


universität ulm

# Heterogeneity and Incompressibility in the Evolution of Elastic Wires

## Dissertation

zur Erlangung des Doktorgrades (Dr. rer. nat.)
der Fakultät für Mathematik und Wirtschaftswissenschaften der Universität Ulm

vorgelegt von

**Leonie Langer**

geboren in Freising

September 2024

| | |
|---|---|
| **Amtierender Dekan:** | Prof. Dr. Gunter Löffler |
| **Erstgutachterin:** | Prof. Dr. Anna Dall'Acqua |
| **Zweitgutachter:** | Prof. Dr. Oliver Schnürer |
| **Mitbetreut von:** | Dr. Fabian Rupp |
| **Tag der Promotion:** | 13. Dezember 2024 |

# Contents

**Preface** v

Abstract . . . v
Deutsche Zusammenfassung . . . vi
About this thesis . . . vii
Acknowledgments . . . viii

**1. Introduction** 1

1.1. Modeling with elastic wires . . . 2
1.2. The elastic energy . . . 4
1.2.1. The elastic energy of curves in Euclidean space . . . 4
1.2.2. Elasticae in Euclidean space . . . 8
1.2.3. The relevance of the elastic energy . . . 10
1.2.4. The elastic energy of curves in Riemannian manifolds . . . 11
1.2.5. The Willmore energy . . . 15
1.3. (Constrained) gradient flows in Hilbert spaces . . . 18
1.3.1. Gradient flows . . . 19
1.3.2. Constrained gradient flows . . . 20
1.3.3. Key questions . . . 22
1.4. The classical elastic flow and other geometric flows . . . 24
1.4.1. The $L^2$-gradient flow of the elastic energy . . . 24
1.4.2. Results on the classical elastic flow of closed curves . . . 27
1.4.3. Other geometric flows . . . 28
1.5. Elastic flow with modulated stiffness . . . 31
1.6. Area preserving elastic flow . . . 34

**2. Elastic flow with modulated stiffness** 37

2.1. Preliminaries . . . 40
2.1.1. Describing planar curves . . . 40
2.1.2. The Lagrange multipliers . . . 41
2.1.3. Decrease of the energy . . . 42
2.2. The static problem . . . 43
2.2.1. The Euler–Lagrange equations . . . 43
2.2.2. Existence of minimizers and smoothness of critical points . . . 46
2.2.3. Homogeneous elastica . . . 48

2.3. Local well-posedness and regularity . . . . . 56
2.3.1. Transformation to a periodic setting . . . . . 57
2.3.2. Short time existence and continuous dependence . . . . . 59
2.3.3. Auxiliary results for Section 2.3.2 . . . . . 61
2.3.4. Smoothness and uniqueness of solutions . . . . . 69
2.3.5. Proof of local well-posedness . . . . . 74
2.3.6. Appendix to Section 2.3: Function Spaces . . . . . 74
2.4. Global existence . . . . . 78
2.4.1. Controlling the matrix $\Pi^{-1}$ . . . . . 78
2.4.2. Energy estimates . . . . . 79
2.4.3. Proof of global existence . . . . . 85
2.5. Convergence result . . . . . 87
2.5.1. The constrained Łojasiewicz–Simon gradient inequality . . . . . 88
2.5.2. Convergence . . . . . 90
2.6. Qualitative properties of solutions . . . . . 92
2.6.1. Numerical experiments . . . . . 93
2.6.2. The two parts of the energy . . . . . 93
2.6.3. Zeros of the curvature . . . . . 96
2.6.4. Convexity . . . . . 98
2.6.5. Nonnegativity of the density . . . . . 101
2.6.6. Embeddedness . . . . . 103
2.6.7. Symmetry . . . . . 104
2.7. Asymptotic behavior . . . . . 113
2.7.1. Convergence to a homogeneous elastica under growth assumptions on $\beta$ . . . . . 113
2.7.2. Convergence to a homogeneous elastica for special $c_0$ . . . . . 116
2.7.3. Convergence to a homogeneous elastica for large $\mu$ . . . . . 117

**3. Area preserving elastic flow** **121**
3.1. Derivation of the flow equation . . . . . 124
3.1.1. An appropriate Hilbert space setting . . . . . 124
3.1.2. The area preserving elastic flow as a gradient flow . . . . . 129
3.2. Basic properties of the flow . . . . . 130
3.3. Short time existence . . . . . 133
3.3.1. A crucial reparametrization argument . . . . . 133
3.3.2. Parabolic Hölder spaces and the linear problem . . . . . 134
3.3.3. The nonlinear problem . . . . . 135
3.3.4. Proof of short time existence . . . . . 140

3.4. Global existence and subconvergence . . . 141
3.4.1. Geometrical lemmata and interpolation inequalities . . . 141
3.4.2. Bounds on the length . . . 143
3.4.3. Proof of global existence and subconvergence . . . 143
3.5. Convergence . . . 147
3.5.1. Stationary solutions as constrained critical points . . . 147
3.5.2. The constrained Łojasiewicz–Simon gradient inequality . . . 148
3.5.3. Proof of full convergence . . . 152
3.5.4. Blow-up in infinite time . . . 158
3.6. Stationary solutions . . . 159
3.6.1. The defect in the elastica equation . . . 159
3.6.2. Existence of non-elastica stationary solutions . . . 160
3.6.3. Symmetries . . . 161
3.6.4. Explicite parametrization . . . 164

**4. Outlook** **187**

**A. Formalities** **191**
Declaration of First Submission . . . 191
Declaration of co-authorship . . . 192
Declaration of consent . . . 193
Declaration of independent scientific work . . . 194
Curriculum vitae . . . 195

**Bibliography** **199**

# Preface

## Abstract

Elastic wires are mathematical curves composed of matter. They have no thickness but exhibit bending stiffness. Although elastic wires do not exist in the real world, they are widely used in modeling. Examples include the modeling of plant stems, polymers such as DNA, marine cables, and hair movement in digital simulations.
The elastic energy of a sufficiently smooth regular curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ describing an elastic wire is defined as

$$\mathcal{E}(\gamma) = \frac{1}{2}\int_\gamma |\kappa|^2 \mathrm{d}s. \tag{0.0.1}$$

Here, $\kappa = \partial_s^2\gamma$ is the curvature of $\gamma$, $\partial_s = |\partial_x\gamma|^{-1}\partial_x$, and $\mathrm{d}s = |\partial_x\gamma|\mathrm{d}x$ is the arclength element. In the last decades, several authors have studied the $L^2$-gradient flow of the elastic energy in different variants. In this monograph, we briefly present the underlying theory and background, summarize their results, and then focus on two new variants.
First, we consider elastic wires with a heterogeneity described by a density function. We define a generalization of (0.0.1), which depends on material parameters, captures the interplay between curvature and density effects and resembles the Canham–Helfrich functional. Describing the closed planar curve by its inclination angle, the $L^2$-gradient flow of this energy is a nonlocal coupled parabolic system of second order. We discuss local well-posedness, global existence and convergence. Then, we show that the (non)preservation of quantities such as convexity as well as the asymptotic behavior of the system depend delicately on the choice of material parameters.
Second, we study the evolution of elastic wires under the assumption of incompressiblity and derive a gradient flow of (0.0.1) which preserves the enclosed area of the evolving planar curves. Contrary to an earlier approach based on a fourth order equation featuring nonlocal Lagrange multipliers, we give priority to the locality of the evolution and propose a sixth order gradient flow equation with no nonlocal terms. For this flow, we establish a global existence result. When including an additional term penalizing length, we prove convergence to an area constrained critical point of the elastic energy.

## Deutsche Zusammenfassung

Näherungsweise eindimensionale elastische Objekte wie Haare, dünne Pflanzenstängel, Polymere oder Unterwasserkabel können durch sogenannte elastische Drähte modelliert werden. Die elastische Energie einer ausreichend glatten, regulären Kurve, die einen solchen elastischen Draht beschreibt, ist definiert als das Integral über die quadrierte Krümmung. In den letzten Jahrzehnten haben einige Mathematiker und Mathematikerinnen den $L^2$-Gradientenfluss der elastischen Energie in verschiedensten Varianten untersucht. Wir fassen kurz die zugrundeliegende Theorie und ihre Ergebnisse zusammen und konzentrieren uns dann auf zwei neue Varianten.

Zum einen betrachten wir elastische Drähte mit einer Heterogenität, die durch eine Dichtefunktion beschrieben wird. Wir definieren eine Verallgemeinerung der elastischen Energie, die von Materialparametern abhängt und das Zusammenspiel von Krümmung und Dichteeffekten berücksichtigt. Dadurch dass wir die geschlossenen, ebenen Kurven durch ihre Winkelfunktionen beschreiben, ist der $L^2$-Gradientenfluss dieser Energie ein nichtlokales, gekoppeltes, parabolisches System zweiter Ordnung. Wir studieren die lokale Wohlgestelltheit, globale Existenz und Konvergenz. Dann zeigen wir, dass die (Nicht-)Erhaltung von Größen wie Konvexität sowie das asymptotische Verhalten des Systems empfindlich von der Wahl der Materialparameter abhängt.

Zum anderen untersuchen wir elastische Drähte unter Annahme von Inkompressibilität und leiten einen Gradientenfluss für die elastische Energie her, der die eingeschlossene Fläche erhält. Im Gegensatz zu einem früheren Ansatz mit Lagrange-Multiplikatoren, priorisieren wir die Lokalität der Evolutionsgleichung. Der Preis dafür ist, dass wir eine Gleichung sechster Ordnung erhalten. Wir beweisen, dass eine globale Lösung existiert und zeigen Konvergenz zu einer stationären Lösung, die ein kritische Punkt der Energie unter Annahme von gegebener Fläche ist.

## About this thesis

The main part of this thesis consists of the contents of the following three articles.

**Article 1:** A. Dall'Acqua, L. Langer, and F. Rupp: *A dynamic approach to heterogeneous elastic wires*, J. Differential Equations 392 (2024), 1-42.
Licensed under CC BY 4.0 (`https://creativecommons.org/licenses/by/4.0`).
Cited in this thesis as [37].

**Article 2:** A. Dall'Acqua, G. Jankowiak, L. Langer, and F. Rupp: *Conservation, convergence, and computation for evolving heterogeneous elastic wires*, SIAM J. Math. Anal. 56:4 (2024), 4494-4529.
Reproduced with permission from SIAM.
Cited in this thesis as [36].

**Article 3:** L. Langer: *Dynamics of Elastic Wires: Preserving Area without Nonlocality*, Calc. Var. Partial Differential Equations 64, 106 (2025).
Licensed under CC BY 4.0 (`https://creativecommons.org/licenses/by/4.0`).
Cited in this thesis as [99].

For those who are unfamiliar with the elastic flow, we provide an introduction in beforehand. We introduce both the differential geometric concepts of curve theory and the basics of gradient flows. With these two fundamentals, we formulate the elastic flow and summarize the most important results with regard to the main part of the thesis.

## Acknowledgments

*Die Danksagung wurde aus Gründen des Datenschutzes entfernt.*

*The acknowledgments have been removed for privacy reasons.*

CHAPTER 1

# Introduction

Geometric flows are a significant and highly active field of research in mathematics. Their study bridges the theory of partial differential equations, geometry, calculus of variations, and numerical methods. Geometric flows are a powerful tool that can tackle not only important theoretical problems across various fields but also numerous real-world applications. In material science, for example, geometric flows are used to model elastic deformations. This often involves studying the gradient flow of an energy that measures the bending of a geometric object in terms of its curvature. In this dissertation, we focus on physically and biologically motivated geometric flows that arise as gradient flows of bending energies for curves.

In the following, we first give an overview of the classical theory of the elastic flow. The elastic energy on which the elastic flow is based is a bending energy for elastic wires. First, we clarify what elastic wires are and where they find practical application in modeling. Then, we review the basic concepts of curve theory and discuss the elastic energy for curves in Euclidean space. To provide a broader context, we also briefly consider the elastic energy of curves in Riemannian manifolds as well as the Willmore energy. We then give a definition of (constrained) gradient flows in an abstract Hilbert space setting, followed by deriving the classical elastic flow as the $L^2$-gradient flow of the elastic energy. The classical elastic flow is an important and well-studied problem. We summarize the key results that are most relevant to this dissertation. Additionally, we offer a brief overview of some other gradient flows. Chapter 1 concludes with introducing two new variants of the elastic flow, which are studied in detail in Chapter 2 and Chapter 3.

Following the overview in Chapter 1, Chapter 2 explores the first new variant of the elastic flow: an elastic flow for heterogeneous elastic wires. We define a modified elastic energy that accounts for the interplay between heterogeneity and curvature and study the $L^2$-gradient flow of this energy. Our investigation covers local well-posedness, global in time existence, convergence, and the study of properties of the solutions and the limit.

Chapter 3 explores the second new variant of the elastic flow: an area preserving elastic flow without nonlocality. We first derive this flow as a gradient flow. Then, we establish global in time existence and convergence and discuss the stationary solutions.

Finally, in Chapter 4, we offer a brief outlook on further research questions that would be interesting to explore.

The contents of Chapter 2 are published in the articles [37] and [36], while the contents of Chapter 3 are taken from the article [99]. All definitions, theorems, propositions, and lemmas are unchanged in their formulation and references to the original articles where they first appeared are included. However, the notation is slightly revised for consistency, and the order of results and comments may differ. We note that not all of the content from [36] is included in this thesis.

## 1.1. Modeling with elastic wires

In the title of this thesis, we refer to *elastic wires*. But what do we mean by this? Elastic wires are mathematical curves composed of matter. They can be bent, stretched, kinked and wildly deformed, but they cannot break. Elastic wires have no thickness but exhibit bending stiffness. Although they do not exist in the real world, they find wide application in modeling.

The history of modeling with elastic wires dates back to the 17th century. In 1691, James Bernoulli posed the following problem: Given a lamina that is clamped at a point $A$ and has a weight hanging at its end point $B$ (see Figure 1.1), what shape does the lamina take?

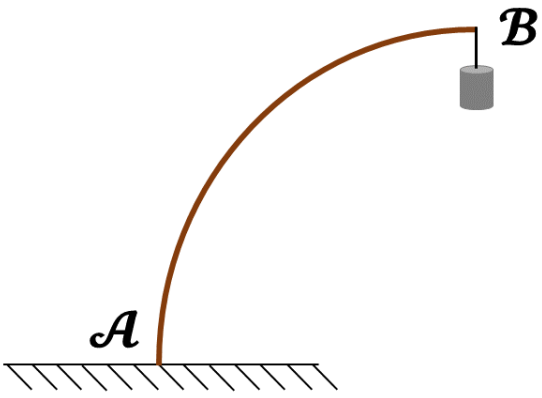



Figure 1.1: Problem of a bent lamina posed by James Bernoulli.

James Bernoulli also published a first solution in 1694. His nephew Daniel Bernoulli revisited the problem in 1742 and proposed variational techniques for its solution. He expressed the energy of the bent lamina as $\int \frac{\mathrm{d}s}{RR}$, where $R$ is the radius of curvature and $s$ refers to arclength. In modern notation and describing the shape of the lamina by the curve $\gamma$, this is

$$\int_\gamma k^2 \,\mathrm{d}s \tag{1.1.1}$$

with $k$ the curvature and $\mathrm{d}s$ the arclength element. Daniel Bernoulli brought the problem

of minimizing this energy to Euler, who was able to characterize the solutions. In honor of their contributions, the energy (1.1.1) is also called Euler–Bernoulli bending energy. We introduce it rigorously as the elastic energy in Section 1.2. More about the history of the Euler–Bernoulli bending energy can be found in [105], [117], [153], and [72].

With the problem of the bent lamina, we have seen an early example where elastic wires appear. From that time until today, elastic wires have always played an important role for modeling. For example, in biology, elastic wires are used to model plant stems, see [71]. Moreover, polymers, and especially DNA and its functions, can be described with elastic wires, see [78], [80], [62], and [127]. Another interesting application of elastic wires is in the explanation of crumbling processes, see for example [107]. In marine research, they have practical applications for modeling marine cables to prevent kinks and loops, see [74]. Elastic wires are also utilized in medicine for modeling purposes. We can model needle insertion procedures or describe the navigation of a catheter through the vessel tree, see [147]. We can also use elastic wires to create decorative elements in computer graphics, see [166] or to design extraordinary objects, see [77]. Moreover, in recent years, a particularly significant application has emerged in simulation software to simulate one-dimensional objects and, for instance, model the movement of hair, ropes, or threads. Here, we find examples in [15], [148], [90], or [26].

When considering not one dimensional but two dimensional objects, elastic shells are the corresponding analog to elastic wires. It is easy to imagine that these also play an important role in modeling. A simple everyday situation that can be modeled with elastic shells is a sports ball being deflated and collapsing, see [91]. Elastic shells are pivotal in architecture, or more generally for manufacturing-oriented design, see for example [3]. Moreover, elastic shells are especially important in the biological field for modeling biomembranes with negligible thickness. This allows for a better understanding of the properties and behavior of vesicles, cells, and, for example, red blood cells, see for example [131]. A key concept here is the so-called Canham–Helfrich energy. In 1970, Canham ([27]) first proposed that the bending energy of elastic membranes should depend on the square of the mean curvature. Combined with contributions from Helfrich ([81]) of a few years later, the bending energy for a closed membrane $\Sigma$ was formulated into

$$E_{\mathrm{CH}}(\Sigma) = \int_\Sigma \left( \frac{\beta_1}{2}(H - H_0)^2 + \beta_2 K \right) \mathrm{d}S, \tag{1.1.2}$$

where $H$ and $K$ are the mean and the Gauss curvature and the bending rigidities $\beta_1 > 0$, $\beta_2 < 0$ and the spontaneous curvature $H_0 \in \mathbb{R}$ are material parameters. The equilibrium configuration of a vesicle modeled by an elastic shell and governed by (1.1.2) reproduces the biconcave shape found in red blood cells, see [27]. If we take a step back and consider the cross-section of a (rotationally symmetric) vesicle, its boundary can again be modeled as an elastic wire (see [142] or [155] for the example of cylindrical shapes). We return to

this in Section 1.5.
Summing up the last paragraphs, we see that modeling with elastic wires has not only played a role in the past but also continues to be important in addressing current issues across a wide range of fields. And if one keeps an observant eye, one finds numerous objects in everyday life that can be modeled as elastic wires (see Figure 1.2).




Figure 1.2: On the way to university, one finds various objects that can be modeled as elastic wires.

## 1.2. The elastic energy

In the following Section 1.2.1, we rigorously define the Euler–Bernoulli bending energy of an elastic wire and discuss important properties. We refer to this energy as *elastic energy*. Before doing so, we introduce some basic curve theory, which can be found, for example, in [11] or [51]. In Section 1.2.2, we take a closer look at the critical points of elastic energy, so-called elasticae. Section 1.2.3 is dedicated to briefly discussing the relevance of the elastic energy. Afterwards, we provide a brief outlook on the elastic energy of curves in Riemannian manifolds in Section 1.2.4 and on the Willmore energy in Section 1.2.5.

### 1.2.1. The elastic energy of curves in Euclidean space

We consider an elastic wire which is modeled by a sufficiently smooth curve $\gamma\colon I \to \mathbb{R}^n$, where $\emptyset \neq I \subset \mathbb{R}$ and $n \in \mathbb{N}$. We assume $\gamma$ to be *regular*, i.e. $\partial_x \gamma(x) \neq 0$ for all $x \in I$.

Given $x_0 \in I$, the arclength from the point $x_0$ is

$$s(x) = \int_{x_0}^{x} |\partial_x \gamma| \, \mathrm{d}x \tag{1.2.1}$$

for $x \in I$. Since we assume $\gamma$ to be regular, (1.2.1) is strictly increasing and therefore invertible with smooth derivative $\partial_x s(x) = |\partial_x \gamma|$. The composition of $\gamma$ and the inverse of $s$ yields a reparametrization $\tilde{\gamma}$ with $|\partial_x \tilde{\gamma}| \equiv 1$. This reparametrization is called *arclength parametrization*. By (1.2.1), the *length* of the curve $\gamma$ is given by

$$\mathcal{L}(\gamma) = \int_I |\partial_x \gamma| \, \mathrm{d}x = \int_\gamma \mathrm{d}s. \tag{1.2.2}$$

Here, we already introduced the *arclength measure* $\mathrm{d}s = |\partial_x \gamma| \, \mathrm{d}x$, which we will primarily use in the following. Differentiation with respect to the arclength is denoted by the *arclength derivative* $\partial_s = |\partial_x \gamma|^{-1} \partial_x$. The *tangent vector* of $\gamma$ is given by

$$\tau = \frac{\partial_x \gamma}{|\partial_x \gamma|} = \partial_s \gamma.$$

The *curvature* of $\gamma$ is defined as

$$\kappa = \partial_s^2 \gamma = \partial_s \tau.$$

Since $2\langle \kappa, \tau \rangle = \partial_s \langle \tau, \tau \rangle = \partial_s |\partial_s \gamma|^2 = 0$ for $n > 1$, it directly follows that $\kappa$ is orthogonal to $\tau = \partial_s \gamma$.

**Example 1.2.1.** One of the simplest examples is an $\omega$-fold covered circle of radius $r > 0$ with midpoint in $(a_1, a_2) \in \mathbb{R}^2$ parametrized by

$$\gamma \colon [0, 2\pi\omega] \to \mathbb{R}^2, \quad x \mapsto \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + r \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$$

for $\omega \in \mathbb{Z}$ with $\omega > 0$. For this parametrization, we have

$$|\partial_x \gamma| \equiv r, \quad \mathrm{d}s = r \, \mathrm{d}x, \quad \text{and} \quad \partial_s = r^{-1} \partial_x.$$

The length of the circle is

$$\mathcal{L}(\gamma) = \int_0^{2\pi\omega} r \, \mathrm{d}x = 2\pi\omega r,$$

the tangential vector is given by

$$\tau(x) = \partial_s \gamma(x) = \frac{\partial_x \gamma}{|\partial_x \gamma|} = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}, \tag{1.2.3}$$

and the curvature is

$$\kappa(x) = \partial_s^2 \gamma(x) = -\frac{1}{r} \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}. \tag{1.2.4}$$

In particular, we obtain $|\kappa(x)| \equiv r^{-1}$, as one expects.

Before we come to the definition of the elastic energy, we want to establish some additional preliminaries on curves, which we will need later.

First: Closedness of curves. In Chapter 2 and Chapter 3 we work with closed curves. A curve $\gamma$ is called closed, if $I = [a, b]$, $-\infty < a < b < \infty$, and $\gamma(a) = \gamma(b)$ (as in Example 1.2.1) or if $I = \mathbb{R}$ and there exists a regular periodic reparametrization. When working with closed curves, we sometimes define them on the unit circle $\mathbb{S}^1$. We identify $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ such that $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^n$ can be lifted to a periodic function $\gamma\colon \mathbb{R} \to \mathbb{R}^n$. When defining the length as in (1.2.2) (and later the elastic energy as in (1.2.8)), we think of the periodic parametrization restricted to $I = [0, 1]$.

Second: If $n = 2$, i.e. if we consider a planar curve $\gamma\colon I \to \mathbb{R}^2$, the unit normal vector $\nu$ to $\gamma$ is given by the counterclockwise rotation of $\tau = \partial_s \gamma$ by $\frac{\pi}{2}$. In this case, we have $\kappa = k\nu$ with the *signed scalar curvature*

$$k = \langle \kappa, \nu \rangle \tag{1.2.5}$$

of $\gamma$. If the planar curve is closed, we define the signed enclosed area $\mathcal{A}$ by

$$\mathcal{A}(\gamma) = -\frac{1}{2} \int_\gamma \langle \gamma, \nu \rangle \, \mathrm{d}s. \tag{1.2.6}$$

Changing the orientation of $\gamma$ changes the sign of $\mathcal{A}$. Further, for closed planar curves, the rotation index $\omega \in \mathbb{Z}$ of $\gamma$ measures the number of turns of the tangent vector along the curve. It is given by

$$\int_\gamma k \, \mathrm{d}s = 2\pi\omega. \tag{1.2.7}$$

We observe that, like the signed enclosed area, the rotation index changes sign when the orientation of the curve is reversed. Figure 1.3 shows examples for planar curves with different rotation indices and enclosed area.

Let us shortly come back to Example 1.2.1.

**Example 1.2.2.** Let the planar closed curve $\gamma$ be given as in Example 1.2.1, describing an $\omega$-fold covering of a circle. Then the unit normal vector is given by the counterclockwise

rotation of $\tau$ as in (1.2.3), i.e.

$$\nu(x) = -\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}.$$

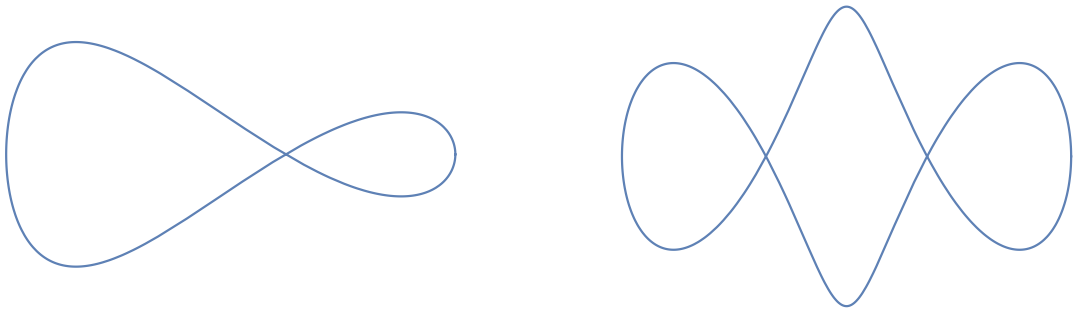
With (1.2.4), this yields $k = \langle \kappa, \nu\rangle = r^{-1}$ and thus

$$\int_0^{2\pi\omega} k\, r \,\mathrm{d}x = 2\pi\omega.$$

So, as expected, the rotation index is given by the number of the coverings of the circle. The enclosed area is given by

$$\mathcal{A}(\gamma) = -\frac{1}{2}\int_0^{2\pi\omega}\langle\gamma,\nu\rangle r\,\mathrm{d}x = \frac{1}{2}\int_0^{2\pi\omega}(a_1\cos x + a_2\sin x + r)r\,\mathrm{d}x = \pi\omega r^2,$$

which is also consistent with our expectations.



(A) (B)

Figure 1.3: Examples for closed planar curves with $\omega = 0$ and $\mathcal{A} \neq 0$ in (A) and with $\omega = 1$ or $\omega = -1$ and $\mathcal{A} = 0$ in (B).

Having reviewed the most important concepts of curve theory, we now proceed to define the *elastic energy* of the sufficiently smooth regular curve $\gamma\colon I \to \mathbb{R}$, $I$ bounded. It is given by

$$\mathcal{E}(\gamma) = \frac{1}{2}\int_\gamma |\kappa|^2\,\mathrm{d}s = \frac{1}{2}\int_I |\kappa|^2|\partial_x\gamma|\,\mathrm{d}x. \tag{1.2.8}$$

In the literature, the factor of one-half is sometimes omitted. Clearly, (1.2.8) is invariant under translation, reflection and rotation of $\gamma$.
The elastic energy is defined for regular curves $\gamma \in W^{2,p}(I;\mathbb{R}^n)$ with $p \geq 2$. For curves merely in $W^{2,p}(I;\mathbb{R}^n)$, $s$ defined in (1.2.1) is still continuous and stricly increasing since $W^{2,p} \hookrightarrow C^1$ by Sobolev embedding. The curvature has to be defined weakly.
If $n = 2$ and $\kappa = k\nu$, then $|\kappa|^2 = |\langle\kappa,\nu\rangle\nu| = |\langle\kappa,\nu\rangle| = k^2$. Thus, in this case, (1.1.1) and (1.2.8) coincide.

Let us note the following properties of the elastic energy.

① *Invariance under reparametrization.* Let $\Phi\colon \tilde{I} \to I$ be a smooth diffeomorphism and define $\tilde{\gamma}\colon \tilde{I} \to \mathbb{R}^n$, $\tilde{\gamma} = \gamma \circ \Phi$. Then

$$\begin{aligned}\mathcal{E}(\tilde{\gamma}) &= \frac{1}{2}\int_{\tilde{\gamma}} |\tilde{\kappa}|^2 \,\mathrm{d}s = \frac{1}{2}\int_{\tilde{I}} |\tilde{\kappa}(x)|^2 |\partial_x \tilde{\gamma}(x)| \,\mathrm{d}x \\ &= \frac{1}{2}\int_{\tilde{I}} |\kappa(\Phi(x))|^2 |\partial_x \gamma(\Phi(x))||\partial_x \Phi(x)| \,\mathrm{d}x = \frac{1}{2}\int_I |\kappa(y)|^2 |\partial_x \gamma(y)| \,\mathrm{d}y = \mathcal{E}(\gamma).\end{aligned}$$

② *Scaling behavior.* If $\tilde{\gamma} = r\gamma\colon I \to \mathbb{R}^n$ with a scaling factor $r > 0$, then

$$\mathcal{E}(r\gamma) = \frac{1}{2}\int_I \left|\frac{\kappa}{r}\right|^2 r \,\mathrm{d}s = \frac{1}{r}\mathcal{E}(\gamma). \tag{1.2.9}$$

We see that by blowing up the curve $\gamma$, the elastic energy can be made arbitrarily small. This property motivates to take into account a penalization of the length and to consider the *length penalized elastic energy*

$$\mathcal{E}_\lambda(\gamma) = \mathcal{E}(\gamma) + \lambda\mathcal{L}(\gamma) = \frac{1}{2}\int_\gamma \left(|\kappa|^2 + 2\lambda\right) \mathrm{d}s$$

for some $\lambda \geq 0$. Note that in contrast to (1.2.9), the length scales proportional to the scaling factor of the curve. That means that beyond a certain size of $r$, the energy increases as we further inflate the curve.

### 1.2.2. Elasticae in Euclidean space

Critical points of $\mathcal{E}_\lambda$ in $W^{2,p}(I;\mathbb{R}^n)$ are called *elasticae*. They satisfy the weak Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}_\lambda(\gamma + \varepsilon\phi)\Big|_{\varepsilon=0} = 0$$

for all $\phi \in C_0^\infty(I;\mathbb{R}^n)$ if we consider given first order boundary data or $\phi \in C^\infty(\mathbb{S}^1;\mathbb{R}^n)$ if we consider closed curves (see for example [167, Section 40]). Note that for $\varepsilon > 0$ small enough, the perturbed curve $\gamma_\varepsilon := \gamma + \varepsilon\phi$ is also regular. A computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\,\mathrm{d}s_\varepsilon)\Big|_{\varepsilon=0} = \langle \tau, \partial_s\phi\rangle \,\mathrm{d}s \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\kappa_\varepsilon\Big|_{\varepsilon=0} = \partial_s\nabla_s\phi - \langle \tau, \partial_s\phi\rangle\kappa.$$

Here, $\nabla_s = P^\perp\partial_s$, where $P^\perp$ is the normal projection along $\gamma$ defined by $P^\perp\phi = \phi - \langle \phi, \partial_s\gamma\rangle\partial_s\gamma$ for some vector field $\phi : \mathbb{S}^1 \to \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product in $\mathbb{R}^n$. We obtain for the first variation

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}_\lambda(\gamma + \varepsilon\phi)\Big|_{\varepsilon=0} = \int_I \langle \kappa, \partial_s\nabla_s\phi\rangle \,\mathrm{d}s - \frac{1}{2}\int_I |\kappa|^2\langle \tau, \partial_s\phi\rangle \,\mathrm{d}s + \lambda\int_I \langle \tau, \partial_s\phi\rangle \,\mathrm{d}s,$$

see for example [43, Lemma A.1]. If $\gamma$ is sufficiently smooth, we can integrate by parts and obtain the strong Euler–Lagrange equation

$$\nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa - \lambda\kappa = 0. \tag{1.2.10}$$

Note that since we consider $\phi \in C_0^\infty(I;\mathbb{R}^n)$ or $\phi \in C^\infty(\mathbb{S}^1;\mathbb{R}^n)$, there are no boundary terms. In the following, (1.2.10) is also referred to as the *classical elastica equation*.
If $n = 2$, we can use the signed scalar curvature $k = \langle \kappa, \nu\rangle$ (defined in (1.2.5)) to write (1.2.10) equivalently as

$$\partial_s^2 k + \frac{1}{2}k^3 - \lambda k = 0. \tag{1.2.11}$$

Here, we use that $\partial_s^2 k = \partial_s\langle\partial_s\kappa,\nu\rangle = \partial_s\langle\nabla_s\kappa,\nu\rangle = \langle\partial_s\nabla_s\kappa,\nu\rangle = \langle\nabla_s^2\kappa,\nu\rangle$ since $\partial_s\nu = -k\tau$. It is immediately clear that straight lines with $\kappa \equiv 0$ are elasticae. To see another example, let us again come back to the circle of Example 1.2.1.

**Example 1.2.3.** Let $\gamma$ be given as in Example 1.2.1. Then

$$\nabla_s\kappa = \partial_s\kappa - \langle\partial_s\kappa,\partial_s\gamma\rangle\partial_s\gamma = \frac{1}{r^2}\begin{pmatrix}\sin x\\ -\cos x\end{pmatrix} - \Big\langle\frac{1}{r^2}\begin{pmatrix}\sin x\\ -\cos x\end{pmatrix},\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}\Big\rangle\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix} = 0$$

and

$$\frac{1}{2}|\kappa|^2\kappa = -\frac{1}{2r^3}\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}.$$

So we obtain

$$\nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa - \lambda\kappa = \Big(-\frac{1}{2r^3} + \frac{\lambda}{r}\Big)\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}.$$

Thus, a multifold covering of a circle with radius $r$ is an elastica if $\lambda > 0$ and $r = \frac{1}{\sqrt{2\lambda}}$.

For $\lambda > 0$, solutions to the elastica equation (1.2.10) have been classified in several works, see for instance [98] or [48]. If $n = 2$ and $\gamma$ solves (1.2.10), then $\gamma$ is either a

- *linear elastica*, i.e. a line with scalar curvature $k \equiv 0$, or
- *wavelike elastica*, i.e. $k$ is a periodic function that changes sign (see Figure 1.4), or
- *borderline elastica*, i.e. $k$ is not periodic and asymptotically zero (see Figure 1.5), or
- *orbitlike elastica*, i.e. $k$ is a periodic function that does not change sign (see Figure 1.6), or
- *circular elastica*, i.e. a (multifold) covered circle with constant $k$,

see for example [124, Proposition B.3]. We only want to provide an idea of what these types look like. For the explicit parametrizations using Jacobi elliptic functions, see [124, Proposition B.3].

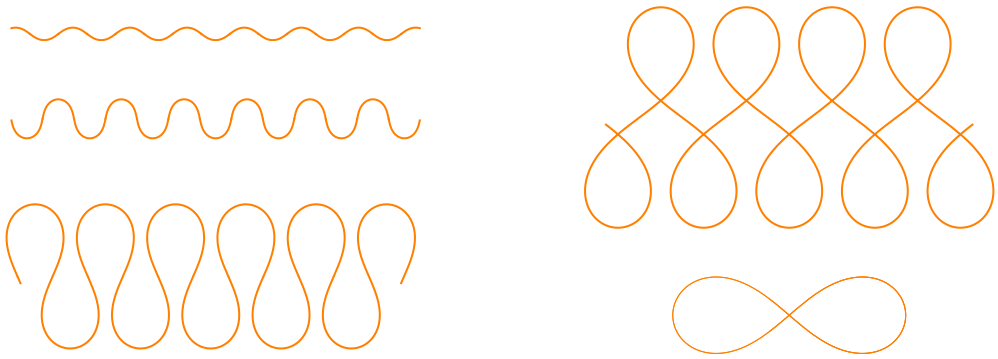


Figure 1.4: Wavelike elasticae. The figure eight elastica (bottom right) is rotated.

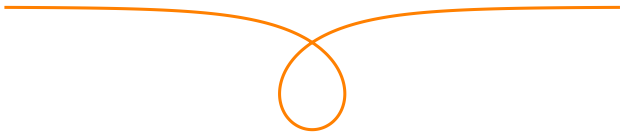


Figure 1.5: Borderline elastica.



Figure 1.6: Orbitlike elasticae.

In [124, Lemma 5.4], it is shown that linear, borderline and orbitlike elasticae cannot close up. Circular elasticae trivially close up and wavelike elasticae close up if both components of $\gamma$ are periodic with the same period, which yields exactly the figure eight defined in [124, Definition 5.1] and shown in Figure 1.4. A proper rescaling of the figure eight solves (1.2.10), so we can indeed call it the figure eight *elastica*.
We especially emphasize the result that the only closed planar elasticae are multifold coverings of the circle and of the figure eight elastica.
For $\lambda = 0$, and considering planar closed curves, no critical points of $\mathcal{E}_\lambda$ exist, see for example [122, Section 2.2].

### 1.2.3. The relevance of the elastic energy

With today's rapid advances in science and technology, one might wonder if the elastic energy is still a relevant concept. We believe that it is. Several modeling approaches lead to the consideration of the elastic energy, showing its continuing importance.

An example of the relevance of the elastic energy is found in one-dimensional discrete mechanical systems, which are used to model, for example, one-dimensional atomic chains, such as those found in graphene structures. These models are related to the Hencky bar chain model, see [156]. Consider a chain of atoms where the connections between atoms have a bond length of $\varepsilon$. This interatomic bond length is assumed to be small, and as $\varepsilon \to 0$, the atomic chain converges to a curve. The discrete bending energy of the chain $\Gamma$-converges to the elastic energy. This is studied in [60] using a description of the chain and the curve by discrete and continuous angle functions, respectively. In [2], the authors further show the $\Gamma$-convergence of a discrete bending energy to a continuous bending energy under the additional assumption of potential material nonlinearities. For a similar result that also considers torsion, see [52].

The elastic energy also appears as the $\Gamma$-limit in the modeling of lipid bilayer membranes. In [132], the authors study an energy functional that appears in a simplified model of these membranes. They show that in the zero-thickness limit, this energy functional $\Gamma$-converges to the elastic energy of a system of curves.

We also want to mention the articles [123] and [12], which establish the elastic energy as dimensionally reduced limits from three-dimensional hyperelasticity. Both articles incorporate also torsion effects.

Regarding the Canham–Helfrich energy (consider the connection to the elastic energy described in Section 1.1) we also want to mention the articles [86] and [13] where the role of this energy in biological and physical contexts is discussed. We come back to these articles in Chapter 2 when discussing spontaneous curvatures.

### 1.2.4. The elastic energy of curves in Riemannian manifolds

In this section, we define the elastic energy more generally for curves that live in a Riemannian manifold. By this extension, we can apply our previous concepts to a broader range of geometric contexts.

We consider an $n$-dimensional Riemannian manifold $(M, g)$, that is, a differentiable manifold with a Riemannian metric (see for example [85]), and a curve $\gamma$ mapping from an interval $\emptyset \neq I \subset \mathbb{R}$ (or from $\mathbb{S}^1$) to $M$.

The simplest example is $M = \mathbb{R}^n$, $n \in \mathbb{N}$, with the Euclidean metric, bringing us back to the case in Section 1.2.1. Another common example is the unit sphere $M = \mathbb{S}^2 \subset \mathbb{R}^3$ with the metric induced by the Euclidean scalar product in $\mathbb{R}^3$ (see Figure 1.7 and for details see [102, p. 33] or [38, Section 2.1]). Also important is the hyperbolic half plane $M = \mathbb{H}^2 = \mathbb{R} \times (0, \infty)$ with global coordinates $(y_1, y_2) \mapsto (y_1, y_2)$ and the metric

$$g_{(y_1,y_2)} = \frac{1}{y_2^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.12}$$

(for details see for example [45, Section 2] or [50, p. 46]).

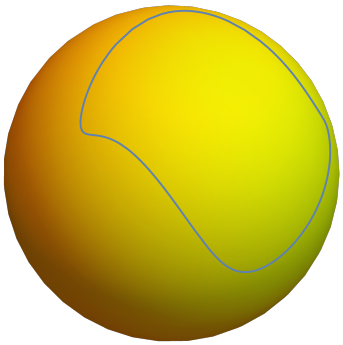


Figure 1.7: Example of an immersed curve $\gamma\colon \mathbb{S}^1 \to \mathbb{S}^2$ (in blue).

We assume $\gamma\colon I \to M$ to be a sufficiently smooth immersion. This means that the velocity vector $\partial_x\gamma(x)$, which is defined as the push-forward $\gamma_*(\partial_x)$ acting on smooth functions $f\colon M \to \mathbb{R}$ by

$$\partial_x\gamma(x)f = \frac{\mathrm{d}}{\mathrm{d}x}(f \circ \gamma)(x),$$

does not vanish for any $x \in I$. The arclength $s$ of $\gamma$ starting from a fixed $x = x_0$ is given by

$$s(x) = \int_{x_0}^{x} \sqrt{g_{\gamma(x)}(\partial_x\gamma(x), \partial_x\gamma(x))}\,\mathrm{d}x = \int_{x_0}^{x} |\partial_x\gamma|_{g(\gamma(x))}\,\mathrm{d}x$$

for $x \in I$ (see [50, Section 3.2]) and the length of $\gamma$ by

$$\mathcal{L}(\gamma) = \int_I |\partial_x\gamma|_{g(\gamma(x))}\,\mathrm{d}x = \int_\gamma \mathrm{d}s$$

(compare to (1.2.1) and (1.2.2)). Here and in the following, $\mathrm{d}s = \mathrm{d}s_g = |\partial_x\gamma|_{g(\gamma(x))}\,\mathrm{d}x$. Differentiation with respect to arclength is written as

$$\partial_s = |\gamma_*(\partial_x)|^{-1}_{g(\gamma(x))}\partial_x$$

which allows as to write the tangent vector of $\gamma$ as $\tau = \partial_s\gamma$ like in (1.2.3). By the Fundamental Lemma of Riemannian Geometry (see for example [104, Theorem 5.10]), there exists a unique linear connection $\nabla$ on $M$ that is compatible with $g$ and symmetric. With this so called *Levi–Civita connection* of $g$, we define the *geodesic curvature vector field* as

$$\kappa = \nabla_\tau\tau\colon I \to TM,$$

where $TM$ is the tangent bundle of $M$.

As in Section 1.2.1, let us take a look at the example of a circle. We choose $M = \mathbb{H}^2$ with the metric defined in (1.2.12).

**Example 1.2.4.** Consider $\gamma\colon \mathbb{S}^1 \to \mathbb{H}^2$ parametrized by

$$\gamma(x) = \begin{pmatrix} \gamma_1(x) \\ \gamma_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix} + r \begin{pmatrix} \cos(2\pi x) \\ \sin(2\pi x) \end{pmatrix} \tag{1.2.13}$$

for some $a > r > 0$. As explained in [46, Lemma 2.1], this is the boundary of a ball in the hyperbolic plane with radius $\tanh^{-1}(\frac{r}{a})$ centered in $(0, \sqrt{a^2 - r^2})$. For the example of $r = \sinh(\frac{1}{\sqrt{2}})$ and $a = \cosh(\frac{1}{\sqrt{2}})$, see Figure 1.8.

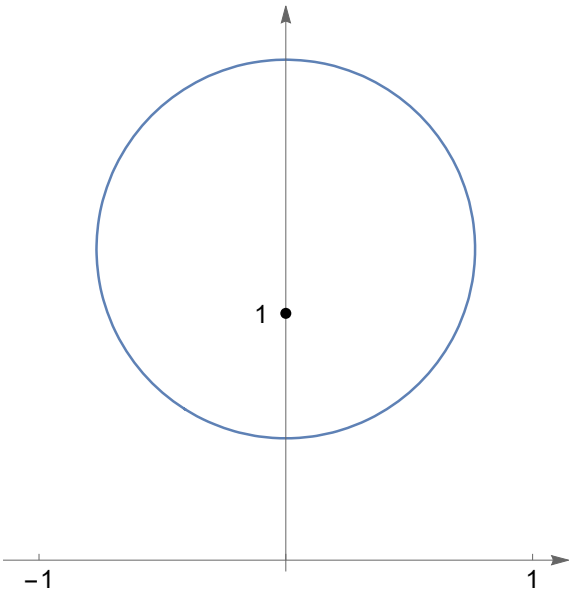



Figure 1.8: The ball of radius $\frac{1}{\sqrt{2}}$ centered at $(0, 1)$ in the hyperbolic geometry. For more explanations, see [46, Lemma 2.1].

With (1.2.12), we compute

$$|\partial_x \gamma|_g = \frac{1}{\gamma_2} |\partial_x \gamma|_e = \frac{2\pi r}{a + r\sin(2\pi x)}. \tag{1.2.14}$$

For clarity, we denote here the Euclidean norm with the index $e$. Using (1.2.14), we compute the tangential vector

$$\tau = \partial_s \gamma = \frac{a + r\sin(2\pi x)}{2\pi r} \partial_x \gamma = (a + r\sin(2\pi x)) \begin{pmatrix} -\sin(2\pi x) \\ \cos(2\pi x) \end{pmatrix}.$$

With a formula for the the Levi–Civita connection of $(\mathbb{H}^2, g)$ which can be found, for

example in [45, (10)] or in [46, (2.1)], we compute

$$\begin{aligned}
\kappa(x) &= \begin{pmatrix} \partial_s^2\gamma_1(x) - \frac{2}{\gamma_2(x)}\partial_s\gamma_1(x)\partial_s\gamma_2(x) \\ \partial_s^2\gamma_2(x) + \frac{1}{\gamma_2(x)}((\partial_s\gamma_1(x))^2 - (\partial_s\gamma_2(x))^2) \end{pmatrix} \\
&= (a + r\sin(2\pi x))\cos x \begin{pmatrix} -\sin(2\pi x) \\ \cos(2\pi x) \end{pmatrix} + \frac{(a + r\sin(2\pi x))^2}{r} \begin{pmatrix} -\cos(2\pi x) \\ -\sin(2\pi x) \end{pmatrix} \\
&\quad + (a + r\sin(2\pi x)) \begin{pmatrix} 2\sin(2\pi x)\cos(2\pi x) \\ \sin^2(2\pi x) - \cos^2(2\pi x) \end{pmatrix} \\
&= -(a + r\sin(2\pi x))\frac{a}{r} \begin{pmatrix} \cos(2\pi x) \\ \sin(2\pi x) \end{pmatrix}.
\end{aligned}$$

In particular, we see that

$$|\kappa|_g = \frac{1}{\gamma_2}|\kappa|_e = \frac{1}{a + r\sin(2\pi x)} \frac{a(a + r\sin(2\pi x))}{r} \equiv \frac{a}{r}, \tag{1.2.15}$$

i.e. the absolute value of the curvature is constant, as we might expect for a circle, though this is not obvious in hyperbolic geometry.

In accordance with the special case of the Euclidean space, we define the length penalized elastic energy as

$$\mathcal{E}_\lambda(\gamma) = \frac{1}{2}\int_\gamma |\kappa|_g^2 \,\mathrm{d}s + \lambda\mathcal{L}(\gamma) = \frac{1}{2}\int_I \left(|\kappa(x)|^2_{g(\gamma(x))} + 2\lambda\right) |\partial_x\gamma(x)|_{g(\gamma(x))} \,\mathrm{d}x.$$

We assume that $\lambda \geq 0$, but point out that the case $\lambda < 0$ is also of interest in a certain range (see [46]).

Being an elastica, or more precisely, the Euler–Lagrange equation for critical points of the elastic energy, now depends on the curvature of the manifold. Again, we consider closed curves or curves which satisfy given first order boundary data. Let $R$ be the Riemann curvature tensor of $(M, g)$ acting on vector fields $X, Y, Z$ on $M$ by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where $[X, Y] = \nabla_X Y - \nabla_Y X$ is the Lie bracket. Then sufficiently smooth critical points of $\mathcal{E}_\lambda$ satisfy the elastica equation

$$2\nabla_\tau\nabla_\tau\kappa + \nabla_\tau(3|\kappa|_g^2 - 2\lambda)\tau + 2R(\kappa, \tau)\tau = 0,$$

see [98, (1.1)].

If $\dim(M) = 2$, there is a unit normal field $\nu\colon I \to TM$ along $\gamma$. In this case, we define

the signed scalar curvature

$$k = g(\nabla_\tau \tau, \nu) = g(\kappa, \nu)$$

such that $\kappa = \nabla_\tau \tau = k\nu$. With the sectional curvature $G$ of $M$ (see for example [85, Definition 4.3.4] or [103, p. 145]), the elastica equation then becomes

$$\partial_s^2 k + \frac{1}{2}k^3 + (G \circ \gamma)k - \lambda k = 0, \tag{1.2.16}$$

see [98, (1.2)].
For manifolds with constant sectional curvature $G \equiv G_0$, (1.2.16) can be solved explicitly, see [98]. In particular, this is the case for our previously mentioned examples. When considering the sphere ($M = \mathbb{S}^2$), we have constant sectional curvature $G_0 \equiv 1$ (see [103, p. 148]), and for the hyperbolic half plane ($M = \mathbb{H}^2$) with the metric in (1.2.12), constant sectional curvature $G_0 \equiv -1$ (see [50, Chapter 8]). If we take $M = \mathbb{R}^2$ with the Euclidean metric, then $G_0 \equiv 0$ and we come back to (1.2.11).
As in Section 1.2.2, let us briefly consider the example of a circle.

**Example 1.2.5.** Consider the setting of Example 1.2.4. Then $\gamma\colon \mathbb{S}^1 \to \mathbb{H}^2$ parametrized by (1.2.13) satisfies

$$\partial_s^2 k + \frac{1}{2}k^3 - k - \lambda k = \frac{a}{r}\left(\frac{1}{2}\frac{a^2}{r^2} - 1 - \lambda\right).$$

Here we used that

$$k = g(\kappa, \nu) = \frac{1}{\gamma_2^2}\langle \kappa, \nu \rangle_e \equiv \frac{a}{r}$$

(see also (1.2.15)). Comparing this equation to (1.2.16) with $G \equiv -1$, we see that $\gamma$ is an elastica if

$$\frac{a}{r} \equiv \sqrt{2(\lambda + 1)}.$$

As we might expect, unlike in Example 1.2.3 for the Euclidean case, the condition for a circle to be an elastica now also incorporates the position of the circle.

At this point, we do not want to go further. For more on elasticae in Riemannian manifolds with constant sectional curvature, we refer to [98], to [125] for $M = \mathbb{H}^2$, and to [21] or [146, Section 2.1] for $M = \mathbb{S}^2$.

### 1.2.5. The Willmore energy

The elastic energy can be seen as the one-dimensional analogue to the famous *Willmore energy*. To understand this statement and to put the elastic energy in a broader context,

we define the Willmore energy in this section. We are relatively brief and instead provide references where further information is given. An overview can also be found in [95], whose notation we largely follow.

We consider a smooth surface $\Sigma$ and a smooth immersion $f\colon \Sigma \to \mathbb{R}^n$, $n \in \mathbb{N}$ (see [50, Definition 3.1]). The pullback metric of the Euclidean metric under $f$ is denoted by $g$. In local coordinates, we have

$$g_{ij} = \langle \partial_i f, \partial_j f \rangle, \quad i, j \in \{1, 2\},$$

see [95, Section 1.1]. We write $g = \det(g_{ij})_{ij}$ and $(g^{ij})_{ij}$ for the inverse of the matrix $(g_{ij})_{ij}$. The pullback metric $g$ induces an area measure $\mu_f$. To define an extrinsic notion of curvature we first have to clarify what we mean by the normal projection. For a vector field $X$ along $f$, the tangential projection is given by $g^{ij}\langle X, \partial_i f\rangle \partial_j f$ and the normal projection by the difference

$$P^\perp X = X - g^{ij}\langle X, \partial_i f\rangle \partial_j f.$$

Here and in the following, we use the Einstein summation convention, i.e. we sum over repeated indices. The normal projections of the second derivatives of $f$ define the *vector valued second fundamental form*

$$A_{ij} = P^\perp \partial_i \partial_j f.$$

With this, we define the *mean curvature vector* as the trace of the second fundamental form given by

$$\mathbf{H} = g^{ij} A_{ij},$$

see [95, Section 1.1]. The *Willmore energy* is now given by the integral of the square of the mean curvature. More precisely, we define it as

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |\mathbf{H}|^2 \, \mathrm{d}\mu_f. \tag{1.2.17}$$

These definitions become clearer if we assume that $\Sigma$ is oriented and $f$ has codimension 1, meaning $n = 2 + 1 = 3$. In this case, a smooth unit normal field $\nu\colon \Sigma \to \mathbb{S}^2$ along $f$ exists, defined in local coordinates by

$$\nu = \frac{\partial_1 f \times \partial_2 f}{|\partial_1 f \times \partial_2 f|},$$

see [102, Section 8]. Using this, we can express the mean curvature vector $\mathbf{H}$ as $\mathbf{H} = H\nu$ with the scalar mean curvature $H$. With the principal curvatures $k_1$ and $k_2$, which are

the eigenvalues of the symmetric Weingarten matrix $(g^{ik}A_{kj})_{ij}$, we have $H = k_1 + k_2$. In this case, the Willmore energy (1.2.17) is given by

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma H^2 \,\mathrm{d}\mu_f = \frac{1}{4}\int_\Sigma (k_1 + k_2)^2 \,\mathrm{d}\mu_f$$

(compare to [164, Section 7.2]).

**Example 1.2.6.** Let us look at the example that $f$ parametrizes a round sphere $r\mathbb{S}^2$ for some $r > 0$. Then $k_1 = k_2 \equiv \frac{1}{r}$ and $\mathrm{d}\mu = r^2 \,\mathrm{d}A_{\mathbb{S}^2}$, where $\mathrm{d}A_{\mathbb{S}^2}$ is the area measure of the unit sphere $\mathbb{S}^2$. With this, we obtain

$$\mathcal{W}(f) = 4\pi. \tag{1.2.18}$$

Let us summarize several selected important properties of the Willmore energy.

① *Invariance under reparametrization.* Like the elastic energy, the Willmore energy is invariant under reparametrizations. For a smooth diffeomorphism $\phi\colon \Sigma \to \Sigma$ it holds that $\mathcal{W}(f) = \mathcal{W}(f \circ \phi)$.

② *Conformal invariance.* The fact, that the Willmore energy in (1.2.18) does not depend on $r$ has a deeper reason. If $\Sigma$ is closed (i.e. $\Sigma$ is compact without boundary), then $\mathcal{W}(f) = \mathcal{W}(\phi \circ f)$ for all conformal transformations $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ which are non singular on $f(\Sigma)$, see [95, Section 1.2]. In particular, unlike the elastic energy of curves in Euclidean space, the Willmore energy is scaling invariant.

③ *Lower bound.* If $\Sigma$ is a closed surface and $f$ is an embedding, then $\mathcal{W}(f) \geq 4\pi$. Equality holds if and only if $f$ parametrizes a round sphere, see Example 1.2.6. A proof for this can be found in [163, Theorem 1]. As we observed in (1.2.9), such a statement does not hold for the elastic energy of curves in Euclidean space.

④ *Embeddedness.* In [106, Theorem 6], it is proven that if $\mathcal{W}(f) < 8\pi$, then $f$ is an embedding. This means that there are no self-intersections of $f(\Sigma)$. An analogous result for closed planar curves in Euclidean space is shown in [124, Theorem 1.2]. Since the elastic energy is not scaling invariant, the authors here prove an explicit smallness condition on the product $\mathcal{LE}$. For higher codimension, see also [120].

We do not go into detail here, but we want to recall an interesting connection between the Willmore energy of surfaces of revolution and the elastic energy of curves in $\mathbb{H}^2$, see [22]. If we take a smooth immersed closed curve $\gamma\colon \mathbb{S}^1 \to \{(x, z) \in \mathbb{R}^2 : z > 0\}$ and rotate it around the $x$-axis, we obtain an immersion $f_\gamma$ describing a surface of revolution. Considering the curve as $\gamma\colon \mathbb{S}^1 \to \mathbb{H}^2$, we compute the elastic energy to be a multiple of

the Willmore energy of $f_\gamma$. More precisely,

$$\mathcal{E}(\gamma) = \frac{1}{\pi}\mathcal{W}(f_\gamma). \tag{1.2.19}$$

The exact computation can be found in [45, Section 2.1.1]. This also shows that, like the Willmore energy, the elastic energy of curves in $\mathbb{H}^2$ must also be scaling invariant (compare to ②). The proof can be found, for example, in [45, Remark 2.2].

We would also like to briefly mention the following. In the situation where $\Sigma$ is oriented and $f$ has codimension 1 such that a unit normal field $\nu$ along $f$ exists, a generalization of the Willmore energy is the *Helfrich energy*. It is defined as

$$\mathcal{H}_{H_0}(f) = \frac{1}{4}\int_\Sigma (H - H_0)^2 \,\mathrm{d}\mu, \tag{1.2.20}$$

where $H_0 \in \mathbb{R}$ is the spontaneous curvature, see [27], [81], and for example [140]. As explained in Section 1.1, it is used to characterized the equilibrium shapes of elastic membranes as (constrained) minimizers. The selection of constants is irrelevant for this, but it is immediately apparent that, compared to (1.1.2), the second integral term is missing. This second term in (1.1.2) is a topological invariant by virtue of the Gauss–Bonnet theorem and can therefore be essentially ignored if a fixed genus is prescribed.

## 1.3. (Constrained) gradient flows in Hilbert spaces

The following section aims to provide an introduction to the framework in which we later derive the gradient flows that we want to investigate. We consider a real Banach space $V$ with norm $\|\cdot\|_V$. The dual space of $V$ is denoted by $V'$ and for the value of $u' \in V'$ at $u \in V$, we write $u'(u)$, $u'u$ or sometimes also $\langle u', u\rangle_{V',V}$.

Let $U \subset V$ be an open subset of $V$ and let $\mathcal{F}\colon U \to \mathbb{R}$ be a continuously (Fréchet-) differentiable function. This means that for every $u \in U$, there exists a unique $u' \in V'$ such that

$$\lim_{\|h\|_V \to 0} \frac{\mathcal{F}(u+h) - \mathcal{F}(u) - u'(h)}{\|h\|_V} = 0 \tag{1.3.1}$$

and $\mathcal{F}'\colon U \to V'$, $u \mapsto u'$ with $u'$ from (1.3.1), is continuous.

Moreover, let $H$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$. We assume that $V$ is continuously and densely embedded into $H$, written as $V \hookrightarrow H$. So we can identify $V$ with a dense subset of $H$ and there exists a constant $C > 0$ such that $\|u\|_H \le C\|u\|_V$ for all $u \in V$.

### 1.3.1. Gradient flows

In $H$, we define the gradient of $\mathcal{F}$ as follows (see also [29]).

**Definition 1.3.1** (Gradient)**.** Let $\mathcal{F} \in C^1(U)$ and $u \in U$. The *(H-)gradient* $\nabla_H \mathcal{F}(u)$ is the unique element in $H$ (if it exists) which represents $\mathcal{F}'(u)$ with respect to the inner product in $H$, i.e.

$$\mathcal{F}'(u)\phi = \langle \nabla_H \mathcal{F}(u), \phi \rangle_H \quad \text{for all } \phi \in V.$$

The set of points $u \in U$ where $\mathcal{F}$ possesses an $H$-gradient is the domain $D(\nabla_H \mathcal{F})$ of the $H$-gradient of $\mathcal{F}$.

Let us consider a simple example for this definition. More details can be found for a similar energy functional in [29, Section 3.4].

**Example 1.3.2.** Consider $V = H_0^1(0,1)$, which is the closure of $C_c^\infty(0,1)$ in $W^{1,2}(0,1)$, with the $W^{1,2}$-norm, $H = L^2(0,1)$ equipped with the usual inner product

$$\langle u, v \rangle_{L^2} = \int_0^1 u(x)v(x)\,\mathrm{d}x \quad \text{for } u, v \in L^2(0,1)$$

and the energy $\mathcal{F}_\varepsilon \colon V \to \mathbb{R}$ given by

$$\mathcal{F}_\varepsilon(u) = \int_0^1 \Big( W(u(x)) + \frac{1}{2}\varepsilon (u'(x))^2 \Big)\,\mathrm{d}x.$$

Here, $\varepsilon > 0$ and $W \colon \mathbb{R} \to \mathbb{R}$ is a smooth function. Then $D(\nabla_H \mathcal{F}_\varepsilon) = W^{2,2}(0,1) \cap H_0^1(0,1)$ and $\nabla_H \mathcal{F}_\varepsilon(u) = -\varepsilon u'' + W'(u)$ for $u \in D(\nabla_H \mathcal{F}_\varepsilon)$.

As in Example 1.3.2, $D(\nabla_H \mathcal{F})$ does not always coincide with $U$. Since the embedding $H' \to V'$ is in general not surjective, the derivative $\mathcal{F}'(u)$, $u \in U$, does not necessarily extend to a continuous linear functional on $H$. But if it does, then $u \in D(\nabla_H \mathcal{F})$ by the Riesz representation theorem.
In the course of this thesis, however, we mostly consider the case where $D(\nabla_H \mathcal{F}) = U$ and regard $\nabla_H \mathcal{F}$ as a mapping $U \to H$. This assumption is already used in the following definition of the $H$-gradient flow of $\mathcal{F}$.

**Definition 1.3.3** (Gradient flow)**.** The $H$-gradient flow of $\mathcal{F} \in C^1(U)$ is the differential equation

$$\partial_t u(t) = -\nabla_H \mathcal{F}(u(t)). \tag{1.3.2}$$

A solution of the $H$-gradient flow is a function $u \in W^{1,2}(0,T;H) \cap L^2(0,T;V)$, $T > 0$ such that $u(t) \in U$ for almost every $t \in (0,T)$ and (1.3.2) holds almost everywhere on $(0,T)$.

The definition of the $H$-gradient flow can be motivated by the following observation: If $u$ is a sufficiently smooth solution of (1.3.2), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u(t)) = \langle \nabla_H \mathcal{F}(u(t)), \partial_t u(t)\rangle_H = -\|\nabla_H \mathcal{F}(u(t))\|_H^2 \leq 0,$$

i.e. $\mathcal{F}$ during the evolution. This property is not only useful when studying existence of solutions to (1.3.2), but also makes gradient flows attractive for modeling physical processes. In a physical context, $\mathcal{F}$ is usually given as an energy and (1.3.2) describes the evolution towards an energetically preferred state.

### 1.3.2. Constrained gradient flows

Especially for physically motivated problems, it is often the case that a solution should satisfy certain constraints. To meet these constraints, we define the constraint functional $\mathcal{G}\colon U \to \mathbb{R}^k$, $k \in \mathbb{N}$. We assume that $\mathcal{G} \in C^1(U;\mathbb{R}^k)$ and that $\mathcal{G}$ is a submersion for all $u \in U$ with $\mathcal{G}(u) = 0$. This means that

- $\mathcal{G}'(u)\colon V \to \mathbb{R}^k$ is surjective and
- the kernel $\ker(\mathcal{G}'(u))$ splits $V$, i.e. there exists a continuous projection operator $P$ of $V$ on $\ker(\mathcal{G}'(u))$. Therefore, $V = \ker(\mathcal{G}'(u)) \oplus (I-P)(V)$,

see [167, Section 43.6d]. We note that the latter is satisfied if $V$ is a Hilbert space. The set of all functions that satisfy the given constraints, that is, the set on which we seek solutions, is defined by the elements of $U$ that satisfy $\mathcal{G} = 0$. Under the aforementioned assumptions on $\mathcal{G}$, this set has the following advantageous structure.

**Proposition 1.3.4** (Ljusternik (1934) in [167, Theorem 43.C])**.** *Let the above assumptions on the constraint functional $\mathcal{G}$ hold and define $\mathcal{M} := \{u \in U : \mathcal{G}(u) = 0\}$. Then*

*(i) $\mathcal{M}$ is a $C^1$-manifold and*

*(ii) the tangent space at $u_0 \in \mathcal{M}$ is given by $\mathcal{T}_{u_0}\mathcal{M} = \ker(\mathcal{G}'(u_0))$.*

Before we define a gradient flow that remains in the manifold $\mathcal{M}$, we recall the characterization of constraint critical points, i.e. critical points under the constraint $\mathcal{G} = 0$. This characterization can be found either using tangent vectors (see [135, Definition 2.12]) or using Lagrange multipliers (see [167, Section 43.9]). Let us take a look at the latter. In [167, Theorem 43.21], we find that the function $\mathcal{F}$ has a critical point with respect to $\mathcal{M}$ at $u_0$ if and only if

$$\mathcal{F}'(u_0)\phi - \sum_{i=1}^{k} \lambda_i (\mathcal{G}^i)'(u_0)\phi = 0 \quad \text{for all } \phi \in V \tag{1.3.3}$$

for some $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$. Equation (1.3.3) is called *weak Euler–Lagrange equation*. We assume that the $H$-gradients of $\mathcal{G}^i$, $i = 1, \dots, k$, exist for all $u \in U$. Then we can write (1.3.3) as the equivalent *strong Euler–Lagrange equation*, given by

$$\nabla_H \mathcal{F}(u_0) - \Lambda \cdot \nabla_H \mathcal{G}(u_0) := \nabla_H \mathcal{F}(u_0) - \sum_{i=1}^{k} \lambda_i \nabla_H \mathcal{G}^i(u_0) = 0 \quad \text{in } H. \tag{1.3.4}$$

If $k = 1$, the Lagrange multiplier $\Lambda = \lambda$ in (1.3.4) is given by

$$\lambda = \frac{\langle \nabla_H \mathcal{F}(u_0), \nabla_H \mathcal{G}(u_0) \rangle_H}{\|\nabla_H \mathcal{G}(u_0)\|_H^2}.$$

Note that $\nabla_H \mathcal{G}(u_0) \neq 0$ due to the surjectivity of $\mathcal{G}'(u_0)$. For $k > 1$, the representations of the Lagrange multipliers become more involved and, in particular, interdependent. We consider an example in Chapter 2 for $k = 3$.
With regard to Proposition 1.3.4, the right hand side of (1.3.4) is the projection of the full $H$-gradient $\nabla_H \mathcal{F}(u_0)$ onto the $H$-closure of the tangent space $\mathcal{T}_{u_0}\mathcal{M}$. We define the *constrained gradient flow* as the flow with velocity not $-\nabla_H \mathcal{F}$ but the negative of the gradient projected onto the tangent space of $\mathcal{M}$. Let us make this concrete for $k = 1$.

**Definition 1.3.5** (Constrained gradient flow)**.** Let $k = 1$. The ($\mathcal{G}$-)constrained gradient flow of $\mathcal{F} \in C^1(U)$ is the differential equation

$$\partial_t u(t) = -\nabla_H \mathcal{F}(u(t)) + \lambda(u(t)) \nabla_H \mathcal{G}(u(t)), \tag{1.3.5}$$

where

$$\lambda(u(t)) = \frac{\langle \nabla_H \mathcal{F}(u(t)), \nabla_H \mathcal{G}(u(t)) \rangle_H}{\|\nabla_H \mathcal{G}(u(t))\|_H^2}.$$

A solution of the constrained gradient flow is a function $u \in W^{1,2}(0, T; H) \cap L^2(0, T; V)$, $T > 0$ such that $u(t) \in U$, $\nabla_H \mathcal{G}(u(t)) \neq 0$ and (1.3.5) holds for almost every $t \in (0, T)$ .

The constraint $\mathcal{G} = 0$ is indeed preserved along the evolution, as for a sufficiently smooth solution $u$ of (1.3.5),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{G}(u(t)) &= \langle \nabla_H \mathcal{G}(u(t)), \partial_t u(t) \rangle_H \\ &= -\langle \nabla_H \mathcal{G}(u(t)), \nabla_H \mathcal{F}(u(t)) \rangle_H + \lambda(u(t)) \|\nabla_H \mathcal{G}(u(t))\|_H^2 = 0 \end{aligned} \tag{1.3.6}$$

holds. At the same time, we see from (1.3.6) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}(u(t)) &= \langle \nabla_H \mathcal{F}(u(t)), \partial_t u(t) \rangle_H = \langle \nabla_H \mathcal{F}(u(t)) - \lambda(u(t)) \nabla_H \mathcal{G}(u(t)), \partial_t u(t) \rangle_H \\ &= -\|\partial_t u(t)\|_H^2, \end{aligned}$$

i.e. the energy is strictly decreasing in time unless the solution is constant. We come back to Example 1.3.2 and impose the condition that the integral over the solution remains constant. This is a typical physically motivated condition describing, for example, conservation of a total mass.

**Example 1.3.6.** Let $V = W^{2,2}(0,1) \cap H_0^1(0,1) \cap \{u'(0) = u'(1)\}$, $H = L^2(0,1)$ and

$$\mathcal{F}_\varepsilon \colon V \to \mathbb{R}, \quad \mathcal{F}_\varepsilon(u) = \int_0^1 \left( W(u(x)) + \frac{1}{2}\varepsilon(u'(x))^2 \right) \mathrm{d}x$$

as in Example 1.3.2. Define

$$\mathcal{G} \colon V \to \mathbb{R}, \quad \mathcal{G}(u) = \int_0^1 u(x)\,\mathrm{d}x - M$$

for $M \in \mathbb{R}$. Then $\nabla_H \mathcal{G}(u) = 1$ and the $H$-gradient flow of $\mathcal{F}_\varepsilon$ subject to the constraint $\mathcal{G} = 0$ is given by

$$\partial_t u(t,x) = \varepsilon \partial_x^2 u(t,x) - W'(u(t,x)) + \int_0^1 W'(u(t,x))\,\mathrm{d}x.$$

In physics, for example for two-phase problems, it is common to choose $W$ as a double-well potential, for example $W'(u) = u^3 - u$. For this choice, we obtain

$$\partial_t u(t,x) = \varepsilon \partial_x^2 u(t,x) - (u(t,x))^3 + u(t,x) + \int_0^1 (u(t,x))^3\,\mathrm{d}x.$$

We emphasize that a different choice of $H$ would result in a different evolution equation. Depending on the physics we want to describe, or the properties we want the flow to have, we have to choose $H$ accordingly. In Example 1.3.6, it is possible to satisfy the constraint $\mathcal{G} = 0$ without nonlocal term by a clever choice of $H$. This means considering an unconstrained flow and still maintaining the constraint. Here, this could be achieved with choosing $H$ to be the zero-average subspace of the dual of $H^1$. This way, we would obtain the Cahn–Hilliard equation in one dimension, see [61]. However, when the constraints become more complex and can no longer be described by open subsets of Banach spaces, constrained gradient flows may be a more appropriate framework. We consider an example for elaborated constraints in Section 1.5.

### 1.3.3. Key questions

At the end of this section, we summarize the key questions commonly explored in the study of gradient flows. Here, we are brief and general. For specific equations, we address these questions in detail in Chapter 2 and Chapter 3.

**① Local well-posedness**

Given a suitable initial datum, does there exist a solution to (1.3.3), or to (1.3.5), for small times? Is this solution unique? And how does it depend on the initial datum? If the answer to the first two questions is yes and the answer to the third question is that the dependence is continuous, then the problem is locally well-posed in the sense of Hadamard. The answers generally depend not only on the flow equation but also on the initial datum, particularly its regularity. In the context of geometric evolution equations, one often considers smooth initial data, and assumes local well-posedness to be given. In Chapter 2, however, we thoroughly address this issue and establish local well-posedness for initial data in a weak regularity class.

**② Global in time existence**

Once we know that a solution exists, it is natural to investigate whether it can be extended beyond any finite time. The answer strongly depends on the given equation. In Section 1.4.3, we see examples where the maximal existence time is finite. Whereas for the evolution equations we study in Chapter 2 and Chapter 3, we show that the solution exists for all times. To do so, we prove that suitable norms of the solution remain bounded for finite times. From this, it follows that the solution can be extended and hence must exist globally in time.

**③ Convergence**

In the case where the solution exists for all times, the question of convergence arises. If we show that suitable norms of the solution are globally bounded, we can deduce subconvergence. By this, we mean that given a sequence of times, there is a subsequence along which the solution converges (in some sense). However, it is important to note that, in general, different subsequences may converge to different limits. Therefore, additional work is required to establish full convergence. One approach employs a Łojasiewicz–Simon gradient inequality or an appropriate constrained version of it. More details on this can be found in Chapter 2 and Chapter 3, where we apply this method to prove convergence.

**④ Properties of the solution**

Examining qualitative properties of solutions is a question of equal interest. For instance, one might investigate whether and under what conditions properties such as positivity of the initial datum are preserved along the evolution or how the set of zeroes of the solution develops. The answers again depend strongly on the precise structure of the equation, particularly on its order. In the geometric context and especially when studying the evolution of curves, one might ask whether the curve remains embedded along the evolution or whether a solution for a convex initial datum remains convex. In Chapter 2, we provide a detailed (non)preservation analysis of these and other properties.

**⑤ Characterization of the limit**

If the solution converges to a critical point of the energy, the following natural question arises: can we determine the limit? There are cases where all stationary solutions can be explicitly characterized, allowing us to precisely describe the limit. In other cases, by carefully choosing certain parameters and the initial data, we can also characterize the limit. This point is also closely related to the previous one, as properties that are preserved along the evolution often carry over to the limit.

To address Questions ④ and ⑤, numerical experiments can be quite helpful. They can provide insights into which directions one should guide the research. We make use of this tool to some extent in Chapter 2.

## 1.4. The classical elastic flow and other geometric flows

In this chapter, we derive the elastic flow as the $L^2$-gradient flow of the elastic energy and summarize the main results of research. Additionally, we briefly introduce other geometric flows to get an impression of how the solutions to other equations behave.
Since the main results of this thesis relate to closed curves, we focus on curves defined on $\mathbb{S}^1$ also in this section. Remember that we identify $\mathbb{S}^1 \cong \mathbb{R}/(L\mathbb{Z})$, $L > 0$ (in most cases, $L = 1$) such that a curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^n$ can be lifted to an $L$-periodic function $\gamma\colon \mathbb{R} \to \mathbb{R}^n$.

