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A Parallelizable Decomposition Approach for Constrained Optimal Control Problems

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Abstract—This paper describes a decomposition approach for the solution of constrained optimal control problems (OCPs) that is implementable in a parallel manner. A transformation method is applied to the constrained OCP by means of saturation functions resulting in a new extended problem formulation with additional subdynamics and equality constraints. The structure of the reformulated OCP is then exploited to derive a decomposition method in order to split the entire OCP into smaller subproblems which can be solved independently. Moreover, an algorithm is presented to solve the decomposed problem which is based on a hierarchical optimization scheme. An example system is used to demonstrate the performance of the presented approach.

I. INTRODUCTION

Many control problems in science and engineering can be formulated as optimal control problems (OCP) such as load changes in process control or point-to-point motion of industrial robots in a time-optimal or energy-optimal way while accounting for physical or security constraints. Moreover, the advancement of computational power and the development of efficient techniques and algorithms to solve optimal control problems make a real-time solution possible, as the field of model predictive control (MPC) demonstrates, see e.g. [1], [2], [3].

A common approach to solve constrained optimal control problems is to discretize the OCP, for instance by means of a multiple shooting scheme [4] or full discretization [5]. In this way, the problem is reduced to a nonlinear programming (NLP) problem which can be solved by using sequential quadratic programming (SQP) methods or interior point (IP) methods [6].

A more classical way to solve an OCP is to use the calculus of variations, see i.e. [7]. This method is based on the necessary optimality conditions forming a two-point boundary value problem (BVP). The main drawback of this method is, however, that state constraints are in general difficult to incorporate due to the occurrence of discontinuities between constrained and unconstrained arcs [7].

From the numerical point of view, an OCP can be solved by applying distributed optimal control methods to reduce the high complexity as it is preferably done for large-scale systems. Various approaches were developed to deal with this problem from a hierarchical perspective [8], [9], [10] or are based on an Augmented Lagrangian approach [11]. The basic idea is to decompose the problem into subproblems and to manipulate the subproblems in such a way that the objective of the overall problem is met.

This paper presents an approach to solve constrained OCPs by means of decomposing the problem. The considered

constrained optimal control problem is firstly reformulated via the transformation method developed in [12]. The new structure of the problem can then be used to derive a decomposition method to split the overall problem into smaller subproblems, that are suitable for a parallel implementation. To this end, a parallelizable algorithm based on a hierarchical optimization scheme with two layers is presented.

Finally, the performance of the distributed method is demonstrated for a three-degrees-of-freedom helicopter with state and input constraints.

II. GENERAL PROBLEM FORMULATION

In this paper, the following inequality-constrained optimal control problem is considered (see also [12]):

$$\min_{u(\cdot)} J(u) = V(x(T)) + \int_0^T L(x, u) dt \quad (1a)$$

$$\text{s.t.} \quad \dot{x} = f(x, u), \quad x(0) = x_0 \quad (1b)$$

$$c_i(x) \in [c_i^-, c_i^+], \quad i = 1, \dots, p \quad (1c)$$

$$u_j \in [u_j^-, u_j^+], \quad j = 1, \dots, q. \quad (1d)$$

The cost functional (1a) to be minimized consists of a sufficiently smooth Mayer term $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and Lagrange term $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$. The end time T may be fixed or unspecified. Moreover, (1b) describes the nonlinear multiple-input system dynamics with states $x \in \mathbb{R}^n$ and controls $u \in \mathbb{R}^m$. It is assumed, that the vector field $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is also sufficiently smooth. State constraints (1c) are considered as constraint functions $c_i(x)$ with constraint order [7] or relative degree [13] r_i , which is defined as follows

$$\begin{aligned} \frac{\partial}{\partial u} c_i^{(k)}(x) &= 0, \quad k = 0, \dots, r_i - 1, \quad \frac{\partial}{\partial u} c_i^{(r_i)}(x, u) \neq 0 \\ c_i^{(k)}(x) &:= L_f^k c_i(x), \quad c_i^{(r_i)}(x, u) := L_f^{r_i} c_i(x) \end{aligned} \quad (2)$$

with the Lie derivative L_f along the vector field f . The order r_i reveals how many times $c_i(x)$ has to be differentiated until at least one component of the control vector u appears. In addition, input constraints (1d) are also taken into account. Note that no terminal conditions on the state x are considered. However, the approach can easily be adapted to account for these conditions.

