

DIFFUSION SYSTEMS
AND
HEAT EQUATIONS ON NETWORKS

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STEFANO CARDANOBILO
aus Bari

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Dekan: Prof. Frank Stehling

Erstgutachter: Prof. Wolfgang Arendt

Zweitgutachter: Prof. Frank Steiner

Drittgutachter: Prof. Joachim von Below

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Introduction

Sesquilinear forms have played an important role in the analysis of elliptic equations for a long time. A possible starting point can be seen in Bernhard Riemann's doctoral thesis [53, Art. 18], where the celebrated Dirichlet's Principle is formulated. In short, the latter states that given an open, bounded domain Ω , the solution of the Laplace equation $\Delta\psi(x) = 0$ with homogeneous boundary conditions $\psi|_{\partial\Omega} = 0$ is the function for which the Dirichlet integral

$$\int_{\Omega} |\nabla\psi(x)|^2 dx \tag{DI}$$

attains a minimum in the set of the once differentiable functions that vanish on the boundary of Ω .

The Dirichlet integral of ψ can be seen as the value in the point ψ of the quadratic form $a : H_0^1(\Omega) \rightarrow \mathbb{C}$ defined by

$$a(\psi) := \int_{\Omega} (\nabla\psi(x) | \overline{\nabla\psi(x)}) dx, \quad \psi \in H_0^1(\Omega).$$

An important question is whether it is possible to connect properties of the Laplace equation and of its quadratic form with properties of the heat equation

$$\begin{cases} \frac{\partial}{\partial t}\psi(t, x) = \Delta\psi(t, x), & t \geq 0, \\ \psi(t, x) = 0, & x \in \partial\Omega, \\ \psi(0, x) = f(x) & x \in \Omega. \end{cases} \tag{HE}$$

After the second world war, Tosio Kato ([39], [40, Chap. 6]) recognised that for this purpose it is convenient to consider the *sesquilinear form* $a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{C}$ defined by

$$a(\psi, \psi') := \int_{\Omega} (\nabla\psi(x) | \overline{\nabla\psi'(x)}) dx, \quad \psi, \psi' \in H_0^1(\Omega).$$

He also formulated the general concept of sesquilinear forms and constructed an autonomous theory for such mappings.

We shortly explain the basic idea, in a slightly different manner from that introduced by Tosio Kato. Having fixed a complex Hilbert space V , a sesquilinear form (a, V) is a mapping $a : V \times V \rightarrow \mathbb{C}$ which is linear in the first component and antilinear in the second one. If the mapping $a : V \times V \rightarrow \mathbb{C}$ is continuous and a second Hilbert space H is given, such that V is densely embedded in H , it is possible to canonically associate an operator $(A, D(A))$ to the form (a, V) by a result due to Peter Lax and Arthur Milgram, see [42]. Under suitable assumptions on the form (a, V) there exists a family of bounded operators $(S_t)_{t \geq 0}$ such that $(S_t f)_{t \geq 0}$ is the solution of the abstract Cauchy problem

$$\begin{cases} \frac{d}{dt} \psi(t) = A\psi(t), & t \geq 0, \\ \psi(0) = f, & f \in H, \end{cases} \quad (\text{ACP})$$

where $(A, D(A))$ is the operator associated with the form. For this reason, we write $(e^{ta})_{t \geq 0} := (e^{tA})_{t \geq 0} := (S_t)_{t \geq 0}$. Once this family has been obtained, it is possible to show that some properties of the family of operators $(e^{ta})_{t \geq 0}$ can be characterised in terms of properties of the sesquilinear form (a, V) .

After the fundamental work of Tosio Kato, the use of sesquilinear forms for the analysis of partial differential equations has developed autonomously. Among other important results, a striking theorem was proved by El-Maati Ouhabaz in [51]. There, a closed convex set $C \subset H$ is considered and the property

$$\psi \in C \implies e^{ta}\psi \in C, \quad t \geq 0,$$

is studied. We will refer to this property as to the *invariance* of C under $(e^{ta})_{t \geq 0}$. El-Maati Ouhabaz proved that under suitable assumptions on the form (a, V) the invariance of C is equivalent to $P_C V \subset V$ and $a(\psi, \psi - P_C \psi) \leq 0$ for all $\psi \in V$. This result will be the key tool of our investigations.

We start our work observing that a sesquilinear form (a, V) on the direct sum

$$V := \bigoplus_{i \in I} V_i$$

of Hilbert spaces V_i admits a matrix representation obtained by setting

$$a_{ij}(\psi, \psi') := a(\mathbb{1}_j \otimes \psi, \mathbb{1}_i \otimes \psi').$$

Following this idea, it is possible to develop a matrix theory for such sesquilinear forms and to characterise or, at least, to give sufficient conditions for properties of (a, V) in terms of properties of the individual mappings.

In the case that $V_i = V_j$ for all $i, j \in I$, this matrix theory can be used to investigate *symmetry properties* of the semigroup $(e^{ta})_{t \geq 0}$ associated with

the form (a, V) on a certain Hilbert space H , i.e., to investigate invariance of subspaces of a certain form under the action of $(e^{ta})_{t \geq 0}$.

This is due to the fact that the invariance criterion proved by El-Maati Ouhabaz can be simplified in the case of closed subspaces. Denote $P_{\mathcal{Y}}$ the orthogonal projection of H onto a closed subspace \mathcal{Y} . Then, $e^{ta}\mathcal{Y} \subset \mathcal{Y}$ for all $t \geq 0$ if and only if

$$P_{\mathcal{Y}}V \subset V \tag{1}$$

and

$$a(\psi, \psi') = 0, \quad \psi \in \mathcal{Y}, \psi' \in \mathcal{Y}^{\perp}. \tag{2}$$

On the one hand, it turns out that the condition (1) is always satisfied in the case of subspaces representing symmetries, if V is a direct sum of Hilbert spaces. Thus, invariance is equivalent to the orthogonality condition (2), and this can be characterised in terms of properties of the mappings a_{ij} .

On the other hand, if the form domain contains coupling terms, i.e., if $V \subset \bigoplus_{i \in I} V_i$ but V is not an ideal, it is more difficult to develop a general matrix theory for forms. However, it is possible to systematically investigate special classes of spaces and couplings.

A possible example is the case of the Laplace operator on a network. Here, the form domain is defined by

$$V := \left\{ \psi \in \bigoplus_{i \in I} H^1(0, 1) : \psi(0) \oplus \psi(1) \in Y \right\}$$

where Y is a suitable subspace of $\ell^2(I) \oplus \ell^2(I)$, and the action of the form is defined by the one-dimensional Dirichlet integral.

Again, it is possible to systematically study symmetry properties of the associated semigroup $(e^{ta})_{t \geq 0}$. For the heat equation, in particular, the orthogonality condition (2) is always satisfied. Invariance is thus equivalent to the admissibility condition (1). A natural question is whether it is possible to characterise admissibility by graph theoretic properties. This task is not trivial, but it is also possible to extend some results to more general cases than network equations.

We start by observing that a sesquilinear form (a, V) on a Hilbert space V of the form $V := \bigoplus_{i \in I} V_i$ can be thought of as a matrix of sesquilinear mappings $(a_{ij})_{i, j \in I}$. Thus, some sufficient conditions for properties of the form (a, V) can be proved applying linear algebraic (or elementary functional analytic) arguments to suitable complex-valued matrices constructed from the form (a, V) . This is done in Section 1.1.

However, these arguments fail in many cases even in the easy task of characterising continuity or coercivity of infinite form matrices. In Section 1.2 we

consider the case of an infinite strongly coupled system and we characterise for such systems both continuity and coercivity.

In Section 1.3 we move back to the investigation of coercivity in the most general case, and we show that it suffices to analyse the finite restrictions of the form (a, V) . In the case of two-dimensional matrices it is even possible to characterise coercivity in terms of properties of the single mappings.

We start in Section 1.4 to address the issue of evolution equations, and we try to identify the domain of the operator associated with the form (a, V) . We continue our investigations of evolution equations in Section 1.5, where we characterise well-posedness and contractive properties of the semigroup $(e^{ta})_{t \geq 0}$ in terms of properties of the single mappings a_{ij} . As an example, we prove in Theorem 27 that the semigroup $(e^{ta})_{t \geq 0}$ generated by a matrix of sesquilinear mappings is positive if and only if the semigroups generated by the sesquilinear forms on the diagonal are positive and the mappings off-diagonal are negative mappings.

We completely devote Section 1.6 to symmetry properties, i.e., to invariance of subspaces of a certain form. Theorem 34 gives a complete characterisation of these invariance properties.

Section 1.7 and Section 1.8 contain two illustrative applications of the techniques developed in the first part of Chapter 1. In Section 1.7 we consider a strongly damped wave equation, and in Section 1.8 a heat equation with dynamic boundary conditions. In both cases, we first obtain well-posedness for a large choice of parameters by means of Proposition 25, and then we investigate qualitative properties of the solutions.

Finally, in Section 1.9 we introduce the topic of non-diagonal domain. Network equations represent maybe the most important case of non-diagonal domains and this will be the object of Chapter 2.

The results of Chapter 1 are motivated and discussed in Section 1.10. In particular, we shortly discuss the advantages of a matrix formalism for sesquilinear forms compared to the matrix formalism for operators. Further, we shortly describe the history of the different issues that we have addressed in Chapter 1.

We begin Chapter 2 by explaining informally the basic ideas of defining sesquilinear forms on networks. Section 2.1 can be seen as a quick course in integration by parts on networks.

Since we want to consider infinite networks in a L^2 -setting, we show in Section 2.2 that all definitions that are usual for finite networks also make sense for infinite networks. In particular, we investigate operator theoretic properties of the incidence matrices. Once we have done this, we can define in (2.4)-(2.6) the form domain and the action of the form (a, V) that will be the object of the following sections.

In Section 2.4 we investigate symmetry properties for systems of diffusion equations on networks. This section can be seen as a continuation of Section 1.2. We are able in Theorem 66 to completely characterise the admissibility of projections connected to symmetries.

Both Section 2.5 and Section 2.6 are an application of the results in the previous section. In the first one, we address the issue of irreducibility for the heat equation on infinite networks, and we characterise those networks for which the heat equation is irreducible. In Section 2.6 we investigate symmetry properties of special types of networks.

In Section 2.7 we leave the framework of networks, and we show that systems of diffusion equations on $H^1(0, 1)$ can be seen as a heat equation on a network, if the boundary conditions satisfy some properties.

The concept of symmetry is explained from the physical point of view in Section 2.8. We distinguish between *space* and *gauge* symmetries and we show that the symmetries that we have investigated in our work are gauge symmetries in a genuine physical sense.

Finally, we discuss in the rest of Section 2.8 the history of the different issues that we have discussed in the previous sections.

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Chapter 1

Sesquilinear forms on product Hilbert spaces

This Chapter is devoted to the study of sesquilinear forms that are defined on the direct sum of Hilbert spaces. We start proving general properties of these forms (Sections 1.1-1.3) and then we move to the investigation of properties of the associated evolution equation (Sections 1.4-1.6). Two illustrative applications are also discussed (Sections 1.7-1.8). Finally, we devote a part of the Chapter (Sections 1.9 and 1.10) to general considerations.

1.1 Finite dimensional arguments

The goal of this introductory section is twofold. First, we introduce the matrix representation of a form defined on the direct sum of Hilbert spaces $V := \bigoplus_{i \in I} V_i$. Then, we show that it is possible to deduce properties of such a form by properties of its matrix representation. We recall the definition of the direct sum of Hilbert spaces.

Definition 1. Let $I = \{1, \dots, m\}$ or $I = \mathbb{N}$ be an index set and V_i be Hilbert spaces for all $i \in I$. We consider on the space

$$V := \bigoplus_{i \in I} V_i = \{(\psi_i)_{i \in I} : \psi_i \in V_i \text{ and } \sum_{i \in I} \|\psi_i\|_{V_i}^2 < \infty\}, \quad (1.1)$$

the scalar product $(\cdot | \cdot) : V \times V \rightarrow \mathbb{C}$ by

$$(\psi | \psi') := \sum_{i \in I} (\psi_i | \psi'_i)_{V_i}. \quad (1.2)$$

The inner product space $(V, (\cdot | \cdot))$ is complete and thus it is Hilbert space.

In the case of a finite set I , the direct sum coincides with the set theoretical Cartesian product. Therefore, we call the Hilbert space V a *product Hilbert space*.

Before we formally introduce the concept of a sesquilinear form on V induced by a family of mappings, we recall how it is possible to define a matrix associated with a sesquilinear form on $\mathbb{R}^m \times \mathbb{R}^m$. Later in this section, we show that the same idea also applies for sesquilinear forms on infinite-dimensional Hilbert spaces.

Consider a linear mapping L on the Hilbert space \mathbb{C}^m , on which we have fixed an orthonormal basis $B = \{b_1, \dots, b_m\}$. Further, we consider the matrix representation $A = (a_{ij})_{i,j=1,\dots,m}$ of this linear mapping L . Since $(Lb_j | b_i) = (a_{.j} | b_i) = a_{ij}$, it is possible to determine the entries of the matrix representation (with respect to B) of L in terms of the scalar product.

Moreover, the sesquilinear form $a : \mathbb{C}^m \times \mathbb{C}^m \rightarrow \mathbb{C}$ defined by

$$a(u, v) := (Lu | v)$$

can be expressed in terms of the one-dimensional sesquilinear mappings

$$a_{ij} : \langle b_j \rangle \times \langle b_i \rangle \rightarrow \mathbb{C}, \quad a_{ij}(\lambda_j b_j, \mu_i b_i) := \lambda_j \bar{\mu}_i a_{ij}.$$

To see this, fix two vectors $u = \sum_{i=1}^m \lambda_i b_i$ and $v = \sum_{i=1}^m \mu_i b_i$. Computing now

$$\begin{aligned} a(u, v) &= a\left(\sum_{i=1}^m \lambda_i b_i, \sum_{j=1}^m \mu_j b_j\right) \\ &= \left(L \sum_{i=1}^m \lambda_i b_i \mid \sum_{j=1}^m \mu_j b_j\right) \\ &= \sum_{i,j=1}^m a_{ij}(\lambda_j b_j, \mu_i b_i), \end{aligned}$$

shows how it is possible to define a sesquilinear form on the space H by means of a family of sesquilinear forms a_{ij} . This formal computation also holds if the (possibly countably many) vectors b_j are elements of an infinite dimensional Hilbert space.

We prove two simple results. First, we define a sesquilinear form on the space V by means of a family of sesquilinear mapping. Second, we show that each form on such a product space always induces a family of sesquilinear mappings $a_{ij} : V_j \times V_i \rightarrow \mathbb{C}$. Observe that the case of operator matrices

is much more involved, see Section 1.10 for historical considerations on this topic.

We start by discussing the first implication, i.e., we show under which conditions a family of continuous sesquilinear mappings induces a continuous sesquilinear form on V . Recall that the mapping $a_{ij} : V_j \times V_i \rightarrow \mathbb{C}$ is said to be *continuous* if for some $M_{ij} \geq 0$ the estimate

$$|a_{ij}(\psi, \psi')| \leq M_{ij} \|\psi\|_{V_j} \|\psi'\|_{V_i}, \quad \psi \in V_j, \psi' \in V_i$$

holds. Fix now a family of continuous mappings $a_{ij} : V_j \times V_i \rightarrow \mathbb{C}$ and define an infinite matrix $M := (M_{ij})_{i,j \in I}$. We associate to the matrix M an operator (which we also denote by M) defining it on its maximal domain

$$D(M_{max}) := \{x \in \ell^2(I) : (M_{ij}x_j)_{j \in I} \in \ell^1(I) \text{ for all } i \text{ and } \sum_{i \in I} \left| \sum_{j \in I} M_{ij}x_j \right|^2 < \infty\}.$$

Since the mappings a_{ij} are continuous, we formally compute

$$\left| \sum_{i,j \in I} a_{ij}(\psi_j, \psi_i) \right| \leq \sum_{i,j \in I} M_{ij} \|\psi_j\|_{V_j} \|\psi_i\|_{V_i} = M(\|\psi_i\|_{V_i})_{i \in I},$$

and this makes sense if and only if $(\|\psi_i\|_{V_i})_{i \in I} \in D(M_{max})$. In fact, we have proved the following result.

Lemma 2. *Consider a family of continuous sesquilinear mappings $a_{ij} : V_j \times V_i \rightarrow \mathbb{C}$, and assume a_{ij} to be continuous with continuity constants M_{ij} for all $i, j \in I$, i.e.,*

$$|a_{ij}(\psi, \psi')| \leq M_{ij} \|\psi\|_{V_j} \|\psi'\|_{V_i}, \quad \psi \in V_j, \psi' \in V_i.$$

If the linear operator M on $\ell^2(I)$ defined by $M := (M_{ij})_{i,j \in I}$ is bounded, then the sesquilinear form $a : V \times V \rightarrow \mathbb{C}$ defined by

$$a(\psi, \psi') := \sum_{i,j \in I} a_{ij}(\psi_j, \psi'_i), \quad \psi, \psi' \in V \tag{1.3}$$

is continuous with constant $\|M\|_{\ell^2(I)}$.

Conversely, if a is a continuous sesquilinear form on V , then it is possible to give a matrix representation of the form a .

Lemma 3. *Let $a : V \times V \rightarrow \mathbb{C}$ be a continuous sesquilinear form on a product Hilbert space V . Then there exist uniquely determined continuous sesquilinear mappings $a_{ij} : V_j \times V_i \rightarrow \mathbb{C}$ such that $a(\psi) = \sum_{i,j \in I} a_{ij}(\psi_j, \psi_i)$.*

Proof. We start defining the projection $\pi_i \in \mathcal{L}(V, V_i)$ by

$$\pi_i(\psi) := \psi_i.$$

Observe now that π_i does possess a (not uniquely determined) isometric right inverse $\pi_{i,0}^{-1,r} \in \mathcal{L}(V_i, V)$ defined by

$$(\pi_{i,0}^{-1,r}(\psi))_j := \begin{cases} \psi & j = i, \\ 0 & \text{otherwise,} \end{cases} \quad \psi \in V_i.$$

One sees that the mapping $\pi_{i,0}^{-1,r}$ satisfies the identity

$$\sum_{i \in I} \pi_{i,0}^{-1,r}(\pi_i(\psi)) = \psi, \quad \psi \in V. \quad (1.4)$$

Define now mappings $a_{ij} : V_j \times V_i \rightarrow \mathbb{C}$ by

$$a_{ij}(\psi, \psi') := a(\pi_{j,0}^{-1,r}(\psi), \pi_{i,0}^{-1,r}(\psi')), \quad (1.5)$$

whose sesquilinearity is clear because of the analogous property of (a, V) .

We prove that the mappings a_{ij} are continuous. Denote M the continuity constant of (a, V) and compute for $\psi \in V_j, \psi' \in V_i$

$$\begin{aligned} |a_{ij}(\psi, \psi')| &= |a(\pi_{j,0}^{-1,r}(\psi), \pi_{i,0}^{-1,r}(\psi'))| \\ &\leq M \|\pi_{j,0}^{-1,r}(\psi)\|_V \|\pi_{i,0}^{-1,r}(\psi')\|_V \\ &= M \|\psi\|_{V_j} \|\psi'\|_{V_i}. \end{aligned}$$

We now show that the series $\sum_{i,j \in I} a_{ij}(\psi_j, \psi_i) = a(\psi)$ for all $\psi \in V$. By the definition of a_{ij} and the sesquilinearity of (a, V) we compute

$$\begin{aligned} \sum_{i,j \in I} a_{ij}(\psi_j, \psi_i) &= \sum_{i,j \in I} a(\pi_{j,0}^{-1,r}(\psi_j), \pi_{i,0}^{-1,r}(\psi_i)) \\ &= a\left(\sum_{j \in I} \pi_{j,0}^{-1,r}(\psi_j), \sum_{i \in I} \pi_{i,0}^{-1,r}(\psi_i)\right) \\ &= a(\psi). \end{aligned}$$

Here all series converge since (a, V) is continuous. To see that the mappings a_{ij} are unique, assume that there exists a second family $b_{ij} : V_j \times V_i \rightarrow \mathbb{C}$ satisfying $a(\psi, \psi') = \sum_{i,j \in I} b_{ij}(\psi_j, \psi'_i)$. Computing

$$\begin{aligned} a(\pi_{j,0}^{-1,r}(\psi), \pi_{i,0}^{-1,r}(\psi')) &= \sum_{i,j \in I} b_{ij}(\pi_{j,0}^{-1,r}(\psi)_j, \pi_{i,0}^{-1,r}(\psi')_i) \\ &= b_{ij}(\psi, \psi') \end{aligned}$$

shows that $a_{ij} = b_{ij}$. □

At a first glance, both above results seem quite satisfying. In fact, Lemma 3 shows that the analogy between matrices of complex numbers and matrices of sesquilinear mappings also holds in general. Therefore, we sometimes write with an abuse of notation $a =: (a_{ij})_{i,j \in I}$ for the sesquilinear form defined in (1.3). However, the continuity bound given in Lemma 2 is far away from being optimal, even in the case of optimal bounds M_{ij} and finite dimensional Hilbert spaces $V_1 = V_2 = \mathbb{C}$.

Example 4. Consider the sesquilinear mappings $a_{ij} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$a_{11}(z_1, z_2) := a_{12}(z_1, z_2) := a_{22}(z_1, z_2) := z_1 \overline{z_2},$$

and

$$a_{21}(z_1, z_2) := -z_1 \overline{z_2}.$$

Then the bound given in Lemma 2 on the continuity constant is the norm of the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

whereas the optimal continuity norm of (a, V) is given by the norm of the matrix

$$M' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

It turns out that $\sqrt{2} = \|M'\| < \|M\| = 2$.

In Section 1.10 we shortly discuss the theoretical background of this phenomenon.

Despite such unsatisfactory phenomena, it is possible to use both arguments given in the Lemmas 2 and 3 and obtain a characterisation of continuous sesquilinear forms in the case of a finite index set I .

Corollary 5. Consider a family of sesquilinear mappings $\{a_{ij} : i, j \in I\}$ on a finite index set I . Then the matrix of sesquilinear forms $a = (a_{ij})_{i,j \in I}$ is continuous if and only if all mappings a_{ij} are continuous.

We now want to give some finite dimensional arguments for the coercivity of a form. Recall that a sesquilinear form $a : V \times V \rightarrow \mathbb{C}$ is said to be *coercive*, if there exists an $\alpha > 0$ such that $\operatorname{Re} a(\psi) \geq \alpha \|\psi\|_V^2$ for all $\psi \in V$. Analogously, we say that a bounded linear operator L on $\ell^2(I)$ is coercive if $\operatorname{Re}(Lv | v) \geq \alpha \|v\|_{\ell^2(I)}^2$, and this is equivalent to the positive definiteness of the symmetric operator $L + L^*$. Consider the matrix representation $A =$

$(a_{ij})_{i,j \in I}$ of the linear operator L with respect to an orthonormal basis $(b_i)_{i \in I}$. For the vectors of the orthonormal basis B compute

$$\operatorname{Re} a_{ii} = \operatorname{Re}(Lb_i | b_i) \geq \alpha \|b_i\|_{\ell^2(I)} = \alpha.$$

As a consequence, the elements on the main diagonal are bounded from below by the coercivity constant α . The same argument also works in the for a form matrix (a, V) .

Lemma 6. *Consider a sesquilinear form $a : V \times V \rightarrow \mathbb{C}$. If $a = (a_{ij})_{i,j \in I}$ is coercive with constant α , then the sesquilinear forms $a_{ii} : V_i \times V_i \rightarrow \mathbb{C}$ are coercive with constant α .*

Proof. Let $\psi \in V_i$ and compute

$$a_{ii}(\psi_i) = a(\pi_{i,0}^{-1,r}(\psi_i)) \geq \alpha \|\pi_{i,0}^{-1,r}(\psi_i)\|_V^2 = \alpha \|\psi_i\|_{V_i}^2.$$

This is the claim. □

Using the same argument as in Lemma 2, we can also give a sufficient condition for the coercivity of a form $a = (a_{ij})_{i,j \in I}$.

Proposition 7. *Consider a sesquilinear form $a = (a_{ij})_{i \in I}$ of continuous mappings a_{ij} and denote the continuity constants by M_{ij} . Assume the diagonal forms a_{ii} to be coercive with constants α_{ii} . Define the matrix A by*

$$A_{ij} := \begin{cases} \alpha_{ii} & i = j \\ -M_{ij} & i \neq j. \end{cases}$$

If the matrix A is a bounded operator, and it is coercive with constant α , then the form (a, V) is coercive with the same constant.

Proof. The result is a consequence of the following computation

$$\begin{aligned} \operatorname{Re} a(\psi) &= \operatorname{Re} \sum_{i \in I} a_{ii}(\psi_i) + \operatorname{Re} \sum_{i \neq j \in I} a_{ij}(\psi_j, \psi_i) \\ &\geq \sum_{i \in I} \alpha_{ii} \|\psi_i\|_{V_i}^2 - \sum_{i \neq j \in I} M_{ij} \|\psi_j\|_{V_j} \|\psi_i\|_{V_i} \\ &= A(\|\psi_i\|_{V_i})_{i \in I} \geq \alpha \|\psi\|_V^2. \end{aligned}$$

□

As in the above result about boundedness, the coercivity estimate is not optimal. In the next section we will discuss a standard example for which it is possible to characterise both continuity and coercivity.

1.2 Systems of diffusion equations

We have seen that it is not a trivial task to characterise continuity and coercivity of form matrices, even in the case of a finite index set I . Assuming that all Hilbert spaces V_i are equal and that the mappings a_{ij} have the same structure, i.e., considering the case of a strongly coupled system allows us to give better results.

To be more precise, we assume $V_i = H_0^1(\Omega)$ or $V_i = H^1(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^d$. Moreover, C will denote some bounded operator-valued function $C : \Omega \rightarrow \mathcal{L}(\ell^2(I))$. In this setting, we define the form $a : V \times V \rightarrow \mathbb{C}$ by

$$a(\psi, \psi') := \int_{\Omega} (C(x)\nabla\psi(x) \mid \nabla\psi'(x))dx. \quad (1.6)$$

We want to discuss two different topics in this section. First, we address the issue of continuity for systems defined on $H^1(\Omega)$. Second, we investigate coercivity properties for systems defined on $H_0^1(\Omega)$. We want to focus our attention on infinite systems and therefore, we restrict ourselves to the case $I = \mathbb{N}$.

In order to fix the ideas, let us summarise the standing assumptions for this section.

Assumptions 8. *In this section we always assume the following.*

- *The set Ω is a domain in \mathbb{R}^d .*
- *The function $C : \Omega \rightarrow \mathcal{L}(\ell^2(\mathbb{N})), x \mapsto (c_{ij}(x))_{i,j \in \mathbb{N}}$ is componentwise measurable.*
- *If there is no danger of confusion, we denote by C also the range of the function C , i.e., the operator family $C := \{C(x) : x \in \Omega\}$.*
- *The Hilbert space H is defined as in (1.1) by*

$$H := \bigoplus_{i \in \mathbb{N}} L^2(\Omega). \quad (1.7)$$

Since the function C is an operator for all x and since $\psi(x) \in \ell^2(\mathbb{N})$ for almost every x , it is possible to define for almost every $x \in \Omega$ and for all $n \in \mathbb{N}$

$$(M_C\psi)_n(x) := \sum_{m \in \mathbb{N}} c_{nm}(x)\psi_m(x). \quad (1.8)$$

The function C is componentwise measurable. Thus, also $M_C\psi : x \mapsto (\sum_{m \in \mathbb{N}} c_{nm}(x)\psi_m(x))_{n \in \mathbb{N}}$ is componentwise measurable. As a consequence, we can define M_C on its maximal domain

$$D(M_{C_{max}}) := \{\psi \in \ell^2(E) : M_C\psi \in \ell^2(E)\}.$$

In the following, we will need estimates for the gradient of ψ . We thus introduce a multiplication operator M_C^d on H^d . This is defined on

$$D(M_{C_{max}}^d) := \bigoplus_{i=1, \dots, d} D(M_{C_{max}})$$

by

$$(M_C^d\psi) := (M_C\psi^1, \dots, M_C\psi^d), \quad \psi = (\psi^i)_{i=1, \dots, d} \in D(M_{C_{max}}^d). \quad (1.9)$$

In the next Lemma, we show that H admits different representations.

Lemma 9. *Assume that X_i are σ -finite measure spaces. Then*

$$\bigoplus_{i \in \mathbb{N}} L^2(X_i) \simeq L^2\left(\bigoplus_{i \in \mathbb{N}} X_i\right). \quad (1.10)$$

If $X_i = X$ for all $i \in \mathbb{N}$ both spaces are also isometrically isomorphic to $L^2(X, \ell^2(\mathbb{N}))$. In particular,

$$\|\psi\|_{L^2(\bigoplus_{i \in \mathbb{N}} X)}^2 = \int_X \|\psi(x)\|_{\ell^2}^2 dx. \quad (1.11)$$

Proof. For the sake of the completeness, we recall the construction of the measure space $\bigoplus_{i \in \mathbb{N}} X_i$. The direct sum of sets is defined by

$$X := \bigoplus_{i \in \mathbb{N}} X_i := \{(i, x) : i \in \mathbb{N}, x \in X_i\}.$$

We define a σ -algebra Σ on X . Observe that each subset $A \subset X$ has the form

$$A = \bigoplus_{i \in \mathbb{N}} A_i, \quad A_i := \{x \in X_i : \exists y \in A, y = (i, x)\}.$$

We define a σ -algebra Σ by requiring that $A \in \Sigma$ if and only if A_i is Lebesgue measurable for each $i \in \mathbb{N}$. It is a simple exercise to check that Σ is indeed a σ -algebra on X . For all $A \in \Sigma$ define

$$\lambda(A) := \sum_{i \in \mathbb{N}} \lambda_i(A_i).$$

Since all terms in the above sum are positive, the series converges if and only if it converges absolutely. Thus, convergence is independent of the order of summation and the expression $\sum_{i \in I} \lambda_i(A_i)$ makes sense. Then, $\lambda(A) = 0$ if and only if $\lambda_i(A_i) = 0$ for all $i \in \mathbb{N}$. Since the Cartesian product of countable sets is countable, we see that the triple (X, Σ, λ) is a σ -finite measure space. Define a mapping $\phi : L^2(X) \rightarrow H$ by

$$(\phi(\psi))_i(x) := \psi_i(x) \quad x \in X_i.$$

The mapping ϕ is bijective. We prove that it is isometric. Fix to this aim $\psi \in L^2(X)$ and observe that the sets X_i are a measurable partition of X . Thus, by the definition of the norm in $L^2(X)$ and the monotone convergence theorem, we obtain

$$\begin{aligned} \|\psi\|_{L^2(X)}^2 &= \int_X |\psi(x)|^2 dx = \int_X \sum_{i \in \mathbb{N}} |\mathbb{1}_{X_i} \psi(x)|^2 dx \\ &= \sum_{i \in I} \int_X |\mathbb{1}_{X_i} \psi(x)|^2 dx = \sum_{i \in \mathbb{N}} \int_{X_i} |\psi_i(x)|^2 dx. \end{aligned}$$

The latter is the definition of the norm in H .

Assume now that $X_i = \Omega$ for all $i \in \mathbb{N}$ and define a mapping $\phi : H \rightarrow L^2(\Omega, \ell^2(\mathbb{N}))$ by

$$(\phi(\psi))_i(x) := \psi_i(x) \quad x \in X_i.$$

Since $\psi_i \in L^2(\Omega)$ for all $i \in \mathbb{N}$, the function $f : x \mapsto \sum_{i \in \mathbb{N}} |\psi_i(x)|^2$ is defined almost everywhere and is a measurable function as series of positive measurable functions. Thus, the mapping ϕ is well-defined and it is clearly bijective. The formula (1.11) follows again from the monotone convergence theorem, and so the claim is proven. \square

We are now in the position of attacking the main problem. As first result, we characterise the continuity of the form (a, V) in the case $V_i = H^1(\Omega)$ for all $i \in \mathbb{N}$. As usual, we denote V the Hilbert space defined in (1.1). In order to achieve the goal of characterising continuity, we need to obtain sharp estimates for the L^2 norm of the gradient of functions in V , and for this optimal bounds for the operator M_C^d are needed. We start showing that the norm boundedness of the range of C is equivalent to the boundedness of the operator M_C .

Lemma 10. *Recall the definition (1.7) of the Hilbert space H . The following assertions are equivalent for an arbitrary $M \in \mathbb{R}$.*

a) *The operator M_C is bounded by M , i.e., $\|M_C\|_{\mathcal{L}(H)} \leq M$.*

b) The operator family C is uniformly essentially bounded, i.e., $\|C(x)\|_{\mathcal{L}(\ell^2)} \leq M$ for almost all $x \in \Omega$.

Proof. Assume the family C to be uniformly essentially bounded. Use the definition of the norm in H to see

$$\|M_C\psi\|_H^2 = \sum_{n \in \mathbb{N}} \int_{\Omega} |(C(x)\psi(x))_n|^2 dx.$$

Since the series converges absolutely, this yields

$$\|M_C\psi\|_H^2 \leq M \int_{\Omega} \|\psi(x)\|_{\ell^2}^2 dx = M \|\psi\|_H^2,$$

and this implies a).

We now prove the converse implication. Assume M_C to be bounded by M and fix an arbitrary normed sequence $y := (y_i)_{i \in \mathbb{N}} \in \ell^2$. Define now functions in $L^2(\Omega)$ by

$$\psi_i := \chi_{\omega} y_i,$$

where $\lambda(\omega) < \infty$.

Since $y \in \ell^2$ the vector $\psi := (\psi_i)_{i \in \mathbb{N}}$ is an element of H , whose norm is given by

$$\|\psi\|_H = \sum_{i \in \mathbb{N}} \|\psi_i\|_{L^2}^2 = \lambda(\omega) \sum_{i \in \mathbb{N}} |y_i|^2 = \lambda(\omega).$$

As a consequence, the boundedness of M_C yields

$$\|M_C\psi\| \leq M\lambda(\omega). \tag{1.12}$$

Observe that

$$M_C\psi(x) = \begin{cases} C(x)y & x \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Now are we in the position of completing the proof. Fix an arbitrary $x_0 \in \Omega$ and set $\omega_j = B_{\frac{1}{j}}(x_0)$. Let us denote $\beta(d)$ the volume of the unit ball in \mathbb{R}^d .