### 1.4.1. The $L^2$-gradient flow of the elastic energy

We define $V := W^{2,2}(\mathbb{S}^1;\mathbb{R}^n)$ and $U := \{\gamma \in W^{2,2}(\mathbb{S}^1;\mathbb{R}^n) : \gamma \text{ regular curve}\}$. By Sobolev embedding, $U$ is an open subset of $V$. The space with respect to which we take the gradient shall be $H := L^2(\mathbb{S}^1;\mathbb{R}^n)$. Consider now

$$\begin{aligned}
&\mathcal{E}\colon U \to \mathbb{R}, \quad \gamma \mapsto \mathcal{E}(\gamma) = \frac{1}{2}\int_{\mathbb{S}^1} |\kappa|^2 \,\mathrm{d}s = \frac{1}{2}\int_0^1 |\kappa(x)|^2 |\partial_x\gamma| \,\mathrm{d}x \quad \text{and}\\
&\mathcal{L}\colon U \to \mathbb{R}, \quad \gamma \mapsto \mathcal{L}(\gamma) = \int_{\mathbb{S}^1} \mathrm{d}s = \int_0^1 |\partial_x\gamma| \,\mathrm{d}x.
\end{aligned}$$

The $H$-gradients and their domains in the sense of Definition 1.3.1 are given by

$$\begin{aligned}
&D(\nabla_H\mathcal{E}) = U \cap W^{4,2}(\mathbb{S}^1;\mathbb{R}^n), \quad \nabla_H\mathcal{E}(\gamma) = \Big(\nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa\Big)|\partial_x\gamma| \quad \text{and}\\
&D(\nabla_H\mathcal{L}) = U, \quad \nabla_H\mathcal{L}(\gamma) = -\kappa\,|\partial_x\gamma|,
\end{aligned}$$

see also [44, Section 2]. The gradients as defined here play a role when working with functional analytical arguments (see for example Section 3.5 or [44, Section 3]). To define the elastic flow we switch to the geometric setting and work with the following geometric

spaces. For a fixed $\gamma \in U$, we consider $L^2(\mathbb{S}^1, \mathrm{d}s_\gamma; \mathbb{R}^n)$ with the inner product

$$\langle \cdot, \cdot \rangle_{L^2(\mathrm{d}s_\gamma)} \colon (X_1, X_2) \mapsto \int_{\mathbb{S}^1} \langle X_1, X_2 \rangle \, \mathrm{d}s_\gamma, \tag{1.4.1}$$

where $\mathrm{d}s_\gamma = |\partial_x \gamma| \, \mathrm{d}x$. Moreover, we consider $W^{2,2}(\mathbb{S}^1, \mathrm{d}s_\gamma; \mathbb{R}^n)$, the $W^{2,2}$-Sobolev space with respect to the arclength differentiation $\partial_s = |\partial_x \gamma|^{-1} \partial_x$. For $\gamma \in U$, we find that $L^2(\mathbb{S}^1, \mathrm{d}s_\gamma; \mathbb{R}^n) = H$ and $W^{2,2}(\mathbb{S}^1, \mathrm{d}s_\gamma; \mathbb{R}^n) = V$. Therefore, it is not essential whether we define the function spaces with respect to the measure $\mathrm{d}s$ or $\mathrm{d}x$. However, the choice of the inner product is important. With respect to the inner product defined in (1.4.1), we obtain the geometric gradients

$$\nabla_{L^2(\mathrm{d}s_\gamma)} \mathcal{E}(\gamma) = \nabla_s^2 \kappa + \frac{1}{2} |\kappa|^2 \kappa \quad \text{and} \quad \nabla_{L^2(\mathrm{d}s_\gamma)} \mathcal{L}(\gamma) = -\kappa. \tag{1.4.2}$$

We refer to them as *geometric* gradients because they behave appropriately under reparameterizations. This means that for a reparameterization $\tilde{\gamma} = \gamma \circ \Phi \in U$,

$$\nabla_{L^2(\mathrm{d}s_{\tilde{\gamma}})} \mathcal{E}(\tilde{\gamma}) = \nabla_{L^2(\mathrm{d}s_\gamma)} \mathcal{E}(\gamma) \circ \Phi \quad \text{and} \quad \nabla_{L^2(\mathrm{d}s_{\tilde{\gamma}})} \mathcal{L}(\tilde{\gamma}) = \nabla_{L^2(\mathrm{d}s_\gamma)} \mathcal{L}(\gamma) \circ \Phi.$$

In the sequel, we omit the index $\gamma$ in $s_\gamma$ where it is not necessary. In the literature, the following variants of the elastic flow are common.

#### ① The free elastic flow

With (1.4.2), the $L^2$-gradient flow of the elastic energy $\mathcal{E}$ is defined by

$$\partial_t^\perp \gamma = -\nabla_{L^2(\mathrm{d}s)} \mathcal{E}(\gamma) = -\nabla_s^2 \kappa - \frac{1}{2} |\kappa|^2 \kappa. \tag{FEL}$$

This flow is also called the *free elastic flow*, see for example [122]. Here, $\partial_t^\perp = P^\perp \partial_t$, where $P^\perp$ is the normal projection along $\gamma$ as in Section 1.2. It is more or less a matter of preference whether one considers only the normal component of the velocity ([4], [114], [139], ...) or also prescribes the tangential component ([40], [43], [53], ...). The same applies to the flows defined in the following paragraphs. One could argue that, to do justice to the geometric nature of the energy, we should consider only the normal component. In this case, a reparameterization of a solution is again a solution, as the next remark shows.

**Remark 1.4.1** (Invariance under reparametrization)**.** Let $T > 0$ and $\gamma \colon (0,T) \times \mathbb{S}^1 \to \mathbb{R}^n$ be a smooth solution to (FEL). Then

$$\partial_t \gamma = -\nabla_{L^2(\mathrm{d}s)} \mathcal{E}(\gamma) + h \partial_s \gamma$$

for some smooth $h \colon (0,T) \times \mathbb{S}^1 \to \mathbb{R}$. Let $\Phi \colon (0,T) \times \mathbb{S}^1 \to \mathbb{S}^1$ be a smooth one-parameter

family of diffeomorphisms. Then the reparametrization $\tilde{\gamma}(t,x) := \gamma(t,\Phi(t,x))$ solves

$$\begin{aligned}\partial_t\tilde{\gamma} &= \partial_t\gamma(t,\Phi(t,x)) + \partial_t\Phi(t,x)\partial_x\gamma(t,\Phi(t,x))\\ &= -\nabla_{L^2(\mathrm{d}s_\gamma)}\mathcal{E}(\gamma(t,\Phi(t,x))) + h(t,\Phi(t,x))\partial_s\gamma(t,\Phi(t,x)) + \partial_t\Phi(t,x)\partial_x\gamma(t,\Phi(t,x))\\ &= -\nabla_{L^2(\mathrm{d}s_{\tilde{\gamma}})}\mathcal{E}(\tilde{\gamma}) + \tilde{h}\partial_s\tilde{\gamma}\end{aligned}$$

for some smooth $\tilde{h}\colon [0,T)\times\mathbb{S}^1\to\mathbb{R}$. Thus, $\partial_t^\perp\tilde{\gamma} = -\nabla_{L^2(\mathrm{d}s)}\mathcal{E}(\tilde{\gamma})$.

As observed in Equation (1.2.9), the elastic energy can be made arbitrarily small by blowing up the curve. This characteristic reappears within the framework of (FEL) as the following example shows.

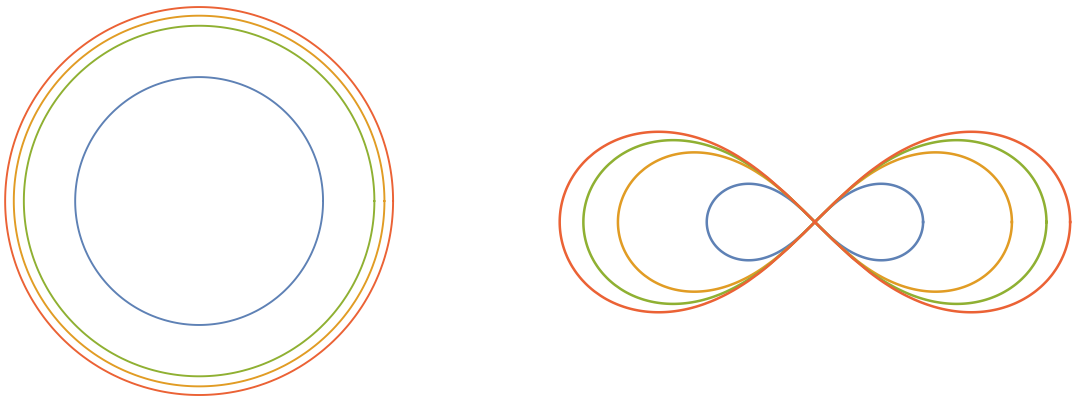
**Example 1.4.2** (Selfsimilar expanding solutions to (FEL))**.** The family of evolving $\omega$-fold covered circles

$$c(t,x) = (1+2t)^{\frac{1}{4}}\begin{pmatrix}\cos(2\pi\omega x)\\ \sin(2\pi\omega x)\end{pmatrix}$$

(see Figure 1.9A) and the evolving lemniscate of Bernoulli

$$\beta(t,x) = (1+30t)^{\frac{1}{4}}\frac{1}{1+\sin^2 x}\begin{pmatrix}\cos x\\ \frac{1}{2}\sin(2x)\end{pmatrix}$$

(see Figure 1.9B) are selfsimilar expanding solutions to (FEL). This can be shown by a straightforward computation which can be found, for example, in [122, Section 2.2.2 and Section 2.2.3].



(A) (B)

Figure 1.9: Selfsimilar expanding circle $c$ with $\omega = 1$ for $t = 0, 2, 4, 6$ in (A) and selfsimilar expanding lemniscate of Bernoulli $\beta$ for $t = 0, \frac{1}{3}, \frac{2}{3}, 1$ in (B).

#### ② The length penalized elastic flow

The *length penalized elastic flow* is the $L^2$-gradient flow of the length penalized elastic energy $\mathcal{E}_\lambda = \mathcal{E} + \lambda\mathcal{L}$ for some constant parameter $\lambda > 0$, see for example [134] or [53]. It is given by

$$\partial_t^\perp \gamma = -\nabla_{L^2(\,\mathrm{d}s)}\mathcal{E}(\gamma) - \lambda\nabla_{L^2(\,\mathrm{d}s)}\mathcal{L}(\gamma) = -\nabla_s^2\kappa - \frac{1}{2}|\kappa|^2\kappa + \lambda\kappa \tag{PEF}$$

(see (1.4.2)). The length penalization ensures that the length of a solution is bounded, meaning that in particular, selfsimilarly expanding solutions like in Example 1.4.2 cannot exist. This can be easily observed from the energy decay

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\gamma(t)) = \langle\nabla_{L^2(\,\mathrm{d}s)}\mathcal{E}_\lambda(\gamma(t)), \partial_t\gamma(t)\rangle_{L^2(\,\mathrm{d}s)} = -\|\partial_t^\perp\gamma(t)\|^2_{L^2(\,\mathrm{d}s)} \leq 0, \tag{1.4.3}$$

for a sufficiently smooth solution $\gamma$ of (PEF). Due to (1.4.3), the length

$$\mathcal{L}(\gamma(t)) \leq \frac{1}{\lambda}(\mathcal{E}(\gamma(t)) + \lambda\mathcal{L}(\gamma(t))) = \frac{1}{\lambda}\mathcal{E}_\lambda(\gamma(t))$$

is bounded.

#### ③ The length constrained elastic flow

Instead of penalizing the length, we can also keep it fix, see for example [161], [53], or [139]. For this, we define the constraint functional $\mathcal{G}(\gamma) := \mathcal{L}(\gamma) - L$ for some given $L > 0$ and consider the *length constrained gradient flow*

$$\partial_t^\perp \gamma = -\nabla_{L^2(\,\mathrm{d}s)}\mathcal{E}(\gamma) + \lambda(\gamma)\nabla_{L^2(\,\mathrm{d}s)}\mathcal{G}(\gamma) = -\nabla_s^2\kappa - \frac{1}{2}|\kappa|^2\kappa + \lambda(\gamma)\kappa \tag{CEF}$$

(see Definition 1.3.5). Here, the Lagrange multiplier $\lambda$ is given by

$$\lambda(\gamma(t)) = \frac{\langle\nabla_{L^2(\,\mathrm{d}s)}\mathcal{E}(\gamma), \nabla_{L^2(\,\mathrm{d}s)}\mathcal{G}(\gamma)\rangle_{L^2(\,\mathrm{d}s)}}{\|\nabla_{L^2(\,\mathrm{d}s)}\mathcal{G}(\gamma)\|^2_{L^2(\,\mathrm{d}s)}} = \frac{\int_{\mathbb{S}^1}\langle\nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa, \kappa\rangle\,\mathrm{d}s}{\int_{\mathbb{S}^1}|\kappa|^2\,\mathrm{d}s}.$$

As in (1.3.6), one checks that for a solution $\gamma$ to (CEF), the length $\mathcal{L}(\gamma)$ is constant for all times.

When we refer to *the classical elastic flow*, we mean the $L^2(\,\mathrm{d}s)$-gradient flow of the elastic energy, either with or without penalized length or with fixed length. Thus, we are referring to one of the variants (FEL), (PEF) or (CEF) which we all call the classical elastic flow.

### 1.4.2. Results on the classical elastic flow of closed curves

In the last decades, several authors have studied (FEL), (PEF) and (CEF) in different variants. To our knowledge, the first to address the topic was Polden, who investigated the

length penalized elastic flow in his PhD thesis ([134]). He proves in [134, Theorem 3.2.2.5] that for any planar smooth immersed initial curve, there exists a global in time solution to (PEF). Moreover, he shows that a solution converges along subsequences (see [134, Theorem 3.2.3.1]) and if the rotation index is $\pm 1$, the limit is a unique circle (see [134, Corollary 3.2.3.3]). Eight years later, Dziuk, Kuwert and Schätzle (see [53]) generalized the results of Polden to curves in $\mathbb{R}^n$. They study (FEL), (PEF) as well as (CEF). In summary, we have the following.

**Proposition 1.4.3** (Global existence and subconvergence, [53, Section 3])**.** *Let $\gamma_0 \colon \mathbb{S}^1 \to \mathbb{R}^n$ be a smooth regular curve. Then* (FEL)*,* (PEF) *and* (CEF) *with initial datum $\gamma_0$ have a global solution $\gamma \colon [0,\infty) \times \mathbb{S}^1 \to \mathbb{R}^n$. For* (PEF) *and* (CEF)*, as $t_i \to \infty$, the curves $\gamma(t_i, \cdot)$ subconverge, when suitably reparametrized and translated, to an elastica.*

Let us recall that by subconverging, we mean that given the sequence $(t_i)_{i\in\mathbb{N}}$, $t_i \to \infty$, there exists a subsequence $(t_{i_k})_{k\in\mathbb{N}}$ such that $\gamma(t_{i_k}, \cdot)$ converges smoothly. After seeing Example 1.4.2, it is evident that we cannot expect subconvergence for the free elastic flow. A closer examination of the existence proof, for instance by following the arguments in [114, Section 3], reveals geometric uniqueness. This means that the solutions are unique up to reparametrizations.
With the help of a Łojasiewicz–Simon gradient inequality for the length penalized elastic energy, Mantegazza and Pozzetta prove full convergence of the solution to (PEF) in [115].

**Proposition 1.4.4** (Convergence, [115])**.** *Let $\gamma \colon [0,\infty) \times \mathbb{S}^1 \to \mathbb{R}^n$ be the solution to* (PEF) *with smooth regular initial datum $\gamma_0$. Then there exists an elastica $\gamma_\infty$ such that $\gamma(t, \cdot) \to \gamma_\infty$, $t \to \infty$, smoothly, up to reparametrization.*

This result implies in particular that the solution remains in a compact set for all times. Since no maximum principle is available for higher-order flows like (PEF), which would allow us to conclude this property by comparison arguments, this is a remarkable result. With a suitable version of a constrained Łojasiewicz–Simon gradient inequality based on [135], the same convergence result can also be shown for the solution to (CEF). The arguments for this are similar to those in [139, Section 4].

### 1.4.3. Other geometric flows

We derived the elastic flow as the $L^2$-gradient flow of the elastic energy. Considering the $L^2$-gradient of the length as the velocity, we arrive at the so-called *curve shortening flow*

$$\begin{cases} \partial_t^\perp \gamma = \kappa & \text{in } (0,T) \times \mathbb{S}^1, \\ \gamma(0,\cdot) = \gamma_0 & \text{on } \mathbb{S}^1, \end{cases}$$

for a sufficiently smooth regular initial curve $\gamma_0 \colon \mathbb{S}^1 \to \mathbb{R}^2$, see for example [33]. The curve evolves by moving in the direction of its normal vector, with speed proportional to its

scalar curvature — where it bends the most it moves the fastest. The curve shortening flow (CSF) originated in the 1950s in materials science and was more rigorously investigated in the mid-to-late 1980s by mathematicians such as Michael Gage, Richard Hamilton, and Matthew Grayson. Here, we only mention a few key aspects of the CSF. For a detailed introduction and further information, we refer to [33]. First, we note that along the flow, the length of a solution $\gamma\colon (0,T)\times\mathbb{S}^1\to\mathbb{R}^2$ decreases like

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) = -\|\partial_t^\perp\gamma\|^2_{L^2(\,\mathrm{d}s)} = -\int_{\mathbb{S}^1}|\kappa|^2\,\mathrm{d}s \leq 0.$$

The signed enclosed area, see (1.2.6), changes like

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(\gamma(t)) = -\int_{\mathbb{S}^1}\langle\partial_t\gamma,\nu\rangle\,\mathrm{d}s = -\int_{\mathbb{S}^1}\langle\kappa,\nu\rangle\,\mathrm{d}s = -\int_{\mathbb{S}^1}k\,\mathrm{d}s = -2\pi\omega. \tag{1.4.4}$$

Here, $k$ is the signed scalar curvature of the solution, see (1.2.5), and $\omega$ is the rotation index, see (1.2.7), which does not change during the evolution. An important property of the CSF is that it preserves embeddedness, see [65, Theorem 3.2.1]. The proof primarily relies on the maximum principle applicable to the CSF, which is a second order flow. The maximum principle can also be applied to establish so-called avoidance principles. For instance, it can be shown that if a regular smooth embedded curve lies within another such curve, the curves will never touch each other. Similarly, a convex initial curve that is contained within a convex set will never leave that set. It is not difficult to see that if we take a circle of radius $r_0$ as initial datum, the solution to the CSF is given by a circle of radius $r(t)=\sqrt{r_0^2-2t}$, see also (1.2.4). In particular, this means that in this case, the flow develops a singularity in finite time. Along with the avoidance principles, this shows that the CSF causes any embedded curve contained within a compact set to vanish in finite time. The CSF has been studied in various forms, such as Gage's area-preserving variant (see [64]), and, for example, also for curves on Riemannian manifolds (see Section 1.2.4), for instance in [75].

If we consider not the $L^2$-gradient but an $H^{-1}$-gradient of the length, then we obtain the so-called *curve diffusion flow*

$$\begin{cases}\partial_t^\perp\gamma = -\nabla_s^2\kappa & \text{in } (0,T)\times\mathbb{S}^1,\\ \gamma(0,\cdot)=\gamma_0 & \text{on } \mathbb{S}^1.\end{cases}$$

We do not intend to give a precise definition of the Hilbert space in which the gradient is computed at this point. In Section 3.1.1, we derive an $H^{-1}$-gradient flow of the elastic energy and define an appropriate Hilbert space setting for this. Within this framework, the curve diffusion flow (CDF) can be derived analogously. The CDF was studied in various articles by Taylor and Cahn ([151]), Elliott and Garcke ([57]), Polden ([134, Section 3]),

Giga, Escher, and Ito ([68], [69], [58]) and many others. Along a solution $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ of the CDF, the length decreases with velocity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\gamma(t)) = -\int_{\mathbb{S}^1} |\nabla_s \kappa|^2 \,\mathrm{d}s \leq 0 \tag{1.4.5}$$

and the signed enclosed area is constant. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(\gamma(t)) = -\int_{\mathbb{S}^1} \langle \partial_t \gamma, \nu\rangle \,\mathrm{d}s = \int_{\mathbb{S}^1} \langle \nabla_s^2 \gamma, \nu\rangle \,\mathrm{d}s = -\int_{\mathbb{S}^1} \langle \nabla_s \gamma, \partial_s \nu\rangle \,\mathrm{d}s = 0, \tag{1.4.6}$$

where we integrated by parts and used that $\partial_s \nu = -k\partial_s \gamma$ is tangential. The CDF is a fourth order equation, and as such we do not expect that a comparison principle holds. Indeed, in contrast to the curve shortening flow, the CDF does not preserve embeddedness in general. There are examples of embedded curves developing a self-intersection in finite time (see [69]). Moreover, it is known that the CDF can develop finite curvature singularities starting from smooth regular initial data. For a blow-up criterion, see for example [32, Proposition B].

It is also possible to study not only the evolution of curves, but also the evolution of $n$-dimensional submanifolds in $\mathbb{R}^{n+1}$. The so-called *mean curvature flow* evolves a hypersurface with velocity given by the mean curvature vector. It is known that under the mean curvature flow, surfaces usually develop singularities in finite time. For more on the mean curvature flow, we refer to [54], [111] and [79].
Transferring the curve diffusion flow to hypersurfaces, we come to the so-called *surface diffusion flow*, see for example [59]. This fourth order flow decreases the area of an evolving embedded closed hypersurface (compare to (1.4.5)) while keeping the enclosed volume fixed (compare to (1.4.6)).

We also want to mention the so-called *Willmore flow*, the $L^2$-gradient flow of the Willmore energy (1.2.17). In recent years, this flow has been studied extensively. First existence and convergence results were established by Simonett ([144]) and Kuwert and Schätzle ([93] and [94]). In [41], the Willmore flow of tori of revolution is studied. Here, the relationship between curves in $\mathbb{H}^2$ and the Willmore energy of surfaces of revolution (see (1.2.19) and the text above) is explored for the first time for a problem involving time. We want to note that the Willmore flow has also been studied as a constrained gradient flow, see Section 1.3.2. In [136] and [137], the constraints of fixed volume and fixed isoperimetric ratio are realized using suitable nonlocal Lagrange multipliers.
The study of the $L^2$-gradient flow of the Helfrich energy (1.2.20), known as the *Helfrich flow*, is just starting to get more attention. In [140], the influence of a nonzero spontaneous curvature is considered for the first time.

To conclude this section, we note that all the flows mentioned so far are *extrinsic* geometric flows, meaning they describe the evolution of an immersed submanifold. However, there

are also *intrinsic* geometric flows, which describe how the metric on a Riemannian manifold changes. These flows are another significant area of research not covered in this thesis.

## 1.5. Elastic flow with modulated stiffness

In Chapter 2, we study the elastic flow with modulated bending stiffness. What is this generalization, and what are the underlying motivations and ideas behind it?

In shape optimization, the energy and thus the equilibrium configuration of an elastic shell can depend both on its geometry as well as the distribution of some physical densities, representing, for example mass, electrostatic charge, or temperature differences. For instance, the shape can depend on the material composition and vice versa, as it is the case for certain biomembranes [86, 13, 119]. This phenomenon is not accounted for in the classical Canham–Helfrich model that we introduced in Section 1.1. Thus, we aim to adjust and generalize the model. Let us briefly review the progress that has already been made on this topic.

For the case that the physical densities are discrete, a variational existence theory has already been established, relying on either rotational symmetry [31, 83] or weak formulations using curvature varifolds [19].

In [18], a one-dimensional model which attributes for the aforementioned interplay between curvature and density-related effects has been introduced. Brazda, Jankowiak, Schmeiser and Stefanelli equip closed planar curves with a (nondiscrete) density function. We use this model in a slightly more general form in the following.

We mention that in [76], the Euler–Lagrange equations for the energy introduced in [18] are examined for a given density function. Here, the authors assume a one-sided effect: the density influences the curvature, but not vice versa.

Moreover, for the discrete setting, we also refer to [82] and mention a related discrete-to-continuum $\Gamma$-limit result in [52].

Let us now define the energy we work with. We consider a heterogeneous elastic wire, described by a planar closed curve $\gamma$ and a density function $\rho$. The density function describes the distribution of the physical densities that affect the curve's geometry, and vice versa. For the moment, we assume $\gamma$ and $\rho$ to be smooth. The energy of the heterogeneous elastic wire is given by a Canham–Helfrich energy (1.1.2) for curves, including a spontaneous curvature. But instead of a constant bending stiffness, the bending stiffness depends on the density $\rho$. In this way, the interplay between geometry and density effects is taken into account. Moreover, we add the Dirichlet energy of the density. To summarize, we define the generalized elastic energy

$$\mathcal{E}_\mu(\gamma,\rho) = \frac{1}{2}\int_\gamma \left(\beta(\rho)(k-c_0)^2 + \mu\,(\partial_s\rho)^2\right)\,\mathrm{d}s. \tag{1.5.1}$$

As in Section 1.2.1, $k$ is the signed curvature of $\gamma$, $\mathrm{d}s = |\partial_x \gamma(x)|\,\mathrm{d}x$ is the arc-length element, and $\partial_s = |\partial_x \gamma(x)|^{-1}\,\partial_x$. The density-modulated bending stiffness $\beta\colon \mathbb{R} \to \mathbb{R}$ is assumed to be a smooth positive function, the parameter $c_0 \in \mathbb{R}$ denotes the spontaneous curvature, and the constant $\mu > 0$ models the diffusivity of the density.

In this model (see also Figure 1.10), the curve strives for a preferred curvature $c_0 \in \mathbb{R}$ (determined by the material, see [119]), and the defect is penalized depending on the continuous density distribution. The diffusivity coefficient $\mu$ causes the energy to strive for homogeneity. The higher the diffusivity coefficient, the more the material tends to diffuse, meaning it aims to reach a constant density, i.e., become homogeneous. In the special case where $\rho$ is constant and $c_0 = 0$, we essentially retrieve the classical elastic energy (1.2.8). Note that for a multifold covered circle with constant curvature equal to $c_0$ and constant density, the energy (1.5.1) is zero.

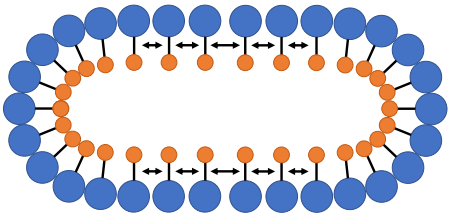


Figure 1.10: Modeling a planar heterogeneous elastic wire with constant curvature different from zero. Here, the heterogeneity is due to the molecules being packed differently dense (represented by the arrows). The structure of the molecules induces a preferred curvature, which is achieved in the curved regions.

For planar curves, the order of the energy (1.5.1) may be reduced as follows. Since (1.5.1) is invariant under orientation preserving reparametrisations, we assume that the planar curve $\gamma$ with prescribed length $L$ is parametrized by arclength. Then there exists a smooth (inclination) angle function $\theta\colon [0, L] \to \mathbb{R}$ such that

$$\partial_s \gamma = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{1.5.2}$$

Modulo isometries of $\mathbb{R}^2$, the curve $\gamma$ together with its orientation is uniquely determined by $\theta$. More precisely, for some $\gamma(0) \in \mathbb{R}^2$ we have

$$\gamma\colon [0, L] \to \mathbb{R}^2, \quad \gamma(s) = \int_0^s \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mathrm{d}r + \gamma(0). \tag{1.5.3}$$

For more details on describing a planar curve by an angle function $\theta$, see also Section 2.1.1.

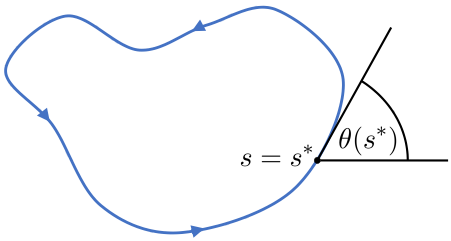



Figure 1.11: Describing a planar curve by its inclination angle $\theta$.

With $\theta$ and the density function $\rho\colon [0, L] \to \mathbb{R}$, the energy (1.5.1) can be expressed by

$$\mathcal{E}_\mu(\theta, \rho) = \frac{1}{2} \int_0^L \left( \beta(\rho)(\partial_s \theta - c_0)^2 + \mu \, (\partial_s \rho)^2 \right) \mathrm{d}s, \tag{1.5.4}$$

using that $k = \partial_s \theta$, so that $\mathcal{E}_\mu(\gamma, \rho) = \mathcal{E}_\mu(\theta, \rho)$.

Naturally, to achieve compactness, the variational discussion of (1.5.4) involves prescribing the length $L$, the rotation index $\omega$ of the curve, and the integral of the density, i.e.

$$\int_0^L \rho \, \mathrm{d}s = mL \tag{1.5.5}$$

for $m \in \mathbb{R}$. We refer to (1.5.5) also as the condition of fixed total mass. The length constraint and the preferred curvature $c_0 \in \mathbb{R}$ are competing forces for minimizing $\mathcal{E}_\mu$, in general, since prescribing the length may not allow for a curve with constant curvature $k \equiv c_0$. The constrained minimization problem $\min \mathcal{E}_\mu(\theta, \rho)$ with zero spontaneous curvature and rotation index equal to one was studied in [18]. In Section 2.2, we continue the discussion of the static problem. However, our primary focus lies on adopting a dynamic approach to decrease the elastic energy $\mathcal{E}_\mu$ by evolving $(\theta, \rho)$ by the associated $L^2$-gradient flow. This flow can be used to describe, in the simplified quasistatic and viscous regime, the continuous deformation towards an energetically more favourable state. In addition to the constraints on length and rotation index, we need to ensure that the angle function describes a closed curve and that the integral of the density does not change. This is why we work with a constrained gradient flow containing nonlocal Lagrange multipliers (see Section 1.3).

At the beginning of Chapter 2, the evolution equations and the equations for the nonlocal Lagrange multipliers can be found. We propose a suitable initial value problem, for which we then study the key questions of Section 1.3.3. We show local well-posedness and global existence of solutions for initial data in a weak regularity class. Moreover, we prove full convergence of the flow. We analyze the preservation and nonpreservation of quantities such as convexity, positivity, embeddedness, and symmetry. Finally, we also study the

asymptotic behavior of the system and find conditions under which we can characterize the limit.

## 1.6. Area preserving elastic flow

In Chapter 3, we derive an unconstrained gradient flow of the elastic energy that preserves the area enclosed by the evolving planar closed curve. What is the motivation behind this? The $L^2$-gradient flow of the elastic energy is a well-studied problem. In recent years, various authors have explored numerous variants. One version of the elastic flow is the length constrained elastic flow (CEF), which we introduce in Section 1.4.1. It keeps the length of the evolving curves fixed. On the modeling level, this means we assume that the elastic wires are inextensible. The length constrained elastic flow has been studied in several works, see, for example [92], [39], [40], [53], or [139].
In contrast, to the best of our knowledge, there is only one work ([129]) in which the elastic flow under the constraint of fixed enclosed area is studied. The area preserving scenario is physically relevant, for instance, to describe an interface between an inner domain and an outer domain filled with two kinds of incompressible viscous fluids, see Figure 1.12. In particular, there are several works exploring area preserving variants for other flows of curves, see for example [64], [154], or the recent articles [157], [87]. In the survey article [114, Section 6] on the elastic flow of curves, the authors explicitly point out the absence of studies of an area preserving elastic flow.
The higher-dimensional analogue to the elastic flow with the constraint of fixed enclosed area is the Willmore flow with the constraint of fixed enclosed volume. This flow is studied in [136].

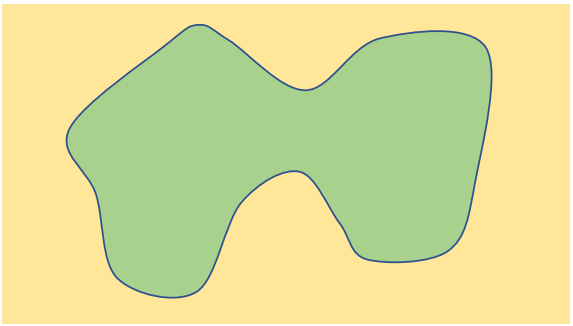


Figure 1.12: Illustration of an elastic wire describing an interface between to domains filled with incompressible fluids.

Let us come back to the aforementioned article [129], in which an area constrained elastic flow is studied. In [129], Okabe derives a system of equations with a nonlocal Lagrange multiplier that governs the $L^2$-gradient of the elastic energy (1.2.8) under the constraints of inextensiblity and preserved enclosed area. For this nonlocal system, existence of a

unique solution and convergence is proven.
In some models, local evolution equations that do not depend on any action at a distance are preferred (see for example [61]). This motivates to search for an evolution equation which decreases the elastic energy (1.2.8) and preserves the enclosed area without the need for nonlocal Lagrange multipliers. Inspired by the Cahn–Hilliard equation, which can be seen as an $H^{-1}$-gradient flow preserving a fixed total mass without any nonlocal Lagrange multiplier, one could also consider an $H^{-1}$-gradient approach for the elastic energy (1.2.8). We note that this approach also works for the length functional. In this context, we have the curve shortening flow as the $L^2$-gradient flow, see Section 1.4.3. We have seen in (1.4.4) that it decreases the enclosed area. If the enclosed area is to remain constant, this can be achieved with the constrained $L^2$-gradient flow with a nonlocal Lagrange multiplier. We obtain the area constrained curve shortening flow which is studied in several articles, see [64], [28], [47] and others. On the other hand, we can derive the curve diffusion flow as an $H^{-1}$-gradient flow of the length, see Section 1.4.3. It preserves the enclosed area without nonlocality.
The $H^{-1}$-gradient approach increases the order of the evolution equation by two in comparison to the $L^2$-gradient flow. This effect is also observed when applying the approach to the elastic energy. It transforms the classical elastic flow with fourth order velocity into the sixth order flow

$$\partial_t^\perp \gamma = \nabla_s^2 \left( \nabla_s^2 \kappa + \tfrac{1}{2} |\kappa|^2 \kappa - \lambda \kappa \right). \tag{1.6.1}$$

We are imprecise here and only present the underlying idea. A derivation of (1.6.1) is given in Chapter 3. The flow given by (1.6.1) is an unconstrained gradient flow for the elastic energy (1.2.8). The gradient flow structure ensures that, along the flow (1.6.1), the energy decreases. Moreover, the signed enclosed area $\mathcal{A}$, see (1.2.6), is preserved. Therefore, (1.6.1) is indeed an area preserving flow without nonlocality. Unlike in [129], we consider an optional penalization of the length of the evolving curves, rather than fixing it.
The goal of Chapter 3 is to rigorously derive and then to study (1.6.1). We show that compared to the classical elastic flow, the behavior of solutions does not deteriorate. First, we prove global in time existence of the geometrically unique smooth solution to (1.6.1) with a suitable initial datum. Then, we establish convergence of the solution. Finally, we discuss the stationary solutions and see that a variety of limits is possible.

CHAPTER 2

# Elastic flow with modulated stiffness

In Section 1.5, we introduce an elastic energy for heterogeneous elastic wires, which captures the interplay between curvature and effects induced by heterogeneity. When describing the equilibrium configurations of certain biomembranes, this mutual influence cannot be ignored, which motivates this generalization of the elastic energy. We describe the elastic wire by an inclination angle function and express the energy in terms of this function and a density function $\rho$ that represents the heterogeneity. In this chapter, we study the $L^2$-gradient flow of this energy.

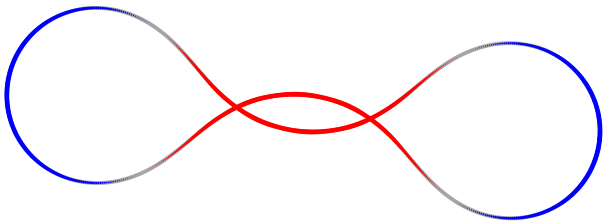


Figure 2.1: Illustration of a heterogeneous elastic wire. The color symbolizes the sign of the density, while the thickness of the line represents its absolute value.

We consider a heterogeneous elastic wire which is described by a sufficiently smooth regular closed planar curve $\gamma$ with density $\rho$. We assume that the curve is parametrized by arclength and has length $L > 0$. Then, as described in Section 1.5, the energy of the heterogeneous elastic wire can be expressed in terms of an inclination angle $\theta\colon [0, L] \to \mathbb{R}$ and the density $\rho\colon [0, L] \to \mathbb{R}$ by

$$\mathcal{E}_\mu(\theta, \rho) = \frac{1}{2} \int_0^L \left( \beta(\rho)(\partial_s \theta - c_0)^2 + \mu \, (\partial_s \rho)^2 \right) \mathrm{d}s, \tag{2.0.1}$$

see (1.5.4). As in the previous chapter, $\mathrm{d}s$ is the arclength element, and $\partial_s$ is the arclength derivative. Note that since we work with the arclength parametrization, in (2.0.1), the differential operator $\partial_s$ is just the ordinary derivative with respect to $s$, the arclength parameter. Similarly, $\mathrm{d}s$ just denotes integration with respect to the variable $s \in [0, L]$.

Let us specify the assumptions on the model. For the density modulated bending stiffness $\beta$, we assume

$$\beta \in C^\infty(\mathbb{R}) \text{ and } \beta > 0.$$

In particular, asymptotic decay to zero is permitted. Conditions such as analyticity, convexity or other growth conditions are not imposed unless explicitly stated in the results. The constant

$$\mu > 0$$

is arbitrary but strictly greater than zero. This provides us with control over the density function, which ensures both the existence of minimizers and the parabolic nature of the evolution equation for $\rho$. We show that for sufficiently large values of $\mu$, the minimizer of the energy is a multifold covered circle with constant density. The constant

$$c_0 \in \mathbb{R}, \tag{2.0.2}$$

which accounts for a spontaneous curvature, is arbitrary. In particular, the spontaneous curvature can be either positive or negative. We note that reversing the orientation of the curve corresponds to a change of the sign of $c_0$. When studying the qualitative properties of the flow, the spontaneous curvature plays a crucial role in preserving convexity. Another constant, which is fixed for the minimization problem and determined by the initial condition for the dynamic approach, is the ratio of total mass to length

$$\mathbb{R} \ni m = \frac{1}{L}\int_0^L \rho \,\mathrm{d}s.$$

Unless we are specifically discussing the preservation of positivity of the density, the constant $m$ can take any positive or negative value. The rotation index

$$\omega \in \mathbb{Z},$$

see (1.2.7), is also fixed for the minimization problem and determined by the initial condition in the dynamic approach. We allow $\omega$ to be any integer, including zero. By allowing $\omega$ to be negative, we keep track of the orientation of the curve. For the following, we fix the bending stiffness $\beta$ and the constants

$$L > 0,\ \mu > 0,\ c_0 \in \mathbb{R},\ m \in \mathbb{R},\ \text{and } \omega \in \mathbb{Z},$$

and refer to them as model parameters.

The main focus of this chapter is to study the constrained $L^2$-gradient flow associated to (2.0.1) which evolves $(\theta, \rho)$ under the conditions that $\theta$ describes a closed curve and that

the integral of the density is fixed. Since an angle function $\theta\colon [0,L] \to \mathbb{R}$ represents a zeroth order closed curve if and only if

$$\int_0^L \sin\theta \,\mathrm{d}s = \int_0^L \cos\theta \,\mathrm{d}s = 0, \tag{2.0.3}$$

we require that (2.0.3) and

$$\int_0^L \rho \,\mathrm{d}s = mL$$

hold along a solution. If we write the constraints as a constraint functional, as described in Section 1.3.2, and compute the gradients of the energy and the constraint functional, we can derive the constrained $L^2$-gradient flow. It is given by the quasilinear parabolic system

$$\begin{cases} \partial_t\theta = \partial_s(\beta(\rho)(\partial_s\theta - c_0)) + \lambda_{\theta 1}\sin\theta - \lambda_{\theta 2}\cos\theta, \\ \partial_t\rho = \mu\partial_s^2\rho - \frac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2 - \lambda_\rho. \end{cases}$$

Here, $\lambda_{\theta 1}$, $\lambda_{\theta 2}$, and $\lambda_\rho$ are nonlocal Lagrange multipliers ensuring the closedness constraint and the constraint of fixed total mass. The explicit formulas are given in Section 2.1.2. Observe that the evolution equations are coupled so that we cannot treat them separately. Another important observation is that by working with the angle function $\theta$, the order of the equation for the curve is reduced to second order. Working with $\gamma$, i.e. considering the $L^2$-gradient flow of (1.5.1), would yield a fourth order equation as for the classical elastic flow.

We require that along the evolution, the curve remains closed up to second order and that the density is $C^1$-periodic. With (1.5.2), we see that a curve described by an angle function is closed to first order if $\theta(L) - \theta(0)$ is an integer multiple of $2\pi$ and (1.2.7) shows that this multiple corresponds to the fixed rotation index $\omega$. Therefore, we consider the boundary conditions

$$\theta(s=L) - \theta(s=0) = 2\pi\omega, \quad \rho(s=L) = \rho(s=0),$$

and

$$\partial_s\theta(s=L) = \partial_s\theta(s=0), \quad \partial_s\rho(s=L) = \partial_s\rho(s=0).$$

Note that we do not prescribe a fixed value at the boundary, which results in having two boundary conditions for both $\theta$ and $\rho$.

In addition, we take an initial datum $(\theta_0, \rho_0) \in C^1([0,L])$ that belongs to a $C^2$-closed curve with $C^1$-periodic density. This means that we assume that the initial datum satisfies the

boundary conditions. For the zeroth order closedness, we additionally require that $\theta_0$ satisfies (2.0.3). To sum up, we obtain the initial boundary value problem

$$\begin{cases} \partial_t\theta = \partial_s\big(\beta(\rho)(\partial_s\theta - c_0)\big) + \lambda_{\theta 1}\sin\theta - \lambda_{\theta 2}\cos\theta & \text{in } (0,T)\times[0,L], \\ \partial_t\rho = \mu\partial_s^2\rho - \dfrac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2 - \lambda_\rho & \text{in } (0,T)\times[0,L], \\ \theta(\cdot,L) - \theta(\cdot,0) = 2\pi\omega, \qquad \rho(\cdot,L) = \rho(\cdot,0) & \text{on } [0,T), \\ \partial_s\theta(\cdot,L) = \partial_s\theta(\cdot,0), \qquad \partial_s\rho(\cdot,L) = \partial_s\rho(\cdot,0) & \text{on } [0,T), \\ \theta(0,\cdot) = \theta_0, \qquad \rho(0,\cdot) = \rho_0 & \text{on } [0,L], \end{cases} \tag{2.0.4}$$

with $(\theta_0,\rho_0)\in C^1([0,L])$ satisfying the boundary conditions and (2.0.3). The main goal of this chapter is to study (2.0.4).

After some preliminary considerations in Section 2.1, we shortly consider the variational problem associated to (2.0.1) in Section 2.2. The goal of Section 2.3 is then to establish local well-posedness of (2.0.4) for initial data in a weak regularity class. In Section 2.4, we prove global in time existence of the solution and in Section 2.5, we establish convergence. In Section 2.6 we analyze qualitative properties of the solution and in Section 2.7, we discuss assumptions under which the limit is determined.

The contents of this chapter are taken from [37] and [36].

## 2.1. Preliminaries

In the following, we first briefly discuss the description of a planar curve via an angle function, see Section 2.1.1. Next, we provide formulas for the Lagrange multipliers (see Section 2.1.2) and verify that the energy decreases along the evolution (see Section 2.1.3).

### 2.1.1. Describing planar curves

In Section 1.5, we use that a planar curve $\gamma$ with prescribed length $L$ can be described by an inclination angle function $\theta\colon[0,L]\to\mathbb{R}$. If we consider regular $W^{2,2}(0,L;\mathbb{R}^2)$-curves with $W^{1,2}(0,L;\mathbb{R})$-density, for which the the energy (2.0.1) is well-defined, such an angle function indeed exists. This can be proven analogously to the case of smooth curves in elementary differential geometry, see for example [20, Lemma 2.2.5].

**Lemma 2.1.1** ([124, Lemma A.2])**.** *For a regular curve $\gamma\in W^{2,2}(0,L,\mathbb{R}^2)$, there exists $\theta\in W^{1,2}(0,L;\mathbb{R})$ such that*

$$\partial_s\gamma = \frac{\partial_x\gamma}{|\partial_x\gamma|} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \tag{2.1.1}$$

*for $x\in[0,L]$. Any two angle functions satisfying* (2.1.1) *differ by an integer multiple of $2\pi$.*

We work with an parametrization by arclength, so $|\partial_x \gamma| \equiv 1$ in (2.1.1). For regular $W^{2,2}(0,L;\mathbb{R}^2)$-curves, such a parametrization exists. Using the Sobolev embedding $W^{2,2}(0,L;\mathbb{R}^2) \hookrightarrow C^1([0,L];\mathbb{R}^2)$, this follows like in [20, Proposition 2.1.13]. See also [124, Lemma A.6].

**Lemma 2.1.2.** *For a regular curve curve $\gamma \in W^{2,2}(0,L,\mathbb{R}^2)$, there exists an arclength parametrization $\tilde{\gamma} \in W^{2,2}(0,L,\mathbb{R}^2)$.*

At this point, we also refer to (1.5.3), which explains how to derive the curve $\gamma$ from a given angle function $\theta$.

### 2.1.2. The Lagrange multipliers

Let us take a closer look at the three Lagrange multipliers in (2.0.4). In the first equation in (2.0.4), $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$ ensure that $\gamma$ remains closed along the flow if the initial curve is closed. We define

$$\begin{pmatrix} \lambda_{\theta 1}(t) \\ \lambda_{\theta 2}(t) \end{pmatrix} := \Pi^{-1}(\theta)(t) \int_0^L \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \partial_s(\beta(\rho)(\partial_s\theta - c_0))\,\mathrm{d}s, \tag{2.1.2}$$

where $\Pi^{-1}(\theta)(t)$ denotes the inverse of the matrix

$$\Pi(\theta)(t) := \begin{pmatrix} \int_0^L \sin^2\theta\,\mathrm{d}s & -\int_0^L \cos\theta\sin\theta\,\mathrm{d}s \\ -\int_0^L \cos\theta\sin\theta\,\mathrm{d}s & \int_0^L \cos^2\theta\,\mathrm{d}s \end{pmatrix}. \tag{2.1.3}$$

Then, for a sufficiently smooth solution $(\theta,\rho)$ of (2.0.4) we indeed have

$$\begin{aligned} \partial_t \int_0^L \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mathrm{d}s &= \int_0^L \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \partial_t\theta\,\mathrm{d}s \\ &= \int_0^L \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \Big(\partial_s\left(\beta(\rho)(\partial_s\theta - c_0)\right) + \lambda_{\theta 1}\sin\theta - \lambda_{\theta 2}\cos\theta\Big)\,\mathrm{d}s \\ &= \int_0^L \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \partial_s\left(\beta(\rho)(\partial_s\theta - c_0)\right)\,\mathrm{d}s - \Pi(\theta)(t)\begin{pmatrix} \lambda_{\theta 1}(t) \\ \lambda_{\theta 2}(t) \end{pmatrix} \\ &= 0. \end{aligned} \tag{2.1.4}$$

With (2.0.3) we see that this means that if the initial angle function corresponds to a closed curve, the solution is also closed.

**Remark 2.1.3.** The boundary conditions in (2.0.4) make it possible to use integration by parts to write the Lagrange multipliers $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$ in the form

$$\begin{pmatrix} \lambda_{\theta 1}(t) \\ \lambda_{\theta 2}(t) \end{pmatrix} = \Pi^{-1}(\theta)(t) \int_0^L \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \partial_s\theta\,\beta(\rho)(\partial_s\theta - c_0)\,\mathrm{d}s. \tag{2.1.5}$$

One might wonder whether the inverse of the matrix $\Pi$ is well-defined. The answer to this question is provided by the following remark.

**Remark 2.1.4.** As already observed in [160, Lemma 2.1] we have

$$\det \Pi(\theta)(t) = \frac{1}{2}\int_0^L \int_0^L \sin^2 \psi(t,s,\tilde{s})\,\mathrm{d}s\,\mathrm{d}\tilde{s},$$

where $\psi(t,s,\tilde{s}) := \theta(t,s) - \theta(t,\tilde{s})$. Hence, as long as $\theta(t,\cdot)$ is not constant, the matrix $\Pi(\theta)(t)$ is invertible. In particular, the Lagrange multipliers in (2.1.2) are well-defined, as long as the angle function describes a closed curve.

The third Lagrange multiplier $\lambda_\rho$ is chosen so that the total mass remains fixed during the flow. Defining

$$\lambda_\rho(t) := -\frac{1}{2L}\int_0^L \beta'(\rho)(\partial_s\theta - c_0)^2\,\mathrm{d}s, \tag{2.1.6}$$

a sufficiently smooth solution $(\theta,\rho)$ of (2.0.4) satisfies

$$\begin{aligned}\partial_t \int_0^L \rho\,\mathrm{d}s &= \int_0^L \partial_t\rho\,\mathrm{d}s\\ &= \int_0^L \Big(\mu\partial_s^2\rho(t,s) - \frac{1}{2}\beta'(\rho(t,s))(\partial_s\theta(t,s)-c_0)^2 - \lambda_\rho\Big)\,\mathrm{d}s\\ &= -\frac{1}{2}\int_0^L \beta'(\rho(t,s))(\partial_s\theta(t,s)-c_0)^2\,\mathrm{d}s - L\lambda_\rho = 0.\end{aligned} \tag{2.1.7}$$

To sum up, the Lagrange multipliers $\lambda_{\theta 1}$, $\lambda_{\theta 2}$, $\lambda_\rho$ are chosen such that

$$\int_0^L \cos\theta(t,s)\,\mathrm{d}s = 0, \quad \int_0^L \sin\theta(t,s)\,\mathrm{d}s = 0, \quad \int_0^L \rho(t,s)\,\mathrm{d}s = \int_0^L \rho_0(s)\,\mathrm{d}s \tag{2.1.8}$$

along a solution so that the evolution indeed corresponds to an evolution of closed curves with fixed total mass.

### 2.1.3. Decrease of the energy

Let us note that a direct computation reveals a constrained $L^2$-gradient flow structure, see Section 1.3.2. This means that for a sufficiently smooth solution $(\theta,\rho)\colon (0,T)\times[0,L]\to\mathbb{R}^2$ of (2.0.4), the energy (2.0.1) decreases.
We also observe this by the direct computation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\mu(\theta,\rho) &= \int_0^L \Big(\frac{1}{2}\beta'(\rho)(\partial_s\theta-c_0)^2\partial_t\rho + \beta(\rho)(\partial_s\theta-c_0)\partial_t\partial_s\theta\Big)\,\mathrm{d}s + \int_0^L \mu\,\partial_s\rho\,\partial_s\partial_t\rho\,\mathrm{d}s\\ &= -\int_0^L \partial_s\left(\beta(\rho)(\partial_s\theta-c_0)\right)\partial_t\theta\,\mathrm{d}s + \int_0^L \Big(\frac{1}{2}\beta'(\rho)(\partial_s\theta-c_0)^2 - \mu\partial_s^2\rho\Big)\partial_t\rho\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^L \partial_t\theta\,(\partial_t\theta - \lambda_{\theta 1}\sin\theta + \lambda_{\theta 2}\cos\theta)\,\mathrm{d}s - \int_0^L \partial_t\rho\,(\partial_t\rho + \lambda_\rho)\,\mathrm{d}s \\
&= -\int_0^L (\partial_t\theta)^2\,\mathrm{d}s - \int_0^L (\partial_t\rho)^2\,\mathrm{d}s \leq 0
\end{aligned} \tag{2.1.9}
$$

for all $t \in (0,T)$.

## 2.2. The static problem

Before addressing the dynamic problem, this section focuses on analyzing the minimizers and constrained critical points of the energy functional (2.0.1). As indicated by the form of the energy (2.0.1), a global minimizer has to have constant density if $\mu$ is sufficiently large, cf. also [18] for the case $\omega = 1$, $c_0 = 0$. Remarkably, this is generically not true on the level of critical points, see Example 2.2.15. We provide sharp sufficient conditions for constrained critical points to be homogeneous elasticae, i.e. elasticae with constant density.

Since the constrained critical points are precisely the stationary solutions to (2.0.4), the better understanding of the static problem provides us with insights, which are useful for the asymptotic analysis in Section 2.7.

### 2.2.1. The Euler–Lagrange equations

For $k \in \mathbb{N}$, we denote the Hilbert space of periodic Sobolev functions on $[0,L]$ by

$$
W^{k,2}_{\mathrm{per}}(0,L) := \{u \in W^{k,2}(0,L) : \partial_s^\ell u(L) = \partial_s^\ell u(0) \text{ for } \ell = 0,\dots,k-1\}. \tag{2.2.1}
$$

The standard angle function of an $\omega$-fold covering of the circle is given by

$$
\phi(s) = \frac{2\pi\omega s}{L} \quad \text{for } s \in [0,L]. \tag{2.2.2}
$$

If $\theta \in W^{k,2}(0,L)$ is an angle function for a $C^k$-closed curve with rotation index $\omega \in \mathbb{Z}$, there exists a unique $u \in W^{k,2}_{\mathrm{per}}(0,L)$ such that $\theta = \phi + u$. Consequently, it is natural to study the energy $\mathcal{E}_\mu$ defined in (2.0.1) on the set

$$
\begin{aligned}
\mathcal{U} &:= W^{1,2}_{\mathrm{per}}(0,L;\mathbb{R}^2) + (\phi,0) \\
&= \left\{(\theta,\rho) \in W^{1,2}(0,L;\mathbb{R}^2) :\ \theta(L) - \theta(0) = 2\pi\omega,\ \rho(L) = \rho(0)\right\}.
\end{aligned}
$$

By introducing the constraint functional

$$
\mathcal{G}(\theta,\rho) = \Big(\int_0^L \cos\theta\,\mathrm{d}s,\ \int_0^L \sin\theta\,\mathrm{d}s,\ \int_0^L \rho\,\mathrm{d}s - mL\Big), \tag{2.2.3}
$$

we see that $(\theta, \rho) \in \mathcal{U}$ corresponds to a $C^1$-closed curve with rotation index $\omega \in \mathbb{Z}$ and a density with total mass $mL$ if and only if $\mathcal{G}(\theta, \rho) = 0$. We thus define the side condition

$$\mathcal{A} := \{(\theta, \rho) \in \mathcal{U} : \ \mathcal{G}(\theta, \rho) = 0\}.$$

The set $\mathcal{U}$ is not a vector space (unless $\omega = 0$), but only an affine subspace of the Hilbert space $W^{1,2}(0, L; \mathbb{R}^2)$. However, this causes only some minor technical difficulties when applying results from functional analysis that are typically formulated on open subsets of Banach spaces, see also [37]. To circumvent this, we define the shifted functionals

$$\begin{aligned} &E_\mu : W^{1,2}_{\mathrm{per}}(0, L) \to \mathbb{R}, \ E_\mu(u, \rho) = \mathcal{E}_\mu(u + \phi, \rho), \\ &G : W^{1,2}_{\mathrm{per}}(0, L) \to \mathbb{R}^3, \ G(u, \rho) = \mathcal{G}(u + \phi, \rho), \end{aligned} \tag{2.2.4}$$

with $\phi$ as in (2.2.2). If $(\theta, \rho) = (u + \phi, \rho) \in \mathcal{U}$, then we have $\mathcal{E}_\mu(\theta, \rho) = E_\mu(u, \rho)$ and $\mathcal{G}(\theta, \rho) = G(u, \rho)$. The Fréchet derivatives of $E_\mu$ and $G$ at $(u, \rho) \in W^{1,2}_{\mathrm{per}}(0, L; \mathbb{R}^2)$ are linear mappings $E'_\mu(u, \rho) \colon W^{1,2}_{\mathrm{per}}(0, L; \mathbb{R}^2) \to \mathbb{R}$, $G'(u, \rho) \colon W^{1,2}_{\mathrm{per}}(0, L; \mathbb{R}^2) \to \mathbb{R}^3$, given by

$$\begin{aligned} E'_\mu(u, \rho)(v, \sigma) = \int_0^L \Big( &\beta(\rho)(\partial_s u + \partial_s \phi - c_0)\partial_s v \\ &+ \frac{1}{2}\beta'(\rho)(\partial_s u + \partial_s \phi - c_0)^2 \sigma + \mu \partial_s \rho \partial_s \sigma \Big) \, \mathrm{d}s, \end{aligned} \tag{2.2.5}$$

$$G'(u, \rho)(v, \sigma) = \Big( -\int_0^L v \, \sin(u + \phi) \, \mathrm{d}s, \ \int_0^L v \, \cos(u + \phi) \, \mathrm{d}s, \ \int_0^L \sigma \, \mathrm{d}s \Big). \tag{2.2.6}$$

Moreover, for $(v, \sigma) \in W^{1,2}_{\mathrm{per}}(0, L; \mathbb{R}^2)$, we have

$$\begin{aligned} E''_\mu(u, \rho)((v, \sigma)^2) = \int_0^L \Big( &\beta(\rho)(\partial_s v)^2 + 2\beta'(\rho)(\partial_s u + \partial_s \phi - c_0)\sigma \partial_s v \\ &+ \frac{1}{2}\beta''(\rho)(\partial_s u + \partial_s \phi - c_0)^2 \sigma^2 + \mu(\partial_s \sigma)^2 \Big) \, \mathrm{d}s, \end{aligned} \tag{2.2.7}$$

$$G''(u, \rho)((v, \sigma)^2) = \Big( -\int_0^L v^2 \cos(u + \phi) \, \mathrm{d}s, \ -\int_0^L v^2 \sin(u + \phi) \, \mathrm{d}s, \ 0 \Big). \tag{2.2.8}$$

In the following, we write in a slight abuse of notation

$$\begin{aligned} &\mathcal{E}'_\mu(\theta, \rho) := E'_\mu(u, \rho), && \mathcal{E}''_\mu(\theta, \rho) := E''_\mu(u, \rho), \\ &\mathcal{G}'(\theta, \rho) := G'(u, \rho), && \mathcal{G}''(\theta, \rho) := G''(u, \rho) \end{aligned} \tag{2.2.9}$$

for $(u, \rho) = (\theta - \phi, \rho) \in W^{1,2}_{\mathrm{per}}(0, L; \mathbb{R}^2)$. Moreover, we say that $\mathcal{E}_\mu$ has a critical point subject to the constraint $\mathcal{G} = 0$ at some $(\theta, \rho) \in \mathcal{A}$ if $E_\mu$ has a critical point subject to the constraint $G = 0$ at $(u, \rho) = (\theta - \phi, \rho)$ (cf. [167, Definition 43.20]). This is the first instance where the lack of the vector space structure of $\mathcal{U}$ causes technicalities.

**Proposition 2.2.1.** *The energy $\mathcal{E}_\mu$ has a critical point subject to the constraint $\mathcal{G} = 0$ at some $(\theta, \rho) \in \mathcal{A}$ if and only if there exists $\Lambda = (\lambda_{\theta 1}, \lambda_{\theta 2}, \lambda_\rho) \in \mathbb{R}^3$ such that*

$$\mathcal{E}'_\mu(\theta,\rho)(v,\sigma) + \Lambda \cdot \mathcal{G}'(\theta,\rho)(v,\sigma) = 0 \quad \text{for all } (v,\sigma) \in W^{1,2}_{\mathrm{per}}(0,L;\mathbb{R}^2). \tag{2.2.10}$$

Here, $\Lambda \cdot \mathcal{G}'(\theta,\rho) = \sum_{i=1}^2 \lambda_{\theta i} \left(\mathcal{G}'\right)^i (\theta,\rho) + \lambda_\rho \left(\mathcal{G}'\right)^3 (\theta,\rho)$. Equation (2.2.10) is called the *weak Euler–Lagrange equation*. Written as two separate equations, (2.2.10) is

$$0 = \int_0^L \left(\beta(\rho)(\partial_s\theta - c_0)\partial_s v - \lambda_{\theta 1} \sin\theta\, v + \lambda_{\theta 2} \cos\theta\, v\right) \mathrm{d}s, \tag{2.2.11}$$

$$0 = \int_0^L \left(\mu \partial_s\rho \partial_s\sigma + \frac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2\sigma + \lambda_\rho \sigma\right) \mathrm{d}s. \tag{2.2.12}$$

Choosing appropriate test functions shows that $\lambda_{\theta 1}$, $\lambda_{\theta 2}$ and $\lambda_\rho$ are given as in (2.1.5), and (2.1.6). Proposition 2.2.1 follows from formulating the problem in the periodic setting by [167, Proposition 43.21] and the following Lemma.

**Lemma 2.2.2.** *For $(\theta,\rho) \in \mathcal{A}$, the map $\mathcal{G}'(\theta,\rho)\colon W^{1,2}_{\mathrm{per}}(0,L;\mathbb{R}^2) \to \mathbb{R}^3$ is surjective.*

*Proof.* Let $(x_1,x_2,x_3) \in \mathbb{R}^3$. Choosing $\bar\sigma \equiv \frac{x_3}{L}$ yields $(\mathcal{G}')^3(\theta,\rho)(v,\bar\sigma) = x_3$ for all $v \in W^{1,2}_{\mathrm{per}}(0,L)$ (see (2.2.6) and (2.2.9)). Moreover, choose $\bar v := -c_1 \sin\theta + c_2 \cos\theta \in W^{1,2}_{\mathrm{per}}(0,L)$ for some $c_1, c_2 \in \mathbb{R}$. Then, from (2.2.6) and (2.2.9) we obtain

$$\left((\mathcal{G}')^1, (\mathcal{G}')^2\right)(\theta,\rho)(\bar v,\bar\sigma) = \Pi(\theta)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

with $\Pi$ defined as in (2.1.3) for time-independent $\theta$. Since $(\theta,\rho) \in \mathcal{A}$, the matrix $\Pi(\theta)$ is invertible (see Remark 2.1.4). Hence, there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\left((\mathcal{G}')^1, (\mathcal{G}')^2\right)(\theta,\rho)(\bar v,\bar\sigma) = (x_1,x_2)$$

and the surjectivity follows. □

For $(u,\rho) \in W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2)$, the energy $E_\mu$ possesses an $L^2$-gradient, i.e. there exists

$$\nabla E_\mu(u,\rho) = (\nabla_u E_\mu(u,\rho), \nabla_\rho E_\mu(u,\rho)) \in L^2(0,L;\mathbb{R}^2)$$

such that for all $(v,\sigma) \in W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2)$ one has

$$\begin{aligned} E'_\mu(u,\rho)(v,\sigma) &= \langle \nabla E_\mu(u,\rho), (v,\sigma)\rangle_{L^2(0,L;\mathbb{R}^2)} \\ &= \langle \nabla_u E_\mu(u,\rho), v\rangle_{L^2(0,L)} + \langle \nabla_\rho E_\mu(u,\rho), \sigma\rangle_{L^2(0,L)}. \end{aligned}$$

Integrating (2.2.5) by parts, we obtain the explicit representation

$$\nabla E_\mu(u,\rho) = \Big( -\partial_s(\beta(\rho)(\partial_s u + \partial_s\phi - c_0)),\ -\mu\partial_s^2\rho + \frac{1}{2}\beta'(\rho)(\partial_s u + \partial_s\phi - c_0)^2\Big). \tag{2.2.13}$$

It is not difficult to see that the components $G^j$ of $G$ also possess $L^2$-gradients for $j = 1, 2, 3$. For all $(u,\rho), (v,\sigma) \in W^{2,2}_{\mathrm{per}}(0, L; \mathbb{R}^2)$, we have

$$G'(u,\rho)(v,\sigma) = \Big( \big\langle \nabla G^1(u,\rho), (v,\sigma)\big\rangle_{L^2(0,L;\mathbb{R}^2)},\ \big\langle \nabla G^2(u,\rho), (v,\sigma)\big\rangle_{L^2(0,L;\mathbb{R}^2)},\ \big\langle \nabla G^3(u,\rho), (v,\sigma)\big\rangle_{L^2(0,L;\mathbb{R}^2)} \Big)$$

for the $L^2$-gradients of the components $G^j$ given as

$$\nabla G^1(u,\rho) = (-\sin(u+\phi), 0)\,,\ \nabla G^2(u,\rho) = (\cos(u+\phi), 0)\,,\ \nabla G^3(u,\rho) = (0,1)\,. \tag{2.2.14}$$

Again, for simplicity we abuse notation and work with

$$\nabla\mathcal{E}_\mu(\theta,\rho) := \nabla E_\mu(\theta-\phi,\rho), \quad \nabla\mathcal{G}(\theta,\rho) := \nabla G(\theta-\phi,\rho) \tag{2.2.15}$$

for $(\theta,\rho) \in W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2) + (\phi, 0)$ with $\phi$ as in (2.2.2) and $(u,\rho) = (\theta,\rho) - (\phi,0)$. If $(\theta,\rho)$ is a critical point of $\mathcal{E}_\mu$ subject to $\mathcal{G} = 0$ with $(\theta,\rho) \in W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2) + (\phi,0)$, Proposition 2.2.1 implies the existence of $(\lambda_{\theta1}, \lambda_{\theta2}, \lambda_\rho) \in \mathbb{R}^3$ such that

$$\nabla\mathcal{E}_\mu(\theta,\rho) + \lambda_{\theta1}\nabla\mathcal{G}^1(\theta,\rho) + \lambda_{\theta2}\nabla\mathcal{G}^2(\theta,\rho) + \lambda_\rho\nabla\mathcal{G}^3(u,\rho) = 0 \tag{2.2.16}$$

in $L^2(0,L;\mathbb{R}^2)$. We call (2.2.16) the *Euler–Lagrange equation*. Written as two separate equations (2.2.16) is

$$0 = \partial_s\left(\beta(\rho)(\partial_s\theta - c_0)\right) + \lambda_{\theta1}\sin\theta - \lambda_{\theta2}\cos\theta, \tag{2.2.17}$$

$$0 = \mu\partial_s^2\rho - \frac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2 - \lambda_\rho. \tag{2.2.18}$$

### 2.2.2. Existence of minimizers and smoothness of critical points

The existence of a solution to the minimization problem

$$\inf_{(\theta,\rho)\in\mathcal{A}} \mathcal{E}_\mu(\theta,\rho) \longrightarrow \min! \tag{2.2.19}$$

can be shown via the direct method following the arguments in [18, Prop. 3.2].

**Proposition 2.2.3** (Existence of a minimizer)**.** *There exists* $(\theta^*, \rho^*) \in \mathcal{A}$ *such that*

$$\mathcal{E}_\mu(\theta^*, \rho^*) = \inf_{(\theta,\rho)\in\mathcal{A}} \mathcal{E}_\mu(\theta, \rho).$$

*Proof.* Since $\mathcal{E}_\mu(\theta, \rho) \geq 0$, there exists an infimizing sequence $(\theta_n, \rho_n) \subset \mathcal{A}$ for $\mathcal{E}_\mu$. As $\mathcal{E}_\mu$ does not depend on $\theta$ but only on $\partial_s\theta$, without loss of generality, we assume that $\theta_n(0) = 0$ for all $n \in \mathbb{N}$. Let $M \in \mathbb{R}$ be such that $\mathcal{E}_\mu(\theta_n, \rho_n) < M$ for all $n \in \mathbb{N}$.
Firstly, the definition of $\mathcal{E}_\mu$ implies that $\mu \|\partial_s\rho_n\|^2_{L^2(0,L)} < 2M$. Due to the fixed mean value of $\rho_n$, the Poincaré inequality allows to bound $\|\rho_n\|_{L^2(0,L)}$ by the $L^2$-norm of the derivative. Thus, $\rho_n$ is uniformly bounded in $W^{1,2}(0, L)$ and hence there is $K \in \mathbb{R}$ such that $\|\rho_n\|_{W^{1,2}(0,L)}, \|\rho_n\|_{C^0([0,L])} \leq K$ for all $n \in \mathbb{N}$. This enables us to bound $\|\partial_s\theta_n\|_{L^2(0,L)}$ using again the definition of $\mathcal{E}_\mu$ by

$$\|\partial_s\theta_n\|^2_{L^2(0,L)} \leq \|\partial_s\theta_n - c_0\|^2_{L^2(0,L)} + 4\pi\omega c_0 < \frac{2M}{\inf_{[-K,K]} \beta} + 4\pi\omega c_0.$$

Due to $\theta_n(0) = 0$, we can bound $\|\theta_n\|_{L^2(0,L)}$ by the $L^2$-norm of the derivative. Thus, $\theta_n$ is also bounded in $W^{1,2}(0, L)$. The boundedness and the compactness of the Sobolev embedding $W^{1,2}(0, L) \subset C([0, L])$ yields the existence of a subsequence $(\theta_{n_k}, \rho_{n_k})$ such that $(\theta_{n_k}, \rho_{n_k}) \rightharpoonup (\theta^*, \rho^*)$ in $W^{1,2}(0, L)$ and $(\theta_{n_k}, \rho_{n_k}) \to (\theta^*, \rho^*)$ in $C([0, L])$. On the one hand, the limit $(\theta^*, \rho^*)$ satisfies $\theta^*(L) - \theta^*(0) = 2\pi\omega$, $\rho^*(L) = \rho^*(0)$ and

$$\int_0^L \rho^* \,\mathrm{d}s = \lim_{k\to\infty} \int_0^L \rho_{n_k} \,\mathrm{d}s = \int_0^L \rho_0 \,\mathrm{d}s,$$
$$\int_0^L \sin\theta^* \,\mathrm{d}s = \lim_{k\to\infty} \int_0^L \sin\theta_{n_k} \,\mathrm{d}s = 0, \quad \int_0^L \cos\theta^* \,\mathrm{d}s = \lim_{k\to\infty} \int_0^L \cos\theta_{n_k} \,\mathrm{d}s = 0.$$

Consequently $(\theta^*, \rho^*) \in \mathcal{A}$. On the other hand, we obtain

$$\lim_{k\to\infty} \|\beta(\rho_{n_k}) - \beta(\rho^*)\|_{C([0,L])} \leq \sup_{[-K,K]} |\beta'| \lim_{k\to\infty} \|\rho_{n_k} - \rho^*\|_{C([0,L])} = 0.$$

It follows that $(\beta(\rho_{n_k}))^{\frac{1}{2}} (\partial_s\theta_{n_k} - c_0) \rightharpoonup (\beta(\rho^*))^{\frac{1}{2}} (\partial_s\theta^* - c_0)$ in $L^2(0, L)$ and

$$\int_0^L \beta(\rho^*)(\partial_s\theta^* - c_0)^2 \,\mathrm{d}s \leq \liminf_{k\to\infty} \int_0^L \beta(\rho_{n_k})(\partial_s\theta_{n_k} - c_0)^2 \,\mathrm{d}s.$$

Together with the weak $L^2$-convergence $\partial_s\rho_{n_k} \rightharpoonup \partial_s\rho^*$ and

$$\int_0^L \mu(\partial_s\rho^*)^2 \,\mathrm{d}s \leq \liminf_{k\to\infty} \int_0^L \mu(\partial_s\rho_{n_k})^2 \,\mathrm{d}s,$$

we conclude that $\mathcal{E}_\mu(\theta^*, \rho^*) \leq \liminf_{k\to\infty} \mathcal{E}_\mu(\theta_{n_k}, \rho_{n_k}) = \inf_{(\theta,\rho)\in\mathcal{A}} \mathcal{E}_\mu(\theta, \rho)$. □

We prove now that constrained critical points (and in particular minimizers) are smooth.

**Proposition 2.2.4** (Smoothness of critical points, [36, Proposition 2.2])**.** *If $(\theta, \rho)$ is a constrained critical point, the $L$-periodic extension of $(\partial_s\theta, \partial_s\rho)$ to $\mathbb{R}$ is smooth. In particular, $(\theta, \rho) \in C^\infty([0, L])$ and $\partial_s\theta(L) = \partial_s\theta(0)$, $\partial_s\rho(L) = \partial_s\rho(0)$.*

*Proof.* Let $(\theta, \rho) \in \mathcal{U}$ be a constrained critical point. Then there exist $\lambda_{\theta 1}, \lambda_{\theta 2}, \lambda_\rho \in \mathbb{R}$ such that $(\theta, \rho)$ satisfies (2.2.11) and (2.2.12).
*Step 1:* $(\theta, \rho) \in W^{2,2}(0, L) \subset C^1([0, L])$. Since $\rho \in W^{1,2}(0, L) \subset C([0, L])$, $\|\beta' \circ \rho\|_{C([0,L])}$ and $\|\partial_s\rho\|_{L^2(0,L)}$ are bounded. Thus, (2.2.11) implies that there is $C = C(\theta, \rho)$ such that

$$\left| \int_0^L \beta(\rho)\partial_s\theta\partial_s v \,\mathrm{d}s \right| = \left| \int_0^L (\lambda_{\theta 1} \sin\theta - \lambda_{\theta 2} \cos\theta) v \,\mathrm{d}s - \int_0^L \beta'(\rho)\partial_s\rho c_0 v \,\mathrm{d}s \right| \leq C \|v\|_{L^2(0,L)}$$

for all $v \in C_c^1((0, L)) \subset W^{1,2}_{\mathrm{per}}(0, L)$. So, $\beta(\rho)\partial_s\theta \in W^{1,2}(0, L)$ by [20, Prop. 8.3]. Hence, $\|\beta(\rho)(\partial_s\theta - c_0)\|_{C([0,L])}$ and also $\|\beta'(\rho)(\partial_s\theta - c_0)\|_{C([0,L])}$ are bounded, so (2.2.12) implies that

$$2\mu \left| \int_0^L \partial_s\rho\partial_s\sigma \,\mathrm{d}s \right| = \left| \int_0^L \left( -\beta'(\rho)(\partial_s\theta - c_0)^2\sigma - 2\lambda_\rho\sigma \right) \mathrm{d}s \right| \leq C \|\sigma\|_{L^2(0,L)}$$

for all $\sigma \in C_c^1((0, L))$. It follows that $\rho \in W^{2,2}(0, L) \subset C^1([0, L])$. We thus obtain $\partial_s(\beta(\rho)\partial_s\theta) = \beta'(\rho)\partial_s\rho\partial_s\theta + \beta(\rho)\partial_s^2\theta$ in the sense of distributions, so $\partial_s^2\theta \in L^2(0, L)$ as $\inf_{[0,L]} \beta(\rho) > 0$.
*Step 2:* $(\theta, \rho) \in W^{3,2}(0, L) \subset C^2([0, L])$. The increased regularity yields that $(\theta, \rho)$ satisfies (2.2.17) and (2.2.18) in $L^2(0, L)$. Using the same ideas as in Step 1, we can deduce that $(\theta, \rho) \in W^{3,2}(0, L)$.
*Step 3: Smooth $L$-periodic extension.* Testing (2.2.11) and (2.2.12) with $W^{1,2}_{\mathrm{per}}$-functions not vanishing at the boundary results in the natural boundary conditions

$$\partial_s\theta\big|_0^L = 0 \quad \text{and} \quad \partial_s\rho\big|_0^L = 0. \tag{2.2.20}$$

Since $(\theta, \rho)$ satisfies (2.2.17) and (2.2.18) pointwise we conclude with (2.2.20) that $\partial_s^2\theta(L) = \partial_s^2\theta(0)$ and $\partial_s^2\rho(L) = \partial_s^2\rho(0)$. The claim follows by bootstrapping. □

### 2.2.3. Homogeneous elastica

The structure of the energy functional $\mathcal{E}_\mu$ suggests that for large values of $\mu$, minimizers favor almost constant density, cf. [18]. For constant density, $\mathcal{E}_\mu$ is essentially the elastic energy whose critical points are called elasticae (see Section 1.2.2). For $\mu$ large, these elasticae also play an important role for the heterogeneous elastic energy (2.0.1).
Since $\mathcal{E}_\mu$ only depends on the derivative of $\theta$, if $(\theta, \rho) \in \mathcal{A}$ is a critical point of $\mathcal{E}_\mu$ subject to the constraint $\mathcal{G} = 0$, then $(\theta + c, \rho) \in \mathcal{A}$, is also a constrained critical point for any $c \in \mathbb{R}$. This motivates working with the curvature $k = \partial_s\theta$.