III. TRANSFORMATION OF AN OPTIMAL CONTROL PROBLEM AND DISTRIBUTED ALGORITHM

In this section, a method is presented to decompose the constrained OCP (1) suitable for a parallel solution. The approach uses a transformation technique presented in [12], which is briefly discussed in Section III-A. The transformation technique is based on the introduction of saturation functions and a system extension and allows to reformulate (1) in an equality-constrained OCP. The new formulation of OCP (1) provides a basis for a decomposition method

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to split the problem into subproblems which can be solved independently. Following the ideas of dual decomposition and the Augmented Lagrangian method [14], an algorithm is presented at the end of this section to solve the decomposed problem in a hierarchical manner [8], [9].

A. Saturation function approach

1) *Incorporation of state constraints:* The main idea of the approach developed in [12] is to take the state constraints (1c) into account by successively introducing saturation functions and differentiating up to the corresponding constraint order r_i . To this end, each constraint function $c_i(x)$ is replaced as follows

$$c_i(x) = \psi_i(\xi_{i,1}) =: h_{i,0}(\xi_{i,1}), \quad i = 1, \dots, p \quad (3)$$

where $\xi_{i,1} \in \mathbb{R}$ denotes a new unconstrained variable. Note that in the following the first subscript i in each new function or variable relates to the i -th constraint $c_i(x)$. Moreover, the saturation function $\psi_i : \mathbb{R} \rightarrow (c_i^-, c_i^+)$ is assumed to be strictly monotonically increasing (i.e. $\partial\psi_i/\partial\xi_{i,1} > 0$) and the corresponding saturation limits are set to the constraint values $c_i^\pm$.

Equation (3) is now differentiated up to the corresponding constraint order r_i and new variables $\xi_{i,k}$ are successively introduced in the following way¹:

$$\begin{aligned} c_i^{(1)}(x) &= \psi_i' \xi_{i,2} =: h_{i,1}(\xi_{i,1}, \xi_{i,2}) \\ c_i^{(2)}(x) &= \psi_i'' \xi_{i,2}^2 + \psi_i' \xi_{i,3} \\ &= \gamma_{i,2}(\xi_{i,1}, \xi_{i,2}) + \psi_i' \xi_{i,3} =: h_{i,2}(\xi_{i,1}, \dots, \xi_{i,3}) \\ c_i^{(3)}(x) &= \psi_i^{(3)} \xi_{i,2}^3 + 3\psi_i'' \xi_{i,2} \xi_{i,3} + \psi_i' \xi_{i,4} \\ &= \gamma_{i,3}(\xi_{i,1}, \dots, \xi_{i,3}) + \psi_i' \xi_{i,4} =: h_{i,3}(\xi_{i,1}, \dots, \xi_{i,4}) \\ &\vdots \\ c_i^{(r_i-1)}(x) &= \gamma_{i,r_i-1}(\xi_{i,1}, \dots, \xi_{i,r_i-1}) + \psi_i' \xi_{i,r_i} =: h_{i,r_i-1}(\xi_i) \end{aligned} \quad (4)$$

where the arising time derivatives $\dot{\xi}_{i,k} = \xi_{i,k+1}$, $k = 1, \dots, r_i - 1$, are systematically replaced by new unconstrained variables $\xi_{i,k+1} \in \mathbb{R}$. The vector $\xi_i = [\xi_{i,1}, \dots, \xi_{i,r_i}]^T$ comprises all new introduced variables with regard to the constraint function $c_i(x)$. Furthermore, differentiating the constraint function $c_i(x)$ reveals a particular structure as it can be observed in (4). A term with a first derivative of the saturation function and a new substituted variable appears in each differentiation level. The remaining terms involving higher derivatives of $\psi_i(\xi_{i,1})$ and nonlinear expressions are summarized in the function $\gamma_{i,k}(\cdot)$.

Finally, a further differentiation of the last equation in (4) leads to

$$c_i^{(r_i)}(x, u) = \gamma_{i,r_i}(\xi_i) + \psi_i'(\xi_{i,1})v_i =: h_{i,r_i}(\xi_i, v_i) \quad (5)$$

where the last derivative $\dot{\xi}_{i,r_i} = v_i$ is substituted by the new input $v_i \in \mathbb{R}$.

In summary, the introduction of a saturation function (3) for each constraint function $c_i(x)$ and the subsequent differentiation results in the algebraic equation (5) in combination with the dynamical subsystem for all $i = 1, \dots, p$

$$\left. \begin{aligned} \dot{\xi}_{i,k} &= \xi_{i,k+1}, \quad k = 1, \dots, r_i - 1, \\ \dot{\xi}_{i,r_i} &= v_i, \end{aligned} \right\} \dot{\xi}_i = f_{\xi_i}(\xi_i, v_i). \quad (6)$$

¹In the remainder of the paper, the function arguments are omitted where it is convenient to maintain readability.

