q_1 z_1, z_2 z_1^r) \\ \gamma_2 &: (z_1, z_2) \mapsto (z_1, q_2 z_2) \end{aligned}$$

For $r = 0$, the automorphism $\gamma_1$ is just multiplication by $q_1$ in the first coordinate, so the quotient $\mathbb{G}_m^2/\Gamma$ is an analytic torus which is algebraic as a product of two elliptic curves; i.e.

$$\mathbb{G}_m^2/\Gamma \cong \mathbb{G}_m/q_1^{\mathbb{Z}} \times \mathbb{G}_m/q_2^{\mathbb{Z}}.$$

This is just a special case of section 5.2. In the following, we will assume $r \neq 0$. We will see that this changes the situation drastically.

**Proposition 5.5.1.** *The quotient* $X_K = \mathbb{G}_m^2/\Gamma$ *exists as a rigid-analytic variety. It is a general polytopal domain, which has a totally degenerated formal model* $X$ *over a finite extension* $R'$ *over* $R$. *We call* $X_K$ *a* sheared torus.

*Proof.* As in the previous example, we translate the action of $\Gamma$ on $\mathbb{G}_m^2$ into an action on $\mathbb{R}^2 = \mathrm{val}(\mathbb{G}_m^2)$. Namely, $\gamma_1, \gamma_2$ act on $\mathbb{R}^2$ via

$$\begin{aligned}\gamma_1 :& (x_1, x_2) \mapsto (c_1 + x_1, rx_1 + x_2)\\ \gamma_2 :& (x_1, x_2) \mapsto (x_1, x_2 + c_2),\end{aligned}$$

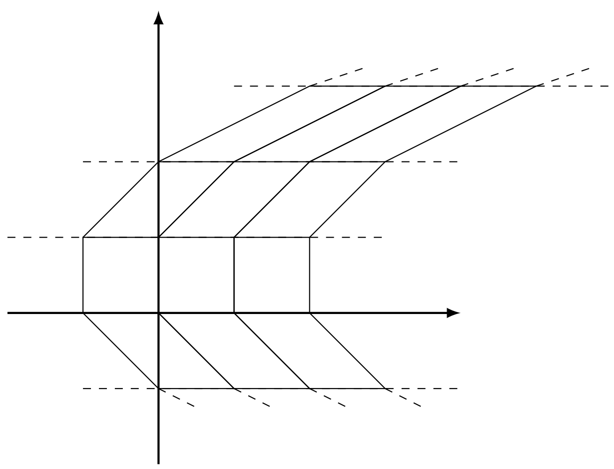
where $c_i = -\log |q_i|$. Let

$$F_K = \{(z_1, z_2)\ ;\ |q_1| \leqslant |z_1| \leqslant 1, |q_2| \leqslant |z_2| \leqslant 1\}.$$

Its image $\mathrm{val}(F_K)$ in $\mathbb{R}^2$ is the rectangle with vertices $(0,0)$, $(c_1, 0)$, $(0, c_2)$ and $(c_1, c_2)$. Now, let $\gamma = \gamma_1^{k_1}\gamma_2^{k_2}$, then

$$\begin{aligned}&\gamma(0,0) = (k_1c_1, k_2c_2), && \gamma(0, c_2) = (k_1c_1, (k_2+1)c_2)\\ &\gamma(c_1, 0) = ((k_1+1)c_1, rk_1c_1 + k_2c_2), && \gamma(c_1, c_2) = ((k_1+1)c_1, rk_1c_1 + (k_2+1)c_2)\end{aligned}$$

From this, one can check that $\mathrm{val}(F_K)$ is a fundamental domain for the action of $\Gamma$ on $\mathbb{R}^2$. The covering of $\mathbb{R}^2$ by images of $\mathrm{val}(F_K)$ looks as follows:



Thus, $F_K$ is a fundamental comain for the action of $\Gamma$ on $\mathbb{G}_m^2$, and the quotient $\mathbb{G}_m^2/\Gamma$ can be constructed by identifying the subsets

$$\{|z_1| = 1\}, \{|z_1| = |q_1|\}, \{|z_2| = 1\}, \{|z_2| = |q_2|\}$$

of $F_K$ via $\gamma_1, \gamma_2$. Hence, the quotient exists as a general polytopal domain. The rest follows with Proposition 4.5.9. □

**Remark 5.5.2.** As $\gamma_1$ and $\gamma_2$ commute, $\Gamma$ is free abelian of rank $2$.

**Theorem 5.5.3.** *For the rigid analytic Picard variety of the sheared torus, we have*

$$\mathrm{Pic}^0_{X_K/K} = \mathbb{G}_{m,K}/q_1^{\mathbb{Z}} \times \mathbb{G}_{m,K}.$$

*Proof.* As $\Gamma \cong \mathbb{Z}^2$, the Picard variety $\mathrm{Pic}^0_{X_K/K}$ will be a quotient of $\mathbb{G}_{m,K}^2$ by a lattice $M$. Let $f$ be a unit on $\mathbb{G}_{m,K}^2$, then $f = c\zeta_1^{k_1}\zeta_2^{k_2}$. For simplicity, we assume $c = 1$. Then

$$\gamma_1^* f = q_1^{k_1}\zeta_1^{k_1+rk_2}\zeta_2^{k_2}$$
$$\gamma_2^* f = q_2^{k_2}\zeta_1^{k_1}\zeta_2^{k_2}.$$

From this, we see that $f$ is an automorphic form with constant factor of automorphy if and only if $k_2 = 0$. In that case, $f = \zeta_1^{k_1}$ is automorphic with factor of automorphy $c(\gamma_1) = q_1^{k_1}$, $c(\gamma_2) = 1$. This proves the claim. □

As the Picard variety of an algebraic variety is always proper, this shows the following:

**Corollary 5.5.4.** *The sheared torus is not algebraizable.*

## 5.6 The General Case

In this section, we will generalize the examples of the last two sections. We will assume that $X_K = \mathbb{G}_m^n/\Gamma$, where $\Gamma$ is a suitable subgroup of the automorphism group of $\mathbb{G}_m^n$.

The automorphism group of $\mathbb{G}_m^n$ is a semi-direct product

$$\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n, \mathbb{Z}) \ltimes (K^\times)^n,$$

where a tuple $\tau := \tau(A, q)$ with $A = (a_{ij})$, $q = (q_i)$ acts on $\mathbb{G}_m^n$ via

$$(z_1, \ldots, z_n) \mapsto (q_1 \cdot z_1^{a_{11}} \cdot \ldots \cdot z_n^{a_{1n}}, \ldots, q_i \cdot z_1^{a_{i1}} \cdot \ldots \cdot z_n^{a_{in}}, \ldots, q_n \cdot z_1^{a_{n1}} \cdot \ldots \cdot z_n^{a_{nn}})$$

**Example 5.6.1.** For the Klein Surface resp. the sheared torus, the corresponding automorphisms are represented as follows:

(i) For the Klein Surface:

$$\gamma_1 = \tau(A_1, q^{(1)}) : \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \, q^{(1)} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$
$$\gamma_2 = \tau(A_2, q^{(2)}) : \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, q^{(2)} = \begin{pmatrix} 1 \\ q_2 \end{pmatrix}$$

(ii) For the sheared torus:

$$\gamma_1 = \tau(A_1, q^{(1)}), \quad A_1 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \, q^{(1)} = \begin{pmatrix} q_1 \\ 1 \end{pmatrix}$$
$$\gamma_2 = \tau(A_2, q^{(2)}), \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, q^{(2)} = \begin{pmatrix} 1 \\ q_2 \end{pmatrix}$$

As the valuation of $K$ is discrete, we may assume without loss of generality that the valuation group of $K^\times$ is $\mathbb{Z}$. Let

$$\mathrm{Aff}(n, \mathbb{Z}) \cong \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$$

denote the group of affine linear transformations

$$\tau = \tau(A, b) : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad x \longmapsto Ax + b$$

with $A \in \mathrm{GL}(n, \mathbb{Z})$, $b \in \mathbb{Z}^n$. Under the valuation map $\mathrm{val}$, the action $\tau(A, q)$ on $\mathbb{G}_{m,K}^n$ pulls back to an affine linear action $\tau(A, \mathrm{val}(q))$ on $\mathbb{R}^n$. This yields a surjective group morphism $\mathrm{Aut}(\mathbb{G}_{m,K}^n) \twoheadrightarrow \mathrm{Aff}(n, \mathbb{Z})$.

Now, let $\Gamma$ be a subgroup of $\mathrm{Aut}(\mathbb{G}_m^n)$. We assume that $\Gamma$ is mapped injectively to a subgroup of $\mathrm{Aff}(n, \mathbb{Z})$, which we will again denote by $\Gamma$. Thus, we can identify the action on $\mathbb{G}_m^n$ with the action on the valuation space $\mathbb{R}^n$.

In the following, we will further assume that the action of $\Gamma$ on $\mathbb{R}^n$ satisfies the following conditions:

**Assumption 5.6.2.** (i) $\Gamma$ has a fundamental polytope $\sigma$ of dimension $n$; i.e. $\gamma\sigma \cap \sigma$ is either empty or a proper face of $\sigma$ for $\gamma \neq 1$, and $\mathbb{R}^n = \bigcup_{\gamma \in \Gamma} \gamma(\sigma)$.

(ii) For $\gamma \neq 1$, the action of $\gamma$ on $\mathbb{R}^n$ has no fixed points.

(iii) The fundamental polytope $\sigma$ is a parallelotope. Let $\sigma_1^{(0)}, \sigma_1^{(1)}, \ldots, \sigma_n^{(0)}, \sigma_n^{(1)}$ denote pairs of opposite facets of $\sigma$. Then $\Gamma$ is generated by elements $\gamma_1, \ldots, \gamma_n$ such that $\gamma_i = \tau(A_i, b_i)$ induces an isomorphism $\gamma_i : \sigma_i^{(0)} \xrightarrow{\sim} \sigma_i^{(1)}$

(iv) Let $H^{(i)}$ denote the halfspace which contains $\sigma$ and whose supporting hyperplane contains $\sigma_i^{(0)}$. Then $\gamma_i$ induces an isomorphism $\gamma_i : H^{(i)} \xrightarrow{\sim} H^{(i)} + b_i$ with $b_i$ as in (iii).

**Remark 5.6.3.** Due to Assumption 5.6.2, the quotient $X_K = \mathbb{G}_m^n/\Gamma$ is a proper general polytopal domain. Hence, after a suitable finite extension of $K$, we find a totally degenerated formal model of $X_K$. Obviously, the universal covering is $\Omega_K \cong \mathbb{G}_{m,K}^n$.

Let $\Gamma_1$ be the subgroup of $\Gamma$ consisting of all translations; i.e. elements $\tau(I_n, q)$ where $I_n$ denotes the $n \times n$-unity matrix. We may identify the automorphism $\tau(I_n, q)$ with the vector $q$ itself. This way, $\Gamma_1$ yields a lattice in $\mathbb{G}_m^n$. The translation subgroup will be the key in determining the structure of the Picard variety of $\mathbb{G}_m^n/\Gamma$.