**Definition 2.2.5** ([36, Definition 2.3])**.** Let $(\theta, \rho) \in \mathcal{A}$. We say that $\theta$ describes a *(length-constrained) elastica* if the curvature $k = \partial_s \theta$ is smooth and satisfies the *constrained elastica equation*

$$\partial_s^2 k + \frac{1}{2} k^3 - \lambda k = 0 \quad \text{for some } \lambda \in \mathbb{R}. \tag{2.2.21}$$

If further $\rho$ is constant, we say that $(\theta, \rho)$ describes a *homogeneous elastica*.

Solutions of (2.2.21) have been classified explicitely in several previous works, see Section 1.2.2. We emphasize once again that in the case of closed curves, the elasticae can be characterized as follows.

**Lemma 2.2.6** ([98])**.** *The only closed constrained elasticae are multifold coverings of circles and multifold coverings of the figure eight (elastica).*

The figure eight is illustrated in Figure 2.2. For an explicit parametrization, see [124, Definition 5.1].

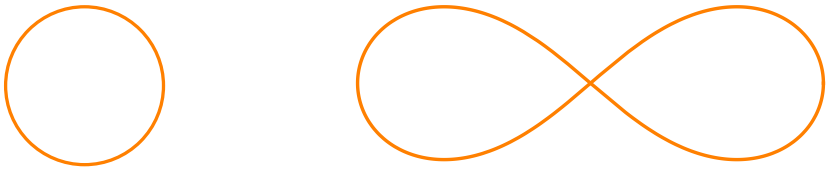


Figure 2.2: The only closed planar elasticae are multifold coverings of a circle (left) and of the figure eight elastica (right).

**Remark 2.2.7.** Let $(\theta, \rho) \in \mathcal{A}$. If $\omega \neq 0$ and $\theta$ describes an elastica, then $\theta$ is the angle function of an $\omega$-fold covering of a circle with curvature $k = \frac{2\pi\omega}{L}$. Thus, $\theta$ is determined up to an additive constant. If $(\theta, \rho)$ describes a homogeneous elastica and we require that $\theta(0) = 0$ or that $\int_0^L \theta(s)\,\mathrm{d}s = \pi\omega L$, then

$$(\theta, \rho) = (\theta_c, \rho_c) := \left(\frac{2\pi\omega}{L} s, m\right) \in \mathcal{A}.$$

#### Minimizers and critical points for large $\mu$

We now show that for large values of $\mu$ and $\omega \neq 0$, a minimizer $(\theta, \rho)$ in (2.2.19) describes a homogeneous elastica. To state a uniqueness result, we define

$$\mathcal{I}(\theta) := \int_0^L \theta(s)\,\mathrm{d}s$$

and fix $\mathcal{I}(\theta) = \pi\omega L$.

**Proposition 2.2.8** (Unique minimizer for large $\mu$, [36, Proposition 2.6])**.** *Let $\omega \neq 0$. Then there exists $\mu_0 \in (0,\infty)$ such that if $\mu \geq \mu_0$,*

$$\mathcal{E}_\mu(\theta_c,\rho_c) < \mathcal{E}_\mu(\theta,\rho) \quad \text{for all} \quad (\theta,\rho) \in \mathcal{A} \cap \{\mathcal{I}(\theta) = \pi\omega L\} \setminus \{(\theta_c,\rho_c)\}\,.$$

*Proof.* We basically follow the ideas of [18, Prop. 3.3]. In a first step, we show that for $\mu$ large enough, the second variation $\mathcal{E}''_\mu(\theta_c,\rho_c)$ is strictly positive. It follows (for example with [167, Theorem 43.D]) that $(\theta_c,\rho_c)$ is a strict local minimizer in the sense that there exists $\delta = \delta(\mu) > 0$ such that $\mathcal{E}_\mu(\theta_c,\rho_c) < \mathcal{E}_\mu(\theta,\rho)$ for all $(\theta,\rho) \in \mathcal{A}\cap\{\mathcal{I}(\theta) = \pi\omega L\}\setminus\{(\theta_c,\rho_c)\}$ with norm $\|(\theta,\rho)-(\theta_c,\rho_c)\|_{W^{1,2}(0,L)} < \delta$. Now, note that $\mathcal{E}_\mu(\theta_c,\rho_c)$ is independent of $\mu$ whereas $\mathcal{E}_\mu(\theta,\rho)$ is increasing in $\mu$. Consequently, the neighborhood in which $(\theta_c,\rho_c)$ is a strict local minimizer can be chosen only depending on a lower bound $\mu_1$ on $\mu$. In a second step, we prove that there exists $\mu_2$ such that for $\mu > \mu_2$ all global minimizers are contained in $B_\delta(\theta_c,\rho_c) \subset W^{1,2}(0,L)$. Together with the first step, for $\mu > \max\{\mu_1,\mu_2\}$, $(\theta_c,\rho_c)$ is the unique global minimizer.

*Step 1: Strict local minimizer.* To prove the first part, we switch to the periodic setting of Section 2.2.1, which allows us to apply [167, Theorem 43.D]. So, let $(u_c,\rho_c) := (\theta_c,\rho_c) - (\phi,0) = (0,m)$. Fix $\mu > 0$. To begin with, choose

$$\Lambda_c := \Big(0,\, 0,\, \frac{1}{2}\beta'(m)(\partial_s\phi - c_0)^2\Big). \tag{2.2.22}$$

Then $E'_\mu(u_c,\rho_c)(v,\sigma) - \Lambda_c \cdot G'(u_c,\rho_c)(v,\sigma) = 0$ for all $(v,\sigma) \in W^{1,2}_{\text{per}}(0,L;\mathbb{R}^2)$, which means that $(u_c,\rho_c)$ is a critical point of $E_\mu$ subject to the constraint $G = 0$. Next, we prove that there exist $\alpha > 0$ and $\mu_1$ such that for $\mu \geq \mu_1$,

$$E''_\mu(u_c,\rho_c)((v,\sigma)^2) - \Lambda_c \cdot G''(u_c,\rho_c)((v,\sigma)^2) \geq \alpha\,\|(v,\sigma)\|^2_{W^{1,2}(0,L)} \tag{2.2.23}$$

for all $(v,\sigma) \in W^{1,2}_{\text{per}}(0,L;\mathbb{R}^2) \cap \{\mathcal{I}(v) = 0\}$ such that $G'(u_c,\rho_c)(v,\sigma) = 0$. Let $(v,\sigma) \in W^{1,2}_{\text{per}}(0,L;\mathbb{R}^2) \cap \{\mathcal{I}(v) = 0\}$ such that $G'(u_c,\rho_c)(v,\sigma) = 0$. With (2.2.8) and (2.2.22), we observe first that

$$\Lambda_c \cdot G''(u_c,\rho_c)((v,\sigma)^2) = 0. \tag{2.2.24}$$

Hence, we concentrate on $E''_\mu(u_c,\rho_c)$, see (2.2.7). For a clearer representation we use $\tilde{c} := \partial_s\phi - c_0$. With the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} E''_\mu(u_c,\rho_c)((v,\sigma)^2) \geq \beta(m)\,\|\partial_s v\|^2_{L^2(0,L)} &- 2\,|\beta'(m)\,\tilde{c}|\,\|\sigma\|_{L^2(0,L)}\,\|\partial_s v\|_{L^2(0,L)} \\ &- \frac{\tilde{c}^2}{2}\,|\beta''(m)|\,\|\sigma\|^2_{L^2(0,L)} + \mu\,\|\partial_s\sigma\|^2_{L^2(0,L)}. \end{aligned}$$

Further, Young's inequality yields

$$2\left|\beta'(m)\,\tilde{c}\right|\|\sigma\|_{L^2(0,L)}\|\partial_s v\|_{L^2(0,L)} \leq \frac{2\left(\beta'(m)\right)^2\tilde{c}^2}{\beta(m)}\|\sigma\|^2_{L^2(0,L)} + \frac{\beta(m)}{2}\|\partial_s v\|^2_{L^2(0,L)}.$$

Remember that $\mathcal{I}(v) = \int_0^L v\,\mathrm{d}s = 0$ and note that $G'(u_c,\rho_c)(v,\sigma) = 0$ implies that $\int_0^L \sigma\,\mathrm{d}s = 0$. Therefore, the $L^2$-norms of $v$ and $\sigma$ can be estimated by the $L^2$-norms of the derivatives, which in turn are equivalent to the $W^{1,2}$-norms. So, combining the last two estimates, we get

$$\begin{aligned}E''_\mu(u_c,\rho_c)((v,\sigma)^2) \geq \left(\mu - \left(\frac{L\tilde{c}}{2\pi}\right)^2\left(\frac{|\beta''(m)|}{2} + \frac{2\left(\beta'(m)\right)^2}{\beta(m)}\right)\right)\|\partial_s\sigma\|^2_{L^2(0,L)}\\ + \left(\beta(m) - \frac{\beta(m)}{2}\right)\|\partial_s v\|^2_{L^2(0,L)},\end{aligned}$$

and (2.2.23) follows for suitable $\mu_1$ and $\alpha$.

*Step 2: Convergence of minimizers to* $(\theta_c,\rho_c)$. Let $(\mu_j)_{j\in\mathbb{N}}$ with $\mu_j \to \infty$ for $j\to\infty$ be a monotonically increasing sequence and define

$$\mathcal{M}_j := \left\{(\theta,\rho):\ \text{global minimizer of } \mathcal{E}_{\mu_j} \text{ subject to } \mathcal{G}=0 \text{ and } \mathcal{I}(\theta) = \pi\omega L\right\}.$$

We claim that for all $\delta > 0$ there is $J\in\mathbb{N}$ such that $\mathcal{M}_j$ is contained in the $W^{1,2}$-ball $B_\delta(\theta_c,\rho_c)$ for all $j \geq J$. In the following, $C$ denotes a constant that can vary from line to line without being renamed. Furthermore, we write $\mathcal{E}(\theta_c,\rho_c)$ for $\mathcal{E}_\mu(\theta_c,\rho_c)$ to make clear that there is no dependence on $\mu$. Let $(\theta,\rho)\in\mathcal{M}_j$ for some $j\in\mathbb{N}$. As $\int_0^L(\theta-\theta_c)\,\mathrm{d}s = 0$ and $\int_0^L(\rho-\rho_c)\,\mathrm{d}s = 0$, there is $C>0$ such that

$$\|(\theta,\rho)-(\theta_c,\rho_c)\|_{W^{1,2}(0,L)} \leq (1+C)\,\|\partial_s\theta - \partial_s\theta_c\|_{L^2(0,L)} + (1+C)\,\|\partial_s\rho\|_{L^2(0,L)}.$$

First, we notice that for any $\delta_1 > 0$ there is $J\in\mathbb{N}$ such that

$$\|\partial_s\rho\|^2_{L^2(0,L)} \leq \frac{2}{\mu_j}\mathcal{E}_{\mu_j}(\theta,\rho) \leq \frac{2}{\mu_j}\mathcal{E}(\theta_c,\rho_c) < \delta_1^2$$

for all $j\geq J$. Hence, by Sobolev-embedding and the Poincaré inequality, there is $C = C(L)$ such that $\|\rho-\rho_c\|_{C([0,L])} < C\delta_1$. Thus, there exists $K\in\mathbb{R}$ such that

$$\|\beta(\rho)-\beta(\rho_c)\|_{C([0,L])} \leq \sup_{[-K,K]}|\beta'|\,\|\rho-\rho_c\|_{C([0,L])} < C\delta_1 \tag{2.2.25}$$

for all $(\theta,\rho)\in\mathcal{M}_j$, $j\geq J$. Since $\inf_{[-K,K]}\beta > 0$, we have that

$$\int_0^L(\partial_s\theta - c_0)^2\,\mathrm{d}s \leq \frac{1}{\inf_{[-K,K]}\beta}\int_0^L\beta(\rho)(\partial_s\theta-c_0)^2\,\mathrm{d}s \leq \frac{2\mathcal{E}(\theta_c,\rho_c)}{\inf_{[-K,K]}\beta} \tag{2.2.26}$$

is uniformly bounded. With a short computation using $\partial_s\theta_c \equiv$ constant and $\int_0^L \partial_s\theta \,\mathrm{d}s = \int_0^L \partial_s\theta_c \,\mathrm{d}s = 2\pi\omega$ we see that

$$\begin{aligned}\|\partial_s\theta - \partial_s\theta_c\|^2_{L^2(0,L)} &= \int_0^L (\partial_s\theta - c_0)^2 - (\partial_s\theta_c - c_0)^2 \,\mathrm{d}s \\ &= \frac{1}{\beta(\rho_c)}\Big( \int_0^L (\beta(\rho_c) - \beta(\rho))(\partial_s\theta - c_0)^2 \,\mathrm{d}s + \int_0^L \beta(\rho)(\partial_s\theta - c_0)^2 \,\mathrm{d}s - 2\mathcal{E}(\theta_c, \rho_c)\Big).\end{aligned}$$

Since $\int_0^L \beta(\rho)(\partial_s\theta - c_0)^2 \,\mathrm{d}s \leq 2\mathcal{E}(\theta_c, \rho_c)$, using (2.2.25) and (2.2.26), we have $\|\partial_s\theta - \partial_s\theta_c\|^2_{L^2(0,L)} < C\delta_1$ for some $C$ depending on the model parameters but not on $\mu$. Choosing $\delta_1 > 0$ small enough, the claim follows. □

This result does not extend to $\omega = 0$ and $\theta$ describing the figure eight elastica as we see in Remark 2.2.12.
The next example shows that $(\theta_c, \rho_c)$ does not always need to be a global minimizer.

**Example 2.2.9** ([36, Example 2.7])**.** For the bending stiffness, we consider a double-well potential given by $\beta(x) := (x^2 - 1)^2 + c$ with $c > 0$. Let $L = 2\pi$, $\omega = 1$, $c_0 = 0$ and $m = 0$. Consider $\theta_c = s$, $\rho_c = 0$ and $\rho := \sin(2s)$. Then $(\theta_c, \rho) \in \mathcal{A}$ and

$$\mathcal{E}_\mu(\theta_c, \rho_c) = (1 + c)\,\pi > \left(\frac{3}{8} + c + 2\mu\right)\pi = \mathcal{E}_\mu(\theta_c, \rho)$$

for $\mu < \frac{5}{16}$.

Nonetheless, for large $\mu$, $(\theta_c, \rho_c)$ is locally the unique constrained critical point. This basically follows from the continuity of $\mathcal{E}''_\mu$ and $\mathcal{G}''$ and the positive definiteness of $\mathcal{E}''_\mu(\theta_c, \rho_c)$.

**Corollary 2.2.10** ([36, Corollary 2.8])**.** *Let $\omega \neq 0$. There is $\mu_0 > 0$ and a $C^1$-neighborhood $\mathcal{O}$ of $(\theta_c, \rho_c)$ such that if $(\theta, \rho) \in \mathcal{O}$ is a constrained critical point with $\mathcal{I}(\theta) = \pi\omega L$ and $\mu > \mu_0$, then $(\theta, \rho) = (\theta_c, \rho_c)$.*

*Proof.* Again, we switch to the periodic setting. We have seen in the proof of Proposition 2.2.8, that there exist $\alpha > 0$ such that for

$$\mu > \left(\omega - \frac{c_0 L}{2\pi}\right)^2 \left(\frac{|\beta''(m)|}{2} + \frac{2\,(\beta'(m))^2}{\beta(m)}\right)$$

and $\Lambda_c$ as in (2.2.22),

$$E''_\mu(u_c, \rho_c)((v, \sigma)^2) - \Lambda_c \cdot G''(u_c, \rho_c)((v, \sigma)^2) \geq \alpha\,\|(v, \sigma)\|^2_{W^{1,2}(0,L)}$$

for all $(v, \sigma) \in W^{1,2}_{\mathrm{per}}(0, L; \mathbb{R}^2) \cap \{\mathcal{I}(v) = 0\}$ such that $G'(u_c, \rho_c)(v, \sigma) = 0$. So $E_\mu$ has a *strict* local minimum at $(u_c, \rho_c)$. Now, we claim that for $\varepsilon > 0$ there exists $\delta > 0$ such that

if we take $(u,\rho) \in W^{1,2}_{\mathrm{per}}(0,L;\mathbb{R}^2)$ with $\|(u,\rho)-(u_c,\rho_c)\|_{C^1([0,L])} < \delta$, then

$$\left|(E''_\mu(u_c,\rho_c) - \Lambda_c \cdot G''(u_c,\rho_c) - E''_\mu(u,\rho) + \Lambda \cdot G''(u,\rho))((v,\sigma)^2)\right| \leq \varepsilon \, \|(v,\sigma)\|^2_{W^{1,2}(0,L)} \,. \tag{2.2.27}$$

This is a continuity type property of the second derivatives at $(u_c,\rho_c)$. Here, $\Lambda = (\lambda_{u1},\lambda_{u2},\lambda_\rho) = (\lambda_{\theta 1}(\theta-\phi,\rho),\lambda_{\theta 2}(\theta-\phi,\rho),\lambda_\rho(\theta-\phi,\rho))$ is explicitly given as in (2.1.2), and (2.1.6). By [167, Theorem 43.D] and choosing $\varepsilon < \alpha$, this implies that if $(u,\rho)$ is a critical point of $E_\mu$ subject to the constraint $G=0$ and $\mathcal{I}(u)=0$, then $E_\mu$ has a *strict* local minimum at $(u,\rho)$ subject to the constraint $G=0$ and $\mathcal{I}(u)=0$. We conclude that there cannot exist other constrained critical points in a sufficiently small $C^1$-neighborhood of $(u_c,\rho_c)$.
It remains to prove (2.2.27). We take $0<\bar\delta<1$ small enough and consider $(u,\rho)$ such that $\|(u,\rho)-(u_c,\rho_c)\|_{C^1([0,L])} < \bar\delta$. Then, there exists $C = C(\bar\delta)$ such that

$$\begin{aligned}
&\left|(E''_\mu(u_c,\rho_c) - E''_\mu(u,\rho))((v,\sigma)^2)\right| \\
&\quad\leq \int_0^L |\beta(\rho_c)-\beta(\rho)|\,(\partial_s v)^2 \,\mathrm{d}s + 2\int_0^L \left|(\beta'(\rho_c)-\beta'(\rho))\,(\partial_s\phi - c_0)\,\sigma\,\partial_s v\right| \,\mathrm{d}s \\
&\qquad + 2\int_0^L \left|\beta'(\rho)\,(\partial_s u_c - \partial_s u)\,\sigma\,\partial_s v\right| \,\mathrm{d}s + \frac{1}{2}\int_0^L |\beta''(\rho_c)-\beta''(\rho)|\,(\partial_s\phi - c_0)^2\sigma^2 \,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^L \left|\beta''(\rho)\left((\partial_s\phi - c_0)^2 - (\partial_s u + \partial_s\phi - c_0)^2\right)\right| \sigma^2 \,\mathrm{d}s \\
&\quad\leq C\,\|(u_c,\rho_c)-(u,\rho)\|_{C^1([0,L])}\,\|(v,\sigma)\|^2_{W^{1,2}(0,L)} \,,
\end{aligned}$$

cf. (2.2.7). Since $\Lambda_c \cdot G''(u_c,\rho_c) = 0$ (see (2.2.24)), it remains to prove that

$$\left|\Lambda \cdot G''(u,\rho)((v,\sigma)^2)\right| \leq C\,\|(u_c,\rho_c)-(u,\rho)\|_{C^1([0,L])}\,\|(v,\sigma)\|^2_{W^{1,2}(0,L)} \tag{2.2.28}$$

for some constant $C$. With (2.2.8), $|G''(u,\rho)((v,\sigma)^2)| \leq \|(v,\sigma)\|^2_{W^{1,2}(0,L)}$. Moreover, using Lemma 2.4.2 in Section 2.4 to bound the inverse matrix and $u_c = 0$,

$$\begin{aligned}
|(\lambda_{u1},\lambda_{u2})| &= \left\|\Pi^{-1}(u+\phi)\right\| \left| \int_0^L \begin{pmatrix} -\cos(u+\phi) \\ -\sin(u+\phi)\end{pmatrix} \partial_s u\,\beta(\rho)(\partial_s u + \partial_s\phi - c_0)\,\mathrm{d}s\right| \\
&\leq C\,\|\partial_s u\|_{C([0,L])} \leq C\,\|(u_c,\rho_c)-(u,\rho)\|_{C^1([0,L])} \,.
\end{aligned}$$

Since $(G'')^3(u,\rho) = 0$, we conclude (2.2.28) and the proof is complete. □

#### Conditions for homogeneous elastica

If a constrained critical point has constant density, this already implies that it describes a homogeneous elastica.

**Lemma 2.2.11** ([36, Lemma 2.9]). *If $(\theta, \rho)$ describes a constrained critical point and $\rho \equiv m$, then $\theta$ describes an elastica.*

*Proof.* By Proposition 2.2.4, $(\theta, \rho)$ is smooth. Since $\rho$ is constant, the Euler–Lagrange equation (2.2.17) reads

$$0 = \beta(m)\partial_s k + \lambda_{\theta 1} \sin\theta - \lambda_{\theta 2} \cos\theta, \tag{2.2.29}$$

with $k = \partial_s \theta$. As in [128, Remark 2.2], we multiply (2.2.29) with $k$. This yields

$$0 = \frac{1}{2}\beta(m)\partial_s(k^2) + \partial_s\left(-\lambda_{\theta 1}\cos\theta - \lambda_{\theta 2}\sin\theta\right).$$

So there is $\tilde{\lambda} \in \mathbb{R}$ such that

$$\tilde{\lambda} = \frac{1}{2}\beta(m)k^2 - \lambda_{\theta 1}\cos\theta - \lambda_{\theta 2}\sin\theta. \tag{2.2.30}$$

Differentiation of (2.2.29) and inserting (2.2.30) leads to (2.2.21) with $\lambda = \frac{\tilde{\lambda}}{\beta(m)}$. □

**Remark 2.2.12** ([36, Remark 2.10]). In the case $\omega = 0$, a constrained critical point with constant density exists only if $\beta'(m) = 0$. Indeed, if $\rho \equiv m$ and $\beta'(m) \neq 0$, it follows from (2.2.18) that $(\partial_s\theta - c_0)^2$ is constant. For $\omega = 0$, this contradicts the closedness of the curve described by $\theta$.

For $\omega \neq 0$, we provide suitable assumptions on $\beta$, under which the converse implication of Lemma 2.2.11 also holds.

**Lemma 2.2.13** ([36, Lemma 2.11]). *Let $\omega \neq 0$ and $c_0 \neq \frac{2\pi\omega}{L}$. Let $(\theta, \rho)$ be a constrained critical point and suppose $\theta$ describes an elastica. If $\beta$ is such that*

*(a) $\beta$ is convex or*

*(b) $|\beta'(x)| < 2\mu\left(\dfrac{2\pi}{2\pi\omega - Lc_0}\right)^2 |m - x|$ or*

*(c) $\sup|\beta''| < 2\mu\left(\dfrac{2\pi}{2\pi\omega - Lc_0}\right)^2$,*

*then $\rho$ is constant. In particular, $(\theta, \rho)$ describes a homogeneous elastica.*

*Proof.* Since $\theta$ describes a constrained closed elastica and $\omega \neq 0$, we have $k = \partial_s\theta \equiv \frac{2\pi\omega}{L}$ by Remark 2.2.7. The Euler–Lagrange equation for $\rho$ (cf. (2.2.18)) simplifies to

$$\partial_s^2\rho = \frac{(k - c_0)^2}{2\mu}\left(\beta'(\rho) - \frac{1}{L}\int_0^L \beta'(\rho)\,\mathrm{d}s\right). \tag{2.2.31}$$

*(a)* Using integration by parts, (2.2.31), $\int_0^L \rho \,\mathrm{d}s = mL$, and the convexity of $\beta$, we have

$$\begin{aligned}\int_0^L (\partial_s \rho)^2 \,\mathrm{d}s &= -\frac{(k-c_0)^2}{2\mu} \int_0^L \beta'(\rho)(\rho - m) \,\mathrm{d}s \\ &= -\frac{(k-c_0)^2}{2\mu} \int_0^L \left(\beta'(\rho) - \beta'(m)\right)(\rho - m) \,\mathrm{d}s \leq 0.\end{aligned}$$

*(b)* First, we proceed as in (a), then we obtain with

$$(k-c_0)^2 = \left(\frac{2\pi}{L}\right)^2 \left(\frac{2\pi\omega - Lc_0}{2\pi}\right)^2,$$

the assumption on $\beta'$, and the Wirtinger inequality that

$$\int_0^L (\partial_s \rho)^2 \,\mathrm{d}s = -\frac{(k-c_0)^2}{2\mu} \int_0^L \beta'(\rho)(\rho - m) \,\mathrm{d}s < \left(\frac{2\pi}{L}\right)^2 \int_0^L (\rho - m)^2 \,\mathrm{d}s \leq \int_0^L (\partial_s \rho)^2 \,\mathrm{d}s.$$

*(c)* We write $\overline{\beta'(\rho)} = \frac{1}{L}\int_0^L \beta'(\rho) \,\mathrm{d}s$ and use the Wirtinger inequality twice to get

$$\begin{aligned}\int_0^L (\partial_s^2 \rho)^2 \,\mathrm{d}s &= \left(\frac{(k-c_0)^2}{2\mu}\right)^2 \int_0^L \left(\beta'(\rho) - \overline{\beta'(\rho)}\right)^2 \,\mathrm{d}s \\ &\leq \frac{(k-c_0)^4}{(2\mu)^2} \left(\frac{L}{2\pi}\right)^2 \int_0^L \left(\beta''(\rho)\partial_s \rho\right)^2 \,\mathrm{d}s \\ &< \left(\frac{2\pi}{L}\right)^2 \int_0^L (\partial_s \rho)^2 \,\mathrm{d}s \leq \int_0^L (\partial_s^2 \rho)^2 \,\mathrm{d}s.\end{aligned}$$

In all cases, the periodic boundary conditions imply that $\rho \equiv m$. □

For clarity, we also present a corollary now, which follows from the analysis of the asymptotic behavior. Under the assumptions of Lemma 2.2.13, we can already prove it using the same ideas as in the proof of Part (a). Later (in Theorem 2.7.1), we see that it also holds for $\omega = 0$.

**Corollary 2.2.14** ([36, Corollary 2.12])**.** *Let $(\theta, \rho)$ be a constrained critical point. If $\beta$ is such that $\beta'(x) \leq 0$ for $x < m$ and $\beta'(x) \geq 0$ otherwise, then $(\theta, \rho)$ describes a homogeneous elastica.*

Without additional assumptions on $\beta$, the density of a constrained critical point describing an elastica might be nonconstant.

**Example 2.2.15** ([36, Example 2.13])**.** Let $L = 2\pi$, $\omega = 2$, $m = 0$ and let $\mu > 0$ and $c_0 \neq 2$ be chosen such that $(2 - c_0)^2 = 2\mu$. Let $\beta(x) = -\frac{x^2}{2} + 1$ for $-1 \leq x \leq 1$. Then both $(\theta = \theta_c, \rho \equiv m) \in \mathcal{A}$ and $(\theta = \theta_c, \rho = \sin s) \in \mathcal{A}$ are constrained critical points. In this case, inequalities (b) and (c) in Lemma 2.2.13 are attained with equality, so the assumptions are sharp.

**Remark 2.2.16** (Special choice of $c_0$, [36, Remark 2.14])**.** For $\omega \neq 0$, $c_0 = \frac{2\pi\omega}{L}$ and $(\theta, \rho)$ a constrained critical point with $\theta$ describing an elastica, it follows directly from (2.2.31) and the periodic boundary conditions that $\rho$ is constant.

## 2.3. Local well-posedness and regularity

In the literature, short time existence and uniqueness for quasilinear parabolic systems is widely accepted if the initial datum is smooth. The goal of this section is to provide a complete proof of local well-posedness for the system (2.0.4), including continuous dependence on the data, even for initial data with low regularity and despite the presence of nonlocal terms, in a fairly concise way.
First, we summarize the compatibility conditions for the initial datum.

**Hypothesis 2.3.1** ([37, Hypothesis 1.1])**.** Let $\theta_0, \rho_0 \colon [0, L] \to \mathbb{R}$ be $C^1$-functions such that

$$\begin{aligned} &\theta_0(L) - \theta_0(0) = 2\pi\omega, \qquad && \partial_s\theta_0(L) = \partial_s\theta_0(0), \\ &\rho_0(L) = \rho_0(0), && \partial_s\rho_0(L) = \partial_s\rho_0(0). \end{aligned} \tag{2.3.1}$$

Moreover, let $\theta_0$ satisfy (2.0.3).

The goal of this section is to prove the following existence result. Here, $h^{1+\alpha}$ is the little Hölder space, cf. Section 2.3.6.

**Theorem 2.3.2** (Local well-posedness, [37, Theorem 1.2])**.** *Let $(\theta_0, \rho_0) \in h^{1+\alpha}([0, L])$ for some $\alpha \in (0, 1)$ and assume Hypothesis 2.3.1. Then there exists $T_0 > 0$ and a unique solution $(\theta, \rho) \in C^\infty((0, T_0) \times [0, L])$ of (2.0.4) on $(0, T_0) \times [0, L]$ which satisfies*

$$\lim_{t\to 0}(\theta(t), \rho(t)) = (\theta_0, \rho_0) \quad \text{in } C^{1+\alpha}([0, L]).$$

*Moreover, the solution depends continuously on the initial datum.*

The idea of the proof is to meet the boundary conditions by formulating the problem in a periodic setting (see Section 2.3.1). Then, following the ideas in [17], the Inverse Function Theorem between appropriate time-dependent little Hölder spaces yields existence and continuous dependence on the initial datum (see Section 2.3.2). We deduce the local invertibility of the Fréchet derivative from the maximal regularity theory developed in [101] and a perturbation argument (see Proposition 2.3.12). The precise formulation of the continuous dependence is given in Proposition 2.3.5 and Remark 2.3.6 below. To show instantaneous smoothing, we rely on the theory of parabolic systems in [96] and a bootstrap argument (see Section 2.3.4). Finally, exploiting the dissipative structure of the system, uniqueness follows from an energy argument (see Section 2.3.4).

### 2.3.1. Transformation to a periodic setting

Proving Theorem 2.3.2 in a periodic setting is quite natural due to the geometric interpretation of the problem. In the periodic setting, we get rid of the boundary conditions which enables us to apply an existence result in [101] for a suitable linearization of our problem. We first prove short time existence and continuous dependence on the data as well as uniqueness and smoothness of the solution to the periodic problem. Then, by a suitable transformation, we can transfer the results to problem (2.0.4).

For the moment, assume that $(\theta, \rho)$ is a sufficiently smooth function in $(t, s)$. We observe that the boundary conditions in (2.0.4) imply that $\partial_s\theta$ is a $C^0$-periodic function, i.e. it can be extended to a continuous $L$-periodic function on $\mathbb{R}$. Similarly, $\rho$ is a $C^1$-periodic function. The angle function $\theta$ is itself not periodic if the rotation number is different from zero. We hence first perform a suitable transformation. We work with the function

$$\phi\colon [0, L] \to \mathbb{R}, \quad \phi(s) := \frac{2\pi\omega s}{L}$$

defined in (2.2.2) and consider the function $u(t, s) := \theta(t, s) - \phi(s)$. The boundary conditions in (2.0.4) imply that $u$ and $\rho$ are $C^1$-periodic functions on $[0, L]$. We thus consider the $L$-periodic extensions of $u_0 := \theta_0 - \phi$ and $\rho_0$ to all of $\mathbb{R}$ and look for $L$-periodic solutions $(u, \rho)$ defined on $(0, T) \times \mathbb{R}$ for some $T > 0$. Now, instead of Hypothesis 2.3.1, it is sufficient to assume that

$$\int_0^L \cos(u_0 + \phi)\,\mathrm{d}s = \int_0^L \sin(u_0 + \phi)\,\mathrm{d}s = 0. \tag{2.3.2}$$

In terms of $(u, \rho)$ $L$-periodic, (2.0.4) can be equivalently written as

$$\begin{cases} \partial_t u = \partial_s\left[\beta(\rho)\left(\partial_s u + \partial_s\phi - c_0\right)\right] + \Lambda_u(u, \rho) & \text{in } (0, T) \times \mathbb{R}, \\ \partial_t\rho = \mu\partial_s^2\rho - \dfrac{1}{2}\beta'(\rho)\left(\partial_s u + \partial_s\phi - c_0\right)^2 + \Lambda_\rho(u, \rho) & \text{in } (0, T) \times \mathbb{R}, \\ u(0, \cdot) = u_0, \qquad \rho(0, \cdot) = \rho_0 & \text{on } \mathbb{R}. \end{cases} \tag{2.3.3}$$

Here, to simplify the notation, instead of working with $(\lambda_{\theta 1}, \lambda_{\theta 2})$ (see Remark 2.1.3) we consider $\Lambda_u$ given by

$$\begin{aligned} \Lambda_u(u, \rho)(t, s) &:= \begin{pmatrix} \lambda_{\theta 1}(u + \phi, \rho)(t) \\ \lambda_{\theta 2}(u + \phi, \rho)(t) \end{pmatrix} \cdot \begin{pmatrix} \sin(u(t, s) + \phi(s)) \\ -\cos(u(t, s) + \phi(s)) \end{pmatrix} \\ &= \Pi^{-1}(u + \phi) \int_0^L \begin{pmatrix} \cos(u + \phi) \\ \sin(u + \phi) \end{pmatrix} (\partial_s u + \partial_s\phi)\,\beta(\rho)\left(\partial_s u + \partial_s\phi - c_0\right)\mathrm{d}s \\ &\quad \cdot \begin{pmatrix} \sin(u + \phi) \\ -\cos(u + \phi) \end{pmatrix} =: \Pi^{-1}(u + \phi)\,\mathcal{J}(u, \rho) \cdot \begin{pmatrix} \sin(u + \phi) \\ -\cos(u + \phi) \end{pmatrix} \end{aligned} \tag{2.3.4}$$

for $(t,s) \in (0,T) \times \mathbb{R}$ and, instead of $\lambda_\rho$, $\Lambda_\rho$ given by

$$\Lambda_\rho(u,\rho)(t) := \lambda_\rho(u+\phi,\rho)(t) = \frac{1}{2L}\int_0^L \beta'(\rho)\left(\partial_s u + \partial_s \phi - c_0\right)^2 \,\mathrm{d}s,$$

for $t \in (0,T)$, see (2.1.6).
We now define our concept of a solution. To that end, we work with time-weighted little Hölder spaces (see Section 2.3.6 for a precise definition), which provide an appropriate framework for the local well-posedness of the problem.

**Definition 2.3.3** ([37, Definition 3.1])**.** Let $T > 0$, $\alpha \in (0,1)$ and $\eta \in (\frac{1}{2}, 1)$ be such that $2\eta + \alpha \notin \mathbb{Z}$. Moreover, let $(u_0, \rho_0) \in h_L^{2\eta+\alpha}(\mathbb{R};\mathbb{R}^2)$ be such that $u_0$ satisfies (2.3.2). We call $(u,\rho)$ a *solution of* (2.3.3) *on* $(0,T)\times\mathbb{R}$ *with initial datum* $(u_0,\rho_0)$ if

$$(u,\rho) \in \mathrm{BUC}^1_{1-\eta}\left([0,T']; h_L^\alpha(\mathbb{R};\mathbb{R}^2)\right) \cap \mathrm{BUC}_{1-\eta}\left([0,T']; h_L^{2+\alpha}(\mathbb{R};\mathbb{R}^2)\right)$$

for all $T' < T$, if $(u,\rho)$ solves the flow equations in (2.3.3) on $(0,T)\times[0,L]$ and if

$$\lim_{t\to 0}(u,\rho)(t,\cdot) = (u_0,\rho_0) \quad \text{in } C^{2\eta+\alpha}(\mathbb{R};\mathbb{R}^2).$$

The time weight (described by the parameter $\eta$) allows us to consider less regular initial data. For $T > 0$, $\alpha \in (0,1)$ and $\eta \in \left(\frac{1}{2},1\right)$ such that $2\eta+\alpha \notin \mathbb{Z}$, we set

$$\mathbb{E}_0([0,T]) := \mathrm{BUC}_{1-\eta}\left([0,T]; h_L^\alpha(\mathbb{R};\mathbb{R}^k)\right), \tag{2.3.5}$$
$$\mathbb{E}_1([0,T]) := \mathrm{BUC}^1_{1-\eta}\left([0,T]; h_L^\alpha(\mathbb{R};\mathbb{R}^k)\right) \cap \mathrm{BUC}_{1-\eta}\left([0,T]; h_L^{2+\alpha}(\mathbb{R};\mathbb{R}^k)\right) \tag{2.3.6}$$

for $k = 1,2$. For $k = 2$, $\mathbb{E}_1$ is the space of solutions in the sense of Definition 2.3.3. For $(u,\rho) \in \mathbb{E}_1$ the right hand side of the evolution equations in (2.3.3) belongs to $\mathbb{E}_0$, cf. Remark 2.3.20. We define $\gamma\mathbb{E}_1 := h_L^{2\eta+\alpha}(\mathbb{R};\mathbb{R}^k)$, the space for initial data (see (2.3.37)). In the following, the value of $k = 1,2$ is clear from the context.
We introduce the open set

$$\mathbb{U}([0,T]) := \{(u,\rho) \in \mathbb{E}_1([0,T]) : \inf_{[0,T]} \det \Pi(u+\phi) > 0\}$$

and define the nonlinear differential operator

$$\begin{aligned}
&\mathbf{F}\colon \mathbb{U}([0,T]) \to \mathbb{E}_0([0,T]), \quad \mathbf{F}(u,\rho) = (\mathbf{F}_1(u,\rho), \mathbf{F}_2(u,\rho)),\\
&\mathbf{F}_1(u,\rho) := \partial_s\left[\beta(\rho)\left(\partial_s u + \partial_s\phi - c_0\right)\right] + \Lambda_u(u,\rho),\\
&\mathbf{F}_2(u,\rho) := \mu\,\partial_s^2\rho - \frac{1}{2}\beta'(\rho)\left(\partial_s u + \partial_s\phi - c_0\right)^2 + \Lambda_\rho(u,\rho).
\end{aligned} \tag{2.3.7}$$

The evolution equations in (2.3.3) can be written as $\partial_t(u,\rho) = \mathbf{F}(u,\rho)$ on $(0,T)\times\mathbb{R}$.

**Remark 2.3.4** ([37, Remark 3.2])**.** The choice $\eta > \frac{1}{2}$ ensures that the initial datum has finite energy, but also turns out to be essential to control the nonlinearities in the equation (cf. Lemma 2.3.9, Proposition 2.3.12) and to show uniqueness (cf. Lemmas 2.3.14, 2.3.21).

### 2.3.2. Short time existence and continuous dependence

To prove short time existence and continuous dependence on the data, we essentially follow the ideas of [17, Theorem 2]. However, since we work with periodic little Hölder spaces, we have to justify some steps differently.

**Proposition 2.3.5** (Existence and continuous dependence, [37, Proposition 3.3])**.** *Let $\alpha \in (0,1)$ and $\eta \in (\frac{1}{2}, 1)$. Let $(u_0, \rho_0) \in \gamma\mathbb{E}_1$ and let $u_0$ satisfy (2.3.2). Then there exist $T_0 \in (0,\infty)$ and $r > 0$ such that for every $(\tilde{u}_0, \tilde{\rho}_0) \in \gamma\mathbb{E}_1$ with $\|(u_0,\rho_0) - (\tilde{u}_0,\tilde{\rho}_0)\|_{C^{2\eta+\alpha}([0,L])} < r$ the problem*

$$\begin{cases} \partial_t \tilde{u} = \partial_s \left[\beta(\tilde{\rho})\left(\partial_s \tilde{u} + \partial_s \phi - c_0\right)\right] + \Lambda_u(\tilde{u},\tilde{\rho}) & \text{in } (0,T_0)\times\mathbb{R}, \\ \partial_t \tilde{\rho} = \mu\partial_s^2\tilde{\rho} - \frac{1}{2}\beta'(\tilde{\rho})\left(\partial_s\tilde{u} + \partial_s\phi - c_0\right)^2 + \Lambda_\rho(\tilde{u},\tilde{\rho}) & \text{in } (0,T_0)\times\mathbb{R}, \\ \tilde{u}(0,\cdot) = \tilde{u}_0, \qquad \tilde{\rho}(0,\cdot) = \tilde{\rho}_0 & \text{on } \mathbb{R}, \end{cases} \tag{2.3.8}$$

*admits a solution $(\tilde{u},\tilde{\rho}) \in \mathbb{E}_1([0,T_0])$. Moreover, there is a $C^1$-map which associates to the initial datum a solution of the problem. In particular, there exists a solution $(u,\rho) \in \mathbb{E}_1([0,T_0])$ of (2.3.3) on $(0,T_0)\times\mathbb{R}$ with initial datum $(u_0,\rho_0)$.*

*Proof.* We divide the proof into several steps. For the sake of readability, the details are partially moved to Section 2.3.3. We will refer to this at the respective points.
*Step 1: Construction of a "good" function.* We show that there exist $\bar{T} > 0$ and $(\bar{u},\bar{\rho}) \in \mathbb{E}_1([0,\bar{T}])$ with $(\bar{u},\bar{\rho})(0,\cdot) = (u_0,\rho_0)$, such that $(\bar{u},\bar{\rho}) \in \mathbb{U}([0,\bar{T}])$, i.e.

$$\inf_{[0,\bar{T}]} \det\left(\Pi(\bar{u}+\phi)\right) > 0, \tag{2.3.9}$$

and having some extra regularity. More precisely, $\partial_s(\bar{u},\bar{\rho}) \in \tilde{\mathbb{E}}_1([0,\bar{T}])$ where

$$\tilde{\mathbb{E}}_1([0,\bar{T}]) := \mathrm{BUC}^1_{1-\tilde{\eta}}\left([0,\bar{T}]; h^\alpha_L(\mathbb{R})\right) \cap \mathrm{BUC}_{1-\tilde{\eta}}\left([0,\bar{T}]; h^{2+\alpha}_L(\mathbb{R})\right)$$

for $\tilde{\eta} := \eta - \frac{1}{2} > 0$.
First, we take the solution $(\bar{u},\bar{\rho}) \in \mathbb{E}_1([0,\hat{T}])$, for $\hat{T} > 0$ arbitrary, of

$$\begin{cases} \partial_t(v,\sigma) - \partial_s^2(v,\sigma) = (0,0) & \text{in } (0,\hat{T})\times\mathbb{R}, \\ (v,\sigma)(0,\cdot) = (u_0,\rho_0) & \text{on } \mathbb{R}. \end{cases}$$

Such a solution exists by [101, Theorem 6.4]. As solution of the heat equation, we know that $(\bar{u},\bar{\rho}) \in C^\infty((0,\hat{T})\times\mathbb{R})$. Due to the fact that $\eta > \frac{1}{2}$, $\partial_s(u_0,\rho_0) \in h^{2\tilde{\eta}+\alpha}_L(\mathbb{R}) = \gamma\tilde{\mathbb{E}}_1$

and hence it is an admissible initial datum. We look at

$$\begin{cases} \partial_t(v,\sigma) - \partial_s^2(v,\sigma) = (0,0) & \text{in } (0,\hat{T}) \times \mathbb{R}, \\ (v,\sigma)(0,\cdot) = \partial_s(u_0,\rho_0) & \text{on } \mathbb{R}. \end{cases} \tag{2.3.10}$$

Again, we use [101, Theorem 6.4] and obtain a unique solution in $\tilde{\mathbb{E}}_1([0,\hat{T}])$ that is smooth for positive times. Since this solution coincides with $\partial_s(\bar{u},\bar{\rho})$ for $t=0$ and $\partial_s(\bar{u},\bar{\rho})$ solves the evolution equation in (2.3.10), we conclude by uniqueness that $\partial_s(\bar{u},\bar{\rho})$ is the solution of (2.3.10), so that $\partial_s(\bar{u},\bar{\rho}) \in \tilde{\mathbb{E}}_1([0,\hat{T}])$.
It remains to check (2.3.9). Since $u_0+\phi$ satisfies Hypothesis 2.3.1, using Remark 2.1.4 and Lemma 2.3.18, by continuity there exists $\bar{T} \leq \hat{T}$ such that (2.3.9) is satisfied.
*Step 2:* We claim that there exists $T' \in (0,\bar{T}]$ such that the nonlinear operator

$$\begin{aligned} &\Phi\colon \mathbb{U}([0,T']) \to \mathbb{E}_0([0,T']) \times \gamma\mathbb{E}_1, \\ &\Phi(\tilde{u},\tilde{\rho}) := \Big(\partial_t(\tilde{u},\tilde{\rho}) - \mathbf{F}(\tilde{u},\tilde{\rho}),\ (\tilde{u},\tilde{\rho})\,(0,\cdot)\Big) \end{aligned}$$

is well defined and a local diffeomorphism at $(\bar{u},\bar{\rho})$ (restricted to $[0,T']$). Here $\mathbf{F}$ is the map defined in (2.3.7).
It is sufficient to prove the existence of $T' \in (0,\bar{T}]$ such that the Fréchet derivative

$$\begin{aligned} &\mathrm{D}\Phi(\bar{u},\bar{\rho})\colon \mathbb{E}_1([0,T']) \to \mathbb{E}_0([0,T']) \times \gamma\mathbb{E}_1, \\ &\mathrm{D}\Phi(\bar{u},\bar{\rho})(v,\sigma) := \Big(\partial_t(v,\sigma) - \mathrm{D}\mathbf{F}(\bar{u},\bar{\rho})(v,\sigma), (v,\sigma)(0,\cdot)\Big), \end{aligned} \tag{2.3.11}$$

is a linear isomorphism. It then follows, since $\Phi \in C^1$, by the Local Inverse Function Theorem that $\Phi$ is a local diffeomorphism at $(\bar{u},\bar{\rho})$. The proof that the map in (2.3.11) is a linear isomorphism is given in Proposition 2.3.12 (with $\tilde{\mathbf{J}} = \mathrm{D}\Phi(\bar{u},\bar{\rho})$). The idea is to use a maximal regularity result in time-weighted little Hölder spaces [101], where the time weight allows for rough initial data. Since the results in [101] are applicable to linear systems with time-independent coefficients, we first consider an auxiliary linear problem, cf. Proposition 2.3.7. A perturbation argument then gives that the map in (2.3.11) is an isomorphism for small times.
*Step 3:* By Step 2, there is a neighborhood $V \subset \mathbb{U}([0,T'])$ of $(\bar{u},\bar{\rho})$ and a neighborhood $W \subset \mathbb{E}_0([0,T']) \times \gamma\mathbb{E}_1$ of $\Phi(\bar{u},\bar{\rho}) =: ((\bar{f}_1,\bar{f}_2),(u_0,\rho_0))$ such that $\Phi : V \to W$ is a diffeomorphism. Choose $r>0$ small enough such that $B_r(\bar{f}_1,\bar{f}_2) \times B_r(u_0,\rho_0) \subset W$. Moreover, choose $T'' \in (0,T')$ small enough so that

$$\|(\bar{f}_1,\bar{f}_2)\|_{\mathbb{E}_0([0,T''])} < r. \tag{2.3.12}$$

Here it is crucial that the $\mathbb{E}_0$-norm becomes arbitrary small for small times (see the definition of $\mathrm{BUC}_{1-\eta}$ in Section 2.3.6).

*Step 4:* Define $(\tilde{f}_1, \tilde{f}_2) \in \mathbb{E}_0([0,T'])$ by

$$(\tilde{f}_1, \tilde{f}_2) := \begin{cases} (0,0) & \text{on } \left[0, \frac{1}{2}T''\right], \\ \frac{2}{T''}(t - \frac{1}{2}T'')(\bar{f}_1, \bar{f}_2) & \text{on } \left(\frac{1}{2}T'', T''\right), \\ (\bar{f}_1, \bar{f}_2) & \text{on } [T'', T']. \end{cases}$$

Then, by a short computation and the choice of $T''$ in (2.3.12), we have

$$\|(\tilde{f}_1, \tilde{f}_2) - (\bar{f}_1, \bar{f}_2)\|_{\mathbb{E}_0([0,T'])} < r.$$

This implies that, with $(\tilde{u}_0, \tilde{\rho}_0) \in \gamma\mathbb{E}_1$ as in the statement, we have

$$((\tilde{f}_1, \tilde{f}_2), (\tilde{u}_0, \tilde{\rho}_0)) \in B_r(\bar{f}_1, \bar{f}_2) \times B_r(u_0, \rho_0) \subset W.$$

Since $\Phi : V \to W$ is a diffeomorphism, there exists $(\tilde{u}, \tilde{\rho}) \in V \subset \mathbb{U}([0,T'])$ such that $\Phi(\tilde{u}, \tilde{\rho}) = ((\tilde{f}_1, \tilde{f}_2), (\tilde{u}_0, \tilde{\rho}_0))$. We restrict $(\tilde{u}, \tilde{\rho})$ to $[0, \frac{1}{2}T''] \times \mathbb{R}$, set $T_0 := \frac{1}{2}T''$ and obtain a solution of (2.3.8) on $(0, T_0) \times \mathbb{R}$.
*Step 5:* We constructed a mapping which maps every $(\tilde{u}_0, \tilde{\rho}_0) \in B_r(u_0, \rho_0)$ to a solution $(\tilde{u}, \tilde{\rho}) \in \mathbb{E}_1([0, T_0])$ of the problem. It consists essentially of the composition of the $C^1$-mapping $\Phi^{-1}$ and a restriction operator and is thus continuously differentiable. □

**Remark 2.3.6** ([37, Remark 3.4])**.** In Section 2.3.4 we prove uniqueness of solutions. Together with Proposition 2.3.5 this yields the continuity — or more precisely even the continuous differentiability — of the mapping which maps any initial datum $(u_0, \rho_0) \in \gamma\mathbb{E}_1$ to the unique solution $(u, \rho) \in \mathbb{E}_1([0, T_0])$ of (2.3.3).

## 2.3.3. Auxiliary results for Section 2.3.2

In the following, we show Step 2 of the proof of Proposition 2.3.5, i.e. that $\mathrm{D}\Phi(\bar{u}, \bar{\rho})$ (see (2.3.11)) is a linear isomorphism. This is equivalent to show existence and uniqueness of a solution for a certain inhomogeneous linear initial value problem.
In this section, $\alpha \in (0,1)$ and $\eta \in (\frac{1}{2}, 1)$ are fixed.

### The linearization

We compute here $\mathrm{D}\mathbf{F}(\bar{u}, \bar{\rho})$ needed in (2.3.11), where $\mathbf{F}$ is given as in (2.3.7). By a direct calculation one finds

$$\begin{aligned} \mathrm{D}\mathbf{F}_1(\bar{u}, \bar{\rho})(v, \sigma) = {} & \beta(\bar{\rho})\, \partial_s^2 v + \beta'(\bar{\rho})\, \partial_s^2 \bar{u}\, \sigma + \beta'(\bar{\rho})\, \partial_s \bar{\rho}\, \partial_s v \\ & + \beta'(\bar{\rho})\, (\partial_s \bar{u} + \partial_s \phi - c_0)\, \partial_s \sigma + \beta''(\bar{\rho})\, \partial_s \bar{\rho}\, (\partial_s \bar{u} + \partial_s \phi - c_0)\, \sigma \\ & + \mathrm{D}\Lambda_u(\bar{u}, \bar{\rho})(v, \sigma), \end{aligned} \tag{2.3.13}$$

$$\mathrm{D}\mathbf{F}_2(\bar{u},\bar{\rho})(v,\sigma) = \mu\,\partial_s^2\sigma - \frac{1}{2}\beta''(\bar{\rho})\,(\partial_s\bar{u}+\partial_s\phi-c_0)^2\,\sigma - \beta'(\bar{\rho})\,(\partial_s\bar{u}+\partial_s\phi-c_0)\,\partial_s v + \mathrm{D}\Lambda_\rho(\bar{u},\bar{\rho})(v,\sigma). \tag{2.3.14}$$

Let $T > 0$. We first observe that by (2.3.4), in the Lagrange multipliers only spatial derivatives of first order appear. More specifically, $\Lambda_u$ is well defined with $\Lambda_u(u,\rho) \in \mathbb{E}_0([0,T])$ for $(u,\rho) \in \mathbb{V}([0,T])$ where

$$\mathbb{V}([0,T]) := \Big\{(u,\rho)\in \mathbb{F}([0,T]) := \mathrm{BUC}^1_{1-\eta}([0,T];h^\alpha_L(\mathbb{R}))\cap \mathrm{BUC}_{1-\eta}([0,T];h^{1+\alpha}_L(\mathbb{R})) \text{ such that } \inf_{[0,T]}\det\Pi(u+\phi)>0\Big\}. \tag{2.3.15}$$

This is an open subset of $\mathbb{F}([0,T])$ by continuity. Moreover, using $\eta > \frac{1}{2}$ and (2.3.37), any initial datum $(v_0,\sigma_0) \in \gamma\mathbb{E}_1$ is in $\mathbb{F}([0,T])$ if we identify it with the function $(t,s) \mapsto (v_0(s),\sigma_0(s))$. It is not difficult to see that

$$\mathbb{V}([0,T]) \ni (u,\rho) \mapsto \Lambda_u(u,\rho) \in \mathbb{E}_0([0,T]) \tag{2.3.16}$$

is of class $C^1$. Moreover, it follows from the product rule that

$$\begin{aligned}\mathrm{D}\Lambda_u(\bar{u},\bar{\rho})(v,\sigma) = {}& \mathrm{D}\Pi^{-1}(\bar{u}+\phi)(v)\,\mathcal{J}(\bar{u},\bar{\rho})\cdot\begin{pmatrix}\sin(\bar{u}+\phi)\\-\cos(\bar{u}+\phi)\end{pmatrix}\\ &+\Pi^{-1}(\bar{u}+\phi)\,\mathrm{D}\mathcal{J}(\bar{u},\bar{\rho})(v,\sigma)\cdot\begin{pmatrix}\sin(\bar{u}+\phi)\\-\cos(\bar{u}+\phi)\end{pmatrix}\\ &+\Pi^{-1}(\bar{u}+\phi)\,\mathcal{J}(\bar{u},\bar{\rho})\cdot\begin{pmatrix}\cos(\bar{u}+\phi)\\\sin(\bar{u}+\phi)\end{pmatrix}\end{aligned} \tag{2.3.17}$$

for all $(v,\sigma) \in \mathbb{F}([0,T])$. A similar argument yields that $\Lambda_\rho\colon \mathbb{V}([0,T]) \to \mathbb{E}_0([0,T])$ is well defined and $C^1$ with derivative

$$\mathrm{D}\Lambda_\rho(\bar{u},\bar{\rho})(v,\sigma) = \frac{1}{2L}\int_0^L \beta''(\bar{\rho})\,(\partial_s\bar{u}+\partial_s\phi-c_0)^2\sigma + 2\beta'(\bar{\rho})\,(\partial_s\bar{u}+\partial_s\phi-c_0)\,\partial_s v\,\mathrm{d}s.$$

#### Linear evolution problem with time-independent coefficients

We consider an auxiliary linear evolution problem. Since we rely on [101], where time-independent coefficients are used, we would like to freeze the coefficients of the terms in (2.3.13) and (2.3.14) in $t = 0$. For the second term in (2.3.13), the regularity of the initial datum does not allow this. Therefore, we treat this term differently from the others and move it together with $\mathrm{D}\Lambda_u$ and $\mathrm{D}\Lambda_\rho$, which are nonlocal, to the right hand side of the inhomogeneous initial value problem we want to solve. This explains the structure of the differential operator chosen below.

**Proposition 2.3.7** ([37, Proposition B.1])**.** *Let $(u_0, \rho_0) \in \gamma\mathbb{E}_1$ such that $u_0$ satisfies (2.3.2). Let $(\bar{u}, \bar{\rho}) \in \mathbb{E}_1([0, \bar{T}])$ be the function constructed in the proof of Proposition 2.3.5. Then for all $T \in (0, \bar{T}]$ the mapping*

$$\mathbf{J}\colon \mathbb{E}_1([0,T]) \to \mathbb{E}_0([0,T]) \times \gamma\mathbb{E}_1, \quad (v,\sigma) \mapsto \big(\partial_t(v,\sigma) - \mathcal{A}(v,\sigma) - \Psi,\ (v,\sigma)(0,\cdot)\big)$$

*where*

$$\begin{aligned}\mathcal{A}(v,\sigma) :=& \Big(\beta(\rho_0)\partial_s^2 v + \beta'(\rho_0)\partial_s\rho_0\partial_s v + \beta'(\rho_0)(\partial_s u_0 + \partial_s\phi - c_0)\partial_s\sigma \\ &\quad + \beta''(\rho_0)\partial_s\rho_0(\partial_s u_0 + \partial_s\phi - c_0)\sigma, \\ &\quad \mu\partial_s^2\sigma - \frac{1}{2}\beta''(\rho_0)(\partial_s u_0 + \partial_s\phi - c_0)^2\sigma - \beta'(\rho_0)(\partial_s u_0 + \partial_s\phi - c_0)\partial_s v\Big), \\ \Psi = \Psi(v_0,\sigma_0) :=& \Big(\beta'(\bar{\rho})\partial_s^2\bar{u}\,\sigma_0 + \mathrm{D}\Lambda_u(\bar{u},\bar{\rho})(v_0,\sigma_0),\ \mathrm{D}\Lambda_\rho(\bar{u},\bar{\rho})(v_0,\sigma_0)\Big)\end{aligned}$$

*is well defined and an isomorphism. Here $(v_0, \sigma_0) = (v,\sigma)(0,\cdot) \in \mathbb{F}([0,T])$, cf. (2.3.15). Equivalently[1], for any pair $(\varphi_1, \varphi_2) \in \mathbb{E}_0([0,T])$ and $(v_0, \sigma_0) \in \gamma\mathbb{E}_1$ there exists a unique solution $(u, \rho) \in \mathbb{E}_1([0,T])$ of the linear problem*

$$\begin{cases} \partial_t(v,\sigma) - \mathcal{A}(v,\sigma) = (\varphi_1,\varphi_2) + \Psi & \text{in } (0,T)\times\mathbb{R}, \\ (v,\sigma)(0,\cdot) = (v_0,\sigma_0) & \text{on } \mathbb{R}. \end{cases} \tag{2.3.18}$$

*Proof.* We want to use [101], which works with functions on the one-dimensional torus, or equivalently for periodic functions, cf. [101, Proposition 1.1]. With (2.3.37) and Lemma 2.3.17, the time-independent coefficients of the operator $\mathcal{A}$ are in $h_L^\alpha(\mathbb{R})$. Because $\beta(\rho_0)$ is bounded away from zero and $\mu > 0$, $\mathcal{A}$ is a uniformly elliptic operator. Apart from this we have to check that $\Psi \in \mathbb{E}_0([0,T])$. Using Lemma 2.3.17 and Remark 2.3.20, we find $\beta'(\bar{\rho}) \in \mathbb{E}_1([0,T])$ and thus $\beta'(\bar{\rho})\partial_s^2\bar{u}\,\sigma_0 \in \mathbb{E}_0([0,T])$. From (2.3.16), we conclude that $\mathrm{D}\Lambda_u(\bar{u},\bar{\rho})(v_0,\sigma_0) \in \mathbb{E}_0([0,T])$ and similarly for $\Lambda_\rho$. Applying [101, Theorem 6.4] yields a unique solution $(v,\sigma) \in \mathbb{E}_1([0,T])$ of (2.3.18). □

From the fact that the mapping $\mathbf{J}$ from Proposition 2.3.7 is an isomorphism, an estimate of the norm of the solution follows implicitly. We now show that the constant in this estimate can be chosen independent of $T$.

**Lemma 2.3.8** ([37, Lemma B.2])**.** *Let $(u_0, \rho_0) \in \gamma\mathbb{E}_1$ such that $u_0$ satisfies (2.3.2). Let $(\bar{u}, \bar{\rho}) \in \mathbb{E}_1([0, \bar{T}])$ be the map constructed in the proof of Proposition 2.3.5. Let $T \in (0, \bar{T}]$.*

[1] Here we slightly abuse notation, denoting by $(v_0, \sigma_0)$ both the initial datum of the problem (2.3.18) and the evaluation at $t = 0$ of $(v,\sigma) \in \mathbb{E}_1([0,T])$. However, for solving (2.3.18), this does not make a difference.

*Then the solution operator* $\mathbf{J}^{-1}$ *satisfies the estimate*

$$\left\|\mathbf{J}^{-1}((\varphi_1,\varphi_2),(v_0,\sigma_0))\right\|_{\mathbb{E}_1([0,T])} \leq C\left(\|(\varphi_1,\varphi_2)\|_{\mathbb{E}_0([0,T])} + \|(v_0,\sigma_0)\|_{\gamma\mathbb{E}_1}\right)$$

*for all* $(\varphi_1,\varphi_2)\in\mathbb{E}_0([0,T])$ *and* $(v_0,\sigma_0)\in\gamma\mathbb{E}_1$ *with* $C=C(u_0,\rho_0,\bar{T})$.

Notice that here we do not write explicitely the dependence on $T$ of the solution operator. This abuse of notation is justified in the proof below.

*Proof of Lemma 2.3.8.* Proposition 2.3.7 already yields the claimed inequality with $C = C(u_0,\rho_0,T)$. We show that $C$ can be chosen only depending on $\bar{T} > T$ by an extension argument. Let $(v,\sigma) = \mathbf{J}^{-1}((\varphi_1,\varphi_2),(v_0,\sigma_0)) \in \mathbb{E}_1([0,T])$. Consider

$$(\varphi_1^*,\varphi_2^*)(t,s) = \begin{cases} (\varphi_1,\varphi_2)(t,s) & \text{for } (t,s)\in[0,T]\times\mathbb{R}, \\ (\varphi_1,\varphi_2)(T,s) & \text{for } (t,s)\in(T,\bar{T}]\times\mathbb{R}. \end{cases}$$

Then $(\varphi_1^*,\varphi_2^*)\in\mathbb{E}_0([0,\bar{T}])$ and we can apply Proposition 2.3.7 and obtain $(v^*,\sigma^*) = \mathbf{J}^{-1}((\varphi_1^*,\varphi_2^*),(v_0,\sigma_0))\in\mathbb{E}_1([0,\bar{T}])$, the solution of (2.3.18) on $(0,\bar{T}]$ with $(\varphi_1^*,\varphi_2^*)$ instead of $(\varphi_1,\varphi_2)$. The same proposition gives that $\mathbf{J}^{-1}$ is an isomorphism from $\mathbb{E}_1([0,\bar{T}])$ to $\mathbb{E}_0([0,\bar{T}])$. Hence, there exists a constant $C = C(u_0,\rho_0,\bar{T})$ such that

$$\|(v^*,\sigma^*)\|_{\mathbb{E}_1([0,\bar{T}])} \leq C\left(\|(\varphi_1^*,\varphi_2^*)\|_{\mathbb{E}_0([0,\bar{T}])} + \|(v_0,\sigma_0)\|_{\gamma\mathbb{E}_1}\right).$$

Because of the uniqueness of the solution and using that (2.3.18) is local in time, we find $(v^*,\sigma^*)|_{[0,T]} = (v,\sigma)$. The claim follows using that $\|(v,\sigma)\|_{\mathbb{E}_1([0,T])} \leq \|(v^*,\sigma^*)\|_{\mathbb{E}_1([0,\bar{T}])}$, together with $\|(\varphi_1^*,\varphi_2^*)\|_{\mathbb{E}_0([0,\bar{T}])} = \|(\varphi_1,\varphi_2)\|_{\mathbb{E}_0([0,T])}$. □

### Some auxiliary estimates

We collect here some technical results, which we use to examine the differences appearing when comparing the linear operators in (2.3.11) and $\mathbf{J}$ from Proposition 2.3.7.
Throughout this subsection, we fix $(u_0,\rho_0)\in\gamma\mathbb{E}_1$ such that $u_0$ satisfies (2.3.2) and denote by $(\bar{u},\bar{\rho})\in\mathbb{E}_1([0,\bar{T}])$ the function constructed in the proof of Proposition 2.3.5. In particular, $\eta\in(\frac{1}{2},1)$, $\tilde{\eta}=\eta-\frac{1}{2}$ and $\partial_s(\bar{u},\bar{\rho})\in\tilde{\mathbb{E}}_1([0,\bar{T}])$.

**Lemma 2.3.9** ([37, Lemma B.3])**.** *Let* $t\in(0,\bar{T}]$. *Then we have*

$$\|\bar{u}(t)-u_0\|_{C^\alpha} \leq \|\bar{u}\|_{\mathbb{E}_1}\frac{t^\eta}{\eta}, \qquad \|\partial_s\bar{u}(t)-\partial_s u_0\|_{C^\alpha} \leq \|\partial_s\bar{u}\|_{\tilde{\mathbb{E}}_1}\frac{t^{\tilde{\eta}}}{\tilde{\eta}},$$

*and similarly for* $\bar{\rho}$.

*Proof.* Inspired by (2.3.36), we compute

$$\begin{aligned}\|\bar{u}(t)-u_0\|_{C^\alpha} &= \|\lim_{t'\to 0+}(\bar{u}(t)-\bar{u}(t'))\|_{C^\alpha} = \|\lim_{t'\to 0+}\int_{t'}^t \partial_\tau \bar{u}(\tau)\,\mathrm{d}\tau\|_{C^\alpha}\\ &\le \lim_{t'\to 0+}\int_{t'}^t \left\|\tau^{\eta-1}\tau^{1-\eta}\partial_\tau\bar{u}(\tau)\right\|_{C^\alpha}\,\mathrm{d}\tau \le \|\bar{u}\|_{\mathbb{E}_1}\lim_{t'\to 0+}\int_{t'}^t \tau^{\eta-1}\,\mathrm{d}\tau = \|\bar{u}\|_{\mathbb{E}_1}\frac{t^\eta}{\eta}.\end{aligned}$$

For $\partial_s\bar{u}$, we proceed completely analogously. □

In the sequel, we estimate terms involving $\beta^{(i)}(\bar{\rho})$. Since $\bar{\rho}\in \mathrm{BUC}([0,\bar{T}];h_L^{2\eta+\alpha}(\mathbb{R}))$, see Lemma 2.3.18, there exists $J\subset\mathbb{R}$, a compact interval, such that $\bar{\rho}(t,s)\in J$ for all $(t,s)\in[0,\bar{T}]\times\mathbb{R}$. Since $\beta\in C^\infty(\mathbb{R})$, there is a constant $M\in(0,\infty)$ such that

$$\begin{aligned}&\sup_{t\in[0,\bar{T}]}\left\|\Pi^{-1}(\bar{u}+\phi)\right\|\le M,\quad \sup_J|\beta^{(i)}|\le M \ \text{ for } i=0,\dots,4,\\ &\sup_{t\in[0,\bar{T}]}\|\bar{u}\|_{C^1},\ \|\bar{u}\|_{\mathbb{E}_1([0,\bar{T}])},\ \sup_{t\in[0,\bar{T}]}\|\bar{\rho}\|_{C^1}\le M.\end{aligned}\tag{2.3.19}$$

**Lemma 2.3.10** ([37, Lemma B.4])**.** *Let $M>0$ be as in* (2.3.19)*. There is $C=C(M)$ such that for all $i\in\{0,1,2\}$ we have*

$$\|\beta^{(i)}(\rho_0)-\beta^{(i)}(\bar{\rho}(t))\|_{C^\alpha}\le C(M)\,\|\bar{\rho}\|_{\mathbb{E}_1}\frac{t^\eta}{\eta}\quad \text{for all } t\in[0,\bar{T}].$$

*Proof.* For the supremum norm we find

$$\|\beta(\rho_0)-\beta(\bar{\rho})\|_{C^0}=\sup_{s\in\mathbb{R}}|\beta(\rho_0)-\beta(\bar{\rho})|\le\sup_J|\beta'|\sup_{s\in\mathbb{R}}|\rho_0-\bar{\rho}|\,.\tag{2.3.20}$$

To estimate the Hölder semi-norm, we note that for $s_1,s_2\in\mathbb{R}$

$$\begin{aligned}&|\beta(\rho_0(s_1))-\beta(\bar{\rho}(s_1))-\beta(\rho_0(s_2))+\beta(\bar{\rho}(s_2))|\\ &=\left|\int_0^1\partial_\lambda\left[\beta\left(\lambda\rho_0(s_1)+(1-\lambda)\bar{\rho}(s_1)\right)-\beta\left(\lambda\rho_0(s_2)+(1-\lambda)\bar{\rho}(s_2)\right)\right]\mathrm{d}\lambda\right|\\ &\le\int_0^1|\beta'\left(\lambda\rho_0(s_1)+(1-\lambda)\bar{\rho}(s_1)\right)|\,|\rho_0(s_1)-\bar{\rho}(s_1)-\rho_0(s_2)+\bar{\rho}(s_2)|\,\mathrm{d}\lambda\\ &+\int_0^1|\beta'\left(\lambda\rho_0(s_1)+(1-\lambda)\bar{\rho}(s_1)\right)-\beta'\left(\lambda\rho_0(s_2)+(1-\lambda)\bar{\rho}(s_2)\right)|\,|\rho_0(s_2)-\bar{\rho}(s_2)|\,\mathrm{d}\lambda\\ &\le\sup_J|\beta'|\,[\rho_0-\bar{\rho}]_\alpha\,|s_1-s_2|^\alpha\\ &\quad+\sup_J|\beta''|\left(|\rho_0(s_1)-\rho_0(s_2)|+|\bar{\rho}(s_1)-\bar{\rho}(s_2)|\right)|\rho_0(s_2)-\bar{\rho}(s_2)|\\ &\le\Big[\sup_J|\beta'|\,[\rho_0-\bar{\rho}]_\alpha+\sup_J|\beta''|\left([\rho_0]_\alpha+[\bar{\rho}]_\alpha\right)\sup_{s\in\mathbb{R}}|\rho_0-\bar{\rho}|\Big]\,|s_1-s_2|^\alpha\,.\end{aligned}$$

This estimate together with (2.3.20) and Lemma 2.3.9 yields the claim for $i=0$. Since $\beta$ is smooth, we can proceed in the same way for $i\ge 1$. □

**Lemma 2.3.11** ([37, Lemma B.5]). *Let $M > 0$ be as in (2.3.19). Then there exists a constant $C$, only depending on $M$, $\eta$, $\alpha$ and the model parameters, such that for all $T \in (0, \bar{T}]$, $(v, \sigma) \in \mathbb{E}_1([0,T])$ and $(v_0, \sigma_0) := (v, \sigma)(0, \cdot) \in \gamma\mathbb{E}_1$ we have*

$$\begin{aligned} \|\mathrm{D}\Lambda_u(\bar{u}, \bar{\rho})(v, \sigma) - \mathrm{D}\Lambda_u(\bar{u}, \bar{\rho})(v_0, \sigma_0)\|_{\mathbb{E}_0} &\leq C \left( \|v\|_{\mathbb{E}_1} + \|\sigma\|_{\mathbb{E}_1} \right) \max\{T, T^\eta\}, \\ \|\mathrm{D}\Lambda_\rho(\bar{u}, \bar{\rho})(v, \sigma) - \mathrm{D}\Lambda_\rho(\bar{u}, \bar{\rho})(v_0, \sigma_0)\|_{\mathbb{E}_0} &\leq C \left( \|v\|_{\mathbb{E}_1} + \|\sigma\|_{\mathbb{E}_1} \right) \max\{T, T^\eta\}. \end{aligned} \tag{2.3.21}$$

Here $\mathrm{D}\Lambda_u$ is the Fréchet derivative of the map defined in (2.3.16), thus may be applied to $(v_0, \sigma_0)$, identified with an element of $\mathbb{F}([0,T])$, cf. (2.3.15). Similarly for $\Lambda_\rho$.