As a consequence $\lambda(\omega_j) = \frac{\beta(d)}{j^d}$. By the Lebesgue-Besicovitch Theorem (see e.g. [31, Theo. 1.7.1]) we obtain for almost all $x_0 \in \Omega$

$$\lim_{j \rightarrow \infty} \frac{j^d}{\beta(d)} \int_{\omega_j} (M_C\psi)_i(x) dx = \lim_{j \rightarrow \infty} \frac{j^d}{\beta(d)} \int_{\omega_j} (C(x)y)_i dx = (C(x_0)y)_i.$$

for all $i \in \mathbb{N}$. Since (1.12) holds, by the monotone convergence theorem we obtain

$$\|C(x_0)y\| = \lim_{j \rightarrow \infty} \frac{j^d}{\beta(d)} \|M_C\psi\|.$$

Now, passing to the limit yields

$$\|C(x_0)y\| = \lim_{j \rightarrow \infty} \frac{j^d}{\beta(d)} \|M_C \psi\| \leq \lim_{j \rightarrow \infty} \frac{j^d}{\beta(d)} \lambda(\omega_j) M = M.$$

what we had to prove. \square

Remark 11. *The operator M_C^d defined in (1.9) is a bounded operator on H^d if and only if M_C is a bounded operator on H .*

We are now going to define the sesquilinear form whose continuity we want to characterise. We introduce to this aim some notations. For a function $\psi \in V$ define the gradient of ψ by

$$\nabla \psi := (\nabla \psi_n)_{n \in \mathbb{N}}, \quad \psi \in V.$$

We observe that, by virtue of the definition of V , for all $\psi \in V$

$$(\|\nabla \psi_n\|_{L^2(\Omega)})_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}),$$

and so $\nabla \psi \in \mathcal{L}(V, H^d)$. Recall that for functions ψ, ψ' the scalar product on H^d is defined by

$$(\psi \mid \psi')_{H^d} = \sum_{k=1}^d (\psi^k \mid \psi'^k)_H = \sum_{k=1}^d \sum_{i=1}^{\infty} (\psi_i^k \mid \psi_i'^k)_{L^2(\Omega)}$$

On finite sequences of V it is possible to define a sesquilinear form a_C by

$$a_C(\psi, \psi') := (M_C^d \nabla \psi \mid \nabla \psi')_{H^d}. \quad (1.13)$$

In the next Proposition we characterise the continuity of the form a_C on the whole space V .

Proposition 12. *Assume the set Ω to have finite measure. Then the form a_C defined in (1.13) is continuous as a mapping $a_C : V \times V \rightarrow \mathbb{C}$ if and only if the operator family C is essentially uniformly bounded.*

Proof. Assume the form a_C to be continuous with constant M . In particular, for each $k = 1, \dots, d$ the form $a_{C,k} : V \times V \rightarrow \mathbb{C}$ defined by

$$a_{C,k}(\psi, \psi') := \left(M_C \frac{\partial}{\partial x_k} \psi \mid \frac{\partial}{\partial x_k} \psi' \right)_H \quad (1.14)$$

is continuous. Thus,

$$\left| \left(M_C \frac{\partial}{\partial x_k} \psi \mid \frac{\partial}{\partial x_k} \psi' \right)_H \right| \leq M_1, \quad \text{for all } \psi, \psi' \in B_1^V(0).$$

The operator $\frac{\partial}{\partial x_k} \in \mathcal{L}(H^1(\Omega), L^2(\Omega))$ is a bounded and surjective linear operator. So it is an open mapping and this implies

$$|(M_C \psi | \psi')_H| \leq M_2, \quad \text{for all } \psi, \psi' \in B_1^H(0).$$

The formula

$$\|M_C\| = \sup_{\psi, \psi' \in B_1^H(0)} (M_C \psi | \psi'),$$

together with the Lemma 10 shows the claim.

Conversely, assume the operator family C to be bounded. By Lemma 10 and Remark 11 the operator M_C^d is bounded. Thus, the estimate

$$|a_C(\psi, \psi')| = |(M_C^d J\psi | J\psi')| \leq \|M_C^d\| |(J\psi | J\psi')| \leq \|M_C^d\| \|\psi\|_V \|\psi'\|_V$$

yields the continuity of a_C , thus completing the proof. \square

We now turn our attention to coercivity properties of the form defined in (1.6) in the case $V_i = H_0^1(\Omega)$ for all $i \in \mathbb{N}$ and V as in (1.1). Analogously as in the previous part of the section we want to prove the equivalence between the coercivity of (a, V) and the that of essential uniform coercivity of C . In order to prove this result the equivalence of the norm on V and the scalar product defined by $[u | v] := (\nabla u | \nabla v)_{L^2}$ is needed. Domains having this property are said to be of *Poincaré type*. In particular, bounded domains and domains lying in a strip are of Poincaré type, see [4, Prop. 3.4.1].

Theorem 13. *Assume the operator-valued function C to be of class $C(\overline{\Omega})$, uniformly bounded and accretive. The form (a, V) is coercive if and only if the operator family C is uniformly coercive.*

As before, we split the proof into two steps. First, we prove the equivalence between the uniform coercivity of C and the coercivity of M_C . Second, we transfer the information about M_C to the form (a, V) .

Lemma 14. *If Ω is of Poincaré type and C is essentially uniformly bounded, the following assertions are equivalent.*

a) *The family C is essentially uniformly coercive with constant α , i.e.,*

$$\operatorname{Re}(C(x)y | y)_{\ell^2} \geq \alpha \|y\|_{\ell^2}^2 \quad \text{for almost all } x \in \Omega;$$

b) *The operator M_C is coercive with constant α , i.e.,*

$$\operatorname{Re}(M_C \psi | \psi)_H \geq \alpha \|\psi\|_H^2, \quad \text{for all } \psi \in H.$$

Proof. Assume the family C to be essentially uniformly coercive. We thus compute

$$\begin{aligned}
\operatorname{Re}(M_C \psi \mid \psi)_H &= \operatorname{Re} \sum_{n \in \mathbb{N}} \int_{\Omega} \sum_{m \in \mathbb{N}} c_{nm}(x) \psi_m(x) \overline{\psi_n(x)} dx \\
&= \operatorname{Re} \int_{\Omega} \sum_{n, m \in \mathbb{N}} c_{nm}(x) \psi_m(x) \overline{\psi_n(x)} dx \\
&= \operatorname{Re} \int_{\Omega} (C(x) \psi(x) \mid \psi(x)) dx \\
&\geq \int_{\Omega} \alpha \|\psi(x)\|_{\ell^2}^2 dx = \alpha \|\psi\|_H^2,
\end{aligned}$$

where in the last step we used the formula (1.11).

The converse implication is a consequence of the Lebesgue-Besicovitch Theorem, and can be obtained exactly in the same way as in the analogous implication of Lemma 10. \square

Exactly as before, M_C is coercive if and only if M_C^d is coercive. Observe that since Ω is of Poincaré type, the inequality $\|\nabla \psi_i\|_{L^2(\Omega)}^2 \geq M_1 \|\psi_i\|_{H_0^1(\Omega)}^2$ holds for all $\psi_i \in H_0^1(\Omega)$. Thus,

$$\sum_{i \in \mathbb{N}} \|\nabla \psi_i\|_{L^2(\Omega)}^2 = \|\nabla \psi\|_H^2 \geq M_1 \|\psi\|_V^2.$$

We can now move to the proof of the main result.

Proof of Theorem 13. First, we assume the family C to be essentially uniformly coercive. Because of Lemma 14 and of the Poincaré property

$$\operatorname{Re} a(\psi) = (M_C^d \nabla \psi \mid \nabla \psi) \geq \alpha \|\nabla \psi\|_H^2 \geq \alpha M_1 \|\psi\|_V^2.$$

Thus, the form a is coercive with constant αM_1 .

Conversely, assume that C is not uniformly coercive. Thus, there exists a sequence $(x_k)_{k \in \mathbb{N}} \subset \Omega$ and a sequence $(y^k)_{k \in \mathbb{N}} \subset \ell^2(\mathbb{N})$, $\|y^k\|_{\ell^2(\mathbb{N})} = 1$ such that

$$\operatorname{Re}(C(x_k) y^k \mid y^k) \leq \frac{1}{k} \|y^k\|_{\ell^2(\mathbb{N})}^2.$$

In particular, since C is of class $C(\overline{\Omega})$, it is also uniformly continuous and there exists an uniform radius $r > 0$ such that

$$\operatorname{Re}(C(x) y^k \mid y^k) \leq \frac{2}{k} \|y^k\|_{\ell^2(\mathbb{N})}^2, \quad x \in B_r(x_k) \cap \Omega. \quad (1.15)$$

Consider now real valued functions $\psi^k \in C_c^\infty(\Omega)$ such that $\text{supp } \psi^k \subset B_r(x_k)$. Define $\widehat{\psi^k} \in V$ by

$$(\widehat{\psi^k})_i := y_i^k \psi^k, \quad i, k \in \mathbb{N}.$$

Observe that $\|\widehat{\psi^k}\|_V^2 = \|\psi^k\|_{H^1(\Omega)}^2$ since $\|y^k\|_{\ell^2(\mathbb{N})} = 1$. Furthermore

$$\frac{\partial \widehat{\psi^k}(x)}{\partial x_\ell} = \frac{\partial \psi^k(x)}{\partial x_\ell} y^k, \quad \ell = 1, \dots, d, k \in \mathbb{N}, x \in \Omega,$$

and so by (1.15)

$$\begin{aligned} (C(x) \frac{\partial \widehat{\psi^k}(x)}{\partial x_\ell} \mid \frac{\partial \widehat{\psi^k}(x)}{\partial x_\ell}) &= (C(x) \frac{\partial \psi^k(x)}{\partial x_\ell} y^k \mid \frac{\partial \psi^k(x)}{\partial x_\ell} y^k) \\ &\leq \frac{2}{k} \left| \frac{\partial \psi^k(x)}{\partial x_\ell} \right|^2 \|y^k\|_{\ell^2(\mathbb{N})}^2 \\ &= \frac{2}{k} \left| \frac{\partial \psi^k(x)}{\partial x_\ell} \right|^2 \end{aligned}$$

for all $\ell = 1, \dots, d, k \in \mathbb{N}$ and all $x \in B_r(x_k) \cap \Omega$. The estimate

$$\begin{aligned} \text{Re } a(\widehat{\psi^k}) &= \text{Re} \sum_{\ell=1}^d \int_{\Omega} (C(x) \frac{\partial \widehat{\psi^k}(x)}{\partial x_\ell} \mid \frac{\partial \widehat{\psi^k}(x)}{\partial x_\ell}) dx \\ &= \text{Re} \sum_{\ell=1}^d \int_{B_r(x_k) \cap \Omega} (C(x) \frac{\partial \widehat{\psi^k}(x)}{\partial x_\ell} \mid \frac{\partial \widehat{\psi^k}(x)}{\partial x_\ell}) \\ &\leq \frac{2}{k} \sum_{\ell=1}^d \int_{B_r(x_k) \cap \Omega} \left| \frac{\partial \psi^k(x)}{\partial x_\ell} \right|^2 dx \\ &\leq \frac{2}{k} \|\psi^k\|_{H^1(\Omega)}^2 = \frac{2}{k} \|\widehat{\psi^k}\|_V^2 \end{aligned}$$

shows that the form (a, V) is not coercive. \square

In the special case of systems we have characterised continuity and coercivity properties of the form a . In the next section we characterise coercivity in another special case.

1.3 Coercivity

The characterisation of the coercivity of a form matrix is in general a complex task. We present two results in this section: the first gives a characterisation

in terms of the finite restrictions of the form. In the second one, we discuss the coercivity in the case of a two-dimensional matrix.

We start by fixing the standing assumptions for this section.

Assumptions 15. *During this section we always assume the following.*

- The form (a, V) is continuous.
- Fix $I' \subset I$. We denote $a_{|I'}$ the restriction of the form to $V_{|I'} := \prod_{i \in I'} V_i$ and we call it the restriction of the form (a, V) induced by I' .

Theorem 16. *The following assertions are equivalent.*

- The form (a, V) is coercive with constant α
- Each finite $I' \subset I$ induces a coercive restriction with constant α .

Proof. The necessity of the condition b) can be seen by defining the projection $\pi_{I'} \in \mathcal{L}(V, V_{|I'})$

$$(\pi_{I'}(\psi))_i := \psi_i, \quad \text{for all } i \in I',$$

and arguing as in the proof of Lemma 6. The sufficiency of the condition b) in the case of a finite index set I is tautological. So, we only have to prove the sufficiency in the case $I = \mathbb{N}$.

To this aim, let $\psi \in V, \|\psi\|_V = 1$. We have to show $\operatorname{Re} a(\psi) \geq \alpha$. First observe that each $\psi \in V$ can be uniquely decomposed as

$$\psi = \psi' + \psi'', \quad \psi' \in \bigoplus_{i \in I'} V_i, \psi'' \in \bigoplus_{i \in I''} V_i.$$

Here $I'' = \mathbb{N} \setminus I'$. By sesquilinearity we obtain

$$a(\psi) = a(\psi') + a(\psi'') + a(\psi', \psi'') + a(\psi'', \psi'). \quad (1.16)$$

Since $a(\psi') = a_{I'}(\pi_{I'}(\psi))$, we obtain

$$\operatorname{Re} a(\psi) \geq \alpha \|\psi'\|_V^2. \quad (1.17)$$

Further, by the continuity of (a, V) , we also see that

$$|a(\psi'') + a(\psi', \psi'') + a(\psi'', \psi')| \leq M \|\psi''\|_V^2 + 2M \|\psi'\|_V \|\psi''\|_V. \quad (1.18)$$

Thus, using $\|\psi'\| \leq 1$ and estimating as in the proof of Proposition 7,

$$-(\operatorname{Re} a(\psi'') + \operatorname{Re} a(\psi', \psi'') + \operatorname{Re} a(\psi'', \psi')) \geq -M \|\psi''\|_V (\|\psi''\|_V + 2). \quad (1.19)$$

Observe now that $\|\psi'\|_V^2 + \|\psi''\|_V^2 = 1$, i.e.,

$$\|\psi'\|_V = \sqrt{1 - \|\psi''\|_V^2}. \quad (1.20)$$

Combining equations (1.16)-(1.17)-(1.18)-(1.19)-(1.20) we obtain the estimate

$$\operatorname{Re} a(\psi') \geq \alpha \|\psi'\|_V^2 - M(1 - \|\psi'\|_V) - 2M\sqrt{1 - \|\psi'\|_V^2}. \quad (1.21)$$

We want to estimate the right hand-side of (1.21) from below. Define the auxiliary function

$$f(t) := \alpha t^2 - M(1 - t) - 2M\sqrt{1 - t^2}.$$

Thus, we can reformulate (1.21) as $\operatorname{Re} a(\psi') \geq f(\|\psi'\|_V)$. Since f is continuous and $f(1) = \alpha$, we obtain

$$\limsup_{n \rightarrow \infty} \operatorname{Re} a(\psi') \geq \lim_{t \rightarrow 1} f(t) \geq \alpha,$$

where $I' := \{1, \dots, n\}$. Since $\lim_{n \rightarrow \infty} \psi' = \psi$ by definition and (a, V) is a continuous form

$$\operatorname{Re} a(\psi) = \lim_{n \rightarrow \infty} \operatorname{Re} a(\psi') = \limsup_{n \rightarrow \infty} \operatorname{Re} a(\psi') \geq \alpha,$$

and the proof is complete. \square

Next, we show that for two by two matrices it is possible to characterise the coercivity in terms of property of the single mappings a_{ij} . To this aim, for a finite index set I and a vector $w := (w_i)_{i \in I}, w_i > 0$ we define a weighted equivalent norm $\|\cdot\|_w$ on V by

$$\|\psi\|_w^2 := \sum_{i \in I} w_i \|\psi\|_{V_i}^2.$$

Proposition 17. *Consider as index set $I := \{1, 2\}$ and coercive sesquilinear forms $a_{ii} : V_i \times V_i \rightarrow \mathbb{C}$ for $i \in I$, with optimal constants α_1, α_2 , respectively. Then the following assertions are equivalent.*

a) *The form (a, V) is coercive.*

b) *There exist $\alpha > 0$ such that for all $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in V$ and for $w = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$*

$$\operatorname{Re} a_{12}(\psi_2, \psi_1) + \operatorname{Re} a_{21}(\psi_1, \psi_2) \geq \alpha \|\psi\|_w^2 - 2\sqrt{\alpha_1 \alpha_2} \|\psi_1\|_{V_1} \|\psi_2\|_{V_2}. \quad (1.22)$$

Proof. If there is no danger of confusion, we do not specify in which space we are computing the norm. For a real parameter $\lambda > 0$

$$\begin{aligned}
\operatorname{Re} a(\psi) &= \operatorname{Re} a\left(\lambda\psi, \frac{1}{\lambda}\psi\right) \\
&= \operatorname{Re} \lambda^2 a_{11}(\psi_1, \psi_1) + \operatorname{Re} \frac{1}{\lambda^2} a_{22}(\psi_2, \psi_2) \\
&\quad + \operatorname{Re} a_{12}(\psi_2, \psi_1) + \operatorname{Re} a_{21}(\psi_1, \psi_2) \\
&\geq \alpha_1 \lambda^2 \|\psi_1\|^2 + \alpha_2 \frac{1}{\lambda^2} \|\psi_2\|^2 + \operatorname{Re} a_{12}(\psi_2, \psi_1) + \operatorname{Re} a_{21}(\psi_1, \psi_2) \\
&=: f(\lambda)
\end{aligned}$$

The auxiliary function $f : (0, \infty) \rightarrow \mathbb{R}$ has derivative

$$f'(\lambda) = 2 \left(\alpha_1 \|\psi_1\|^2 \lambda - \alpha_2 \|\psi_2\|^2 \lambda^{-3} \right),$$

which vanishes only at

$$\lambda_0 = \sqrt[4]{\frac{\alpha_2 \|\psi_2\|^2}{\alpha_1 \|\psi_1\|^2}} = \sqrt{\frac{\sqrt{\alpha_2} \|\psi_2\|}{\sqrt{\alpha_1} \|\psi_1\|}}.$$

Inserting λ_0 into f yields:

$$f(\lambda_0) = 2\sqrt{\alpha_1 \alpha_2} \|\psi_1\| \|\psi_2\| + \operatorname{Re} a_{12}(\psi_2, \psi_1) + \operatorname{Re} a_{21}(\psi_1, \psi_2).$$

The strict convexity of f implies that λ_0 is the global minimum of the function f . Since the constants are optimal and ψ_1, ψ_2 can be chosen independently of each other, the coercivity is equivalent to

$$f(\lambda) \geq \alpha \|\psi\|_V^2$$

for some $\alpha > 0$. Since $\alpha_1, \alpha_2 > 0$, the norm in V is of course equivalent to the weighted norm $\|\cdot\|_w$, where $w := (\alpha_1, \alpha_2)$. We deduce that the coercivity with respect to $\|\cdot\|$ is equivalent to the coercivity with respect to $\|\cdot\|_w$, i.e., to the existence of $\alpha > 0$ such that

$$2\sqrt{\alpha_1 \alpha_2} \|\psi_1\| \|\psi_2\| + \operatorname{Re} a_{12}(\psi_2, \psi_1) + \operatorname{Re} a_{21}(\psi_1, \psi_2) \geq \alpha (\alpha_1 \|\psi_1\|^2 + \alpha_2 \|\psi_2\|^2),$$

what we had to prove. \square

We illustrate a possible use of this criterion.

Example 18. On a domain Ω of class C^∞ consider the operator matrix

$$H := \begin{pmatrix} \Delta & M_V \\ M_V & \Delta \end{pmatrix}$$

on $D(H) := D(\Delta_{\text{Dirichlet}}) \times D(\Delta_{\text{Dirichlet}})$. Here M_V denotes a bounded multiplication operator which is defined by

$$M_V f(x) := V(x)f(x).$$

The form matrix associated with the operator is defined on the Hilbert space $V := H_0^1(\Omega) \times H_0^1(\Omega)$ by setting

$$a_{11}(\psi, \psi') := a_{22}(\psi, \psi') := \int_{\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi'(x)} dx, \quad \psi, \psi' \in H_0^1(\Omega),$$

$$a_{12}(\psi, \psi') := - \int_{\Omega} V(x) \psi(x) \overline{\psi'(x)} dx, \quad \psi, \psi' \in H_0^1(\Omega),$$

and

$$a_{21}(\psi', \psi) := - \int_{\Omega} V(x) \psi'(x) \overline{\psi(x)} dx, \quad \psi, \psi' \in H_0^1(\Omega).$$

Assume now that $V(x) \in i\mathbb{R}$ for all $x \in \mathbb{R}$. Compute for $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in V$

$$\begin{aligned} & \operatorname{Re}(a_{12}(\psi_2, \psi_1) + a_{21}(\psi_1, \psi_2)) \\ &= - \operatorname{Re} \int_{\Omega} V(x) \left(\psi_1(x) \overline{\psi_2(x)} + \psi_2(x) \overline{\psi_1(x)} \right) dx \\ &= 0. \end{aligned}$$

Since the right hand-side of (1.22) is larger than 0 if $\psi \neq 0 = \psi'$, the condition b) in Proposition 17 is not satisfied. As a consequence, the form (a, V) is not coercive.

1.4 Identification of the operator domain

Our main interest are evolution equations, and we will address this topic in details in Section 1.5. In that context, we will relate properties of the form (a, V) to properties of solutions of evolution equations governed by the associated operator $(A, D(A))$, as described in Appendix A.1. Therefore, it is important to determine the domain of the operator associated with the sesquilinear form. This the object of this section.

In order to define the operator associated with the form (a, V) we have to specify a Hilbert space H such that $V \hookrightarrow H$ densely. To obtain such a space in our context, we assume that the Hilbert spaces V_i are uniformly and densely embedded into Hilbert spaces H_i , i.e., we choose a family H_i such that $V_i \hookrightarrow H_i$ for all $i \in I$, and the norms of the canonical injections are bounded from above. In order to fix the ideas, we state the standing assumptions.

Assumptions 19. *During the rest of this chapter we always assume the following.*

- *The Hilbert spaces V_i are continuously and densely embedded into the Hilbert spaces H_i for all $i \in I$.*
- *The Hilbert space H is defined as*

$$H := \bigoplus_{i \in I} H_i.$$

- *The canonical injections are denoted $\phi_j : V_j \rightarrow H_j$.*
- *The canonical, set theoretical injection $\phi : \prod_{j \in I} V_j \rightarrow \prod_{j \in I} H_j$ is defined componentwise by*

$$\pi_j(\phi(\psi)) := \phi_j(\pi_j(\psi)).$$

In the case of a finite set I there are no difficulties connected with the boundedness of the injection ϕ . In the general case, however, the mapping ϕ is an unbounded operator from V to H , see Example 21. We recall some embedding results, proving them for the sake of completeness.

Lemma 20. *The following equivalences hold.*

- a) *The injection ϕ is continuous if and only if the injections ϕ_j are uniformly continuous, i.e., if and only if there exists $M \geq 0$ such that*

$$\|\psi\|_{H_i} \leq M \|\psi\|_{V_i}, \quad \text{for all } i \in I, \psi \in V_i. \quad (1.23)$$

- b) *Assume I to be infinite. The injection ϕ is a compact operator if and only if the injections ϕ_j are compact operators and*

$$\lim_{j \rightarrow \infty} \|\phi_j\|_{\mathcal{L}(V_j, H_j)} = 0.$$

Proof. a) If the estimate (1.23) holds, then

$$\|\psi\|_H^2 = \sum_{i \in I} \|\psi_i\|_{H_i}^2 \leq M \sum_{i \in I} \|\psi_i\|_{V_i}^2 = M \|\psi\|_V^2$$

yields the continuity of the injection ϕ .

Assume now the injection ϕ to be continuous with constant M and fix arbitrary $i \in I$ and $\psi_i \in V_i$. Convince yourself that the identity

$$\phi_i(\psi_i) = \phi(\pi_{i,0}^{-1,r}(\psi_i))$$

holds and observe that $\pi_{i,0}^{-1,r} : H_i \rightarrow H$ is also isometric. Thus,

$$\|\phi_i(\psi_i)\|_{H_i} = \|\phi(\pi_{i,0}^{-1,r}(\psi_i))\|_H \leq M \|\pi_{i,0}^{-1,r}(\psi_i)\|_V = M \|\psi_i\|_{V_i}.$$

b) First assume that $\lim_{j \rightarrow \infty} \|\phi_j\|_{\mathcal{L}(V_j, H_j)} = 0$ and approximate V by $V^{(m)} := \bigoplus_{j \leq m} V_j$ and $H^{(m)}$ analogously. Observe that the canonical injections $\phi^{(m)} \in \mathcal{L}(V^{(m)}, H^{(m)})$ are compact operators. Using this fact, we define a compact operator $\phi_m \in \mathcal{K}(V, H)$ by

$$\phi_m \psi := \pi_{\{1, \dots, m\}, 0}^{-1,r}(\phi^{(m)}(\pi_{\{1, \dots, m\}} \psi)).$$

This operator satisfies $\pi_i(\phi_m \psi) = \psi_i$, $i \leq m$, and $\pi_i(\phi_m \psi) = 0$, $i > m$. Computing

$$\|(\phi - \phi_m)\psi\|_H^2 \leq \sum_{j \in I} \|\phi_j\|_{\mathcal{L}(V_j, H_j)}^2 \|\psi_j\|_{V_j}^2 \leq \sup_{j \geq m} \|\phi_j\|_{\mathcal{L}(V_j, H_j)}^2 \|\psi\|_V^2.$$

shows that ϕ_m converges to ϕ in the operator norm. Since $\mathcal{K}(V, H)$ is a closed ideal the first implication is proved.

Assume now that $V \hookrightarrow H$ compactly and fix an arbitrary infinite subset $I' \subset I$. Consider $\psi_j \in V_j$ such that

$$\|\psi_j\|_{V_j} = 1 \quad \text{and} \quad \|\psi_j\|_{H_j} \geq \frac{\|\phi_j\|_{\mathcal{L}(V_j, H_j)}}{2}.$$

For the sake of the simplicity of the notation denote for the rest of the proof $\psi^j := \pi_{j,0}^{-1,r}(\psi_j)$.

Since $\pi_{j,0}^{-1,r}$ is isometric, ψ^j lies in a bounded subset of $\bigoplus_{j \in I'} V_j$ and since $\bigoplus_{j \in I'} V_j \hookrightarrow \bigoplus_{j \in I'} H_j$ compactly, there exists a subsequence j_k such that ψ^{j_k} converges in H .

The sequence $(\psi^j)_{j \in I}$ converges pointwise to 0, i.e., $\lim_{j \rightarrow \infty} \pi_k(\psi^j) = 0$ for all $k \in I'$. In fact, each component is eventually 0. Thus, ψ^{j_k} converges in H to 0. Since $\pi_{j,0}^{-1,r}$ is isometric $\|\phi_{j_k}\|_{\mathcal{L}(V_{j_k}, H_{j_k})}$ converges to 0, too.

Since I' is infinite, it is possible to identify it with the subsequence

$$(\|\phi_{j_k}\|_{\mathcal{L}(V_{j_k}, H_{j_k})})_{j_k \in I'}, I' = \{j_k : k \in I\}$$

of $(\|\phi_j\|_{\mathcal{L}(V_j, H_j)})_{j \in I}$.

We have proved that there exists a subsubsequence

$$\|\phi_{j_{k_\ell}}\|_{\mathcal{L}(V_{j_{k_\ell}}, H_{j_{k_\ell}})}, \quad \ell = 1, 2, \dots$$

converging to 0. This means that $(\|\phi_j\|_{\mathcal{L}(V_j, H_j)})_{j \in I}$ converges to 0. \square

It has to be stressed that the Hilbert space H in which the form domain V is embedded determines the operator associated with the sesquilinear form: changing the state space H and considering the same form (a, V) leads to a different operator on H . By this method, sesquilinear forms can be used in order to investigate also second-order operators that are not in divergence form, see Section 1.10 for references. The following is an illustrative example of the problems that can arise with such techniques in our context.

Example 21 (Multiplicative perturbations of the Laplacian). *For a bounded domain Ω consider $V_i = H^1(\Omega)$ for all $i \in \mathbb{N}$. Let $M \in C^1(\overline{\Omega})$ be a strictly positive, bounded function whose strictly positive minimum is given by $m := \min_{x \in \Omega} M(x)$. Denote $(\Delta_{\text{Neumann}}, D(\Delta_{\text{Neumann}}))$ the Laplace operator with Neumann boundary condition on Ω and define the operators $(A_i, D(A_i))$ by*

$$A_i f(x) := \frac{M(x)}{i} \Delta_N f(x), \quad D(A_i) := D(\Delta_{\text{Neumann}}).$$

Set $H_i := L^2(\Omega, \frac{i}{M(x)} d\lambda)$. Then the operators A_i are associated with the sesquilinear forms $a_i : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$

$$a_i(\psi, \psi') := \int_{\Omega} \nabla \psi(x) \overline{\nabla \psi'(x)} dx.$$

Although $V_i \hookrightarrow H_i$ for all $i \in \mathbb{N}$, V is not continuously embedded into H . As a consequence V is not suitable as a form domain for the form (a, V) in the renormed Hilbert space H . In fact, this is due to the degeneracy of the diagonal multiplication operator $\frac{M}{i} \otimes Id$.

We want to identify the domain of the operator associated with the form (a, V) . In the following we give a possible characterisation of the domain.

Proposition 22. *Consider a continuous, densely defined form (a, V) such that $V \hookrightarrow H$. The domain of the associated operator $(A, D(A))$ satisfies*

$$D(A) = \left\{ (\psi_i)_{i \in I} \in V : \exists f \in H, \forall \psi' \in V, i \in I : \sum_{j \in I} a_{ij}(\psi_j, \psi'_i) = (f, \psi'_i)_{H_i} \right\}$$

Proof. Denote by X the space in the claim. By the definition, the domain of the operator $D(A)$ is given by

$$D(A) := \{\psi \in V : \exists f \in H, \forall \psi' \in V : a(\psi, \psi') = (f, \psi')_H\}.$$

In particular, if $\psi \in D(A)$ there exists a f as in the above definition for all $\psi' := \pi_{i,0}^{-1,f}(\psi'_i)$ with $\psi'_i \in V_i$. Plugging ψ' into the expression of the form (a, V) shows that $\psi \in X$.

Conversely, fix $f \in H$. Fix an arbitrary $\psi' \in V$. Using the embedding of $V \hookrightarrow H$, we see that the sum $\sum_{i \in I} (f_i, \psi'_i) = (f, \psi)$ converges. Further,

$$(f, \psi') = \sum_{i \in I} (f_i, \psi'_i) = \sum_{i \in I} \sum_{j \in I} a_{ij}(\psi_j, \psi'_i), = a(\psi, \psi') \quad \text{for all } \psi' \in V$$

shows that $\psi \in D(A)$. □

1.5 Evolution equations

During this section we always assume that the Assumptions 19 hold and that the canonical injections ϕ_j are continuous, uniformly on j . In the Hilbert space H , consider the abstract Cauchy problem

$$\begin{cases} \frac{d}{dt}\psi(t) = A\psi(t), & t \geq 0, \\ \psi(0) = f, & f \in H, \end{cases} \quad (\text{ACP})$$

where the operator $(A, D(A))$ is the operator associated in H with a sesquilinear form (a, V) . Then, it is possible to deduce properties of the solution of the equations from properties of the sesquilinear forms. In particular, we are interested in deducing properties of the solution of (ACP) by arguments of linear algebraic type applied on properties of the *single* mappings a_{ij} . All general results about sesquilinear forms that we will need in the following are stated without proof in Appendix A.1.

We start by discussing the well-posedness of the problem (ACP). By Theorem 93 the operator $(A, D(A))$ associated with a form (a, V) generates an analytic semigroup $(e^{tA})_{t \geq 0}$ on H if the form (a, V) is H -elliptic, i.e., if there exists $\omega \in \mathbb{R}$ such that

$$\operatorname{Re} a(\psi) + \omega \|\psi\|_H^2 \geq \alpha \|\psi\|_V^2$$

holds for all $\psi \in V$. In this case the system (ACP) is well-posed and the solutions $\psi(t)$ to the initial data f are given by $e^{tA}f$ for all $t \geq 0$.

Our first goal is to give sufficient conditions for the ellipticity. We use the same idea as in Lemma 2 and Proposition 7. Assume that the form (a, V) is continuous. Since the form a_{ij} are also continuous for all i, j , there exist $\alpha_{ij} < 0, \omega_{ij} \in \mathbb{R}$ such that the estimate

$$|a_{ij}(\psi, \psi')| \leq -\alpha_{ij} \|\psi\|_{V_j} \|\psi'\|_{V_i} + \omega_{ij} \|\psi\|_{H_j} \|\psi'\|_{H_i} \quad (1.24)$$

holds for all $\psi \in V_j, \psi' \in V_i$ and all $i \neq j$. Assume now that the forms a_{ii} are H_i -elliptic. Thus, there exist $\alpha_{ii} > 0, \omega_{ii} \in \mathbb{R}$ such that

$$\operatorname{Re} a_{ii}(\psi) + \omega_{ii} \|\psi\|_{H_i}^2 \geq \alpha_{ii} \|\psi\|_{V_i}^2 \quad (1.25)$$

holds for all $\psi \in V_i$. Combining these facts, we obtain the following result.