**Example 5.6.4.** For the Klein Surface resp. the sheared torus, the translation subgroup $\Gamma_1$ is given as follows:

(i) For the Klein Surface:

$$\Gamma_1 := \langle \gamma_1^2, \gamma_2 \rangle = \langle (q_1^2, 1), (1, q_2) \rangle$$

(ii) For the sheared torus:

$$\Gamma_1 := \langle \gamma_2 \rangle = \langle (1, q_2) \rangle.$$

**Lemma 5.6.5.** *The quotient group* $\Gamma/\Gamma_1$ *is finite if and only if* $\operatorname{rk} \Gamma_1 = n$.

*Proof.* We can construct a fundamental domain $\sigma_1$ for the action of $\Gamma_1$ on $\mathbb{R}^n$ by setting

$$\sigma_1 := \bigcup_{[\gamma] \in \Gamma/\Gamma_1} \gamma(\sigma),$$

where we choose a representative $\gamma$ for each coset $[\gamma] \in \Gamma/\Gamma_1$. Note that $\sigma$ has a finite non-zero volume. Due to Assumption 5.6.2 (i), for $\gamma \neq \gamma'$, the intersection $\gamma\sigma \cap \gamma'\sigma$ has volume $0$. Thus, we have $\operatorname{vol} \sigma_1 = |\Gamma/\Gamma_1| \cdot \operatorname{vol} \sigma$; i.e. $\sigma_1$ has a finite volume if and only if $\Gamma/\Gamma_1$ is finite. On the other hand, the volume of a fundamental domain $\sigma_1$ of $\Gamma_1$ is independent of the choice of $\sigma_1$; it is finite if and only if $\operatorname{rk} \Gamma_1 = n$. This proves the claim. □

**Theorem 5.6.6.** *Let $\Gamma \subset \operatorname{Aut}(\mathbb{G}^n_{m,K})$ be a subgroup satisfying Assumption 5.6.2. Assume that the translation subgroup $\Gamma_1$ satisfies $\operatorname{rk} \Gamma_1 = n$. Then $X_K$ is algebraizable if and only if there exists a group morphism $\lambda : \Gamma_1 \to M' := \operatorname{Hom}(\mathbb{G}^r_{m,K}, \mathbb{G}_{m,K})$ such that the quadratic form $\langle \lambda(m), m \rangle$ is positive definite on $\Gamma_1$; i.e. $|\langle \lambda(m), m \rangle| < 1$ for every $m \in \Gamma_1$ with $m \neq 0$.*

*Proof.* The canonical morphism $\mathbb{G}^n_m/\Gamma_1 \to \mathbb{G}^n_m/\Gamma = X_K$ is just the quotient morphism by the quotient group $\Gamma/\Gamma_1$. If $\operatorname{rk} \Gamma_1 = n$, this quotient group is finite, so $\mathbb{G}^n_m/\Gamma_1$ is algebraizable if and only if $X_K$ is algebraizable. As $\Gamma_1$ is a lattice of full rank, $\mathbb{G}^n_m/\Gamma_1$ is an analytic torus. Thus, the claim follows with Theorem 5.2.4. □

We will see later that $\operatorname{rk} \Gamma_1 = n$ is a necessary condition for $X_K$ to be algebraizable.

In the following, we will introduce further assumptions on $\Gamma$ and $\Gamma_1$:

**Assumption 5.6.7.** Assume the following:

(i) $\Gamma/\Gamma_1$ is abelian.

(ii) After renumbering the generators of $\Gamma$, we have

$$\Gamma_1 = \langle \gamma_1^{k_1}, \ldots, \gamma_r^{k_r} \rangle$$

for some $k_i \in \mathbb{N}$.

(iii) $\operatorname{rk} \Gamma_1 = r$, with $r$ as in (ii); i.e. $\gamma_1^{k_1}, \ldots, \gamma_r^{k_r}$, considered as elements of $\mathbb{R}^n$, are linearly independent over $\mathbb{R}$.

(iv) The free part of $\Gamma/\Gamma_1$ has rank $n - r$ and is generated by $\gamma_{r+1}, \ldots, \gamma_n$.

**Remark 5.6.8.** One checks easily that these conditions are satisfied for the Klein surface and the sheared torus. They are also trivially satisfied for the analytic torus $\mathbb{G}^n_{m,K}/M$, where $M$ is a lattice.

If $\Gamma/\Gamma_1$ is abelian, then $[\Gamma, \Gamma] \subset \Gamma_1$; i.e. $[\Gamma, \Gamma]$ is a free abelian group.

From Theorem 4.4.12, we get the following result for the Picard variety $\operatorname{Pic}^0_{X_K/K}$:

**Proposition 5.6.9.** *If Assumptions 5.6.2 and 5.6.7 are fulfilled, then* $\mathrm{Pic}^0_{X_K/K}$ *is represented by an analytic quotient* $\mathbb{G}^g_{m,K}/M$ *, where* $g = n - \mathrm{rk}[\Gamma, \Gamma]$ .

*Proof.* It only remains to check the assertion for the dimension. From Theorem 4.4.12, we see that $\mathrm{Pic}^0_{X_K/K}$ has dimension $g = \mathrm{rk}\,\Gamma/[\Gamma, \Gamma]$ . Now, consider the following exact sequence of finitely generated abelian groups:

$$0 \to \Gamma_1/[\Gamma, \Gamma] \to \Gamma/[\Gamma, \Gamma] \to \Gamma/\Gamma_1 \to 0$$

As $\Gamma/\Gamma_1$ has rank $n - r$ , we have

$$\mathrm{rk}\,\Gamma/[\Gamma, \Gamma] = \mathrm{rk}\,\Gamma/\Gamma_1 + \mathrm{rk}\,\Gamma_1/[\Gamma, \Gamma] = n - \mathrm{rk}[\Gamma, \Gamma].$$

Hence, $g = \mathrm{rk}\,\Gamma/[\Gamma, \Gamma] = n - \mathrm{rk}[\Gamma, \Gamma]$ , and the claim follows. □

**Definition 5.6.10.** Let

$$N := \{a\zeta^m \ ; \ a \in K^\times, m \in \mathbb{Z}^n\} = \mathcal{O}(\mathbb{G}^n_{m,K})^\times$$

denote the character group of the universal covering $\mathbb{G}^n_{m,K}$ . Let $N^{[\Gamma,\Gamma]}$ denote the subgroup of characters which are $[\Gamma, \Gamma]$ -invariant; i.e.

$$N^{[\Gamma,\Gamma]} = \{\chi \in N \ ; \ \gamma^*\chi = \chi \text{ for all } \gamma \in [\Gamma, \Gamma]\}$$

Furthermore, let

$$\Theta^\times = \{\chi \in N \ ; \ \gamma^*\chi = c(\gamma) \cdot \chi, \ c(\gamma) \in K^\times \text{ for all } \gamma \in [\Gamma, \Gamma]\}$$

denote the subgroup of characters which are $\Gamma$ -automorphic.

**Remark 5.6.11.** Every $\Gamma$ -automorphic character is invariant under $[\Gamma, \Gamma]$ ; i.e. $\Theta^\times \subset N^{[\Gamma,\Gamma]}$ .

**Lemma 5.6.12.** $\mathrm{Pic}^0_{X_K/K}$ *is proper if and only if* $\Theta^\times = N^{[\Gamma,\Gamma]}$ *; i.e. if every* $[\Gamma, \Gamma]$ *-invariant character is* $\Gamma$ *-automorphic.*

*Proof.* At first, note that a character $a\zeta^m \in N$ is $[\Gamma, \Gamma]$ -invariant if and only if $\langle m, u\rangle = 0$ for every $u \in [\Gamma, \Gamma]$ , considered as a lattice in $\mathbb{R}^n$ . This yields $\mathrm{rk}[\Gamma, \Gamma]$ linearly independent conditions on $m$ ; hence $N^{[\Gamma,\Gamma]}$ has rank $n - \mathrm{rk}[\Gamma, \Gamma]$ .

Using $\Theta^\times \subset N^{[\Gamma,\Gamma]}$ and applying Theorem 4.4.13 yields

$$\operatorname{rk} M = \operatorname{rk} \Theta^\times \leqslant \operatorname{rk} N^{[\Gamma,\Gamma]} = n - \operatorname{rk}[\Gamma,\Gamma].$$

From Proposition 5.6.9, we see that $\operatorname{Pic}^0_{X_K/K}$ is proper if and only if equality holds for the ranks. However, both $N^{[\Gamma,\Gamma]}$ and $\Theta^\times$ are saturated in $N$; i.e. if $\chi \in N$ satisfies $\chi^r \in N^{[\Gamma,\Gamma]}$ (resp. $\Theta^\times$) for some $r > 0$, then already $\chi \in N^{[\Gamma,\Gamma]}$. From this, we see that equality holds for the ranks if and only if $\Theta^\times = N^{[\Gamma,\Gamma]}$. □

Lemma 5.6.12 is the essential tool the prove the central result of this section:

**Theorem 5.6.13.** *Under Assumptions 5.6.2 and 5.6.7, the following holds:*

$$\operatorname{Pic}^0_{X_K} \text{ is proper if and only if } \operatorname{rk}\Gamma_1 = n.$$

*Proof.* At first, assume $\operatorname{rk}\Gamma_1 = n$. We will show that any unit $f := \zeta^m$ on $\mathbb{G}_m^n$ which is invariant under $[\Gamma,\Gamma]$ is already $\Gamma$-automorphic. Now, let $\gamma := \tau(A,c) \in \Gamma$, and let $\tau_b := \tau(I_n, b) \in \Gamma_1$. We have

$$[\gamma,\tau_b](x) = \gamma \circ \tau_b \circ \gamma^{-1} \circ \tau_b^{-1}(x) = x + (A-I)b = \tau(I,(A-I)b)(x)$$

If $\zeta^m$ is $[\Gamma,\Gamma]$-invariant, we have

$$m^t x = m^t[\gamma,\tau_b](x) = m^t x + m^t(A-I)b$$

for all $\tau_b \in \Gamma_1$; i.e. $m^t(A-I)b = 0$. As $\Gamma_1$ has rank $n$, this yields $m^t(A-I) = 0$. From this, we get

$$m^t\gamma(x) = m^t Ax + m^t c = m^t x + m^t c$$

for all $x \in \mathbb{R}^n$. Going back to $\mathbb{G}_m^n$ via $\operatorname{val}$, this shows that $f$ is $\gamma$-automorphic. Using Lemma 5.6.12, it follows that $\operatorname{Pic}^0_{X_K/K}$ is proper.

For the converse, assume that $\operatorname{rk}\Gamma_1 < n$. If the $\gamma_1,\dots,\gamma_n$ are numbered as in Assumption 5.6.7 (ii), then $\gamma_n$ has infinite order in $\Gamma/\Gamma_1$. We will show in Proposition 5.6.22 that there exists a character $f := \zeta^m$ which is invariant under $[\Gamma,\Gamma]$, but not automorphic with respect to $\gamma_n$. Again, using Lemma 5.6.12, it follows that $\operatorname{Pic}^0_{X_K/K}$ is not proper. The proof involves some explicit computations, which will be done in several lemmata. □

**Remark 5.6.14.** If $X_K$ is algebraizable, then $\mathrm{Pic}^0_{X_K/K}$ is proper. Namely, due to the GAGA-principle [25, 2.8], the rigid-analytic Picard variety if the analytification of the classical algebraic Picard variety, as $X_K$ is proper. The properness of $\mathrm{Pic}^0_{X_K/K}$ follows then from the smoothness of $X_K$, using [10, 8.4/3]. Thus, under Assumption 5.6.7, Theorem 5.6.13 implies that $\mathrm{rk}\,\Gamma_1 = n$ is a necessary condition in Theorem 5.6.6 for $X_K$ to be algebraizable.