*Proof of Lemma 2.3.11.* In the following, $C$ denotes a constant which varies from line to line but only depends on $M$ and the model parameters. First, we prove (2.3.21) considering each term in (2.3.17) separately. By (2.1.3) with $\theta = \bar{u} + \phi$, we find

$$\mathrm{D}\Pi\left(\bar{u} + \phi\right)(v) = \begin{pmatrix} \int_0^L \sin\left(2\left(\bar{u} + \phi\right)\right) v \, \mathrm{d}s & -\int_0^L \cos\left(2\left(\bar{u} + \phi\right)\right) v \, \mathrm{d}s \\ -\int_0^L \cos\left(2\left(\bar{u} + \phi\right)\right) v \, \mathrm{d}s & -\int_0^L \sin\left(2\left(\bar{u} + \phi\right)\right) v \, \mathrm{d}s \end{pmatrix} \tag{2.3.22}$$

and obtain using (2.3.19) and (2.3.22) (considering the operator norm of the matrix)

$$\begin{aligned} &\left\| \mathrm{D}\Pi^{-1}\left(\bar{u} + \phi\right)(v) - \mathrm{D}\Pi^{-1}\left(\bar{u} + \phi\right)(v_0) \right\| \\ &\quad = \left\| -\Pi^{-1}\left(\bar{u} + \phi\right) \left( \mathrm{D}\Pi\left(\bar{u} + \phi\right)(v) - \mathrm{D}\Pi\left(\bar{u} + \phi\right)(v_0) \right) \Pi^{-1}\left(\bar{u} + \phi\right) \right\| \\ &\quad \leq M^2 \sqrt{2} L \sup_{s \in [0,L]} |v - v_0| \leq C \|v - v_0\|_{C^\alpha}. \end{aligned}$$

Hence, with Lemma 2.3.9 the difference coming from the first term on the right hand side of (2.3.17) can be estimated by

$$\begin{aligned} &\left\| \left( \mathrm{D}\Pi^{-1}\left(\bar{u} + \phi\right)(v) - \mathrm{D}\Pi^{-1}\left(\bar{u} + \phi\right)(v_0) \right) \mathcal{J}(\bar{u}, \bar{\rho}) \cdot \begin{pmatrix} \sin\left(\bar{u} + \phi\right) \\ -\cos\left(\bar{u} + \phi\right) \end{pmatrix} \right\|_{\mathbb{E}_0} \\ &\quad \leq C \sup_{t \in (0,T]} t^{1-\eta} \|v - v_0\|_{C^\alpha} \leq C \|v\|_{\mathbb{E}_1} T. \end{aligned}$$

For the second term of (2.3.17) we integrate once by parts and obtain

$$\begin{aligned} \mathrm{D}\mathcal{J}(\bar{u}, \bar{\rho})(v, \sigma) = &\int_0^L \begin{pmatrix} -\sin\left(\bar{u} + \phi\right) \\ \cos\left(\bar{u} + \phi\right) \end{pmatrix} v \left(\partial_s \bar{u} + \partial_s \phi\right) \beta(\bar{\rho}) \left(\partial_s \bar{u} + \partial_s \phi - c_0\right) \mathrm{d}s \\ &+ \int_0^L \begin{pmatrix} \cos\left(\bar{u} + \phi\right) \\ \sin\left(\bar{u} + \phi\right) \end{pmatrix} \left(\partial_s \bar{u} + \partial_s \phi\right) \beta'(\bar{\rho}) \sigma \left(\partial_s \bar{u} + \partial_s \phi - c_0\right) \mathrm{d}s \\ &- \int_0^L \begin{pmatrix} -\sin\left(\bar{u} + \phi\right) \\ \cos\left(\bar{u} + \phi\right) \end{pmatrix} \left(\partial_s \bar{u} + \partial_s \phi\right) \left(2\partial_s \bar{u} + 2\partial_s \phi - c_0\right) \beta(\bar{\rho}) \, v \, \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&- \int_0^L \begin{pmatrix} \cos(\bar{u}+\phi) \\ \sin(\bar{u}+\phi) \end{pmatrix} 2\partial_s^2 \bar{u}\, \beta(\bar{\rho})\, v \,\mathrm{d}s \\
&- \int_0^L \begin{pmatrix} \cos(\bar{u}+\phi) \\ \sin(\bar{u}+\phi) \end{pmatrix} (2\partial_s \bar{u} + 2\partial_s \phi - c_0)\, \beta'(\bar{\rho}) \partial_s \bar{\rho}\, v \,\mathrm{d}s.
\end{aligned}
$$

So the difference we need to consider can be estimated by

$$
\begin{aligned}
&\left\| \mathrm{D}\mathcal{J}(\bar{u},\bar{\rho})(v_0,\sigma_0) - \mathrm{D}\mathcal{J}(\bar{u},\bar{\rho})(v,\sigma) \right\|_{C^0} \\
&\quad \le C \left(1 + \left\| \partial_s^2 \bar{u} \right\|_{C^\alpha}\right) \|v - v_0\|_{C^\alpha} + C \|\sigma - \sigma_0\|_{C^\alpha} \\
&\quad \le C t^\eta \left(1 + \left\| \partial_s^2 \bar{u} \right\|_{C^\alpha}\right) \left( \|v\|_{\mathbb{E}_1} + \|\sigma\|_{\mathbb{E}_1} \right).
\end{aligned}
$$

In the last step we used Lemma 2.3.9. Using (2.3.19), we obtain

$$
\begin{aligned}
&\left\| \Pi^{-1}(\bar{u}+\phi) \left( \mathrm{D}\mathcal{J}(\bar{u},\bar{\rho})(v,\sigma) - \mathrm{D}\mathcal{J}(\bar{u},\bar{\rho})(v_0,\sigma_0) \right) \cdot \begin{pmatrix} \sin(\bar{u}+\phi) \\ -\cos(\bar{u}+\phi) \end{pmatrix} \right\|_{\mathbb{E}_0} \\
&\le C \sup_{t\in(0,T]} t^{1-\eta}\, t^\eta (1 + \left\| \partial_s^2 \bar{u} \right\|_{C^\alpha}) \left( \|v\|_{\mathbb{E}_1} + \|\sigma\|_{\mathbb{E}_1} \right) \le C \left( \|v\|_{\mathbb{E}_1} + \|\sigma\|_{\mathbb{E}_1} \right) (T + T^\eta).
\end{aligned}
$$

Finally, we look at the difference coming from the third term of (2.3.17). We have

$$
\left\| \begin{pmatrix} \cos(\bar{u}+\phi) \\ \sin(\bar{u}+\phi) \end{pmatrix} (v - v_0) \right\|_{C^\alpha} \le C \|v - v_0\|_{C^\alpha}.
$$

Thereby we obtain using (2.3.19) and Lemma 2.3.9

$$
\begin{aligned}
&\left\| \Pi^{-1}(\bar{u}+\phi)\ \mathcal{J}(\bar{u},\bar{\rho}) \cdot \begin{pmatrix} \cos(\bar{u}+\phi) \\ \sin(\bar{u}+\phi) \end{pmatrix} (v - v_0) \right\|_{\mathbb{E}_0} \\
&\qquad \le C \sup_{t\in(0,T]} t^{1-\eta} \|v - v_0\|_{C^\alpha} \le C \|v\|_{\mathbb{E}_1} T.
\end{aligned}
$$

This proves the first part of the claim. The second part follows similarly. □

#### Solving the linearized problem

We show that the map in (2.3.11) is an isomorphism, now considering a linear initial value problem with time-dependent coefficients.

**Proposition 2.3.12** ([37, Proposition B.6])**.** *Let $(u_0, \rho_0) \in \gamma\mathbb{E}_1$ such that $u_0$ satisfies* (2.3.2) *and let $(\bar{u}, \bar{\rho}) \in \mathbb{E}_1([0,\bar{T}])$ be the function constructed in the proof of Proposition 2.3.5. Then there exists $T' \in (0,\bar{T}]$ such that the mapping*

$$
\tilde{\mathbf{J}} \colon \mathbb{E}_1([0,T']) \to \mathbb{E}_0([0,T']) \times \gamma\mathbb{E}_1,\ (v,\sigma) \mapsto \left( \partial_t(v,\sigma) - \mathrm{D}\mathbf{F}(\bar{u},\bar{\rho})(v,\sigma),\ (v,\sigma)(0,\cdot) \right)
$$

*is an isomorphism. Equivalently, for any right-hand side* $(\varphi_1, \varphi_2) \in \mathbb{E}_0([0,T'])$ *and any initial datum* $(v_0, \sigma_0) \in \gamma\mathbb{E}_1$ *there is a unique solution* $(v,\sigma) \in \mathbb{E}_1([0,T'])$ *of*

$$\begin{cases} \partial_t(v,\sigma) - \mathrm{D}\mathbf{F}(\bar{u},\bar{\rho})(v,\sigma) = (\varphi_1,\varphi_2) & \text{in } (0,T')\times\mathbb{R}, \\ (v,\sigma)(0,\cdot) = (v_0,\sigma_0) & \text{on } \mathbb{R}. \end{cases}$$

*Proof.* Let $T \in (0,\bar{T}]$ to be chosen. We compare $\tilde{\mathbf{J}}$ with $\mathbf{J}$ from Proposition 2.3.7, which we already know is an isomorphism. We have

$$\begin{aligned} \tilde{\mathbf{J}}(v,\sigma) &= \mathbf{J}(v,\sigma) + \big(\mathcal{A}(v,\sigma) + \Psi(v_0,\sigma_0) - \mathrm{D}\mathbf{F}(\bar{u},\bar{\rho})(v,\sigma),\, (0,0)\big) \\ &= \mathbf{J}\Big(\mathbf{I}(v,\sigma) + \mathbf{J}^{-1}(v,\sigma)\,\big(\mathbf{S}(v,\sigma),\,(0,0)\big)\Big), \end{aligned}$$

where $\mathcal{A}, \Psi$ are given as in Proposition 2.3.7, $\mathbf{I}$ is the operator identity and

$$\begin{aligned} &\mathbf{S} := (\mathbf{S}_1, \mathbf{S}_2)\colon \mathbb{E}_1([0,T]) \to \mathbb{E}_0([0,T]), \\ &\mathbf{S}(v,\sigma) = \mathcal{A}(v,\sigma) + \Psi(v_0,\sigma_0) - \mathrm{D}\mathbf{F}(\bar{u},\bar{\rho})(v,\sigma). \end{aligned}$$

Thus, by a Neumann series argument, it is sufficient to show that the operator norm of $\mathbf{S}$ becomes arbitrarily small for $T$ getting small and at the same time the operator norm $\|\mathbf{J}^{-1}\|$ remains bounded, independent of $T \le \bar{T}$. This last part has already been proven in Lemma 2.3.8. Hence, the statement follows if we show

$$\|\mathbf{S}\| \le C \max\left\{T, T^{\eta}, T^{\tilde{\eta}}\right\}. \tag{2.3.23}$$

By definition of $\mathcal{A}, \Psi$ and using (2.3.13), (2.3.14) we find

$$\begin{aligned} \mathbf{S}_1(v,\sigma) = {} & \left(\beta(\rho_0) - \beta(\bar{\rho})\right)\partial_s^2 v + \beta'(\bar{\rho})\,\partial_s^2\bar{u}\,(\sigma_0 - \sigma) + \left(\beta'(\rho_0)\partial_s\rho_0 - \beta'(\bar{\rho})\partial_s\bar{\rho}\right)\partial_s v \\ & + \left(\beta'(\rho_0)\left(\partial_s u_0 + \partial_s\phi - c_0\right) - \beta'(\bar{\rho})\left(\partial_s\bar{u} + \partial_s\phi - c_0\right)\right)\partial_s\sigma \\ & + \left(\beta''(\rho_0)\partial_s\rho_0\left(\partial_s u_0 + \partial_s\phi - c_0\right) - \beta''(\bar{\rho})\partial_s\bar{\rho}\left(\partial_s\bar{u} + \partial_s\phi - c_0\right)\right)\sigma \\ & + \mathrm{D}\Lambda_u(\bar{u},\bar{\rho})(v_0,\sigma_0) - \mathrm{D}\Lambda_u(\bar{u},\bar{\rho})(v,\sigma) \end{aligned} \tag{2.3.24}$$

and

$$\begin{aligned} \mathbf{S}_2(v,\sigma) = {} & \Big(-\frac{1}{2}\beta''(\rho_0)\left(\partial_s u_0 + \partial_s\phi - c_0\right)^2 + \frac{1}{2}\beta''(\bar{\rho})\left(\partial_s\bar{u} + \partial_s\phi - c_0\right)^2\Big)\sigma \\ & + \left(-\beta'(\rho_0)\left(\partial_s u_0 + \partial_s\phi - c_0\right) + \beta'(\bar{\rho})\left(\partial_s\bar{u} + \partial_s\phi - c_0\right)\right)\partial_s v \\ & + \mathrm{D}\Lambda_\rho(\bar{u},\bar{\rho})(v_0,\sigma_0) - \mathrm{D}\Lambda_\rho(\bar{u},\bar{\rho})(v,\sigma). \end{aligned} \tag{2.3.25}$$

We now take a look at the terms of (2.3.24) and (2.3.25). In the following, $C$ denotes a constant, only depending on the model parameters, $\alpha$, $\eta$ and $M$ as in (2.3.19), which

is allowed to change from line to line. The $\mathbb{E}_0$-norm of the first term of (2.3.24) can be estimated using Lemma 2.3.17 and Lemma 2.3.10 by

$$\left\| \left(\beta(\rho_0) - \beta(\bar{\rho})\right) \partial_s^2 v \right\|_{\mathbb{E}_0} \leq C \sup_{t\in(0,T]} t^{1-\eta} \left\|\beta(\rho_0) - \beta(\bar{\rho})\right\|_{C^\alpha} \|v\|_{C^{2+\alpha}} \leq C \|v\|_{\mathbb{E}_1} T^\eta.$$

Here we used the time-weight of the $\mathbb{E}_0$-norm to bound the second derivative of $v$. For the second term of (2.3.24) we have to use the time-weight to bound the second derivative of $\bar{u}$. An additional factor depending on $T$ and going to zero for $T$ going to zero is given by Lemma 2.3.9. This motivates the choice of the first term in $\Psi$ in Proposition 2.3.7. We obtain

$$\begin{aligned}\left\|\beta'(\bar{\rho})\, \partial_s^2 \bar{u}\, (\sigma_0 - \sigma)\right\|_{\mathbb{E}_0} &\leq C \sup_{t\in(0,T]} t^{1-\eta} \left\|\beta'(\bar{\rho})\right\|_{C^\alpha} \|\bar{u}\|_{C^{2+\alpha}} \|\sigma_0 - \sigma\|_{C^\alpha} \\ &\leq C \|\sigma\|_{\mathbb{E}_1} T^\eta.\end{aligned}$$

The third term of (2.3.24) can be estimated using Lemma 2.3.10 and Lemma 2.3.9 by

$$\begin{aligned}&\left\|\left(\beta'(\rho_0)\partial_s\rho_0 - \beta'(\bar{\rho})\partial_s\bar{\rho}\right) \partial_s v\right\|_{\mathbb{E}_0} \\ &\leq C \sup_{t\in(0,T]} t^{1-\eta} \left\|\beta'(\rho_0) - \beta'(\bar{\rho})\right\|_{C^\alpha} \|\rho_0\|_{C^{1+\alpha}} \|v\|_{C^{1+\alpha}} \\ &\quad + C \sup_{t\in(0,T]} t^{1-\eta} \left\|\beta'(\bar{\rho})\right\|_{C^\alpha} \|\partial_s\rho_0 - \partial_s\bar{\rho}\|_{C^\alpha} \|v\|_{C^{1+\alpha}} \leq C \|v\|_{\mathbb{E}_1} T^\eta + C \|v\|_{\mathbb{E}_1} T^{\tilde{\eta}}.\end{aligned}$$

The same also applies for the fourth term of (2.3.24) and the second term of (2.3.25). For the fifth term of (2.3.24) we proceed similarly and estimate

$$\begin{aligned}&\left\|\left(\beta''(\rho_0)\partial_s\rho_0 \left(\partial_s u_0 + \partial_s\phi - c_0\right) - \beta''(\bar{\rho})\partial_s\bar{\rho} \left(\partial_s\bar{u} + \partial_s\phi - c_0\right)\right) \sigma\right\|_{\mathbb{E}_0} \\ &\leq C \sup_{t\in(0,T]} t^{1-\eta} \left\|\beta''(\rho_0) - \beta''(\bar{\rho})\right\|_{C^\alpha} \|\rho_0\|_{C^{1+\alpha}} \left\|\partial_s u_0 + \partial_s\phi - c_0\right\|_{C^\alpha} \|\sigma\|_{C^\alpha} \\ &\quad + C \sup_{t\in(0,T]} t^{1-\eta} \left\|\beta''(\bar{\rho})\right\|_{C^\alpha} \|\partial_s\rho_0 - \partial_s\bar{\rho}\|_{C^\alpha} \left\|\partial_s u_0 + \partial_s\phi - c_0\right\|_{C^\alpha} \|\sigma\|_{C^\alpha} \\ &\quad + C \sup_{t\in(0,T]} t^{1-\eta} \left\|\beta''(\bar{\rho})\right\|_{C^\alpha} \|\bar{\rho}\|_{C^{1+\alpha}} \left\|\partial_s u_0 - \partial_s\bar{u}\right\|_{C^\alpha} \|\sigma\|_{C^\alpha} \leq C \|\sigma\|_{\mathbb{E}_1} \left(T^\eta + T^{\tilde{\eta}}\right).\end{aligned}$$

Analogously, we can treat the first term of (2.3.25). Together with Lemma 2.3.11, (2.3.23) and thus the claim follows for $T' = T < \bar{T}$ sufficiently small. □

### 2.3.4. Smoothness and uniqueness of solutions

We now show that a solution is smooth for positive times.

**Lemma 2.3.13** (Smoothing, [37, Lemma 3.5])**.** *Let $0 < T \leq \infty$ and $(u, \rho)$ be a solution, in the sense of Definition 2.3.3, of* (2.3.3) *on $(0, T) \times \mathbb{R}$. Then $(u, \rho) \in C^\infty((0, T) \times \mathbb{R})$.*

The idea is to define a smooth cut-off function, which vanishes at $t = 0$ and outside of a spatial interval of length larger than $L$. The product of $u$ with this cut-off function is the solution of a new initial value problem — this time with zero boundary values and zero initial data. This enables us to use [96, Theorem IV.5.2], which gives us higher regularity of $u$ in the scale of parabolic Hölder spaces, cf. Section 2.3.6. We proceed similarly for $\rho$ and iterate this procedure to conclude smoothness.

*Proof of Lemma 2.3.13.* By Definition 2.3.3 the solution $(u, \rho)$ satisfies $(u, \rho) \in \mathbb{E}_1([0, T'])$ for all $T' < T$. Let $0 < \delta < T' < T$ and let $\psi_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function satisfying

$$\begin{aligned}
&\psi_1(t, s) \equiv 1 \text{ for } (t, s) \in \left[\delta/2, T'\right] \times \left[-L/4, 5L/4\right], \\
&\psi_1(t, s) \equiv 0 \text{ for } (t, s) \in \left(\left[0, \delta/4\right] \times \mathbb{R}\right) \cup \left(\left[0, T'\right] \times \left(\mathbb{R} \setminus \left[-L/2, 3L/2\right]\right)\right).
\end{aligned}$$

We define $u_1 := \psi_1 u \in \mathbb{E}_1([0, T'])$. By (2.3.3) we have

$$\partial_t u_1 - a_{11}\partial_s^2 u_1 + a_1 \partial_s u_1 = f_1,$$

where $a_{11}(t, s) := \beta(\rho)$, $a_1(t, s) := -\beta'(\rho)\partial_s \rho$, and

$$\begin{aligned}
f_1(t, s) := &- 2\beta(\rho)\partial_s u \partial_s \psi_1 - \beta(\rho) u \partial_s^2 \psi_1 - \beta'(\rho) u \partial_s \rho \partial_s \psi_1 + \psi_1 \beta'(\rho) \partial_s \rho \left(\partial_s \phi - c_0\right) \\
&+ \psi_1 \Lambda_u(u, \rho) + u \partial_t \psi_1.
\end{aligned}$$

Defining $\zeta := \eta + \frac{\alpha}{2} - \frac{1}{2} \in (0, 1)$, Lemma 2.3.22 yields

$$\partial_s u, \partial_s \rho \in H^{\frac{\zeta}{2}, \zeta}\left([0, T'] \times [-L, 2L]\right).$$

Since $\beta \in C^\infty(\mathbb{R})$, we conclude $a_{11}, a_1, f_1, \in H^{\frac{\zeta}{2}, \zeta}([0, T'] \times [-L, 2L])$. By [96, Theorem IV.5.2] there exists a unique solution $w \in H^{\frac{\zeta+2}{2}, \zeta+2}([0, T'] \times [-L, 2L])$ of

$$\begin{cases}
\partial_t w - a_{11}\partial_s^2 w + a_1 \partial_s w = f_1 & \text{in } (0, T') \times (-L, 2L), \\
w = 0 & \text{on } (0, T') \times \{-L, 2L\}, \\
w(0, \cdot) = 0 & \text{on } (-L, 2L).
\end{cases} \tag{2.3.26}$$

By uniqueness $u_1 = w \in H^{\frac{\zeta+2}{2}, \zeta+2}([0, T'] \times [-L, 2L])$. Analogously for $\rho$, we obtain

$$u, \rho \in H^{\frac{\zeta+2}{2}, \zeta+2}\left(\left[\delta/2, T'\right] \times \left[-L/4, 5L/4\right]\right).$$

Since $u$ and $\rho$ are periodic with period length $L$, we even have

$$u, \rho \in H^{\frac{\zeta+2}{2}, \zeta+2}\left(\left[\delta/2, T'\right] \times [-L, 2L]\right). \tag{2.3.27}$$

Let $\psi_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ be another smooth cut-off function satisfying

$$\psi_2(t,s) \equiv 1 \text{ for } (t,s) \in \left[3\delta/4, T'\right] \times \left[-L/4, 5L/4\right],$$
$$\psi_2(t,s) \equiv 0 \text{ for } (t,s) \in \left(\left[0, 2\delta/3\right] \times \mathbb{R}\right) \cup \left(\left[0,T'\right] \times \mathbb{R} \setminus \left[-L/2, 3L/2\right]\right).$$

We set $u_2 := \psi_2 u \in H^{\frac{\zeta+2}{2},\zeta+2}\left(\left[\delta/2, T'\right] \times [-L, 2L]\right)$. Now, $u_2$ solves an initial boundary value problem similar to (2.3.26) where $f_2$ is given by $f_1$ with $\psi_1$ replaced by $\psi_2$ and where the initial condition is prescribed at $t = \delta/2$. By (2.3.27), we find that $a_{11}, a_1, f_2 \in H^{\frac{\zeta+1}{2},\zeta+1}\left(\left[\delta/2, T'\right] \times [-L, 2L]\right)$, so by [96, Theorem IV.5.2], we conclude $u_2 \in H^{\frac{\zeta+3}{2},\zeta+3}\left(\left[\delta/2, T'\right] \times [-L, 2L]\right)$. Proceeding similarly for $\rho$, we find

$$u, \rho \in H^{\frac{\zeta+3}{2},\zeta+3}\left(\left[3\delta/4, T'\right] \times [-L, 2L]\right).$$

Iterating the above procedure, by periodicity and since $0 < \delta < T' < T$ were arbitrary, it follows $(u, \rho) \in C^\infty((0,T) \times \mathbb{R})$. □

We are now able to prove that solutions are unique as long as they exist. To do this, we use the energy method, which we can apply once we have established smoothness.

**Lemma 2.3.14** (Uniqueness, [37, Lemma 3.6])**.** *Let $(u_0, \rho_0) \in \gamma\mathbb{E}_1$. Let $0 < T_1, T_2 \leq \infty$ and let $(u_1, \rho_1)$ and $(u_2, \rho_2)$ be solutions of (2.3.3) on $(0, T_1) \times \mathbb{R}$ and $(0, T_2) \times \mathbb{R}$ with initial datum $(u_0, \rho_0)$. Then $(u_1, \rho_1) \equiv (u_2, \rho_2)$ on $[0, \min\{T_1, T_2\})$.*

*Proof.* Without loss of generality we assume $0 < T_1 \leq T_2$. Let $0 < T < T_1$. We show that the difference $(u_d, \rho_d) := (u_1, \rho_1) - (u_2, \rho_2)|_{[0,T]}$ of the two solutions is zero using Gronwall's lemma. By Lemma 2.3.13 $(u_d, \rho_d) \in C^\infty((0,T] \times \mathbb{R})$, so

$$\varphi(t) := \frac{1}{2} \int_0^L \left((\partial_s u_d)^2 + (\partial_s \rho_d)^2 + u_d^2 + \rho_d^2\right) \mathrm{d}s, \quad t \in (0,T],$$

is differentiable. In the following, $C$ denotes a constant, only depending on $u_1$, $u_2$, $\rho_1$, $\rho_2$, $T$, and the model parameters $\beta, c_0, L, \mu, \omega$, which may change from line to line. Other dependencies are given explicitly. Lemma 2.3.18 and Remark 2.3.19 yield

$$\sup_{t\in[0,T]} \|(u_i, \rho_i)\|_{C^1} \leq C, \quad \sup_{[0,T]\times\mathbb{R}} |\beta^{(k)}(\rho_i)| \leq C, \quad \sup_{[0,T]} \left\|\Pi^{-1}(u_i + \phi)\right\| \leq C \tag{2.3.28}$$

for $i = 1, 2$, $k = 0, 1, 2$. In particular, there exists a compact interval $J \subset \mathbb{R}$ such that $\rho_i(t,s) \in J$ for all $(t,s) \in [0,T] \times \mathbb{R}$ and $i = 1, 2$. Consequently, we have

$$\inf_{[0,T]\times\mathbb{R}} \beta(\rho_i) \geq c > 0 \quad \text{for } i = 1, 2. \tag{2.3.29}$$

After integration by parts and using $\int_0^L \rho_d \,\mathrm{d}s = 0$ by (2.1.7), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = &-\int_0^L \beta(\rho_1)(\partial_s^2 u_d)^2 \,\mathrm{d}s - \mu\int_0^L (\partial_s^2\rho_d)^2 \,\mathrm{d}s - \mu\int_0^L (\partial_s\rho_d)^2 \,\mathrm{d}s\\
&-\int_0^L \partial_s^2 u_d \left(\beta(\rho_1)-\beta(\rho_2)\right)\partial_s^2 u_2 \,\mathrm{d}s\\
&-\int_0^L \partial_s^2 u_d \Big(\beta'(\rho_1)\partial_s\rho_1\left(\partial_s u_1+\partial_s\phi-c_0\right)+\Lambda_u(u_1,\rho_1)\\
&\qquad\qquad -\beta'(\rho_2)\partial_s\rho_2\left(\partial_s u_2+\partial_s\phi-c_0\right)-\Lambda_u(u_2,\rho_2)\Big)\,\mathrm{d}s\\
&+\int_0^L \partial_s^2\rho_d \Big(\frac{1}{2}\beta'(\rho_1)\left(\partial_s u_1+\partial_s\phi-c_0\right)^2-\frac{1}{2}\beta'(\rho_2)\left(\partial_s u_2+\partial_s\phi-c_0\right)^2\Big)\,\mathrm{d}s\\
&+\int_0^L u_d \Big(\beta(\rho_1)\partial_s^2 u_1+\beta'(\rho_1)\partial_s\rho_1\left(\partial_s u_1+\partial_s\phi-c_0\right)+\Lambda_u(u_1,\rho_1)\\
&\qquad\qquad -\beta(\rho_2)\partial_s^2 u_2-\beta'(\rho_2)\partial_s\rho_2\left(\partial_s u_2+\partial_s\phi-c_0\right)-\Lambda_u(u_2,\rho_2)\Big)\,\mathrm{d}s\\
&+\int_0^L \rho_d \Big(-\frac{1}{2}\beta'(\rho_1)\left(\partial_s u_1+\partial_s\phi-c_0\right)^2+\frac{1}{2}\beta'(\rho_2)\left(\partial_s u_2+\partial_s\phi-c_0\right)^2\Big)\,\mathrm{d}s.\end{aligned} \tag{2.3.30}$$

We estimate everything but the first three terms using Young's inequality to absorb higher order derivatives and bound the remaining ones by $\varphi(t)$ using (2.3.28). Indeed, since $\int_0^L \rho_d \,\mathrm{d}s = 0$, for each $t$ there is $s_t \in [0,L]$ with $\rho_d(t,s_t) = 0$ by the Mean Value Theorem for Integrals. The Fundamental Theorem of Calculus thus yields

$$|\rho_d(t,s)| \le \int_0^L |\partial_s\rho_d| \,\mathrm{d}s. \tag{2.3.31}$$

Hence, the term on the second line in (2.3.30) can be estimated by a constant times

$$\sup_{\mathbb{R}}|\rho_d|\int_0^L |\partial_s^2 u_d||\partial_s^2 u_2| \,\mathrm{d}s \le \delta\int_0^L (\partial_s^2 u_d)^2 \,\mathrm{d}s + C(\delta)\int_0^L (\partial_s^2 u_2)^2 \,\mathrm{d}s\int_0^L (\partial_s\rho_d)^2 \,\mathrm{d}s.$$

In the third line of (2.3.30), we add, subtract, and use Young's inequality to find

$$\begin{aligned}&-\int_0^L \partial_s^2 u_d \left(\beta'(\rho_1)\partial_s\rho_1\left(\partial_s u_1+\partial_s\phi-c_0\right)-\beta'(\rho_2)\partial_s\rho_2\left(\partial_s u_2+\partial_s\phi-c_0\right)\right)\mathrm{d}s\\
&=-\int_0^L \partial_s^2 u_d\beta'(\rho_1)\partial_s\rho_d\left(\partial_s u_1+\partial_s\phi-c_0\right)\,\mathrm{d}s-\int_0^L \partial_s^2 u_d\beta'(\rho_1)\partial_s\rho_2\partial_s u_d \,\mathrm{d}s\\
&\quad-\int_0^L \partial_s^2 u_d\left(\beta'(\rho_1)-\beta'(\rho_2)\right)\partial_s\rho_2\left(\partial_s u_2+\partial_s\phi-c_0\right)\,\mathrm{d}s\\
&\quad\le \delta\int_0^L \left(\partial_s^2 u_d\right)^2 \,\mathrm{d}s + C(\delta)\Big(\int_0^L (\partial_s\rho_d)^2+(\partial_s u_d)^2+\rho_d^2\,\mathrm{d}s\Big).\end{aligned} \tag{2.3.32}$$

On the other hand, for the term in the third line of (2.3.30) involving $\Lambda_u$ we have

$$\begin{aligned}
&-\int_0^L \partial_s^2 u_d \left(\Lambda_u(u_1,\rho_1)-\Lambda_u(u_2,\rho_2)\right)\,\mathrm{d}s\\
&=-\int_0^L \partial_s^2 u_d \Big[\Pi^{-1}\left(u_1+\phi\right)\,\mathcal{J}(u_1,\rho_1)\cdot\begin{pmatrix}\sin\left(u_1+\phi\right)\\-\cos\left(u_1+\phi\right)\end{pmatrix}\\
&\qquad\qquad-\Pi^{-1}\left(u_2+\phi\right)\,\mathcal{J}(u_2,\rho_2)\cdot\begin{pmatrix}\sin\left(u_2+\phi\right)\\-\cos\left(u_2+\phi\right)\end{pmatrix}\Big]\,\mathrm{d}s. \qquad (2.3.33)
\end{aligned}$$

We write the term in square brackets as

$$\begin{aligned}
&\Big[\Pi^{-1}\left(u_1+\phi\right)-\Pi^{-1}\left(u_2+\phi\right)\Big]\,\mathcal{J}(u_1,\rho_1)\cdot\begin{pmatrix}\sin\left(u_1+\phi\right)\\-\cos\left(u_1+\phi\right)\end{pmatrix}\\
&\quad+\Pi^{-1}\left(u_2+\phi\right)\left(\mathcal{J}(u_1,\rho_1)-\mathcal{J}(u_2,\rho_2)\right)\cdot\begin{pmatrix}\sin\left(u_1+\phi\right)\\-\cos\left(u_1+\phi\right)\end{pmatrix}\\
&\quad+\Pi^{-1}\left(u_2+\phi\right)\mathcal{J}(u_2,\rho_2)\cdot\begin{pmatrix}\sin\left(u_1+\phi\right)-\sin\left(u_2+\phi\right)\\-\cos\left(u_1+\phi\right)+\cos\left(u_2+\phi\right)\end{pmatrix}.
\end{aligned}$$

Using the explicit formula for the inverse matrix, it can be shown that

$$\left\|\Pi^{-1}(u_1+\phi)-\Pi^{-1}(u_2+\phi)\right\|\le C\int_0^L |u_d|\,\mathrm{d}s.$$

Proceeding as in (2.3.32), after adding and subtracting, we have

$$|\mathcal{J}(u_1,\rho_1)-\mathcal{J}(u_2,\rho_2)|\le C\int_0^L \left(|u_d|+|\partial_s u_d|+|\rho_d|\right)\,\mathrm{d}s.$$

Consequently, (2.3.33) may be estimated by

$$\delta\int_0^L(\partial_s^2 u_d)^2\,\mathrm{d}s+C(\delta)\int_0^L u_d^2\,\mathrm{d}s+C(\delta)\int_0^L(\partial_s u_d)^2+C(\delta)\int_0^L\rho_d^2\,\mathrm{d}s.$$

For the remaining terms, we proceed similarly. Using (2.3.29) and choosing $\delta>0$ sufficiently small, the first two terms in (2.3.30) can be used to absorb all terms involving a factor of $\delta$. Altogether, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\le C\Big(1+\int_0^L(\partial_s^2 u_2)^2\,\mathrm{d}s\Big)\varphi(t)\quad\text{for } t\in(0,T).$$

Using Gronwall's lemma, we conclude that for all $t'\in[0,T]$ we have

$$\varphi(t')\le C\varphi(0)\exp\Big(\int_0^{t'}\Big(1+\int_0^L(\partial_s^2 u_2)^2\,\mathrm{d}s\Big)\,\mathrm{d}t\Big).$$

By Lemma 2.3.21, the integral is finite and uniqueness follows from $\varphi(0) = 0$. □

### 2.3.5. Proof of local well-posedness

It remains to transfer the results of Section 2.3.2 and Section 2.3.4 to the initial nonperiodic setting and to prove Theorem 2.3.2.

*Proof of Theorem 2.3.2.* We define $(u_0, \rho_0)$ as the $L$-periodic extension of $(\theta_0 - \phi, \rho_0)$ to all of $\mathbb{R}$. Then $(u_0, \rho_0) \in h_L^{1+\alpha}(\mathbb{R})$. Now we choose $\bar{\alpha} \in (0,1)$ and $\bar{\eta} \in \left(\frac{1}{2}, 1\right)$ such that $2\bar{\eta} + \bar{\alpha} = 1 + \alpha$. Since $(u_0, \rho_0) \in h_L^{2\bar{\eta}+\bar{\alpha}}(\mathbb{R})$, Proposition 2.3.5 yields $\bar{T} > 0$ and a solution

$$(u, \rho) \in \mathrm{BUC}^1_{1-\bar{\eta}}\left([0, \bar{T}]; h_L^{\bar{\alpha}}(\mathbb{R})\right) \cap \mathrm{BUC}_{1-\bar{\eta}}\left([0, \bar{T}]; h_L^{2+\bar{\alpha}}(\mathbb{R})\right) \tag{2.3.34}$$

of (2.3.3) on $(0, \bar{T}) \times \mathbb{R}$ that satisfies $\lim_{t\to 0}(u, \rho)(t) = (u_0, \rho_0)$ in $C^{2\bar{\eta}+\bar{\alpha}}(\mathbb{R})$ by Lemma 2.3.18. According to Proposition 2.3.14, the solution is unique and due to Lemma 2.3.13 $(u, \rho) \in C^\infty((0, \bar{T}) \times \mathbb{R})$. Now we define $(\theta, \rho)$ as the restriction of $(u+\phi, \rho)$ on $[0, \bar{T}) \times [0, L]$. By construction $(\theta, \rho)$ solves (2.0.4) on $(0, \bar{T}) \times [0, L]$. Moreover, the $C^1$-periodicity of $(u, \rho)$ ensures, that $(\theta, \rho)$ satisfies the boundary conditions in (2.0.4). Furthermore, the choice of $\bar{\alpha}$ and $\bar{\eta}$ as $2\bar{\eta} + \bar{\alpha} = 1 + \alpha$ gives us $\lim_{t\to 0}(\theta(t), \rho(t)) = (\theta_0, \rho_0)$ in $C^{1+\alpha}([0, L]; \mathbb{R}^2)$. The uniqueness of the solution, its smoothness and the continuous dependence on the initial datum transfer from $(u, \rho)$ to $(\theta, \rho)$. □

**Remark 2.3.15.** In the above proof of Theorem 2.3.2 we chose $\bar{\alpha} \in (0,1)$ and $\bar{\eta} \in \left(\frac{1}{2}, 1\right)$ such that $2\bar{\eta} + \bar{\alpha} = 1 + \alpha \in (1, 2)$. This yields the convergence of the solution $(\theta, \rho)$ to the initial datum $(\theta_0, \rho_0)$ in $C^{1+\alpha}([0, L])$. But in fact, in Proposition 2.3.5 we can allow for arbitrary $\bar{\alpha} \in (0,1)$ and $\bar{\eta} \in \left(\frac{1}{2}, 1\right)$ with just the condition that $2\bar{\eta} + \bar{\alpha} \notin \mathbb{Z}$. By exploiting this freedom we observe the following:
If we even have $(\theta_0, \rho_0) \in h^{2+\alpha}([0, L])$ for $\alpha \in (0,1)$ (with periodic second derivatives), we get convergence of the solution $(\theta, \rho)$ to the initial datum $(\theta_0, \rho_0)$ in $C^{2+\alpha}([0, L])$. This is an important ingredient in proving global existence, cf. Section 2.4.3 below.

### 2.3.6. Appendix to Section 2.3: Function Spaces

The main result of Section 2.3 involves time-weighted little Hölder spaces and in the proof of Lemma 2.3.13 additionally parabolic Hölder spaces are used. Here we collect the definitions of these spaces and list some properties.

#### Little Hölder functions

For $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$, we denote by $\lfloor \alpha \rfloor$ the largest integer less than $\alpha$ and $\{\alpha\} := \alpha - \lfloor \alpha \rfloor \in (0,1)$. Let $k \in \mathbb{N}$. The space of $L$-periodic $\mathbb{R}^k$-valued functions over $\mathbb{R}$ which are continuous and

$\lfloor\alpha\rfloor$-times continuously differentiable is denoted by $C_L^{\lfloor\alpha\rfloor}(\mathbb{R};\mathbb{R}^k)$. The *periodic Hölder space with exponent* $\alpha$ is defined as

$$C_L^\alpha(\mathbb{R};\mathbb{R}^k) := \Big\{ f \in C_L^{\lfloor\alpha\rfloor}(\mathbb{R};\mathbb{R}^k) : \Big[f^{(\lfloor\alpha\rfloor)}\Big]_{\alpha,\mathbb{R}} := \sup_{\substack{x,y\in\mathbb{R},\\ x\neq y}} \frac{\left|f^{(\lfloor\alpha\rfloor)}(x) - f^{(\lfloor\alpha\rfloor)}(y)\right|}{|x-y|^{\{\alpha\}}} < \infty \Big\}$$

and equipped with the norm $\|f\|_{C^\alpha(\mathbb{R})} := \|f\|_{C^{\lfloor\alpha\rfloor}(\mathbb{R})} + \Big[f^{(\lfloor\alpha\rfloor)}\Big]_{\alpha,\mathbb{R}}$. Therewith, we define the *periodic little Hölder space with exponent* $\alpha$ over $\mathbb{R}$ by

$$h_L^\alpha(\mathbb{R};\mathbb{R}^k) := \Big\{ f \in C_L^\alpha(\mathbb{R};\mathbb{R}^k) : \lim_{\delta\to 0} \sup_{\substack{x,y\in\mathbb{R},\\ 0<|x-y|<\delta}} \frac{\left|f^{(\lfloor\alpha\rfloor)}(x) - f^{(\lfloor\alpha\rfloor)}(y)\right|}{|x-y|^{\{\alpha\}}} = 0 \Big\}. \tag{2.3.35}$$

This is a closed subspace of $C_L^\alpha(\mathbb{R};\mathbb{R}^k)$ and therefore a Banach space.

In the following we collect some properties of periodic little Hölder functions. This type of functions plays a fundamental role in the classical maximal regularity theory, see for example [110].

**Lemma 2.3.16.** [101, Prop. 1.2] *Let $\alpha_1, \alpha_2 \in \mathbb{R}_+ \setminus \mathbb{Z}$ such that $\alpha_1 < \alpha_2$. Then $h_L^{\alpha_1}(\mathbb{R})$ is the closure of $C_L^{\alpha_2}(\mathbb{R})$ in $\big(C_L^{\alpha_1}(\mathbb{R}), \|\cdot\|_{C^{\alpha_1}}\big)$ and $h_L^{\alpha_2}(\mathbb{R}) \hookrightarrow h_L^{\alpha_1}(\mathbb{R})$ densely.*

**Lemma 2.3.17** (Product of periodic little Hölder functions)**.** *Let $\alpha_1, \alpha_2 \in \mathbb{R}_+ \setminus \mathbb{Z}$. Let $g_1 \in h_L^{\alpha_1}(\mathbb{R})$ and let $g_2 \in h_L^{\alpha_2}(\mathbb{R})$. Then*

$$g_1 g_2 \in h_L^{\min\{\alpha_1,\alpha_2\}}(\mathbb{R}) \quad \text{and} \quad \|g_1 g_2\|_{C^{\min\{\alpha_1,\alpha_2\}}} \leq C(\alpha_1,\alpha_2,L)\, \|g_1\|_{C^{\alpha_1}}\, \|g_2\|_{C^{\alpha_2}}.$$

*Proof.* This follows from (2.3.35) and a lengthy computation. □

### Time-weighted BUC spaces

Let $E$ be an arbitrary Banach space. Let $\eta \in (0,1)$ and let $T > 0$. We define

$$\mathrm{BUC}_{1-\eta}([0,T];E) := \Big\{ \xi \in C((0,T];E) : \Big[t \mapsto t^{1-\eta}\xi(t)\Big] \text{ bounded and uniformly continuous on } (0,T] \text{ with } \lim_{t\to 0^+} t^{1-\eta}\|\xi(t)\|_E = 0 \Big\}.$$

Moreover, we define the subspace

$$\mathrm{BUC}^1_{1-\eta}([0,T];E) := \Big\{ \xi \in C^1((0,T];E) : \xi, \partial_t\xi \in \mathrm{BUC}_{1-\eta}([0,T];E) \Big\}.$$

In Section 2.3, we use this space with $E$ a periodic little Hölder space. In particular, the spaces $\mathbb{E}_0$ and $\mathbb{E}_1$, defined in (2.3.5) and (2.3.6) are Banach spaces with the norms

$$\|\xi\|_{\mathbb{E}_0} = \|\xi\|_{\mathbb{E}_0([0,T])} := \sup_{t\in(0,T]} t^{1-\eta}\,\|\xi(t)\|_{C^\alpha(\mathbb{R})} \quad \text{and}$$
$$\|\xi\|_{\mathbb{E}_1} = \|\xi\|_{\mathbb{E}_1([0,T])} := \sup_{t\in(0,T]} t^{1-\eta}\left(\|\partial_t\xi(t)\|_{C^\alpha(\mathbb{R})} + \|\xi(t)\|_{C^{2+\alpha}(\mathbb{R})}\right),$$

respectively. We use that the mapping

$$\gamma\colon \mathbb{E}_1([0,T]) \to h^\alpha_L(\mathbb{R};\mathbb{R}^k), \quad \xi(t,\cdot)\mapsto \xi(0,\cdot)$$

is well defined, linear and continuous. This follows from the observation that for $\xi \in \mathbb{E}_1([0,T])$ and $0<s<t$

$$\xi(t)-\xi(s) = \int_s^t \partial_t\xi(\tau)\,\mathrm{d}\tau, \tag{2.3.36}$$

and hence there exists $\lim_{t\to 0}\xi(t)$ in $h^\alpha_L(\mathbb{R})$, see [34, Remark 2.1]. We introduce the trace space $\gamma\mathbb{E}_1 := \mathrm{Im}(\gamma)$. In particular, $\gamma\mathbb{E}_1$ does not depend on $T>0$. In [101, Section 5.1] it is shown that up to equivalent norms

$$\gamma\mathbb{E}_1 = h^{2\eta+\alpha}_L(\mathbb{R};\mathbb{R}^k) \tag{2.3.37}$$

provided $2\eta+\alpha\notin\mathbb{Z}$. For simplicity, we mostly omit the index $k$.
If $\eta>\frac{1}{2}$, we have continuity in space and time of the first order $s$-derivative by the following lemma. This is essential in Section 2.3.

**Lemma 2.3.18.** [34, Lemma 2.2] *Let $T>0$. We have*

$$\mathbb{E}_1([0,T]) \hookrightarrow \mathrm{BUC}\big([0,T];h^{2\eta+\alpha}_L(\mathbb{R})\big).$$

*In particular, this implies that for $\xi\in\mathbb{E}_1([0,T])$ there exists $\lim_{t\to0}\xi(t)$ in $C^{2\eta+\alpha}_L(\mathbb{R})$ and $\sup_{t\in[0,T]}\|\xi\|_{C^{2\eta+\alpha}} \le C(T)\,\|\xi\|_{\mathbb{E}_1}<\infty$.*

**Remark 2.3.19.** Lemma 2.3.18 implies that if $\eta\ge\frac{1}{2}$, for $\xi\in\mathbb{E}_1([0,T])$, there exists $\lim_{t\to0}\partial_s\xi(t)$ in $C^\alpha_L(\mathbb{R})$ and $\partial_s\xi\in\mathrm{BUC}\big([0,T];h^\alpha_L(\mathbb{R})\big)$.

**Remark 2.3.20.** We observe that from the definition of the spaces $\mathbb{E}_0([0,T])$ and $\mathbb{E}_1([0,T])$ we can derive the following properties.

(i) If $f\in\mathbb{E}_1([0,T])$, then $\partial_s f,\partial_s^2 f\in\mathbb{E}_0([0,T])$.

(ii) If $f_1\in\mathbb{E}_1([0,T])$ and $f_2\in\mathbb{E}_0([0,T])$, then $f_1f_2\in\mathbb{E}_0([0,T])$.

(iii) If $f\in\mathbb{E}_1([0,T])$ and $g\in C^\infty(\mathbb{R})$, then $g\circ f\in\mathbb{E}_1([0,T])$.

**Lemma 2.3.21.** *Let $T>0$, $\eta\in(\frac{1}{2},1)$, $\xi\in\mathbb{E}_1([0,T])$. Then $\partial_s^2\xi\in L^2([0,T]\times[0,L])$.*

*Proof.* Since the $\mathbb{E}_1$-norm is finite, there exists $K > 0$ such that for all $t \in [0,T]$ we have $\sup_{s\in\mathbb{R}} |\partial_s^2 \xi(t,s)| \leq K t^{\eta-1}$. It thus follows

$$\int_0^T \int_0^L (\partial_s^2 \xi)^2 \,\mathrm{d}s \,\mathrm{d}t \leq K^2 L \int_0^T t^{2\eta-2} \,\mathrm{d}t = \frac{K^2 L}{2\eta-1} T^{2\eta-1} < \infty. \qquad \square$$

### Parabolic Hölder spaces of order 2

At the end of Section 2.3 we show that the solution of (2.3.3) from Proposition 2.3.5 is smooth. For this purpose we consider two additional function spaces. First, we set for $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$, $\zeta \in [0,1)$ and $T > 0$

$$\begin{aligned}\mathrm{BUC}^\zeta\left([0,T]; h_L^\alpha(\mathbb{R})\right) := \Big\{ &\xi : [0,T] \to h_L^\alpha(\mathbb{R}) \text{ bounded and uniformly continuous} \\ &\text{on } [0,T], \sup_{t_1 \neq t_2 \in [0,T]} \frac{\|\xi(t_1,s) - \xi(t_2,s)\|_{C^\alpha}}{|t_1 - t_2|^\zeta} < \infty \Big\}.\end{aligned}$$

Moreover, we work with parabolic Hölder spaces, cf. [96]. Let $l \in \mathbb{R}_+ \setminus \mathbb{Z}$, $a, b \in \mathbb{R}$ with $a < b$, $T > 0$ and let $Q_T := (0,T) \times (a,b)$. The parabolic Hölder space $H^{\frac{l}{2},l}\left(\overline{Q_T}\right)$ is the space of all functions $f : \overline{Q_T} \to \mathbb{R}$ with continuous derivatives $\partial_t^m \partial_s^n f$ for $2m + n < l$ and finite norm

$$\begin{aligned}\|f\|_{H^{\frac{l}{2},l}\left(\overline{Q_T}\right)} := &\sum_{j=0}^{\lfloor l \rfloor} \sum_{2m+n=j} \sup_{(t,s)\in Q_T} |\partial_t^m \partial_s^n f| \\ &+ \sum_{2m+n=\lfloor l \rfloor} \sup_{\substack{(t,s_1),(t,s_2)\in Q_T \\ s_1 \neq s_2}} \frac{|\partial_t^m \partial_s^n f(t,s_1) - \partial_t^m \partial_s^n f(t,s_2)|}{|s_1 - s_2|^{l - \lfloor l \rfloor}} \\ &+ \sum_{0<l-2m-n<2} \sup_{\substack{(t_1,s),(t_2,s)\in Q_T \\ t_1 \neq t_2}} \frac{|\partial_t^m \partial_s^n f(t_1,s) - \partial_t^m \partial_s^n f(t_2,s)|}{|t_1 - t_2|^{\frac{l-2m-n}{2}}}.\end{aligned}$$

The following lemma shows how $\mathbb{E}_1$ relates to these parabolic Hölder spaces.

**Lemma 2.3.22.** *Let $\alpha \in (0,1)$ and $\eta \in \left[\frac{1}{2}, 1\right)$ such that $2\eta + \alpha \notin \mathbb{Z}$. Let $a, b \in \mathbb{R}$ with $a < b$ and let $T > 0$. Let $f \in \mathbb{E}_1([0,T])$. Then $\partial_s f \in H^{\frac{\zeta}{2},\zeta}([0,T] \times [a,b])$ for $\zeta := \eta + \frac{\alpha}{2} - \frac{1}{2} \in (0,1)$.*

*Proof.* Since $0 < \frac{\eta}{2} - \frac{\alpha}{4} + \frac{1}{4} < \eta$, [34, Lemma 2.2 d)] together with [101, Prop. 1.2 b)] give

$$\mathbb{E}_1([0,T]) \hookrightarrow \mathrm{BUC}^{\eta - \left(\frac{\eta}{2} - \frac{\alpha}{4} + \frac{1}{4}\right)}\Big([0,T]; h_L^{2\left(\frac{\eta}{2} - \frac{\alpha}{4} + \frac{1}{4}\right) + \alpha}(\mathbb{R})\Big).$$

With $\zeta = \eta + \frac{\alpha}{2} - \frac{1}{2} \in (0,1)$, we thus have (in particular) $\partial_s f \in \mathrm{BUC}^{\frac{\zeta}{2}}\left([0,T]; h_L^\zeta(\mathbb{R})\right)$. The

continuity of $\partial_s f$ in $[0,T]\times[a,b]$ (since $\eta \geq \frac{1}{2}$) and the finiteness of the norm

$$\begin{aligned}\|\partial_s f\|_{H^{\frac{\zeta}{2},\zeta}(\overline{Q_T})} = &\sup_{(t,s)\in Q_T} |\partial_s f| + \sup_{\substack{(t,s_1),(t,s_2)\in Q_T \\ s_1\neq s_2}} \frac{|\partial_s f(t,s_1)-\partial_s f(t,s_2)|}{|s_1-s_2|^{\zeta}} \\ &+ \sup_{\substack{(t_1,s),(t_2,s)\in Q_T \\ t_1\neq t_2}} \frac{|\partial_s f(t_1,s)-\partial_s f(t_2,s)|}{|t_1-t_2|^{\frac{\zeta}{2}}}\end{aligned}$$

imply that $\partial_s f \in H^{\frac{\zeta}{2},\zeta}([0,T]\times[a,b])$. □

## 2.4. Global existence

For the classical elastic flow, we have long time existence of the solution (see Proposition 1.4.3). In this section, we prove such a result in the much more general model for heterogeneous elastic wires.

**Theorem 2.4.1** (Global existence, [37, Theorem 1.3]). *The solution $(\theta,\rho)$ in Theorem 2.3.2 exists for all times, i.e. $T=\infty$.*

The flow equations (2.0.4) are a parabolic system, so we cannot use arguments based on the maximum principle to show global existence. Instead, we rely on energy estimates to prove that suitable Sobolev norms along the solution cannot blow up in finite time. To that end, we first show boundedness of the velocity $\partial_t(\theta,\rho)$ in order to deal with the nonlinear coupling which limits the direct application of interpolation inequalities. To allow for $\omega = 0$, a new argument is needed to show a priori boundedness of the nonlocal Lagrange multipliers, compared to the strategy used in [160]. In the following section, we start the proof of global existence with the presentation of this argument.

### 2.4.1. Controlling the matrix $\Pi^{-1}$

As a first step, we show that the inverse of the matrix in (2.1.3), which defines the Lagrange-multipliers in (2.1.2), does not degenerate if the $L^2$-norm of the curvature $k=\partial_s\theta$ is bounded. In contrast to [160, Lemma 2.1], our proof is variational and does not depend on the rotation index.

**Lemma 2.4.2** ([37, Lemma 4.1]). *Let $K>0$. Then there exists $M=M(K,L)>0$ such that for all functions $\theta\in W^{1,2}(0,L)$ satisfying (2.0.3) and*

$$\int_0^L (\partial_s\theta)^2 \,\mathrm{d}s \leq K \tag{2.4.1}$$

*we have*

$$\left\|\Pi^{-1}(\theta)\right\| := \sup_{X\in\mathbb{R}^2,|X|=1} \left|\Pi^{-1}(\theta)(X)\right| \leq M.$$

*Proof.* By (2.1.3), all entries of the matrix $\Pi(\theta)$ are bounded by $L$, thus it suffices to bound $\det \Pi(\theta)$ from below for all $\theta \in W^{1,2}(0,L)$ satisfying (2.0.3) and (2.4.1) by some constant $m = m(K,L) > 0$. Let $(\theta_k)_{k\in\mathbb{N}} \subset W^{1,2}(0,L)$ be a minimizing sequence for

$$m := \inf \left\{ \det \Pi(\theta) \mid \theta \in W^{1,2}(0,L) \text{ satisfying (2.0.3) and (2.4.1)} \right\}.$$

Note that $m \geq 0$ by Remark 2.1.4. Without loss of generality, we may assume $\theta_k(0) = 0$, after replacing $\theta_k$ by $\theta_k - \theta_k(0)$. Indeed, by direct computation (2.4.1) and also (2.0.3) are invariant under addition of a constant. Moreover, Remark 2.1.4 directly implies that $\det \Pi(\theta_k - \theta_k(0)) = \det \Pi(\theta_k)$.
Consequently, $(\theta_k)_{k\in\mathbb{N}} \subset W^{1,2}(0,L)$ is bounded, so that we may assume $\theta_k \rightharpoonup \theta$ in $W^{1,2}(0,L)$ and $\theta_k \to \theta$ in $C([0,L])$ by Rellich–Kondrachov. The limit $\theta$ clearly satisfies (2.0.3) and, by weak lower semicontinuity of the $L^2$-norm, also (2.4.1). From (2.1.3), we find $\det \Pi(\theta_k) \to \det \Pi(\theta)$ and thus conclude that $m = \det \Pi(\theta)$ is a minimum. Since $\theta$ is continuous, by Remark 2.1.4 we find that $m = 0$ if and only if $\theta$ is constant. However, the latter is impossible due to (2.0.3), so $m > 0$ and the claim follows. □

### 2.4.2. Energy estimates

We now prove norm estimates for our solution, which by Lemma 2.3.13 instantaneously becomes smooth. Since we are only interested in long time existence here, for the rest of this subsection we assume that we have a fixed *smooth* solution $(\theta, \rho) \in C^\infty([0,T) \times [0,L])$ of (2.0.4). This means that the initial datum $(\theta_0, \rho_0)$ is smooth and satisfies Hypothesis 2.3.1.
Furthermore, throughout this subsection $C \in (0,\infty)$ denotes a generic constant that may vary from line to line. The constant is only allowed to depend on the fixed model parameters as well as the initial datum $(\theta_0, \rho_0)$. In particular, $C$ does not depend on the (not necessarily maximal) existence time $T$. This is essential for proving the convergence in Section 2.5.2.
First, we observe that due to $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$ ensuring the curve remains closed under the flow, it follows that the integral over the inclination angle remains preserved along the evolution.

**Lemma 2.4.3.** *For all $t \in (0,T)$ we have*

$$\int_0^L \theta \, \mathrm{d}s = \int_0^L \theta_0 \, \mathrm{d}s.$$

*Proof.* Integration of the evolution equation for $\theta$ in (2.0.4) yields

$$\partial_t \int_0^L \theta \, \mathrm{d}s = \beta(\rho) \left( \partial_s \theta - c_0 \right) \Big|_0^L + \lambda_{\theta 1} \int_0^L \sin\theta \, \mathrm{d}s - \lambda_{\theta 2} \int_0^L \cos\theta \, \mathrm{d}s.$$

By (2.1.8), the integrals vanish. The boundary conditions in (2.0.4) yield the claim. □

Next, we gather some direct consequences of the constrained gradient flow structure.

**Proposition 2.4.4** ([37, Proposition 4.2])**.** *We have*

*(i)* $\|\theta\|_{L^\infty([0,T)\times[0,L])} \leq C, \quad \|\rho\|_{L^\infty([0,T)\times[0,L])} \leq C,$

*(ii)* $\sup_{t\in[0,T)} \|\partial_s\theta\|_{L^2(0,L)} \leq C, \quad \sup_{t\in[0,T)} \|\partial_s\rho\|_{L^2(0,L)} \leq C,$

*(iii)* $\|\partial_t\theta\|_{L^2([0,T)\times[0,L])} \leq C, \quad \|\partial_t\rho\|_{L^2([0,T)\times[0,L])} \leq C.$

*Proof.* Statement (iii) follows from integrating (2.1.9) in time and using $\mathcal{E}_\mu(\theta,\rho) \geq 0$. Moreover, for any $t \in [0,T)$ we obtain

$$\mathcal{E}_\mu(\theta_0,\rho_0) \geq \frac{\mu}{2}\int_0^L (\partial_s\rho)^2 \,\mathrm{d}s$$

and the second estimate in (ii) is proven.
We will now use this to get an $L^\infty$-bound on $\rho$. Since the total mass is conserved along the evolution by (2.1.8), the Mean Value Theorem for integrals implies the existence of $s_t \in [0,L]$ with $\rho(t,s_t) = \frac{1}{L}\int_0^L \rho_0 \,\mathrm{d}s$ for any $t \in [0,T)$. Similar to (2.3.31), the Fundamental Theorem of Calculus yields

$$|\rho(t,s) - \rho(t,s_t)| \leq \int_0^L |\partial_s\rho| \,\mathrm{d}s \leq L^{\frac{1}{2}}\Big(\int_0^L (\partial_s\rho)^2 \,\mathrm{d}s\Big)^{\frac{1}{2}} \leq C \tag{2.4.2}$$

and the second estimate of (i) is proven.
Consequently, there exists a compact interval $J \subset \mathbb{R}$ only depending on $\theta_0, \rho_0$ and the model parameters such that

$$\rho(t,s) \in J \quad \text{for all } t \in [0,T),\ s \in [0,L]. \tag{2.4.3}$$

By continuity, we find that

$$\inf_J \beta \geq C > 0. \tag{2.4.4}$$

Therefore, using (2.1.9) again, we obtain

$$2\mathcal{E}_\mu(\theta_0,\rho_0) \geq \inf_J \beta \int_0^L (\partial_s\theta - c_0)^2 \,\mathrm{d}s \geq C\Big(\int_0^L (\partial_s\theta)^2 \,\mathrm{d}s - 2c_0\int_0^L \partial_s\theta \,\mathrm{d}s + c_0^2 L\Big).$$

By (2.0.3), we have $\int_0^L \partial_s\theta \,\mathrm{d}s = 2\pi\omega$, and hence the first part of (ii) follows.
This together with the fact that $\int_0^L \theta \,\mathrm{d}s = \int_0^L \theta_0 \,\mathrm{d}s$ by Lemma 2.4.3 yields the first estimate in (i) by proceeding in the same fashion as in (2.4.2). □

**Proposition 2.4.5** ([37, Proposition 4.3])**.** *For the Lagrange multipliers we have*

$$\sup_{t\in[0,T)} |\lambda_{\theta 1}| \le C, \quad \sup_{t\in[0,T)} |\lambda_{\theta 2}| \le C, \quad \sup_{t\in[0,T)} |\lambda_{\rho}| \le C.$$

*Proof.* Because of Proposition 2.4.4, Lemma 2.4.2 can be applied with $K = C$, only depending on the model parameters and the initial datum. Hence, for the Lagrange multipliers $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$ by Remark 2.1.3 we have

$$\left|\begin{pmatrix}\lambda_{\theta 1}\\ \lambda_{\theta 2}\end{pmatrix}\right| \le C \sup_J \beta\Big( \int_0^L (\partial_s\theta)^2 \,\mathrm{d}s + \int_0^L |c_0\,\partial_s\theta| \,\mathrm{d}s\Big),$$

where $J$ is as in (2.4.3). The right hand side is bounded for all $t \in [0,T)$ by Proposition 2.4.4 and the Cauchy–Schwarz inequality. This way, we obtain the desired bounds for $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$. For $\lambda_\rho$ we use (2.1.6) and proceed similarly, using that $\sup_J |\beta'| \le C$ by continuity. □

As a next step, we would like to bound the $L^2$-norm of $\partial_s^2(\theta,\rho)$ uniformly in time. Directly pursuing this idea, we encounter difficulties in controlling the nonlinear coupling of the system (2.0.4), which seems to be incompatible with a direct application of interpolation inequalities. Instead, we control the $L^2$-norm of the velocity $\partial_t(\theta,\rho)$ first.

**Proposition 2.4.6** ([37, Proposition 4.4])**.** *We have*

$$\sup_{t\in[0,T)} \|\partial_t\theta\|_{L^2(0,L)} \le C, \qquad \sup_{t\in[0,T)} \|\partial_t\rho\|_{L^2(0,L)} \le C.$$

*Proof.* By continuity, Proposition 2.4.4 and with $J$ as in (2.4.3) we have that

$$\sup_J |\beta'| \le C, \qquad \sup_J |\beta''| \le C. \tag{2.4.5}$$

We consider the smooth function

$$\varphi(t) := \frac{1}{2}\int_0^L \Big((\partial_t\theta)^2 + (\partial_t\rho)^2\Big)\,\mathrm{d}s, \quad \text{for } t\in[0,T).$$

We now use (2.0.4) and integration by parts to differentiate $\varphi$. The boundary terms dissapear as a consequence of the boundary conditions in (2.0.4). We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = &-\int_0^L \beta(\rho)(\partial_t\partial_s\theta)^2\,\mathrm{d}s - \int_0^L \mu(\partial_t\partial_s\rho)^2\,\mathrm{d}s\\ &- 2\int_0^L \partial_t\partial_s\theta\,\beta'(\rho)\,\partial_t\rho\,(\partial_s\theta - c_0)\,\mathrm{d}s - \frac{1}{2}\int_0^L (\partial_t\rho)^2\beta''(\rho)\,(\partial_s\theta - c_0)^2\,\mathrm{d}s\\ &+ \int_0^L (\partial_t\theta)^2\lambda_{\theta 1}\cos\theta\,\mathrm{d}s + \int_0^L (\partial_t\theta)^2\lambda_{\theta 2}\sin\theta\,\mathrm{d}s.\end{aligned}$$

By (2.1.4) and (2.1.7) no time derivatives of the Lagrange multipliers appear. Using (2.4.5), Young's inequality, Proposition 2.4.4 and Proposition 2.4.5, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) \leq & -\inf_J \beta \int_0^L (\partial_t\partial_s\theta)^2 \,\mathrm{d}s - \mu\int_0^L (\partial_t\partial_s\rho)^2\,\mathrm{d}s + \delta\int_0^L (\partial_t\partial_s\theta)^2\,\mathrm{d}s \\ & + C(\delta)\sup_{s\in[0,L]}(\partial_t\rho)^2 + C\int_0^L (\partial_t\theta)^2\,\mathrm{d}s \end{aligned} \tag{2.4.6}$$

for $\delta > 0$ to be chosen and some correspondingly adjusted $C(\delta) > 0$. Using (2.1.7) and arguing as in (2.4.2), we find there exists $s_t \in [0, L]$ with $\partial_t\rho(t, s_t) = 0$ and thus

$$\begin{aligned}\sup_{s\in[0,L]}|\partial_t\rho|^2 &= \sup_{s\in[0,L]}\left|(\partial_t\rho)^2 - (\partial_t\rho(t,s_t))^2\right| \leq \int_0^L \left|\partial_s\left(\partial_t\rho\right)^2\right|\,\mathrm{d}s \\ &\leq 2\left\|\partial_t\rho\right\|_{L^2(0,L)}\left\|\partial_t\partial_s\rho\right\|_{L^2(0,L)} \leq \delta\left\|\partial_t\partial_s\rho\right\|^2_{L^2(0,L)} + C(\delta)\left\|\partial_t\rho\right\|^2_{L^2(0,L)}, \end{aligned} \tag{2.4.7}$$

using Young's inequality in the last step. Using (2.4.4) and choosing $\delta > 0$ small enough in (2.4.6) and (2.4.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) \leq C\varphi(t) \quad \text{for all } t \in [0, T). \tag{2.4.8}$$

By Proposition 2.4.4, we find that $\int_0^T \varphi(t)\,\mathrm{d}t$ is bounded independently of $T$ and thus the claim follows from integrating (2.4.8). □

**Proposition 2.4.7** ([37, Proposition 4.5])**.** *We have*

$$\sup_{t\in[0,T)}\left\|\partial_s^2\theta\right\|_{L^2(0,L)} \leq C, \qquad \sup_{t\in[0,T)}\left\|\partial_s^2\rho\right\|_{L^2(0,L)} \leq C.$$

*Proof.* To prove the required estimates, we need to consider simultaneously the evolution equations of $\theta$ and $\rho$. Recall from (2.0.4) that we have

$$\begin{aligned}\beta(\rho)\partial_s^2\theta &= \partial_t\theta - \beta'(\rho)\partial_s\rho(\partial_s\theta - c_0) - \lambda_{\theta 1}\sin\theta + \lambda_{\theta 2}\cos\theta, \\ \mu\partial_s^2\rho &= \partial_t\rho + \frac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2 + \lambda_\rho\end{aligned}$$

on $[0, T)$. Using (2.4.4) and arguing as in (2.4.5), by Young's inequality we find

$$\begin{aligned}&\left\|\partial_s^2\theta\right\|^2_{L^2(0,L)} + \left\|\partial_s^2\rho\right\|^2_{L^2(0,L)} \\ &\leq C\Big(\int_0^L (\partial_t\theta)^2\,\mathrm{d}s + \int_0^L (\partial_s\rho)^2(\partial_s\theta - c_0)^2\,\mathrm{d}s + \lambda_{\theta 1}^2\int_0^L \sin^2\theta\,\mathrm{d}s + \lambda_{\theta 2}^2\int_0^L \cos^2\theta\,\mathrm{d}s \\ &\quad + \int_0^L (\partial_t\rho)^2\,\mathrm{d}s + \int_0^L (\partial_s\theta - c_0)^4\,\mathrm{d}s + \lambda_\rho^2 L\Big)\end{aligned}$$

for all $t \in [0, T)$. By Propositions 2.4.5 and 2.4.6 it is sufficient to consider the second

term and the sixth term on the right side of the inequality. Using Hölder's inequality, interpolation [20, p. 233] and $\int_0^L \partial_s\rho \,\mathrm{d}s = 0$, we find

$$\begin{aligned}\left\|(\partial_s\rho)^2(\partial_s\theta)^2\right\|_{L^1(0,L)} &\leq \|\partial_s\rho\|^2_{L^4(0,L)} \|\partial_s\theta\|^2_{L^4(0,L)} \\ &\leq C\|\partial_s^2\rho\|^{\frac{1}{2}}_{L^2(0,L)} \|\partial_s\rho\|^{\frac{3}{2}}_{L^2(0,L)} \left(\|\partial_s^2\theta\|^{\frac{1}{4}}_{L^2(0,L)} \|\partial_s\theta\|^{\frac{3}{4}}_{L^2(0,L)} + \|\partial_s\theta\|_{L^2(0,L)}\right)^2.\end{aligned}$$

Due to Proposition 2.4.4 the norm $\|\partial_s\theta\|_{L^2(0,L)}$ is bounded for all $t \in [0,T)$ and similarly for $\rho$. Therefore, using twice Young's inequality we estimate

$$\left\|(\partial_s\rho)^2(\partial_s\theta)^2\right\|_{L^1(0,L)} \leq \delta\|\partial_s^2\rho\|^2_{L^2(0,L)} + \delta\|\partial_s^2\theta\|^2_{L^2(0,L)} + C(\delta),$$

for $\delta > 0$ to be chosen. In an analogous way we also get

$$\left\|(\partial_s\theta)^4\right\|_{L^1(0,L)} \leq \delta\|\partial_s^2\theta\|^2_{L^2(0,L)} + C(\delta).$$

Finally, taking $\delta > 0$ sufficiently small and absorbing, the claim follows. □

**Proposition 2.4.8** ([37, Proposition 4.6])**.** *We have*

$$\sup_{t\in[0,T)} \|\partial_s^3\theta\|_{L^2(0,L)} \leq C, \qquad \sup_{t\in[0,T)} \|\partial_s^3\rho\|_{L^2(0,L)} \leq C.$$

*Proof.* Again, we use a Gronwall argument. To that end, we define

$$\varphi(t) := \frac{1}{2}\int_0^L \left((\partial_s^3\theta)^2 + (\partial_s^3\rho)^2\right) \mathrm{d}s$$

for $t \in [0,T)$ and obtain after integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) + \varphi(t) = -\int_0^L \partial_s^4\theta\,\partial_t\partial_s^2\theta + \partial_s^4\rho\,\partial_t\partial_s^2\rho \,\mathrm{d}s - \frac{1}{2}\int_0^L \partial_s^4\theta\partial_s^2\theta + \partial_s^4\rho\partial_s^2\rho \,\mathrm{d}s. \quad (2.4.9)$$

Differentiating twice the evolution equations (2.0.4) with respect to $s$ yields

$$\begin{aligned}\partial_s^2\partial_t\theta &= 3\beta'(\rho)\partial_s\rho\partial_s^3\theta + \beta(\rho)\partial_s^4\theta + 3\beta''(\rho)(\partial_s\rho)^2\partial_s^2\theta + 3\beta'(\rho)\partial_s^2\rho\partial_s^2\theta \\ &\quad + \beta'''(\rho)(\partial_s\rho)^3(\partial_s\theta - c_0) + 3\beta''(\rho)\partial_s^2\rho\partial_s\rho(\partial_s\theta - c_0) + \beta'(\rho)\partial_s^3\rho(\partial_s\theta - c_0) \\ &\quad - \lambda_{\theta 1}\sin\theta(\partial_s\theta)^2 + \lambda_{\theta 1}\cos\theta\partial_s^2\theta + \lambda_{\theta 2}\cos\theta(\partial_s\theta)^2 + \lambda_{\theta 2}\sin\theta\partial_s^2\theta, \\ \partial_s^2\partial_t\rho &= \mu\partial_s^4\rho - \frac{1}{2}\beta'''(\rho)(\partial_s\rho)^2(\partial_s\theta - c_0)^2 - \frac{1}{2}\beta''(\rho)\partial_s^2\rho(\partial_s\theta - c_0)^2 \\ &\quad - 2\beta''(\rho)\partial_s\rho(\partial_s\theta - c_0)\partial_s^2\theta - \beta'(\rho)(\partial_s^2\theta)^2 - \beta'(\rho)(\partial_s\theta - c_0)\partial_s^3\theta.\end{aligned}$$

By continuity, Proposition 2.4.4 and (2.4.3), we find that $|\beta^{(k)}(\rho)|$ is bounded by $C$ for $k = 0,\dots,3$. Moreover, due to the embedding $W^{2,2}(0,L) \hookrightarrow C^1([0,L])$, Proposition 2.4.4 and Proposition 2.4.7 we know that $\sup_{s\in[0,L]} |\partial_s\theta|$ is bounded by $C$ for all $t \in [0,T)$. The

same applies to $\sup_{s\in[0,L]}|\partial_s\rho|$. This implies that after plugging the last two equations into (2.4.9) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)+\varphi(t)\leq -\inf_J \beta(\rho)\int_0^L(\partial_s^4\theta)^2\,\mathrm{d}s-\mu\int_0^L(\partial_s^4\rho)^2\,\mathrm{d}s+CR(t), \tag{2.4.10}$$

where $R(t)$ is a sum of terms of the form

$$\int_0^L|\partial_s^4 f||\partial_s^3 g|\,\mathrm{d}s,\ \int_0^L|\partial_s^4 f||\partial_s^2 g|\,\mathrm{d}s,\ \int_0^L|\partial_s^4 f|\,\mathrm{d}s,\ \int_0^L|\partial_s^4 f||\partial_s^2 f||\partial_s^2 g|\,\mathrm{d}s \tag{2.4.11}$$

for $f,g\in\{\theta,\rho\}$. For the first term, we use Young's inequality, integration by parts and Proposition 2.4.7 and estimate

$$\begin{aligned}\int_0^L|\partial_s^4 f||\partial_s^3 g|\,\mathrm{d}s&\leq\delta\int_0^L\left(\partial_s^4 f\right)^2\mathrm{d}s+C(\delta)\int_0^L\left(\partial_s^3 g\right)^2\mathrm{d}s\\&\leq\delta\int_0^L\left(\partial_s^4 f\right)^2\mathrm{d}s+\delta\int_0^L\left(\partial_s^4 g\right)^2\mathrm{d}s+C(\delta).\end{aligned}$$

Similarly, for terms of the second and third type we have

$$\int_0^L\left(|\partial_s^4 f||\partial_s^2 g|+|\partial_s^4 f|\right)\mathrm{d}s\leq\delta\int_0^L\left(\partial_s^4 f\right)^2\mathrm{d}s+C(\delta).$$

The terms of the fourth type in (2.4.11) can be controlled by interpolation, cf. [20, p. 233] (using $\int_0^L\partial_s^k f\,\mathrm{d}s=0$ for $k\geq 2$, $f\in\{\theta,\rho\}$) to estimate

$$\begin{aligned}\int_0^L|\partial_s^4 f||\partial_s^2 f||\partial_s^2 g|\,\mathrm{d}s&\leq\|\partial_s^4 f\|_{L^2(0,L)}\|\partial_s^2 f\|_{L^4(0,L)}\|\partial_s^2 g\|_{L^4(0,L)}\\&\leq C\|\partial_s^4 f\|_{L^2(0,L)}\|\partial_s^4 f\|^{\frac{1}{8}}_{L^2(0,L)}\|\partial_s^2 f\|^{\frac{7}{8}}_{L^2(0,L)}\|\partial_s^4 g\|^{\frac{1}{8}}_{L^2(0,L)}\|\partial_s^2 g\|^{\frac{7}{8}}_{L^2(0,L)}\\&\leq\delta\|\partial_s^4 f\|^2_{L^2(0,L)}+\delta\|\partial_s^4 g\|^2_{L^2(0,L)}+C(\delta)\end{aligned}$$

as above by Proposition 2.4.7 and Young's inequality for $\delta>0$ to be chosen. This way, we have estimated the remainder term $R(t)$ in (2.4.10). Consequently, if we now use $\inf_J\beta\geq C$ by (2.4.4) and choose $\delta>0$ sufficiently small, after absorbing we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)+\varphi(t)\leq C.$$

With Gronwall's lemma, we conclude that $\varphi(t)\leq C$ for all $t\in[0,T)$. □

**Remark 2.4.9** ([37, Remark 4.7])**.** It is also possible to bound

$$\sup_{t\in[0,T)}||(\theta,\rho)||_{W^{m,2}(0,L)}\leq C(m),\quad m\in\mathbb{N},$$

by essentially the same arguments as in Proposition 2.4.8.

