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Semigroup Methods in Finance

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I hail a semigroup when I see one and I seem to see them everywhere.

Einar Hille

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Introduction

The aim of this thesis is to highlight the role of semigroup theory in mathematical finance and to provide a class of useful methods. Apart from this special task, several new results concerning invariant subsets of strongly continuous semigroups and regular perturbations of sesquilinear forms (related to the classical Kato class) are given.

In financial mathematics one is interested in today's prices of financial derivatives written, for instance, on a stock which are exercised at a future date. Now two questions arise: What is a fair price and how can we determine it? Several attempts have been made to this question, but the following idea seems the most natural one: Whenever we can replicate the cash flow of the financial derivative with a self-financing trading strategy in the market, then the initial value of this stratey gives us the fair price of the derivative. So far the theory behind the pricing procedure, but what are the techniques one shall use? Due to the uncertainty of future prices it seems obvious that probability theory has to be the daily work in mathematical finance. As a matter of fact the fast growing interest of the latter in recent years has influenced the research in a wide area of fields. For instance, the martingale theory has seen a boost in its popularity and research progress, since Harrison and Pliska demonstrated in 1981 (cf. [HP81]) that option prices can be written as the conditional expectation with respect to a martingale measure, i.e. an equivalent probaility measure under which the discounted price process is a martingale. Another important aspect is stochastic calculus and the famous Itô-integral.

On the other hand, in many models of financial mathematics there is an underlying semigroup describing evolution in time. Our aim is to investigate this semigroup. More precisely, we are in the following situation:

A major step in finance was the pioneering work of Black and Scholes in 1973. They showed with an replication argument that prices of financial derivatives can be obtained as solutions of partial differential equations. Solving this equation "by hand", they derived the famous Black-Scholes formula which is still in use today, although certain major disadvantages of the model (constant volatility, continuous paths) cannot be rationalized. But with those partial differential equations we are already deep in the theory of strongly continuous semigroups. In fact, rewriting the equations in terms of a differential operator A one can interpret this as a Cauchy problem. If the operator A now generates a strongly continuous semigroup T on a suitable Banach space X, then the solution of the Cauchy problem and thus the price of the derivative is obtained from the semigroup: the semigroup gives us the price! Therefore, it is of great interest to determine whether the Black-Scholes operator (or other differential operators arising in mathematical finance) generates a strongly continuous semigroup or not. This will be one of our goals in this thesis.

Going a step further, one observes that the derivation of Black-Scholes partial differential equation was based on stochastic calculus. Thus, one might ask whether there is no functionalanalytic replacement for this procedure.

Mark Garman was probably the first who did a step into this direction (cf. [Gar85]). He simply assumed that the prices of derivatives were evolving like a evolution family and that there exists a generator of this evolution family. Then he could derive a (non-autonomous) Cauchy problem from this, which finally lead to the Black-Scholes equation or other known equations in finance. However, he was not able to overcome the crucial point in his reasoning and did not provide any condition under which his assumptions are fulfilled. As a matter of fact, nowadays we are still missing a (more or less) pure functionalanalytic pricing procedure in the literature, although several techniques from functional analysis (e.g. existence proofs of equivalent martingale measures or the stochastic integral itself; see, for instance, the work of Freddy Delbaen and Walter Schachermayer) have their prominent place in finance. The stated methods in this thesis might help to go a step further into the pricing direction.

After this entry into the subject we want to give a more detailed account of the concrete goals and results of the thesis. The starting point is chapter 2. We demonstrate the idea of Black and Scholes and describe the price of a European option as a solution of a partial differential equation. In this connection, we observe a change of drift in the price process, which can be interpreted as a perturbation of the associated differential operator. In addition, we take a kind of reverse point of view and study the structure of price operators in an arbitrage-free market. It turns out that these form an evolution family of linear, positive, injective operators. Moreover, the order interval $[-\infty, \mathrm{id}]$ is invariant under this family.

We take these statements as a motivation inspiring the following four main topics of the thesis:

1. invariant subsets of strongly continuous semigroup,
2. semigroups of injective operators,
3. perturbation results for differential operators,
4. generation results for the Black-Scholes operator.

Clearly, the stated results of each topic are of their own interest and importance independent from their use in finance.

We start with invariant subsets of strongly continuous semigroups in chapters 3 and 4. The relevance of this theory is out of question. Given an abstract Cauchy problem for some operator A and initial value x_0 in a Banach space X one is interested which properties of x_0 are transfered to the solution. If A generates a strongly continuous semigroup T, this corresponds to invariant subsets of T, since the unique solution is given by $T(t)x_0$, $t \geq 0$.

In chapter 3 we consider closed, convex sets in a Banach space X as possible prototypes for invariant subsets. We remark that in the Hilbert space case Brézis has treated the subject to a satisfying degree. However, our more complicated general Banach space case seems to be new. Our method will be the following: With help of the Hahn-Banach separation theorem we show that for each $x \in X$ the subdifferential of the distance function to the closed, convex set C in x is a non-empty set. This enables us to generalize the notion of Φ-dissipativity for half-norms Φ introduced by Arendt, Chernoff and Kato to the distance function of arbitrary closed, convex sets. We call this property C-dissipativity and show that the set is invariant under the semigroup T if its generator A is C-dissipative. For quasi-contractive semigroups we even have equivalence of these statements. In order to make this theory more applicable we introduce afterwards normally projectable and proximinal sets and show that a set is normally projectable if and only if it is proximinal and convex. In particular, normally projectable sets are closed and convex. We show that elements of the subdifferential of the distance function to a convex, proximinal set are written in terms of best approximation points. This makes the theory highly applicable. For instance, we recover the famous characterisations of positive or contractive semigroups. Additionally, we include an extensive treatment for the invariance of order intervals in Banach lattices. Here, we also encounter Kato-type inequalities similar to the characterisation of positive semigroups provided by Arendt (cp. [Nag86, Theorem C-II.3.8]).

Meanwhile chapter 4 is dedicated to invariant subsets of semigroups on Hilbert spaces associated to densely defined, continuous, elliptic, sesquilinear forms. Here, a beautiful result due to Ouhabaz is known characterising the invariance of a closed, convex set under contractive semigroups with conditions on the form related to the orthogonal projection onto the closed, convex set. We replace the assumption of contractivity by the existence of a common fixed point of the semigroup in the invariant subset and show that the theorem is still valid. Thanks to a result of Browder this does, in fact, generalise the theorem of Ouhabaz. In particular, the situations where the set contains the origin or the generator has compact resolvent fall into our new framework. We close the section by recovering the famous Beurling-Deny criteria.

In chapter 5 we turn our attention to semigroups of injective operators. Using a variation of the Phragmen-Lindelöf principle we give a condition in terms of the resolvent of the generator of a strongly continuous semigroup T, under which each operator $T(t)$, $t \geq 0$, is injective. In particular, every holomorphic semigroup is injective. By giving a counterexample we show, however, that this condition does not characterise semigroups of injective semigroups.

Chapter 6 concentrates on the third goal, regular perturbations of differential operators. Again, we focus on sesquilinear forms and their associated semigroups. If the semigroup T belongs to some abstract regularity space X, we ask for perturbations of the associated form such that the perturbed semigroup is still in X. As this problem is connected to the classical Kato class we take the freedom to define the abstract Kato class for our form $(\mathfrak{a}, D(\mathfrak{a}))$ as the set of all $\varphi \in D(\mathfrak{a})'$ such that $R(\lambda, \mathcal{A})\varphi \in X$. Here, $\mathcal{A} : D(\mathfrak{a}) \to D(\mathfrak{a})'$ is the operator associated to the form $\mathfrak{a}$. Next we introduce local versions of the spaces $D(\mathfrak{a})$ and $D(\mathfrak{a})'$ and the operator $\mathcal{A}$. This is essential to define a local version of the Kato class. We will also prove several properties of the Kato class and the local Kato class and in particular address the independence of the Kato class from the parameter λ. Afterwards, we introduce Kato perturbations, which are the appropriate generalisation of potentials and measures belonging to the classical Kato class. However, even in the classical situation, there can be Kato perturbations which are not associated to a measure. Finally, we consider the space X_0 of regular functions vanishing at infinity. As belonging to X_0 is in general not a local property, there is no local Kato class for X_0. To obtain semigroups on X_0, we present a theorem in the spirit of Lyapunov functions. In order to prove the Theorem, one needs a certain approximation result, which is equivalent to some abstract sort of Dirichlet boundary condition. The final part is devoted to applications. We introduce deGiorgi-Nash forms, for which many elements of the Kato class for $X = C(\Omega)$ are known from the famous deGiorgi-Nash Theorem. We prove that for any deGiorgi-Nash form and any bounded $\Omega \subset \mathbb{R}^N$, there exists a potential $V \in L^\infty_{\text{loc}}$ such that the semigroup associated to the perturbed form on $L^\infty(\Omega)$ leaves the space $C_0(\Omega)$ invariant.

In the closing chapter 7 we finally provide generation results for the Black-Scholes operator. Here, we use two different techniques. By a variational approach we realize strongly continuous semigroups T on $L^p(0,\infty)$, $2 \leq p < \infty$. With the criterion of chapter 4 we show that T leaves the order interval $[-\infty, \mathrm{id}]$ invariant. Here we really need our extension of Ouhabaz' Theorem since the semigroup is not contractive. However, in finance the price dictates the impact of a theory and so we use a different approach in order to provide an explicit pricing formula. Working on spaces of continuous functions we write the Black-Scholes operator as a simple perturbation of the square of a generator of a strongly continuous group. Now the powerful semigroup theory implies that the Black-Scholes operator generates a strongly continuous semigroup and we are provided with an explicit formula for this semigroup. We use this formula at the end to reconstruct well-known price formulas for European options in the Black-Scholes market.

The recommended starting point for the reader depends on his or her choice of subject. The chapters 3 to 7 can be read (relatively) independent from each other. Nevertheless, chapter 2 has an introductory nature, where we motivate the different goals of the thesis from a financial point of view, and is therefore a good point to start. In order to make the thesis self-contained we have also included in chapter 1 several definitions and results (partly with proof) from Functionalanalysis and semigroup theory, which will be used frequently throughout the thesis.

Chapter 1

Preliminaries

The aim of this chapter is to make the thesis more self-contained for the reader's convenience. We state several definitions and results so that the reader will always have a reference at hand if needed. For any result, if not proven, we give a concrete reference from the standard literature.

1.1 Functionalanalysis

In this section we recall some functional analytic background used in this thesis. Included are subdifferentials, duality mappings or results concerning the geometry of Banach spaces. Good references in the literature are, for instance, [Miy92], [Cio90] and [Ist81]. Although the results are all well-known, we could not resist the temptation to give a proof from time to time.

If not stated differently, X will always be a normed vector space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ with dual space X'. The duality pairing of a normed vector space S and its dual X' is given by $\langle x', x \rangle := x'(x)$ for $x' \in X'$, $x \in X$.

1.1.1 Geometry of Banach spaces

Our aims of this section are characterisations of strictly convex, uniformly convex and reflexive spaces and their relationships. We show that every uniformly convex Banach space is reflexive and strictly convex. While strict und uniform convexity are properties of the norm, the notion of reflexivity is of topological nature. Therefore, it is interesting that any uniformly convex space is reflexive.

We start with strictly convex spaces.

Definition 1.1.1. A normed vector space X is called *strictly convex*, if for all $x, y \in X$ with $\|x\| = \|y\| = 1$ the equality $\|x + y\| = 2$ implies $x = y$.

Thus, X being strictly convex means that the points on the segment connecting two distint points on the surface of the unit sphere in X are contained in the interior of the unit sphere.

In order to get different type of characterisations we first prove the following Lemma.

Lemma 1.1.2. *Let X be a normed vector space and $x, y \in X$ with $x \neq y$ and $\|y\| = \|x\| = 1$. Then the following assertions are equivalent:*

1. $\|\lambda x + (1 - \lambda)y\| < 1$ *for all* $\lambda \in (0, 1)$;

2. $\|\lambda x+(1-\lambda)y\|<1$ *for some* $\lambda\in(0,1)$.

Proof. Assume that $\|x_0\|<1$ with $x_0:=\lambda_0 x+(1-\lambda_0)y$ for some $\lambda_0\in(0,1)$. Let $\lambda\in(0,1)$ and define $x_\lambda:=\lambda x+(1-\lambda)y=y+\lambda(x-y)$. If $\lambda<\lambda_0$, we put $\mu:=1-\frac{\lambda}{\lambda_0}\in(0,1)$. Then

$$\mu y+(1-\mu)x_0=\left(1-\frac{\lambda}{\lambda_0}\right)y+\frac{\lambda}{\lambda_0}(y+\lambda_0(x-y))=y+\lambda(x-y)=x_\lambda$$

and hence $\|x_\lambda\|\le\mu+(1-\mu)\|x_0\|<1$. If $\lambda>\lambda_0$, we put instead $\mu:=\frac{\lambda-\lambda_0}{1-\lambda_0}\in(0,1)$ and obtain

$$x_0+\mu(x-x_0)=y+\lambda_0(x-y)+\frac{\lambda-\lambda_0}{1-\lambda_0}(x-y-\lambda_0(x-y))=y+(\lambda_0+\lambda-\lambda_0)(x-y)=x_\lambda.$$

Again, $\|x_\lambda\|\le(1-\mu)\|x_0\|+\mu<1$ by the tringular inequality. Since $\lambda\in(0,1)$ was arbitrarily chosen, the Lemma is proved. □

Next we can derive several other characterizations for strict convexity.

Proposition 1.1.3. *Let X be a normed vector space. The following assertions are equivalent:*

1. *X is strictly convex;*
2. $\|\lambda x+(1-\lambda)y\|<1$ *for all* $\lambda\in(0,1)$ *and* $x,y\in X$, $x\neq y$, *with* $\|x\|=\|y\|=1$;
3. *if* $\|x\|=\|y\|=\|\lambda x+(1-\lambda)y\|$ *for some* $\lambda\in(0,1)$, *then* $x=y$;
4. $\|x+y\|<\|x\|+\|y\|$ *for all linearly independent elements x and y of X.*

Proof. "(1) ⇒ (2)": Let $x,y\in X$ with $x\neq y$ and $\|x\|=\|y\|=1$. Since X is strictly convex, we have $\frac{1}{2}\|x+y\|<1$. Now Lemma 1.1.2 implies assertion (2).

"(2) ⇒ (3)": Let $x,y\in X$ and $\lambda_0\in(0,1)$ with $\|x\|=\|y\|=\|\lambda_0 x+(1-\lambda_0)y\|$. Then

$$1=\left\|\lambda_0\frac{x}{\|x\|}+(1-\lambda_0)\frac{y}{\|y\|}\right\|$$

and thus $\frac{x}{\|x\|}=\frac{y}{\|y\|}$ by assumption (2). This implies $x=y$.

"(3) ⇒ (4)": Let $x,y\in X$ be linearly independent. Let $\tilde{x}:=\frac{x}{\|x\|}$, $\tilde{y}:=\frac{y}{\|y\|}$. Furthermore, we define

$$\mu:=\frac{1}{2}(\|x\|+\|y\|),\quad \lambda:=\frac{\|y\|}{2\mu}\in(0,1).$$

It follows

$$\mu(\tilde{x}+\lambda(\tilde{y}-\tilde{x}))=\frac{1}{2}(x+\|y\|\tilde{x}+\|y\|(\tilde{y}-\tilde{x}))=\frac{1}{2}(x+y),$$

Now assume that $\|x+y\|=\|x\|+\|y\|$. Then $\frac{1}{2}\|x+y\|=\mu$ and so

$$\|\tilde{x}+\lambda(\tilde{y}-\tilde{x})\|==\frac{1}{2\mu}\|x+y\|=1=\|\tilde{x}\|=\|\tilde{y}\|.$$

From assertion (3) we get $\tilde{x}=\tilde{y}$ contradicting the linear independence of x and y. Hence, $\|x+y\|<\|x\|+\|y\|$ and (4) is shown.

"(4) ⇒ (1)": Let $x,y\in X$, $x\neq y$, with $\|x\|=\|y\|=1$. Then x and y are linearly independent and so we obtain from (4) the inequality $\|x+y\|<\|x\|+\|y\|=2$. □

Next we focus on uniformly convex spaces.

Definition 1.1.4. A normed vector space X is called *uniformly convex* if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x+y\| \le 2(1-\delta)$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon$.

The function $(0,\infty) \ni \varepsilon \mapsto \delta(\varepsilon)$ is said to be the modulus of convexity of the normed vector space X. Following Lindenstrauss one can define the modulus of convexity of a Banach space X as the function

$$\delta_X(\varepsilon) := \frac{1}{2} \inf \left\{2 - \|x+y\| \;\middle|\; \|x\| = 1, \|y\| = 1, \|x-y\| \ge \varepsilon\right\}, \quad \varepsilon > 0.$$

Obviously, a Banach space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$.

Lemma 1.1.5. *1. Every Hilbert space is uniformly convex.*

2. Every uniformly convex normed vector space is strictly convex.

Proof. (1) Let H be a Hilbert space. Let $\varepsilon \in (0,2)$ and put $\delta = \delta(\varepsilon) := 1 - \frac{1}{2}\sqrt{4-\varepsilon^2}$. Let $x, y \in H$ with $\|x\| = 1 = \|y\|$ and $\|x-y\| \ge \varepsilon$. Due to the parallelogramm law we have

$$4 = 2\left(\|x\|^2 + \|y\|^2\right) = \|x+y\|^2 + \|x-y\|^2 \ge \|x+y\|^2 + \varepsilon^2.$$

It follows $\|x+y\| \le \sqrt{4-\varepsilon^2} = 2(1-\delta)$. Hence, H is uniformly convex.

(2) Let $x, y \in X$, $x \ne y$ and $\|x\| = \|y\| = 1$. We put $\varepsilon := \|x-y\| > 0$. Since X is uniformly convex, there exists $\delta > 0$ such that $\|x+y\| \le 2(1-\delta) < 2$. Hence, X is strictly convex. $\square$

Thanks to the famous Clarkson inequalities (cf. [Ist81, p. 50]), namely

$$\begin{aligned} \forall\, p \in [2,\infty): \quad \forall\, f, g \in L^p(\Omega): \quad & \left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \le \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right), \\ \forall\, p \in (1,2]: \quad \forall\, f, g \in L^p(\Omega): \quad & \|f+g\|_p^{\frac{p}{p-1}} + \|f-g\|_p^{\frac{p}{p-1}} \le \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p-1}}, \end{aligned}$$

one can show that $L^p(\Omega)$, $1 < p < \infty$, is uniformly convex, where (Ω, Σ, μ) is a measure space, A similar proof applies for ℓ_p with $1 < p < \infty$.

In a next step we are going to prove that every uniformly convex Banach space is reflexive. Recall that a Banach space X is *reflexive*, if the mapping $j : X \to X''$, $(j(x))(x') := \langle x', x\rangle$ is surjective. We start with a useful characterisation of uniform convexity.

Proposition 1.1.6. *A normed vector space X is uniformly convex if and only if for sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset X$ with*

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = \lim_{n\to\infty} \frac{1}{2}\|x_n + y_n\|$$

one has $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. First, we assume that X is uniformly convex. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be sequences in X such that

$$M := \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = \lim_{n\to\infty} \frac{1}{2}\|x_n + y_n\|.$$

If $M = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| \leq \lim_{n\to\infty} \|x_n\| + \|y_n\| = 0$ and the claim is proved. So let us assume $M > 0$. Then we may define

$$\tilde{x}_n := \frac{x_n}{\|x_n\|}, \quad \tilde{y}_n := \frac{y_n}{\|y_n\|}$$

for sufficiently large $n \in \mathbb{N}$. We claim that $\lim_{n\to\infty} \|\tilde{x}_n + \tilde{y}_n\| = 2$. In fact, for $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$2 \leq \varepsilon + \frac{\|x_n + y_n\|}{\|x_n\|}, \quad |\|y_n\| \|x_n\| - 1| \leq \varepsilon, \quad n \geq n_\varepsilon.$$

It follows

$$2 - \|\tilde{x}_n + \tilde{y}_n\| \leq \varepsilon + \left\| \tilde{x}_n + \frac{y_n}{\|x_n\|} \right\| - \|\tilde{x}_n - \tilde{y}_n\| \leq \varepsilon + \left\| \tilde{y}_n \left(\frac{\|y_n\|}{x_n} - 1 \right) \right\| \leq 2\varepsilon.$$

Hence, $\|\tilde{x}_n + \tilde{y}_n\| \to 2$ as $n \to \infty$.

Next we will show that $\lim_{n\to\infty} \|\tilde{x}_n - \tilde{y}_n\| = 0$. Let $\varepsilon > 0$. Since X is uniformly convex, there exists $\delta > 0$ such that $\|\tilde{x}_n + \tilde{y}_n\| \leq 2(1 - \delta)$ whenever $\|\tilde{x}_n - \tilde{y}_n\| \geq \varepsilon$. Due to $\lim_{n\to\infty} \|\tilde{x}_n + \tilde{y}_n\| = 2$, we find $n_\delta \in \mathbb{N}$ such that $2 - \|\tilde{x}_n + \tilde{y}_n\| < 2\delta$, i.e. $\|\tilde{x}_n + \tilde{y}_n\| > 2(1-\delta)$, for all $n \geq n_\delta$. Hence, $\|\tilde{x}_n - \tilde{y}_n\| < \varepsilon$ for all $n \geq n_\delta$. We have shown that $\|\tilde{x}_n - \tilde{y}_n\| \to 0$ as $n \to \infty$.

From this it follows

$$\|x_n - y_n\| = \|\|y_n\| \tilde{x}_n - y_n + x_n - \|y_n\| \tilde{x}_n\| \leq \|y_n\| \|\tilde{x}_n - \tilde{y}_n\| + |\|x_n\| - \|y_n\|| \to 0$$

as $n \to \infty$. That finally proves the first implication.

Conversely, assume that X is not uniformly convex. Then there exists $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$ we find $x_n, y_n \in \mathbb{N}$ with $\|x_n\| = \|y_n\| = 1$, $\|x_n - y_n\| \geq \varepsilon_0$ and $\|x_n + y_n\| \geq 1 - \frac{1}{n}$. Hence, $\lim_{n\to\infty} \|x_n + y_n\| = 2$. By assumption, it would follow $\varepsilon_0 \leq \|x_n - y_n\| \to 0$ as $n \to \infty$, which is a contradiction. Therefore, X has to be uniformly convex. □

Corollary 1.1.7. *Let X be uniformly convex Banach space and $(x_n)_{n\in\mathbb{N}} \subset X$ such that*

$$\lim_{n\to\infty} \|x_n\| = \lim_{n,m\to\infty} \frac{1}{2} \|x_n + x_m\|.$$

Then $(x_n)_{n\in\mathbb{N}}$ is convergent in X.

Proof. We will show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is assumed to be complete, this will imply the convergence of $(x_n)_{n\in\mathbb{N}}$ in X. Assume the converse, i.e. there exists $\varepsilon_0 > 0$ and integers

$$n_1 < m_1 < n_2 < m_2 < \dots$$

such that $\|x_{n_k} - x_{m_k}\| \geq \varepsilon_0$ for all $k \in \mathbb{N}$. By our assumption on the sequence $(x_n)_{n\in\mathbb{N}}$, we have

$$\lim_{k\to\infty} \|x_{n_k}\| = \lim_{k\to\infty} \|x_{m_k}\| = \lim_{k\to\infty} \frac{1}{2} \|x_{n_k} + x_{m_k}\|.$$

Since X is uniformly convex, Proposition 1.1.6 implies $\varepsilon_0 \leq \|x_{n_k} - x_{m_k}\| \to 0$ as $k \to \infty$, which is a contradiction. Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and thus convergent. □

We will use these results to show that every uniformly convex Banach space is reflexive.

Proposition 1.1.8 (Milman-Pettis). *Every uniformly convex Banach space is reflexive.*

Proof. Let X be a uniformly convex Banach space and let $\phi \in X''$. We need to find $x_0 \in X$ such that $\langle \phi, x' \rangle_{X'',X'} = \langle x', x_0 \rangle_{X',X}$ for all $x' \in X'$.

We may assume that $\|\phi\| = 1$. Then there exists a sequence $(x'_n)_{n\in\mathbb{N}} \subset X'$, $\|x'_n\| = 1$, such that

$$\lim_{n\to\infty} \left|\langle \phi, x'_n \rangle\right| = 1.$$

We will prove the existence of some $x_0 \in X$, $\|x_0\| = 1$, such that $\langle x'_n, x_0 \rangle = \langle \phi, x'_n \rangle$ for all $n \in \mathbb{N}$. Thanks to Helly's Theorem we find $(x_m)_{m\in\mathbb{N}} \subset X$ such that $\|x_m\| \leq 1 + \frac{1}{m}$ and $\langle x'_n, x_m \rangle = \langle \phi, x'_n \rangle$ for all $m \in \mathbb{N}$ and $n \in \{1, \ldots, m\}$. Let $m \in \mathbb{N}$, $n \in \{1, \ldots, m\}$. Then

$$\|x_n + x_m\| \geq \left|\langle x'_n, x_n + x_m \rangle\right| = 2\left|\langle \phi, x'_n \rangle\right|.$$

Hence,

$$2 \geq \lim_{n,m\to\infty} \|x_n + x_m\| \geq \lim_{n\to\infty} 2\left|\langle \phi, x'_n \rangle\right| = 2.$$

We obtain

$$\lim_{m\to\infty} \|x_m\| = 1 = \lim_{n,m\to\infty} \frac{1}{2}\|x_n + x_m\|.$$

Now Corollary 1.1.7 implies the convergence of $(x_m)_{m\in\mathbb{N}}$ in X. Let $x_0 \in X$ be the limit. Then $\|x_0\| = 1$ and

$$\langle x'_n, x_0 \rangle = \lim_{m\to\infty} \langle x'_n, x_m \rangle = \langle \phi, x'_n \rangle, \quad n \in \mathbb{N}.$$

Now let $x'_0 \in X'$ with $\|x'_0\| = 1$ and consider the sequence $(x'_n)_{n\in\mathbb{N}_0}$. As before we find $\hat{x}_0 \in X$, $\|\hat{x}_0\| = 1$, such that $\langle x'_n, \hat{x}_0 \rangle = \langle \phi, x'_n \rangle$ for all $n \in \mathbb{N}$. That leads to

$$\frac{1}{2}\|x_0 + \hat{x}_0\| \geq \frac{1}{2}\left|\langle x'_n, x_0 + \hat{x}_0 \rangle\right| = \left|\langle \phi, x'_n \rangle\right|$$

for all $n \in \mathbb{N}$ and thus

$$\frac{1}{2}\|x_0 + \hat{x}_0\| = \lim_{n\to\infty} \left|\langle \phi, x'_n \rangle\right| = 1.$$

Since X is uniformly convex (and thus strictly convex), Proposition 1.1.3 now implies $x_0 = \hat{x}_0$. In particular, $\langle x'_0, x_0 \rangle = \langle \phi, x'_0 \rangle$. Since x'_0 was an arbitrary element of the surface of the unit sphere of X', the Proposition is proved. □

Therefore, we have the inclusions:

$$\{X \text{ Hilbert space}\} \subset \{X \text{ uniformly convex}\} \subset \{X \text{ strictly convex, reflexive}\}.$$

We close this first section with the following two results:

Proposition 1.1.9. *(cf. [Wer00, Theorem III.3.7]) In a reflexive Banach space any bounded sequence has a weakly convergent subsequence.*

Corollary 1.1.10. *Let X be a uniformly convex Banach space and let $(x_n)_{n\in\mathbb{N}}, x \in X$, such that*

$$x_n \rightharpoonup x, \ \ \|x_n\| \to \|x\|$$

as $n \to \infty$. Then $x_n \to x$ in X.

Proof. Let $x' \in X'$ such that $\langle x', x\rangle = \|x\|$. From our assumptions we have

$$\|x\| \geq \overline{\lim}_{n\to\infty}\frac{1}{2}\|x_n + x\| \geq \overline{\lim}_{n\to\infty}\frac{1}{2}\left|\langle x', x_n + x\rangle\right| = \left|\langle x', x\rangle\right| = \|x\|.$$

Hence,

$$\lim_{n\to\infty}\|x_n\| = \|x\| = \lim_{n\to\infty}\frac{1}{2}\|x_n + x\|$$

and since X is uniformly convex, Proposition 1.1.6 implies $\|x_n - x\| \to 0$ as $n \to \infty$. □

1.1.2 Convex Analysis

This section is devoted to convex analysis on normed vector spaces. We will state two classical versions of the theorem of Hahn-Banach concerning the separation of convex sets. Then we focus on convex and lower semicontinuous functions. Using the theorem of Hahn-Banach we show that any convex, lower semicontinuous function is weakly lower-semicontinuous. From this we deduce useful existence results of minima in convex sets for certain convex, lower semicontinuous functions.

We start with the separation theorems. A subset C of a normed vetor space X is called *convex* if for all $x, y \in C$ and $\lambda \in [0,1]$ the vector $\lambda x + (1-\lambda)y$ belongs to C. Using the Eich function

$$p(x) := \inf\{\alpha > 0 \mid x \in \alpha C\}, \quad x \in X,$$

for a open, convex set $C \subseteq X$, $0 \in C$, one can show the following separation theorem for convex sets:

Theorem 1.1.11. *(cf. [Wer00]) Let X be a normed vector space and $C_1, C_2 \subseteq X$ be two disjoint, convex subsets. If C_1 is open, there are $x' \in X' \setminus \{0\}$ and $\gamma \in \mathbb{R}$ such that*

$$\sup_{y\in C_1} \operatorname{Re}\langle x', y\rangle < \gamma \leq \inf_{y\in C_2} \operatorname{Re}\langle x', y\rangle.$$

For a strict separation we need to assume compactness for one of the sets:

Theorem 1.1.12. *(cf. [Wer00]) Let X be a normed vector space and C_1, C_2 be two disjoint, nonempty, convex subsets. Assume that C_1 is compact and C_2 is closed. Then there exists $x' \in X'$ and constants $\gamma_1, \gamma_2 \in \mathbb{R}$ such that*

$$\operatorname{Re}\langle x', x\rangle < \gamma_1 < \gamma_2 < \operatorname{Re}\langle x', y\rangle$$

for all $(x, y) \in C_1 \times C_2$.

The *domain* of a function $\phi : X \to (-\infty, +\infty]$ is defined as $D(\phi) := \{x \in X \mid \phi(x) < \infty\}$. The function is called *proper* if $D(\phi) \neq \emptyset$. Moreover, the function ϕ is said to be *convex* if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

for all $\lambda \in [0,1]$ and $x, y \in X$ and is called lower semicontinuous if the sets

$$X_c := \{x \in X \mid \phi(x) \leq c\}.$$

are closed for all $c \in \mathbb{R}$.

Lemma 1.1.13. *Let X be a normed vector space. The following assertions are equivalent:*

1. *ϕ is lower semicontinuous;*
2. *$\phi(x) \leq \underline{\lim}_{n\to\infty}\phi(x_n)$ whenever $x_n \to x$ as $n \to \infty$;*

Proof. Let ϕ be lower semicontinuous and let $x_n \to x$ as $n \to \infty$. We assume $\phi(x) > \underline{\lim}_{n\to\infty}\phi(x_n)$. We choose some $\phi(x) > c > \underline{\lim}_{n\to\infty}\phi(x_n)$. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\phi(x_{n_k}) \leq c$ for all $k \in \mathbb{N}$. Hence, $x_{n_k} \in X_c$ for all $k \in \mathbb{N}$. It follows $x \in X_c$ and so, in contradiction, $c \geq \phi(x) > c$. Thus, $\phi(x) \leq \underline{\lim}_{n\to\infty}\phi(x_n)$.

Conversely, let $c \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_c converging to x. Then $\phi(x) \leq \underline{\lim}_{n\to\infty}\phi(x_n) \leq c$ and hence $x \in X_c$. Therefore, ϕ is lower semicontinuous. □

It is of interest that every convex, lower semicontinuous function is weakly lower semicontinuous. Here, a function $\phi : X \to \mathbb{R}$ is called *weakly lower semicontinuous* if the sets X_c are weakly closed for all $c \in \mathbb{R}$. For the proof we use the separation theorem 1.1.12.

Lemma 1.1.14. *Let X be normed vector space and let $\phi : X \to (-\infty, +\infty]$ be a convex, lower semicontinuous function. Then ϕ is weakly lower semicontinuous.*

Proof. Let $c \in \mathbb{R}$ and consider a sequence $(x_n)_{n\in\mathbb{N}} \subset X_c$ such that $x_n \rightharpoonup x_0$ as $n \to \infty$. Assume $x_0 \notin X_c$. Since ϕ is convex and lower semicontinuous, the set X_c is closed and convex. Thus, due to the separation theorem 1.1.12, there exists a functional $x' \in X'$ and $\varepsilon > 0$ such that

$$\mathrm{Re}\ \langle x', x_0\rangle < \mathrm{Re}\ \langle x', x_0\rangle + \varepsilon \leq \inf_{y\in X_c} \mathrm{Re}\ \langle x', y\rangle.$$

By assumption, we have $x_n \in X_c$ for all $n \in \mathbb{N}$. That leads to the contradiction

$$\varepsilon \leq \mathrm{Re}\ \langle x', x_n - x_0\rangle \to 0 \ \text{ as } \ n \to \infty.$$

Therefore, $x_0 \in X_c$. □

Corollary 1.1.15. *Any convex, closed subset of a normed vector space is weakly closed.*

We use this result to deduce the following theorem, which guarantees the existence of minima in convex subsets $C \subseteq X$ for suitable functions on C. This will be useful in section 3.4.1, when we consider the existence of best approximations of elements in closed, convex sets.

Theorem 1.1.16. *Let X be a reflexive Banach space and let $C \subset X$ be a convex, closed subset. Let $\psi : C \to (-\infty, \infty]$ be a convex, lower semicontinuous function such that $\lim_{\|x\|\to\infty} \psi(x) = \infty$. Then ψ has a minimum in C.*

Proof. We may assume $\psi \not\equiv \infty$. Put $m := \inf_{x\in C} \psi(x) \in (-\infty, \infty]$. Since $\psi(x) \to \infty$ if $\|x\| \to \infty$, there exists $r > 0$ such that $\psi(x) > m + 1$ whenever $\|x\| \leq r$. Hence, we find a sequence $(x_n)_{n\in\mathbb{N}} \subset C \cap \overline{B}(0, r)$ such that $\psi(x_n) \to m$ as $n \to \infty$. By passing to a subsequence, x_n converges weakly to some x as $n \to \infty$. Since C is convex and closed, the set C is weakly closed and we obtain $x \in C$. It further follows from Lemma 1.1.14 that ψ is weakly lower semicontinuous. Hence, $\psi(x) \leq \underline{\lim}_{n\to\infty}\psi(x_n) = m$ and ψ attains its minimum in x. □

1.1.3 The duality mapping

Here we introduce duality mappings which are motivated from the following fact: given a vector $x \neq 0$ in a normed vector space X, we know from the Hahn-Banach theorem that there exists a functional $x' \in X'$, with norm $\|x'\| = 1$ such that $\langle x', x\rangle = \|x\|$. Naturally one is interested in properties of the set of all functionals fulfilling this condition. For instance, it would be advantageous to know when this set is a singleton. Those questions bring us to the notion of duality mappings. In order to have a broader basis we consider duality mappings for gauge functions and show that they are single valued if the dual space X' is strictly convex.

A *gauge function* is a strictly increasing function $\mu : [0, \infty) \to \mathbb{R}$ with the properties $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = +\infty$. For instance, the identity function is a gauge function.

Definition 1.1.17. Let X be a normed vector space and $\mu : [0, \infty) \to \mathbb{R}$ be a gauge function. The *μ-duality set* of $x \in X$ is given by the subset

$$J_\mu(x) := \left\{x' \in X' \mid \left\|x'\right\| = \mu(\|x\|) \text{ and } \langle x, x'\rangle = \|x\| \left\|x'\right\|\right\}.$$

We call the (multivalued) mapping J_μ the *μ-duality mapping.* If μ is the identity function, then $J = J_\mu$ is simply called *duality mapping.*

Thanks to the Hahn-Banach theorem the set $J_\mu(x)$ is non-empty for each $x \in X$. Furthermore, one easily sees that $J_\mu(x)$ is closed and convex and fulfills $J_\mu(\alpha x) = \alpha J_\mu(x)$ for all $\alpha \in \mathbb{K}$.

As mentioned before, we are interested in situations, where J_μ is single valued. A sufficient condition is the strict convexity of the dual space.

Lemma 1.1.18. *Let X be a normed vector space with strictly convex dual X' and let μ : $[0, \infty) \to \mathbb{R}$ be a gauge function. Then $J_\mu(x)$ is a singleton for all $x \in X$.*

Proof. . Let $x \in X$ and consider $x', y' \in J_\mu(x)$. We may assume $x \neq 0$ since $J_\mu)(0) = \{0\}$. Then

$$\left\|x' + y'\right\| \|x\| \geq \langle x' + y', x\rangle = \|x\| \left(\left\|x'\right\| + \left\|y'\right\|\right).$$

Hence, $\|x' + y'\| = \|x'\| + \|y'\|$. Since X' is strictly convex, this implies linear dependence of x' and y' (see Proposition 1.1.3), i.e. there exists $\alpha \in \mathbb{C} \setminus 0$ such that $y' = \alpha x'$. It follows

$$\langle x', x\rangle = \|x\| \, \mu(\|x\|) = \langle y', x\rangle = \overline{\alpha}\langle x', x\rangle.$$

Thus, $\alpha = 1$ and so $x' = y'$. The map J_μ is indeed single valued. □

Thanks to the representation theorem of Fréchet-Riesz (cf. [Wer00, Theorem V.3.6]) the duality mapping in a Hilbert space is the identity:

Lemma 1.1.19. *If X is a Hilbert space, then $J(x) = x$ for all $x \in X$.*

Proof. As usual we identity X' with X. Then X' is strictly convex and hence the duality mapping is single-valued. Let $x \in X$ and $x' = J(x)$. There exists $x_0 \in X$ such that $(\cdot|x_0) = x'$ on X, $\|x_0\| = \|x'\| = \|x\|$. We obtain $(x|x_0) = x'(x) = \|x\|^2$ and hence

$$\|x - x_0\|^2 = \|x\|^2 - 2 \text{ Re } (x|x_0) + \|x_0\|^2 = 0.$$

Thus, $x = x_0$ and by identification $J(x) = x$. □

At the end of the section we characterise the existence of an element in the duality set with certain properties with conditions on the norm. Since we will prove a similar theorem for the distance function in section 3.3.1 (cp. Lemma 3.3.7), we omit a proof here and give only a reference.

Lemma 1.1.20. *(cf. [Miy92, Corollary 2.7]) Let X be a normed vector space. For $x, y \in X$ the following assertions are equivalent:*

1. $\mathrm{Re}\,\langle x', y\rangle \leq 0$ *for some* $x' \in J(x)$*;*
2. $\|x - \lambda y\| \geq \|x\|$ *for all* $\lambda \geq 0$.
3. $\|x - \lambda y\| \geq \|x\|$ *for all* $\lambda \in [0, 1]$.

1.1.4 The subdifferential

In the final section of this chapter we study subdifferentials of functions on X with values in $\mathbb{R}$. This notion generalises the property of differentiability to a broader class of functions. We prove the famous theorem of Asplund connecting the notions of subdifferentials and duality mappings.

A function $f : X \to (-\infty, +\infty]$ is said to be *subdifferentiable at a point* $x \in D(f)$ if there exists a functional $x \in X'$, called *subgradient* of f at x, such that

$$\mathrm{Re}\,\langle x', y - x\rangle \leq f(y) - f(x)$$

for all $y \in X$. The set of all subgradients of f at x is denoted by $(\partial f)(x)$ and the set-valued mapping ∂f is called the *subdifferential* of f.

For a better insight we illustrate this notion for a convex function $\varphi : \mathbb{R} \to \mathbb{R}$. The subgradients of φ at some point $x_0 \in \mathbb{R}$ are all $c \in \mathbb{R}$ such that $\varphi(x) - \varphi(x_0) \geq c(x - x_0)$ for all $x \in \mathbb{R}$ or, equivalently,

$$c \leq \frac{\varphi(x) - \varphi(x_0)}{x - x_0}, \quad x \in \mathbb{R}.$$

Taking the one-sided limits

$$a := \lim_{x \nearrow x_0} \frac{\varphi(x) - \varphi(x_0)}{x - x_0}, \; b := \lim_{x \searrow x_0} \frac{\varphi(x) - \varphi(x_0)}{x - x_0},$$

the subdifferential of φ in x_0 is indeed the interval $[a, b]$. For instance, the subdifferential of $\varphi(x) := |x|$, $x \in \mathbb{R}$, at the origin is $[-1, 1]$.

We would like to compare the notion of the subdifferential with the duality mappings from the previous section. This is done by the famous theorem of Asplund:

Theorem 1.1.21 (Asplund)**.** *Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous gauge function. We define*

$$\psi(t) := \int_0^t \phi(s)\; ds, \quad t \geq 0.$$

Then $\psi : [0, \infty) \to [0, \infty)$ is a convex function and $(\partial\psi)(\|x\|) = J_\phi(x)$ for all $x \in X \setminus \{0\}$.

Proof. First, we will prove the convexity of the function ψ. Since ϕ is strictly increasing, we have

$$\phi(t+h) - \psi(t) \geq \psi(t)h \geq \phi(t) - \phi(t-h)$$

for all $0 \leq h \leq t$. Let $0 \leq t_1 < t_2$, $\lambda \in [0,1]$ and consider $t := t_2 + \lambda(t_1 - t_2)$ Then

$$\lambda = \frac{t - t_2}{t_1 - t_2}, \;\; 1 - \lambda = \frac{t_1 - t}{t_1 - t_2}.$$

Hence, we have $t_2 - t = \lambda(t_2 - t_1) \geq 0$ and $t - t_1 = (1-\lambda)(t_2 - t_1) \geq 0$. From the above inequalities it follows

$$\begin{array}{rcl} \psi(t_2) - \psi(t) & \geq & (t_2 - t)\phi(t), \\ \psi(t) - \psi(t_1) & \leq & (t - t_1)\phi(t). \end{array}$$

In conclusion, we obtain

$$\begin{array}{rcl} (1-\lambda)\psi(t_2) + \lambda\psi(t_1) - \psi(t) & = & (1-\lambda)(\psi(t_2) - \psi(t)) - \lambda(\psi(t) - \psi(t_1)) \\ & \geq & (1-\lambda)(t_2 - t)\phi(t) + \lambda(t_1 - t)\phi(t) \\ & = & \phi(t)\left((1-\lambda)(t_2 - t) + \lambda(t_1 - t)\right) = 0 \end{array}$$

and therefore

$$(1-\lambda)\psi(t_2) + \lambda\psi(t_1) \geq \psi(t_2 + \lambda(t_1 - t_2)), \quad 0 \leq t_1 < t_2, \lambda \in [0,1],$$

i.e. ψ is convex. Now let $x \in X \setminus \{0\}$. Let $x' \in J_\phi(x)$, i.e. $\langle x', x\rangle = \|x'\| \, \|x\| = \phi(\|x\|) \, \|x\|$. For $y \in X$ with $\|y\| > \|x\|$ we have

$$\left\|x'\right\| = \phi(\|x\| = \psi'(\|x\|) \leq \frac{\psi(\|y\|) - \psi(\|x\|)}{\|y\| - \|x\|}.$$

That leads to

$$\psi(\|y\|) - \psi(\|x\|) \geq \left\|x'\right\| (\|y\| - \|x\|) \geq \text{Re } \langle x', y - x\rangle.$$

For $y \in X$ with $\|y\| < \|x\|$ one uses the inequality

$$\psi'(\|x\|) \geq \frac{\psi(\|x\|) - \psi(\|y\|)}{\|x\| - \|y\|}$$

and obtain

$$\psi(\|x\|) - \psi(\|y\|) \leq \text{Re } \langle x', x - y\rangle.$$

Hence, $x' \in (\partial\psi)(\|x\|)$. Conversely, let $x' \in (\partial\psi)(\|x\|)$. For $y \in X$ with $\|y\| = \|x\|$ we have

$$\text{Re } \langle x', y - x\rangle \leq \psi(\|y\|) - \psi(\|x\|) = 0$$

and hence $\text{Re } \langle x', y\rangle \leq \text{Re } \langle x', x\rangle$. It follows

$$\left\|x'\right\| \|x\| = \sup_{\|y\| = \|x\|} \left|\langle x', y\rangle\right| = \sup_{\|y\| = \|x\|} \text{Re } \langle x', y\rangle \leq \text{Re } \langle x', x\rangle.$$

With the usual trick

$$\|x'\| \|x\| \le \mathrm{Re}\, \langle x', x\rangle \le |\langle x', x\rangle| \le \|x'\| \|x\|$$

we obtain $\langle x', x\rangle = \|x'\| \|x\|$. It remains to prove that $\|x'\| = \phi(\|x\|)$. Therefore, we consider $y = \frac{t}{\|x\|}x$ with $t > 0$. Then

$$\begin{aligned} \psi(t) - \psi(\|x\|) &= \psi(\|y\|) - \psi(\|x\|) \\ &\ge \mathrm{Re}\, \langle x', y - x\rangle \\ &= \left(\frac{t}{\|x\| - 1}\right) \mathrm{Re}\, \langle x', x\rangle. \\ &= (t - \|x\|) \|x'\|, \end{aligned}$$

i.e.

$$\sup_{t>\|x\|} \frac{\psi(t) - \psi(\|x\|)}{t - \|x\|} \ge \|x'\| \ge \sup_{t<\|x\|} \frac{\psi(t) - \psi(\|x\|)}{t - \|x\|}.$$

We finally obtain

$$\phi(\|x\|) = \psi'(\|x\|) = \|x'\|$$

and so $x' \in J_\phi(x)$. □

For the duality mapping, i.e. $\phi = \mathrm{id}$, we have:

Corollary 1.1.22. *The function $\psi(x) := \frac{1}{2}\|x\|^2$, $x \in X$, is convex and $(\partial\psi)(x) = J(x)$.*

As a matter of fact, a Banach space X is strictly convex if and only if the map $X \ni x \mapsto \|x\|^2$ is strictly convex.

1.2 Semigroups, their generators and associated forms

In this chapter we take a tour through the theory of strongly continuous semigroup of linear operators. Motivated as solutions of abstract Cauchy problems, we give the definition of the infinitesimal generator, state the famous Hille-Yosida Theorem for generators of strongly continuous semigroups and formulate the relationship between sesquilinear forms and semigroups. The stated results are all well-known and extensively studied in the literature, so that we will omit the proofs and cite only related references. For a more detailed view onto the subject we refer to [Are06], [?], [EN00], [Ouh04], [Paz83] or [Yos65].

Throughout this chapter let X be a Banach space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ with dual space X'. We denote by $L(X)$ the space of all bounded, linear operators on X and use the notation $\langle\cdot,\cdot\rangle$ for the duality pairing on X, i.e. $\langle x', x\rangle = x'(x)$ for $x' \in X'$, $x \in X$.

1.2.1 Abstract Cauchy-Problems

In this first section of the chapter we consider abstract Cauchy problems for operators on Banach space and define classical and mild solutions. Furthermore, we clearify what we mean by well-posedness of the problem.

Let X be a Banach space and $(A, D(A))$ be a linear operator on X. The initial value problem

$$(\text{ACP}) \begin{cases} \dot{u}(t) &= Au(t) \quad , \quad t \geq 0, \\ u(0) &= x_0 \end{cases}$$

is called the (abstract) *Cauchy problem* for the operator A on X with initial value $x_0 \in X$.

Definition 1.2.1. Let $u : [0, \infty) \to X$ be a continuous function.

1. u is said to be a *classical solution of (ACP)* if $u \in C^1([0, \infty); X)$, $u(t) \in D(A)$ for all $t \geq 0$ and u solves (ACP).

2. u is said to be a *mild solution of (ACP)* if $\int_0^t u(s)\, ds \in D(A)$ and $u(t) = A \int_0^t u(s)\, ds + x_0$ for all $t \geq 0$.

There are several ways to define well-posedness of the Cauchy problem (ACP). Here we will follow the notes of Engel and Nagel (cp. [EN00]).

Definition 1.2.2. Let $(A, D(A))$ be a closed operator on a Banach space X. We say that the (abstract) Cauchy problem (ACP) for the operator A is *well-posed* if the following conditions are fulfilled:

1. For each initial value $x \in D(A)$ there exists a unqiue classical solution $u : [0, \infty) \to X$ of the Cauchy problem (ACP);

2. $D(A)$ is dense in X;

3. For every sequence $(x_n)_{n \in \mathbb{N}} \subset D(A)$ satisfying $\lim_{n \to \infty} x_n = 0$ one has $\lim_{n \to \infty} u(t, x_n) = 0$ uniformly in compact intervals $[0, t_0]$.

Intuitively, well-posedness expresses that the conditions "existence", "uniqueness" and "continuous dependence on the data" are fulfilled by the solution. In the next section we will see that (ACP) is well-posed for a closed operator A if and only if A is the generator of a strongly continuous semigroup.

1.2.2 Strongly continuous semigroups

Linear semigroup of operators will play the central role in this section. We define strongly continuous semigroups and their infinitesimal generator, which uniquely determines the semigroup. It turns out that strongly continuous semigroups are closely connected to the Cauchy problems from the previous section. In fact, the semigroup solves the Cauchy problem for its generator. Moreover, the Cauchy problem for a closed operator A is well-posed if and only if the A generates a strongly continuous semigroup.

A *linear semigroup* of operators on X is a family $T = (T(t))_{t \geq 0} \subset L(X)$ such that $T(0) = I$ and $T(t + s) = T(t)T(s)$ for all $s, t \geq 0$. The semigroup T is said to be *strongly continuous* (or a C_0-semigroup), if $T(t)x \to x$ in X as $t \searrow 0$ for each $x \in X$.

Proposition 1.2.3. *(cf. [EN00, Proposition I.5.3 + Theorem I.5.8]) Let $T = (T(t))_{t \geq 0} \subset L(X)$ be a semigroup on a Banach space X. The following assertions are equivalent:*

1. *T is strongly continuous;*

2. *$[0, \infty) \ni t \mapsto T(t)x$ is continuous for all $x \in X$;*

3. *T is weakly continuous, i.e. the maps $[0, \infty) \ni t \mapsto \langle x', T(t)x\rangle$ are continuous for all $x \in X$, $x' \in X'$;*

4. *there are constants $\delta > 1$, $M > 0$ and a dense subset $D \subseteq X$ such that $\|T(t)\| \leq M$ for all $t \geq 0$ and $\lim_{t\searrow 0} T(t)x = x$ for all $x \in D$.*

If the map $t \mapsto T(t)$ is norm-continuous, then T is called *immediately norm-continuous.* A strongly continuous semigroup $T = (T(t))_{t\geq 0}$ is said to be *eventually compact* if there exists $t_0 > 0$ such that $T(t_0)$ is a compact operator. In case, one obtains compactness of the operators $T(t)$ for all $t \geq t_0$ due to the ideal property of compact operators and the semigroup law.

Thanks to the uniform boundedness principle each strongly continuous semigroup is uniformly bounded on compact intervals in $[0, \infty)$. This leads to the following result:

Proposition 1.2.4. *(cf. [EN00, Proposition I.5.5]) Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. There exists constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.*

We call a semigroup T *contractive* if $\|T(t\| \leq 1$ for all $t \geq 0$ and *quasi-contractive* if $\|T(t)\| \leq e^{\omega t}$ for some $\omega \in \mathbb{R}$ and all $t \geq 0$.

The *growth bound* of a strongly continuous semigroup T is given by

$$\omega(T) := \inf\left\{\omega \in \mathbb{R} \mid \exists\, M \geq 1 : \ \|T(t)\| \leq Me^{\omega t}, \ t \geq 0\right\}.$$

Note that $\omega(T) = -\infty$ is possible and the infimum may not be attained (cp. [EN00, p. 40]).

The *(infinitesimal) generator of a strongly continuous semigroup T* on X is the (possibly unbounded) linear operator

$$\begin{aligned} D(A) &:= \left\{x \in X \mid \lim_{t\searrow 0} \frac{1}{t}\left(T(t)x - x\right) \text{ exists in } X\right\}, \\ Ax &:= \lim_{t\searrow 0} \frac{1}{t}\left(T(t)x - x\right). \end{aligned}$$

The generator of a strongly continuous semigroup is a closed and densely defined linear operator which determines the semigroup uniquely (cf. [EN00, Theorem II.1.4]). We have the following obvious properties:

Remark 1.2.5. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$.*

1. *Let $\omega \in \mathbb{C}$. Then $S(t) := e^{-\omega t}T(t)$, $t \geq 0$, defines a strongly continuous semigroup on X with generator $A - \omega I$.*

2. *Let $\alpha > 0$. Then $S(t) := T(\alpha t)$, $t \geq 0$, defines a strongly continuous semigroup on X with generator αA.*

Furthermore, the following relations between generator and semigroup hold:

Proposition 1.2.6. *(cf. [EN00, Lemma II.1.3]) Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$.*

1. *For all $x \in D(A)$ and $t \geq 0$ one has:*
   (a) *$T(t)x \in D(A)$ and $\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x$;*
   (b) *$T(t)x - x = \int_0^t T(s)Ax\, ds$.*
2. *For all $x \in X$ and $t \geq 0$ one has:*
   (a) *$\int_0^t T(s)x\, ds \in D(A)$;*
   (b) *$T(t)x - x = A\int_0^t T(s)x\, ds$.*

Proposition 1.2.7. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let $x, y \in X$. Then $x \in D(A)$ and $Ax = y$ if and only if $T(t)x - x = \int_0^t T(s)y\, ds$ for all $t \geq 0$.*

Next we focus on the proposed relationship between generators of strongly continuous semigroups and well-posed Cauchy problems.

Proposition 1.2.8. *(cf. [EN00, Proposition II.6.4]) Let $(A, D(A))$ be the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on a Banach space X. Let $x_0 \in X$. We consider the abstract Cauchy problem (ACP) for the operator A with initial value x_0. Then $u(t) := T(t)x_0$, $t \geq 0$, is a mild solution of (ACP) and a classical solution of (ACP) if $x \in D(A)$.*

As a matter of fact, we even have a characterisation of the following type:

Proposition 1.2.9. *(cf. [EN00, Theorem II.6.7]) Let $(A, D(A))$ be a closed operator on a Banach space X. The related Cauchy problem (ACP) for the operator A is well-posed if and only if A generates a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on X. In case, the unique classical solution of (ACP) for the initial value $x_0 \in D(A)$ is given by $u(t) := T(t)x_0$, $t \geq 0$.*

So far we have only treated Cauchy problems for initial values. But often we have to solve a *backward Cauchy problem* for an operator A and a final value $x \in X$ of the following kind:

$$\text{(BCP)} \quad \begin{cases} \dot{u}(t) &= Au(t) \quad , \quad t \in [0, \tau], \\ u(\tau) &= x \end{cases}$$

Here, τ is an arbitrary positive number. We see, however, that the unique solution of (BCP) is again given in terms of the semigroup:

Proposition 1.2.10. *Let A be the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on a Banach space X. Let $x \in D(A)$. Then $u(t) := T(\tau - t)x$, $t \in [0, \tau]$, is the unqiue solution of the backward Cauchy problem for the operator A with final value x.*

Proof. One easily sees that u solves (BCP). So we concentrate on the uniqueness of the solution. Let v be another solution of (BCP). For $t \in [0, \tau]$ we define

$$\Phi : [0, \tau - t] \longrightarrow X, \;\; \Phi(s) := T(\tau - t - s)v(\tau - s).$$

It follows for $s \in [0, \tau - t]$ and $h \in [0, s]$:

$$\begin{aligned}
& \frac{1}{h}\left(\Phi(s) - \Phi(s-h)\right) \\
= \;& \frac{1}{h}\left(T(\tau - t - s)v(\tau - s) - T(\tau - t - s + h)v(\tau - s + h)\right) \\
= \;& -T(\tau - t - s)\left(\frac{T(h)v(\tau - s) - v(\tau - s)}{h} + T(h)\left(\frac{v(\tau - s + h) - v(\tau - s)}{h}\right)\right).
\end{aligned}$$

From the uniform continuity of the map $(t, x) \mapsto T(t)x$ in t on the compact interval $[0, s]$ and the fact that v is a solution of (BCP) we now obtain

$$\lim_{h \searrow 0} \frac{\Phi(s) - \Phi(s-h)}{h} = -T(\tau - t - s)\left(Av(\tau - s) + \dot{v}(\tau - s)\right) = 0$$

In a similar manner we get

$$\lim_{h \nearrow 0} \frac{1}{h}\left(\Phi(s+h) - \Phi(s)\right) = 0.$$

We conclude $\Phi \in C^1([0, \tau - t]; X)$ and $\dot{\Phi} \equiv 0$. Hence, Φ is constant and it follows

$$u(t) = T(\tau - t)x = T(\tau - t)v(\tau) = \Phi(0) = \Phi(\tau - t) = T(0)v(t) = v(t).$$

Since $t \in [0, \tau]$ was arbitrary chosen, the functions u and v have to coincide implying the proposed uniqueness of the solution. $\square$

1.2.3 Hille-Yosida Generation Theorems

Given a linear operator $(A, D(A))$ in a Banach space X it is desirable to find criteria which imply that A is the generator of a strongly continuous semigroup. Most characterizations are based on the resolvent of the operator. Therefore, we recall the definition of the *resolvent set* of an operator $(A, D(A))$:

$$\rho(A) := \left\{\lambda \in \mathbb{C} \mid \lambda - A : D(A) \to X \text{ bijective, } (\lambda - A)^{-1} \in L(X)\right\}.$$

For $\lambda \in \rho(A)$ we define the *resolvent* $R(\lambda, A) := (\lambda - A)^{-1} \in L(X)$. The *spectrum* $\sigma(A)$ of A is the complement of the resolvent set, i.e. $\sigma(A) = \mathbb{C} \setminus \rho(A)$. The *spectral bound* of A is given by

$$s(A) := \sup\left\{\mathrm{Re}\lambda \mid \lambda \in \rho(A)\right\}.$$

If A is the generator of a strongly continuous semigroup T, then one has $-\infty \leq s(A) \leq \omega(T) < +\infty$, where $\omega(T)$ is the growth bound of T.

Lemma 1.2.11. *(cf. [EN00, Lemma 3.4] Let $(A, D(A))$ be a closed, densely defined, linear operator on a Banach space X. Suppose that there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that $[\omega, \infty) \subset \rho(A)$ and $\|\lambda R(\lambda, A)\| \leq M$ for all $\lambda \geq \omega$. Then $\lambda R(\lambda, A)x \to x$ for all $x \in X$ and $\lambda AR(\lambda, A)u \to Au$ for all $u \in D(A)$ as $\lambda \to \infty$.*

It is extremely useful that the resolvent can be written as the Laplace-transform of the semigroup:

Proposition 1.2.12. *(cf. [Are06, Proposition 2.4.1]) Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$ and growth bound $\omega(T)$. Then $N := \{\lambda \in \mathbb{C} \mid \mathrm{Re}\ \lambda > \omega(T)\} \subseteq \rho(A)$ and*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda s} T(s)x\, ds, \quad x \in X,$$

for all $\lambda \in N$.