In the following, we will perform a change of coordinates as follows: Assume the unique common vertex of the facets $\sigma_1^{(0)}, \dots, \sigma_n^{(0)}$ is the origin $0 := (0, \dots, 0)$ of $\mathbb{R}^n$. Let $u_1, \dots, u_n$ denote the vertices of $\sigma$ which have a common edge with $0$, such that $u_i \in \sigma_i^{(1)}$. Then $u_1, \dots, u_n$ are a basis of $\mathbb{R}^n$, and, with respect to this basis, $\sigma$ is just the unit hypercube. Note that, with respect to that basis, the matrices $A_i$ are not necessarily integer matrices. However, only very few entries will be non-integral, as the following lemma shows:

**Lemma 5.6.15.** *With respect to the basis $u_1, \dots, u_n$, the generators $\gamma_1, \dots, \gamma_n$ are given by $\gamma_i = \tau(A_i, v_i)$, where*

$$A_i := \begin{pmatrix} B_i^{(11)} & b_i^{(1)} & B_i^{(12)} \\ 0 & 1 & 0 \\ B_i^{(21)} & b_i^{(2)} & B_i^{(22)} \end{pmatrix}, \quad v_i := \begin{pmatrix} w_i^{(1)} \\ 1 \\ w_i^{(2)} \end{pmatrix},$$

*with the $1$ sitting at the entry $(i,i)$. Define*

$$B_i := \begin{pmatrix} B_i^{(11)} & B_i^{(12)} \\ B_i^{(21)} & B_i^{(22)} \end{pmatrix}, \quad b_i := \begin{pmatrix} b_i^{(1)} \\ b_i^{(2)} \end{pmatrix}, \quad w_i := \begin{pmatrix} w_i^{(1)} \\ w_i^{(2)} \end{pmatrix}.$$

*Then the following holds: The rows of $B_i$ are given by $\delta_1 e^t_{\tau(1)}, \dots, \delta_{n-1} e^t_{\tau(n-1)}$, where $\tau \in S_{n-1}$ is a permutation, $e_j$ denotes the $j$-th unit vector, and $\delta_j = \pm 1$. Especially, $B_i$ is orthogonal and satisfies $B_i^k = I$ for some $k \geqslant 1$, where $I$ is the $(n-1) \times (n-1)$ unit matrix. For $w_i = (w_{i,1}, \dots, w_{i,n-1})$, the $j$-th entry is given by*

$$w_{i,j} = \begin{cases} 0, & \text{if } \delta_j = 1, \\ 1, & \text{if } \delta_j = -1. \end{cases}$$

*Moreover, we have $w_i \in \mathrm{Im}(B_i - I)$.*

*Proof.* We do the proof for $i = n$; the rest follows in complete analogy. Due to Assumption 5.6.2 (iii), $\gamma_n$ maps the hyperplane spanned by $0, u_1, \dots, u_{n-1}$ bijectively onto the affine hyperplane spanned by $u_n, u_1 + u_n, \dots, u_{n-1} + u_n$. Thus, $\gamma_n$ restricts to an isomorphism of the linear subspace generated by $u_1, \dots, u_{n-1}$. Hence, the pair $(A_n, v_n)$ is given by

$$A_n := \begin{pmatrix} B_n & b_n \\ 0 & 1 \end{pmatrix}, \quad v_n := \begin{pmatrix} w_n \\ 1 \end{pmatrix},$$

with $B_n \in \mathrm{GL}(n-1, \mathbb{R})$. Let $u$ be a vertex of $\sigma_n^{(0)}$ with

$$u = \lambda_1 u_1 + \cdots + \lambda_{n-1} u_{n-1}, \quad \lambda_i \in \{0, 1\}.$$

Then

$$\gamma_n' : \sigma_n^{(0)} \to \sigma_n^{(0)}, \quad \lambda \mapsto B_n \lambda + w_n$$

is an automorphism of $\sigma_n^{(0)}$ which permutes the vertices. Writing $B_n = (b_{ij})$, the $i$-th coordinate of $\gamma_n'(u)$ is given by

$$\lambda_1 b_{i,1} + \cdots + \lambda_{n-1} b_{i,n-1} + w_{n,i}. \tag{5.1}$$

As $\gamma_n'(u)$ is again a vertex of $\sigma_n^{(0)}$, the $i$-th coordinate of $\gamma_n'(u)$ is either $0$ or $1$ for every vertex $u$ of $\sigma_n^{(0)}$. For $\lambda = 0$, we see therefore that $w_{n,i} \in \{0, 1\}$. Looking at the values of (5.1) for every $\lambda \in \{0, 1\}^{n-1}$, we see that there is exactly one non-zero entry $b_{i,j}$, which is either $-1$ if $w_{n,i} = 1$ or $+1$ if $w_{n,i} = 0$. Thus, the rows of $B_n$ are, up to sign, unit vectors; i.e. $B_n$ is, up to sign, a permutation matrix. Hence, $B_n$ is orthogonal. On the other hand, $\gamma_n'$ acts as a permutation of the vertices; hence it has finite order. So there exists a $k \geqslant 1$ such that, for all $y \in \mathbb{R}^{n-1}$, we have

$$y = (\gamma_n')^k(y) = B_n^k y + (B_n^{k-1} + B_n^{k-2} + \cdots + I) w_n.$$

Thus, $B_n^k = I$, and

$$w_n \in \mathrm{Ker}(B_n^{k-1} + \cdots + I) = \mathrm{Im}(B_n - I).$$

This proves the last claim. □

**Lemma 5.6.16.** *Let $B_i, b_i$ be as in Lemma 5.6.15. Then $b_i \in \mathrm{Im}(B_i - I)$ if and only if $\gamma_i$ has finite order in $\Gamma/\Gamma_1$.*

*Proof.* Again, consider only $\gamma_n$. Then $\gamma_n$ has finite order in $\Gamma/\Gamma_1$ if and only if there exists $k \geqslant 1$ such that $\gamma_n^k$ is a translation; i.e. $A_n$ satisfies $A_n^k = I$. We have

$$A_n^k = \begin{pmatrix} B_n & b_n \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} B_n^k & (B_n^{k-1} + \cdots + I)b_n \\ 0 & 1 \end{pmatrix}$$

Thus, $A_n^k = I$ if and only if $B_n^k = I$ and $(B_n^{k-1} + \cdots + I)b_n = 0$. But for $B_n^k = I$, the second condition is equivalent to $b_n \in \operatorname{Im}(B_n - I)$; see the proof of Lemma 5.6.15. □

**Lemma 5.6.17.** *Let $\gamma_1, \ldots, \gamma_n$ be numbered according to Assumption 5.6.7; i.e. $\gamma_1, \ldots, \gamma_r$ have finite order in $\Gamma/\Gamma_1$; $\gamma_{r+1}, \ldots, \gamma_n$ have infinite order. Then $A_i$ and $v_i$ have the following form:*

$$A_i = \left(\begin{array}{c|ccccc} C_i & 0 & \cdots & b_i & \cdots & 0 \\ \hline 0 & 1 & & & & \\ \vdots & & \ddots & & & \\ \vdots & & & 1 & & \\ \vdots & & & & \ddots & \\ 0 & & & & & 1 \end{array}\right), \quad v_i = \begin{pmatrix} w_i \\ \hline 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

*where $C_i \in \mathbb{R}^{r \times r}$ satisfies $C_i^k = I$ for some $k \geqslant 1$, and $b_i \in \mathbb{R}^r$ such that $b_i = 0$ for $i \leqslant r$, $b_i \notin \operatorname{Im}(C_i - I)$ for $i \geqslant r+1$. Moreover, $\Gamma_1 \otimes_{\mathbb{Z}} \mathbb{R}$ is generated as a vector space over $\mathbb{R}$ by the vectors $u_1, \ldots, u_r$.*

*Proof.* If $r = n$, there is nothing to show. Thus, assume $r < n$, so that $\gamma_n$ has infinite order in $\Gamma/\Gamma_1$. For $i < r$, we write

$$A_i = \begin{pmatrix} C_i & d_i \\ c_i^t & a_i \end{pmatrix}, \quad A_n = \begin{pmatrix} B_n & b_n \\ 0 & 1 \end{pmatrix},$$

with $C_i \in \mathbb{R}^{(n-1)\times(n-1)}$, $c_i, d_i \in \mathbb{R}^{n-1}$. Let $k \in \mathbb{N}$ with $B_n^k = I$. As in Lemma 5.6.16, $A_n^k$ is given by

$$A_n^k = \begin{pmatrix} I & b_n' \\ 0 & 1 \end{pmatrix}, \text{ where } b_n' := (B_n^{k-1} + \cdots + I)b_n.$$

As $\gamma_n$ has infinite order in $\Gamma/\Gamma_1$, we have $b_n' \neq 0$ by Lemma 5.6.16. Due to Assumption 5.6.7 (i), $\Gamma/\Gamma_1$ is abelian; i.e. we have $A_i A_n^k = A_n^k A_i$. Writing out the products yields

$$\begin{pmatrix} C_i & C_i b_n' \\ c_i^t & c_i^t b_n' + a_i \end{pmatrix} = A_i A_n^k = A_n^k A_i = \begin{pmatrix} C_i + b_n' c_i^t & d_i + a_i b_n' \\ c_i^t & a_i \end{pmatrix}$$

Hence, we have $b_n' c_i^t = 0$. However, as $b_n' \neq 0$, we have $c_i = 0$. Lemma 5.6.15 implies that $a_i = \pm 1$ and $d_i = 0$ as well; i.e. $A_i$ has the following form:

$$A_i = \begin{pmatrix} C_i & 0 \\ 0 & a_i \end{pmatrix}.$$

We claim that $a_i = +1$. In order to prove this, we will first show that

$$\Gamma_1 \otimes_{\mathbb{Z}} \mathbb{R} = \langle u_1, \ldots, u_r \rangle_{\mathbb{R}}.$$

Fix $i \in \{1, \ldots, r\}$, and let $k_i$ such that $\gamma_i^{k_i} \in \Gamma_1$; i.e. $A_i^{k_i} = I$. Due to Lemma 5.6.15, we can write $v_i = (w_i, \beta_i)^t$ with $\beta_i = 1$ if $a_i = -1$, and $\beta_i = 0$ if $a_i = 1$. Considered as a vector in $\mathbb{R}^n$, we have

$$\gamma_i^{k_i} = (A_i^{k_i - 1} + \cdots + I) v_i = \begin{pmatrix} (C_i^{k_i - 1} + \cdots + I) w_i \\ (a_i^{k_i - 1} + \cdots + 1) \beta_i \end{pmatrix}.$$

If $a_i = 1$, then $\beta_i = 0$, so the last coordinate of $\gamma_i^{k_i}$ vanishes. If $a_i = -1$, then $k_i$ is even, and hence $a_i^{k_i - 1} + \cdots + 1 = 0$. Again, the last coordinate of $(A_i^{k_i - 1} + \cdots + I) v_i$ vanishes. This shows $\gamma_i^{k_i} \in \langle u_1, \ldots, u_{n-1} \rangle_{\mathbb{R}}$. Repeating this argument for $\gamma_{r+1}, \ldots, \gamma_{n-1}$ instead of $\gamma_n$, we get $\gamma_i^{k_i} \in \langle u_1, \ldots, u_r \rangle_{\mathbb{R}}$.