### 2.4.3. Proof of global existence

Let $(\theta_0, \rho_0) \in h^{1+\alpha}([0,L])$, $\alpha \in (0,1)$ be such that Hypothesis 2.3.1 is satisfied. As a consequence of Theorem 2.3.2, a standard argument for initial value problems yields the existence of a *maximal existence time* $T_{\max} \in (0,\infty]$ such that the unique solution $(\theta,\rho)$ to (2.0.4) with this initial datum can be smoothly extended to $(0,T_{\max})$. The goal of this section is to prove Theorem 2.4.1, i.e. that $T_{\max} = \infty$.

*Proof of Theorem 2.4.1.* Let $0 < T_{\max} \leq \infty$ be the maximal existence time of the smooth solution of (2.0.4) with initial datum $(\theta,\rho)(0,\cdot) = (\theta_0,\rho_0)$ in $C^{1+\alpha}([0,L])$. We want to show that the assumption $T_{\max} < \infty$ yields a contradiction as we can extend the solution past $T_{\max}$.
Assume $T_{\max} < \infty$ and let $\delta \in (0,T_{\max})$. We have $(\theta,\rho) \in C^\infty([\delta,T_{\max}) \times [0,L])$. By Propositions 2.4.4 and 2.4.8, there is a constant $C > 0$ depending only on $(\theta,\rho)(\delta,\cdot)$ and the model parameters such that

$$\sup_{t\in[\delta,T_{\max})} \|\theta\|_{W^{3,2}(0,L)} \leq C \quad \text{and} \quad \sup_{t\in[\delta,T_{\max})} \|\rho\|_{W^{3,2}(0,L)} \leq C. \tag{2.4.12}$$

We consider an arbitrary sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \nearrow T_{\max}$. Let $\tilde{\alpha} \in (0,\frac{1}{2})$. The bounds in (2.4.12) together with the compact embedding $W^{3,2}(0,L) \hookrightarrow C^{2+\tilde{\alpha}}([0,L])$ and the closedness of $h^{2+\tilde{\alpha}}([0,L])$ yield the existence of $\tilde{\theta}_0 \in h^{2+\tilde{\alpha}}([0,L])$ and $\tilde{\rho}_0 \in h^{2+\tilde{\alpha}}([0,L])$ such that after passing to a subsequence

$$\theta(t_n,\cdot) \to \tilde{\theta}_0 \quad \text{and} \quad \rho(t_n,\cdot) \to \tilde{\rho}_0 \quad \text{in } C^{2+\tilde{\alpha}}([0,L]) \quad \text{as } n \to \infty. \tag{2.4.13}$$

Suppose $t_n' \nearrow T_{\max}$ denotes another sequence such that (2.4.13) holds with $t_n'$ instead of $t_n$ and $(\tilde{\theta}_0',\tilde{\rho}_0')$ instead of $(\tilde{\theta}_0,\tilde{\rho}_0)$. Using (2.0.4), Propositions 2.4.4, 2.4.5 and (2.4.12), we find $\|\partial_t\theta\|_{W^{1,2}(0,L)} \leq C$ for $t \in [\delta,T_{\max})$, where $C > 0$ again only depends on the model parameters and $(\theta,\rho)(\delta,\cdot)$. Using that $W^{1,2}(0,L) \hookrightarrow C^{\tilde{\alpha}}([0,L])$, we find

$$\|\theta(t_n,\cdot) - \theta(t_n',\cdot)\|_{C^{\tilde{\alpha}}([0,L])} \leq \Big\| \int_{t_n}^{t_n'} \partial_t\theta(\tau,\cdot)\,\mathrm{d}\tau \Big\|_{C^{\tilde{\alpha}}([0,L])} \leq C|t_n - t_n'|,$$

after suitably modifying $C > 0$. Taking $n \to \infty$ yields $\tilde{\theta}_0' = \tilde{\theta}_0$. With exactly the same argument we find $\tilde{\rho}_0' = \tilde{\rho}_0$. Consequently,

$$\theta(t,\cdot) \to \tilde{\theta}_0 \quad \text{and} \quad \rho(t,\cdot) \to \tilde{\rho}_0 \quad \text{in } C^{2+\tilde{\alpha}}([0,L]) \quad \text{as } t \nearrow T_{\max}. \tag{2.4.14}$$

It is clear that $(\tilde{\theta}_0,\tilde{\rho}_0)$ satisfies (2.3.1). Thus, Theorem 2.3.2 and Remark 2.3.15 yield the existence of $\tilde{T} > 0$ and a unique solution $(\tilde{\theta},\tilde{\rho}) \in C^\infty((0,\tilde{T}) \times [0,L])$ of (2.0.4) with

$$\tilde{\theta}(t,\cdot) \to \tilde{\theta}_0 \quad \text{and} \quad \tilde{\rho}(t,\cdot) \to \tilde{\rho}_0 \quad \text{in } C^{2+\tilde{\alpha}}([0,L]) \quad \text{as } t \searrow 0. \tag{2.4.15}$$

We take this solution to extend the solution $(\theta, \rho)$ past $T_{\max}$ by defining

$$(\bar{\theta}, \bar{\rho})(t,s) := \begin{cases} (\theta,\rho)(t,s) & \text{for } t \in (0, T_{\max}), \\ (\tilde{\theta}_0, \tilde{\rho}_0)(s) & \text{for } t = T_{\max}, \\ (\tilde{\theta}, \tilde{\rho})(t - T_{\max}, s) & \text{for } t \in (T_{\max}, T_{\max} + \tilde{T}). \end{cases}$$

We now claim that

$$(\bar{\theta}, \bar{\rho}) \in C^\infty((0, T_{\max} + \tilde{T}) \times [0, L]). \tag{2.4.16}$$

Fix $T \in (T_{\max}, T_{\max} + \tilde{T})$. Examining $t \nearrow T_{\max}$ and using (2.4.14), respectively $t \searrow T_{\max}$ and using (2.4.15), we conclude

$$(\bar{\theta}, \bar{\rho}) \in \mathrm{BUC}\left([\delta, T]; C^{2+\tilde{\alpha}}([0, L])\right).$$

By (2.0.4), we may write $\partial_t(\theta, \rho)$ in terms of $\partial_s^2(\theta, \rho), \partial_s(\theta, \rho)$ and $(\theta, \rho)$ on $(0, T_{\max})$ and similarly for $\partial_t(\tilde{\theta}, \tilde{\rho})$ on $(0, \tilde{T})$. Hence $\partial_t(\bar{\theta}, \bar{\rho})$ exists on $[\delta, T_{\max}) \cup (T_{\max}, T]$ and possesses a continuous extension at $t = T_{\max}$, so that we find

$$\partial_t(\bar{\theta}, \bar{\rho}) \in \mathrm{BUC}\left([\delta, T]; C^{\tilde{\alpha}}([0, L])\right).$$

Consequently, we obtain

$$(\bar{\theta}, \bar{\rho}) \in \mathrm{BUC}^1\left([\delta, T]; C^{\tilde{\alpha}}([0, L])\right) \cap \mathrm{BUC}\left([\delta, T]; C^{2+\tilde{\alpha}}([0, L])\right). \tag{2.4.17}$$

By (2.3.34), for every $T' < T_{\max}$ the solution $(\theta, \rho)$ satisfies

$$(\theta, \rho) \in \mathrm{BUC}^1_{1-\bar{\eta}}\left([0, T']; h^{\bar{\alpha}}([0, L])\right) \cap \mathrm{BUC}_{1-\bar{\eta}}\left([0, T']; h^{2+\bar{\alpha}}([0, L])\right), \tag{2.4.18}$$

where $\bar{\alpha} \in (0, 1)$ and $\bar{\eta} \in (\frac{1}{2}, 1)$ are chosen such that $2\bar{\eta} + \bar{\alpha} = 1 + \alpha$. Since the time weight $t^{1-\bar{\eta}}$ only plays a role near $t = 0$, from (2.4.17) and (2.4.18) we conclude that

$$(\bar{\theta}, \bar{\rho}) \in \mathrm{BUC}^1_{1-\bar{\eta}}\left([0, T]; h^{\hat{\alpha}}([0, L])\right) \cap \mathrm{BUC}_{1-\bar{\eta}}\left([0, T]; h^{2+\hat{\alpha}}([0, L])\right),$$

for $0 < \hat{\alpha} < \min\{\bar{\alpha}, \tilde{\alpha}\}$, using Lemma 2.3.16. If we transfer this back to the periodic setting as in Section 2.3.1, we may apply Lemma 2.3.13 to deduce that

$$(\bar{\theta}, \bar{\rho}) \in C^\infty((0, T) \times [0, L]).$$

Since $T \in (T_{\max}, T_{\max} + \tilde{T})$ was arbitrary, we conclude that (2.4.16) is satisfied.

Consequently, we have extended the solution $(\theta, \rho)$ smoothly past $T_{\max}$, a contradiction, so $T_{\max} = \infty$ has to hold. □

## 2.5. Convergence result

Once global existence is established, it is natural to study the asymptotic behavior of the flow. Although we do not have a classification of the stationary solutions to (2.0.4) at hand, we can prove full convergence. The goal of this section is to prove the following theorem.

**Theorem 2.5.1** (Long time behavior, [37, Theorem 1.4]). *Let $\beta\colon \mathbb{R} \to \mathbb{R}$ be real analytic. Then, the global solution $(\theta, \rho)$ of Theorem 2.4.1 converges in $C^{2+\tilde{\alpha}}([0, L])$ for all $\tilde{\alpha} \in (0, \frac{1}{2})$ to a stationary solution of (2.0.1) as $t \to \infty$, i.e. to a solution of*

$$\begin{cases} 0 = \partial_s \left[\beta(\rho)(\partial_s\theta - c_0)\right] + \lambda_{\theta 1} \sin\theta - \lambda_{\theta 2} \cos\theta & \text{in } [0, L], \\ 0 = \mu\partial_s^2\rho - \dfrac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2 - \lambda_\rho & \text{in } [0, L], \\ \theta(L) - \theta(0) = 2\pi\omega, \qquad \rho(L) = \rho(0), \\ \partial_s\theta(L) = \partial_s\theta(0), \qquad \partial_s\rho(L) = \partial_s\rho(0) \end{cases} \tag{2.5.1}$$

*for some $\lambda_{\theta 1}, \lambda_{\theta 2}, \lambda_\rho \in \mathbb{R}$.*

Geometrically, this result can be interpreted as follows. The solution $(\theta(t), \rho(t))_{t\geq 0}$ describes a family of closed curves $(\gamma(t))_{t\geq 0}$ with density $(\rho(t))_{t\geq 0}$ that evolves in time decreasing the energy (1.5.1) and converging to an equilibrium as $t \to \infty$.
For the proof, we first deduce subconvergence of the solution from the energy estimates used in the proof of Theorem 2.4.1. This means that given a sequence of times $t_n \to \infty$, we can find a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ along which the solution converges. However, for different subsequences, the limit may be different. Moreover, the set of possible limits is non-discrete. Therefore, we need more to prove full convergence. We assume $\beta$ to be analytic and rely on a suitable version of the Łojasiewicz–Simon gradient inequality [135]. As Chapter 1 already suggests, inequalities of this kind are a powerful functional analytic tool for studying the asymptotic behavior of (geometric) PDEs, even in the constrained setting; see, for instance, [143, 30, 44] and [139, 136, 138].
In our case, the constraint is given by the zero set of the functional (2.2.3), which is

$$\mathcal{G}(\theta, \rho) = \Big( \int_0^L \cos\theta \,\mathrm{d}s,\ \int_0^L \sin\theta \,\mathrm{d}s,\ \int_0^L \rho \,\mathrm{d}s - mL \Big),$$

where $m \in \mathbb{R}$ corresponds to the fixed total mass, determined by the initial density, i.e. $mL = \int_0^L \rho_0 \,\mathrm{d}s$. We consider the Banach space of periodic Sobolev functions $W^{k,2}_{\mathrm{per}}(0, L)$ defined in (2.2.1). By the choice of the Lagrange multipliers, cf. (2.1.4) and (2.1.7), the gradient flow remains in the closed set

$$\mathcal{B} := \{(\theta, \rho) \in W^{2,2}_{\mathrm{per}}(0, L; \mathbb{R}^2) + (\phi, 0) : \mathcal{G}(\theta, \rho) = 0\} \tag{2.5.2}$$

for all $t \geq 0$ with $\phi$ as in (2.2.2). To apply the results in [135], we need to work in Banach spaces which is why we consider the shifted energies $E_\mu(u,\rho)$ and $G(u,\rho)$ for $(u,\rho) \in W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2)$, defined as in (2.2.4). Here, we work only in the domain of the $L^2$-gradient of the functionals, and not in the energy space $W^{1,2}(0,L)$. In (2.2.13) and (2.2.14), we have computed the $L^2$-gradients of $E_\mu$ (with $\nabla E_\mu(u,\rho) = (\nabla_u E_\mu(u,\rho), \nabla_\rho E_\mu(u,\rho))$) and of the components of $G$. For the convenience of the reader, we repeat that they are given by

$$\begin{aligned}
&\nabla_u E_\mu(u,\rho) = -\partial_s(\beta(\rho)(\partial_s u + \partial_s\phi - c_0)),\\
&\nabla_\rho E_\mu(u,\rho) = -\mu\partial_s^2\rho + \frac{1}{2}\beta'(\rho)(\partial_s u + \partial_s\phi - c_0)^2,\\
&\nabla G^1(u,\rho) = (-\sin(u+\phi),0)\,,\ \nabla G^2(u,\rho) = (\cos(u+\phi),0)\,,\ \nabla G^3(u,\rho) = (0,1)\,.
\end{aligned}$$

### 2.5.1. The constrained Łojasiewicz–Simon gradient inequality

We first discuss analyticity properties of the energy and the constraint functional. A concise overview of the relevant properties of analytic functions on Banach spaces can be found in [135, Section 2.1]. In the following, $C_{\mathrm{per}}([0,L])$ denotes the Banach space of $L$-periodic continuous functions on $[0,L]$, equipped with the supremum norm.

**Lemma 2.5.2** ([37, Lemma 5.1])**.** *Let $\beta\colon \mathbb{R} \to \mathbb{R}$ be analytic. The maps*

$$\begin{aligned}
&W^{1,2}_{\mathrm{per}}(0,L) \to C_{\mathrm{per}}([0,L]), \quad \rho \mapsto \beta(\rho),\\
&W^{k,2}_{\mathrm{per}}(0,L) \to L^2(0,L), \quad \theta \mapsto \partial_s^k\theta \text{ for } k = 1,2
\end{aligned}$$

*are analytic. Moreover, the energy and the constraint functional*

$$\begin{aligned}
&W^{2,2}_{\mathrm{per}}([0,L],\mathbb{R}^2) \to \mathbb{R}, \quad (u,\rho) \mapsto E_\mu(u,\rho),\\
&W^{2,2}_{\mathrm{per}}([0,L];\mathbb{R}^2) \to \mathbb{R}^3, \quad (u,\rho) \mapsto G(u,\rho)
\end{aligned}$$

*and their gradients given by*

$$\begin{aligned}
&W^{2,2}_{\mathrm{per}}([0,L];\mathbb{R}^2) \to L^2(0,L;\mathbb{R}^2), \quad (u,\rho) \mapsto \nabla E_\mu(u,\rho),\\
&W^{2,2}_{\mathrm{per}}([0,L];\mathbb{R}^2) \to L^2(0,L;\mathbb{R}^2), \quad (u,\rho) \mapsto \nabla G^j(u,\rho)
\end{aligned}$$

*are analytic for $j = 1,2,3$.*

*Proof.* The first part follows from the Sobolev embedding $W^{1,2}_{\mathrm{per}}(0,L) \to C_{\mathrm{per}}([0,L])$ and since the Nemytskii operator $C_{\mathrm{per}}([0,L]) \ni u \mapsto \beta(u) \in C_{\mathrm{per}}([0,L])$ is analytic by [7, Theorem 6.8].
The rest follows using that the composition and sum of analytic maps is again analytic and that any bounded multilinear map is analytic, see [135, Section 2.1]. □

Besides analyticity, we also need to verify certain Fredholm and compactness properties of the derivatives of $\nabla E_\mu$ and $\nabla G$, cf. [135, Corollary 5.2]. First, we compute these derivatives. Let $(u,\rho),(v,\sigma) \in W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2)$. Then we have

$$\begin{aligned}(\nabla_u E_\mu)'(u,\rho)(v,\sigma) &= -\beta''(\rho)\partial_s\rho(\partial_s u + \partial_s\phi - c_0)\sigma - \beta'(\rho)\partial_s^2 u\ \sigma \\ &\quad - \beta'(\rho)(\partial_s u + \partial_s\phi - c_0)\partial_s\sigma - \partial_s(\beta(\rho)\partial_s v),\\ (\nabla_\rho E_\mu)'(u,\rho)(v,\sigma) &= \frac{1}{2}\beta''(\rho)(\partial_s u + \partial_s\phi - c_0)^2\sigma \\ &\quad + \beta'(\rho)(\partial_s u + \partial_s\phi - c_0)\partial_s v - \mu\partial_s^2\sigma. \end{aligned} \tag{2.5.3}$$

Moreover, we have

$$\begin{aligned}(\nabla G^1)'(u,\rho)(v,\sigma) &= (-\cos(u+\phi)v, 0),\\ (\nabla G^2)'(u,\rho)(v,\sigma) &= (-\sin(u+\phi)v, 0),\\ (\nabla G^3)'(u,\rho)(v,\sigma) &= (0,0).\end{aligned}$$

**Lemma 2.5.3** ([37, Lemma 5.2])**.** *The operator*

$$(\nabla E_\mu)'(u,\rho)\colon W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2) \to L^2(0,L;\mathbb{R}^2)$$

*is Fredholm of index zero for all* $(u,\rho) \in W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2)$.

*Proof.* We observe that by (2.5.3), we may write

$$(\nabla E_\mu)'(u,\rho)(v,\sigma) = A(v,\sigma) + K(v,\sigma),$$

where $A(v,\sigma) = \left(-\partial_s\left(\beta(\rho)\partial_s v\right), -\mu\partial_s^2\sigma\right)$ and $K\colon W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2) \to L^2(0,L;\mathbb{R}^2)$ only involves first and zeroth order expressions in $(v,\sigma)$. Hence $K$ is compact by the Rellich–Kondrachov theorem. Moreover, $\beta(\rho) \in C^1([0,L])$ and $\min_{[0,L]}\beta(\rho) > 0$ by Sobolev embedding. Thus, for every $f \in L^2(0,L)$ there exists a unique solution $v \in W^{2,2}_{\mathrm{per}}(0,L)$ to $\partial_s\left(\beta(\rho)\partial_s v\right) = f$ as a consequence of the Riesz representation theorem. Consequently, we find that $A\colon W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2) \to L^2(0,L;\mathbb{R}^2)$ is an isomorphism. The claim then follows using [97, XVII, Corollaries 2.6 and 2.7]. □

For simplicity, we again abuse the notation and write $\nabla\mathcal{E}_\mu(\theta,\rho)$ and $\nabla\mathcal{G}(\theta,\rho)$, see (2.2.15). Moreover, setting $\Lambda(\theta,\rho) := (\lambda_{\theta 1}(\theta,\rho), \lambda_{\theta 2}(\theta,\rho), \lambda_\rho(\theta,\rho))$, with $\lambda_{\theta 1}(\theta,\rho)$, and $\lambda_{\theta 2}(\theta,\rho)$, $\lambda_\rho(\theta,\rho)$ as in (2.1.2) and (2.1.6), we define

$$\Lambda(\theta,\rho)\cdot\nabla\mathcal{G}(\theta,\rho) := \lambda_{\theta 1}(\theta,\rho)\nabla\mathcal{G}^1(\theta,\rho) + \lambda_{\theta 2}(\theta,\rho)\nabla\mathcal{G}^2(\theta,\rho) + \lambda_\rho(\theta,\rho)\nabla\mathcal{G}^3(\theta,\rho).$$

In this notation, the evolution in (2.0.4) may abstractly be written as

$$\partial_t(\theta,\rho) = -\nabla\mathcal{E}_\mu(\theta,\rho) - \Lambda(\theta,\rho)\cdot\nabla\mathcal{G}(\theta,\rho). \tag{2.5.4}$$

where $(\theta,\rho)\colon [0,T)\times[0,L]\to\mathbb{R}^2$. This way, stationary solutions to (2.5.4) (i.e. solutions to (2.5.1)) are constrained critical points of $\mathcal{E}$ subject to the constraint $\mathcal{G}=0$ by the method of Lagrange multipliers, see (2.2.16).

**Theorem 2.5.4** (Constrained Łojasiewicz–Simon gradient inequality, [37, Theorem 5.3])**.** *Let $\beta$ be real analytic and suppose that $(\bar\theta,\bar\rho)\in\mathcal{B}$ (cf. (2.5.2)) is a critical point of $\mathcal{E}_\mu$ subject to the constraint $\mathcal{G}=0$. Then there exist $C,r>0$ and $\vartheta\in(0,\frac12]$ such that for all $(\theta,\rho)\in\mathcal{B}$ with $\|(\theta,\rho)-(\bar\theta,\bar\rho)\|_{W^{2,2}(0,L)}\le r$ we have*

$$|\mathcal{E}_\mu(\theta,\rho)-\mathcal{E}_\mu(\bar\theta,\bar\rho)|^{1-\vartheta}\le C\|\nabla\mathcal{E}_\mu(\theta,\rho)+\Lambda(\theta,\rho)\cdot\nabla\mathcal{G}(\theta,\rho)\|_{L^2(0,L)}.$$

*Proof.* We verify that assumptions (i)–(vi) in [135, Corollary 5.2] are satisfied for the energy functional $E_\mu\colon W^{2,2}_{\mathrm{per}}(0,L;\mathbb{R}^2)\to\mathbb{R}$ subject to the constraint $G=0$ at the constrained critical point $(\bar u,\bar\rho) := (\bar\theta-\phi,\bar\rho)$. Assumption (i) is clearly satisfied. Assumptions (ii) and (iv), and also analyticity of $E_\mu$ and $G$ follow from Lemma 2.5.2, whereas (iii) is precisely the statement of Lemma 2.5.3. Assumption (v) is satisfied by (2.5.3) and using the Rellich–Kondrachov theorem. For (vi), it only remains to show that $\nabla G^1(\bar u,\bar\rho),\nabla G^2(\bar u,\bar\rho),\nabla G^3(\bar u,\bar\rho)$ are linearly independent, which can be verified using $\int_0^L\cos\bar\theta\,\mathrm{d}s=\int_0^L\sin\bar\theta\,\mathrm{d}s=0$.
By [135, Corollary 5.2], $E_\mu|_{\{G=0\}}$ satisfies the constrained Łojasiewicz–Simon gradient inequality near $(\bar u,\bar\rho)$, which immediately transfers to $\mathcal{E}_\mu|_{\{\mathcal{G}=0\}}$ near $(\bar\theta,\bar\rho)$ as in the statement. The explicit form of the projected gradient in terms of the Lagrange multipliers follows from [135, Proposition 3.3] and a computation. □

## 2.5.2. Convergence

Equipped with the Łojasiewicz–Simon gradient inequality in (2.5.4), we can now prove full convergence, i.e. Theorem 2.5.1.

*Proof of Theorem 2.5.1.* Since we are only interested in the long time behavior, we may without loss of generality assume $\theta,\rho\in C^\infty([0,\infty)\times[0,L])$.
*Step 1: Subconvergence.* We first prove that for any sequence $t_n\to\infty$, there exist a subsequence $(t_n')_{n\in\mathbb{N}}$ and a stationary solution $(\theta_\infty,\rho_\infty)\in C^{2+\tilde\alpha}([0,L])$ such that

$$\lim_{n\to\infty}(\theta(t_n'),\rho(t_n')) = (\theta_\infty,\rho_\infty) \text{ in } C^{2+\tilde\alpha}([0,L]) \text{ for all } \tilde\alpha\in(0,\tfrac12).$$

Let $(t_n)_{n\in\mathbb{N}}$ be any sequence with $t_n\to\infty$. First, note that the constant $C$ in (2.4.12) does not depend on the existence time, so (2.4.12) remains valid for $T_{\max}=\infty$. The compact

embedding $W^{3,2}(0,L) \hookrightarrow C^{2+\tilde{\alpha}}([0,L])$, where $\tilde{\alpha} \in (0,\frac{1}{2})$, and a subsequence argument yields that there exists $(t_n')_{n\in\mathbb{N}}$, a subsequence of $(t_n)_{n\in\mathbb{N}}$, and $(\theta_\infty,\rho_\infty)$ such that for all $\tilde{\alpha} \in (0,\frac{1}{2})$ we have

$$\lim_{n\to\infty}(\theta(t_n'),\rho(t_n')) = (\theta_\infty,\rho_\infty) \quad \text{in } C^{2+\tilde{\alpha}}([0,L]). \tag{2.5.5}$$

It remains to prove that $(\theta_\infty,\rho_\infty)$ is stationary, i.e. it satisfies (2.5.1). We consider

$$\varphi(t) = \frac{1}{2}\int_0^L \Big((\partial_t\theta)^2 + (\partial_t\rho)^2\Big)\,\mathrm{d}s, \quad t \geq 0,$$

which is bounded by Proposition 2.4.6, and thus by (2.4.8) its derivative $\frac{\mathrm{d}}{\mathrm{d}t}\varphi$ is bounded from above. Since $\varphi$ is integrable on $[0,\infty)$ by Proposition 2.4.4, this is sufficient to guarantee that $\lim_{t\to\infty}\varphi(t) = 0$. In particular by (2.5.5), we have $\partial_t\theta(t_n',\cdot) \to 0$, $\partial_t\rho(t_n',\cdot) \to 0$ in $C^{\tilde{\alpha}}([0,L])$ for all $\tilde{\alpha} \in (0,\frac{1}{2})$, so $(\theta_\infty,\rho_\infty)$ is stationary, thus a solution of (2.5.1).

*Step 2: Full convergence.* By Step 1, there exists a sequence $t_n \to \infty$ and a stationary solution $(\theta_\infty,\rho_\infty)$ of (2.0.4) such that $(\theta_\infty,\rho_\infty) = \lim_{n\to\infty}(\theta(t_n),\rho(t_n))$ in $C^{2+\tilde{\alpha}}([0,L])$ for all $\tilde{\alpha} \in (0,\frac{1}{2})$.

By (2.1.9), we have $\mathcal{E}_\mu(\theta(t),\rho(t)) \searrow \mathcal{E}_\mu(\theta_\infty,\rho_\infty)$ as $t \to \infty$. In fact, we may assume $\mathcal{E}_\mu(\theta(t),\rho(t)) > \mathcal{E}_\mu(\theta_\infty,\rho_\infty)$ for all $t \in [0,\infty)$, since otherwise the solution must be eventually constant, hence converges trivially.

The stationary solution $(\theta_\infty,\rho_\infty)$ is a constrained critical point as observed directly after (2.5.4). Thus, we may apply Theorem 2.5.4 to $(\theta_\infty,\rho_\infty)$. Let $C,r > 0$ and $\vartheta \in (0,\frac{1}{2}]$ be as in Theorem 2.5.4. After passing to a subsequence, we may assume $\|(\theta(t_n),\rho(t_n)) - (\theta_\infty,\rho_\infty)\|_{W^{2,2}(0,L)} < r$ for all $n \in \mathbb{N}$. We define

$$\tau_n := \sup\{\tau \geq t_n \mid \|(\theta(t),\rho(t)) - (\theta_\infty,\rho_\infty)\|_{W^{2,2}(0,L)} < r \text{ for all } t \in [t_n,\tau]\}. \tag{2.5.6}$$

Since the solution is smooth, we find that $\tau_n > t_n$ for all $n \in \mathbb{N}$. Define

$$\mathcal{H}(t) := \left(\mathcal{E}_\mu(\theta(t),\rho(t)) - \mathcal{E}_\mu(\theta_\infty,\rho_\infty)\right)^\vartheta.$$

By a direct computation and using Theorem 2.5.4 we have

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} &= \vartheta\Big(\mathcal{E}_\mu(\theta,\rho) - \mathcal{E}_\mu(\theta_\infty,\rho_\infty)\Big)^{\vartheta-1}\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\mu(\theta,\rho)\Big) \\
&= \vartheta\Big(\mathcal{E}_\mu(\theta,\rho) - \mathcal{E}_\mu(\theta_\infty,\rho_\infty)\Big)^{\vartheta-1}\|\nabla\mathcal{E}_\mu(\theta,\rho)\|_{L^2(0,L)}\|\partial_t(\theta,\rho)\|_{L^2(0,L)} \\
&\geq \frac{\vartheta}{C}\|\partial_t(\theta,\rho)\|_{L^2(0,L)}
\end{aligned} \tag{2.5.7}$$

on $[t_n, \tau_n)$. After integration, for $t' \in [t_n, \tau_n)$ we obtain

$$\|(\theta(t'), \rho(t')) - (\theta(t_n), \rho(t_n))\|_{L^2(0,L)} \leq \int_{t_n}^{t'} \|\partial_t(\theta, \rho)\|_{L^2} \, \mathrm{d}t \leq \frac{C}{\vartheta} \mathcal{H}(t_n) \to 0, \quad (2.5.8)$$

as $n \to \infty$. We now assume that all of the $\tau_n$ are finite. By continuity, (2.5.8) also remains valid for $t' = \tau_n$. By Step 1, after passing to a subsequence, we may assume that $(\theta(\tau_n), \rho(\tau_n)) \to (\hat{\theta}, \hat{\rho})$ in $C^{2+\tilde{\alpha}}([0, L])$ as $n \to \infty$ for all $\tilde{\alpha} \in (0, \frac{1}{2})$. By continuity and (2.5.6), we find that

$$\|(\hat{\theta}, \hat{\rho}) - (\theta_\infty, \rho_\infty)\|_{W^{2,2}(0,L)} = r > 0, \quad (2.5.9)$$

whereas on the other hand by (2.5.8) with $t' = \tau_n$, we have

$$\|(\hat{\theta}, \hat{\rho}) - (\theta_\infty, \rho_\infty)\|_{L^2(0,L)} = \lim_{n \to \infty} \|(\theta(\tau_n), \rho(\tau_n)) - (\theta(t_n), \rho(t_n))\|_{L^2(0,L)} = 0,$$

a contradiction to (2.5.9).
Consequently, there exist some $n_0$ with $\tau_{n_0} = \infty$, and thus (2.5.7) is satisfied on $[t_{n_0}, \infty)$, which implies that $t \mapsto \|\partial_t(\theta(t), \rho(t))\|_{L^2(0,L)} \in L^1(0, \infty)$. Therefore, $(\theta(t), \rho(t))$ is Cauchy in $L^2(0, L)$ as $t \to \infty$. Hence the limit $\lim_{t\to\infty}(\theta(t), \rho(t))$ exists in $L^2(0, L)$ and thus necessarily equals $(\theta_\infty, \rho_\infty)$. From Step 1 and a subsequence argument, it follows that we have $\lim_{t\to\infty}(\theta(t), \rho(t)) = (\theta_\infty, \rho_\infty)$ in $C^{2+\tilde{\alpha}}([0, L])$ for all $\tilde{\alpha} \in (0, \frac{1}{2})$. □

**Remark 2.5.5** ([37, Remark 5.4])**.** Note the following.

(i) In Step 1 above, we have not used that $\beta$ is analytic. Thus, subconvergence also holds if merely $\beta \in C^\infty(\mathbb{R})$.

(ii) Together with stronger global bounds on the solution as in Remark 2.4.9, it can be shown that the convergence in Theorem 2.5.1 is smooth. The same applies to the subconvergence in Step 1 of its proof.

## 2.6. Qualitative properties of solutions

In the following, we study qualitative properties of the solution. We observe that the preservation and nonpreservation of properties of the initial datum along (2.0.4) depends delicately on the choice of the model parameters. The decisive advantage of working with the angle function $\theta$ is that the equation is of second order. In contrast, working with the curve $\gamma$ yields a fourth order equation, like the classical elastic flow. Due to the reduction to second order, parabolic maximum principles are available for both evolution equations in (2.0.4) individually, but of course not for the full system. A key difficulty in applying maximum principles is the structure of the relevant evolution equations, explaining the fundamentally different behavior for a different choice of parameters.

In this section, we assume the initial datum $(\theta_0, \rho_0)$ to be smooth, to satisfy Hypothesis 2.3.1 and $\partial_s^2\theta_0(L) = \partial_s^2\theta_0(0)$ as well as $\partial_s^2\rho_0(L) = \partial_s^2\rho_0(0)$. Theorem 2.3.2, Remark 2.3.15 and Theorem 2.4.1 then yield a unique global solution $(\theta, \rho) \in C^\infty((0,\infty) \times [0,L]) \cap C^0([0,\infty); C^2([0,L]))$ of (2.0.4). We refer to a function $(\theta_0, \rho_0)$ satisfying the assumptions above as an *admissible initial datum* in the sequel.

Before we begin applying maximum principles, we want to first explain the origin of the figures in the following sections. Then, in Section 2.1.3, we want to note that the two parts of the energy $\mathcal{E}_\mu$ do not decrease separately, which makes investigating the properties of the solution and the limit more challenging. In Section 2.6.3, we study the evolution of the zeroset of the curvature which is related to the preservation of convexity, see Section 2.6.4. With similiar ideas as for the preservation of convexity, we examine the preservation of nonnegativity of the density in Section 2.6.5. Unlike the preservation of convexity and nonnegativity of the density, the preservation of embeddedness cannot be proven using maximum principles. Instead, we employ an energy-based argument to explore the preservation of embeddedness in Section 2.6.6. In Section 2.6.7 we prove that for a rotationally symmetric or axially symmetric initial datum, the solution preserves the rotational or axial symmetry. This result enables us to generalize the result on convexity preservation, see Theorem 2.6.22.

### 2.6.1. Numerical experiments

As described in Section 1.3.3, it can be helpful to guide one's analysis through numerical experiments. For the system of evolution equations (2.0.4), Gaspard Jankowiak from the University of Graz developed a numerical scheme approximating the solutions. The scheme takes advantage of the gradient flow structure of the problem and is based on De Giorgi's Minimizing Movements and finite differences, extending the idea of [18]. We do not go into the details here at all; these can be found in [36, Section 5]. However, we use figures from the numerical experiments to illustrate our analytically obtained results. The implementation and configuration files used for most of the figures are available online[2]. We note that in the numerically obtained figures, a hat marks a space discrete quantity, i.e. $(\hat{\theta}, \hat{\rho}) \in \mathbb{R}^{2N}$ is the piecewise constant approximation of $(\theta, \rho)$, as well as the corresponding energy $\hat{\mathcal{E}}_\mu$ and so on. Positive values of $\rho$ are shown in blue, negative values in red. The thickness of the line increases with $|\rho|$.

### 2.6.2. The two parts of the energy

We have already seen that the $L^2$-gradient structure of the flow equations in (2.0.4) ensures that the energy $\mathcal{E}_\mu$ decreases along the evolution. On the other hand, the two parts of the

[2] `https://github.com/gjankowiak/dAJLR.2024` (code licensed under the GPLv3)

energy,

$$\mathcal{E}^\theta(\theta,\rho) := \frac{1}{2}\int_0^L \beta(\rho)(\partial_s\theta - c_0)^2\,\mathrm{d}s \quad \text{and} \quad \mathcal{E}^\rho_\mu(\rho) := \frac{\mu}{2}\int_0^L (\partial_s\rho)^2\,\mathrm{d}s$$

are not monotonically decreasing individually as the computation in (2.1.9) already suggests. This fact significantly complicates the discussion of the limit in Section 2.7. We give concrete examples where either $\mathcal{E}^\theta$ or $\mathcal{E}^\rho_\mu$ grows.

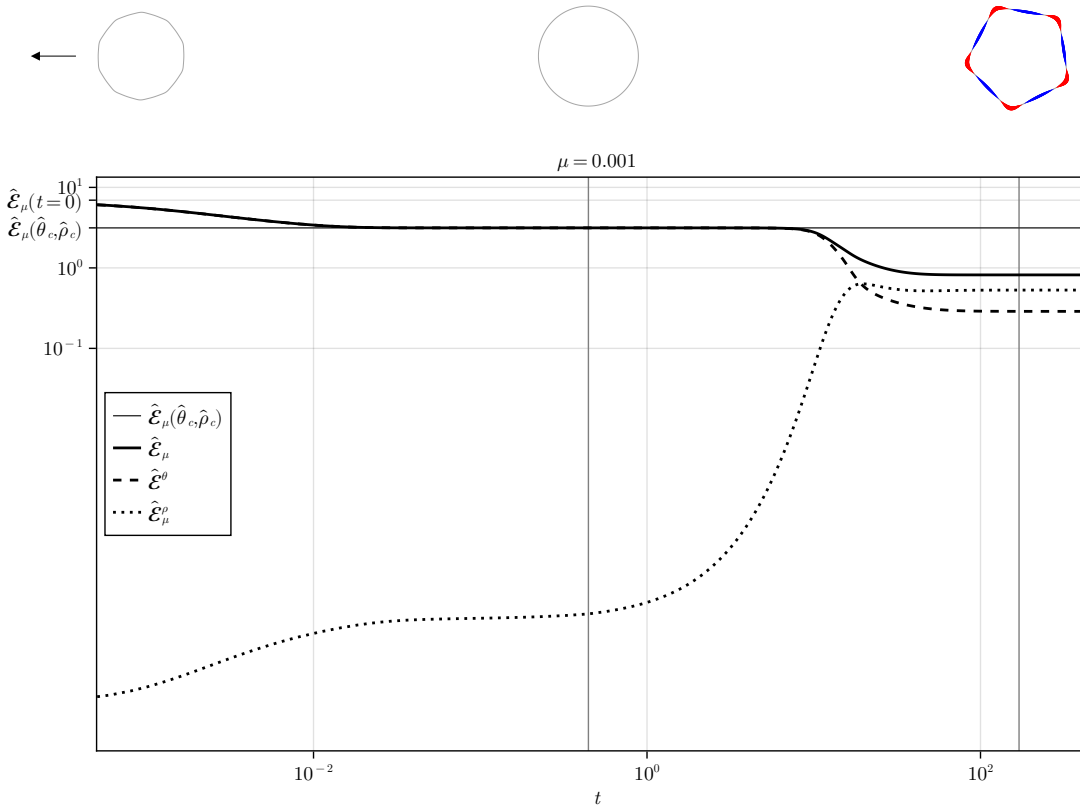



Figure 2.3: Energy evolution starting with relatively low $\|\hat{\kappa} - 1\|_\infty$. Here, $L = 2\pi$, $\mu = 10^{-3}$, $c_0 = 0$, $\beta(x) = e^x$, $m = 0$, $\omega = 1$. From [36, Figure 8], reprinted with permission.

**Example 2.6.1** ([36, Example 3.1])**.** Consider the double-well potential $\beta(x) := (x^2 - 1)^2 + c$ with $c > 0$. Let $L = 2\pi$, $\omega = 1$, $m = 0$, and $c_0 \neq 1$. Take $\theta_0 := \theta_c = s$ and $\rho_0 := \sin s$. Then, for the solution $(\theta, \rho)$ of (2.0.4), an elementary computation yields $\lambda_{\theta 1}(0) = \lambda_{\theta 2}(0) = \lambda_\rho(0) = 0$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\theta(\theta,\rho)\Big|_{t=0} &= (1-c_0)^2\int_0^{2\pi}\left(\frac{\mu}{2}\beta'(\rho_0)\,\partial_s^2\rho_0 - \left(\partial_s(\beta(\rho_0))\right)^2\right)\mathrm{d}s \\ &\quad - \frac{(1-c_0)^4}{4}\int_0^{2\pi}(\beta'(\rho_0))^2\,\mathrm{d}s\end{aligned}$$

$$= \frac{\pi}{2}(1-c_0)^2\left(\mu - (1-c_0)^2 - \frac{5}{2}\right),$$

which is positive for $\mu > (1-c_0)^2 + \frac{5}{2}$. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\mu^\rho(\rho)\Big|_{t=0} = (1-c_0)^2\,\frac{\mu}{2}\int_0^{2\pi}\beta'(\rho_0)\,\partial_s^2\rho_0\,\mathrm{d}s - \mu^2\int_0^{2\pi}(\partial_s^2\rho_0)^2\,\mathrm{d}s = \mu\pi\left(\frac{1}{2}(1-c_0)^2 - \mu\right)$$

is positive for $\mu < \frac{1}{2}(1-c_0)^2$.

In Figure 2.3 and Figure 2.4 we see numerical examples for the growth of $\mathcal{E}_\mu^\rho$.

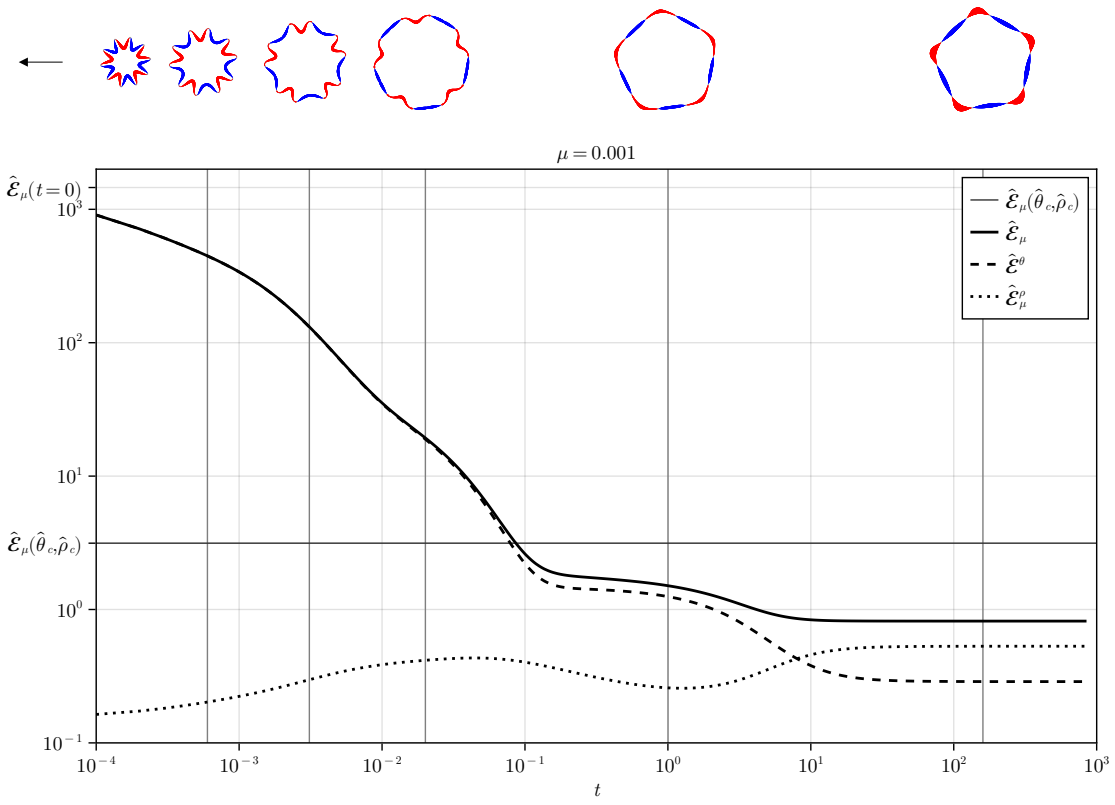



Figure 2.4: Energy evolution starting with relatively large $\|\hat{\kappa} - 1\|_\infty$. Here, $L = 2\pi$, $\mu = 10^{-3}$, $c_0 = 0$, $\beta(x) = e^x$, $m = 0$, $\omega = 1$. From [36, Figure 9], reprinted with permission.

**Remark 2.6.2** ([36, Remark 3.2])**.** In view of Proposition 2.2.8 and Corollary 2.2.10, for $\mu$ large, $\mathcal{E}_\mu^\rho$ can be seen as the dominant term in $\mathcal{E}_\mu$. However, even for arbitrary large $\mu$, $\mathcal{E}_\mu^\rho$ might still not be monotonically decreasing. This can be seen with the evolution equations in (2.0.4). Indeed, taking $\rho_0 \equiv m$ yields $\mathcal{E}_\mu^\rho(\rho_0) = 0$. On the other hand, $\partial_t\rho|_{t=0} \neq 0$ for all $\mu > 0$ as long as $\partial_s\theta_0$ is not constant and $\beta'(m) \neq 0$.

### 2.6.3. Zeros of the curvature

In this section, we adapt the methods in [5] to study the zeros of the curvature and the inflection points of the curve, i.e. the sign changes of the curvature. The results are summarized in the following theorem.

**Theorem 2.6.3** ([36, Theorem 1.2]). *Let $c_0 = 0$ and let $(\theta, \rho)$ be a global solution of (2.0.4). Then both the number of zeros of $k = \partial_s\theta$ and the number of inflection points of the associated curve are nonincreasing in time.*

Here, it is crucial that the equation for $\theta$ is of second order. Differentiating (2.0.4) we find that the curvature $k = \partial_s\theta$ satisfies the second order equation

$$\partial_t k = \beta(\rho)\partial_s^2 k + 2\,\partial_s(\beta(\rho))\partial_s k + \partial_s^2(\beta(\rho))k + (\lambda_{\theta 1}\cos\theta + \lambda_{\theta 2}\sin\theta)\,k - \partial_s^2(\beta(\rho))c_0. \tag{2.6.1}$$

The structure of this evolution equation already indicates that the behavior of $k$ strongly depends on $c_0$. In case $c_0 = 0$, (2.6.1) may be written as a linear second order parabolic equation for $k$. Indeed, for $c_0 = 0$, we have

$$\partial_t k = a\partial_s^2 k + b\partial_s k + ck, \tag{2.6.2}$$

where we define the nonconstant coefficients

$$a := \beta(\rho) > 0, \quad b := 2\partial_s(\beta(\rho)), \quad \text{and} \quad c := \partial_s^2(\beta(\rho)) + \lambda_{\theta 1}\cos\theta + \lambda_{\theta 2}\sin\theta.$$

This allows us to use the techniques in [5] to study the evolution of the zeroset of the curvature. Sign-changing zeros of the curvature are *inflection points* of the curve, i.e. points where the curve locally changes from being convex to being concave or vice versa. Zeros at which the curvature does not change sign are called *undulation points* of the curve. First, we do not distinguish between inflection points and undulation points and show that the total number of zeros of $k$ decreases. In [160, Remark 3.2], this idea was also indicated, without proof, for the $L^2$-gradient flow of the angle function of the classical elastic energy without density-modulated stiffness.
For all $t \geq 0$, we denote by $z_k(t) \in \mathbb{N}_0 \cup \{\infty\}$ the number of zeros in $[0, L)$ of the curvature $k(t, \cdot)$ of the global solution of (2.0.4). First we note that this number is finite, which follows by [5, Theorem C, (a)].

**Lemma 2.6.4** ([36, Lemma 3.3]). *Let $c_0 = 0$ and $(\theta, \rho)$ be the global solution of (2.0.4). Then $z_k(t)$ is finite for any $t > 0$.*

Further, the number of zeros of the curvature (inflection points and undulation points) does not increase along the evolution.

**Proposition 2.6.5** ([36, Proposition 3.4]). *Let $c_0 = 0$ and $(\theta, \rho)$ be the global solution of (2.0.4). Then $z_k$ is a nonincreasing function on $[0, \infty)$.*

*Proof.* Consider the smooth $L$-periodic extension of $k = \partial_s \theta$ to $\mathbb{R}$, which we do not rename for simplicity. The function $z_k$ still denotes the number of zeros of the curvature in the interval $[0, L)$.
Let $t_2 > 0$. By Lemma 2.6.4, there exists $n \in \mathbb{N}_0$ such that $z_k(t_2) = n$. Without loss of generality, we assume that $n \geq 1$. Let $0 \leq s_1 < s_2 < \cdots < s_n < L$ be the zeros of $k$ at time $t_2$. By [5, Lemma 5.5], there are continuous curves $x_i(t)$, $i = 1, \ldots, n$ in the zeroset $Z_k = \{(t, s) \in (0, \infty) \times \mathbb{R} : k(t, s) = 0\}$ defined for $t \in (0, t_2]$ such that $x_i(t_2) = s_i$. Moreover, define $x_{n+1}(t) := x_1(t) + L$, $t \in (0, t_2]$. Then $x_{n+1}(t)$ is also in $Z_k$.
For $t_1 \in (0, \infty)$ with $t_1 < t_2$, [5, Lemma 5.3] tells us that if $x_i(t_2) < x_j(t_2)$, then $x_i(t) < x_j(t)$ for all $t \in [t_1, t_2]$, $i, j = 1, \ldots n + 1$. Hence, considering $I_t := [x_1(t), x_{n+1}(t))$ for $t \in [t_1, t_2]$, we find $x_i(t) \in I_t$ for $t \in [t_1, t_2]$ and $i = 1, ..., n$. Thus, there are at least $n$ zeros of $k$ on $\{t_1\} \times I_t$. Since by periodicity, the number of zeros on $\{t_1\} \times [0, L)$ equals the number of zeros on $\{t_1\} \times I_t$, it follows that $z_k$ is nonincreasing on $(0, \infty)$.
It remains to consider the transition from $t = 0$ to positive times. Due to Lemma 2.6.4 we assume without loss of generality that $z_k(0) < \infty$. Similarly as in [5, Lemma 5.2] it follows that $\lim_{t \searrow 0} x_i(t)$ exists for $i = 1, \ldots n$. Suppose that $\lim_{t \searrow 0} x_i(t) = \lim_{t \searrow 0} x_j(t)$ for $i < j$ and consider the nonempty open set $G := \{(t, s) : 0 < t < t_2,\, x_i(t) < s < x_j(t)\}$. Since $k \equiv 0$ on the parabolic boundary of $G$, the parabolic maximum principle implies that $k \equiv 0$ in $G$. This contradicts Lemma 2.6.4. Thus, $\lim_{t \searrow 0} x_i(t) \neq \lim_{t \searrow 0} x_j(t)$ for $i \neq j$ and it follows that $z_k(0) \geq n$. □

In the following, we specifically consider the inflection points and show that the number of sign-changing zeros of $k$ does not increase. This means geometrically that the number of 'dents' of a curve like in Figure 2.5 is not increasing along the evolution. This is supported by numerical experiments, see Figure 2.4, while for $c_0 \neq 0$, Figure 2.7 in Section Section 2.6.6 gives an example for growing number of inflection points.

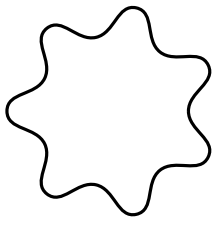


Figure 2.5: Curve with dents. From [36, Figure 1], reprinted with permission.

We denote the number of sign-changing zeros in $[0, L)$ of the $L$-periodic extension of $k(t, \cdot)$ by $\bar{z}_k(t) \in \mathbb{N}_0 \cup \{\infty\}$, $t \in [0, \infty)$.

**Proposition 2.6.6** ([36, Proposition 3.5]). *Let $c_0 = 0$ and $(\theta, \rho)$ be the global solution of (2.0.4). Then $\bar{z}_k$ is a nonincreasing function on $[0, \infty)$.*

*Proof.* Consider the smooth $L$-periodic extension of $(k, \rho)$ to $\mathbb{R}$. First we observe, that the zeros of $\tilde{k}(t, s) := e^{-\lambda t} k(t, s)$, $\lambda \in \mathbb{R}$, $(t, s) \in [0, \infty) \times \mathbb{R}$, coincide with the zeros of $k(t, \cdot)$. By choosing $\lambda$ sufficiently large, we can thus assume that $k$ satisfies (2.6.2) with $c(t, s) < 0$ on $[0, \infty) \times \mathbb{R}$.
Let $0 \leq t_1 < t_2 < \infty$. By Lemma 2.6.4, there is $n \in \mathbb{N}_0$ such that $z_k(t_2) = n$. In view of Proposition 2.6.5 and due to periodicity of $k$, we may assume without loss of generality that $n > 1$. As in the proof of Proposition 2.6.5, let $0 \leq s_1 < \cdots < s_n < L$ be such that $k(t_2, s_i) = 0$, $i = 1, \ldots, n$. By the arguments in the proof of Proposition 2.6.5, there exist continuous curves $x_i(t)$ in the zeroset of $k$ defined for $t \in [0, t_2]$ such that $x_i(t_2) = s_i$, $i = 1, \ldots, n$, and $x_i(t) < x_{i+1}(t)$ for $t \in [0, t_2]$, where $x_{n+1} := x_1 + L$.
Let $i \in \{1, \ldots, n\}$. Either $k(t_2, \cdot) < 0$ or $k(t_2, \cdot) > 0$ on $(x_i(t_2), x_{i+1}(t_2))$. We assume the latter. Consider the open set $G := \{(t, s) : t_1 < t < t_2, x_i(t) < s < x_{i+1}(t)\}$, whose boundary is composed of the curves $x_i, x_{i+1}$ and two vertical lines at $t = t_1, t_2$. By assumption and continuity, $k$ attains a positive maximum on $\bar{G}$, more precisely at the parabolic boundary by the maximum principle. But since $k(t, x_i(t)) = 0$ for $t \in [t_1, t_2]$ and similarly for $x_{i+1}$, the maximum is attained at the vertical line $t = t_1$. Consequently, there exists an interval $I \subset (x_i(t_1), x_{i+1}(t_1))$ with $k(t_1, s) > 0$ for $s \in I$. Thus, for each of the disjoint intervals $(x_i(t_2), x_{i+1}(t_2))$, $i = 1, \ldots, n$, with $k > 0$ (or $k < 0$), there is an interval in $(x_i(t_1), x_{i+1}(t_1))$ with $k > 0$ (or $k < 0$). Since $x_i(t) < x_{i+1}(t)$, $t \in [t_1, t_2]$, there are at least as many sign changes of $k$ in $[x_1(t_1), x_1(t_1) + L)$ as in $[x_1(t_2), x_1(t_2) + L)$. By periodicity, it follows that $\bar{z}_k(t_1) \geq \bar{z}_k(t_2)$. □

For the sake of completeness, we note the following.

*Proof of Theorem 2.6.3.* The result directly follows from Proposition 2.6.5 and Proposition 2.6.6. □

### 2.6.4. Convexity

Combining Theorem 2.6.3 from the last section with further maximum principle arguments, we examine (strict) convexity along the evolution. More precisely, we prove the following result.

**Theorem 2.6.7** (Preservation of convexity for $c_0 = 0$, [36, Theorem 1.3]). *Let $(\theta, \rho)$ be the global solution of (2.0.4) with admissible initial datum $(\theta_0, \rho_0)$ and $c_0 = 0$. Then*

*(i) $k_0 \geq 0$ ($k_0 \leq 0$) on $[0, L]$ implies $k \geq 0$ ($k \leq 0$) on $[0, \infty) \times [0, L]$;*

*(ii) $k_0 > 0$ ($k_0 < 0$) on $[0, L]$ implies $k > 0$ ($k < 0$) on $[0, \infty) \times [0, L]$.*

We refer to convexity because the nonnegativity (or nonpositivity) of the curvature is closely linked to the convexity of the corresponding curve, see Remark 2.6.8 below. Like Theorem 2.6.3, Theorem 2.6.7 relies heavily on the vanishing of $c_0$. Indeed, even for $|c_0| \neq 0$ small, convexity is not preserved in general (see Example 2.6.9). This behavior is not easily predicted from the energy, since for example a large positive $c_0$ clearly favors positive curvature.

*Proof of Theorem 2.6.7.* Case (ii) follows from Proposition 2.6.5. For case (i), we now show that if $\partial_s\theta_0 \geq 0$, then $\partial_s\theta \geq 0$ for all $t \in (0,\infty)$. The proof for the nonpositive case works analogously. We consider the $L$-periodic extension of the global solution $(\theta,\rho)$ to all of $\mathbb{R}$, which we do not rename for simplicity. Note that $\partial_s\theta \in C^\infty((0,\infty)\times\mathbb{R})$ and $\rho \in C^\infty((0,\infty)\times\mathbb{R})$ by Lemma 2.3.13. Since $c_0 = 0$, $k$ satisfies (2.6.2) on $(0,\infty)\times\mathbb{R}$. Even though $\theta$ itself is not even continuous on $\mathbb{R}$, due to the boundary condition $\theta(L) - \theta(0) = 2\pi\omega$, the coefficients $a$, $b$ and $c$ in (2.6.2) are smooth on $\mathbb{R}$ for all $t \geq 0$. Moreover, since the initial datum is attained in the $C^2([0,L])$-norm (see Remark 2.3.15 and the assumptions at the beginning of Section 2.6) and $c$ is bounded globally in $(0,\infty)\times\mathbb{R}$ (see Section 2.4), there is $K > 0$ such that $\sup_{t\in[0,\infty),\, s\in\mathbb{R}} c(t,s) < K$. Defining $k_\varepsilon := k + \varepsilon\exp(Kt)$ for $\varepsilon > 0$ it follows that

$$\partial_t k_\varepsilon > a(t,s)\partial_s^2 k_\varepsilon + b(t,s)\partial_s k_\varepsilon + c(t,s)k_\varepsilon \quad \text{ on } (0,\infty)\times\mathbb{R} \tag{2.6.3}$$

and $\min_{s\in\mathbb{R}} k_\varepsilon(0,s) = \min_{s\in[0,L]} \partial_s\theta_0(s) + \varepsilon > 0$. Now we set

$$t^* := \sup\big\{t \geq 0\colon \min_{s\in\mathbb{R}} k_\varepsilon(\tau,s) > 0 \text{ for all } \tau \in (0,t)\big\}.$$

Clearly, $t^* > 0$. We assume that $t^* < \infty$. This implies that there is $s^* \in \mathbb{R}$ such that

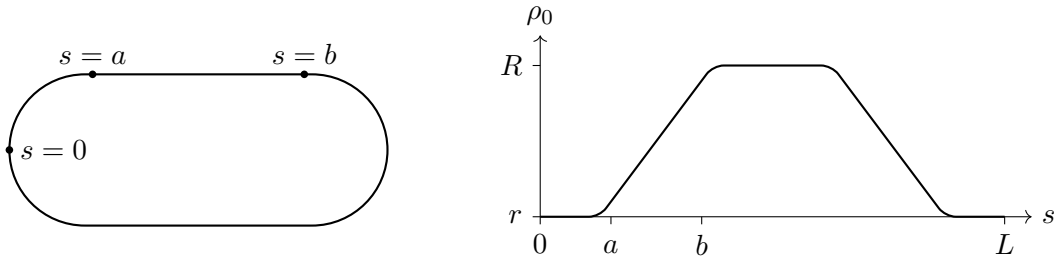
$$0 = \min_{s\in\mathbb{R}} k_\varepsilon(t^*,s) = k_\varepsilon(t^*,s^*). \tag{2.6.4}$$

The necessary conditions for a local minimum yield $\partial_s^2 k_\varepsilon(t^*,s^*) \geq 0$, $\partial_s k_\varepsilon(t^*,s^*) = 0$ and $\partial_t k_\varepsilon(t^*,s^*) \leq 0$. Together with (2.6.4) this is a contradiction to (2.6.3). Hence, $k_\varepsilon > 0$ on $(0,\infty)\times\mathbb{R}$. Sending $\varepsilon \to 0$ yields $\partial_s\theta = k \geq 0$ on $(0,\infty)\times\mathbb{R}$. □

**Remark 2.6.8** ([36, Remark 3.6])**.** A planar closed curve $\gamma$ is called convex if it is *simple* and its curvature $k = \partial_s\theta$ does not change sign. This is equivalent to $\gamma$ parametrizing the boundary of a convex subset of $\mathbb{R}^2$. If the initial datum $\theta_0$ of the flow (2.0.4) describes a convex curve and if $c_0 = 0$, then the corresponding curve remains convex for all times. Indeed, by Theorem 2.6.7, the sign of $k_0 = \partial_s\theta_0$ is preserved. Since the initial curve is simple and the rotation index is preserved along the flow, we have $\omega = \pm 1$ for all $t \geq 0$ by Hopf's Umlaufsatz. Thus, $\int_0^L |k(t,s)|\,\mathrm{d}s = 2\pi$ for any $t \geq 0$. Fenchel's Theorem yields that the evolving curve remains convex for all $t \geq 0$, in particular it remains simple.

For $c_0 = 0$, this is a distinctive advantage of the second order system (2.0.4) over the classical fourth order elastic flow, as simple curves generically can become nonsimple, see [121].

Remarkably, the statement of Theorem 2.6.7 can, in general, not be extended to the case $c_0 \neq 0$. We give an example with $|c_0| > 0$ arbitrarily small.




(A) (B)

Figure 2.6: Cigar-shaped curve (A) with linear density on $[a, b]$ (B). From [36, Figure 2], reprinted with permission.

**Example 2.6.9** ([36, Example 3.7])**.** Let $c_0 \neq 0$. If $c_0 > 0$, let $\beta'' > 0$ on $[r, R]$ for some $r, R \in \mathbb{R}$. Otherwise, let $\beta'' < 0$ on $[r, R]$. We consider an initial curve consisting of a straight line smoothly connected to the rest of the curve in such a way that $\partial_s \theta_0 \geq 0$ on $[0, L]$. For example, a cigar-shaped curve as shown in Figure 2.6A. Let $[a, b] \subset [0, L]$, $a < b$, such that

$$k_0(s) = \partial_s \theta_0(s) = 0, \quad \rho_0(s) \in (r, R), \quad \partial_s \rho_0(s) \neq 0, \quad \partial_s^2 \rho_0(s) = 0 \quad \text{for } s \in [a, b],$$

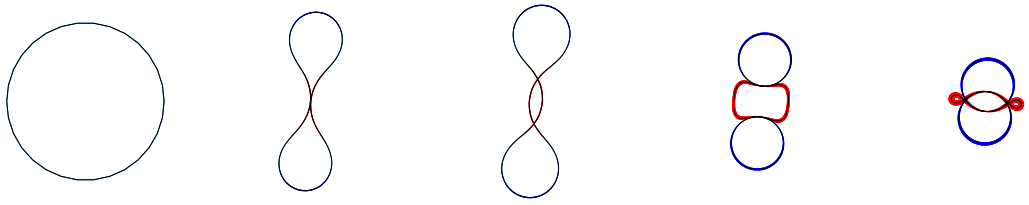
cf. Figure 2.6. Then (2.6.1) gives

$$\partial_t k(t, s)\big|_{t=0} = -\beta''(\rho_0(s))(\partial_s \rho_0(s))^2 c_0 < 0 \text{ for } s \in [a, b].$$

Thus, $k$ becomes negative on $[a, b]$ for $t > 0$ instantaneously and for any $c_0$.

Numerically, we can also observe loss of convexity. For an example, see Figure 2.7. To directly compare the behavior for $c_0 = 0$ and $c_0 \neq 0$, we also provide two videos from the numerical experiments by Gaspard Jankowiak. In the one video[3], we see the evolution of a convex initial datum for $c_0 = 0$. We use exponential $\beta$, $\mu$ relatively small and $m = 0$. In the other video[4], we see the evolution of the same initial datum with the same parameters, except now $c_0 \neq 0$, and observe the loss of convexity.

[3] `https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.010/DallAcqua/movie_c.mp4`
[4] `https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.010/DallAcqua/movie_nc.mp4`



$t = 0$ $t \approx 1.82$ $t \approx 2$ $t \approx 8$ $t \approx 6500$

Figure 2.7: Snapshots of the solution showing the loss of convexity for $c_0 = 3$, starting from $\hat{\theta}_0$ close to the angle function of a circle and $\hat{\rho}_0$ corresponding to $\cos(4\pi/Ls)$. Here, $L = 2\pi$, $\mu = 10^{-3}$, $\beta(x) = e^x$, $m = 0$, $\omega = 1$. From [36, Figure 6], reprinted with permission.

### 2.6.5. Nonnegativity of the density

For a physical mass density $\rho$, only a positive sign is meaningful. We state a condition on $\beta$ which gives a lower bound on $\rho$ and particularly enables us to control its sign. Compared to the situation of Theorem 2.6.7, the evolution equation for $\rho$ in (2.0.4) is not linear in $\rho$. This asks for stricter assumptions on $\beta$, which we again show to be sharp, see Example 2.6.12.

**Proposition 2.6.10** ([36, Proposition 3.8])**.** *Let $x_0 \in \mathbb{R}$. Let $\beta \in C^\infty(\mathbb{R})$ be such that $\beta' \geq 0$ on $[x_0, \infty)$ and $\beta'(x_0) = 0$. Let $(\theta, \rho)$ be the global solution of (2.0.4) with admissible initial datum $(\theta_0, \rho_0)$. If $\rho_0 \geq x_0$ on $[0, L]$, then $\rho \geq x_0$ on $(0, \infty) \times [0, L]$.*

*Proof.* As in the proof of Theorem 2.6.7, consider the $L$-periodic extension of the global solution $(\theta, \rho)$ to all of $\mathbb{R}$. Since $\beta'(x_0) = 0$, the function

$$f(x) := \begin{cases} \frac{\beta'(x)}{x - x_0} & \text{for } x \neq x_0, \\ \beta''(x) & \text{for } x = x_0 \end{cases}$$

is continuous. Choose $R > 0$ such that $\sup_{t \in [0,\infty), s \in \mathbb{R}} |\rho| < R$ and $K > 0$ such that

$$\sup_{t \in [0,\infty),\, s \in \mathbb{R}} \Big( \Big( \sup_{x \in [-R,R]} |\beta''(x)| - f(\rho(t,s)) \Big) (\partial_s \theta(t,s) - c_0)^2 \Big) < 2K. \tag{2.6.5}$$

Let $\varepsilon, T > 0$ be arbitrary. Define $\rho_\varepsilon := \rho - x_0 + \varepsilon \exp(Kt)$ and

$$t^* := \sup \{ t \in [0,T] \colon \min_{s \in \mathbb{R}} \rho_\varepsilon(\tau, s) > 0 \text{ for all } \tau \in (0,t) \}.$$

We assume that $t^* < T$. Observe that with (2.0.4) and (2.1.6),

$$\partial_t \rho_\varepsilon = \mu \partial_s^2 \rho_\varepsilon - \frac{1}{2} f(\rho)(\partial_s \theta - c_0)^2 \rho_\varepsilon + \frac{1}{2} f(\rho)(\partial_s \theta - c_0)^2 \varepsilon e^{Kt} + K \varepsilon e^{Kt}$$

$$+\frac{1}{2L}\int_0^L \beta'(\rho_\varepsilon + x_0)(\partial_s\theta - c_0)^2\,\mathrm{d}s + \frac{1}{2L}\int_0^L \left(\beta'(\rho) - \beta'(\rho_\varepsilon + x_0)\right)(\partial_s\theta - c_0)^2\,\mathrm{d}s. \tag{2.6.6}$$

For $t \in [0, t^*]$, $\rho_\varepsilon(t,s) + x_0 \geq x_0$ on $\mathbb{R}$ and hence $\beta'(\rho_\varepsilon(t,s) + x_0) \geq 0$. Thus, the first integral in (2.6.6) has positive sign. Choosing $\varepsilon$ so small that

$$\sup_{t\in[0,T],s\in\mathbb{R}} (\rho + \varepsilon e^{Kt}) \leq R,$$

we estimate the second integral in (2.6.6) for $t \in [0, t^*]$ by

$$\frac{1}{2L}\int_0^L \left(\beta'(\rho) - \beta'(\rho_\varepsilon + x_0)\right)(\partial_s\theta - c_0)^2\,\mathrm{d}s \geq -\frac{1}{2}\sup_{[-R,R]}|\beta''|\,\varepsilon e^{Kt}\sup_{t\in[0,\infty),s\in\mathbb{R}}(\partial_s\theta - c_0)^2.$$

Thus it follows with the definition of $K$ in (2.6.5) that for $t \in [0, t^*]$,

$$\partial_t\rho_\varepsilon > \mu\partial_s^2\rho_\varepsilon - \frac{1}{2}f(\rho)(\partial_s\theta - c_0)^2\rho_\varepsilon.$$

Arguing as in the proof of Theorem 2.6.7 we find $t^* = T$, which implies that $\rho_\varepsilon \geq 0$ on $[0,T]\times\mathbb{R}$. The limit $\varepsilon \to 0$ yields $\rho \geq x_0$ on $[0,T]\times\mathbb{R}$. Since $T$ was chosen arbitrarily, the claim follows. □

**Remark 2.6.11** ([36, Remark 3.9])**.** With the same arguments it can be shown that for $\beta \in C^\infty(\mathbb{R})$ such that $\beta' \leq 0$ on $(-\infty, x_0]$ and $\beta'(x_0) = 0$ for some $x_0 \in \mathbb{R}$, $\rho_0 \leq x_0$ on $[0,L]$ implies that $\rho \leq x_0$ on $(0,\infty)\times[0,L]$.

The following example shows that it is necessary to impose some conditions on $\beta$ in order to control the sign of $\rho$.

**Example 2.6.12** ([36, Example 3.10])**.** Let $x_0, R \in \mathbb{R}$ with $x_0 < R$ and suppose that $\beta'(x_0) \geq \beta'(x)$ for all $x \in [x_0, R]$. Assume $\omega \neq 0$ and $c_0 \neq \frac{2\pi\omega}{L}$. Consider an admissible initial datum with

$$k_0 \equiv \frac{2\pi\omega}{L}, \quad x_0 \leq \rho_0 \leq R, \quad \rho_0 \equiv x_0 \text{ on } [a,b], \quad \rho \not\equiv x_0 \text{ on } [0,L]$$

for some nontrivial $[a,b] \subset [0,L]$. For $s \in [a,b]$, (2.0.4) and (2.1.6) imply

$$\partial_t\rho(t,s)\big|_{t=0} = -\frac{1}{2}\left(\frac{2\pi\omega}{L} - c_0\right)^2\left(\beta'(x_0) - \frac{1}{L}\int_0^L \beta'(\rho_0(s))\,\mathrm{d}s\right) < 0.$$

Thus, on $[a,b]$ the density is not bounded from below by $x_0$ for $t > 0$.

### 2.6.6. Embeddedness

We now discuss the (non)preservation of embeddedness. The curve $\gamma$ described by the angle function $\theta$ does no longer satisfy a second order parabolic equation (not even one with Lagrange multipliers), but a nonstandard integro-differential equation instead. Indeed, if $\theta$ evolves according to (2.0.4), we may integrate (1.5.3) to describe the evolution of $\gamma$. Denoting by $\nu = (-\sin\theta, \cos\theta)$ the usual normal vector field along $\gamma$, by integration by parts we find

$$\partial_t \gamma = \beta(\rho)(\partial_s^2 \gamma - c_0 \nu) + \int_0^s \partial_s \gamma\, \beta(\rho) k (k - c_0)\, \mathrm{d}r + \int_0^s \nu(\lambda_{\theta 1} \sin\theta - \lambda_{\theta 2} \cos\theta)\, \mathrm{d}r + v(t).$$

Here $v(t) \in \mathbb{R}^2$ attributes for $\partial_t \gamma(t, 0)$ and the boundary term arising at $s = 0$. Note that the second term on the right hand side still contains $k$, a term of order two, in a nonlocal way. While it is possible to express $\theta$ and thus the right hand side entirely in terms of $\gamma$ and its derivatives, the resulting evolution equation is rather complicated and of nonstandard structure.

In particular, in contrast to the curve shortening flow (see Section 1.4.3), we cannot rely on classical maximum principles to show the preservation of embeddedness. Instead, we use a recent energy-based argument in [121] to conclude preservation of embeddedness for explicitly small initial energy. Note that $\omega = \pm 1$ for any embedded curve by Hopf's Umlaufsatz.