In particular, the half-plane $\{\mathrm{Re}\ \lambda > \omega(T)\}$ is in the resolvent set of an operator A, if A generates a strongly continuous semigroup.

Next we come to a general generation theorem for strongly continuous semigroups:

Theorem 1.2.13. *(cf. [EN00, Theorem II.3.8]) Let $(A, D(A))$ be a linear operator in a Banach space X, $\omega \in \mathbb{R}$ and $M \geq 1$. The following assertions are equivalent:*

1. *A is the generator of a strongly continuous semigroup T on X such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.*
2. *A is closed, densely defined and fulfills $(\omega, \infty) \subset \rho(A)$ with*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \quad \lambda > \omega, n \in \mathbb{N}.$$

3. *A is closed, densely defined and for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$ is $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\mathrm{Re}(\lambda) - \omega)^n}, \quad n \in \mathbb{N}.$$

For the case of a contraction semigroup this reads:

Theorem 1.2.14. *(Hille-Yosida) A linear operator $(A, D(A))$ on a Banach space X generates a contractive strongly continuous semigroup T if and only if A is closed, densely defined and fulfills $(\omega, \infty) \subset \rho(A)$ with $\|\lambda R(\lambda, A)\| \leq 1$.*

This is the classical *Hille-Yosida Theorem*, while Theorem 1.2.13 is essentially due to Feller, Miyadera and Phillips (1952).

The difficult part in the proof of Corollary 1.2.14 ist the converse implication, i.e. the verification of A being a generator. One has to construct the semigroup out of the resolvent. Independent from each other Yosida and Hille attacked the problem on different ways. Yosida considered the operators

$$A_\lambda := \lambda^2 R(\lambda, A) - \lambda = \lambda A R(\lambda, A) \in L(X), \quad \lambda > 0,$$

nowadays called the *Yosida-Approximation* of A, and showed that the limit $\lim_{\lambda\to\infty} e^{tA_\lambda}x$ exists for all $x \in X$ and equals $T(t)x$. On the contrary, Hille was inspired by Euler's formula for the exponential function and showed the following:

Proposition 1.2.15. *(Euler's formula; cf. [Nag86, Proposition A-II.1.10]) Let A be the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on a Banach space X. Then*

$$T(t)x = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} x = \lim_{n\to\infty}\left(\frac{n}{t}R(\frac{n}{t}, A)\right)^{n} x$$

for all $x \in X$, $t \geq 0$.

Next we come to generation theorems for strongly continuous groups. A *strongly continuous linear group* on X is a family $G = (G(t))_{t\in\mathbb{R}} \subset L(X)$ such that $G(0) = I$, $G(t+s) = G(t)G(s)$ for all $t, s \in \mathbb{R}$ and the map $\mathbb{R} \ni t \mapsto G(t)x$ is continuous for all $x \in X$. For a group G we define the strongly continuous semigroups $G_+ := (G(t))_{t\geq 0}$ and $G_-(t) := (G(-t))_{t\geq 0}$. The *(infinitesimal) generator* of G is the linear operator

$$\begin{aligned} D(A) \ &:= \ \left\{x \in X \mid \lim_{t\to 0}\frac{1}{t}\left(G(t)x - x\right) \ \text{ exists in } \ X\right\}, \\ Ax \ &:= \ \lim_{t\to 0}\frac{1}{t}\left(G(t)x - x\right). \end{aligned}$$

It turns out that a linear operator $(A, D(A))$ is the generator of a strongly continuous group G if and only if A generates G_+ and $-A$ generates G_-.

Theorem 1.2.16. *([EN00, p. 79]) Let $(A, D(A))$ be a linear operator on a Banach space X. For constants $M \geq 1$, $\omega \in \mathbb{R}$ the following assertions are equivalent:*

1. *A generates a strongly continuous group $G = (G(t))_{t\in\mathbb{R}}$ such that $\|G(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.*
2. *A and $-A$ generate the strongly continuous semigroups G_+ and G_-, respectively, and $\|G_+(t)\|, \|G_-(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.*
3. *A is closed, densely defined and for all $\lambda \in \mathbb{R}$ with $|\lambda| > \omega$ one has $\lambda \in \rho(A)$ and $\|(|\lambda| - \omega)^n R(\lambda, A)^n\| \leq M$, $n \in \mathbb{N}$.*
4. *A is closed, densely defined and for all $\lambda \in \mathbb{C}$ with $|\mathrm{Re}\ \lambda| > \omega$ one has $\lambda \in \rho(A)$ and $\|(|\mathrm{Re}\ \lambda| - \omega)^n R(\lambda, A)^n\| \leq M$, $n \in \mathbb{N}$.*

1.2.4 Holomorphic semigroups

Naturally one seeks for different types continuity or regularity for semigroups instead of strong continuity. We have already encountered immediately norm-continuous and compact semigroups in section 1.2.2. Another example are differentiable semigroups (see [EN00, Chapter II.4.b]). We will focus here on holomorphic semigroups. One can speak about them as strongly continuous semigroups on complex Banach spaces who have a holomorphic extension to some sector. Their generators are sectorial, i.e. the resolvent set contains some sector of the complex plane. In addition, we give a Hille-Yosida generation theorem for holomorphic semigroups. At the end of the section we also show that the square of a group generator is the generator of a holomorphic semigroup.

We start with the notion of a sector in the complex plane: for $\delta \in (0, \pi]$ we define the sector Σ_δ as the set

$$\Sigma_\delta := \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \delta\} \setminus \{0\}$$

A family of operators $T = (T(z))_{z \in \Sigma_\delta \cup \{0\}} \subseteq L(X)$ is called an *holomorphic semigroup of angle* $\delta \in (0, \frac{\pi}{2}]$ if

1. $T(0) = I$, $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$;
2. the map $\Sigma_\delta \ni z \mapsto T(z)$ is holomorphic;
3. $\lim_{\Sigma_{\delta'} \ni z \to 0} T(z)x = x$ for all $z \in X$ and $\delta' \in (0, \delta)$.

If, in addition, $\|T(z)\|$ is bounded in $\Sigma_{\delta'}$ for every $\delta' \in (0, \delta)$, we call T a *bounded holomorphic semigroup* (of angle δ).

The notion of holomorphic semigroups is closely connected to sectorial operators: Here, a densely defined, closed, linear operator $(A, D(A))$ on a Banach space X is called *sectorial of angle* $\delta \in (0, \frac{\pi}{2}]$ if the sector $\Sigma_{\delta + \frac{\pi}{2}}$ is contained in the resolvent set $\rho(A)$ and if for each $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon \geq 1$ such that $\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|}$ for all $0 \neq \lambda \in \overline{\Sigma}_{\delta + \frac{\pi}{2} - \varepsilon}$.

That leads to the following Hille-Yosida type generation theorem:

Theorem 1.2.17. *(cf. [EN00, Theorem II.4.6]) For a linear operator $(A, D(A))$ on a Banach space X the following statements are equivalent:*

1. *A generates a bounded holomorphic semigroup $T = (T(z))_{z \in \Sigma_\delta \cup \{0\}}$ on X.*
2. *A generates a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$ on X such that $T(t)X \subset D(A)$, $t \geq 0$, and $M := \sup_{t>0} \|tAT(t)\| < \infty$.*
3. *A generates a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$ on X and there exists a constant $C > 0$ such that $\|R(r + is, A)\| \leq \frac{C}{|s|}$ for all $r > 0$ and $0 \neq s \in \mathbb{R}$.*
4. *A is sectorial.*

Using this characterisation one can show that the square of a group generator is the generator of a holomorphic semigroup.

Proposition 1.2.18. *(cf. [ABHN01]) Let A be the generator of a bounded strongly continuous group $G = (G(t))_{t \in \mathbb{R}}$ on a Banach space X. Then the operator A^2 with domain $D(A^2) := \{x \in D(A) \mid Ax \in D(A)\}$ generates a bounded holomorphic semigroup T of angle $\frac{\pi}{2}$ on X given by*

$$T(t)x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} G(s\sqrt{2t})x \, ds, \quad t \geq 0, x \in X.$$

1.2.5 Semigroups associated to sesquilinear forms

In this final section we treat semigroups on Hilbert spaces which are associated to densely defined, continuous, elliptic sesquilinear forms. We will see that those semigroups are automatically holomorphic.

Let V and H both be Hilbert spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, where V is continuously injected into H, i.e. there exists $C > 0$ such that $\|u\|_H \leq C \|u\|_V$ for all $u \in V$. We denote this by $V \hookrightarrow H$ and with $V \overset{\hookrightarrow}{d} H$ if V is dense in H.

A *sesquilinear form* is a mapping $\mathfrak{a} : V \times V \to \mathbb{K}$, which is linear in the first and antilinear in the second component. To indicate the dependence of its domain V we usually write $(\mathfrak{a}, V)$. We call the form $(\mathfrak{a}, V)$

- *densely defined* if V is dense in H,
- *continuous* if there exists $m > 0$ such that $|\mathfrak{a}[u, v]| \leq M \|u\|_V \|v\|_V$ for all $u, v \in V$;
- H-elliptic or *elliptic* if there exist constants $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\operatorname{Re} \mathfrak{a}[u, u] + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2$$

  for all $u \in V$,
- *accretive* if $\operatorname{Re} \mathfrak{a}[u, u] \geq 0$ for all $u \in V$,
- *coercive* if there exists $\alpha > 0$ such that $\operatorname{Re} \mathfrak{a}[u, u] \geq \alpha \|u\|_V^2$ for all $u \in V$.

If $\mathbb{K} = \mathbb{R}$, then V' denotes the dual space of V. In the case $\mathbb{K} = \mathbb{C}$ we consider antilinear functional, i.e. mappings $f : V \to \mathbb{C}$ such that $f(u + v) = f(u) + f(v)$ and $f(\lambda u) = \overline{\lambda} f(u)$ for all $u, v \in V$, $\lambda \in \mathbb{C}$. We call it the *antidual* of V. We identify the Hilbert space H as a subspace of V' and obtain the continuous injections $V \hookrightarrow H \hookrightarrow V'$.

Theorem 1.2.19. *(Lax-Milgram; cf. [Are06, Theorem 7.1.1]) Let $(\mathfrak{a}, V)$ be a continuous, coercive form on a Hilbert space H. Then there exists an isomorphism $\mathcal{A} : V \to V'$ such that $(\mathcal{A}u)(v) = \mathfrak{a}[u, v]$ for all $u, v \in V$.*

From this isomorphism we obtain a holomorphic semigroup on V':

Proposition 1.2.20. *(cf. [Are06, Theorem 7.1.4]) Let $(\mathfrak{a}, V)$ be a continuous, coercive, densely defined form on a Hilbert space H. Let $\mathcal{A} : V \to V'$ be the isomorphism given by the Theorem of Lax-Milgram. Then $-\mathcal{A}$ generates a bounded holomorphic semigroup on V'.*

In order to get a semigroup on H we define the *associated operator to the form* $(\mathfrak{a}, V)$ on H by

$$\begin{aligned} D(A) &:= \{u \in V \mid \exists\, f \in H : \ (f|v)_H = \mathfrak{a}[u, v] \text{ for all } v \in V\}, \\ Au &:= f. \end{aligned}$$

Since V is dense in H, the operator A is well defined.

Theorem 1.2.21. *(cf. [Are06, Theorem 7.1.5 + Proposition 7.1.7]) Let $(\mathfrak{a}, V)$ be a continuous, coercive, densely defined form on a Hilbert space H. Then $-A$ generates a bounded holomorphic semigroup T on H, which leaves V invariant und whose restriction to V is a bounded holomorphic semigroup on V. In particular, $D(A)$ is dense in V.*

For an elliptic, continuous form $(\mathfrak{a}, V)$ with

$$\operatorname{Re} \mathfrak{a}[u, u] + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2, \quad u \in V,$$

the perturbed form $(\mathfrak{a}+\omega, V)$, i.e. $(\mathfrak{a}+\omega)[u,v] = \mathfrak{a}[u,v] + \omega(u|v)_H$ for all $u, v \in V$, is coercive and continuous. Therefore, we can pass from the class of coercive forms to elliptic forms in the theorem above. The obtained semigroup T on H (generated by the operator $-A-\omega I$) will be called the *semigroup associated to the form* $(\mathfrak{a}, V)$.

We close the section and thus the chapter with a representation of the form $(\mathfrak{a}, V)$ in terms of resolvent of its associated operator. This result will be of use in chapter 4, when we consider invariant subsets of semigroups associated to forms.

Proposition 1.2.22. *(cf. [Are06, Lemma 9.1.4]) Let* $(\mathfrak{a}, V)$ *be a densely defined, continuous, elliptic form on a Hilbert space* H *with associated operator* A*. Then one has*

$$\mathfrak{a}[u,v] = \lim_{\lambda\to\infty} \lambda(u - \lambda R(\lambda, -A)u|v)_H$$

for all $u, v \in V$*.*

Chapter 2

Motivation: Semigroups arising in Mathematical Finance

2.1 Introduction

In mathematical finance one is interested in today's prices of financial derivatives written, for instance, on a stock which are exercised at a future date. Now two questions arises: What is a fair price and how to determine it? In this chapter we will first recall the idea of Black and Scholes who replicated European options in a financial market by self-financing trading strategies. The option is thereby written on an underlying which evolves like a geometric Brownian motion. Then the value of the trading strategy at each time gives the fair price of the option. More interesting, they could prove that the stock price is the solution of a backward Cauchy problem. Here, for the first time the semigroup theory comes into play. A second attempt of this chapter is to explore the semigroup (or evolutionary) structure behind the pricing procedure a bit further. In a arbitrage-free setting of simple trading strategies we show that the price operators form an evolution family of positive, injective operators. This approach seems to be unexplored in the literature so far. In particular, we can derive the well-known Put-Call parity from this. We close the chapter with several conclusions determining the goals of the up-coming chapters from a financial point of view.

2.2 Option prices as solutions of Cauchy problems

In this section we will recall the idea of Black and Scholes of pricing European options by replicating trading strategies. We will see that under their assumptions the price is the solution of a backward Cauchy problem. The techniques we will use come from the theory of stochastic calculus.

We will allow continuous trading up to some expiration date $\tau > 0$. That means the investors are allowed to trade at any time t in the time interval $I := [0, \tau]$. For simplicity we will ignore transaction costs at any time.

For the modelling of the finanical market we have to rely on probability theory since we cannot be certain about future prices. Thus, the uncertainty in the markt will be modelled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in I}, \mathbb{P})$. Here, the filtration $\mathbb{F}$ shall reflect the developement of the information structure in the market: For $t \in I$ the filtration $\mathcal{F}_t$ consists of all information, e.g. prices of financial products, available in the market up to time t. Since we know today prices, $\mathcal{F}_0$ is trivial. Further, at time $t = \tau$ all information needed is given within the market. Thus, the following assumptions on the filtration are reasonable:

1. (completeness) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$;
2. (right continuity) For all $t \geq 0$ one has $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{u>t} \mathcal{F}_u$;
3. $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_\tau = \mathcal{F}$.

We fix two components in our market model: a stock and a bond. Black and Scholes assume that the price process of the stock S can be written as a geometric Brownian motion,

$$S(t) := S(0) \exp\left(\sigma W(t) + (\mu - \frac{1}{2}\sigma^2)t\right), \quad t \in I,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ are constants and $(W(t))_{t \in I}$ is a one-dimensional Brownian motion (with respect to the filtration $\mathbb{F}$). In the notion of stochastic differentials that means

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t)\right).$$

Moreover, for the bond B we will have the description.

$$dB(t) = rB(t)dt.$$

We want to evaluate European options written on the stock S and exercised at time τ, that means, we are interested in random variables $H = h(S(\tau))$ for Borel-measurable functions $h : (0, \infty) \to \mathbb{R}$. For instance, $H = (K - S(\tau))^+$ stands for an European Put option with strike price $K > 0$ giving the owner of the option the right (but not the obligation) to sell the option at time τ for the fixed price K.

In the following we denote by $P(t)H$ the price of the option H at time $t \in I$.

Before going into the details we will give a brief summary of the idea of Black and Scholes in their celebrated paper (cp. [BS73]). Let us fix some Borel-measurable function $h : (0, \infty) \to \mathbb{R}$. Black and Scholes assumed the existence of a function $f^{(h)} \in C^{1,2}(I \times (0, \infty))$ such that one has $P(t)h(S(\tau)) = f^{(h)}(t, S(t))$ for every $t \in I$, i.e. the option price at time t depends on the stock price $S(t)$ and the time t only. Next they had the idea to construct a self-financing trading strategy $(\alpha(t), \beta(t))_{t \in I}$ in the stock S and the bond B such that the corresponding value process (including one sold option)

$$Y(t) := \alpha(t)S(t) + \beta(t)B(t) - f_h(t, S(t))$$

underlies no longer risky fluctuations. That means, we replicate the option with a self-financing strategy. In the absence of arbitrage the value process $Y(t)$ has to develop like the bond and this leads finally to a Cauchy problem.

To step into details, a *trading strategy* $(\alpha, \beta) = (\alpha(t), \beta(t))_{t \in I}$ in the stock and the bond is a $\mathbb{R}^2$-valued stochastic process, which is progressively measurable with respect to $\mathbb{F}$ and fulfills the properties

1. $\mathbb{P}\left(\int_0^\tau |\beta(t)| \; dt < \infty\right) = 1$,
2. $\mathbb{P}\left(\int_0^\tau (\alpha(t)S(t))^2 \; dt < \infty\right) = 1$.

Those conditions ensure that the stochastic integrals of α and β with respect to the processes S and B, respectively, are well-defined. The *value process* of (α, β) is given by

$$V(t) := \alpha(t)S(t) + \beta(t)B(t), \quad t \in I.$$

We call a trading strategy (α, β) *self-financing* if for all $t \in I$ holds:

$$\begin{aligned} V(t) &= V(0) + \int_0^t \alpha(s)\ dS(s) + \int_0^t \beta(s)\ dB(s) \\ &= V(0) + \int_0^t \sigma\alpha(s)S(s)\ dW(s) + \int_0^t (\mu\alpha(s)S(s) + r\beta(s)B(s))\ ds. \end{aligned}$$

That means there is no income or outcome of money during trading. In terms of stochastic differential:

$$dV(t) = \sigma\alpha(t)S(t)dW(t) + (\mu\alpha(t)S(t) + r\beta(t)B(t))dt.$$

An *arbitrage opportunity* is a self-financing strategy (α, β) such that $V(0) = 0$, $V(\tau) \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(V(\tau) > 0) > 0$. From now on we will always assume that no such strategy exists.

For convenience, we write $f = f^{(h)}$ from now on. Now Ito's formula implies that

$$df(t, S(t)) = (f_t(t, S(t)) + f_x(t, S(t))\mu S(t) + \frac{1}{2}\sigma^2 f_{xx}(t, S(t))S^2(t))dt + (\sigma f_x(t, S(t))S(t))dW(t).$$

Hence, for a self-financing trading stratey (α, β) in the stock and the bond we obtain for the related value process (including one sold option):

$$\begin{aligned} dY(t) &= \sigma\alpha(t)S(t)dW(t) + (\mu\alpha(t)S(t) + r\beta(t)B(t))dt - df(t, S(t)) \\ &= \left(\mu\alpha(t)S(t) + r\beta(t)B(t) - (f_t(t, S(t)) + f_x(t, S(t))\mu S(t) + \frac{1}{2}\sigma^2 f_{xx}(t, S(t))S^2(t))\right) dt \\ &\quad + (\sigma\alpha(t)S(t) - \sigma f_x(t, S(t))S(t))\, dW(t). \end{aligned}$$

In order to get a riskless portfolio the diffusion term has to vanish. Thus, we set

$$\alpha(t) := f_x(t, S(t)), \quad t \in I.$$

Then the value process $(Y(t))_{t\in I}$ is riskless, so it has to develop like the bond to omit arbitrage possibilities:

$$dY(t) = rY(t)dt.$$

It follows:

$$\begin{aligned} rY(t) &= \mu f_x(t, S(t))S(t) + r\beta(t)B(t) - (f_t(t, S(t)) + f_x(t, S(t))\mu S(t) + \frac{1}{2}\sigma^2 f_{xx}(t, S(t))S^2(t)) \\ &= r\beta(t)B(t) - f_t(t, S(t)) - \frac{1}{2}\sigma^2 f_{xx}(t, S(t))S^2(t) \\ &= rY(t) - rf_x(t, S(t))S(t) + rf(t, S(t)) - f_t(t, S(t)) - \frac{1}{2}\sigma^2 f_{xx}(t, S(t))S^2(t) \end{aligned}$$

We obtain the condition

$$f_t(t, x) + \frac{1}{2}\sigma^2 x^2\ f_{xx}(t, x) + rxf_x(t, x) - rf(t, x) = 0$$

for all $t \in I$ and $x \in (0, \infty)$.

In conclusion, under the assumption of no-arbitrage the option price $P(t)h(S(\tau)) = f(t, S(t))$ at time $t \in I$ solves the backward Cauchy problem:

$$\text{(BS)} \quad \begin{cases} f_t(t,x) + \frac{1}{2}\sigma^2 x^2 \ f_{xx}(t,x) + rxf_x(t,x) - rf(t,x) = 0 & , \quad (t,x) \in I \times (0,\infty) \\ f(\tau, x) = h(x) & , \quad x \in (0,\infty) \end{cases} .$$

Here, we have mainly followed the notes of [KK99].

Now the importance of semigroup methods are obvious. For the differential operator $Af := \frac{1}{2}\sigma^2 x^2 \ f_{xx} + rxf_x - rf$ we rewrite (BS) as (abstract) Cauchy problem:

$$\text{(ABS)} \quad \begin{cases} f_t + Af = 0 & , \quad t \in I \\ f(\tau) = h \end{cases} .$$

If the operator A generates a strongly continous semigroup T in a suitable Banach space X and $h \in D(A)$, then $(T(\tau - t)h)(x) = f(t, x)$ gives the price of the contingent claim $h(S(\tau))$ at time $t \in [0, \tau]$ when the stock price $S(t)$ equals x.

2.3 Semigroup Pricing Methods

We have seen that prices of European options can be obtained as solutions of Cauchy problems. Therefore, the theory of semigroups comes into play. Now we want to examine the semigroup structure behind the pricing procedure a bit further. It is not the aim of our current approach to show ways of pricing. We show that the price operators form an evolution family of positive, injective operators in an arbitrage-free market.

As before we allow continuous trading up to some finite horizon $\tau > 0$ and ignore transaction costs. The uncertainty of the financial market is again modelled within filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in I}, \mathbb{P})$ such that

1. (completeness) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$;
2. (right continuity) For all $t \geq 0$ it holds $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{u > t} \mathcal{F}_u$;
3. $\mathcal{F}_0$ is trivial and $\mathcal{F}_\tau = \mathcal{F}$.

For any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ we will denote by $L^0(\Omega, \mathcal{G}, \mathbb{P})$ the vector space of all measurable mappings $X : (\Omega, \mathcal{G}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ identifying mappings which coincide $\mathbb{P}$-a.s.. Regarding the filtration $\mathbb{F}$ one has $L^0(\Omega, \mathcal{F}_s, \mathbb{P}) \subset L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ for $0 \leq s \leq t \leq \tau$. We write $X \geq Y$ for $X, Y \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ if $\mathbb{P}(X \geq Y) = 1$ and define

$$L^0_+ := \left\{ X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid X \geq 0 \right\}, \ \ K := L^0_+ \setminus \{0\} .$$

Note that for $X \in K$ we have $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X > 0) > 0$.

Coming back to our modelling of the financial market we will denote by M_t for $t \in I$ the space of all financial products in the market which are priced for all times in $[0, t]$. We identify $H \in M_t$ with its price $H(t)$ at time t. By our interpretation of the filtration $\mathbb{F}$ as information structure we obtain for $H \in M_t$ that its price $H(s)$ at time $s \in [0, t]$ need to belong to $L^0(\Omega, \mathcal{F}_s, \mathbb{P})$. Moreover, it holds $M_t \subset M_s$ for $0 \leq s \leq t \leq \tau$. We will assume the existence of a strictly positive process $B = (B(t))_{t\in I}$ in M_τ giving the possibility of riskless investment. It is further a reasonable assumption that M_t, $t \in I$, is a real vector space. In the following we will refer to this market model by $\mathcal{M}$.

We remark that in the standard approach of pricing theory for financial products written on a stock S the model $\mathcal{M}$ consists of the assumed stock price process $(S(t))_{t\geq 0}$ and the numeraire B solely. The prices of products are then obtained by replication strategies. However, the stated approach has a different intention as indicated in the beginning. Assuming the existence of a price process of an option, e.g. a call, we will derive conditions on this price development by arbitrage-reasons. That is the slight advantage of this definition. Nevertheless it is surely applicable for the standard approach as well.

Our only additional assumption on $\mathcal{M}$ will be the absence of arbitrage. Shortly, an arbitrage opportunity is a trading strategy in the market which makes something out of nothing. We will give a precise definition in the context of simple trading strategies.

Definition 2.3.1. A stochastic process $\varphi = (\varphi(t))_{t\geq 0}$ is said to be *simply predictable*, if there exist a finite sequence of stopping times $0 = \tau_0 < \tau_1 < \ldots < \tau_{n+1} < \infty$ and random variables $\varphi_i \in L^0(\Omega, \mathcal{F}_{\tau_i}, \mathbb{P})$ with $|\varphi_i| < \infty$ for $i \in \{0, 1, \ldots, n\}$ such that

$$\varphi(t) = \mathbb{1}_0(t)\ \varphi_0 + \sum_{i=1}^{n} \mathbb{1}_{(\tau_i,\tau_{i+1}]}(t)\ \varphi_i, \quad t \geq 0.$$

Then we say that φ has the representation $((\varphi_i)_{0\leq i\leq n}, (\tau_i)_{0\leq i\leq n+1})$. We denote by $\mathcal{S}$ the set of all simply predictable processes.

For a stochastic process $X = (X(t))_{t\in I}$ we define the *(simple) stochastic integral* with respect to X as the linear mapping $I_X : \mathcal{S} \to L^0(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$\varphi \mapsto \varphi_0 X(0) + \sum_{i=1}^{n} \varphi_i\ (X_{\tau_{i+1}\wedge\tau} - X_{\tau_i\wedge\tau})$$

if $\varphi \in \mathcal{S}$ has the representation $((\varphi_i)_{0\leq i\leq n}, (\tau_i)_{0\leq i\leq n+1})$.

Definition 2.3.2. Let $t \in I$ and $(H_k)_{1\leq k\leq m} \subset M_t$.

1. Let $k \in \{1, \ldots, m\}$. A *simple trading strategy in* H_k is a stochastic process $\varphi \in \mathcal{S}$ with representation $((\varphi_i)_{0\leq i\leq n}, (\tau_i)_{0\leq i\leq n+1})$ where $\tau_{n+1} = t$. The value of φ at time $s \in [0, t]$ is defined by
$$V_\varphi(s) := \varphi(s) H_k(s).$$

2. A *simple trading strategy in* $(H_k)_{1\leq k\leq m}$ is a $\mathbb{R}^m$-valued stochastic process $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(m)})$ where for each $k \in \{1, \ldots, m\}$ the process $\varphi^{(k)} \in \mathcal{S}$ is a simple trading strategy in H_k. The value of φ at time $s \in [0, t]$ is defined by
$$V_\varphi(s) := \sum_{k=1}^{n} \varphi^{(k)}(s)\ H_k(s).$$

For a simple trading strategy φ in $H \in M_t$, $t \in I$, with representation $((\varphi_i)_{0\leq i\leq n}, (\tau_i)_{0\leq i\leq n+1})$ the stopping times τ_i reflect the trading points. During the interval $(\tau_i, \tau_{i+1}]$ we hold φ_i portions of H in our portfolio. At time τ_{i+1} we will trade again, from now on holding φ_{i+1} portion of H up to time τ_{i+2}, and so on. A strategy where we de not invest or borrow money after an initial investment is called self-financing.

Definition 2.3.3. Let $t \in I$ and $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(m)})$ a simple trading strategy in $(H_k)_{1 \leq k \leq m} \subset M_t$. For $s \in I$ the strategy φ is called *self-financing* in $[s,t]$ if for the assembled trading points $s \leq \tau_1 < \tau_2 < \ldots < \tau_{n+1} = t$ (coming from the representation of the trading strategies $\varphi^{(k)}$) holds:

$$\sum_{k=1}^{m} (\varphi^{(k)}(\tau_i) - \varphi^{(k)}(\tau_{i+1}))\ H_k(\tau_{i+1}) = 0, \quad i = 0, \ldots, n.$$

We denote by $\mathcal{H}[s,t]$ the space of all simple trading strategies which are self-financing in $[s,t]$ and, shortly, $\mathcal{H} := \mathcal{H}[0,\tau]$.

As an example of a strategy $\varphi \in \mathcal{H}[s,t]$ consider a stochastic process $\varphi = (\varphi(r))_{r \in I}$ which is constant over the time, i.e. $\varphi(r) \equiv c$ for all $r \in I$. Then φ is a simple predictable process and applied to $H \in M_t$ we obtain at any different times $r, u \in [s,t]$: $(\varphi(r) - \varphi(u))H(u) = 0$. Hence, $\varphi \in \mathcal{H}[s,t]$.

By trading we can extend the spaces M_t of marketed products.

Definition 2.3.4. Let $t \in I$. A contingent claim $H \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ is called (simply) attainable (in $\mathcal{M}$) if there exists a simple trading strategy $\varphi \in \mathcal{H}[0,t]$ such that $V_\varphi(t) = H$. The space of all (simply) attainable claims at time t is denoted by A_t.

Remark 2.3.5. *For $t \in I$ it holds $M_t \subset A_t$. In fact, let $H \in M_t$ with price process $(H(s))_{0 \leq s \leq t}$. As seen before the stochastic process $\varphi \equiv 1$ lies in $\mathcal{H}[0,t]$ and it holds $V_\varphi(t) = H_t$ implying $H \in A_t$.*

Let $t \in I$. For $s \in [0,t]$ we define the price operators $P(s,t) : A_t \to A_s$ by setting $P(s,t)H := V_\varphi(s)$ whenever $H = V_\varphi(t)$ for $\varphi \in \mathcal{H}[0,]$. For instance, if $H \in M_t$ then $P(s,t)H = H(s)$. However, we cannot be sure by now if those operators are well-defined. But this will be guaranteed by the notion of arbitrage, i.e. making something out of nothing:

Definition 2.3.6. Let $0 \leq s \leq t \leq \tau$. An *simple arbitrage opportunity* in $[s,t]$ is a strategy $\varphi \in \mathcal{H}[s,t]$ such that $V_\varphi(s) = 0$ and $V_\varphi(t) \in K$. If no simple arbitrage opportunity exists, then we say that $(NSA)_{s,t}$ is fulfilled. We write shortly (NSA) for $(NSA)_{0,\tau}$.

Remark 2.3.7. *Assume (NSA). Then $(NSA)_{s,t}$ for all $0 \leq s \leq t \leq \tau$.*

As stated in the beginning the absence of arbitrage opportunities is a fundamental (and natural) assumption on $\mathcal{M}$. It is surprising how many properties of price operators can be obtained from this:

Proposition 2.3.8. *Assume (NSA). Then for $0 \leq s \leq t \leq \tau$ the price operators*

$$P(s,t) : A_t \to A_s, \ \ V_\varphi(t) \mapsto V_\varphi(s)$$

are well-defined, linear, positive and injective. Moreover, they form an evolution family, i.e. $P(t,t) = I$ and $P(r,s)P(s,t) = P(r,t)$ for $0 \leq r \leq s \leq t \leq \tau$.

Proof. Let $0 \leq s \leq t \leq \tau$. Let $\varphi, \psi \in \mathcal{H}[0,t]$ with $V_\varphi(t) = V_\psi(t)$ and assume for

$$A := \{\omega \in \Omega \mid V_\varphi(s)(\omega) > V_\psi(s)(\omega)\}$$

that $\mathbb{P}(A) > 0$. We define the simple strategy $\phi = (\phi_0, \phi_1)$ on $[s,t]$ by setting $\phi_0(r) := \mathbb{1}_A(\varphi(r) - \psi(r))$ and $\phi_1(r) := \mathbb{1}_A B(s)^{-1}(V_\varphi(s) - V_\psi(s))$ for $r \in [s,t]$. Then $\phi \in \mathcal{H}[s,t]$ with $V_\phi(s) = 0$ and $V_\phi(t) = \mathbb{1}_A \frac{B(t)}{B(s)}(V_\varphi(s) - V_\psi(s)) \in K$. That contradicts (NSA). Therefore $P(s,t)$ is well-defined.

Similar proofs apply for positivity and injectivity. The evolution property is clear from the definition. □

Corollary 2.3.9. *Assume (NSA). For $t \in [0,\tau]$ let $S = (S(s))_{s\in[0,t]}$ and $T = (T(s))_{s\in[0,t]}$ belong to M_t. If $S(t) \leq T(t)$, then $S(s) \leq T(s)$ for all $s \in [0,t]$.*

We take a closer look to this fact, when we consider the price of an European option $h(S(\tau))$ for some Borel-measurable function h and an underlying $(S(t))_{t\in[0,\tau]}$. If $h(S(\tau)) \leq S(\tau)$, then it follows from Corollary 2.3.9 that the price $P(t,\tau)h(S(\tau))$ at time $t \in [0,\tau]$ of the option has to be less than or equal to $S(t)$. This corresponds to the invariance of the order interval $[-\infty, \mathrm{id}]$ under the operators $P(t,\tau)$, $t \in [0,\tau]$.

From this we obtain, in addition, the well-known *Put-Call-Parity*:

Corollary 2.3.10. *Let $S = (S(t))_{t\in I} \in M_\tau$ and $K > 0$. Assume $B(t) = e^{rt}$ for all $t \geq 0$ and some $r > 0$. Let $C(t)$ be the price at time $t \in I$ of the call option $(S(\tau) - K)^+$ and $P(t)$ be the corresponding price of the put option $(K - S(\tau))^+$. Then the equality*

$$C(t) + Ke^{-r(\tau - t)} = P(t) + S(t)$$

holds for all $t \in [0,\tau]$.

Proof. We know $S(\tau) + (K - S(\tau))^+ = K + (S(\tau) - K)^+$. Now the injectivity of the price operator $P(t,\tau)$ implies for $t \in [0,\tau]$ the desired equality. □

We have observed that there is indeed a evolutionary structure hidden in the pricing procedure which is caused by the no-arbitrage assumption and can be reflected via positive, injective evolution families of operators.

2.4 Conclusions and open tasks

We want to summarize the results of this chapter and clearify the tasks for the up-coming chapters from a financial point of view.

Following Black and Scholes we have to solve a Cauchy problem

$$\begin{cases} \dot{u}(t) &= Au(t) \quad , \quad t \geq 0, \\ u(\tau) &= h \end{cases}$$

in order to get the price of the European option $h(S(\tau))$, where $(S(t))_{t\geq 0}$ is the considered underlying. The operator A depends on the model of the financial market. In the Black-Scholes market we have, for instance,

$$Au = \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru.$$

If this operator generates a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on a suitable Banach space X, i.e. $h \in X$, then $(T(\tau - t)h)(x)$ gives the option at time t, whenever the underlying has the value x at time t. In case, we call T a pricing semigroup. Therefore, one goal are generation results for the Black-Scholes operator in suitable Banach spaces.

Taking a closer look we observe that during the development of the Cauchy problem the dynamics of the underlying have changed in the following perspective: We started with the assumption

$$S(t) = S(0)\exp\left(\sigma W(t) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right), \quad t \in [0,\tau].$$

In correspondence to the semigroup $(T(t)f)(x) = \mathbb{E}[f(S(t))|S(0) = x]$, $t \in [0,\tau]$, $x \in (0,\infty)$, for suitable functions $f : (0,\infty) \to \mathbb{R}$, this process is related to the operator (cp. [App04])

$$A_0 u = \frac{1}{2}\sigma^2 x^2 u_{xx} + \mu x u_x.$$

Compared with the Black-Scholes operator A, we can see A as a perturbation of A_0. Assuming that both operators A and A_0 are generators of semigroups T and T_0, it would be of interest to know what properties are transfered from T_0 to the perturbed semigroup T. In particular, if T_0 belongs to some regularity space, e.g. continuous functions, is T still in this regularity space? This effect will be taken care of in chapter 6.

Next we concentrate on properties the pricing semigroup T has to fulfill. The investigation of the evolutionary structure in the market has shown that each of the operators $T(t)$ has to be positive and injective. Furthermore, the order interval $[-\infty, \mathrm{id}]$ is invariant under T. While semigroups of injective operators will be treated in chapter 5, the two other properties are related to invariant closed, convex subsets of semigroups. They are the subject of the chapters 3 and 4.

At the end let us summarize the four tasks we want to attack in the next chapters. As we have seen, each aim has its relevance in mathematical finance.

1. invariant subsets of strongly continuous semigroup (chapters 3 and 4)
2. semigroups of injective operators (chapter 5),
3. perturbation results for differential operators (chapter 6),
4. generation results for the Black-Scholes operator (chapter 7).

Chapter 3

A generalization of dissipativity and invariant subsets of semigroups

3.1 Introduction

It is well-known that the solution of an (abstract) Cauchy problem

$$\begin{cases} u_t &= Au \\ u(0) &= u_0 \end{cases}$$

for an (unbounded) linear operator $(A, D(A))$ on a Banach space X and an inital value $u_0 \in D(A)$ is given by $u(t) := T(t)u_0$, $t \geq 0$, whenever A generates a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on X. One is interested in common properties of the initial value u_0 and the solution $u(t)$, $t \geq 0$, e.g. whether positivity of u_0 does imply a positive solution $u(t)$. This corresponds to invariance properties for certain subsets under the semigroup T. Regarding our example for positivity this will mean that the positive cone of X is invariant under T. Other examples could be the invariance of the unit ball in X ($\|u(t)\| \leq \|u_0\|$ for all $t \geq 0$) or the invariance of order intervals $[x, y]$ in Banach lattices, i.e. $x \leq u_0 \leq y$ implies $x \leq u(t) \leq y$ for all $t \geq 0$. Therefore, the study of invariant subsets of strongly continuous semigroup is a fruitful subject and due to its importance several results can be found in the literature. We would like to mention some of them before we turn to our results.

A remarkable step has been done by Haim Brézis (cf. [Bre73]) in 1973. He studied nonlinear semigroups of contractions on a real Hilbert space and answered the question when a closed, convex subset is invariant under the semigroup (see [Bre73, Proposition 4.5]).

Theorem (Brézis)**.** Let H be a real Hilbert space and C be a closed, convex subset of H. Let P_C be the orthogonal projection of H onto C. Let A be an m-accretive operator on H and let $S = (S(t))_{t\geq 0}$ be the (nonlinear) semigroup of contractions generated by $-A$ on $\overline{D(A)}$. Assume $P_{\overline{D(A)}}C \subseteq C$. Then the following assertions are equivalent:

1. $(I + \lambda A)^{-1}C \subseteq C$ for all $\lambda > 0$;

2. $(A^\circ x|x - P_C x) \geq 0$ for all $x \in D(A)$, where $A^\circ x$ is the orthogonal projection of the origin onto the closed, convex set Ax;

3. $(y|x - Px) \geq 0$ for all $x \in D(A)$ and all $y \in Ax$;

4. $d(S(t)x, C) \leq d(x, C)$ for all $x \in \overline{D(A)}$ and all $t \geq 0$;

5. $S(t)\left(\overline{D(A)} \cap C\right) \subseteq C$ for all $t \geq 0$.

We remark that in the linear case the generator A is densely defined, i.e. $\overline{D(A)} = H$, and hence the assumption $P_{\overline{D(A)}}C \subseteq C$ is superfluous. It is remarkable that this theorem turned out as a corollary of an even more general result. As a matter of fact the starting point for Brézis results are invariance properties for proper, convex, lower semicontinuous functions $\varphi : H \to \mathbb{R}$, i.e. $\varphi(S(t)x) \leq \varphi(x)$ for all $t \geq 0$, $x \in X$ (cp. [Bre73, Théorème 4.4]). Then turning its attention to closed, convex sets, he considered the indicator function of a set C, namely

$$I_C(x) = \begin{cases} 0 & , \quad x \in C \\ \infty & , \quad x \in H \setminus C, \end{cases} \quad , \quad x \in H.$$

As long as the set C is closed, convex and nonempty, the function I_C is proper, convex and lower semicontinuous. Thus, I_C falls into the general framework provided by Brézis and with some effort one can deduce the stated theorem.

Later on Brézis' results and arguments have been extended by Yokota to quasi-contractive (nonlinear) semigroups on complex Hilbert space (cf. [Yok01]). In conclusion, one might say that in a Hilbert space the situation is studied to a satisfying degree by Brézis. But we are still lacking a theory for Banach spaces.

We will keep Brézis' result in mind as an inspiration, while we turn our focus to a second major achievement in the study of invariant subsets: Φ-dissipative operators. This notion was brought up by Wolfgang Arendt, Paul R. Chernoff and Tosio Kato in 1982 (see [ACK82]). They considered on a real Banach space X a continuous half-norm Φ, i.e. the function $\Phi : X \to \mathbb{R}$ is continuous and fulfills the properties

- $\Phi(x + y) \leq \Phi(x) + \Phi(y)$ for all $x, y \in X$ (subadditivity),
- $\Phi(\lambda x) = \lambda\Phi(x)$ for all $\lambda \geq 0$, $x \in X$ (positive homogeneity),
- $\Phi(x) + \Phi(-x) > 0$ for all $x \in X$, $x \neq 0$.

In particular, Φ is sublinear. The probably most prominent examples for half-norms are

1. $\Phi(x) = \|x\|$, $x \in X$;
2. $\Phi(x) = \|\sup\{x, 0\}\|$, $x \in X$, where X is a Banach lattice;
3. (canonical half-norm) $\Phi(x) = d(-x, X_+)$, $x \in X$, where X is an ordered Banach space with closed positive cone X_+.

Since a half-norm is positive homogeneous, the subdifferential of Φ in $x \in X$ is given by

$$(\partial\Phi)(x) = \left\{x' \in X' \mid \langle x', x\rangle = \Phi(x) \text{ and } \langle x', y\rangle \leq \Phi(y) \text{ for all } y \in X\right\}.$$

The Hahn-Banach Theorem shows that $(\partial\Phi)(x) \neq \emptyset$ for all $x \in X$. In fact, let $x \in X$ and define the subspace $U := \{\lambda x \mid \lambda \in \mathbb{R}\}$. The linear function $l(\lambda x) := \lambda\Phi(x)$, $\lambda \in \mathbb{R}$, is dominated by the half-norm Φ on U. Hence, there exists a linear function $L : X \to \mathbb{R}$ such that $L(\lambda x) = \lambda\Phi(x)$ for all $\lambda \in \mathbb{R}$ and $L(y) \leq \Phi(y)$ for all $y \in X$. This shows $L(x) = \Phi(x)$ and, since Φ is continuous, one has $L \in X'$. We conclude $L \in (\partial\Phi)(x)$.

Now they called an operator $(A, D(A))$ Φ-dissipative if for each $x \in D(A)$ there exists some $x' \in (\partial\Phi)(x)$ such that $\langle x', Ax\rangle \leq 0$ and strictly Φ-dissipative if for each $x \in D(A)$ and all $x' \in (\partial\Phi)(x)$ one has $\langle x', Ax\rangle \leq 0$.

With this terminology at hand, they could prove the following result (see [ACK82, Theorem 4.1]):

Theorem (Arendt, Chernoff, Kato). Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a real Banach space X with generator $(A, D(A))$. Let Φ be a continuous half-norm on X. Then $\Phi(T(t)x) \leq \Phi(x)$ for all $t > 0$, $x \in X$, if and only if the operator A is Φ-dissipative.

In context with invariant subsets, this theorem enables us, for instance, to charaterise positive semigroups by taking the canonical half-norm as a candidate for Φ. The major drawback is, however, that we cannot treat the invariance of general subsets within this theory, since the distance function of a closed, convex set does not always fulfill the properties of a half-norm. For instance, if $C = \{x\}$ for some $x \neq 0$, then $d(2x, C) = \|x\| > 0 = d(x, C) + d(x, C)$, i.e. $d(\cdot, C)$ is neither subadditive nor positive homogeneous. In fact, the distance function of a set C is a half-norm in the sense of Arendt, Chernoff and Kato if and only if C is a proper cone (see Remark 3.3.9).

Summarizing the known results we have two inspirations at hand: the theory of Brézis and the notion of Φ-dissipative operators introduced by Arendt, Chernoff and Kato. While the first one is only known in Hilbert spaces, the second approach, although acting in a general Banach space setting, does not allow for invariance criterions for arbitrary closed, convex sets. In this chapter we are going to fill in the missing link, a theory for invariant subsets of strongly continuous linear semigroups in Banach spaces.

Therefore, we will proceed as follows: Our starting point is the theory of Arendt, Chernoff and Kato. Treating more general functions than half-norms we will put their results on a broader basis. For a lower semicontinuous function φ on a Banach space X, whose subdifferential has a full domain $D(\partial\varphi) = X$, we define (strict) φ-dissipativity for operators in X in analogy to Arendt, Chernoff and Kato. Given a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on a Banach space X we show in Theorem 3.2.4 that $\varphi(T(t)x) \leq \varphi(x)$ if and only if its generator is φ-dissipative. This extends the Theorem of Arendt, Chernoff and Kato.

In section 3.3 we treat for the first time the invariance of closed, convex sets under strongly continuous semigroups. Since a set C is invariant under a contractive semigroup $T = (T(t))_{t\geq 0}$ if and only if one has $d(T(t)x, C) \leq d(x, C)$ for all $t \geq 0$, $x \in X$, we study the distance function $d(\cdot, C)$ of the set C as a possible candidate for the function φ in our framework. In Proposition 3.3.3 and Theorem 3.3.4 we show that the subdifferential of the distance function of a closed set has full domain if and only if the set is convex. Here, Theorem 3.3.3 is well-known (cf. [Bre83, p.13]). In addition, we give a proof of Moreau's theorem in case of a distance function.

Applying the theorems of section 3.2 for the function $\varphi = d(\cdot, C)$ we prove in Theorem 3.3.20 that a closed, convex set C of a Banach space X is invariant under a contractive strongly continuous semigroup $T = (T(t))_{t\geq 0}$ if and only if the generator is (strictly) $d(\cdot, C)$-dissipative. As a matter of fact the theorem is valid for quasi-contractive semigroups. Theorem 3.3.20 is probably the centre of this chapter. It provides us with a satisfying theory for invariant subsets of strongly continuous linear semigroups in Banach spaces. We make the following observations: While $d(\cdot, C)$-dissipativity of the generator always implies the invariance of a closed, convex set C under the semigroup T, the (quasi)-contractivity of T is mandatory for the converse implication as we can show using an example of El Maati Ouhabaz (cp. Example 3.4.26).

Despite its beauty, Theorem 3.3.20 lacks some concreteness regarding its practicality. We have derived the existence of functionals in the subdifferential of the distance function from a version of Hahn-Banach's separation theorem and have no concrete description at hand. We overcome this problem in section 3.4 introducing the notion of normally projectable and proximinal sets. In Corollary 3.4.10 we conclude that a proximinal set is normally projectable if and only if it is convex. The subdifferential of the distance function for a proximinal, convex set can now be given in terms of best approximations making the theory highly applicable. In Theorem 3.4.20 we conclude that a proximinal convex set C is invariant under a contractive semigroup T if and only if for all elements x in the domain of the generator A there exists a best approximation x_0 of x in C such that Re $\langle x_0', Ax\rangle \leq 0$ for some $x_0' \in J(x-x_0) \cap N(x_0, C)$. Here, J is the duality mapping and $N(x_0, C)$ the normal cone of x_0 in C. In particular, any closed, convex subset C of a Hilbert space H is proximinal and the unique proximum of some $x \in X$ is given by the orthogonal projection P of H onto C. Hence, the condition reads Re $(x - Px|Ax) \leq 0$, we have, in fact, recovered Brézis' Theorem in the linear case. In addition, we further recover in sections 3.4.2.1 and 3.4.2.2 the famous characterisations of contractive and positive semigroups corresponding to dissipative and dispersive generators, respectively.

Prime examples for proximinal, convex sets are order intervals in Banach lattices, which we treat extensively in section 3.4.4. Here we encounter also a new Kato-type inequality characterising invariant subsets of positive semigroups (see Proposition 3.4.38). The obtained results are afterwards applied to L^p-spaces and $C_0(\Omega)$.

We would like to remark that parts of the theory are also applicable for the theory of non-linear semigroups in uniformly convex Banach spaces as Brézis did for Hilbert spaces. However, this is still work in progress, so that we concentrate only on linear semigroups in this chapter.

3.2 φ-dissipative operators

Taking the notion of Φ-dissipativity of Arendt, Chernoff and Kato as a lead we define (strictly) ϕ-dissipative operators for a broad class of functions $\phi : X \to (-\infty, +\infty]$ on a Banach space X. The essential requirement is that the subdifferential of φ has full domain, i.e. $(\partial\varphi)(x) \neq \emptyset$ for all $x \in X$. Our aim is to charaterise the property $\varphi(T(t)x) \leq e^{\omega t}\varphi(x)$, $x \in X$, for a semigroup $T = (T(t))_{t\geq 0}$ and $\omega \in \mathbb{R}$. If φ is lower semicontinuous, we show that a condition on the resolvent of the generator implies this property. Connecting this to φ-dissipative operators, we see that the conditions are indeed equivalent and fulfilled if and only if the generator A is (strictly) φ-dissipative. This is a new result extending former theorems of Arendt, Chernoff and Kato (cf. [ACK82, Theorem 4.1]) or Brézis (cf. [Bre73, Thérème 4.4]).

Throughout this section let X be a (complex) Banach space with dual space X'. As mentioned we are interested in conditions under which the property $\varphi(T(t)x) \leq e^{\omega t}\varphi(x)$, $x \in X$, is fulfilled for a function $\varphi : X \to (-\infty, +\infty]$, some $\omega \in \mathbb{R}$ and a strongly continuous semigroup $T = (T(t))_{t\geq 0}$. We start with a sufficient condition in terms of the resolvent of the generator:

Lemma 3.2.1. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let $\varphi : X \to \mathbb{R}$ be a lower semicontinuous function and $\omega \in \mathbb{R}$. We* *assume* $\varphi(\lambda R(\lambda, A)x) \leq \frac{\lambda}{\lambda-\omega}\varphi(x)$ *for all* $x \in X$ *and all sufficiently large real* $\lambda > \max\{\omega, 0\}$. *Then* $\varphi(T(t)x) \leq e^{\omega t}\varphi(x)$ *for all* $t > 0$, $x \in X$.

Proof. Let $x \in X$, $t > 0$. For $n \in \mathbb{N}$ such that $n > \omega t$ and $\frac{n}{t} \in \rho(A)$ we put

$$x_n := \left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^n x \in D(A).$$

From our assumption it follows

$$\varphi(x_n) \leq \frac{\frac{n}{t}}{\frac{n}{t}-\omega}\,\varphi\left(\left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^{n-1} x\right) \leq \ldots \leq \left(\frac{n}{n-t\omega}\right)^n \varphi(x) = \left(1-\frac{\omega t}{n}\right)^{-n} \varphi(x).$$

Now Euler's formula (cp. [Nag86, Proposition A-II.1.10]) states $T(t)x = \lim_{n\to\infty} x_n$ and so we end up with

$$\varphi(T(t)x) \leq \underline{\lim}_{n\to\infty}\varphi(x_n) \leq \underline{\lim}_{n\to\infty}\left(1-\frac{\omega t}{n}\right)^{-n}\varphi(x) = e^{\omega t}\varphi(x)$$

using the lower semicontinuity of φ. This finally proves the Lemma. □

A matter of particular interest are conditions directly related to the generator of the semigroup. Here the notion of φ-dissipativity (like by Arendt, Chernoff and Kato) comes into play. However, we again emphasize that we do not assume φ to be a half-norm.

Definition 3.2.2. Let $(A, D(A))$ be an operator on the Banach space X and let $\varphi : X \to \mathbb{R}$ be a function such that $D(\partial\varphi) = X$. Let $\omega \in \mathbb{R}$.

- A is said to be (φ, ω)-*dissipative*, if for all $x \in D(A)$ there exists $x' \in (\partial\varphi)(x)$ such that $\operatorname{Re}\langle x', Ax\rangle \leq \omega\varphi(x)$.
- A is said to be *strictly* (φ, ω)-*dissipative*, for all $x \in D(A)$ and all $x' \in (\partial\varphi)(x)$ one has $\operatorname{Re}\langle x', Ax\rangle \leq \omega\varphi(x)$.

Remark 3.2.3. *This notion is in correspondence to the one of Arendt, Chernoff and Kato. For, let* Φ *be a half-norm on a real Banach space* X. *From the description of its subdifferential (see "Introduction") it follows for* $\omega \in \mathbb{R}$

$$\langle x', (A-\omega)x\rangle = \langle x', Ax\rangle - \omega\ \Phi(x)$$

for all $x \in D(A)$, $x' \in (\partial\Phi)(x)$. *Hence, an operator* A *is* (Φ, ω)-*dissipative (in our terminology) if and only if the operator* $A - \omega$ *is* Φ-*dissipative (in the terminology of Arendt, Chernoff and Kato).*

At first glance the notion of (φ, ω)-dissipativity of an operator seems a bit awkward. Why don't we stick to the known terminology and discuss the $(\varphi, 0)$-dissipativity of the operator $A - \omega$? As we have seen there is no distinction between these two notions if φ is a half-norm in the sense of Arendt, Chernoff and Kato. However, we are working in a far more general setting and, as we will reveal in a second, the notions have no longer to be equivalent in general.

To this end we take a view ahead to the results of the up-coming sections. Let φ be the distance function of the unit ball B_X in X, i.e.

$$\varphi(x) := \begin{cases} \|x\| - 1 & , \quad x \in X \setminus B_X \\ 0 & , \quad x \in B_X. \end{cases}$$

Note that φ is not a half-norm in the sense of Arendt, Chernoff and Kato (for instance, φ is not positive homogeneous). Nevertheless the function φ is continuous and fulfills $D(\partial\varphi) = X$. For $\omega \in \mathbb{R}$ an operator $(A, D(A))$ on X is (φ, ω)-dissipative if and only if for each $x \in D(A) \setminus B_X$ there exists $x' \in (\partial\varphi)(x)$ such that $\mathrm{Re}\, \langle x', Ax\rangle \le \omega\, (\|x\| - 1)$. Let $x \in D(A) \setminus B_X$ and consider $x' \in (\partial\varphi)(x)$. One can prove

$$\left(1 - \frac{1}{\|x\|}\right) \langle x', x\rangle = \langle x', x - \frac{x}{\|x\|}\rangle = \left\| x - \frac{x}{\|x\|} \right\| = \left(1 - \frac{1}{\|x\|}\right) \|x\|$$

i.e $\langle x', x\rangle = \|x\|$ (cp. Proposition 3.4.9). Hence, we have

$$\mathrm{Re}\, \langle x', (A - \omega)x\rangle = \mathrm{Re}\, \langle x', Ax\rangle - \omega\ \ \|x\| \neq \mathrm{Re}\, \langle x', Ax\rangle - \omega\ (\|x\| - 1)$$

as long as $\omega \neq 0$. In conclusion, for any generator A of a strongly continuous semigroup T, which fulfills $\|T(t)\| \le e^{\omega t}$ for some $\omega > 0$ but is not contractive, we know that A is not (B_X, ω)-dissipative although the operator $A - \omega = 0$ is $(B_X, 0)$-dissipative.

For a better illustration we consider the strongly continuous semigroup $T(t) := e^{\omega t}$, $t \ge 0$, for some $\omega > 0$ on X. The generator of T is the bounded operator $A = \omega I$. Hence, $A - \omega = 0$ is trivially $(B_X, 0)$-dissipative but A is, in fact, not (B_X, ω)-dissipative:

$$\mathrm{Re}\, \langle x', Ax\rangle = \omega\, \|x\| > \omega\ \ (\|x\| - 1)$$

for all $x \in X$ and $x' \in (\partial\varphi)(x)$. This may finally justify our choice of terminology.

As stated in the introduction Arendt, Chernoff and Kato have shown that the property $\Phi(T(t)x) \le \Phi(x)$, $x \in X$, $t \ge 0$, for a strongly continuous semigroup $T = (T(t))_{t\ge 0}$ and a half-norm Φ is equivalent to the (strict) Φ-dissipativity of its generator. We can prove that, in fact, more is true. Instead of a half-norm we consider a lower semicontinuous function, whose subdifferential has full domain, and obtain the following extension:

Theorem 3.2.4. *Let $T = (T(t))_{t\ge 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let $\varphi : X \to \mathbb{R}$ be a lower semicontinuous function such that $D(\partial\varphi) = X$. For $\omega \in \mathbb{R}$ the following assertions are equivalent:*

1. *$\varphi(T(t)x) \le e^{\omega t}\varphi(x)$ for all $t > 0$, $x \in X$;*
2. *$\varphi(\lambda R(\lambda, A)x) \le \frac{\lambda}{\lambda - \omega}\varphi(x)$ for all $x \in X$ and all sufficiently large $\lambda > \max\{\omega, 0\}$;*
3. *A is (φ, ω)-dissipative;*
4. *A is strictly (φ, ω)-dissipative.*

Proof. "(1) $\Rightarrow$ (4)": Let $x \in D(A)$ and $x' \in (\partial\varphi)(x)$. Due to assertion (1), we have

$$\mathrm{Re}\, \langle x', T(t)x - x\rangle \le \varphi(T(t)x) - \varphi(x) \le \left(e^{\omega t} - 1\right) \varphi(x)$$

for all $t > 0$. It follows

$$\mathrm{Re}\,\langle x', Ax\rangle = \lim_{t\searrow 0} \frac{\mathrm{Re}\,\langle x', T(t)x - x\rangle}{t} \le \varphi(x) \lim_{t\searrow 0} \frac{e^{\omega t}-1}{t} = \omega\varphi(x).$$

This shows (4).

The implication "(4) $\Rightarrow$ (3)" is clear.

"(3) $\Rightarrow$ (2)": Let $x \in X$, $\lambda \in \rho(A)$ such that $\lambda > \max\{\omega, 0\}$ and put $u := \lambda R(\lambda, A)x \in D(A)$. Thanks to assertion (2) there exists $x' \in (\partial\varphi)(u)$ such that $\mathrm{Re}\,\langle x', Au\rangle \le \omega\varphi(u)$. Since

$$\lambda(u - x) = \lambda(\lambda - (\lambda - A))R(\lambda, A)x = Au$$

and $x' \in (\partial\varphi)(u)$, it follows

$$\lambda(\varphi(u) - \varphi(x)) \le \lambda\ \mathrm{Re}\,\langle x', u - x\rangle = \mathrm{Re}\,\langle x', Au\rangle \le \omega\varphi(u).$$

From this we obtain (2).

Finally, the implication "(2) $\Rightarrow$ (1)" follows from Lemma 3.2.1. □

We need to remark that Brézis has proven a similar result for proper, convex, lower semicontinuous functions φ acting on Hilbert spaces, although he did not use the notion of φ-dissipativity and treated nonlinear semigroups (see [Bre73, Théorème 4.4]).