Fig. 1. Basic structure of the extended OCP (9).

In addition, the relations (3) and (4) can be used to recursively determine the necessary initial conditions $\xi_i(0) = \xi_{i,0}$ for the new variables ξ_i according to

$$\xi_{i,0} := \begin{bmatrix} h_{i,0}^{-1}(c_i(x_0)) \\ \vdots \\ h_{i,r_i-1}^{-1}(\xi_{i,1}(0), \dots, \xi_{i,r_i-1}(0), c_i^{(r_i-1)}(x_0)) \end{bmatrix}. \quad (7)$$

2) *Incorporation of input constraints:* The controls u_j are directly replaced by further saturation functions in order to incorporate the input constraints (1d), i.e.

$$u_j = \phi_j(w_j), \quad j = 1, \dots, q \quad (8)$$

where $w_j \in \mathbb{R}$ denote new unconstrained inputs. Moreover, the input saturation functions $\phi_j : \mathbb{R} \rightarrow (u_j^-, u_j^+)$ possess the same properties as the state saturation functions ψ_i .

3) *New optimal control problem:* As a result of the transformation approach, the constraints (1c) and (1d) in OCP (1) can now be replaced by the algebraic equations (5), (8) and the chain of integrators (6). In view of the original OCP (1), the new problem formulation reads as follows:

$$\min_{\tilde{u}(\cdot)} J_\varepsilon(\tilde{u}) = J(u) + \int_0^T \varepsilon (\|v\|^2 + \|w\|^2) dt \quad (9a)$$

$$\text{s.t.} \quad \dot{x} = f(x, u), \quad x(0) = x_0 \quad (9b)$$

$$\dot{\xi}_i = f_{\xi_i}(\xi_i, v_i), \quad \xi_i(0) = \xi_{i,0}, \quad i = 1, \dots, p \quad (9c)$$

$$0 = c_i^{(r_i)}(x, u) - h_{i,r_i}(\xi_i, v_i), \quad i = 1, \dots, p \quad (9d)$$

$$0 = u_j - \phi_j(w_j), \quad j = 1, \dots, q \quad (9e)$$

where $\tilde{u} = [u^T, v^T, w^T]^T$ with $v = [v_1, \dots, v_p]^T$ and $w = [w_1, \dots, w_q]^T$ denotes the extended input. Note that the new introduced inputs v_i and w_j relate to one of the state constraints (1c) and the input constraints (1d) and hence to one of the separate algebraic equations (5) and (8) and the subdynamics (6), respectively, and can be seen as additional optimization variables. Moreover, a regularization term $\varepsilon (\|v\|^2 + \|w\|^2)$ with penalty parameter $\varepsilon > 0$ is added to the original cost (1a) in order to counteract the occurrence of singular arcs. In practice, OCP (9) is either solved with a decreasing sequence of values for $\varepsilon \rightarrow 0$ or set sufficiently small to accept suboptimal solutions. Furthermore, it can be shown under some certain conditions that the optimal solution of the extended problem (9) converges for $\varepsilon \rightarrow 0$ to the optimal solution of the original OCP (1). More details on these points and the singular arc behavior can be found in [12].

B. Decomposition and hierarchical algorithm

The extended OCP (9) has a favourable structure for a decomposition into smaller subproblems. The cost (9a) and the corresponding system dynamics (9b) and (9c) can be split and are only coupled due to the algebraic equations (9d) and (9e), as it is illustrated in Figure 1, where the components of the norm $\|v\|^2 = \sum_{i=1}^p v_i^2$ are used (respectively for $\|w\|^2$).

To this end, the structure of OCP (9) is exploited in the following to solve the overall problem in a hierarchical manner. Hence, the Augmented Lagrangian approach from static optimization [14] is adapted to the optimal control case as it was similarly done in [11]. The algebraic couplings (9d) and (9e) are adjoined to the cost (9a) via the multipliers $\mu \in \mathbb{R}^p$ and $\nu \in \mathbb{R}^q$ and a quadratic penalty term, i.e.