Proposition 23 (Well-posedness of (ACP)). *Assume the following.*

- *The form (a, V) is continuous and (1.24) holds.*
- *The forms a_{ii} are elliptic and (1.25) holds.*
- *The matrix $A = (\alpha_{ij})_{i,j \in I}$ defines a positive definite operator, i.e., there exists $\alpha > 0$ such that $\operatorname{Re}(Av | v) \geq \alpha \|v\|_{\ell^2(I)}^2$ for all $v \in \ell^2(I)$.*
- *The matrix $\Omega = (\omega_{ij})_{i,j \in I}$ defines an operator in $\ell^2(I)$.*

Then (a, V) is elliptic and satisfies

$$\operatorname{Re} a(\psi) + \|\Omega\|_{\mathcal{L}(\ell^2(I))} \|\psi\|_H^2 \geq \alpha \|\psi\|_V^2.$$

Remark 24. *If the form (a, V) is elliptic, computing $a(\psi_i) = a_{ii}(\psi)$ shows that also a_{ii} is elliptic, with the same constants. Thus, the ellipticity with uniform constants of a_{ii} is a necessary condition for the form (a, V) to be elliptic.*

Proof of Prop. 23. The ellipticity is a consequence of the estimate

$$\begin{aligned} \operatorname{Re} a(\psi) &= \operatorname{Re} \sum_{i \in I} a_{ii}(\psi_i) + \operatorname{Re} \sum_{j \neq i} a_{ij}(\psi_j, \psi_i) \\ &\geq \alpha_{ij} \|\psi_j\|_{V_j} \|\psi_i\|_{V_i} - \omega_{ij} \|\psi_j\|_{H_j} \|\psi_i\|_{H_i} \\ &\geq \alpha \|\psi\|_V^2 - \|\Omega\| \|\psi\|_H^2. \end{aligned}$$

□

We now discuss a second way to prove ellipticity. Assume the diagonal entries to be uniformly elliptic and the off-diagonal entries to be bounded as mappings $a_{ij} : H_j \times H_i \rightarrow \mathbb{C}$. Then the resulting matrix of forms will be elliptic. In fact, it holds also if the off-diagonal entries are bounded in some interpolation space. Recall that an interpolation space between V and H of order $\alpha \in [0, 1)$ is any linear space H^α such that

$$V \hookrightarrow H^\alpha \hookrightarrow H, \quad \text{and} \quad \|\psi\|_V^\alpha \|\psi\|_H^{1-\alpha} \leq M \|\psi\|_{H^\alpha}, \quad \psi \in V.$$

For finite index sets, Lemma 98 yields a criterion for the ellipticity of the form (a, V) .

Proposition 25. *For a finite index set I , assume that $H^\alpha = \bigoplus_{i \in I} H_i^\alpha$ is an interpolation space between V and H and that the mappings $a_{ij} : H_j^\alpha \times V_i \rightarrow \mathbb{C}$ are continuous. If (a, V) is continuous and a_{ii} are elliptic for all $i \in I$, the form (a, V) is elliptic.*

Proof. Observe that for finite index sets I , $H^\alpha := \bigoplus_{i \in I} H_i^\alpha$ is an interpolation space of order α between V and H , whenever H_i^α are such between V_i and H_i .

So the sesquilinear mapping $b = (a_{ij})_{i \neq j} : H^\alpha \times V \rightarrow \mathbb{C}$ is bounded. Thus, we can apply Lemma 98 and see that the form (a, V) is elliptic. \square

We now turn our attention to the cosine operator functions, see Appendix A.1. We give a sufficient condition for $(A, D(A))$ to generate a cosine operator function. Two applications of this result can be found in Section 1.7 and Section 1.8.

Proposition 26. *Consider a finite index set I and an elliptic, continuous form $a = (a_{ij})_{i, j \in I}$. The following assertions hold.*

(1) *Assume that there exists $M \geq 0$ such that for all $\psi \in V_j$ and $\psi' \in V_i$*

(i) *$|\operatorname{Im} a_{ii}(\psi)| \leq M \|\psi\|_{V_i} \|\psi\|_{H_i}$ for all $i \in I$ and moreover*

(ii) *There exists a set $J \subset K := \{(i, j) \in I \times I : i \neq j\}$ such that*

- *for all $i \neq j$ either $(i, j) \in J$ or $(j, i) \in J$, and*
- *either $|\operatorname{Im}(a_{ij}(\psi, \psi') + a_{ji}(\psi', \psi))| \leq M \|\psi\|_{V_j} \|\psi'\|_{H_i}$, for all $(i, j) \in J$,*
- *or $|\operatorname{Im}(a_{ij}(\psi, \psi') + a_{ji}(\psi', \psi))| \leq M \|\psi'\|_{H_j} \|\psi\|_{V_i}$, for all $(i, j) \in K \setminus J$.*

Then the operator A associated with (a, V) generates a cosine operator function with associated phase space $V \times H$. In particular, A generates an analytic semigroup of angle $\frac{\pi}{2}$ on H .

Proof. We want to apply Proposition 99. Under the assumptions in (1), we have

$$\begin{aligned}
|\operatorname{Im} a(\psi)| &\leq \left| \sum_{i \in I} \operatorname{Im} a_{ii}(\psi_i, \psi_i) \right| + \left| \sum_{i \neq j} \operatorname{Im} a_{ij}(\psi_j, \psi_i) \right| \\
&= \left| \sum_{i \in I} \operatorname{Im} a_{ii}(\psi_i, \psi_i) \right| + \left| \sum_{(i,j) \in J} \operatorname{Im}(a_{ij}(\psi_j, \psi_i) + a_{ji}(\psi_i, \psi_j)) \right| \\
&\leq \sum_{i \in I} |\operatorname{Im} a_{ii}(\psi_i, \psi_i)| + \sum_{(i,j) \in J} |\operatorname{Im}(a_{ij}(\psi_j, \psi_i) + a_{ji}(\psi_i, \psi_j))| \\
&\leq \begin{cases} M \sum_{i \in I} \|\psi_i\|_{V_i} \|\psi_i\|_{H_i} + M \sum_{(i,j) \in J} \|\psi_j\|_{V_j} \|\psi_i\|_{H_i} \\ M \sum_{i \in I} \|\psi_i\|_{V_i} \|\psi_i\|_{H_i} + M \sum_{(i,j) \in J} \|\psi_i\|_{V_i} \|\psi_j\|_{H_j} \end{cases} \\
&\leq \tilde{M} \|\psi\|_V \|\psi\|_H
\end{aligned}$$

for some constant $\tilde{M} \geq 0$. So, the first claim follows from Proposition 99. The second claim is a consequence of [5, Thm. 3.14.17]. \square

These criteria give sufficient conditions to prove well-posedness of the problem (ACP). Our next aim is to characterise properties of the solutions. First, we show under which conditions the semigroup generated by a form (a, V) is real, positive or L^∞ -contractive. To this end we restrict ourselves to the case $H_i = L^2(X_i)$ and we introduce the set

$$\begin{aligned}
C_i^\infty &:= B_{X_1}^\infty \times \dots \times B_{X_{i-1}}^\infty \times L^2(X_i) \times B_{X_{i+1}}^\infty \times \dots \\
&= \{(\psi_j)_{j \in I} : \psi_j \in B_{X_j}^\infty \text{ for all } j \neq i, \text{ and } \psi_i \in L^2(X_i)\}.
\end{aligned}$$

Here we denote by B_X^∞ the unitary ball of the space $L^\infty(X)$.

Theorem 27. *Consider a family of Hilbert spaces $H_i := L^2(X_i)$, $V_i \hookrightarrow H_i$ uniformly, such that X_i is a σ -finite measure space for all $i \in I$. Assume that (a, V) is a continuous, elliptic and accretive form. The following assertions hold.*

- a) *The semigroup $(e^{ta})_{t \geq 0}$ is real if and only the two following conditions hold*
- *For all $i \in I$, $\psi \in V_i \Rightarrow \operatorname{Re} \psi \in V_i$.*
 - *For all $i, j \in I$ the mapping a_{ij} is real, i.e., $\operatorname{Im} a_{ij}(\psi, \psi') = 0$ for all real functions $\psi \in V_j, \psi' \in V_i$.*
- b) *If the semigroup $(e^{ta})_{t \geq 0}$ is real, then it is positive if and only if the two following conditions hold*

- For all $i \in I$ the semigroup $(e^{ta_{ii}})_{t \geq 0}$ is positive.
- For all $i \neq j$ the sesquilinear mapping $-a_{ij}$ is positive, i.e.,

$$a_{ij}(\psi, \psi') \leq 0, \quad \text{for all } \psi \in V_j^+, \psi' \in V_i^+.$$

c) If the semigroup $(e^{ta})_{t \geq 0}$ is contractive, then it is L^∞ -contractive if and only if the two following conditions hold

- For all $i \in I$, if $\psi \in V_i$, $(1 \wedge |\psi|) \text{ sign } \psi \in V_i$
- The estimate

$$\begin{aligned} & \sum_{j \neq i} |a_{ij}(\psi_j, (|\psi_i| - 1)^+ \text{sign } \psi)| \\ & \leq \text{Re } a_{ii}((|\psi_i| \wedge 1) \text{sign } \psi_i, (|\psi_i| - 1)^+ \text{sign } \psi) \end{aligned}$$

holds for all $\psi \in V \cap C_i^\infty$.

We recall that by definition $\psi \in H$ is real, respectively, positive, if and only ψ_i is almost everywhere real, respectively, positive, for all $i \in I$.

Proof. In order to apply the characterisations of real, positive and L^∞ -contractive semigroups arising from Theorem 95 (see also [52]), represent H as $L^2(X)$ where X is a σ -finite measure space, as discussed in Lemma 9.

a) Reality. First assume that both conditions in the claim hold. Observe that the projection on real cone of $L^2(X)$ is given by $P(\psi_i)_{i \in I} = (\text{Re } \psi_i)_{i \in I}$. We have to prove $\psi \in V \Rightarrow \text{Re } \psi \in V$, but this is clear since V is the direct sum and the first condition holds. Further, computing

$$\text{Im } a(\text{Re } \psi, \text{Im } \psi') = \sum_{i,j \in I} \text{Im } a_{ij}(\text{Re } \psi_j, \text{Im } \psi'_i) = 0$$

shows that also the second condition in Theorem 95 holds.

b) Positivity. We first observe that the projection on the positive cone of the Hilbert space H is given by $P(\psi_i)_{i \in I} = ((\text{Re } \psi_i)^+)_{i \in I}$. Denote P_i the projection of H_i on its positive cone. Then, the domain condition in 95 is equivalent to the condition $P_i V_i \subset V_i$ for all $i \in I$.

In order to prove the equivalence to the algebraic condition of Theorem 95,

we compute

$$\begin{aligned}
\operatorname{Re} a(P\psi, (I - P)\psi) &= -\operatorname{Re} \sum_{i,j \in I} a_{ij}(\operatorname{Re} \psi_j^+, \operatorname{Re} \psi_i^-) \\
&= -\left(\operatorname{Re} \sum_{i \in I} a_{ii}(\operatorname{Re} \psi_i^+, \operatorname{Re} \psi_i^-) \right. \\
&\quad \left. + \operatorname{Re} \sum_{i \neq j} a_{ij}(\operatorname{Re} \psi_j^+, \operatorname{Re} \psi_i^-) \right).
\end{aligned}$$

The conditions stated in the theorem are sufficient. To prove that they are also necessary, fix arbitrary $\psi \in H_j, \varphi \in H_i$, recall that setting $a_{ij}(\psi, \varphi) = a(\pi_{j,0}^{-1,r}(\psi), \pi_{i,0}^{-1,r}(\varphi))$ yields a representation for (a, V) and distinguish the following two cases. If $i = j$, then

$$0 \leq \operatorname{Re} a(\operatorname{Re}(\pi_{i,0}^{-1,r}(\psi))^+, \operatorname{Re}(\pi_{i,0}^{-1,r}(\varphi))^-) = a_{ii}(\operatorname{Re}(\pi_{i,0}^{-1,r}(\psi))^+, \operatorname{Re}(\pi_{i,0}^{-1,r}(\varphi))^-)$$

is equivalent to the algebraic condition for the positivity of the semigroups $(e^{ta_{ii}})_{t \geq 0}$, due to the surjectivity and positivity of $\pi_{i,0}^{-1,r}$. If, conversely $i \neq j$, then

$$0 \leq \operatorname{Re} a(\operatorname{Re}(\pi_{j,0}^{-1,r}(\psi))^+, \operatorname{Re}(\pi_{i,0}^{-1,r}(\varphi))^-) = a_{ij}(\operatorname{Re}(\pi_{j,0}^{-1,r}(\psi))^+, \operatorname{Re}(\pi_{i,0}^{-1,r}(\varphi))^-)$$

is equivalent to off-diagonal condition in the theorem, against by the surjectivity and positivity of $\pi_{i,0}^{-1,r}, \pi_{j,0}^{-1,r}$.

c) Contractivity in $L^\infty(X)$. We use [52, Thm. 2.14]. The semigroup is L^∞ -contractive if and only if

$$\begin{aligned}
\psi \in V &\Rightarrow (1 \wedge |\psi|) \operatorname{sign} \psi \in V \\
&\text{and} \\
\operatorname{Re} a((1 \wedge |\psi|) \operatorname{sign} \psi, (|\psi| - 1)^+ \operatorname{sign} \psi) &\geq 0.
\end{aligned} \tag{1.26}$$

One sees that $\psi \in V \Rightarrow (1 \wedge |\psi|) \operatorname{sign} \psi \in V$ if and only if $\psi \in V_i \Rightarrow (1 \wedge |\psi|) \operatorname{sign} \psi \in V_i$ for all $i \in I$. We have to prove the equivalence of the estimates in b) and (1.26). Let first $\psi \in C_i^\infty$. Then

$$(1 \wedge |\psi|) \operatorname{sign} \psi = \pi_{i,\psi}^{-1,r}(1 \wedge \psi_i \operatorname{sign} \psi_i)$$

and

$$(|\psi| - 1)^+ \operatorname{sign} \psi = \pi_{i,0}^{-1,r}((|\psi_i| - 1)^+ \operatorname{sign} \psi_i).$$

Accordingly,

$$\begin{aligned} 0 &\leq \operatorname{Re} a((1 \wedge |\psi|) \operatorname{sign} \psi, (|\psi| - 1)^+ \operatorname{sign} \psi) \\ &= \sum_{j \neq i} \operatorname{Re} a_{ij}(\psi_j, (|\psi_i| - 1)^+ \operatorname{sign} \psi_i) \\ &\quad + \operatorname{Re} a_{ii}((1 \wedge |\psi_i|), (|\psi_i| - 1)^+ \operatorname{sign} \psi_i) \end{aligned}$$

for all $\psi \in C_i^\infty \cap V$ and all $i \in I$. Due to the sesquilinearity of a_{ij} , this also implies

$$0 \leq \sum_{j \neq i} \operatorname{Re} a_{ij}(\pm \psi_j, (|\psi_i| - 1)^+ \operatorname{sign} \psi_i) + \operatorname{Re} a_{ii}((1 \wedge |\psi_i|), (|\psi_i| - 1)^+ \operatorname{sign} \psi_i)$$

for all $\psi \in C_i^\infty \cap V$, all $i \in I$, and all $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$. This yields the claimed criterion. The converse implication can be proven analogously. \square

Remarks 28. (1) *Perturbing the form (a, V) by a multiple of the scalar product, i.e., defining $a_\omega : V \times V \rightarrow \mathbb{C}$ by*

$$a_\omega(\psi, \psi') := a(\psi, \psi') - \omega(\psi \mid \psi')_H$$

leads to a semigroup $(e^{ta_\omega})_{t \geq 0} = (e^{\omega t} e^{ta})_{t \geq 0}$.

Observe that since $e^{\omega t} > 0$ for all $\omega, t \in \mathbb{R}$ the semigroup e^{ta} is real, respectively, positive, if and only if the semigroup e^{ta_ω} is real, respectively, positive. Since every elliptic form is also accretive up to a translation, one sees that the accretivity condition in Theorem 27 can be dropped in part a) and b).

(2) In the case of systems, i.e., in the case $X_i = \Omega$ for all $i \in I$, the off-diagonal mappings are in fact sesquilinear forms. So, the condition of positivity becomes quite restrictive. For a discussion of the positivity of sesquilinear forms see Section 1.10.

The adjoint of the operator A is associated with the form $a^* : V \times V \rightarrow \mathbb{C}$ defined by $a^*(\psi, \psi') := \overline{a(\psi', \psi)}$. If (a^*, V) is contractive in L^∞ , then the semigroup e^{ta} extrapolates to a family of strongly continuous semigroups on all spaces $L^p(X)$. Thus, applying the criterion c) in Proposition 27 to the adjoint form (a^*, V) , we obtain a characterisation of form matrices generating an extrapolating semigroup.

Corollary 29. *Consider a family of Hilbert spaces $H_i := L^2(X_i)$, $V_i \hookrightarrow H_i$ uniformly, such that X_i is a σ -finite measure space for all $i \in I$. Assume $a = (a_{ij})_{i,j \in I} : V \times V \rightarrow \mathbb{C}$ to be accretive. Then the semigroup $(e^{ta})_{t \geq 0}$ extrapolates to a family of contractive C_0 -semigroups $(e^{ta_p})_{t \geq 0}$ on $L^p(X)$ for all $p \in [1, \infty)$ if and only if for all $i \in I$*

- $\psi \in V_i \implies (1 \wedge |\psi|) \text{sign } \psi \in V_i$;
- For all $\psi \in V \cap C_i^\infty$,

$$\begin{aligned} & \sum_{j \neq i} |a_{ij}(\psi_j, (|\psi_i| - 1)^+ \text{sign } \psi_i)| \\ & \leq \text{Re } a_{ii}((1 \wedge |\psi_i|) \text{sign } \psi_i, (|\psi_i| - 1)^+ \text{sign } \psi_i); \end{aligned}$$

- For all $\psi \in V \cap C_i^\infty$

$$\begin{aligned} & \sum_{j \neq i} |a_{ji}((|\psi_i| - 1)^+ \text{sign } \psi_i, \psi_j)| \\ & \leq \text{Re } a_{ii}((|\psi_i| - 1)^+ \text{sign } \psi_i, (1 \wedge |\psi_i|) \text{sign } \psi_i). \end{aligned}$$

Proof. Let us first assume the semigroup $(e^{ta})_{t \geq 0}$ to extrapolate to a family of contractive C_0 -semigroups on $L^p(X)$, $p \in [1, \infty)$, and hence in particular to be L^∞ -contractive. Since also the unit ball of $L^1(X)$ is left invariant, it follows by duality that the semigroup $(e^{ta^*})_{t \geq 0}$ is L^∞ -contractive. Here a^* denotes the adjoint form of (a, V) , which by definition is given by $a^*(\psi, \psi') = \overline{a(\psi', \psi)} = \sum_{i,j=1}^m \overline{a_{ji}(\psi'_i, \psi_j)}$, $\psi, \psi' \in V$. Since a^* is accretive if and only if (a, V) is accretive, we can apply Theorem 27 to $(e^{ta})_{t \geq 0}$ and $(e^{ta^*})_{t \geq 0}$ and obtain the conditions in the claim.

Conversely, since both (a, V) and a^* are accretive, it follows from the first two conditions and Theorem 27 that $(e^{ta})_{t \geq 0}$ is L^∞ -contractive. Moreover, since also a^* is accretive, it follows from the above conditions and Theorem 27 that $(e^{ta^*})_{t \geq 0}$ is L^∞ -contractive, too. Thus, by standard interpolation and duality arguments $(e^{ta})_{t \geq 0}$ extrapolates to a family of contractive semigroups on $L^p(X)$, $p \in [1, \infty)$, that are strongly continuous for all $p > 1$. Finally, contractivity implies strongly continuity of the extrapolated semigroup also in $L^1(X)$, cf. [3, 7.2.1]. \square

A second corollary of the above theorem is the characterisation of *ultracontractive* semigroups. Assume the semigroup $(T(t))_{t \geq 0}$ to have consistent realisations on all $L^p(X)$ spaces, $p \in [1, \infty]$. Then the semigroup is *ultracontractive of dimension d* if there is a constant $c > 0$ such that for all $p, q \in [1, \infty]$, $f \in L^p(X)$ the estimate

$$\|T(t)f\|_{L^q(X)} \leq ct^{\frac{-d}{2}|p^{-1}-q^{-1}|} \|f\|_{L^p(X)}$$

holds.

Corollary 30. *Assume that the three conditions in Corollary 29 hold. Let $d > 2$ a real number. Then $(e^{ta})_{t \geq 0}$ is ultracontractive of dimension d if and only if V_i is continuously embedded in $L^{\frac{2d}{d-2}}(X_i)$ for all $i \in I$.*

Proof. This is a consequence of Corollary 29 and [3, Thm. 7.3.2]. \square

Again using criteria derived from Theorem 95 it is possible to investigate domination properties of semigroups. Recall that the semigroup $(e^{ta})_{t \geq 0}$ dominates the semigroup $(e^{tb})_{t \geq 0}$ if $(e^{ta})_{t \geq 0}$ is a positive semigroup and if for all $t \geq 0$ $e^{ta} \geq |e^{tb}|$.

Proposition 31. *Consider a family of Hilbert spaces $H_i := L^2(X_i)$, $V_i \hookrightarrow H_i$ uniformly, such that X_i is a σ -finite measure space for all $i \in I$. Assume $a = (a_{ij})_{i,j \in I} : V \times V \rightarrow \mathbb{C}$ to be continuous, elliptic and accretive and $(e^{ta})_{t \geq 0}$ to be positive. Consider another densely defined, continuous, H -elliptic sesquilinear form $b := (b_{ij})_{i,j \in I} : W \times W \rightarrow \mathbb{C}$, $W = \bigoplus_{i \in I} W_i$. Then $(e^{ta})_{t \geq 0}$ dominates $(e^{tb})_{t \geq 0}$ if and only if the following conditions hold.*

- W_i is an ideal of V_i for all $i \in I$,
- $\operatorname{Re} b_{ii}(\psi, \psi') \geq a_{ii}(|\psi|, |\psi'|)$ for all $\psi, \psi' \in V_i$ such that $\psi, \overline{\psi'} \geq 0$, $i \in I$, and
- $|\operatorname{Re} b_{ij}(\psi, \psi')| \leq -a_{ij}(|\psi|, |\psi'|)$ for all $\psi \in V_j, \psi' \in V_i, i \in I$.

Proof. First recall that from Theorem 95 it is possible to derive the following equivalent conditions for domination:

- \mathcal{W} is an ideal of V ;
- $\operatorname{Re} b(\psi, \psi') \geq a(|\psi|, |\psi'|)$ for all $\psi, \psi' \in \mathcal{W}$ such that $\psi, \overline{\psi'} \geq 0$.

Assume that $(e^{ta})_{t \geq 0}$ dominates $(e^{tb})_{t \geq 0}$.

Since W has diagonal form, W is an ideal of V if and only if each component W_i is an ideal of V_i .

Let $i_0 = j_0$ and $\psi, \psi' \in V_{i_0}$ such that $\psi, \overline{\psi'} \geq 0$. So,

$$\pi_{i_0,0}^{-1,r}(\psi) \overline{\pi_{i_0,0}^{-1,r}(\psi')} \geq 0.$$

Computing

$$\begin{aligned} \operatorname{Re} b_{i_0 i_0}(\psi, \psi') &= \operatorname{Re} b(\pi_{j_0,0}^{-1,r}(\psi), \pi_{i_0,0}^{-1,r}(\psi')) \\ &\geq a(|\pi_{j_0,0}^{-1,r}(\psi)|, |\pi_{i_0,0}^{-1,r}(\psi')|) \\ &= a_{i_0 i_0}(|\psi|, |\psi'|) \end{aligned}$$

shows that the second condition is necessary.

For $i_0 \neq j_0$, let $\psi \in V_{j_0}$ and $\psi' \in V_{i_0}$, so that $\pi_{j_0,0}^{-1,r}(\psi)\overline{\pi_{i_0,0}^{-1,r}(\psi')} = 0 = (-\pi_{j_0,0}^{-1,r}(\psi))\overline{\pi_{i_0,0}^{-1,r}(\psi')}$. Then,

$$\begin{aligned} \pm \operatorname{Re} b_{i_0 j_0}(\psi, \psi') &= \operatorname{Re} b_{i_0 j_0}(\pm\psi, \psi') \\ &= \operatorname{Re} b(\pm\pi_{j_0,0}^{-1,r}(\psi), \pi_{i_0,0}^{-1,r}(\psi')) \\ &\geq a(|\pi_{j_0,0}^{-1,r}(\psi)|, |\pi_{i_0,0}^{-1,r}(\psi')|) \\ &= a_{i_0 j_0}(|\psi|, |\psi'|), \end{aligned}$$

thus proving that also the third condition is necessary.

To check the converse implication let $\psi, \psi' \in W$ and compute

$$\operatorname{Re} b(\psi, \psi') = \operatorname{Re} \sum_{i,j \in I} b_{ij}(\psi_j, \psi'_i) \geq \sum_{i,j \in I} a_{ij}(|\psi_j|, |\psi'_i|) = a(|\psi|, |\psi'|).$$

The proof is now complete since we have already shown that the conditions involving ideals are equivalent. \square

1.6 Symmetry properties

In this section we are going to study *symmetry* properties of the semigroup $(e^{\mathfrak{t}a})_{\mathfrak{t} \geq 0}$ in the case that $H_i = L^2(X)$ for all $i \in I$. Recall that $H = \bigoplus_{i \in I} H_i$. Thus, in this case H can be interpreted as $H = L^2(X, \ell^2(I))$, as discussed in Lemma 9, and, in this way, functions in H are vector-valued functions. The word *symmetry* is used to denote the invariance of a particular class of subspaces of the state space H , whose connection to the physical use of the word symmetry is discussed in Section 2.8.

Fix a Hilbert basis $(e_i)_{i \in I}$ of the space $\ell^2(I)$. Since all functions $\psi \in H$ are vector-valued it is possible to decompose such functions using the Hilbert basis of the space $\ell^2(I)$.

Proposition 32. *Consider a family of Hilbert spaces $H_i := L^2(X)$, such that X is a σ -finite measure space and consider a Hilbert basis $(e_i)_{i \in I}$ of the space $\ell^2(I)$. Then, for all $\psi \in H$ there exist uniquely determined functions $c_i \in L^2(X)$, $i \in I$ that satisfy following conditions.*

- *The decomposition*

$$\psi(x) = \sum_{i \in I} c_i(x) e_i \tag{1.27}$$

holds in $\ell^2(I)$ for almost every $x \in \Omega$.

- The estimate $\|c_i\|_{L^2(X)} \leq \|\psi\|_H$ holds for all $i \in I$.

The uniquely determined coefficients are denoted by

$$c_i^\psi(x) := c_i(x). \quad (1.28)$$

Proof. Fix a $\psi \in H$. By the definition of the norm in H ,

$$\|\psi\|_H^2 = \sum_{i \in I} \|\psi_i\|_{L^2}^2 = \sum_{i \in I} \int_{\Omega} |\psi_i(x)|^2 d\mu = \int_{\Omega} \sum_{i \in I} |\psi_i(x)|^2 d\mu.$$

It follows that $\psi(x) \in \ell^2(I)$ for almost every $x \in \Omega$. Define

$$c_i(x) := (\psi(x) | e_i)_{\ell^2}.$$

For almost every $x \in \Omega$ decompose

$$\psi(x) = \sum_{i \in I} c_i(x) e_i,$$

and we have obtained (a). We obtain (b) using the Cauchy–Schwarz inequality and estimating

$$\|c_i\|_{L^2(X)} = \int_{\Omega} |c_i(x)|^2 d\mu \leq \int_{\Omega} \|\psi(x)\|_{\ell^2}^2 d\mu = \|\psi\|.$$

To see that c_i is a measurable function for all $i \in I$, observe that if χ_n is a sequence of simple functions converging to ψ , then for all $i \in I$, χ_n^i defined by

$$\chi_n^i(x) := (\chi_n(x) | e_i), \quad i, n \in \mathbb{N}$$

is a sequence of simple functions converging to c_i .

Assume now that a second family $(d_i)_{i \in I} \subset L^2(X)$ satisfies both conditions. Then it follows

$$(c_i(x) - d_i(x))e_i = 0 \quad \text{for all } i \in I \text{ and almost all } x \in X.$$

Denote by N_i the exception set of the above equality, i.e.,

$$N_i := \{x \in X : (c_i(x) - d_i(x))e_i \neq 0\}.$$

Since N_i has measure 0 for all $i \in I$ and I is countable, also $N := \cup_{i \in I} N_i$ has measure 0. So, $\sum_{i \in I} (c_i(x) - d_i(x))e_i = 0$ pointwise almost everywhere.

By the dominated convergence theorem for vector-valued functions, we obtain $c_i = d_i$.

Since $(e_i)_{i \in I}$ is a Hilbert basis, $c_i(x) = d_i(x)$ almost everywhere in X and the claim follows. \square

We want to investigate the invariance of subspaces of H induced by subspaces of $\ell^2(Y)$.

Proposition 33. *Fix a closed linear subspace $Y \subset \ell^2(I)$. The subspace $\mathcal{Y} \subset H$ defined by*

$$\mathcal{Y} := \{\psi \in H : \psi(x) \in Y \text{ almost everywhere in } X\}$$

is closed.

Proof. Consider a sequence ψ_n converging to ψ in H . So, there exists a subsequence such that

$$\lim_{k \rightarrow \infty} \psi_{n_k}(x) = \psi(x) \quad \text{almost everywhere in } X.$$

In particular, $\psi(x) \in Y$ almost everywhere since Y is closed. \square

Since \mathcal{Y} is a closed subspace it is possible to decompose $H = \mathcal{Y} \oplus \mathcal{Y}^\perp$. One sees that

$$\mathcal{Y}^\perp = \{\psi \in H : \psi(x) \in Y^\perp \text{ almost everywhere in } X\}.$$

We now want to characterise the invariance of such subspaces \mathcal{Y} .

Theorem 34. *Consider a family of Hilbert spaces $H_i := H_0 := L^2(X)$, $V_i = V_0 \hookrightarrow H_0$, such that X is a σ -finite measure space. Fix a closed linear subspace $Y \subset \ell^2(I)$. Assume $a = (a_{ij})_{i,j \in I} : V \times V \rightarrow \mathbb{C}$ to be a continuous and H -elliptic form. Then for all Hilbert bases $(v_i)_{i \in I}$ of H such that $(v_\ell)_{\ell \in I'}$ is a Hilbert basis of Y and $(v_k)_{k \in I''}$ is a Hilbert basis of Y^\perp the following assertions are equivalent.*

a) *The subspace \mathcal{Y} is invariant under the action of $(e^{ta})_{t \geq 0}$, i.e., $e^{ta}Y \subset Y$ for all $t \geq 0$.*

b) *The identity*

$$\sum_{i,j \in I} \sum_{\ell \in I'} \sum_{k \in I''} (v_\ell)_j \overline{(v_k)_i} a_{ij}(\psi_\ell, \psi'_k) = 0 \quad (1.29)$$

holds for all $\psi_\ell, \psi'_k \in V_0$.

Proof. Observe that there exists a Hilbert basis $(v_\ell)_{\ell \in I}$ of the space $\ell^2(I)$ such that $(v_\ell)_{\ell \in I'}$ is a Hilbert basis of Y and $(v_k)_{k \in I''}$ is a Hilbert basis of Y^\perp . In particular, $I' \cup I'' = I$ and $I' \cap I'' = \emptyset$. Denote $P_{\mathcal{Y}}$ the orthogonal projection of H onto \mathcal{Y} . According to Corollary 96 the invariance property is equivalent to $P_{\mathcal{Y}}V \subset V$ and $a(\psi, \psi') = 0$ for all $\psi \in Y \cap V, \psi' \in Y^\perp \cap V$.

Denote P_Y the orthogonal projection of $\ell^2(I)$ onto Y , and observe that the orthogonal projection \mathcal{P} onto \mathcal{Y} satisfies

$$(P_{\mathcal{Y}}\psi)(x) = (P_Y(\psi(x))), \quad \psi \in H, x \in X.$$

So, it is in each component a convergent series of vectors in V . Since V is a Hilbert space, then $P_Y V \subset V$ is satisfied.

We derive the algebraic relation (1.29). Fix $\psi \in \mathcal{Y} \cap V$, $\psi' \in \mathcal{Y}^\perp \cap V$ and compute

$$\begin{aligned} a(\psi, \psi') &= \sum_{i,j \in I} a_{ij} \left(\left(\sum_{\ell \in I'} c_\ell^\psi v_\ell \right)_j, \left(\sum_{k \in I''} c_k^{\psi'} v_k \right)_i \right) \\ &= \sum_{i,j \in I} \sum_{\ell \in I'} \sum_{k \in I''} (v_\ell)_j \overline{(v_k)_i} a_{ij} (c_\ell^\psi, c_k^{\psi'}) \end{aligned}$$

Since all coefficients may occur, this is equivalent to the relation in b). \square

As an illustrative example we prove an irreducibility result.

Example 35. Consider a family of Hilbert spaces $H_i := H_0 := L^2(X)$, $V_i = V_0 \hookrightarrow H_0$, such that X is a σ -finite measure space. Assume the form $a = (a_{ij})_{i,j \in I} : V \times V \rightarrow \mathbb{C}$ to be continuous and H -elliptic. Then the subspace

$$\mathcal{Y}' = \{\psi \in H : \psi_i(x) = 0, i \in I', x \in X\}$$

corresponds to the subspace $Y' := \ell^2(I')$ of $\ell^2(I)$. The projection on the subspace is given, of course, by

$$(Pv)_i = \begin{cases} v_i & i \in I', \\ 0 & \text{otherwise.} \end{cases}$$

So, the subset $(e_\ell)_{\ell \in I'}$ of the canonical Hilbert basis $(e_\ell)_{\ell \in I}$ of $\ell^2(I)$ can be used as the Hilbert basis used in Theorem 34

Observe that in this case $v_{ij} = \delta_{ij}$ and so the sum (1.29) in Theorem 34 can be computed by

$$\begin{aligned} \sum_{i,j \in I} \sum_{\ell \in I'} \sum_{k \in I''} (v_\ell)_j \overline{(v_k)_i} a_{ij} (\psi_\ell, \psi'_k) &= \sum_{i,j \in I} \sum_{\ell \in I'} \sum_{k \in I''} \delta_{\ell j} \delta_{ki} a_{ij} (\psi_\ell, \psi'_k) \\ &= \sum_{\ell \in I'} \sum_{k \in I''} a_{k\ell} (\psi_\ell, \psi'_k) \end{aligned}$$

where $I'' = I \setminus I'$. Since all functions ψ_ℓ, ψ'_k may occur, then the sum only vanishes if $a_{ij} = 0$ for all $i \in I'', j \in I'$. So, the space $Y' = \bigoplus_{i \in I'} H_i$ is invariant if and only if $a_{ij} = 0$ for all $i \in I'', j \in I'$.