We explore Theorem 3.2.4 a bit further: If the function φ is positive, then the proof already indicates that we do not need to consider all elements in the domain of A in assertion (3). As a matter of fact, the following variation holds, which will be of great use, when we characterise C-dissipative operators in section 3.3.2.

Theorem 3.2.5. *Let $T = (T(t))_{t\ge 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let $\varphi : X \to [0, \infty)$ be a positive, lower semicontinuous function such that $D(\partial\varphi) = X$. For $\omega \in \mathbb{R}$ the following assertions are equivalent:*

1. *$\varphi(T(t)x) \le e^{\omega t}\varphi(x)$ for all $t > 0$, $x \in X$;*
2. *$\varphi(\lambda R(\lambda, A)x) \le \frac{\lambda}{\lambda-\omega}\varphi(x)$ for all $x \in X$ and all sufficiently large real $\lambda > \max\{\omega, 0\}$;*
3. *for all $x \in D(A)$, $\varphi(x) \neq 0$, there exists $x' \in (\partial\varphi)(x)$ such that* $\mathrm{Re}\,\langle x', Ax\rangle \le \omega\ \varphi(x)$*;*
4. *A is (φ, ω)-dissipative;*
5. *for all $x \in D(A)$, $\varphi(x) \neq 0$, and all $x' \in (\partial\varphi)(x)$ one has* $\mathrm{Re}\,\langle x', Ax\rangle \le \omega\ \varphi(x)$*;*
6. *A is strictly (φ, ω)-dissipative.*

Proof. The equivalence of (1), (2), (4) and (6) is shown in Theorem 3.2.4. The implications "(6) $\Rightarrow$ (5) $\Rightarrow$ (3)" and "(4) $\Rightarrow$ (3)" are obvious. It remains to prove that (3) implies (2). Let $x \in X$, $\lambda \in \rho(A)$ such that $\lambda > \max\{\omega, 0\}$ and put $u := \lambda R(\lambda, A)x \in D(A)$. If $\varphi(u) = 0$, we have $\varphi(u) \le \frac{\lambda}{\lambda-\omega}\ \varphi(x)$, since φ is positive and we are done. Hence, let us assume $\varphi(u) > 0$. Then, by assumption, there exists $x' \in (\partial\varphi)(x)$ such that $\mathrm{Re}\,\langle x', Ax\rangle \le \omega\varphi(x)$. From now on we can proceed as in the proof of Theorem 3.2.4. □

The Theorems 3.2.4 and 3.2.5 are our fundament for the next chapters, where we explore bit by bit the theory of invariant subsets of strongly continuous semigroups. In this connection the focus will be on the distance function $d(\cdot, C)$ of a subset $C \subseteq X$ in the role of φ.

3.3 Invariance of closed, convex sets under C_0-semigroups

In this section we discuss invariance criterions for closed, convex sets in general Banach spaces X under strongly continuous semigroups. If X is a Hilbert space, then Brézis has given a satisfying characterisation in terms of the generator using the orthogonal projection of X onto the closed, convex set (cf. [Bre73, Proposition 4.5]). Curiously, a related theory for Banach spaces seems to be missing in the literature. This section will close this gap.

One easily sees that a subset C of a Banach space X is invariant under a contractive semigroup T if and only if one has $d(T(t)x, C) \leq d(x, C)$ for all $t \geq 0$, $x \in X$. We take this as a motivation and consider - in view of section 3.2 - the distance function $d(\cdot, C)$ of closed sets. In section 3.3.1 we give a full treatment of its subdifferential like we have not found it in the literature. It is well-known that the subdifferential of a distance function to a closed set has full domain in X, if C is convex (e.g. [Bre83, p. 13]). We give an own proof of this statement in Theorem 3.3.4. In addition, we show that the set C indeed has to be convex if the subdifferential has full domain. We further note, if C is convex, then the function $d(\cdot, C)$ is convex and one can apply Moreau's Theorem. We cannot resist the temptation to state an own proof of Moreau's result in Lemma 3.3.6.

In conclusion, the distance function of a closed, convex set is a proper candidate for the framework in section 3.2. We make use of this fact and define (strictly) C-dissipative operators as (strictly) $d(\cdot, C)$-dissipative operators. Moreau's Theorem enables us to find equivalent useful characterisations to this notion. We further show that C-dissipativity of the generator implies the invariance of the closed, convex set C under the semigroup T. If the semigroup T is (quasi)-contractive, then the converse is true as well.

3.3.1 The distance function

Here we consider the distance function of closed sets in normed linear spaces. In particular, we are interested in its subdifferential for which we provide a full treatment. We show that the subdifferential has full domain if and only if the closed set is convex. In addition, we prove the classical theorem of Moreau for the distance function of a closed, convex set and show that the distance function of a set C is a half-norm in the sense of Arendt, Chernoff and Kato if and only if C is a proper cone.

Let X be a normed linear space with dual space X'. For a subset A of X we introduce the *distance function*

$$d(x, A) := \inf_{y \in A} \|x - y\|, \quad x \in X,$$

for a subset A of X. We state some obvious properties:

- $d(\cdot, A) : X \to [0, \infty)$ is Lipschitz-continuous, i.e. $|d(x, A) - d(y, A)| \leq \|x - y\|$ for all $x, y \in X$.
- $d(x, A) = 0$ if and only if $x \in \overline{A}$.

Now let C be a closed subset of X. With regard to the previous section we are interested in the subdifferential of the function $d(\cdot, C)$. In particular, we would like to know if $D(\partial d(\cdot, C)) = X$. It will turn out that this condition holds if and only if the set C is convex (cp. Corollary 3.3.5).

But first let us introduce the *normal cone to the set C at some $x \in C$* which is defined by

$$N(x, C) := \left\{x' \in X' \mid \text{Re } \langle x', y - x\rangle \le 0 \text{ for all } y \in C\right\}.$$

We have the following simple but useful results regarding the subdifferential of $d(\cdot, C)$:

Lemma 3.3.1. *Let C be a closed subset of a normed linear space X and let $x \in C$. Then*

$$(\partial d(\cdot, C))(x) = B_{X'} \cap N(x, C) = \left\{x' \in N(x, C) \mid \|x'\| \le 1\right\}.$$

In particular, one has $0 \in (\partial d(\cdot, C))(x)$.

Proof. Let $x' \in (\partial d(\cdot, C))(x)$. By definition, one has

$$\text{Re } \langle x', y - x\rangle \le d(y, C) - d(x, C) = d(y, C) \le \|x - y\|$$

for all $y \in X$. Hence, $\|x'\| \le 1$ and $\text{Re } \langle x', y - x\rangle \le d(y, C) = 0$ for all $y \in C$, i.e. $x' \in N(x, C)$.

Conversely, let $x' \in B_{X'} \cap N(x, C)$. For arbitrary $y \in X$ and $z \in C$ one has

$$\text{Re } \langle x', y - x\rangle = \text{Re } \langle x', y - z\rangle + \text{Re } \langle x', z - x\rangle \le \text{Re } \langle x', y - z\rangle \le \|y - z\|.$$

Taking the infimum over all $z \in C$, this yields $\text{Re } \langle x', y - x\rangle \le d(y, C)$ for all $y \in X$. We have shown $x' \in (\partial d(\cdot, C))(x)$. $\square$

Lemma 3.3.2. *Let C be a closed subset of a normed linear space X and let $x \in X \setminus C$.*

1. *One has*

$$\begin{aligned} & \{x' \in X' \mid \|x'\| = 1,\ d(x, C) = \inf_{y \in C} \text{Re } \langle x', x - y\rangle\} \\ \subseteq\ & (\partial d(\cdot, C))(x) \\ \subseteq\ & \{x' \in X' \mid \|x'\| \ge 1,\ d(x, C) \le \inf_{y \in C} \text{Re } \langle x', x - y\rangle\}. \end{aligned}$$

2. *For $x' \in X'$, $\|x'\| \le 1$ one has:*

$$x' \in (\partial d(\cdot, C))(x) \Leftrightarrow d(x, C) = \inf_{y \in C} \text{ Re } \langle x', x - y\rangle.$$

Proof. (1) Let $x' \in X'$, $\|x'\| = 1$, such that $d(x, C) = \inf_{y \in C} \text{Re } \langle x', x - y\rangle$. For an arbitrary $y \in X$ we obtain

$$\begin{aligned} d(y, C) &= \inf_{z \in C} \|y - z\| \\ &\ge \inf_{z \in C} \text{Re } \langle x', y - z\rangle \\ &= \text{Re } \langle x', y - x\rangle + \inf_{z \in C} \text{ Re } \langle x', x - z\rangle \\ &= \text{Re } \langle x', y - x\rangle + d(x, C). \end{aligned}$$

This proves $x' \in (\partial d(\cdot, C))(x)$.

Next let $x' \in (\partial d(\cdot, C))(x)$. Then Re $\langle x', y - x\rangle \leq d(y, C) - d(x, C)$ for all $y \in X$. For $y \in C$ this yields $d(x, C) \leq$ Re $\langle x', x - y\rangle$. It follows

$$d(x,C) \leq \inf_{y\in C} \mathrm{Re}\ \langle x', x-y\rangle \leq \left\|x'\right\| \ \inf_{y\in C} \|x-y\| = \left\|x'\right\| d(x,C).$$

Hence, $d(x, C) \leq \inf_{y\in C}$ Re $\langle x', x - y\rangle$ and $\|x'\| \geq 1$.

(2) Let $x' \in X'$, $\|x'\| \leq 1$. If $x' \in (\partial d(\cdot, C))(x)$, then it follows from (1) that $\|x'\| = 1$ and, subsequently,

$$d(x,C) \leq \inf_{y\in C} \mathrm{Re}\ \langle x', x-y\rangle \leq d(x,C).$$

This proves one direction.

For the converse direction, assume $d(x, C) = \inf_{y\in C}$ Re $\langle x', x - y\rangle$. We obtain

$$d(x,C) = \inf_{y\in C}\ \mathrm{Re}\ \langle x', x-y\rangle \leq \left\|x'\right\| d(x,C) \leq d(x,C)$$

implying $\|x'\| = 1$. Now assertion (1) shows that $x' \in (\partial d(\cdot, C))(x)$. □

This result already implies that the closed set C has to be convex whenever the condition $D(\partial d(\cdot, C)) = X$ holds.

Proposition 3.3.3. *Let C be a closed subset of a normed linear space X. Assume $D(\partial d(\cdot, C)) = X$. Then C is convex.*

Proof. Let $x, y \in C$, $\lambda \in (0, 1)$ and assume $z := \lambda x + (1-\lambda)y \notin C$. By assumption, there exists $x' \in (\partial d(\cdot, C))(z)$. Thanks to Lemma 3.3.2 we know $0 < d(z, C) \leq \inf_{w\in C}$ Re $\langle x', z - w\rangle$. In particular, we have

$$\lambda\ \mathrm{Re}\ \langle x', x-y\rangle = \mathrm{Re}\ \langle x', z-y\rangle > 0,$$

i.e. Re $\langle x', x - y\rangle > 0$. But on the other hand we have also

$$(\lambda - 1)\ \mathrm{Re}\ \langle x', x-y\rangle = \mathrm{Re}\ \langle x', z-x\rangle > 0,$$

which is a contradiction. Therefore, $z \in C$ and C is convex. □

As the following remarkable statement shows, the converse implication is also true:

Theorem 3.3.4. *Let C be a closed, convex subset of a normed linear space X. Then for every $x \in X \setminus C$ there exists $x'_0 \in (\partial d(\cdot, C))(x)$ with norm $\|x'_0\| = 1$.*

Proof. Let $x \in X \setminus C$. According to Lemma 3.3.2 it is sufficient to find a $x'_0 \in X'$, $\|x'_0\| = 1$ such that $d(x, C) = \inf_{y\in C}$ Re $\langle x'_0, x - y\rangle$. In general, for $x' \in X'$, $\|x'\| \leq 1$, we have

$$d(x,C) = \inf_{y\in C} \|x-y\| \geq \inf_{y\in C} \left\|x'\right\| \|x-y\| \geq \inf_{y\in C} \mathrm{Re}\ \langle x', x-y\rangle.$$

For the converse inequality, we will construct a functional $x'_0 \in X'$, $\|x'_0\| = 1$, such that $d(x, C) \leq$ Re $\langle x'_0, x - y\rangle$ for all $y \in C$. Therefore, we consider the ball

$$B := \{z \in X \mid \ \|x-z\| < d(x,C)\}$$

centered at x, which is open and convex. By definition of the distance function, we get $B \cap C \neq \emptyset$. Due to the separation theorem 1.1.11 we can separate the open, convex set $x - B$ from the disjoint, convex set $x - C$ by a hyperplane, i.e. there are $x_0' \in X'$, $x_0' \neq 0$, and $\gamma \in \mathbb{R}$ such that

$$\text{Re } \langle x_0', x - z \rangle < \gamma \leq \text{Re } \langle x_0', x - y \rangle, \quad (z, y) \in B \times C. \tag{3.1}$$

We may assume $\|x_0'\| = 1$ (otherwise replace x_0' and γ by $\frac{x_0'}{\|x_0'\|}$ and $\frac{\gamma}{\|x_0'\|}$, respectively). Thanks to the continuity of x_0', the inequality (3.1) extends to

$$\sup_{z \in \overline{B}} \text{Re } \langle x_0', x - z \rangle \leq \gamma.$$

We claim that $\gamma = d(x, C)$. From the inequality (3.1) we get

$$\gamma \leq \inf_{y \in C} \text{Re } \langle x_0', x - y \rangle \leq \inf_{y \in C} \|x - y\| = d(x, C).$$

Now assume that $\gamma < d(x, C)$. Then $\|x_0'\| = 1 > \frac{\gamma}{d(x,C)}$ and there exists a $x_1 \in X$, $\|x_1\| = 1$, such that $\text{Re } \langle x_0', x_1 \rangle > \frac{\gamma}{d(x,C)}$. The vector $z := x - d(x, C)x_1$ belongs to the closure of B and so we obtain the contradiction

$$\gamma \geq \text{Re } \langle x_0', x - z \rangle = d(x, C) \text{ Re } \langle x_0', x_1 \rangle > \gamma.$$

Hence, $\gamma = \text{dist}(x, C)$. Again from (3.1) it follows

$$d(x, C) = \gamma \leq \inf_{y \in C} \text{Re } \langle x_0', x - y \rangle.$$

This finishes the proof. □

Thus, one goal of this section is fulfilled: the subdifferential for the distance function of a closed, convex set has full domain:

Corollary 3.3.5. *For a closed subset C of a normed linear space X holds $D(\partial d(\cdot, C)) = X$ if and only if C is convex.*

In addition, we would like to use Theorem 3.3.4 to prove Moreau's theorem for the distance function. Moreau (cf. [Mor66]) has shown that for a convex, lower semicontinuous function $\varphi : X \to \mathbb{R}$ on a real Banach space X and vectors $x \in \text{int } D(\varphi)$, $y \in X$, there exists a functional $x_0' \in (\partial\varphi)(x)$ such that

$$\langle x_0', y \rangle = \sup_{x' \in (\partial\varphi)(x)} \langle x', y \rangle = \lim_{\lambda \searrow 0} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda}.$$

Although already known, we cannot resist the temptation to give a proof of this statement for the distance function:

Lemma 3.3.6. *Let C be a closed, convex subset of a normed linear space X and $x, y \in X$.*

1. *There exists $x_0' \in (\partial d(\cdot, C))(x)$ such that*

$$\text{Re } \langle x_0', y \rangle = \sup_{x' \in (\partial d(\cdot, C))(x)} \text{Re } \langle x', y \rangle = \lim_{\lambda \searrow 0} \frac{d(x + \lambda y, C) - d(x, C)}{\lambda}.$$

2. There exists $x_1' \in (\partial d(\cdot, C))(x)$ such that

$$\mathrm{Re}\ \langle x_1', y\rangle = \inf_{x' \in (\partial d(\cdot,C))(x)} \mathrm{Re}\langle x', y\rangle = \lim_{\lambda \nearrow 0} \frac{d(x+\lambda y, C) - d(x,C)}{\lambda}.$$

Proof. (1) Since the function $d(\cdot, C)$ is convex, we have for $\mu > \lambda > 0$:

$$\begin{aligned} \frac{\lambda}{\mu}\left(d(x+\mu y, C) - d(x,C)\right) &= \frac{\lambda}{\mu} d(x+\mu y, C) + \left(1 - \frac{\lambda}{\mu}\right) d(x,C) - d(x,C) \\ &\geq d(x+\lambda y, C) - d(x,C). \end{aligned}$$

Thus, the function $(0,\infty) \ni \lambda \mapsto \frac{1}{\lambda}\left(d(x+\lambda y, C) - d(x,C)\right)$ is decreasing as $\lambda \searrow 0$. Furthermore, for $x' \in (\partial d(\cdot, C))(x)$ one has by definition $\mathrm{Re}\ \langle x', z - x\rangle \leq d(z,C) - d(x,C)$ for all $z \in C$. In particular, for $z = x + \lambda y$ with $\lambda > 0$ the estimate

$$\mathrm{Re}\ \langle x', \lambda y\rangle \leq d(x+\lambda y, C) - d(x,C)$$

holds. In conclusion, the limit

$$L := \lim_{\lambda \searrow 0} \frac{d(x+\lambda y, C) - d(x,C)}{\lambda} = \inf_{\lambda > 0} \frac{d(x+\lambda y, C) - d(x,C)}{\lambda}.$$

exists and satisfies $\mathrm{Re}\ \langle x', y\rangle \leq L$ for all $x' \in (\partial d(\cdot, C))(x)$. In a final step, we will now construct a functional $x_0' \in (\partial d(\cdot, C))(x)$ such that $\mathrm{Re}\ \langle x_0', y\rangle \geq L$.

Therefore, we consider $x_\lambda := x + \lambda y$ for $\lambda > 0$. We may assume $x_\lambda \in X \setminus C$ (otherwise we would obtain $x \in C$ by an approximation argument, implying $L = 0$ and hence the null functional fulfills the required estimate). From Theorem 3.3.4 we deduce for every $\lambda > 0$ the existence of $x_\lambda' \in (\partial d(\cdot, C))(x_\lambda) \cap S_{X'}$ such that

$$d(x+\lambda y, C) = \inf_{z \in C} \mathrm{Re}\ \langle x_\lambda', x + \lambda y - z\rangle.$$

We know from the Theorem of Alaoglu that the unit ball $B_{X'}$ is weak*-compact (cf. [Wer00, Korollar VIII.3.12]). Therefore, the net $(x_\lambda')_{\lambda > 0}$ has a weak*-limit point x_0' in $B_{X'}$ as $\lambda \searrow 0$. We will show that $x_0' \in (\partial d(\cdot, C))(x)$ and $\mathrm{Re}\ \langle x_0', y\rangle \geq L$.

By definition, for every $\lambda > 0$ and $z \in X$ we know

$$\mathrm{Re}\ \langle x_\lambda', z - x - \lambda y\rangle \leq d(z, C) - d(x + \lambda y, C).$$

This leads to

$$\mathrm{Re}\ \langle x_\lambda', z - x\rangle \leq d(z,C) - d(x+\lambda, C) + \lambda\ \mathrm{Re}\ \langle x_\lambda', y\rangle \leq d(z,C) - d(x+\lambda, C) + \lambda \left\|y\right\|.$$

Letting $\lambda \searrow 0$, we obtain $\mathrm{Re}\ \langle x_0', z - x\rangle \leq d(z,C) - d(x,C)$ for all $z \in X$, i.e. $x_0' \in (\partial d(\cdot, C))(x)$. Furthermore, one has for each $\lambda > 0$ and $z \in C$

$$\begin{aligned} \frac{1}{\lambda}\left(d(x+\lambda y, C) - d(x,C)\right) &\leq \frac{1}{\lambda}\big(\mathrm{Re}\ \langle x_\lambda', x + \lambda y - z\rangle - d(x,C)\big) \\ &= \mathrm{Re}\ \langle x_\lambda', y\rangle + \frac{1}{\lambda}\big(\mathrm{Re}\ \langle x_\lambda', x - z\rangle - d(x,C)\big)\big\rangle \\ &\leq \mathrm{Re}\ \langle x_\lambda', y\rangle + \frac{1}{\lambda}\left(\left\|x - z\right\| - d(x,C)\right). \end{aligned}$$

Taking the infimum over all $z \in C$, we obtain

$$\frac{1}{\lambda}\left(d(x+\lambda y, C) - d(x, C)\right) \leq \mathrm{Re}\ \langle x'_\lambda, y\rangle$$

for all $\lambda > 0$. Letting $\lambda \searrow 0$, this finally shows $\mathrm{Re}\ \langle x'_0, y\rangle \geq L$.

(2) We consider $-y$ instead of y in assertion (1). There exists $x'_1 \in (\partial d(\cdot, C))(x)$ such that

$$\mathrm{Re}\ \langle x'_1, -y\rangle = \sup_{x' \in (\partial d(\cdot,C))(x)} \mathrm{Re}\ \langle x', -y\rangle = \lim_{\lambda \searrow 0} \frac{d(x-\lambda y, C) - d(x, C)}{\lambda}.$$

It follows

$$\mathrm{Re}\ \langle x'_1, y\rangle = -\sup_{x' \in (\partial d(\cdot,C))(x)} \mathrm{Re}\ \langle x', -y\rangle = \inf_{x' \in (\partial d(\cdot,C))(x)} \mathrm{Re}\ \langle x', y\rangle$$

and

$$\mathrm{Re}\ \langle x'_1, y\rangle = \lim_{\lambda \searrow 0} \frac{d(x-\lambda y, C) - d(x, C)}{\lambda} = \lim_{\lambda \nearrow 0} \frac{d(x+\lambda y, C) - d(x, C)}{\lambda}.$$

This shows (2). □

Now one easily deduces the following result:

Lemma 3.3.7. *Let C be a closed, convex subset of a normed linear space X and $x, y \in X$. Let $\omega \in \mathbb{R}$. There exists $x'_0 \in (\partial d(\cdot, C))(x)$ such that* $\mathrm{Re}\ \langle x'_0, y\rangle \leq \omega\ d(x, C)$ *if and only if $d(x - ty, C) \geq (1 - t\omega)\ d(x, C)$ for all $t > 0$.*

Proof. First we take a functional $x'_0 \in (\partial d(\cdot, C))(x)$ such that $\mathrm{Re}\ \langle x'_0, y\rangle \leq \omega\ d(x, C)$. For $t > 0$ we know, by definition, that

$$d(x, C) - d(x - ty, C) \leq t\ \mathrm{Re}\ \langle x'_0, y\rangle \leq t\ \omega\ d(x, C).$$

Hence, $d(x - ty, C) \geq (1 - t\omega)\ d(x, C)$ for all $t > 0$.

The converse direction is now a direct consequence of Lemma 3.3.6. In fact, let us assume $d(x - ty, C) \geq (1 - t\omega)\ d(x, C)$ for all $t > 0$. Due to Lemma 3.3.6 we find $x'_0 \in (\partial d(\cdot, C))(x)$ such that

$$\mathrm{Re}\ \langle x'_0, y\rangle = \lim_{t \nearrow 0} \frac{d(x+ty, C) - d(x, C)}{t}.$$

It follows

$$\begin{aligned} \mathrm{Re}\ \langle x'_0, -y\rangle &= -\lim_{t \nearrow 0} \frac{d(x+ty, C) - d(x, C)}{t} \\ &= \lim_{t \searrow 0} \frac{d(x-ty, C) - d(x, C)}{t} \\ &\geq -\omega\ d(x, C). \end{aligned}$$

Hence, $\mathrm{Re}\ \langle x'_0, y\rangle \leq \omega\ d(x, C)$. □

At the end of this section we will treat the important special case of a closed, convex cone. A subset K of a normed linear space is called a *cone* with vertex $\tilde{x} \in X$ if $\tilde{x} + \lambda(x - \tilde{x}) \in K$ for all $\lambda > 0$, $x \in K$. A *proper cone* is a convex cone K with vertex 0 such that $K \cap (-K) = \{0\}$.

Remark 3.3.8. *Let K be a nonempty cone with vertex $\tilde{x}$ in a normed linear space X.*

1. *K is convex if and only if $x + y - \tilde{x} \in K$ for all $x, y \in K$;*

2. *if K is closed, then $\tilde{x} \in K$.*

Proof. (1) Let K be convex. It follows for $x, y \in K$ that $\frac{1}{2}(x+y) \in K$ and thus $x + y - \tilde{x} = x_0 + 2\left(\frac{1}{2}(x+y) - \tilde{x}\right) \in K$. Conversely, let $x, y \in K$, $\lambda \in [0,1]$ and put $z := \lambda x + (1-\lambda)y$. We know that the vectors $z + (1-\lambda)(\tilde{x} - y) = \tilde{x} + \lambda(x - \tilde{x})$ and $z + \lambda(\tilde{x} - x) = \tilde{x} + (1-\lambda)(y - \tilde{x})$ both belong to K. Hence, we obtain from our assumption

$$z = (z + (1-\lambda)(\tilde{x} - y)) + (z + \lambda(\tilde{x} - x)) - \tilde{x} \in K,$$

i.e. K is convex.

(2) Let K be closed and consider $x_0 \in K$. Then $x_n := \tilde{x} + \frac{1}{n}(x_0 - \tilde{x}) \in K$ for all $n \in \mathbb{N}$, which implies $\tilde{x} = \lim_{n\to\infty} x_n \in K$. □

We recall from the introduction that a continuous function $\Phi : X \to \mathbb{R}$ is a half-norm in the sense of Arendt, Chernoff and Kato if it fulfills the properties

- $\Phi(x+y) \leq \Phi(x) + \Phi(y)$ for all $x, y \in X$ (subadditivity),
- $\Phi(\lambda x) = \lambda\Phi(x)$ for all $\lambda \geq 0$, $x \in X$ (positive homogeneity),
- $\Phi(x) + \Phi(-x) > 0$ for all $x \in X$, $x \neq 0$.

One easily sees that the distance function of a closed, convex set C does, in general, not satisfy these properties. In fact, it does if and only if the set C is a proper cone.

Remark 3.3.9. *Let C be a subset of a normed linear space X. Then the distance function $d(\cdot, C)$ is a half-norm in the sense of Arendt, Chernoff and Kato if and only if C is a proper cone.*

Proof. Assume that $d(\cdot, C)$ is a half-norm in the sense of Arendt, Chernoff and Kato. Then $d(0, C) = 0$, i.e. $0 \in C$, and for $\lambda > 0$, $x \in C$, we have $d(\lambda x, C) = \lambda d(x, C) = 0$, i.e. $\lambda x \in C$. Hence, C is a cone with vertex 0. Moreover, $d(x+y, C) \leq d(x, C) + d(y, C) = 0$ for all $x, y \in C$ implying the convexity of C according to the remark above. Finally, we have to show $C \cap (-C) = \{0\}$. Let $x \in C \cap (-C)$. Then $d(x, C) + d(-x, C) = 0$ and we obtain from the third property of a half-norm that $x = 0$. In conclusion, C is a proper cone.

Now assume that C is a proper cone. For $x, y \in X$ one has

$$d(x+y, C) = \inf_{z\in C} \|x + y - z\| = \inf_{z\in C} \|x + y - 2z\| \leq \inf_{z\in C} (\|x - z\| + \|y - z\|) \leq d(x, C) + d(y, C).$$

Hence, $d(\cdot, C)$ is subadditive. Moreover, the equation

$$d(\lambda x, C) = \inf_{z\in C} \|\lambda x - z\| = \inf_{z\in C} \|\lambda x - \lambda z\| = \lambda d(y, C)$$

holds for all $x \in X$, $\lambda > 0$. Therefore, $d(\cdot, C)$ is positive homogeneous. Finally, assume $d(x, C) + d(-x, C) = 0$ for some $x \in X$. Then $x \in C \cap (-C)$ and we get $x = 0$. In conclusion, $d(\cdot, C)$ is a half-norm in the sense of Arendt, Chernoff and Kato. □

Example 3.3.10. *Let $(X, \leq)$ be an ordered, normed linear space and $\tilde{x} \in X$. Then the set $K := \{x \in X \mid x \leq \tilde{x}\}$ is a convex cone with vertex $\tilde{x}$.*

Proof. We may assume that $\tilde{x} = 0$. For $x \in K$, i.e. $x \leq 0$, and $\lambda > 0$ we have $\lambda x \leq 0$. Hence, K is a cone. Furthermore, if $x, y \in K$, then $x + y \leq 0$ and thus, by the previous remark, the cone K is convex. □

In virtue of Lemma 3.3.2 we have the following result for closed, convex cones closing this section:

Lemma 3.3.11. *Let K be a closed cone with vertex $\tilde{x}$ in a normed linear space X and let $x \in X \setminus K$.*

1. *One has*

$$\begin{aligned} & \left\{x' \in X' \mid \|x'\| = 1,\ x' \in N(\tilde{x}, K),\ d(x, K) = \mathrm{Re}\ \langle x', x - \tilde{x}\rangle\right\} \\ \subseteq\ & (\partial d(\cdot, K))(x) \\ \subseteq\ & \left\{x' \in X' \mid \|x'\| \geq 1,\ x' \in N(\tilde{x}, K),\ d(x, K) \leq \mathrm{Re}\ \langle x', x - \tilde{x}\rangle\right\} \end{aligned}$$

2. *Let $x' \in X'$, $\|x'\| \leq 1$. Then $x' \in (\partial d(\cdot, K))(x)$ if and only if* Re $\langle x', x - \tilde{x}\rangle = d(x, K)$ *and $x' \in N(\tilde{x}, K)$.*

Proof. (1) Let $x' \in S_{X'} \cap N(\tilde{x}, K)$ such that $d(x, K) = \mathrm{Re}\ \langle x', x - \tilde{x}\rangle$. It follows

$$d(x, K) = \mathrm{Re}\ \langle x', x - \tilde{x}\rangle \leq \mathrm{Re}\ \langle x', x - y\rangle$$

for all $y \in K$ and hence $d(x, K) = \inf_{y \in K}\ \mathrm{Re}\ \langle x', x - y\rangle$. From Lemma 3.3.2 we obtain $x' \in (\partial d(\cdot, K))(x)$.

Next we consider $x' \in (\partial d(\cdot, K))(x)$. From Lemma 3.3.2 we know $\|x'\| \geq 1$ and $d(x, K) \leq \inf_{y \in K}\ \mathrm{Re}\ \langle x', x - y\rangle$. Since $\tilde{x} \in K$, this already implies $d(x, K) \leq \mathrm{Re}\ \langle x', x - \tilde{x}\rangle$. It remains to prove $x' \in N(\tilde{x}, K)$, i.e. Re $\langle x', y - \tilde{x}\rangle \leq 0$ for all $y \in K$. Therefore, let $y \in C$. For all $\lambda > 0$ is $\tilde{x} + \lambda(y - \tilde{x}) \in K$ and it follows

$$\mathrm{Re}\langle x', \tilde{x} + \lambda(y - \tilde{x}) - x\rangle \leq d(\tilde{x} + \lambda(y - \tilde{x}), K) - d(x, K) = -d(x, K), \quad \lambda > 0.$$

That is equivalent to

$$\mathrm{Re}\ \langle x', y - \tilde{x}\rangle \leq \frac{1}{\lambda}\left(\mathrm{Re}\ \langle x', x - \tilde{x}\rangle - d(x, K)\right), \quad \lambda > 0.$$

Letting $\lambda \to \infty$, we get Re $\langle x', y - \tilde{x}\rangle \leq 0$. Since $y \in K$ was arbitrarily chosen, this shows $x' \in N(\tilde{x}, K)$.

(2) Let $x' \in X'$, $\|x'\| \leq 1$. Assume $x' \in (\partial d(\cdot, K))(x)$. By assertion (1), we get $x' \in S_{X'} \cap N(\tilde{x}, K)$ and $d(x, K) \leq \mathrm{Re}\ \langle x', x - \tilde{x}\rangle$. From Lemma 3.3.2 we further know

$$d(x, K) = \inf_{y \in K}\ \mathrm{Re}\ \langle x', x - y\rangle.$$

Since $x' \in N(\tilde{x}, K)$, we have Re $\langle x', x - \tilde{x}\rangle \leq \mathrm{Re}\ \langle x', x - y\rangle$ for all $y \in K$ and thus $d(x, K) = \mathrm{Re}\ \langle x', x - \tilde{x}\rangle$.

For the converse direction, assume $x' \in N(\tilde{x}, K)$ and Re $\langle x', x - \tilde{x}\rangle = d(x, K)$. Then $d(x, K) = \inf_{y \in K}\ \mathrm{Re}\ \langle x', x - y\rangle$ and the statement follows from Lemma 3.3.2. □

3.3.2 C-dissipative operators and invariance conditions

This section is probably the centre of the chapter: we provide invariance criterions for closed, convex sets in Banach spaces under strongly continuous semigroups. Stepping into details, we show that a closed set C is invariant under a semigroup T with generator A if it is invariant under $\lambda R(\lambda, A)$ for large $\lambda > 0$. This is due to Euler's formula. The converse holds, if the set C is additionally convex. Those results and the used methods are all well-known (e.g. [Ouh04]). However, in extension to the common literature on this subject we are able to provide an example of a closed set C which is invariant under the shift semigroup, but not under $\lambda R(\lambda, A)$. Hence, convexity of C is needed for this implication. In view of section 3.2 we define in a next step (strictly) C-dissipative operators for closed, convex sets C as (strictly) $d(\cdot, C)$-dissipative operators. Since the subdifferential has full domain in this case, this notion is well-defined. We use Moreau's Theorem to obtain several other characterisations. Now a closed, convex set C is invariant under a semigroup T, if its generator A is (strictly) C-dissipative. For (quasi)-contractive semigroups the converse is true as well. However, the theorem is false for arbitrary semigroups.

Let X be a Banach space with dual space X'. We are interested in conditions under which a subset C of X is invariant under a strongly continuous semigroup $T = (T(t))_{t\geq 0}$, i.e. $T(t)x \in C$ for all $t \geq 0$, $x \in C$. Our idea is to apply the results of section 3.2 to the indicator function of the set C, namely

$$I_C(x) = \begin{cases} 0 & , \quad x \in C \\ \infty & , \quad x \in X \setminus C, \end{cases}$$

and the distance function

$$d(x, C) = \inf_{y \in C} \|x - y\|, \quad x \in X,$$

from the previous section.

Note that C is invariant under T if and only if $I_C(T(t)x) \leq I_C(x)$ for all $t \geq 0$, $x \in X$. Moreover, the function I_C is lower semicontinuous if and only if the set C is closed. Thanks to Lemma 3.2.1 this leads to the following first result:

Lemma 3.3.12. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let C be a closed subset of X.*

1. *If $\lambda R(\lambda, A)C \subseteq C$ for all sufficiently large real $\lambda > 0$, then C is invariant under T.*
2. *Assume that C is, in addition, convex. Then the invariance of C under T implies $\lambda R(\lambda, A)C \subseteq C$ for all sufficiently large real $\lambda > 0$.*

Proof. For (1) we only have to apply Lemma 3.2.1 to the characteristic function I_C of C.

For (2), let C be closed, convex and invariant under T. We assume that there exist $f \in C$ and $\lambda \in \rho(A)$, $\lambda > 0$, such that $u := \lambda R(\lambda, A)f \notin C$. Due to the Hahn-Banach Theorem we find $x' \in X'$ and $\alpha \in \mathbb{R}$ such that

$$\text{Re } \langle x', u \rangle > \alpha > \text{Re } \langle x', v \rangle, \quad v \in C.$$

Since $T(t)f \in C$ for all $t \geq 0$ and due to the strict positivity of $\lambda > 0$, we can deduce

$$\begin{aligned}
\mathrm{Re}\ \langle x', u\rangle &> \alpha = \int_0^\infty \lambda e^{-\lambda t} \alpha \ dt \\
&\geq \int_0^\infty \lambda e^{-\lambda t} \mathrm{Re}\ \langle x', T(t)f\rangle \ dt \\
&= \mathrm{Re}\ \langle x', \int_0^\infty \lambda e^{-\lambda t} T(t)f \ dt\rangle \\
&= \mathrm{Re}\ \langle x', u\rangle,
\end{aligned}$$

which is a contradiction. □

As the following example shows convexity in assertion (2) of Lemma 3.3.12 is indeed necessary:

Example 3.3.13 ("Convexity is needed."). *Consider the shift semigroup $(T(t)f)(x) := f(x+t)$, $t \geq 0$, with generator $(A, D(A))$ on the real Hilbert space $H := L^2([0,\infty))$ and the set*

$$C := \{0\} \cup \left\{ f \in H \mid \exists\ y \in [0,1] : \ f = \mathbb{1}_{[0,y]} \ \textit{a.e.} \right\}.$$

For $y \geq 0$ and $t \geq 0$ one has

$$T(t)\mathbb{1}_{[0,y]} = \begin{cases} \mathbb{1}_{[0,y-t]} & , \ y \geq t \\ 0 & , \ y < t \end{cases},$$

i.e. C is invariant under T. Note that C is not convex (for instance, $\frac{1}{2}\mathbb{1}_{[0,1]}$ does not belong to C) but one can show that C is closed. For, let $(f_n)_{n\in\mathbb{N}} \subset C$ such that $f_n \to f$ in H as $n \to \infty$. Leaving the constant null function out of the reasoning, we may assume that for every $n \in \mathbb{N}$ there exists $y_n \in [0,1]$ such that $f_n = \mathbb{1}_{[0,y_n]}$ a.e.. Clearly, $f = 0$ a.e. on $(1,\infty]$. The bounded sequence $(y_n)_{n\in\mathbb{N}} \subset [0,1]$ has a convergent subsequence $y_{n_k} \nearrow y_0$ as $k \to \infty$ for some $y_0 \in [0,1]$. It follows:

$$\int_0^\infty |f_{n_k} - f|^2 \ dx = \int_0^{y_{n_k}} |1-f|^2 \ dx + \int_{y_{n_k}}^1 |f|^2 \ dx \to \int_0^{y_0} |1-f|^2 \ dx + \int_{y_0}^1 |f|^2 \ dx$$

as $k \to \infty$. Since $f_{n_k} \to f$ in H as $k \to \infty$ as well, we conclude

$$\int_0^{y_0} |1-f|^2 \ dx + \int_{y_0}^1 |f|^2 \ dx = 0 \ \textit{ a.e. on } [0,\infty).$$

Hence, $f = \mathbb{1}_{[0,y_0]}$ a.e. and we obtain $f \in C$. Therefore, the set C is closed.

However, C is not invariant under $\lambda R(\lambda, A)$ for $\lambda > 0$. For, let $\lambda > 0$ and $\phi \in C_c^\infty([0,\infty))$ be a test function. Then $\lambda \in \rho(A)$ (since T is a contraction semigroup) and

$$\begin{aligned}
(R(\lambda, A)\mathbb{1}_{[0,1]}|\phi)_H &= \int_0^\infty \phi(x) \int_0^\infty e^{-\lambda t}(T(t)\mathbb{1}_{[0,1]})(x) \ dt \ dx \\
&= \int_0^\infty \phi(x) \int_0^{1-x} e^{-\lambda t} \ dt \ dx \\
&= \int_0^\infty \phi(x) \frac{1}{\lambda}\left(1 - e^{-\lambda(1-x)}\right) \ dx,
\end{aligned}$$

i.e. $\lambda R(\lambda, A)\mathbb{1}_{[0,1]} = 1 - e^{-\lambda(1-\cdot)} \notin C$.

In section 3.2 we introduced the notion of (φ, ω)-dissipative operators and showed the equivalence of $\varphi(T(t)x) \leq e^{\omega t}\varphi(x)$, $x \in X$, $t \geq 0$, and (φ, ω)-dissipativity for the generator of the semigroup T, whenever the subdifferential of the lower semicontinuous function φ has full domain in the Banach space X. Now we want to find a condition of this type in order to characterise the invariance of closed, convex sets under T. While the indicator function is not a suitable candidate (one has $D(\partial I_C) \subset C$), we have seen that the distance function does fulfill the requirements and will thus be the function to consider. In a first step into this direction we slightly adjust the notion of (φ, ω)-dissipativity to our now more set-based terminology:

Definition 3.3.14. Let $(A, D(A))$ be an operator in a Banach space X and let C be a closed, convex subset of X. Let $\omega \in \mathbb{R}$.

- A is said to be (C, ω)-*dissipative* if A is $(d(\cdot, C), \omega)$-dissipative.
- A is said to be *strictly* (C, ω)-*dissipative* if A is strictly $(d(\cdot, C), \omega)$-dissipative

Like Arendt, Chernoff and Kato (see [ACK82, Theorem 3.1]) we can derive a different kind of characterisation of (C, ω)-dissipativity using Moreau's Theorem. However, we point out once more that the distance function of a set C is a half-norm in their sense if and only if C is proper cone.

Lemma 3.3.15. *Let $(A, D(A))$ be an operator in a Banach space X and let C be a closed, convex subset of X. Let $\omega \in \mathbb{R}$. The following assertions are equivalent:*

1. *A is (C, ω)-dissipative;*
2. *for each $x \in D(A) \setminus C$ there exists $x' \in (\partial d(\cdot, C))(x)$ such that* $\operatorname{Re} \langle x', Ax \rangle \leq \omega\ d(x, C)$*;*
3. *for each $x \in D(A)$ one has $d(x - tAx, C) \geq (1 - t\omega)\ d(x, C)$ for all $t > 0$.*

Proof. The implication "(2) $\Rightarrow$ (1)" follows from Lemma 3.3.1, more precisely, from the fact that $0 \in (\partial d(\cdot, C))(x)$ for all $x \in C$. Finally, the implications "(1) $\Rightarrow$ (3) $\Rightarrow$ (2)" are obtained from Lemma 3.3.7. □

Corollary 3.3.16. *Let $(A, D(A))$ be an operator in a Banach space X and let C be a closed, convex subset of X. Let $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$. Then A is (C, ω)-dissipative if and only if $d(\lambda R(\lambda, A)x, C) \leq \frac{\lambda}{\lambda - \omega}\ d(x, C)$ for all $\lambda > \max\{\omega, 0\}$ and $x \in X$.*

Proof. Let $\lambda > \max\{\omega, 0\}$, $x \in X$ and put $u := \lambda R(\lambda, A)x$. Then $Au = \lambda(u - x)$ and we have $d(\lambda R(\lambda, A)x, C) \leq \frac{\lambda}{\lambda - \omega}\ d(x, C)$ if and only if $d(u - \frac{1}{\lambda}Au, C) \geq \left(1 - \frac{\omega}{\lambda}\right)\ d(u, C)$. Now the result follows from Lemma 3.3.15. □

Proposition 3.3.17. *Let C be a closed, convex subset of a Banach space X and let $\omega \in \mathbb{R}$. Let $(A, D(A))$ be a closable, (C, ω)-dissipative operator on X. Then its closure $\overline{A}$ is also (C, ω)-dissipative.*

Proof. Let $x \in D(\overline{A})$, $t \geq 0$. There exists $x_n \in D(A)$, $n \in \mathbb{N}$, such that $x_n \to x$ and $Ax_n \to \overline{A}x$ in X as $n \to \infty$. Since A is (C,ω)-dissipative, Lemma 3.3.15 implies that $d(x_n - tAx_n, C) \geq (1-t\omega)\, d(x_n, C)$ for all $n \in \mathbb{N}$. Now the continuity of the distance function shows $d(x - t\overline{A}x, C) \geq (1-t\omega)\, d(x,C)$ and, again by Lemma 3.3.15, we conclude that $\overline{A}$ is (C,ω)-dissipative. □

For strict (C,ω)-dissipativity we have:

Lemma 3.3.18. *Let $(A, D(A))$ be an operator in a Banach space X and let C be a closed, convex subset of X. Let $\omega \in \mathbb{R}$. Then A is strictly (C,ω)-dissipative if and only if for all $x \in D(A)$ one has*

$$\lim_{t \searrow 0} \frac{d(x + tAx, C) - d(x,C)}{t} \leq \omega\, d(x,C).$$

Proof. Follows directly from Lemma 3.3.6. □

We come back to the invariance of subsets under strongly continuous semigroups. We obtain as a direct consequence of Theorem 3.2.5 for $\varphi = d(\cdot, C)$:

Theorem 3.3.19. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let C be a closed, convex subset of X. For $\omega \in \mathbb{R}$ the following assertions are equivalent:*

1. *$d(T(t)x, C) \leq e^{\omega t} d(x, C)$ for all $t > 0$, $x \in X$;*
2. *$d(\lambda R(\lambda, A)x, C) \leq \frac{\lambda}{\lambda - \omega} d(x,C)$ for all sufficiently large $\lambda > \max\{\omega, 0\}$ and $x \in X$;*
3. *for each $x \in D(A) \setminus C$ there exists $x' \in (\partial d(\cdot, C))(x)$ such that* $\operatorname{Re} \langle x', Ax\rangle \leq \omega\, d(x,C)$*;*
4. *A is (C,ω)-dissipative;*
5. *for all $x \in D(A) \setminus C$ and all $x' \in (\partial d(\cdot, C))(x)$ one has* $\operatorname{Re} \langle x', Ax\rangle \leq \omega\, d(x,C)$*;*
6. *A is strictly (C,ω)-dissipative.*

In either one of these cases, the set C is invariant under T.

It is natural to ask whether the converse is true as well. More precisely, if the invariance of a closed, convex set $C \subseteq X$ under the semigroup $T = (T(t))_{t\geq 0}$ implies one (and thus each) of the assertions in Theorem 3.3.19. In fact, one has

$$d(T(t)x, C) = \inf_{y \in C} \|T(t)x - y\| \leq \inf_{y\in C} \|T(t)x - T(t)y\| \leq \|T(t)\|\, d(x,C)$$

for all $t \geq 0$, $x \in X$. Thus, if the semigroup T is quasi-contractive, i.e. $\|T(t)\| \leq e^{\omega t}$ for some $\omega \in \mathbb{R}$ and all $t \geq 0$, and leaves C invariant, then we obtain assertion (1). A similar result is true for the resolvent. We conclude:

Theorem 3.3.20. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let $\omega_0 \in \mathbb{R}$ such that $\|T(t)\| \leq e^{\omega_0 t}$ for all $t \geq 0$. Let C be a closed, convex subset of X. For $\omega \geq \omega_0$ the following assertions are equivalent:*

1. *C is invariant under T;*
2. *$d(T(t)x, C) \leq e^{\omega t} d(x, C)$ for all $t > 0$, $x \in X$;*
3. *$\lambda R(\lambda, A)C \subseteq C$ for all sufficiently large $\lambda > \max\{\omega, 0\}$ and $x \in X$;*
4. *$d(\lambda R(\lambda, A)x, C) \leq \frac{\lambda}{\lambda - \omega} d(x, C)$ for all sufficiently large $\lambda > \max\{\omega, 0\}$ and $x \in X$;*
5. *for each $x \in D(A) \setminus C$ there exists $x' \in (\partial d(\cdot, C))(x)$ such that $\mathrm{Re}\,\langle x', Ax\rangle \leq \omega\, d(x, C)$;*
6. *A is (C, ω)-dissipative;*
7. *for all $x \in D(A) \setminus C$ and all $x' \in (\partial d(\cdot, C))(x)$ one has $\mathrm{Re}\,\langle x', Ax\rangle \leq \omega\, d(x, C)$;*
8. *A is strictly (C, ω)-dissipative.*

Corollary 3.3.21. *Let C be a closed, convex subset of a Banach space X and $\omega \in \mathbb{R}$.*

1. *$e^{\omega t} C \subseteq C$ for all $t \geq 0$ if and only if for all $x \in X \setminus C$ there exists $x' \in (\partial d(\cdot, C))(x)$ such that $\omega\,(\mathrm{Re}\,\langle x', x\rangle - d(x, C)) \leq 0$.*
2. *Assume $0 \in C$. Then $e^{\omega t} C \subseteq C$ for all $t \geq 0$ if and only if $\omega \leq 0$.*

Proof. We consider the quasi-contractive C_0-semigroup $T(t) := e^{\omega t}$, $t \geq 0$, on X with the bounded generator $Ax := \omega x$, $x \in X$. Then (1) is a direct consequence of the equivalent assertions (1) and (5) of Theorem 3.3.20. For (2) we assume $0 \in C$. This implies

$$\mathrm{Re}\,\langle x', -x\rangle \leq d(0, C) - d(x, C) = -d(x, C)$$

for all $x \in X \setminus C$ and $x' \in (\partial d(\cdot, C))(x)$. Thus, $\mathrm{Re}\,\langle x', x\rangle - d(x, C) \geq 0$ and (2) follows from (1). □

We have established several conditions under which a closed, convex set $C \subseteq X$ is invariant under a semigroup T. The notion of (C, ω)-dissipative operators is presumably most prominent among them. Unfortunately, a proof the converse implication, i.e. deriving (C, ω)-dissipativity of the generator from the invariance of C under T, could only be provided for the quasi-contractive case. Therefore, it is natural to ask whether Theorem 3.3.20 holds for all strongly continuous semigroups. Clearly, the assertions would have to be adjusted to the general case, since the set $\{0\}$ is always invariant under T and thus assertion (2) of Theorem 3.3.20 reads $\|T(t\| \leq e^{\omega t}$, $t > 0$, indicating a quasi-contractive semigroup. However, there still might be a notation which fits for the general case.

But we will see later on (using an example of El Maati Ouhabaz) that Theorem 3.3.20 is false for arbitrary semigroups (cp. Example 3.4.26).

3.4 Invariance of proximinal, convex sets under C_0-semigroups

It is the aim of this section to make the abstract theory of section 3.3 more accessible. Although we could characterise the invariance of closed, convex sets C in general Banach spaces under a (quasi)-contractive strongly continuous semigroup with the (C, ω)-dissipativity of its generator, a practicable description of the functionals in the subdifferential of the distance function is still missing. Therefore, we introduce the notions of normally projectable and proximinal sets, which for convex sets coincide. The subdifferential of the distance function can now be described via elements of the normal cone and the duality mapping. Thus, the notion of (C,ω)-dissipativity can be formulated in a more applicable way. We give several examples for this theory recovering for instance the famous characterisations of contractive and positive semigroups and an extensive treatment of order intervals in real Banach lattices.

3.4.1 Normally projectable and proximinal sets

Here we introduce normally projectable and proximinal sets in normed linear spaces. While the notion of proximinal sets and best approximation is widely known in the standard literature (see, for instance, [Sin70]), a treatment of its natural counterpart, the normally projectable sets, or even a name seems to be missing. Therefore, we have taken the freedom to call them normally projectable in correspondence to the normal cone used in their definition. We show that a set is normally projectable if and only if it is proximinal and convex. The subdifferential of the distance function for a proximinal, convex set can be described in terms of best approximations. We finally extend these results to convex cones.

Let X be a normed linear space with dual space X'. For a subset $C \subseteq X$ we recall the definitions of the normal cone

$$N(x,C) := \left\{x' \in X' \mid \text{Re } \langle x', y-x\rangle \leq 0 \text{ for all } y \in C\right\}, \quad x \in C,$$

and of the duality mapping

$$J(x) := \left\{x' \in X' \mid \left\|x'\right\| = \|x\|, \ \langle x', x\rangle = \|x\|^2\right\}, \quad x \in X.$$

We define the (possibly multivalued) operator P_C of x as

$$P_C x := \{y \in C \mid J(x-y) \cap N(y,C) \neq \emptyset\}$$

for all $x \in X$. For an example of a set C, where $P_C(x)$ is multivalued for some $x \in X$, we refer to Example 3.4.16.

Remark 3.4.1. *Let X be a Hilbert space and C a closed, convex subset of H. Then P_C is the orthogonal projection P of H onto C. In fact, let $y \in P_C x$ for some $x \in X$. Since $J(x-y) = x-y$, it follows $x-y \in N(y,C)$, i.e.* Re $(x-y|z-x) \leq 0$ *for all $z \in C$. Hence, y has to equal Px due to the uniqueness of the orthogonal projection.*

Like the orthogonal projection in the Hilbert space case our operator P_C enjoys the following properties:

Remark 3.4.2. *Let C be a subset of a normed linear space X.*

1. *$P_C x = \{x\}$ for all $x \in C$.*
2. *Let $x_0, x_1 \in X$. Then*
$$\text{Re } \langle x_0' - x_1', y_0 - y_1\rangle \geq 0$$
for all $y_0 \in P_C(x_0)$, $y_1 \in P_C(x_1)$ and functionals $x_0' \in J(x_0 - y_0) \cap N(y_0, C)$, $x_1' \in J(x_1 - y_1) \cap N(y_1, C)$.

Proof. For (1) let $x \in C$ and $x_0 \in P_C x$. Then there exits $x' \in J(x - x_0) \cap N(x_0, C)$ and it follows $\|x - x_0\|^2 = \text{Re}\,\langle x', x - x_0\rangle \leq 0$, since $x \in C$.

For (2) we consider $y_0 \in P_C(x_0)$, $y_1 \in P_C(x_1)$ and functionals $x_0' \in J(x_0 - y_0) \cap N(y_0, C)$, $x_1' \in J(x_1 - y_1) \cap N(y_1, C)$. Then

$$\text{Re}\,\langle x_0' - x_1', y_0 - y_1\rangle = -\text{Re}\,\langle x_1', y_0 - y_1\rangle - \text{Re}\,\langle x_0', y_1 - y_0\rangle \geq 0$$

since $x_0' \in N(y_0, C)$, $x_1' \in N(y_1, C)$. □

The subsets C for which $P_C x \neq \emptyset$ for all $x \in X$ will play an important role in the subsequent sections.

Definition 3.4.3. We say that a subset C of a normed linear space X is *normally projectable* if $P_C x \neq \emptyset$ for every $x \in X$.

Lemma 3.4.4. *Every normally projectable set is convex.*

Proof. Let C be a normally projectable subset of a normed linear space X. For $x, y \in C$ we define $z := \lambda x + (1 - \lambda) y$. Since C is normally projectable, there are $z_0 \in C$ and $x' \in J(z - z_0) \cap N(z_0, C)$. It follows

$$\|z - z_0\|^2 = \langle x', z - z_0\rangle = \lambda\ \text{Re}\,\langle x', x - z_0\rangle + (1 - \lambda)\ \text{Re}\,\langle x', y - z_0\rangle \leq 0,$$

i.e. $z = z_0 \in C$. In conclusion, the set C is convex. □

For instance, the sets X and $\{x\}$ for $x \in X$ are normally projectable. It will turn out that the notion of normal projectability (like in the Hilbert space case) is closely connected to the existence of a best approximation in C for all $x \in X$. In order to prove this fact, we introduce the best *approximation operator* (or *metric projection*)

$$\begin{aligned} \pi(x, C) &:= \{y \in C \mid \|x - y\| \leq \|x - z\| \ \text{ for all } \ z \in C\} \\ &= \{y \in C \mid \|x - y\| = d(x, C)\} \end{aligned}$$

for $x \in X$ and a subset $C \subseteq X$. Obviously, $\pi(x, C) = \{x\}$ for $x \in C$. An element $y \in \pi(x, C)$ is called *best approximation* (or *proximum*) of x in C.

Definition 3.4.5. A subset C of a normed linear space X is called

- *proximinal* if $\pi(x, C) \neq \emptyset$ for all $x \in X$,
- a *semi-Chebyshev set* if $\pi(x, C)$ is a singleton at most,
- a *Chebyshev set* if C is proximinal and a semi-Chebyshev set.

Remark 3.4.6. *Any proximinal subset of a normed linear space is closed.*

Proof. Let X be a normed linear space and $C \subseteq X$ a proximinal subset. We assume that C is not closed. Hence, there exists some $x \in \overline{C} \subset C$. Since C is proximinal, there exists some $x_0 \in \pi(x, C)$, i.e. $\|x - x_0\| \leq \|x - y\|$ for all $y \in C$. Let $(x_n)_{n \in \mathbb{N}} \subset C$ converge to x as $n \to \infty$. Then $\|x - x_0\| \leq \|x - x_n\|$ for all $n \in \mathbb{N}$ implies $x = x_0$ which is a contradiction. Hence, C is indeed closed. □

As a first result we can establish that every normally projectable is proximinal. In particular, it is a closed, convex set.

Lemma 3.4.7. *Let C be a subset of a normed linear space X and let $x \in X$.*

1. $P_C x \subseteq \pi(x, C)$*;*
2. *For $x_0, x_1 \in \pi(x, C)$ is $J(x - x_0) \cap N(x_0, C) = J(x - x_1) \cap N(x_1, C)$;*
3. *Let $x' \in N(x_0, C) \cap N(x_1, C)$ for $x_1, x_2 \in C$. Then* $\mathrm{Re}\,\langle x', x_0 \rangle = \mathrm{Re}\,\langle x', x_1 \rangle$.

Proof. (1) For $x \in C$ the statement is obviously true. Let $x \in X \setminus C$ and $x_0 \in P_C x$, $x' \in J(x - x_0) \cap N(x_0, C)$. For $y \in C$ one has

$$\begin{aligned} \|x - y\| \, \|x - x_0\| &= \|x - x_0 - (y - x_0)\| \, \|x'\| \\ &\geq \mathrm{Re}\,\langle x', x - x_0 \rangle - \mathrm{Re}\,\langle x', y - x_0 \rangle \\ &\geq \mathrm{Re}\,\langle x', x - x_0 \rangle = \|x - x_0\|^2 . \end{aligned}$$

Hence, $x_0 \in \pi(x, C)$.

(2) Let $x_0, x_1 \in \pi(x, C)$ and consider $x' \in J(x - x_0) \cap N(x_0, C)$. Since $x_1 \in C$, we have $\mathrm{Re}\,\langle x', x_1 - x_0 \rangle \leq 0$ and so

$$\|x - x_0\|^2 \geq \left|\langle x', x - x_1 \rangle\right| \geq \mathrm{Re}\,\langle x', x - x_1 \rangle = \mathrm{Re}\,\langle x', x - x_0 \rangle + \mathrm{Re}\,\langle x', x_0 - x_1 \rangle \geq \|x - x_0\|^2 .$$

Thus, $\langle x', x - x_1 \rangle$ is real und equals $\|x - x_0\|^2 = \|x - x_1\|^2$. In conclusion, $x' \in J(x - x_1)$ and

$$\langle x', x_0 - x_1 \rangle = \langle x', x_0 - x \rangle + \langle x', x - x_1 \rangle = -d(x, C) + d(x, C) = 0.$$

In order to prove that $x' \in N(x_1, C)$ let $y \in C$ be arbitrarily chosen. It follows

$$\mathrm{Re}\,\langle x', y - x_1 \rangle = \mathrm{Re}\,\langle x', y - x_0 \rangle + \mathrm{Re}\,\langle x', x_0 - x_1 \rangle = \mathrm{Re}\,\langle x', y - x_0 \rangle \leq 0.$$

Hence, $x' \in N(x_1, C)$.

(3) follows directly from the definition. □

Corollary 3.4.8. *Every normally projectable set is convex, proximinal (and hence closed).*

The necessary condition of convexity already implies that there are proximinal sets, which are not normally projectable. For instance, let x_0 and x_1 be two disjoint vectors in a normed linear space X and consider the closed, non-convex set $C := \{x_0\} \cup \{x_1\}$. For $x \in C$ we have

$$\pi(x, C) = \arg\min_{x_0, x_1} \left\{ \|x - x_0\| , \|x - x_1\| \right\} ,$$

i.e. C is proximinal. But since C is not convex, the set is not normally projectable. Therefore, the set of normally projectable sets is properly included in the set of proximinal sets.