**Proposition 2.6.13** ([36, Proposition 3.11])**.** *Let $C_{2T} \approx 146.628$ be as in [121, Theorem 1.1]. Let $(\theta, \rho)$ be the global solution of (2.0.4) with admissible initial datum $(\theta_0, \rho_0)$. Suppose that $\theta_0$ describes an embedded curve with rotation index $\omega = 1$ and $(\theta_0, \rho_0)$ satisfies*

$$\mathcal{E}_\mu(\theta_0, \rho_0) \leq \frac{\inf_{\mathbb{R}} \beta}{2} \Big( \frac{C_{2T}}{L} - 4\pi c_0 + L c_0^2 \Big). \tag{2.6.7}$$

*Then $\theta(t, \cdot)$ describes an embedded curve for all $t > 0$.*

**Remark 2.6.14** ([36, Remark 3.12])**.** If $\inf_{\mathbb{R}} \beta > 0$, then the above threshold is nontrivial, i.e. there exist an admissible initial datum $(\theta_0, \rho_0)$ satisfying (2.6.7). Indeed, let $\omega = 1$, $\varepsilon > 0$ and let $m \in \mathbb{R}$ such that $\beta(m) \leq \inf_{\mathbb{R}} \beta + \varepsilon$. Consider $\rho_0 \equiv m$ and let $\theta_0(s) = \frac{2\pi s}{L}$. Then, we have

$$\mathcal{E}_\mu(\theta_0, \rho_0) = \frac{\beta(m)}{2} \Big( \frac{4\pi^2}{L} - 4\pi c_0 + L c_0^2 \Big).$$

Using that $4\pi^2 < C_{2T}$, it follows that the assumptions of Proposition 2.6.13 are satisfied for $\varepsilon > 0$ small enough. Moreover, $4\pi^2 < C_{2T}$ also yields that the right hand side of (2.6.7) is positive if $\inf_{\mathbb{R}} \beta > 0$.

*Proof of Proposition 2.6.13.* Let $t > 0$. If $(\theta_0, \rho_0)$ is stationary, then there is nothing

to show. Thus, by (2.1.9), we may assume that the energy is strictly decreasing, i.e. $\mathcal{E}_\mu(\theta(t,\cdot),\rho(t,\cdot)) < \mathcal{E}_\mu(\theta_0,\rho_0)$. Moreover, (2.1.9) and the assumption yield that

$$\begin{aligned}\int_0^L k(t,s)^2\,\mathrm{d}s &\leq \frac{2}{\inf_{\mathbb{R}}\beta}\mathcal{E}_\mu(\theta(t,\cdot),\rho(t,\cdot)) + 4\pi c_0 - Lc_0^2 \\ &< \frac{2}{\inf_{\mathbb{R}}\beta}\mathcal{E}_\mu(\theta_0,\rho_0) + 4\pi c_0 - Lc_0^2 \leq \frac{C_{2T}}{L}.\end{aligned}$$

Now, $\theta(t,\cdot)$ describes a closed curve with rotation index one, and hence this curve must be embedded by [121, Theorem 1.4]. □

In Figure 2.7, we have seen a numerical example for the loss of convexity and the loss of embeddedness. In the following, we give a second example, for which the discrete energy $\hat{\mathcal{E}}_\mu(\hat{\theta}_0,\rho_0)$ is above the threshold given by Proposition 2.6.13, and for which embeddedness is lost along the flow.

**Example 2.6.15** (Numerical example, [36, Section 5.4.2])**.** We consider the choice of parameters $\beta : x \mapsto 0.03 + bx^2, b > 0$ and $L = 2\pi$, $\mu = 1$, $\nu = \frac{1}{\pi}$, $\omega = 1$, and $c_0 = 0$. The energy threshold in Proposition 2.6.13 is then

$$\varepsilon := \frac{\inf\beta}{2}\left(\frac{C_{2T}}{L} - 4\pi c_0 + Lc_0^2\right) \approx 0.35\,,$$

where we recall that it holds $C_{2T} \approx 146.628$. As initial datum, we pick a curve which can be described as consisting of two lateral drop-shaped lobes which are connected by a long, narrow neck. The initial datum $\hat{\rho}_0$ is chosen positive and distributed in the concave parts of the lobes. Heuristically, the concave parts of the lobes concentrate a large part of the energy, and are quickly "flattened" by the flow, making the two sides of the middle channel cross. We show numerically that this crossing does occur for $b = 8$. A representation of the corresponding curve and initial distribution $\hat{\rho}_0$ is shown in Figure 2.8A and 2.8B. For $b = 8$, we have $\hat{\mathcal{E}}_\mu(\hat{\theta}_0,\hat{\rho}_0) \approx 16 > \varepsilon$. Loss of embeddedness occurs at $t_1 \approx 4.1 \times 10^{-3}$ with $\hat{\mathcal{E}}_\mu(\hat{\theta}(t_1),\hat{\rho}(t_1)) \approx 10$. The curve becomes simple again at $t_2 \approx 2.91 \times 10^{-2}$ with $\hat{\mathcal{E}}_\mu(\hat{\theta}(t_2),\hat{\rho}(t_2)) \approx 4.9 > \varepsilon$. See Figure 2.8D. Note that at time $t_2$ the energy is still one order of magnitude larger than the threshold given by Proposition 2.6.13. Simplicity is then kept for $t > t_2$.

### 2.6.7. Symmetry

Since the evolution equations in (2.0.4) arise from a geometric energy functional, it is natural to ask if symmetries of the initial datum are preserved. The following section is inspired by [108]. We show that both $N$-fold rotational symmetry and axial symmetry of the initial datum are preserved along the evolution (see Propositions 2.6.20 and 2.6.25).

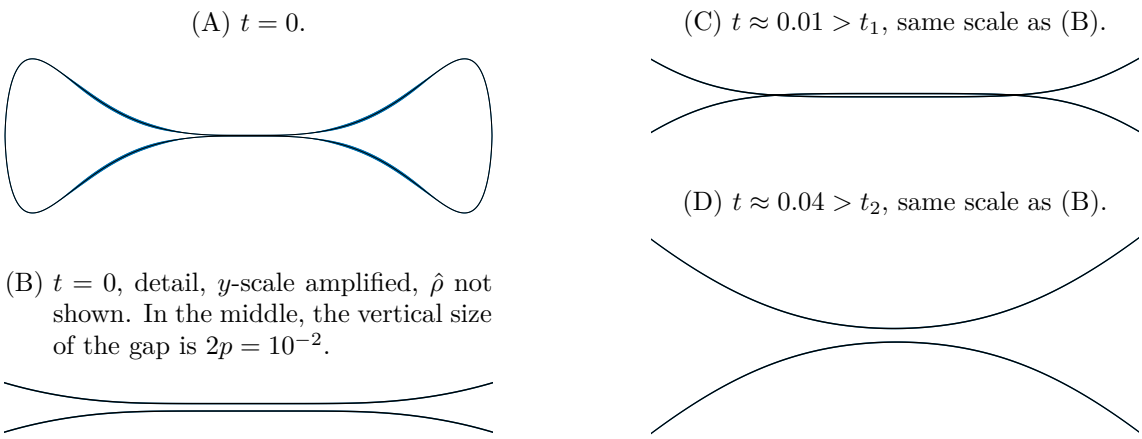



Figure 2.8: Snapshots showing how an embedded curve loses embeddedness. Here, $\beta(x) = 0.03 + 8x^2$, $L = 2\pi$, $\mu = 1$, $\nu = \frac{1}{\pi}$, $\omega = 1$, and $c_0 = 0$. From [36, Figure 7], reprinted with permission.

For simplicity, we restrict ourselves here to the case $\omega = 1$. First, let us examine rotational symmetry.

### Rotational symmetry

Without further comment, in this section, we frequently identify $(\theta - \phi, \rho)$ with its $L$-periodic extension to $\mathbb{R}$. Here, $\phi$ as in (2.2.2).

**Definition 2.6.16** ([36, Definition 3.13])**.** Let $\omega = 1$, $(\theta, \rho) \in C^\infty([0, L])$ and $N \in \mathbb{N}$ with $N \geq 2$. If $(\theta - \phi, \rho)$ is $\frac{L}{N}$-periodic, we call the heterogeneous curve described by $(\theta, \rho)$ *$N$-fold rotationally symmetric*.

**Remark 2.6.17.** The $\frac{L}{N}$-periodicity of $\theta - \phi$ is equivalent to demand that

$$\theta(s) = \theta\left(s - \tfrac{nL}{N}\right) + \tfrac{2\pi n}{N} \quad \text{for } s \in \left[\tfrac{nL}{N}, \tfrac{(n+1)L}{N}\right] \text{ and } n = 0, 1, \dots, N-1. \tag{2.6.8}$$

Indeed, (2.6.8) implies that

$$\theta(s) - \phi(s) = \theta\left(s - \tfrac{nL}{N}\right) + \tfrac{2\pi n}{N} - \tfrac{2\pi}{L}s = \theta\left(s - \tfrac{nL}{N}\right) - \phi\left(s - \tfrac{nL}{N}\right)$$

for $s \in \left[\frac{nL}{N}, \frac{(n+1)L}{N}\right]$ and $n = 0, 1, \dots, N-1$. Moreover, the $\frac{L}{N}$-periodicity of $(\theta - \phi, \rho)$ implies $\frac{L}{N}$-periodicity of $\partial_s\theta$ and thus

$$\begin{aligned}\theta(s) - \theta(0) &= \int_0^s \partial_s\theta \,\mathrm{d}r = \sum_{i=0}^{n-1} \int_{\frac{iL}{N}}^{\frac{(i+1)L}{N}} \partial_s\theta \,\mathrm{d}r + \int_{\frac{nL}{N}}^{s} \partial_s\theta \,\mathrm{d}r = n\int_0^{\frac{L}{N}} \partial_s\theta \,\mathrm{d}r + \int_0^{s-\frac{nL}{N}} \partial_s\theta \,\mathrm{d}r \\ &= n\Big(\theta(\tfrac{L}{N}) - \theta(0)\Big) + \theta(s - \tfrac{nL}{N}) - \theta(0)\end{aligned}$$

for $s \in [\frac{nL}{N}, \frac{(n+1)L}{N}]$. Combining this with

$$2\pi = \theta(L) - \theta(0) = \int_0^L \partial_s\theta \,\mathrm{d}r = \sum_{i=0}^{N-1} \int_{\frac{iL}{N}}^{\frac{(i+1)L}{N}} \partial_s\theta \,\mathrm{d}r = \sum_{i=0}^{N-1} \int_0^{\frac{L}{N}} \partial_s\theta \,\mathrm{d}r = N\big(\theta(\tfrac{L}{N}) - \theta(0)\big)$$

yields (2.6.8).

We also note the following immediate consequence of $\frac{L}{N}$-periodicity of $(\theta - \phi, \rho)$.

**Remark 2.6.18** ([36, Remark 3.14 (ii)])**.** If $\theta - \phi$ is $\frac{L}{N}$-periodic, then $\theta$ describes a closed curve. Indeed, using (2.6.8) we have

$$\begin{aligned}\int_0^L \sin\theta(s)\,\mathrm{d}s &= \sum_{n=0}^{N-1} \int_{\frac{nL}{N}}^{\frac{(n+1)L}{N}} \sin\theta(s)\,\mathrm{d}s = \sum_{n=0}^{N-1} \int_{\frac{nL}{N}}^{\frac{(n+1)L}{N}} \sin\big(\theta(s - \tfrac{nL}{N}) + \tfrac{2\pi n}{N}\big)\,\mathrm{d}s \\ &= \int_0^{\frac{L}{N}} \sum_{n=0}^{N-1} \sin\big(\theta(s) + \tfrac{2\pi n}{N}\big),\end{aligned}$$

which is zero for $N \geq 2$ since

$$\sum_{n=0}^{N-1} \sin\Big(\theta + \tfrac{2\pi n}{N}\Big) = \mathrm{Im}\Big(\exp(i\theta) \sum_{n=0}^{N-1} \exp(i\tfrac{2\pi n}{N})\Big) = \mathrm{Im}\Big(\exp(i\theta)\frac{1 - \exp(2\pi i)}{1 - \exp\left(\frac{2\pi i}{N}\right)}\Big) = 0.$$

Analogously, we obtain $\int_0^L \cos\theta(s)\,\mathrm{d}s = 0$.

Let us explain why we speak of rotational symmetry.

**Remark 2.6.19.** If $(\theta, \rho) \in C^\infty([0, L])$ describes an $N$-fold rotationally symmetric heterogeneous curve and we choose

$$\gamma(0) := \frac{1}{L}\int_0^L s \begin{pmatrix}\cos\theta(s)\\ \sin\theta(s)\end{pmatrix} \mathrm{d}s, \tag{2.6.9}$$

a computation shows that for $s \in [\frac{nL}{N}, \frac{(n+1)L}{N}]$ with $n = 0, 1, \ldots, N-1$, we have

$$\mathrm{Rot}(\tfrac{2\pi n}{N})\gamma(s - \tfrac{nL}{N}) := \begin{pmatrix}\cos\left(\frac{2\pi n}{N}\right) & -\sin\left(\frac{2\pi n}{N}\right)\\ \sin\left(\frac{2\pi n}{N}\right) & \cos\left(\frac{2\pi n}{N}\right)\end{pmatrix} \gamma\big(s - \tfrac{nL}{N}\big) = \gamma(s).$$

Indeed, with Remark 2.6.17 and using trigonometric identities, we have

$$\begin{aligned}\gamma(s) &= \int_0^{\frac{nL}{N}} \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \mathrm{d}r + \int_{\frac{nL}{N}}^{s} \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \mathrm{d}r + \gamma(0) \\ &= \gamma(\tfrac{nL}{N}) + \int_{\frac{nL}{N}}^{s} \begin{pmatrix}\cos\left(\theta(r - \frac{nL}{N}) + \frac{2\pi n}{N}\right)\\ \sin\left(\theta(r - \frac{nL}{N}) + \frac{2\pi n}{N}\right)\end{pmatrix} \mathrm{d}r = \gamma(\tfrac{nL}{N}) + \int_0^{s - \frac{nL}{N}} \begin{pmatrix}\cos\left(\theta + \frac{2\pi n}{N}\right)\\ \sin\left(\theta + \frac{2\pi n}{N}\right)\end{pmatrix} \mathrm{d}r\end{aligned}$$

$$= \gamma(\tfrac{nL}{N}) + R(\tfrac{2\pi n}{N}) \int_0^{s-\frac{L}{N}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mathrm{d}r = \gamma(\tfrac{nL}{N}) + R(\tfrac{2\pi n}{N})(\gamma(s-\tfrac{nL}{N}) - \gamma(0)).$$

Moreover, using (2.6.8) repeatedly and with the closedness of $\gamma$, we obtain

$$\begin{aligned}
R(\tfrac{2\pi n}{N})\gamma(0) &= \frac{1}{L}\int_0^L r \begin{pmatrix} \cos(\theta + \frac{2\pi n}{N}) \\ \sin(\theta + \frac{2\pi n}{N}) \end{pmatrix} \mathrm{d}r \\
&= \frac{1}{L}\left( \int_0^{\frac{(N-n)L}{N}} r \begin{pmatrix} \cos(\theta + \frac{2\pi}{N}) \\ \sin(\theta + \frac{2\pi}{N}) \end{pmatrix} \mathrm{d}r + \int_{\frac{(N-n)L}{N}}^L r \begin{pmatrix} \cos(\theta + \frac{2\pi}{N}) \\ \sin(\theta + \frac{2\pi}{N}) \end{pmatrix} \mathrm{d}r \right) \\
&= \frac{1}{L}\int_{\frac{nL}{N}}^L (r - \tfrac{nL}{N}) \begin{pmatrix} \cos(\theta(r - \frac{nL}{N}) + \frac{2\pi n}{N}) \\ \sin(\theta(r - \frac{nL}{N}) + \frac{2\pi n}{N}) \end{pmatrix} \mathrm{d}r \\
&\quad + \frac{1}{L}\int_0^{\frac{nL}{N}} (r - \tfrac{nL}{N} + L) \begin{pmatrix} \cos(\theta(r + L - \frac{nL}{N}) + \frac{2\pi n}{N}) \\ \sin(\theta(r + L - \frac{nL}{N}) + \frac{2\pi n}{N}) \end{pmatrix} \mathrm{d}r \\
&= \frac{1}{L}\int_{\frac{nL}{N}}^L (r - \tfrac{nL}{N}) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mathrm{d}r + \frac{1}{L}\int_0^{\frac{nL}{N}} (r - \tfrac{nL}{N} + L) \begin{pmatrix} \cos(\theta + 2\pi) \\ \sin(\theta + 2\pi) \end{pmatrix} \mathrm{d}r \\
&= \gamma(0) + \gamma(\tfrac{nL}{N}) - \gamma(0).
\end{aligned}$$

Note that $\gamma$ is chosen such that $\int_0^L \gamma(s)\,\mathrm{d}s = 0$. Since additionally $\rho(s - \frac{nL}{N}) = \rho(s)$, the graph of the heterogeneous curve described by $(\theta, \rho)$ indeed possesses an $N$-fold rotational symmetry.

For an example of an 3-fold rotationally symmetric heterogeneous curve, see Figure 2.9.

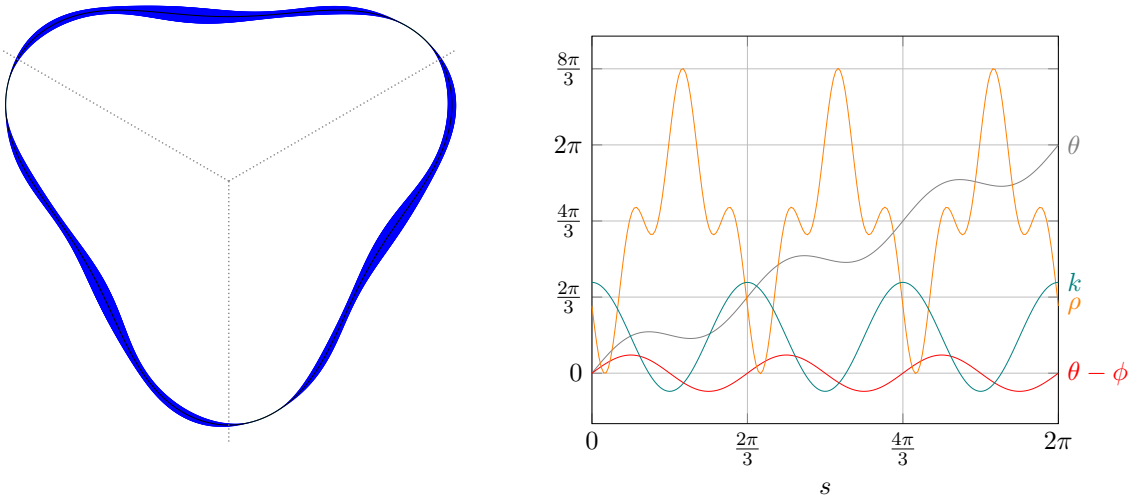



Figure 2.9: Example of a 3-fold rotationally symmetric configuration $(\theta, \rho)$.

We now show that the flow (2.0.4) preserves $N$-fold rotational symmetry for all $N \in \mathbb{N}$, $N \geq 2$.

**Proposition 2.6.20** ([36, Proposition 3.15])**.** *Let $\omega = 1$ and $N \geq 2$. Let $(\theta, \rho)$ be the solution of (2.0.4) with admissible initial datum $(\theta_0, \rho_0)$. If $(\theta_0, \rho_0)$ describes a $N$-fold rotationally symmetric heterogeneous curve, then so does $(\theta, \rho)$ for all $t \in (0, \infty)$.*

*Proof.* We consider the $L$-periodic extension of $(u, \rho) := (\theta - \phi, \rho)$ to $\mathbb{R}$. Notice that $(u, \rho) \in C^\infty((0,\infty)\times\mathbb{R})$. We define $(\tilde{u}, \tilde{\rho}) \in C^\infty((0,\infty)\times\mathbb{R})$ by $(\tilde{u}, \tilde{\rho})(t, s) := (u, \rho)(t, s - \frac{L}{N})$ and $(\tilde{\theta}, \tilde{\rho}) \in C^\infty((0,\infty)\times\mathbb{R})$ by $(\tilde{\theta}, \tilde{\rho}) := (\tilde{u} + \phi, \tilde{\rho})$. Notice, that $\tilde{\theta} \in C^\infty((0,\infty)\times\mathbb{R})$ is not periodic. Our intention now is to show that the restriction of $(\tilde{\theta}, \tilde{\rho})$ to $(0,\infty)\times[0, L]$ solves (2.0.4) with initial datum $(\theta_0, \rho_0)$. By uniqueness of the solution (see Lemma 2.3.14), it then follows that $(\tilde{\theta} - \phi, \tilde{\rho}) = (\theta - \phi, \rho)$ and thus

$$(\theta - \phi, \rho)(t, s) = (\tilde{u}, \tilde{\rho})(t, s) = (u, \rho)(t, s - \tfrac{L}{N}) = (\theta - \phi, \rho)(t, s - \tfrac{L}{N})$$

for $s \in \mathbb{R}$. Thus, the heterogeneous curve described by $(\theta, \rho)$ is $N$-fold rotationally symmetric.
It is clear that $(\tilde{\theta}, \tilde{\rho}) \in C^\infty((0,\infty)\times[0, L])$. With (2.2.2) and the $L$-periodicity of $u$, we have

$$\begin{aligned}
\tilde{\theta}(t, s) &= \tilde{u}(t, s) + \phi(s) = u(t, s - \tfrac{L}{N}) + \phi(s - \tfrac{L}{N}) + \phi(\tfrac{L}{N}) = \theta(t, s - \tfrac{L}{N}) + \tfrac{2\pi}{N} \\
&\qquad \text{for } (t, s) \in (0,\infty)\times(\tfrac{L}{N}, L], \\
\tilde{\theta}(t, s) &= \theta(t, s - \tfrac{L}{N} + L) + \tfrac{2\pi}{N} - 2\pi \quad \text{for } (t, s) \in (0,\infty)\times[0, \tfrac{L}{N}].
\end{aligned} \tag{2.6.10}$$

Using this and trigonometric identities, we see that for $t \geq 0$ and $s \in [0, L]$, $(\tilde{\theta}, \tilde{\rho})$ solves

$$\partial_t\tilde{\theta} = \partial_s(\beta(\tilde{\rho})(\partial_s\tilde{\theta} - c_0)) + \alpha_1 \sin\tilde{\theta} - \alpha_2 \cos\tilde{\theta} \tag{2.6.11}$$

with $\alpha_1 := \lambda_{\theta1}(\theta, \rho)\cos(\frac{2\pi}{N}) - \lambda_{\theta2}(\theta, \rho)\sin(\frac{2\pi}{N})$, $\alpha_2 := \lambda_{\theta1}(\theta, \rho)\sin(\frac{2\pi}{N}) + \lambda_{\theta2}(\theta, \rho)\cos(\frac{2\pi}{N})$. We use (2.6.10) and the fact that $\theta$ describes a closed curve to obtain

$$\begin{aligned}
\int_0^L \sin\tilde{\theta}\,\mathrm{d}s &= \int_0^{\frac{L}{N}} \sin\left(\theta(s - \tfrac{L}{N} + L) + \tfrac{2\pi}{N}\right)\mathrm{d}s + \int_{\frac{L}{N}}^L \sin\left(\theta(s - \tfrac{L}{N}) + \tfrac{2\pi}{N}\right)\mathrm{d}s \\
&= \int_0^L \sin\left(\theta + \tfrac{2\pi}{N}\right)\mathrm{d}s = \cos(\tfrac{2\pi}{N})\int_0^L \sin\theta\,\mathrm{d}s + \sin(\tfrac{2\pi}{N})\int_0^L \cos\theta\,\mathrm{d}s = 0
\end{aligned}$$

(cf. (2.0.3)) and $\int_0^L \cos\tilde{\theta}\,\mathrm{d}s = 0$ for all $t \geq 0$. In particular, this implies

$$\int_0^L \sin\tilde{\theta}\,\partial_t\tilde{\theta}\,\mathrm{d}s = \int_0^L \cos\tilde{\theta}\,\partial_t\tilde{\theta}\,\mathrm{d}s = 0. \tag{2.6.12}$$

Inserting (2.6.11) into (2.6.12) and comparing to (2.1.2), a short computation yields $\alpha_1 = \lambda_{\theta1}(\tilde{\theta}, \tilde{\rho})$ and $\alpha_2 = \lambda_{\theta2}(\tilde{\theta}, \tilde{\rho})$. Moreover, we check that $\lambda_\rho(\tilde{\theta}, \tilde{\rho}) = \lambda_\rho(\theta, \rho)$ and

$$\partial_t\tilde{\rho} = \mu\partial_s^2\tilde{\rho} - \frac{1}{2}\beta'(\tilde{\rho})(\partial_s\tilde{\theta} - c_0)^2 + \lambda_\rho(\tilde{\theta}, \tilde{\rho}).$$

Next, we notice that $(\tilde{\theta}, \tilde{\rho})$ satisfies the boundary conditions

$$\tilde{\theta}(t, L) - \tilde{\theta}(t, 0) = \tilde{u}(t, L) + \phi(L) - \tilde{u}(t, 0) = u(t, L - \tfrac{L}{N}) + 2\pi - u(t, -\tfrac{L}{N}) = 2\pi,$$

$\tilde{\rho}(t, L) = \tilde{\rho}(t, 0)$, and $\partial_s(\tilde{\theta}, \tilde{\rho})(t, L) = \partial_s(\tilde{\theta}, \tilde{\rho})(t, 0)$ for all $t > 0$. Since $(\theta_0 - \phi, \rho_0)$ describes a $N$-fold rotationally symmetric heterogeneous curve, we know with (2.6.10) and Remark 2.6.17 that

$$\lim_{t\to 0}(\tilde{\theta}, \tilde{\rho})(t, s) = \lim_{t\to 0}\left(\theta(t, s - \tfrac{L}{N}) + \tfrac{2\pi}{N}, \rho(t, s - \tfrac{L}{N})\right) = (\theta_0, \rho_0)(s)$$

(in $C^{2+\alpha}([0, L])$ for all $\alpha \in (0, 1)$) for $s \in [\frac{L}{N}, L]$ and analogously for $s \in [0, \frac{L}{N}]$. Hence, $(\tilde{\theta}, \tilde{\rho})$ solves (2.0.4) with initial datum $(\theta_0, \rho_0)$. □

In Figure 2.10, a numerical example for the preservation of 5-fold rotational symmetry is shown. Observe that while the initial angle function has 10-fold symmetry, the initial density exhibits only 5-fold symmetry.

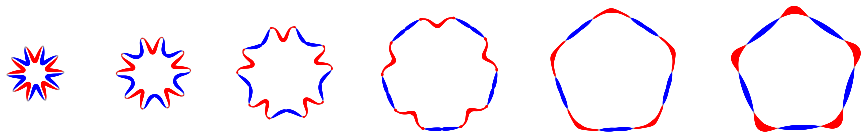


Figure 2.10: Preservation of 5-fold rotational symmetry. Time increases from left to right. Here, $\hat{\theta}_0$ is $L/10$-periodic, $\hat{\rho}_0$ is $L/5$-periodic, $\beta(x) = e^x$, $c_0 = 0$, $\mu = 10^{-3}$, $\nu = 0$, $\omega = 1$. Adapted from [36, Figure 9].

If we take the same parameters and the same 10-fold symmetric initial angle function as for Figure 2.10, but an 20-fold rotationally symmetric initial density, the 10-fold rotational symmetry is preserved, see Figure 2.11.

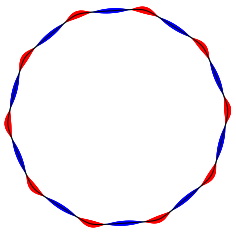


Figure 2.11: 10-fold rotational limit of the solution with parameters identical to those of Figure 2.10, except that $\hat{\rho}_0$ is $L/20$-periodic. From [36, Figure 10], reprinted with permission.

A distinctive feature of rotationally symmetric configurations is that the Lagrange multipliers $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$ vanish.

**Lemma 2.6.21** ([36, Lemma 3.16]). *Let $N \geq 2$ and let $(\theta, \rho) \in C^\infty([0, L])$ describe a $N$-fold rotationally symmetric heterogeneous curve. Then $\lambda_{\theta 1}(\theta, \rho) = \lambda_{\theta 2}(\theta, \rho) = 0$.*

*Proof.* With the $\frac{L}{N}$-periodicity of $\rho$ and with (2.6.8) we obtain that

$$\int_0^L \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \partial_s(\beta(\rho)(\partial_s\theta - c_0))\,\mathrm{d}s = \sum_{n=0}^{N-1} \int_0^{\frac{L}{N}} \begin{pmatrix} -\sin\left(\theta + \frac{2\pi n}{N}\right) \\ \cos\left(\theta + \frac{2\pi n}{N}\right) \end{pmatrix} \partial_s(\beta(\rho)(\partial_s\theta - c_0))\,\mathrm{d}s.$$

This sum is zero as we have seen in Remark 2.6.18. With (2.1.2), this yields that $\lambda_{\theta 1} = \lambda_{\theta 2} = 0$. □

The vanishing of $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$ gives the following extension of Lemma 2.2.13 in Section 2.2. If $\omega = 1$, $c_0 \neq \frac{2\pi}{L}$ and $(\theta, \rho)$ is a constrained critical point with constant curvature, which describes a $N$-fold rotationally symmetric configuration, then $\mathcal{L}^1(\{x : \beta'(x) = 0\}) = 0$ implies $\rho \equiv m$.
Furthermore, Lemma 2.6.21 together with Proposition 2.6.20 enables us to generalize Theorem 2.6.7 on the preservation of convexity for a $N$-fold rotationally symmetric initial datum as follows.

**Theorem 2.6.22** ([36, Theorem 1.4]). *Let $\omega = 1$, $N \geq 2$ and $c_0 \in \mathbb{R}$. Let $(\theta_0, \rho_0)$ be an admissible initial datum corresponding to a $N$-fold rotationally symmetric heterogeneous curve with $k_0 \geq c_0$ on $[0, L]$ and let $(\theta, \rho)$ be the solution of (2.0.4). Then $k \geq c_0$ on $[0, \infty) \times [0, L]$. Similarly, if $k_0 \leq c_0$ on $[0, L]$, then $k \leq c_0$ on $[0, \infty) \times [0, L]$.*

*Proof of Theorem 2.6.22.* With Proposition 2.6.20 we know that the solution $(\theta - \phi, \rho)$ remains $\frac{L}{N}$-periodic for all $t \geq 0$. Hence, Lemma 2.6.21 implies that $\lambda_{\theta 1}(t) = \lambda_{\theta 2}(t) = 0$ for all $t \geq 0$. This is why (2.6.1) can be written as

$$\partial_t(k - c_0) = \beta(\rho)\partial_s^2(k - c_0) + 2\partial_s(\beta(\rho))\partial_s(k - c_0) + \partial_s^2(\beta(\rho))(k - c_0). \tag{2.6.13}$$

Since (2.6.13) has the same structure as (2.6.2), we can proceed as in the proof of Theorem 2.6.7 to show the claim. □

In Example 2.6.9, we saw that convexity may not be preserved for $c_0 \neq 0$. With a small modification of the density, the example can be made rotationally symmetric, so that convexity ($k_0 \geq 0$) is still lost, while the bound $k_0 \geq c_0$ is preserved by Theorem 2.6.22.

### Axial symmetry

Besides rotational symmetry, axial symmetry is likewise preserved along the evolution. We characterize axial symmetry of heterogeneous curves (after a possible shift in the $s$-argument) as follows.

**Definition 2.6.23** ([36, Definition 3.17])**.** Let $(\theta, \rho) \in C^\infty([0, L])$. We call the heterogeneous curve described by $(\theta, \rho)$ *axially symmetric*, if

$$(\partial_s \theta, \rho)(s) = (\partial_s \theta, \rho)(L - s) \quad \text{for } s \in [0, L]. \tag{2.6.14}$$

**Remark 2.6.24** ([36, Remark 3.18])**.** Axial symmetry implies the following properties.

(i) Integrating (2.6.14) we obtain the equivalent condition

$$\theta(s) = \theta(L) + \theta(0) - \theta(L - s) \quad \text{for } s \in [0, L]. \tag{2.6.15}$$

(ii) Let $(\theta, \rho) \in C^\infty([0, L])$ satisfy (2.6.14) and $\gamma(0) \in \mathbb{R}^2$ be as in (2.6.9). A computation shows that

$$\mathrm{Ref}(\theta(0) - \tfrac{\pi}{2})\, \gamma(s) = \gamma(L - s),$$

where $\mathrm{Ref}(\alpha)$ represents the reflection about an axis through the origin with angle $\alpha$ to the $x$-axis. Since additionally $\rho(s) = \rho(L - s)$, (2.6.14) indeed implies that the corresponding heterogeneous curve is axially symmetric. For an example, see Figure 2.12.

(iii) A short computation, relying on (2.6.15) and the preservation of the integral of the inclination angle [37, Lemma 2.3], implies that for a solution $(\theta, \rho)$ describing an axially symmetric curve for all times $t \geq 0$, we have $\theta(t, 0) = \theta_0(0)$.

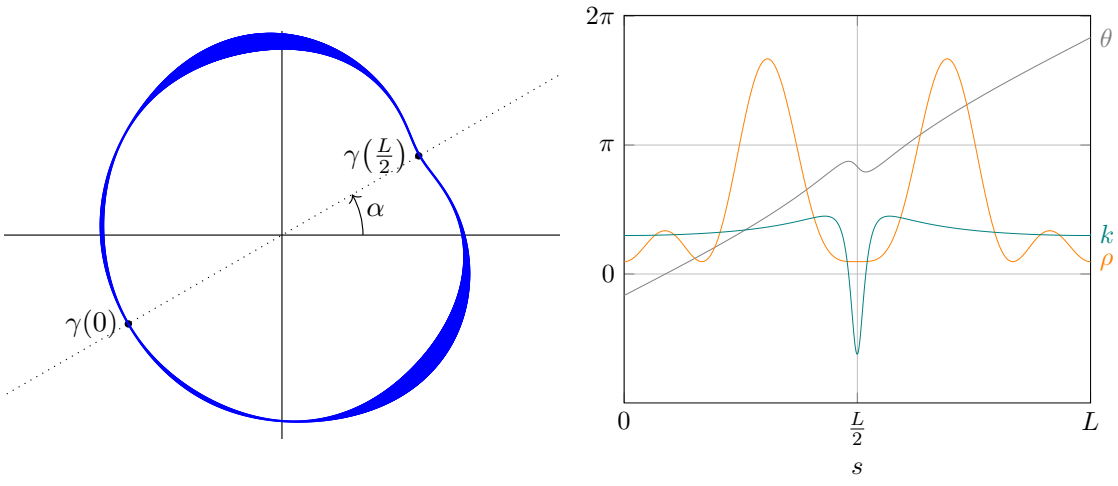



Figure 2.12: Example of an axially symmetric configuration $(\theta, \rho)$.

We show that if the initial datum describes an axially symmetric heterogeneous curve, then this also applies to the solution for all $t > 0$.

**Proposition 2.6.25** ([36, Proposition 3.19])**.** *Let $\omega = 1$ and $(\theta, \rho)$ be the solution of (2.0.4) with admissible initial datum $(\theta_0, \rho_0)$. If $(\theta_0, \rho_0)$ describes a heterogeneous curve which is axially symmetric, then so does $(\theta, \rho)$ for all $t \in (0, \infty)$.*

*Proof.* Let $(\theta, \rho)$ be the solution of (2.0.4) with initial datum $(\theta_0, \rho_0)$. We define

$$(\tilde{\theta}, \tilde{\rho})(t, s) := \big(\theta_0(L) + \theta_0(0) - \theta(t, L - s),\, \rho(t, L - s)\big)$$

for $(t, s) \in (0, \infty) \times [0, L]$ and show that $(\tilde{\theta}, \tilde{\rho})$ solves (2.0.4) with the same initial datum $(\theta_0, \rho_0)$. By uniqueness of the solution, it follows that $(\theta, \rho) = (\tilde{\theta}, \tilde{\rho})$ in $[0, \infty) \times [0, L]$. This yields $(\partial_s \theta, \rho)(t, s) = (\partial_s \theta, \rho)(t, L - s)$ in $[0, \infty) \times [0, L]$. Thus, $(\theta, \rho)$ describes an axially symmetric heterogeneous curve for all $t > 0$.
It is clear that $(\tilde{\theta}, \tilde{\rho}) \in C^\infty((0, \infty) \times [0, L])$. Since $\partial_s \tilde{\theta}(s) = (\partial_s \theta)(L - s)$ as well as $\partial_s^2 \tilde{\theta} = -(\partial_s^2 \theta)(L - s)$ and $\partial_s \tilde{\rho}(s) = -(\partial_s \rho)(L - s)$, we see that

$$\partial_t \tilde{\theta} = \beta(\tilde{\rho}) \partial_s^2 \tilde{\theta} + \beta'(\tilde{\rho}) \partial_s \tilde{\rho} (\partial_s \tilde{\theta} - c_0) + \alpha_1 \sin \tilde{\theta} - \alpha_2 \cos \tilde{\theta}.$$

Here, we have

$$\begin{aligned}
\alpha_1(t) &= \lambda_{\theta 1}(\theta, \rho) \cos\big(\theta_0(L) + \theta_0(0)\big) - \lambda_{\theta 2}(\theta, \rho) \sin\big(\theta_0(L) + \theta_0(0)\big) \text{ and} \\
\alpha_2(t) &= \lambda_{\theta 1}(\theta, \rho) \sin\big(\theta_0(L) + \theta_0(0)\big) + \lambda_{\theta 2}(\theta, \rho) \cos\big(\theta_0(L) + \theta_0(0)\big)
\end{aligned}$$

(by using trignometric identities). Similarly,

$$\int_0^L \sin \tilde{\theta} \, \mathrm{d}s = \sin\big(\theta_0(L) - \theta_0(0)\big) \int_0^L \cos \theta \, \mathrm{d}s - \cos\big(\theta_0(L) - \theta_0(0)\big) \int_0^L \sin \theta \, \mathrm{d}s = 0.$$

Analogously, we obtain $\int_0^L \cos \tilde{\theta} \, \mathrm{d}s = 0$. Thus, $\tilde{\theta}$ describes a closed curve. As in the proof of Proposition 2.6.20 one shows that this implies that $\alpha_1 = \lambda_{\theta 1}(\tilde{\theta}, \tilde{\rho})$ and $\alpha_2 = \lambda_{\theta 2}(\tilde{\theta}, \tilde{\rho})$. Moreover, $\lambda_\rho(\tilde{\theta}, \tilde{\rho}) = \lambda_\rho(\theta, \rho)$ and

$$\partial_t \tilde{\rho} = \mu \partial_s^2 \tilde{\rho} - \frac{1}{2} \beta'(\tilde{\rho}) (\partial_s \tilde{\theta} - c_0)^2 + \lambda_\rho(\tilde{\theta}, \tilde{\rho}).$$

It is readily checked that $(\tilde{\theta}, \tilde{\rho})$ satisfies the boundary conditions in (2.0.4). Using that $(\theta_0, \rho_0)$ describes an axially symmetric configuration, we further have

$$\lim_{t \to 0} (\tilde{\theta}, \tilde{\rho})(t, s) = \big(\theta_0(L) + \theta_0(0) - \theta_0(L - s), \rho_0(L - s)\big) = (\theta_0, \rho_0)(s)$$

(in $C^{2+\alpha}([0, L])$ for all $\alpha \in (0, 1)$). Hence, $(\tilde{\theta}, \tilde{\rho})$ is a solution of (2.0.4) with initial datum $(\theta_0, \rho_0)$ and the claim follows. □

**Remark 2.6.26** ([36, Remark 3.20])**.** We note that the proof of Proposition 2.6.25 shows that for an axially symmetric initial datum $(\theta_0, \rho_0)$, the solution $(\theta, \rho)$ keeps $\theta(t, 0) = \theta_0(0)$

for all $t \in [0, \infty)$.

For a numerical example for the preservation of axial symmetry, see Figure 2.4 or Figure 2.7 in the previous sections.

## 2.7. Asymptotic behavior

The convergence result in Theorem 2.5.1 naturally raises the question of a characterization of the limit, which we can answer despite the large number of selectable parameters under assumptions on $\beta$, $m$, and $\mu$.
Since stationary solutions of (2.0.4) are precisely the constrained critical points (compare (2.2.17)–(2.2.18) with (2.5.1)), we can already derive some properties of the limit $(\theta_\infty, \rho_\infty)$ in Theorem 2.5.1 by using the classification of constrained critical points in Sections 2.2.2 and 2.2.3.
In this section, we present several additional assumptions on the initial datum and the model parameters, under which the limit of (2.0.4) is a homogeneous elastica (see Definition 2.2.5).

### 2.7.1. Convergence to a homogeneous elastica under growth assumptions on the bending stiffness $\beta$

First, we focus on specific assumptions on the bending stiffness $\beta$, under which we can determine the asymptotic behavior.

**Theorem 2.7.1** (Growth assumptions on $\beta$, [36, Theorem 1.5])**.** *Suppose there exists $\bar{C} \geq 0$ such that*

$$\beta'(x)(m-x) \leq \bar{C}\beta(x)(m-x)^2 \quad \text{for all } x \in \mathbb{R}. \tag{2.7.1}$$

*Let $(\theta_0, \rho_0)$ be an admissible initial datum with $\bar{C}L\mathcal{E}_\mu(\theta_0, \rho_0) < \mu$. Then the density $\rho$ of the solution $(\theta, \rho)$ to (2.0.4) converges exponentially fast to $\rho_\infty \equiv m$ in $C^2([0, L])$ as $t \to \infty$. Moreover,*

*(i) if $\omega \neq 0$, then $\theta(t) \to \theta_\infty$ in $C^2([0, L])$, where $\theta_\infty$ describes a $\omega$-fold covered circle;*

*(ii) if $\omega = 0$ and $\beta$ is analytic, then $\theta(t) \to \theta_\infty$ in $C^2([0, L])$, where $\theta_\infty$ describes a multifold covered figure eight elastica.*

Assumption (2.7.1) is clearly satisfied if we have $\beta'(m) = 0$, $||\beta''||_\infty < \infty$, and $\inf_{\mathbb{R}} \beta > 0$. If $\beta$ is such that $\beta'(x)(m-x) \leq 0$ for $x \in \mathbb{R}$, we may choose $\bar{C} = 0$ in (2.7.1). In this case, there is no assumption on the initial energy. We highlight that, remarkably, analyticity is not needed for the case $\omega \neq 0$, see the discussion after the proof.

*Proof of Theorem 2.7.1.* With (2.0.4), (2.1.6), and integration by parts we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (\rho-m)^2\,\mathrm{d}s = 2\int_0^L \rho\,\partial_t\rho\,\mathrm{d}s = -2\mu\int_0^L (\partial_s\rho)^2\,\mathrm{d}s + \int_0^L (m-\rho)\beta'(\rho)(\partial_s\theta - c_0)^2\,\mathrm{d}s. \tag{2.7.2}$$

We use (2.7.1) to estimate the second term on the right hand side of (2.7.2) and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (\rho-m)^2\,\mathrm{d}s &\le -2\mu\int_0^L (\partial_s\rho)^2\,\mathrm{d}s + \bar{C}\int_0^L (m-\rho)^2\beta(\rho)(\partial_s\theta - c_0)^2\,\mathrm{d}s\\ &\le -2\mu\int_0^L (\partial_s\rho)^2\,\mathrm{d}s + \bar{C}\sup_{s\in[0,L]}(m-\rho)^2\int_0^L \beta(\rho)(\partial_s\theta - c_0)^2\,\mathrm{d}s\\ &\le -2\mu\int_0^L (\partial_s\rho)^2\,\mathrm{d}s + 2L\bar{C}\mathcal{E}_\mu(\theta_0,\rho_0)\int_0^L (\partial_s\rho)^2\,\mathrm{d}s,\end{aligned}$$

using (2.1.9), $\sup_{s\in[0,L]}|m-\rho| \le \int_0^L |\partial_s\rho|\,\mathrm{d}s$ and Cauchy–Schwarz. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (\rho-m)^2\,\mathrm{d}s \le -2\Big(\frac{2\pi}{L}\Big)^2(\mu - L\bar{C}\mathcal{E}_\mu(\theta_0,\rho_0))\int_0^L (\rho-m)^2\,\mathrm{d}s.$$

By Gronwall's inequality we conclude that

$$\int_0^L (\rho-m)^2\,\mathrm{d}s \le \Big(\int_0^L \rho_0^2\,\mathrm{d}s - m^2L\Big)\,\exp\Big(-\frac{8\pi^2}{L^2}(\mu - L\bar{C}\mathcal{E}_\mu(\theta_0,\rho_0))t\Big).$$

Since $\bar{C}L\mathcal{E}_\mu(\theta_0,\rho_0) < \mu$, it follows that $\rho \to m$ in $L^2(0,L)$ exponentially fast. By the subconvergence result in Section 2.5.2, there is a sequence $t_n \to \infty$ and $\theta_\infty \in C^\infty([0,L])$ such that $(\theta_\infty, \rho_\infty = m)$ is a solution of (2.5.1) and $\theta(t_n) \to \theta_\infty$ in $C^2([0,L])$. By Lemma 2.2.11, we find that $(\theta_\infty,\rho_\infty)$ describes a homogeneous elastica.
If $\omega \neq 0$, Lemma 2.2.6 yields that $\theta_\infty$ describes an $\omega$-fold covering of a circle, so that necessarily $\partial_s\theta_\infty = \frac{2\pi\omega}{L}$. From Lemma 2.4.3, we conclude that $\int_0^L \theta_\infty\,\mathrm{d}s = \int_0^L \theta_0\,\mathrm{d}s$, which implies $\theta_\infty(s) = \phi(s) + \frac{1}{L}\int_0^L \theta_0\,\mathrm{d}s - \pi\omega L$, $s\in[0,L]$. In particular, $\theta_\infty$ does not depend on the sequence $(t_n)_{n\in\mathbb{N}}$, and statement (i) follows from a subsequence argument.
In the case $\omega = 0$, the analyticity assumption on $\beta$ and Theorem 2.5.1 imply that $(\theta(t),\rho(t)) \to (\theta_\infty, \rho_\infty = m)$ in $C^2([0,L])$ as $t\to\infty$, where $(\theta_\infty,\rho_\infty)$ satisfies (2.5.1). Again, Lemma 2.2.11 implies that $\theta_\infty$ describes an elastica, so necessarily a multifold covered figure eight elastica by Lemma 2.2.6. Statement (ii) follows. □

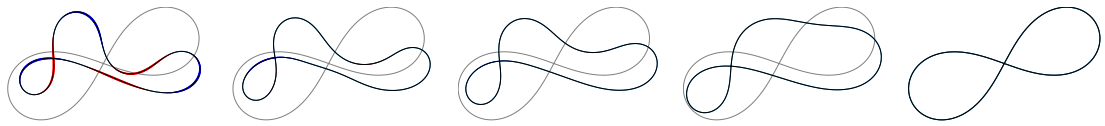
The necessity for stronger assumptions in the case $\omega = 0$ arises from the parametrization invariance of the energy, a general issue for geometric flows which occurs here despite working only with arclength parametrizations. Suppose that $(\theta,\rho)$ is a solution to (2.0.4) and $(\theta_\infty,\rho_\infty)$ is a solution to (2.5.1) originating from the subconvergence result in Section 2.5.2, i.e. $(\theta_\infty,\rho_\infty) = \lim_{n\to\infty}(\theta(t_n),\rho(t_n))$ for some sequence $t_n\to\infty$. With $\phi$ as in (2.2.2), we write $\theta_\infty = u_\infty + \phi$. Identifying $u_\infty$ and $\rho_\infty$ with their smooth $L$-periodic

extensions to $\mathbb{R}$, we find that for any $s_0 \in \mathbb{R}$, the pair

$$(\hat{\theta}_\infty, \hat{\rho}_\infty)(s) = (\phi(s) + u_\infty(s - s_0), \rho_\infty(s - s_0)), \quad s \in [0, L], \tag{2.7.3}$$

is also stationary for any $s_0 \in \mathbb{R}$. In fact, any other arclength parametrization of the corresponding curve leads to an angle function of this form. In particular, the set of possible limits (i.e. solutions to (2.5.1)) is nondiscrete, so that Łojasiewicz–Simon gradient inequalities are generically needed for deducing convergence from subconvergence. Hence, it is somehow surprising that this argument is not needed in case (i) of Theorem 2.7.1. The reason for this is that if $(\theta_\infty, \rho_\infty)$ describes a circle, then any reparametrization of the form (2.7.3) with $s_0 \neq 0$ results in adding a constant to the original angle function since $\theta_\infty$ is affine. Since by Lemma 2.4.3, $\int_0^L \theta_\infty \, \mathrm{d}s$ is determined by the initial datum, this degree of freedom is not present in the case $\omega \neq 0$, resulting in full convergence. On the curve level, adding a constant to $\theta$ corresponds to a rotation of the associated curve about a fixed angle, i.e. for a circle there is a one-to-one correspondence between arclength reparametrizations and rotations.
On the other hand, in the case $\omega = 0$, the classification of solutions to the elastica equation in [109, Proposition 3.3] allows us to determine all the parameters, except for the invariance due to (2.7.3), see also [124, Proposition B.8]. Hence the Łojasiewicz inequality (and consequently analyticity of $\beta$, cf. [135, Corollary 6.3]) is necessary to ensure convergence.

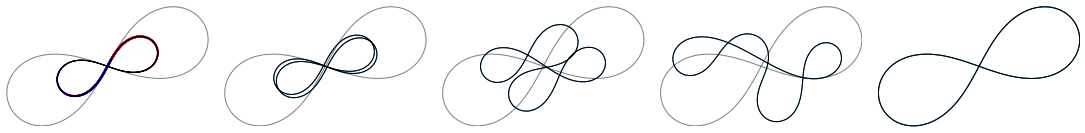
In Figure 2.13 and Figure 2.14 we present two numerical examples for Case (ii) of Theorem 2.7.1. In both situations we take $m = 0$ and $\beta(x) = 0.1 + x^2$, a choice which fits the assumptions of Theorem 2.7.1 with $\bar{C} = 0$, so that the limit is necessarily a (potentially multiple) covering of the figure eight elastica.



$t = 0$ $\quad$ $t \approx 0.5$ $\quad$ $t \approx 1$ $\quad$ $t \approx 2$ $\quad$ $t \approx 135$

Figure 2.13: Convergence to the homogeneous figure eight elastica. Here, $L = 2\pi$, $\mu = 10^{-1}$, $c_0 = 2$ and $\beta(x) = 0.1 + x^2$, $m = 0$. From [36, Figure 12], reprinted with permission.

We observe convergence to the 1-fold covering of the homogeneous figure eight in both situations. In the second case, this suggests that the multiple coverings of the figure eight are not stable under the flow, at least for this quadratic choice of $\beta$.



$t = 0$ $t \approx 36$ $t \approx 60$ $t \approx 70$ $t \approx 100$

Figure 2.14: Evolution of the perturbation of a 2-fold covering of the figure eight, where the second is rotated by an angle of $2\pi/1000$ with respect to the first one. Here, $L = 2\pi$, $\mu = 10^{-1}$, $c_0 = 0$ and $\beta(x) = 0.1 + x^2$, $m = 0$. From [36, Figure 13], reprinted with permission.

### 2.7.2. Convergence to a homogeneous elastica for special $c_0$

For $\omega = 1$, we have dealt with rotational symmetry of solutions. In this case we can prove exponential convergence of the curvature to a constant if the length allows for a circle with constant curvature $k \equiv c_0$.

**Proposition 2.7.2** ([36, Proposition 4.1])**.** *Let $\beta$ be real analytic, $\omega = 1$, $c_0 = \frac{2\pi}{L}$, and let $(\theta_0, \rho_0) \in C^\infty([0, L])$ be an admissible initial datum describing an $N$-fold rotationally symmetric heterogeneous curve for some $N \geq 2$. Then, as $t \to \infty$, the solution $(\theta, \rho)$ to (2.0.4) converges exponentially fast to $(\theta_\infty, \rho_\infty)$ with $\partial_s \theta_\infty \equiv c_0$, $\rho_\infty \equiv m$. In particular, the limit describes a circle with constant density.*

*Proof.* Let $(\theta, \rho)$ be the solution to (2.0.4) and recall $\mathcal{E}^\theta(t) = \frac{1}{2}\int_0^L \beta(\rho)(k - c_0)^2 \,\mathrm{d}s$ with $k = \partial_s \theta$. Due to Proposition 2.6.20 and Lemma 2.6.21, $\lambda_{\theta 1}(t) = \lambda_{\theta 2}(t) = 0$ for all $t \geq 0$. Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\theta &= \frac{1}{2}\int_0^L \beta'(\rho)\partial_t\rho(k - c_0)^2 \,\mathrm{d}s + \int_0^L \beta(\rho)(k - c_0)\partial_t k \,\mathrm{d}s \\ &\leq \frac{\sup_{(t,s)} |\beta'(\rho)|}{2\inf_{(t,s)}(\beta(\rho))^2}\|\partial_t\rho\|_{L^\infty(0,L)}\int_0^L (\beta(\rho))^2(k - c_0)^2 \,\mathrm{d}s - \int_0^L \left(\partial_s(\beta(\rho)(k - c_0))\right)^2 \mathrm{d}s\end{aligned} \tag{2.7.4}$$

Note that $\sup_{(t,s)\in[0,\infty)\times[0,L]} |\beta'(\rho)| < \infty$ and $\inf_{(t,s)\in[0,\infty)\times[0,L]} \beta(\rho) > 0$ since by convergence of the flow, $\rho(t, s)$ lies in a compact set for all $(t, s)$. By the assumptions on $\omega$ and $c_0$, the function $k(t) - c_0$ has a zero in $[0, L]$ for all $t \geq 0$. Therefore, (2.7.4) and Wirtinger's inequality imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\theta \leq \left(C\|\partial_t\rho\|_{L^\infty(0,L)} - \frac{4\pi^2}{L^2}\right)\int_0^L (\beta(\rho))^2(k - c_0)^2 \,\mathrm{d}s,$$

where $C = C(\beta, \theta, \rho) \in (0, \infty)$ is a constant independent of $t \geq 0$. Now, $(\theta(t), \rho(t)) \to (\theta_\infty, \rho_\infty)$ in $C^2([0, L])$ and (2.0.4) imply that $\|\partial_t\rho\|_{L^\infty(0,L)} \to 0$ as $t \to \infty$. Consequently,

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\theta(t) \leq -\frac{4\pi^2 \inf_{(t,s)} \beta(\rho)}{2L^2}\mathcal{E}^\theta(t),$$

for $t \geq T$ large enough, whence Gronwall's lemma yields $\mathcal{E}^\theta(t) \leq Ce^{-\alpha t}$ for some appropriate $C, \alpha > 0$. It follows that $k = \partial_s\theta \to c_0$ in $L^2(0,L)$ for $t \to \infty$ exponentially fast. For the exponential convergence of $\rho$, we use (2.7.2) to conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (\rho - m)^2 \,\mathrm{d}s \leq -2\mu\frac{4\pi^2}{L^2}\int_0^L (\rho - m)^2 \,\mathrm{d}s + \sup_{(t,s)} |m - \rho| \frac{\sup_{(t,s)} |\beta'(\rho)|}{\inf_{(t,s)} \beta(\rho)}\mathcal{E}^\theta(t).$$

Using $\mathcal{E}^\theta(t) \leq Ce^{-\alpha t}$, the exponential convergence $\rho \to m$ in $L^2(0,L)$ follows with a Gronwall argument. Since $\int_0^L \theta \,\mathrm{d}s$ is preserved by Lemma 2.4.3, we have $\theta \to \phi + \frac{1}{L}\int_0^L \theta_0 \,\mathrm{d}s - \pi$ as $t \to \infty$ exponentially fast by the Poincaré–Wirtinger inequality. Theorem 2.5.1 and an interpolation argument imply that $(\theta, \rho) \to (\phi + \frac{1}{L}\int_0^L \theta_0 \,\mathrm{d}s - \pi, m)$ exponentially fast in $C^{2+\tilde{\alpha}}([0,L])$ for all $\tilde{\alpha} \in (0, \frac{1}{2})$. □

Proposition 2.7.2 implies that if $\omega = 1$ and $c_0 = \frac{2\pi}{L}$, there exists no nontrivial constrained critical point which is $N$-fold rotationally symmetric. Moreover, in this setting, Proposition 2.7.2 implies that for $t$ large enough, $\mathcal{E}^\theta(\theta, \rho)$ is eventually monotonically decreasing (compare to Section 2.6.2).

### 2.7.3. Convergence to a homogeneous elastica for large $\mu$

Furthermore, we can determine the limit for large values of $\mu$ and constant initial density. In Proposition 2.2.8, we have seen that for $\omega \neq 0$ and large $\mu$, the $\omega$-fold covering of the circle with constant density is the unique global minimizer. Theorem 2.7.3 present a time-dependent version of this result if $\rho_0 \equiv m$.

**Theorem 2.7.3** (Asymptotic behavior for large $\mu$, [36, Theorem 1.6])**.** *Let $\omega \neq 0$ and suppose that $\beta$ is real analytic. Let $(\theta_0, \rho_0)$ be an admissible initial datum with $\rho_0 \equiv m$. There exists $\mu_0 \in (0, \infty)$ such that if $\mu \geq \mu_0$, then the limit $(\theta_\infty, \rho_\infty)$ of Theorem 2.5.1 describes an $\omega$-fold covered circle with constant density.*

A major difficulty here is that the two parts of the energy do not decrease individually. Note that this also means that a constant initial density does not necessarily remain constant, unless $\beta'(m) = 0$ or $\partial_s\theta_0 \equiv c_0$. Example 2.2.15 shows that the assumption of constant initial density is necessary.

*Proof of Theorem 2.7.3.* We assume that $\int_0^L \theta_0(s) \,\mathrm{d}s = \pi\omega L$. This is no loss of generality because if $(\theta, \rho)$ is the solution to (2.0.4) with initial datum $(\theta_0, \rho_0) \in C^\infty([0,L])$, then by a direct computation using trigonometric identities, it is readily checked that $(\theta + r, \rho)$ is the solution to (2.0.4) with initial datum $(\theta_0 + r, \rho_0)$ for $r \in \mathbb{R}$.

Consider $(\mu_j)_{j\in\mathbb{N}}$ such that $\mu_j \to \infty$ for $j \to \infty$. By assumption, $\mathcal{E}_{\mu_j}(\theta_0, \rho_0) =: K$ is independent of $j \in \mathbb{N}$. For any $\mu_j$, there exists a unique global solution $(\theta_j, \rho_j)$ with initial datum $(\theta_0, \rho_0)$ and this solution converges to some $(\theta_{\infty,j}, \rho_{\infty,j}) \in C^\infty([0, L])$ in $C^2([0, L])$ for $t \to \infty$ (see Theorem 2.5.1 and Proposition 2.2.4). Since the integral of the angle is preserved by Lemma 2.4.3, we have

$$\int_0^L \theta_{\infty,j}(s)\,\mathrm{d}s = \int_0^L \theta_j(t,s)\,\mathrm{d}s = \int_0^L \theta_0(s)\,\mathrm{d}s = \pi\omega L \quad \text{for all } t \in (0,\infty),\ j \in \mathbb{N}. \tag{2.7.5}$$

Thus, we want to show that

$$(\theta_{\infty,j}, \rho_{\infty,j}) \to (\theta_c, \rho_c) \quad \text{in } C^1([0, L]) \quad \text{for } j \to \infty, \tag{2.7.6}$$

cf. Remark 2.2.7. Corollary 2.2.10 and (2.7.5) then allow us to conclude that for $j$ large enough, $(\theta_{\infty,j}, \rho_{\infty,j}) = (\theta_c, \rho_c)$ and the statement follows.

*Step 1: Uniform boundedness of* $\|(\theta_{\infty,j}, \rho_{\infty,j})\|_{W^{1,2}(0,L)}$, $\lambda_{\theta 1}(\theta_{\infty,j}, \rho_{\infty,j})$, $\lambda_{\theta 2}(\theta_{\infty,j}, \rho_{\infty,j})$. For this, we first observe that

$$\int_0^L (\partial_s \rho_{\infty,j})^2\,\mathrm{d}s \le \frac{2}{\mu_j} K \to 0, \quad j \to \infty. \tag{2.7.7}$$

Since the integral of the density is fixed (see (1.5.5)), this yields $\rho_{\infty,j} \to m$ in $W^{1,2}(0, L)$ and in particular uniform boundedness of $\|\rho_{\infty,j}\|_{W^{1,2}(0,L)}$ and $\|\rho_{\infty,j}\|_{C([0,L])}$. Thus, there is $M \in \mathbb{R}$ (not depending on $j$) such that

$$\int_0^L (\partial_s \theta_{\infty,j})^2\,\mathrm{d}s \le \frac{K}{\inf_{[-M,M]} \beta} + 4\pi c_0 \omega. \tag{2.7.8}$$

With (2.7.5), we conclude that also $\|\theta_{\infty,j}\|_{W^{1,2}(0,L)}$ is uniformly bounded. By Lemma 2.4.2 and (2.7.8), the matrix $\Pi^{-1}(\theta_{\infty,j})$ is bounded uniformly in $j$. Hence the bounds on $\|\theta_{\infty,j}\|_{W^{1,2}(0,L)}$ and $\|\rho_{\infty,j}\|_{C([0,L])}$ imply that

$$\begin{pmatrix} \lambda_{\theta 1} \\ \lambda_{\theta 2} \end{pmatrix} (\theta_{\infty,j}, \rho_{\infty,j}) = \Pi^{-1}(\theta_{\infty,j}) \int_0^L \begin{pmatrix} \cos\theta_{\infty,j} \\ \sin\theta_{\infty,j} \end{pmatrix} \partial_s \theta_{\infty,j} \beta(\rho_{\infty,j})(\partial_s \theta_{\infty,j} - c_0)\,\mathrm{d}s$$

is bounded uniformly in $j \in \mathbb{N}$.

*Step 2: Uniform boundedness of* $\|(\theta_{\infty,j}, \rho_{\infty,j})\|_{W^{2,2}(0,L)}$. To show boundedness of the $L^2$-norm of the second derivatives, we use that for all $j \in \mathbb{N}$, $(\theta_{\infty,j}, \rho_{\infty,j})$ is a stationary solution, i.e. a solution of (2.5.1). This allows to use similar arguments as in the proof of Proposition 2.2.4. First, we observe that

$$\left\|\partial_s(\beta(\rho_{\infty,j})(\partial_s \theta_{\infty,j} - c_0))\right\|_{L^2(0,L)} = \left\|\lambda_{\theta 1} \sin\theta_{\infty,j} - \lambda_{\theta 2} \cos\theta_{\infty,j}\right\|_{L^2(0,L)}.$$

From Step 1, it follows that $\beta(\rho_{\infty,j})(\partial_s\theta_{\infty,j} - c_0)$ is bounded in $W^{1,2}(0,L)$ uniformly in $j \in \mathbb{N}$ and hence also in $C([0,L])$. By (2.1.6), this implies boundedness of $\lambda_\rho(\theta_{\infty,j}, \rho_{\infty,j})$ and with that, (2.5.1) implies that $\partial_s^2\rho_{\infty,j}$ is uniformly bounded in $L^2(0,L)$. Now, we know that $\rho_{\infty,j}$ is uniformly bounded in $C^1([0,L])$ and since $\beta(\rho_{\infty,j}) \geq \inf_{[-M,M]}\beta$, this implies that $\partial_s^2\theta_{\infty,j}$ is uniformly bounded in $L^2(0,L)$. It follows that $(\theta_{\infty,j}, \rho_{\infty,j})$ is bounded in $W^{2,2}(0,L)$, independently in $j \in \mathbb{N}$.
*Step 3:* $(\theta_{\infty,j}, \rho_{\infty,j}) \to (\theta_c, \rho_c)$ *in* $C^1([0,L])$. Due to Step 2 and (2.7.7), there exists a (not relabeled) subsequence such that $(\theta_{\infty,j}, \rho_{\infty,j}) \rightharpoonup (\theta_\infty, m)$ in $W^{2,2}(0,L)$ and $(\theta_{\infty,j}, \rho_{\infty,j}) \to (\theta_\infty, m)$ in $C^1([0,L])$. The limit $(\theta_\infty, m)$ satisfies the Euler–Lagrage equations (2.2.11) and (2.2.12). Hence, $(\theta_\infty, m)$ is a constrained critical point. With Proposition 2.2.4 it follows that $\theta_\infty \in C^\infty([0,L])$. Further, Lemma 2.2.11 implies that $\theta_\infty$ describes an elastica. More precisely, (2.7.5) together with Remark 2.2.7 yields $\theta_\infty = \frac{2\pi\omega}{L}s = \theta_c$. Finally, a standard subsequence argument yields (2.7.6). □

To conclude the sections on the classification of the limit, we present another numerical example by Gaspard Jankowiak that demonstrates the influence of the model parameters $\mu$ and $\beta$ on the evolution for a given initial data with $\omega = 2$.

**Example 2.7.4** (Influence of the model parameters for $\omega = 2$, [36, Section 5.4.4])**.** In Figure 2.15, starting from the same initial datum (in the center), we illustrate how the shapes assumed by the limit $(\hat{\theta}_\infty, \hat{\rho}_\infty)$ change with the model parameters $\mu$ and $\beta$. The spontaneous curvature $c_0$ is taken to be zero, and $\beta$ is of the form $\beta(x) = e^{ax}$.
For relatively small values of $\mu$ and large values of $a$, the solution is cigar- (or stadium-) shaped, with a small additional loop at one end, which accounts for $\omega = 2$, see Figure 2.15A and 2.15D. In those cases, $\hat{\mathcal{E}}^\theta$ can be made small for a curve with flat sections corresponding to large values of $\beta(\hat{\rho})$ (and highly curved sections corresponding to small values) without making $\hat{\mathcal{E}}_\mu^\rho$ large thanks to the small value of $\mu$.
As $\mu$ increases, large changes of $\hat{\rho}$ are penalized and the geometric part dominates, so that the curve becomes rounder as a result, cf. Figure 2.15B.
Increasing $\mu$ further, in view of Theorem 2.7.3, it seems plausible that the solution should converge to $(\hat{\theta}_c, \hat{\rho}_c)$, although here the density $\hat{\rho}_0$ is not constant. This is what can be observed in Figure 2.15C.
For what concerns $a$, i.e. $\beta'$, as it gets smaller (with $\mu$ kept small), the gain in $\hat{\mathcal{E}}^\theta$ coming from a given oscillation of density distribution diminishes, so the oscillation increases, see Figure 2.15E. Eventually, as $a$ becomes very small, this is balanced by the increase in $\hat{\mathcal{E}}_\mu^\rho$, and the solution converges to the trivial state $(\hat{\theta}_c, \hat{\rho}_c)$, see Figure 2.15F.

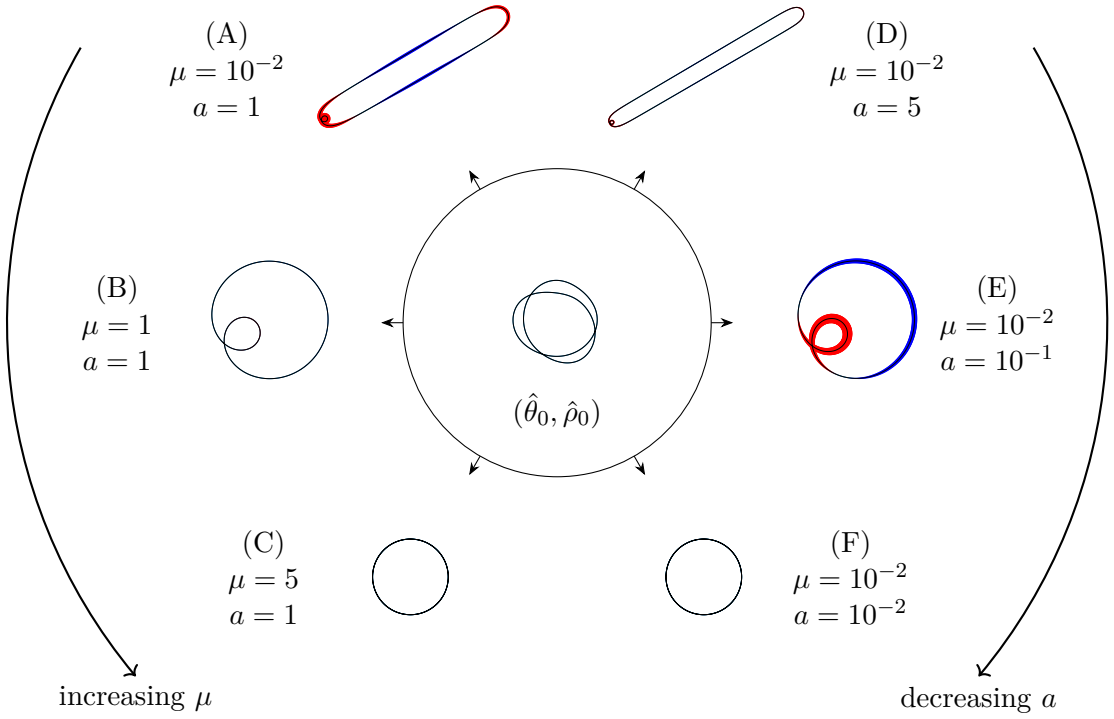



Figure 2.15: Dependency of $(\hat{\theta}_\infty, \hat{\rho}_\infty)$ on $\mu$ and $\beta$. Here, $\omega = 2$, $c_0 = 0$ and $\beta(x) = e^{ax}$. Left: $(\hat{\theta}_\infty, \hat{\rho}_\infty)$ as $\mu$ increases in $\{10^{-2}, 1, 5\}$ with $a = 1$. Right: $(\hat{\theta}_\infty, \hat{\rho}_\infty)$ as $a$ decreases in $\{5, 10^{-1}, 10^{-2}\}$ with $\mu = 10^{-2}$. Center: Initial datum. From [36, Figure 11], reprinted with permission.

CHAPTER 3

# Area preserving elastic flow

In Section 1.6, we motivate the study of a local gradient flow of the elastic energy that preserves the enclosed area of the evolving curves. Such a flow, on a modeling level, describes the evolution of elastic wires under the assumption of incompressibility. Inspired by the curve diffusion flow that can be viewed as an $H^{-1}$-gradient flow, we intend to apply the same idea to the elastic energy. Our goal is to derive an evolution equation that preserves the enclosed area, without introducing nonlocal Lagrange multipliers. This approach yields a sixth order evolution equation, which we aim to study in this chapter.

The elastic wire that we want to evolve is modeled by a regular closed planar curve. This means we consider $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$. Recall that the (length penalized) elastic energy of the elastic wire is given by

$$\mathcal{E}_\lambda(\gamma) = \mathcal{E}(\gamma) + \lambda\mathcal{L}(\gamma) = \frac{1}{2}\int_{\mathbb{S}^1} \left(|\kappa|^2 + 2\lambda\right) \mathrm{d}s \tag{3.0.1}$$

for $\lambda > 0$ or $\lambda = 0$. Here, as introduced in Section 1.2.1, $\kappa = \partial_s^2\gamma$ (with the arclength derivative $\partial_s = |\partial_x\gamma|^{-1}\partial_x$) is the curvature vector and $\mathrm{d}s = |\partial_x\gamma|\,\mathrm{d}x$ denotes the arclength measure of $\gamma$. The most straightforward dynamic approach to study (3.0.1), is to consider its $L^2$-gradient flow. As derived in Section 1.4.1, this is the classical elastic flow given by

$$\partial_t^\perp\gamma = -\nabla_s^2\kappa - \frac{1}{2}|\kappa|^2\kappa + \lambda\kappa \tag{3.0.2}$$

with $\partial_t^\perp = P^\perp\partial_t$, where $P^\perp$ denotes the normal projection along $\gamma$. If $\lambda = 0$, (3.0.2) is the free elastic flow (FEL) and if $\lambda > 0$, (3.0.2) is the length penalized elastic flow (PEF), see Section 1.4.1. In Section 1.4.2, we state that given a suitable initial datum, we have global in time existence of the geometrically unique solution to (3.0.2) and if $\lambda > 0$, the solution converges to an elastica.

In this chapter, we consider the gradient with respect to another Hilbert space than $L^2(\mathrm{d}s)$. This results in a gradient flow that follows the steepest descent with respect to the norm of this other Hilbert space. The properties of a solution generally differ from those of the $L^2$-gradient flow. As mentioned above, we choose a type of $H^{-1}$-space as the Hilbert space in which to compute the gradient. This choice of Hilbert space ensures that the area enclosed by the evolving curves remains constant along the flow. At this point, we intentionally remain imprecise and refer to the exact definition in Section 3.1.1.

The $H^{-1}$-gradient approach results in two additional derivatives of the velocity in (3.0.2). More precisely, we obtain

$$\begin{cases} \partial_t^\perp \gamma = \nabla_s^2(\nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa - \lambda\kappa) & \text{in } (0,T)\times\mathbb{S}^1, \\ \gamma(0,\cdot) = \gamma_0 & \text{on } \mathbb{S}^1. \end{cases} \tag{3.0.3}$$

The evolution equation is now of sixth order. In (3.0.3), $\gamma_0\colon \mathbb{S}^1 \to \mathbb{R}^2$ is a smooth regular initial datum with rotation index $\omega_0 \in \mathbb{Z}$, see (1.2.7), and enclosed area $\mathcal{A}(\gamma_0) = A_0 \in \mathbb{R}$, see (1.2.6). The enclosed area $A_0 \in \mathbb{R}$ of the initial datum is considered arbitrary, including the case where $A_0 = 0$. We also consider arbitrary rotation index $\omega_0 \in \mathbb{Z}$, in particular allowing the case $\omega = 0$. The curves in Figure 1.3 are possible examples for an initial datum.

The curve shortening flow, as the $L^2$-gradient flow of length functional, and the curve diffusion flow, as the $H^{-1}$-gradient flow of it, behave similarly in the sense that both flows can develop singularities in finite time, see Section 1.4.3. How does the behavior of the classical elastic flow (3.0.1) compare to that of the area preserving sixth order variant? It will turn out that the area preserving flow behaves just as well as the classical flow, the behavior of solutions does not deteriorate. We have global in time existence of solutions and, if we add a term penalizing the length, convergence. We summarize the main results of Chapter 3 in the following theorem, which is to be compared with Proposition 1.4.3 (on global existence) and Proposition 1.4.4 (on convergence) for the classical elastic flow in Section 1.4.2.