Conversely, the invariance of the subspace $\mathcal{Y}'' = \bigoplus_{i \in I''} H_i$ induced by $\ell^2(I'')$ is equivalent to $a_{ij} = 0$ for all $i \in I', j \in I''$.

Summing up, Y' and Y'' are both invariant if and only if (a, V) has block diagonal form, i.e., if and only if

$$a_{ij} = a_{ji} = 0, \quad (i, j) \in I' \times I''.$$

1.7 A strongly damped wave equation

As an application of the theory developed so far, we investigate a strongly damped wave equation. In Section 1.10 references for the problem are given. Let Ω be a bounded domain with boundary of class C^∞ , $\alpha \in \mathbb{C}$, and consider the following second order problem.

$$\begin{cases} \ddot{u}(t, x) = \Delta(\alpha u + \dot{u})(t, x), & t \geq 0, x \in \Omega, \\ \frac{\partial u}{\partial \nu}(t, z) = \frac{\partial \dot{u}}{\partial \nu}(t, z) = 0, & t \geq 0, z \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \\ \dot{u}(0, x) = v_0(x), & x \in \Omega, \end{cases} \quad (1.30)$$

The problem (1.30) can be reformulated as an abstract Cauchy problem of the form that we have presented in (ACP). To see this, denote by $D(\Delta_N)$ the domain of the Laplacian with Neumann boundary conditions as in Example 21. The operator governing the abstract Cauchy problem is given by

$$A = \begin{pmatrix} 0 & I \\ \alpha \Delta & \Delta \end{pmatrix}, \quad D(A) = \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in H^1(\Omega) \times H^1(\Omega) : \alpha \psi_1 + \psi_2 \in D(\Delta_N) \right\}.$$

We want to investigate the problem by means of our theory. To this end we introduce a form matrix $a : V \times V \rightarrow \mathbb{C}$, where $V_1 = V_2 = H_1 = H^1(\Omega)$, $H_2 = L^2(\Omega)$. Set now

$$\begin{aligned} a_{11}(\psi, \psi') &:= 0, \\ a_{12}(\psi, \psi') &:= - \int_{\Omega} \psi(x) \overline{\psi'(x)} dx - \int_{\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi'(x)} dx, \\ a_{21}(\psi, \psi') &:= \alpha \int_{\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi'(x)} dx, \\ a_{22}(\psi, \psi') &:= \int_{\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi'(x)} dx. \end{aligned}$$

We prove that (a, V) is the form associated with the operator $(A, D(A))$.

Proposition 36. *The operator $(A, D(A))$ is associated with the form (a, V) .*

Proof. Denote B the operator associated with (a, V) and consider $\psi \in D(A)$, $\varphi \in V$. By definition

$$\begin{aligned} a(\psi, \varphi) &= a_{12}(\psi_2, \varphi_1) + a_{21}(\psi_1, \varphi_2) + a_{22}(\psi_2, \varphi_2) \\ &= -(\psi_2, \varphi_1)_{H_1} + \alpha \int_{\Omega} \nabla \psi_1(x) \cdot \overline{\nabla \varphi_2(x)} dx \\ &\quad + \int_{\Omega} \nabla \psi_2(x) \cdot \overline{\nabla \varphi_2(x)} dx. \end{aligned}$$

Applying on both second terms in right hand-side Green's first identity we obtain

$$\begin{aligned} a(\psi, \varphi) &= -(\psi_2, \varphi_1)_{H_1} - \alpha \int_{\Omega} \Delta \psi_1(x) \overline{\varphi_2(x)} dx + \alpha \int_{\partial\Omega} \frac{\partial \psi_1(x)}{\partial \nu} \overline{\varphi_2(x)} dx \\ &\quad - \int_{\Omega} \Delta \psi_2(x) \overline{\varphi_2(x)} dx + \int_{\partial\Omega} \frac{\partial \psi_2(x)}{\partial \nu} \overline{\varphi_2(x)} dx \\ &= -(\psi_2, \varphi_1)_{H_1} + \int_{\partial\Omega} \left(\alpha \frac{\partial \psi_1(x)}{\partial \nu} + \frac{\partial \psi_2(x)}{\partial \nu} \right) \overline{\varphi_2(x)} dx \\ &\quad - \int_{\Omega} \Delta \psi_2(x) \overline{\varphi_2(x)} dx - \alpha \int_{\Omega} \Delta \psi_1(x) \overline{\varphi_2(x)} dx \\ &= -(A\psi, \varphi)_H, \end{aligned}$$

since $\alpha\psi_1 + \psi_2 \in D(\Delta_N)$. We have proved $A \subset B$.

To see that the converse inclusion also holds, fix $\psi \in D(B)$. Since $\psi \in D(B)$, there exists $f = (f_1, f_2) \in H$ such that $a(\psi, \varphi) = (f, \varphi)$ for all $\varphi = (0, \varphi_2) \in V$. This yields

$$a(\psi, \varphi) = \int_{\Omega} \nabla(\alpha\psi_1(x) + \psi_2(x)) \overline{\varphi_2(x)} dx = \int_{\Omega} f_2(x) \overline{\varphi_2(x)} dx.$$

By definition of the weak Laplacian, this means $\alpha\psi_1 + \psi_2 \in D(\Delta_N)$ and $f_2 = \Delta(\alpha\psi_1 + \psi_2)$. Choosing $\varphi = (\varphi_1, 0) \in V$ yields $f_1 = \psi_2$ and so $B \subset A$. This completes the proof. \square

The following properties hold.

- The form $a_{11} = 0$, and so it is continuous on $V_1 \times V_1$ and is H_1 -elliptic with constants (ω, ω) for any $\omega \geq 0$.
- The form a_{22} is associated with the Laplace operator on H_2 with Neumann boundary conditions, thus it is elliptic and continuous.

- The sesquilinear mappings a_{21} and a_{12} are continuous on $H_1 \times V_2$ and on $V_2 \times H_1$, respectively.

Combining these facts, we see that the problem (ACP) is well-posed by Proposition 25.

Proposition 37. *The following assertions hold.*

- The sesquilinear form (a, V) is continuous and H -elliptic for each $\alpha \in \mathbb{C}$.*
- The semigroup generated by $(A, D(A))$ is analytic of angle $\frac{\pi}{2}$.*

Proof. Observe now that $\text{Im } a_{ii}(\psi) = 0$ for all $\psi \in V_i$, $i = 1, 2$. Furthermore, there holds

$$\begin{aligned} |\text{Im}(a_{12}(\psi, \psi') + a_{21}(\psi', \psi))| &= |\text{Im}(-\alpha \int_{\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi'(x)} dx \\ &\quad + \int_{\Omega} \nabla \psi'(x) \cdot \overline{\nabla \psi(x)} dx)| \\ &\leq (1 + |\alpha|) \|\psi\|_{V_2} \|\psi'\|_{H_1}. \end{aligned}$$

Thus, Proposition 26 applies and A generates a cosine operator function, hence also an analytic semigroup of angle $\frac{\pi}{2}$. \square

We discuss now a particular invariance property of the problem (1.30).

Proposition 38. *Consider a closed product subspace $Y = Y_1 \oplus Y_2$ of H and denote by P_1, P_2 the projections of H_1, H_2 onto Y_1, Y_2 respectively. Then, $Y_1 \oplus Y_2$ is invariant under e^{ta} if and only if*

- *The subspace Y_2 is invariant under the semigroup $(e^{ta_{22}})_{t \geq 0}$ on H_2 .*
- *For all $\psi \in Y, \psi' \in Y^\perp$*

$$(\psi_2 | \psi'_1)_{H_1} = 0, \quad \alpha(\nabla \psi_1 | \nabla \psi'_2)_{H_2} = 0.$$

Proof. The projection of H onto Y is given by

$$P\psi = (P_1 \oplus P_2) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} P_1 \psi_1 \\ P_2 \psi_2 \end{pmatrix}$$

By Proposition 96 we deduce that Y is invariant under the action of $(e^{ta})_{t \geq 0}$ if and only if

- $P_1 V_1 \subset V_1$ and $P_2 V_2 \subset V_2$;

ii) for all $\psi \in (Y_1 \times Y_2) \cap V, \psi' \in (Y_1^\perp \times Y_2^\perp) \cap V$

$$a(\psi, \psi') = (\psi_2, \psi'_1)_{H_1} + \alpha(\nabla\psi_1 | \nabla\psi'_2)_{H_2} + (\nabla\psi_2 | \nabla\psi'_2)_{H_2} = 0.$$

We prove the necessity of the conditions.

The first condition in i) is empty, since $V_1 = H_1$. Fix $\psi_1 = \psi'_1 = 0$. The second condition in i) and ii) imply that $e^{ta_{22}}Y_2 \subset Y_2$.

Choosing now $\psi_1 = \psi'_2 = 0$, we obtain

$$(\psi_2, \psi'_1)_{H_1} = 0, \quad \psi_2 \in Y_2, \psi'_1 \in Y_1^\perp,$$

and choosing $\psi_2 = \psi'_1 = 0$, we obtain

$$\alpha(\nabla\psi_1 | \nabla\psi'_2)_{H_2} = 0, \quad \psi_1 \in Y_1, \psi'_2 \in Y_2^\perp.$$

So, we have shown that the conditions are necessary. The sufficiency is analogous. \square

Using Proposition 38 we finally show that the space of radial functions is invariant under the action of the semigroup $(e^{ta})_{t \geq 0}$.

Corollary 39. *Assume $\Omega = B_R(0)$. Then the solution u of (1.30) is radial provided that the initial data (u_0, v_0) are radial.*

Proof. Define $Y_1 := \{\psi \in L^2(\Omega) : \psi \text{ is radial}\}$. Then the claim is equivalent to the invariance of the subspace $Y = Y_1 \cap H^1(\Omega) \times Y_1$. The semigroup $(e^{ta_{22}})_{t \geq 0}$ is the heat semigroup with Neumann boundary conditions, and therefore it leaves invariant radial functions. Observe that

$$Y_1^\perp = \{\psi \in L^2 : \int_{\partial B_r(0)} \psi(x) d\sigma(x) = 0\}.$$

Let $\psi \in Y, \psi' \in Y^\perp$. In spherical coordinates $\psi_2(\theta, r) = f(r)$. We may thus compute

$$\begin{aligned} (\psi_2 | \psi'_1)_{H_1} &= (\psi_2 | \psi'_1)_{L_2(\Omega)} + (\nabla\psi_2 | \nabla\psi'_1)_{L_2(\Omega)} \\ &= \int_{\Omega} \nabla\psi_2(x) \cdot \overline{\nabla\psi'_1(x)} dx \\ &= \int_0^R f'(r) \int_{\partial B_r(0)} \nu(x) \cdot \overline{\nabla\psi'_1(x)} d\sigma(x) dr \\ &= \int_0^R f'(r) \int_{\partial B_r(0)} \frac{\partial\psi'_1(x)}{\partial\nu} d\sigma(x) dr. \end{aligned}$$

Since $\int_{\partial B_r(0)} \psi'_1(x) d\sigma(x) = 0$ for all r , in particular it is constant, and so

$$\frac{\partial}{\partial r} \int_{\partial B_r(0)} \psi'_1(x) d\sigma(x) = \int_{B_r(0)} \frac{\partial \psi'_1(x)}{\partial \nu} = 0.$$

Plugging in this in the above equation, we obtain $(\psi_2 | \psi'_1)_{H_1} = 0$. Exactly in the same way it is possible to see the second condition in Proposition 38. So, the claimed invariance holds. \square

1.8 A heat equation with dynamic boundary conditions

As a second application of our theory, we investigate a heat equation with dynamic boundary conditions. For references, see Section 1.10.

Consider a bounded open domain $\Omega \subset \mathbb{R}^n$ with C^∞ boundary $\partial\Omega$ and set

$$V_1 := H^1(\Omega), \quad H_1 := L^2(\Omega), \quad V_2 := H^1(\partial\Omega), \quad H_2 := L^2(\partial\Omega).$$

Consider now the initial-boundary value problem

$$\begin{cases} \dot{u}(t, x) = \Delta u(t, x), & t \geq 0, x \in \Omega, \\ \dot{w}(t, z) = u(t, z) + \Delta_{\partial\Omega} w(t, z), & t \geq 0, z \in \partial\Omega, \\ w(t, z) = \frac{\partial u}{\partial \nu}(t, z), & t \geq 0, z \in \partial\Omega, \\ u(0, x) = f(x), & x \in \Omega, \\ w(0, z) = h(z), & z \in \partial\Omega. \end{cases} \quad (1.31)$$

Here, $\Delta_{\partial\Omega}$ denotes the Laplace–Beltrami operator, which is defined weakly as the operator associated with the form

$$a_{22}(\psi, \psi') := \int_{\partial\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi'(x)} d\sigma.$$

Moreover, define the forms

$$\begin{aligned} a_{11}(\psi, \psi') &:= \int_{\Omega} \nabla \psi(x) \cdot \overline{\nabla \psi'(x)} dx, \\ a_{12}(\psi, \psi') &:= - \int_{\partial\Omega} \psi(x) \overline{\psi'(x)}|_{\partial\Omega} d\sigma, \\ a_{21}(\psi, \psi') &:= - \int_{\partial\Omega} \psi|_{\partial\Omega}(x) \overline{\psi'(x)} d\sigma. \end{aligned}$$

Define now

$$A := \begin{pmatrix} \Delta & 0 \\ \cdot|_{\partial\Omega} & \Delta_{\partial\Omega} \end{pmatrix}, \quad D(A) = \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in H^2(\Omega) \times H^2(\partial\Omega) : \frac{\partial \psi_1}{\partial \nu} = \psi_2 \right\}.$$

Observe that the system (1.31) is governed by the operator $(A, D(A))$. Our aim is to prove well-posedness of the problem (1.31) on all L^p -spaces.

Proposition 40. *The form $a = (a_{ij})_{i,j \in I} : V \times V \rightarrow \mathbb{C}$ is associated with the operator $(A, D(A))$.*

Proof. As in Proposition 36, we denote $(B, D(B))$ the operator associated with (a, V) . We first prove $D(A) \subset D(B)$. Fix arbitrary $\psi \in D(A), \psi' \in V$. Compute

$$\begin{aligned}
-(A\psi' | \psi) &= - \int_{\Omega} \Delta\psi_1(x) \overline{\psi'_1(x)} dx \\
&\quad - \int_{\partial\Omega} \psi_{1|\partial\Omega}(x) \overline{\psi'_2(x)} d\sigma(x) - \int_{\partial\Omega} \Delta_{\partial\Omega}\psi_2(x) \overline{\psi'_2(x)} d\sigma(x) \\
&= \int_{\Omega} \nabla\psi_1(x) \overline{\nabla\psi'_1(x)} dx - \int_{\partial\Omega} \frac{\partial\psi_1(x)}{\partial\nu} \overline{\psi'_1(x)} d\sigma(x) \\
&\quad - \int_{\partial\Omega} \psi_{1|\partial\Omega}(x) \overline{\psi'_2(x)} d\sigma(x) + \int_{\partial\Omega} \nabla\psi_2(x) \overline{\nabla\psi'_2(x)} d\sigma(x) \\
&= \int_{\Omega} \nabla\psi_1(x) \overline{\nabla\psi'_1(x)} dx - \int_{\partial\Omega} \psi_2(x) \overline{\psi'_{1|\partial\Omega}(x)} d\sigma(x) \\
&\quad - \int_{\partial\Omega} \psi_{1|\partial\Omega}(x) \overline{\psi'_2(x)} d\sigma(x) + \int_{\partial\Omega} \nabla\psi_2(x) \overline{\nabla\psi'_2(x)} d\sigma(x) \\
&= a(\psi, \psi').
\end{aligned}$$

To see that $D(B) \subset D(A)$ also holds, fix $\psi \in D(B)$. Then there exists $f \in H$ such that for all $\psi' \in V$ holds

$$(f, \psi')_H = a(\psi, \psi').$$

Choose $\psi' = (\psi'_1, 0)^\top$ or $\psi' = (0, \psi'_2)^\top$.

In the first case, choose $\psi'_1 \in C_c^\infty(\Omega)$. So,

$$\int_{\Omega} f_1(x) \overline{\psi'_1(x)} dx = \int_{\Omega} \nabla\psi_1(x) \overline{\nabla\psi'_1(x)} dx$$

and, since ψ'_1 is arbitrary, this means $\psi \in D(\Delta)$, and $f_1 = -\Delta\psi_1$ weakly. Choose now an arbitrary ψ'_1 . Now

$$\int_{\Omega} f_1(x) \overline{\psi'_1(x)} dx = \int_{\Omega} \nabla\psi_1(x) \overline{\nabla\psi'_1(x)} dx - \int_{\partial\Omega} \psi_2(x) \overline{\psi'_1(x)} d\sigma(x).$$

Since $f_1 = \Delta\psi_1$ and the Green's first identity applies to the right hand-side, one obtains

$$\begin{aligned}
- \int_{\Omega} \Delta\psi_1(x) \overline{\psi'_1(x)} dx &= - \int_{\Omega} \Delta\psi_1(x) \overline{\psi'_1(x)} \\
&\quad + \int_{\partial\Omega} \frac{\partial\psi_1(x)}{\partial\nu} \overline{\psi'_1(x)} d\sigma(x) - \int_{\partial\Omega} \psi_2(x) \overline{\psi'_1(x)} d\sigma(x).
\end{aligned}$$

So, $\frac{\partial \psi_1(x)}{\partial \nu}(x) = \psi_2(x)$ weakly.

For the second choice compute

$$\int_{\partial\Omega} f_2(x) \overline{\psi_2'(x)} dx = - \int_{\partial\Omega} \psi_{1|\partial\Omega}(x) \overline{\psi_2'(x)} d\sigma(x) + \int_{\partial\Omega} \nabla \psi_2(x) \overline{\nabla \psi_2'(x)} d\sigma(x).$$

This means $-f_2 = \psi_{1|\partial\Omega} + \Delta \psi_2$ weakly. \square

We now address well-posedness properties of the system (1.31).

Proposition 41. *Following assertions hold.*

- a) *The form (a, V) is continuous and elliptic. Thus, it generates an analytic semigroup $(e^{ta})_{t \geq 0}$.*
- b) *The semigroup $(e^{ta})_{t \geq 0}$ is analytic of angle $\frac{\pi}{2}$.*
- c) *The semigroup $(e^{ta})_{t \geq 0}$ is real, positive.*
- d) *Denote $(e^{ta_0})_{t \geq 0}$, where a_0 is associated with the uncoupled system, i.e., $a_{012} = a_{021} = 0$. Then $(e^{ta})_{t \geq 0}$ dominates $(e^{ta_0})_{t \geq 0}$.*
- e) *The semigroup $(e^{ta})_{t \geq 0}$ does not leave invariant the unitary ball of L^∞ .*

Proof. a) Observe that a_{11} (resp. a_{22}) are continuous and H_1 - (resp. H_2 -) elliptic. Moreover, due to boundedness from $H^1(\Omega)$ to $L^2(\partial\Omega)$ of the trace operator, the forms a_{12} and a_{21} are bounded on $H_2 \times V_1$ and on $H_1 \times V_2$, respectively. Accordingly, (a, V) is continuous and by Proposition 25 also $H_1 \times H_2$ -elliptic.

b) In order to estimate the angle of analyticity, we want to apply Proposition 26. To this end, observe that $\text{Im } a_{ii}(\psi) = 0$ for all $\psi \in V_i$, $i = 1, 2$. Moreover

$$\begin{aligned} |\text{Im}(a_{12}(\psi, \psi') + a_{21}(\psi', \psi))| &= |\text{Im}(\int_{\partial\Omega} \psi(x) \overline{\psi'_{|\partial\Omega}(x)} d\sigma(x) \\ &\quad + \int_{\partial\Omega} \psi'_{|\partial\Omega}(x) \overline{\psi(x)} d\sigma(x))| \\ &= |\text{Im}(\int_{\partial\Omega} \psi(x) \overline{\psi'(x)}_{|\partial\Omega} d\sigma(x) \\ &\quad + \int_{\partial\Omega} \psi(x) \overline{\psi'_{|\partial\Omega}(x)} d\sigma(x))| \\ &= 0. \end{aligned}$$

and so Proposition 26 applies. So, A generates a cosine family and, thus, the analyticity angle is the maximal one.

c) Theorem 27 promptly yields that the semigroup is real. To see that it is positive, observe that a_{11} is associated with the Laplace operator with Neumann boundary conditions and a_{22} with the Laplace–Beltrami operator on $\partial\Omega$. Therefore they generate positive semigroups and the first condition of Theorem 27 is satisfied. The second condition is also clear, since $\psi|_{\partial\Omega}$ is positive whenever ψ is positive.

d) It is a direct consequence of Proposition 31.

e) It also follows from Proposition 27 that $(e^{ta})_{t \geq 0}$ is not $L^\infty(\Omega) \times L^\infty(\partial\Omega)$ -contractive, since for non-constant $\psi \in H^1(\partial\Omega)$ such that $|\psi| \leq 1$ and for $\psi' \in H^1(\Omega)$ with $\psi'|_{\partial\Omega} = 1 + \psi$ the estimate $a_{12}(\psi, \psi') = -\int_{\partial\Omega} |\nabla\psi|^2 d\sigma < 0$ holds. \square

The usual method to prove well-posedness in L^p -spaces is to prove L^∞ -contractivity of a semigroup and its adjoint. In this case, an additional perturbation argument is needed.

Theorem 42. *The semigroup $(e^{ta})_{t \geq 0}$ extrapolates to a consistent family of semigroups on all L^p -spaces.*

Proof. The strategy is to write the generator as a relatively compact perturbation of the generator of an ultracontractive semigroup and then obtain consistent semigroups on all L^p -spaces by a perturbation theorem.

If $p > 2$, we define operators \tilde{A}, B by

$$\tilde{A} := \begin{pmatrix} \Delta - C^* & 0 \\ \cdot|_{\partial\Omega} & \Delta_{\partial\Omega} - Id \end{pmatrix}, \quad B := \begin{pmatrix} C^* & 0 \\ 0 & Id \end{pmatrix}.$$

Here, C^* is the adjoint of the linear operator from $H^1(\Omega)$ to $L^2(\Omega)$ defined by

$$C\psi(x) := \nabla\psi(x) \cdot \overline{\nabla D_N 1(x)}, \quad \psi \in H^1(\Omega), x \in \Omega,$$

where $D_N 1$ denotes the unique (modulo constants) solution u of the inhomogeneous Neumann problem

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial \nu}(z) = 1, & z \in \partial\Omega. \end{cases}$$

The operator \tilde{A} is associated with the matrix form \tilde{a} whose diagonal entries are given by

$$\begin{aligned} \tilde{a}_{11}(\psi, \psi') &:= \int_{\Omega} \nabla\psi(x) \cdot \overline{\nabla\psi'(x)} dx + \int_{\Omega} \psi \overline{\nabla\psi' \cdot \nabla D_N 1} dx, \\ \tilde{a}_{22}(\psi, \psi') &:= \int_{\partial\Omega} \nabla\psi(x) \cdot \overline{\nabla\psi'(x)} d\sigma + \int_{\partial\Omega} \psi(x) \overline{\psi'(x)} d\sigma. \end{aligned}$$

The off-diagonal entries are given by

$$\tilde{a}_{12} := a_{12}, \quad \tilde{a}_{21} := a_{21}.$$

One sees that the perturbation $\tilde{a}_{11} - a_{11}$ is bounded on $H_1 \times V_1$, thus by Proposition 25 \tilde{a} is associated with a semigroup $(e^{t\tilde{a}})_{t \geq 0}$ on $H_1 \times H_2$.

Compute now for any $\psi' \in H^1(\partial\Omega)$ such that $|\psi'| \leq 1$ and all $\psi \in H^1(\Omega)$

$$\begin{aligned} |\tilde{a}_{12}(\psi', (|\psi| - 1)^+ \text{sign}\psi)| &\leq \int_{\partial\Omega} |\psi'(x)| (|\psi(x)| - 1)^+ d\sigma(x) \\ &\leq \int_{\partial\Omega} (|\psi(x)| - 1)^+ d\sigma(x) \\ &= \int_{\partial\Omega} \frac{\partial D_N 1}{\partial \nu} (|\psi(x)| - 1)^+ d\sigma(x) \\ &= \int_{\Omega} \nabla(|\psi(x)| - 1)^+ \cdot \nabla D_N 1 dx. \end{aligned}$$

Splitting now the integral in two parts, the last expression can be written as

$$\begin{aligned} &\int_{\Omega} \left[(1 \wedge |\psi(x)|) (\nabla(|\psi(x)| - 1)^+ \cdot \nabla D_N 1) \mathbb{1}_{\{|\psi(x)| \geq 1\}} \right] dx \\ &+ \int_{\Omega} \left[(1 \wedge |\psi(x)|) (\nabla(|\psi(x)| - 1)^+ \cdot \nabla D_N 1) \mathbb{1}_{\{|\psi(x)| \leq 1\}} \right] dx. \end{aligned}$$

The latter is

$$\text{Re } \tilde{a}_{11}((1 \wedge |\psi|) \text{sign}\psi, (|\psi| - 1)^+ \text{sign}\psi).$$

since $\nabla(|\psi| - 1)^+ = 0$ a.e. on $\{x \in \Omega : |\psi(x)| \leq 1\}$.

Likewise, for all $\psi \in H^1(\Omega)$ such that $|\psi| \leq 1$ and all $\psi' \in H^1(\partial\Omega)$ compute

$$\begin{aligned} |\tilde{a}_{21}(\psi, (|\psi'| - 1)^+ \text{sign}\psi')| &\leq \int_{\partial\Omega} |\psi(x)| (|\psi'(x)| - 1)^+ d\sigma(x) \\ &\leq \int_{\partial\Omega} (|\psi'(x)| - 1)^+ d\sigma(x) \\ &= \int_{\partial\Omega} (1 \wedge |\psi'(x)|) (|\psi'(x)| - 1)^+ \mathbb{1}_{\{|\psi'(x)| \geq 1\}} dx \\ &\quad + \int_{\partial\Omega} (1 \wedge |\psi'(x)|) (|\psi'(x)| - 1)^+ \mathbb{1}_{\{|\psi'(x)| \leq 1\}} dx \end{aligned}$$

since $|\psi|_{\partial\Omega} \leq 1$.

The last term corresponds to

$$\text{Re } \tilde{a}_{22}((1 \wedge |\psi'(x)|) \text{sign}\psi'(x), (|\psi'(x)| - 1)^+ \text{sign}\psi'(x)).$$

Thus, Theorem 27 applies, the semigroup $(e^{t\tilde{a}})_{t \geq 0}$ is L^∞ -contractive, and we conclude that extrapolates to a consistent family of semigroups on $L^p(\Omega) \times L^p(\partial\Omega)$, $p \geq 2$. The generator of the semigroup in $L^p(\Omega) \times L^p(\partial\Omega)$ is the part of \tilde{A} in $L^p(\Omega) \times L^p(\partial\Omega)$.

The part of B is a compact operator from $W^{2,p}(\Omega) \times W^{2,p}(\partial\Omega)$ to $L^p(\Omega) \times L^p(\partial\Omega)$ for all $p = [1, \infty)$, and so we conclude that also (the part of) $A = \tilde{A} + B$ generates a semigroup on $L^p(\Omega) \times L^p(\partial\Omega)$, $p \geq 2$.

The case of $p \in (1, 2)$ can be carried out in the following manner. Introduce an operator \tilde{A} by replacing C^* by C and the corresponding form \tilde{a} . By the same arguments as in the first case, the semigroup associated with the adjoint \tilde{a}^* is L^∞ -contractive. By duality and perturbation arguments we conclude as above that A generates a semigroups on $L^p(\Omega) \times L^p(\partial\Omega)$ also for $p \in [1, 2]$. \square

Remark 43. *The standard way to deduce L^p -well-posedness of the semigroup $(e^{t\tilde{a}})_{t \geq 0}$ would be via ultracontractivity. Observe now that $|\tilde{a}_{12}((|\psi'| - 1)^+ \text{sign}\psi', \psi)| \leq \text{Re } \tilde{a}_{22}((|\psi'| - 1)^+ \text{sign}\psi', (1 \wedge |\psi'|) \text{sign}\psi')$ does not hold for all $(\psi, \psi') \in H^1(\Omega) \times H^1(\partial\Omega)$ such that $|\psi| \leq 1$. Thus condition (iii) in Theorem 29.(1) is not satisfied and it is not possible to deduce ultracontractivity of $(e^{t\tilde{a}})_{t \geq 0}$ from Theorem 29 and the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, $H^1(\partial\Omega) \hookrightarrow L^{\frac{2n-2}{n-3}}(\partial\Omega)$.*

1.9 Some remarks on non-diagonal domains

Assumptions 44. *During the rest of this section we always assume the following.*

- *The subspace Y is closed in the Hilbert space V .*
- *The Hilbert spaces Y and V are densely continuously embedded into the Hilbert space H .*
- *The linear operator P_Y denotes the orthogonal projection of V onto Y .*
- *We denote by a_Y the restriction of the form a to the subspace Y , A_Y the operator associated with the form a_Y and $A|_Y$ the restriction of the operator A to the subspace $Y \cap D(A)$.*

Under these assumptions the following easy result holds.

Lemma 45. *If the sesquilinear form a is continuous, then a_Y is continuous. The same holds for accretivity, coercivity and ellipticity.*

In the general case, it is difficult to identify the domain of the operator associated with the form a_Y . Although the operator A_Y is contained in the operator $A|_Y$, in the sense of operator theoretic inclusion, it is difficult to say whether the operator A_Y coincides with the part of the operator A in Y . In Chapter 2 we will investigate the case in which the subspace Y reflects the topological structure of a graph: for all those forms the domain of A_Y does not, in fact, coincide with the part of the operator A in the subspace Y .

We turn our attention to characterisations of coercivity properties of the restricted form. This ensures that the operator A_Y generates an analytic semigroups, even if it is not possible to explicitly identify the domain $D(A)$. The following result is less useful than it might appear at a first glance.

Lemma 46. *Assume that the subspace Y is invariant under the action of $(e^{ta})_{t \geq 0}$, where a is a continuous, elliptic form. Then the form a_Y is coercive with constant α if and only if*

$$\operatorname{Re} a(\psi, P_Y \psi) \geq \alpha \|P_Y \psi\|_V^2, \quad \text{for all } \psi \in V.$$

Proof. By definition, the form is coercive if and only if $\operatorname{Re} a(\psi) \geq \alpha \|\psi\|_V^2$ for all $\psi \in Y$. Since the projection P_Y is, by definition, surjective onto Y , the last is equivalent to

$$\operatorname{Re} a(P_Y \psi) \geq \alpha \|P_Y \psi\|_V^2, \quad \text{for all } \psi \in V.$$

Using Corollary 96, compute

$$\begin{aligned} \operatorname{Re} a(P_Y \psi) &= \operatorname{Re} a(P_Y \psi, P_Y \psi) \\ &= \operatorname{Re} a(\psi, P_Y \psi) - \operatorname{Re} a(\psi - P_Y \psi, P_Y \psi) \\ &= \operatorname{Re} a(\psi, P_Y \psi), \end{aligned}$$

and the claim holds. □

In fact, we assumed in the Lemma that the form (a, V) is H -elliptic, and we derived a condition for the coercivity on Y . In other words, Lemma 46 only yields a sufficient condition for the exponential stability along an invariant subspace. The very difficult challenge, however, is to find non-trivial conditions on a non-elliptic form (a, V) which guarantee the ellipticity of a_Y .

Remark 47. *Assume the form (a, V) to be continuous, accretive and symmetric. It is tempting to use a curve integral in the Hilbert space H to characterise the coercivity of a_Y . Recall that since (a, V) is continuous and*

densely defined, there exists a densely defined operator $A : D(A) \rightarrow H$ associated with (a, V) . For an arbitrary $\psi \in D(A)$, denote $\gamma : [0, 1] \rightarrow H$ the curve defined by $\gamma : t \mapsto (1 - t)\psi + tP_Y\psi$. Formally, we compute

$$\begin{aligned}
\frac{1}{2}a(P_Y\psi) &= \int_0^1 (\partial a(\gamma(t)), \frac{d\gamma(t)}{dt}) dt \\
&= \int_0^1 (A\gamma(t), P_Y\psi - \psi) dt \\
&= \int_0^1 a((1 - t)\psi + tP_Y\psi, P_Y\psi - \psi) dt \\
&= a(P_Y\psi, P_Y\psi - \psi),
\end{aligned}$$

where $\partial a(\psi)$ has to be understood in the sense of monotone operators, see [16]. Thus, for all $\psi \in D(A)$ the above computation, if rigorously justified, yields an estimate for $a(P_Y\psi)$, for all $\psi \in D(A)$.

1.10 Discussion and remarks

In contrast to matrices of operators, matrices of sesquilinear forms have not received until now a great attention in the literature. However, it seems to us that even in the simple case of diagonal domains, form matrices present some fundamental advantages, if compared to operator matrices. We now want to comment two papers that already contained the main ideas of this work.

In 1989, Rainer Nagel pointed out (see [50]) that “*many linear evolution equations [...] can be written formally as a first order Cauchy problem [...] with values in a product of different Banach spaces.*” There the generator properties of an operator matrix

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := D(A) \times D(D)$$

are discussed, under the conditions that C is relatively A -bounded and B is relatively D -bounded.