However, if we additionally assume that the proximinal set is convex, then the notion of normal projectability and best approximation are equivalent. In order to prove this result, we first discuss the subdifferential of the distance function for a proximinal set.

Proposition 3.4.9. *Let C be a closed subset of a normed linear space X and let $x \in X \setminus C$ such that $\pi(x, C) \neq \emptyset$. For $x' \in X'$ the following assertions are equivalent:*

1. $x' \in (\partial d(\cdot, C))(x)$;
2. $\|x'\| = 1$ *and* $d(x, C) = \inf_{y \in C} \operatorname{Re} \langle x', x - y \rangle$;
3. $\exists\, x_0 \in \pi(x, C)$: $\|x - x_0\|\, x' \in J(x - x_0) \cap N(x_0, C)$;
4. $\forall\, y \in \pi(x, C)$: $\|x - y\|\, x' \in J(x - y) \cap N(y, C)$.

Proof. "(1) $\Rightarrow$ (2)": Let $x' \in (\partial d(\cdot, C))(x)$ and choose some arbitrary $y \in \pi(x, C)$. Due to assertion (1) of Lemma 3.3.2, we know $\|x'\| \geq 1$. From the definition of the subdifferential we further obtain

$$\operatorname{Re} \langle x', z - x \rangle \leq d(z, C) - d(x, C) = d(z, C) - \|x - y\| \leq \|z - y\| - \|x - y\| \leq \|z - x\|$$

for all $z \in X$. Thus, $\|x'\| = 1$ and, thanks to assertion (2) of Lemma 3.3.2, we get $d(x, C) = \inf_{y \in C} \operatorname{Re} \langle x', x - y \rangle$.

The implication "(2) $\Rightarrow$ (1)" is shown in Lemma 3.3.2.

"(2) $\Rightarrow$ (4)": Let $x' \in X'$ with norm $\|x'\| = 1$ such that $d(x, C) = \inf_{y \in C} \operatorname{Re} \langle x', x - y \rangle$. Let $y \in \pi(x, C) \subseteq C$. Then we have

$$\|x - y\| = d(x, C) \leq \operatorname{Re} \langle x', x - y \rangle \leq \left| \langle x', x - y \rangle \right| \leq \|x - y\| .$$

Hence, $\langle x', x - y \rangle$ is real and equals $\|x - y\|$. Therefore, the functional $x_0' := \|x - y\|\, x'$ belongs to $J(x - y)$. This implies

$$\langle x', x - y \rangle = \|x - y\| = d(x, C) \leq \operatorname{Re} \langle x', x - z \rangle$$

for all $z \in C$ showing that x' (and thus x_0') belongs to $N(y, C)$.

The implication "(4) $\Rightarrow$ (3)" is clear.

"(3) $\Rightarrow$ (2)": Let $x' \in X'$ and assume that $x_0' := \|x - x_0\|\, x' \in J(x - x_0) \cap N(x_0, C)$ for some $x_0 \in \pi(x, C)$. Note that $x' \in N(x_0, C)$ and $\|x'\| = 1$. One has

$$d(x, C) = \|x - x_0\| = \langle x', x - x_0 \rangle \leq \operatorname{Re} \langle x', x - y \rangle \leq \|x - y\|$$

for all $y \in C$. Taking the infimum over all $y \in C$, we conclude $d(x, C) = \inf_{y \in C} \operatorname{Re} \langle x', x - y \rangle$. This shows (2). □

Coming back to the problem of establishing the equivalence of notion for convex, proximinal sets and normally projectable sets, we recall from Theorem 3.3.4 that for each $x \in X \setminus C$, where $C \subseteq X$ is a closed, convex set, there exists $x' \in (\partial d(\cdot, C))(x)$ with norm $\|x'\| = 1$. Thus, we get as a direct corollary of Proposition 3.4.9 and Corollary 3.4.8:

Corollary 3.4.10. *A subset C of a normed linear space X is normally projectable if and only if it is convex and proximinal.*

Moreover, we have the following description of the set of best approximinations for a closed, convex set.

Proposition 3.4.11. *Let C be a closed, convex subset of a normed linear space X and let $x \in X \setminus C$, $x' \in (\partial d(\cdot, C))(x)$ with $\|x'\| = 1$. Then one has*

$$\pi(x, C) = \left\{ y \in C \mid \|x - y\| \, x' \in J(x - y) \cap N(y, C) \right\}.$$

Proof. For $y \in \pi(x, C)$ Proposition 3.4.9 already shows $\|x - y\| \, x' \in J(x - y) \cap N(y, C)$. The converse direction is meanwhile a direct consequence of Lemma 3.4.7. In fact, let $y \in C$ such that $x'_y := \|x - y\| \, x' \in J(x - y) \cap N(y, C)$. Then $y \in P_C x$ and since $P_C x \subseteq \pi(x, C)$ (see Lemma 3.4.7) we have $y \in \pi(x, C)$. $\square$

Moreover, if the duality mapping is single-valued, then Proposition 3.4.11 already shows $J(x - x_0) \subseteq N(x_0, C)$ for a closed, convex set $C \subseteq X$, $x \in X$ and $x_0 \in \pi(x, C)$. While the case is obvious for $x \in C$ (then one has $x = x_0$ and the null functional always belongs to $N(x_0, C)$), let $x \in X \setminus C$ and consider an arbitrary $x'_0 \in J(x - x_0)$. Since the duality mapping is single-valued, x'_0 equals $\|x - x_0\| \, x'$, where the functional $x' \in (\partial d(\cdot, C))(x)$, $\|x'\| = 1$, comes from Theorem 3.3.4. By Proposition 3.4.11, this implies $x'_0 = \|x - x_0\| \, x' \in N(x_0, C)$. Hence, $J(x - x_0) \subseteq N(x_0, C)$.

Here, we assumed closedness of the convex set C. As the following Proposition shows, this condition is in fact superfluous.

Proposition 3.4.12. *Let C be a convex subset of a normed linear space X and assume that X' is strictly convex. Let $x \in X$. Then one has $J(x - x_0) \subseteq N(x_0, C)$ for all $x_0 \in \pi(x, C)$.*

Proof. Let $x_0 \in \pi(x, C)$, $x' \in J(x - x_0)$ and $y \in C$. Since C is convex, the vector $\lambda y + (1 - \lambda) x_0$ belongs to C for all $\lambda \in [0, 1]$. We obtain

$$\|x - x_0 - \lambda(y - x_0)\| = \|x - (\lambda y + (1 - \lambda) x_0)\| \geq \|x - x_0\|, \quad \lambda \in [0, 1].$$

From Lemma 1.1.20 it follows the existence of some $x'_y \in J(x - x_0)$ such that $\mathrm{Re}\, \langle x'_y, y - x_0 \rangle \leq 0$. But the set $J(x - x_0)$ is single-valued, since X' is assumed to be strictly convex (see Lemma 1.1.18). Hence, $x'_y = x'$ and due to the arbitrary choice of $y \in C$ we conclude

$$\mathrm{Re}\, \langle x', y - x_0 \rangle \leq 0, \quad y \in C,$$

i.e. $x' \in N(x_0, C)$. $\square$

Thus, in the situation of a strictly convex dual space (or more general, if the duality mapping is single-valued), e.g. $X = L^p(\Omega)$ for $1 < p < \infty$, the search for some $x' \in J(x - x_0) \cap N(x_0, C)$ is particularly easy.

Regarding cones we have:

Lemma 3.4.13. *Let K be a cone with vertex $\tilde{x}$ in a normed linear space X. Let $x_0 \in K$. Then $x' \in N(x_0, K)$ fulfills* $\mathrm{Re}\, \langle x', y - \tilde{x} \rangle \leq 0$ *for all $y \in K$. In particular, one has $N(x_0, K) \subseteq N(\tilde{x}, K)$ if $\tilde{x} \in K$.*

Proof. Let $x' \in N(x_0, K)$, i.e. $\mathrm{Re}\, \langle x', y - x_0 \rangle \leq 0$ for all $y \in K$. By definition, with $y \in K$ the vector $\tilde{x} + \lambda(y - \tilde{x})$ also belongs to the cone K for all $\lambda > 0$. We conclude

$$\mathrm{Re}\, \langle x', y - \tilde{x} \rangle \leq \frac{1}{\lambda} \, \mathrm{Re}\, \langle x', x - \tilde{x} \rangle, \quad y \in K, \lambda > 0.$$

Hence, Re $\langle x', y - \tilde{x}\rangle \leq 0$ for all $y \in K$. □

In virtue of Proposition 3.4.9 we obtain for a closed cone:

Proposition 3.4.14. *Let K be a closed cone with vertex $\tilde{x}$ in a normed linear space X and let $x \in X \setminus K$ such that $\pi(x, K) \neq \emptyset$. Then for $x' \in X'$ the following assertions are equivalent:*

1. $x' \in (\partial d(\cdot, K))(x)$
2. $x' \in S_{X'} \cap N(\tilde{x}, K)$ *and* $d(x, K) = \mathrm{Re}\ \langle x', x - \tilde{x}\rangle$;
3. $\exists\ x_0 \in \pi(x, K)$: $\|x - x_0\|\, x' \in J(x - x_0) \cap N(x_0, K)$;
4. $\exists\ x_0 \in \pi(x, K)$: $\|x - x_0\|\, x' \in J(x - x_0) \cap N(x_0, K) \cap N(\tilde{x}, K)$;
5. $\forall\ y \in \pi(x, K)$: $\|x - y\|\, x' \in J(x - y) \cap N(y, K)$;
6. $\forall\ y \in \pi(x, K)$: $\|x - y\|\, x' \in J(x - y) \cap N(y, K) \cap N(\tilde{x}, K)$.

Proof. The implications "(6) $\Rightarrow$ (5) $\Rightarrow$ (3)" and "(6) $\Rightarrow$ (4) $\Rightarrow$ (3)" are obvious. Proposition 3.4.9 further shows "(3) $\Rightarrow$ (1)". It remains to prove that (1) implies (2) and (2) implies (6).

"(1) $\Rightarrow$ (2)": Let $x' \in (\partial d(\cdot, K))(x)$. From Proposition 3.4.9 we get $\|x'\| = 1$. Now Lemma 3.3.11 shows $x' \in N(\tilde{x}, K)$ and $d(x, K) = \mathrm{Re}\ \langle x', x - \tilde{x}\rangle$.

"(2) $\Rightarrow$ (6)": Let $x' \in S_{X'} \cap N(\tilde{x}, K)$ such that $d(x, K) = \mathrm{Re}\ \langle x', x - \tilde{x}\rangle$. For an arbitrary $y \in \pi(x, K)$ we define $x'_0 := \|x - y\|\, x' \in X'$. Then $x'_0 \in N(\tilde{x}, K)$ and $\|x'\| = \|x - y\|$. We have

$$\mathrm{Re}\ \langle x'_0, x - y\rangle \leq \|x - y\|^2 = \mathrm{Re}\ \langle x'_0, x - \tilde{x}\rangle.$$

Thus, Re $\langle x'_0, \tilde{x} - y\rangle \leq 0$. Since $x'_0 \in N(\tilde{x}, K)$ and $y \in K$, we additionally know Re $\langle x'_0, y - \tilde{x}\rangle = 0$. We conclude Re $\langle x'_0, y\rangle =$ Re $\langle x'_0, \tilde{x}\rangle$ and it follows $x'_0 \in N(y, K)$. Furthermore, we have

$$\|x - y\|^2 = \mathrm{Re}\ \langle x'_0, x - \tilde{x}\rangle = \mathrm{Re}\ \langle x'_0, x - y\rangle \leq \left|\langle x'_0, x - y\rangle\right| \leq \|x - y\|^2 .$$

Hence, $\langle x'_0, x - y\rangle$ is real and equals $\|x - y\|^2$, i.e. $x'_0 \in J(x - y)$. This shows assertion (6). □

Like for general closed, convex sets (see Lemma 3.4.11) we additionally derive a description of the set of best approximations for closed, convex cones.

Proposition 3.4.15. *Let K be a closed, convex cone with vertex $\tilde{x}$ in a normed linear space X. Let $x \in X \setminus K$ and $x' \in (\partial d(\cdot, K))(x)$ with $\|x'\| = 1$. Then one has*

$$\pi(x, K) = \left\{ y \in K \mid \mathrm{Re}\ \langle x', y - \tilde{x}\rangle = 0 \ \textit{ and }\ \mathrm{Re}\ \langle x', x - y\rangle = \|x - y\| \right\}$$

Proof. First we recall from Lemma 3.3.11 that Re $\langle x', x - \tilde{x}\rangle = d(x, K)$. Now let $x_0 \in \pi(x, K)$. Thanks to Proposition 3.4.11 we know $d(x, K) = \|x - x_0\| = \langle x', x - x_0\rangle$. It follows Re $\langle x', x_0 - \tilde{x}\rangle = 0$. Conversely, let $y \in K$ such that Re $\langle x', \tilde{x}\rangle =$ Re $\langle x', y\rangle$ and $\|x - y\| =$ Re $\langle x', x - y\rangle$. We conclude

$$d(x, K) = \mathrm{Re}\ \langle x', x - \tilde{x}\rangle = \mathrm{Re}\ \langle x', x - y\rangle = \|x - y\| ,$$

i.e. $y \in \pi(x, K)$. □

In the subsequent sections the closed, convex sets considered will often be balls or cones. Therefore, we show their proximinality.

Example 3.4.16. *1. Let $\alpha > 0$. Then the closed, convex ball $B := \{x \in X \mid \|x\| \leq \alpha\}$ is proximinal and hence normally projectable. However, B is not necessarily a Chebysev set.*

2. Let $(X, \leq)$ be a Banach lattice, i.e. the ordered vector space $(X, \leq)$ is a lattice and the norm on X is a lattice norm. For $\tilde{x} \in X$ the set $K := \{x \in X \mid x \leq \tilde{x}\}$ is a closed, convex cone with vertex $\tilde{x}$ and a proximum of $x \in X$ in K is given by $\inf\{x, \tilde{x}\}$.

Proof. (1) For $x \in X \setminus B$ we define $x_0 := \frac{\alpha x}{\|x\|} \in B$. Then

$$\|x - x_0\| = \left(1 - \frac{\alpha}{\|x\|}\right)\|x\| = \|x\| - \alpha \leq \|x\| - \|y\| \leq \|x - y\|$$

for all $y \in B$, i.e. $x_0 \in \pi(x, B) = P_B x$.

In order to show that B is not necessarily a Chebyshev set, we provide a simple counterexample. Therefore, we assume for simplicity $\alpha = 1$ and consider the unit ball B in ℓ_p where $1 \leq p < \infty$. Let e_n, $n \in \mathbb{N}$, be the unit vectors in ℓ_p. Then $x = e_1 + e_2 \in \ell_p \setminus B$ and $d(x, B) = \|x\|_p - 1 = 1$. However, we have, for instance, $\|x - e_1\|_p = 1 = \|x - e_2\|_p$. In conclusion, B is not a Chebyhev set.

(2) In view of Example 3.3.10 and Remark 3.4.6 we only need to show the proximinality of K. Let $x \in X$ and put $x_0 := \inf\{x, \tilde{x}\} \in K$. Since the norm on X is a lattice norm, it is sufficient to prove $x - x_0 = |x - x_0| \leq |x - y|$ for all $y \leq \tilde{x}$. Let $y \leq \tilde{x}$. Then $x = y + x - y \leq \tilde{x} + |x - y|$ and since the right-hand side is bigger than $\tilde{x}$ we obtain $\sup\{x, \tilde{x}\} \leq \tilde{x} + |x - y|$. Thanks to the general representation

$$x + \tilde{x} = \sup\{x, \tilde{x}\} + \inf\{x, \tilde{x}\}$$

we end up with

$$x - x_0 = \sup\{x, \tilde{x}\} - \tilde{x} \leq |x - y|\,.$$

Due to the arbitrary choice of $y \in K$ we have shown $x_0 \in \pi(x, K)$. □

Furthermore, with the characterisation of Corollary 3.4.10 at hand, we can determine a large class of (automatically) normally projectable sets: closed, convex subsets of reflexive Banach spaces. This result is well-known and be found for instance in [Sin70].

Proposition 3.4.17. *Let X be a Banach space and $C \subseteq X$ a closed, convex, nonempty subset.*

1. If X is reflexive, then C is proximinal (and hence normally projectable).

2. If X is strictly convex, then C is semi-Chebyshev.

Proof. For (1) assume that X is reflexive and let $x \in X \setminus C$. The function $f(y) := \frac{1}{2}\|x - y\|^2$, $y \in C$, is convex (see Corollary 1.1.22), continuous and fulfills $f(y) \to \infty$ if $\|y\| \to \infty$ for $y \in C$. By virtue of Theorem 1.1.16 the function f has a minimum x_0 in C. Thus, $x_0 \in \pi(x, C) = P_C x$.

For (2) let X be strictly convex, $x \in X \setminus C$, and $x_1, x_2 \in \pi(x, C)$. Since C is convex, the convex combination $u := \frac{1}{2}(x_1 + x_2)$ belongs to C. Now the vectors $u_1 := x - x_1$ and $u_2 := x - x_2$ satisfy $\|u_1\| = \|u_2\|$ and so we obtain

$$\|u_j\| \geq \frac{1}{2}\|u_1 + u_2\| = \|x - u\| \geq \|x - x_j\| = \|u_j\|\,, \quad j \in \{1, 2\}\,,$$

i.e. $\|u_1\| = \|u_2\| = \frac{1}{2}\|u_1 + u_2\|$. Since X is strictly convex, it follows $u_1 = u_2$ and thus $x_1 = x_2$. $\square$

3.4.2 Invariance results in a Banach space

In this section we will treat the invariance results of section 3.3.2 for proximinal, convex sets. In doing so Theorems 3.3.19 and 3.3.20 will be our main focus. We have seen in the previous that the subdifferential can now be described in terms of best approximation. Thus, we are able to reformulate the notion of (C,ω)-dissipativity in a more practicable way. As a first application we derive the famous characterisations of contractive and positive semigroups wit dissipative and dispersive generators, respectively.

Let X be a Banach space with dual space X'. We start the section with a reformulation of the notion of (C,ω)-dissipative operators, where C is now a proximinal, convex set. The used results are provided in section 3.4.1.

Proposition 3.4.18. *Let $(A, D(A))$ be an operator in a Banach space X and let C be a convex, proximinal subset of X. Let $\omega \in \mathbb{R}$.*

1. *A is (C,ω)-dissipative if and only if for each $x \in D(A) \setminus C$ there exists $x_0 \in \pi(x, C)$ such that* Re $\langle x_0', Ax\rangle \le \omega \|x - x_0\|^2$ *for some $x_0' \in J(x - x_0) \cap N(x_0, C)$.*
2. *Assume, in addition, that A is a generator of a strongly continuous semigroup on X. Then A is strictly (C,ω)-dissipative if and only if for each $x \in D(A) \setminus C$ there exists $x_0 \in \pi(x, C)$ such that* Re $\langle x', Ax\rangle \le \omega \|x - x_0\|^2$ *for all $x' \in J(x - x_0) \cap N(x_0, C)$.*
3. *Assume that X' is strictly convex. Then A is (C,ω)-dissipative if and only if for each $x \in D(A) \setminus C$ there exists $x_0 \in \pi(x, C)$ such that* Re $\langle x_0', Ax\rangle \le \omega \|x - x_0\|^2$ *for some $x_0' \in J(x - x_0)$.*

Proof. (1) First we recall from Lemma 3.3.15 that A is (C,ω)-dissipative if and only if for each $x \in D(A) \setminus C$ there exists $x' \in (\partial d(\cdot, C))(x)$ such that Re $\langle x', Ax\rangle \le \omega\ d(x, C)$.

Let $x \in D(A) \setminus C$. Assume the existence of some $x' \in (\partial d(\cdot, C)(x)$ such that Re $\langle x', Ax\rangle \le \omega\ d(x, C)$. Due to Proposition 3.4.9 we have $x_0' := \|x - x_0\|\, x' \in J(x - x_0) \cap N(x_0, C)$ for some (and thus all) $x_0 \in \pi(x, C)$. Hence,

$$\mathrm{Re}\ \langle x_0', Ax\rangle = \|x - x_0\|\ \ \mathrm{Re}\ \langle x', Ax\rangle \le \omega \|x - x_0\|^2 .$$

This shows one direction. For the converse direction we assume Re $\langle x_0', Ax\rangle \le \omega \|x - x_0\|^2$ for some $x_0 \in \pi(x, C)$ and some $x_0' \in J(x - x_0) \cap N(x_0, C)$. Put $x' := \frac{x_0'}{\|x - x_0\|} \in S_{X'}$. Now Proposition 3.4.9 shows $x' \in (\partial d(\cdot, C))(x)$. Since we have

$$\mathrm{Re}\ \langle x', Ax\rangle = \frac{1}{\|x - x_0\|}\ \mathrm{Re}\ \langle x_0', Ax\rangle \le \omega \|x - x_0\| = \omega\ d(x, C),$$

the operator A is (C,ω)-dissipative.

(2) Let A be the generator of a strongly continuous semigroup on X. We recall from Theorem 3.3.19 that A is strictly (C,ω)-dissipative if and only if for all $x \in D(A) \setminus C$ and all $x' \in (\partial d(\cdot, C))(x)$ one has Re $\langle x', Ax\rangle \le \omega\ d(x, C)$. Now the statement follows in a similar manner as assertion (1) from Proposition 3.4.9.

(3) follows from Proposition 3.4.12. $\square$

Theorem 3.3.19 connects the invariance of a closed, convex set under a semigroup T to the (C,ω)-dissipativity of its generator. From a practical point of view the concrete form of the subdifferential of the distance function is missing in this characterisation. For proximinal convex sets we have now a concrete description of the subdifferential at hand, whenever we know the best approximations, which is often the case. We have already used this to reformulate the notion of (C,ω)-dissipativity and now we can deduce directly from Theorem 3.3.19:

Theorem 3.4.19. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let C be a convex, proximinal subset of X. For $\omega \in \mathbb{R}$ the following assertions are equivalent:*

1. *$d(T(t)x, C) \leq e^{\omega t} d(x, C)$ for all $t > 0$, $x \in X$;*
2. *$d(\lambda R(\lambda, A)x, C) \leq \frac{\lambda}{\lambda-\omega} d(x, C)$ for all sufficiently large $\lambda > \max\{\omega, 0\}$;*
3. *for each $x \in D(A) \setminus C$ there exists $x_0 \in \pi(x, C)$ such that* $\operatorname{Re}\langle x_0', Ax\rangle \leq \omega \|x - x_0\|^2$ *for some $x_0' \in J(x - x_0) \cap N(x_0, C)$;*
4. *A is (C, ω)-dissipative;*
5. *for each $x \in D(A) \setminus C$ there exists $x_0 \in \pi(x, C)$ such that* $\operatorname{Re}\langle x', Ax\rangle \leq \omega \|x - x_0\|^2$ *for all $x' \in J(x - x_0) \cap N(x_0, C)$;*
6. *A is strictly (C, ω)-dissipative.*

In either one of these cases, the set C is invariant under T.

For quasi-contractive semigroups we have proven the converse implication in Theorem 3.3.20. This is, of course, true as well in our special setting here.

Theorem 3.4.20. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let $\omega_0 \in \mathbb{R}$ such that $\|T(t)\| \leq e^{\omega_0 t}$ for all $t \geq 0$. Let C be a convex, proximinal subset of X. For $\omega \geq \omega_0$ the following assertions are equivalent:*

1. *C is invariant under T,*
2. *$d(T(t)x, C) \leq e^{\omega t} d(x, C)$ for all $t > 0$, $x \in X$,*
3. *$\lambda R(\lambda, A)C \subseteq C$ for all sufficiently large $\lambda > \max\{\omega, 0\}$,*
4. *$d(\lambda R(\lambda, A)x, C) \leq \frac{\lambda}{\lambda-\omega} d(x, C)$ for all sufficiently large $\lambda > \max\{\omega, 0\}$,*
5. *for each $x \in D(A) \setminus C$ there exists $x_0 \in \pi(x, C)$ such that* $\operatorname{Re}\langle x_0', Ax\rangle \leq \omega \|x - x_0\|^2$ *for some $x_0' \in J(x - x_0) \cap N(x_0, C)$;*
6. *A is (C, ω)-dissipative;*
7. *for each $x \in D(A) \setminus C$ there exists $x_0 \in \pi(x, C)$ such that* $\operatorname{Re}\langle x_0', Ax\rangle \leq \omega \|x - x_0\|^2$ *for all $x_0' \in J(x - x_0) \cap N(x_0, C)$;*
8. *A is strictly (C, ω)-dissipative.*

Often the closed, convex sets considered are indeed proximinal, convex cones. We have seen at the end of section 3.4.1 that elements of the subdifferential for the distance function of a proximinal, convex cone can be written in a special way. In particular, they fulfill some conditions in relation to the vertex of the cone. Therefore, we provide, in addition, a reformulation of (C,ω)-dissipativity for proximinal, convex cones similar to Proposition 3.4.21. The used methods are provided in Proposition 3.4.14.

Proposition 3.4.21. *Let $(A, D(A))$ be an operator in a Banach space X and let K be a convex, proximinal cone with vertex $\tilde{x}$ in X. Let $\omega \in \mathbb{R}$.*

1. *A is (K,ω)-dissipative if and only if for each $x \in D(A) \setminus K$ there exists $x_0 \in \pi(x, K)$ such that* Re $\langle x_0', (A-\omega)x\rangle \leq -\omega$ Re $\langle x_0', \tilde{x}\rangle$ *for some $x_0' \in J(x-x_0) \cap N(x_0, K) = J(x-x_0) \cap N(x_0, K) \cap N(\tilde{x}, K)$.*
2. *Assume, in addition, that A is a generator of a strongly continuous semigroup on X. Then A is strictly (K,ω)-dissipative if and only if for each $x \in D(A) \setminus K$ and all $x' \in S_{X'} \cap N(\tilde{x}, K)$ with* Re $\langle x', x-\tilde{x}\rangle = d(x,K)$ *one has* Re $\langle x', Ax\rangle \leq \omega d(x,K)$.
3. *Let X' be strictly convex. Then A is (K,ω)-dissipative if and only if for each $x \in D(A)\setminus K$ there exists $x_0 \in \pi(x,K)$ such that* Re $\langle x_0', (A-\omega)x\rangle \leq -\omega$ Re $\langle x_0', \tilde{x}\rangle$ *for some $x_0' \in J(x-x_0)$.*

Proof. Let $x \in D(A) \setminus K$, $x_0 \in \pi(x,K)$ and $x_0' \in J(x-x_0) \cap N(x_0,K)$. Thanks to Proposition 3.4.14 we know

$$\text{Re } \langle x_0', x-\tilde{x}\rangle = \|x-x_0\| \ d(x,K) = \|x-x_0\|^2 .$$

Now the statements follow from their analogues in Proposition 3.4.21. ☐

From these results we can deduce well-known characterisation theorems. For instance, the connection of contraction semigroups and dissipative operators and the one of positive semigroups and dispersive operators.

3.4.2.1 Contraction semigroups and dissipative operators

An operator $(A, D(A))$ on X is said to be *dissipative* if for all $x \in D(A)$, $x \neq 0$, there exists $x' \in X'$, $\|x'\| = 1$, such that Re $\langle x', Ax\rangle \leq 0$.

Considering the proximinal, convex set $\{0\}$ this means in our terminology:

Lemma 3.4.22. *Let $(A, D(A))$ be an operator on a Banach space X and let $\omega \in \mathbb{R}$. Then $A-\omega$ is dissipative if and only if A is $(\{0\},\omega)$-dissipative.*

This follows directly from Proposition 3.4.21. Equivalently, we could have used the definition of (C,ω)-dissipativity and the fact that $d(x,\{0\}) = \|x\|$, $x \in X$, in combination with the well-known description of the subdifferential

$$(\partial \|\cdot\|)(x) = \left\{x' \in X' \mid \left\|x'\right\| = 1,\ \langle x', x\rangle = \|x\|\right\}, \quad x \in X, x \neq 0,$$

(see [Cio90, Proposition 3.4]). We deduce from Theorem 3.4.19:

Theorem 3.4.23. *Let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$. Then $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$ if and only if the operator $A - \omega$ is dissipative.*

This fact is well-known and dates back to the generation theorems of Hille-Yosida and Lumer-Philipps.

3.4.2.2 Positive semigroups and dispersive operators

Now let X be a Banach lattice with an ordering "$\leq$". We define the negative cone

$$X_- := \{x \in X \mid x \leq 0\}.$$

Thanks to Example 3.4.16 we know that X_- is a convex, proximinal cone with vertex 0 and $\inf\{x, 0\}$ is a proximum of $x \in X$ in X_-. Additionally, one can verify that X_- is a proper cone.

An operator $(A, D(A))$ on X is called *dispersive* if for all $x \in D(A)$ there exists a positive functional $x' \in J(x^+)$, such that $\mathrm{Re}\,\langle x', Ax\rangle \leq 0$. Since $x^+ = 0$ for all $x \in X_-$, we may restrict ourselves in the definition to $x \in D(A) \setminus X_-$.

Since $x^+ = \sup\{x, 0\} = x - \inf\{x, 0\}$ for $x \in X$, this means in our terminology:

Lemma 3.4.24. *Let $(A, D(A))$ be an operator on a Banach lattice $(X, \leq)$ and let $\omega \in \mathbb{R}$. Then $A - \omega$ is dispersive if and only if A is (X_-, ω)-dissipative.*

Proof. We recall from Proposition 3.4.21 that A is (X_-, ω)-dissipative if and only if for each $x \in D(A) \setminus X_-$ one has

$$\mathrm{Re}\,\langle x_0', (A - \omega)x\rangle \leq -\omega\ \mathrm{Re}\,\langle x_0', 0\rangle = 0$$

for some $x_0' \in J(x^+) \cap N(-x^-, X^-) \subseteq J(x^+) \cap N(0, X^-)$. Now note that by definition

$$N(0, X_-) = \left\{x' \in X' \mid \mathrm{Re}\,\langle x', y\rangle \leq 0 \text{ for all } y \in X_-\right\} = \left\{x' \in X' \mid x' \geq 0\right\}.$$

This proves the Lemma. □

A semigroup T on a Banach lattice is called *positive* if X_- is invariant under T. Hence, we can characterize generators of positive, quasi-contractive semigroups as dispersive operators. This characterisation is due to Philipps (cf. [Phi62]).

Theorem 3.4.25. *Let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Banach lattice $(X, \leq)$ with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$. Then $A - \omega$ is dispersive if and only if T is positive and $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$.*

Proof. We may assume $\omega = 0$. Let A be dispersive. Then A is $(X_-, 0)$-dissipative (see Lemma 3.4.24) and so we get from Theorem 3.4.19 the invariance of X_- under T, i.e. T is positive, and

$$\left\|(T(t)x)^+\right\| = d(T(t)x, X_-) \leq d(x, X_-) = \left\|x^+\right\|, \quad x \in X, t \geq 0. \tag{3.2}$$

Now let $x \in X$, $t \geq 0$. Since $T(t)$ is positive, we have $|T(t)x| \leq T(t)\,|x| = (T(t)\,|x|)^+$. The norm on X is a lattice norm, so it follows $\|T(t)x\| \leq \|(T(t)\,|x|)^+\|$. From (3.2) we finally get $\|T(t)x\| \leq \|x\|$, i.e. $T(t)$ is a contraction. This shows one direction. The converse direction can meanwhile directly be obtained via Theorem 3.4.20 and Lemma 3.4.24. □

With this Theorem we can finally provide the proposed counterexample for a general version of Theorem 3.3.20. The example is due to El Maati Ouhabaz (cf. [Ouh92]).

Example 3.4.26 (Ouhabaz)**.** *Consider the real Hilbert space $H = L^2(0,1)$ and the continuous, elliptic, densely defined sesquilinear form*

$$\mathfrak{a}[u,v] := \int_0^1 DuDv \, dx + \int_0^1 \sqrt{x} Duv \, dx$$

with domain $V = H_0^1(0,1)$ and associated operator $(A_2, D(A_2))$. The operator $-A_2$ generates a strongly continuous semigroup T_2 on H. Since the interval $(0,1)$ is of finite measure, the semigroup T_2 extrapolates, in particular, to a strongly continuous semigroup T_1 on $L^1(0,1)$ (cf. [Are04, p.77]), i.e. $T_1(t)f = T_2(t)f$ for all $f \in L^1(0,1) \cap L^2(0,1)$ and $t \geq 0$, with generator $(A_1, D(A_1))$. Then the semigroup T_1 is positive but not quasi-contractive. In particular, $A_1 - \omega$ is non-dispersive for all $\omega \in \mathbb{R}$.

Proof. Since positivity of T_2 is inherited by T_1 (cf. [Are04, p. 78]), we first show that T_2 is positive. Thanks to the Beurling-Deny criteria T_2 is positive if and only if $u^+ \in V$ and $\mathfrak{a}[u^+, u^-] \leq 0$ for all $u \in V$ (see Proposition 4.4.2). Let $u \in V$. Then $u^+ \in V$ with $Du^+ = \mathbb{1}_{\{u>0\}} Du$ and we have

$$\mathfrak{a}[u^+, u^-] = \int_0^1 \mathbb{1}_{\{u>0\}} \mathbb{1}_{\{u<0\}} (Du)^2 \, dx + \int_0^1 \sqrt{x} \, \mathbb{1}_{\{u>0\}} Du \, \mathbb{1}_{\{u<0\}} u \, dx = 0.$$

Hence, T_2 and, subsequently, T_1 is positive.

Next assume that $\|T_1(t)\| \leq e^{\omega t}$ for some $\omega \in \mathbb{R}$ and all $t \geq 0$. Then the operator $A_1 - \omega$ is strictly $(\{0\}, \omega)$-dissipative, i.e. for all $u \in D(A_1) \setminus \{0\}$ and all $x' \in J(u)$ one has $\langle x', A_1 u \rangle \leq \omega \|u\|^2$. Let $\varphi \in \mathcal{D}(0,1) \subset D(A_1) \cap D(A_2)$, $\varphi \geq 0$, be a positive test function and put $v := \|\varphi\|$. Then $v \in J(\varphi)$ and hence

$$\omega \|\varphi\| \geq \int_0^1 A_1 \varphi \, dx = \int_0^1 \varphi''(x) - \sqrt{x}\varphi'(x) \, dx = \int_0^1 -\sqrt{x}\varphi'(x) \, dx.$$

Now integration by parts on the right-hand side leads to

$$\int_0^1 \left(\omega - \frac{1}{2\sqrt{x}} \right) \varphi(x) \, dx \geq 0,$$

which is a contradiction. Thus, T_1 is not quasi-contractive. □

3.4.3 Invariance results for closed, convex sets in a Hilbert space

In this subsection we will focus on the case, where X is a Hilbert space H with scalar product $(\cdot|\cdot)$. Thanks to Proposition 3.4.17 we know that any closed, convex subset C of H is a Chebyshev set. We denote the orthogonal projection of H onto C by P and recover within our theory Brézis' Theorem in the linear case. Since the essential steps of the proof are particularly enlightening in this case, we cannot resist to state them.

The duality mapping in a Hilbert space is the identity (cf. Lemma 1.1.19). So we get from Lemma 3.3.1 and Proposition 3.4.9 the following description of the subdifferential for the distance function which may be found as well in [Bau96, Proposition 3.2.5].

Lemma 3.4.27. *Let C be a closed, convex subset of a Hilbert space H and let P be the orthogonal projection of H onto C. Then one has*

$$(\partial d(\cdot, C))(x) = \begin{cases} \frac{x-Px}{\|x-Px\|} & , \quad x \in H \setminus C, \\ B_H \cap N(x, C) & , \quad x \in C. \end{cases}$$

This description leads to the following particularly simple characterisation of (C, ω)-dissipativity in Hilbert spaces.

Proposition 3.4.28. *Let $(A, D(A))$ be an operator in a Hilbert space H and let P be the orthogonal projection onto a closed, convex subset C of H. Let $\omega \in \mathbb{R}$. Then the following are equivalent:*

1. *A is (C, ω)-dissipative;*
2. *for all $x \in D(A)$ is $\mathrm{Re}\,(Ax|x - Px) \leq \omega \|x - Px\|^2$;*
3. *for all $x \in D(A) \setminus C$ is $\mathrm{Re}\,(Ax|x - Px) \leq \omega \|x - Px\|^2$.*

As usual, we apply our fundamental Theorem 3.4.19 to this setting and obtain directly:

Theorem 3.4.29. *Let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Hilbert space H with generator $(A, D(A))$. Let C be a closed, convex subset of H and P be the orthogonal projection of H onto C. For $\omega \in \mathbb{R}$ the following assertions are equivalent:*

1. *$\|T(t)x - P(T(t)x)\| \leq e^{\omega t} \|x - Px\|$ for all $t > 0$, $x \in H$;*
2. *$\|\lambda R(\lambda, A)x - P(\lambda R(\lambda, A)x)\| \leq \frac{\lambda}{\lambda - \omega} \|x - Px\|$ for all $x \in H$ and all sufficiently large real $\lambda > \max\{\omega, 0\}$;*
3. *$\mathrm{Re}\,(Ax|x - Px) \leq \omega \|x - Px\|^2$ for all $x \in D(A)$;*
4. *$\mathrm{Re}\,(y|Ax) \leq \omega \|x - Px\|^2$ for all $x \in D(A)$ and all $y \in (\partial d(\cdot, C))(x)$.*

In either one of these cases, the set C is invariant under T.

Furthermore, we obtain from Theorem 3.4.20 for quasi-contractive semigroups in H:

Theorem 3.4.30. *Let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Hilbert space H with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$. Let C be a closed, convex subset of H and P be the orthogonal projection of H onto C. For $\omega \geq \omega_0$ the following assertions are equivalent:*

1. *C is invariant under T;*
2. *$\|T(t)x - P(T(t)x)\| \leq e^{\omega t} \|x - Px\|$ for all $t > 0$, $x \in H$;*
3. *$\lambda R(\lambda, A)C \subseteq C$ for all sufficiently large real $\lambda > \max\{\omega, 0\}$;*
4. *$\|\lambda R(\lambda, A)x - P(\lambda R(\lambda, A)x)\| \leq \frac{\lambda}{\lambda - \omega} \|x - Px\|$ for all $x \in H$ and all sufficiently large real $\lambda > \max\{\omega, 0\}$;*

5. Re $(Ax|x - Px) \leq \omega \|x - Px\|^2$ *for all* $x \in D(A)$;

6. Re $(y|Ax) \leq \omega \|x - Px\|^2$ *for all* $x \in D(A)$ *and all* $y \in (\partial d(\cdot, C))(x)$.

For contractive semigroups this is indeed the Theorem of Brézis we have stated at the beginning of the chapter. As said Theorem 3.4.30 is a direct corollary of Theorem 3.4.20. However, due to the particularly nice setting in a Hilbert space we would like to state the essential steps of the proof.

Proof. (Sketch) For simplicity, we concentrate on contractive semigroups. First assume that the closed, convex set C is invariant under the contractive semigroup $T = (T(t))_{t\geq 0}$. For $x \in D(A)$ and $t > 0$ we have

$$\begin{aligned}
\text{Re } (T(t)x - x|x - Px) &= \text{Re } (T(t)(x - Px) - (x - Px)|x - Px) + \text{Re } (T(t)Px - Px|x - Px) \\
&\leq \text{Re } (T(t)(x - Px) - (x - Px)|x - Px) \\
&\leq \|T(t)(x - Px)\| \, \|x - Px\| - \|x - Px\|^2 \leq 0.
\end{aligned}$$

In the second line we have used the property of the orthogonal projection and the fact that $T(t)Px$ lies in C. Now it follows

$$\text{Re } (Ax|x - Px) = \lim_{t \searrow 0} \text{ Re } \left(\frac{T(t)x - x}{t} |x - Px \right) \leq 0.$$

We have shown assertion (5).

Next let us assume that assertion (5) holds. We will prove that C is invariant under $\lambda R(\lambda, A)$ for all $\lambda > 0$, i.e. assertion (3). Let $u \in C$, $\lambda > 0$ and $x := \lambda R(\lambda, A)u \in D(A)$. Then one has $Ax = \lambda(x - u)$ and it follows from (5):

$$\begin{aligned}
0 \geq \text{Re } (Ax|x - Px) &= \text{Re } \lambda \, (x - u|x - Px) \\
&= \lambda \left((x - Px|x - Px) - \text{Re } (u - Px|x - Px) \right) \\
&\geq \lambda \|x - Px\|^2 .
\end{aligned}$$

Hence, $x = Px \in C$ and assertion (3) is proven.

The implication "(3) $\Rightarrow$ (1)" is finally obtained as in Lemma 3.3.12 using Euler's formula. □

So far the theory behind Theorem 3.4.30. The arguments are surprisingly simple, in particular, because Brézis proof in the nonlinear case is much more sophisticated.

Like in Corollary 3.3.21 we give a simple application of Theorem 3.4.30, which will be of use in chapter 4.

Corollary 3.4.31. *Let C be a closed, convex subset of a Hilbert space H and let P be the orthogonal projection of H onto C. Let $\omega \in \mathbb{R}$. Then $e^{\omega t}C \subseteq C$ for all $t \geq 0$ if and only if* ω Re $(Px|x - Px) \leq 0$ *for all* $x \in X$.

We close this section with the following observation, which deals with the special case when $P_C D(A) \subseteq D(A)$:

Proposition 3.4.32. *Let $T = (T(t))_{t\geq 0}$ be a quasi-contractive, strongly continuous semigroup on a Hilbert space H with generator $(A, D(A))$. Let C be a closed, convex subset of H and P be the orthogonal projection of H onto C. If $Px \in D(A)$ and* Re $(APx|x - Px) \leq 0$ *for all $x \in D(A)$, then C is invariant under T.*

Proof. Since T is quasi-contractive, there exists $\omega_0 \in \mathbb{R}$ such that $\|T(t)\| \leq e^{\omega_0 t}$ for all $t \geq 0$. Now let us assume $Px \in D(A)$ and Re $(APx|x-Px) \leq 0$ for all $x \in D(A)$. Since $\|T(t)\| \leq e^{\omega_0 t}$, $t \geq 0$, the operator $A - \omega_0$ is dissipative (see Theorem 3.4.23). It follows

$$\text{Re } (Ax|x - Px) = \text{Re } (APx|x - Px) + \text{Re } (A(x - Px)|(x - Px)) \leq \omega_0 \ \|x - Px\|^2 .$$

Now Theorem 3.4.30 shows that C is invariant under T. □

3.4.4 Invariance of order intervals in Banach lattices

Prime examples for proximinal, convex sets are order intervals in Banach lattices. In this section we use our theory in order to characterise their invariance under strongly continuous semigroups. To ease our notation we stick to the real case, the complex case can be obtained from this by the usual procedures. We start by recalling some definitions and results from the theory of Banach lattices in section 3.4.4.1. In the next section we give a complete characterisation of the invariance for order intervals under strongly continuous semigroups introducing the new notion of order-admissible pair of spaces. Here, we also encounter quite naturally Kato-type inequalities. We prove a new version characterising the invariance of closed, convex sets under positive semigroups and apply this to order intervals. At the end of the section we use our results and discuss the invariance of order intervals under semigroups in L^p-spaces, $1 \leq p < \infty$, and $C_0(\Omega)$. Well-known characterisations of submarkovian and positive semigroups are encountered thereby.

3.4.4.1 A short reminder on vector lattices

Here we recall some facts about real Banach lattices following [Nag86, Chapter C-I] and [Sch71]. An *ordered vector space* is a (real) vector space X with a reflexive, transitive and anti-symmetric ordering "$\leq$" satisfying the following *axioms of compatibility*:

- (translation invariance) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$,
- $x \leq 0$ implies $\lambda x \leq 0$ for all $x \in X$, $\lambda \geq 0$.

An ordered vector space $(X, \leq)$ is called a *vector lattice* if any two elements $x, y \in X$ have a *supremum*, which is denoted by $\sup\{x, y\}$ or $x \vee y$, and an *infimum*, denoted by $\inf\{x, y\}$ or $x \wedge y$. This implies that X is directed under the order relation "$\leq$". If the suprema and infima of countably majorized subsets exist, then the vector lattice $(X, \leq)$ is said to be σ-order complete. If they exist for infinite majorized subsets, the vector lattice is called *order complete.*

For elements x of a vector lattice $(X, \leq)$ we define

- $|x| := \sup\{x, -x\}$ as the the absolute value of x,
- $x^+ := \sup\{x, 0\}$ as the positive part of x,
- $x^- := \sup\{-x, 0\}$ as the negative part of x.

Due to the relations

$$z + \sup\{x, y\} = \sup\{x + z, y + z\}, \quad x, y, z \in X,$$

and

$$\sup\{x, y\} = -\inf\{-x, -y\}, \quad x, y \in X$$

one has

$$z + w - \inf\{x, y\} = \sup\{z + w - x, z + w - y\}, \quad x, y, z, w \in X.$$

This implies for $x, y \in X$:

1. $x + y = \sup\{x, y\} + \inf\{x, y\}$,
2. $x = \inf\{x, y\} + (x - y)^+$,
3. $x = x^+ - x^-$,
4. $|x| = x^+ + x^-$.

Moreover, one has $|\lambda x| \leq |\lambda| \, |x|$ for $\lambda > 0$ and $|x + y| \leq |x| + |y|$.

Finally, a (real) *Banach lattice* is a Banach space $(X, \|\cdot\|)$ endowed with an ordering $\leq$ such that $(X, \leq)$ is a vector lattice and the norm on X is a *lattice norm*, i.e.

$$|x| \leq |y| \text{ implies } \|x\| \leq \|y\|$$

for all $x, y \in X$.

3.4.4.2 Order intervals

In this section we want to discuss invariant subsets of strongly continuous semigroups in a real Banach lattice. The subsets we are interested in are the so-called order intervals. In order to provide a framework we define order-admissible pair of space. We show the proximinality and convexity of order intervals in a Banach lattice by stating a proximum. This enables us to apply Theorems 3.4.19 and 3.4.20 to our setting. Next we discuss Kato-type inequalities. We give a new version characterising the invariance of closed, convex sets under positive semigroups and apply this to order intervals.

Throughout this section we consider a real vector lattice $(\tilde{X}, \leq)$ and a real Banach space X which is a sublattice of $\tilde{X}$ and the norm on X is a lattice norm with respect to the ordering in $\tilde{X}$, i.e. $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in X$. In particular, $(X, \leq)$ is a Banach lattice. We will say that $(X, \tilde{X}, \leq)$ is a *order-admissible pair of spaces*. In this situation we consider the sets

$$\begin{aligned} \hat{X}_{\inf} &:= \left\{ \tilde{x} \in \tilde{X} \mid x \wedge \tilde{x} = \inf\{x, \tilde{x}\} \in X \text{ for all } x \in X \right\}, \\ \hat{X}_{\sup} &:= \left\{ \tilde{x} \in \tilde{X} \mid x \vee \tilde{x} = \sup\{x, \tilde{x}\} \in X \text{ for all } x \in X \right\}. \end{aligned}$$

Since X is a sublattice of X, we have $X = \hat{X}_{\inf} \cap \hat{X}_{\sup}$. As a consequence of the description $\sup\{\tilde{x}, \tilde{y}\} = -\inf\{-\tilde{x}, -\tilde{y}\}$ for $\tilde{x}, \tilde{y} \in \tilde{X}$ we further obtain $\hat{X}_{\inf} = -\hat{X}_{\sup}$.

Next we define the *order intervals*

$$\begin{aligned}
[-\infty,\tilde{x}] &:= \{z\in X \mid z\le \tilde{x}\}\,,\\
[\tilde{y},+\infty] &:= \{z\in X \mid \tilde{y}\le z\}\,,\\
[\tilde{y},\tilde{x}] &:= [-\infty,\tilde{x}]\cap[\tilde{y},+\infty]
\end{aligned}$$

for elements $\tilde{x}\in\hat{X}_{\mathrm{inf}}$ and $\tilde{y}\in\hat{X}_{\mathrm{sup}}$ such that $\tilde{y}\le\tilde{x}$. Note that the order intervals are nonempty due to the choice of $\tilde{x}$ and $\tilde{y}$. Moreover, one immediately determines their convexity from the axioms of compatibility for the ordering in $\tilde{X}$. The next Lemma shows their proximinality:

Lemma 3.4.33. *Let $(X,\tilde{X},\le)$ be a order-admissible pair of spaces and let $\tilde{x}\in\hat{X}_{\mathrm{inf}}$, $\tilde{y}\in\hat{X}_{\mathrm{sup}}$ such that $\tilde{y}\le\tilde{x}$.*

1. *$[-\infty,\tilde{x}]$ is a convex, proximinal cone with vertex $\tilde{x}$ and $z\wedge\tilde{x}=z-(z-\tilde{x})^+$ is a proximum of $z\in X$ in $[-\infty,\tilde{x}]$.*
2. *$[\tilde{y},+\infty]$ is a convex, proximinal cone and $z\vee\tilde{y}$ is a proximum of $z\in X$ in $[\tilde{y},+\infty]$.*
3. *$[\tilde{y},\tilde{x}]$ is a convex, proximinal set and $\tilde{y}\vee(z\wedge\tilde{x})=z-(z-\tilde{x})^++(\tilde{y}-z)^+$ is a proximum of $z\in X$ in $[\tilde{y},\tilde{x}]$.*

Proof. We will start with some useful observations: Let $z\in X$. For $w\in[-\infty,\tilde{x}]$ one has $z=w+z-w\le\tilde{x}+|z-w|$ and since the right-hand side is bigger than $\tilde{x}$ we obtain $\sup\{z,\tilde{x}\}\le\tilde{x}+|z-w|$. Similarly, one shows $\inf\{z,\tilde{y}\}\ge\tilde{y}-|z-w|$ for all $w\in[\tilde{y},+\infty]$.

Next we prove assertion (3). The assertions (1) and (2) are obtained in a similar manner, thus, we leave their proofs to the reader. Let $z\in X$ and put $z_0:=\sup\{\tilde{y},\inf\{z,\tilde{x}\}\}$. Since $\tilde{x}\in\hat{X}_{\mathrm{inf}}$ we know $\inf\{z,\tilde{x}\}\in X$ and so $z_0=\sup\{\tilde{y},\inf\{z,\tilde{x}\}\}\in X$ thanks to $\tilde{y}\in\hat{X}_{\mathrm{sup}}$. Now one easily sees $z_0\in[\tilde{y},\tilde{x}]$. Finally, we have to show that $\|z-z_0\|\le\|z-w\|$ for all $w\in[\tilde{y},\tilde{x}]$. Since the norm on X is a lattice norm, it is sufficient to prove $|z-z_0|\le|z-w|$ for all $w\in[\tilde{y},\tilde{x}]$. Let $w\in[\tilde{y},\tilde{x}]=[\tilde{y},+\infty]\cap[-\infty,\tilde{x}]$. From our previous observations we already know $\sup\{z,\tilde{x}\}\le\tilde{x}+|z-w|$ and $\inf\{z,\tilde{y}\}\ge\tilde{y}-|z-w|$. It follows

$$z_0\ge\inf\{z,\tilde{x}\}=z+\tilde{x}-\sup\{z,\tilde{x}\}\ge z-|z-w|\,.$$

and

$$z_0\le\sup\{\tilde{y},z\}=z+\tilde{y}-\inf\{z,\tilde{y}\}\le z+|z-w|\,.$$

We conclude $|z-z_0|=\sup\{z-z_0,z_0-z\}\le|z-w|$ and since $w\in[\tilde{y},\tilde{x}]$ was arbitrarily chosen, the vector z_0 is indeed a proximum of z in $[\tilde{y},\tilde{x}]$. □

We recall that a linear operator T on a Banach lattice $(X,\le)$ is called positive if it leaves the *positive cone* $X_+:=[0,+\infty]$ or, equivalently, the *negative cone* $X_-:=[-\infty,0]$ invariant.

Lemma 3.4.34. *Let $(X,\tilde{X},\le)$ be a order-admissible pair of spaces, $\tilde{x}\in\hat{X}_{\mathrm{inf}}$, $\tilde{x}\ge 0$ and let T be a linear operator on X.*

1. *If $[-\infty,\tilde{x}]$ is invariant under T, then T is positive.*
2. *Assume in addition $\tilde{x}\in X$. Then $[-\infty,\tilde{x}]$ is invariant under T if and only if T is positive and satisfies $T\tilde{x}\le\tilde{x}$.*

Proof. For (1) we assume that $[-\infty, \tilde{x}]$ is invariant under T. Let $x \in [-\infty, 0]$. Then $nx \leq 0 \leq \tilde{x}$ for all $n \in \mathbb{N}$, i.e. $nx \in [-\infty, \tilde{x}]$, and we get from our assumption $nT(x) = T(nx) \leq \tilde{x}$ for all $n \in \mathbb{N}$. This implies $Tx \leq 0$. Therefore, T is positive.

For (2) we assume additionally $\tilde{x} \in X$. Let $[-\infty, \tilde{x}]$ be invariant under T. Thanks to (1) the operator T is positive. Furthermore, $T\tilde{x} \leq \tilde{x}$ since $\tilde{x} \in [-\infty, \tilde{x}]$. Conversely, we have $Tx \leq T\tilde{x} \leq \tilde{x}$ for all $x \in [-\infty, \tilde{x}]$. Hence, $[-\infty, \tilde{x}]$ is invariant under T. □

Remark 3.4.35. *Let $(X, \tilde{X}, \leq)$ be a order-admissible pair of spaces, T be a linear operator on X and $\tilde{x} \in \hat{X}_{\inf}$. Then $[-\infty, \tilde{x}]$ is invariant under T if and only if $[-\tilde{x}, +\infty]$ is invariant under T.*

With regard to this result, we will omit from now on the discussion of the order intervals $[\tilde{y}, +\infty]$ since their invariance properties under linear operators are fully described by those of the order intervals $[-\infty, \tilde{x}]$.

In addition, we point out that the invariance of the intervals $[\tilde{y}, +\infty]$ and $[-\infty, \tilde{x}]$ under a linear operator T already imply the invariance of $[\tilde{y}, \tilde{x}]$ under T due to the representation $[\tilde{y}, \tilde{x}] = [\tilde{y}, +\infty] \cap [-\infty, \tilde{x}]$.

Now we have in virtue of Theorem 3.4.19 and Proposition 3.4.21:

Proposition 3.4.36. *Let $(X, \tilde{X}, \leq)$ be a order-admissible pair of spaces and let $\tilde{x} \in \hat{X}_{\inf}$, $\tilde{y} \in \hat{X}_{\sup}$ with $\tilde{y} \leq \tilde{x}$. Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$.*

1. *Assume that for all $x \in D(A)$ there exists $x' \in J((x - \tilde{x})^+) \cap N(x \wedge \tilde{x}, [-\infty, \tilde{x}])$ such that $\langle x', (A - \omega)x \rangle \leq -\omega \ \langle x', \tilde{x} \rangle$. Then $[-\infty, \tilde{x}]$ is invariant under T.*
2. *Assume that for all $x \in D(A)$ there exists $x' \in J((x - \tilde{x})^+ - (\tilde{y} - x)^+) \cap N(\tilde{y} \vee (\tilde{x} \wedge x), [\tilde{y}, \tilde{x}])$ such that $\langle x', Ax \rangle \leq \omega \ \|(x - \tilde{x})^+ - (\tilde{y} - x)^+\|^2$. Then $[\tilde{y}, \tilde{x}]$ is invariant under T.*

If, in addition, $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$, then the converse implications in the assertions (1) and (2) hold as well.

Remark 3.4.37. *Thanks to Proposition 3.4.21 the assumption in assertions (1) and (2) in Proposition 3.4.36 that the functional x' belongs to the normal cone for the best approximation is superfluous if the dual space X' is strictly convex.*

Next we come to Kato-type inequalities. Therefore, let $(X, \leq)$ be a σ-order complete real Banach lattice. We know (see [Nag86, Proposition C-II.2.1]) that for any $x \in X$ there exists a unique linear operator $\operatorname{sign}(x) : X \to X$ such that

1. $\operatorname{sign}(x)x = |x|$,
2. $\operatorname{sign}(x)y = 0$ if $\inf\{|x|, |y|\} = 0$,
3. $|\operatorname{sign}(x)y| \leq |y|$ for all $y \in X$.

We always consider a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on X with generator $(A, D(A))$. The adjoint of A is denoted by A'. One has $\rho(A) = \rho(A')$, $R(\lambda, A') = R(\lambda, A)'$ for $\lambda \in \rho(A)$.

Proposition 3.4.38. *Let $(X, \tilde{X}, \leq)$ be a order-admissible pair of spaces and assume that X is σ-order complete. Let T be a positive strongly continuous semigroup on X with generator $(A, D(A))$. Let C be a closed subset of X and P a projection of X onto C such that*

$$\operatorname{sign}(x - Px)(y - Px) \leq 0, \quad x \in X, y \in C.$$

Then the following assertions are equivalent:

1. *C is invariant under T;*
2. *for all $x \in D(A)$ and all $x' \in D(A')$, $x' \geq 0$, one has*

$$\langle x', \operatorname{sign}(x - Px)Ax \rangle \leq \langle A'x', |x - Px| \rangle.$$

Proof. Assume (1). Let $x \in D(A)$, $x' \in D(A')$, $x' \geq 0$, and $t \geq 0$. Then $T(t)Px \in C$. From our assumption on P and the positivity of $T(t)$ we obtain the estimate

$$\begin{aligned}
\langle x', \operatorname{sign}(x - Px)(T(t)x - x) \rangle &= \langle x', \operatorname{sign}(x - Px)\,(T(t)(x - Px) - (x - Px)) \rangle \\
&\quad + \langle x', \operatorname{sign}(x - Px)(T(t)Px - Px) \rangle \\
&\leq \langle x', |T(t)(x - Px)| - |x - Px| \rangle \\
&\leq \langle x', T(t)\,|x - Px| - |x - Px| \rangle \\
&= \langle T(t)'x' - x', |x - Px| \rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle x', \operatorname{sign}(x - Px)Ax \rangle &= \lim_{t \searrow 0} \frac{1}{t} \langle x', \operatorname{sign}(x - Px)(T(t)x - x) \rangle \\
&\leq \lim_{t \searrow 0} \frac{1}{t} \langle T(t)x' - x', |x - Px| \rangle \\
&= \langle A'x', |x - Px| \rangle.
\end{aligned}$$

This shows (2).

For the converse implication we assume (2). Since C is closed, it is sufficient to prove that $\lambda R(\lambda, A)C \subset C$ for all sufficiently large $\lambda > 0$ (cp. Lemma 3.3.12).