$$\begin{aligned} \tilde{J}(\tilde{u}) = & J_\varepsilon(\tilde{u}) + \underbrace{\int_0^T \mu^\top [c^{(r)}(x, u) - h_r(\xi, v)] + \nu^\top [u - \Phi(w)] dt}_{=: P(\tilde{u})} \\ & + \int_0^T \rho \|c^{(r)}(x, u) - h_r(\xi, v)\|^2 + \rho \|u - \Phi(w)\|^2 dt \end{aligned} \quad (10)$$

where the vectors $c^{(r)}(x, u) = [c_1^{(r_1)}(x, u), \dots, c_p^{(r_p)}(x, u)]^\top$, $h_r(\xi, v) = [h_{1,r_1}(\xi_1, v_1), \dots, h_{p,r_p}(\xi_p, v_p)]^\top$, $\Phi(w) = [\phi_1(w_1), \dots, \phi_q(w_q)]^\top$ and $\xi = [\xi_1^\top, \dots, \xi_p^\top]^\top$ comprise all constraint and saturation functions and all new variables in order to obtain the compact cost formulation (10). Note that quadratic penalty terms with the additional parameter $\rho > 0$ are added to the cost according to the Augmented Lagrangian approach [14], [15]. It can be shown for the static optimization case that convergence properties can be improved by including the quadratic penalty [14].

The idea is now to solve a corresponding max-min-problem for (10) arising from the dual problem (see [14]) subject to the dynamics (9b) and (9c), i.e.

$$\max_{\mu(\cdot), \nu(\cdot)} \min_{\tilde{u}(\cdot)} \left\{ \begin{array}{l} \tilde{J}(\tilde{u}) \\ \text{s.t. (9b) \& (9c)} \end{array} \right\}. \quad (11)$$

In particular, the quadratic penalty terms $\|c^{(r)}(x, u) - h_r(\xi, v)\|^2$ and $\|u - \Phi(w)\|^2$ induce a nonlinear coupling of the subsystems (9b) and (9c) and $\tilde{u}$.

An obvious procedure to circumvent this difficulty is to use a quasi-linearization approach [16] based on previous iterates which allows separating the cost (10) into decoupled parts that can be minimized independently and hence in parallel.

Another method to split the cost (10) is to assign to each subproblem a corresponding penalty term with appropriate trajectories $(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{\xi})$, i.e. (10) is modified according to

$$\begin{aligned} \tilde{J}(\tilde{u}) = & J_\varepsilon(\tilde{u}) + P(\tilde{u}) \\ & + \int_0^T \frac{\rho}{2} \|c^{(r)}(x, u) - h_r(\bar{\xi}, \bar{v})\|^2 + \frac{\rho}{2} \|u - \Phi(\bar{w})\|^2 dt \\ & + \int_0^T \frac{\rho}{2} \|c^{(r)}(\bar{x}, \bar{u}) - h_r(\xi, v)\|^2 + \frac{\rho}{2} \|\bar{u} - \Phi(w)\|^2 dt \end{aligned} \quad (12)$$

with the consistency constraints

$$0 = x - \bar{x}, \quad 0 = \xi - \bar{\xi}, \quad (13a)$$

$$0 = u - \bar{u}, \quad 0 = v - \bar{v}, \quad 0 = w - \bar{w}. \quad (13b)$$

The algorithm presented in the next section performs a Jacobi-type iteration such that the variables with the bar notation are replaced by previously computed iterates. Hence, the cost (12) is separated into the following parts

$$J^u(u; \mu, \nu, \rho) = J(u) + \int_0^T \mu^\top c^{(r)}(x, u) + \nu^\top u dt \quad (14a)$$

$$+ \int_0^T \frac{\rho}{2} \|c^{(r)}(x, u) - h_r(\bar{\xi}, \bar{v})\|^2 + \frac{\rho}{2} \|u - \Phi(\bar{w})\|^2 dt$$

$$J^{v_i}(v_i; \mu_i, \rho, \varepsilon) = \int_0^T \varepsilon v_i^2 - \mu_i h_{i,r_i}(\xi_i, v_i) dt \quad (14b)$$

$$+ \int_0^T \frac{\rho}{2} (c_i(\bar{x}, \bar{u}) - h_{i,r_i}(\xi_i, v_i))^2 dt, \quad i = 1, \dots, p$$

$$J^{w_j}(w_j; \nu_j, \rho, \varepsilon) = \int_0^T \varepsilon w_j^2 - \nu_j \phi_j(w_j) dt \quad (14c)$$

$$+ \int_0^T \frac{\rho}{2} (\bar{u}_j - \phi_j(w_j))^2 dt, \quad j = 1, \dots, q$$

where the related components μ_i and ν_j of the multiplier vectors μ and ν are used, respectively. The cost function (10) follows directly from summing up all separated functions (14) in combination with the consistency conditions (13). As a result, the minimization in (11) with respect to the decomposed cost functionals (14), the corresponding dynamics (9b), (9c), and the consistency constraints (13) can now be solved in a distributed manner. Note that the superscript notation of the cost functions (14) denotes the different subproblems and also indicates the associated inputs.