Observe that in the case of operator matrices, diagonal domains lead to uncomfortable phenomena. Consider an arbitrary Banach space X and an arbitrary closed operator $(A, D(A))$ on X , the operator matrix

$$\mathcal{A} := \begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

is closed for the domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in X \times X : \psi_1 + \psi_2 \in D(A) \right\}.$$

However, the operator \mathcal{A} on the diagonal domain $D(A) \times D(A)$ is not closed. This observation led Rainer Nagel to claim that the diagonality of the domain “*seem to be artificial*”. In fact, in subsequent works of Klaus-Jochen Engel (see e.g. [28]), the case of non-diagonal domain was investigated.

One can ask the question whether diagonal domains are artificial also for forms. We do not intend to answer this somehow philosophical question but do observe that the same example does not work in the case of forms. To see this, assume that X is a Hilbert space and that the operator $(A, D(A))$ on X is associated with a continuous form (a, Y) on a dense Y . Set $H_1 = H_2 = X$, $V_1 = V_2 = Y$ and $a_{ij} = a$ for $i, j \in \{1, 2\}$. Denote $(B, D(B))$ the operator

associated with (a, V) and fix a vector $\psi = (\psi_i)_{i=1,2} \in D(B)$. By definition, there exists $f = (f_i)_{i=1,2} \in H$ such that

$$a(\psi, \psi') = (f \mid \psi')_H, \quad \psi' \in H.$$

In particular, for $\psi'_2 = 0$ we obtain $a(\psi_1 + \psi_2, \psi'_1) = (f_1 \mid \psi'_1)$, i.e. $\psi_1 + \psi_2 \in D(A)$. The proof of the converse inclusion is analogous. So the closed operator $(\mathcal{A}, D(\mathcal{A}))$ has non-diagonal domain, but it is associated with a form that has diagonal domain.

A further important work to Felix Ali Mehmeti and Serge Nicaise, see [44]. The work is concerned with the analysis of “*evolution phenomena on a (possibly infinite) family of domains*”. As they are interested in parabolic problems, they work in the framework of bilinear forms (and of maximal monotone operators for the nonlinear case). In fact, they introduce the form domain V as in Section 1.1, i.e., they consider the case of diagonal domains. They also develop a theory for diagonal forms, i.e., $a_{ij} = 0$ for all $i \neq j$ and they consider in the Example 1.2.3 a particular strongly coupled form, observing that it is elliptic. However, the major goal of that work was to prove well-posedness and regularity properties of non-linear systems; further, the invariance criterion of El-Maati Ouhabaz was not yet known.

In fact, our work can be seen as an extension of [44] to the case of non-diagonal sesquilinear forms, in which we are also able to investigate qualitative properties by Theorem 95.

Section 1.1

The results in this section are a generalisation of results contained in [17]. In Example 4 we exhibited an elementary example in which the boundedness constant is not optimal. This phenomenon has been deeply investigated in the literature. Assume that the form is of the type $a(\psi, \psi') = (C\nabla\psi \mid \nabla\psi')$, where C is an operator on $\ell^2(I)$. Now, the estimate in Lemma 2 in this case means $|a(\psi, \psi')| \leq \|C\| \|\nabla\psi\| \|\nabla\psi'\|$. So, this argument only makes sense if C is a regular operator. Consider a matrix C which is a bounded, non regular operator on $\ell^2(\mathbb{N})$, then the criterion in Lemma 2 fails, but the form (a, V) is continuous as a consequence of Proposition 12. Observe that a matrix which is an operator on $\ell^2(I)$ but whose modulus is unbounded has been constructed by Wolfgang Arendt and Jürgen Voigt in [7]. There, they also show that the space of regular operators is not even dense in $\mathcal{L}(\ell^2)$.

Section 1.2

The results in this section are unpublished. Systems of PDE’s are a major topic in the literature and we refer to [2] for a systematic presentation of the

theory in the finite case. There, also the case of non-linear, non-autonomous equations is discussed.

It is worth mentioning that *weighted positivity* has been discussed in the literature, see e.g. [43], and references therein.

In [43] is considered the case of a strongly coupled operator $(A, D(A))$ in non-divergence form. The operator $(A, D(A))$ is said to be weighted positive with weight M , if

$$\int_{\Omega} (A\psi \mid M\psi) dx \geq 0,$$

with a positive definite matrix M . This, in fact, equivalent to the accretivity of the form in the rescaled Hilbert space $L^2(\Omega, \ell^2(I); M dx)$, and so the theory developed for systems can be applied in this case. See also Example 21.

We also mention that it is possible to generalise the theory developed in Section 1.10 substituting the operator M_C^d by a more general operator of the form $\text{id} \otimes (M_{C_1}, \dots, M_{C_d})$.

Section 1.3

The results in this section are unpublished. The idea of the proof of Proposition 17 is due to Wolfgang Arendt. In Section 1.2 we have investigated coercivity properties for matrices in the case that the single forms have a particular expression. Proposition 17 is an example of a characterisation exploiting a particular structure of the form matrix. An open question is whether it is possible to translate other arguments holding for scalar-valued matrices in the context of forms, e.g. symmetric matrices or Toeplitz matrices.

Section 1.4

Proposition 20.b) is due to Felix Ali-Mehmeti and Serge Nicaise in [44]. We slightly simplified the proof. Proposition 22 is already contained in [17]. Example 21 uses Ulm's recipe for the investigation of multiplicative perturbations of the Laplacian by means of sesquilinear forms, see [4]. See also the comments to Section 1.2.

Section 1.5

The results in this section mostly come from [17], with some minor generalisations. We chose not to give the proofs of any general results concerning the theory of sesquilinear forms, and we refer to the Appendix.

However, the reader interested in the general theory of evolution equations governed by operators associated with an energy form should consult the introductory manuscript [4] and the fundamental monograph [52].

Invariance of sets and subspaces have already been investigated in the literature, see e.g. [1]. Most works focus on systems of PDE's, that are a special case of our theory.

Theorem 27 deals with positivity of form matrices, when the single mappings are in general defined on the product of different $H^1(X_i)$ -spaces. If $X_i = X$ for all i and a_{ij} are bounded mappings from $H_j \times H_i$, then we obtain the case of *weakly coupled* parabolic systems. Such systems have been extensively investigated in the literature. For the case $C(\bar{\Omega})$ see [55] and [45], and [14] for the case of fourth order uncoupled operators in L^2 .

Observe that in Theorem 27 positivity of the off-diagonal forms is required in order to achieve positivity. In the case of systems this already implies that the coupling is of order 0, although it can also work with unbounded coefficients, see [6]. In fact, weak coupling is a necessary condition.

Section 1.6

The results in this section are a generalisation of results contained in [17]. We have always assumed that the set I is a countable set. Indeed, this condition can in principle be weakened by requiring that for all $i \in I$, $a_{ij} = 0$ vanishes for all but countable j and that for all $j \in I$, $a_{ij} = 0$ vanishes for all but countable i . Then, due to a graph theoretic argument (see Section 2.8), the uncountable matrix $a = (a_{ij})_{i,j \in I}$ can be decomposed in the uncountable direct sum countable matrices $a_s = (a_{ij}^s)_{i,j=1}$, each of them being defined on a separable ideal I_s of $\ell^2(I)$. Here s is an index in an uncountable set S . Then, the parabolic equations leaves each of these ideals invariants. As a consequence, if the non-countable matrix $a = (a_{ij})_{i,j \in I}$ has only countable many non vanishing terms in each of the rows and columns, then all properties of $(e^{ta})_{t \geq 0}$ can be studied on the level of the single separable ideals.

Section 1.7

The ideas contained in this section were introduced in [17]. The problem is discussed in less generality and with the same techniques in [25, § XVIII.6]. The case of a strictly positive damping term α was investigated in [23, Thm. 1.1]. See also [49] for a thorough treatment of the problem with techniques of sesquilinear forms. There also well-posedness issues of related non-linear problems are discussed.

Section 1.8

The results in this section are taken from [17]. In particular, the perturbation argument was used there. The definition of the Laplace-Beltrami operator is, in general, quite difficult and involves concepts coming from the differential geometry. However, in the case of hypersurfaces it can be elementary defined, see [32]. The same system has been investigated in [19] and [48]. There, well-posedness and exponential stability of (1.31) has been proved by different methods.

Section 1.9

The case of non-diagonal domains is more difficult and we did not success in developing a satisfactory theory for the general case. In fact, also for operators on product spaces it is quite difficult to develop an unified theory for operator matrices with non-diagonal domain. To be more precise, only in the case that the operator allows a matrix representation, it is indeed possible to ask formulate the question whether the domain is diagonal or not. An important class of operators in this context are the *one-sided coupled* operator matrices, see [29]. Another approach usual in the context of operator theory is that of perturbation of domains, see [33], and also [34]. However, no general results for sesquilinear forms are known to us.

Chapter 2

Network equations

In this chapter we will describe and investigate the theory of parabolic equations on network-shaped structures. To be more precise, we want to consider differential equations taking place on the edges of a network.

Our main goal is to prove relations between graph theoretic properties and features of the solutions of strongly coupled diffusion equations on networks. To this end, we start discussing basic graph theoretic issues in the first section.

2.1 Wolfgang's first network tutorial

In this section, we want to introduce the basic tools necessary to define and study diffusion equations in network-shaped structures. We use the word *graph* and the word *network* with two different meanings. A graph is a combinatorial object, whereas a network is the measure theoretic object on which we define functions.

We start discussing the relations between these two objects in Figure 2.1.

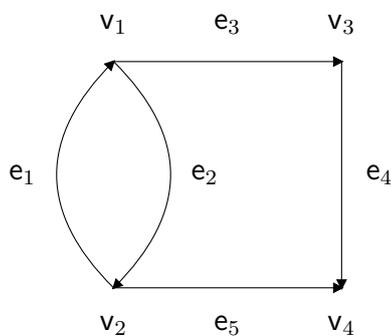


Figure 2.1: A simple example

One sees that we have fixed some labels on the graph, $\mathbf{e}_i, \mathbf{v}_k$. This is always the case: we only consider labelled graphs, even if it were possible to develop a theory for diffusion equations on networks avoiding these combinatorial issues, see Section 2.8 for these considerations. We assume that labels of the type \mathbf{v}_k always refer to *vertexes* of the graph, whereas labels of the type \mathbf{e}_i always refer to *edges* of the graph. In fact in order to avoid confusion while juggling with a large number of indexes, we always use the indexes k, ℓ, n when we are handling with vertexes and the indexes i, j, m if handling with edges.

Since we are interested in second order problems, the orientation of the network could seem redundant or artificial. However, since we want to consider functions taking values at points of edges, we need to fix a system of coordinates on the network. In this formalism each edge is assumed to have length one (possibly rescaling the coefficients of the equation), i.e., each edge is identified with a copy of the interval $[0, 1]$. When we represent the network by a picture like in Figure 2.1, the arrows are oriented in such a way that the terminal endpoint of an arrow represents the origin of the coordinate system, i.e., the point 0, and the initial endpoint of an arrow represents the end of the coordinate system, i.e., the point 1.

So far, we gave an intuitive definition of a graph. We now want to identify a *graph* with a combinatorial (or algebraic object) and a *network* with a measure-theoretic one. Again, we start discussing the graph in Figure 2.1. In this graph, there is an edge from \mathbf{v}_1 coming into the vertex \mathbf{v}_3 . So, identifying the vertexes with their order numbers, we say that the couple $(3, 1)$ belongs to the graph. Following this idea, we define a function $a : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{N}$ encoding the number of edges connecting two vertexes. In this case $a(3, 1) = 1$. This is the idea underlying the adjacency representation.

Adjacency representation

Each edge can be identified by specifying the vertex where it is starting, say \mathbf{v}_ℓ and the vertex where it is ending, say \mathbf{v}_k . So, each edge can be identified with a couple $(\mathbf{v}_k, \mathbf{v}_\ell)$. This identification can be made more formal by specifying a natural valued function $a : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{N}$ by

$$a(\mathbf{v}_\ell, \mathbf{v}_k) := a_{k\ell} := \#\{\mathbf{e}_j : \text{the edge } \mathbf{e}_j \text{ connects } \mathbf{v}_k \text{ to } \mathbf{v}_\ell\}. \quad (2.1)$$

Definition 48. *The adjacency matrix of the graph \mathbf{G} is the matrix*

$$A := (a_{ij})_{i,j=1,\dots,n}$$

defined as in (2.1).

The adjacency representation for the graph in Figure 2.1 is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Incidence representation

The adjacency representation has the disadvantage that it is not possible to determine the labels of the edges of the corresponding graph. We want to fix indexes for the edges in order to uniquely determine functions on the networks, and for that the adjacency representation is not suitable. In the incidence representation the *incoming incidence matrix* \mathcal{I}^+ is defined by

$$l_{kj}^+ := \begin{cases} 1 & \text{if the edge } j \text{ ends in the node } k, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

and its companion, the *outgoing incidence matrix* \mathcal{I}^- , defined by

$$l_{kj}^- := \begin{cases} 1 & \text{if the edge } j \text{ starts in the node } k, \\ 0 & \text{otherwise.} \end{cases} \quad (2.3)$$

Definition 49. *The incoming (respectively, outgoing) incidence matrix of the graph G is the matrix \mathcal{I}^+ , respectively \mathcal{I}^- , defined as in (2.2), respectively (2.3). The incidence matrix of the graph G is the matrix $\mathcal{I} = \mathcal{I}^+ - \mathcal{I}^-$.*

As an example, we compute the incidence matrix of the graph in Figure 2.1

$$\mathcal{I} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The incidence matrix has also a disadvantage: in contrast to the adjacency matrix, it is not possible to represent loops. The incoming and outgoing term would cancel each other, making impossible the detection of a loop in the matrix \mathcal{I} . However, it is possible to allow loops in the incidence representation, if only the ingoing and the outgoing incidence matrices are considered.

Integration by parts on networks

We now want to make a *network* out of the graph in Figure 2.1. We define different spaces.

The Hilbert space $H := (L^2(0, 1))^5$, i.e., the space of square integrable functions defined on the edges of the graph will later be our state space.

Denote $\mathbf{G} := \bigoplus_{i=1}^5 [0, 1]$. Via Lemma 9, H is isomorphic to $L^2(\mathbf{G})$, and so \mathbf{G} can be identified with the measure-theoretic network.

Moreover, we want to define continuous functions on the network. To this aim define

$$C(\mathbf{G}) := \left\{ \psi \in (C(0, 1))^5 : \exists d^\psi \in \mathbb{C}^4, \begin{array}{l} (\mathcal{I}^+)^\top d^\psi = \psi(0) \\ (\mathcal{I}^-)^\top d^\psi = \psi(1) \end{array} \right\}.$$

Later in this paragraph, we will show how the conditions involving the incidence matrices lead to functions that are continuous on the whole graph.

Now we can define weakly differentiable function on \mathbf{G} by

$$H^1(\mathbf{G}) := \left\{ \psi \in (H^1(0, 1))^5 : \psi \in C(\mathbf{G}) \right\}.$$

Finally, we define a space

$$D(A) := \left\{ \psi \in H^1(\mathbf{G}) \cap (H^2(0, 1))^5 : \mathcal{I}^+ \frac{d}{dx} \psi(0) - \mathcal{I}^- \frac{d}{dx} \psi(1) = 0 \right\}.$$

The two conditions involving the incidence matrices now lead to functions that are continuous in the nodes and satisfy the classical Kirchhoff node condition.

We now show that functions in $C(\mathbf{G})$ are actually continuous in the nodes. We fix a node, say $k = 2$ and a function $\psi \in C(\mathbf{G})$. Since it is in $C(\mathbf{G})$ there exists d^ψ as in the definition of $C(\mathbf{G})$. With an abuse of notation, we can write

$$\mathbf{e}_1(1) = \mathbf{v}_2, \quad \mathbf{e}_5(1) = \mathbf{v}_2, \quad \mathbf{e}_2(0) = \mathbf{v}_2.$$

Thus, we have to show that $\psi_1(1) = \psi_5(1) = \psi_2(0)$ for all $\psi \in C(\mathbf{G})$. Since $\psi \in C(\mathbf{G})$,

$$\psi(0) = (\mathcal{I}^+)^\top d^\psi.$$

From this, compute

$$d_2^\psi = \psi_2(0) = \psi_5(1) = \psi_1(1),$$

and so we are done. It is clear that it works also for the others vertexes.

We now define a form on the space $H^1(\mathbf{G})$ and observe that the corresponding operator has domain $D(A)$. We are mainly interested in the Laplacian, it is thus natural to define $a : H^1(\mathbf{G}) \times H^1(\mathbf{G}) \rightarrow \mathbb{C}$ by

$$a(\psi, \psi') := \sum_{i=1}^5 \int_0^1 \frac{d}{dx} \psi_i(x) \overline{\frac{d}{dx} \psi'_i(x)} dx =: \int_{\mathbf{G}} \frac{d}{dx} \psi(x) \overline{\frac{d}{dx} \psi'(x)} dx,$$

for all $\psi, \psi' \in H^1(\mathbf{G})$. Our claim is the following.

Proposition 50. *The form (a, V) is associated with the Laplacian on the edges of \mathbf{G} with domain $D(A)$.*

Proof. Denote $(B, D(B))$ the operator associated with (a, V) . We show $D(A) \subset D(B)$. Fix an arbitrary $\psi \in D(A)$. Recall that whenever \mathbf{e}_i end in \mathbf{v}_k , the relation $f_i(1) = \iota_{ki}^+ f(\mathbf{v}_k)$ holds. For all $\psi' \in H^1(\mathbf{G})$

$$\begin{aligned} a(\psi, \psi') &= \sum_{i=1}^5 \int_0^1 \frac{d}{dx} \psi_i(x) \overline{\frac{d}{dx} \psi'_i(x)} dx \\ &= \sum_{i=1}^5 \left[\frac{d}{dx} \psi_i(x) \overline{\psi'_i(x)} \right]_0^1 - \sum_{i=1}^5 \int_0^1 \frac{d^2}{dx^2} \psi_i(x) \overline{\psi'_i(x)} dx \\ &= \sum_{i,k} (\iota_{ki}^+ - \iota_{ki}^-) \frac{d}{dx} \psi_i(\mathbf{v}_k) \overline{\psi'_i(\mathbf{v}_k)} - \sum_{i=1}^5 \int_0^1 \frac{d^2}{dx^2} \psi_i(x) \overline{\psi'_i(x)} dx. \end{aligned}$$

Since $\psi' \in C(\mathbf{G})$, there exists $d^{\psi'} \in \mathbb{C}^5$ encoding the values of ψ' in the nodes. Thus, we can compute

$$\sum_{i,k} (\iota_{ki}^+ - \iota_{ki}^-) \frac{d}{dx} \psi_i(\mathbf{v}_k) \overline{\psi'_i(\mathbf{v}_k)} = \sum_{k=1}^4 d_k^{\psi'} \sum_{i=1}^5 (\iota_{ki}^+ - \iota_{ki}^-) \frac{d}{dx} \psi_i(\mathbf{v}_k) = 0,$$

since $\psi \in D(A)$. Choose $f = -(\frac{d^2}{dx^2} \psi_i)_{i=1, \dots, 5} \in L^2(\mathbf{G})$. Then $a(\psi, \psi') = (f | \psi')_H$, i.e., $\psi \in D(B)$. Moreover, the operator B acts as the Laplacian. The proof of the converse inclusion is also elementary. \square

2.2 Graph and operator theoretic settings

We state in this section the general assumptions that we will use during the rest of this chapter.

Intuitively, a graph is a set of *vertices* belonging to a set V connected by *edges*, which we assume to belong to a set E . We only consider edges of finite length. The extremal points of an edge are given by a map $\partial : E \rightarrow V \times V$. Our final goal is to study global symmetry properties. For this, it is necessary to fix an orientation of the graph, which is implicitly given by the map ∂ .

Definition 51. Consider sets V, E and a map $\partial : E \rightarrow V \times V$. Then, (V, E, ∂) is said to be an oriented graph with vertex set V and edges set E .

This definition is a standard one in combinatorial graph theory, and it only requires V and E to be disjoint, without any cardinality assumption on the sets. However, we will only consider finite or countable graphs.

Definition 52. Consider a graph G . We fix the graph theoretic notation we are going to use.

- Consider $v, v' \in V$. If $(v, v') \in \text{Range}(\partial)$ or $(v', v) \in \text{Range}(\partial)$, we say that v and v' are adjacent and we write $v \sim v'$. The edge e such that the above property holds is said to be incident on both v, v' . We write indifferently $e \sim v, v'$ or $v, v' \sim e$.
- The edge e is said to be outgoing from v if $\pi_1(\partial e) = v$ and incoming in v if $\pi_2(\partial e) = v$.
- The incoming (respectively, outgoing) degree of a node is the number of incoming (respectively, outgoing) edges. The degree of a node is the sum of the incoming and outgoing degree. Notice that loops are counted twice. They are denoted by $\deg v$, $\deg^+(v)$, $\deg^-(v)$, respectively.
- Consider subsets $V' \subset V, E' \subset E$. The subgraph G' induced by (V', E') is defined as following

$$v \in G' \Leftrightarrow v \in V' \text{ or } v \sim E',$$

and

$$e \in G' \Leftrightarrow \begin{cases} e \in E' \text{ or} \\ \exists v, v' \in V' \text{ such that } \partial e \in \{(v, v'), (v', v)\}. \end{cases}$$

We denote $G' = g(V', E')$.

- Consider a subgraph $G' = (V', E')$ and the subgraph $G'' = (V'', E'')$ induced by $E \setminus E'$. We denote

$$\partial G' = \partial G'' = V' \cap V''.$$

- The sets $V_{\text{fin}}^+, V_{\infty}^+ \subset V$ are defined as

$$V_{\text{fin}}^+ := \{v \in V : \deg^+(v) < \infty\}, \quad V_{\infty}^+ := \{v \in V : \deg^+(v) = \infty\}.$$

The sets $V_{\text{fin}}^-, V_{\text{fin}}, V_{\infty}^-, V_{\infty}$ are defined analogously.

- The graph G is finite if both V and E are finite.
- The finite part G_{fin} of a graph is the subgraph induced by $(V_{\text{fin}}, \emptyset)$, and $E_{\text{fin}} := \{e \in E : \partial e \subset V_{\text{fin}}\}$.
- The graph G is uniformly locally finite with degree k if $\deg(v) \leq k$ for all $v \in V$.
- The incoming incidence matrix is the map $\mathcal{I}^+ : E \times V \rightarrow \{0, 1\}$ defined by

$$\iota(e, v) := \iota_{e,v} := \#\{e \in E, \pi_2(\partial(e)) = v\}$$

The outgoing incidence matrices \mathcal{I}^- is defined analogously, and the incidence matrices \mathcal{I} is defined by $\mathcal{I} = \mathcal{I}^+ - \mathcal{I}^-$.

- The incoming incidence list is the map $\Gamma^+ : V \rightarrow \mathcal{P}(E)$ defined by

$$\Gamma^+(v) := \{e \in E : e(0) = v\}.$$

The outgoing incidence list Γ^- and the incidence list of the graph is defined analogously.

- Fix two nodes $v, v' \in V$. A path P is defined as a sequence of nodes $P := (v = v_0, \dots, v_n = v')$ such that v_k and v_{k+1} are adjacent. The length $l(P)$ of the path is n .
- Analogously, fix two edges $e, e' \in E$. A path P is defined as a sequence of nodes $P := (e \sim v_1, \dots, v_n \sim e')$ such that v_k and v_{k+1} are adjacent. The length $l(P)$ of the path is n .
- We denote $P[v, v']$ the set of all paths from v to v' and $P[e, e']$ the set of all paths from e to e' .
- The distance $d(v, v')$ of two nodes is defined as

$$d(v, v') := \min_{P \in P[v, v']} l(P).$$

If no path connects v to v' , then $d(v, v') := \infty$.

- The distance between two edges is defined analogously.

- The distance of two subgraphs $d(\mathbf{G}', \mathbf{G}'')$ is defined by

$$d(\mathbf{G}', \mathbf{G}'') := \inf_{\mathbf{v}' \in \mathbf{G}', \mathbf{v}'' \in \mathbf{G}''} d(\mathbf{v}', \mathbf{v}'').$$

- A graph is connected if $d(\mathbf{v}, \mathbf{v}') < \infty$ for all $\mathbf{v}, \mathbf{v}' \in \mathbf{V}$.

This purely combinatorial setting is not suitable to define a network diffusion equation, due to lack of a metric structure. To this aim, we identify each edge with a bounded interval, which we parametrize as $[0, 1]$. The endpoints of the interval are consequently identified with the vertexes. Fix now a graph \mathbf{G} , with finite or countably many edges $\mathbf{e} \in \mathbf{E}$. Following the approach of the first chapter, we can define $L^2(\mathbf{G})$ as the product space of the $L^2(0, 1)$ spaces corresponding to the single intervals.

Definition 53. Consider a finite or countable graph \mathbf{G} . We define

$$L^2(\mathbf{G}) := \bigoplus_{\mathbf{e} \in \mathbf{E}} L^2(0, 1).$$

For functions $\psi \in L^2(\mathbf{G})$ we consequently write $\psi = (\psi_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}}$.

We already observed in Section 1.9 that sesquilinear forms defined on graphs represent an important class of form matrices with non-diagonal domain. However, Definition 53 does not yet contain any information about the topology of the graph, that is instead involved in the definition of the form domain.

Consider now the space

$$V_0 := \bigoplus_{\mathbf{e} \in \mathbf{E}} H^1(0, 1).$$

As a consequence of the boundedness of the trace operator on $H^1(0, 1)$ both $\psi(0)$ and $\psi(1)$ are in $\ell^2(\mathbf{E})$, and so the Definitions 48 and 49 can be interpreted also in the case of a countable graph, if the corresponding matrices are allowed to be infinite. In fact, they have to be understood as (possibly unbounded) operators on the space from $\ell^2(\mathbf{E})$ to $\ell^2(\mathbf{V})$.

We now define a form domain by

$$V := \left\{ \psi \in V_0 : \exists d^\psi \in \ell^2(\mathbf{V}), \begin{array}{l} (\mathcal{I}^+)^{\top} d^\psi = \psi(0) \\ (\mathcal{I}^-)^{\top} d^\psi = \psi(1) \end{array} \right\}. \quad (2.4)$$

In the case of an infinite graph, $\mathcal{I}^{+\top}, \mathcal{I}^{-\top}$ denote the transpose of the incidence operators. Consequently, the existence of a d^ψ with the required properties has to be understood as the existence of d^ψ in the domain of the incidence operators. We discuss in the next paragraph some issues concerning the operator theoretical properties of the incidence operators.

Incidence matrices as operators

We start by specifying in which space the vectors $\psi(0), \psi(1)$ must be thought.

Lemma 54. *The spaces*

$$\partial_0 V := \{\psi(0) : \psi \in V\}, \quad \partial_1 V := \{\psi(1) : \psi \in V\}$$

are isomorphic to subspaces of $\ell^2(\mathbf{E})$.

Proof. We show that $V \hookrightarrow C([0, 1], \ell^2)$. The inclusion is clear by the definition of V . Further, the identity $i : V \rightarrow C([0, 1], \ell^2)$ is a continuous mapping since it is continuous in each component with the same continuity constant,

So, we obtain $\psi(x) \in \ell^2$ for all $x \in [0, 1]$ and the claim follows. \square

It is a natural question whether the incidence operators have or not dense domain. This is the case, independently of the graph.

Proposition 55. *Consider a graph \mathbf{G} . Both \mathcal{I}^+ and \mathcal{I}^- have dense domain as operators from $\ell^2(\mathbf{E})$ to $\ell^2(\mathbf{V})$.*

Proof. We split the proof in several cases. First, assume that the graph G is locally finite. Then,

$$\mathbf{G} = \bigcup_{n \in \mathbb{N}} \mathbf{G}_n := \bigcup_{n \in \mathbb{N}} g(\mathbf{V}_n, \emptyset), \quad \mathbf{V}_n := \{\mathbf{v} \in \mathbf{V} : \deg(\mathbf{v}) \leq n\}.$$

Further, for all $y \in \ell^2(\mathbf{E})$

$$\lim_{n \rightarrow \infty} \|y - \pi_{\mathbf{G}_n}(y)\|_{\ell^2(\mathbf{E})} = 0.$$

The estimate

$$\begin{aligned} \|\mathcal{I}^+ \pi_{\mathbf{G}_n}(y)\|_{\ell^2(\mathbf{V})}^2 &= \sum_{\mathbf{v} \in \mathbf{V}_n} \left| \sum_{\mathbf{e} \in \Gamma^+(\mathbf{v})} (\pi_{\mathbf{G}_n}(y))_{\mathbf{e}} \right|^2 \\ &\leq M \sum_{\mathbf{v} \in \mathbf{V}_n} \sum_{\mathbf{e} \in \Gamma^+(\mathbf{v})} |(\pi_{\mathbf{G}_n}(y))_{\mathbf{e}}|^2 \\ &= M \|\pi_{\mathbf{G}_n}(y)\|_{\ell^2(\mathbf{E})}^2 \end{aligned}$$

yields that $\pi_{\mathbf{G}_n}(y) \in \text{Dom}(\mathcal{I}^+)$ for all n , and so the claim is proved for a locally finite graph.

If the graph consist of a single, incoming infinite star S (see Definition 79), then $\ell^1(\mathbf{E}) \subset \text{Dom}(\mathcal{I}^+)$, and so $\text{Dom}(\mathcal{I}^+)$ is dense in $\ell^2(\mathbf{E})$. So, the claim is true for such graphs.

Finally, for a general graph, fix a numbering $\mathbf{V}_\infty^+ := \{\mathbf{v}_k : k \in \mathbb{N}\}$ decompose the graph as

$$\mathbf{G} = \left(\bigcup_{k \in \mathbb{N}} S_k \right) \cup g(\mathbf{V}_{\text{fin}}, \emptyset), \quad S_k := \{\mathbf{e} \in \mathbf{E} : \mathbf{e} \in \Gamma^+(\mathbf{v}_k)\}, \quad (2.5)$$

and define for all $n \in \mathbb{N}$ the approximations

$$\mathbf{G}_n = \bigcup_{k \leq n-1} S_k \cup g(\mathbf{V}_{\text{fin}}, \emptyset).$$

Fix an arbitrary $x \in \ell^2(\mathbf{E})$. We define

$$v_0 := (\pi_{g(\mathbf{V}_{\text{fin}}, \emptyset)}(x)), \quad v_k := (\pi_{S_k}(x)).$$

Since $g(\mathbf{V}_{\text{fin}}, \emptyset)$ is locally finite, there exists a sequence

$$(v_0^n)_{n \in \mathbb{N}} \in \text{Dom}(\mathcal{I}_{|g(\mathbf{V}_{\text{fin}}, \emptyset)}^+) \cap \ell^2(\mathbf{E}_{\text{fin}})$$

such that

$$\|v_0^n - v_0\| \leq \frac{1}{2^n}, \quad n \in \mathbb{N}.$$

In particular, $\pi_{\mathbf{E}_{\text{fin},0}}^{-1,r}(v_0) \in \text{Dom}(\mathcal{I}^+)$, since $\mathcal{I}^+ \ell^2(\mathbf{E}_{\text{fin}}) \subset \ell^2(\mathbf{V}_{\text{fin}})$.

Since the domain on infinite stars is also dense, for all $k \geq 1$ there is a sequence $(v_k^n)_{n \in \mathbb{N}} \in \text{Dom}(\mathcal{I}_{|S_k}^+)$ such that

$$\|v_k^n - v_k\| \leq \frac{1}{2^{n+k}}, \quad k \geq 1, n \in \mathbb{N}.$$

Again by the same arguments as for finite part, $\pi_{\mathbf{E}_{\text{fin},0}}^{-1,r}(v_k) \in \text{Dom}(\mathcal{I}^+)$

Summing up, for all $k \in \mathbb{N}$ there is a sequence $(v_k^n)_{n \in \mathbb{N}} \in \text{Dom}(\mathcal{I}^+)$ such that

$$\|v_k^n - v_k\| \leq \frac{1}{2^{n+k}}, \quad k, n \in \mathbb{N}.$$

Define $x^n := \sum_{k \leq n} v_k$ and fix $\varepsilon < 0$. Since (2.5) holds, there exists $n_0 \in \mathbb{N}$, $\|x - \pi_{\mathbf{G}_n}(x)\| < \varepsilon$ for all $n \geq n_0$. For such an n estimate

$$\begin{aligned} \|x - x^n\| &= \|x - \pi_{\mathbf{G}_n}(x) + \pi_{\mathbf{G}_n}(x) - x^n\| \\ &\leq \|x - \pi_{\mathbf{G}_n}(x)\| + \|\pi_{\mathbf{G}_n}(x) - x^n\| \\ &< \varepsilon + \frac{1}{2^n}, \end{aligned}$$

and so $\lim_{n \rightarrow \infty} x^n = x$.

We still have to prove $x^n \in \text{Dom}(\mathcal{I}^+)$, but this is clear since x^n is a finite linear combination of elements in the domain. \square

In the next result we investigate the boundedness of the incidence matrices.