By Assumption 5.6.7, the elements $\gamma_1^{k_1}, \ldots, \gamma_r^{k_r}$ are a basis of $\Gamma_1$. Hence, over $\mathbb{R}$, they generate $\langle u_1, \ldots, u_r \rangle_{\mathbb{R}}$. This proves $\Gamma_1 \otimes_{\mathbb{Z}} \mathbb{R} = \langle u_1, \ldots, u_r \rangle_{\mathbb{R}}$.

It still remains to show that $a_i = 1$ for all $i = 1, \ldots, n$. We compute the commutator of $\gamma_i$ and $\gamma_n$, using $A_i A_n = A_n A_i$:

$$[\gamma_i, \gamma_n](x) = \gamma_i \circ \gamma_n \circ \gamma_i^{-1} \circ \gamma_n^{-1}(x) = x + (A_i - I) v_n - (A_n - I) v_i.$$

Again, write $v_i = (w_i, \beta_i)$ , $v_n = (w_n, 1)$ . We compute

$$\begin{aligned}
&(A_i - I)v_n - (A_n - I)v_i \\
&= \begin{pmatrix} C_i - I & 0 \\ 0 & a_i - 1 \end{pmatrix} \begin{pmatrix} w_n \\ 1 \end{pmatrix} - \begin{pmatrix} B_n - I & b_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_i \\ \beta_i \end{pmatrix} \\
&= \begin{pmatrix} (C_i - I)w_n - (B_n - I)w_i - \beta_i b_n \\ a_i - 1 \end{pmatrix}
\end{aligned}$$

However, as $\Gamma_1 \otimes_{\mathbb{Z}} \mathbb{R} = \langle u_1, \ldots, u_r \rangle_{\mathbb{R}}$ by the second claim and $[\Gamma, \Gamma] \subset \Gamma_1$ , the last coordinate vanishes; hence $a_i = 1$ . Using the last assertion of Lemma 5.6.15, this implies $\beta_i = 0$ . Repeating the same argument for $\gamma_{r+1}, \ldots, \gamma_{n-1}$ instead of $\gamma_n$ , the first claim follows. □

**Lemma 5.6.18.** *In the situation of Lemma 5.6.17, we have* $b_i \in \operatorname{Ker}(C_j - I)$ *for* $j \neq i$ *.*

*Proof.* Again, consider only the case $j = n$ . Using $A_i A_n = A_n A_i$ , we get

$$\left(\begin{array}{c|ccccc} C_i C_n & \cdots & b_i & \cdots & C_i b_n \\ \hline 0 & & & & \\ \vdots & & & I & \\ 0 & & & & \end{array}\right) = A_i A_n = A_n A_i = \left(\begin{array}{c|ccccc} C_n C_i & \cdots & C_n b_i & \cdots & b_n \\ \hline 0 & & & & \\ \vdots & & & I & \\ 0 & & & & \end{array}\right).$$

Hence, $(C_n - I)b_i = (C_i - I)b_n = 0$ . This proves the claim. □

**Lemma 5.6.19.** *Let* $f := \langle m, \cdot \rangle \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ *for some* $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ *. Then* $f$ *is invariant under* $[\Gamma, \Gamma]$ *if and only if* $m' := (m_1, \ldots, m_r) \in \mathbb{Z}^r$ *annihilates the vector space*

$$V := \operatorname{Im}(C_1 - I) + \cdots + \operatorname{Im}(C_n - I)$$

*Proof.* At first, let $\tau_b = \tau(I, b)$ in $\Gamma_1$ , $b = (b', 0, \ldots, 0)^t \in \langle u_1, \ldots, u_r \rangle_{\mathbb{R}}$ with $b' \in \mathbb{R}^r$ . Then

$$[\gamma_i, \tau_b] := \gamma_i \circ \tau_b \circ \gamma_i^{-1} \circ \tau_b^{-1}(x) = x + (A_i - I)b = \tau(I, (A_i - I)b).$$

However, using Lemma 5.6.17, we have

$$(A_i - I)b = \begin{pmatrix} (C_i - I)b' \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

As $b'$ runs through $\langle u_1, \ldots, u_r \rangle_{\mathbb{R}}$, we see that $f$ annihilates all commutators of type $[\gamma_i, \tau_b]$ if and only if $m'$ satisfies $(m')^t(C_i - I) = 0$; i.e. $m'$ annihilates $\operatorname{Im}(C_i - I)$ for all $i$. As in the proof of Lemma 5.6.17, we see that the commutator $[\gamma_i, \gamma_j]$ for $i \neq j$ is given by the following vector:

$$(A_i - I)v_j - (A_j - I)v_i = \begin{pmatrix} (C_i - I)w_j - (C_j - I)w_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Again, $f$ is invariant under $[\gamma_i, \gamma_j]$ if and only if $m'$ annihilates $(C_i - I)w_j - (C_j - I)w_i$. Thus, we see that $(m')^t(C_i - I) = 0$ for all $i = 1, \ldots, n$ already implies that $f$ is invariant under $[\Gamma, \Gamma]$. This proves the claim. □

**Lemma 5.6.20.** *Let $C_i, b_i$ as in Lemma 5.6.17. Then, for $j \geqslant r + 1$, we have*

$$b_j \notin V = \operatorname{Im}(C_1 - I) + \cdots + \operatorname{Im}(C_n - I).$$

*Proof.* Again, consider only the case $j = n$. The matrices $C_i$ are diagonalizable and satisfy $C_iC_j = C_jC_i$. Thus, there exists a common basis of eigenvectors $z_1, \ldots, z_r$ such that $C_iz_j = \lambda_j^{(i)} z_j$ with $\lambda_n^{(1)}, \ldots, \lambda_n^{(s)} = 1$, $\lambda_n^{(s+1)}, \ldots, \lambda_n^{(r)} \neq 1$. Write

$$b_n = b_n^{(1)} z_1 + \cdots + b_n^{(r)} z_r$$

Due to Lemma 5.6.16, $b_n$ is not contained in $\operatorname{Im}(C_n - I)$, so we have $b_n^{(k)} \neq 0$ for some $k \in \{1, \ldots, s\}$. Without loss of generality, we may assume $b_n^{(1)} \neq 0$. However, Lemma 5.6.18 yields $b_n \in \operatorname{Ker}(C_i - I)$ for $i < n$, so we have $\lambda_i^{(1)} = 1$ for all $i < n$. But then $b_n \notin V$. □

**Definition 5.6.21.** In analogy to Definition 4.4.1, we say that a linear function $f := \langle m, \cdot \rangle \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ is $\Gamma$-automorphic, if there exists a group homomorphism $c : \Gamma \to \mathbb{R}$ such that $f(\gamma(x)) = f(x) + c(\gamma)$ for all $x \in \mathbb{R}^n$.

**Proposition 5.6.22.** *Let $\operatorname{rk} \Gamma_1 < n$. Then there exists $f := \langle m, \cdot \rangle \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ such that $f$ is invariant under $[\Gamma, \Gamma]$, but $f$ is not $\Gamma$-automorphic.*

*Proof.* Note that $f$ is $\Gamma$-automorphic if and only if $m^t(A_i - I) = 0$ for all $i = 1, \ldots, n$, as

$$m^t \gamma_i(x) = m^t(A_i x + v_i) = m^t A_i x + m^t v_i.$$

Consider the case $i = n$. Using Lemma 5.6.17, we have

$$m^t(A_n - I) = m^t \begin{pmatrix} C_n - I & \cdots & b_n \\ \vdots & \ddots & \\ 0 & & 0 \end{pmatrix} = \begin{pmatrix} (m')^t(C_n - I) & \cdots & (m')^t b_n \\ \vdots & \ddots & \\ 0 & & 0 \end{pmatrix}$$

with $m' = (m_1, \ldots, m_r)$. Thus, $m$ annihilates $A_n - I$ if and only if $m'$ annihilates $C_n - I$ and $b_n$. However, $b_n \notin V := \mathrm{Im}(C_1 - I) + \cdots + \mathrm{Im}(C_n - I)$ due to Lemma 5.6.20, so we find some $m'$ which annihilates $V$ but not $b_n$. This choice of $m$ yields a linear function $f$ which is invariant under $[\Gamma, \Gamma]$, but which is not $\Gamma$-automorphic. This proves the claim. □

# Chapter 6

# Affinoid Polytopal Domains are Factorial

Again, let $\sigma$ be a $\Gamma$-rational polytope. Let $X_{\sigma,K} := \mathrm{val}^{-1}(\sigma) \subset \mathbb{G}_m^n$ be the corresponding affinoid polytopal domain. For simplicity, we will write $X = X_{\sigma,K}$ in this section; keeping in mind that $X$ is the *affinoid* $K$-space, not its affine formal model (which we will not need here). The goal of this chapter is to prove the following:

**Theorem 6.0.1.** *If $X \subset \mathbb{G}_m^n$ is an affinoid polytopal domain, then*

$$H^i(X, \mathcal{O}^\times) = 0 \textit{ for all } i \geqslant 1.$$

In [32], van der Put proved a similar result for generalized polyannuli (i.e. affinoid domains of the form $X = D_1 \times \cdots \times D_r$, where $D_i \subset \mathbb{D}^1$ is a standard domain), and monomial convex subsets, which are described by a finite number of inequalities $|\zeta^m| \leqslant a_m$, $m \in \mathbb{N}^r$. In the following, we will modify van der Put's proof for our situation.

From now on, we will no longer assume that $K$ is algebraically closed. Let $\Gamma$ denote the value group of the additive valuation $v(z) := -\log|z|$ on the algebraic closure $\bar{K}$ of $K$. The notions of $\Gamma$-rational polytope and affinoid polytopal domain are defined as in 3.1: If $\sigma$ is a $\Gamma$-rational polytope in $\mathbb{R}^n$ given by inequalities $\langle m_i, x\rangle + c_i \geqslant 0$ with $m_i \in \mathbb{Z}^n$, $c_i \in \Gamma$, let $X_\sigma$ denote the corresponding affinoid polytopal domain $\mathrm{val}^{-1}(\sigma)$. In order to prove the above theorem, we will need van der Put's Base Change Theorem, see [32]. The following section will gather the theory needed to formulate and apply the Base Change Theorem.

## 6.1 Van der Put's Base Change Theorem

Let $X = \mathrm{Sp}(A)$ be an affinoid variety over $K$. In a certain sense, $X$ does not have "enough" points; there are sheaves $\mathcal{F}$ on $X$ such that $\mathcal{F}_x = 0$ for all $x \in X$, but $\mathcal{F} \neq 0$. To remedy this fact, it can be useful to allow a broader definition of *point* of $X$. The main ideas in this section have been established among others by van der Put, Schneider, and Berkovich; see for instance [32, 33, 3, 31]. For a detailed treatment, we refer to [14, § 7.1].