Let $\lambda \in \rho(A)$, $\lambda > 0$ and $y \in C$ and $x' \in X'$, $x' \geq 0$. We define

$$x := \lambda R(\lambda, A)y \in D(A), \;\; x_0' := \lambda R(\lambda, A)'x' \in D(A').$$

Since $\lambda R(\lambda, A)'$ is positive, we obtain $x_0' \geq 0$. Moreover, $\lambda(x - y) = Ax$ and $\lambda(x_0' - x') = A'x_0'$. Now (2) implies

$$\begin{aligned}
\lambda \langle x_0', \operatorname{sign}(x - Px)(x - y) \rangle &= \langle x_0', \operatorname{sign}(x - Px)Ax \rangle \\
&\leq \langle A'x_0', |x - Px| \rangle \\
&= \lambda \langle x_0' - x', |x - Px| \rangle.
\end{aligned}$$

Since $\lambda > 0$, it follows

$$\langle x', |x - Px| \rangle \leq \langle x_0', |x - Px| - \operatorname{sign}(x - Px)(x - y) \rangle.$$

Note that

$$|x - Px| - \operatorname{sign}(x - Px)(x - y) = \operatorname{sign}(x - Px)(x - Px - x + y) = \operatorname{sign}(x - Px)(y - Px) \leq 0$$

by the assumption on the projection P. Hence, $\langle x', |x - Px|\rangle = 0$ for all $x' \in X'$, $x' \geq 0$. It follows $|x - Px| = 0$ and so $\|x - Px\| = 0$ implying $x = Px \in C$. As indicated before, this shows (1). □

Since the set $\{0\}$ is always invariant under T und $\operatorname{sign}(x)0 = 0$ for all $x \in X$, we get as a corollary:

Corollary 3.4.39. *Let $(X, \tilde{X}, \leq)$ be a order-admissible pair of spaces and assume that X is σ-order complete. Let T be a positive strongly continuous semigroup on X with generator $(A, D(A))$. Then one has*

$$\langle x', \operatorname{sign}(x)Ax\rangle \leq \langle A'x', |x|\rangle.$$

for all $x \in D(A)$ and all $x' \in D(A')$, $x' \geq 0$.

This is indeed a result of Arendt (cf. [Nag86, Theorem C-II.2.4]). It is motivated by Kato's classical inequality stating that

$$\langle \operatorname{sign}(f)\triangle f, \varphi\rangle \leq \langle \triangle\,|f|\,, \varphi\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n), \varphi \geq 0,$$

for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $\triangle f \in L^1_{\text{loc}}(\mathbb{R}^n)$. For the converse implication, i.e. the characterisation of positive semigroups in terms of Kato-type inequalities, one has to assume an additional property as Arendt has shown (for a proof see [Nag86, Theorem C-II.3.8]):

Theorem 3.4.40 (Arendt)**.** *Let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on a σ-order complete real Banach lattice X with generator $(A, D(A))$. Then the following assertions are equivalent:*

1. *T is positive;*
2. *there exists a core D of A and a strictly positive set M' of subeigenvectors of A' such that*

   $$\langle x', \operatorname{sign}(x)Ax\rangle \leq \langle A'x', |x|\rangle$$

   for all $x \in D$ and $x' \in M'$.

Here, an element $x' \in X'$ is called a *positive subeigenvector* of A' if $0 < x' \in D(A')$ and $A'x' \leq \lambda x'$ for some $\lambda \in \mathbb{R}$. A subset M' of X' is called *strictly positive* if for every $x \in X$, $x \geq 0$, such that $\langle x', x\rangle = 0$ for all $x' \in M'$ one has $x = 0$.

We leave this characterisation for a moment and come back to Proposition 3.4.38 applying it to the invariance of order intervals.

Corollary 3.4.41. *Let $(X, \tilde{X}, \leq)$ be a order-admissible pair of spaces and assume that X is σ-order complete. Let T be a positive strongly continuous semigroup on X with generator $(A, D(A))$. Let $\tilde{x} \in \hat{X}_{\text{inf}}$ and assume*

$$\operatorname{sign}((x - \tilde{x})^+)(y - \tilde{x}) \leq 0, \quad x \in X, y \in [-\infty, \tilde{x}].$$

Then the following assertions are equivalent:

1. $[-\infty, \tilde{x}]$ *is invariant under* T*;*

2. *for all* $x \in D(A)$ *and all* $x' \in D(A')$*,* $x' \geq 0$*, one has*

$$\langle x', \operatorname{sign}((x-\tilde{x})^+)Ax\rangle \leq \langle A'x', (x-\tilde{x})^+\rangle.$$

Proof. From Lemma 3.4.33 we know that $Px := x \wedge \tilde{x}$ is a projection of X onto $[-\infty, \tilde{x}]$ for $x \in X$. Since $\inf\{(x-\tilde{x})^+, (\tilde{x}-x)^+\} = 0$, it follows from the properties of the signum operator:

$$\begin{aligned}
\operatorname{sign}(x-Px)(y-Px) &= \operatorname{sign}((x-\tilde{x})^+)(y-(x\wedge\tilde{x})) \\
&= \operatorname{sign}((x-\tilde{x})^+)(y-\tilde{x}+\tilde{x}-(x\wedge\tilde{x})) \\
&= \operatorname{sign}((x-\tilde{x})^+)(y-\tilde{x}) + \operatorname{sign}((x-\tilde{x})^+)((\tilde{x}-x)^+) \\
&= \operatorname{sign}((x-\tilde{x})^+)(y-\tilde{x}).
\end{aligned}$$

Now the statement follows directly from Proposition 3.4.38. □

In the next section we will apply these results to order intervals in L^p-spaces ($1 \leq p < \infty$) and $C_0(\Omega)$.

3.4.4.3 Order intervals in L^p, $1 \leq p < \infty$

Let (Ω, Σ, μ) be a σ-finite measure space. We consider the real vector space $\tilde{X} := L^0(\Omega) = L^0(\Omega, \mu; \mathbb{R})$ of all real-valued μ-measurable functions on Ω and the real Banach space $X_p := L^p(\Omega) = L^p(\Omega, \mu; \mathbb{R})$, $1 \leq p < \infty$, with norm $\|f\|_p := \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}}$. As usual, we identify functions in $\tilde{X}$ and X_p whenever they coincide μ-a.e. on Ω. The dual space of X_p is given by $X_{p'} = L^{p'}(\Omega)$ where $p' := \frac{p}{p-1}$ for $p > 1$ and $p' = \infty$ for $p = 1$. The duality pairing of X_p and $X_{p'}$ is given by

$$\langle u, v\rangle := \int_\Omega u(x)v(x) \, d\mu(x).$$

It is well-known that $(X, \tilde{X}, \leq)$ is a order-admissible pair of spaces for the ordering

$$f \leq g :\Leftrightarrow f(x) \leq g(x) \;\; \mu - \text{a.e. on } \; \Omega.$$

With regard to the terminology of the previous section we define

$$\begin{aligned}
\hat{X}_{p,\inf} &:= \left\{ g \in \tilde{X} \mid f \wedge g \in X_p \text{ for all } \; f \in X_p \right\}, \\
\hat{X}_{p,\sup} &:= \left\{ g \in \tilde{X} \mid f \vee g \in X_p \text{ for all } \; f \in X_p \right\}
\end{aligned}$$

for all $1 \leq p < \infty$.

Remark 3.4.42. *Let* $g \in \tilde{X}$*,* $g \geq 0$*. Then* $g \in \hat{X}_{p,\inf}$ *for all* $p \in [1, \infty)$*. In fact, for* $f \in X_p$ *one has*

$$|f \wedge g| = \mathbb{1}_{\{f \leq g\}} |f| + \mathbb{1}_{\{f \geq g\}} g \leq |f|.$$

Hence, $\|f \wedge g\|_p \leq \|f\|_p$*. Similarly, one shows that* $h \in \tilde{X}$*,* $h \leq 0$*, belongs to* $\hat{X}_{p,\sup}$ *for all* $p \in [1, \infty)$*.*

We ask for the invariance of order intervals under a strongly continuous semigroup T on X_p, $1 \leq p < \infty$, with generator $(A, D(A))$.

In particular, we call the semigroup T positive if $[-\infty, 0]$ (with the constant zero function) is invariant under T and *submarkovian* if $[-\infty, \mathbb{1}]$ is invariant under T. From Lemma 3.4.34 we see that a submarkovian semigroup is positive and L^∞-*contractive* in the sense that for every $t \geq 0$ and any $u \in L^p(\Omega) \cap L^\infty(\Omega)$ one has $\|T(t)u\|_{L^\infty} \leq \|u\|_{L^\infty}$.

Next we would like to apply the invariance theorems from the previous section. Therefore, let $1 \leq p < \infty$ and define for $f \in X_p$ the mulitplication operator $\operatorname{sign}(f) : X_p \to X_p$, $f \mapsto g_f f$, by

$$g_f(x) := \begin{cases} \frac{f(x)}{|f(x)|} & , \quad f(x) \neq 0 \\ 0 & , \quad f(x) = 0. \end{cases} \quad . \tag{3.3}$$

Then $\operatorname{sign}(f)$ is the signum operator of f in X_p. Furthermore, we point out that $f^0 = \mathbb{1}_{\{f \neq 0\}}$ for $f \in \tilde{X}$. In view of the duality mapping we have:

Lemma 3.4.43. *Let $p \in [1, \infty)$. For $0 \neq f \in X_p$ we define the function*

$$\hat{f} := \operatorname{sign}(f) \|f\|_p^{2-p} |f|^{p-1} \in X_{p'}.$$

Then $\hat{f} = J(f)$ for $1 < p < \infty$ and $\hat{f} \in J(f)$ for $p = 1$.

Proof. Let $1 < p < \infty$. Then

$$\left\|\hat{f}\right\|_{p'} = \left(\int_\Omega \left|\hat{f}\right|^{p'} \, d\mu\right)^{\frac{1}{p'}} = \|f\|_p^{2-p} \left(\int_\Omega |f|^{(p-1)p'} \, d\mu\right)^{\frac{1}{p'}} = \|f\|_p^{2-p} \|f\|_p^{\frac{p}{p'}} = \|f\|_p$$

and

$$\langle \hat{f}, f \rangle = \|f\|_p^{2-p} \int_\Omega f(x) \, |f(x)|^{p-2} \, f(x) \, d\mu = \|f\|_p^{2-p} \int_\Omega |f(x)|^p \, d\mu = \|f\|_p^2 .$$

Hence, $\hat{f} \in J(f)$. Since the dual space $X_{p'}$ is uniformly convex (cf. [Ist81, Chapter 2]), the duality mapping J is single valued due to Lemma 1.1.18 and we end up with $\hat{f} = J(f)$. Finally, for $p = 1$ we have

$$\left\|\hat{f}\right\|_\infty = \|f\|_1 \sup_{x \in \Omega} \left|\frac{f(x)}{f(x)}\right| = \|f\|_1$$

and

$$\langle \hat{f}, f \rangle = \|f\|_1 \int_\Omega f(x) \frac{f(x)}{|f(x)|} \, dx = \|f\|_1^2 .$$

Hence, $\tilde{f} \in J(f)$. ☐

Since the order intervals are convex, we may apply Proposition 3.4.12 and get from Lemma 3.4.33:

Corollary 3.4.44. *Let $p \in (1, \infty)$ and $f \in X_p$. Let $g \in \hat{X}_{p,\inf}$, $h \in \hat{X}_{p,\sup}$ such that $h \leq g$. Then one has*

1. $\|(f-g)^+\|_p^{2-p} \, ((f-g)^+)^{p-1} = J((f-g)^+) = J((f-g)^+) \cap N(f \wedge g, [-\infty, g])$,
2. $\|(f-g)^+ - (h-f)^+\|_p^{2-p} \, |(f-g)^+ - (h-f)^+|^{p-1} \, \operatorname{sign}((f-g)^+ - (h-f)^+) = J((f-g)^+ - (h-f)^+) = J((f-g)^+ - (h-f)^+) \cap N(h \vee (f \wedge g), [h, g])$.

Next we apply Proposition 3.4.36 to this situation and we obtain:

Proposition 3.4.45. *Let $1 < p < \infty$ and let $g \in \hat{X}_{p,\inf}$, $h \in \hat{X}_{p,\sup}$ such that $h \leq g$. Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on X_p with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$.*

1. *Assume that for all $f \in D(A)$ one has*

$$\int_\Omega \left((f-g)^+\right)^{p-1} \ (Af - \omega(f-g)) \ d\mu \leq 0.$$

Then $[-\infty, g]$ is invariant under T.

2. *Assume that for all $f \in D(A)$ one has*

$$\int_\Omega \left|(f-g)^+ - (h-f)^+\right|^{p-1} \operatorname{sign}((f-g)^+ - (h-f)^+) \ Af \ d\mu \leq \omega \ \left\|(f-g)^+ - (h-f)^+\right\|_p^p.$$

Then $[h, g]$ is invariant under T.

If, in addition, $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$, then the converse implications in the assertions (1) and (2) hold as well.

For the case $p = 1$ we have the following result:

Proposition 3.4.46. *Let $g \in \hat{X}_{1,\inf}$, $h \in \hat{X}_{1,\sup}$ such that $h \leq g$. Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on X_1 with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$.*

1. *Assume that for all $f \in D(A)$ one has*

$$\int_\Omega \mathbb{1}_{\{f\geq g\}} \left(Af - \omega(f-g)\right) \ d\mu \leq 0.$$

Then $[-\infty, g]$ is invariant under T.

2. *Assume that for all $f \in D(A)$ one has*

$$\int_\Omega (\mathbb{1}_{\{f\geq g\}} - \mathbb{1}_{\{f<h\}}) \ Af \ d\mu \leq \omega \ \left\|(f-g)^+ - (h-f)^+\right\|_1.$$

Then $[h, g]$ is invariant under T.

If, in addition, $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$, then the converse implications in the assertions (1) and (2) hold as well.

Proof. (1) For an arbitrary $f \in D(A)$ we consider the function $\hat{f} := \|(f-g)^+\|_1 \mathbb{1}_{\{f\geq g\}}$. From Lemma 3.4.43 we know $\hat{f} \in J((f-g)^+)$. Next we will show that $\hat{f} \in N(f \wedge g, [-\infty, g])$. Therefore, let $\varphi \in [-\infty, g]$. Then

$$\int_\Omega \hat{f}(\varphi - (f\wedge g)) \ d\mu = \left\|(f-g)^+\right\|_1 \int_\Omega \mathbb{1}_{\{f\geq g\}}(\varphi - g) \ d\mu \leq 0.$$

Hence, $\hat{f} \in N(f \wedge g, [-\infty, g])$. Now we have by assumption

$$\langle \hat{f}, (A-\omega)f\rangle \leq -\omega \ \langle \hat{f}, g\rangle$$

and so we obtain from Proposition 3.4.36 the invariance of $[-\infty, g]$ under T.

If, in addition, $\|T(t)\| \le e^{\omega t}$ for all $t \ge 0$ and $[-\infty, g]$ is invariant under T, then we know from Theorem 3.4.20 that A is strictly $([-\infty, g], \omega)$-dissipative. Since $\hat{f} \in J((f-g)^+) \cap N(f \wedge g, [-\infty, g])$ for $f \in D(A)$, this shows the converse implication.

(2) Now we consider for $f \in D(A)$ the function

$$\begin{aligned} \hat{f} &:= \left\|(f-g)^+ - (h-f)^+\right\|_1 \ \mathbb{1}_{\{f \ge g\} \cup \{f<h\}} \operatorname{sign}((f-g)^+ - (h-f)^+) \\ &= \left\|(f-g)^+ - (h-f)^+\right\|_1 \ \left(\mathbb{1}_{\{f \ge g\}} - \mathbb{1}_{\{f<h\}}\right) \in J((f-g)^+ - (h-f)^+). \end{aligned}$$

We show that $\hat{f} \in N(h \vee (f \wedge g), [h, g])$. Let $\varphi \in [h, g]$. Then one has

$$\begin{aligned} &\int_\Omega \hat{f} \ (\varphi - (h \vee (f \wedge g))) \ d\mu \\ = \ &\left\|(f-g)^+ - (h-f)^+\right\|_1 \left(\int_{\{f \ge g\}} (\varphi - g) \ d\mu + \int_{\{f \le h\}} (h - \varphi) \ d\mu \right) \\ \le \ &0. \end{aligned}$$

Hence, $\hat{f} \in J((f-g)^+ - (h-f)^+) \cap N(h \vee (f \wedge g), [h, g])$. In combination with Proposition 3.4.36 this shows one direction. For the converse direction we argue like in the proof of assertion (1). □

We come back to the description of positive and submarkovian semigroups in X_p.

Proposition 3.4.47. *Let $p \in [1, \infty)$ and let $T = (T(t))_{t \ge 0}$ be a strongly continuous semigroup on X_p with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$. Then the following assertions are equivalent:*

1. *$A - \omega$ is dispersive;*
2. *for all $f \in D(A)$ one has $\int_\Omega (f^+)^{p-1} \ (A - \omega) f \ d\mu \le 0$;*
3. *T is positive and fulfills $\|T(t)\| \le e^{\omega t}$ for all $t \ge 0$.*

Proof. The equivalence of (1) and (3) is shown in Theorem 3.4.25. Since $\|f^+\|^{2-p} (f^+)^{p-1} \in J(f^+)$ for $f \in D(A)$ (see Lemma 3.4.43), assertion (2) implies assertion (1). Finally, the implication "(3) ⇒ (2)" is shown in the Propositions 3.4.45 (for $p > 1$) and 3.4.46 (for $p = 1$). □

For the characterisation of submarkovian semigroups we have as an application of the Propositions 3.4.45 (for $p > 1$) and 3.4.46 (for $p = 1$).

Proposition 3.4.48. *Let $p \in [1, \infty)$ and let $T = (T(t))_{t \ge 0}$ be a strongly continuous semigroup on X_p with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$. We consider the assertions:*

1. *For all $f \in D(A)$ one has*

$$\int_\Omega \left((f - \mathbb{1})^+\right)^{p-1} \ ((A - \omega) f - \omega) \ d\mu \le 0.$$

2. *T is submarkovian, i.e. $[-\infty, \mathbb{1}]$ is invariant under T.*

Then (1) implies (2). If $\|T(t)\| \le e^{\omega t}$ for all $t \ge 0$, then (1) and (2) are equivalent.

Next we will discuss Kato-type inequalities for the invariance of order intervals. In particular, we would like to apply Corollary 3.4.41. It is well-known that $L^p(\Omega)$ is order complete (cp. [Sch71, p. 210]) for $1 \leq p < \infty$.

Proposition 3.4.49. *Let $p \in [1, \infty)$ and let $T = (T(t))_{t \geq 0}$ be a positive, strongly continuous semigroup on X_p with generator $(A, D(A))$. Let $g \in \hat{X}_{p,\inf}$. Then $[-\infty, g]$ is invariant under T if and only if*

$$\langle \phi, \mathbb{1}_{\{f \geq g\}} Af \rangle \leq \langle A'\phi, (f-g)^+ \rangle.$$

for all $f \in D(A)$ and all $\phi \in D(A') \subset X_{p'}$, $\phi \geq 0$, where A' is the adjoint of A.

Proof. In view of Corollary 3.4.41 we only have to show

$$\operatorname{sign}((f-g)^+)(\varphi - g) \leq 0, \quad f \in X_p, \varphi \in [-\infty, g].$$

Let $f \in X_p$ and $\varphi \in [-\infty, g]$. Then $\operatorname{sign}((f-g)^+) = \mathbb{1}_{\{f \geq g\}}$ and so $\operatorname{sign}((f-g)^+)(\varphi - g) = \mathbb{1}_{\{f \geq g\}}(\varphi - g) \leq 0$ a.e.. This finishes the proof. □

Without assuming positivity for the semigroup T we have to assume more conditions than the mere Kato-type inequality. At first we recall the following result which characterizes positive semigroups in L^p-spaces in terms of the Kato inequality. For a version in arbitrary real Banach lattice we refer to Theorem 3.4.40.

Theorem 3.4.50. *(Arendt) Let $p \in [1, \infty)$ and let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on X_p with generator $(A, D(A))$. Let A' be the adjoint of A. The semigroup T is positive if and only if its generator A fulfills the following two properties:*

1. *for all $f \in D(A)$ and all $\phi \in D(A')$, $\phi \geq 0$, one has*

$$\langle \phi, \operatorname{sign}(f) Af \rangle \leq \langle A'\phi, |f| \rangle;$$

2. *A' has a positive subeigenvector, i.e. there exists $\phi \in D(A')$, $\phi > 0$, and $\lambda \in \mathbb{R}$ such that $A'\phi \leq \lambda\phi$.*

For a proof we refer to [Nag86, Theorem C-II.3.8 and Corollary C-II.3.9]. Now we have in the spirit of [Ouh04, Theorem 3.7]:

Proposition 3.4.51. *Let $p \in [1, \infty)$ and let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on X_p with generator $(A, D(A))$. Let $g \in \tilde{X}$, $g \geq 0$. Then $[-\infty, g]$ is invariant under the semigroup T if and only if its generator A fulfills the following two properties:*

1. *for all $f \in D(A)$ and all $\phi \in D(A')$, $\phi \geq 0$, one has*

$$\langle \phi, \mathbb{1}_{\{f \geq g\}} Af \rangle \leq \langle A'\phi, (f-g)^+ \rangle.$$

2. *the adjoint A' has a positive subeigenvector, i.e. there exists $\phi \in D(A')$, $\phi > 0$, and $\lambda \in \mathbb{R}$ such that $A'\phi \leq \lambda\phi$.*

Proof. Note that $g \in \hat{X}_{p,\inf}$ due to Remark 3.4.42. Now assume $[-\infty, g]$ is invariant under T. Since $g \geq 0$, the semigroup T is positive (see Lemma 3.4.34) and the properties (1) and (2) follow from Proposition 3.4.49 and Theorem 3.4.50.

Next let us assume that the generator A fulfills the properties (1) and (2). We will show that (1) implies property (1) in Theorem 3.4.50. Then the semigroup would be automatically positive and the invariance of $[-\infty, g]$ under T follows directly from Proposition 3.4.49. Therefore, let $f \in D(A)$ and $\phi \in D(A')$, $\phi > 0$. For any $n \in \mathbb{N}$ is $nf \in D(A)$. The representation $(nf - g)^+ = n\left(f - \frac{g}{n}^+\right)^+$, $n \in \mathbb{N}$, gives in property (1)

$$n\langle \phi, \mathbb{1}_{\{f \geq \frac{g}{n}\}} Af \rangle \leq \langle A'\phi, (nf - g)^+ \rangle = n\langle A'\phi, \left(f - \frac{g^+}{n}\right)^+ \rangle$$

for all $n \in \mathbb{N}$. As $n \to \infty$ the functions $\left(f - \frac{g}{n}^+\right)^+$ converge to f^+ and $\mathbb{1}_{\{f \geq \frac{g}{n}\}}$ converge to $\mathbb{1}_{\{f \geq 0\}}$. It follows

$$\langle \phi, \mathbb{1}_{\{f \geq 0\}} Af \rangle \leq \langle A'\phi, f^+ \rangle.$$

Applying this inequality to $-f \in D(A)$ we get

$$-\langle \phi, \mathbb{1}_{\{f < 0\}} Af \rangle \leq \langle A'\phi, (-f)^+ \rangle = \langle A'\phi, f^- \rangle.$$

We finally combine these two inequalities and end up with

$$\langle \phi, \operatorname{sign}(f) Af \rangle = \langle \phi, \left(\mathbb{1}_{\{f \geq 0\}} - \mathbb{1}_{\{f < 0\}}\right) Af \rangle \leq \langle A'\phi, f^+ + f^- \rangle = \langle A'\phi, |f| \rangle.$$

The property (1) in Theorem 3.4.50 and the invariance of $[-\infty, g]$ follows on the way mentioned before. $\square$

It is remarkable that those characterisations for the invariance of order intervals relies not on the quasi-contractiveness of the semigroup T in contrast to the Propositions 3.4.45 and 3.4.46.

3.4.4.4 Order intervals in C_0

Let $\Omega \subset \mathbb{R}^n$ be an open set. In this section we consider the real Banach space

$$X := C_0(\Omega; \mathbb{R}) := \{f \in C(\Omega; \mathbb{R}) \mid \forall\, \varepsilon > 0\ \exists\, K \Subset \Omega : \ |f(x)| \leq \varepsilon \text{ for all } x \in \Omega \setminus K\}$$

equipped with the supremum norm $\|\cdot\|_\infty$ on Ω. As vector lattice $\tilde{X}$ we take $C(\Omega; \mathbb{R})$ with the canonical ordering

$$f \leq g :\Leftrightarrow f(x) \leq g(x)\ \ \forall\, x \in \Omega.$$

It is well-known that $(X, \tilde{X}, \leq)$ is an order-admissible pair of spaces. We recall the definition of the sets

$$\begin{aligned} \hat{X}_{\inf} &:= \left\{g \in \tilde{C}(\Omega) \mid f \wedge g \in C_0(\Omega) \text{ for all } f \in C_0(\Omega)\right\}, \\ \hat{X}_{\sup} &:= \left\{h \in \tilde{C}(\Omega) \mid f \vee h \in C_0(\Omega) \text{ for all } f \in C_0(\Omega)\right\}. \end{aligned}$$

and remark:

Remark 3.4.52. *Let $g \in \tilde{X}$, $g \geq 0$. Then $g \in \hat{X}_{\inf}$. In fact, for $f \in X$ one knows $f \wedge g \in C(\Omega)$ and $|f \wedge g| \leq |f|$ (see Remark 3.4.42). Hence, $f \wedge g \in X$. Similarly, one proves that $h \in \tilde{X}$, $h \leq 0$, belongs to $\hat{X}_{\sup}$.*

Like in the previous section we start with elements of the duality mapping. Here δ_x denotes the point measure in $x \in \Omega$.

Lemma 3.4.53. *Let $0 \neq f \in X$. Then*

$$\{f(x_0)\delta_{x_0} \mid x_0 \in \Omega,\ \|f\|_\infty = |f(x_0)|\} \subset J(f).$$

Proof. Let $x_0 \in \Omega$ such that $\|f\|_\infty = |f(x_0)|$. Then $\mu := f(x_0)\delta_{x_0} \in X'$ and

$$\langle \mu, f\rangle = \int_\Omega f(x)\ d\mu(x) = |f(x_0)|^2 = \|f\|_\infty^2 .$$

Furthermore, we have

$$\|\mu\| = \sup_{f \in X, \|f\| \leq 1} |\langle \mu, f\rangle| \leq |f(x_0)| = \|f\|_\infty$$

and, in particular, $\|\mu\| \geq \langle \mu, \frac{f}{\|f\|_\infty}\rangle = \frac{|f(x_0)|^2}{\|f\|_\infty^2} = \|f\|_\infty$. Hence, $\mu \in J(f)$. □

Coming to order intervals this means:

Lemma 3.4.54. *Let $0 \neq f \in X$ and $g \in \hat{X}_{\inf}$, $h \in \hat{X}_{\sup}$ such that $h \leq g$.*

1. *Let $x_0 \in \Omega$ such that $(f-g)^+(x_0) = \|(f-g)^+\|_\infty$. Then one has*

$$\mu_0 := (f-g)^+\delta_{x_0} \in J((f-g)^+) \cap N(f \wedge g, [-\infty, g]).$$

2. *Let $x_0 \in \Omega$ such that $|(f-g)^+(x_0) - (h-f)^+(x_0)| = \|(f-g)^+ - (h-f)^+\|_\infty$. Then one has*

$$\mu_1 := \left((f-g)^+(x_0) - (h-f)^+(x_0)\right)\delta_{x_0} \in J((f-g)^+(x_0) - (h-f)^+) \cap N(h \vee (f \wedge g), [h, g]).$$

Proof. For (1) we know from Lemma 3.4.53 that $\mu_0 \in J((f-g)^+)$. It remains to prove $\mu_0 \in N(f \wedge g, [-\infty, g])$. Let $\varphi \in [-\infty, g]$. We have

$$\langle \mu_0, \varphi - (f \wedge g)\rangle = (f-g)^+(x_0) - (\varphi(x_0) - (f(x_0) \wedge g(x_0))) = (f-g)^+(x_0) - (\varphi(x_0) - g(x_0)) \leq 0.$$

Hence, $\mu_0 \in N(f \wedge g, [-\infty, g])$. Assertion (2) can be shown in a similar way. □

Now we can apply Proposition 3.4.36. We use the function $\operatorname{sign}(f)$ for $f \in X$, which is defined by

$$\operatorname{sign}(f)(x) := \begin{cases} \frac{f(x)}{|f(x)|} & , \quad f(x) \neq 0 \\ 0 & , \quad f(x) = 0 \end{cases} .$$

Proposition 3.4.55. *Let $T = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$. Let $\omega \in \mathbb{R}$.*

1. *Assume that for all $f \in D(A) \setminus [-\infty, g]$ one has*

$$(Af)(x_0) \leq \omega\ (f-g)(x_0),$$

for some $x_0 \in \Omega$ such that $(f-g)(x_0) = \|(f-g)^+(x_0)\|$. Then $[-\infty, g]$ is invariant under T.

2. *Assume that for all $f \in D(A)$ one has*

$$\operatorname{sign}((f-g)^+ - (h-f)^+)(x_0)\ (Af)(x_0) \leq \omega\ \left|(f-g)^+(x_0) - (h-f)^+(x_0)\right|$$

for some $x_0 \in \Omega$ such that $|(f-g)^+(x_0) - (h-f)^+(x_0)| = \|(f-g)^+ - (h-f)^+\|_\infty$. Then $[h, g]$ is invariant under T.

If, in addition, $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$, then the converse implications in the assertions (1) and (2) hold as well.

Of special interest is the invariance of the following intervals under the semigroup T:

- $[-\infty, 0]$, i.e. T is positive;
- $[-\infty, \mathbb{1}]$, i.e. T is contractive;

From Lemma 3.4.34 we see that a contractive semigroup T on X is automatically positive.

At first we discuss the positivity of the semigroup T. An operator B on X with domain $D(B)$ is said to satisfy the *positive maximum principle* if, whenever $f \in D(B)$ and there exists $x_0 \in \Omega$ such that $f(x_0) = \sup_{x\in\Omega} f(x) \geq 0$ we have $(Bf)(x_0) \leq 0$. It is well-known that this notion is connected to the positivity of the semigroup and we obtain this result as a corollary to assertion (1) of Proposition 3.4.55.

Proposition 3.4.56. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$. If A satifies the positive maximum principle, then T is positive and contractive.*

For contractive semigroups we have the following characterisations as a corollary of assertion (1) in Proposition 3.4.55.

Proposition 3.4.57. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$. The semigroup T is contractive if and only if for all $x \in D(A) \setminus [-\infty, \mathbb{1}]$ one has $(Af)(x_0) \leq 0$ where $x_0 \in \Omega$ fulfills $(f-\mathbb{1})^+(x_0) = \|(f-\mathbb{1})^+\|_\infty$.*

This finishes our treatment of order intervals in $C_0(\Omega)$.

Chapter 4

An extension of Ouhabaz' invariance criterion for C_0-semigroups

4.1 Introduction

As in chapter 3 we want to discuss invariance criterions for closed, convex sets under strongly continuous semigroups. Here we focus on semigroups acting on Hilbert spaces, but in difference to chapter 3 we will assume the quite common situation that the semigroup is associated to a form. This means, its generator comes from an elliptic, densely defined, continuous sesquilinear form $(\mathfrak{a}, V)$. In this setting it is of interest to find (equivalent) conditions on the form guaranteeing the invariance of a closed, convex set C under the semigroup.

This matter has been discussed frequently in the literature culminating in the following theorem of El Maati Ouhabaz (cf. [Ouh04, Theorem 2.2]):

Theorem (Ouhabaz)**.** Let H be a Hilbert space and $(\mathfrak{a}, V)$ be an elliptic, densely defined, continuous, accretive sesquilinear form on H. Let $T = (T(t))_{t\geq 0}$ be the bounded holomorphic C_0-semigroup on H associated to $(\mathfrak{a}, V)$. Let C be a closed, convex subset of H and P be the orthogonal projection of H onto C. Then the following assertions are equivalent:

1. C is invariant under T;
2. for all $u \in V$ is $Pu \in V$ and $\operatorname{Re} \mathfrak{a}[u, u - Pu] \geq 0$;
3. for all $u \in V$ is $Pu \in V$ and $\operatorname{Re} \mathfrak{a}[Pu, u - Pu] \geq 0$.

The impact of this theorem is enormous. For instance, one can deduce the famous Beurling-Deny criteria characterising positive and submarkovian semigroups solely in conditions on the form. In addition, an appropriate choice of a convex set leads to conditions for irreducible semigroups or domination of semigroups, always stated in terms of the form, which are most often easier to check. For the details we refer to the monograph of Ouhabaz (cf. [Ouh04]).

In this chapter we would like to extend Ouhabaz' Theorem to a more general setting. In particular, we want to get rid of the accretivity assumption. It will turn out that the accretivity assumption on the form $(\mathfrak{a}, V)$ can be replaced by the existence of a fixed point of the semigroup T in the invariant subset C. Since any contraction semigroup T has a fixed point in a invariant closed, convex subset C, this, in fact, generalises the Theorem of Ouhabaz.

We will proceed as follows: In section 4.2 we introduce the set $F(T)$ of common fixed points of a strongly continuous semigroup T on a Banach space. For their sheer beauty we state and prove two Theorems of Suzuki and Bruck, respectively, describing the set $F(T)$ in a surprisingly simple manner. Regarding the proposed extension of Ouhabaz' invariance criterion we show at the end of the section that the existence of a fixed point in an invariant subset C for one member $T(t_0)$ already implies the existence of a common fixed point in C for the whole semigroup T. That given, we can extend the known fixed point theorems of Schauer and Browder for single operators to our semigroup situation. In section 4.3 we concentrate on the proof of the extended invariance criterion. Using characterisations of section 3.4.3 we show that the implications "(2) $\Rightarrow$ (1)" and "(3) $\Rightarrow$ (1)" in Ouhabaz' Theorem are valid without assuming accretivity of the form $(\mathfrak{a}, V)$. Meanwhile the implication "(1) $\Rightarrow$ (2)" can only hold for contractive semigroups in a general setting, since the set $\{0\}$ is invariant under the semigroup. Therefore, we focus on the implication "(1) $\Rightarrow$ (3)". Here the crucial point is the invariance of the form domain under the orthogonal projection. We state Ouhabaz' proof of this condition for accretive forms and extend it step by step to our fixed point argument. Thanks to a result of Browder this generalises Ouhabaz' theorem. Other embedded interesting cases are generators with compact resolvent or when the set C is a neighbourhood of the origin. In particular, we can now describe invariance conditions for a large class of order intervals, which is done in section 4.4, where we also recover the famous Beurling-Deny criteria.

4.2 Common fixed points of C_0-semigroups and existence results

This section is devoted to a fixed point theory for strongly continuous semigroups $T = (T(t))_{t\geq 0}$ in Banach spaces. We are interested in answers to the following questions:

1. How can we describe the set of common fixed points for a semigroup T?
2. Let C be a subset of the Banach space, which is invariant under the semigroup T. Does T have a fixed point in C?

The first topic has been studied to a wide extent. Here we will state two results (and proofs) of Suzuki and Bruck characterising the set of common fixed points of T. Suzuki has shown that this set equals the intersection of the fixed point sets of $T(\alpha)$ and $T(\beta)$ where α, β are arbitrary positive numbers whose quotient is not a rational number. While this result is valid in general, Bruck concentrates on contractive semigroups in strictly convex Banach spaces. Then any fixed point of a convex combination of the operators $T(\alpha)$ and $T(\beta)$ is already a common fixed point for the semigroup. At the end of the section we show with regard to the second topic that the existence of a fixed point in an invariant subset C for one member $T(t_0)$ of the semigroup already implies the existence of a common fixed point in C for the whole semigroup T.

Let X be a Banach space. For any operator $B : X \to X$ we denote by

$$F(B) := \{x \in X \mid Bx = x\}$$

the *set of all fixed points of* B.

We will always consider a strongly continuous semigroup T on X with generator $(A, D(A))$. A point $x \in F(T) := \bigcap_{t>0} F(T(t))$ is called a *common fixed point for the semigroup* T. From the semigroup law we see immediately that $x \in X$ is a common fixed point for T if and only if x is a common fixed point for the family $\{T(t) \mid t \in (0, 1]\}$.

We start with the first topic, a characterisation of $F(T)$. At first we show the following interesting characterisation of $F(T)$ which is due to Suzuki (cf. [Suz05]):

Proposition 4.2.1 (Suzuki). *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Let $\alpha, \beta > 0$ such that $\frac{\alpha}{\beta} \notin \mathbb{Q}$. Then:*

$$F(T) = F(T(\alpha)) \cap F(T(\beta)).$$

Proof. Let $x \in F(T(\alpha)) \cap F(T(\beta))$. In a first step, we define the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ inductively by

- $\alpha_1 := \max\{\alpha, \beta\}$, $\alpha_2 := \min\{\alpha, \beta\}$;
- $k_n := [\frac{\alpha_n}{\alpha_{n+1}}]$, $n \in \mathbb{N}$;
- $\alpha_{n+2} := \alpha_n - k_n\alpha_{n+1}$, $n \in \mathbb{N}$.

Here, we denote by $[z]$ the maximum integer not exceeding the real number z. We claim $T(\alpha_n)x = x$ for all $n \in \mathbb{N}$. For the proof we proceed inductively. For $n = 1, 2$ the claim follows from our assumption. Let $n > 2$ such that the claim holds for $n-1$ and $n-2$. It follows

$$T(\alpha_n)x = T(\alpha_n)T(\alpha_{n-1})^{k_{n-2}}x = T(\alpha_n + k_{n-2}\alpha_{n-1})x = T(\alpha_{n-2})x = x.$$

Thus, the claim is proved.

Next we will prove:

$$\forall\, n \in \mathbb{N} :\ \ 0 < \alpha_{n+1} < \alpha_n,\ \ \frac{\alpha_n}{\alpha_{n+1}} \notin \mathbb{Q},\ \ k_n \in \mathbb{N}. \tag{4.1}$$

We use induction on $n \in \mathbb{N}$. For $n = 1$ the claim is true due to the assumption $\frac{\alpha}{\beta} \notin \mathbb{Q}$. Assume that the claim holds for some $n \in \mathbb{N}$. We put $\varepsilon_n := \frac{\alpha_n}{\alpha_{n+1}} - k_n$. By assumption, $\varepsilon_n \in (0, 1) \setminus \mathbb{Q}$. We obtain

$$\frac{\alpha_{n+2}}{\alpha_{n+1}} = \frac{\alpha_n}{\alpha_{n+1}} - k_n = \varepsilon_n \notin \mathbb{Q}.$$

It follows $\frac{\alpha_{n+1}}{\alpha_{n+2}} \notin \mathbb{Q}$ and $\alpha_{n+2} = \varepsilon_n\alpha_{n+1} \leq \alpha_{n+1}$. This shows the claim.

In particular, $\alpha_n \searrow 0$ as $n \to \infty$. In fact, the claim implies $\alpha_n \searrow a$ for some $a \geq 0$. Assume $a > 0$. Then there exists $n \in \mathbb{N}$ such that $a < \alpha_{n+1} < \alpha_n < 2a$. Hence, $k_n = 1$ and so $\alpha_{n+2} = \alpha_n - \alpha_{n+2} < 2a - a = a$, which is a contradicition. Thus, $a = 0$.

Now let $t > 0$. We claim that there exists a sequence $(c_n)_{n\in\mathbb{N}} \subset \mathbb{N} \cup \{0\}$ such that $\sum_{n=1}^{\infty} c_n\alpha_n = t$. Therefore, we define the sequences $(\delta_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ by

- $\delta_1 := t$,
- $c_n := \left[\frac{\delta_n}{\alpha_n}\right]$, $n \in \mathbb{N}$;
- $\delta_{n+1} := \delta_n - c_n\alpha_n$, $n \in \mathbb{N}$.

We put $b_n := \frac{\delta_n}{\alpha_n} - c_n \in [0, 1)$ for all $n \in \mathbb{N}$. Then $\delta_{n+1} = \delta_n - c_n\alpha_n = b_n\alpha_n \in [0, \alpha_n)$ for $n \in \mathbb{N}$. Hence, $\delta_n \searrow 0$ as $n \to \infty$. Next we will prove

$$\forall\, n \in \mathbb{N} :\ \ t = \sum_{j=1}^{n} c_j\alpha_j + \delta_{n+1}.$$

For $n = 1$ this follows from the definition of the sequences. Assume that the claim is true for some $n \in \mathbb{N}$. It follows

$$\sum_{j=1}^{n+1} c_j\alpha_j + \delta_{n+2} = \sum_{j=1}^{n+1} c_j\alpha_j + \delta_{n+1} - c_{n+1}\alpha_{n+1} = \sum_{j=1}^{n} c_j\alpha_j + \delta_{n+1} = t.$$

Thus, the claim is obtained by induction. We finally have that

$$\sum_{j=1}^{\infty} c_j\alpha_j = \lim_{n\to\infty}\left(\sum_{j=1}^{n} c_j\alpha_j + \delta_{n+1}\right) = t.$$

From the continuity of the semigroup we now obtain

$$T(t)z = \lim_{n\to\infty} T(\sum_{j=1}^{n} c_j\alpha_j)z = \lim_{n\to\infty} T(\alpha_1)^{c_1}\cdots T(\alpha_n)^{c_n}z = z.$$

This finally proves $z \in F(T)$. □

It is natural to ask for generalisations of this result. The following one is a combination of Proposition 4.2.1 and a result of Bruck (cf. [Bru73]):

Proposition 4.2.2. *Let $T = (T(t))_{t\geq 0}$ be a contractive C_0-semigroup on a strictly convex Banach space X with generator $(A, D(A))$. Let $\alpha, \beta > 0$ such that $\frac{\alpha}{\beta} \notin \mathbb{Q}$ and let $\lambda \in (0,1)$. Then one has*

$$F(T) = F(\lambda\ T(\alpha) + (1-\lambda)\ T(\beta)).$$

Proof. We put $Sx := \lambda T(\alpha)x + (1-\lambda)T(\beta)x$, $x \in X$. Then S is a contraction and

$$F(T) = F(T(\alpha)) \cap F(T(\beta)) \subset F(S).$$

For the converse implication, let $x \in F(S)$. It follows

$$\begin{aligned}\|x\| &= \|\lambda T(\alpha)x + (1-\lambda)T(\beta)x\| \\ &\leq \lambda\|T(\alpha)x\| + (1-\lambda)\|T(\beta)x\| \\ &\leq \lambda\|x\| + (1-\lambda)\|x\| = \|x\|.\end{aligned}$$

Hence, $\|x\| = \lambda\|T(\alpha)x\| + (1-\lambda)\|T(\beta)x\|$. Now assume $\|T(\alpha)x\| < \|x\|$. Then we have $\|x\| < \lambda\|x\| + (1-\lambda)\|x\| = \|x\|$, which is a contradiction. Thus, $\|T(\alpha)x\| = \|x\|$ and, similarly, $\|T(\beta)x\| = \|x\|$. Therefore, we have

$$\|T(\alpha)x\| = \|T(\beta)x\| = \|\lambda T(\alpha)x + (1-\lambda)T(\beta)x\|.$$

Now the strict convexity of X implies $T(\alpha)x = T(\beta)x$ (see Proposition 1.1.3). It follows

$$x = Sx = \lambda T(\alpha)x + (1-\lambda)T(\beta)x = T(\alpha)x = T(\beta)x,$$

and so $x \in F(T(\alpha)) \cap F(T(\beta)) = F(T)$. □

As mentioned, all of the stated results so far are known in the literature and they are not of greater value in our approach to provide an extension of Ouhabaz' invariance criterion.

Nevertheless they naturally fall into the framework of fixed point theory for semigroups.

Next we will turn our attention to the second task, i.e. the existence of common fixed points in invariant subsets. We start with some simple oberservations:

Lemma 4.2.3. *Let T be a C_0-semigroup on a Banach space X with generator $(A, D(A))$. Let $C \subseteq X$ be invariant under T. Then:*

$$F(T) \cap C = \operatorname{Ker}(A) \cap C$$

Moreover,

$$A\left(\int_0^t T(s)x \, ds\right) + x \in C, \quad x \in C, t \geq 0.$$

Proof. The implication "$\subseteq$" is clear. For the converse direction let $x \in \operatorname{Ker}(A) \cap C$. Let $\lambda > 0$ such that $\lambda \in \rho(A)$. Then one has $(\lambda - A)x = \lambda x$ and so $\lambda R(\lambda, A)x = x \in C$. It follows from Lemma 3.3.12 that $x \in C \cap F(T)$.

Now let $x \in C$, $t \geq 0$. Then

$$A\left(\int_0^t T(s)x \, ds\right) + x = T(t)x - x + x = T(t)x \in C.$$

This finishes the proof. $\square$

Thus, the set of common fixed points can be described in terms of the generator. We will use this fact and classical semigroup theory to show the following existence result. Here, we denote by $\overline{\operatorname{co}}(C)$ the closed, convex hull of a subset $C \subseteq X$.

Proposition 4.2.4. *Let X be a Banach space and $T = (T(t))_{t \geq 0}$ be a C_0-semigroup on X with generator $(A, D(A))$. Let C be a subset of X which is invariant under T. For $t > 0$ we define*

$$S(t)x := \frac{1}{t} \int_0^t T(s)x \, ds \in D(A), \quad x \in X.$$

Then one has:

1. *$S(t)C \subset \overline{\operatorname{co}}(C)$ for all $t > 0$;*
2. *if $x \in C \cap F(T(t_0))$ for some $t_0 > 0$, then $S(t_0)x$ is a fixed point of T in $\overline{\operatorname{co}}(C)$.*

Proof. (1) Let $x \in C$ and $t > 0$. For $s \in [0, t]$ we define $u(s) := T(s)x$. Note that $u \in C([0, t], C)$. Let π be a partition $0 = t_0 < t_1 < \ldots < t_n = t$ of $[0, t]$ with intermediate points $s_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$. By $|\pi| := \max_{i=1,\ldots,n}(t_i - t_{i-1})$ we denote the norm of π and by

$$S(\pi, u) := \sum_{i=1}^n (t_i - t_{i-1}) u(s_i)$$

the Riemann sum of u with respect to π. Since $\sum_{i=1}^n \frac{t_i - t_{i-1}}{t} = \frac{1}{t}(t_n - t_0) = 1$, we obtain $\frac{1}{t} S(\pi, u) \in \operatorname{co}(C)$. It finally follows

$$S(t)x = \frac{1}{t} \int_0^t u(s) \, ds = \lim_{|\pi| \to 0} \frac{1}{t} S(\pi, u) \in \overline{\operatorname{co}}(C).$$

(2) Let $x \in C \cap F(T(t_0))$ for some $t_0 > 0$. Then $S(t_0)x \in D(A)$ and

$$t_0\ AS(t_0)x = A \int_0^{t_0} T(s)x\ ds = T(t_0)x - x = 0.$$

Thus, $S(t_0)x \in \operatorname{Ker}(A) = \bigcap_{t\geq 0} F(T(t))$. Moreover, $S(t_0)x \in \overline{\text{co}}(C)$ due to (a). □

Remark 4.2.5. *Assertion (a) can also be proven in a different way: Assume $S(t)x_0 \notin D := \overline{\text{co}}(C)$ for some $x_0 \in C$. Thanks to the Hahn-Banach-Theorem there exists a functional $x' \in X'$ and $\alpha \in \mathbb{R}$ such that*

$$\text{Re}\ \langle x', S(t)x_0\rangle > \alpha > \text{Re}\ \langle x', x\rangle$$

for all $x \in D$. Since $T(s)x_0 \in D$ for all $s \in [0,t]$, it follows the contradiction

$$\alpha \geq \text{Re}\ \frac{1}{t}\int_0^t \langle x', T(s)x_0\rangle\ ds = \text{Re}\ \langle x', S(t)x_0\rangle > \alpha.$$

Hence, $S(t)C \subseteq D$.

The consequences of Proposition 4.2.4 are surprising: The existence of a common fixed point for the semigroup $(T(t))_{t\geq 0}$ can be reduced to that of a single member $T(t_0)$. In particular, all known existence results from the fixed point theory of single operators easily carry over to the semigroup case:

Proposition 4.2.6. *Let X be a Banach space, $C \subseteq X$ be a non-empty, closed, convex set and $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on X leaving C invariant. In either one of the following cases, T has a fixed point in C:*

1. *C is compact;*
2. *T is eventually compact;*
3. *T is immediately norm continuous and its generator has compact resolvent;*
4. *X is uniformly convex and T is contractive.*

Proof. We cite the relevant fixed point theorems for single operators: (1) is due to Schauder (cf. [Ist81, Theorem 5.1.2 and 5.1.3]), (2) is the generalized Schauder Theorem (cf. [DG03, Theorem 7.9]) and (3) is a corollary of (2) thanks to [EN00, Theorem II.4.29]. For (4) Browder has shown that the contraction $T(t)$ for an arbitrary $t > 0$ has a fixed point in C, if C is a bounded, closed, convex subset of a uniformly convex Banach space X (cf. [Bro65]). We remark that Browder focused on contractive mappings on C. In our situation, we assume contractivity for the semigroup on the whole space X and are thus able to dismiss the assumption of boundedness. In fact, there exists $n \in \mathbb{N}$ such that $C_n := \{x \in C \mid \|x\| \leq n\}$ is non-empty. The set C_n is bounded, closed and convex. We show that $T(t)$ leaves C_n invariant. Let $x \in C_n$. Then $T(t)x \in C$ and $\|T(t)x\| \leq \|x\| \leq n$, i.e. $T(t)x \in C_n$. Thanks to Browder's result $T(t)$ has a fixed point in C_n and thus in C. □

In the next section we will use these results in order to extend the famous invariance criterion of El Maati Ouhabaz.

4.3 An extension of Ouhabaz' invariance criterion

In this section we concentrate on invariance conditions for subsets of a Hilbert space under strongly continuous semigroups associated to sesquilinear forms. We extend Ouhabaz' invariance criterion to non-contractive semigroups. Our method takes its inspiration in the fixed point theory for semigroups. We show that Ouhabaz' criterion is still valid if one replaces the contractivity assumption by the existence of a common fixed point of the semigroup in the invariant subset. Thanks to Proposition 4.2.6 this extends Ouhabaz' theorem.

Let H be a Hilbert space. Let $(\mathfrak{a}, V)$ be an elliptic, densely defined, continuous sesquilinear form on X with associated operator $(A, D(A))$. In particular, there exist constants $\omega_0 \in \mathbb{R}$, $\alpha_0 > 0$ such that

$$\text{Re } \mathfrak{a}[u,u] + \omega_0 \|u\|_H^2 \geq \alpha_0 \|u\|_V^2, \quad u \in V. \tag{4.2}$$

It is well-known that $-A$ generates a bounded holomorphic C_0-semigroup T on H known as the associated semigroup of $\mathfrak{a}$.

We show that T is quasi-contractive: The form $(\mathfrak{a} + \omega_0, V)$, which is defined by

$$(\mathfrak{a} + \omega_0)[u,v] := \mathfrak{a}[u,v] + \omega_0 \, (u|v)_H \;, \quad u, v \in V,$$

is accretive and the associated operator $-A - \omega_0$ generates the contractive C_0-semigroup $S(t) := e^{-\omega_0 t} T(t)$, $t \geq 0$ (cp. [Are06, Proposition 7.3.2]). Hence, $\|T(t)\| \leq e^{\omega_0 t}$ for all $t \geq 0$..

We obtain as a corollary from Theorem 3.4.30:

Proposition 4.3.1. *Let H be a Hilbert space, C be a closed, convex subset and let P be the orthogonal projection of H onto C. Let T be the semigroup associated to the elliptic, densely defined, continuous sesquilinear form $(\mathfrak{a}, V)$ on H and assume (4.2). Then the following assertions are equivalent:*

1. *C is invariant under T;*
2. *$\lambda R(\lambda, -A) C \subseteq C$ for all sufficiently large real $\lambda > \max\{\omega_0, 0\}$;*
3. *$\text{Re } (Au|u - Pu) \geq -\omega_0 \|u - Pu\|^2$ for all $u \in D(A)$.*

It is of interest to find conditions in terms of the form. We start our discussion with two sufficient conditions inspired by Ouhabaz' invariance criterion.

Proposition 4.3.2. *Let H be a Hilbert space, C be a closed, convex subset and let P be the orthogonal projection of H onto C. Let T be the semigroup associated to the elliptic, densely defined, continuous sesquilinear form $(\mathfrak{a}, V)$ on H and assume (4.2). Whenever one of the following conditions is fulfilled,*

1. *$Pu \in V$ and $\text{Re } \mathfrak{a}[Pu, u - Pu] \geq 0$ for all $u \in V$;*
2. *$Pu \in V$ and $\text{Re } \mathfrak{a}[u, u - Pu] \geq 0$ for all $u \in V$.*

then C is invariant under T.

Proof. Assume (1) is fulfilled. Let $u \in D(A) \subseteq V$. Then $Pu \in V$ and hence

$$\begin{aligned}\text{Re } (Au|u - Pu) + \omega_0 \|u - Pu\|^2 &= \text{Re } \mathfrak{a}[u - Pu + Pu, u - Pu] + \omega_0 \|u - Pu\|^2 \\ &\geq \alpha_0 \|u - Pu\|^2 + \text{Re } \mathfrak{a}[Pu, u - Pu] \geq 0.\end{aligned}$$

Due to Proposition 4.3.1 this shows the invariance of C under T.

Now assume (2) is fulfilled. Due to Proposition 4.3.1 it is sufficient to prove $\lambda R(\lambda, -A)C \subseteq C$ for all $\lambda > \max\{\omega_0, 0\}$. Let $\lambda > \max\{\omega_0, 0\}$, $f \in C$ and put $u := \lambda R(\lambda, -A)f \in D(A)$. Then $u, Pu \in V$. Moreover, one has $(\lambda + A)u = \lambda f$ and it follows

$$\begin{aligned}0 &= \text{Re } ((\lambda + A)u - \lambda f|u - Pu) \\ &= \text{Re } (Au|u - Pu) + \lambda \text{ Re } (u - Pu + Pu - f|u - Pu) \\ &= \text{Re } \mathfrak{a}[u, u - Pu] + \lambda \|u - Pu\|^2 + \lambda \text{ Re } (Pu - f|u - Pu) \\ &\geq \lambda \|u - Pu\|^2.\end{aligned}$$

Hence, $u = Pu \in C$. □

Remark 4.3.3. *If $(\mathfrak{a}, V)$ is accretive, then Ouhabaz (cf. [Ouh04, Theorem 2.2]) has shown that (1) and (2) of Proposition 4.3.2 are both equivalent to the invariance of C under T.*

It is natural to ask if the converse is true as well, i.e. if the invariance of C under T implies the assertions (1) and (2). Here, it is obvious that for the implication of (2) the form $(\mathfrak{a}, V)$ has to be accretive. For, $\{0\}$ is a closed, convex subset which is always invariant under T and so assertion (2) reads Re $\mathfrak{a}[u, u] \geq 0$ for all $u \in V$, i.e. $(\mathfrak{a}, V)$ is accretive.

However, for the implication of (1) we can weaken the assumption of accretivity in terms of the existence of a fixed point for the semigroup in C:

Theorem 4.3.4. *Let H be a Hilbert space, C a closed, convex subset and P be the orthogonal projection of H onto C. Let $T = (T(t))_{t\geq 0}$ be the semigroup associated to the elliptic, densely defined, continuous sesquilinear form $(\mathfrak{a}, V)$ on H and assume (4.2). Assume further that C is invariant under T.*

1. *If $PV \subseteq V$, then* Re $\mathfrak{a}[Pu, u - Pu] \geq 0$.
2. *If T has a fixed point in C, then $Pu \in V$ and thus* Re $\mathfrak{a}[Pu, u - Pu] \geq 0$ *for all $u \in V$.*

Proof. (1) Let us assume that $PV \subseteq V$ and let $u \in V$. Due to the representation formula (cp. Section 1.2), the invariance of C under $\lambda R(\lambda, -A)$ for sufficiently large real $\lambda > 0$ (cp. Theorem 3.4.30) and the property of the orthogonal projection we have

$$\text{Re } \mathfrak{a}[Pu, u - Pu] = \lim_{\lambda \to \infty} \lambda \left(Pu - \lambda R(\lambda, -A)Pu|u - Pu\right)_H \geq 0.$$

This shows assertion (1).

For the proof of assertion (2) we will proceed stepwise.

Step 1: Assume $(\mathfrak{a}, V)$ is accretive. Then $P(V) \subset V$.

We follow the notes of Ouhabaz (cp. [Ouh04, Theorem 2.2]). Let $u \in V$. We show that $Pu \in V$. Since $(\mathfrak{a}, V)$ is accretive, we may assume $\|v\|_V^2 = \|v\|_{\mathfrak{a}}^2 = \text{Re } \mathfrak{a}[v,v] + \|v\|_H^2$ for all $v \in V$. Due to the continuity of the form, there exists a constant $C > 0$ such that

$$|\mathfrak{a}[u,v]| \leq C \|u\|_V \|v\|_V , \quad u, v \in V.$$

Let $\lambda > 0$. Since T is contractive, one has $\lambda \in \rho(-A)$. We put $v := \lambda R(\lambda, -A) Pu \in D(A) \subset V$. Then $\lambda(Pu - v) = Av$ and $v \in C$, because C is invariant under $\lambda R(\lambda, -A)$ (cp. Theorem 3.4.30). Moreover, $\|v\|_H \leq \|Pu\|_H$, since T and thus $\lambda R(\lambda, -A)$ is contractive. Note that v depends on λ. The crucial step is now to provide a bound for v in V which is independent from λ so that the family $(\mu R(\mu, -A) u)_{\mu > 0}$ is uniformly bounded in V. Therefore, we make the following estimates:

$$\begin{aligned}
\text{Re } \mathfrak{a}[v,v] &= \text{Re } (Av|v)_H \\
&= \lambda \text{ Re } (Pu - v|v)_H \\
&= -\lambda \ \|Pu - v\|_H^2 + \lambda \text{ Re } (Pu - v|Pu)_H \\
&\leq \lambda \text{ Re } (Pu - v|Pu)_H \\
&= \lambda \text{ Re } (Pu - v|Pu - u)_H + \lambda \text{ Re } (Pu - v|u)_H \\
&\leq \lambda \text{ Re } (Pu - v|u)_H \\
&= \text{Re } (Av|u)_H \\
&= \text{Re } \mathfrak{a}[v,u] \\
&\leq C \|v\|_V \|u\|_V \\
&\leq \frac{1}{2} \left(\|v\|_V^2 + C^2 \|u\|_V^2 \right) \\
&= \frac{1}{2} \left(\|v\|_{\mathfrak{a}}^2 + C^2 \|u\|_V^2 \right) \\
&= \frac{1}{2} \text{ Re } \mathfrak{a}[v,v] + \frac{1}{2} \left(\|v\|_H^2 + C^2 \|u\|_V^2 \right)
\end{aligned}$$

It follows

$$\begin{aligned}
\|v\|_V^2 &= \|v\|_{\mathfrak{a}}^2 \\
&= \text{Re } \mathfrak{a}[v,v] + \|v\|_H^2 \\
&\leq 2 \|v\|_H^2 + C^2 \|u\|_V^2 \\
&\leq 2 \|Pu\|_H^2 + C^2 \|u\|_V^2 .
\end{aligned}$$

We have indeed obtained a bound for v in V which is independent from λ. Since $\lambda > 0$ was arbitrarily chosen, the sequence $(nR(n, -A)Pu)_{n \in \mathbb{N}}$ is bounded in V. Moreover, $nR(n, -A)Pu \to Pu$ in H as $n \to \infty$. This implies $Pu \in V$.