An algorithm to solve the max-min-problem (11) with the decomposed cost functions (14) is summarized in Table I. Thereby, the superscript notation denotes the current iteration index of the variables. The algorithm starts with a suitable choice of the penalty parameters ε and ρ and with initial trajectories for the controls (u^0, v^0, w^0) as well as for the multipliers (μ^0, ν^0) . Subsequently, the derived subproblems are minimized independently in Step II.1. Moreover, this minimization step can also be performed in parallel due to the separation of the problems and it can be repeated for a fixed number of iterations $l_{\max}$ before the multiplier update is performed. Note that this step contains two iteration indices. The first one denotes the solution of the entire max-min-problem (index k) and the second one describes the iteration of the minimization of the subproblems only (index l). The maximization step in (11) is performed via the steepest ascent approach (16), where the penalty parameter ρ is applied as step size (see [14]). If the algebraic equations (5) and (8) are not fulfilled accordingly, the max-min-problem is solved again with the updated multipliers until the iteration limit $k_{\max}$ is reached. The entire optimization procedure can then be restarted with a decreased penalty parameter ε and the last solution for reinitialization until ε is negligibly small. Alternatively, a suboptimal solution for a fixed parameter ε is obtained. Furthermore, if the algorithm is restarted with a new ε , the parameter ρ regarding the quadratic penalty terms can also be modified in an appropriate way, e.g. by increasing the value as proposed in [14] for the static optimization case.

Convergence results regarding a Jacobi-type iteration scheme, as the minimization steps (15), are reported in [17] for static optimization problems. Moreover, convergence properties of the maximization step in a distributed optimiza-

TABLE I
HIERARCHICAL ALGORITHM

I) **Initialization:**

I.1 Choose initial parameters $\varepsilon, \rho > 0$ and initial trajectories for the controls (u^0, v^0, w^0) and for the multipliers (μ^0, ν^0) .

II) **Max-min-problem for $k = 0, \dots, k_{\max}$:**

II.1 Minimize subproblems for $l = 0, \dots, l_{\max}$:

II.1-1 Set $[x^{k|0}, \xi^{k|0}, u^{k|0}, v^{k|0}, w^{k|0}]^T = [x^k, \xi^k, u^k, v^k, w^k]^T$

II.1-2 Update controls:

$$u^{k|l+1} = \arg \min \left\{ J^u(u; \mu^k, \nu^k, \rho) \right. \quad \text{s.t. (9b)} \quad \left. \right\} \quad (15a)$$

$$v_i^{k|l+1} = \arg \min \left\{ J^{v_i}(v_i; \mu_i^k, \rho, \varepsilon) \right\}, \quad i = 1, \dots, p \quad (15b)$$

$$w_j^{k|l+1} = \arg \min \left\{ J^{w_j}(w_j; \nu_j^k, \rho, \varepsilon) \right\}, \quad j = 1, \dots, q \quad (15c)$$

with previous iterates $\bar{x} = x^{k|l}$, $\bar{u} = u^{k|l}$, $\bar{\xi} = \xi^{k|l}$, $\bar{v} = v^{k|l}$, $\bar{w} = w^{k|l}$ (cf. (12) - (14))

II.1-3 Iteration stopping criterion:

If $l < l_{\max}$ then set $l \leftarrow l+1$ and go back to II.1-2 or otherwise set $[u^{k+1}, v^{k+1}, w^{k+1}]^T = [u^{k|l+1}, v^{k|l+1}, w^{k|l+1}]^T$

II.2 Multiplier update (steepest ascent, cf. maximization in (11)):

$$\mu^{k+1} = \mu^k + \frac{\rho}{2} [c^{(r)}(x^{k+1}, u^{k+1}) - h_r(\xi^{k+1}, v^{k+1})] \quad (16a)$$

$$\nu^{k+1} = \nu^k + \frac{\rho}{2} [u^{k+1} - \Phi(w^{k+1})] \quad (16b)$$

II.3 Compute residual $\Delta = \left\| \begin{bmatrix} c^{(r)}(x^{k+1}, u^{k+1}) - h_r(\xi^{k+1}, v^{k+1}) \\ u^{k+1} - \Phi(w^{k+1}) \end{bmatrix} \right\|$

II.4 Iteration stopping criterion:

If $\Delta \geq \varepsilon_c$ then set $k \leftarrow k+1$ and go back to II.1

III) **Penalty parameter update:**

III.1 Optimization stopping criterion:

If $\varepsilon \geq \varepsilon_g$ then set $\varepsilon \leftarrow \delta \varepsilon$ with $\delta \in (0, 1)$, modify ρ and use the last solutions for reinitialization and go back to II

tion algorithm for a similar decomposition scheme and linear systems can be found in [18]. The extension of these results to general optimal control problems and nonlinear systems is under current research.