**Definition 6.1.1.** A *prime filter* on $X = \mathrm{Sp}(A)$ is a set $p$ of admissible subsets of $X$, such that

(i) $X \in p$, $\emptyset \notin p$;

(ii) $U_1 \cap U_2 \in p$ if $U_1, U_2 \in p$;

(iii) $V \in p$ if $U \in V$ and $V \supset U$;

(iv) If $U_1$, $U_2$ are admissible subsets of $X$ such that $U_1 \cup U_2 \in p$, then $U_1 \in p$ or $U_2 \in p$.

A *maximal filter* is a prime filter which is not contained in any larger prime filter. The set of all prime filters will be denoted by $\mathcal{P}(X)$, the subset of maximal filters by $\mathcal{M}(X)$.

Note that any ordinary point $x \in X$ induces a maximal Filter

$$p(x) := \{U \subset X \text{ admissible } : x \in U\}.$$

Let $U = R(f_0, \ldots, f_n) \subset X$ be a rational domain. A rational domain $U' \subset X$ is called a *neighbourhood* of $U$, if $U' \supset R_\rho := R(\rho f_0, f_1, \ldots, f_n)$ for some $\rho \in \sqrt{|K^\times|}$, $\rho > 1$. We write $U' \Supset_X U$.

If $p$ is a prime filter, we define a maximal Filter $r(p)$ containing $p$ as follows: An admissible subset $U \subset X$ is in $r(p)$ if and only if $U$ contains a rational domain $R$ such that $R_\rho \in p$ for all $\rho \in \sqrt{|K^\times|}$, $\rho > 1$. This is the unique maximal filter such that $r(p) \supset p$.

Let $\mathcal{F}$ be a sheaf of $X$. For any prime filter $p$, we define the *stalk* of $\mathcal{F}$ in $p$ via

$$\mathcal{F}_p := \varinjlim \{\mathcal{F}(U) : U \in p\}$$

If $p$ corresponds to an ordinary point, this coincides with the classical definition of a stalk.

Now, let $\mathcal{F}$ be a presheaf on $X$. Define a presheaf $\mathcal{F}^+$ on $X$ via

$$\mathcal{F}^+(U) := \check{H}^0(U, \mathcal{F}) = \varinjlim \check{H}^0(\mathcal{U}, \mathcal{F})$$

where $U$ is an admissible subset of $X$, $\mathcal{U}$ an admissible covering of $U$. For any prime filter $p$, we have $\mathcal{F}_p \cong \mathcal{F}_p^+$, see [32, 1.2.1]. Applying this construction twice yields a sheaf $\mathcal{F}^{++}$. We call $\mathcal{F}^{++}$ the *sheafification* of $\mathcal{F}$.

The following result shows why it is sometimes helpful to consider prime filters instead of ordinary points:

**Theorem 6.1.2.** (i) *Let $\mathcal{F}$ be a sheaf on $X$. Then $\mathcal{F} = 0$ holds if and only if $\mathcal{F}_p = 0$ for all prime filters $p$.*

(ii) *A sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of sheaves is exact, if and only if the sequence of sheaves $0 \to \mathcal{F}_p \to \mathcal{F}'_p \to \mathcal{F}''_p \to 0$ is exact at every prime filter $p$ of $X$.*

Any maximal filter can be described equivalently by a seminorm on $\mathcal{O}(X)$. Namely, if $p$ is a maximal filter, we can define

$$|f|_p := \inf\{\|f\|_U : U \in p\},$$

where $\|f\|_U := \sup\{f(x) : x \in U\}$ is the supremum seminorm on $U \subset X$. The seminorm $|f|_p$ has the following properties; see [32, Lem. 1.3.1]:

(i) $|f|_p \leqslant \|f\|_X$.

(ii) $|f+g|_p \leqslant \max\{|f|_p, |g|_p\}$.

(iii) $|fg|_p = |f|_p |g|_p$.

(iv) $|\lambda|_p = |\lambda|$ for $\lambda \in K$.

A mapping $|\cdot| : \mathcal{O}(X) \to \mathbb{R}_{\geqslant 0}$ which satisfies conditions (i) – (iv) is also called a *rank 1 valuation* or *analytic point* on $\mathcal{O}(X)$.

On the other hand, any analytic point $|\cdot|$ as above induces a maximal filter $p$; see [32, 1.3.2]. Thus, $p \mapsto |\cdot|_p$ yields a one-to-one correspondence between maximal filters and analytic points. We may thus use the notions of *analytic points* and *maximal filters* interchangeably.

As with ordinary points, one can define a *residue field* $K_p$ for an analytic point $p$. Namely, let $L_p := \mathcal{O}_{X,p}/\mathfrak{m}_p$, where $\mathfrak{m}_p := \{f \in \mathcal{O}_{X,p} : |f|_p = 0\}$. This is is an extension of $K$ with a non-archimedean valuation $|\cdot|_p$ which extends the valuation of $K$. We can then define $K_p$ as the completion of $L_p$ with respect to $|\cdot|_p$. If $x$ is an ordinary point, then $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,p}$ corresponding to $x$, and $K_p = L_p = \mathcal{O}_{X,p}/\mathfrak{m}_x$ is a finite extension of $K$, namely the usual residue field of $x$.

Again, let $\mathcal{M}(X)$ denote the set of maximal filters on $X$. The *Berkovich topology* on $\mathcal{M}(X)$ is the weakest topology such that $p \mapsto |f|_p$ is continuous for all $f \in \mathcal{O}(X)$. This topology makes $\mathcal{M}(X)$ a compact Hausdorff space.

Let $\varphi : X \to Y$ be a morphism of affinoid spaces over $K$. Then $\varphi$ induces a continuous morphism $\mathcal{M}(\varphi) : \mathcal{M}(X) \to \mathcal{M}(V)$ by sending a seminorm $\mathcal{O}_X(X) \to \mathbb{R}_{\geqslant 0}$ to the composition $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X) \to \mathbb{R}_{\geqslant 0}$. If $U \subset X$ is an affinoid subdomain, we can use this construction to identify $\mathcal{M}(U)$ with a subset of $\mathcal{M}(X)$.

**Definition 6.1.3.** A (pre-)sheaf $\mathcal{F}$ on $X$ is called *overconvergent*, if

$$\mathcal{F}(U) \cong \varinjlim \{\mathcal{F}(U') : U' \Supset U\}$$

holds for all rational subdomains $U \subset X$.

**Example 6.1.4.** Let $G$ be an abelian group. For any non-empty admissible open $U \subset X$, we set $P(U) := G$. This defines a presheaf on $X$, which is called the *constant presheaf*. The sheafification $P^{++}$ of $P$ is called the *constant sheaf* on $X$ and will be denoted by $G_X$. It is overconvergent.

In the following, we will gather some important results for overconvergent (pre-)sheaves; see [14, Lem. 7.4.1] and [32, 1.4.6 – 1.4.12]:

**Lemma 6.1.5.**
(i) *If $\mathcal{F}$ is an overconvergent presheaf, then the presheaf $\mathcal{F}^+$ is also overconvergent. Especially, the sheafification $\mathcal{F}^{++}$ of $\mathcal{F}$ is overconvergent.*
(ii) *If $\mathcal{F}$ is an overconvergent sheaf, then the presheaf given by $U \mapsto H^i(U, \mathcal{F})$ is also overconvergent.*
(iii) *Let $\varphi : X \to Y$ be a morphism between affinoid spaces, and let $\mathcal{F}$ be an overconvergent sheaf on $X$. Then the direct image sheaves $\varphi_*\mathcal{F}$ and $R^i\varphi_*\mathcal{F}$ are also overconvergent.*
(iv) *Let $\varphi : X \to Y$ be a morphism between affinoid spaces, and let $\mathcal{F}$ be an overconvergent sheaf on $Y$. Then $\varphi^{-1}\mathcal{F}$ is also overconvergent.*

Overconvergent sheaves have the following central property; see [14, Thm. 7.17]:

**Theorem 6.1.6.** *For an overconvergent sheaf $\mathcal{F}$ and any prime filter $p \in \mathcal{P}(X)$, we have $\mathcal{F}_p \cong \mathcal{F}_{r(p)}$.*

Thus, an overconvergent sheaf $\mathcal{F}$ is already determined by its stalks in analytic points. Hence, it suffices to check the conditions of Theorem 6.1.2 for analytic points.

**Definition 6.1.7.** Let $\varphi : X \to Y$ be a morphism of affinoid spaces, and let $p \in Y$ be an analytic point. We can define the fibre of $\varphi$ over $p$ as follows:

Let $K_p$ denote the residue field of $p$. Then $\mathcal{O}(X)\hat{\otimes}_{\mathcal{O}(Y)}K_p$ is an affinoid $K_p$-algebra; see [32, Lem. 2.1]. We define

$$X \times_Y p := \mathrm{Sp}\left(\mathcal{O}(X)\hat{\otimes}_{\mathcal{O}(Y)}K_p\right).$$

The morphism $\mathcal{O}(X) \to \mathcal{O}(X \times_Y p)$ induces a homeomorphism

$$\alpha : \mathcal{M}(X \times_Y p) \xrightarrow{\sim} \mathcal{M}(\varphi)^{-1}p \subset \mathcal{M}(X).$$

If $p$ is not an ordinary point of $Y$, then $K_p/K$ is not a finite extension, so in general this does not yield a morphism $X \times_Y p \to X$ between affinoid spaces. However, one can interpret $\alpha$ as a *general morphism*; see [11, 2.6].

Now, let $\mathcal{F}$ be an overconvergent sheaf on $X$. We can identify $\mathcal{F}$ with a sheaf on $\mathcal{M}(X)$. By restriction, this yields a sheaf on $\mathcal{M}(X \times_Y p)$, which we will denote by $\alpha^{-1}\mathcal{F}$. If $U \subset X$ is a finite union of open affinoid subdomains of $X$, then

$$H^0(U \times_Y p, \alpha^{-1}\mathcal{F}) = \varinjlim\{\mathcal{F}(U \cap \varphi^{-1}V);\ U \in p\}.$$

The central result of this section is the following Base-Change Theorem; see [32, Th. 2.3]. For a generalized version, see also [11, Th. 2.7.4].

**Theorem 6.1.8** (Base-Change Theorem). *Let $\varphi : X \to Y$ be a morphism of affinoid spaces, and let $\mathcal{F}$ be a sheaf on $X$.*

- (i) *If $\mathcal{F}$ is overconvergent, then $(R^i\varphi_*\mathcal{F})_p \cong H^i(X \times_Y p, \alpha^{-1}\mathcal{F})$ holds for all $i$ and all analytic points $p$ of $Y$.*
- (ii) *If $R^i\varphi_*\mathcal{F} = 0$ for all $i \geqslant 1$, then $H^i(X, \mathcal{F}) \cong H^i(Y, \varphi_*\mathcal{F})$ holds for all $i$.*
- (iii) *If $\mathcal{F}$ is overconvergent and $H^i(X \times_Y p, \alpha^{-1}\mathcal{F}) = 0$ holds for all $i \geqslant 1$ and all analytic points $p$ of $Y$, then $H^i(X, \mathcal{F}) \cong H^i(Y, \varphi_*\mathcal{F})$ holds for all $i$.*

## 6.2 The Main Theorem

Again, let $X = X_{\sigma,K}$ be an affinoid polytopal domain.