Step 2: Assume that $0 \in C$. Then $P(V) \subset V$.

Let $0 \in C$. If $\omega_0 \leq 0$, then $(\mathfrak{a}, V)$ is accretive and we get from step 1 that $P(V) \subset V$. Let $\omega_0 \geq 0$. Then $e^{-\omega_0 t} C \subset C$ for all $t \geq 0$ thanks to Corollary 3.3.21. It follows that the semigroup $S(t) := e^{-\omega_0 t} T(t)$, $t \geq 0$, associated to the accretive form $(\mathfrak{a} + \omega_0, V)$, leaves C invariant. Hence, $P(V) \subset V$ due to step 1.

Step 3: Assume that T has a fixed point in C. Then $P(V) \subset V$.

Let $z \in C$ be the fixed point of T in C. In particular, $z \in D(A) \subset V$ (see Lemma 4.2.3).

We define the subset
$$D := \{u \in H \mid u + z \in C\}.$$
Note that $0 \in D$. Furthermore, one easily sees that D is closed and convex. The orthogonal projection of H onto D is given by

$$P_D u := P(u+z) - z, \quad u \in H.$$

For, let $v \in H$. Then $P_D(v) + z = P(v+z) \in C$, i.e. $P_D(v) \in D$, and for $u \in D$ one has

$$\mathrm{Re}\,(v - P_D v | u - P_D v) = \mathrm{Re}\,(v + z - P(v+z) | u + z - P(v+z)) \leq 0.$$

We claim that D is invariant under T. Therefore, let $t \geq 0$ and $u \in D$. Then

$$T(t)u + z = T(t)u + T(t)z = T(t)(u+z) \in C$$

and so $T(t)u \in D$, i.e. D is invariant under T.

From step 2 we now obtain $P_D(V) \subset V$. Since $z \in V$ this implies $P(u+z) = P_D(u) + z \in V$ for all $u \in V$. This shows $P(V) \subset V$. □

Remark 4.3.5. *The existence of a common fixed point in the invariant subset is not optimal in the setting of Theorem 4.3.4. For instance, let $\omega > 0$ and consider the semigroup $T(t) := e^{\omega t}$ on the Hilbert space $H := L^2([0,1])$ associated to the sesquilinear form $\mathfrak{a}[u,v] := -\omega(u|v)$ with domain H. The closed, convex set $C := \{u \in L^2([0,1]) \mid u \geq 1 \text{ a.e.}\}$ is invariant under T and the orthogonal projection P of H onto C, namely $Pu = u \vee 1$, trivially fulfills $PH \subset H$. Thus, we have (like in the proof of Theorem 4.3.4) $\mathrm{Re}\,\mathfrak{a}[Pu, u - Pu] \geq 0$ for all $u \in H$. However, the semigroup T has no common fixed point in C.*

Remark 4.3.6. *Another idea in order to extend Ouhabaz' invariance criterion in the third step of the proof was the following: One easily sees that the set $D := \overline{co}(C \cup \{0\})$ is invariant under the semigroup T. Since $0 \in D$, we obtain $P_D V \subseteq V$ from the second step. However, one can construct simple examples where $P_C u_0$ for some $u_0 \in V$ cannot be described by linear combinations of $P_D u_0$ and u_0 and therefore, unfortunately, this natural approach seems to be inapplicable.*

We summarize the results of Proposition 4.3.2 and Theorem 4.3.4:

Corollary 4.3.7. *Let H be a Hilbert space, C a closed, convex subset and P be the orthogonal projection of H onto C. Let T be the semigroup associated to the elliptic, densely defined, continuous sesquilinear form $(\mathfrak{a}, V)$ on H and assume (4.2). Assume that T has a fixed point in C. Then the following assertions are equivalent:*

1. *C is invariant under T;*
2. *$\lambda R(\lambda, -A)C \subseteq C$ for all $\lambda > \max\{\omega_0, 0\}$;*
3. *$\mathrm{Re}\,(Au|u - Pu) \geq -\omega_0 \|u - Pu\|^2$ for all $u \in D(A)$;*
4. *For all $u \in V$ is $Pu \in V$ and $\mathrm{Re}\,\mathfrak{a}[Pu, u - Pu] \geq 0$.*

We finally obtain from Proposition 4.2.6 several conditions under which the semigroup T has indeed a fixed point in its invariant subset C, so that Theorem 4.3.4 is applicable:

Lemma 4.3.8. *Let H be a Hilbert space, C a closed, convex subset and P be the orthogonal projection of H onto C. Let T be the semigroup associated to the elliptic, densely defined, continuous sesquilinear form $(\mathfrak{a}, V)$ on H and assume (4.2). Assume that C is invariant under T. Then T has a fixed point in C in either one of the following cases:*

1. *T is contractive or, equivalently, $(\mathfrak{a}, V)$ is accretive;*
2. *$0 \in C$;*
3. *$e^{-\omega_0 t} C \subseteq C$ for all $t \geq 0$;*
4. *the generator $(A, D(A))$ has compact resolvent.*

Proof. Assertion (1) is a Corollary of Proposition 4.2.6 (note that a Hilbert space is uniformly convex). If $0 \in C$, then 0 is a fixed point of T in C. That shows (2). For (3) the closedness of C implies $0 \in C$ and (2) applies. For (4) we know that T is holomorphic and thus immediately norm-continuous. Therefore, (4) follows directly from Proposition 4.2.6. □

In particular, we see that Theorem 4.3.4 (in combination with Proposition 4.3.2) is indeed an extension of Ouhabaz' invariance criterion.

4.4 Invariance of order intervals and Beurling-Deny criteria

In this final section we would like to apply Theorem 4.3.4 and Proposition 4.3.2 to the invariance of order intervals under the semigroup T. In relation to the generator we have extensively discussed this matter in section 3.4.4. We will now focus on conditions for the sesquilinear form. For simplicity, we restrict ourselves to order intervals in $L^2(\Omega) = L^2(\Omega, \mu; \mathbb{R})$, where (Ω, Σ, μ) is a measure space. Versions for the complex case or the situation in arbitrary Hilbert lattices can be easily deduced from the stated results and are therefore left to the reader. We derive invariance criterions which are in the spirit of the famous Beurling-Deny criteria.

Throughout this section let (Ω, Σ, μ) be a measure space and $H = L^2(\Omega, \mu; \mathbb{R})$ the Hilbert space of all square-integrable, μ-measurable, real-valued functions on Ω. As in section 3.4.4.3 the ordering in H is defined by

$$f \leq g :\Leftrightarrow f \leq g \;\; \mu-\text{a.e. on } \Omega$$

for $f, g \in H$. For μ-measurable functions $g : \Omega \to [0, \infty)$ and $h : \Omega \to (-\infty, 0]$ we will consider the order intervals

$$\begin{aligned} [-\infty, g] &:= \{f \in H \mid f \leq g\}, \\ [h, g] &:= \{f \in H \mid h \leq f \leq g\}. \end{aligned}$$

Note that $f \wedge g \in H$ for all $f \in H$, since g is positive, and similarly $h \vee f \in H$ for all $f \in H$ (cp. Remark 3.4.42). From Lemma 3.4.33 we further know that $f \wedge g = f - (f - g)^+$ and $h \vee (f \wedge g) = f - (f - g)^+ + (h - f)^+$ are the orthogonal projections of $f \in H$ onto $[-\infty, g]$ and $[h, g]$, respectively.

In virtue of Theorem 4.3.4 and Proposition 4.3.2 we obtain the following invariance result:

Proposition 4.4.1. *Let $g : \Omega \to [0, \infty)$ and $h : \Omega \to (-\infty, 0]$ both be μ-measurable functions. Let $(\mathfrak{a}, V)$ be an elliptic, densely defined, continuous sesquilinear form on $L^2(\Omega)$ and T be the associated semigroup on H.*

1. *$[-\infty, g]$ is invariant under T if and only if $f \wedge g \in V$ and* $\operatorname{Re} \mathfrak{a}[f \wedge g, (f-g)^+] \geq 0$ *for all $f \in V$;*
2. *$[h, g]$ is invariant under T if and only if $h \vee (f \wedge g) \in V$ and* $\operatorname{Re} \mathfrak{a}[h \vee (f \wedge g), (f-g)^+ - (h-f)^+] \geq 0$ *for all $f \in V$.*

Proof. In both cases the constant zero function belongs to the order interval. Thus, the stated result follows from Corollary 4.3.7. □

As a corollary we can derive the famous Beurling-Deny criteria characterising positive and submarkovian semigroups. We recall that the semigroup T is said to be positive if it leaves $[-\infty, 0]$ invariant and submarkovian if $[-\infty, \mathbb{1}]$ is invariant under T.

Proposition 4.4.2. *(Beurling-Deny criteria) Let $(\mathfrak{a}, V)$ be an elliptic, densely defined, continuous sesquilinear form on $L^2(\Omega)$ with associated semigroup T.*

1. *T is positive if and only if $f^+ \in V$ and* $\operatorname{Re} \mathfrak{a}[f^-, f^+] \leq 0$ *for all $f \in V$;*
2. *T is submarkovian if and only if $(f - \mathbb{1})^+ \in V$ and* $\operatorname{Re} \mathfrak{a}[f \wedge \mathbb{1}, (f-\mathbb{1})^+] \geq 0$ *for all $f \in V$.*

Chapter 5

Semigroups of injective operators

5.1 Introduction

In this chapter we concentrate on strongly continuous semigroups $T = (T(t))_{t\geq 0}$, for which each operator $T(t)$ is injective. In case, we call T injective. From the semigroup law we say that T is injective if and only if the operators $T(t)$ are injective for all $t \in [0,1]$. In fact, let $t > 0$. There exist $n \in \mathbb{N}$ and $r \in (0,1)$ such that $t = n + r$. One easily shows by induction that $T(n)$ is injective. Hence, $T(t) = T(n)T(r)$ is injective as the combination of two injective mappings.

We are interested in characterisations of injective semigroups. Following Lasiecka, Renardy and Triggiani (cf. [LRT01]) we will derive a condition in terms of the resolvent $R(\lambda, A)$, where A is the generator of T, under which T is injective. In particular, we will see that every holomorphic semigroup is injective. However, the condition does not characterise injective semigroups, what can be seen by a simple counterexample.

5.2 A sufficient condition for injective semigroups

We would like to prove the following statement, which is due to Lasiecka, Renardy and Triggiani (cf. [LRT01, Theorem 3.1]).

Theorem 5.2.1. *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Assume that there are constants $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $R > 0$ and $C > 0$ such that for all $r \geq R$ one has*

$$re^{i\theta} \in \rho(A), \quad \left\| R(re^{i\theta}, A) \right\| \leq C.$$

Then T is injective.

The proof is based on the Phargmen-Lindelöf method, a kind of maximum principle for holomorphic functions on unbounded domains. We will proceed stepwise in order to provide a proof of Theorem 5.2.1 and start with a useful lemma:

Lemma 5.2.2. *Let $\alpha \in (1, \infty)$ amd $\theta_1, \theta_2 \in [0, 2\pi)$ such that $\theta_1 - \theta_2 = \frac{\pi}{\alpha}$. Let f be a holomorphic function on the region*

$$\Omega := \left\{ z = re^{i\theta} \in \mathbb{C} \mid r > 0,\ \theta_1 < \theta < \theta_2 \right\},$$

continuous the closure of Ω *and* $f(re^{i\theta}) = O(e^r)$ *for* $r \to \infty$. *If there exists a constant* $M > 0$ *such that* f *is bounded by* M *on the rays* $\{re^{i\theta_1}\}$ *and* $\{re^{i\theta_2}\}$, *i.e.* $|f(re^{i\theta_k})| \leq M$ *for all* $r \geq 0$ *and* $k \in \{1,2\}$, *then* $|f| \leq M$ *on* Ω.

Proof. We may assume $\theta_1 = \frac{\pi}{2\alpha}$ and $\theta_2 = -\frac{\pi}{2\alpha}$. We fix some $\beta \in (1,\alpha)$ and define for $\varepsilon > 0$ on the closure of Ω the function

$$F_\varepsilon(z) := e^{-\varepsilon z^\beta} f(z), \quad z \in \overline{\Omega}.$$

Then one has $F \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$ and

$$|F_\varepsilon(z)| = e^{-\varepsilon r^\beta \cos\beta\theta} |f(z)|, \quad z = re^{i\theta} \in \Omega.$$

Let $\varepsilon > 0$. We claim that for sufficiently large $R > 0$ the inequality $|F_\varepsilon| \leq M$ holds on the region $\Omega_R := \Omega \cap \{z \in \mathbb{C} \mid |z| < R\}$. Therefore, let $R > 0$. For $z = re^{i\theta}$ with $0 \leq r \leq R$ and $|\theta| = \frac{\pi}{2\alpha}$ the estimate $\beta < \alpha$ implies

$$\beta\theta \in \left\{-\frac{\beta\pi}{2\alpha}, \frac{\beta\pi}{2\alpha}\right\} \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Hence, $\cos(\beta\theta) > 0$. Furthermore, we get from the boundedness of f on the rays $\{re^{i\theta_1}\}$ and $\{re^{i\theta_2}\}$ the estimate

$$|F_\varepsilon(z)| = e^{-\varepsilon r^\beta \cos\beta\theta} |f(z)| \leq |f(z)| \leq M.$$

For $z = Re^{i\theta}$ with $|\theta| \leq \frac{\pi}{2\alpha}$ we have additionally

$$|F_\varepsilon(z)| \leq e^{-\varepsilon R^\beta \cos\beta\frac{\pi}{2\alpha}} |f(z)| \leq Ce^{R - \varepsilon R^\beta \cos\beta\frac{\pi}{2\alpha}} \to_{R\to\infty} 0,$$

for some constant $C > 0$ due to assumed behaviour of f as $R \to \infty$. Thus, we have shown $|F_\varepsilon| \leq M$ on $\partial\Omega_R$ for sufficiently large $R > 0$. Now the maximum principle implies $|F_\varepsilon| \leq M$ on the whole of Ω_R, which finally proves the claim above.

Since $\Omega = \bigcup_{R\geq 0} \Omega_R$ we know $|F_\varepsilon| \leq M$ on Ω for all $\varepsilon > 0$. We conclude

$$|f(z)| \leq e^{\varepsilon r^\beta} |F_\varepsilon(z)| \leq e^{\varepsilon r^\beta} M, \quad z = re^{i\theta} \in \Omega, \varepsilon > 0.$$

Letting $\varepsilon \searrow 0$ we obtain $|f| \leq M$ on Ω. ☐

Corollary 5.2.3. *Let* $f \in \mathcal{O}(\mathbb{C})$ *such that* $f(re^{i\theta}) \in O(e^r)$ *for* $r \to \infty$ *and* F *is bounded on the rays* $\{re^{i\theta_1}\}, \{re^{i\theta_2}\}$ *and* $\{re^{i\theta_3}\}$, *where* $0 \leq \theta_1 < \theta_2 < \theta_3 < 2\pi$. *Then* f *is constant.*

Proof. We apply Lemma 5.2.2 successively to the regions

$$\begin{aligned}
\Omega_{1,2} &:= \left\{re^{i\theta} \mid r > 0, \theta_1 < \theta < \theta_2\right\},\\
\Omega_{2,3} &:= \left\{re^{i\theta} \mid r > 0, \theta_2 < \theta < \theta_3\right\},\\
\Omega_{3,1} &:= \left\{re^{i\theta} \mid r > 0, \theta_3 < \theta < 2\pi + \theta_1\right\}.
\end{aligned}$$

It follows the boundedness of f on $\mathbb{C} = \Omega_{1,2} \cup \Omega_{2,3} \cup \Omega_{3,1}$. Now Liouville's Theorem implies that f is constant. ☐

So far we have only worked with complex-valued, holomorphic functions on the complex plane. However, the situation we are interested in takes place in the Banach space X. But we can use the well-known equiavelence of holomorphy and weak holomorphy of X-valued functions.

Corollary 5.2.4. *Let X be a complex Banach space with dual space X' and $f \in \mathcal{O}(\mathbb{C}; X)$. Assume*

1. $\lim_{r\to\infty} f(r) = 0$ *in X;*
2. $\lim_{r\to\infty} \frac{f(re^{i\theta})}{e^r} = 0$ *in X for all $\theta \in [0, 2\pi)$;*
3. *f is bounded on the rays $\{re^{i\theta_1}\}, \{re^{i\theta_2}\}$ and $\{re^{i\theta_3}\}$ mit $0 \le \theta_1 < \theta_2 < \theta_3 < 2\pi$.*

Then one has $f \equiv 0$.

Proof. For $x' \in X'$ we define

$$f_0 : \mathbb{C} \to \mathbb{C}, \quad f_0(\lambda) := \langle x', f(\lambda)\rangle.$$

Since $f : \mathbb{C} \to X$ is holomorphic, the function f_0 is a scalar-valued holomorphic function on the complex plane $\mathbb{C}$. Thanks to the estimate $|f_0(\lambda)| \le \|x'\| \, \|f(\lambda)\|$, $\lambda \in \mathbb{C}$, we conclude that f_0 fulfills the assertions (1), (2) and (3) (now in $\mathbb{C}$ instead of X). From Corollary 5.2.3 we deduce that f_0 is constant and subsequently, $f_0 \equiv 0$ thanks to property (1). Since $x' \in X'$ was arbitrarily chosen, the Theorem of Hahn-Banach finally shows $f \equiv 0$. $\square$

Finally, we are able to prove Theorem 5.2.1.

Proof. (of Theorem 5.2.1) Let $t_0 \in (0, 1]$, $x_0 \in X$ such that $T(t_0)x_0 = 0$. We will show that $x_0 = 0$. For $t \ge t_0$ is $T(t)x_0 = T(t - t_0)T(t_0)x_0 = 0$. We obtain for the Laplace-transform

$$f(\lambda) := \int_0^\infty e^{-\lambda t} T(t)x_0 \, dt = \int_0^{t_0} e^{-\lambda t} T(t)x_0 \, dt, \quad \lambda \in \mathbb{C}.$$

Note that $f \in \mathcal{O}(\mathbb{C}; X)$.

Since T is a strongly continuous semigroup, there are constants $\omega \in \mathbb{R}$, $M \ge 1$, such that $\|T(t)\| \le Me^{\omega t}$ for all $t \ge 0$. In particular, one has $N := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > \omega\} \subset \varrho(A)$ and for $\lambda \in N$ the resolvent $R(\lambda, A)$ is the Laplace-transform of T satisfying the estimate

$$\|(\mathrm{Re}(\lambda) - \omega)^n R(\lambda, A)^n\| \le M$$

for all $n \in \mathbb{N}$. Thanks to the uniqueness of the Laplace-transform we conclude $R(\cdot, A)x_0 = f$ on N. Since N is an open subset of $\varrho(A)$ and f and $R(\cdot, A)x_0$ both are holomorphic on $\varrho(A)$, it follows $f(\lambda) = R(\lambda, A)x_0$ for all $\lambda \in \varrho(A)$. In particular, the function f is now bounded on the rays $\{re^{i\theta}\}, \{re^{-i\theta}\}, \{r\}$ and satisfies $f(\lambda) \to 0$ as $\lambda \to \infty$. Moreover, one has for $\beta \in [0, 2\pi)$

and $r \geq 0$:

$$\begin{aligned} e^{-r}\left\|f(re^{i\beta})\right\| &\leq \frac{M}{e^r}\int_0^{t_0} e^{t(\omega - r\cos\beta)}\,dt \\ &= \frac{M\left(e^{t_0(\omega - r\cos\beta)} - 1\right)}{e^r(\omega - r\cos\beta)} \\ &= \frac{Me^{t_0\omega}}{\omega - r\cos\beta}e^{-r(t_0\cos\beta + 1)} - \frac{M}{e^r(\omega - r\cos\beta)} \to_{r\to\infty} 0. \end{aligned}$$

Note at this that $t_0 \cos\beta \geq -1$ for all $\beta \in [0, 2\pi)$. Due to Corollary 5.2.4 it follows $f \equiv 0$. This implies

$$0 = \lambda f(\lambda) = \lambda R(\lambda, A)x_0 \to_{\lambda\to\infty} x_0,$$

hence $x_0 = 0$. The semigroup T is injective. □

Corollary 5.2.5. *Any holomorphic semigroup on X is injective.*

We would like to remark that the converse statement of Theorem 5.2.1 is wrong. As a matter of fact there is no way to characterize injective semigroups by properties of the resolvent set for its generator, since any multiplication semigroup is injective but the spectrum of the generator can be constructed in an arbitrary fashion. We give a simple counterexample.

On the domain $\Omega := \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 1\}$ we consider the continuous function $q : \Omega \to \mathbb{C}$, $q(z) := z$. Then the multiplication operator

$$D(A_q) := \{f \in C_0(\Omega) \mid qf \in C_0(\Omega)\}\,, \quad A_q f := qf$$

on $C_0(\Omega)$ satisfies $\sigma(A_q) = \overline{q(\Omega)} = \overline{\Omega}$. Hence, the assumption in Theorem 5.2.1 is not fulfilled. Nevertheless, the operator A_q generates the strongly continuous semigroup $T = (T(t))_{t\geq 0}$ with $(T(t)f)(z) := e^{tq(z)}f(z) = e^{tz}f(z)$, $z \in \Omega$, in $C_0(\Omega)$ (cf. [EN00, p. 27]), which is in fact injective:

$$(T(t)f)(z) = 0 \text{ for all } z \in \Omega \;\Rightarrow\; f(z) = 0 \text{ for all } z \in \Omega.$$

Therefore, the converse statement of Theorem 5.2.1 is not true.

Chapter 6

Regular form perturbations and the generalized Kato class

6.1 Introduction

Form methods provide excellent means to define realisations of second order differential operators on L^2-spaces and obtain generator properties of such operators. Using extrapolation techniques it is also possible to extend the semigroup associated to such operators to other L^p-spaces, in particular to the space L^∞. However, often – in particular in connection with stochastic processes – one is interested in semigroups on spaces of continuous functions, such as C_b or C_0. In particular, it is interesting to perturb such "regular forms", i.e. forms where one has a semigroup on a regularity space as C_b or C_0, and obtain a regular form again.
This problem is connected to the Kato class, which was introduced by Aizenman and Simon in [AS82] in connection with Schrödinger operators, i.e. perturbations of the Laplacian by a potential $V \in L^1_{\text{loc}}$. There, the (local) Kato class is defined as the set of all $V \in L^1_{\text{loc}}$ satisfying a certain integrability condition (which itself goes back to Kato [Kat72]). It is then proved that V belongs to the local Kato class if and only if $R(\lambda, \Delta)Vg$ is a continuous function for any bounded and measurable g (see [AS82, Theorem 1.5]). Thus the Kato class is connected with the continuity of solutions to elliptic problems.
Later, Stollmann and Voigt replaced the Laplacian by general regular symmetric Dirichlet considered also measures instead of locally integrable functions, see [SV96, Voi95]. Consequences for the semigroups generated by such perturbed operators where investigated in [Sim82, DvC00] using a probabilistic approach. We also mention the connection of the Kato class with Miyadera perturbation [Voi86, OSSV96].

We will replace the space of continuous functions by some abstract regularity space X. This allows for greater flexibility in the regularity looked for, e.g. when working on some domain $\Omega \subset \mathbb{R}^N$, one can require regularity also on the boundary by choosing $X = C(\overline{\Omega})$. Also, we consider general sub-Markovian forms $\mathfrak{a}$, dropping the requirement that $\mathfrak{a}$ be symmetric. Then the abstract Kato class is defined as the set of all $\varphi \in D(\mathfrak{a})'$ such that $R(\lambda, \mathcal{A})\varphi \in X$. Here, $\mathcal{A} : D(\mathfrak{a}) \to D(\mathfrak{a})'$ is the operator associated to the form $\mathfrak{a}$, see section 6.1.1.
We note that we do *not* seek to describe the elements of the Kato class by some integrability condition. We rather assume that already sufficiently many elements of the abstract Kato class are known.
In section 6.2.1 we introduce local versions of the spaces $D(\mathfrak{a})$ and $D(\mathfrak{a})'$ and the operator $\mathcal{A}$.

This is essential to define a local version of the Kato class in section 6.2.2. There, we will also prove several properties of the Kato class and the local Kato class and in particular address the independence of the Kato class from the parameter λ. This does not always hold, see section 6.3.1. Afterwards, we introduce Kato perturbations, which are the appropriate generalisation of potentials and measures belonging to the classical Kato class. However, even in the classical situation, there can be Kato perturbations which are not associated to a measure.
In section 6.2.3, we consider the space X_0 of regular functions vanishing at infinity. As belonging to X_0 is in general not a local property, there is no local Kato class for X_0. To obtain semigroups on X_0, we present a Theorem in the spirit of Lyapunov functions, cf. [BL07, Theorem 4.3.2]. In order to prove the Theorem, one needs a certain approximation result, which is equivalent to some abstract sort of Dirichlet boundary condition.
The third part is devoted to applications. We introduce deGiorgi-Nash forms, for which many elements of the Kato class for $X = C(\Omega)$ are known from the deGiorgi-Nash Theorem. In section 6.3.3 we prove that for any deGiorgi-Nash form and any bounded $\Omega \subset \mathbb{R}^N$, there exists a potential $V \in L^\infty_{\text{loc}}$ such that the semigroup associated to the perturbed form on $L^\infty(\Omega)$ leaves the space $C_0(\Omega)$ invariant.

6.1.1 Notation and Setting for this chapter

In the sequel we will always work on the Hilbert space $L^2(M, dm)$, where M is a locally compact topological space which is countable at infinity and m is a positive Radon measure on M. We will often write L^p for $L^p(M, dm)$, $\|\cdot\|_p$ for the canonical norm in L^p and $\langle\,\cdot\,,\,\cdot\,\rangle_{p,q}$ for the canonical duality between L^p and L^q, where q is the conjugate index to p. For $p = 2$ we just write $\|\cdot\|$ for the canonical L^2-norm and $(\,\cdot\,,\,\cdot\,)$ for the scalar product in L^2.
On L^2, we will consider densely defined sectorial forms. We briefly recall some notions and facts about sectorial forms. For more details we refer to [Kat95, Ouh04].

A *densely defined sesquilinear form* on L^2 is a mapping $\mathfrak{a} : D(\mathfrak{a})\times D(\mathfrak{a}) \to \mathbb{C}$ which is linear in the first component and antilinear in the second; $D(\mathfrak{a})$ is a dense subspace of L^2 and is called *the domain* of $\mathfrak{a}$. $\mathfrak{a}$ is called *sectorial*, if the numerical range $\Theta(\mathfrak{a}) := \{\mathfrak{a}[u,u] \;:\; u \in D(\mathfrak{a})\;,\; \|u\| \le 1\}$ is contained in some right open sector of angle $\theta < \frac{\pi}{2}$ around the real axis. In this case, there exists $\gamma \in \mathbb{R}$ such that $(f,g)_\mathfrak{a} := \gamma(f,g)_H + \operatorname{Re}\mathfrak{a}[f,g]$ is a scalar product on $D(\mathfrak{a})$. If $(D(\mathfrak{a}), (\,\cdot\,,\,\cdot\,)_\mathfrak{a})$ is a Hilbert space, $\mathfrak{a}$ is called *closed.* In the sequel we assume without loss that $\gamma = 1$. We write $\|\cdot\|_\mathfrak{a}$ for the norm induced by $(\,\cdot\,,\,\cdot\,)_\mathfrak{a}$. We will call $\mathfrak{a}$ *local*, if (i) $\mathfrak{a}[u,v] = 0$, whenever u and v have disjoint support and (ii) if ω is an open subset of M, then the space $D(a,\omega) := \{u \in D(\mathfrak{a}) \;:\; u = 0 \text{ a.e. on } M \setminus \omega\}$ is dense in $L^2(\omega, dm)$.

We also consider the space $D(\mathfrak{a})'$ of bounded antilinear functionals on $(D(\mathfrak{a}), \|\cdot\|_\mathfrak{a})$. However, we do not identify this space with $(D(\mathfrak{a}), \|\cdot\|_\mathfrak{a})$ but we use L^2 as a pivot space: $D(\mathfrak{a}) \hookrightarrow L^2 \hookrightarrow D(\mathfrak{a})'$. That is, we identify $f \in L^2$ with the bounded antilinear functional $\varphi_f : g \mapsto (f,g)$. We denote the duality pairing between $D(\mathfrak{a})'$ and $D(\mathfrak{a})$ by $\langle\,\cdot\,,\,\cdot\,\rangle$.

Given a densely defined, closed sectorial form $\mathfrak{a}$, we may associate an operator $\mathcal{A}$ on $D(\mathfrak{a})'$ with the form $\mathfrak{a}$ by defining

$$D(\mathcal{A}) := D(\mathfrak{a}), \qquad -\langle \mathcal{A}u, v\rangle := \mathfrak{a}[u,v]\;. \tag{6.1}$$

It is well-known (cf. [Ouh04, Theorems 1.55 and 1.52]) that $\mathcal{A}$ defined in this way generates

a holomorphic, strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $D(\mathfrak{a})'$. Furthermore, $\mathcal{T}$ leaves L^2 invariant and the restricted semigroup $T(t) := \mathcal{T}(t)|_{L^2}$ is also holomorphic and strongly continuous. The generator A_2 of T is the part of $\mathcal{A}$ in L^2.
A *sub-Markovian form* is a densely defined, closed, sectorial form $\mathfrak{a}$ on L^2, such that the associated semigroup T is real, positive and L^∞- contractive. The Beurling-Deny Criteria (cf. [Ouh04, Section 2.2] and Proposition 4.4.2) give a useful characterisation of sub-Markovian forms.
It is well-known (cf. [Ouh04, p. 56 ff.]) that if $\mathfrak{a}$ is sub-Markovian, then there exists a consistent family $(T_p)_{2\leq p\leq\infty}$ of semigroups on L^p (i.e. for $f \in L^p \cap L^q$ we have $T_p(t)f = T_q(t)f$ for all $t \geq 0$) such that T_2 is the semigroup associated to $\mathfrak{a}$ on L^2 as introduced above. Furthermore, T_p is strongly continuous for $2 \leq p < \infty$ and T_∞ is an adjoint semigroup, in particular it is $\sigma(L^\infty, L^1)$-continuous. In the following, we will denote by A_p the generator of T_p. This is the strong generator for $2 \leq p < \infty$ and the weak*-generator for $p = \infty$. It is known that the holomorphy of T_2 is inherited by the semigroups T_p for $2 \leq p < \infty$. For a proof of these facts and other properties of consistent families of semigroups we refer to [Are04, Chapter 7.2].
Since M is locally compact and countable at infinity, there exists a sequence $(\omega_n)_{n\geq 0}$ of open sets such that $\omega_n \Subset \omega_{n+1} \Subset M$ for any $n \geq 0$ (where $A \Subset B$ means $\overline{A}$ is compact and contained in B) and $\bigcup_n \omega_n = M$. We fix – once and for all – such a sequence. It is easy to see that $D(\mathfrak{a}, \omega_n)$ as defined above is a closed subspace of $D(\mathfrak{a})$. Thus, if $\mathfrak{a}$ is local, then $(\mathfrak{a}_n, D(\mathfrak{a}, \omega_n))$ defined by $\mathfrak{a}_n[u, v] := \mathfrak{a}[u, v]$, is a densely defined, closed, sectorial form on $L^2(\omega_n)$. It is also possible to consider $\mathfrak{a}_n$ as a non-densely defined form on $L^2(M)$. For this we refer to [Ouh04, Chapter 2.6]. We will denote by $\mathcal{A}_n : D(\mathfrak{a}, \omega_n) \to D(\mathfrak{a}, \omega_n)'$ the associated operator. We note that using the Beurling-Deny criteria, we see that $\mathfrak{a}_n$ is a sub-Markovian form if $\mathfrak{a}$ is.

6.2 Abstract results

6.2.1 Local forms

In this section we are given a local, sub-Markovian form $\mathfrak{a}$ on $L^2(M, dm)$. We introduce local versions of the spaces $D(\mathfrak{a})$ and $D(\mathfrak{a})'$ and extend the associated operator $\mathcal{A}$ to an operator $\tilde{\mathcal{A}}$ defined on $D(\mathfrak{a})_{\mathrm{loc}}$ taking values in $D(\mathfrak{a})'_{\mathrm{loc}}$. Then we investigate the connection between the semigroup generators A_p and the extended operator $\tilde{\mathcal{A}}$.
To localise the spaces, we use the spaces $D(\mathfrak{a}, \omega_n)$ introduced in the previous section. We will denote by $D(\mathfrak{a})_c$ the vector space of all elements of $D(\mathfrak{a})$ having compact support in M. It is obviously $D(\mathfrak{a})_c = \bigcup D(\mathfrak{a}, \omega_n)$. As a local version of the antidual $D(\mathfrak{a})'$ we consider

$$\begin{aligned} D(\mathfrak{a})'_{\mathrm{loc}} &:= \bigcap_{n\geq 0} D(\mathfrak{a}, \omega_n)' \\ &= \{\varphi : D(\mathfrak{a})_c \to \mathbb{C} \text{ antilinear} \mid \forall\, n \geq 0\ \exists\, C_n \text{ such that} \\ &\qquad |\varphi(u)| \leq C_n \cdot \|u\|_{\mathfrak{a}}\ \forall\, u \in D(\mathfrak{a}, \omega_n)\ \}\ . \end{aligned}$$

Last, the local version of $D(\mathfrak{a})$ will be

$$D(\mathfrak{a})_{\mathrm{loc}} := \left\{ u \in L^2_{\mathrm{loc}}(M) \,:\, \forall n \geq 0\ \exists\, u_n \in D(\mathfrak{a}) \text{ such that } u = u_n \text{ a.e. on } \omega_n \ \right\}\ .$$

Note, that $D(\mathfrak{a})'_{\mathrm{loc}}$ is *not* the dual of $D(\mathfrak{a})_{\mathrm{loc}}$, but a local version of $D(\mathfrak{a})'$.
To extend the operator $\mathcal{A}$ to an operator $\tilde{\mathcal{A}}$ defined on $D(\mathfrak{a})_{\mathrm{loc}}$ and taking values in $D(\mathfrak{a})'_{\mathrm{loc}}$ we

make use of the locality of the form $\mathfrak{a}$:

Lemma 6.2.1. *Let $\mathfrak{a}$ be a local form on $L^2(M)$. Then the operator $\mathcal{A}$ has an extension to an operator $\tilde{\mathcal{A}}$ from $D(\mathfrak{a})_{\mathrm{loc}}$ to $D(\mathfrak{a})'_{\mathrm{loc}}$ satisfying the following condition:*
If $u \in D(\mathfrak{a})_{\mathrm{loc}}$ and $u_n \in D(\mathfrak{a})$ satisfies $u = u_n$ a.e. on ω_n for some $n \in \mathbb{N}_0$, then

$$\langle \tilde{\mathcal{A}}u, v \rangle = \langle \mathcal{A}u_n, v \rangle \ , \tag{6.2}$$

for all $v \in D(\mathfrak{a})$ with $\operatorname{supp}\{v\} \Subset \omega_n$. Furthermore, $\tilde{\mathcal{A}}$ is the unique extension of $\mathcal{A}$ with this property.

Proof. Let $u \in D(\mathfrak{a})_{\mathrm{loc}}$. We have to give meaning to $\langle \tilde{\mathcal{A}}u, v \rangle$ for all $v \in D(\mathfrak{a})_c$. So let $\omega \Subset M$ and $v \in D(\mathfrak{a}, \omega)$ be given. There exists $n \geq 0$ such that $\omega \Subset \omega_n$. Moreover, since $u \in D(\mathfrak{a})_{\mathrm{loc}}$ there exists $u_n \in D(\mathfrak{a})$ such that $u = u_n$ a.e. on ω_n. We define $\tilde{\mathcal{A}}$ by equation (6.2), that is $\tilde{\mathcal{A}}u$ acts on $v \in D(\mathfrak{a}, \omega)$ via (6.2). We need only show that this is well defined. So suppose that $\omega \Subset \omega_n$ and $\omega \Subset \omega_m$ for some $n, m \in \mathbb{N}_0$. Suppose further that u_n, u_m are two elements of $D(\mathfrak{a})$ coinciding a.e. with u on ω_n and ω_m respectively. We obtain

$$\langle \mathcal{A}u_n, v \rangle - \langle \mathcal{A}u_m, v \rangle = \mathfrak{a}[u_n - u_m, v] = 0$$

by locality, since $u_n - u_m$ vanishes on $\omega_n \cap \omega_m$ and hence its support is disjoint from $\operatorname{supp} v \subset \overline{\omega}$. $\square$

Of course, there should be some relation between the operator $\tilde{\mathcal{A}}$ and the operators A_p. We start with the following observation:

Proposition 6.2.2. *Let $2 \leq p \leq \infty$. Let B_p be the part of A_2 in $X_p := L^2 \cap L^p$. Then, for $2 \leq p < \infty$, A_p is the closure of B_p and A_∞ is the weak*-closure of B_∞. Furthermore, for $u \in D(A_\infty)$ there exists a sequence $u_n \in D(B_\infty)$ such that $u_n \rightharpoonup^* u$ and $B_\infty u_n \rightharpoonup^* A_\infty u$.*

Proof. Let $u \in D(B_p)$, i.e. $u \in D(A_2) \cap L^p$ and $A_2 u \in L^p$. By consistency we have

$$p - \int_0^t T_p(s) B_p u \, ds = 2 - \int_0^t T_2(t) A_2 u \, ds = T_2(t)u - u = T_p(t)u - u \ , \tag{6.3}$$

where $p - \int$ denotes the Bochner integral in L^p for $2 \leq p < \infty$ and the weak*-integral for $p = \infty$. The assertion now follows from the fact (see [ABHN01, Proposition 3.1.9] for the strongly continuous case, [vNe92, Proposition 1.2.2] for the weak*-case), that for a strongly continuous (weak*-continuous) semigroup T with (weak*-) generator A, we have $x \in D(A)$ and $Ax = y$ if and only if for all $t \geq 0$ we have

$$\int_0^t T(s) y \, ds = T(t)x - x \ .$$

Now we show that A_p is in fact the closure of B_p. First consider the case $2 \leq p < \infty$: By consistency, T_p and T_2 leave the Banach space X_p invariant. The restricted semigroup is strongly continuous and has generator B_p, which follows from a computation as in (6.3). In particular, $D(B_p)$ is dense in X_p and thus dense in L^p. Using the holomorphy of T_2 and consistency, we see that $D(B_p)$ is invariant under T_p. It is well-known (cf. [EN00, Prop. II.1.7]) that this implies that $D(B_p)$ is a core for A_p.

For $p=\infty$ we choose a different approach: Given $u\in D(A_\infty)$, we put $v_n=\mathbb{1}_{\omega_n}(\lambda-A_\infty)u$. Then $v_n\in L^2\cap L^\infty$, whence $u_n:=R(\lambda,A_\infty)v_n\in D(B_\infty)$. Then we have $v_n \rightharpoonup^* (\lambda-A_\infty)u$ and since $R(\lambda,A_\infty)$ is weak*-continuous as an adjoint operator, we have $u_n\rightharpoonup^* u$. Also, we have

$$\begin{aligned} A_\infty u_n &= A_\infty R(\lambda,A_\infty)v_n \\ &= \lambda R(\lambda,A_\infty)v_n - v_n \\ &\rightharpoonup^* \lambda R(\lambda,A_\infty)(\lambda-A_\infty)u-(\lambda-A_\infty)u = A_\infty u\ . \end{aligned}$$

This proves the claim. □

Remark 6.2.3. *Under more restrictive assumptions on $\mathfrak{a}$, one obtains consistent semigroups T_p for $1\le p\le\infty$. In this case, Proposition 6.2.2 also holds for $1\le p\le\infty$.*

It follows from Proposition 6.2.2 that if M has finite measure so that $L^p\subset L^2$ for $p\ge 2$, then A_p is the part of A_2 in L^p. In particular, $\mathcal{A}$ is an extension of A_p. If $m(M)=\infty$, then L^p is not a subset of L^2 and hence we cannot expect $\mathcal{A}$ to be an extension of A_p. However, we may ask whether $\tilde{\mathcal{A}}$ is an extension of A_p, i.e. $D(A_p)\subset D(\mathfrak{a})_{\mathrm{loc}}$ and

$$\langle\tilde{\mathcal{A}}u,v\rangle=\int_M A_pu\cdot v\,dm\ , \tag{6.4}$$

for all $v\in D(\mathfrak{a})_c\cap L^q$, where q is the conjugate index to p.

As a sufficient condition for $\tilde{\mathcal{A}}$ to be an extension of A_p is the following:

Definition 6.2.4. Let $\mathfrak{a}$ be a closed sectorial form. We say that $\mathfrak{a}$ has *rich domain* if there exists constants $(C_n)_{n\in\mathbb{N}}$ such that for every $u\in D(\mathfrak{a})$ and $n\in\mathbb{N}$ there exists $v\in D(\mathfrak{a})$ with the following properties:

1. $v\in D(\mathfrak{a},\omega_n)$ and $u=v$ a.e. on ω_{n-1};
2. $\|v\|_{L^2(\omega_n)}\le C_n\|u\|_{L^2(\omega_n)}$;
3. $\|\mathcal{A}v\|_{D(\mathfrak{a},\omega_n)'}\le C_n\left(\|u\|_{L^2(\omega_n)}+\|\mathcal{A}u\|_{D(\mathfrak{a},\omega_n)'}\right)$.

In the proof of the following theorem and also in the sequel, we will treat the cases of norm convergence and weak*-convergence together. Given $f_n,f\in L^p$ we will write $p-\lim f_n=f$. This is to be understood as "f is the norm limit of f_n" for $p<\infty$, whereas for $p=\infty$ it stands for "f is the weak*-limit of f_n".

Theorem 6.2.5. *Let $\mathfrak{a}$ be a local sub-Markovian form with rich domain. Then $\tilde{\mathcal{A}}$ is an extension of A_p for any $2\le p\le\infty$.*

Proof. Let $u\in D(A_p)$. By Proposition 6.2.2, there exists a sequence $u_n\in D(A_2|_{L^2\cap L^p})\subset D(\mathfrak{a})$ such that $p-\lim u_n=u$ and $p-\lim A_pu_n=A_pu$. Furthermore, we have $A_pu_n\equiv\mathcal{A}u_n$. The sequences u_n and A_pu_n are bounded in L^p. (For $p<\infty$ this is clear, for $p=\infty$ it follows from the uniform boundedness principle.)

Now let $k \in \mathbb{N}$ be fixed. Since $\mathfrak{a}$ has rich domain, there exists a sequence $v_n \in D(\mathfrak{a}, \omega_k) \cap L^p$ such that $v_n = u_n$ a.e. on ω_{k-1}. Furthermore, we have

$$\|v_n\|_{L^2(\omega_k)} \leq C_k \|u_n\|_{L^2(\omega_k)} \leq \tilde{C}_k \|u_n\|_{L^p(\omega_k)} \leq M_1 < \infty \tag{6.5}$$

and

$$\begin{aligned} \|\mathcal{A} v_n\|_{D(\mathfrak{a},\omega_k)'} &\leq C_k \left(\|u_n\|_{L^2(\omega_k)} + \|\mathcal{A} u_n\|_{D(\mathfrak{a},\omega_k)'} \right) \\ &\leq \tilde{C}_k \left(\|u_n\|_{L^p(\omega_k)} + \|A_p u_n\|_{L^p(\omega_k)} \right) \leq M < \infty . \end{aligned} \tag{6.6}$$

Here we have used the inclusions $L^p(\omega_k) \hookrightarrow L^2(\omega_k) \hookrightarrow D(\mathfrak{a}, \omega_k)'$ and the boundedness of the sequences u_n and $A_p u_n$ in L^p.

It follows from (6.5) that – after possibly passing to a subsequence – v_n converges weakly in L^2 to some $v \in L^2$. However, as a sequence in $D(\mathfrak{a}, \omega_k)'$ it also converges weakly to (the same) v.
Similarly, (6.6) and the reflexivity of $D(\mathfrak{a}, \omega_k)'$ imply that – possibly passing to yet another subsequence – $\mathcal{A} v_n$ converges weakly to some $w \in D(\mathfrak{a}, \omega_k)'$.
Since $\mathcal{A}_k$ is a generator, its graph is closed and hence by the Hahn-Banach theorem also weakly closed. Thus $v \in D(\mathfrak{a}, \omega_k)$ and $\mathcal{A} v = w$.

Now let $\omega \Subset \omega_{k-1}$ and $f \in D(A_2) \cap L^2(\omega) \subset D(\mathfrak{a}, \omega_k) \cap L^q$. Here q is the conjugate index to p. By Proposition 6.2.2, $D(A_2) \cap L^2(\omega)$ is $\sigma(L^p, L^q)$-dense in $L^p(\omega)$. We obtain:

$$\langle u, f \rangle_{p,q} = \lim \int u_n \cdot f = \lim \int v_n \cdot f = \langle v, f \rangle_{p,q} .$$

It follows by density that $u = v$ a.e. on ω.
Furthermore, we have

$$\langle A_p u, f \rangle_{p,q} = \lim \langle A_p u_n, f \rangle_{q,p} = \lim \langle \mathcal{A} v_n, \rangle f = \langle \mathcal{A} v, f \rangle .$$

Here the second equality follows from the fact that $u_n = v_n$ a.e. on ω_{k-1} and the locality of $\mathfrak{a}$. Since $D(A_2) \cap L^2(\omega)$ is the domain of the operator associated to the form $(\mathfrak{a}, D(\mathfrak{a}, \omega))$, it is dense in $D(\mathfrak{a}, \omega)$. It follows that $A_p u = \mathcal{A} v$ in $D(\mathfrak{a}, \omega)'$. Since ω was arbitrary, it follows that $u \in D(\mathfrak{a})_{\mathrm{loc}}$ and $A_p u = \mathcal{A} u$. □

6.2.2 Kato perturbations

In this section we consider again the Hilbert space $L^2(M, dm)$ as in the previous section and a local sub-Markovian form $\mathfrak{a}$ on $L^2(M, dm)$. In this whole section we fix $\lambda_0 \in -\Theta(\mathfrak{a})^c \subset \rho(\mathcal{A})$. We are interested in the elliptic equation

$$\lambda_0 u - \tilde{\mathcal{A}} u = \varphi \tag{6.7}$$

where φ is an element of $D(\mathfrak{a})'_{\mathrm{loc}} \supset D(\mathfrak{a})'$. In particular, we want to investigate, whether solutions to (6.7) have a certain regularity, i.e. whether u belongs to some function space X. If $\varphi \in D(\mathfrak{a})'$, then (6.7) has a unique solution $u \in D(\mathfrak{a})$ (note however, that there may be more solutions of (6.7) in $D(\mathfrak{a})_{\mathrm{loc}}$). However, if $\varphi \in D(\mathfrak{a})'_{\mathrm{loc}}$, then we cannot expect solutions u of (6.7) in $D(\mathfrak{a})$. But there may be several solutions of the elliptic equation in $D(\mathfrak{a})_{\mathrm{loc}}$. We build our theory in such a way, that we just need information about "local" solutions of (6.7), i.e. we consider $u_n = R(\lambda_0, \mathcal{A}_n)\varphi$. We call this a "local" solution, since u_n satisfies

$$\lambda_0(u_n, v) + \mathfrak{a}[u_n, v] = \langle \varphi, v \rangle \ ,$$

for all $v \in D(\mathfrak{a}, \omega_n)$, that is, $\lambda_0 u_n + \tilde{A} u_n = \varphi$ on $D(\mathfrak{a}, \omega_n)$. For φ to belong to the local Kato class, we will require these "local" solutions of (6.7) to belong to X "locally".

Definition 6.2.6. Let X and $(X(\omega_n))_{n\geq 0}$ be vector spaces of (equivalence classes of) measurable functions on M. We say that X is *localised by* $(X(\omega_n))_{n\geq 0}$ if

1. $X(\omega_n) \downarrow X$, i.e. $X(\omega_n) \subset X(\omega_{n+1})$ for all $n \geq 0$ and $X = \bigcap_{n\geq 0} X(\omega_n)$ and
2. If $u \in X(\omega_n)$ and v is a measurable function such that $u = v$ a.e. on ω_n, then $v \in X(\omega_n)$.

Definition 6.2.7. Let X be a vector space and $\mathfrak{a}$ be a local, sub-Markovian form on $L^2(M, dm)$.

1. The *X-Kato class* $\mathrm{Kat}(\mathfrak{a}, \lambda_0, X)$ of $\mathfrak{a}$ is defined as

$$\mathrm{Kat}(\mathfrak{a}, \lambda_0, X) := \{\varphi \in D(\mathfrak{a})' \mid R(\lambda_0, \mathcal{A})\varphi \in X \ \} \ .$$

2. Now assume that X is localised by $X(\omega_n)$. The *local X-Kato class* is defined by

$$\mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X) := \bigcap_{n\in\mathbb{N}_0} \mathrm{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n)) \ ,$$

i.e. $\mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ consists of those functionals $\varphi \in D(\mathfrak{a})'_{\mathrm{loc}}$ such that for all $n \in \mathbb{N}_0$ we have $R(\lambda_0, \mathcal{A}_n)\varphi \in X(\omega_n)$.

Note that the local Kato class depends on the spaces $X(\omega_n)$ used to localise X. It is clear from the definition that $\mathrm{Kat}(\mathfrak{a}, \lambda_0, X)$ and $\mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ are vector spaces. We will see in section 6.3.1 that the Kato class may heavily depend on λ_0. In the following proposition we characterise λ_0-independence of the Kato class. Note that this also characterises λ_0-independence of the local Kato class, if we apply it to $\mathrm{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n))$.

Theorem 6.2.8. *Let $\mathfrak{a}$ be a local sub-Markovian form and X be a vector space.*

1. *Let $\lambda, \mu \in \rho(\mathcal{A})$ with $\lambda \neq \mu$. The following are equivalent:*
   - *(a)* $\mathrm{Kat}(\mathfrak{a}, \lambda, X) \subset \mathrm{Kat}(\mathfrak{a}, \mu, X)$.
   - *(b)* $D(\mathfrak{a}) \cap X \subset \mathrm{Kat}(\mathfrak{a}, \mu, X)$.
2. *Let $\Lambda \subset \rho(\mathcal{A})$ be a set containing at least two elements. The following are equivalent:*
   - *(a)* $\mathrm{Kat}(\mathfrak{a}, \lambda, X) = \mathrm{Kat}(\mathfrak{a}, \mu, X)$ *for all* $\lambda, \mu \in \Lambda$.
   - *(b)* $D(\mathfrak{a}) \cap X \subset \bigcap_{\lambda\in\Lambda} \mathrm{Kat}(\mathfrak{a}, \lambda, X)$.

Proof. (1) Assume (a) and let $u \in D(\mathfrak{a}) \cap X$. Then $\varphi := \lambda u - \mathcal{A}u \in \mathrm{Kat}(\mathfrak{a}, \lambda, X) \subset \mathrm{Kat}(\mathfrak{a}, \mu, X)$. The resolvent equation implies

$$R(\mu, \mathcal{A})\varphi - u = (\lambda - \mu) R(\mu, \mathcal{A}) u \ .$$

By assumption, the lefthand side belongs to X. Since X is a vector space and $\lambda \neq \mu$ it follows $R(\mu, \mathcal{A})u \in X$ proving (b). Now assume (b) and let $\varphi \in \mathrm{Kat}(\mathfrak{a}, \lambda, \mathcal{A})$. Then $u := R(\lambda, \mathcal{A})\varphi \in D(\mathfrak{a}) \cap X$, whence $R(\mu, \mathcal{A})\varphi = u + (\lambda - \mu)R(\mu)u \in X$, i.e. $\varphi \in \mathrm{Kat}(\mathfrak{a}, \mu, X)$. (2) Follows from (1) since Λ contains at least two elements. $\square$

If for $\lambda > 0$ the (local) Kato class is independent of λ, then we will omit the dependence on λ and just write $\mathrm{Kat}(\mathfrak{a}, X)$ and $\mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, X)$.
To obtain regularity of solutions of (6.7) in $D(\mathfrak{a})_{\mathrm{loc}}$ for $\varphi \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$, we require some connection between the spaces $X(\omega_n)$ and our operator $\mathcal{A}$:

Definition 6.2.9. We say that a local sub-Markovian form $\mathfrak{a}$ has *local kernel belonging to* X, if for all ω_n and $u \in D(\mathfrak{a})$ the relation $\lambda_0 u - \mathcal{A}u = 0$ on $D(\mathfrak{a}, \omega_n)$ implies that $u \in X(\omega_n)$.

Theorem 6.2.10. *Let $\mathfrak{a}$ be a local sub-Markovian form and X be a vector space localised by $X(\omega_n)$. Assume that $\mathfrak{a}$ has local kernel belonging to X. Then*

1. $\mathrm{Kat}(\mathfrak{a}, \lambda_0, X) \subset \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ *and for $n \geq 0$ we have* $\mathrm{Kat}(\mathfrak{a}_{n+1}, \lambda_0, X(\omega_{n+1})) \subset \mathrm{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n))$.
2. *If $\varphi \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$, $u \in D(\mathfrak{a})_{\mathrm{loc}}$ and $\lambda_0 u - \tilde{\mathcal{A}}u = \varphi$, then $u \in X$.*

Proof. (1) Let $\varphi \in \mathrm{Kat}(\mathfrak{a}, \lambda_0, X)$ and $n \geq 0$. Then $u := R(\lambda_0, \mathcal{A})\varphi \in X \subset X(\omega_n)$. Put $u_n := R(\lambda_0, \mathcal{A}_n)\varphi$. We have to show $u_n \in X(\omega_n)$. However, $\lambda_0(u - u_n) - \mathcal{A}(u - u_n) = 0$ on $D(\mathfrak{a}, \omega_n)$. Since $\mathfrak{a}$ has local kernel belonging to X we obtain $u - u_n \in X(\omega_n)$. But then also $u_n = u - (u - u_n) \in X(\omega_n)$. The proof of the second statement is similar.
(2) Fix $n \in \mathbb{N}_0$. By definition of $D(\mathfrak{a})_{\mathrm{loc}}$ there exists $v \in D(\mathfrak{a})$ such that $u = v$ a.e. on ω_{n+1}. By the definition of $\tilde{\mathcal{A}}$ we have $\lambda_0 v - \mathcal{A}v = \varphi$ on $D(\mathfrak{a}, \omega_n)$. Since $\varphi \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ we have $u_n := R(\lambda_0, \mathcal{A}_n)\varphi \in X(\omega_n)$. However $\lambda_0(v - u_n) - \mathcal{A}(v - u_n) = 0$ on $D(\mathfrak{a}, \omega_n)$. It follows that $v - u_n$ and hence also v and u belong to $X(\omega_n)$. Since n was arbitrary, $u \in X$. $\square$

In the sequel, we will be particularly interested in perturbing a local, sub-Markovian form $\mathfrak{a}$ by a sesquilinear form $\mathfrak{b}$. We do not require $\mathfrak{b}$ to be sectorial (and in particular not to be closed). We will call $\mathfrak{b}$ a *sub-Markovian perturbation* of $\mathfrak{a}$, if $\mathfrak{a} + \mathfrak{b}$, defined by $D(\mathfrak{a} + \mathfrak{b}) := D(\mathfrak{a}) \cap D(\mathfrak{b})$, $(\mathfrak{a} + \mathfrak{b})[u, v] := \mathfrak{a}[u, v] + \mathfrak{b}[u, v]$ is sub-Markovian. Such a perturbation will be called *local* , if $\mathfrak{a} + \mathfrak{b}$ is local. To obtain regularity for the perturbed form, we introduce Kato-perturbations:

Definition 6.2.11. Let $\mathfrak{a}$ be a local sub-Markovian form on $L^2(M, dm)$, $2 \leq p \leq \infty$ and $\mathfrak{b}$ be a local sub-Markovian perturbation of $\mathfrak{a}$ such that $D(\mathfrak{a})_c \subset D(\mathfrak{b})$. We denote for $u \in \mathfrak{b}$ by $\mathcal{B}u$ the linear map

$$D(\mathfrak{b}) \ni v \mapsto \langle \mathcal{B}u, v \rangle := -\mathfrak{b}[u, v]$$

1. $\mathfrak{b}$ is called a (p, X)-*Kato perturbation of* $\mathfrak{a}$, if $D(\mathfrak{a}) \subset D(\mathfrak{b})$ and $\mathcal{B}u \in \mathrm{Kat}(\mathfrak{a}, \lambda_0, X)$ for all $u \in D(\mathfrak{a}) \cap L^p(M)$.
2. Now let X be localised by $X(\omega_n)$. Then $\mathfrak{b}$ is called a *local* (p, X)- *Kato perturbation* of $\mathfrak{a}$ if $\mathcal{B}u \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ for all $u \in D(\mathfrak{a})_c \cap L^p(M)$.

Lemma 6.2.12. *Let X be a vector space localised by $X(\omega_n)$ and $\mathfrak{a}$ be a local sub-Markovian form on $L^2(M)$ having local kernel belonging to X. Then $\mathfrak{b}$ is a local (p, X)-Kato perturbation of $\mathfrak{a}$ if and only if $\mathfrak{b}$ is a $(p, X(\omega_n))$-Kato perturbation of $\mathfrak{a}_n$ for all $n \geq 0$.*

Proof. Let $\mathfrak{b}$ be a local (p, X)-Kato perturbation of $\mathfrak{a}$ and $u \in D(\mathfrak{a}, \omega_n) \cap L^p$. Then $u \in D(\mathfrak{a})_c \cap L^p$ whence $\mathcal{B}u \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X) \subset \mathrm{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n))$. That is, $\mathfrak{b}$ is a $(p, X(\omega_n))$-Kato perturbation of $\mathfrak{a}_n$.
Conversely, assume that $\mathfrak{b}$ is a $(p, X(\omega_n))$-Kato perturbation of $\mathfrak{a}_n$ for every $n \geq 0$. Let $u \in D(\mathfrak{a})_c$. Then there exists n_0, such that $u \in D(\mathfrak{a}, \omega_n)$ for all $n \geq n_0$. By hypothesis, $\mathcal{B}u \in \mathrm{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n)$ for all $n \geq n_0$. However, by Theorem 6.2.10 (1), we see $\mathcal{B}u \in \mathrm{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n))$ for all $n \geq 0$. □

Theorem 6.2.13. *Let $2 \leq p \leq \infty$, $\mathfrak{a}$ be a local sub-Markovian form on $L^2(M)$, and $\mathfrak{b}$ be a local sub-Markovian perturbation of $\mathfrak{a}$. Denote by $\mathcal{S}$ and S_2 the operators associated to $\mathfrak{s} := \mathfrak{a} + \mathfrak{b}$ on $D(\mathfrak{s})'$ and L^2 respectively and by S_p the (if $p = \infty$: weak*-) generator of the extrapolated semigroup on L^p. Further suppose that $R(\lambda_0, A_p)L^p \cap Y \subset X$ for some space Y. Then the following hold:*

1. *If $\mathfrak{b}$ is a (p, X)-Kato perturbation of $\mathfrak{a}$, then $R(\lambda_0, S_p)(L^2 \cap L^p \cap Y) \subset X \cap L^p$.*
2. *Additionally assume that X is localised by $X(\omega_n)$, $\mathfrak{a}$ has local kernel belonging to X and given $u \in D(\mathfrak{a})$ and $n \in \mathbb{N}$ we find $v \in D(\mathfrak{a}, \omega_{n+1})$ such that $u = v$ a.e. on ω_n. If $\mathfrak{b}$ is a local (p, X)-Kato perturbation of $\mathfrak{a}$ and $\tilde{\mathcal{S}}$ is an extension of S_p, then $R(\lambda_0, S_p)(L^p \cap Y) \subset X \cap L^p$.*

Proof. Let $f \in L^p(M) \cap Y$. Then $u = R(\lambda_0, S_p)f \in L^p$. In both cases we have to show that $u \in X$.
(1) If $f \in L^2 \cap L^p$, then $u \in D(S_2) \cap L^p \subset D(\mathfrak{a}) \cap L^p$ and $S_p u = \mathcal{A}u + \mathcal{B}u$ by Proposition 6.2.2. We see that $u = R(\lambda_0, \mathcal{A})(f + \mathcal{B}u)$. By assumption $f \in \mathrm{Kat}(\mathfrak{a}, \lambda_0, X)$ and also $\mathcal{B}u \in \mathrm{Kat}(\mathfrak{a}, \lambda_0, X)$, since $u \in D(\mathfrak{a}) \cap L^p$. Thus, $u \in X$.