Proposition 56. *The following assertions hold.*

- a) *The incidence matrices $\mathcal{I}^+, \mathcal{I}^-$ are bounded operators from $\ell^2(\mathbf{E})$ to $\ell^2(\mathbf{V})$ if and only if the graph \mathbf{G} is uniformly locally finite.*
- b) *The incidence matrices $\mathcal{I}^+, \mathcal{I}^-$ are bounded operators in $\ell^\infty(\mathbf{E})$ to $\ell^\infty(\mathbf{V})$ if and only if the graph \mathbf{G} is uniformly locally finite.*
- c) *The operators $\mathcal{I}^+, \mathcal{I}^-$ are contractive from $\ell^1(\mathbf{E})$ to $\ell^\infty(\mathbf{V})$ for every countable graph \mathbf{G} .*

Proof. We start proving a). Assume the graph \mathbf{G} to be uniformly locally finite with a degree bound D , fix $x \in \ell^2(\mathbf{E})$ and compute

$$\begin{aligned} \|\mathcal{I}^+ x\|_{\ell^2(\mathbf{V})}^2 &= \sum_{\mathbf{v} \in \mathbf{V}} \left| \sum_{\mathbf{e} \in \Gamma^+(\mathbf{v})} x_{\mathbf{e}} \right|^2 \\ &\leq \sum_{\mathbf{v} \in \mathbf{V}} \|(x_{\mathbf{e}})_{\mathbf{e} \in \Gamma^+(\mathbf{v})}\|_{\ell^1(\Gamma^+(\mathbf{v}))}^2 \\ &\leq \sum_{\mathbf{v} \in \mathbf{V}} \deg^+(\mathbf{v}) \|(x_{\mathbf{e}})_{\mathbf{e} \in \Gamma^+(\mathbf{v})}\|_{\ell^2(\Gamma^+(\mathbf{v}))}^2 \\ &\leq D \|x\|_{\ell^2(\mathbf{E})}^2. \end{aligned}$$

Assume now that the graph is not uniformly locally finite. We distinguish two cases. First, assume that there exists a node \mathbf{v} such that $\deg^+(\mathbf{v}) = \infty$. It suffices to prove the claim for an inward star \mathbf{G} with center \mathbf{v}_1 and infinitely many leaves. In this case, all vectors $0 \leq x \in \ell^2 \setminus \ell^1$ are not in the domain of \mathcal{I}^+ . Alternatively, assume that there exists a sequence of nodes $(\mathbf{v}_\ell)_{\ell \in \mathbb{N}}$ such that $\lim_{\ell \rightarrow \infty} \deg(\mathbf{v}_\ell) = \infty$. Consider the vectors $x_\ell := \frac{1}{\sqrt{\deg^+(\mathbf{v}_\ell)}} \mathbb{1}_{I(\mathbf{v}_\ell)}$. Then $\|x_\ell\|_{\ell^2(\mathbf{E})} = 1$, but $\|\mathcal{I}^+ x_\ell\|_{\ell^2(\mathbf{V})} = \deg^+(\mathbf{v}_\ell)$. This shows that \mathcal{I}^+ is not bounded.

To see that b) holds observe that the operator \mathcal{I}^+ is a positive operator. Thus, it is sufficient to compute $\mathcal{I}^+ \mathbb{1}_{\mathbf{e}} = (\deg^+ \mathbf{v})_{\mathbf{v} \in \mathbf{V}}$.

To see that c) holds observe that \mathcal{I}^+ is a positive operator from $\ell^1(\mathbf{E})$ to $\ell^1(\mathbf{V})$, which is isometric on the positive cone. Thus, for arbitrary $x \in \ell^1$,

$$\|\mathcal{I}^+ x\|_{\ell^\infty} \leq \|\mathcal{I}^+ x\|_{\ell^1} = \|x\|_{\ell^1}.$$

□

We turn now our attention to the transpose operators $(\mathcal{I}^+)^*, (\mathcal{I}^-)^*$.

Proposition 57. *Consider a graph \mathbf{G} . The transpose incidence operators $(\mathcal{I}^+)^*$, $(\mathcal{I}^-)^*$ are densely defined if and only if \mathbf{G} is locally finite.*

Proof. Assume that \mathbf{G} is locally finite and fix a vector $x \in c_{00}(\mathbf{V})$. So, also $(\mathcal{I}^+)^* \in c_{00}(\mathbf{E}) \subset \ell^2(\mathbf{E})$. This implies $x \in D((\mathcal{I}^+)^*)$ and so the operator is densely defined. Conversely, assume that $(\mathcal{I}^+)^*$ is densely defined. In particular, $\mathbb{1}_{\mathbf{v}}$ has to be in the domain for all $\mathbf{v} \in \mathbf{V}$. Observe that $(\mathcal{I}^+)^*(\mathbb{1}_{\mathbf{v}}) = \mathbb{1}_{\Gamma^+(\mathbf{v})}$ which is in ℓ^2 if and only if $\Gamma^+(\mathbf{v})$ is finite. \square

Continuity properties

We can now investigate continuity properties of the sesquilinear form (a, V) .

Assumptions 58. *During the rest of the section we always assume the following.*

- C is an operator valued function $C : x \mapsto (c_{ij}(x))_{i,j \in \mathbf{E}}$ of class C^1 .
- $M = (m_{k\ell})_{k,\ell \in \mathbf{V}}$ is an operator in $\ell^2(\mathbf{V})$.
- The sesquilinear form $a : V \times V \rightarrow \mathbb{C}$ is defined by

$$a(\psi, \psi') = \left(C \frac{d}{dx} \psi \mid \frac{d}{dx} \psi' \right)_{L^2(\mathbf{G})} - (M d^\psi \mid d^{\psi'})_{\ell^2(\mathbf{V})}. \quad (2.6)$$

We observe that on nodes with infinite degree Dirichlet node conditions are automatically imposed for all functions in the form domain.

Proposition 59. *Consider a graph \mathbf{G} such that there exists $\mathbf{v} \in \mathbf{V}$ with $\deg(\mathbf{v}) = +\infty$. Then $\psi(\mathbf{v}) = 0$ for all $\psi \in V$.*

Proof. Recall that $H^1(0, 1) \hookrightarrow C[0, 1]$ and so $\|\psi\|_{H^1(0,1)} \geq M \|\psi\|_\infty$.

Fix an arbitrary node $\mathbf{v} \in \mathbf{V}$ and assume that there exists $\psi \in V$, $\psi(\mathbf{v}) \neq 0$. Then

$$\|\psi\|_V^2 = \sum_{\mathbf{e} \in \mathbf{E}} \|\psi_{\mathbf{e}}\|_{V_{\mathbf{e}}}^2 \geq M \sum_{\mathbf{e} \in \Gamma(\mathbf{v})} \|\psi_{\mathbf{e}}\|_\infty^2 \geq M \sum_{\mathbf{e} \in \Gamma(\mathbf{v})} |\psi(\mathbf{v})|^2,$$

and so $\Gamma(\mathbf{v})$ has to be finite. \square

Consequences of this results are discussed in Section 2.5

Proposition 60. *The sesquilinear form (a, V) is continuous if the function C is essentially bounded in $\mathcal{L}(\ell^2(\mathbf{E}))$ and $M \in \mathcal{L}(\ell^2(\mathbf{V}_{\text{fin}}))$.*

Proof. First observe that, since $d_V^\psi = 0$ whenever $\deg(\mathbf{v}) = \infty$, then M has only to be investigate on the orthogonal complement of $\ell^2(\mathbf{V}_\infty)$. So, the sufficiency of the two conditions for the continuity of the form is clear. \square

The next step is to identify the operator associated with the sesquilinear form (a, V) . Observe that the matrix C causes difficulties in the integration by parts: on the one hand, the node conditions are requiring some boundary coupling of the different components ψ_i of a function $\psi \in V$. On the other hand, for non-diagonal C some internal coupling of the functions is prescribed. These issues are the object of the next section.

2.3 Identification of the network operator domain

Aim of this section is to identify the domain of the operator associated with the form defined in (2.6), thus deriving the boundary conditions that can be used in the formulation of a parabolic equation on the network.

In this section we fix a numbering both of the vertices and of the edges, i.e. $\mathbf{V} := \{\mathbf{v}_1, \mathbf{v}_2, \dots\}$ and $\mathbf{E} := \{\mathbf{e}_1, \mathbf{e}_2, \dots\}$, and we want to derive the expression of the *incidence tensor* for a strongly coupled network equation.

Integrate by part the expression of the sesquilinear form

$$\begin{aligned} a(\psi, \psi') &= \sum_{i,j \in \mathbf{E}} \int_0^1 c_{ji}(x) \frac{d}{dx} \psi_i(x) \overline{\frac{d}{dx} \psi'_j(x)} dx - \sum_{k,\ell \in \mathbf{V}} m_{k\ell} d_k^\psi \overline{d_\ell^{\psi'}} \\ &= \sum_{i,j \in \mathbf{E}} [c_{ji}(x) \frac{d}{dx} \psi_i(x) \overline{\psi'_j(x)}]_0^1 \\ &\quad - \sum_{i,j \in \mathbf{E}} \int_0^1 \frac{d}{dx} (c_{ji} \frac{d}{dx} \psi_i)(x) \overline{\psi'_j(x)} dx - \sum_{k,\ell \in \mathbf{V}} m_{k\ell} d_\ell^\psi \overline{d_k^{\psi'}}. \end{aligned}$$

It remains to derive the dependence on the incidence matrices of the first term in the sum. To this aim we define tensors

$$\mathfrak{I}^- = (\hat{t}_{\ell i}^{kj})_{i,j \in \mathbf{E}, k,\ell \in \mathbf{V}}, \quad \mathfrak{I}^+ = (\hat{t}_{\ell i}^{kj})_{i,j \in \mathbf{E}, k,\ell \in \mathbf{V}}$$

by setting

$$\hat{t}_{\ell i}^{kj} := \begin{cases} 1, & \text{if } i_i(1) = \ell_\ell \text{ and } i_j(1) = \ell_k, \\ 0, & \text{otherwise.} \end{cases}$$

The tensor \mathfrak{I}^- is defined analogously. In a more compact way we also write $\mathfrak{I}^+ = \mathcal{I}^+ \otimes \mathcal{I}^+$ and $\mathfrak{I}^- = \mathcal{I}^- \otimes \mathcal{I}^-$. For this reason, we call \mathfrak{I}^+ the *incoming incidence tensor* and \mathfrak{I}^- the *outgoing incidence tensors*. In fact, the incoming incidence tensor contains all possible pairs of couples (i_i, ℓ_ℓ) such that $i_i(1) = \ell_\ell$, and the outgoing can be interpreted analogously.

Further, we introduce the *weighted incidence tensor*

$$\mathfrak{W} = (\omega_{\ell i}^{kj})_{i,j \in E, k, \ell \in V}, \quad \omega_{\ell i}^{kj} := c_{ij}(\ell_\ell) t_{\ell i}^{kj}, \quad k, \ell \in V, i, j \in E.$$

Using these conventions, we obtain

$$\begin{aligned} \sum_{i,j \in E} [c_{ji}(x) \frac{d}{dx} \psi_i(x) \overline{\psi'_j(x)}]_0^1 &= \sum_{i,j \in E} \sum_{\ell, k \in V} c_{ji}(\ell_\ell) (t_{\ell i}^{kj} - t_{\ell i}^{kj}) \frac{d}{dx} \psi_i(\ell_\ell) \overline{\psi'_j(\ell_k)} \\ &= \sum_{i,j \in E} \sum_{k, \ell \in V} \omega_{\ell i}^{kj} \frac{d}{dx} \psi_i(\ell_\ell) \overline{\psi'_j(\ell_k)} \\ &= \sum_{k \in V} \overline{d_k^{\psi'}} \sum_{i,j \in E} \sum_{\ell \in V} \omega_{\ell i}^{kj} \frac{d}{dx} \psi_i(\ell_\ell) \\ &= \sum_{k \in V} \overline{d_k^{\psi'}} \sum_{\ell \in V} \sum_{i,j \in E} \omega_{\ell i}^{kj} \frac{d}{dx} \psi_i(\ell_\ell). \end{aligned}$$

Now we have derived the correct condition. Assume that

$$\sum_{\ell \in V} \sum_{i,j \in E} \omega_{\ell i}^{kj} \frac{d}{dx} \psi_i(\ell_\ell) = \sum_{\ell \in V} m_{k\ell} d_\ell^\psi, \quad \psi \in D(A), k \in V. \quad (2.7)$$

Then,

$$\sum_{k \in V} \overline{d_k^{\psi'}} \sum_{\ell \in V} \sum_{i,j \in E} \omega_{\ell i}^{kj} \frac{d}{dx} \psi_i(\ell_\ell) = \sum_{k, \ell \in V} m_{k\ell} d_\ell^\psi d_k^{\psi'}.$$

Define now

$$D(A) := \{ \psi \in V \cap H^2(\mathbf{G}) : \psi \text{ satisfies (2.7)} \}. \quad (2.8)$$

We have shown that $D(A) \subset D(B)$, where B is the operator associated with the sesquilinear form a and A is the operator

$$A\psi := \left(\sum_{j \in E} \frac{d}{dx} \left(\frac{d}{dx} c_{ji}(x) \psi_i(x) \right) \right)_{i \in E}^\top, \quad \psi \in D(A). \quad (2.9)$$

Since $d^{\psi'}$ is arbitrary, also the converse inclusion can be proved.

Theorem 61. *Assume C to be an uniformly bounded operator-valued function and M to be a bounded operator. Then, the operator $(A, D(A))$ defined as in (2.8), (2.9) is the operator associated with the form a .*

Proof. Denote $(B, D(B))$ the operator associated with the form (a, V) . We have already proved that $D(A) \subset D(B)$. Consider now $\psi \in D(B)$. By definition, there exists $f \in H$ such that $a(\psi, \psi') = -(f | \psi')$ for all $\psi' \in V$. Thus,

$$\sum_{i,j \in E} \int_0^1 c_{ji}(x) \frac{d}{dx} \psi_i(x) \overline{\frac{d}{dx} \psi'_j(x)} dx - \sum_{\ell, k \in V} m_{\ell k} d_k^\psi \overline{d_\ell^{\psi'}} = - \sum_{i \in E} \int_0^1 f_i(x) \overline{\psi'_i(x)} dx.$$

Integrating by part the left hand side, we obtain that

$$\begin{aligned} - \sum_{i \in E} \int_0^1 f_i(x) \overline{\psi'_i(x)} dx &= \sum_{\ell \in V} \overline{d_\ell^{\psi'}} \sum_{i,j \in E} \sum_{k \in V} \omega_{kl}^{\ell j} \frac{d}{dx} \psi_i(\mathbf{v}_k) \\ &\quad - \sum_{i,j \in E} \int_0^1 \frac{d}{dx} (c_{ji} \frac{d}{dx} \psi_i)(x) \overline{\psi'_j(x)} dx \\ &\quad - \sum_{\ell, k \in V} m_{\ell k} d_k^\psi \overline{d_\ell^{\psi'}}. \end{aligned}$$

This holds for all $\psi' \in V$. In particular, considering $\psi' \in V$ vanishing on all but one edge of the network, we conclude that

$$f_i(x) = \sum_{j \in E} \frac{d}{dx} (c_{ji} \frac{d}{dx} \psi_j)(x) \quad \text{for all } x \in (0, 1) \text{ and all } i \in E.$$

Similarly, considering ψ' with arbitrary nodal values and arbitrary small H -norm, we obtain

$$\sum_{i,j \in E} \sum_{k \in V} \omega_{ki}^{\ell j} \frac{d}{dx} \psi_i(\mathbf{v}_k) - \sum_{k \in V} m_{\ell k} d_k^\psi = 0 \quad \text{for all } \ell \in V.$$

This shows that $\psi \in D(A)$ and completes the proof. \square

2.4 Systems of network equations

In this section we want to extend the arguments presented in Section 1.2 to the case diffusion taking place on a network. We start characterizing the well-posedness in $H := \bigoplus_{e \in E} L^2(0, 1)$ of the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) &= Au(t), & t \geq 0, \\ u(0) &= u_0, & u_0 \in H. \end{cases} \quad (2.10)$$

Here A is the operator associated with the form (a, V) defined in (2.6) on the domain defined in (2.4). According to the theory of sesquilinear forms, the problem (2.10) is well-posed if the form (a, V) is continuous, elliptic and densely defined. First observe that the form domain contains $\ell^2(\mathbf{E}, H_0^1(0, 1))$. Since the latter is dense in H , the form (a, V) is densely defined. The continuity of the form has already been discussed in Section 2.2, whereas the characterization of ellipticity given in Section 1.2 also works in the case of infinite networks. We thus assume that the form (a, V) is densely defined, continuous and elliptic and we turn our attention to the analysis of the qualitative properties of (2.10). In particular, we want to discuss symmetry properties of the system. To this end, we introduce a concept of symmetry which is suitable for network equations.

Recall that in the elementary calculus a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called even if it is symmetric with respect to the y -axis, i.e., if $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Assume now the function f to be in $H^1(-1, 1)$. Then f is also continuous, and thus the definition makes sense also for functions in $H^1(-1, 1)$. More generally, assume that the graph \mathbf{G} is an outward star, see Definition 79. We denote \mathbf{v}_0 the centre of the star and we number the other edges such that $\mathbf{v}_\ell \sim \mathbf{e}_\ell$ for all $\ell = 1, \dots, |\mathbf{E}|$. We say that a function $f : \mathbf{G} \rightarrow \mathbb{C}$ is *even* if $f_i(x) = f_j(x)$ for all $x \in [0, 1], i, j = 1, \dots, |\mathbf{E}|$.

Lemma 62. *A function $f \in V$ on a star \mathbf{G} is even if and only if $f(x) \in \langle \mathbf{1} \rangle$ for all $x \in [0, 1]$.*

Proof. Let $f \in V$ even. Then $f_i(x) = f_j(x)$ for all $i, j \in \mathbf{E}, x \in (0, 1)$. So, there exists $\lambda \in H^1(0, 1)$ such that $f_i(x) = \lambda(x)$ for all $i \in \mathbf{E}$. This means $f(x) \in \langle \mathbf{1} \rangle$ for all x .

Conversely, if $f(x) \in \langle \mathbf{1} \rangle$ for all x , then for all x there exists $\lambda(x)$, $f(x) = \lambda(x)\mathbf{1}$ and so $f_j(x) = f_i(x)$ for all $i, j \in \mathbf{E}, x \in (0, 1)$. \square

Observe now that this concept also makes sense for L^2 -functions if it is understood almost everywhere

We now generalise this concept. For an arbitrary linear subspace $Y \subset \ell^2(\mathbf{E})$, we interpret Y also as a subspace \mathcal{Y} of L^2 by considering those functions that belong to Y almost everywhere.

Definition 63. *Consider a function $\psi \in H$ and a closed linear subspace $Y \subset \ell^2(\mathbf{E})$. We say that ψ is Y -symmetric if*

$$\psi(x) \in Y, \quad \text{for almost all } x \in (0, 1).$$

Moreover, we define

$$\mathcal{Y} := \{\psi \in H : \psi \text{ is } Y\text{-symmetric}\}.$$

Further, consider the semigroup of the solution operators $(e^{ta})_{t \geq 0}$ of the problem (2.10). We say that the semigroup $(e^{ta})_{t \geq 0}$ is Y -symmetric if the subspace \mathcal{Y} is invariant under the action of e^{ta} for all $t \geq 0$.

In the following, H will always denote the product $\bigoplus_{e \in E} L^2(0, 1)$ and V the form domain as in (2.4). It is useful to decompose the space $H = \mathcal{Y} \oplus \mathcal{Y}^\perp$. We formulate a result analogous to Proposition 32.

Lemma 64. *Let $Y \subset \ell^2(\mathbf{E})$ a closed linear subspace. Fix orthonormal bases $(x'_e)_{e \in E'}$ of Y and $(x''_e)_{e \in E''}$ of Y^\perp . Then*

$$\mathcal{Y} = \left\{ \sum_{e' \in E'} \psi_{e'} x'_{e'} : (\psi_{e'})_{e' \in E'} \in \ell^2(E', L^2(0, 1)) \right\},$$

and

$$\mathcal{Y}^\perp = \left\{ \sum_{e \in E''} \psi_{e''} x''_{e''} : (\psi_{e''})_{e'' \in E''} \in \ell^2(E'', L^2(0, 1)) \right\}.$$

Proof. Since the subspace Y is closed, it is an Hilbert space with orthonormal basis $\{(x_{e'})_{e' \in E'}\}$. So, the claim follows by Proposition 32. \square

Observe that having computed $\mathcal{Y}, \mathcal{Y}^\perp$ allows us to apply Corollary 96 for the investigation of the invariance properties of \mathcal{Y} . In particular, for a system of network equations, the second condition in Corollary 96 is equivalent to the statement b) in Theorem 34.

In contrast to the case of systems associated with forms with diagonal domains, however, the condition $P_Y V \subset V$ is not automatically fulfilled.

In the case of network equations, one has in particular to prove that if ψ is a function which is continuous in the nodes, then also $P_Y \psi$ is continuous in the nodes. If this is the case, we call the projection P_Y *admissible*.

We show now how it is possible to reduce the issue of admissibility to linear algebraic statements. We start by proving a linear algebraic auxiliary result.

Lemma 65. *Consider an orthogonal projection $K \in \mathcal{L}(\ell^2(\mathbf{E}))$ and a closed linear subspace $Y \subset \ell^2(\mathbf{E})$.*

Then the following assertions are equivalent.

- a) $KY \subset Y$.
- b) $Y = (\text{Ker } K \cap Y) \oplus (\text{Range } K \cap Y)$.

Proof. To see that b) implies a) fix an arbitrary $y \in Y$. By hypothesis $y = y_1 + y_2$, where $y_1 \in \text{Ker } K \cap Y$ and $y_2 \in \text{Range } K \cap Y$. Then $Ky = Ky_1 + Ky_2 = y_2 \in Y$, which proves the claim.

We now prove that a) implies b). Since $KY \subset Y$, the restriction $K|_Y$ is a linear operator on Y . In particular, since Y is a closed subspace, then Y is an Hilbert space and $K|_Y$ is the orthogonal projection onto its own range. As a consequence $Y = \text{Ker } K|_Y \oplus \text{Range } K|_Y$. Obviously $\text{Ker } K|_Y = \text{Ker } K \cap Y$ and, by a), $\text{Range } K|_Y = \text{Range } K \cap Y$ because K is a projection. The claim follows. \square

To characterise admissibility of projections we introduce some additional notation. We define the operators

$$\tilde{\mathcal{I}} : \ell^2(\mathbf{V}) \rightarrow \ell^2(\mathbf{E}) \times \ell^2(\mathbf{E}) \quad \text{and} \quad \widehat{P}_Y : \ell^2(\mathbf{E}) \times \ell^2(\mathbf{E}) \rightarrow \ell^2(\mathbf{E}) \times \ell^2(\mathbf{E})$$

by

$$\tilde{\mathcal{I}} := (\mathcal{I}^+, \mathcal{I}^-)^\top = \begin{pmatrix} (\mathcal{I}^+)^\top \\ (\mathcal{I}^-)^\top \end{pmatrix} \quad \text{and} \quad \widehat{P}_Y := \begin{pmatrix} P_Y & 0 \\ 0 & P_Y \end{pmatrix}. \quad (2.11)$$

Observe that \widehat{P}_Y is an orthogonal projection.

Theorem 66. *If the graph \mathbf{G} is connected, then the following assertions are equivalent.*

- a) *The projection P_Y is admissible.*
- b) *The range of $\tilde{\mathcal{I}}$ is invariant under \widehat{P}_Y , i. e.,*

$$\begin{pmatrix} P_Y & 0 \\ 0 & P_Y \end{pmatrix} \text{Range } \tilde{\mathcal{I}} \subset \text{Range } \tilde{\mathcal{I}}.$$

- c) *There exists a basis of $\text{Range } \tilde{\mathcal{I}}$ consisting of eigenvectors of \widehat{P}_Y .*

Proof. We start by proving the equivalence of (a) and (b). Recall that for every $\psi \in V$ there exists a vector $d^\psi \in \ell^2(\mathbf{E})$ such that

$$(\mathcal{I}^+)^\top d^\psi = \psi(0), \quad (\mathcal{I}^-)^\top d^\psi = \psi(1).$$

The admissibility of the projection is equivalent to the fact that for every $\psi \in V$ there exists a vector $d^{P_Y\psi} \in \ell^2(\mathbf{V})$ such that

$$(\mathcal{I}^+)^\top d^{P_Y\psi} = P_Y\psi(0) = P_Y\psi(0), \quad (\mathcal{I}^-)^\top d^{P_Y\psi} = P_Y\psi(1) = P_Y\psi(1).$$

Inserting the first equation into the second and observing that for all $x \in \ell^2(\mathbf{V})$ there exists a function $\psi \in \mathbf{V}$ which is continuous in the nodes and such that $d^\psi = x$, we obtain that (a) is equivalent to the fact that for all $x \in \ell^2(\mathbf{V})$ there exists $y \in \ell^2(\mathbf{V})$ such that

$$(\mathcal{I}^+)^\top y = P_Y(\mathcal{I}^+)^\top x, \quad (\mathcal{I}^-)^\top y = P_Y(\mathcal{I}^-)^\top x,$$

which can equivalently be stated as

$$\widehat{P}_Y \text{Range } \tilde{\mathcal{I}} \subset \text{Range } \tilde{\mathcal{I}}.$$

To see the second equivalence, observe first that the existence of the claimed basis is equivalent to $\text{Range } \tilde{\mathcal{I}}$ being decomposable into $\text{Range } \tilde{\mathcal{I}} = (\text{Ker } \widehat{P}_Y \cap \text{Range } \tilde{\mathcal{I}}) \oplus (\text{Range } \widehat{P}_Y \cap \text{Range } \tilde{\mathcal{I}})$. Now we can apply Lemma 65. \square

The conditions in Theorem 66 are difficult to check. In practice one maybe wants to use criteria which are easier. The next two Lemmas are criteria of this type.

Lemma 67. *Consider a connected finite graph \mathbf{G} . If P_Y is admissible, then $\mathbb{1}$ is an eigenvector of P_Y .*

Proof. First observe that $\mathbb{1} \in V$, since it is differentiable and continuous on the graph. Since the continuity conditions on the nodes must be satisfied, there exists $\lambda \in \mathbb{C}$ such that $P_Y \mathbb{1} = \lambda \mathbb{1}$, i.e., $P_Y \mathbb{1} = \lambda \mathbb{1}$. So, $\mathbb{1}$ is an eigenvector. \square

Lemma 68. *Consider a subset $E' \subset E$ of the edge set, the induced graph $G' := (\emptyset, E')$, and a non-admissible orthogonal projection $\mathcal{P}_{Y'}$ on $\ell^2(E')$. Then the projection \mathcal{P}_Y of $\ell^2(E)$ onto $Y' \oplus \ell^2(E \setminus E')$ is given by*

$$P_Y := \begin{pmatrix} P_{Y'} & 0 \\ 0 & \text{id} \end{pmatrix}$$

and is not admissible.

Proof. Define $V' := \bigoplus_{e \in E'} H^1(0, 1)$ with continuity node conditions in the nodes of G' . Since $\mathcal{P}_{Y'}$ is not admissible, there exists a function $\psi \in V'$ such that $\mathcal{P}_{Y'} \psi \notin V'$, i.e., such that the continuity condition is violated in a node \mathbf{v}_{k_0} . It is possible to extend the function ψ to a function $\tilde{\psi}$ on the whole graph, such that $d^{\tilde{\psi}} = 0$ in all nodes of $\mathbf{G} \setminus G'$. Then the function $\mathcal{P}_Y \tilde{\psi}$ does not satisfy the continuity condition in \mathbf{v}_{k_0} , either. \square

2.5 Irreducibility in infinite networks

It is well known that the heat semigroup on a finite, connected graph (as well as on smooth domains) is positive and irreducible. A consequence of Proposition 59 is that the same does not need to hold for infinite graphs that are not locally finite.

Roughly speaking, since on nodes with infinite degree Dirichlet boundary conditions are automatically imposed, initial data localised on a side of such nodes cannot propagate to the other side. Thus, irreducibility is reduced to the purely combinatorial problem whether the graph is still connected if such nodes are “cut out” of the graph.

In order to characterise this property precisely, we need some auxiliary results, which we shortly discuss now, for the sake of readability. Lemma 69 states that Dirichlet boundary conditions are only imposed on nodes with infinite degree; Proposition 70 deals with invariance of dual decompositions; in Proposition 71 we show that a graph is path-connected if and only if it is topologically connected; Definition 72 sets up a language for the main theorem, which is stated in Theorem 73. The Lemmas 76–77 and Proposition 78 are the last steps to the proof.

Lemma 69. *The following implication holds for each graph \mathbf{G} :*

$$[\forall \mathbf{v} \in \mathbf{V}, \psi \in V : \pi_{\mathbf{v}}(d^\psi) = 0] \Rightarrow [\deg(\mathbf{v}) = \infty].$$

Proof. We prove the equivalent statement

$$[\deg(\mathbf{v}) < \infty] \Rightarrow [\exists \mathbf{v} \in \mathbf{V}, \psi \in V : \pi_{\mathbf{v}}(d^\psi) \neq 0].$$

To see this, choose $\mathbb{C} \ni \lambda \neq 0$ and set

$$\psi(\mathbf{v}') = \begin{cases} \lambda, & \mathbf{v}' = \mathbf{v}, \\ 0, & \text{otherwise.} \end{cases}$$

For all $x \in \mathbf{G} \setminus \mathbf{V}$ interpolate ψ by affine functions. Then $\psi_e \in H^1(0, 1)$ for all $e \in \mathbf{E}$ and moreover

$$\|\psi\|_{L^2}^2 = \deg(\mathbf{v}) \frac{|\lambda|^2}{3}, \quad \|\psi\|_{H^1}^2 = \deg(\mathbf{v}) \frac{4|\lambda|^2}{3}.$$

Finally, ψ is continuous in the nodes. So, $\psi \in V$ has the claimed properties. \square

Proposition 70. *Consider the form (a, V) associated with the Laplacian on an graph \mathbf{G} with Kirchhoff node conditions, i.e., V is defined in (2.4) and*

in the Definition (2.6) we choose $c_{ij} = \mathbb{1} \otimes \delta_{ij}$ and $M = 0$. Fix a nodal decomposition of the graph $\mathbf{G} = \mathbf{G}_1 \cup \mathbf{G}_2$. Then the following assertions are equivalent.

- a) The ideal $I_1 := L^2(\mathbf{G}_1)$ is invariant.
- b) If $\mathbf{v}_1 \in \mathbf{G}_1, \mathbf{v}_2 \in \mathbf{G}_2$ are adjacent, then $\deg(\mathbf{v}_2) = \infty$.

Proof. To see that b) implies a), we have to prove that the conditions in Theorem 96 hold. We only prove that $P_{I_1}V \subset V$. The second condition is clear, since $P_{I_1}\psi$ and $(\text{id} - P_{I_1})\psi$ have disjoint support.

First observe that P_{I_1} acts on a function ψ by

$$P_{I_1}\psi(x) = \begin{cases} \psi(x) & x \in \mathbf{G}_1, \\ 0 & x \in \mathbf{G}_2. \end{cases}$$

So, we only have to prove that $P_{I_1}\psi$ is continuous in each node of \mathbf{G}_2 . Fix an arbitrary node $\mathbf{v}_2 \in \mathbf{G}_2$. If \mathbf{v}_2 is not adjacent to a node in \mathbf{G}_1 , then $P_{I_1}\psi(x) = 0$ for all x in a neighbourhood of \mathbf{v}_2 and so it is continuous. If \mathbf{v}_2 is adjacent to a node on \mathbf{G}_1 , then $\deg(\mathbf{v}_2) = \infty$ and thus $\psi(\mathbf{v}_2) = 0$ by Proposition 59. As a consequence, also $P_{I_1}\psi$ is continuous in \mathbf{v}_2 .

To prove the converse implication, observe that the boundary space $\partial V \subset \ell^2(V)$ satisfies

$$\partial V := \{d^\psi : \psi \in V\} \subset \{(x_{\mathbf{v}})_{\mathbf{v} \in V} \in \ell^2(V) : [\deg(\mathbf{v}') = \infty \Rightarrow x_{\mathbf{v}'} = 0]\}.$$

Assume that $L^2(\mathbf{G}_1)$ is invariant. By Theorem 95 $P_{I_1}\psi$ is continuous in the nodes if ψ is continuous in the nodes. In particular, $P_{I_1}\psi$ has to be continuous in all nodes $\mathbf{v}_2 \in \mathbf{G}_2$ which are adjacent to \mathbf{G}_1 . For those nodes $P_{I_1}\psi(\mathbf{v}_2) = 0$. Fix now an arbitrary \mathbf{v}_2 of this kind, a neighbourhood N of \mathbf{v}_2 and an arbitrary $\psi \in V$. On each point $x \in (N \cap \mathbf{G}_1) \setminus \{\mathbf{v}_2\} =: N_1$, the projection P_{I_1} acts as the identity. Further, the ideal I_1 is invariant and so P_{I_1} is a continuous function. We compute

$$0 = \lim_{N_1 \ni x \rightarrow \mathbf{v}_2} P_{I_1}\psi(x) = \lim_{N_1 \ni x \rightarrow \mathbf{v}_2} \psi(x) = \psi(\mathbf{v}_2).$$

Since ψ is arbitrary, $\deg(\mathbf{v}_2) = \infty$ follows from Lemma 69. \square

Proposition 70 allows us to characterize irreducible semigroups. For finite graphs, irreducibility is equivalent to the graph \mathbf{G} being connected in the sense of Definition 52.

Before proving similar results for infinite graphs, we show that the definition of connectedness in Definition 52 is equivalent to the topological one.

Proposition 71. *The following assertions are equivalent.*

a) *The graph G is connected in the sense of Definition 52, i.e.,*

$$d(v, v') < \infty, \quad v, v' \in V.$$

b) *The graph G is topologically connected, i.e., if $\emptyset \neq V_1, V_2 \subset V$ are sets such that*

$$V_1 \cap V_2 = \emptyset, \quad \text{and} \quad V_1 \cup V_2 = V,$$

then there exists $e \in E$ such that $e \sim V_1, e \sim V_2$.

Proof. Assume that a) holds and fix a decomposition $V = V_1 \cup V_2$. Since the graph is connected, for every $v_1 \in V_1, v_2 \in V_2$ there exists a path P of finite length connecting v_1 to v_2 . In other words, P does not lie entirely in either G_1 or G_2 . In particular, there exists an edge $e \in P$ connecting G_1 to G_2 .

Assume that a) does not hold. Fix an arbitrary $v \in V$ and define

$$V_1 := \bigcup_{n=1}^{\infty} \{v' \in V : d(v, v') = n\}.$$

Obviously, V_1 is connected subgraph of V . Since a) does not hold, $V \neq V_1$. Denote $V_2 := V \setminus V_1$. We observe that the subgraphs induced by V_1 and V_2 are disjoint.