**Remark 6.2.1.** If $\varphi : X \to \mathbb{G}_m^k$ is the projection onto the first $k$ coordinates, then $\varphi(X) = X_\tau = \mathrm{val}^{-1}(\tau)$, where $\tau \subset \mathbb{R}^k$ is the projection of $\sigma$ onto the first $k$ coordinates. If $p$ is an analytic point of $X_\tau$, then the fibre $X_\sigma \times_{X_\tau} p$ is given over $K_p$ by

$$\left| z_1^{\beta_1} \cdots z_{n-k}^{\beta_{n-k}} \right|_p \leqslant \rho \left| z_{n-k+1}^{-\beta_{n-k+1}} \cdots z_n^{-\beta_n} \right|_p .$$

The term on the right side is a constant in $K_p$, as $z_{n-k+1}, \ldots, z_n \in K_p$.

Hence, $X_\sigma \times_{X_\tau} p = \mathrm{val}_{K_p}^{-1}(\sigma_p)$, where $\sigma_p$ is the fibre of $\tau$ over the point

$$(-\log|z_{n-k+1}|_p, \ldots, -\log|z_n|_p) \in \tau.$$

Hence, $\sigma_p$ is again a $\Gamma_p$-rational polytope, where $\Gamma_p$ is the additive valuation group of $\overline{K}_p$.

**Notation 6.2.2.** On $X$, consider the sheaf $\mathcal{O}(r)$ given by

$$\mathcal{O}(r)(U) := \{f : |f(x)| < r \text{ for all } x \in U\}.$$

It is not overconvergent. For $0 < r < s \leqslant \infty$, we define $\mathcal{O}(r,s)$ as the quotient $\mathcal{O}(s)/\mathcal{O}(r)$; by [14, Ex. 7.4.2], $\mathcal{O}(r,s)$ is overconvergent.

Now, let $\mathcal{O}^\times(1) := 1+\mathcal{O}(1)$. We take again the quotient $S_X := \mathcal{O}^\times/\mathcal{O}^\times(1)$; it is overconvergent by [32, 1.5.2]. It contains the subsheaf $A_X$, which is the constant sheaf associated to the group $A = K^\times/\{1+h : |h| < 1\}$. We take the sheaf $T_X$ to be the quotient $T_X := S_X/A_X$.

If $\dim X_\sigma = 1$, then Theorem 6.0.1 follows already from the following result:

**Theorem 6.2.3.** *Let $X \subset \mathbb{D}^1$ be a rational subdomain. Then the following holds:*

(i) $H^i(X, B_X) = 0$ *for all* $i \geqslant 1$, *all constant sheaves* $B_X$.

(ii) $H^i(X, \mathcal{O}_X(r)) = H^i(X, \mathcal{O}_X(r,s)) = 0$ *for all* $i \geqslant 1$, $0 < r < s \leqslant \infty$.

(iii) $H^i(X, \mathcal{O}_X^\times) = 0$ *for all* $i \geqslant 1$.

*Proof.* See [32, Cor. 3.8]. □

For the general case, we will proceed by induction on $\dim X_\sigma$.

As a first step, we will show that any constant sheaf on a polytopal domain has trivial cohomology. This assertion is analogous to [32, Th. 3.10].

**Proposition 6.2.4.** *Let $X \subset \mathbb{G}_{m,K}^n$ be an affinoid polytopal domain. Then $H^i(X, B_X) = 0$ holds for all $i \geqslant 1$ and all constant sheaves $B_X$.*

*Proof.* Let $\varphi : X \to D$ be the projection onto the last coordinate. Then the image $D \subset \mathbb{G}_m^1$ is an annulus given by $0 < r_1 \leqslant |z_n| \leqslant r_2$. $B_X$ is overconvergent, so is $\varphi_* B_X$. Let $p$ be an analytic point of $D$, then $X \times_D p \subset \mathbb{G}_{m,K_p}^{n-1}$ is a polytopal domain $\mathrm{val}^{-1}(\tau)$ for a polytope $\tau \subset \mathbb{R}^{n-1}$. By induction, we have $H^i(X \times_D p, \alpha^{-1} B_X) = 0$ for $i \geqslant 1$, since $\alpha^{-1} B_X \cong B_{X \times_D p}$. Theorem 6.1.8 now yields $H^i(X, B_X) = H^i(D, \varphi_* B_X)$. Again due to 6.1.8, we have

$$(\varphi_* B_X)_p = H^0(X \times_D p, B_{X \times_D p}) = B,$$

as $X \times_D p$ is connected. This proves $\varphi_* B_X \cong B_D$, and hence

$$H^i(X, B_X) = H^i(D, B_D) = 0$$

by Theorem 6.2.3 (i). □

As a next step, we will prove the following:

**Proposition 6.2.5.** *Let $X$ be a polytopal domain; $0 < r < s \leqslant \infty$. Then*

$$H^i(X, \mathcal{O}_X(r)) = H^i(X, \mathcal{O}_X(r,s)) = 0 \text{ for all } i \geqslant 1.$$

For $\dim X = 1$, this is exactly Theorem 6.2.3 (ii). Now, let $\varphi : X \to D$ be the projection of $X$ onto the last coordinate.

**Lemma 6.2.6.** *For $0 < r < s \leqslant \infty$, we have*

$$R^i \varphi_*(X, \mathcal{O}_X(r,s)) = 0 \quad \text{for all } i \geqslant 1.$$

*Proof.* $\mathcal{O}_X(r,s)$ is overconvergent; so is $\varphi_* \mathcal{O}_X(r,s)$. Let $p$ be an analytic point of $D$. Then $X \times_D p \subset \mathbb{G}_{m,K_p}^{n-1}$ is again a polytopal domain. Due to [32, Lem. 3.16], we have $\alpha^{-1} \mathcal{O}_X(r,s) \cong \mathcal{O}_{X \times_D p}(r,s)$. Proceding inductively and using Proposition 6.2.5 yields

$$H^i(X \times_D p, \alpha^{-1} \mathcal{O}_X(r,s)) = H^i(X \times_D p, \mathcal{O}_{X \times_D p}(r,s)) = 0 \text{ for all } i \geqslant 1.$$

Due to Theorem 6.1.8, this implies

$$(R^i\varphi_*\mathcal{O}_X(r,s))_p \cong H^i(X \times_D p, \alpha^{-1}\mathcal{O}_X(r,s)) = 0;$$

so $R^i\varphi_*\mathcal{O}(r,s) = 0$ holds as claimed. □

**Lemma 6.2.7.** *Let $\varphi : X \to Y$ be a morphism of affinoid spaces. If*

$$R^i\varphi_*\mathcal{O}(r,\infty) = 0$$

*holds for all $i \geqslant 1$, then*

$$H^i(X,\mathcal{O}_X(r)) \cong H^i(Y,\varphi_*\mathcal{O}_X(r)).$$

*Proof.* This is proven in [32, 3.17]. We sketch the proof for completeness.

Consider the following exact sequence of sheaves on $X$:

$$0 \to \mathcal{O}_X(r) \to \mathcal{O}_X \to \mathcal{O}_X(r,\infty) \to 0.$$

This induces a long exact sequence on $Y$:

$$0 \to \varphi_*\mathcal{O}_X(r) \to \varphi_*\mathcal{O}_X \to \varphi_*\mathcal{O}_X(r,\infty) \to R^1\varphi_*\mathcal{O}_X(r) \to R^1\varphi_*\mathcal{O}_X \to \cdots$$

$\mathcal{O}_X$ is acyclic on $X$; so $R^i\varphi_*\mathcal{O}_X = 0$ for $i \geqslant 1$. Using $R^i\varphi_*\mathcal{O}(r,\infty) = 0$ on the above exact sequence yields $R^i\varphi_*\mathcal{O}_X(r) = 0$ for $i \geqslant 2$.

It remains to prove $R^1\varphi_*\mathcal{O}_X(r) = 0$. Note that $\mathcal{O}(r)$ and $R^1\varphi_*\mathcal{O}_X(r)$ are not necessarily overconvergent; so we have to show that the stalks vanish at each prime filter $p_0$. Let $\delta : (\varphi_*\mathcal{O}_X)_{p_0} \to (\varphi_*\mathcal{O}_X(r,\infty))_{p_0}$; we will show that $\delta$ is surjective. For the prime filter $p_0$, let $p := r(p_0)$ denote the unique analytic point with $p \supset p_0$. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\varphi_*\mathcal{O}_X(r))_{p_0} & \longrightarrow & (\varphi_*\mathcal{O}_X)_{p_0} & \xrightarrow{\delta} & (\varphi_*\mathcal{O}_X(r,\infty))_{p_0} & & \\
 & & \big\downarrow{\scriptstyle\gamma_1} & & \big\downarrow{\scriptstyle\gamma_2} & & \big\downarrow{\scriptstyle\gamma_3} & & \\
0 & \longrightarrow & \mathcal{O}_{X\times_Y p}(r)(X\times_Y p) & \longrightarrow & \mathcal{O}_{X\times_Y p}(X\times_Y p) & \longrightarrow & \mathcal{O}_{X\times_Y p}(r,\infty)(X\times_Y p) & \longrightarrow & 0
\end{array}$$

$\varphi_*\mathcal{O}_X(r,\infty)$ is overconvergent, so

$$(\varphi_*\mathcal{O}_X(r,\infty))_{p_0} \cong (\varphi_*\mathcal{O}_X(r,\infty))_p = H^0(X \times_Y p, \mathcal{O}_{X\times_Y p}(r,\infty)).$$

Hence, $\gamma_3$ is bijective. Due to [32, Lem 3.16.1], $\gamma_1$ and $\gamma_2$ have identical kernel and cokernel. By diagram chasing, we find that $\delta$ is surjective. So $R^1\varphi_*\mathcal{O}_X(r) = 0$, and the claim follows. □

**Lemma 6.2.8.** *Let $\rho \in \sqrt{|K^\times|}$, and let $D = \{z : |z| = \rho\}$. Let $Y$ be an affinoid space such that $H^i(Y, \mathcal{O}_Y(r)) = 0$ holds for all $i \geqslant 1$ and all $r > 0$. Then*

$$H^i(Y \times D, \mathcal{O}_{Y\times D}(r)) = 0 \text{ for all } i \geqslant 1, r > 0.$$

*Proof.* See [32, 3.21]. □

For the next result, see also [32, Lem. 3.25]:

**Lemma 6.2.9.** *Let $X = X_\sigma$, and let $\rho \in \sqrt{|K^\times|}$. We set $X_1 := \{z \in X : |z_n| \leqslant \rho\}$, $X_2 := \{z \in X : |z_n| \geqslant \rho\}$, $X_3 := \{z \in X : |z_n| = \rho\}$. Then the map*

$$\mathcal{O}(r)(X_1) \oplus \mathcal{O}(r)(X_2) \to \mathcal{O}(r)(X_3), \quad (f_1, f_2) \mapsto f_1 - f_2$$

*is surjective.*

*Proof.* We define

$$\begin{aligned}
\sigma_1 &:= \{x \in \sigma : x_n \geqslant -\log\rho\},\\
\sigma_2 &:= \{x \in \sigma : x_n \leqslant -\log\rho\},\\
\sigma_3 &:= \sigma_1 \cap \sigma_2 = \{x \in \sigma : x_n = -\log\rho\}.
\end{aligned}$$

Then $X_i := \mathrm{val}^{-1}(\sigma_i)$. Now, let $f := \sum a_m z^m \in \mathcal{O}(r)(X_3)$. We have to show that every term $a_m z^m$ is either in $\mathcal{O}(r)(X_1)$ or in $\mathcal{O}(r)(X_2)$.