(2) Since $\tilde{\mathcal{S}}$ is an extension of S_p, we have $u \in D(\mathfrak{s})_{\mathrm{loc}}$ and $(\lambda_0 - \tilde{\mathcal{A}})u = f + \tilde{\mathcal{B}}u$. By Theorem 6.2.10 (2), it suffices to prove $f + \tilde{\mathcal{B}}u \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$. Let $n \in \mathbb{N}_0$ be given. By hypothesis, there exists $v \in D(\mathfrak{a}, \omega_{n+1})$ such that $u = v$ a.e. on ω_n. We may assume that $v \in L^p$ (otherwise we replace v by $w := u^+ \wedge v^+ - u^- \wedge u^-$ which is an element of $D(\mathfrak{a})_c$ since $\mathfrak{a}$ is submarkovian and satisfies $|w| \leq |u|$ and is hence an element of L^p). By definition, $\tilde{\mathcal{B}}u = \mathcal{B}v$ on $D(\mathfrak{a}, \omega_n)$ and $\mathcal{B}v \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$. It follows that $\tilde{\mathcal{B}}u \in \mathrm{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n)$. Since n was arbitrary, the claim follows. □

The previous Theorem gives sufficient conditions for the resolvent $R(\lambda_0, S_p)$ to map L^p into $L^p \cap X$ and hence – in particular – for the domain of S_p to be a subset of X. It is also interesting to know, when also the semigroup T_p generated by S_p maps L^p to $L^p \cap X$.
For $2 \leq p < \infty$, there is no problem, since the holomorphy of the semigroup T_2 is inherited by the semigroup T_p for such p (see [Are04, Chapter 7.2]). However, for $p = \infty$ holomorphy and not even differentiability of the semigroup T_∞ can be expected. In fact, it follows from [Kun02] that there exists an open bounded set $\Omega \subset \mathbb{R}^N$ such that the spectrum of the Neumann Laplacian on $L^\infty(\Omega)$ contains a vertical line. Thus, the semigroup generated by it cannot be holomorphic or differentiable and hence does *not* map L^∞ into the domain of the generator.

However, we give a sufficient condition for the semigroup on L^∞ to leave X invariant.

Theorem 6.2.14. *Let Y be a closed subspace of $L^\infty(M)$ such that $D(S_\infty)$ is norm dense in Y and assume that $R(\lambda, S_\infty)Y \subset Y$ for all $\lambda > 0$. Then Y is invariant under the semigroup T_∞ and the restricted semigroup $T_\infty|_Y$ is strongly continuous.*

Proof. For $u \in D(S_\infty)$ the map $t \mapsto T_\infty(t)u$ is strongly continuous. Since $D(S_\infty)$ is norm dense in Y, the same is true for $u \in Y$. In particular, for $u \in Y$ we have

$$R(\lambda, S_\infty)u = \int_0^\infty e^{-\lambda t} T_\infty(t)u \, dt$$

as a Bochner integral, not just as a weak* integral.
Consider the quotient map $Q : L^\infty(M) \to L^\infty(M)/Y$. It is a bounded operator, even though not neccesarily weak* continuous. We obtain:

$$0 = QR(\lambda, S_\infty)u = Q \int_0^\infty e^{-\lambda t} T_\infty(t)u \, dt = \int_0^\infty e^{-\lambda t} Q T_\infty(t)u \, dt \ .$$

By the uniqueness of the Laplace transform (see [ABHN01, Theorem 1.7.3]) we obtain $QT_\infty(t)u = 0$ a.e. that is, $T_\infty(t)u \in Y$ for almost every t But since $t \mapsto T_\infty(t)u$ is strongly continuous, we have $T_\infty(t)u \in Y$ for every $t \geq 0$. □

6.2.3 Invariance of X_0

In this section we consider again a local sub-Markovian form $\mathfrak{a}$. We are interested in the subspace X_0 of X consisting of those elements of X vanishing at infinity, i.e.

$$X_0 := \{ f \in X \ : \forall\, \varepsilon > 0 \ \exists\, K \Subset M \text{ s. t. } |f(x)| \leq \varepsilon \ \ \forall\, x \in M \setminus K \}$$

In particular, we want to know, whether X_0 is invariant under $R(\lambda, A_\infty)$. However, belonging to X_0 is usually *not* a local property:

Example 6.2.15. *The space $X = C_0(\mathbb{R}^N) := \{u \in C(\mathbb{R}^N) \ : \ u(x) \to 0 \text{ as } x \to \infty\}$ cannot be localised. Indeed, if we assume that X was localised by some spaces $X(\omega_n)$, we may consider the function $\mathbb{1} : x \mapsto 1$. Then for every $k \geq 0$ there exists a function $f_k \in C_0(\mathbb{R}^N)$ such that $f_k = \mathbb{1}$ on ω_k. Property (1) in the definition of localised implies $f_k \in X(\omega_k)$ and now property (2) yields that $\mathbb{1} \in X(\omega_k)$. However, since k was arbitrary, it would follow that $\mathbb{1} \in C_0(\mathbb{R}^N)$, which is a contradiction.*

Thus, to obtain semigroups on X_0, one has to use different techniques. One possibility is to use domination and we will sketch how to use it in section 6.3.3. In this section we introduce a different possibility which makes use of Lyapunov functions. For this approach to work, one has to assume more about the form $\mathfrak{a}$:

Definition 6.2.16. Let $\mathfrak{a}$ be a local form. We say that $\mathfrak{a}$ satisfies the *local maximum principle* if the following holds:
If $\lambda > 0$, $0 \leq \varphi \in D(\mathfrak{a})'_{\text{loc}}$ and $v \in D(\mathfrak{a})^+_{\text{loc}}$ satisfies $\lambda v - \tilde{\mathcal{A}} v = \varphi$, then $u_n \leq v$, where $u_n = R(\lambda, \mathcal{A}_n)\varphi$. In other words, for any nonnegative $\varphi \in D(\mathfrak{a})'_{\text{loc}}$ the smallest nonnegative solution of

$$\lambda u - \tilde{\mathcal{A}} u = \varphi \quad \text{on } D(\mathfrak{a}, \omega_n)$$

is the one in $D(\mathfrak{a}, \omega_n)$.

Here, we call an element $\varphi \in D(\mathfrak{a})_{\mathrm{loc}}$ *positive*, if $\langle \varphi, u \rangle \geq 0$ for all $u \in D(\mathfrak{a})_c^+$.

Theorem 6.2.17. *Let $\mathfrak{a}$ be a local sub-Markovian form satisfying the local maximum principle and assume that $\tilde{A}$ is an extension of A_p for every $p \in [2, \infty]$. The the following are equivalent:*

1. *$D(\mathfrak{a})_c$ is dense in $D(\mathfrak{a})$.*
2. *For some (equivalently all) $p \in [2, \infty]$ we have $p - \lim_{n\to\infty} R(\lambda, \mathcal{A}_n)f = R(\lambda, A_p)f$ for all $f \in L^p$.*
3. *For some (equivalently all) $p \in [2, \infty]$ we have that if $f \in L^p_+$ and $v \in D(\mathfrak{a})^+_{\mathrm{loc}}$ satisfies $\lambda v - \tilde{A}v = f$ then $R(\lambda, A_p)f \leq v$.*

Proof. (1) $\Rightarrow$ (2) for $p = 2$: We have $D(\mathfrak{a}_n) \subset D(\mathfrak{a})$ and $\mathfrak{a}_n - \mathfrak{a} = 0$ is uniformly sectorial. Condition (1) states that $D := D(\mathfrak{a})_c$ is a core for $\mathfrak{a}$. Clearly, $D \subset \underline{\lim} D(\mathfrak{a}_n)$ and $\mathfrak{a}_n[u] \to \mathfrak{a}[u]$ for all $u \in D$.
Thus (2) for $p = 2$ follows directly from a version of the convergence theorem "from above" (cf. [Kat95, Theorem VIII.3.6]) for nondensely defined forms.
Now assume that (2) holds true for some $p \in [2, \infty]$. We show that the same holds for any $q \in [2, \infty]$. It suffices to prove this for nonnegative $f \in L^q$. Since $(\lambda - A_q)R(\lambda, A_q)f = (\lambda - \tilde{A})R(\lambda, A_q)f = f$ we obtain from the local maximum principle $R(\lambda, \mathcal{A}_n)f \leq R(\lambda, \mathcal{A}_{n+1})f \leq R(\lambda, A_q)f$ for all $n \geq 0$. Hence $R(\lambda, \mathcal{A}_n)f$ converges pointwise a.e. to some function $g \in L^q$.
If $f \in L^p \cap L^q$ then we have by consistency $R(\lambda, A_p)f = R(\lambda, A_q)f$. By our assumption we have $p - \lim R(\lambda, \mathcal{A}_n)f = R(\lambda, A_q)f$ and hence $g = R(\lambda, A_q)f$. Now the dominated convergence theorem implies $q - \lim R(\lambda, \mathcal{A}_n)f = R(\lambda, A_q)f$. The result for general $f \in L^p$ follows by approximation, using the uniform boundedness of $R(\lambda, \mathcal{A}_n)$, which holds, since the forms $\mathfrak{a}_n$ are uniformly sectorial.
(2) $\Rightarrow$ (3): Let $v \in D(\mathfrak{a})^+_{\mathrm{loc}}$ be given such that $\lambda v - \tilde{A}v = f$ for some $f \in L^p_+$. By the local maximum principle, we have $R(\lambda, \mathcal{A}_n)f \leq v$ for all n. But now (2) implies $R(\lambda, A_p)f = \lim R(\lambda, \mathcal{A}_n)f \leq v$.
Now assume (3) holds for some p. We prove, that it holds for any $q \in [2, \infty]$. By density, there exists an increasing sequence $f_n \in L^p \cap L^q$, such that $q - \lim f_n = f$. Using consistency and positivity we obtain

$$R(\lambda, A_q)f_n = R(\lambda, A_p)f_n \leq R(\lambda, A_p)f \leq v \ ,$$

by assumption. The continuity of $R(\lambda, A_q)$ now implies $R(\lambda, A_q)f = q - \lim R(\lambda, A_q)f_n \leq v$.
(3) $\Rightarrow$ (1): Define the form $\mathfrak{b}$ by $\mathfrak{b}[u, v] = \mathfrak{a}[u, v]$ and$D(\mathfrak{b}) = \overline{D(\mathfrak{a})_c}^{D(\mathfrak{a})}$. Then $\mathfrak{b}$ is a closed sectorial form and the continuity of the lattice operations imply that it is also sub-Markovian. Furthermore, the local spaces and operators associated to the forms $\mathfrak{a}$ and $\mathfrak{b}$ agree, in particular, $\mathfrak{b}$ satisfies the local maximum principle. However, $\mathfrak{b}$ satisfies condition (1) of this theorem and therefore (3) of this theorem holds true for $\mathfrak{b}$. We obtain $R(\lambda, B_2)f \leq R(\lambda, A_2)f$ for all $f \in L^2_+$. Since we assumed that (4) holds also for $\mathfrak{a}$ we obtain the reversed inequality and thus $R(\lambda, A_2) = R(\lambda, B_2)$. In particular $D(A_2) = D(B_2)$. However, by general theory $D(A_2)$ and $D(B_2)$ are cores of the forms $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Hence $\mathfrak{a}$ and $\mathfrak{b}$ coincide on a common core and thus have to be equal. In particular $D(\mathfrak{b}) = D(\mathfrak{a})$. □

Definition 6.2.18. Let $\mathfrak{a}$ be a local sub-Markovian form. We say that $\mathfrak{a}$ has *abstract Dirichlet boundary conditions* if $\mathfrak{a}$ satisfies the local maximum principle and $D(\mathfrak{a})_c$ is dense in $D(\mathfrak{a})$.

Lemma 6.2.19. *Let $\mathfrak{a}$ be a local sub-Markovian form which has abstract Dirichlet boundary conditions, $p \in [2,\infty]$ and $\lambda > 0$. Further suppose that $\tilde{A}$ is an extension of A_p. If $f, g \in D(\mathfrak{a})^+_{\mathrm{loc}}$ satisfy*

$$g \leq \lambda f - \tilde{A} f$$

and $g \in L^p$, then $R(\lambda, A_p) g \leq f$.

Proof. First note that if $\mathfrak{a}$ is any sub-Markovian form, then for $\lambda > 0$ also the resolvent of $\mathcal{A}$ is positive on $D(\mathfrak{a})'$. Indeed, if $\varphi \in D(\mathfrak{a})'_+$ and $u = R(\lambda, \mathcal{A})\varphi$. Since $\mathfrak{a}$ is submarkovian, $u^- \in D(\mathfrak{a})$ and $\mathfrak{a}[u^+, u^-] \leq 0$. Thus

$$0 \leq \langle \varphi, u^- \rangle = \lambda(u, u-) + \mathfrak{a}[u, u^-] \leq -\lambda \|u^-\|^2 - \mathfrak{a}[u^-, u^-] \leq -\lambda \|u - \|^2 \ .$$

Hence, $u^- = 0$.
From this observation, we obtain $R(\lambda, \mathcal{A}_n) g \leq R(\lambda, \mathcal{A}_n)(\lambda f - \tilde{A} f)$ for any $n \geq 0$. By the local maximum principle we have $R(\lambda, \mathcal{A}_n)(\lambda f - \tilde{A} f) \leq f$, whence $R(\lambda, \mathcal{A}_n) g \leq f$, for all $n \geq 0$. Since $\mathfrak{a}$ has abstract Dirichlet boundary conditions $R(\lambda, \mathcal{A}_n) g \to R(\lambda, A_p) g$ by Theorem 6.2.17 and the statement follows. □

We are now prepared to tackle the invariance of X_0. We shall again consider the space $X_b := X \cap L^\infty$ and assume that X_b is closed in L^∞. Clearly, X_0 is a closed subspace of X_b. By X_c we denote the vectorspace of all elements of X_b having compact support.

Theorem 6.2.20. *Let $\mathfrak{a}$ be a local sub-Markovian form which has abstract Dirichlet boundary conditions. Assume that X_b is a closed subspace of L^∞, that X_c is dense in X_0 and that for large λ we have $R(\lambda, A_\infty) X_c \subset X_b$. If there exists $\lambda_0 > 0$ and a strictly positive function $\varphi \in X_0 \cap D(\mathfrak{a})_{\mathrm{loc}}$ such that*

$$\lambda_0 \varphi - \tilde{A}\varphi \geq 0 \ , \tag{6.8}$$

then for $\lambda > \lambda_0$ we have $R(\lambda, A_\infty) X_0 \subset X_0$. If $\tilde{A}\varphi \in L^\infty_{\mathrm{loc}}$ then $R(\lambda, A_\infty) X_0 \subset X_0$ for $\lambda > \lambda_0$ large enough, provided that $\lambda_0 \varphi - \tilde{A}\varphi \geq 0$ on $M \setminus K$ for some compact set $K \Subset M$.

Proof. It follows from (6.8) that we have

$$(\lambda_0 + \varepsilon)\varphi - \tilde{A}\varphi \geq \varepsilon \varphi \ .$$

Thus, by Lemma 6.2.19 we obtain $\varepsilon R(\lambda, A_\infty)\varphi \leq \varphi$. For $f \in X_c$ we may find $c > 0$ such that $|f| \leq c\varphi$ since φ is strictly positive. It follows that

$$0 \leq R(\lambda, A_\infty)|f| \leq R(\lambda, A_\infty) c\varphi \leq \frac{c}{\varepsilon}\varphi \ .$$

Since $|R(\lambda, A_\infty) f| \leq R(\lambda, A_\infty)|f|$ we obtain $R(\lambda, A_\infty) X_c \subset X_0$. The general case follows by approximation. For the addendum observe, that if $\tilde{A}\varphi \in L^\infty_{\mathrm{loc}}$, then $\lambda\varphi - \tilde{A}\varphi \geq 0$, if $\lambda - \lambda_0 > \|\lambda_0 \varphi - \tilde{A}\varphi\|_{L^\infty(K)}$. □

6.3 Applications

6.3.1 The $C(\overline{\Omega})$-Kato class for multiplication operators

In this section we work on the space $L^2(\Omega, dx)$, where Ω is an open set in $\mathbb{R}^N$. We consider sub-Markovian forms $\mathfrak{a}$ defined by

$$\mathfrak{a}[u,v] := \int_\Omega u(x)\overline{v(x)}m(x)\,dx \ , \quad D(\mathfrak{a}) = L^2(\Omega) \ ,$$

where $0 \le m \in L^\infty(\Omega)$. In this case, $D(\mathfrak{a}) = D(\mathfrak{a})' = L^2(\Omega)$. Furthermore, the associated operator $\mathcal{A}$ is the multiplication operator given by $\mathcal{A}u = -mu$. In particular, $\rho(\mathcal{A}) = \{\lambda : (\lambda+m)^{-1} \in L^\infty\}$ and for $\lambda \in \rho(\mathcal{A})$ we have $R(\lambda,\mathcal{A})f = (\lambda+m)^{-1}f$. We will consider the regularity space $X = \{u \in L^2(\Omega) \ : \ \exists \tilde{u} \in C(\overline{\Omega}) \text{ such that } \tilde{u} = u \ \textit{a.e.}\}$. In the sequel we will denote for $u \in X$ its unique continuous version by $\tilde{u}$.

Proposition 6.3.1. *With the above definitions we have:*

1. *For $\lambda \in \rho(\mathcal{A})$ we have* $\mathrm{Kat}(\mathfrak{a},\lambda,X) = \{u(\lambda+m) \ : \ u \in X\}$. *In particular,* $\mathrm{Kat}(\mathfrak{a},\lambda,X)$ *is dense in $L^2(\Omega)$.*
2. *If $\lambda, \mu \in \rho(\mathcal{A})$ with $\lambda \neq \mu$ then*

$$\mathrm{Kat}(\mathfrak{a},\lambda,X) \cap \mathrm{Kat}(\mathfrak{a},\mu,X) = \{u(\lambda+m) \ : \ u \in X \ , \ \tilde{u}|_U \equiv 0\} \ ,$$

where U is the set of all points $x \in \Omega$ such that no version of m is continuous at x.

Proof. (1) is clear.
(2) Define m_0 by

$$m_0(x) := \overline{\lim}_{r\to 0+} \fint_{B(x,r)\cap\Omega} m(y)\,dy \ .$$

Since almost every $x \in \Omega$ is a Lebesgue point of m, m_0 is a version of m. It has the following additional property:
If m has a version $\bar{m}$ which is continuous at x_0, then $\bar{m}(x_0) = m_0(x_0)$. Furthermore, m_0 is continuous at x_0. This means, m_0 is continuous at every point $x \in \Omega \setminus U$. Now let $f \in \mathrm{Kat}(\mathfrak{a},\lambda,X) \cap \mathrm{Kat}(\mathfrak{a},\mu,X)$. Then there exist $u, v \in X$ with

$$f = \tilde{u}(\lambda+m_0) = \tilde{v}(\mu+m_0) \ \ \textit{a.e.}$$

We see that $(\tilde{u}-\tilde{v})(\lambda+m_0) = \tilde{v}(\mu-\lambda)$ a.e. This implies that m_0 is continuous on the open set $\mathcal{O} := \{x \in \Omega \ : \ \tilde{u}(x) \neq \tilde{v}(x) \ \}$. Indeed, m_0 agrees almost everywhere on $\mathcal{O}$ with the continuous function $(\tilde{u}-\tilde{v})^{-1}\tilde{v}(\mu-\lambda)-\lambda$. Since $\mathcal{O}$ is open, it follows that m has a version which is continuous at every point in $\mathcal{O}$. But the properties of m_0 imply that in fact m agrees with this continuous version everywhere on $\mathcal{O}$. It now follows that $w := (\tilde{u}-\tilde{v})(\lambda+m_0)$ is a continuous function. It is clear that w is continuous at every point $x \in \mathcal{O}$, since there it is the product of two functions continuous at x. On the other hand, if $x \in \Omega\setminus\mathcal{O}$, then $\tilde{u}(y)-\tilde{v}(y) \to \tilde{u}(x)-\tilde{v}(x) = 0$ if $y \to x$, whereas $(\lambda+m_0)$ is bounded, which shows that w is continuous at every x. We see that $(\tilde{u}-\tilde{v})(\lambda+m_0)$ and $v(\mu-\lambda)$ are two continuous functions which are equal a.e., hence, they are equal everywhere, in particular, $v = 0$ on $\mathcal{O}^c$. As above, it follows that $\tilde{v}(\lambda+m_0)$ is a continuous function. Hence we have showed that f has a continuous version which vanishes on

$\mathcal{O}^c$ and thus in particular on U. This proves one inclusion in the statement, the other inclusion is obvious. $\square$

Lemma 6.3.2. *There exists a measurable function $m : [0,1] \to [0,1]$ such that if $\bar{m}$ is a function such that $\bar{m} = m$ on $[0,1] \setminus N$, where N is a set of measure 0, then $\bar{m}$ is not continuous on $[0,1] \setminus N$.*

Proof. Let O_n be a sequence of open sets which are dense in $[0,1]$, such that $|O_n| \le \frac{1}{n}$ and $\bigcap O_n = \emptyset$. Such a sequence may be obtained as follows:
Let $\{q_k : k \in \mathbb{N}\} = \mathbb{Q} \cap [0,1]$ and $\{r_k : k \in \mathbb{N}\} = (\mathbb{Q} + \pi) \cap [0,1]$. Then define

$$O_n := \begin{cases} \bigcup_{k\in\mathbb{N}} B\left(q_k, \frac{1}{n2^{-k}}\right) & , \quad n \text{ even} \\ \bigcup_{k\in\mathbb{N}} B\left(r_k, \frac{1}{n2^{-k}}\right) & , \quad n \text{ odd} \end{cases} .$$

Now we define

$$m(t) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{1}_{O_n}(t) .$$

Clearly, m is a bounded, measurable function with values in $[0,1]$. Let $\bar{m}$ be a version of m, say $\bar{m} = m$ on $[0,1] \setminus N$ for a null set N, and x_0 be a continuity point of $\bar{m}$. Given $j \in \mathbb{N}$, we find δ_j such that

$$|\bar{m}(x_0) - \bar{m}(y)| < \frac{1}{2^{j+1}} \quad \text{for all } y \in B(x_0, \delta_j) .$$

It follows from the triangle inequality that

$$|m(x) - m(y)| < \frac{1}{2^j} \quad \text{for all } x, y \in B(x_0, \delta_j) \setminus N .$$

But now we see that for $n = 1, \dots, j-1$ and $x, y \in B(x_0, \delta_j) \setminus N$ we have $x \in O_n$ if and only if $y \in O_n$. Indeed, if $x \in O_k$ whereas $y \notin O_k$, then $|f(x) - f(y)| \ge 2^{-k}$. Thus $B(x_0, \delta_j) \setminus N$ is either a subset of O_n or of O_n^c for $1 \le n \le j-1$. However, if $B(x_0, \delta_j) \setminus N \subset O_n^c$, then we have $B(x_0, \delta_j) \subset O_n^c$, since O_n^c is closed and hence contains the closure of every set it contains. But we cannot have $B(x_0, \delta_j) \subset O_n^c$, since O_n is dense. Thus for any $j \in \mathbb{N}$ we have $B(x_0, \delta_j) \setminus N \subset O_n$ for $n = 1, \dots, j-1$. Hence if $x_0 \notin N$ it follows that $x_0 \in \bigcap O_n = \emptyset$. Thus x_0 can only lie in N. $\square$

Corollary 6.3.3. *There exists a local sub-Markovian form $\mathfrak{a}$ and a regularity space X such that* $\mathrm{Kat}(\mathfrak{a}, \lambda, X)$ *is dense in L^2 for every $\lambda \in \rho(\mathcal{A})$ whereas for $\lambda, \mu \in \rho(\mathcal{A})$ with $\lambda \ne \mu$ we have* $\mathrm{Kat}(\mathfrak{a}, \lambda, X) \cap \mathrm{Kat}(\mathfrak{a}, \mu, X) = \{0\}$.

Proof. Take $\mathfrak{a}$ as above with the function m from Lemma 6.3.2. If m_0 is defined as in the proof of Proposition 6.3.1, then it follows that m_0 is a version of m which is continuous in every point such that m has a version being continuous in that point. Lemma 6.3.2 implies that m_0 is only continuous on a null set. Now Proposition 6.3.1 (2) proves the claim. $\square$

6.3.2 Local regularity of deGiorgi-Nash forms

In this section we introduce a special class of sub-Markovian forms on the Hilbert space $L^2(\Omega, dx)$, where Ω is a domain in $\mathbb{R}^N$. For these forms, many elements of the local $C(\Omega)$-Kato class are known courtesy of the deGiorgi-Nash Theorem.

Definition 6.3.4. Let $\Omega \subset \mathbb{R}^N$ be a domain and $a_{ij}, b_i, c \in L^\infty(\Omega, dx, \mathbb{R})$ for $1 \leq i, j \leq N$. Further suppose that $c \geq 0$. Assume that there exist constants $\eta > 0$ and $M \geq 0$ such that

$$\mathrm{Re} \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\overline{\xi_i} \geq \eta|\xi|^2 \quad \text{and} \quad \left| \mathrm{Im} \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\overline{\xi_j} \right| \leq M \mathrm{Re} \sum a_{ij}(x)\xi_i\overline{\xi_j}$$

hold for all $\xi \in \mathbb{C}^d$ and almost every x. A *deGiorgi-Nash form* is a form $(\mathfrak{a}, D(\mathfrak{a}))$ satisfying the following conditions:

1. $D(\mathfrak{a})$ is a closed subspace of $H^1(\Omega)$ containing $H^1_0(\Omega)$ such that if $f \in D(\mathfrak{a})$ then also $\mathrm{Re}\, f, f^+, \mathrm{sgn} f \cdot (1 \wedge f) \in D(\mathfrak{a})$.

2. For $f, g \in D(\mathfrak{a})$ we have

$$\mathfrak{a}[f,g] = \int_\Omega \sum_{i,j=1}^{N} a_{ij} D_i f \overline{D_j g} + \sum_{i=1}^{N} b_i (D_i f)\overline{g} + cf\overline{g}\, dx$$

   for all $f, g \in D(\mathfrak{a})$.

Clearly, deGiorgi-Nash forms are densely defined and local. It is not hard to see that they are also sectorial and closed, in fact, $\| \cdot \|_{\mathfrak{a}}$ is equivalent to the Sobolev norm $\| \cdot \|_{H^1}$ (this is clear if the b_i are vanishing, however, the drift part may be treated as a form perturbation of the rest, which yields an equivalent scalar product). It follows from [Ouh04, 4.1, 4.2 and 4.9] that deGiorgi-Nash forms are sub-Markovian.
We note that $D(\mathfrak{a})_{\mathrm{loc}} = H^1_{\mathrm{loc}}(\Omega)$ and that if $D(\mathfrak{a}) = H^1_0(\Omega)$ then $D(\mathfrak{a})' = H^{-1}(\Omega)$. Otherwise $D(\mathfrak{a})'$ is a subspace of $H^{-1}(\Omega)$. It follows from Hölder's inequality that for bounded Ω we have $W^{-1,p}(\Omega) \hookrightarrow H^{-1}(\Omega)$ for $p > \frac{N}{2}$. Recall that any $\varphi \in W^{-1,p}$ may be represented as $g + \sum_{i=1}^N D_i f_i$, where $g, f_i \in L^p$, see [Ada75, Chapter III]. Thus in the injection $W^{-1,p} \hookrightarrow H^{-1}(\Omega)$ we identify φ with the functional

$$H^1_0(\Omega) \ni u \mapsto \int_\Omega \left(gu - \sum_{i=1}^{N} f_i D_i u \right) dx \ .$$

We will be interested in the regularity space $X = C(\Omega)$, more precisely

$$X = \{ u \in L^1_{\mathrm{loc}}(\Omega) \,:\, u \text{ has a version which is continuous on } \Omega \} \ .$$

For localisation we will use a sequence ω_n of open, bounded sets such that $\overline{\omega_n} \subset \Omega$. This corresponds to choosing $M = \Omega$ in the previous sections. We will discuss an application of choosing M differently in the next section. In this case, $D(\mathfrak{a}, \omega_n) = \tilde{H}^1_0(\omega) := \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ a.e. on } \omega_n^c\}$. However, $\tilde{H}^1_0(\omega_n) = H^1_0(\omega_n)$ if ω_n satisfies a mild regularity assumption, e.g. ω_n has Lipschitz boundary. It is no loss of generality to assume that $D(\mathfrak{a}, \omega_n) = H^1_0(\omega_n)$ since every domain may be exhausted by an increasing sequence of open sets having Lipschitz

boundary.
We localise our regularity space X by by the spaces

$$X(\omega_n) := \{u \in L^1_{\mathrm{loc}}(\Omega) \,:\, u \text{ has a version which is continuous on } \omega_n\}\ .$$

Now elements of the Kato class for $X = C(\Omega)$ are easily available courtesy of the deGiorgi-Nash Theorem (cf. [GT01, Theorems 8.22 and 8.24], which we restate in our terminology:

Theorem 6.3.5. *(deGiorgi-Nash)*
Let $\mathfrak{a}$ be a deGiorgi-Nash form on $L^2(\Omega)$, ω be an open subset of Ω and $\lambda \in \mathbb{C}$. Furthermore, let $f_1, \dots, f_d \in L^p(\Omega, dx)$, $g \in L^{\frac{p}{2}}(\Omega, dx)$ for some $p > N$ and $\psi \in H^1(\Omega)$ be given. If $u \in D(\mathfrak{a})$ is a solution of the generalised Dirichlet Problem

$$\mathbf{D}_{\mathfrak{a},\lambda,\omega} \begin{cases} \lambda u - \mathcal{A}u &= g + \sum D_i f_i \quad \text{on } H^1_0(\omega) \\ u &= \psi \quad \text{on } \partial\omega \end{cases}$$

then u is locally Hölder continuous on ω.

Corollary 6.3.6. *Let $(\mathfrak{a}, D(\mathfrak{a}))$ be a deGiorgi-Nash form, $\lambda \in \rho(\mathcal{A})$ and $p \geq 2$.*

1. *If Ω is bounded and $D(\mathfrak{a}) = H^1_0(\Omega)$, then $L^p(\Omega) \subset \mathrm{Kat}(\mathfrak{a}, \lambda, X)$ for $p \in (\frac{N}{2}, \infty]$ and $W^{-1,p}(\Omega) \subset \mathrm{Kat}(\mathfrak{a}, \lambda, X)$ for $p \in (N, \infty]$.*
2. *If $p \in (\frac{N}{2}, \infty]$, then $L^p_{\mathrm{loc}}(\Omega) \subset \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$. If $p \in (N, \infty]$, then $W^{-1,p}_{\mathrm{loc}}(\Omega) := \bigcap W^{-1,p}(\omega_n) \subset \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$.*
3. *$\mathfrak{a}$ has local kernel belonging to X and $\mathrm{Kat}(\mathfrak{a}, \lambda, X)$ and $\mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$ are independent of $\lambda \in \rho(\mathcal{A})$.*

We remark that the definition of $W^{-1,p}_{\mathrm{loc}}(\Omega) := \bigcap W^{-1,p}(\omega_n)$ does not depend on the sequence ω_n.

Proof. (1) The assumption that Ω be bounded yields $L^p(\Omega), W^{-1,p}(\Omega) \subset H^{-1}(\Omega)$ for the respective values of p. Since $D(\mathfrak{a}) = H^1_0(\Omega)$ we have $H^{-1}(\Omega) = D(\mathfrak{a})'$. The assertion now follows immediately from Theorem 6.3.5 noting that $u = R(\lambda, \mathcal{A})\varphi$, is a solution of $\mathbf{D}_{\mathfrak{a},\lambda,\Omega}$ for $\psi = 0$ and right hand side φ.
(2) Follows from (1) and the definition of the local Kato class, observing that $\mathfrak{a}_n$ is just the form $\mathfrak{a}$ restricted to $H^1_0(\omega_n)$.
(3) If $u \in D(\mathfrak{a})$ satisfies $\lambda_0 u - \mathcal{A}u = 0$ on $D(\mathfrak{a}, \omega_n)$, then u is a solution of $\mathbf{D}_{\mathfrak{a},\lambda_0,\omega_n}$ with right hand side $0 \in L^\infty(\omega_n)$ and boundary values $\psi = u$. It follows from Theorem 6.3.5 that $u \in X(\omega_n)$. To see that the Kato classes are independent of λ observe that since $D(\mathfrak{a}) \cap X \subset L^\infty_{\mathrm{loc}}(\Omega)$ we have $D(\mathfrak{a}) \cap X \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$. Since $\mathfrak{a}$ has local kernel belonging to X we have $R(\lambda, \mathcal{A})\varphi \in X$ for any $\varphi \in D(\mathfrak{a}) \cap X$ by Theorem 6.2.10 (2). Thus $D(\mathfrak{a}) \cap X \subset \bigcap_{\lambda \in \rho(\mathcal{A})} \mathrm{Kat}(\mathfrak{a}, \lambda, X)$ and $D(\mathfrak{a}_n) \cap X(\omega_n) \subset \bigcap_{\lambda \in \rho(\mathcal{A})} \mathrm{Kat}(\mathfrak{a}_n, \lambda, X(\omega_n))$. Now Theorem 6.2.10 (2) implies that the Kato classes are independent of λ. □

We now turn to Kato perturbations of deGiorgi-Nash forms. We will focus on perturbing a deGiorgi-Nash form by a measure. Viewed as an operator, a measure μ should be associated with the form $\mathfrak{m}[u,v] = \int_\Omega u\overline{v}\,d\mu$. However, if μ is not absolutely continuous with respect to Lebesgue measure, then the meaning of the latter integral is not clear. This leads to the following definition:

Definition 6.3.7. Let $(\mathfrak{a}, D(\mathfrak{a}))$ be a deGiorgi-Nash form on $L^2(\Omega, dx)$. A positive measure μ on $\overline{\Omega}$ is called *admissible for* $\mathfrak{a}$ if there is a continuous linear mapping

$$\begin{array}{rccc} J: & D(\mathfrak{a}) & \to & L^1_{\mathrm{loc}}(\overline{\Omega}, d\mu) \\ & u & \mapsto & \tilde{u} \end{array}$$

such that the following hold:

(A1) J preserves positivity, i.e. $u \geq 0$ dx-a.e. implies $\tilde{u} \geq 0$ μ-a.e.

(A2) J is multiplicative, in particular, if $u \cdot v = 0$ dx-a.e then $\tilde{u} \cdot \tilde{v} = 0$ μ-a.e.

(A3) $J\mathbb{1} = \mathbb{1}$.

(A4) For $\omega \Subset \Omega$ we have $JD(\mathfrak{a}, \omega) \hookrightarrow L^2(\overline{\Omega}, d\mu)$, i.e. there exists a constant C_ω such that $\|\tilde{u}\|_{L^2(\mu)} \leq C_\omega \cdot \|u\|_{\mathfrak{a}}$.

Examples 6.3.8. *1. If μ is absolutely continuous with repect to Lebesgue measure, then it is admissible for any deGiorgi-Nash form; one may choose J as the identity mapping.*

2. If $N = 1$, then $D(\mathfrak{a}) \subset H^1(\Omega) \hookrightarrow C(\overline{\Omega})$. Thus if we choose J as this injection restricted to $D(\mathfrak{a})$, we see that any locally finite measure on $\overline{\Omega}$ is admissible for $\mathfrak{a}$.

If we are given a deGiorgi-Nash form and an admissible measure μ, we define the form $\mathfrak{m}$ as

$$\mathfrak{m}[u,v] = \int_{\overline{\Omega}} \tilde{u}\overline{\tilde{v}}\,d\mu \qquad D(\mathfrak{m}) = \left\{ u \in D(\mathfrak{a}) \,:\, \tilde{u} \in L^2(\overline{\Omega}, d\mu) \right\} \,. \tag{6.9}$$

Theorem 6.3.9. *Let $\mathfrak{a}$ be a deGiorgi Nash form and μ be an admissible measure for $\mathfrak{a}$. Define $\mathfrak{m}$ by (6.9). Then we have:*

1. *$\mathfrak{m}$ is a local sub-Markovian perturbation of $\mathfrak{a}$.*

2. *If the coefficients a_{ij} belong to $W^{1,\infty}(\Omega)$, then $\mathfrak{a} + \mathfrak{m}$ has rich domain.*

3. *If the sets ω_n are chosen such that $H^1_0(\omega_n) = \{u \in H^1(\omega) \;:\; u = 0\, a.e.\ on\ \omega_n^c\}$, then $\mathfrak{a} + \mathfrak{m}$ satisfies the local maximum principle.*

Proof. (1) We prove that $\mathfrak{a} + \mathfrak{m}$ is a closed sectorial form. Since J is positivity preserving, the numerical range of $\mathfrak{m}$ is a subinterval of the positive real axis, whence $\mathfrak{a} + \mathfrak{m}$ is sectorial. We prove that $\mathfrak{a} + \mathfrak{m}$ is closed. First observe that

$$\|u\|_{\mathfrak{a}+\mathfrak{m}} \simeq \|u\|_{H^1} + \|\tilde{u}\|_{L^2(d\mu)} \,.$$

Hence, given a $\|\cdot\|_{\mathfrak{a}+\mathfrak{m}}$ Cauchy sequence, we see that it is a Cauchy sequence in $(D(\mathfrak{a}), \|\cdot\|_{H^1})$ and in $L^2(\overline{\Omega}, d\mu)$. By completeness of these spaces, there exist $u \in D(\mathfrak{a})$ and $v \in L^2(\overline{\Omega}, d\mu)$ such that $u_n \to u$ with respect to $\|\cdot\|_{\mathfrak{a}}$ and $\tilde{u}_n \to v$ with respect to $\|\cdot\|_{L^2(d\mu)}$. Since $\tilde{u}_n \to \tilde{u}$ in $L^1_{\mathrm{loc}}(d\mu)$, we have $\tilde{u} = v$. This proves $u \in D(\mathfrak{a} + \mathfrak{m})$. Clearly, $u_n \to u$ with respect to $\|\cdot\|_{\mathfrak{a}+\mathfrak{m}}$.

That $\mathfrak{a}+\mathfrak{m}$ is sub-Markovian follows from checking the Beurling-Deny criteria.

(2) Let $n \in \mathbb{N}$ and choose $\varphi \in C_c^\infty(\omega_n)$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on ω_{n-1}. Using the admissibility of μ, it is easily seen, that multiplication with such a function is a bounded operator on $D(\mathfrak{a}+\mathfrak{m})$. Conditions (1) and (2) in the definition of rich domain are obvious. It remains to show, that there exists a constant $\tilde{C}_n$ such that

$$\|(\mathcal{A}+\mathcal{M})v\|_{D(\mathfrak{a}+\mathfrak{m},\omega_n)'} \leq \tilde{C}_n \left(\|u\|_{L^2(\omega_n)} + \|(\mathcal{A}+\mathcal{M})u\|_{D(\mathfrak{a}+\mathfrak{m},\omega_n)'}\right) . \tag{6.10}$$

To this end, first observe that

$$(\mathfrak{a}+\mathfrak{m})[\varphi u, w] = (\mathfrak{a}+\mathfrak{m})[u,\varphi w] \tag{6.11}$$
$$+\int_{\omega_n} u \cdot \left(\sum_{i,j=1}^N a_{ij} D_i\varphi \overline{D_j w} + \sum_{i=1}^N b_i \overline{w} D_i \varphi\right) dx \tag{6.12}$$
$$+\int_{\omega_n} u \cdot \sum_{i,j=1}^N D_i\left(\overline{w} a_{ij} D_j \varphi\right) dx , \tag{6.13}$$

which is easily verified using differntiation rules and integration by parts (which uses that $a_{ij} \in W^{1,\infty}$. Now let $B := \{v \in D(\mathfrak{a}+\mathfrak{m},\omega_n) : \|v\|_{\mathfrak{a}+\mathfrak{m}} \leq 1\}$. By definition, we have $\|(\mathcal{A}+\mathcal{M})\varphi u\|_{D(\mathfrak{a},\omega_n)} = \sup_{w\in B} |(\mathfrak{a}+\mathfrak{m})[\varphi u, w]|$. To estimate this norm, it thus suffices to estimate the absolute value of the terms in (6.11) , (6.12) and (6.13). Since multiplication with φ is a bounded operation on $D(\mathfrak{a}+\mathfrak{m})$, there exists a constant $C_{n,1}$ such that

$$\sup_{w\in B} |(\mathfrak{a}+\mathfrak{m})[u,\varphi w]| \leq C_{n,1} \|(\mathcal{A}+\mathcal{M})u\|_{D(\mathfrak{a}+\mathfrak{m},\omega_n)} .$$

The absolute values of the terms in (6.12) and (6.13) may be estimated using the Cauchy Schwarz inequality by $C_{n,2} \cdot \|u\|_{L^2(\omega_n,dx)}$, where $C_{n,2}$ is a constant depending only on the coefficients a_{ij}, b_i and φ. Together, estimate (6.10) follows.

(3) Let $0 \leq \varphi \in D(\mathfrak{a}+\mathfrak{m})'_{\mathrm{loc}}$ and $0 \leq v \in D(\mathfrak{a}+\mathfrak{m})_{\mathrm{loc}}$ with $\lambda v - (\tilde{\mathcal{A}}+\tilde{\mathcal{M}})v = \varphi$ be given. Fix $n \geq 0$ and put $u_n = R(\lambda, (\mathcal{A}+\mathcal{M})_n)\varphi$. By the definition of $D(\mathfrak{a}+\mathfrak{m})_{\mathrm{loc}}$ there exists $v_{n+1} \in D(\mathfrak{a}+\mathfrak{m})$ such that $v = v_{n+1}$ a.e. on ω_{n+1}. We obtain

$$\lambda(u_n - v_{n+1}, w) + (\mathfrak{a}+\mathfrak{m})[u_n - v_{n+1}, w] = \langle \varphi, w\rangle - \langle \varphi, w\rangle = 0 , \tag{6.14}$$

for all $w \in D(\mathfrak{a}+\mathfrak{m},\omega_n)$. Arguing as in the proof of the weak maximum principle (cf. [GT01, Theorem 8.1]), this implies

$$\sup_{\omega_n}(u_n - v_{n+1}) \leq \sup_{\partial\omega_n}(u_n - v_{n+1})^+ . \tag{6.15}$$

However, u_n vanishes on the boundary of ω_n, whereas v_{n+1} is positive there, whence $(u_n - v_{n+1})^+ = 0$. Thus, (6.15) implies that $u_n \leq v_{n+1} = v$ on ω_n. But since u_n vanishes almost everywhere outside ω_n, we have $u_n \leq v$ a.e. on Ω. $\square$

Remark 6.3.10. *If one drops the requirement that $c \geq 0$ in the definition of deGiorgi-Nash form, then one obtains quasi sub-Markovian forms, i.e. forms $\mathfrak{a}$ such that $\gamma+\mathfrak{a}$ is sub-Markovian* *for some* $\gamma > 0$. *All of our theory works also for quasi sub-Markovian forms. However, perturbing a quasi sub-Markovian form by a signed measure* μ, *i.e. allowing signed measures in (6.9), one cannot expect* $\mathfrak{a} + \mathfrak{m}$ *to be quasi sub-Markovian again, unless the negative part* μ^- *has an* L^∞ *density with respect to Lebesgue measure. Indeed, the form*

$$(\mathfrak{a} + \mathfrak{m})[u, v] = \int_0^1 u'\overline{v'}\, dx - f(1)\overline{v(1)}$$

is not quasi sub-Markovian.

Now we obtain information about when $\mathfrak{m}$ is a Kato perturbation of $\mathfrak{a}$ from Corollary 6.3.6.

Theorem 6.3.11. *Let* $\mathfrak{a}$ *be a deGiorgi Nash form and* $\mathfrak{m}$ *as in (6.9).*

1. *Assume that* $\mu = V dx$. *I If* $V \in L^q(\Omega, dx)$ *(* $V \in L^q_{\mathrm{loc}}(\Omega, dx)$ *) for some* $q > \frac{N}{2}$, *then* $\mathfrak{m}$ *is a (local)* $(p, C(\Omega))$*-Kato perturbation of* $\mathfrak{a}$ *for* $p > \frac{Nq}{2q-N}$.
2. *If for some* $q > N$ *the mapping* J *injects* $W_0^{1,q}(\omega)$ *into* $L^1(\Omega, d\mu)$ *for every* $\omega \Subset \Omega$, *then* $\mathfrak{m}$ *is a local* $(\infty, C(\Omega))$*-Kato perturbation of* $\mathfrak{a}$. *If* J *injects* $W^{1,q}(\Omega)$ *into* $L^1(\Omega, d\mu)$, *then* $\mathfrak{m}$ *is a* $(\infty, C(\Omega))$*-Kato perturbation of* $\mathfrak{a}$.

Proof. (1) If $V \in L^q_{(\mathrm{loc})}$ and $u \in L^p$, then by Hölder's inequality $\mathcal{M}u = V \cdot u \in L^r_{(\mathrm{loc})}$ where

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < \frac{1}{q} + \frac{2q - N}{Nq} = \frac{2}{N} .$$

Now the assertion follows from Corollary 6.3.6.
(2) Let $\omega \Subset \Omega$, $v \in W_0^{1,q}(\omega) \subset H_0^1(\omega) \subset D(\mathfrak{a})$ and $u \in D(\mathfrak{m}) \cap L^\infty$. Note that by (A3), we have $\tilde{u} \in L^\infty$ and hence we obtain

$$|\mathfrak{m}[u, v]| \le \int_\omega |\tilde{u}\overline{\tilde{v}}|\, d\mu \le C\|\tilde{v}\|_{L^1(\omega, d\mu)} \le C_1\|v\|_{W^{1,q}(\omega)} .$$

Since ω was arbitrary, we see that $\mathcal{M}u \in W^{-1,q}_{\mathrm{loc}}(\Omega)$. It follows from Corollary 6.3.6 that $\mathfrak{m}$ is a $(\infty, C(\Omega)$-Kato perturbation of $\mathfrak{a}$.

□

Remark 6.3.12. *Kato perturbations of deGiogi-Nash forms are not necessarily of this type. Indeed, using Sobolev embeddings and a perturbation result for forms (see [Kat95, Theorem VI.1.33]) it can be shown that*

$$\mathfrak{d}[u, v] := \int_\Omega \sum_{i=1}^N d_i D_i u \cdot \overline{v}, \quad D(\mathfrak{d}) := H_0^1(\Omega) ,$$

is a sub-Markovian perturbation of any deGiorgi-Nash form $\mathfrak{a}$, *provided* $q > \max\{2, N\}$ *and* $D(\mathfrak{a}) = H_0^1(\Omega)$. *However,* $\mathcal{D}u \in L^r$ *where* $\frac{1}{r} = \frac{1}{2} + \frac{1}{q}$. *Thus, if* $r > \frac{N}{2}$, *then* $\mathfrak{d}$ *is a* $(p, C(\Omega))$*-Kato perturbation of* $\mathfrak{a}$ *for any* $p \in [2, \infty]$.

6.3.3 Perturbation by a Potential and Semigroups on C_0

Having studied interior regularity properties of perturbations of deGiorgi-Nash forms in the previous section, we now want to obtain a semigroup on the space $C_0(\Omega)$. We will always consider forms $\mathfrak{a}+\mathfrak{m}$, where $\mathfrak{a}$ is a deGiorgi-Nash form with $D(\mathfrak{a}) = H_0^1(\Omega)$ and $\mathfrak{m}$ is defined by (6.9) with $\mu = V\,dx$ for an L^∞_{loc} function $V \geq 0$. Using [Ouh04, Theorem 2.21], it is easy to see that the semigroup P_∞ associated to $\mathfrak{a}+\mathfrak{m}$ is dominated by the semigroup T_∞ associated to $\mathfrak{a}$, i.e. $|P_\infty(t)f| \leq T_\infty(t)|f|$ for all $f \in L^\infty$. Thus, if T_∞ leaves $C_0(\Omega)$ invariant – which is equivalent to Ω being Dirichlet regular, cf. [AB98, Theorem 4.1] – then so does P_∞.
We will prove in this section, that given any bounded Ω and any deGiorgi-Nash form $\mathfrak{a}$ on $H_0^1(\Omega)$, there exists a potential $V \in L^\infty_{\text{loc}}(\Omega)$ such that P_∞ leaves $C_0(\Omega)$ invariant and the restriction is strongly continuous.

Theorem 6.3.13. *Let $\mathfrak{a}$ be a deGiorgi-Nash form with $D(\mathfrak{a}) = H_0^1(\Omega)$ and $a_{ij} \in W^{1,\infty}(\Omega)$. If there exists some $g \in C^2(\Omega) \cap C_0(\Omega)$ which is strictly positive and satisfies*

$$|D^\alpha g| \leq C|g|^{1-|\alpha|} \quad \text{for } 1 \leq |\alpha| \leq 2 \tag{6.16}$$

for some constant $C \geq 0$, then $V = g^{-2}$ is a local $(\infty, C(\Omega))$-Kato perturbation of $\mathfrak{a}$ and the perturbed semigroup on L^∞ leaves $C_0(\Omega)$ invariant. Furthermore, the restriction of the perturbed semigroup to $C_0(\Omega)$ is strongly continuous.

Proof. Clearly, $V \in L^\infty_{\text{loc}}(\Omega)$. Hence $\mathfrak{m}$ defined by (6.9) is a local $(\infty, C(\Omega))$-Kato perturbation of $\mathfrak{a}$. Furthermore, Theorem 6.3.9 implies that $\mathfrak{a}+\mathfrak{m}$ has rich domain and satisfies the local maximum principle. It is easy to see that $\mathfrak{a}+\mathfrak{m}$ has Dirichlet boundary conditions. Denote by S_∞ the weak* -generator of the semigroup T_∞ associated to $\mathfrak{a}+\mathfrak{m}$ on L^∞. It follows from Theorem 6.2.13, that for $\lambda > 0$, $R(\lambda, S_\infty)$ leaves the space $C_b(\Omega)$ invariant. To prove invariance of $C_0(\Omega)$, we apply Theorem 6.2.20:
We try to use $\varphi = g^\gamma$ as a Lyapunov function. Here, γ is a positive constant to be specified later. Then $\varphi \in C^2(\Omega) \cap C_0(\Omega)$ is strictly positive. Using integration by parts, we see that

$$\tilde{\mathcal{A}}\varphi = \sum_{i,j=1}^N a_{ij} D_{ij}\varphi - \sum_{i=1}^N \tilde{b}_i D_i \varphi - c\varphi\ .$$

Here, $\tilde{b}_i$ are modified coefficients, depending on b_i and partial derivatives of a_{ij} obtained from integration by parts. Rewriting this in terms of g we have

$$\tilde{\mathcal{A}}\varphi = \gamma g^{\gamma-1}\tilde{\mathcal{A}}_0 g + \gamma(\gamma-1)g^{\gamma-2}\langle A\nabla g, \nabla g\rangle - cg^\gamma\ ,$$

where $\tilde{\mathcal{A}}_0 u := \tilde{\mathcal{A}}u + cu$ and A is the matrix containing the entries a_{ij}. Thus, we see that

$$\lambda\varphi - (\tilde{\mathcal{A}} - V)\varphi = g^{\gamma-2}\Big((\lambda+c)g^2 - \gamma g\tilde{\mathcal{A}}_0 g - \gamma(\gamma-1)\langle A\nabla g, \nabla g\rangle + 1\Big)\ .$$

It follows from the assumptions on g, that $g\tilde{\mathcal{A}}_0 g$ is a bounded function, say $|g\tilde{\mathcal{A}}_0 g| \leq M$. Choose $0 < \gamma < \min\{\frac{1}{2M}, 1\}$. Since $\langle A\nabla g, \nabla g\rangle \geq 0$, we obtain

$$\lambda\varphi - \tilde{\mathcal{A}}\varphi + V\varphi \geq g^{\gamma-2}\left((\lambda+c)g^2 + \frac{1}{2}\right) \geq 0\ .$$

It follows from Theorem 6.2.20, that $R(\lambda, S_\infty)C_0(\Omega) \subset C_0(\Omega)$ for $\lambda > 0$. Clearly, $C_c^\infty(\Omega) \subset D(\mathfrak{a}+\mathfrak{m})_c \subset D(S_\infty)$. Thus it follows from Theorem 6.2.14, that T_∞ leaves $C_0(\Omega)$ invariant and the restricted semigroup $T_\infty|_{C_0(\Omega)}$ is strongly continuous. □

Corollary 6.3.14. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\mathfrak{a}$ be a deGiorgi-Nash form on $H_0^1(\Omega)$ with $a_{ij} \in W^{1,\infty}(\Omega)$. Then there exists a potential $V \in L^\infty_{\text{loc}}(\Omega)$ such that the semigroup T_∞ associated to $\mathfrak{a}+\mathfrak{m}$ leaves $C_0(\Omega)$ invariant and the restriction to that space is strongly continuous.*

Proof. The idea of the proof is to use as g in Theorem 6.3.13 the regularized distance to the boundary. Let $\rho(x) := \inf\{|x-y|\, y \in \partial\Omega\}$. Then ρ is Lipschitz continuous, strictly positive and $\rho \in C_0(\Omega)$. It follows from [Ste70, Theorem VI.2], that there exists $g \in C^\infty(\Omega)$ such that $c_1\rho \leq g \leq c_2\rho$ and the estimates (6.16) hold. Theorem 6.3.13 applied to $\mathfrak{a}$ and g yields the thesis. □

Thus, if we perturb an operator associated to deGiorgi-Nash forms with $a_{ij} \in W^{1,\infty}$ with a potential which grows near the boundary as the square of the distance to the boundary, then a realisation of the perturbed operator on $C_0(\Omega)$ generates a strongly continuous semigroup on $C_0(\Omega)$.
However, not every boundary point of an open set is "bad". Define the "good" boundary Γ_0 by

$$\Gamma_0 := \{x \in \partial\Omega \,:\, \exists\, g_x \in L^\infty(\Omega) \text{ strictly positive, s. t. } T(t)g_x(y) \to 0 \text{ as } y \to x \;\forall\, t \geq 0\}\ .$$

If $x \in \Gamma_0$, then we have $T(t)f(y) \to 0$ as $y \to x$ for all $f \in C_0(\Omega)$ and all $t \geq 0$. Indeed, if g_x is strictly positive and $f \in C_c(\Omega)$, then there exists a constant c such that $|f| \leq c \cdot g_x$. The positivity of $T(t)$ yields $|T(t)f| \leq cT(t)g_x$, whence $T(t)f(y) \to 0$ as $y \to x$ if $T(t)g_x(y) \to 0$ as $y \to x$. Now the density of $C_c(\Omega)$ in $C_0(\Omega)$ proves the assertion.
Thus, in order to prove invariance of $C_0(\Omega)$, it remains to take care of the "bad boundary" $\Gamma_1 := \partial\Omega \setminus \Gamma_0$. The question arises, whether it suffices to perturb $\mathfrak{a}$ near Γ_1, or else, to perturb $\mathfrak{a}$ with a potential which grows near the "good boundary" Γ_0 slower than ρ^{-2}. Indeed this is possible as the following consideration show:
We consider $\mathfrak{a}$ as a form on $M := \Omega \cup \Gamma_0$. Our regularity space X however is unchanged: $X := \{u \in L^1_{\text{loc}}(\Omega) \,:\, \exists \text{ a version of } u \text{ continuous on } \Omega\}$. The approximating sequence ω_n has to be chosen such that $\bigcup \omega_n = M$, i.e. ω_n has to contain some of the boundary of Ω. However, for $X(\omega_n)$ we still only demand a version continuous in the interior.

Example 6.3.15. *We consider $\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}$. Then the "good boundary" is the sphere $\partial B(0,1)$, whereas the "bad" boundary consists of the point 0. Thus $M := \{x \in \mathbb{R}^N \,:\, 0 < |x| \leq 1\}$. For localisation we consider $\omega_n := \{x \in \mathbb{R}^N \,:\, \frac{1}{n} < |x| \leq 1\}$ and*

$$X(\omega_n) = \{u \in L^1_{\text{loc}}(M) \,:\, u \textit{ has a version continuous on } \frac{1}{n} < |x| < 1\}\ .$$

Thus we have changed what we consider a "compact subset of Ω" whereas our notions of continuity remain unchanged (we do *not* require continuity on the boundary). It should be noted that concerning the Kato-class nothing has changed. Only "local" now means local with respect to M (e.g. $L^\infty_{\text{loc}}(M)$ is the space of functions bounded on compact subsets of M, they may not explode near the good boundary). This change in compact subsets now yields a different space X_0:

$$X_0 := \{u \in C(\Omega) \,:\, u(x) \to 0 \text{ as } x \to \Gamma_1\} \;.$$

The proofs of Theorem 6.3.13 remains unchanged when replacing $C_0(\Omega)$ by X_0. Using as g a regular version of $\rho_1(x) = \mathrm{dist}(x, \Gamma_1)$ we see that perturbing with a potential exploding near the bad boundary implies that P_∞ leaves invariant the continuous functions vanishing on the bad boundary. Combining this with the domination result above, we see that $P_\infty C_0(\Omega) \subset C_0(\Omega)$ for such perturbations.

Chapter 7

Application: Generation results for the Black-Scholes Operator

7.1 Introduction

Since the pioneering work of Black and Scholes (cf. [BS73]) the theory of partial differential equations plays an important role in the pricing theory of options. In their market model the price of an European option $H = h(S(\tau))$, where h is a suitable measurable function, written on a stock $S = (S(t))_{0\leq t\leq \tau}$ is the solution of the Cauchy problem

$$\text{(BS)} \quad \begin{cases} u_t(x,t) + \frac{1}{2}x^2\sigma^2 u_{xx}(t) + rxu_x(x,t) - ru(x,t) = 0 & , \quad (x,t) \in (0,\infty)\times \in [0,\tau] \\ u(x,\tau) = h(x) & , \quad x \in (0,\infty) \end{cases} ,$$

where σ and r are constants coming from the modelling assumptions and τ is the expiration date of the option (see, for instance, chapter 2). Thus, by solving the Cauchy problem (BS) one can determine the price of the option. Black and Scholes solved the problem with the aid of several transformations ending up with the classical heat equation for which a solution is well-known.