**Theorem 3.0.1** (Global existence and convergence, [99, Theorem 1.1])**.** *For any $\lambda \geq 0$, there exists a geometrically unique global smooth solution $\gamma\colon [0,\infty)\times\mathbb{S}^1 \to \mathbb{R}^2$ of (3.0.3). If $\lambda > 0$, there exists a family of smooth diffeomorphisms $\Phi(t)\colon \mathbb{S}^1 \to \mathbb{S}^1$, $t \in (0,\infty)$, such that $\gamma(t,\Phi(t,\cdot))$ converges smoothly for $t \to \infty$ to a stationary solution $\gamma_\infty$.*

The theorem refers to geometric uniqueness, which means uniqueness up to reparametrization — more cannot be expected since we only prescribe the normal component of the velocity.

It turns out to be interesting to have a closer look at the stationary solutions. Let us first clarify that by a stationary solution, we mean a smooth regular curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ with rotation index $\omega_0$ and $\mathcal{A}(\gamma) = A_0$ (like the initial datum) such that

$$\nabla_s^2(\nabla_s^2\kappa + \tfrac{1}{2}|\kappa|^2\kappa - \lambda\kappa) = 0. \tag{3.0.4}$$

Since $\gamma$ describes a closed curve, the constant obtained when integrating the equation once must be zero. Thus, (3.0.4) is equivalent to

$$\nabla_s(\nabla_s^2\kappa + \tfrac{1}{2}|\kappa|^2\kappa - \lambda\kappa) = 0. \tag{3.0.5}$$

Integrating this equation once more, we obtain the classical elastica equation (1.2.10) with a possibly nonzero constant multiple of the normal vector on the right hand side. It is immediately clear that for nonzero rotation index, multifold coverings of a circle satisfy (3.0.5). Compared to the classical elastic equation and Example 1.2.3, the radius no longer matters. More precisely, regardless of the scaling of the multifold covered circle, (3.0.5) is satisfied. For $\omega_0 = 0$, we obtain the multiple covered figure eight elastica as a solution of (3.0.5). The presence of the nonzero term on the right hand side of the elastica equation, however, implies that there are stationary solutions which do not correspond to a multiple covering of a circle or of the figure eight elastica. This follows solely from proving convergence of the solution and taking an initial curve with rotation index and enclosed area that cannot correspond to an elastica, for example $\omega_0 \neq 0$ and $A_0 = 0$ or $\omega_0 = 0$ and $A_0 \neq 0$.

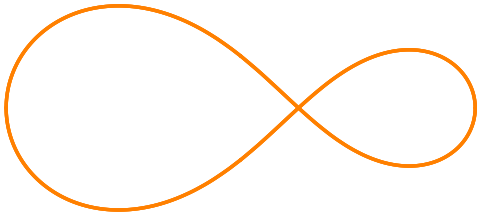


Figure 3.1: Possible choice as an initial datum with $\omega_0 = 0$ and $A_0 \neq 0$ and stationary solution for $\lambda \approx 17.661$. Threefold covered, length is $2\pi$.

Before we begin to study the area preserving elastic flow (3.0.3), we want to point out that in [4], an evolution equation similar to (3.0.3) is studied. The authors consider the $L^2$-gradient flow of the Dirichlet energy of the scalar curvature (without penalization of length) which also yields a sixth order evolution equation. However, this equation does not preserve the enclosed area in general and shows a completely different asymptotic behavior. In [4], the authors classify the stationary solutions as $\omega$-fold covered circles. In particular, this shows that the solution for an initial datum with $\omega = 0$ cannot converge. Convergence to an $\omega$-fold covered circle is shown under the assumption that the length of the evolving curve remains bounded, which is the case for small initial energy. The length preserving and area preserving variants (with nonlocal Lagrange multipliers) of the flow discussed in [4], are studied in [118] and [165].

In Section 3.1.1, we first provide a properly derivation of the area preserving flow as an $H^{-1}$-gradient flow. Section 3.2 explores the basic properties of this flow, including the proof that the enclosed area of the evolving curves is indeed preserved. In Section 3.3, we show short time existence, though the focus is not on this aspect; we basically only cover what is necessary for the later convergence result. Section 3.4 provides a proof of

global in time existence by employing appropriate interpolation inequalities. Section 3.5 establishes convergence using a suitable Łojasiewicz–Simon gradient inequality, which we derive beforehand. Finally, Section 3.6 is dedicated to the stationary solutions.

The contents of this chapter are mostly taken from [99]. Section 3.6.3 and Section 3.6.4 are new.

## 3.1. Derivation of the flow equation

Since we have not found a rigorous derivation of geometric $H^{-1}$-gradient flows in the literature, we will address this topic in detail here. The Hilbert spaces that we introduce can also be used to derive the curve diffusion flow (see Section 1.4.3) as the $H^{-1}$-gradient flow of the length functional.

### 3.1.1. An appropriate Hilbert space setting

For a given smooth regular curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$, we define the vector space

$$H_\gamma := \Big\{ Y\colon \mathbb{S}^1 \to \mathbb{R}^2 : Y = \langle Y, \nu_\gamma\rangle \nu_\gamma,\ Y, \nabla_{s_\gamma} Y \in L^2(\,\mathrm{d}s_\gamma),\ \int_{\mathbb{S}^1} \langle \nu_\gamma, Y\rangle \,\mathrm{d}s_\gamma = 0 \Big\}.$$

Here, $\mathrm{d}s_\gamma$ denotes the arclength measure of $\gamma$, $\nabla_{s_\gamma}$ is the normal projection of the arclength derivative $\partial_{s_\gamma}$ along $\gamma$ and $\nu_\gamma$ is the normal vector of $\gamma$. On $H_\gamma$, we consider the inner product

$$\langle Y_1, Y_2\rangle_{H_\gamma} = \langle \nabla_{s_\gamma} Y_1, \nabla_{s_\gamma} Y_2\rangle_{L^2(\,\mathrm{d}s_\gamma)}$$

which induces the norm

$$\|Y\|_{H_\gamma} = \|\nabla_{s_\gamma} Y\|_{L^2(\,\mathrm{d}s_\gamma)}. \tag{3.1.1}$$

This is justified by the following lemma, which also shows that on $H_\gamma$, $\|\cdot\|_{H_\gamma}$ is equivalent to the Sobolev norm $\|\cdot\|_{W^{1,2}} = \|\cdot\|_{W^{1,2}(\,\mathrm{d}x)}$.

**Lemma 3.1.1** ([99, Lemma 2.1])**.** *There exist constants $c_1, c_2 > 0$, depending on $\mathcal{L}(\gamma)$, $\|\kappa_\gamma\|_{C^0}$, $\|\partial_x \gamma\|_{C^0}$, and $\min_{\mathbb{S}^1} |\partial_x \gamma|$, such that*

$$c_1 \|Y\|_{W^{1,2}} \le \|Y\|_{H_\gamma} \le c_2 \|Y\|_{W^{1,2}} \quad \text{for } Y \in H_\gamma. \tag{3.1.2}$$

*In particular, $(H_\gamma, \|\cdot\|_{H_\gamma})$ is a Hilbert space.*

*Proof.* Let $y := \langle \nu_\gamma, Y\rangle$. Then $\partial_{s_\gamma} y \in L^2(\,\mathrm{d}s_\gamma)$ and $\int_{\mathbb{S}^1} y \,\mathrm{d}s_\gamma = 0$. Thus, with Poincaré's

inequality, we obtain

$$\|Y\|^2_{L^2(\mathrm{d}s_\gamma)} = \int_{\mathbb{S}^1} y^2 \,\mathrm{d}s_\gamma \leq C \int_{\mathbb{S}^1} (\partial_{s_\gamma} y)^2 \,\mathrm{d}s_\gamma = C \int_{\mathbb{S}^1} \langle \nabla_{s_\gamma} Y, \nu_\gamma \rangle^2 \,\mathrm{d}s_\gamma = C \|\nabla_{s_\gamma} Y\|^2_{L^2(\mathrm{d}s_\gamma)} \tag{3.1.3}$$

for a constant $C$ depending on $\mathcal{L}(\gamma)$. Moreover, $Y = \langle Y, \nu_\gamma \rangle \nu_\gamma$ implies that

$$\partial_{s_\gamma} Y = \nabla_{s_\gamma} Y - \langle Y, \nu_\gamma \rangle \langle \kappa_\gamma, \nu_\gamma \rangle \partial_{s_\gamma} \gamma. \tag{3.1.4}$$

Here, we use $\partial_{s_\gamma} \nu_\gamma = -\langle \kappa_\gamma, \nu_\gamma \rangle \partial_{s_\gamma} \gamma$. Equation (3.1.4) yields together with (3.1.3) that

$$C_1 \|\nabla_{s_\gamma} Y\|^2_{L^2(\mathrm{d}s_\gamma)} \leq \|Y\|^2_{L^2(\mathrm{d}s_\gamma)} + \|\partial_{s_\gamma} Y\|^2_{L^2(\mathrm{d}s_\gamma)} \leq C_2 \|\nabla_{s_\gamma} Y\|^2_{L^2(\mathrm{d}s_\gamma)} \tag{3.1.5}$$

for constants $C_1 > 0$ depending on the inverse of $\max_{\mathbb{S}^1} |\kappa_\gamma|^2$ and $C_2 > 0$ depending on $\mathcal{L}(\gamma)$ and $\max_{\mathbb{S}^1} |\kappa_\gamma|^2$. From $\partial_{s_\gamma} Y = |\partial_x \gamma|^{-1} \partial_x Y$ and $\mathrm{d}s_\gamma = |\partial_x \gamma| \,\mathrm{d}x$, we conclude

$$\|Y\|^2_{W^{1,2}} = \|Y\|^2_{L^2} + \|\partial_x Y\|^2_{L^2} \leq C \big( \|Y\|^2_{L^2(\mathrm{d}s_\gamma)} + \|\partial_{s_\gamma} Y\|^2_{L^2(\mathrm{d}s_\gamma)} \big)$$

for some $C$ depending on $\min_{\mathbb{S}^1} |\partial_x \gamma|$ and $\max_{\mathbb{S}^1} |\partial_x \gamma|$. From this, the left inequality of (3.1.2) follows with (3.1.3) and (3.1.5). The right inequality follows similarly leading to the equivalence of the norms. Moreover, $H_\gamma$ is a closed subspace of $W^{1,2}(\mathbb{S}^1; \mathbb{R}^2)$. Thus, $(H_\gamma, \|\cdot\|_{H_\gamma})$ is a Hilbert space. □

The Hilbert space $H_\gamma$ is geometric in the following sense.

**Lemma 3.1.2** ([99, Lemma 2.2])**.** *Let $\tilde{\gamma} = \gamma \circ \varphi \colon \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth reparametrization of $\gamma$. If $Y \in H_\gamma$, then $Y \circ \varphi \in H_{\tilde{\gamma}}$ and $\|Y\|_{H_\gamma} = \|Y \circ \varphi\|_{H_{\tilde{\gamma}}}$. For $Y_1, Y_2 \in H_\gamma$, we have $\langle Y_1 \circ \varphi, Y_2 \circ \varphi \rangle_{H_{\tilde{\gamma}}} = \langle Y_1, Y_2 \rangle_{H_\gamma}$.*

*Proof.* Let $Y \in H_\gamma$. If the smooth diffeomorphism $\varphi$ of the reparametrization is orientation preserving, then $\nu_{\tilde{\gamma}} = \nu_\gamma \circ \varphi$. If $\varphi$ changes the orientation of $\gamma$, then $\nu_{\tilde{\gamma}} = -\nu_\gamma \circ \varphi$. In both cases, we obtain

$$\tilde{Y} := Y \circ \varphi = \langle Y \circ \varphi, \nu_\gamma \circ \varphi \rangle \, \nu_\gamma \circ \varphi = \langle \tilde{Y}, \nu_{\tilde{\gamma}} \rangle \, \nu_{\tilde{\gamma}}.$$

Moreover, substitution and $|\partial_x \tilde{\gamma}(x)| = |\varphi'(x) \, \partial_x \gamma(\varphi(x))|$ yields

$$\|\tilde{Y}\|^2_{L^2(\mathrm{d}s_{\tilde{\gamma}})} = \int_{\mathbb{S}^1} |Y(\varphi(x))|^2 |\partial_x \tilde{\gamma}(x)| \,\mathrm{d}x = \int_{\mathbb{S}^1} |Y|^2 \,\mathrm{d}s_\gamma < \infty$$

as well as

$$\|\nabla_{s_{\tilde{\gamma}}} \tilde{Y}\|_{L^2(\mathrm{d}s_{\tilde{\gamma}})} = \int_{\mathbb{S}^1} (\langle \partial_{s_{\tilde{\gamma}}} \tilde{Y}, \nu_{\tilde{\gamma}} \rangle)^2 \,\mathrm{d}s_{\tilde{\gamma}} = \int_{\mathbb{S}^1} \frac{1}{|\partial_x \tilde{\gamma}(x)|} \big( \langle \partial_x (Y(\varphi(x))), \nu_\gamma(\varphi(x)) \rangle \big)^2 \,\mathrm{d}x$$

$$= \int_{\mathbb{S}^1} \frac{1}{|\partial_x \gamma(z)|} (\langle \partial_x Y(z), \nu_\gamma(z) \rangle)^2 \, \mathrm{d}z = \int_{\mathbb{S}^1} (\langle \partial_{s_\gamma} Y, \nu_\gamma \rangle)^2 \, \mathrm{d}s_\gamma = \|\nabla_{s_\gamma} Y\|_{L^2(\mathrm{d}s_\gamma)} < \infty. \tag{3.1.6}$$

Along the same lines, we obtain

$$\int_{\mathbb{S}^1} \langle \nu_{\tilde{\gamma}}, \tilde{Y} \rangle \, \mathrm{d}s_{\tilde{\gamma}} = \int_{\mathbb{S}^1} \langle \nu_\gamma(\varphi(x)), Y(\varphi(x)) \rangle |\partial_x \tilde{\gamma}(x)| \, \mathrm{d}x = 0,$$

since $\int_{\mathbb{S}^1} \langle \nu_\gamma, Y \rangle \, \mathrm{d}s_\gamma = 0$. Everything together results in $\tilde{Y} \in H_{\tilde{\gamma}}$. Moreover, with (3.1.1), (3.1.6) yields $\|Y\|_{H_\gamma} = \|\tilde{Y}\|_{H_{\tilde{\gamma}}}$. The rest of the statement follows similarly. □

In the following, we denote by $H_\gamma^{-1}$ the dual space of $H_\gamma$.

**Definition 3.1.3** ([99, Definition 2.3])**.** Let $\phi \in H_\gamma^{-1}$. We call $X \in H_\gamma$ a *weak solution* of

$$-\nabla_{s_\gamma}^2 X = \phi \tag{3.1.7}$$

if $\phi(Y) = \langle X, Y \rangle_{H_\gamma} = \langle \nabla_{s_\gamma} X, \nabla_{s_\gamma} Y \rangle_{L^2(\mathrm{d}s_\gamma)}$ for all $Y \in H_\gamma$.

Note that by the Riesz representation theorem, for all $\phi \in H_\gamma^{-1}$, there exists a unique weak solution of (3.1.7).

We define an inner product on $H_\gamma^{-1}$ by

$$\langle \phi_1, \phi_2 \rangle_{H_\gamma^{-1}} := \langle X_1, X_2 \rangle_{H_\gamma} = \langle \nabla_{s_\gamma} X_1, \nabla_{s_\gamma} X_2 \rangle_{L^2(\mathrm{d}s_\gamma)},$$

where $X_i \in H_\gamma$ is the unique weak solution of $-\nabla_{s_\gamma}^2 X_i = \phi_i$, $i = 1, 2$. Moreover, we have

$$\|X\|_{H_\gamma} = \|\phi\|_{H_\gamma^{-1}}. \tag{3.1.8}$$

**Remark 3.1.4** ([99, Remark 1])**.** If the weak solution $X$ of (3.1.7) is sufficiently smooth, precisely $\nabla_{s_\gamma}^2 X \in L^2(\mathrm{d}s_\gamma)$, then $\phi \in H_\gamma^{-1}$ is represented by $\phi = \langle -\nabla_{s_\gamma}^2 X, \cdot \rangle_{L^2(\mathrm{d}s_\gamma)}$. We identify the functional $\phi$ with the function $-\nabla_{s_\gamma}^2 X$ using this $L^2$-pairing. With this identification, $H_\gamma \subseteq L^2(\mathrm{d}s_\gamma) \subseteq H_\gamma^{-1}$. Moreover, this identification allows us to express the $H_\gamma^{-1}$-norm as

$$\|\phi\|_{H_\gamma^{-1}} = \sup_{Y \in H_\gamma \setminus \{0\}} \frac{1}{\|Y\|_{H_\gamma}} \int_{\mathbb{S}^1} \langle \phi, Y \rangle \, \mathrm{d}s_\gamma. \tag{3.1.9}$$

**Remark 3.1.5** ([99, Remark 2])**.** Let $\phi \in C^\infty(\mathbb{S}^1; \mathbb{R}^2) \cap H_\gamma$. Then $\phi$ is interpreted as an element of $H_\gamma^{-1}$ as $H_\gamma \ni Y \mapsto \langle \phi, Y \rangle_{L^2(\mathrm{d}s_\gamma)}$. In this case, the weak solution $X$ of (3.1.7) is smooth (and coincides with the strong solution). Indeed, with (3.1.4), for all $Y \in W^{1,2}(\mathrm{d}x)$, we have

$$C(\gamma) \left| \langle \partial_x X, \partial_x Y \rangle_{L^2(\mathrm{d}x)} \right| \leq \left| \langle \partial_{s_\gamma} X, \partial_{s_\gamma} Y \rangle_{L^2(\mathrm{d}s_\gamma)} \right|$$

$$
\begin{aligned}
&\leq \left|\langle \nabla_{s_\gamma} X, \nabla_{s_\gamma} Y\rangle_{L^2(\mathrm{d}s_\gamma)}\right| + \left|\int_{\mathbb{S}^1} |\kappa_\gamma| \langle X, \nu_\gamma\rangle \langle \partial_{s_\gamma}\gamma, \partial_{s_\gamma} Y\rangle \,\mathrm{d}s_\gamma\right| \\
&\leq |\phi(Y)| + \left|\int_{\mathbb{S}^1} (\partial_{s_\gamma}|\kappa_\gamma|) \langle X, \nu_\gamma\rangle \langle \partial_{s_\gamma}\gamma, Y\rangle \,\mathrm{d}s_\gamma\right| + \left|\int_{\mathbb{S}^1} |\kappa_\gamma| \langle \partial_{s_\gamma} X, \nu_\gamma\rangle \langle \partial_{s_\gamma}\gamma, Y\rangle \,\mathrm{d}s_\gamma\right| \\
&\quad + \left|\int_{\mathbb{S}^1} |\kappa_\gamma| \langle X, \partial_{s_\gamma}\nu_\gamma\rangle \langle \partial_{s_\gamma}\gamma, Y\rangle \,\mathrm{d}s_\gamma\right| + \left|\int_{\mathbb{S}^1} |\kappa_\gamma| \langle X, \nu_\gamma\rangle \langle \kappa_\gamma, Y\rangle \,\mathrm{d}s_\gamma\right| \\
&\leq C\|Y\|_{L^2(\mathrm{d}s_\gamma)} \leq C\|Y\|_{L^2(\mathrm{d}x)},
\end{aligned}
$$

for some $C = C(\gamma, \|\phi\|_{L^2(\mathrm{d}s_\gamma)}, \|X\|_{H_\gamma})$, that changes along the lines. It follows that $X \in W^{2,2}(\mathrm{d}x) \subset C^1(\mathbb{S}^1;\mathbb{R}^2)$. With similar arguments we achieve $X \in C^\infty(\mathbb{S}^1;\mathbb{R}^2)$.

On $H_\gamma$, we have the following interpolation inequality.

**Lemma 3.1.6** ([99, Lemma 2.4])**.** *Let $\phi \in C^\infty(\mathbb{S}^1;\mathbb{R}^2) \cap H_\gamma$. Then*

$$\|\phi\|^2_{L^2(\mathrm{d}s_\gamma)} \leq \|\phi\|_{H_\gamma^{-1}} \|\phi\|_{H_\gamma}.$$

*Proof.* Let $X \in C^\infty(\mathbb{S}^1;\mathbb{R}^2) \cap H_\gamma$ be the solution of $-\nabla^2_{s_\gamma} X = \phi$. With (3.1.8), we have

$$
\begin{aligned}
\|\phi\|^2_{L^2(\mathrm{d}s_\gamma)} &= -\int_{\mathbb{S}^1} \langle \nabla^2_{s_\gamma} X, \phi\rangle \,\mathrm{d}s_\gamma = \int_{\mathbb{S}^1} \langle \nabla_{s_\gamma} X, \nabla_{s_\gamma}\phi\rangle \,\mathrm{d}s_\gamma \\
&\leq \|\nabla_{s_\gamma} X\|_{L^2(\mathrm{d}s_\gamma)} \|\nabla_{s_\gamma}\phi\|_{L^2(\mathrm{d}s_\gamma)} = \|\phi\|_{H_\gamma^{-1}} \|\phi\|_{H_\gamma}.
\end{aligned}
$$

□

Moreover, on $H_\gamma$, the $H_\gamma^{-1}$-norm is equivalent to another norm, which we use in Section 3.5 as an auxiliary norm.

**Lemma 3.1.7** ([99, Lemma 2.5])**.** *Let $\phi \in C^\infty(\mathbb{S}^1;\mathbb{R}^2) \cap H_\gamma$. Define*

$$\|\phi\|_\star := \sup_{Y \in W^{1,2}(\mathbb{S}^1;\mathbb{R}^2)\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}} \int_{\mathbb{S}^1} \langle \phi, Y\rangle \,\mathrm{d}x.$$

*Then there exist constants $c_1, c_2 > 0$ depending on $\|\kappa_\gamma\|_{C^0}$, $\mathcal{L}(\gamma)$ as well as $\min_{\mathbb{S}^1} |\partial_x\gamma|$ and $\|\partial_x\gamma\|_{C^0}$ such that*

$$c_1\|\phi\|_\star \leq \|\phi\|_{H_\gamma^{-1}} \leq c_2\|\phi\|_\star.$$

*Proof.* For the proof, we consider

$$\|\phi\|_{\mathrm{aux}} := \sup_{Y \in W^{1,2}(\mathbb{S}^1;\mathbb{R}^2)\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}} \int_{\mathbb{S}^1} \langle \phi, Y\rangle \,\mathrm{d}s_\gamma$$

and show the equivalence of $\|\cdot\|_{\mathrm{aux}}$ and $\|\cdot\|_{H_\gamma^{-1}}$ on $C^\infty(\mathbb{S}^1;\mathbb{R}^2) \cap H_\gamma$. Estimating the arclength element $|\partial_x\gamma|$ from below and from above, this equivalence yields the claim.
For $Y \in H_\gamma \subset W^{1,2}(\mathbb{S}^1;\mathbb{R})$, Lemma 3.1.1 yields the existence of a constant $c_2$ depending

on $\gamma$ such that

$$\begin{aligned}\|\phi\|_{H_\gamma^{-1}} &= \sup_{Y\in H_\gamma\setminus\{0\}} \frac{1}{\|Y\|_{H_\gamma}}\int_{\mathbb{S}^1}\langle\phi,Y\rangle\,\mathrm{d}s_\gamma \le c_2 \sup_{Y\in H_\gamma\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}}\int_{\mathbb{S}^1}\langle\phi,Y\rangle\,\mathrm{d}s_\gamma\\ &\le c_2\|\phi\|_{\mathrm{aux}}.\end{aligned}$$

For the other inequality, we first note that if $\phi\in H_\gamma$, then $\phi$ is orthogonal to $\partial_s\gamma$ and thus $\langle\phi,Y\rangle = \langle\phi,Y^\perp\rangle$, where $Y^\perp := \langle Y,\nu_\gamma\rangle\nu_\gamma$ for any $Y\in W^{1,2}(\mathbb{S}^1;\mathbb{R}^2)$. Moreover, $|Y^\perp|^2\le|Y|^2$ and

$$\begin{aligned}|\partial_{s_\gamma}(Y^\perp)|^2 &= |\partial_{s_\gamma}Y - \langle\partial_{s_\gamma}Y,\partial_s\gamma\rangle\partial_s\gamma - \langle Y,\kappa_\gamma\rangle\partial_s\gamma - \langle Y,\partial_s\gamma\rangle\kappa_\gamma|^2\\ &= |\partial_{s_\gamma}Y|^2 - 2(\langle\partial_{s_\gamma}Y,\partial_s\gamma\rangle)^2 - 2\langle Y,\partial_s\gamma\rangle\langle\partial_{s_\gamma}Y,\kappa_\gamma\rangle + (\langle\partial_{s_\gamma}Y,\partial_s\gamma\rangle)^2\\ &\quad + (\langle Y,\kappa_\gamma\rangle)^2 + (\langle Y,\partial_s\gamma\rangle)^2|\kappa_\gamma|^2\\ &= |\partial_{s_\gamma}Y|^2 - (\langle\partial_{s_\gamma}Y,\partial_s\gamma\rangle)^2 - 2\langle Y,\partial_s\gamma\rangle\langle\partial_{s_\gamma}Y,\kappa_\gamma\rangle + |\kappa_\gamma|^2|Y|^2\\ &\le C(|Y|^2+|\partial_{s_\gamma}Y|^2)\end{aligned}\tag{3.1.10}$$

for a constant $C$ depending on bounds on $\kappa_\gamma$. Thus, $\|Y^\perp\|_{W^{1,2}}\le C\|Y\|_{W^{1,2}}$ for a constant $C$ depending on bounds on $\kappa_\gamma$ and the arclength element $|\partial_x\gamma|$. As $\langle\phi,Y\rangle=\langle\phi,Y^\perp\rangle$, we conclude that

$$\|\phi\|_{\mathrm{aux}} = \sup_{Y\in W^{1,2}\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}}\int_{\mathbb{S}^1}\langle\phi,Y\rangle\,\mathrm{d}s_\gamma \le C \sup_{\substack{Y\in W^{1,2}\setminus\{0\},\\ Y=\langle Y,\nu_\gamma\rangle\nu_\gamma}} \frac{1}{\|Y\|_{W^{1,2}}}\int_{\mathbb{S}^1}\langle\phi,Y\rangle\,\mathrm{d}s_\gamma\tag{3.1.11}$$

with $C$ depending on the same quantities as above. Now we note that $\phi\in H_\gamma$ also implies that

$$\int_{\mathbb{S}^1}\langle\phi,Y\rangle\,\mathrm{d}s_\gamma = \int_{\mathbb{S}^1}\langle\phi,Y-\frac{\nu_\gamma}{\mathcal{L}(\gamma)}\int_{\mathbb{S}^1}\langle Y,\nu_\gamma\rangle\,\mathrm{d}s_\gamma\rangle\,\mathrm{d}s_\gamma =: \int_{\mathbb{S}^1}\langle\phi,Y-C_\gamma(Y)\nu_\gamma\rangle\,\mathrm{d}s_\gamma$$

for $Y\in W^{1,2}(\mathbb{S}^1;\mathbb{R}^2)$. Moreover, for $Y\in W^{1,2}(\mathbb{S}^1;\mathbb{R}^2)\cap\{Y=\langle Y,\nu_\gamma\rangle\nu_\gamma\}$, we have

$$\int_{\mathbb{S}^1}|Y-C_\gamma(Y)\nu_\gamma|^2\,\mathrm{d}s_\gamma = \int_{\mathbb{S}^1}|Y|^2\,\mathrm{d}s_\gamma - \mathcal{L}(\gamma)C_\gamma(Y)^2 \le \int_{\mathbb{S}^1}|Y|^2\,\mathrm{d}s_\gamma\tag{3.1.12}$$

and

$$\begin{aligned}\int_{\mathbb{S}^1}|\partial_{s_\gamma}(Y-C_\gamma(Y)\nu_\gamma)|^2\,\mathrm{d}s_\gamma &= \int_{\mathbb{S}^1}|\partial_{s_\gamma}Y|^2\,\mathrm{d}s_\gamma + 2C_\gamma(Y)\int_{\mathbb{S}^1}\langle\partial_{s_\gamma}Y,\partial_s\gamma\rangle\langle\kappa_\gamma,\nu_\gamma\rangle\,\mathrm{d}s_\gamma\\ &\quad + C_\gamma(Y)^2\int_{\mathbb{S}^1}|\kappa_\gamma|^2\,\mathrm{d}s_\gamma.\end{aligned}\tag{3.1.13}$$

Since

$$C_\gamma(Y)^2 \leq \frac{1}{\mathcal{L}(\gamma)^2}\|Y\|^2_{L^1(\,\mathrm{d}s_\gamma)} \leq \frac{1}{\mathcal{L}(\gamma)}\|Y\|^2_{L^2(\,\mathrm{d}s_\gamma)},$$

(3.1.12) and (3.1.13) imply that $\|Y - C_\gamma(Y)\nu_\gamma\|_{W^{1,2}} \leq C\|Y\|_{W^{1,2}}$ for a constant $C$ depending on bounds on $\kappa_\gamma$ and $|\partial_x\gamma|$ and the energy $\mathcal{E}(\gamma)$. We conclude from (3.1.11) that

$$\|\phi\|_{\mathrm{aux}} \leq C \sup_{\substack{Y\in W^{1,2},\\ Y=\langle Y,\nu_\gamma\rangle\nu_\gamma}} \frac{1}{\|Y\|_{W^{1,2}}}\int_{\mathbb{S}^1}\langle\phi,Y\rangle\,\mathrm{d}s_\gamma \leq C \sup_{\substack{Y\in W^{1,2}, Y=\langle Y,\nu_\gamma\rangle\nu_\gamma,\\ \int_{\mathbb{S}^1}\langle Y,\nu_\gamma\rangle\,\mathrm{d}s_\gamma=0}} \frac{1}{\|Y\|_{W^{1,2}}}\int_{\mathbb{S}^1}\langle\phi,Y\rangle\,\mathrm{d}s_\gamma,$$

with $C$ depending on the quantities listed above. By the definition of $H_\gamma$ and Lemma 3.1.1, this is $\|\phi\|_{\mathrm{aux}} \leq C\|\phi\|_{H_\gamma^{-1}}$. □

### 3.1.2. The area preserving elastic flow as a gradient flow

Now, we are ready to derive flow equation (1.6.1) as the $H_\gamma^{-1}$-gradient flow of the elastic energy. For this, see also Section 1.3.1. We consider a smooth regular curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ and a smooth normal variation $\phi\colon \mathbb{S}^1 \to \mathbb{R}^2$ such that $\phi \in H_\gamma \subseteq L^2(\,\mathrm{d}s_\gamma) \subseteq H_\gamma^{-1}$. For $\varepsilon > 0$ small enough, the perturbed curve $\gamma_\varepsilon := \gamma + \varepsilon\phi$ is also regular. With

$$\nabla_{L^2}\mathcal{E}_\lambda(\gamma) = \nabla^2_{s_\gamma}\kappa_\gamma + \frac{1}{2}|\kappa_\gamma|^2\kappa_\gamma - \lambda\kappa_\gamma,$$

we write

$$\mathcal{E}'_\lambda(\gamma)(\phi) = \langle\nabla_{L^2}\mathcal{E}_\lambda(\gamma),\phi\rangle_{L^2(\,\mathrm{d}s_\gamma)}. \tag{3.1.14}$$

In general, $\nabla_{L^2}\mathcal{E}_\lambda(\gamma) \notin H_\gamma$. Therefore, we define

$$W := \nabla_{L^2}\mathcal{E}_\lambda(\gamma) - C_\gamma\nu_\gamma \in H_\gamma \cap C^\infty(\mathbb{S}^1;\mathbb{R}^2) \quad \text{with} \quad C_\gamma := \frac{1}{\mathcal{L}(\gamma)}\int_{\mathbb{S}^1}\langle\nu_\gamma,\nabla_{L^2}\mathcal{E}_\lambda(\gamma)\rangle\,\mathrm{d}s_\gamma.$$

With $X \in H_\gamma \cap C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ as the weak solution of $-\nabla^2_{s_\gamma}X = \phi$ and $w = -\nabla^2_{s_\gamma}W$, we obtain

$$\begin{aligned}\langle w,\phi\rangle_{H_\gamma^{-1}} &= \langle\nabla_{s_\gamma}W,\nabla_{s_\gamma}X\rangle_{L^2(\,\mathrm{d}s_\gamma)} = \int_{\mathbb{S}^1}\langle W,-\nabla^2_{s_\gamma}X\rangle\,\mathrm{d}s_\gamma = \int_{\mathbb{S}^1}\langle\nabla_{L^2}\mathcal{E}_\lambda(\gamma),\phi\rangle\,\mathrm{d}s_\gamma \\ &= \langle\nabla_{L^2}\mathcal{E}_\lambda(\gamma),\phi\rangle_{L^2(\,\mathrm{d}s_\gamma)}.\end{aligned} \tag{3.1.15}$$

Here, we used that

$$\int_{\mathbb{S}^1}\langle C_\gamma\nu_\gamma,\nabla^2_{s_\gamma}X\rangle\,\mathrm{d}s_\gamma = -C_\gamma\int_{\mathbb{S}^1}\langle\partial_{s_\gamma}\nu_\gamma,\nabla_{s_\gamma}X\rangle\,\mathrm{d}s_\gamma = 0,$$

since $\partial_{s_\gamma}\nu_\gamma$ is tangential. Comparing (3.1.14) and (3.1.15), we define the $H^{-1}_\gamma$-gradient

$$\nabla_{H^{-1}_\gamma}\mathcal{E}_\lambda(\gamma) := w = -\nabla^2_{s_\gamma}(\nabla_{L^2}\mathcal{E}_\lambda(\gamma) - C_\gamma\nu_\gamma) = -\nabla^2_{s_\gamma}(\nabla^2_{s_\gamma}\kappa_\gamma + \tfrac{1}{2}|\kappa_\gamma|^2\kappa_\gamma - \lambda\kappa_\gamma).$$

To properly adress the geometric nature of the problem, we set only the normal component of velocity equal to the negative gradient. This yields

$$\partial_t^\perp\gamma = -\nabla_{H^{-1}_\gamma}\mathcal{E}_\lambda(\gamma),$$

which is (1.6.1). Note that, in general, the space with respect to which the gradient is computed changes in time.

## 3.2. Basic properties of the flow

In order to incorporate the invariance of the elastic energy under reparametrizations, we defined (3.0.3) in such a way that a reparametrization of a solution again solves (3.0.3) with reparametrized initial data. See also Remark 1.4.1. Since we need this later, we state it explicitly.

**Remark 3.2.1** (Invariance under reparametrization, [99, Remark 3])**.** Let $0 < T < \infty$ and $\gamma\colon [0,T]\times\mathbb{S}^1 \to \mathbb{R}^2$ be a smooth solution to (3.0.3). Then

$$\partial_t\gamma = -\nabla_{H^{-1}_\gamma}\mathcal{E}_\lambda(\gamma) + h\partial_s\gamma$$

for some smooth $h\colon [0,T)\times\mathbb{S}^1 \to \mathbb{R}$. Let $\Phi\colon [0,T)\times\mathbb{S}^1 \to \mathbb{S}^1$ be a smooth one-parameter family of diffeomorphisms. Then the reparametrization $\tilde\gamma(t,x) := \gamma(t,\Phi(t,x))$ solves

$$\begin{aligned}\partial_t\tilde\gamma &= \partial_t\gamma(t,\Phi(t,x)) + \partial_t\Phi(t,x)\partial_x\gamma(t,\Phi(t,x))\\ &= -\nabla_{H^{-1}_\gamma}\mathcal{E}_\lambda(\gamma(t,\Phi(t,x))) + h(t,\Phi(t,x))\partial_s\gamma(t,\Phi(t,x)) + \partial_t\Phi(t,x)\partial_x\gamma(t,\Phi(t,x))\\ &= -\nabla_{H^{-1}_\gamma}\mathcal{E}_\lambda(\tilde\gamma) + \tilde h\partial_s\tilde\gamma\end{aligned}$$

for some smooth $\tilde h\colon [0,T)\times\mathbb{S}^1 \to \mathbb{R}$. Thus, $\partial_t^\perp\tilde\gamma = -\nabla_{H^{-1}_\gamma}\mathcal{E}_\lambda(\tilde\gamma)$ and $\tilde\gamma(0,\cdot) = \gamma_0(\Phi(0,\cdot))$.

In what follows, we frequently use the following lemma which gives the time derivative of various geometric quantities related to a curve $\gamma$ moving under a general flow.

**Lemma 3.2.2** ([99, Lemma 3.1])**.** *Let $\gamma\colon [0,T)\times\mathbb{S}^1 \to \mathbb{R}^2$, $T > 0$, be a sufficiently smooth solution of the flow $\partial_t\gamma = V + h\partial_s\gamma$, where $V$ is the normal velocity and $h = \langle\partial_t\gamma, \partial_s\gamma\rangle$. Then*

$$\partial_t(\mathrm{d}s) = (\partial_s h - \langle\kappa, V\rangle)\,\mathrm{d}s, \tag{3.2.1}$$

$$\partial_t\partial_s - \partial_s\partial_t = (\langle\kappa, V\rangle - \partial_s h)\partial_s, \tag{3.2.2}$$

$$\partial_t \partial_s \gamma = \nabla_s V + h\kappa, \tag{3.2.3}$$

$$\partial_t \nu = (-\langle \nabla_s V, \nu\rangle - h\langle \kappa, \nu\rangle)\partial_s \gamma, \tag{3.2.4}$$

$$\nabla_t \kappa = \nabla_s^2 V + \langle \kappa, V\rangle \kappa + h\nabla_s \kappa, \tag{3.2.5}$$

$$\nabla_t \nabla_s \phi - \nabla_s \nabla_t \phi = (\langle \kappa, V\rangle - \partial_s h)\nabla_s \phi + \langle \kappa, \phi\rangle \nabla_s V - \langle \nabla_s V, \phi\rangle \kappa, \tag{3.2.6}$$

*for a sufficiently smooth normal field* $\phi\colon [0,T)\times \mathbb{S}^1 \to \mathbb{R}^2$.

Here, $\nabla_t = \partial_t^\perp = P^\perp \partial_t$. For a proof of (3.2.1), (3.2.2), (3.2.3), (3.2.5) and (3.2.6), see for example [53, Lemma 2.1]. Formula (3.2.4) follows from (3.2.3).

As one expects, the flow of Lemma 3.2.2 does not change the rotation index $\omega$ given by

$$\omega = \frac{1}{2\pi}\int_{\mathbb{S}^1} k \,\mathrm{d}s = \frac{1}{2\pi}\int_{\mathbb{S}^1} \langle \kappa, \nu\rangle \,\mathrm{d}s. \tag{3.2.7}$$

**Lemma 3.2.3.** *Let* $\gamma\colon [0,T)\times \mathbb{S}^1 \to \mathbb{R}^2$, $T>0$, *be a sufficiently smooth solution of the flow* $\partial_t \gamma = V + h\partial_s \gamma$, *where* $V$ *is the normal velocity and* $h = \langle \partial_t \gamma, \partial_s \gamma\rangle$. *Then the rotation index* $t \mapsto \omega(t)$ *defined as in* (3.2.7) *is constant in time.*

*Proof.* Using (3.2.1), (3.2.4) and (3.2.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} \langle \kappa, \nu\rangle \,\mathrm{d}s = \int_{\mathbb{S}^1} \langle \nabla_s^2 V + \langle \kappa, V\rangle \kappa + h\nabla_s \kappa, \nu\rangle \,\mathrm{d}s + \int_{\mathbb{S}^1} \langle \kappa, \nu\rangle (\partial_s h - \langle \kappa, V\rangle) \,\mathrm{d}s. \tag{3.2.8}$$

With integration by parts and since $\partial_s \nu$ is tangential, we see that the right hand side of (3.2.8) vanishes. □

Next, we note the basic but crucial fact that the flow (1.6.1) decreases the energy (3.0.1). This follows by a direct computation using (3.2.1), (3.2.5) and integration by parts. All the terms containing a tangential velocity cancel.

**Lemma 3.2.4** ([99, Lemma 3.2])**.** *For a sufficently smooth solution* $\gamma\colon [0,T)\times \mathbb{S}^1 \to \mathbb{R}^2$, $T>0$, *to* (1.6.1), *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\gamma) = -\int_{\mathbb{S}^1} |\nabla_s \left(\nabla_s^2 \kappa + \tfrac{1}{2}|\kappa|^2 \kappa - \lambda \kappa\right)|^2 \,\mathrm{d}s \le 0. \tag{3.2.9}$$

**Remark 3.2.5** ([99, Remark 4])**.** We observe the energy decrease also in the abstract setting of Section 3.1. Here, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\gamma) &= \langle \nabla_{H_\gamma^{-1}} \mathcal{E}_\lambda(\gamma), \partial_t \gamma\rangle_{H_\gamma^{-1}} = -\|\partial_t^\perp \gamma\|_{H_\gamma^{-1}}^2 = -\|\nabla_s^2 \nabla_{L^2} \mathcal{E}_\lambda(\gamma)\|_{H_\gamma^{-1}}^2 \\ &= -\|\nabla_s \nabla_{L^2} \mathcal{E}_\lambda(\gamma)\|_{L^2(\mathrm{d}s)}^2,\end{aligned}$$

which is (3.2.9).

At the same time, the flow (1.6.1) preserves the signed enclosed area (1.2.6).

**Lemma 3.2.6** ([99, Lemma 3.3]). *Let $\gamma\colon [0,T)\times\mathbb{S}^1\to\mathbb{R}^2$ be a sufficently smooth solution to* (1.6.1)*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(\gamma)=0.$$

*Proof.* Using (3.2.4) and integration by parts yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(\gamma)=-\int_{\mathbb{S}^1}\langle\partial_t\gamma,\nu\rangle\,\mathrm{d}s.$$

So, as it should be, only the normal part of the velocity plays a role. With integration by parts and since $\partial_s\nu=-k\,\partial_s\gamma$ is tangential, we immediately obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(\gamma)=\int_{\mathbb{S}^1}\langle\nabla_s\left(\nabla_s^2\kappa+\tfrac{1}{2}|\kappa|^2\kappa-\lambda\kappa\right),\partial_s\nu\rangle\,\mathrm{d}s=0.\qquad\square$$

A more advanced property to study is the preservation of embeddedness. For a sixth order equation, embeddedness is generally not preserved, see [16]. However, by an energy argument developed in [121], the energy decay (3.2.9) suffices to provide an explicit bound on the initial energy which ensures preservation of embeddedness. Note that according to Hopf's Umlaufsatz, this only makes sense for $\omega=\pm1$.

**Proposition 3.2.7** ([99, Proposition 3.4]). *Let $\omega=1$, $\lambda>0$ and $C^*\approx 146.628$ be as in [121, Theorem 1.1]. Let $\gamma\colon[0,T)\times\mathbb{S}^1\to\mathbb{R}^2$ be a solution to* (3.0.3) *with embedded initial datum $\gamma_0$ such that*

$$(\mathcal{E}_\lambda(\gamma_0))^2\le 4\lambda C^*.\tag{3.2.10}$$

*Then $\gamma$ is embedded for all $t\in[0,T)$.*

*Proof.* If $\gamma_0$ is stationary, the claim is trivial. So with (3.2.9) we can assume that $\mathcal{E}_\lambda(\gamma(t))<\mathcal{E}_\lambda(\gamma_0)$ for all $t\in(0,T)$. It follows that

$$\mathcal{E}(\gamma(t))\mathcal{L}(\gamma(t))\le\frac{1}{4\lambda}(\mathcal{E}(\gamma(t))+\lambda\mathcal{L}(\gamma(t)))^2<\frac{1}{4\lambda}(\mathcal{E}_\lambda(\gamma_0))^2\le C^*$$

for all $t\in(0,T)$. With [121, Theorem 1.4] we conclude that $\gamma$ describes an embedded curve for all $t\in(0,T)$. $\square$

A short computation shows that if $\lambda\ge\frac{1}{2}(\frac{\pi}{C^*})^2$, the above threshold is nontrivial. In this case, there exists $A_0>0$ such that a curve describing a circle with enclosed area $A_0$ satisfies (3.2.10).

## 3.3. Short time existence

In this section we show how to prove short time existence. Addressing this question is additionally motivated by the fact that we use several of the arguments again in the proof of convergence in Section 3.5. Our proof works not only for a smooth initial datum, but also for $\gamma_0 \in C^{7,\alpha}(\mathbb{S}^1;\mathbb{R}^2)$, $\alpha \in (0,1)$. This could be improved further, but since we are mainly interested in questions about the long time behavior, we do not focus on this.

In Section 3.3.1 through Section 3.3.3, we provide some preliminary work. We review an essential reparametrization argument that enables us to find the solution as a graph over the initial datum. We define the parabolic Hölder spaces of order six and show the solvability of a scalar sixth order problem within these spaces. This groundwork allows us to prove short time existence in Section 3.3.4. Sections Section 3.3.1 to Section 3.3.3 can be skipped if desired, and one can proceed directly to Section 3.3.4.

### 3.3.1. A crucial reparametrization argument

For the convenience of the reader, we state the following reparametrization argument, which can be found with a detailed proof in [44, Lemma 4.1 and Remark 4.2]. In addition to the proof of short time existence (Theorem 3.3.6), it is also used in the proof of the constrained Łojasiewicz–Simon gradient inequality (Theorem 3.5.5).

**Lemma 3.3.1** ([99, Lemma A.1])**.** *Let $k \geq 4$ and $\hat{\gamma} \in W^{k+1,2}(\mathbb{S}^1;\mathbb{R}^2)$ be a regular curve. We define*

$$V^\perp := W^{4,2,\perp}(\mathbb{S}^1;\mathbb{R}^2) = \{Y \in W^{4,2}(\mathbb{S}^1;\mathbb{R}^2) : \langle Y, \partial_x \hat{\gamma}\rangle = 0\}. \tag{3.3.1}$$

*Then, there exists a constant $r = r(\hat{\gamma}) > 0$ such that for all $X \in B_r(0) \subset W^{k,2}(\mathbb{S}^1;\mathbb{R}^2)$, there exists a $W^{k,2}$-diffeomorphism $\Phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that*

$$(\hat{\gamma} + X) \circ \Phi = \hat{\gamma} + Y \tag{3.3.2}$$

*for some $Y \in V^\perp \cap W^{k,2}(\mathbb{S}^1;\mathbb{R}^2)$.*
*For $r > 0$ given, there exists $\tilde{r} = \tilde{r}(\hat{\gamma}, r) > 0$ such that for all $X \in B_{\tilde{r}}(0) \subset W^{k,2}(\mathbb{S}^1;\mathbb{R}^2)$, (3.3.2) holds for some $Y \in B_r(0) \subset V^\perp \cap W^{k,2}(\mathbb{S}^1;\mathbb{R}^2)$.*

**Remark 3.3.2** ([99, Remark A.2])**.** With the notations in the preceding lemma, it can be shown that smoothness of the reference curve $\hat{\gamma}$ and smoothness of $X \in B_r(0)$ yields smoothness of the diffeomorphism $\Phi$ and the normal variation $Y$. For details on this, see [44, Proof of Lemma 4.1].

### 3.3.2. Parabolic Hölder spaces and the linear problem

For the proof of short time existence, we work with parabolic Hölder spaces of order 6, cf. [56, Section II.3.1] or [96].

Let $l \in \mathbb{R}_+ \backslash \mathbb{Z}$, $a, b \in \mathbb{R}$ with $a < b$, $T > 0$ and let $Q_T := (0,T) \times (a,b)$. The parabolic Hölder space $H^{\frac{l}{6},l}\left(\overline{Q_T}\right)$ is the space of all functions $f \colon \overline{Q_T} \to \mathbb{R}$ with continuous derivatives $\partial_t^m \partial_x^n f$ for $6m + n < l$ and finite norm

$$\begin{aligned}\|f\|_{H^{\frac{l}{6},l}(\overline{Q_T})} := &\sum_{j=0}^{\lfloor l \rfloor} \sum_{6m+n=j} \sup_{(t,x)\in Q_T} |\partial_t^m \partial_x^n f(t,x)| \\ &+ \sum_{6m+n=\lfloor l \rfloor} \sup_{\substack{(t,x_1),(t,x_2)\in Q_T \\ x_1 \neq x_2}} \frac{|\partial_t^m \partial_x^n f(t,x_1) - \partial_t^m \partial_x^n f(t,x_2)|}{|x_1 - x_2|^{l - \lfloor l \rfloor}} \\ &+ \sum_{0<l-6m-n<6} \sup_{\substack{(t_1,x),(t_2,x)\in Q_T \\ t_1 \neq t_2}} \frac{|\partial_t^m \partial_x^n f(t_1,x) - \partial_t^m \partial_x^n f(t_2,x)|}{|t_1 - t_2|^{\frac{l-6m-n}{6}}}.\end{aligned}$$

To define the parabolic Hölder space $H^{\frac{l}{6},l}\left([0,T] \times \mathbb{S}^1\right)$, we identify $f \colon [0,T] \times \mathbb{S}^1 \to \mathbb{R}$ with its periodic counterpart $f \colon [0,T] \times \mathbb{R} \to \mathbb{R}$. By choosing $a, b$ such that the interval $(a,b)$ contains two periods of $f$, $H^{\frac{l}{6},l}\left([0,T] \times \mathbb{S}^1\right)$ consists of the functions with periodic counterpart in $H^{\frac{l}{6},l}\left(\overline{Q_T}\right)$.

We first consider the linear parabolic problem

$$\begin{cases} L\varphi := \partial_t \varphi - \sum_{q=0}^6 a_q \partial_x^q \varphi = f & \text{in } (0,T] \times \mathbb{S}^1, \\ \varphi(0,\cdot) = \varphi_0 & \text{on } \mathbb{S}^1 \end{cases} \tag{3.3.3}$$

for some $f \in H^{\frac{l}{6},l}\left([0,T] \times \mathbb{S}^1\right)$, coefficients $a_q \in H^{\frac{l}{6},l}\left([0,T] \times \mathbb{S}^1\right)$ and $\varphi_0 \in C^{6+l}(\mathbb{S}^1)$.

**Proposition 3.3.3** ([99, Proposition A.3])**.** *Let $L$ be parabolic, $l \in \mathbb{R}_+ \setminus \mathbb{Z}$ and assume that $a_q \in H^{\frac{l}{6},l}\left([0,T] \times \mathbb{S}^1\right)$, $q = 0, \dots, 6$. Then for all $f \in H^{\frac{l}{6},l}\left([0,T] \times \mathbb{S}^1\right)$ and $\varphi_0 \in C^{6+l}(\mathbb{S}^1)$, there exist a unique solution $\varphi \in H^{\frac{6+l}{6},6+l}\left([0,T] \times \mathbb{S}^1\right)$ to (3.3.3) and a constant $C > 0$ independent of $\varphi$, $f$ and $\varphi_0$ such that*

$$\|\varphi\|_{H^{\frac{6+l}{6},6+l}([0,T]\times\mathbb{S}^1)} \leq C\Big(\|f\|_{H^{\frac{l}{6},l}([0,T]\times\mathbb{S}^1)} + \|\varphi_0\|_{C^{6+l}(\mathbb{S}^1)}\Big). \tag{3.3.4}$$

*Proof.* First, assume that $a_6 \equiv 1$ and $a_q \equiv 0$ for $q = 0, 1, \dots, 5$. Like in [73, Section 3] one proves that in this case, a weak solution to (3.3.3) exists. Localizing the problem and using [56, Theorem VI.21] one obtains Hölder regularity and (3.3.4). As in [10, Section 3.1.4], the general linear problem (3.3.3) is solved with the method of continuity. □

### 3.3.3. The nonlinear problem

In this subsection, we proceed as in [67, Section 2.5] and [45, Section 3.3]. For a smooth immersed reference curve $\gamma_*$ with normal $\nu_*$, we define the open set

$$W := \{\varphi \in C^1([0,T]\times\mathbb{S}^1;\mathbb{R}) : |\partial_x(\gamma_* + \varphi(t,\cdot)\nu_*)| \neq 0\}$$

and $F\colon \mathbb{S}^1 \times \mathbb{R}^7 \times (0,\infty) \to \mathbb{R}$ as

$$\begin{aligned} F(\cdot,\varphi,\partial_x\varphi,\dots,\partial_x^6\varphi,|\partial_x(\gamma_*+\varphi\nu_*)|^{-1}) &:= \langle \nabla_s^2 \nabla_{L^2}\mathcal{E}_\lambda(\gamma_*+\varphi\nu_*),\nu_*^\perp\rangle \\ &= \frac{1}{|\partial_x(\gamma_*+\varphi\,\nu_*)|^6}\,\partial_x^6\varphi + Q(\cdot,\varphi,\partial_x\varphi,\dots,\partial_x^5\varphi,|\partial_x(\gamma_*+\varphi\,\nu_*)|^{-1}). \end{aligned} \tag{3.3.5}$$

For more details, see (3.3.15). As $Q$ is a polynomial with smooth coefficients, the mapping

$$\begin{aligned} \mathbf{F}\colon H^{\frac{6+l}{6},6+l}\left([0,T]\times\mathbb{S}^1\right)\cap W &\to H^{\frac{l}{6},l}\left([0,T]\times\mathbb{S}^1\right), \\ \varphi &\mapsto F(\cdot,\varphi,\partial_x\varphi,\dots,\partial_x^6\varphi,|\partial_x(\gamma_*+\varphi\nu_*)|^{-1}), \end{aligned}$$

$l \in \mathbb{R}_+\setminus\mathbb{Z}$, is well-defined. With this, we will write (3.3.16) as

$$\begin{cases} \dot\varphi = \mathbf{F}[\varphi] & \text{in } (0,T)\times\mathbb{S}^1, \\ \varphi(0,\cdot) = \varphi_0 & \text{on } \mathbb{S}^1. \end{cases} \tag{3.3.6}$$

**Proposition 3.3.4** ([99, Proposition A.4])**.** *Take $\varphi_0 \in C^{6+l}(\mathbb{S}^1)$ such that $\gamma_*+\varphi_0\nu_*$ is an immersed curve. Then there exists $T>0$ and a unique solution $\varphi \in H^{\frac{6+l}{6},6+l}\left([0,T]\times\mathbb{S}^1\right)$ to* (3.3.6)*.*

*Proof.* First, we assume that $0<l<1$.

*Step 1: Linearized problem.* Let us make the following observation. The Fréchet derivative of $\mathbf{F}$ at $\xi \in H^{\frac{6+l}{6},6+l}\left([0,T]\times\mathbb{S}^1\right)\cap W$ can be written as

$$\mathrm{D}\mathbf{F}[\xi] = \sum_{q=0}^{6} a_q \partial_x^q$$

with $a_6 = |\partial_x(\gamma_*+\xi\nu_*)|^{-6}$ and $a_q(t,x) = \tilde a_q(t,x,\xi,\partial_x\xi,\dots,\partial_x^5\xi,|\partial_x(\gamma_*+\xi\nu_*)|^{-1})$, where $q=0,\dots,5$, for smooth functions $\tilde a_q\colon [0,T]\times\mathbb{S}^1\times\mathbb{R}^6\times(0,\infty)\to\mathbb{R}$. In particular, $a_q \in H^{\frac{l}{6},l}\left([0,T]\times\mathbb{S}^1\right)$ for $q=0,\dots,6$. Since $\xi\in W$ and by compactness, $a_6$ is uniformly bounded away from zero on $[0,T]\times\mathbb{S}^1$. Hence, $\partial_t - \mathrm{D}\mathbf{F}[\xi]$ is a parabolic operator and Proposition 3.3.3 yields a unique solution $\sigma \in H^{\frac{6+l}{6},6+l}\left([0,T]\times\mathbb{S}^1\right)$ to

$$\begin{cases} \dot\sigma - \mathrm{D}\mathbf{F}[\xi]\sigma = f & \text{in } [0,T]\times\mathbb{S}^1, \\ \sigma(0,\cdot) = \varphi_0 & \text{on } \mathbb{S}^1 \end{cases} \tag{3.3.7}$$

for $f \in H^{\frac{l}{6},l}\left([0,T]\times\mathbb{S}^1\right)$.

*Step 2: Special solution.* Now, we consider (3.3.7) with $\xi \equiv 0$ and $f = \mathbf{F}[\varphi_0] - \mathrm{D}\mathbf{F}[0]\varphi_0$. Here, $\varphi_0 \in C^{6+l}(\mathbb{S}^1)$ is interpreted as function $\varphi_0 \in H^{\frac{6+l}{6},6+l}\left([0,T]\times\mathbb{S}^1\right) \cap W$ constant in time. Since $f \in H^{\frac{l}{6},l}\left([0,T]\times\mathbb{S}^1\right)$, there exists a unique solution in $H^{\frac{6+l}{6},6+l}\left([0,T]\times\mathbb{S}^1\right)$ to (3.3.7). In the following, we denote this solution by $\sigma$. Since $\varphi_0 \in W$, we can make $T > 0$ small enough such that $\sigma \in W$. Now, we define

$$\tilde{f} := \partial_t\sigma - \mathbf{F}[\sigma] \in H^{\frac{l}{6},l}\left([0,T]\times\mathbb{S}^1\right). \tag{3.3.8}$$

Evaluated at $t = 0$, we have

$$\tilde{f}(0,\cdot) = \mathrm{D}\mathbf{F}[0]\sigma|_{t=0} + \mathbf{F}[\varphi_0] - \mathrm{D}\mathbf{F}[0]\varphi_0 - \mathbf{F}[\sigma]|_{t=0} = 0. \tag{3.3.9}$$

Here, we used (3.3.7) and the smoothness of $\mathbf{F}$.

*Step 3: Application of the Inverse Function Theorem.* Let $0 < k < l$ such that $k \notin \mathbb{Z}$ and consider the nonlinear operator

$$\begin{aligned}\Phi\colon H^{\frac{6+k}{6},6+k}\left([0,T]\times\mathbb{S}^1\right)\cap W &\to H^{\frac{k}{6},k}\left([0,T]\times\mathbb{S}^1\right)\times C^{6+k}(\mathbb{S}^1),\\ \xi &\mapsto (\partial_t\xi - \mathbf{F}[\xi],\, \xi(0,\cdot)).\end{aligned}$$

By Step 1, the Fréchet derivative

$$\begin{aligned}\mathrm{D}\Phi[\sigma]\colon H^{\frac{6+k}{6},6+k}\left([0,T]\times\mathbb{S}^1\right) &\to H^{\frac{k}{6},k}\left([0,T]\times\mathbb{S}^1\right)\times C^{6+k}(\mathbb{S}^1),\\ \zeta &\mapsto (\partial_t\zeta - \mathrm{D}\mathbf{F}[\sigma]\zeta,\, \zeta(0,\cdot))\end{aligned}$$

is a linear isomorphism. It follows by the Local Inverse Function Theorem that $\Phi$ is a local diffeomorphism at $\sigma$. So there is a neighborhood $U \subset H^{\frac{6+k}{6},6+k}\left([0,T]\times\mathbb{S}^1\right)\cap W$ of $\sigma$ and a neighborhood $V \subset H^{\frac{k}{6},k}\left([0,T]\times\mathbb{S}^1\right)\times C^{6+k}(\mathbb{S}^1)$ of $\Phi(\sigma) = (\tilde{f},\varphi_0)$ such that $\Phi\colon U \to V$ is a diffeomorphism.

*Step 4: Cut-off in time.* Choose for $0 < \varepsilon < \min\{1, \frac{T}{2}\}$ cut-off functions $\eta_\varepsilon \in C^\infty([0,T])$ such that $0 \le \eta_\varepsilon \le 1$, $0 \le \dot{\eta}_\varepsilon \le 2\varepsilon^{-1}$ and $\eta_\varepsilon \equiv 0$ on $[0,\varepsilon]$ as well as $\eta_\varepsilon \equiv 1$ on $[2\varepsilon,T]$. Define

$$f_\varepsilon := \eta_\varepsilon\tilde{f}. \tag{3.3.10}$$

Analogously as in [67, Lemma 2.5.8], one shows that $f_\varepsilon \in H^{\frac{l}{6},l}\left([0,T]\times\mathbb{S}^1\right)$ with norm bounded independently of $\varepsilon$. Here, we need the assumption $l < 1$. The crucial point is estimating the Hölder seminorm in time. We have

$$|f_\varepsilon(t_1) - f_\varepsilon(t_2)| \le |\tilde{f}(t_1) - \tilde{f}(t_2)| + |\tilde{f}(t_1)||\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)|$$

for $t_1 < t_2 \in [0,T]$. Assume $t_1 < 2\varepsilon$, otherwise $|\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)| = 0$. For $t_2 \leq 3\varepsilon$, we obtain with (3.3.9) that

$$\begin{aligned}|\tilde{f}(t_1)||\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)| = |\tilde{f}(t_1) - \tilde{f}(0)||\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)| &\leq \frac{C}{\varepsilon} t_1^{\frac{l}{6}} |t_1 - t_2| \\ &\leq \frac{C}{\varepsilon} t_1^{\frac{l}{6}} |t_1 - t_2|^{\frac{l}{6}} t_2^{1-\frac{l}{6}} \leq \frac{C}{\varepsilon} t_2 |t_1 - t_2|^{\frac{l}{6}} \leq C|t_1 - t_2|^{\frac{l}{6}}\end{aligned}$$

for a constant $C$ not depending on $\varepsilon$. For $t_2 > 3\varepsilon$, one has

$$|\tilde{f}(t_1)||\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)| \leq C t_1^{\frac{l}{6}} \leq C\varepsilon^{\frac{l}{6}} \leq C|t_1 - t_2|^{\frac{l}{6}}.$$

Again, the constant $C$ does not depend on $\varepsilon$, which allows us to show the claim. Moreover, one proves with standard arguments (see for example [70, Section 6.8]) and the Arzelà–Ascoli theorem, that for $0 < k < l < 1$, $H^{\frac{l}{6},l}\left([0,T] \times \mathbb{S}^1\right)$ is compactly embedded in $H^{\frac{k}{6},k}\left([0,T] \times \mathbb{S}^1\right)$. It follows that there exists a sequence $\varepsilon_n \to 0$, $n \to \infty$, and $\hat{f} \in H^{\frac{l}{6},l}\left([0,T] \times \mathbb{S}^1\right)$ such that

$$f_{\varepsilon_n} \to \hat{f} \text{ in } H^{\frac{k}{6},k}\left([0,T] \times \mathbb{S}^1\right) \text{ for } n \to \infty. \tag{3.3.11}$$

Due to (3.3.9) and the continuity of $\tilde{f}$, we find that $f_\varepsilon \to \tilde{f}$, $\varepsilon \to 0$ uniformly in the sup norm. So we conclude that $\hat{f} = \tilde{f}$.

*Step 5: Existence of solution.* By (3.3.11), there exists some $\varepsilon_* > 0$ small enough such that $(f_{\varepsilon_*}, \varphi_0) \in V$ with $V$ given as in Step 3. This implies the existence of a unique $\varphi \in U$ such that

$$\partial_t \varphi = \mathbf{F}[\varphi] + f_{\varepsilon_*} \quad \text{and} \quad \varphi(0,\cdot) = \varphi_0.$$

With (3.3.10) and the definition of $\eta_\varepsilon$, we conclude that $\varphi$ solves (3.3.6) for $t \in [0, \varepsilon_*]$.

*Step 6: Regularity.* So far, we have $\varphi \in H^{\frac{6+k}{6},6+k}\left([0,\varepsilon_*] \times \mathbb{S}^1\right) \cap W$ for $0 < k < l < 1$. Defining

$$a_6(t,x) := |\partial_x(\gamma_* + \varphi\nu_*)|^{-6} \in H^{\frac{5+k}{6},5+k}\left([0,\varepsilon_*] \times \mathbb{S}^1\right) \subset H^{\frac{l}{6},l}\left([0,\varepsilon_*] \times \mathbb{S}^1\right)$$

and

$$p(t,x) := P(\cdot, \varphi, \ldots, \partial_x^5 \varphi, |\partial_x(\gamma_* + \varphi\nu_*)|^{-1}) \in H^{\frac{1+k}{6},1+k}\left([0,\varepsilon_*] \times \mathbb{S}^1\right) \subset H^{\frac{l}{6},l}\left([0,\varepsilon_*] \times \mathbb{S}^1\right)$$

with $P$ as in (3.3.5), Proposition 3.3.3 yields a unique solution $\xi \in H^{\frac{6+l}{6},6+l}\left([0,\varepsilon_*] \times \mathbb{S}^1\right)$

to

$$\begin{cases} \partial_t \xi - a_6 \partial_x^6 \xi = p & \text{in } [0, \varepsilon_*] \times \mathbb{S}^1, \\ \xi(0, \cdot) = \varphi_0 & \text{on } \mathbb{S}^1. \end{cases} \tag{3.3.12}$$

Since $\varphi$ solves (3.3.12) on $[0, \varepsilon_*] \times \mathbb{S}^1$, we conclude that $\xi = \varphi$ for $t \leq \varepsilon_*$. Hence, we obtain $\varphi \in H^{\frac{6+l}{6}, 6+l}\left([0, \varepsilon_*] \times \mathbb{S}^1\right)$.

*Step 7: Uniqueness.* Assume we have two solutions $\varphi_1, \varphi_2 \in H^{\frac{6+l}{6}, 6+l}\left([0, \varepsilon_i] \times \mathbb{S}^1\right)$, $i = 1, 2$ and $\varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_*$. Define

$$T^* = \sup\left\{t \in [0, \varepsilon_1) \colon \varphi_1 = \varphi_2 \text{ in } C^{6+l}(\mathbb{S}^1) \text{ on } [0, t]\right\}.$$

We consider $\Phi$ of Step 3 for variable $T$ and show that for $T > 0$ small enough,

$$\varphi_1|_{[0,T]}, \varphi_2|_{[0,T]} \in U \quad \text{as well as} \quad (0, \varphi_0) \in V. \tag{3.3.13}$$

From the proof of the Inverse Function Theorem, it follows that there exists $r > 0$ such that

$$B_r(\sigma) \subset U(T) \quad \text{and} \quad B_r(\tilde{f}, \varphi_0) \subset V(T) \quad \text{for all } 0 < T \leq \varepsilon_1. \tag{3.3.14}$$

Here, $\sigma$ and $\tilde{f}$ are given as in Step 2. Now, we first note that with (3.3.8) and (3.3.9), for $q = 0, \dots, 6$,

$$\partial_x^q(\varphi_i - \sigma)|_{t=0} = 0 \quad \text{and} \quad \partial_t(\varphi_i - \sigma)|_{t=0} = \left(\mathbf{F}[\varphi_i] - \mathbf{F}[\sigma] - \tilde{f}\right)|_{t=0} = 0.$$

With Lemma 3.3.5 it follows that

$$\|\varphi_i - \sigma\|_{H^{\frac{6+k}{6}, 6+k}([0,T] \times \mathbb{S}^1)} \to 0 \quad \text{as } T \to 0.$$

Similarly, (3.3.9) implies

$$\|\tilde{f}\|_{H^{\frac{k}{6}, k}([0,T] \times \mathbb{S}^1)} \to 0 \quad \text{as } T \to 0.$$

Due to (3.3.14) we conclude that for $T > 0$ small enough, (3.3.13) holds and thus

$$\varphi_1 = \varphi_2 \text{ in } H^{\frac{6+k}{6}, 6+k}\Big([0, T] \times \mathbb{S}^1\Big).$$

The regularity of $\varphi_1$ and $\varphi_2$ even gives coincidence in $H^{\frac{6+l}{6}, 6+l}\left([0, T] \times \mathbb{S}^1\right)$. It follows that $T^* > 0$. Now, assume $T^* < \varepsilon_1$. Then $\varphi_1(T^*) = \varphi_2(T^*) =: \tilde{\varphi}_0 \in C^{6+l}(\mathbb{S}^1)$. Taking $\tilde{\varphi}_0$ as new initial datum and repeating the arguments yields a contradiction to the definition of

$T^*$. Hence, we get $T^* = \varepsilon_1$ and uniqueness is proven.

Bootstrapping and using the same arguments as in Step 6 shows the claim for $l > 1$. □

In the last step of the proof of Proposition 3.3.4, we use the following observation.

**Lemma 3.3.5** ([99, Lemma A.5])**.** *Let $0 < k < l < 1$ and $f \in H^{\frac{6+l}{6},6+l}\left([0,T] \times \mathbb{S}^1\right)$ such that*

$$f(0,\cdot) = \partial_t f(0,\cdot) = \partial_x f(0,\cdot) = \cdots = \partial_x^6 f(0,\cdot) = 0.$$

*Then $\|f\|_{H^{\frac{6+k}{6},6+k}([0,T]\times\mathbb{S}^1)} \to 0$ as $T \to 0$.*

*Proof.* Consider the periodic counterpart $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ and let the interval $I \subset \mathbb{R}$ contain at least two periods of $f$. Then

$$\begin{aligned}
\|f\|_{H^{\frac{6+k}{6},6+k}([0,T]\times\mathbb{S}^1)} &= \sup_{[0,T]\times I} |f| + \sup_{[0,T]\times I} |\partial_t f| + \sup_{[0,T]\times I} |\partial_x f| + \cdots + \sup_{[0,T]\times I} |\partial_x^6 f| \\
&+ \sup_{\substack{t\in[0,T],x_1,x_2\in I \\ x_1\neq x_2}} \frac{|\partial_t f(t,x_1) - \partial_t f(t,x_2)|}{|x_1 - x_2|^k} + \sup_{\substack{t\in[0,T],x_1,x_2\in I \\ x_1\neq x_2}} \frac{|\partial_x^6 f(t,x_1) - \partial_x^6 f(t,x_2)|}{|x_1 - x_2|^k} \\
&+ \sup_{\substack{t_1,t_2\in[0,T],x\in I \\ t_1\neq t_2}} \frac{|\partial_t f(t_1,x) - \partial_t f(t_2,x)|}{|t_1 - t_2|^{\frac{k}{6}}} + \sup_{\substack{t_1,t_2\in[0,T],x\in I \\ t_1\neq t_2}} \frac{|\partial_x f(t_1,x) - \partial_x f(t_2,x)|}{|t_1 - t_2|^{\frac{5+k}{6}}} \\
&+ \cdots + \sup_{\substack{t_1,t_2\in[0,T],x\in I \\ t_1\neq t_2}} \frac{|\partial_x^6 f(t_1,x) - \partial_x^6 f(t_2,x)|}{|t_1 - t_2|^{\frac{k}{6}}}.
\end{aligned}$$

For the terms coming from the Hölder seminorm in space, interpolation as in [110, Prop. 1.1.3] yields

$$\begin{aligned}
\sup_{\substack{t\in[0,T],x_1,x_2\in I \\ x_1\neq x_2}} \frac{|\partial_t f(t,x_1) - \partial_t f(t,x_2)|}{|x_1 - x_2|^k} &\leq \sup_{t\in[0,T]} \|\partial_t f\|_{C^k(\mathbb{S}^1)} \\
&\leq \sup_{t\in[0,T]} \|\partial_t f\|_{C^l(\mathbb{S}^1)}^{1-\frac{k}{l}} \|\partial_t f\|_{C(\mathbb{S}^1)}^{\frac{k}{l}} \to 0 \quad \text{as } T \to 0.
\end{aligned}$$

For the terms coming from the Hölder seminorm in time, we have

$$\begin{aligned}
\sup_{\substack{t_1,t_2\in[0,T],x\in I \\ t_1\neq t_2}} \frac{|\partial_t f(t,x_1) - \partial_t f(t,x_2)|}{|t_1 - t_2|^{\frac{k}{6}}} &= \sup_{\substack{t_1,t_2\in[0,T],x\in I \\ t_1\neq t_2}} \frac{|\partial_t f(t,x_1) - \partial_t f(t,x_2)|}{|t_1 - t_2|^{\frac{l}{6}}} |t_1 - t_2|^{\frac{l-k}{6}} \\
&\leq C \sup_{\substack{t_1,t_2\in[0,T] \\ t_1\neq t_2}} |t_1 - t_2|^{\frac{l-k}{6}} \to 0 \quad \text{as } T \to 0.
\end{aligned}$$

Treating the other terms with the same arguments yields the claim. □

### 3.3.4. Proof of short time existence

Now, we are ready to proof short time existence. The idea of the proof is to use the reparametrization argument in Section 3.3.1 and to find the solution as a graph over the initial datum. This approach leads to a scalar problem, which we can uniquely solve using Proposition 3.3.4 in Section 3.3.3.

**Theorem 3.3.6** ([99, Theorem 4.1])**.** *Let $\gamma_0\colon \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth regular curve. Then there exists some $T > 0$ and a smooth solution $\gamma\colon [0,T]\times\mathbb{S}^1 \to \mathbb{R}^2$ to* (3.0.3)*. This solution is unique up to reparametrizations.*

*Proof. Step 1: Good neighborhood.* By Lemma 3.3.1, there exists a $W^{5,2}$-neighborhood $V$ of the smooth immersion $\gamma_0$, such that for all $\zeta \in C^\infty(\mathbb{S}^1;\mathbb{R}^2) \cap V$, there exists a diffeomorphism $\Phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ and a reparametrization $\tilde{\zeta} = \zeta \circ \Phi$ such that $\tilde{\zeta} = \gamma_0 + \varphi\nu_0$ for some $\varphi\colon \mathbb{S}^1 \to \mathbb{R}$. By Remark 3.3.2, the smoothness of $\zeta$ and the smoothness of $\gamma_0$ yield $\Phi \in C^\infty(\mathbb{S}^1;\mathbb{S}^1)$ and $\varphi \in C^\infty(\mathbb{S}^1;\mathbb{R})$.