Finally, the principal procedure of the max-min-problem is illustrated in Figure 2 and consists of two layers in a hierarchical optimization structure (see also [8], [9], [10]).

IV. SIMULATION EXAMPLE:
THREE-DEGREES-OF-FREEDOM HELICOPTER

The presented optimal control method in Section III is demonstrated for a three-degrees-of-freedom (3DOF) helicopter [19]. In a first step, the system dynamics and the problem formulation are stated before the decomposition scheme and simulation results are presented.

Fig. 2. Structure of the hierarchical algorithm to solve OCP (9).

A. System dynamics and problem formulation

Figure 3 shows the schematics of the experimental setup of the 3DOF helicopter from [19]. The helicopter consists of three hinge-mounted components which can rotate about the traveling angle q_1 , the elevation angle q_2 and the pitch angle q_3 . Moreover, two propellers are attached at the end of the third component to generate the lifting forces F_1 and F_2 . A nonlinear model of the helicopter is given by [19]

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= b_1 \cos(x_3) \sin(x_5) u_1 \\ \dot{x}_3 &= x_4, & \dot{x}_4 &= a_1 \sin(x_3) + a_2 \cos(x_3) + b_2 \cos(x_5) u_1 \\ \dot{x}_5 &= x_6, & \dot{x}_6 &= a_3 \cos(x_3) \sin(x_5) + b_3 u_2 \end{aligned} \quad (17)$$

with the states $x = [q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3]^T$ and the sum and the difference of the lifting forces F_1 and F_2 as controls $u = [u_1, u_2]^T = [F_1 + F_2, F_1 - F_2]^T$. Furthermore, the values of the model parameters are

$$\begin{aligned} a_1 &= -1.1713 \frac{\text{rad}}{\text{s}^2}, & b_1 &= -0.6354 \frac{\text{rad}}{\text{kg m}}, \\ a_2 &= +0.3946 \frac{\text{rad}}{\text{s}^2}, & b_2 &= -0.6523 \frac{\text{rad}}{\text{kg m}}, \\ a_3 &= -0.5326 \frac{\text{rad}}{\text{s}^2}, & b_3 &= +4.6276 \frac{\text{rad}}{\text{kg m}}. \end{aligned} \quad (18)$$

The optimal control problem to be solved is formulated as

$$\begin{aligned} \min_{u(\cdot)} \quad & J(u) = \Delta x^T(T) P \Delta x(T) \\ & + \int_0^T \Delta x^T Q \Delta x + \Delta u^T R \Delta u \, dt \end{aligned} \quad (19a)$$

$$\text{s.t.} \quad \dot{x} = f(x, u), \quad x(0) = x_0 \quad (19b)$$

$$c_1(x) = x_3 \in [c_1^-, c_1^+], \quad c_2(x) = x_5 \in [c_2^-, c_2^+] \quad (19c)$$

$$u_1 \in [u_1^-, u_1^+], \quad u_2 \in [u_2^-, u_2^+] \quad (19d)$$

where the matrices $P, Q \in \mathbb{R}^{6 \times 6}$ are positive (semi)-definite, the matrix $R \in \mathbb{R}^{2 \times 2}$ is positive definite and the system function f describes the dynamics (17) of the helicopter. Moreover, the constraint order of $c_1(x)$ and $c_2(x)$ is $r_1 = 2$ and $r_2 = 2$, respectively, and the variables

$$\Delta x = x - x_{\text{SP}}, \quad \Delta u = u - u_{\text{SP}} \quad (20)$$

denote the distance to the desired setpoint $(x_{\text{SP}}, u_{\text{SP}})$.

B. Problem reformulation by means of saturation functions

As described in Section III, the original OCP (19) is reformulated in the extended form (9). To this end, the constrained angles $c_1(x) = x_3$ and $c_2(x) = x_5$ are replaced by the two saturation functions

$$c_1(x) = \psi_1(\xi_{1,1}), \quad c_2(x) = \psi_2(\xi_{2,1}) \quad (21)$$

Fig. 3. Schematics of a 3DOF helicopter [19].