In fact, assume that there exists $e \in E$ such that $e(0) \in V_1, e(1) \in V_2$. Then $d(v, e(1)) < \infty$ for all $v \in V$ and so $e(1) \in V_1$, which is a contradiction. Hence the subgraphs induced by V_1, V_2 are disjoint and so the graph G is not topologically connected. \square

In order to characterise irreducibility, we need some additional definitions.

Definition 72. *During the rest of this section, we use the following definitions.*

- *The generalized distance function $d_V : V \times V \rightarrow \mathbb{N} \cup \{\infty\}$ is defined by*

$$d_V(v, v') = \min_{P \in \mathcal{P}[v, v']} \sum_{v'' \in P} \deg(v'').$$

- *The generalised distance function d_E between two edges is defined analogously.*

- The finite span $S_{\text{fin}}(\mathbf{e}) \subset \mathbf{E}$ of \mathbf{e} is defined by

$$S_{\text{fin}}(\mathbf{e}) := \{\mathbf{e}' \in \mathbf{E} : d_{\mathbf{E}}(\mathbf{e}, \mathbf{e}') < \infty\}.$$

We denote, if there is no danger of confusion by $S_{\text{fin}}(\mathbf{e})$ the finite span on the edge set as well as the induced subgraph.

We are now in the position of stating the main theorem.

Theorem 73. *The following assertions are equivalent.*

- a) The semigroup e^{ta} is irreducible.
- b) $S_{\text{fin}}(\mathbf{e}) = \mathbf{G}$ for one $\mathbf{e} \in \mathbf{E}$.
- c) $S_{\text{fin}}(\mathbf{e}) = \mathbf{G}$ for all $\mathbf{e} \in \mathbf{E}$.

Corollary 74. *If \mathbf{G} is a connected, locally finite network, then $(e^{ta})_{t \geq 0}$ is irreducible.*

Remark 75. *The functions d, d_V, d_E are not distances in the usual sense.*

In fact, d is not defined everywhere if the graph is not connected, and d_V, d_E if are not defined everywhere if the graph is not locally finite.

For the sake of the simplicity, we split the proof in several lemmas. The idea is to prove that the invariant ideals of the semigroup $(e^{ta})_{t \geq 0}$ are all of the form $S_{\text{fin}}(\mathbf{e})$ for some $\mathbf{e} \in \mathbf{E}$.

As a preliminary remark, we observe that the only possible invariant ideals are of the form $L^2(\mathbf{G}')$, where $\mathbf{G}' = (V', \emptyset)$ is some subgraph of \mathbf{G} induced by a subset of the node set. In fact, all ideals of $L^2(\mathbf{G})$ have the form $L^2(\omega)$, where $\omega \subset \bigoplus_{\mathbf{e} \in \mathbf{E}} [0, 1]$. As we already observed, $\omega = \bigoplus_{\mathbf{e} \in \mathbf{E}} \omega_{\mathbf{e}}$. Thus, we are claiming that if $L^2(\omega)$ is invariant, then $\omega_{\mathbf{e}} \in \{\emptyset, [0, 1]\}$, but this is a consequence of the irreducibility of the heat semigroup on $L^2[0, 1]$.

Next, we show that ideals of the form $S_{\text{fin}}(\mathbf{e})$ are invariant.

Lemma 76. *Consider a connected graph \mathbf{G} . Fix $\mathbf{e} \in \mathbf{E}$. Then $S_{\text{fin}}(\mathbf{e})$ is invariant under the action of $(e^{ta})_{t \geq 0}$.*

Proof. We use Theorem 95. To prove that both conditions hold, fix $\mathbf{e} \in \mathbf{E}$ and denote P the projection onto $S_{\text{fin}}(\mathbf{e})$. Observe that $P\psi(x) = \mathbb{1}_{S_{\text{fin}}(\mathbf{e})}(x)\psi(x)$ for all $x \in \mathbf{G}$. So, the first condition of Theorem 95 holds since $P\psi$ and $(I - P)\psi$ have disjoint support.

We prove that $PV \subset V$. Recall that the boundary of $S_{\text{fin}}(\mathbf{e})$ consists of those nodes that are adjacent to $S_{\text{fin}}(\mathbf{e})$ and to its complement, according to Definition 52. As usual, one only has to prove continuity in the nodes

$\mathbf{v}' \in \partial S_{\text{fin}}(\mathbf{e})$. To see this, arguing as in the proof of Proposition 70, it suffices to prove that $\partial S_{\text{fin}}(\mathbf{e})$ only consists of nodes of infinite degree. We split the proof in two steps. First, we assume that the boundary only consist of one node \mathbf{v}' . For that node we define

$$d(\mathbf{e}, \mathbf{v}') := \min_{\mathbf{v}' \sim \mathbf{e}' \in S_{\text{fin}}(\mathbf{e})} d_{\mathbb{E}}(\mathbf{e}, \mathbf{e}').$$

This number is finite, since $\mathbf{v}' \in S_{\text{fin}}(\mathbf{e})$ and so there is a path of finite generalized length going from \mathbf{e} to an edge \mathbf{e}' which is incident on \mathbf{v}' . Since the boundary of $S_{\text{fin}}(\mathbf{e})$ only consists of \mathbf{v}' , all paths from an edge in $S_{\text{fin}}(\mathbf{e})$ to an edge in its complement have to pass through \mathbf{v}' . As a consequence, we compute for all edges $\mathbf{e}' \in \Gamma(\mathbf{v}') \setminus S_{\text{fin}}(\mathbf{e})$

$$d_{\mathbb{E}}(\mathbf{e}', \mathbf{e}) = d(\mathbf{e}, \mathbf{v}') + \deg(\mathbf{v}'). \quad (2.12)$$

Assume that the degree of \mathbf{v}' is finite. Then $d_{\mathbb{E}}(\mathbf{e}', \mathbf{e})$ is also finite, and \mathbf{e}' belongs to $S_{\text{fin}}(\mathbf{e})$ which is a contradiction.

If $\partial S_{\text{fin}}(\mathbf{e})$ contains more than a single node, then $d_{\mathbb{E}}(\mathbf{e}, \mathbf{v}')$ equals the minimum of a degree sum, where the minimum is taken over all possible paths. Since $d(\mathbf{e}, \mathbf{v}') = \infty$, then each term in the minimum is infinite, and the claim follows. We omit the precise technical details of this construction.

So we obtained that exactly the nodes on the boundary of $S_{\text{fin}}(\mathbf{e})$ have infinite degree. As a consequence for all $\mathbf{v} \in \partial S_{\text{fin}}, \psi \in V, \psi(\mathbf{v}) = 0$. As a consequence $P\psi$ is continuous in the nodes and the proof is complete. \square

The next step is to identify the subgraphs of the form $S_{\text{fin}}(\mathbf{e})$. Recall that the boundary of a subgraph is introduced in Definition 52.

Lemma 77. *Consider a connected graph G and a connected subgraph G' . The following assertions hold.*

- a) *For each $\mathbf{e} \in \mathbb{E}$, $\deg(\mathbf{v}) = \infty$ for all $\mathbf{v} \in \partial S_{\text{fin}}(\mathbf{e})$.*
- b) *If $d_{\mathbb{E}}(\mathbf{e}, \mathbf{e}') < \infty$ for all $\mathbf{e}, \mathbf{e}' \in G'$ and $\deg(\mathbf{v}) = \infty$ for all $\mathbf{v} \in \partial G'$, then $G' = S_{\text{fin}}(\mathbf{e})$ for all $\mathbf{e} \in G'$.*
- c) *If $\deg(\mathbf{v}) = \infty$ for all $\mathbf{v} \in \partial G'$, then $S_{\text{fin}}(\mathbf{e}) \subset G'$ for all $\mathbf{e} \in G'$.*

Proof. a). This claim is the first step of proof of the Lemma 76.

b) Fix a subgraph with the required properties. First observe that since $d_{\mathbb{E}}(\mathbf{e}, \mathbf{e}') < \infty$ for all $\mathbf{e}, \mathbf{e}' \in G'$, then $G' \subset S_{\text{fin}}(\mathbf{e})$ for all $\mathbf{e} \in G'$.

Assume now that $\exists \mathbf{e}' \in S_{\text{fin}}(\mathbf{e}) \setminus G'$. Without loss of generality, let $\mathbf{e}' \sim G'$, i.e., $\mathbf{e}' \sim \mathbf{v}, \mathbf{v} \in G'$ and assume that the boundary $\partial G'$ only consists of \mathbf{v} .

By hypothesis $\deg(\mathbf{v}) = \infty$ and so $d_{\mathbb{E}}(\mathbf{e}, \mathbf{e}') = \infty$, which is a contradiction.

c) Fix an arbitrary $\mathbf{e} \in \mathbf{G}'$, $\mathbf{e}' \notin \mathbf{G}'$, and a path $P \in P[\mathbf{e}, \mathbf{e}']$. By definition of $\partial \mathbf{G}'$, there exists $\mathbf{v} \in P \cap \partial \mathbf{G}'$. As a consequence $d_{\mathbb{E}}(\mathbf{e}, \mathbf{e}') = \infty$ and the proof is complete. \square

The following is a consequence of the above lemma.

Proposition 78. *Consider a connected graph \mathbf{G} . Then there exists $\mathbf{E}' \subset \mathbf{E}$ such that*

$$\bigcup_{\mathbf{e} \in \mathbf{E}'} \mathbf{S}_{\text{fin}}(\mathbf{e}) = \mathbf{G},$$

and

$$\partial \mathbf{S}_{\text{fin}}(\mathbf{e}) = \mathbf{S}_{\text{fin}}(\mathbf{e}) \cap \mathbf{S}_{\text{fin}}(\mathbf{e}') = \partial \mathbf{S}_{\text{fin}}(\mathbf{e}'), \quad \mathbf{e}, \mathbf{e}' \in \mathbf{E}'.$$

Moreover, for all $\mathbf{e} \in \mathbf{E}'$, $\deg(\mathbf{v}) = \infty$ for all $\mathbf{v} \in \partial \mathbf{S}_{\text{fin}}(\mathbf{e})$.

The same decomposition holds for each subgraph of \mathbf{G} whose boundary only consists of nodes with infinite degree.

Now we can characterise irreducibility.

Proof of Theorem 73. Observe that, as a consequence of Lemma 69, $\psi(\mathbf{v}) = 0$ for all $\psi \in \mathbf{V}$ if and only if $\deg(\mathbf{v}) = \infty$.

By definition, a semigroup is irreducible if there are no non-trivial invariant ideals. Thus, it suffices to prove that all invariant ideals of $(e^{ta})_{t \geq 0}$ have the form $L^2(\mathbf{S}_{\text{fin}}(\mathbf{e}))$, i.e.,

$$\{I \subset L^2(\mathbf{G}) : I \text{ is an invariant ideal}\} = \{L^2(\mathbf{S}_{\text{fin}}(\mathbf{e})) : \mathbf{e} \in \mathbf{E}\}.$$

By Lemma 76, $L^2(\mathbf{S}_{\text{fin}}(\mathbf{e}))$ are invariant and so “ \supset ” holds.

Consider I invariant ideal of $(e^{ta})_{t \geq 0}$. As we have already observed, $I = L^2(\mathbf{G}')$ for some subgraph $\mathbf{G}' \subset \mathbf{G}$.

The projection $P\psi$ of a function ψ on $L^2(\mathbf{G}')$ vanishes in all points of $\mathbf{G} \setminus \mathbf{G}'$ and so, by continuity it vanishes in all points of the boundary of \mathbf{G}' . As a consequence, each function $\psi \in \mathbf{V}$ also has to vanish on all points of the boundary of \mathbf{G}' .

By Lemma 69, all those points \mathbf{v}' satisfy $\deg(\mathbf{v}') = \infty$. So, there exists a decomposition of \mathbf{G}' as in Lemma 77, i.e.,

$$L^2(\mathbf{G}') = \bigoplus_{\mathbf{e} \in \mathbf{E}''} L^2(\mathbf{S}_{\text{fin}}(\mathbf{e})).$$

Now, $L^2(\mathbf{G}')$ has to be an ideal, so \mathbf{E}'' has cardinality one, i.e., $\mathbf{G}' = \mathbf{S}_{\text{fin}}(\mathbf{e})$ for some \mathbf{e} .

Summing up, we have proved that I is an invariant ideal of $(e^{ta})_{t \geq 0}$ if and only if $I = L^2(S_{\text{fin}}(\mathbf{e}))$ for some $\mathbf{e} \in \mathbf{E}$.

Concluding, irreducibility is then equivalent to $S_{\text{fin}}(\mathbf{e}) = \mathbf{G}$ for all $\mathbf{e} \in \mathbf{G}$. In order to see that b) and c) also are equivalent observe that if $\mathbf{e}' \in S_{\text{fin}}(\mathbf{e})$, then $S_{\text{fin}}(\mathbf{e}) = S_{\text{fin}}(\mathbf{e}')$. \square

2.6 Symmetries in special networks

In this section we will study symmetry issues for some special classes of graphs. We start presenting some graph theoretical definitions.

Definition 79. *Let \mathbf{G} a graph with no isolated nodes, i. e., such that $\deg(\mathbf{V}) \geq 1$ for all $\mathbf{v} \in \mathbf{V}$.*

- *We call the graph \mathbf{G} completely unconnected if \mathbf{G} is the union of disjoint compact intervals, i. e., if \mathbf{G} is a regular graph of degree 1.*
- *We call the graph \mathbf{G} an inbound (respectively, outbound) star, if there exists a node \mathbf{V} such that $\mathbf{e}(1) = \mathbf{V}$, (respectively, if $\mathbf{e}(0) = \mathbf{V}$), for all $\mathbf{e} \in \mathbf{E}$. We call the graph \mathbf{G} a star if it is an inbound or outbound star and \mathbf{V} the centre of the star.*
- *We call the graph \mathbf{G} simple if there are no parallel edges, i. e., if ∂ is an injective mapping.*
- *We call the graph \mathbf{G} bipartite if each node has only either ingoing or outgoing edges.*
- *We call the graph \mathbf{G} Eulerian if all nodes have the same number of ingoing and outgoing edges.*
- *We call a graph \mathbf{G} a layer graph if there exist disjoint sets V_1, \dots, V_L such that*

$$- V = \cup_{p=1}^L V_p,$$

$$- \mathbf{e}(0) \in V_p \text{ implies } \mathbf{e}(1) \in V_{p+1} \text{ for all } p = 1, \dots, L-1, \text{ and}$$

$$- \mathbf{e}(0) \in V_L \text{ implies } \mathbf{e}(1) \in V_1.$$

Nodes belonging to V_p are said to lie in the p^{th} layer. Edges outgoing from nodes in the p^{th} layer are also said to lie in the p^{th} layer.

- We call a layer graph symmetric if the ingoing and outgoing degrees of the nodes only depends on the layer, i.e., if there exist numbers $I(p), O(p) \in \mathbb{N}_0$ such that $\deg^+(\mathbf{V}) = I(p), \deg^-(\mathbf{V}) = O(p)$ for all nodes \mathbf{V} in the p^{th} layer.

During this section, it is always assumed that the graph \mathbf{G} is finite, and that numberings of the edges and of the vertices are fixed, i.e.,

$$\mathbf{E} := (\mathbf{e}_1, \dots, \mathbf{e}_m) \quad \text{and} \quad \mathbf{V} := (\mathbf{v}_1, \dots, \mathbf{v}_n)$$

Bipartite and Euler Graphs

It is possible to characterize some classes of graphs by the admissibility of the projection P_Y on $Y = \langle \mathbb{1} \rangle$.

Theorem 80. *Consider a finite graph \mathbf{G} and the orthogonal projection P_Y of $\ell^2(\mathbf{E})$ onto $Y \langle \mathbb{1} \rangle$. Then P_Y is admissible if and only if \mathbf{G} is bipartite or Eulerian.*

Proof. As usual, we have to prove that for a continuous function $\psi \in \mathbf{V}$ also $P_Y \psi$ is continuous in the nodes. Observe that the projection P_Y of $\ell^2(\mathbf{E})$ onto Y satisfies

$$P_Y x = \sum_{i=1}^m \frac{x_i}{m}.$$

Let $\mathbf{V}_1 \subset \mathbf{V}$ denote the set of all vertices having outgoing edges, and let $\mathbf{V}_2 \subset \mathbf{V}$ denote the set of all vertices having ingoing edges. We distinguish two cases. First, assume $\mathbf{V}_1 \cap \mathbf{V}_2 = \emptyset$. Then \mathbf{G} is a bipartite graph.

On the other hand, if $\mathbf{V}_1 \cap \mathbf{V}_2 \neq \emptyset$, then $d^{P_Y \psi}$ exists if and only if

$$\sum_{j=1}^m \frac{\psi_j(0)}{m} = \sum_{j=1}^m \frac{\psi_j(1)}{m}. \quad (2.13)$$

We show now that the equality (2.13) is equivalent to the graph being Eulerian. First, assume that (2.13) holds for every $\psi \in V$. Fix an arbitrary $\mathbf{v}_k \in \mathbf{V}$ and choose $\psi \in V$ such that $d^\psi = \mathbb{1}_{\{i\}}$. Then

$$\frac{1}{m} \deg^+(\mathbf{v}_k) = \sum_{j=1}^m \frac{\psi_j(0)}{m} = \sum_{j=1}^m \frac{\psi_j(1)}{m} = \frac{1}{m} \deg^-(\mathbf{v}_k).$$

Thus it is necessary that $\deg^-(\mathbf{v}_k) = \deg^+(\mathbf{v}_k)$ holds for every $k = 1, \dots, n$.

Conversely, assume that

$$\deg^-(\mathbf{v}_k) = \deg^+(\mathbf{v}_k), \quad k = 1, \dots, n.$$

Then

$$\begin{aligned}
\sum_{j=1}^m \frac{\psi_j(0)}{m} &= \frac{1}{m} \sum_{k=1}^n \deg^+(\mathbf{v}_k) d_k^\psi \\
&= \frac{1}{m} \sum_{k=1}^n \deg^-(\mathbf{v}_k) d_k^\psi \\
&= \sum_{j=1}^m \frac{\psi_j(1)}{m}.
\end{aligned}$$

Hence (2.13) is satisfied, so this condition is also sufficient.

It only remains to show that indeed for every bipartite graph P_Y is admissible. To see this, note that for an arbitrary $\psi \in V$ the vector $d^{P_Y\psi}$ can be chosen to equal $\sum_{i=1}^m \frac{\psi_i(0)}{m}$ in all components belonging to nodes in V_1 and to equal $\sum_{i=1}^m \frac{\psi_i(1)}{m}$ in all components belonging to V_2 . This shows continuity of $P_Y\psi$ in the nodes, thus implying $P_Y\psi \in V$. \square

Stars

The main result of this subsection is a characterization of stars in the class of the simple graphs. We first investigate the admissibility of projections.

Proposition 81. *The following assertions hold.*

- a) *The graph \mathbf{G} is completely unconnected if and only if P_Y is admissible for all linear subspaces $Y \subset \ell^2(E)$.*
- b) *Let \mathbf{G} be a simple, connected graph. Then \mathbf{G} is a star if and only if P_Y is admissible for all linear subspaces $Y \subset \ell^2(E)$ such that $\langle \mathbf{1} \rangle \subset Y$.*

Proof. a) Since the graph \mathbf{G} is completely unconnected, the continuity condition in V is empty, and therefore each P_Y is admissible. Conversely, if \mathbf{G} is not completely unconnected, then it is possible to decompose \mathbf{G} into the disjoint union of a connected graph \mathbf{G}_1 with m_1 edges, $m_1 \geq 2$ and the remaining graph \mathbf{G}_2 . Let K_1 be an orthogonal projection of \mathbb{C}^{m_1} , which does not have $\mathbf{1}$ as an eigenvector. Lemma 67 asserts that the orthogonal projection

$$\begin{pmatrix} K_1 & 0 \\ 0 & \text{id} \end{pmatrix}$$

is not admissible.

b) Let the graph G be a star and Y a subspace containing $\langle \mathbb{1} \rangle$. Without loss of generality, we prove the claim for an outgoing star with centre V_1 and with the natural numbering of the other nodes.

In fact, for this star

$$\tilde{\mathcal{I}} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \text{id}_m \end{pmatrix}.$$

Since now P_Y has $\mathbb{1}$ as eigenvector to the eigenvalue 1,

$$\begin{pmatrix} P_Y \mathbb{1} & 0 \\ 0 & P_Y \text{id}_m \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & P_Y \end{pmatrix}.$$

So, $\text{Range } \widehat{P}_Y \tilde{\mathcal{I}} \subset \text{Range } \tilde{\mathcal{I}}$, and this implies the admissibility of \mathcal{P}_Y .

Conversely, assume that the graph G is not a star. This implies the existence of an undirected path of length 3. We will denote it by e_1, e_2, e_3 , possibly relabelling the edges. Our strategy is the following: for each graph that is a path consisting of 3 edges we construct a non-admissible projection P_L where $L\mathbb{1} = \mathbb{1}$. We then consider the projection \mathcal{P}_Y , where P_Y is

$$P_Y := \begin{pmatrix} P_L & 0 \\ 0 & \text{id} \end{pmatrix}.$$

Then, by Lemma 68, we conclude that \mathcal{P}_Y is not admissible, although $\mathbb{1}$ is an eigenvector of P_Y .

First, consider cycles of length 3. Since each edge can be directed arbitrarily, there are 8 such graphs. Let us start with the case of a not strongly connected graph.

Such graphs are neither Eulerian nor bipartite. Thus, Theorem 80 provides an example of an L as requested. If the graph is a (directed) cycle such that $e_1(0) = v_1$, consider the projection

$$L := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the function ψ defined by $\psi(x) := (x, 1-x, 0)^\top \in V$. Then, $\psi \in V$ but $P_Y \psi \notin V$, since $P_Y \psi(x) = (\frac{1}{2}, \frac{1}{2}, 0)^\top$ for a. e. $x \in (0, 1)$.

Consider now the lines of length 3. We split this into three possible cases: G may be bipartite line, a (directed) line, or neither a (directed) line nor a bipartite graph. In the last two cases the graphs is neither bipartite nor Eulerian, and hence we can use Theorem 80 again.

In the case of a bipartite line, let us consider the projection

$$L := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

for the parametrization $\mathbf{e}_1(0) = \mathbf{v}_1$, $\mathbf{e}_1(1) = \mathbf{e}_2(1) = \mathbf{v}_2$, $\mathbf{e}_2(0) = \mathbf{e}_3(0) = \mathbf{v}_3$, and $\mathbf{e}_3(1) = \mathbf{v}_4$. Consider the function $\psi(x) := (x, x, 0)^\top$. Again, $\psi \in V$ but $P_Y\psi \notin V$, since $P_Y\psi(x) = (\frac{x}{2}, x, \frac{x}{2})^\top$ for a. e. $x \in (0, 1)$. \square

Remark 82. *In Proposition 81, (2) we have assumed the graph \mathbf{G} to have no multiple edges. In fact, it is not possible to relax this condition, since all orthogonal projections with eigenvector $\mathbf{1}$ are admissible on all connected graphs consisting of 2 nodes and m edges for each $m \in \mathbb{N}$ and each orientation of the edges.*

Layer Graphs

In this section we prove an admissibility result for symmetric layer graphs. We start fixing a canonical numbering of the edges of a layer graph. First observe that the node decomposition induces an edge decomposition $\mathbf{E} = \bigcup_{p=1}^L \mathbf{E}_p$ by setting

$$\mathbf{E}_p := \{\mathbf{e} \in \mathbf{E} : \mathbf{e} \text{ lies in the } p^{\text{th}} \text{ layer}\}.$$

After relabelling the edges we may assume that there exist $L_p, p = 1, \dots, L+1$ satisfying

- a) $L_1 = 0$;
- b) $\mathbf{e}_i(0) = \mathbf{e}_j(0)$ or $\mathbf{e}_i(1) = \mathbf{e}_j(1)$ implies $L_{p-1} < i, j \leq L_p$ for some p ;
- c) $\mathbf{e}_i(0) = \mathbf{e}_j(1)$ implies $L_{p-1} < j \leq L_p < i \leq L_{p+1}$ for some p .

The numbering obtained in such a way has the property that \mathbf{e}_i is in the p^{th} layer if and only if $L_p < i \leq L_{p+1}$. In fact, all edges \mathbf{e}_i such that $i \leq L_{p+1}$ are in any of the first p layers.

We are going to exhibit a class of admissible projections.

Proposition 83. *Consider a symmetric layer graph \mathbf{G} and the orthogonal projection P_Y*

$$P_Y = \begin{pmatrix} \left(\frac{1}{|E_1|}\right)_{i,j=1,\dots,|E_1|} & & & 0 \\ & \ddots & & \\ 0 & & & \left(\frac{1}{|E_L|}\right)_{i,j=1,\dots,|E_L|} \end{pmatrix}, \quad (2.14)$$

where $|E_p|$, $p = 1, \dots, L$ denotes the number of edges in the p^{th} layer. Then \mathcal{P}_K is admissible.

Proof. One has to check the continuity condition for each $p = 1, \dots, L - 1$ in every node of the p^{th} layer. Define the auxiliary function

$$\lambda : k \mapsto \text{layer of the node } \mathbf{v}_k.$$

We thus have to check continuity in those nodes \mathbf{v}_k such that $\lambda(k) = p$, $p = 1, \dots, L - 1$.

The set $\lambda^{-1}(p)$ can be represented in the form

$$\lambda^{-1}(p) = \{k : \exists i \in \{L_p + 1, \dots, L_{p+1}\} \text{ s. t. } \mathbf{e}_i(1) = \mathbf{v}_k\},$$

as well as in the form

$$\lambda^{-1}(p) = \{k : \exists i \in \{L_{p+1} + 1, \dots, L_{p+2}\} \text{ s. t. } \mathbf{e}_i(0) = \mathbf{v}_k\},$$

whenever the expression is defined. By the definition of P_Y , for all $p = 1, \dots, L - 1$ and all $i, j = L_p + 1, \dots, L_{p+1}$ the identities

$$\mathcal{P}_Y \psi_i(1) = P_Y \psi_j(1), \quad \mathcal{P}_Y \psi_i(0) = P_Y \psi_j(0). \quad (2.15)$$

hold. As a consequence, for layers having ingoing or outgoing degree 0, the continuity is obvious. Assume now that $I(p) \neq 0$ and $O(p) \neq 0$.

For an edge \mathbf{e}_i in the p^{th} layer and for $\psi \in \mathcal{V}$,

$$(P_Y \psi)_i(1) = \sum_{i=L_p+1}^{L_{p+1}} \frac{\psi_i(1)}{|E_p|} = \sum_{k \in \lambda^{-1}(p)} \text{deg}^-(\mathbf{v}_k) \frac{\psi_i(1)}{|E_p|}.$$

Recall that since our graph is symmetric, the incidence degree $\text{deg}^-(\mathbf{v}_k)$ only depends on the layer, and therefore we can write

$$(P_Y \psi)_i(1) = \sum_{k \in \lambda^{-1}(p)} |I(p)| \frac{\psi(\mathbf{v}_k)}{|E_p|} = \frac{|I(p)|}{|E_p|} \sum_{k \in \lambda^{-1}(p)} \psi(\mathbf{v}_k).$$

With analogous computations we obtain for edges in the $p + 1$ layer

$$(P_Y \psi)_i(0) = \frac{|O(p)|}{|E_{p+1}|} \sum_{k \in \lambda^{-1}(p)} \psi(\mathbf{v}_k).$$

Observe that the identities $|E_{p+1}| = |\lambda^{-1}(p)| |O(p)|$ and $|E_p| = |\lambda^{-1}(p)| |I(p)|$ imply $|O(p)| |E_{p+1}|^{-1} = |I(p)| |E_p|^{-1}$. We have thus proved that $(P_Y \psi)_i(1) = (P_Y \psi)_j(0)$ for all $\mathbf{e}_i, \mathbf{e}_j$ such that $i \in \lambda^{-1}(p), j \in \lambda^{-1}(p + 1)$. This completes the proof. \square

Remarks 84.

- (1) *The above result can be restated as follows. Let \mathbf{G} be a symmetric layer graph. Then, if $M = 0$ and $C = \text{id}$,*

$$\mathcal{Y} := \{\psi \in L^2 : \psi_i = \psi_j \text{ for all } i, j \in \ell^{-1}(p), p = 1, \dots, L\}$$

is invariant under the action of $(e^{tA})_{t \geq 0}$. In fact, this means that “radial” functions are invariant not only for trees but also for layer graphs.

- (2) *The class of the layer graphs is a common object in the graph theoretical literature. In fact, layer graphs are nothing but (directed) p -partite graphs, for which collapsing the components of the graphs to a single vertex leads to a finite line or to a cycle. In particular, homogeneous trees of finite depth are symmetric layer graphs. Such graphs play a role in the investigation of biological neural networks.*
- (3) *The symmetry condition in Proposition 83 cannot be relaxed. To see this, consider the following simple example. Let \mathbf{G} be an outbound star of order two and consider two copies of \mathbf{G} . Identifying two of the external nodes defines a layer graph. It is possible to show that the orthogonal projection defined in (2.14) is not admissible, due to the two free nodes in the second layer.*

2.7 Generalised networks

We want to discuss in this section an alternative setting which contains network equations as a special case. Therefore, we only assume that \mathbf{E} is a countable set. We stress that we are not in graph theoretic setting, and so \mathbf{E} should not be considered as the edge set of a graph. We start with a definition justified by Lemma 54.

Definition 85. *The bounded operators $\partial_0, \partial_1 \in \mathcal{L}(H^1((0, 1), \ell^2(\mathbf{E})))$ are defined by*

$$\partial_0 \psi := \psi(0), \quad \partial_1 \psi := \psi(1), \quad \psi \in H^1((0, 1), \ell^2(\mathbf{E})).$$

Analogously, the operator $\partial \in \mathcal{L}(H^1(0, 1), \ell^2(\mathbf{E}) \oplus \ell^2(\mathbf{E}))$ is defined by

$$\partial \psi := \begin{pmatrix} \psi(0) \\ \psi(1) \end{pmatrix}, \quad \psi \in H^1((0, 1), \ell^2(\mathbf{E})).$$

Fix a subspace Y of $\ell^2(\mathbf{E}) \oplus \ell^2(\mathbf{E})$. We define a form domain by setting

$$V_Y := \{\psi \in H^1((0, 1), \ell^2(\mathbf{E})) : \partial\psi \in Y\}. \quad (2.16)$$

In analogy with (2.6), we define the form $a : V_Y \times V_Y \rightarrow \mathbb{C}$ by

$$a(\psi, \psi') := \left(\frac{d}{dx} \psi \mid \frac{d}{dx} \psi' \right) - (M \partial\psi \mid \partial\psi'), \quad \psi, \psi' \in V_Y. \quad (2.17)$$

Here $M \in \mathcal{L}(\ell^2(\mathbf{E} \oplus \mathbf{E}))$. Exactly as in the sections before, the form (a, V_Y) has domain dense in $L^2((0, 1), \ell^2(\mathbf{E}))$. Further, it is possible to characterise continuity and ellipticity of the form.

We focus on the identification of *nodes* of the form domains. Observe that, since we are not in a graph theoretic setting, there are no nodes in the sense we discussed in the previous sections.

The idea for doing this is the following: the condition $\partial\psi \in Y$ in the definition of the form domain will contain informations about couplings in the boundary values of ψ . We identify several boundary points with a node, if the boundary condition of two different nodes are mutual independent, and if they are minimal in some sense.

This identification relies on the following lemma.

Lemma 86. *Fix a countable index set I and consider a linear bounded operator $A \in \mathcal{L}(\ell^2(I))$. Then there exists an index set \mathbf{V} and an uniquely determined family $(I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ with the following properties:*

- a) $(I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ is a partition of I ;
- b) For all $\mathbf{v} \in \mathbf{V}$, $\ell^2(I_{\mathbf{v}})$ is invariant under the action of A and A^* ;
- c) For all $I' \subset I$, $I' \neq \emptyset$, if $\ell^2(I')$ is invariant under the action of A and A^* , then there exists $\mathbf{v}' \in \mathbf{V}$ such that $I_{\mathbf{v}'} \subset I'$.

We call the $I_{\mathbf{v}}$ the invariant blocks of A .

Proof. We first observe that the statement can be reformulated in the language of infinite matrices. In fact, if A is an infinite matrix (not necessarily countable), i.e., a doubly indexed family $(a_{ij})_{i,j \in I}$, then the statement is equivalent to the existence of an index set \mathbf{V} and an uniquely determined family $(I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ with the following properties:

- 1) $(I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ is a partition of I ;
- 2) For all $\mathbf{v} \in \mathbf{V}$, $i \in I_{\mathbf{v}}$, $j \in I \setminus I_{\mathbf{v}}$, $a_{ij} = a_{ji} = 0$;

3) For all $I' \subset I$, $I' \neq \emptyset$, if b) holds for I' , then there exists $\mathbf{v}' \in \mathbf{V}$ such that $I_{\mathbf{v}'} \subset I'$.

Without loss of generality, we assume that $a_{ij} \in \{0, 1\}$. If this is not the case, we observe that the matrix $B := (b_{ij})_{i,j \in I}$ obtained by setting

$$b_{ij} := \begin{cases} 1, & a_{ij} \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

has the same invariant blocks as A and satisfies the assumption.

Now, we interpret I as the vertex set of a graph \mathbf{G} and A as the adjacency matrix, and we consider the partition $I = (I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ in connected components. We claim that $(I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ has the claimed properties.

First, $(I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ is a partition of I by construction.

Second, fix $\mathbf{v} \in \mathbf{V}$, $i \in I_{\mathbf{v}}$, $j \in I \setminus I_{\mathbf{v}}$. Since i, j are not in the same connected component, there is no edge connecting the two nodes. Thus, $a_{ij} = a_{ji} = 0$.

Finally, assume that $I' \subset I$ has the property 2). Consider the graph with index set I' and adjacency matrix $A|_{I'}$, and decompose it in its connected components. Since I' is disconnected from the rest of the graph, these connected components are also connected components of the graph \mathbf{G} .