By the definition of the supremum norm for a polytopal domain, we have

$$r \geqslant \min_{u\in\sigma_3} |a_m| e^{-\langle m,x\rangle}$$

for all $m \in \mathbb{Z}^n$. Equivalently, $\inf_{x\in\sigma_3}\langle m, x\rangle \geqslant -\log(r/|a_m|)$. We have to prove

$$\inf_{x\in\sigma_3}\langle m,x\rangle = \max(\inf_{x\in\sigma_1}\langle m,x\rangle, \inf_{x\in\sigma_2}\langle m,x\rangle).$$

The "$\geqslant$" part is clear, as $\sigma_3 = \sigma_1 \cap \sigma_2$. For the converse, let $p_1 \in \sigma_1$, $p_2 \in \sigma_2$. The line through $p_1$ and $p_2$ meets $\sigma_3$ in a point $p_3 = tp_1 + (1-t)p_2 \in \sigma_3$ for a $t \in [0,1]$. Then

$$\langle m, p_3\rangle = t\langle m, p_1\rangle + (1-t)\langle m, p_2\rangle \leqslant \max(\langle m, p_1\rangle, \langle m, p_2\rangle).$$

This proves $\inf_{x\in\sigma_3}\langle m,x\rangle \leqslant \max(\inf_{x\in\sigma_1}\langle m,x\rangle, \inf_{x\in\sigma_2}\langle m,x\rangle)$, and thus the claim follows. □

We can now conclude the proof of Proposition 6.2.5 similarly to [32, 3.22].

*Proof of Proposition 6.2.5.* Applying Lemma 6.2.6 and Lemma 6.2.7, we find

$$H^i(X, \mathcal{O}_X(r)) \cong H^i(D, \varphi_*\mathcal{O}_X(r)).$$

As $\dim D = 1$, all higher cohomology groups vanish; so it is enough to show

$$H^1(D, \varphi_*\mathcal{O}_X(r)) = 0.$$

Consider the following exact sequence:

$$H^0(D, \varphi_*\mathcal{O}_X) \xrightarrow{\beta} H^0(D, \varphi_*\mathcal{O}_X(r,\infty)) \to H^1(D, \varphi_*\mathcal{O}_X(r)) \to H^1(D, \varphi_*\mathcal{O}_X) = 0$$

We have to show that $\beta$ is surjective. $D$ is an annulus which is given by $0 < R_1 \leqslant |z_n| \leqslant R_2$. Let $f \in H^0(D, \varphi_*\mathcal{O}_X(r,\infty))$ have image $\xi \in H^1(D, \varphi_*\mathcal{O}_X(r))$.

We claim that there exists a covering of $D$ by annuli $V_i$ such that $\xi|_{V_i} = 0$.

Let $p$ be an analytic point of $D$. As $R^1\varphi_*\mathcal{O}_X(r) = 0$, we have

$$(R^1\varphi_*\mathcal{O}_X(r))_p = \varinjlim\{H^1(U, \varphi_*\mathcal{O}_X(r)) : U \in p\} = 0;$$

and so for each $p$ there exists $U \in p$ with $\xi|_U = 0$. So, the presheaf

$$U \mapsto H^1(U, \varphi_*\mathcal{O}_X(r))$$

has trivial stalks in all analytic points. Moreover, $R^1\varphi_*\mathcal{O}_X(r) = 0$ yields the following exact sequence of sheaves:

$$0 \to \varphi_*\mathcal{O}_X(r) \to \varphi_*\mathcal{O}_X \to \varphi_*\mathcal{O}_X(r,\infty) \to 0.$$

As $\varphi_*\mathcal{O}_X(r,\infty)$ is overconvergent, we conclude that the presheaf $U \mapsto H^1(U, \varphi_*\mathcal{O}_X(r))$ is also overconvergent.

Now, let $\rho \in [R_1, R_2]$, $\rho \in \sqrt{|K^\times|}$. Define $X_\rho := \{x \in X : |x_n| = \rho\}$, then $X_\rho = X'_\rho \times \{|z_n| = \rho\}$ for a suitable $X'_\rho = \mathrm{val}^{-1}(\sigma') \subset \mathbb{G}_m^{n-1}$. By induction $H^i(X'_\rho, \mathcal{O}(r)) = 0$ holds for all $i \geqslant 1$ and all $r > 0$ by Proposition 6.2.5. Applying Lemma 6.2.8 yields $H^i(X_\rho, \mathcal{O}(r)) = 0$. Applying Lemma 6.2.7 yields

$$H^1(\{|z| = \rho\},\ \varphi_*\mathcal{O}_X(r)) = H^1(\{|z| = \rho\},\ \varphi_*\mathcal{O}_{X_\rho}(r)) = H^1(X_\rho, \mathcal{O}(r)) = 0.$$

Hence, $\xi = 0$ on $\{|z| = \rho\}$. As $U \mapsto H^1(U, \varphi_*\mathcal{O}_X(r))$ is overconvergent, $\xi|_{U'} = 0$ for a suitable $U' \Supset U$. This $U'$ contains an annulus $U'' := \{r_1 \leqslant |z| \leqslant r_2\}$ with $r_1 < \rho < r_2$ such that $\xi$ vanishes on $U''$.

Now, for $\rho \notin \sqrt{|K^\times|}$, $\rho \in [R_1, R_2]$, we consider the analytic point $p$ given by the semi-norm $|\sum a_n z^n|_p := \max |a_n|\rho^n$. Let $U \in p$ such that $\xi|_U = 0$. Then $U$ contains an annulus $U' := \{z : r_1 \leqslant |z| \leqslant r_2\}$ with $r_1 < \rho < r_2$ such that $\xi$ vanishes on $U'$.

Continuing as above, we find radii $r_0 := R_1 < r_1 < \ldots, r_s := R_2$ with $r_i \in \sqrt{|K^\times|}$ such that $\xi|_{V_i} = 0$ for the corresponding annuli $V_i := \{r_i \leqslant |z| \leqslant r_{i+1}\}$, $i = 0, \ldots, s-1$.

Now, we want to show that $f \in \mathrm{Im}(\beta)$.

We consider the commutative diagram on page 118; Figure 6.1.

The surjectivity of $\tau$ follows from Lemma 6.2.9 by induction; the case for general $n$ can be reduced to the case $n = 2$ as in the proof of [32, Cor. 3.3]. By diagram chasing, we find $g \in H^0(D, \varphi_*\mathcal{O}_X)$ with $\beta(g) = f$. This proves the claim. □

In the following, we consider the sheaf $S_X = \mathcal{O}^\times/\mathcal{O}^\times(1)$; cf. Notation 6.2.2. The following result can be found already in [32]. We will give the proof for completeness.

**Lemma 6.2.10.** *Let $\varphi : X \to Y$ a morphism of affinoid spaces. Then $\alpha^{-1}S_X \cong S_{X\times_Y p}$ holds for every analytic point $p$.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(D,\varphi_*\mathcal{O}_X(r)) & \longrightarrow & H^0(D,\varphi_*\mathcal{O}_X) & \xrightarrow{\beta} & H^0(D,\varphi_*\mathcal{O}_X(r,\infty)) & \longrightarrow & H^1(D,\varphi_*\mathcal{O}_X(r)) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus H^0(V_i,\varphi_*\mathcal{O}_X(r)) & \longrightarrow & \bigoplus H^0(V_i,\varphi_*\mathcal{O}_X) & \longrightarrow & \bigoplus H^0(V_i,\varphi_*\mathcal{O}_X(r,\infty)) & \longrightarrow & \bigoplus H^1(V_i,\varphi_*\mathcal{O}_X(r)) \\
 & & \downarrow{\scriptstyle \tau} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus H^0(V_i\cap V_{i+1},\varphi_*\mathcal{O}_X(r)) & \longrightarrow & \bigoplus H^0(V_i\cap V_{i+1},\varphi_*\mathcal{O}_X) & \longrightarrow & \bigoplus H^0(V_i\cap V_{i+1},\varphi_*\mathcal{O}_X(r,\infty)) & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Figure 6.1:

*Proof.* For an admissible open subset $A \subset X \times_Y p$, we consider the presheaves

$$P_1(A) := \varinjlim\{\mathcal{O}^\times(U) : \alpha(A) \subset U\}, \qquad P_2(A) := \varinjlim\{\mathcal{O}^\times(1)(U) : \alpha(A) \subset U\}.$$

Denote with $K_i(A)$ resp. $C_i(A)$ the kernel resp. cokernel of the following maps:

$$P_1(A) \to \mathcal{O}^\times_{X\times_Y p}(A), \qquad P_2(A) \to \mathcal{O}^\times_{X\times_Y p}(1)(A).$$

We will show $K_1(A) = K_2(A)$ and $C_1(A) = C_2(A)$. Due to [32, Lem. 2.6], there exists a rational domain $B \subset X$ with $\alpha^{-1}(B) = A$, and

$$\mathcal{O}_{X\times_Y p}(A) \cong \mathcal{O}_X(B)\hat{\otimes}_{\mathcal{O}(Y)} K_p$$

by [32, Lem. 2.4]; so $A = B \times_Y p$. Thus, it is enough to show the above assertion for $A = X \times_Y p$. Due to [32, Lem. 2.5], the image of $\varinjlim\{\mathcal{O}_X(\varphi^{-1}V) : V \in p\}$ lies dense in $\mathcal{O}(X \times_Y p)$.

Now, let $f \in \mathcal{O}^\times(X \times_Y p)$. We can approximate $f$ as $\bar{g}(1+h)$ with $\|h\| < 1$, $\bar{g} \in \mathcal{O}^\times(X \times_Y p)$, such that $\bar{g}$ is the image of $g \in \mathcal{O}_X(\varphi^{-1}V)$ for a suitable $V \in p$. As in the proof of [32, Lem. 2.6], we find a $V' \in p$, $V' \subset V$, such that $g$ is invertible on $\varphi^{-1}V'$.

On the other hand, if $1 + f \in \mathcal{O}^\times(1)(X \times_Y p)$, as in the proof of [32, Lem. 3.16], one finds a $V \in p$, such that $f = \bar{g}(1+h)$ with $\|g\|_{\varphi^{-1}V} < 1$ and $\|h\| < \delta$ for a $\delta > 0$. Then $1 + f = (\overline{1+g})(1+h')$, where $\|h'\| < 1$, if $\delta$ is small enough. This proves $C_1 = C_2$.