where the limits are appropriately set to the bounds $c_1^\pm$ and $c_2^\pm$. Differentiation of (21) according to (4) and (5) leads to

$$c_1^{(1)}(x) = \psi_1' \xi_{1,2} = h_{1,1}(\xi_{1,1}, \xi_{1,2}) \quad (22a)$$

$$c_1^{(2)}(x, u) = \psi_1'' \xi_{1,2}^2 + \psi_1' v_1 = h_{1,2}(\xi_{1,1}, \xi_{1,2}, v_1) \quad (22b)$$

$$c_2^{(1)}(x) = \psi_2' \xi_{2,2} = h_{2,1}(\xi_{2,1}, \xi_{2,2}) \quad (22c)$$

$$c_2^{(2)}(x, u) = \psi_2'' \xi_{2,2}^2 + \psi_2' v_2 = h_{2,2}(\xi_{2,1}, \xi_{2,2}, v_2) \quad (22d)$$

where the new inputs v_1 and v_2 are introduced in (22b), (22d) due to the constraints order $r_1 = r_2 = 2$. The corresponding chains of integrators in terms of (6) follow to

$$\dot{\xi}_{1,1} = \xi_{1,2}, \quad \dot{\xi}_{1,2} = v_1 \quad (23a)$$

$$\dot{\xi}_{2,1} = \xi_{2,2}, \quad \dot{\xi}_{2,2} = v_2 \quad (23b)$$

where (23a) and (23b) correspond to the constraints $c_1(x)$ and $c_2(x)$, respectively. Moreover, the second time derivatives of $c_1(x)$ and $c_2(x)$ follow to (cf. (2))

$$c_1^{(2)}(x, u) = a_1 \sin(x_3) + a_2 \cos(x_3) + b_2 \cos(x_5) u_1 \quad (24a)$$

$$c_2^{(2)}(x, u) = a_3 \cos(x_3) \sin(x_5) + b_3 u_2 \quad (24b)$$

and hence the resulting algebraic equations (9d) are

$$0 = c_1^{(2)}(x, u) - h_{1,2}(\xi_{1,1}, \xi_{1,2}, v_1) \quad (25a)$$

$$0 = c_2^{(2)}(x, u) - h_{2,2}(\xi_{2,1}, \xi_{2,2}, v_2). \quad (25b)$$

As discussed in Section III-A.2, additional saturation functions are introduced in order to account for the input constraints, i.e.

$$u_1 = \phi_1(w_1), \quad u_2 = \phi_2(w_2) \quad (26)$$

where the saturation limits of ϕ_1 and ϕ_2 are chosen appropriately. Therefore, the additional algebraic equations regarding the input constraints (19d) are

$$0 = u_1 - \phi_1(w_1), \quad 0 = u_2 - \phi_2(w_2). \quad (27)$$

Before the extended OCP (9) is stated, the subdynamics (23) are comprised in the following way

$$\dot{\xi}_1 = [\xi_{1,2}, v_1]^T =: f_{\xi_1}(\xi_1, v_1) \quad (28a)$$

$$\dot{\xi}_2 = [\xi_{2,2}, v_2]^T =: f_{\xi_2}(\xi_2, v_2) \quad (28b)$$

with the unconstrained states $\xi_1 = [\xi_{1,1}, \xi_{1,2}]^T$ and $\xi_2 = [\xi_{2,1}, \xi_{2,2}]^T$, respectively.

Finally, in view of the subdynamics (28) and the algebraic equations (25) and (27), the new OCP (9) reads

$$\min_{\tilde{u}(\cdot)} J_\varepsilon(\tilde{u}) = J(u) + \int_0^T \varepsilon (v_1^2 + v_2^2 + w_1^2 + w_2^2) dt \quad (29a)$$

$$\text{s.t. } \dot{x} = f(x, u), \quad x(0) = x_0 \quad (29b)$$

$$\dot{\xi}_1 = f_{\xi_1}(\xi_1, v_1), \quad \xi_1(0) = \xi_{1,0} \quad (29c)$$

$$\dot{\xi}_2 = f_{\xi_2}(\xi_2, v_2), \quad \xi_2(0) = \xi_{2,0} \quad (29d)$$

$$0 = c_1^{(2)}(x, u) - h_{1,2}(\xi_{1,1}, \xi_{1,2}, v_1) \quad (29e)$$

$$0 = c_2^{(2)}(x, u) - h_{2,2}(\xi_{2,1}, \xi_{2,2}, v_2) \quad (29f)$$

$$0 = u_1 - \phi_1(w_1) \quad (29g)$$

$$0 = u_2 - \phi_2(w_2) \quad (29h)$$

with $\tilde{u} = [u^T, v_1, v_2, w_1, w_2]^T$. The initial conditions $\xi_1(0) = \xi_{1,0}$ and $\xi_2(0) = \xi_{2,0}$ can be obtained by using the relations (21) and (22) recursively, i.e.