In this section we solve the Cauchy problem using the theory of C_0-semigroups on Banach spaces. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. On $W^{2,1}_{\text{loc}}(0,\infty)$ we consider the Black-Scholes operator

$$\mathcal{A}u := \alpha x^2 D^2 u + \beta x D u - \beta u$$

and ask for realizations of $\mathcal{A}$ in the Banach spaces $L^p(0,\infty)$, $2 \leq p < \infty$, and $C_0([0,\infty)) := \{f \in C([0,\infty);\mathbb{R}) \mid \lim_{x\to\infty} f(x) = 0\}$. To this end, we use two different approaches.

For a realization on $L^p(0,\infty)$ we start on $L^2(0,\infty)$ using a variational setting. We define an elliptic, densely defined, continuous form $(\mathfrak{a}, V)$ on $L^2(0,\infty)$, where the form domain V is some weighted Sobolev space and the associated operator A equals $-\mathcal{A}$ on $D(A) \cap W^{2,1}_{\text{loc}}(0,\infty)$. It follows that the operator $-A$ generates a bounded holomorphic C_0-semigroup T on $L^2(0,\infty)$. Moreover, we will show that the semigroup T is positive and submarkovian. Thus, by classical theory, it extrapolates to C_0-semigroups on $L^p(0,\infty)$ for $2 \leq p < \infty$. In addition, we prove the invariance of the order interval $[-\infty, \mathrm{id}]$ under the semigroup T using the techniques from chapter 4. The importance of this property from a financial perspective arises from no-arbitrage arguments in pricing theory (see chapter 2).

In [GMV02] and [GGMV07] Giuli, Gozzi, Monte and Vespri have studied the operator $\mathcal{A}$

with non-constant coefficients and an emphasis on the description of the domain of the generator in $L^p(0,\infty)$. However, in difference to them we do not need any conditions on the constants α and β (apart from strict positivity of α). In fact, we show in the notes that their assumptions (applied in a simple manner) lead to a contradiction for the Black-Scholes operator. Moreover, we use in contrary to them the submarkovian property of the semigroup T on $L^2(\Omega)$ for an extrapolation to $L^p(\Omega)$ for $2 \le p < \infty$.

For a realization on $C_0([0,\infty))$ we write $\mathcal{A}$ as $\alpha((B+\gamma)^2-(1+\gamma)^2)$ for a C_0-group generator B and a certain constant $\gamma \in \mathbb{R}$. It follows that the realization of $\mathcal{A}$ generates a holomorphic semigroup T in $C_0([0,\infty))$. The representation provides us further with an explicit formula of T in terms of the C_0-group generated by B. We use this formula to replicate the price of an European Put option in the classical Black-Scholes market.

This approach dates presumably back to Arendt and de Pagter (cf. [AP02]) where it was used for the L^p-case. Recently, Goldstein, Mininni and Romanelli (cf. [GMR07]) extended it to the spaces of continuous functions having finite limits at infinity.

7.2 C_0-semigroup on $L^p(0,\infty)$ by a variational approach

We consider the Black-Scholes Operator on L^p-spaces. Therefore, let $\alpha > 0$, $\beta \in \mathbb{R}$ and put $\psi(x) := x$ for $x \in (0,\infty)$. On $W^{2,1}_{\text{loc}}(0,\infty)$ we consider the degenerate differential operator

$$\mathcal{A}u := \alpha\psi^2 D^2 u + \beta\psi Du - \beta u,$$

and ask for realizations of $\mathcal{A}$ in $L^p(0,\infty)$, $2 \le p < \infty$. After introducing some required notations, we start on $L^2(0,\infty)$. We define a suitable densely defined, elliptic, continuous form $\mathfrak{a}$, whose domain is a weighted Sobolev space and the associated operator A equals $-\mathcal{A}$ on $D(A) \cap W^{2,1}_{\text{loc}}(0,\infty)$. Then it is known that $-A$ generates a holomorphic semigroup T on $L^2(0,\infty)$. Using the theory of chapter 4 we show in the final part that T is positive and submarkovian, thus extrapolates to semigroups on $L^p(0,\infty)$, $2 \le p < \infty$, and leaves the order interval $[-\infty,\psi]$ invariant.

7.2.1 Notation

We recall some notation needed throughout this section. Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be an open set. For two measurable functions $h, g : \Omega \to \mathbb{R}$ we write $h \le g$ if $h(x) \le g(x)$ a.e. on Ω and identify functions which coincide a.e.. We define on Ω the functions

$$\begin{aligned}
h \wedge g &:= \mathbb{1}_{\{h-g\ge 0\}} g + \mathbb{1}_{\{h-g\le 0\}} h,\\
h \vee g &:= \mathbb{1}_{\{h-g\ge 0\}} h + \mathbb{1}_{\{h-g\le 0\}} g,\\
h^+ &:= h \vee 0,\\
h^- &:= -(h \wedge 0).
\end{aligned}$$

Note that $h \wedge g = g - (h-g)^- = h - (h-g)^+$.

We denote by $\mathcal{D}(\Omega) := C^\infty(\Omega) \cap C_c(\Omega)$ the space of all test functions on Ω. Let $N = 1$ and $k \in \mathbb{N}$. A function $g \in L^1_{\text{loc}}(\Omega)$ is called the *k-th weak derivative* of a function $f \in L^1_{\text{loc}}(\Omega)$ if

$$(-1)^k \int_\Omega f D^k \varphi \, dx = \int_\Omega g\varphi \, dx$$

for all $\varphi \in \mathcal{D}(\Omega)$. Then we set $D^k f := g$. Note that the weak derivative is unique, since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$. For $1 \le p < \infty$ and $k \in \mathbb{N}$ we define the vector spaces

$$W^{k,p}_{\mathrm{loc}}(\Omega) := \{f \in L^p_{\mathrm{loc}}(\Omega) \mid D^l f \in L^p_{\mathrm{loc}}(\Omega) \text{ exists for all } 1 \le l \le k\}.$$

One has for $f, g \in W^{1,1}_{\mathrm{loc}}(\Omega)$ that the function $f \wedge g$ belongs to $W^{1,1}_{\mathrm{loc}}(\Omega)$. In fact, $f - g \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and so $(f-g)^+ \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $D(f-g)^+ = 1\!\!1_{\{f-g>0\}}(Df - Dg)$ (cf. [GT01, Lemma 7.6]). From the representation $f \wedge g = f - (f-g)^+ \in W^{1,1}_{\mathrm{loc}}(\Omega)$ we obtain $D(f \wedge g) = Df - D(f-g)^+ = 1\!\!1_{\{f>g\}} Dg + 1\!\!1_{\{f \le g\}} Df$.

In particular, for $k \ge 0$ it holds $f \wedge k \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $D(f \wedge k) = 1\!\!1_{\{f \le k\}} Df$. By Stampacchia's Lemma (cf. [GT01, Lemma 7.7]) $Df(x) = 0$ for almost all $x \in \{y \in \Omega \mid f(x) = k\}$ and hence $D(f \wedge k) = 1\!\!1_{\{f<k\}} Df$.

7.2.2 Generation results

We consider the real Hilbert space $H := L^2(0,\infty)$ and the identity function $\psi(x) := x$, $x > 0$. Let $\alpha > 0$, $\beta \in \mathbb{R}$. We define the sesquilinear form

$$\mathfrak{a}[u,v] := \alpha(\psi Du|\psi Dv)_H + (2\alpha - \beta)(\psi Du|v)_H + \beta(u|v)_H$$

on the form domain

$$V := \{u \in W^{1,1}_{\mathrm{loc}}(0,\infty) \cap H \mid \psi Du \in H\},$$

which is indeed a Hilbert space:

Lemma 7.2.1. *The vector space V is a Hilbert space for the scalar product*

$$(u|v)_V := (u|v)_H + (\psi Du|\psi Dv)_H, \quad u, v \in V.$$

The induced norm on V is given by

$$\|u\|_V^2 := (u|u)_V = \|u\|_H^2 + \|\psi Du\|_H^2, \quad u \in V.$$

Moreover, the test functions are dense in V, i.e. $\overline{\mathcal{D}(0,\infty)}^{\|\cdot\|_V} = V$.

Proof. We consider the separable Hilbert space $H^2 := H \times H$ with norm $\|(u,v)\|_{H^2}^2 := \|u\|_H^2 + \|v\|_H^2$. Let

$$\Psi : V \to H^2, \ u \mapsto (u, \psi Du).$$

Then Ψ is isometric and linear. Thus, it suffices to prove that the image of Ψ is closed. Let $(u_n)_{n\in\mathbb{N}} \subset V$ such that $\lim_{n\to\infty} \Psi(u_n) = (u,v)$ in H^2. Then $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} \psi Du_n = v$ in H. In particular, $u, v \in H$. It remains to prove $u \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ and $\psi Du = v$. Let $\varphi \in \mathcal{D}(0,\infty)$. Then:

$$-(u|D\varphi)_H = -\lim_{n\to\infty} (u_n|D\varphi)_H = \lim_{n\to\infty} (Du_n|\varphi)_H = \lim_{n\to\infty} (\psi Du_n|\frac{1}{\psi}\varphi)_H = (\frac{1}{\psi} v|\varphi)_H.$$

Hence, $u \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ with $Du = \frac{1}{\psi} v$ and therefore V is a Hilbert space.

For the second part we put $V_0 := \overline{\mathcal{D}(0,\infty)}^{\|\cdot\|_V}$. In order to prove $V = V_0$ we claim as a first step:

$$V \cap C_c(0,\infty) \subset V_0. \tag{7.1}$$

Let $u \in V \cap C_c(0,\infty)$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a mollifier, i.e. $0 \le \rho_n \in \mathcal{D}(\mathbb{R})$, $\int_{\mathbb{R}} \rho_n \, dx = 1$ and $\operatorname{supp}(\rho_n) \subset \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then the convolution

$$u_n(x) := (u * \rho_n)(x) := \int_0^\infty \rho_n(x-y)u(y) \, dy, \quad x \in \mathbb{R},$$

belongs to $\mathcal{D}(0,\infty)$ for $n \in \mathbb{N}$ large enough. It is well-known that $u_n \to u$ and $Du_n = Du * \rho_n \to Du$ in H as $n \to \infty$. It follows

$$\|u_n - u\|_V^2 = \|u_n - u\|_H^2 + \|\psi D(u_n - u)\|_H^2 \le \|u_n - u\|_H^2 + C\|Du_n - Du\|_H^2$$

for some $C > 0$ (using the continuity of ψ). Hence, $u_n \to u$ in V as $n \to \infty$ implying $u \in V_0$. This proves (7.1).

Now let $u \in V$. We will construct a sequence $(u_n)_{n\in\mathbb{N}} \subset V \cap C_c(0,\infty)$ converging to u in V. Therefore, we define $\varphi_n \in W^{1,1}_{\text{loc}}(0,\infty) \cap C_c(0,\infty)$, $n \in \mathbb{N}_{\ge 2}$, on the positive real axis by

$$\varphi_n(x) := \begin{cases} 0 & , \quad x \in (0, 1/n] \cup [n+1, \infty) \\ nx - 1 & , \quad x \in [1/n, 2/n] \\ 1 & , \quad x \in [2/n, n] \\ -x + n + 1 & , \quad x \in [n, n+1] \end{cases} .$$

Hence, $\varphi_n(x) \to 1$ as $n \to \infty$ for all $x > 0$. We define $u_n := u\varphi_n \in C_c(0,\infty)$, $n \in \mathbb{N}_{\ge 2}$. Then $u_n \in H$ and from the product formula we see that

$$\psi Du_n = \psi(\varphi_n \, Du + u(n\mathbb{1}_{[\frac{1}{n},\frac{2}{n}]} - \mathbb{1}_{[n,n+1]})) \in H.$$

Hence, $(u_n)_n \subset V \cap C_c(0,\infty)$. For $x > 0$ we further obtain $u_n(x) \to u(x)$ and

$$(\psi Du_n)(x) = x \; (Du)(x) \; \varphi_n(x) + \mathbb{1}_{[\frac{1}{n},\frac{2}{n}]}(x) \; n \; x \; u(x) - \mathbb{1}_{[n,n+1]}(x) \; x \; u(x) \to x \; (Du)(x)$$

as $n \to \infty$. The Dominated Convergence Theorem implies $u_n \to u$ in V as $n \to \infty$ and so $u \in V_0$. Finally, we have proven $V = V_0$. □

In particular, the Hilbert space V is continuously injected into H. We give some examples for functions in V:

Example 7.2.2. *1. For $c \in \mathbb{R}$ is $(c-\psi)^+ \in V$ with $\|(c-\psi)^+\|_V^2 = \frac{2}{3}c^3$.*

2. The function $\log(\psi) \wedge 0$ belongs to V with $\|\log(\psi) \wedge 0\|_V^2 = 3$. In particular, V is not embedded in $L^\infty(0,\infty)$.

Furthermore, one has $V \subset W^{1,2}_{\text{loc}}(0,\infty)$. In fact, let $u \in V$. Then $u \in H \subset L^2_{\text{loc}}(0,\infty)$ and for any compact subset $K \subset (0,\infty)$ we obtain (using the continuity of ψ):

$$\int_K (Du)^2 \, dx = \int_K \frac{1}{\psi^2}(\psi Du)^2 \, dx \le C \int_K (\psi Du)^2 \, dx \le C\|\psi Du\|_H^2 < \infty.$$

for some constant $C > 0$. Hence, $Du \in L^2_{\text{loc}}(0,\infty)$ and so $u \in W^{1,2}_{\text{loc}}(0,\infty)$. In particular, $V \subset C(0,\infty)$.

Concerning our form $(\mathfrak{a}, V)$ we have the result:

Proposition 7.2.3. *$(\mathfrak{a}, V)$ is a continuous, densely defined, elliptic form on H. Moreover, $(\mathfrak{a}, V)$ is accretive for $\beta \geq \frac{2}{3}\alpha$.*

Proof. Since $\mathcal{D}(0,\infty) \subset V$, the form $(\mathfrak{a}, V)$ is densely defined on H. Let $u, v \in V$. Then one has

$$|\mathfrak{a}[u,v]| \leq (\alpha + |2\alpha - \beta| + |\beta|)\, \|u\|_V \|v\|_V,$$

i.e. $(\mathfrak{a}, V)$ is continuous. For ellipticity we have to find constants $\omega \in \mathbb{R}$ and $\alpha > 0$ such that $\mathfrak{a}[u,u] + \omega\|u\|_H^2 \geq \alpha\|u\|_V^2$ for all $u \in V$. Let $u \in V$. There exists $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(0,\infty)$ such that $\varphi_n \to u$ in V as $n \to \infty$. Note that $\psi\varphi_n \in \mathcal{D}(0,\infty)$, $n \in \mathbb{N}$. It follows:

$$\begin{aligned}
2\ (\psi Du|u) &= (\psi Du|u)_H + \ (u|\psi Du)_H \\
&= \lim_{n\to\infty} ((\psi Du|\varphi_n)_H + (u|\psi D\varphi_n)_H) \\
&= \lim_{n\to\infty} (u| - D(\psi\varphi_n) + \psi D\varphi_n)_H \\
&= \lim_{n\to\infty} -(u|\varphi_n)_H \\
&= -\|u\|_H^2.
\end{aligned}$$

From this we obtain

$$\mathfrak{a}[u,u] = \alpha\|\psi Du\|_H^2 + (\beta - \frac{1}{2}(2\alpha-\beta))\|u\|_H^2 = \alpha\|\psi Du\|_H^2 + (\frac{3}{2}\beta - \alpha)\|u\|_H^2$$

for all $u \in V$. In particular, $(\mathfrak{a}, V)$ is elliptic. Moreover, if $\beta \geq \frac{2}{3}\alpha$, then $\mathfrak{a}[u,u] \geq 0$ for all $u \in V$, i.e. $(\mathfrak{a}, V)$ is accretive in this case. ☐

Remark 7.2.4. *The form $(\mathfrak{a}, V)$ is not acccretive for all choices of $\alpha > 0$ and $\beta \in \mathbb{R}$. For instance, let $u(x) := \log(x) \wedge 0$ for $x > 0$. Then $u \in V$ with $\|u\|_H^2 = 2$, $\|\psi Du\|_H^2 = 1$. It follows*

$$\mathfrak{a}[u,u] = \alpha\|\psi Du\|_H^2 + (\frac{3}{2}\beta - \alpha)\|u\|_H^2 = 3\beta - \alpha.$$

Hence, for $\beta < \frac{1}{3}\alpha$ the form $(\mathfrak{a}, V)$ is not accretive.

Let A be the associated operator to $(\mathfrak{a}, V)$ on H. Observe that for $u \in D(A) \cap W^{2,1}_{\mathrm{loc}}(0,\infty)$ and $v \in \mathcal{D}(0,\infty)$ we have

$$\begin{aligned}
(Au|v)_H &= \mathfrak{a}[u,v] \\
&= \alpha(\psi Du|\psi Dv)_H + (2\alpha - \beta)(\psi Du|v)_H + \beta(u|v)_H \\
&= \alpha(Du|\psi^2 Dv + 2\psi v)_H - \beta(\psi Du - u|v)_H \\
&= \alpha(Du|D(\psi^2 v))_H - \beta(\psi Du - u|v)_H \\
&= -\alpha(D^2 u|\psi^2 v)_H - \beta(\psi Du - u|v)_H \\
&= -(\alpha\psi^2 D^2 u + \beta\psi Du - \beta u|v)_H.
\end{aligned}$$

Hence, $-A = \mathcal{A}$ on $D(A) \cap W^{2,1}_{\mathrm{loc}}(0,\infty)$.

Since $(\mathfrak{a}, V)$ is elliptic, densely defined and continuous, $-A$ generates a bounded holomorphic C_0-semigroup $T = (T(t))_{t\geq 0}$ on H (see section 1.2.5).

7.2.3 Invariance of order intervals

We consider the form $(\mathfrak{a}, V)$ from the previous section and show the invariance of certain order intervals under its associated semigroup T. In this connection, we use the technique we have developped in Proposition 4.4.1.

Theorem 7.2.5. *The order intervals $[-\infty, \psi]$ and $[-\infty, k]$ for $k \geq 0$ are invariant under the C_0-semigroup T.*

Proof. We start with the order interval $[-\infty, \psi]$. Due to Proposition 4.4.1 it is sufficient to prove that for all $u \in V$ the function $u \wedge \psi$ belongs to V and $\mathfrak{a}[u \wedge \psi, (u-\psi)^+] \geq 0$. Let $u \in V$. Since $\psi \in W^{1,1}_{\text{loc}}(0,\infty)$, we have $u \wedge \psi \in W^{1,1}_{\text{loc}}(0,\infty)$ with $D(u \wedge \psi) = \mathbb{1}_{\{u>\psi\}} + \mathbb{1}_{\{u\leq\psi\}} Du$. It follows

$$\|u \wedge \psi\|_V^2 = \int_0^\infty \mathbb{1}_{\{u>\psi\}} \left(\psi^2 + \psi^2\right) + \mathbb{1}_{\{u\leq\psi\}} \left(u^2 + \psi^2 (Du)^2\right)\, dx \leq \|u\|_H^2 + \|u\|_V^2.$$

Hence, the functions $u \wedge \psi$ and $(u-\psi)^+ = u - (u \wedge \psi)$ belong to V. Furthermore, we have

$$\begin{aligned}
\mathfrak{a}[u \wedge \psi, (u-\psi)^+] &= \alpha \left(\psi D(u\wedge\psi) | \psi D(u-\psi)^+\right)_H + (2\alpha - \beta) \left(\psi D(u \wedge \psi) | (u-\psi)^+\right)_H \\
&\quad + \beta \left(u \wedge \psi | (u-\psi)^+\right) \\
&= \int_0^\infty \alpha\psi^2 D(u-\psi)^+ + (2\alpha-\beta)\psi(u-\psi)^+ + \beta\psi(u-\psi)^+\, dx \\
&= \alpha \int_0^\infty D(\psi^2(u-\psi)^+)\, dx.
\end{aligned}$$

We claim:

$$\int_0^\infty D(\psi^2(u-\psi)^+)\, dx = 0. \tag{7.2}$$

There exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(0,\infty)$ converging pointwise to u a.e. on $(0,\infty)$ and $\varphi_n \to u$ in V as $n \to \infty$. For any $n \in \mathbb{N}$ and $x > 0$ we have

$$\begin{aligned}
|x\varphi_n(x)| &= \lim_{\varepsilon\to 0} |x\varphi_n(x) - \varepsilon\varphi_n(\varepsilon)| \\
&\leq \lim_{\varepsilon\to 0} \int_\varepsilon^x |D(\psi\varphi_n)(y)|\, dy \\
&\leq \lim_{\varepsilon\to 0} |x-\varepsilon|^{\frac{1}{2}} \|D(\psi\varphi_n)\|_{L^2(\varepsilon,x)} \\
&\leq |x|^{\frac{1}{2}} \|D(\psi\varphi_n)\|_H.
\end{aligned}$$

It follows

$$|\sqrt{x}\varphi_n(x)| \leq \|D(\psi\varphi_n)\|_H = \|\psi D\varphi_n\|_H.$$

Hence,

$$|\sqrt{x}\; u(x)| = \lim_{n\to\infty} |\sqrt{x}\; \varphi_n(x)| \leq \lim_{n\to\infty} \|\psi D\varphi_n\|_H = \|\psi Du\|_H \leq \|u\|_V$$

a.e. on $(0,\infty)$. This implies $\lim_{x\to\infty} u(x) = 0$ and $\lim_{x\to 0} xu(x) = 0$. Coming back to our claim (7.2) note that $\psi^2(u-\psi)^+ \in W^{1,1}_{\text{loc}}(0,\infty)$ and so

$$\int_{\frac{1}{n}}^n D(\psi^2(u-\psi)^+)\, dx = n^2(u(n)-n)^+ - \frac{1}{n^2}\left(u\left(\frac{1}{n}\right) - \frac{1}{n}\right)^+ \to 0 \text{ as } n \to \infty.$$

This proves the claim (7.2). Therefore, one has $\mathfrak{a}[u \wedge \psi, (u-\psi)^+] = 0$ implying the invariance of $[-\infty, \psi]$ under T.

For $k \geq 0$ and $u \in V$ one easily sees that $u \wedge k \in V$. Since $\mathbb{1}_{\{u>k\}} D(u \wedge k) = 0$ it follows $\mathfrak{a}[u \wedge k, (u-k)^+] = 0$ and thus the invariance of $[-\infty, k]$ under T. $\square$

Remark 7.2.6. *By the same reasoning, one sees that the form*

$$\mathfrak{b}[u,v] := \alpha(\psi Du|\psi Dv)_H + (2\alpha - \beta)(\psi Du|v)_H + \gamma(u|v)_H, \quad u, v \in V,$$

with the same domain V and some constant $\gamma > 0$ is elliptic, densely defined and continuous on H. Its associated semigroup $S = (S(t))_{t\geq 0}$ leaves the order interval $[-\infty, \Phi]$ invariant if and only if $\beta \leq \gamma$.

In virtue of the Beurling-Deny criteria (cp. Proposition 4.4.2) we further get:

Corollary 7.2.7. *The semigroup T is positive and submarkovian. In particular, T extrapolates to C_0-semigroups on $L^p(0,\infty)$ for all $2 \leq p < \infty$.*

This finishes the variational part.

7.3 C_0-semigroup on $C_0([0,\infty))$ and an explicit formula

As stated in the beginning we want to derive an explicit formula for the C_0-semigroup generated by the realization of $\mathcal{A} = \alpha\psi^2 Du + \beta\psi Du - \beta u$ in different Banach spaces. To extend our approach to continuous function spaces we consider the Banach space (equipped with the supremum norm)

$$X := C_0([0,\infty)) = \left\{ f \in C([0,\infty);\mathbb{R}) \mid \lim_{x\to\infty} f(x) = 0 \right\}.$$

Let $\gamma := \frac{1}{2}\left(\frac{\beta}{\alpha} - 1\right)$. On X we define the operator

$$\begin{aligned} D(A) &:= \{f \in X \cap C^2(0,\infty) \mid \psi Df, \psi^2 Df \in X\}, \\ Af &:= \psi^2 D^2 f + (2\gamma+1)\psi Df - (2\gamma+1)f. \end{aligned}$$

Note that $\alpha A = \mathcal{A}$ on $D(A)$. We show that A can be written as

$$A = (B+\gamma)^2 - (1+\gamma)^2$$

for a C_0-group generator B. From this it follows that A generates a holomorphic semigroup T on X. In addition, we are provided with an explicit formula for T.

In a first step of this section we introduce the suitable group generator.

Proposition 7.3.1. *Let $(G(t)f)(x) := f(e^t x)$ for $t \in \mathbb{R}, x \geq 0$ and $f \in X$. Then $\|G(t)\| = 1$ for all $t \in \mathbb{R}$ and $G = (G(t))_{t\in\mathbb{R}}$ is a C_0-group on X with generator*

$$D(B) := \{f \in X \cap C^1(0,\infty) \mid \psi Df \in X\}, \quad Bf := \psi Df.$$

Proof. Obviously, G fulfills the group laws and $\|G(t)\| = 1$ for all $t \in \mathbb{R}$. It remains to show that G is strongly continuous. Let $f \in C_c([0,\infty)$. Then:

$$\|G(t)f - f\|_\infty = \sup_{x\geq 0} |f(e^t x) - f(x)| = |f(e^t x_0) - f(x_0)| \to_{t\to 0} 0.$$

Since $C_c([0,\infty))$ is dense in X, it follows that G is a C_0-group on X. Let C be its generator. We define the C_0-semigroups $G_+(t) := G(t)$ and $G_-(t) := G(-t)$ for all $t \geq 0$ with generator C and $-C$, respectively.

Let $f \in D(C)$ and $g := (1 - C)f \in X$. Since $G_+ = (G_+(t))_{t\geq 0}$ is a contraction semigroup, we know $1 \in \rho(C)$. For any $x > 0$ one has

$$f(x) = (R(1,C)g)(x) = \int_0^\infty e^{-t} g(e^t x)\, dt = x \int_x^\infty \frac{g(u)}{u^2}\, du.$$

It follows

$$\begin{aligned}
\frac{1}{h}(f(x+h) - f(x)) &= \frac{x+h}{h}\int_{x+h}^\infty \frac{g(u)}{u^2}\, du - \frac{x}{h}\int_x^\infty \frac{g(u)}{u^2}\, du \\
&= \int_{x+h}^\infty \frac{g(u)}{u^2}\, du - \frac{x}{h}\int_x^{x+h} \frac{g(u)}{u^2}\, du \\
&= \frac{f(x+h)}{x+h} - \frac{x}{h}\int_0^h \frac{g(u+x)}{(u+x)^2}\, du \\
&\to \frac{f(x)}{x} - \frac{g(x)}{x} \quad \text{as } h \to 0.
\end{aligned}$$

Hence, $f \in C^1(0,\infty)$. Moreover, the function $x \mapsto xf'(x) = f(x) - g(x)$ is in X implying that $f \in D(B)$ and $(Bf)(x) = xf'(x) = f(x) - g(x) = (Cf)(x)$. This shows $C \subset B$. Similarly, one can prove that $-C \subset B$.

Conversely, let $f \in D(B)$ and $g := Bf$. For any $t \geq 0$ and $x \geq 0$ one has

$$\begin{aligned}
\int_0^t (G^+(s)f)(x)\, ds &= \int_0^t g(e^s x)\, ds \\
&= \int_0^t e^s\, x\, f'(e^s x)\, ds \\
&= \int_x^{e^t x} f'(r)\, dr = f(e^t x) - f(x).
\end{aligned}$$

Hence, we conclude $f \in D(C)$ with $Cf = g = Bf$ proving $B \subset C$. Again, a similar proof applies for $B \subset -C$. In conclusion, we have shown that $B = C$. □

We recall the following result from section 1.2.4: Let C be the generator of a bounded C_0-group $G = (G(t))_{t\in\mathbb{R}}$ on X. Then the operator C^2 with domain $D(C^2) := \{x \in D(C) \mid Cx \in D(C)\}$ generates a bounded holomorphic C_0-semigroup T of angle $\frac{\pi}{2}$ on X given by

$$T(t)x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{1}{2}s^2} G(s\sqrt{2t})x\, ds, \quad t \geq 0, x \in X.$$

For a proof we refer to the standard literature, e.g. [ABHN01]. That known, we can prove the following result:

Theorem 7.3.2. *The operator* $(\alpha A, D(A))$ *generates a bounded holomorphic* C_0*-semigroup* $T = (T(t))_{t\geq 0}$ *of angle* $\frac{\pi}{2}$ *on* X *where* T *is given by*

$$\begin{aligned}(T(t)f)(x) &:= \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2}\ (G(s\sqrt{2\alpha t} + (\beta-\alpha)t))f)(x)\ ds \\ &= \frac{e^{-\beta t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2}\ f(e^{s\sqrt{2\alpha t}+(\beta-\alpha)t}x)\ ds\end{aligned}$$

for $t \geq 0$, $f \in X$ *and* $x \geq 0$.

Proof. Thanks to Proposition 7.3.1 the operator $(B+\gamma, D(B))$ generates the C_0-group $(e^{\gamma t}G(t))_{t\in\mathbb{R}}$. Thus, the operator $((B+\gamma)^2, D(B^2))$ generates the C_0-semigroup $T_1 = (T_1(t))_{t\geq 0}$ on X with

$$T_1(t)f = \int_{\infty}^{\infty} \varphi(s)e^{\gamma s\sqrt{2t}}G(s\sqrt{2t})f\ dx, \quad t \geq 0, f \in X.$$

where $\varphi(s) := \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}$, $s \in \mathbb{R}$. Now observe that

$$\begin{aligned}D(B^2) &= \{f \in D(B) \mid Bf \in D(B)\} \\ &= \{f \in X \cap C^2(0,\infty) \mid \psi Df, \psi D(\psi Df) \in X\} \\ &= D(A)\end{aligned}$$

and

$$A = \psi^2D^2 + (2\gamma+1)\psi D - (2\gamma+1) = B^2 - B + (2\gamma+1)B - (2\gamma+1) = (B+\gamma)^2 - (1+\gamma)^2.$$

Hence, A generates the bounded holomorphic C_0-semigroup $T_2 = (T_2(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$ on X with the representation

$$\begin{aligned}T_2(t)f &:= e^{-(1+\gamma)^2 t}T_1(t)f \\ &= e^{-(1+\gamma)^2 t}\int_{-\infty}^{\infty} \varphi(s)e^{\gamma s\sqrt{2t}}G(s\sqrt{2t})f\ ds \\ &= e^{-t(1+2\gamma)}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}s^2 + \gamma s\sqrt{2t} - \gamma^2 t\right)\ G(s\sqrt{2t})f\ ds \\ &= e^{-t(1+2\gamma)}\int_{-\infty}^{\infty} \varphi(s-\gamma\sqrt{2t})\ G(s\sqrt{2t})f\ ds \\ &= e^{-t(1+2\gamma)}\int_{-\infty}^{\infty} \varphi(s)\ G(s\sqrt{2t}+2\gamma t)f\ ds.\end{aligned}$$

It is known that αA generates the bounded holomorphic C_0-semigroup S with $T(t) := T_2(\alpha t)$, $t \geq 0$. Recall that $\frac{\beta}{\alpha} = 2\gamma + 1$. It follows for $f \in X$:

$$T(t)f = e^{-\beta t}\int_{-\infty}^{\infty} \varphi(s)\ G(s\sqrt{2\alpha t} + (\beta-\alpha)t)f\ ds,$$

which proves the Theorem. □

This finishes the part with the generation results.

7.4 Prices of European options via semigroup techniques

In this closing section we want to solve the pricing problem (BS) with our semigroup techniques. In order to get an explicit pricing formula we use the method of section 7.3. We demonstrate the impact of this theory by replicating the well-known price of an European Put option in the framework of Black and Scholes.

Let $\alpha := \frac{1}{2}\sigma^2 > 0$, $\beta := r$ and $\gamma := \frac{1}{2}\left(\frac{\beta}{\alpha} - 1\right) = \frac{r}{\sigma^2} - \frac{1}{2}$. In analogy to Section 7.3 we define the degenerate differential operator

$$\begin{aligned} D(A_{BS}) &:= \{f \in X \cap C^2(0,\infty) \mid \psi Df, \psi^2 Df \in X\}, \\ A_{BS}f &:= \psi^2 D^2 f + (2\gamma+1)\psi Df - (2\gamma+1)f. \end{aligned}$$

with $\psi(x) := x$ on the Banach space $X := C_0([0,\infty))$. Note that

$$(\alpha A_{BS} f)(x) = \frac{1}{2}\sigma^2 x^2 \; D^2 f(x) + rxDf(x) - rf(x)$$

for $f \in D(A_{BS})$. Hence, we can write the problem (BS) as abstract backward Cauchy problem on X for the operator A_{BS}:

$$\text{(ABS)} \quad \left\{ \begin{aligned} f'(t) + \alpha A_{BS} f(t) &= 0 \\ f(\tau) &= h \end{aligned} \right.$$

for $h \in X$. But we know from Theorem 7.3.2 that αA_{BS} generates a C_0-semigroup $T = (T(t))_{t\geq 0}$. Therefore, the (unique) option price $P(t,\tau)h(S(\tau))$ at time $t \in I$ is given by

$$\begin{aligned} P(t,\tau)h(S(\tau)) &= (T(\tau - t)h)(x) \\ &= e^{-\beta(\tau-t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \; e^{-\frac{1}{2}s^2} \; f(e^{s\sqrt{2\alpha(\tau-t)} - (\beta-\alpha)(\tau-t)} x) \; ds \\ &= e^{-r(\tau-t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \; e^{-\frac{1}{2}s^2} \; f(e^{s\sigma\sqrt{\tau-t} + (r-\frac{1}{2}\sigma^2)(\tau-t)} x) \; ds \end{aligned}$$

whenever $S(t) = x \in (0,\infty)$. Shortly, $P(t,\tau)h(S(\tau)) = (T(\tau-t)h)(S(t))$. In conclusion, the application of the semigroup gives us the option price.

For a demonstration we finally calculate the price for an European Put Option:

Example 7.4.1. *(European Put Option) Let $h(x) := (K-x)^+$ for some $K \geq 0$ on $[0,\infty)$. Then $h \in X$. As in finance we define*

$$\begin{aligned} d_1(x,t) &:= \frac{\log\left(\frac{K}{x}\right) + (\frac{1}{2}\sigma^2 - r)t}{\sigma\sqrt{t}}, \\ d_2(x,t) &:= d_1(x,t) - \sigma\sqrt{t} \end{aligned}$$

for $x \in (0,\infty)$, $t \in [0,\tau]$. Note that $h(e^{s\sigma\sqrt{t} + (r-\frac{1}{2}\sigma^2)t}x) \geq 0$ if and only if $s \leq d_1(x,t)$. Let

$\varphi(s) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ *and* $\Phi(x) := \int_{-\infty}^{x} \varphi(y)\, dy$ *for* $x \in \mathbb{R}$. *It follows for fixed* $(x,t) \in (0,\infty) \times [0,\tau]$*:*

$$\begin{aligned}
e^{rt}(T(t)h)(x) &= \int_{-\infty}^{\infty} \varphi(s)\ h(e^{s\sigma\sqrt{t}+(r-\frac{1}{2}\sigma^2)t}x)\ ds \\
&= \int_{-\infty}^{d_1(x,t)} \varphi(s)\ \left(K - e^{s\sigma\sqrt{t}+(r-\frac{1}{2}\sigma^2)t}x\right)\ ds \\
&= K\Phi(d_1(x,t)) - xe^{rt} \int_{-\infty}^{d_1(x,t)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}s^2 + s\sigma\sqrt{t} - \frac{1}{2}\sigma^2 t\right)\ ds \\
&= K\Phi(d_1(x,t)) - xe^{rt} \int_{-\infty}^{d_1(x,t)} \varphi(s - \sigma\sqrt{t})\ ds \\
&= K\Phi(d_1(x,t)) - xe^{rt}\Phi(d_2(x,t)).
\end{aligned}$$

Thus, the price of the European Put Option at time $t \in I$ *is given by*

$$P(t,\tau)h(S(\tau)) = K\ e^{-r(\tau-t)}\ \Phi(d_1(S(t),\tau-t)) - S(t)\ \Phi(d_2(S(t),\tau-t)).$$

7.5 Notes

The work of Gozzi, Monte and Vespri:

In [GMV02] Gozzi, Monte and Vespri study the following operator with realisations on $L^2(\mathbb{R})$:

$$(\mathcal{A}u)(x) := \psi^2(x)a(x)(D^2u)(x) + b(x)(Du)(x) - \gamma^2(x)u(x), \quad x \in \mathbb{R},$$

with coefficients

1. $\psi \in C^1(\mathbb{R})$;
2. $a \in C^1(\mathbb{R})$ and there exists $E > 0$ such that $a(x) \geq E$ for all $x \in \mathbb{R}$;
3. b measurable, real-valued function;
4. $\gamma \in L^2_{\text{loc}}(\mathbb{R})$ with $\operatorname{ess\,inf} \gamma \geq 1$;
5. there are constants $B_1, B_2 \geq 0$ with $B_1 + B_2 < 2$ such that

$$|b(x)| \leq B_1\sqrt{E}\,|\psi(x)|\,\gamma(x), \quad x \in \mathbb{R},$$

and

$$\left|(D(\psi^2 a))(x)\right| \leq B_2\sqrt{E}\,|\psi(x)|\,\gamma(x), \quad x \in \mathbb{R}.$$

As a matter of fact, they study the situation in $\mathbb{R}^n$ instead of $\mathbb{R}$ but for our purpose the one-dimensional situation will be sufficient. Under these conditions they could show that the realization of $\mathcal{A}$ in $L^2(\mathbb{R})$ generates a holomorphic semigroup. However, we will demonstrate that the conditions do not apply directly to the Black-Scholes operator

$$(\mathcal{B}u)(x) := \frac{1}{2}\sigma^2x^2(D^2u)(x) + Rx(Du)(x) - Ru(x), \quad x \in \mathbb{R},$$

with $R \geq 1$ and $\sigma \in \mathbb{R}$. In fact, a direct comparison of the coefficients yields

$$\psi^2(x)a(x) := \frac{1}{2}\sigma^2x^2,\ \ b(x) := Rx,\ \ \gamma(x) := \sqrt{R}$$

for all $x \in \mathbb{R}$. Now assume that the coefficients fulfill the conditions above. We obtain for $x \in \mathbb{R}$:

$$\begin{aligned} \sigma^2 |x|^2 &= |x| \left|(D(\psi^2 a))(x)\right| \\ &\leq B_2 \sqrt{E} |\psi(x)| \sqrt{R} |x| \\ &= B_2 \sqrt{E} |\psi(x)| \left| \frac{b(x)}{\gamma(x)} \right| \\ &\leq B_1 B_2 E \psi(x)^2 \\ &\leq B_1 B_2 a(x) \psi(x)^2 \\ &= \frac{1}{2} B_1 B_2 \sigma^2 |x|^2 . \end{aligned}$$

Hence, the inequality $B_1 \geq \frac{2}{B_2}$ has to hold. But we are also assuming that $B_1 + B_2 < 2$. It follows $2 \geq B_2 + \frac{2}{B_2}$ and thus the contradiction

$$0 > B_2^2 - 2B_2 + 2 = (B_2 - 1)^2 + 1.$$

Therefore, at a first glance their approach surprisingly does not apply for the case of constant coefficients.

More general processes:

From chapter 2 we know that Black and Scholes assumed that the underlying evolves like a geometric Brownian motion with constant coefficients. This assumption has two major drawbacks regarding its practicality. At first, it is well-known that the volatiltity is often not constant like Black and Scholes assumed. Therefore, the Black-Scholes formula is frequently used in a reverse manner: Knowing the option price from market data, one calculates the implicit volatility from the Black-Scholes pricing formula, which is a highly regarded market indicator. The second drawback is the assumption of continuous paths. The past has shown that more accurate models shall include the possibility of jumps. This leads us to general Lévy processes (or semimartingales). As a matter of fact, numerous aspects of the theory of Black and Scholes can be transfered to the new general setting. Nevertheless we have some restrictions in the theory, for instance, the equivalent martingale measure is now longer unique. However, we often still have to solve Cauchy-problems related to the Lévy characteristics of the underlying (exponential) Lévy process. Now we want to point out that the semigroup theory is still of use in these situations in order to guarantee a solution.

For, let $X = (X(t))_{t \geq 0}$ be a one-dimensional, real-valued Lévy process with characteristics (a, b, ν) in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. For functions $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, or $f \in C_0(\mathbb{R})$ it is well-known that

$$(T(t)f)(x) := \mathbb{E}[f(X(t)) \mid X(0) = x] = \int_{-\infty}^{\infty} f(y + x) \, dq_t(y), \quad t \geq 0, x \in \mathbb{R},$$

describes a strongly continuous semigroup $T = (T(t))_{t \geq 0}$ of positive contractions on $L^p(\mathbb{R})$ and $C_0(\mathbb{R})$, respectively (cp. [App04, Chapter 3]). For simplicity, let us focus on $C_0(\mathbb{R})$. Let A be the generator of T in $C_0(\mathbb{R})$. Then we have for $f \in C_0^2(\mathbb{R})$ and $x \in \mathbb{R}$ the description

$$(Af)(x) = bf'(x) + \frac{1}{2} a^2 f''(x) + \int_{\mathbb{R} \setminus \{0\}} f(x + y) - f(x) - \mathbb{1}_{(-1,1)}(y) y f'(x) \, d\nu(y).$$

(cp. [App04]). This already shows that semigroup theory and Lévy processes are closely connected. In finance, however, one is interested in positive price processes and therefore one considers exponential Lévy processes. But this step does not change the relevance of semigroup theory. In fact, let $Y(t)$ be the exponential of the Lévy process $X(t)$ for $t \geq 0$. We define the isomorphism

$$\Psi : C_0((0,\infty)) \longrightarrow C_0(\mathbb{R}), g \mapsto g \circ \exp .$$

Observe that $\Psi^{-1} f = f \circ \log$ for $f \in C_0(\mathbb{R})$. Therefore, it holds

$$\begin{aligned}
(\Psi^{-1} T(t) \Psi f)(x) &= (T(t)(\Psi \circ f))(\log(x)) \\
&= \mathbb{E}[(\Psi \circ f)(X(t)) | X(0) = \log(x)] \\
&= \mathbb{E}[f(Y(t)) | Y(0) = x] =: (S(t)f)(x)
\end{aligned}$$

for all $t \geq 0$, $f \in C_0((0,\infty))$ and $x \in (0,\infty)$. The semigroup $S = (S(t))_{t\geq 0}$ on $C_0(0,\infty)$ now plays the role of the pricing semigroup. We have seen that S is similiar to T. By [EN00, Theorem II.2.1], we obtain strong continuity of the semigroup S on $C_0((0,\infty))$ and its generator is

$$B = \Psi^{-1} A \Psi, \quad D(B) = \{ f \in C_0((0,\infty)) \mid \Psi \circ f \in D(A) \} .$$

In conclusion, we are still dealing with semigroup theory.

This admittedly short introduction shall demonstrate the importance of semigroup theory even for general price processes as exponential Lévy processes.

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Zusammenfassung in deutscher Sprache

Die vorliegende Dissertation beschäftigt sich mit der Anwendung von Halbgruppenmethoden in der Finanzmathematik. Unabhängig von dieser speziellen Anwendung werden zudem eine Vielzahl neuer Resultate betreffend invarianter Mengen von stark stetigen Halbgruppen und einer Regularität erhaltenden Störungstheorie für Sesquilinearformen (verbunden mit einer Erweiterung der klassischen Kato-Klasse) präsentiert.

Zur Absicherung gegen Risikien wie Kursschwankungen werden an der Börse unter anderem Optionen gehandelt. Beispielsweise kauft man sich mit einer europäischen Put-Option auf eine Aktie das Recht ein, diese zu einem festgelegten späteren Zeitpunkt zu einem bestimmten Preis zu verkaufen. Ist der Aktienkurs zum Ausübungszeitpunkt dann niedriger als der vereinbarte Preis, kann man die Aktie zu einem höheren Kurs verkaufen, man hat sich gewissermaßen gegen fallende Kurse abgesichert. Die Vielfalt von gehandelten Instrumenten derartiger Gestalt ist immens. Neben der Komponente des direkten Risikomanagements dienen ihre Kursverläufe dabei insbesondere auch als gute Indikatoren, um das Marktgeschehen zu interpretieren. Jeder Händler ist darum daran interessiert, den aus seiner Sicht wahrscheinlichsten Kursverlauf für die Zukunft abzubilden und in diesem Modellrahmen die Optionen "fair" zu bewerten. Als erste Frage stellt sich damit die nach der fairen Bewertung. Oberstes Gebot in sämtlichen theoretischen Modellen ist die Arbitragefreiheit, d.h. es ist unmöglich, sich aus dem Nichts durch reines Handeln einen Gewinn zu erwirtschaften. In Bezug auf eine faire Bewertung scheint es nun der natürlichste Weg zu sein, den prognostizierten Kursverlauf mittels einer selbstfinanzierenden Handelsstrategie im Modell nachzubilden. Den fairen Preis bildet dann der Wert dieser Strategie zum betreffenden Zeitpunkt. Aus mathematischer Sicht führt die Modellbildung und die Unsicherheit über künftige Kursverläufe zwangsgebunden in den Themenbereich der Stochastik. Interessanterweise hat die enorme Nachfrage nach einem theoretischen Gerüst für die Finanzmathematik zu einem neu belebten Interesse an einer Vielzahl von stochastischen Forschungsgebieten geführt. Beispielhaft ist die Arbeit von Harrison und Pliska aus dem Jahr 1981 zu nennen. Sie zeigten, dass der gesuchte Optionspreis als bedingter Erwartungswert unter dem so genannten Martingalmaßermittelt werden kann, was zur Auswendung der Martingaltheorie führte. Ein weiteres Beispiel ist das stochastische Integral.

Die Bedeutung der Stochastik für die Finanzmathematik ist somit nicht von der Hand zu weisen. In dieser Arbeit wollen wir aber auch die Rolle der Funktionalanalysis - und hierbei besonders der Halbgruppentheorie - aufzeigen. Bekanntermaßen gilt die Arbeit von Black und Scholes aus dem Jahr 1973 als Meilenstein in der Optionspreistheorie. Mittels eines Replikationsarguments und dem Itô-Kalkül konnten sie im Rahmen ihres Modells zeigen, dass sich der faire Preis einer Option als Lösung einer partiellen Differentialgleichung mit Endwertbedingung ergibt. Damit sind wir bereits tief in der Halbgruppentheorie angelangt. Wir können nämlich die zu lösende partielle Differentialgleichung als endwertiges Cauchyproblem bzgl. eines Differentialoperators A auffassen. Erzeugt dieser Operator dann eine stark stetige Halbgruppe auf einem adäquaten Banachraum, so liefert uns die Halbgruppe in der Tat die Lösung des Cauchy-Problems und damit auch den gesuchten Optionspreis! Deshalb ist es von großem Interesse nachzuweisen, ob bestimmte Operatoren (wie eben der Black-Scholes Operator) tatsächlich eine Halbgruppe erzeugen. Dies ist ein Ziel dieser Arbeit.

Zur Herleitung des Cauchy-Problems benutzten Black und Scholes die stochastische Analysis. Betrachtet man sich die Optionspreistheorie jedoch in ihrem Fundament, der arbitragefreien Bewertung, so erkennt man dahinter leicht eine Evolutionsstruktur. Es stellt sich somit die Frage, ob nicht die Möglichkeit besteht, die Stochastik weitest möglich außen vor zu lassen und der Funktionalanalysis mehr Spielraum zu gewähren. Garman war einer der ersten, der diesen Umstand in Grundzügen aufzeichnete und in ersten Resultaten zu formulieren verstand. Es ist ein zweites Ziel dieser Arbeit, hilfreiche Techniken zu entwickeln, die weitere Schritte in diese Richtung ermöglichen.

Wenden wir uns nun den konkreten Ergebnissen sowie dem Aufbau dieser Arbeit zu. Der Ausgangspunkt (aus finanzmathematischer Sicht) ist das Kapitel 2. Wir zeichnen die Argumentation von Black und Scholes nach und beschreiben den Preis einer europäischen Option als Lösung einer partiellen Differentialgleichung. Dabei beobachten wir im Verlauf der Herleitung einen Wechsel im Drift-Parameter des angenommenen Kursverlaufes im Underlying, welcher sich als Störung des zugehörigen Differentialoperators interpretieren läßt. Im zweiten Teil des Kapitels nehmen wir eine umgekehrte Blickrichtung ein. Wir studieren die Struktur der Preisoperatoren in einem arbitragefreien Marktmodell. Es zeigt sich, dass diese eine Evolutionsfamilie von linearen, injektiven, positiven Operatoren bilden. Insbesondere ergibt sich die Invarianz des Ordnungsintervalls $[-\infty, \mathrm{id}]$.

Wir nehmen diese Resultate als Motivation, um im weiteren Verlauf der Arbeit die folgenden Ziele - unabhängig vom finanzmathematischen Aspekt - zu verfolgen:

1. Invariante Mengen von stark stetigen Halbgruppen;
2. Halbgruppen von injektiven Operatoren;
3. Störungsresultate für Differentialoperatoren;
4. Generationsresultate für den Black-Scholes-Operator.

Brechen wir nun zu einem Streifzug durch die Kapitel auf.

In den Kapiteln 3 und 4 widmen wir uns invarianten Mengen von stark stetigen Halbgruppen. Die Bedeutung dieser Theorie steht außer Frage und rührt von der natürlichen Frage her, welche Eigenschaften des Anfangswertes für ein Cauchy-Problem an die Lösung desselben vererbt werden. Wir betrachten ein Cauchy-Problem für einen Operator A und einen Anfangswert x_0. Erzeugt A eine stark stetige Halbgruppe $T = (T(t))_{t\geq 0}$ in einem Banachraum X, so ist die eindeutige Lösung des Cauchy-Problems durch $u(t) := T(t)x_0$, $t \geq 0$, gegeben. Fragt man sich demnach, welche Eigenschaften des Anfangswertes x_0 wie z.B. Positivität an die Lösung übertragen werden, ist dies äquivalent zur Invarianz bestimmter Mengen unter der Halbgruppe T.

In Kapitel 3 betrachten wir abgeschlossene, konvexe Mengen C in einem Banachraum X als mögliche Kandidaten für invariante Mengen. Ist X ein reeller Hilbertraum, so hat Brézis eine erschöpfende Theorie zur Invarianz geliefert. Als wesentliches Hilfsmittel zur Charakterisierung benutzt er die orthogonale Projektion auf C. Erstaunlicherweise scheint eine gleich geartete Theorie für Banachräume in der Literatur zu fehlen. In Kapitel 3 schließen wir diese Lücke. Dabei gehen wir wie folgt vor: Mittels des Trennungssatzes von Hahn-Banach zeigen wir, dass das Subdifferential der Abstandsfunktion $d(\cdot, C)$ zu C in jedem Punkt $x \in X$, eine nicht-leere Menge bildet. Damit können wir die Definition von Φ-dissipativen Operatoren, eingeführt von Arendt, Chernoff and Kato für "half-norms" Φ, auf die Abstandsfunktion erweitern. Wir nennen diese Eigenschaft C-Dissipativität und zeigen, dass C invariant unter einer Halbgruppe T ist, wenn ihr Generator C-dissipativ ist. Für quasi-kontraktive Halbgruppen erhalten wir sogar eine Äquivalenz dieser beiden Aussagen. Um dieses Resultat handbarer zu machen, betrachten wir im nächsten Schritt normal projizierbare wie proximinale Mengen und zeigen, dass eine Menge genau dann normal projizierbar ist, wenn sie proximinal und konvex ist. Insbesondere sind normal projizierbare Mengen proximinal und konvex, fallen demnach in das vorab bereitete Feld. Wir zeigen dass sich die Elemente des Subdifferentials der Abstandsfunktion zu einer proximinalen, konvexen Menge in Form von Lotpunkten (bzw. besten Approximationen) schreiben lassen. Damit wird unsere Theorie für diese Mengen in anschaulicher Weise handbar. Beispielsweise gelingt es uns die bekannten Charakterisierungen von positiven oder kontraktiven Halbgruppen nachzubilden. Zusätzlich beschäftigen wir uns intensiv mit der Invarianz von geordneten Intervallen in reellen Banachverbänden. In diesem Zusammenhang stößt man beinahe zwangsläufig auf Kato-Ungleichungen. Wir beweisen eine neue Version, welche die Invarianz von einer großen Zahl von abgeschlossenen, konvexen Mengen (z. B. von geordneten Intervallen) unter positiven Halbgruppen charakterisiert.

Das Kapitel 4 konzentriert sich derweil auf invariante Mengen in Hilberträumen, beschäftigt sich allerdings nunmehr mit Halbgruppen, die zu elliptischen, dicht definierten, stetigen Sesquilinearformen assoziiert sind. In diesem Zusammenhang ist ein bemerkenswertes Resultat von Ouhabaz bekannt und von großem Nutzen: er charakterisiert die Invarianz von abgeschlossenen, konvexen Mengen unter kontraktiven Halbgruppen mittels Bedingungen an die assoziierte Form. Wir ersetzen die Bedingung der Kontraktivität an die Halbgruppe durch die Existenz eines Fixpunktes der Halbgruppe in der invarianten Menge und zeigen, dass die Aussage dann weiterhin Bestand hat. Dank eines Fixpunktsatzes von Browder können wir zudem nachweisen, dass unsere Version in der Tat eine Erweiterung von Ouhabaz' Theorem ist. Andere relevante Fälle, die in unseren neuen Rahmen fallen, sind die Situationen, wo die invariante Menge eine Umgebung des Ursprungs ist oder der Generator kompakte Resolvente besitzt. Insbesondere ist es uns möglich, die Invarianz von geordneten Intervallen und damit auch die berühmten Beurling-Deny-Kriterien erschöpfend zu behandeln.

In Kapitel 5 widmen wir uns Halbgruppen von injektiven Operatoren. Mittels einer Version des Phragmen-Lindelöf-Prinzips geben wir eine Bedingung an den Generator einer Halbgruppe T an, unter der jedes Element $T(t)$, $t \geq 0$, der Halbgruppe ein injektiver Operator ist. In diesem Rahmen fallen insbesondere auch holomorphe Halbgruppen. Mit einem Gegenbeispiel zeigen wir indes gleichzeitig auf, dass die Bedingung derartige Halbgruppen nicht charakterisiert.

Dem dritten Ziel der Dissertation, einer Störungstheorie für Differentialoperatoren, ist das Kapitel 6 zugewandt. Wir betrachten das Problem aus der Sicht der Regularitätserhaltung unter Störungen. Dabei fokussieren wir uns erneut auf Sesquilinearformen und ihre assoziierten Halbgruppen. Ausgehend von der Annahme, dass die Halbgruppe T zu einem abstrakten, regulären Raum X gehört, fragen wir nach Störungen der assoziierten Form $(\mathfrak{a}, D(\mathfrak{a}))$, so dass die gestörte Halbgruppe ebenfalls zu X gehört. Da dieses Problem in Verbindung zur klassischen Kato-Klasse steht, bezeichnen wir als abstrakte Kato-Klasse $\mathrm{Kat}(\mathfrak{a}, \lambda, X)$ für unsere Form $(\mathfrak{a}, D(\mathfrak{a}))$ die Menge all jener Elemente $\phi \in D(\mathfrak{a})'$, welche $R(\lambda, \mathcal{A})\phi \in X$ erfüllen. Dabei ist $\mathcal{A} : D(\mathfrak{a}) \to D(\mathfrak{a})'$ der zu $\mathfrak{a}$ assoziierte Operator auf $D(\mathfrak{a})$. Im weiteren führen wir lokale Versionen der Räume $D(\mathfrak{a})$ und $D(\mathfrak{a})'$ sowie des Operators $\mathcal{A}$ ein, um eine lokale abstrakte Kato-Klasse definieren zu können. Wir beweisen zudem eine Vielzahl von Eigenschaften der Kato-Klasse und zeigen insbesondere die Abhängigkeit von dem Parameter λ auf. Im Anschluss führen wir Kato-Störungen als angemessenene Verallgemeinerung von Potentialen und Maßen der klassischen Kato-Klasse ein. Zum Ende hin betrachten wir letztlich den Raum X_0 aller Funktionen aus X, die gegen unendlich verschwinden. Da die Zugehörigkeit zu X_0 meist keine lokale Eigenschaft darstellt, können wir in diesem Fall die Theorie unserer lokalen Kato-Klasse nicht anwenden. Neue Techniken sind gefragt. Um eine Halbgruppe auf X_0 zu erhalten, bedienen wir uns der Theorie der Lyapunov-Funktionen. Zum Beweis dieser Aussage benötigen wir insbesondere ein bestimmtes Approximationsresultat, welches zu einer Art abstrakter Dirichlet-Bedingung an den Rand äquivalent ist. Um die Theorie im letzten Schritt mit Leben zu erfillen, wenden wir uns Anwendungen zu. Als Prototyp für die zu behandelnden Formen dient uns hierbei der Satz von deGiorgi und Nash. Die zu den von ihnen betrachteten Operatoren in Verbindung stehenden Formen sinnigerweise deGiorgi-Nash-Formen nennend, ist es uns möglich, viele Elemente der Kato-Klasse für $X = C(\Omega)$ anzugeben. Wir beweisen, dass für jede deGiorgi-Nash-Form und jede beschränkte Menge $\Omega \subset \mathbb{R}^N$ ein Potential $V \in L^\infty_{\mathrm{loc}}$ existiert, so dass die zur gestörten Form assoziierte Halbgruppe auf $L^\infty(\Omega)$ den Raum $C_0(\Omega)$ invariant läßt.

Zum Abschluss präsentieren wir in Kapitel 7 Generationsresultate für den Black-Scholes-Operator. Hierfür benutzen wir zwei verschiedene Techniken. Zum einen nutzen wir den variationellen Ansatz, um stark stetige Halbgruppen auf $L^p(0, \infty)$, $2 \leq p < \infty$, zu realisieren. Mit dem neu entwickelten Kriterium aus Kapitel 4 zeigen wir außerdem, dass die Halbgruppe das Ordnungsintervall $[-\infty, \mathrm{id}]$ invariant läßt. Um jedoch eine explizite Darstellung der Halbgruppe (und damit des Optionenpreises) zu erhalten, verwenden wir einen anderen Ansatz. Zur Erweiterung gehen wir dazu auf Räume stetiger Funktionen über. Wir schreiben den Black-Scholes-Operator als einfache Störung eines quadrierten Gruppengenerators. Aus der Halbgruppentheorie ist dann bekannt, dass der Black-Scholes Operator eine holomorphe Halbgruppe erzeugt und zudem ist die explizite Darstellung dieser Halbgruppe bekannt. Wir nutzen diese Darstellung, um im letzten Schritt bekannte Preise für europäische Optionen nachzubilden.

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig angefertigt habe. Es wurden keine außer den angegebenen Quellen verwendet und die aus diesen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht.

Ferner erkläre ich, dass die vorliegende Arbeit in keinem anderen Promotionsverfahren und an keiner anderen Stelle als Prüfungsleistung verwendet wurde.

Ulm, 8. September 2008 Michael Einemann