Now, let $f \in \mathcal{O}^\times(U)$ for a $U$ with $\alpha A \subset U$, such that $\bar{f} = 1 \in \mathcal{O}(X \times_Y p)$; then $f = 1 + gh$ holds for some $g \in \mathcal{O}(U)$, $h \in \mathcal{O}(Y)$ with $|h|_p = 0$. As $\overline{gh} = 0$, [32, Lem. 2.5] shows that there exists $V \in p$ with $\|gh\|_{\varphi^{-1}V} < 1$; thus $f \in \mathcal{O}^\times(1)(\varphi^{-1}V)$. This proves $K_1 = K_2$. □

Now, consider the sheaf $T = S/A$; cf. Notation 6.2.2. We need the following variant of [32, Lem. 3.27] for polytopal domains.:

**Lemma 6.2.11.** *Let $X = \mathrm{val}^{-1}(\sigma) \subset \mathbb{G}_m^n$, and let $\varphi : X \to Y \subset \mathbb{G}_m^{n-1}$ be the projection onto the first $n-1$ coordinates. Then $\varphi_* T_X \cong T_Y \oplus \mathbb{Z}_Y$.*

*Proof.* Let $U \subset Y$ be a connected affinoid subdomain. Define $\psi$ by

$$\psi : \mathcal{O}(U)^\times \times \mathbb{Z} \to \mathcal{O}(\varphi^{-1}U)^\times; (f, m) \mapsto f z_n^m$$

This induces a morphism of sheaves $\beta : S_Y \oplus \mathbb{Z}_Y \to \varphi_* S_X$. As $S$ is overconvergent, $S_Y \oplus \mathbb{Z}_Y$ and $\varphi_* S_X$ are overconvergent as well.

We need to show that $\beta$ is an isomorphism. It is enough to show this for the stalk at each analytic point. Due to Theorem 6.1.8, we have $(\varphi_* S_X)_p = H^0(X \times_Y p, \alpha^{-1} S_X)$. On the other hand, $\alpha^{-1} S_X \cong S_{X \times_Y p}$ by Lemma 6.2.10; thus

$$(\varphi_* S_X)_p = \mathcal{O}^\times (X \times_Y p)/(\mathcal{O}(1)^\times (X \times_Y p)).$$

As $D = X \times_Y p$ is an annulus, every element of $(\varphi_* S_X)_p$ has a unique representation of the form $\lambda z^m$ with $m \in \mathbb{Z}$, $\lambda \in K_p/\{1+h : |h| < 1\}$; cf. [32, 3.26]. On the other hand, $S_{Y,p} = \mathcal{O}_{Y,p}^\times / \mathcal{O}^\times(1)_{Y,p}$. By definition, $L_p = \mathcal{O}_{Y,p}/\{|f|_p = 0\}$ lies dense in $K_p$, so $L_p^\times \cong \mathcal{O}_{Y,p}^\times$. This proves

$$S_{Y,p} = L_p^\times / \{1+h : |h| < 1\} \cong L_p^\times / \{1+h : |h| < 1\}.$$

Hence, $\beta_p$ is surjective, and $S_Y \oplus \mathbb{Z}_Y \cong \varphi_* S_X$. From the exact sequence

$$1 \to A_X \to S_X \to T_X \to 0,$$

we get the following exact sequence:

$$1 \to \varphi_* A_X \to \varphi_* S_X \to \varphi_* T_X \to R^1 \varphi_* A_X \to \cdots$$

Applying Proposition 6.2.4 yields $R^1 \varphi_* A_X = 0$ and $\varphi_* A_X = A_Y$; thus we have an exact sequence

$$1 \to A_Y \to S_Y \oplus \mathbb{Z} \to \varphi_* T_X \to 0,$$

which proves the claim. □

**Theorem 6.2.12.** *For a polytopal domain $X$*

$$H^i(X, T_X) = H^i(X, S_X) = H^i(X, \mathcal{O}^\times) = 0$$

*holds for all $i \geqslant 1$.*

*Proof.* Using Lemma 6.2.11 yields $H^i(X, T_X) = H^i(Y, T_Y) \oplus H^i(Y, \mathbb{Z})$. Due to Proposition 6.2.4, $H^i(Y, \mathbb{Z}) = 0$ holds for all $i \geqslant 1$; so $H^i(X, T_X) = H^i(Y, T_Y)$. The assertion for $T_X$ follows by induction. As $A_X$ has trivial cohomology, the assertion for $S$ follows from the exact sequence $0 \to A \to S \to T \to 0$. Due to Proposition 6.2.5, $\mathcal{O}(r)$ has trivial cohomology for all $r$. An approximation argument then shows that $\mathcal{O}^\times(1)$ has also trivial cohomology on $X$. The assertion for $\mathcal{O}^\times$ follows now from the exact sequence $0 \to \mathcal{O}^\times(1) \to \mathcal{O}^\times \to S \to 0$. □

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# Index

# Zusammenfassung

In dieser Arbeit betrachten wir eine neue Klasse eigentlicher rigid-analytischer Varietäten über einem vollständigen diskret-bewerteten Körper $K$; nämlich diejenigen, die über dem Bewertungsring $R$ ein *total degeneriertes* formales Modell besitzen. Wir zeigen, dass für eine solche rigid-analytische Varietät $X_K$ die Picard-Varietät $\mathrm{Pic}^0(X_K)$ isomorph ist zu einem Quotienten $\mathbb{G}^g_{m,K}/M$, wobei $M$ ein Gitter in $\mathbb{G}^g_{m,K}$ ist.

Die Existenz der Picard-Varietät für eine glatte eigentliche rigid-analytische Varietät $X_K$ wurde erst 2000 von Hartl und Lütkebohmert bewiesen [21], unter der Voraussetzung, dass $X_K$ ein streng semistabiles Modell $X$ besitzt. Dabei spielt die spezielle Faser $X_0$ von $X$ eine entscheidende Bedeutung für die Struktur der Picard-Varietät von $X_K$. Der in dieser Arbeit betrachtete total degenerierte Fall ist dabei die einfachste auftretende Konfiguration:

Ein total degeneriertes formales Schema $X$ ist folgendermaßen charakterisiert: Die irreduziblen Komponenten der speziellen Faser $X_0$ sind rationale Varietäten, die sich normal schneiden; d.h. $X_0$ ist lokal isomorph zum Schnitt von Koordinatenhyperebenen im affinen Raum $\mathbb{A}^r$ (für die genauen Bedingungen siehe Definition 4.1.1). Damit stellen die total degenerierten rigid-analytischen Varietäten eine Verallgemeinerung der bekannten Mumford-Kurven in höherer Dimension dar. Für Mumford-Kurven vom Geschlecht $g$ haben Drinfeld und Manin gezeigt [26], dass die Picard-Varietät ein analytischer Torus $\mathbb{G}^g_{m,K}/M$ ist.

In Theorem 4.3.5 verallgemeinern wir das Resultat von Drinfeld und Manin und zeigen, dass für eine total degenerierte rigid-analytische Varietät $X_K$ die Picard-Varietät $\mathrm{Pic}^0(X_K)$ durch einen Quotienten $\mathbb{G}^g_{m,K}/M$ beschrieben wird, wobei $M$ ein Gitter in $\mathbb{G}^g_{m,K}$ ist. Falls $X_K$ nicht algebraisch ist, hat das Gitter $M$ nicht notwendigerweise vollen Rang; ein bekanntes Gegenbeispiel ist die Hopf-Fläche, die ebenfalls in die Kategorie der total-degenerierten Varietäten fällt (siehe §5.3). In §4.4 geben wir eine explizite Beschreibung der Picard-Varietät mit Hilfe einer universellen Überlagerung $\Omega_K$ von $X_K$.

Eine große Klasse von Beispielen für total degenerierte Varietäten ist durch *verallgemeinerte Polytopbereiche* gegeben. Ein affinoider Teil eines Polytopbereiches ist das Urbild eines

Polytops $\sigma \subset \mathbb{R}^n$ unter der Bewertungsabbildung

$$\mathrm{val} : \mathbb{G}_{m,K}^n \to \mathbb{R}^n, \quad (x_1, \ldots, x_n) \mapsto (-\log|x_1|, \ldots, -\log|x_n|)$$

Dadurch tragen Polytopbereiche eine reiche kombinatorische Struktur; viele Methoden und Resultate über torische Varietäten lassen sich auf die Situation von Polytopbereichen übertragen. Die Existenz eines total degenerierten Modells für Polytopbereiche folgt aus einem kombinatorischen Resultat von Kempf, Knudson, Mumford und Saint-Donat [24]. In Theorem 6.0.1 zeigen wir mit Methoden von van der Put [32], dass auf einem affinoiden Polytopbereich die Picard-Gruppe verschwindet.

In §5.6 behandeln wir den Spezialfall einer total degenerierten Varietät, für die die universelle Überlagerung $\Omega_K$ gerade $\mathbb{G}_{m,K}^g$ ist; d.h. $X_K \cong \mathbb{G}_{m,K}^g/\Gamma$, wobei $\Gamma$ eine geeignete Untergruppe von $\mathrm{Aut}(\mathbb{G}_{m,K}^g)$ ist (für die genauen Bedingungen an $\Gamma$ siehe Assumption 5.6.2 und Assumption 5.6.7). Diese Quotienten sind Beispiele für verallgemeinerte Polytopbereiche. Falls $\Gamma$ ein Gitter ist, so erhält man einen analytischen Torus $\mathbb{G}_{m,K}^g/\Gamma$; diese Situation ist bereits gut untersucht.

Für einen solchen Quotienten $X_K$ lässt sich anhand der Gruppe $\Gamma$ die Struktur der Picard-Varietät explizit bestimmen. Dabei spielt die Translationsuntergruppe $\Gamma_1 \subset \Gamma$ eine zentrale Gruppe; $\Gamma_1$ ist ein Gitter in $\mathbb{G}_{m,K}^g$. In Theorem 5.6.13 zeigen wir, dass die Picard-Varietät $\mathrm{Pic}^0(X_K)$ genau dann eigentlich ist, wenn $\mathrm{rk}\,\Gamma_1 = n$ ist. Im Fall $\mathrm{rk}\,\Gamma_1 = n$ zeigen wir weiterhin, dass $X_K$ genau dann algebraisch ist, wenn $\Gamma_1$ ein Analogon der Riemannschen Periodenrelationen erfüllt. Dieses Resultat ist im Falle von analytischen Tori (d.h. $\Gamma_1 = \Gamma$) bereits bekannt; das allgemeine Restultat lässt sich darauf zurückführen.

Als Anwendung dieser Theorie geben wir zwei neue Beispiele an, bei denen sich die Picard-Varietät leicht beschreiben lässt. In §5.4 beschreiben wir eine rigid-analytische *Kleinsche Fläche*, deren Konstruktion von der klassichen Konstruktion der Kleinschen Flasche inspiriert ist; sie ist algebraisch. Als zweites Beispiel beschreiben wir in §5.5 einen *gescherten Torus*; dies ist ein weiteres Beispiel für eine rigid-analytische Varietät, deren Picard-Varietät nicht eigentlich ist.

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