$$\xi_{1,1}(0) = \psi_1^{-1}(c_1(x(0))), \quad \xi_{1,2}(0) = \frac{c_1^{(1)}(x(0))}{\psi_1'(\xi_{1,1}(0))} \quad (30a)$$

$$\xi_{2,1}(0) = \psi_2^{-1}(c_2(x(0))), \quad \xi_{2,2}(0) = \frac{c_2^{(1)}(x(0))}{\psi_2'(\xi_{2,1}(0))}. \quad (30b)$$

The structure of the extended OCP (29) is now exploited to derive decoupled subproblems. Following the ideas from Section III-B, the algebraic equations (29e) - (29h) are adjoined to the cost (29a). Subsequently, the augmented cost is separated into five parts by means of the consistency conditions (13) with the corresponding dynamics (29b) - (29d), which results in five subproblems.

C. Simulation results

The control task for the 3DOF helicopter is to perform a 180 deg turn starting from the initial point

$$x_0 = [180 \text{ deg}, 0 \frac{\text{deg}}{\text{s}}, 0 \text{ deg}, 0 \frac{\text{deg}}{\text{s}}, 0 \text{ deg}, 0 \frac{\text{deg}}{\text{s}}]^T \quad (31)$$

to the desired state

$$x_{SP} = [0 \text{ deg}, 0 \frac{\text{deg}}{\text{s}}, 0 \text{ deg}, 0 \frac{\text{deg}}{\text{s}}, 0 \text{ deg}, 0 \frac{\text{deg}}{\text{s}}]^T \quad (32)$$

within the optimization time $T = 10$ s. The considered state and input constraints are set to the values

$$c_1^\pm = \pm 15 \text{ deg}, \quad c_2^\pm = \pm 35 \text{ deg} \quad (33a)$$

$$u_1^\pm = \pm 2 \text{ N}, \quad u_2^\pm = \pm 2 \text{ N}. \quad (33b)$$

The weighting matrices in the cost functional (19a) are chosen according to

$$P = I_{6 \times 6}, \quad Q = I_{6 \times 6}, \quad R = 0.1 \cdot I_{2 \times 2} \quad (34)$$

where $I_{j \times j} \in \mathbb{R}^{j \times j}$ denotes the $j \times j$ identity matrix.

The resulting subproblems (15) are solved by means of the gradient method [20] which is adapted to consider the passed multipliers μ and ν as well as the penalty parameters ε and ρ . Moreover, the maximum number of gradient iterations is set to 30 and the minimization step II.1 in the algorithm (cf. Table I) is repeated three times, i.e. $l_{\max} = 2$ is used. Furthermore, the subproblems were solved for the sequence $\varepsilon = \{10^0, 10^{-1}, \dots, 10^{-9}\}$ of decreasing penalty parameters. Thereby, the algorithm is started with the value $\varepsilon = 10^0$ and after a maximum iteration number $k_{\max} = 600$ the algorithm is restarted with the computed solution as new initialization and ε is decreased by one order of magnitude until the value $\varepsilon = 10^{-9}$ is reached. The parameter ρ of the quadratic penalty functions is fixed at a value of $\rho = 10$ during the entire max-min process.

Figure 4 shows the numerical results for the 3DOF helicopter and for selected values of the penalty parameter ε . It can be seen that the distributed solution of (19) converges to the central solution of the problem for decreasing values of ε . Moreover, the results show that the given constraints (33) are satisfied during the entire setpoint change. The slight deviation from the central solution can be explained due to the gradient algorithm in combination with the limited number of iterations. In addition, the last plot in Figure 4 illustrates the value of the cost function for the successively reduced penalty parameter ε .

Fig. 4. Simulation results for the 3DOF helicopter.

V. CONCLUSION

This paper described a decomposition approach for constrained optimal control problems and a suitable algorithm based on a transformation technique with saturation functions and a hierarchical optimization approach. The transformation method extends the original formulated OCP and replaces the considered state and input constraints by new subdynamics and algebraic equations. In this regard, the reformulation of the OCP represents an analytical step that can be performed with computer algebra software such as MATHEMATICA. The structure of the new OCP formulation is exploited to derive a decomposition approach resulting in a max-min-problem. The corresponding minimization step can be split in the minimization of independent subproblems which can also be performed in parallel. Moreover, an algorithm is presented to solve the decoupled OCP consisting of two layers in a hierarchical optimization structure.

The performance of the method was demonstrated for a 3DOF helicopter. The simulation results showed that the state and input constraints were satisfied and that the distributed solution converged to the central solution of the OCP.

Further work concerns the application of the presented method to a more sophisticated distributed algorithm as well as the investigation of convergence as mentioned in Section III-B. In addition, further research includes adapting the discussed method in a model predictive control framework.

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