Conversely, assume that a family $(I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ has the properties 1)–3). We have to show that $I_{\mathbf{v}}$ are the connected components. First observe that by 2) each of the node sets $I_{\mathbf{v}}$ is disconnected from the rest of the graph. Thus, subsets of $I_{\mathbf{v}}$ are connected in the graph $I_{\mathbf{v}}$ with adjacency matrix $A|_{I_{\mathbf{v}}}$ if and only if they are connected in \mathbf{G} . Consider now the decomposition $I_{\mathbf{v}} := (I_{\mathbf{v}_\ell})_{\ell \in L}$ in connected components. Observe that 2) holds for all $I_{\mathbf{v}_\ell}$. By assumption 3), for all ℓ there exists $I_k \subset I_{\mathbf{v}_\ell} \subset I_{\mathbf{v}}$. Since $(I_{\mathbf{v}})_{\mathbf{v} \in \mathbf{V}}$ is a partition by assumption 1), $I_{\mathbf{v}} = I_{\mathbf{v}_\ell}$ for all ℓ . As a consequence $I_{\mathbf{v}}$ is connected. \square

Observe that $\ell^2(\mathbf{E}) \oplus \ell^2(\mathbf{E}) \simeq \ell^2(\mathbf{E} \oplus \mathbf{E})$, thus the above Lemma can be applied in our situation.

Definition 87. Consider the form (a, V_Y) defined in (2.16)–(2.17) and denote by P_Y the orthogonal projection of $\ell^2(\mathbf{E} \oplus \mathbf{E})$ onto Y . The nodes of the form domain V_Y are the invariant blocks $I_{\mathbf{v}}$ of P_Y .

Remark 88. If $Y := \langle \{\mathbb{1}_{I_{\mathbf{v}}} : I_{\mathbf{v}} \text{ finite}\} \rangle \oplus \{0\}$ and $M = 0$, then (a, V_Y) is the form associated with the Laplace operator with Kirchhoff boundary conditions on $L^2(\mathbf{G})$.

Here the graph \mathbf{G} has vertex set \mathbf{V} and it is obtained identifying the boundary points in the same connected component $I_{\mathbf{v}}$.

Next, we identify the domain of the operator associated with (a, V_Y) . To this aim we give the following definition.

Definition 89. For all $\psi \in H^2((0, 1), \ell^2(\mathbf{E}))$ we define the bounded operator $\frac{\partial}{\partial \nu} \in \mathcal{L}(H^2((0, 1), \ell^2(\mathbf{E})), \ell^2(\mathbf{E} \oplus \mathbf{E}))$ by

$$\frac{\partial}{\partial \nu} \psi := \left(-\frac{d}{dx} \psi(0), \frac{d}{dx} \psi(1) \right). \quad (2.18)$$

Proposition 90. The operator $(A, D(A))$ associated with (a, V_Y) . Then, the operator associated with (a, V_Y) is given by

$$D(A) := \{ \psi \in H^2((0, 1), \ell^2(\mathbf{E})) : \left(\frac{\partial}{\partial \nu} - M \right) \partial \psi \in Y^\perp \}, \quad (2.19)$$

and

$$A\psi := \frac{d^2}{dx^2} \psi. \quad (2.20)$$

Proof. Denote $(B, D(B))$ the operator associated with (a, V_Y) . We first show $D(A) \subset D(B)$. Fix arbitrary $\psi \in D(A)$, $\psi' \in V_Y$ and compute

$$\begin{aligned} a(\psi, \psi') &= \sum_{\mathbf{e} \in \mathbf{E}} \int_0^1 \frac{d}{dx} \psi_{\mathbf{e}}(x) \overline{\frac{d}{dx} \psi'_{\mathbf{e}}(x)} dx - (M \partial \psi \mid \partial \psi') \\ &= - \sum_{\mathbf{e} \in \mathbf{E}} \frac{d^2}{dx^2} \psi_{\mathbf{e}}(x) \overline{\psi'_{\mathbf{e}}(x)} dx + \sum_{\mathbf{e} \in \mathbf{E}} \left[\frac{d}{dx} \psi_{\mathbf{e}} \overline{\psi'_{\mathbf{e}}} \right]_0^1 - (M \partial \psi \mid \partial \psi') \\ &= - \sum_{\mathbf{e} \in \mathbf{E}} \frac{d^2}{dx^2} \psi_{\mathbf{e}}(x) \overline{\psi'_{\mathbf{e}}(x)} dx + \left(\frac{\partial}{\partial \nu} \psi \mid \partial \psi' \right) - (M \partial \psi \mid \partial \psi') \\ &= - \sum_{\mathbf{e} \in \mathbf{E}} \frac{d^2}{dx^2} \psi_{\mathbf{e}}(x) \overline{\psi'_{\mathbf{e}}(x)} dx + \left(\left(\frac{\partial}{\partial \nu} - M \right) \psi \mid \partial \psi' \right) \end{aligned}$$

The last term vanishes, since $\psi \in D(A)$. Thus $a(\psi, \psi') = (A\psi \mid \psi')$ for arbitrary $\psi' \in V_Y$. Choosing $f = A\psi$ shows that $\psi \in D(B)$.

Conversely, fix $\psi \in D(B)$. Then, there exists $f \in L^2((0, 1), \ell^2(\mathbf{E}))$ such that $a(\psi, \psi') := (f \mid \psi')$ for all $\psi' \in V_Y$. Integrating by part shows that

$$- \sum_{\mathbf{e} \in \mathbf{E}} \frac{d^2}{dx^2} \psi_{\mathbf{e}}(x) \overline{\psi'_{\mathbf{e}}(x)} dx + \left(\left(\frac{\partial}{\partial \nu} - M \right) \psi \mid \partial \psi' \right) = (f \mid \psi')$$

holds for all $\psi \in V_Y$ and thus in particular for $\psi \in H_0^1((0, 1), \ell^2(\mathbf{E}))$. As a consequence $f := \frac{d^2}{dx^2} \psi$ and

$$\left(\left(\frac{\partial}{\partial \nu} - M \right) \psi \mid \partial \psi' \right) = 0, \quad \psi' \in V_Y.$$

The latter is equivalent to $(\frac{\partial}{\partial \nu} - M)\partial\psi \in Y^\perp$ since $\partial\psi'$ can assume all values in Y . This means $\psi \in D(A)$ and the proof is complete. \square

It is in principle possible to systematically analyse qualitative properties of the heat semigroup associated with the form a_Y . We refer to Section 2.8 for some remarks about this topic.

2.8 Discussion and remarks

The main topic of our work is the investigation of *symmetry properties* of network equations. We shortly discuss how our use of the term symmetry is related to the physical meaning of the word.

In a physical context, it is often useful to work in a Lagrangian framework. This means, a Lagrange function is given for our system, and the solutions are the orbits that minimise this Lagrangian function.

To be more precise, fix a smooth, bounded domain Ω and consider the space $V := H^1(\Omega)$. Choose now a positive time $t > 0$ and consider the space $X := C^1([0, T], V)$. Now, the solutions of the heat equation

$$\begin{cases} \dot{\psi}(t, x) = \Delta\psi(t, x) & t \in [0, T], x \in \Omega, \\ \psi(0, x) = f(x) & f \in C^\infty(\Omega), x \in \Omega, \end{cases}$$

are the minima of the Lagrange function $\mathcal{L} : X \rightarrow \mathbb{C}$

$$\mathcal{L}(\psi) := \int_0^T \int_\Omega \frac{1}{2} \psi(t, x) \dot{\psi}(t, x) - |\nabla\psi(t, x)|^2 dx dt.$$

One sees that the second term in the Lagrangian function is exactly the diagonal term of the sesquilinear form $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ defined by

$$a(\psi, \psi') := \int_\Omega \nabla\psi(x) \overline{\nabla\psi'(x)} dx, \quad \psi, \psi' \in H^1(\Omega).$$

Now, assume that $(S_z)_{z \in Z}$ is a family of mappings $S_z : H^1(\Omega) \rightarrow H^1(\Omega)$ that is a group for the binary operation \circ defined by

$$(S_{z_1} \circ S_{z_2})\psi := S_{z_1}(S_{z_2}\psi), \quad z_1, z_2 \in Z, \psi \in H^1(\Omega).$$

Further, assume that each S_z commutes with the time derivative and that $a(\psi(x)) = a(S_z\psi)$ for all $z \in Z, \psi \in H^1(\Omega)$. Thus, for all $z \in Z$, S_z maps solutions of the problem above into other solutions: $(S_z)_{z \in Z}$ is a *symmetry group* of the physical system.

It is possible to operate a distinction between two main type of symmetries: *space symmetries* and *gauge symmetries*. We now show that our type of symmetries are gauge symmetries.

Space symmetries are symmetries for which S_z is given by a transformation of the domain Ω . This means, we choose a family of mapping $(s_z)_{z \in Z}$ such that $s_z : \Omega \rightarrow \Omega$ and $(s_z)_{z \in Z}$ is a group for the composition. Then, one defines

$$S_z\psi(t, x) := \psi(t, s_z x), \quad z \in Z, \psi \in H^1(\Omega), x \in \Omega, t \in [0, T].$$

Such a transformations obviously commutes with the time derivative. Thus, if $a(S_z\psi) = a(\psi)$ for all $z \in Z, \psi \in H^1(\Omega)$, then $(S_z)_{z \in Z}$ is a space symmetry of the system.

We now turn our attention to the case of gauge symmetries. Assume that $V := H^1(\Omega, H)$, where H is a complex Hilbert space (in the easiest case, $H = \mathbb{C}$). Fix a group of mappings $(s_z)_{z \in Z}$, where $s_z \in \mathcal{L}(H)$ and define

$$S_z\psi(t, x) := s_z\psi(t, x), \quad z \in Z, \psi \in H^1(\Omega, H), x \in \Omega, t \in [0, T].$$

Then $(S_z)_{z \in Z}$ is also a group for the composition, and it also commutes with the time derivative. If $a(S_z\psi) = a(\psi)$ for all $z \in Z, \psi \in H^1(\Omega, H)$, then one has a global gauge symmetry of the system.

The symmetries that we discuss in this work are of this type. Observe that if Y is invariant, then the unitary group $s_z := (e^{izP_Y})_{z \in \mathbb{R}}$ defines a group such that $a(\psi) = a(S_z\psi)$ for all $z \in Z, \psi \in H^1(\Omega)$. That is, it is a gauge symmetry of the system.

To see what is the action of this group, observe that writing the e^{izP_Y} as the exponential series $\sum_{n=0}^{\infty} \frac{(izP_Y)^n}{n!}$ yields

$$e^{izP_Y} = e^{iz}P_Y + (\text{id} - P_Y), \quad z \in Z.$$

Decompose now $\psi := \psi_Y + \psi_{Y^\perp}$. Now, the gauge group acts as

$$e^{izP_Y} : \psi \mapsto e^{iz}\psi_Y + \psi_{Y^\perp}, \quad z \in Z, \psi \in H^1(\Omega, H).$$

In other words, it acts as the standard gauge group on the Y -component of the function and as the identity on the Y^\perp -component, see also [15].

Finally, observe that also the symmetries considered in Chapter 1 were of this type.

Section 2.1

The Wolfgang's first network tutorial was written for my supervisor Wolfgang Arendt. The approach we use to represent graphs and networks is mostly combinatorial. The reader interested in details and in general topics of graph theory should consult the monographs [27], or [58], from which we have borrowed most of the notations. Another approach on the theory of network equations it is based on the theory of CW -complexes, see [59]-[60]. It avoids combinatorial issues and it is, for instance, the approach used in [20].

Section 2.2

The results in this section are unpublished. The issues discussed in Section 2.2 are not new in the mathematical literature. An amusing introduction to the topic can be found in [37, Chap. 4]. There, the boundedness of

infinite matrices in the space $\ell^2(\mathbf{V})$ set is discussed. As Paul Halmos points out, “*there are no elegant and usable necessary and sufficient conditions*” for characterizing the boundedness of such an operator. In Proposition 56 is an example of such a result for a restricted class of operators. It be compared with the well-known fact that the stochastic version of the adjacency matrix is a bounded operator in ℓ^2 if and only if the graph is uniformly locally finite. We reformulate the result in Proposition 56 for infinite matrices. To this end, assume that the set of edges and of vertexes have the same cardinality, i.e., both are infinite countable sets. Then fixing a numbering of edges and vertexes, i.e., identifying \mathbf{V} and \mathbf{E} with \mathbb{N} , shows that a stochastic, Boolean valued, infinite matrix is bounded in $\ell^2(\mathbb{N})$ if and only if there is a uniform bound on the number of non-zero entries of each row. This result is similar to the classical result of Otto Toeplitz (see [56]), that if A is an operator in $\mathcal{L}(\ell^2(\mathbf{V}))$, then there exists an orthonormal basis such that the number of non-zero entries of each row is finite.

On the other side, one could ask whether the question of the boundedness of the incidence operator have been already discussed in the literature. We refer to [47] for a complete survey of the standard result about discrete operators on graphs, and for boundedness issues to [57]. There, the case of a transition matrix corresponding to a reversible Markov chain is discussed. It should be mentioned that in [46] some results about compactness, boundedness and spectral properties of the adjacency operator are investigated

Section 2.3

The results in this section come from [18] for the case of a finite network.

In particular, the identification of the domain of L^2 -realisations of the strongly coupled elliptic operators on networks did not attract the interest of researchers. In [41], however, some issues regarding the domain of the Laplacian are considered; a more general case is discussed there, since the Laplacian is not seen as the operator associated with a sesquilinear form.

A related question is the study of spectral properties of the Laplacian on a network. This is mostly studied in a C^2 -setting, see [9]-[10] or in L^∞ -setting for locally finite networks, see [12]-[13].

Section 2.4

The results in this section are taken from [18], with some generalisations and modifications. In particular, in that paper symmetries were introduced by fixing an orthogonal projection $P \in \mathcal{L}(\ell^2(I))$. Starting from subspaces has the advantage of making immediately clear what is the connection with

the classical meaning of symmetries. In that work is also shown that for the Laplacian is equivalent to study symmetries for the heat or Schrödinger equation. It should be mentioned that other forms of symmetries have been considered in connection with diffusion (or Schrödinger) equations on graphs. In fact, if the graph possesses some symmetry group, then it is possible to exploit this feature in order to construct counterexamples to the Kac's problem [38] for graphs. This has been done in [11]–[35], and a systematic approach has been announced in a forthcoming paper [8]. For a discussion of the issue of symmetries in quantum graphs see also [15].

Section 2.5

The results in this section are unpublished. As we already pointed out in the comments to Section 2.2, most of the authors have only considered spectral issues for the continuous Laplacian on infinite graphs, neglecting the investigation of the parabolic problem. The standard assumption is the uniform locally finiteness of the graph. Observe that by Theorem 73 the heat semigroups is irreducible on every locally finite, connected graph.

On the other hand, many authors considered the case of the discrete Laplacian on infinite graphs. Such considerations are interesting because of their connection with the theory of Markov chains and Riemannian manifolds. For an introduction to the topic, see [54].

Section 2.6

The results in this section come from [18]. Direct graph theoretic characterisations of non-spectral properties of the parabolic equations are rare in the literature. In [21]–[22] heat kernels of the parabolic equation are derived; this derivation involves graph theoretic objects.

Section 2.7

The results in this section are unpublished. Observe that it is possible to relate qualitative properties of the associated semigroup to functional analytic properties of the orthogonal projection P_Y . This is a work in progress with Delio Mugnolo and Olaf Post. The most general boundary condition have been investigated in [41]. Although not all the boundary conditions considered in that work can be obtained as described in Section 2.7, we stress that our method allows us to characterise symmetry and qualitative properties of the semigroup coming from the form.

Appendix A

Appendix

A.1 Sesquilinear forms on complex Hilbert spaces

We state here all results we used in the work. We also give the basic definitions. The reader interested in the theory of sesquilinear forms and analytic semigroups should consult the monographs [4],[26],[52] and the survey [3].

Definition 91 (Sesquilinear mappings and forms). *Consider Hilbert spaces U, V , and a mapping $a : U \times V \rightarrow \mathbb{C}$.*

- a) *The mapping a is a sesquilinear mapping if a is linear in the first component and antilinear in the second one, i.e. if*

$$a(\lambda\psi + \mu\psi', \psi'') = \lambda a(\psi, \psi'') + \mu a(\psi', \psi''), \quad \text{for all } \psi, \psi' \in U, \psi'' \in V$$

and

$$a(\psi'', \lambda\psi + \mu\psi') = \bar{\lambda} a(\psi, \psi'') + \bar{\mu} a(\psi', \psi''), \quad \text{for all } \psi, \psi' \in V, \psi'' \in U.$$

If $U = V$ we use the term sesquilinear form and we specify the Hilbert space writing $a = (a, V)$.

- b) *With an abuse of notation, we denote the energy functional $a : V \rightarrow \mathbb{C}$ associated with the form (a, V) also (a, V) . The energy functional is defined by*

$$a(\psi) := a(\psi, \psi), \quad \text{for all } \psi \in V.$$

- c) *A sesquilinear mapping a is continuous if there exists $M \in \mathbb{R}$ such that*

$$|a(\psi, \psi')| \leq M \|\psi\|_U \|\psi'\|_V, \quad \text{for all } \psi \in V.$$

For a sesquilinear form this is equivalent to the existence of a $M \in \mathbb{R}$ such that

$$a(\psi) \leq M\|\psi\|_V^2, \quad \text{for all } \psi \in V.$$

d) A sesquilinear form (a, V) is coercive if there exists $\alpha > 0$ such that

$$\operatorname{Re} a(\psi) \geq \alpha\|\psi\|_V^2, \quad \text{for all } \psi \in V.$$

e) A sesquilinear form (a, V) is accretive if

$$\operatorname{Re} a(\psi) \geq 0, \quad \text{for all } \psi \in V.$$

f) Fix a second Hilbert space H , $V \hookrightarrow H$. A sesquilinear form (a, V) is H -elliptic if there exist $\alpha > 0$, $\omega \in \mathbb{R}$ such that the form $a_\omega : V \times V \rightarrow \mathbb{C}$ defined by

$$a_\omega(\psi, \psi') := a(\psi, \psi') + \omega(\psi, \psi')_H$$

is coercive.

If we fix a second Hilbert space H such that $V \hookrightarrow H$ densely, we can then canonically associate an operator $(A, D(A))$ on H with the form (a, V) .

Definition 92. Let (a, V) a continuous form. The operator $(A, D(A))$ associated with the form (a, V) is defined on

$$D(A) := \{\psi \in V : \exists f \in H, \text{ such that } a(\psi, \psi') = (f, \psi'), \text{ for all } \psi' \in V\}$$

by

$$A\psi := -f.$$

We are interested in studying evolution equations in Hilbert spaces which have the form of an abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = u_0, & u_0 \in H. \end{cases} \quad (\text{A.1})$$

Assume that the problem (A.1) is well-posed. Then, in analogy with vector-valued ordinary differential equations, we denote the solution $u(t)$ to (A.1) for the initial data u_0 by $e^{tA}u_0$. The family $(e^{tA})_{t \geq 0} \in \mathcal{L}(H)$ is called a *semigroup of operators* (in the following: semigroup). Reader interested in the general theory of semigroups should consult [30] and [5]. There are many possible notions of continuity and differentiability for semigroups. We focus on analytic semigroups and we refer to [4, Chapter 2] for a short introduction in this topic.

Theorem 93. *Assume that the operator $(A, D(A))$ is associated with a sesquilinear form (a, V) . Then, if the form (a, V) is H -elliptic, then $(A, D(A))$ generates an analytic semigroup $(e^{tA})_{t \geq 0}$. In this case we write $(e^{ta})_{t \geq 0} := (e^{tA})_{t \geq 0}$.*

One can characterise properties of $(e^{ta})_{t \geq 0}$ by properties of the form a . As an example, we observe that accretivity and coercivity of the form already imply contractivity estimates for the semigroup.

Theorem 94 (Forms and analytic semigroups). *On an Hilbert space H consider a continuous, H -elliptic sesquilinear form (a, V) with constants α, ω . The following assertions hold.*

- a) *If the form (a, V) is coercive, then there is $\epsilon > 0$ such that $\|e^{ta}\|_{\mathcal{L}(H)} \leq Me^{-\epsilon t}$, for all $t \geq 0$.*
- b) *The estimate $\|e^{ta}\|_{\mathcal{L}(H)} \leq 1$ holds for all $t \geq 0$ if and only if (a, V) is accretive.*

An important tool for the analysis of qualitative properties of semigroups coming from forms is the following result due to El-Maati Ouhabaz, see [52, Thm. 2.1].

Theorem 95 (Ouhabaz's criterion). *The closed convex subset $C \subset H$ is invariant under the action of the contractive semigroup $(e^{ta})_{t \geq 0}$ if and only if for the Hilbert space projection P onto C holds*

$$\psi \in V \implies P\psi \in V, \quad \operatorname{Re} a(P\psi, \psi - P\psi) \geq 0.$$

A consequence of Theorem 95 is the following result.

Corollary 96 (Invariance criterion for closed subspaces). *The closed linear subspace Y is invariant under the action of the semigroup $(e^{ta})_{t \geq 0}$ if and only if*

- a) *The orthogonal projection P_Y of H onto Y satisfies $\psi \in V \implies P_Y\psi \in V$.*
- b) *$a(\psi, \psi') = 0$ for all $\psi \in Y \cap V$, $\psi' \in Y^\perp \cap V$.*

Remark 97. *We will refer to the first condition in Corollary 96 as to the admissibility condition and to the second one as to the orthogonality condition.*

For $\alpha \in [0, 1)$ an interpolation space of order α between V and H is any linear space H^α such that

$$V \hookrightarrow H^\alpha \hookrightarrow H, \quad \text{and} \quad \|\psi\|_{H^\alpha} \leq M \|\psi\|_V^\alpha \|\psi\|_H^{1-\alpha}, \quad \psi \in V.$$

The following result is [49, Lemma 2.1].

Lemma 98. *Let $a : V \times V \rightarrow \mathbb{C}$ a sesquilinear form. Consider two continuous sesquilinear mappings $a_1 : H \times H_\alpha \rightarrow \mathbb{C}$, $a_2 : H_\alpha \times H \rightarrow \mathbb{C}$, where H_α is an interpolation space between H and V of order α . Then a is H -elliptic if and only if $a + a_1 + a_2$ is H -elliptic.*

Cosine operator functions are defined analogously to semigroups of operators. A cosine operator function is a mapping $\text{Cos} : \mathbb{R} \rightarrow \mathcal{L}(H)$ such that

$$\text{Cos}(0) = \text{Id}, \quad 2\text{Cos}(t)\text{Cos}(s) = \text{Cos}(t+s) + \text{Cos}(t-s), \quad t, s \in \mathbb{R}. \quad (\text{A.2})$$

We do not enter the details of the theory of cosine operator functions, for which we refer to [5, Chapter 10]. We only mention that it is possible to define a *generator* $(A, D(A))$ of a cosine operator function. If we define

$$u(t) := \text{Cos}(t)u_0 + \int_0^t \text{Cos}(s)w_0 ds,$$

then u is a weak solution of the second-order abstract Cauchy problem

$$\begin{cases} \ddot{u}(t) &= Au(t), & t \geq 0, \\ u(0) &= u_0, & u_0 \in H, \\ \dot{u}(0) &= w_0, & w_0 \in H. \end{cases}$$

Conversely, if $(A, D(A))$ is a closed operator such that the abstract Cauchy problem is weakly well-posed, then $(A, D(A))$ is the generator of a cosine function. In the case of an operator $(A, D(A))$ associated with a continuous, elliptic, sesquilinear form (a, V) the following result holds, see [36, p. 212], or [24].

Proposition 99. *Assume that*

$$|\text{Im } a(u)| \leq M\|u\|_H\|u\|_V.$$

Then A generates a cosine operator function.

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Zusammenfassung in deutscher Sprache

Wir beginnen mit der Beobachtung, dass eine auf der direkten Summe von Hilberträumen

$$V := \bigoplus_{i \in I} V_i$$

definierte Sesquilinearform (a, V) eine Matrixdarstellung zulässt, die durch

$$a_{ij}(\psi, \psi') := a(\mathbb{1}_j \otimes \psi, \mathbb{1}_i \otimes \psi')$$

hergestellt wird.

Anhand dieser Überlegung ist es möglich, eine linear algebraische Theorie für auf direkten Summen von Hilberträumen definierte Sesquilinearformen entwickeln, um Stetigkeit, Elliptizität und ähnliche Eigenschaften zu charakterisieren, oder um wenigstens hinreichende Bedingungen beweisen zu können.

Wenn $V_i = V_j$ für alle $i, j \in I$, kann man diese linear algebraische Theorie verwenden, um *Symmetrieeigenschaften* der Halbgruppe $(e^{ta})_{t \geq 0}$ assoziiert mit der Sesquilinearform (a, V) in einem Hilbertraum H zu untersuchen, die sich als Invarianz gewisser Teilräume von H formulieren lassen.

Der Schlüssel dazu ist die Tatsache, dass das Kriterium von Ouhabaz für die Invarianz abgeschlossener, konvexer Mengen für Teilräume eine besonders einfache Form annimmt. Um das zu sehen, bezeichne man P_Y die Hilbertraum Projektion von H auf den abgeschlossenen Unterraum Y . Somit, gilt genau dann $e^{ta}\mathcal{Y} \subset \mathcal{Y}$ für alle $t \geq 0$ wenn

$$P_Y V \subset V \tag{A.3}$$

und

$$a(\psi, \psi') = 0, \quad \psi \in \mathcal{Y}, \psi' \in \mathcal{Y}^\perp. \tag{A.4}$$

Es zeigt sich, dass für Teilräume, die mit Symmetrieeigenschaften assoziiert sind, die erste Bedingung immer erfüllt ist, wenn der Formbereich V eine

direkte Summe ist. Die Invarianz von Y ist dann äquivalent zur zweiten Bedingung, und diese kann durch Eigenschaften der einzelnen Abbildungen a_{ij} charakterisiert werden.

Wenn aber Kopplungsterme in der Definition von V vorhanden sind, das heisst, wenn $V \subset \bigoplus_{i \in I} V_i$ und V kein Ideal ist, wird es aufwändiger eine linear algebraische Theorie zu entwickeln. Eine Lösung hierfür ist Spezialklassen von Formbereichen und Kopplungstermen zu untersuchen.

Ein Beispiel dafür sind Netzwerkgleichungen. In diesem Fall, der Formbereich kann als

$$V := \left\{ \psi \in \bigoplus_{i \in I} H^1(0, 1) : \psi(0) \oplus \psi(1) \in Y \right\}$$

geschrieben werden, wobei Y ein geeigneter Unterraum von $\ell^2(I) \oplus \ell^2(I)$ ist.

Auch in diesem Fall ist es möglich die Symmetrieeigenschaften der assoziierten Halbgruppe $(e^{ta})_{t \geq 0}$ systematisch zu studieren.

Für die Wärmeleitungsgleichung, insbesondere, ist die Orthogonalität (A.4) automatisch erfüllt, so dass die Invarianz äquivalent zur Zulässigkeit (A.4) ist.

Die Herausforderung ist Zulässigkeit graphentheoretisch zu charakterisieren, und, möglicherweise, einige Aussagen auf allgemeinere Situationen zu verallgemeinern.

Wir beginnen mit der Beobachtung, dass eine Sesquilinearform (a, V) , die auf einem Hilbertraum $V := \bigoplus_{i \in I} V_i$ definiert ist, als eine Matrix $(a_{ij})_{i, j \in I}$ von Sesquilinearabbildungen betrachtet werden kann. Von dieser Darstellung kann man hinreichende Bedingungen für unterschiedliche Eigenschaften von (a, V) angeben, die linear algebraischer oder elementar funktionalanalytischer Art sind. Diese Überlegung wurde im Abschnitt 1.1 durchgeführt.

Solche Argumente reichen oft nicht aus, um einfache Eigenschaften sowie Stetigkeit und Koerzivität unendlicher Formmatrizen zu charakterisieren. Im Abschnitt 1.2 betrachten wir ein unendliches, stark gekoppeltes System und charakterisieren Stetigkeit und Koerzivität.

Im Abschnitt 1.3 wenden wir uns dem Thema der Koerzivität im allgemeinen Fall zu und zeigen, dass es ausreichend ist, die endlichen Untermatrizen von a zu untersuchen. Eine Charakterisierung der Koerzivität zweidimensionaler Formmatrizen durch Eigenschaften der einzelnen Abbildungen wird bewiesen.

Im Abschnitt 1.4 beginnen wir die Untersuchung von Evolutionsgleichungen und versuchen den Definitionsbereich des Operators zu identifizieren. Wir setzen unsere Untersuchungen im Abschnitt 1.5 fort, in dem wir Wohlgestelltheit- und Kontraktivitätseigenschaften von $((e^{ta})_{t \geq 0})$ untersuchen. Beispielsweise,

beweisen wir in Theorem 27, dass die Halbgruppe $(e^{ta})_{t \geq 0}$ genau dann positiv ist, wenn die Halbgruppen, die von der Sesquilinearformen auf der Hauptdiagonal erzeugt werden, positiv sind, und alle Abbildungen ausserhalb der Hauptdiagonale negativ.

Wir untersuchen im Abschnitt 1.6 Symmetrieeigenschaften und Theorem 34 charakterisiert vollständig diese Eigenschaften.

Die Abschnitte 1.7 und 1.8 enthalten zwei Anwendungsbeispiele. Im ersten betrachten wir eine stark gedämpfte Wellengleichung und im zweiten eine Wärmeleitungsgleichung mit dynamischen Randbedingungen. In beiden Fällen erhalten wir Wohlgestelltheit für eine große Auswahl an Parametern dank Satz 25. Darauf folgend untersuchen wir qualitative Eigenschaften der Gleichungen.

Schliesslich, führen wir im Abschnitt 1.9 das Thema der nicht diagonalen Formbereiche ein. Netzwerkgleichungen sind wahrscheinlich das wichtigste Beispiel von nicht diagonalen Formbereichen, und die werden im Kapitel 2 untersucht.

Die Ergebnisse des Kapitels 1 sind im Abschnitt 1.10 diskutiert. Insbesondere, wir zeigen die Vorteile eines Matrixformalismus für Sesquilinearformen gegenüber einem Formalismus für Operatoren. Im gleichen Abschnitt beschreiben wir die Geschichte der verschiedenen Fragestellungen des Kapitels.

Wir beginnen in Kapitel 2 mit einer informellen Erklärung der Idee, die hinter der Benutzung von Sesquilinearformen für den Laplace Operator auf Netzwerken steht. Der Abschnitt 2.1 stellt eine komprimierte Erläuterung der partiellen Integration in Netzwerken dar.

Da wir an unendliche Netzwerke im L^2 -Fall interessiert sind, zeigen wir im Abschnitt 2.2, dass alle Definitionen, die für endliche Netzwerke üblich sind, auch für unendliche Netzwerke formuliert werden können. Insbesondere untersuchen wir operatorentheoretische Eigenschaften der Inzidenzmatrizen. Nach diesen Untersuchungen definieren wir in (2.4)-(2.6) den Formbereich und die Wirkung der Form (a, V) .

Im Abschnitt 2.4 charakterisieren wir Symmetrieeigenschaften für Diffusionsysteme in Netzwerken. Dieser Abschnitt ist die Fortsetzung des Abschnittes 1.2. In Theorem 66 charakterisieren wir die Zulässigkeit von Projektionen assoziiert mit Symmetrieeigenschaften vollständig.

Die beiden Abschnitte 2.5 und 2.6 sind eine Anwendung dieser Resultate. Im Ersten wenden wir unsere Aufmerksamkeit der Irreduzibilität der Halbgruppe $(e^{ta})_{t \geq 0}$ im Fall unendlicher Netzwerke zu, und charakterisieren die Netzwerke, so dass die Halbgruppe $(e^{ta})_{t \geq 0}$ irreduzibel ist. Im Zweiten untersuchen wir Symmetrieeigenschaften einiger bestimmter Klassen von Netzwerken.

Im Abschnitt 2.7 verlassen wir die Netzwerke und zeigen, dass alle Systeme von Diffusionsgleichungen auf $H^1(0, 1)$ als Diffusionsysteme in Netzwerken interpretiert werden können, wenn die Randbedingungen geeignete Eigenschaften erfüllen.

Der Begriff der Symmetrie wird in seiner physikalischer Bedeutung im Abschnitt 2.8 erläutert, Wir unterscheiden zwischen *Raum-* und *Eichsymmetrien* und wir zeigen, dass die Symmetrien, die wir untersuchen, Eichsymmetrien in der ursprünglichen physikalischen Bedeutung sind. Abschließend diskutieren wir die Geschichte der unterschiedliche Fragestellungen des Kapitels.

Erklärung:

Die vorliegende Arbeit habe ich selbständig angefertigt und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommene Stellen als solche erkenntlich gemacht.

(Stefano Cardanobile)

Lebenslauf

Stefano Cardanobile

- 21.09.1980 geboren in Bari (Italien)
- 1987 - 1991 Scuola elementare in Bari
- 1991 - 1994 Scuola media in Bari
- 1994 - 1998 Liceo classico in Bari
- 1998 - 1999 Studium der Maschinenbau
an dem Politecnico di Bari
- 1999 - 2001 Studium der Mathematik
an der Università di Bari
- 2001 - 2002 Studienaufenthalt
an der Universität Tübingen
- 2002 - 2005 Studium der Mathematik
an der Universität Tübingen
- 2005 Diplom in Mathematik
- 2005-2008 Stipendiat im Promtionskolleg
Mathematische Analyse
von Evolution, Information und Komplexität
an der Universität Ulm.

Publikationsliste

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2. *Well-posedness and symmetries of strongly coupled network equations*, mit Delio Mugnolo und Robin Nittka. Erschienen in “Journal of Physics A: Mathematical and Theoretical”, 41(5):055102 (28pp), 2008.
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4. *Symmetries in Quantum Graphs*, mit Jens Bolte, Delio Mugnolo und Robin Nittka. Im Druck in “Mathematical Analysis of Evolution, Information and Complexity”, editors Wolfgang Arendt und Wolfgang Schleich, Wiley, Berlin.
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