*Step 2: Velocity in terms of $\gamma$.* Let $\bar{\gamma}\colon \mathbb{S}^1 \to \mathbb{R}^2$ be a sufficiently smooth immersed curve. Continuing the explicit calculation from [139, Proposition A.1], we obtain

$$\nabla_s^2 \nabla_{L^2}\mathcal{E}_\lambda(\bar{\gamma}) = P^{\perp_{\bar{\gamma}}}\left(\frac{\partial_x^6\bar{\gamma}}{|\partial_x\bar{\gamma}|^6} + \tilde{Q}(\partial_x\bar{\gamma},\dots,\partial_x^5\bar{\gamma},|\partial_x\bar{\gamma}|^{-1})\right)$$

for $\tilde{Q}$ a vector valued polynomial. If $\bar{\gamma}$ is time-dependent and $\bar{\gamma}(t,x) = \gamma_0(x) + \bar{\varphi}(t,x)\nu_0(x)$, then

$$P^{\perp_{\bar{\gamma}}}\partial_x^6\bar{\gamma} = P^{\perp_{\bar{\gamma}}}(\partial_x^6\gamma_0 + \partial_x^6\bar{\varphi}\nu_0 + \bar{\varphi}\partial_x^6\nu_0 + \text{l.o.t.}).$$

As long as $\nu_0^\perp = P^{\perp_{\bar{\gamma}}}\nu_0$ does not vanish, we obtain

$$\nabla_s^2\nabla_{L^2}\mathcal{E}_\lambda(\bar{\gamma}) = \frac{\partial_x^6\bar{\varphi}\,\nu_0^\perp}{|\partial_x(\gamma_0+\bar{\varphi}\nu_0)|^6} + Q(\cdot,\bar{\varphi},\partial_x\bar{\varphi},\dots,\partial_x^5\bar{\varphi},|\partial_x(\gamma_0+\bar{\varphi}\,\nu_0)|^{-1})\nu_0^\perp \tag{3.3.15}$$

with a scalar valued polynomial $Q$. Here, $\gamma_0$ and its derivatives are considered as fixed functions and are included in $Q$.

*Step 3: Construction of a solution.* The previous steps motivate to consider the system

$$\begin{cases} \partial_t\varphi = \dfrac{\partial_x^6\varphi}{|\partial_x(\gamma_0+\varphi\nu_0)|^6} + Q(\cdot,\varphi,\dots,\partial_x^5\varphi,|\partial_x(\gamma_0+\varphi\nu_0)|^{-1}) & \text{in } (0,T)\times\mathbb{S}^1, \\ \varphi(0,\cdot) = 0 & \text{on } \mathbb{S}^1. \end{cases} \tag{3.3.16}$$

By Proposition 3.3.4, there exists $T > 0$ and a unique solution $\varphi \in C^\infty([0,T]\times\mathbb{S}^1;\mathbb{R}^2)$ to (3.3.16). With this solution, we define $\gamma := \gamma_0 + \varphi\nu_0 \in C^\infty([0,T]\times\mathbb{S}^1;\mathbb{R}^2)$. By possibly making $T > 0$ smaller, we achieve $\gamma(t,\cdot) \in V$ and $P^{\perp_\gamma}\nu_0 = \nu_0^\perp \neq 0$ for all $t \in [0,T]$. Moreover, $\partial_t^\perp\gamma = \partial_t\varphi\,\nu_0^\perp = \nabla_s^2\nabla_{L^2}\mathcal{E}_\lambda(\gamma)$ by (3.3.15) and (3.3.16). Thus, $\gamma$ solves (3.0.3).

*Step 4: Geometric uniqueness.* Let $\tilde{\gamma} \in C^\infty([0,\tilde{T}] \times \mathbb{S}^1; \mathbb{R}^2)$ be another solution to (3.0.3) in the sense that

$$\partial_t^\perp \tilde{\gamma} = \nabla_s^2 \nabla_{L^2} \mathcal{E}_\lambda(\tilde{\gamma}) \quad \text{and} \quad \tilde{\gamma}(0,\cdot) = \gamma_0 \circ \tilde{\Phi}$$

for a smooth orientation preserving diffeomorphism $\tilde{\Phi}\colon \mathbb{S}^1 \to \mathbb{S}^1$. For the sake of readability, we do not change the notation, but we reparametrize $\tilde{\gamma}$ by $\tilde{\Phi}^{-1}$ such that $\tilde{\gamma}(0,\cdot) = \gamma_0$. By Remark 3.2.1, the reparametrization also satisfies $\partial_t^\perp \tilde{\gamma} = \nabla_s^2 \nabla_{L^2} \mathcal{E}_\lambda(\tilde{\gamma})$. Now, make $\tilde{T}$ so small that $\tilde{\gamma}(t,\cdot) \in V$ for all $t \in [0,\tilde{T}]$. Then there exists a reparametrization (which we again do not rename) such that $\tilde{\gamma} = \gamma_0 + \tilde{\varphi}\nu_0$ for some $\tilde{\varphi} \in C^\infty([0,\tilde{T}) \times \mathbb{S}^1; \mathbb{R})$ with $\tilde{\varphi}(0) = 0$. Since $\partial_t^\perp \tilde{\gamma} = \nabla_s^2 \nabla_{L^2} \mathcal{E}_\lambda(\tilde{\gamma})$ and $\tilde{\varphi}(0) = 0$, we see that $\tilde{\varphi}$ solves (3.3.16). As the solution to (3.3.16) is unique, it follows that $\tilde{\varphi} = \varphi$ and therefore $\tilde{\gamma} = \gamma$. We conclude that every solution to (3.0.3) is a reparametrization of the solution $\gamma$ constructed in Step 3. □

**Remark 3.3.7** ([99, Remark 5])**.** More generally, the solution can also be found as a graph over any arbitrary reference immersion $\hat{\gamma} \in W^{5,2}(\mathbb{S}^1; \mathbb{R}^2)$, provided that it satisfies $\|\hat{\gamma} - \gamma_0\|_{W^{4,2}} < r$, where $r$ is the constant of Lemma 3.3.1. The reference curve in (3.3.15) and the initial datum in (3.3.16) then change accordingly.

## 3.4. Global existence and subconvergence

The proof of global in time existence relies on showing that the solution remains bounded in finite time and can, therefore, be extended as long as the maximal existence time is finite. This idea is also used for the classical elastic flow in [53] and [43]. Due to the sixth order structure, we require new geometric lemmata, which we prove in Section 3.4.1. In Section 3.4.2, we see that the length of the evolving curves grows at most linearly and is bounded if $\lambda > 0$. In Section 3.4.3, we then prove global existence. The boundedness of the solution also provides subconvergence if the length is finite, i.e. for $\lambda > 0$.

### 3.4.1. Geometrical lemmata and interpolation inequalities

Using (3.2.1) and integration by parts directly yields the following lemma.

**Lemma 3.4.1** ([99, Lemma 5.1])**.** *For a sufficiently smooth solution $\gamma\colon [0,T) \times \mathbb{S}^1 \to \mathbb{R}^2$ of a flow $\partial_t \gamma = V + h\partial_s \gamma$ and a normal vector field $\phi : [0,T) \times \mathbb{S}^1 \to \mathbb{R}^2$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{S}^1} |\phi|^2 \,\mathrm{d}s + \int_{\mathbb{S}^1} |\nabla_s^3 \phi|^2 \,\mathrm{d}s = \int_{\mathbb{S}^1} \langle \phi, \nabla_t \phi - \nabla_s^6 \phi + \frac{1}{2}\phi\, \partial_s h\rangle \,\mathrm{d}s - \frac{1}{2}\int_{\mathbb{S}^1} |\phi|^2 \langle \kappa, V\rangle \,\mathrm{d}s.$$

For the following, we introduce a quite common notation (see for example [53] or [43]). Let $\phi$ and $\phi_i$, $i \in \{1,\dots,k\}$, be normal vector fields. We denote by $\phi_1 * \phi_2 * \cdots * \phi_k$ a

term of the type

$$\begin{cases} \langle \phi_{i_1}, \phi_{i_2} \rangle \cdots \langle \phi_{i_{k-1}}, \phi_{i_k} \rangle, & \text{if } k \text{ is even,} \\ \langle \phi_{i_1}, \phi_{i_2} \rangle \cdots \langle \phi_{i_{k-2}}, \phi_{i_{k-1}} \rangle \phi_{i_k}, & \text{if } k \text{ is odd,} \end{cases}$$

where $i_1, \dots, i_k$ is some permutation of $1, \dots, k$. By $P_b^a(\phi)$, we denote any linear combination of terms of the type

$$\nabla_s^{j_1} \phi * \cdots * \nabla_s^{j_b} \phi \qquad \text{with } j_k \in \mathbb{N}_0 \text{ for all } k \in \{1, \dots, b\} \text{ and } j_1 + \cdots + j_b = a$$

with coefficients bounded by some universal constant. Observe that

$$\nabla_s P_b^a(\phi) = P_b^{a+1}(\phi). \tag{3.4.1}$$

**Lemma 3.4.2** ([99, Lemma 5.2])**.** *For a sufficiently smooth solution $\gamma\colon [0,T) \times \mathbb{S}^1 \to \mathbb{R}^2$ of*

$$\partial_t \gamma = \nabla_s^2 (\nabla_s^2 \kappa + \frac{1}{2} |\kappa|^2 \kappa - \lambda \kappa) + h \partial_s \gamma = \nabla_s^4 \kappa + P_3^2(\kappa) - \lambda \nabla_s^2 \kappa + h \partial_s \gamma,$$

*we have*

$$\nabla_t \nabla_s^m \kappa - \nabla_s^{m+6} \kappa = P_3^{m+4}(\kappa) + P_5^{m+2}(\kappa) + P_3^{m+2}(\kappa) - \lambda \nabla_s^{m+4} \kappa + h \nabla_s^{m+1} \kappa.$$

*Proof.* The claim follows by induction on $m$. For $m = 0$ we use (3.2.5). The induction step follows with (3.2.6) and (3.4.1). □

For the convenience of the reader, we repeat in the following lemmata two results of [53] and [43] without giving the proof.

**Lemma 3.4.3** ([53, Proposition 2.5])**.** *Let $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ be a sufficiently smooth curve. Then for $P_b^a(\kappa)$ with $b \geq 2$ containing only derivatives of $\kappa$ of order at most $k-1$ with $2k + 1 > a + \frac{1}{2} b$, we have*

$$\int_{\mathbb{S}^1} |P_b^a(\kappa)| \, \mathrm{d}s \leq \varepsilon \int_{\mathbb{S}^1} |\nabla_s^k \kappa|^2 \, \mathrm{d}s + C \varepsilon^{-\frac{\mu}{2-\mu}} \left( \int_{\mathbb{S}^1} |\kappa|^2 \, \mathrm{d}s \right)^{\frac{b-\mu}{2-\mu}} + C \left( \int_{\mathbb{S}^1} |\kappa|^2 \, \mathrm{d}s \right)^{a+b-1}$$

*for any $\varepsilon > 0$, $\mu = (a + \frac{1}{2} b - 1)/k$, and $C = C(k, a, b)$.*

**Lemma 3.4.4** ([43, Lemma 4.6])**.** *Let $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ be a sufficiently smooth curve. Assume that $\|\nabla_s^m \kappa\|_{L^2(\mathrm{d}s)} \leq C_m$ for some $m \geq 1$ and $\|\kappa\|_{L^2(\mathrm{d}s)} \leq C_0$. Then*

$$\|\partial_s^l \kappa\|_{L^2(\mathrm{d}s)} \leq C \quad \text{for } 0 \leq l \leq m$$

*with a constant $C = C(C_m, C_0, m, l)$.*

### 3.4.2. Bounds on the length

To prove a longtime existence result we have to control the length of the evolving curve. A uniform bound from below is given by Fenchel's theorem which yields

$$4\pi^2 \leq \mathcal{L}(\gamma)\|\kappa\|^2_{L^2(\,\mathrm{d}s)} \leq 2\mathcal{L}(\gamma)\mathcal{E}_\lambda(\gamma_0). \tag{3.4.2}$$

Here, for the second estimate, we used Lemma 3.2.4. If $\lambda > 0$, we also have a uniform bound on the length from above by

$$\mathcal{L}(\gamma) \leq \frac{1}{\lambda}\mathcal{E}_\lambda(\gamma) \leq \frac{1}{\lambda}\mathcal{E}_\lambda(\gamma_0). \tag{3.4.3}$$

In order to allow for $\lambda = 0$, we show that the length grows at most linearly.

**Lemma 3.4.5** ([99, Lemma 5.5])**.** *Let $T > 0$ and $\gamma\colon [0,T)\times\mathbb{S}^1 \to \mathbb{R}^2$ be a smooth solution to* (3.0.3)*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\gamma) \leq C(\mathcal{E}_\lambda(\gamma_0)).$$

*Proof.* Using (3.2.1) and integration by parts and applying Lemma 3.4.3 with $a = 2$, $b = 4$, $k = 2$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\gamma) &= -\int_{\mathbb{S}^1}\langle\partial_t\gamma,\kappa\rangle\,\mathrm{d}s = -\int_{\mathbb{S}^1}|\nabla_s^2\kappa|^2\,\mathrm{d}s + \frac{3}{2}\int_{\mathbb{S}^1}|\kappa|^2|\nabla_s\kappa|^2\,\mathrm{d}s - \lambda\int_{\mathbb{S}^1}|\nabla_s\kappa|^2\,\mathrm{d}s\\ &\leq -\int_{\mathbb{S}^1}|\nabla_s^2\kappa|^2\,\mathrm{d}s + \varepsilon\int_{\mathbb{S}^1}|\nabla_s^2\kappa|^2\,\mathrm{d}s + C(\varepsilon)\left(\int_{\mathbb{S}^1}|\kappa|^2\,\mathrm{d}s\right)^5\end{aligned}$$

for all $t \in (0,T)$, where $\varepsilon > 0$ can be chosen arbitrarily small. The claim follows with Lemma 3.2.4. □

### 3.4.3. Proof of global existence and subconvergence

With the preliminary work of Section 3.4.1 and 3.4.2 we are now ready to prove global existence for $\lambda \geq 0$. If $\lambda > 0$, we additionally obtain subconvergence. This result crucially relies on the fact that, in this case, the length of the evolving curve is uniformly bounded from above.

**Theorem 3.4.6** (Global existence and subconvergence, [99, Theorem 5.6])**.** *For any $\lambda \geq 0$, there exists a geometrically unique global smooth solution $\gamma\colon [0,\infty)\times\mathbb{S}^1 \to \mathbb{R}^2$ of* (3.0.3)*. If $\lambda > 0$, then as $t_i \to \infty$ the curves $\gamma(t_i,\cdot)$ subconverge smoothly, when reparametrized with constant speed and suitably translated, to some stationary solution $\gamma_\infty$.*

*Proof.* Theorem 3.3.6 gives a smooth solution $\gamma$ of (3.0.3) in a small time interval. We assume by contradiction that this solution does not exist globally in time and denote by

$0 < T < \infty$ the maximal existence time. Let $h \in C^\infty([0,T]\times\mathbb{S}^1;\mathbb{R})$ be the scalar tangential velocity, i.e. $\partial_t\gamma = \nabla_s^2(\nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa - \lambda\kappa) + h\partial_s\gamma$.

*Step 1: Bound on $\|\nabla_s^m\kappa\|_{L^2(\mathrm{d}s)}$ for $m \in \mathbb{N}_0$.* With Lemma 3.4.1 and Lemma 3.4.2 it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1}|\nabla_s^m\kappa|^2\,\mathrm{d}s + \int_{\mathbb{S}^1}|\nabla_s^{m+3}\kappa|^2\,\mathrm{d}s &= \int_{\mathbb{S}^1}\langle\nabla_s^m\kappa, P_3^{m+4}(\kappa) + P_5^{m+2}(\kappa) + P_3^{m+2}(\kappa)\rangle\,\mathrm{d}s\\ &\quad - \lambda\int_{\mathbb{S}^1}|\nabla_s^{m+2}\kappa|^2\,\mathrm{d}s + \int_{\mathbb{S}^1}h\langle\nabla_s^m\kappa,\nabla_s^{m+1}\kappa\rangle\,\mathrm{d}s\\ &\quad + \frac{1}{2}\int_{\mathbb{S}^1}|\nabla_s^m\kappa|^2\partial_s h\,\mathrm{d}s - \frac{1}{2}\int_{\mathbb{S}^1}|\nabla_s^m\kappa|^2\langle\kappa,V\rangle\,\mathrm{d}s,\end{aligned} \tag{3.4.4}$$

where $V = \nabla_s^2\left(\nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa - \lambda\kappa\right)$. The second integral is negative and can be ignored in the following estimates. The two integrals containing the tangential velocity vanish after integration by parts. The other terms on the right hand side of (3.4.4) have the structure

$$\int_{\mathbb{S}^1}P_4^{2m+4}(\kappa)\,\mathrm{d}s + \int_{\mathbb{S}^1}P_6^{2m+2}(\kappa)\,\mathrm{d}s + \int_{\mathbb{S}^1}P_4^{2m+2}(\kappa)\,\mathrm{d}s.$$

With Lemma 3.4.3 (for $a = 2m+4$, $b = 4$, $k = m+3$ and for $a = 2m+2$, $b = 6$, $k = m+3$), we know that

$$\int_{\mathbb{S}^1}|P_4^{2m+4}(\kappa)|\,\mathrm{d}s, \int_{\mathbb{S}^1}|P_6^{2m+2}(\kappa)|\,\mathrm{d}s \le \varepsilon\int_{\mathbb{S}^1}|\nabla_s^{m+3}\kappa|^2\,\mathrm{d}s + C(m,\varepsilon)\left(\int_{\mathbb{S}^1}|\kappa|^2\,\mathrm{d}s\right)^{2m+7} \tag{3.4.5}$$

for any $\varepsilon > 0$. Further, Lemma 3.4.3 with $a = 2m+2$, $b = 4$, $k = m+3$ yields

$$\begin{aligned}\int_{\mathbb{S}^1}|P_4^{2m+2}(\kappa)|\,\mathrm{d}s &\le \varepsilon\int_{\mathbb{S}^1}|\nabla_s^{m+3}\kappa|^2\,\mathrm{d}s + C(m,\varepsilon)\left(\int_{\mathbb{S}^1}|\kappa|^2\,\mathrm{d}s\right)^{\frac{2}{3}m+3}\\ &\quad + C(m)\left(\int_{\mathbb{S}^1}|\kappa|^2\,\mathrm{d}s\right)^{2m+5}\end{aligned} \tag{3.4.6}$$

for any $\varepsilon > 0$. Lastly, we use Lemma 3.4.3 with $a = 2m$, $b = 2$ and $k = m+3$ to see that

$$\int_{\mathbb{S}^1}|\nabla_s^m\kappa|^2\,\mathrm{d}s \le \varepsilon\int_{\mathbb{S}^1}|\nabla_s^{m+3}\kappa|^2\,\mathrm{d}s + C\varepsilon^{-\frac{m}{3}}\int_{\mathbb{S}^1}|\kappa|^2\,\mathrm{d}s + C\left(\int_{\mathbb{S}^1}|\kappa|^2\,\mathrm{d}s\right)^{2m+1}. \tag{3.4.7}$$

Putting (3.4.4), (3.4.5), (3.4.6) and (3.4.7) together, choosing $\varepsilon$ small enough and using Lemma 3.2.4, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1}|\nabla_s^m\kappa|^2\,\mathrm{d}s + \int_{\mathbb{S}^1}|\nabla_s^m\kappa|^2\,\mathrm{d}s \le C(\mathcal{E}_\lambda(\gamma_0), m).$$

It follows that $\|\nabla_s^m\kappa\|_{L^2(\mathrm{d}s)}(t) \le \|\nabla_s^m\kappa\|_{L^2(\mathrm{d}s)}(0) + C(\mathcal{E}_\lambda(\gamma_0), m)$.

*Step 2: Bound on $\|\partial_s^m \kappa\|_{C^0}$ for $m \in \mathbb{N}_0$.* By Step 1, Lemma 3.2.4 and Lemma 3.4.4 we have

$$\|\partial_s^m \kappa\|_{L^2(\mathrm{d}s)} \leq C(\gamma_0, m)$$

for $m \in \mathbb{N}_0$. As in [24, Theorem 2.2] one shows that

$$\begin{aligned}\|\partial_s^m \kappa\|_{C^0} &\leq \|\partial_s^{m+1} \kappa\|_{L^1(\mathrm{d}s)} + \mathcal{L}(\gamma)^{-1} \|\partial_s^m \kappa\|_{L^1(\mathrm{d}s)} \\ &\leq \mathcal{L}(\gamma)^{\frac{1}{2}} \|\partial_s^{m+1} \kappa\|_{L^2(\mathrm{d}s)} + \mathcal{L}(\gamma)^{-\frac{1}{2}} \|\partial_s^m \kappa\|_{L^2(\mathrm{d}s)}.\end{aligned}$$

The length is bounded in finite time by Lemma 3.4.5 and (3.4.2). Thus, we conclude that

$$\|\partial_s^m \kappa\|_{C^0} \leq C(\gamma_0, m, T). \tag{3.4.8}$$

Note that this also implies that

$$\|\partial_s^m V\|_{C^0} \leq C(\gamma_0, m, T). \tag{3.4.9}$$

*Step 3: Global existence.* Due to Remark 3.2.1, we can consider the reparametrization $\tilde{\gamma}$ of $\gamma$ with constant speed, i.e. such that $|\partial_x \tilde{\gamma}| = \mathcal{L}(\gamma)$. This ensures that $|\partial_x \tilde{\gamma}|$ is bounded away from zero and bounded from above in the finite time interval $[0, T]$, see (3.4.2) and Lemma 3.4.5. Moreover,

$$\partial_x^{m+2} \tilde{\gamma} = \mathcal{L}(\gamma)^{m+2} \partial_s^m \tilde{\kappa}$$

for $m \in \mathbb{N}_0$, which implies by Step 2 that

$$\|\partial_x^{m+2} \tilde{\gamma}\|_{C^0} \leq C(\gamma_0, m, T) \quad \text{for } m \in \mathbb{N}_0.$$

With the control of the normal velocity, we additionally obtain $\|\tilde{\gamma}\|_{C^0} \leq C(\gamma_0, T)$. These bounds allow for a smooth extension of $\tilde{\gamma}$ to $[0, T] \times \mathbb{S}^1$. The short time existence result gives a smooth extension for $t > T$ which contradicts the maximality of $T$. Thus, the maximal existence time must be infinite.

*Step 4: Subconvergence for $\lambda > 0$.* In the case of $\lambda > 0$, the length is bounded independently of $T$ (see (3.4.3)). Thus, the bounds in (3.4.8) and (3.4.9) do not depend on $T$. Again, we consider the reparametrization $\tilde{\gamma}$ with constant speed such that $|\partial_x \tilde{\gamma}|$ is controlled up to $T = \infty$ (in this case independently of $T$). Moreover, adding a suitable translation like

$$\tilde{\gamma}(t, \cdot) - p(t) := \tilde{\gamma}(t, \cdot) - \frac{1}{\mathcal{L}(\tilde{\gamma})} \int_{\mathbb{S}^1} \tilde{\gamma} \, \mathrm{d}s_{\tilde{\gamma}}, \tag{3.4.10}$$

controls $\|\tilde{\gamma} - p\|_{C^0}$ up to $T = \infty$. In total, $\|\partial_x^m(\tilde{\gamma} - p)\|_{C^0} \leq C(\gamma_0, m)$ for all $m \in \mathbb{N}_0$ and $t \in [0, \infty)$. So, given any sequence $(t_i)_{i\in\mathbb{N}}$, $t_i \to \infty$, there exists a subsequence $(t_{i_k})_{k\in\mathbb{N}}$ such that $\tilde{\gamma}(t_{i_k}, \cdot) - p(t_{i_k})$ converges smoothly to a limit curve $\gamma_\infty$.

*Step 5: Limits of convergent subsequences.* We define $\tilde{V} := \nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa - \lambda\kappa$ and $u(t) := \|\nabla_s\tilde{V}\|^2_{L^2(\,\mathrm{d}s)}$. With (3.2.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = 2\int_{\mathbb{S}^1}\langle\nabla_t\nabla_s\tilde{V}, \nabla_s\tilde{V}\rangle\,\mathrm{d}s + \int_{\mathbb{S}^1}|\nabla_s\tilde{V}|^2(\partial_s\tilde{h} - \langle\kappa, V\rangle)\,\mathrm{d}s, \tag{3.4.11}$$

where $\tilde{h} = \langle\partial_t\tilde{\gamma}, \partial_s\tilde{\gamma}\rangle$ is the tangential velocity of $\tilde{\gamma}$. With a computation using (3.2.5), (3.2.6), and Lemma 3.4.2, we see that the term $\nabla_t\nabla_s(|\kappa|^2\kappa)$ appearing in $\nabla_t\nabla_s\tilde{V}$ can be written as

$$\nabla_t\nabla_s(|\kappa|^2\kappa) = P(\kappa) + \tilde{h}\nabla_s^2(|\kappa|^2\kappa),$$

where $P(\kappa)$ represents several polynomials of the form $P_b^a(\kappa)$, $b \in \mathbb{N}$ odd and $5 \geq a \in \mathbb{N}$. Thus, we see with Lemma 3.4.2 that

$$\nabla_t\nabla_s\tilde{V} = P(\kappa) + \tilde{h}\Big(\nabla_s^4\kappa + \frac{1}{2}\nabla_s^2(|\kappa|^2\kappa) - \lambda\nabla_s^2\kappa\Big) = P(\kappa) + \tilde{h}\nabla_s^2\tilde{V}.$$

Here, $P(\kappa)$ represents now polynomials of the form $P_b^a(\kappa)$, $b \in \mathbb{N}$ odd and $7 \geq a \in \mathbb{N}$. With this and integration by parts, we see that all terms containing the tangential velocity $\tilde{h}$ in (3.4.11) cancel. Thus, with the previous steps, $|\frac{\mathrm{d}}{\mathrm{d}t}u(t)|$ is bounded by a constant depending on $\gamma_0$ for $t \in [0, \infty)$. Since $u \in L^1((0, \infty))$ due to Lemma 3.2.4, this is sufficient to deduce $\lim_{t\to\infty} u(t) = 0$. It follows that $\gamma_\infty$ solves (3.0.5) and is therefore stationary. □

**Remark 3.4.7** (Flow without tangential velocity, [99, Remark 6])**.** Like in some other articles, for example [53] or [43], one could also impose that the tangential velocity is zero and study

$$\begin{cases}\partial_t\gamma = V = \nabla_s^2(\nabla_s^2\kappa + \frac{1}{2}|\kappa|^2\kappa - \lambda\kappa) & \text{in } (0,T)\times\mathbb{S}^1,\\ \gamma(0,\cdot) = \gamma_0 & \text{on } \mathbb{S}^1.\end{cases}$$

In this case, Step 3 in the proof of global existence is different, because one does not want to work with a reparametrization. This difficulty can be overcome with the following considerations. First, one proves bounds on $\|\partial_x^m\kappa\|_{C^0}$ for $m \in \mathbb{N}_0$: By induction, it can be shown that for a sufficiently smooth function $h\colon \mathbb{S}^1 \to \mathbb{R}$ or vector field $h\colon \mathbb{S}^1 \to \mathbb{R}^2$, for example $h = \kappa$ or $h = V$, we have

$$\partial_x^m h = |\partial_x\gamma|^m\partial_s^m h + \sum_{j=1}^{m-1} F_{m-1}(|\partial_x\gamma|, \ldots, \partial_x^{m-1}|\partial_x\gamma|)\partial_s^j h, \tag{3.4.12}$$

where $F_{m-1}$ is a polynomial of degree at most $m-1$. With Step 2, it therefore suffices to find a bound on $\|\partial_x^m|\partial_x\gamma|\|_{C^0}$ for $m \in \mathbb{N}_0$. For this, we proceed by induction as in [43, p. 649]. For $m = 0$, we note that $\partial_t|\partial_x\gamma| = -\langle\kappa, V\rangle|\partial_x\gamma|$ by (3.2.1). Since $\|\langle\kappa, V\rangle\|_{C^0}$ is bounded by (3.4.8) and (3.4.9) and $|\partial_x\gamma_0|$ is bounded from below and above, we conclude that

$$C^{-1} < |\partial_x\gamma| < C \quad \text{for some } C = C(\gamma_0, T).$$

For the induction step from $m-1$ to $m$ one uses (3.4.12) for $h = \langle\kappa, V\rangle$ to obtain bounds $\|\partial_x^i\langle\kappa, V\rangle\|_{C^0} \leq C(\gamma_0, m, T)$ for $0 \leq i \leq m$. Differentiating $\partial_t|\partial_x\gamma| = -\langle\kappa, V\rangle|\partial_x\gamma|$, we thus obtain

$$\partial_t\partial_x^m|\partial_x\gamma| \leq -\langle\kappa, V\rangle\partial_x^m|\partial_x\gamma| + C(\gamma_0, m, T)$$

which yields the bound on $\|\partial_x^m|\partial_x\gamma|\|_{C^0}$, that we wanted to find. Note that the bounds on $\|\partial_x^m\kappa\|_{C^0}$ also yield bounds on $\|\partial_x^m V\|_{C^0}$ for $m \in \mathbb{N}_0$. Again, the uniform bounds on $\kappa$, $V$, $\gamma$ and all their derivatives yield a smooth extension up to the maximal existence time and beyond.

## 3.5. Convergence

In this section, we first establish that stationary solutions to (3.0.3) are area constrained critical points of the elastic energy. For these constrained critical points, we then prove a suitable version of a constrained Łojasiewicz–Simon gradient inequality in Section 3.5.2. In Section 3.5.3, this inequality is the main tool for progressing from subconvergence to full convergence. In general, this is only possible if $\lambda > 0$, as we see in Section 3.5.4.

### 3.5.1. Stationary solutions as constrained critical points

Working with curves, that enclose a given area $A_0 \in \mathbb{R}$, we introduce the constraint functional

$$\mathcal{G}\colon W^{2,2}_{\mathrm{Imm}}(\mathbb{S}^1;\mathbb{R}^2) \to \mathbb{R}, \qquad \mathcal{G}(\gamma) = \mathcal{A}(\gamma) - A_0 = -\frac{1}{2}\int_{\mathbb{S}^1}\langle\nu_\gamma, \gamma\rangle\,\mathrm{d}s_\gamma - A_0.$$

Here, $W^{2,2}_{\mathrm{Imm}}(\mathbb{S}^1;\mathbb{R}^2)$ denotes the space of all $W^{2,2}$-immersions. A short computation shows that the Fréchet derivative is given by

$$\mathcal{G}'(\gamma)\colon W^{2,2}(\mathbb{S}^1;\mathbb{R}^2) \to \mathbb{R}, \qquad \mathcal{G}'(\gamma)(Y) = -\int_{\mathbb{S}^1}\langle\nu_\gamma, Y\rangle\,\mathrm{d}s_\gamma,$$

which is a surjective mapping. Hence, the energy $\mathcal{E}_\lambda\colon W^{2,2}_{\mathrm{Imm}}(\mathbb{S}^1;\mathbb{R}^2) \to \mathbb{R}$ has a critical point subject to the constraint $\mathcal{G} = 0$ at $\gamma \in W^{2,2}_{\mathrm{Imm}}(\mathbb{S}^1;\mathbb{R}^2) \cap \{\mathcal{G} = 0\}$ if and only if there

exits $c_1 \in \mathbb{R}$ such that

$$\begin{aligned} 0 &= \mathcal{E}'(\gamma)(Y) + c_1 \mathcal{G}'(\gamma)(Y) \qquad (3.5.1)\\ &= \int_{\mathbb{S}^1} \Big( \langle \nabla^2_{s_\gamma} Y, \kappa_\gamma \rangle - \frac{1}{2} |\kappa_\gamma|^2 \langle \partial_{s_\gamma} Y, \partial_{s_\gamma} \gamma \rangle + \lambda \langle \partial_{s_\gamma} Y, \partial_{s_\gamma} \gamma \rangle \Big) \, \mathrm{d}s_\gamma - c_1 \int_{\mathbb{S}^1} \langle \nu_\gamma, Y \rangle \, \mathrm{d}s_\gamma \end{aligned}$$

for all $Y \in W^{2,2}(\mathbb{S}^1; \mathbb{R}^2)$. For a computation of $\mathcal{E}'(\gamma)$, see for example [43, Lemma A.1]. If $\gamma$ is sufficiently regular, we obtain from (3.5.1) the Euler Lagrange equation

$$0 = \nabla_s^2 \kappa + \frac{1}{2} |\kappa|^2 \kappa - \lambda \kappa - c_1 \nu = \nabla_{L^2} \mathcal{E}_\lambda(\gamma) - c_1 \nabla_{L^2} \mathcal{G}(\gamma), \qquad (3.5.2)$$

where $\kappa = \kappa_\gamma$ and $\nu = \nu_\gamma$. Since a stationary solution is a smooth regular curve satisfying (3.0.5), we see that for any stationary solution, there is $c_1 \in \mathbb{R}$ such that (3.5.2) holds. Thus, any stationary solution is an (area-)constrained critical point.

### 3.5.2. The constrained Łojasiewicz–Simon gradient inequality

The proof of the convergence result in Section 3.5.3 mainly relays on a constrained version of the Łojasiewicz–Simon gradient inequality. We use this inequality for the $L^2$-gradients of the energy functional $\mathcal{E}_\lambda$ and the constraint functional $\mathcal{G}$. So, we work in the domain of these gradients. Moreover, we first work with variations normal to a given regular curve. With this in mind, we define the following Hilbert spaces and functionals.

**Definition 3.5.1** ([99, Definition 6.1])**.** Let $\hat{\gamma} \in W^{4,2}_{\mathrm{Imm}}(\mathbb{S}^1; \mathbb{R}^2)$. The space of normal vector fields along $\hat{\gamma}$ is denoted by

$$V^\perp := W^{4,2,\perp}(\mathbb{S}^1; \mathbb{R}^2) = \{ Y \in W^{4,2}(\mathbb{S}^1; \mathbb{R}^2) : \langle Y, \partial_x \hat{\gamma} \rangle = 0 \}$$

(cf. (3.3.1)). Moreover, we set

$$L^{2,\perp}(\mathbb{S}^1; \mathbb{R}^2) = \{ Y \in L^2(\mathbb{S}^1; \mathbb{R}^2) : \langle Y, \partial_x \hat{\gamma} \rangle = 0 \text{ a.e.} \}.$$

Let $\varepsilon > 0$ be so small such that $\hat{\gamma} + Y$ is immersed for all $Y \in U_\varepsilon := B_\varepsilon(0) \subset V^\perp$. With this, we define

$$E_\lambda \colon U_\varepsilon \to \mathbb{R}, \qquad E_\lambda(Y) := \mathcal{E}_\lambda(\hat{\gamma} + Y), \qquad (3.5.3)$$

$$G \colon U_\varepsilon \to \mathbb{R}, \qquad G(Y) := \mathcal{G}(\hat{\gamma} + Y). \qquad (3.5.4)$$

As in [44] or in [139], we first prove a Łojasiewicz–Simon gradient inequality in $V^\perp$, i.e. for normal directions. Since we work in a constrained setting, the inequality is based on [135]. The following two lemmata cover the assumptions in [135, Cor. 5.2].

**Lemma 3.5.2** ([99, Lemma 6.2])**.** *Let $E_\lambda$ be defined as in* (3.5.3)*. Then the following holds.*

1. *The energy $E_\lambda \colon U_\varepsilon \to \mathbb{R}$ is analytic.*
2. *The $L^{2,\perp}$-gradient $dE_\lambda \colon U_\varepsilon \to L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2)$,*

$$Y \mapsto P^\perp\Big(\nabla^2_{s_{\hat\gamma+Y}}\kappa_{\hat\gamma+Y} + \frac{1}{2}|\kappa_{\hat\gamma+Y}|^2\kappa_{\hat\gamma+Y} - \lambda\kappa_{\hat\gamma+Y}\Big)|\partial_x(\hat\gamma+Y)|,$$

   *with $P^\perp = \langle\cdot,\nu_{\hat\gamma}\rangle\nu_{\hat\gamma}$, is analytic.*

3. *The derivative of the gradient evaluated at zero $(dE_\lambda)'(0)\colon V^\perp \to L^{2,\perp}$ is Fredholm of index zero.*

For the definition and fundamental properties of analytic maps between Banach spaces, see for example [162]. A detailed proof of Lemma 3.5.2 in the case of open elastic curves can be found in [44, Section 3]. By slightly changing the function spaces (no boundary conditions but working in $\mathbb{S}^1$), this proof is transferred to the case of closed curves. A proof of 3. in Lemma 3.5.2 is also given in [115]. We remark that for this Fredholm property of the second variation, it is crucial that we consider only variations in normal direction.

For the constrained version of the Łojasiewicz–Simon gradient inequality, we additionally have to analyze the functional $G$.

**Lemma 3.5.3** ([99, Lemma 6.3])**.** *The constraint functional $G$ defined in* (3.5.4) *satisfies the following.*

1. *The functional $G\colon U_\varepsilon \to \mathbb{R}$ is analytic.*
2. *The $L^{2,\perp}$-gradient $dG\colon U_\varepsilon \to L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2)$,*

$$Y \mapsto -P^\perp\nu_{\hat\gamma+Y}|\partial_x(\hat\gamma+Y)| = -\langle\nu_{\hat\gamma+Y},\nu_{\hat\gamma}\rangle\nu_{\hat\gamma}\,|\partial_x(\hat\gamma+Y)|$$

   *is analytic.*

3. *The derivative of the gradient evaluated at zero $(dG)'(0)\colon V^\perp \to L^{2,\perp}$ is compact.*
4. *If $\hat\gamma$ in Definition 3.5.1 is given such that $\mathcal{A}(\hat\gamma) = A_0$, we have $G(0) = 0$ and $dG(0) \neq 0$.*

*Proof. 1. and 2.* As in [44, Lem. 3.4] and [139, Prop. 4.5], the functions

$$U_\varepsilon \to C(\mathbb{S}^1;\mathbb{R}),\ Y \mapsto |\partial_x(\hat\gamma+Y)| \quad \text{and} \quad U_\varepsilon \to W^{3,2}(\mathbb{S}^1;\mathbb{R}^2),\ Y \mapsto \partial_{s_{\hat\gamma+Y}}(\hat\gamma+Y)$$

are analytic. Since rotation by a fixed angle is a well-defined, continuous, linear operator on $W^{3,2}(\mathbb{S}^1;\mathbb{R}^2) \subset C^2(\mathbb{S}^1;\mathbb{R}^2)$,

$$U_\varepsilon \to W^{3,2}(\mathbb{S}^1;\mathbb{R}^2), \quad Y \mapsto \nu_{\hat\gamma+Y}$$

is also analytic. The function $Y \mapsto \hat{\gamma} + Y$ is analytic as a map from $U_\varepsilon$ to $W^{3,2}(\mathbb{S}^1;\mathbb{R}^2)$. Thus, the scalar product

$$U_\varepsilon \to W^{3,2}(\mathbb{S}^1;\mathbb{R}), \quad Y \mapsto \langle \hat{\gamma} + Y, \nu_{\hat{\gamma}+Y} \rangle$$

is analytic. As $W^{3,2}(\mathbb{S}^1;\mathbb{R}^2)$ is continuously embedded into $L^2(\mathbb{S}^1;\mathbb{R}^2)$, $Y \mapsto \langle \hat{\gamma} + Y, \nu_{\hat{\gamma}+Y} \rangle$ is also analytic as a map from $U_\varepsilon$ to $L^2(\mathbb{S}^1;\mathbb{R}^2)$. Since scalar multiplication $L^2(\mathbb{S}^1;\mathbb{R}) \times C(\mathbb{S}^1;\mathbb{R}) \to L^2(\mathbb{S}^1;\mathbb{R})$ is analytic, so is the map

$$U_\varepsilon \to L^2(\mathbb{S}^1;\mathbb{R}), \quad Y \mapsto \langle \hat{\gamma} + Y, \nu_{\hat{\gamma}+Y} \rangle |\partial_x(\hat{\gamma} + Y)|.$$

Since integration over $\mathbb{S}^1$ is a well-defined continuous and linear operator on $L^2(\mathbb{S}^1;\mathbb{R})$, the analyticity of $G\colon U_\varepsilon \to \mathbb{R}$ follows. Similar arguments and the continuity and linearity of the projection $P^\perp\colon L^2(\mathbb{S}^1;\mathbb{R}^2) \to L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2)$ yields the analyticity of the $L^{2,\perp}$-gradient $dG\colon U_\varepsilon \to L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2)$.

*3.* With the projection $P^\perp = \langle \cdot, \nu_{\hat{\gamma}} \rangle \nu_{\hat{\gamma}}$, we compute

$$\begin{aligned}(dG')(0)Y &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} dG(tY) = -P^\perp \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nu_{\hat{\gamma}+tY} \, |\partial_x \hat{\gamma}| - P^\perp \nu_{\hat{\gamma}} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} |\partial_x(\hat{\gamma} + tY)| \\ &= \nu_{\hat{\gamma}} |\partial_x \hat{\gamma}| \langle Y, \kappa_{\hat{\gamma}} \rangle. \end{aligned} \tag{3.5.5}$$

Here, we used that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \partial_{s_{\hat{\gamma}+tY}}(\hat{\gamma} + tY) = \nabla_{s_{\hat{\gamma}}} Y$$

(see for example [44, (B.5)]), which implies that the first summand in (3.5.5) vanishes. For the second summand, we used

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} |\partial_x(\hat{\gamma} + tY)| = -|\partial_x \hat{\gamma}| \langle Y, \kappa_{\hat{\gamma}} \rangle$$

(see for example [44, (B.2)]). We see that $(dG')(0)\colon V^\perp \to L^{2,\perp}$ is of zeroth order in $Y$. Hence it is compact.

*4.* The assumptions on $\hat{\gamma}$ imply that $G(0) = \mathcal{G}(\hat{\gamma}) = 0$. Moreover, $|\partial_x \hat{\gamma}| \neq 0$ and $\nu_{\hat{\gamma}} \neq 0$. Thus, $dG(0) = \nu_{\hat{\gamma}} |\partial_x \hat{\gamma}| \neq 0$. □

With this, we prove a first version of a constrained Łojasiewicz-Simon gradient inequality which reads as follows.

**Theorem 3.5.4** ([99, Theorem 6.4])**.** *Let $\hat{\gamma} \in C^\infty_{\mathrm{Imm}}(\mathbb{S}^1;\mathbb{R}^2)$ be a constrained critical point of $\mathcal{E}_\lambda$. Then there exist constants $C > 0$, $\vartheta \in (0, \frac{1}{2}]$ and $r > 0$, such that*

$$|E_\lambda(0) - E_\lambda(Y)|^{1-\vartheta} \leq C \| P(Y) dE_\lambda(Y) \|_{L^2(\mathrm{d}x)}$$

*for all* $Y \in B_r(0) \subset U_\varepsilon \subset V^\perp$ *with* $G(Y) = 0$. *Here,* $P(Y)\colon L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2) \to L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2)$ *is the orthogonal projection onto* $\{X \in L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2) : \langle \nu_{\hat\gamma+Y}, X\rangle_{L^2(\mathrm{d}s_{\hat\gamma+Y})} = 0\}$.

*Proof.* We check that the assumptions in [135, Corollary 5.2] are satisfied for $E_\lambda$ and $G$ defined on $U_\varepsilon \subset V^\perp$. In Lemma 3.5.2 and Lemma 3.5.3 we have seen that $E$ and $G$ are analytic. Assumption (i), i.e. $V^\perp(\mathbb{S}^1;\mathbb{R}^2) \hookrightarrow L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2)$ densely, is given. Assumptions (ii) and (iii) follow from Lemma 3.5.2 and assumptions (iv) to (vi) from Lemma 3.5.3. Thus, by [135, Corollary 5.2], the set of area preserving normal variations

$$\mathcal{M} := \{Y \in U_\varepsilon : G(Y) = 0\}$$

is locally a submanifold of codimension 1 near 0. Since $\hat\gamma$ is a constrained critical point of $\mathcal{E}_\lambda$, 0 is a critical point of $E_\lambda$ on $\mathcal{M}$. Thus, [135, Corollary 5.2] further yields the existence of constants $C, r > 0$ and $\vartheta \in (0, \frac{1}{2}]$ such that for all $Y \in \mathcal{M}$ with $\|Y\|_{W^{4,2}(\mathrm{d}x)} \le r$, it holds that

$$|E(Y) - E(0)|^{1-\vartheta} \le C\|P(Y)dE(Y)\|_{L^2(\mathrm{d}x)},$$

where $P(Y)\colon L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2) \to L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2)$ is the orthogonal projection onto the closure of the tangent space $\overline{T_Y\mathcal{M}}^{\|\cdot\|_{L^2(\mathrm{d}x)}}$. With [135, Proposition 3.3] it follows that

$$\begin{aligned}\overline{T_Y\mathcal{M}}^{\|\cdot\|_{L^2(\mathrm{d}x)}} &= \{X \in L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2) : \langle dG(Y), X\rangle_{L^2(dx)} = 0\}\\ &= \{X \in L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2) : \langle \nu_{\hat\gamma+Y}, X\rangle_{L^2(\mathrm{d}s_{\hat\gamma+Y})} = 0\}.\end{aligned}$$ □

To prove a version of the Łojasiewicz–Simon inequality for variations that are not necessarily normal, the reparametrization argument in Section 3.3.1 is crucial. We use it to extend Theorem 3.5.4 to area preserving variations with tangential component.

**Theorem 3.5.5** ([99, Theorem 6.5])**.** *Let the immersion* $\hat\gamma \in C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ *be a constrained critical point of* $\mathcal{E}_\lambda$. *Then there exist constants* $C > 0$, $\vartheta \in (0, \frac{1}{2}]$ *and* $\tilde r > 0$ *such that*

$$|\mathcal{E}_\lambda(\gamma) - \mathcal{E}_\lambda(\hat\gamma)|^{1-\vartheta} \le C\left\|\nabla_{L^2}\mathcal{E}_\lambda(\gamma) - \frac{\nabla_{L^2}\mathcal{G}(\gamma)}{\mathcal{L}(\gamma)}\int_{\mathbb{S}^1}\langle\nabla_{L^2}\mathcal{E}_\lambda(\gamma), \nu_\gamma\rangle\,\mathrm{d}s_\gamma\right\|_{L^2(\mathrm{d}s_\gamma)}$$

*for all* $\gamma \in W^{4,2}(\mathbb{S}^1;\mathbb{R}^2)$ *with* $\mathcal{G}(\gamma) = 0$ *and* $\|\gamma - \hat\gamma\|_{W^{4,2}} \le \tilde r$.

*Proof.* Let $C > 0$, $\vartheta \in (0, \frac{1}{2}]$ and $r > 0$ be given as in Theorem 3.5.4. Moreover, let $\tilde r = \tilde r(\hat\gamma, r)$ be the constant of Lemma 3.3.1. We take some $\gamma \in W^{4,2}(\mathbb{S}^1;\mathbb{R}^2)$ with $\mathcal{G}(\gamma) = 0$ and $\|\gamma - \hat\gamma\|_{W^{4,2}} \le \tilde r$. By Lemma 3.3.1, there exists a $W^{4,2}$-diffeomorphism $\Phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ and $Y \in B_r(0) \subset V^\perp$ such that $\gamma \circ \Phi = \hat\gamma + Y$. Since $0 = \mathcal{G}(\gamma) = \mathcal{G}(\gamma \circ \Phi)$, we know that $G(Y) = 0$. Thus, the invariance of $\mathcal{E}_\lambda$ under reparametrization and Theorem 3.5.4 yields

$$|\mathcal{E}_\lambda(\gamma) - \mathcal{E}_\lambda(\hat\gamma)|^{1-\vartheta} = |\mathcal{E}_\lambda(\hat\gamma + Y) - \mathcal{E}_\lambda(\hat\gamma)|^{1-\vartheta} = |E_\lambda(Y) - E_\lambda(0)|^{1-\vartheta}$$

$$\leq C\|P(Y)dE_\lambda(Y)\|_{L^2(\,\mathrm{d}x)}. \tag{3.5.6}$$

By [135, Remark 5.4], the projection operator $P(Y)\colon L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2) \to L^{2,\perp}(\mathbb{S}^1;\mathbb{R}^2)$ is given by

$$P(Y)X = X - \frac{\langle dG(Y), X\rangle_{L^2(\,\mathrm{d}x)}}{\|dG(Y)\|^2_{L^2(\,\mathrm{d}x)}} dG(Y).$$

In particular, we have $P(Y)(c_1 dG(Y)) = 0$ for

$$c_1 = c_1(\gamma) := -\frac{1}{\mathcal{L}(\gamma)} \int_{\mathbb{S}^1} \langle \nabla_{L^2}\mathcal{E}_\lambda(\gamma), \nu_\gamma\rangle \,\mathrm{d}s_\gamma.$$

Moreover, comparing $dE_\lambda$ and $dG$ to the $L^2$-gradients, we have

$$dE_\lambda(Y) = P^\perp\left(\nabla_{L^2}\mathcal{E}_\lambda(\hat\gamma + Y)\right)|\partial_x(\hat\gamma + Y)|, \quad dG(Y) = P^\perp\left(\nabla_{L^2}\mathcal{G}(\hat\gamma + Y)\right)|\partial_x(\hat\gamma + Y)|,$$

where $P^\perp = \langle\cdot, \nu_{\hat\gamma}\rangle\nu_{\hat\gamma}$. With these two observations, we obtain

$$\begin{aligned}\|P(Y)dE_\lambda(Y)\|_{L^2(\,\mathrm{d}x)} &= \|P(Y)(dE_\lambda(Y) + c_1 dG(Y))\|_{L^2(\,\mathrm{d}x)}\\ &\leq \|dE_\lambda(Y) + c_1 dG(Y)\|_{L^2(\,\mathrm{d}x)}\\ &= \|P^\perp\left(\nabla_{L^2}\mathcal{E}_\lambda(\hat\gamma + Y) + c_1\nabla_{L^2}\mathcal{G}(\hat\gamma + Y)\right)|\partial_x(\hat\gamma + Y)|\|_{L^2(\,\mathrm{d}x)}.\end{aligned} \tag{3.5.7}$$

By Sobolev embedding and possibly making $r$ smaller, we can assume that $|\partial_x(\hat\gamma + Y)|$ is uniformly bounded for $\|Y\|_{W^{4,2}} \leq r$. So, putting (3.5.6) and (3.5.7) together yields

$$|\mathcal{E}_\lambda(\gamma) - \mathcal{E}_\lambda(\hat\gamma)|^{1-\vartheta} \leq C\|\nabla_{L^2}\mathcal{E}_\lambda(\gamma\circ\Phi) + c_1\nabla_{L^2}\mathcal{G}(\gamma\circ\Phi)\|_{L^2(\,\mathrm{d}s_{\gamma\circ\Phi})}$$

for a modified constant $C > 0$. It remains to notice that the gradients are geometric, i.e.

$$\begin{aligned}\nabla_{L^2}\mathcal{E}_\lambda(\gamma\circ\Phi) &= \nabla_{L^2}\mathcal{E}_\lambda(\gamma)\circ\Phi,\\ \nabla_{L^2}\mathcal{G}(\gamma\circ\Phi) &= \nabla_{L^2}\mathcal{G}(\gamma)\circ\Phi.\end{aligned}$$

With this, we obtain

$$\|\nabla_{L^2}\mathcal{E}_\lambda(\gamma\circ\Phi) + c_1\nabla_{L^2}\mathcal{G}(\gamma\circ\Phi)\|_{L^2(\,\mathrm{d}s_{\gamma\circ\Phi})} = \|\nabla_{L^2}\mathcal{E}_\lambda(\gamma) + c_1\nabla_{L^2}\mathcal{G}(\gamma)\|_{L^2(\,\mathrm{d}s_\gamma)}$$

and the claim follows. □

### 3.5.3. Proof of full convergence

We are now prepared to enhance the subconvergence result to a full convergence result. This finally proves Theorem 3.0.1.

**Theorem 3.5.6** (Convergence, [99, Theorem 6.6])**.** *Let $\lambda > 0$ and $\gamma\colon [0,\infty)\times\mathbb{S}^1 \to \mathbb{R}^2$ be the global smooth solution of* (3.0.3)*. Then there exists a family of smooth diffeomorphisms $\Phi(t)\colon \mathbb{S}^1 \to \mathbb{S}^1$, $t \in (0,\infty)$, such that $\gamma(t,\Phi(t,\cdot))$ converges smoothly for $t \to \infty$ to a stationary solution $\gamma_\infty$.*

For the proof of full convergence, we use an argumentation similar to [44, Section 5], [115], or [139]. However, we note that the $H^{-1}$-gradient flow structure partially obstructs the usual procedure. Indeed, the time-dependent Hilbert spaces in which the energy decreases do not coincide as sets, unlike for the $L^2$-gradient flow. By considering suitably defined time-independent norms, we overcome this difficulty. In [66], the authors have to face similar problems.
We note that in [139], an elegant way to shorten the proof of convergence in [44, Section 5] is shown, but it seems that this cannot be transferred to $H^{-1}$-flows.

*Proof of Theorem 3.5.6.* We take a smooth solution $\gamma\colon [0,\infty)\times\mathbb{S}^1 \to \mathbb{R}^2$ of Theorem 3.4.6 and $\tilde\gamma(t,\cdot)-p(t)$ of (3.4.10), which is the reparametrized with constant speed and translated solution.

*Step 1: Convergent sequence.* By the subconvergence result (Theorem 3.4.6), there exists a sequence of times $t_i \to \infty$, such that $\tilde\gamma(t_i,\cdot) - p(t_i)$ converges smoothly to a stationary solution $\gamma_\infty$ (which is kept fix throughout the proof). As discussed in Section 3.5.1, $\gamma_\infty$ is an area constrained critical point of $\mathcal{E}_\lambda$. Due to (3.2.9), we know that

$$\mathcal{E}_\lambda(\gamma(t)) = \mathcal{E}_\lambda(\tilde\gamma(t) - p(t)) \geq \mathcal{E}_\lambda(\gamma_\infty) \quad \text{for all } t \in [0,\infty). \tag{3.5.8}$$

We assume that the inequality is strict. If not, there exists $\bar t \in [0,\infty)$ such that $\mathcal{E}_\lambda(\gamma(\bar t)) = \mathcal{E}_\lambda(\gamma_\infty)$. But due to Lemma 3.2.4, this means $\mathcal{E}_\lambda(\gamma(t)) = \mathcal{E}_\lambda(\gamma_\infty)$ for all $t \geq \bar t$, so $\partial_t^\perp\gamma \equiv 0$ on $[\bar t,\infty)$ by (3.2.9). Thus, only the tangential velocity changes on on $[\bar t,\infty)$ and the claimed convergence follows trivially.

*Step 2: Fixing $\delta = \delta(\gamma_\infty)$.* For the following, we fix $\delta > 0$ such that for all $\zeta\colon \mathbb{S}^1 \to \mathbb{R}^2$ with $\|\zeta - \gamma_\infty\|_{C^1} < \delta$, it holds that

$$|\partial_x\zeta| \geq M := \frac{1}{2}\min|\partial_x\gamma_\infty| = \frac{\mathcal{L}(\gamma_\infty)}{2} > 0, \tag{3.5.9}$$

see (3.4.2), and such that

$$|\alpha^\perp| := \Big|\alpha - \Big\langle \alpha, \frac{\partial_x\zeta}{|\partial_x\zeta|}\Big\rangle \frac{\partial_x\zeta}{|\partial_x\zeta|}\Big| \geq \frac{1}{2}|\alpha| \tag{3.5.10}$$

for all vector fields $\alpha$ normal to $\gamma_\infty$. The latter equation makes sense, because

$$\Big|\Big\langle \alpha, \frac{\partial_x\zeta}{|\partial_x\zeta|}\Big\rangle\Big| = \Big|\Big\langle \alpha, \frac{\partial_x\zeta}{|\partial_x\zeta|} - \frac{\partial_x\gamma_\infty}{|\partial_x\gamma_\infty|}\Big\rangle\Big|$$

becomes small for $\|\zeta - \gamma_\infty\|_{C^1}$ small.

*Step 3: New initial datum and new flow.* Let $0 < \varepsilon < \frac{1}{2}\min\{\delta, \tilde{r}, 1\}$, where $\tilde{r}$ is given as in Theorem 3.5.5 for $\hat{\gamma} = \gamma_\infty$. Then Lemma 3.3.1 yields the existence of $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon, \gamma_\infty) > 0$ such that for all $X \in B_{\tilde{\varepsilon}}(0) \subset W^{7,2}(\mathbb{S}^1; \mathbb{R}^2)$,

$$(\gamma_\infty + X) \circ \Phi = \gamma_\infty + Y$$

for a $W^{7,2}$-diffeomorphism $\Phi$ and $Y \in B_\varepsilon(0) \subset W^{7,2}(\mathbb{S}^1; \mathbb{R}^2)$ normal to $\gamma_\infty$. Due to the smooth subconvergence, there is an index $i \in \mathbb{N}$ such that $\|\tilde{\gamma}(t_i) - p(t_i) - \gamma_\infty\|_{W^{7,2}} < \tilde{\varepsilon}$. Thus, there exists a $W^{7,2}$-diffeomorphism $\Phi_0$ such that

$$(\tilde{\gamma}(t_i) - p(t_i)) \circ \Phi_0 = \gamma_\infty + N_0$$

for some $N_0 \in W^{7,2}(\mathbb{S}^1; \mathbb{R}^2)$ normal to $\gamma_\infty$ with

$$\|N_0\|_{W^{7,2}} < \varepsilon < \frac{1}{2}\min\{\delta, \tilde{r}\}. \tag{3.5.11}$$

For this index $i$, we set $\bar{\gamma}_0 := (\tilde{\gamma}(t_i) - p(t_i)) \circ \Phi_0 = \gamma_\infty + N_0$. Possibly choosing $\varepsilon > 0$ smaller, we can ensure that $\Phi_0$ does not change the orientation. So $\bar{\gamma}_0$ describes a smooth closed curve with $\mathcal{A}(\bar{\gamma}_0) = A_0$. Moreover, by Remark 3.3.2, $\Phi_0$ and $N_0$ are smooth.

The idea now is to take $\bar{\gamma}_0$ as new initial datum and to consider

$$\begin{cases} \partial_t^\perp \bar{\gamma} = \nabla_s^2 \nabla_{L^2} \mathcal{E}_\lambda(\bar{\gamma}) & \text{in } (0,T) \times \mathbb{S}^1, \\ \bar{\gamma}(0,\cdot) = \bar{\gamma}_0 = \gamma_\infty + N_0 = \gamma_\infty + \varphi_0 \nu_\infty & \text{on } \mathbb{S}^1. \end{cases} \tag{3.5.12}$$

By Theorem 3.3.6 and Remark 3.3.7, there exists a a geometrically unique smooth solution $\bar{\gamma}$ of (3.5.12) that can be written as $\bar{\gamma}(t,x) = \gamma_\infty(x) + \varphi(t,x)\nu_\infty(x)$ for some smooth scalar function $\varphi$. Let $0 < T' \leq \infty$ the maximal existence time of this solution. The geometric uniqueness implies that on $[t_i, T')$, $\bar{\gamma}$ is a smooth reparametrization of $\tilde{\gamma} - p(t_i)$. It follows that for proving Theorem 3.5.6, it suffices to prove that $T' = \infty$ and that $\bar{\gamma}$ converges to $\gamma_\infty$ as $t \to \infty$.

*Step 4:* Let $T \in (0, T']$ be maximal such that

$$\|N(t)\|_{W^{6,2}} := \|\bar{\gamma}(t) - \gamma_\infty\|_{W^{6,2}} < \min\{\delta, \tilde{r}\} \quad \text{for all } t \in [0,T). \tag{3.5.13}$$

Due to (3.5.11), such a $T$ exists. Our goal is to show $T = T' = \infty$. By parabolic Schauder estimates (see [30, Section 4]), it can be shown that

$$\sup_{[0,T)} \|\varphi(t)\|_{C^{6,\alpha}} \leq C(\delta, \tilde{r}, \gamma_0) \quad \text{and hence} \quad \sup_{[0,T)} \|\bar{\gamma}(t) - \gamma_\infty\|_{C^{6,\alpha}} \leq C(\delta, \tilde{r}, \gamma_0) \tag{3.5.14}$$

for some $\alpha \in (0, \frac{1}{2})$.

*Step 5: Application of the Łojasiewicz-Simon inequality.* We define

$$\mathcal{H}(t) := (\mathcal{E}_\lambda(\bar\gamma(t)) - \mathcal{E}_\lambda(\gamma_\infty))^\vartheta \quad \text{for } t \in (0,T).$$

Here $\vartheta$ is given as in Theorem 3.5.5 for $\gamma_\infty$. Note that since $\bar\gamma$ is a reparametrization and translation of $\gamma$ and we made the assumption that the inequality in (3.5.8) is strict, $\mathcal{H}(t) \neq 0$ for all $t$. With (3.5.12) and integration by parts, we compute

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) &= -\vartheta(\mathcal{E}_\lambda(\bar\gamma(t)) - \mathcal{E}_\lambda(\gamma_\infty))^{\vartheta-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\bar\gamma(t)) \\
&= -\vartheta(\mathcal{E}_\lambda(\bar\gamma(t)) - \mathcal{E}_\lambda(\gamma_\infty))^{\vartheta-1} \langle \nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t)), \partial_t\bar\gamma(t)\rangle_{L^2(\mathrm{d}s_{\bar\gamma})} \\
&= -\vartheta(\mathcal{E}_\lambda(\bar\gamma(t)) - \mathcal{E}_\lambda(\gamma_\infty))^{\vartheta-1} \langle \nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t)), \partial_t^\perp\bar\gamma(t)\rangle_{L^2(\mathrm{d}s_{\bar\gamma})} \\
&= \vartheta(\mathcal{E}_\lambda(\bar\gamma(t)) - \mathcal{E}_\lambda(\gamma_\infty))^{\vartheta-1} \|\nabla_s\nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t))\|^2_{L^2(\mathrm{d}s_{\bar\gamma})}.
\end{aligned}$$

Since $\bar\gamma$ solves (3.5.12) with initial datum $\bar\gamma_0$ with $\mathcal{G}(\bar\gamma_0) = 0$, we know that $\mathcal{G}(\bar\gamma) = 0$. Thus, on $(0,T)$, we can apply Theorem 3.5.5 and obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) \geq C\vartheta \frac{\|\nabla_s\nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t))\|^2_{L^2(\mathrm{d}s_{\bar\gamma})}}{\|\nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t)) - c_1(\bar\gamma(t))\bar\nu\|_{L^2(\mathrm{d}s_{\bar\gamma})}}, \tag{3.5.15}$$

where

$$c_1(\bar\gamma(t)) = \frac{1}{\mathcal{L}(\bar\gamma(t))}\int_{\mathbb{S}^1} \langle \nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t)), \bar\nu\rangle \,\mathrm{d}s_{\bar\gamma}.$$

Here, remember that $\nabla_{L^2}\mathcal{G}(\bar\gamma) = -\bar\nu$ and note that $\|\nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t)) - c_1(\bar\gamma(t))\bar\nu\|_{L^2(\mathrm{d}s_{\bar\gamma})} > 0$ by Theorem 3.5.5 and $\mathcal{E}_\lambda(\bar\gamma(t)) > \mathcal{E}_\lambda(\gamma_\infty)$. Writing the denominator of (3.5.15) in terms of the scalar curvature $\bar k = \langle \bar\kappa, \bar\nu\rangle$ and using the Poincaré–Wirtinger inequality, we obtain

$$\begin{aligned}
&\|\nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t)) - c_1(\bar\gamma(t))\bar\nu\|_{L^2(\mathrm{d}s_{\bar\gamma})} \\
&\quad= \Big\|\partial_s^2\bar k + \frac{1}{2}\bar k^3 - \lambda\bar k - \frac{1}{\mathcal{L}(\bar\gamma)}\int_{\mathbb{S}^1}\Big(\partial_s^2\bar k + \frac{1}{2}\bar k^3 - \lambda\bar k\Big)\,\mathrm{d}s_{\bar\gamma}\Big\|_{L^2(\mathrm{d}s_{\bar\gamma})} \\
&\quad\leq C(\mathcal{L}(\bar\gamma))\Big\|\partial_s\Big(\partial_s^2\bar k + \frac{1}{2}\bar k^3 - \lambda\bar k\Big)\Big\|_{L^2(\mathrm{d}s_{\bar\gamma})} = C\|\nabla_s\nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t))\|_{L^2(\mathrm{d}s_{\bar\gamma})}.
\end{aligned} \tag{3.5.16}$$

Here, we used that $\langle \nabla_s^n\kappa, \nu\rangle = \partial_s^n k$ for all $n \in \mathbb{N}_0$ and $\langle \nabla_s^2(|\kappa|^2\kappa), \nu\rangle = \partial_s^2 k^3$. With (3.5.16), we get

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) \geq C\vartheta\|\nabla_s\nabla_{L^2}\mathcal{E}_\lambda(\bar\gamma(t))\|_{L^2(\mathrm{d}s_{\bar\gamma})} = C\vartheta\|\partial_t^\perp\bar\gamma\|_{H^{-1}_{\bar\gamma}} \tag{3.5.17}$$

for a constant $C$ depending on bounds on $\mathcal{L}(\gamma)$ and the constant of Theorem 3.5.5. The definition of the latter norm is given in Section 3.1.1, see (3.1.9).

*Step 6: Time-independent norm.* With the same partial integration as in Lemma 3.2.6, we see that $\nabla_s^2 \nabla_{L^2} \mathcal{E}_\lambda(\bar\gamma) \in H_{\bar\gamma}$ and hence $\partial_t^\perp \bar\gamma \in H_{\bar\gamma}$. So Lemma 3.1.7 implies that

$$\|\partial_t^\perp \bar\gamma\|_{H_{\bar\gamma}^{-1}} \geq C \|\partial_t^\perp \bar\gamma\|_\star \tag{3.5.18}$$

for a constant $C$ depending on $\|\bar\kappa\|_{C^0}$, $\mathcal{L}(\bar\gamma)$ as well as $\||\partial_x \bar\gamma|^{-1}\|_{C^0}$ and $\|\partial_x \bar\gamma\|_{C^0}$. We already know that $\bar\gamma$ is a reparametrization and translation of $\gamma$ and from the proof of Theorem 3.4.6, we know that $\|\kappa\|_{C^0}$ and $\mathcal{L}(\gamma)$ are uniformly bounded. Moreover, $\sup |\partial_x \bar\gamma|^{-1}$ is bounded by (3.5.9) and (3.5.13) and $\sup |\partial_x \bar\gamma| \leq \sup |\partial_x \bar\gamma - \partial_x \gamma_\infty| + |\partial_x \gamma_\infty|$ is bounded by (3.5.13). Thus, the constant $C$ does not depend on the choice of $\varepsilon$. Moreover, since $\partial_t \bar\gamma$ is orthogonal to $\gamma_\infty$, we have

$$\begin{aligned}
\|\partial_t \bar\gamma\|_\star &= \sup_{Y \in W^{1,2}\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}} \int_{\mathbb{S}^1} \langle \partial_t \bar\gamma, Y \rangle \,\mathrm{d}x \\
&= \sup_{Y \in W^{1,2}\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}} \int_{\mathbb{S}^1} \langle \partial_t \bar\gamma, Y^{\perp\infty} \rangle \,\mathrm{d}x \\
&\leq \sup_{Y \in W^{1,2}\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}} \int_{\mathbb{S}^1} \langle \partial_t^\perp \bar\gamma, Y^{\perp\infty} \rangle \,\mathrm{d}x \\
&\quad + \sup_{Y \in W^{1,2}\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}} \int_{\mathbb{S}^1} \langle \partial_t \bar\gamma, \partial_s \bar\gamma \rangle \langle Y^{\perp\infty}, \partial_s \bar\gamma \rangle \,\mathrm{d}x \\
&\leq \sup_{Y \in W^{1,2,\perp\infty}\setminus\{0\}} \frac{C(\gamma_\infty)}{\|Y\|_{W^{1,2}}} \int_{\mathbb{S}^1} \langle \partial_t^\perp \bar\gamma, Y \rangle \,\mathrm{d}x \\
&\quad + \sup_{Y \in W^{1,2}\setminus\{0\}} \frac{1}{\|Y\|_{W^{1,2}}} \int_{\mathbb{S}^1} \langle \partial_t \bar\gamma, Y \rangle (\langle \nu_\infty, \partial_s \bar\gamma \rangle)^2 \,\mathrm{d}x
\end{aligned}$$

where $Y^{\perp\infty} = \langle Y, \nu_\infty \rangle \nu_\infty$ and $W^{1,2,\perp\infty} = W^{1,2} \cap \{Y = \langle Y, \nu_\infty \rangle \nu_\infty\}$. For the first term in the last inequality, we used a computation as in (3.1.10) and for the second term we used $\partial_t \bar\gamma = \langle \partial_t \bar\gamma, \nu_\infty \rangle \nu_\infty$. Since $1 = |\nu_\infty|^2 = |\langle \nu_\infty, \bar\nu \rangle|^2 + |\langle \nu_\infty, \partial_s \bar\gamma \rangle|^2$ and $|\langle \nu_\infty, \bar\nu \rangle| \geq \frac{1}{2}$ on $(0,T)$ by (3.5.10), we conclude that

$$\|\partial_t \bar\gamma\|_\star \leq C(\gamma_\infty) \|\partial_t^\perp \bar\gamma\|_\star + \frac{3}{4} \|\partial_t \bar\gamma\|_\star \tag{3.5.19}$$

on $(0,T)$. Putting (3.5.17), (3.5.18) and (3.5.19) together yields

$$-\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(t) \geq C \|\partial_t \bar\gamma\|_\star \tag{3.5.20}$$

for a constant $C$ depending on the bounds of the solution $\gamma$ from the proof of Theorem 3.4.6, the constants from Theorem 3.5.5, $\delta$ and $T$. Integration now yields that on $(0,T)$, we have

$$\begin{aligned}
\|\bar\gamma(t) - \gamma_\infty\|_\star &\leq \|\bar\gamma(t) - \bar\gamma(0)\|_\star + \|\bar\gamma(0) - \gamma_\infty\|_\star \leq \int_0^t \|\partial_t \bar\gamma\|_\star \,\mathrm{d}\tau + \|\bar\gamma(0) - \gamma_\infty\|_\star \\
&\leq C\Big( \big(\mathcal{E}_\lambda(\bar\gamma(0)) - \mathcal{E}_\lambda(\gamma_\infty)\big)^\vartheta - \big(\mathcal{E}_\lambda(\bar\gamma(t)) - \mathcal{E}_\lambda(\gamma_\infty)\big)^\vartheta \Big) + \|\bar\gamma(0) - \gamma_\infty\|_\star
\end{aligned}$$

$$\leq C\|\bar{\gamma}(0)-\gamma_\infty\|_{C^2}^{\vartheta}+\|\bar{\gamma}(0)-\gamma_\infty\|_\star \leq C\|\bar{\gamma}(0)-\gamma_\infty\|_{C^2}^{\vartheta}. \tag{3.5.21}$$

Here, we used $\|\bar{\gamma}(0)-\gamma_\infty\|_\star \leq \|\bar{\gamma}(0)-\gamma_\infty\|_{L^2(\gamma_\infty)} \leq C(\mathcal{L}(\gamma_\infty))\|\bar{\gamma}(0)-\gamma_\infty\|_{C^0}$, (3.5.11), and $\vartheta<1$.

*Step 7: Interpolation.* Since $N$ is normal to $\gamma_\infty$ and since $\mathcal{A}(\bar{\gamma}(t))=\mathcal{A}(\gamma_\infty)=A_0$ implies $\int_{\mathbb{S}^1}\langle N(t),\nu_\infty\rangle\,\mathrm{d}s_{\gamma_\infty}=0$, we have $\bar{\gamma}(t)-\gamma_\infty=N(t)\in H_{\gamma_\infty}$. Thus, Lemma 3.1.6 and Lemma 3.1.7 yield

$$\begin{aligned}\|\bar{\gamma}(t)-\gamma_\infty\|_{L^2(\mathrm{d}s_{\gamma_\infty})}^2 &\leq \|\bar{\gamma}(t)-\gamma_\infty\|_{H_{\gamma_\infty}}\|\bar{\gamma}(t)-\gamma_\infty\|_{H^{-1}_{\gamma_\infty}}\\ &\leq C\|\bar{\gamma}(t)-\gamma_\infty\|_{H_{\gamma_\infty}}\|\bar{\gamma}(t)-\gamma_\infty\|_\star\\ &\leq C\|\bar{\gamma}(t)-\gamma_\infty\|_{H_{\gamma_\infty}}\|\bar{\gamma}(0)-\gamma_\infty\|_{C^2}^{\vartheta}\leq C\delta\|\bar{\gamma}(0)-\gamma_\infty\|_{C^2}^{\vartheta},\end{aligned} \tag{3.5.22}$$

where we used (3.5.21) and $\|\bar{\gamma}(t)-\gamma_\infty\|_{H_{\gamma_\infty}}\leq C\|\bar{\gamma}(t)-\gamma_\infty\|_{W^{6,2}}<\delta$, see (3.5.13). With [110, Prop. 1.1.3], we have

$$\|\bar{\gamma}(t)-\gamma_\infty\|_{W^{6,2}}\leq\|\bar{\gamma}(t)-\gamma_\infty\|_{C^6}\leq C\|\bar{\gamma}(t)-\gamma_\infty\|_{C^0}^{1-\frac{6}{6+\alpha}}\|\bar{\gamma}(t)-\gamma_\infty\|_{C^{6+\alpha}}^{\frac{6}{6+\alpha}}$$

for $\alpha\in(0,\frac{1}{2})$. Moreover, by Sobolev embedding and with interpolation (see for example [14, Thm. 6.4.5 (3)]),

$$\begin{aligned}\|\bar{\gamma}(t)-\gamma_\infty\|_{C^0}\leq\|\bar{\gamma}(t)-\gamma_\infty\|_{C^{0,\frac{1}{2}}}&\leq C\|\bar{\gamma}(t)-\gamma_\infty\|_{W^{1,2}}\\ &\leq C\|\bar{\gamma}(t)-\gamma_\infty\|_{L^2}^{\frac{1}{2}}\|\bar{\gamma}(t)-\gamma_\infty\|_{W^{2,2}}^{\frac{1}{2}}.\end{aligned}$$

Putting the last two equations together and using (3.5.14) yields

$$\begin{aligned}\|\bar{\gamma}(t)-\gamma_\infty\|_{W^{6,2}}&\leq C\|\bar{\gamma}(t)-\gamma_\infty\|_{L^2}^{\frac{1}{2}-\frac{6}{2(6+\alpha)}}\|\bar{\gamma}(t)-\gamma_\infty\|_{W^{2,2}}^{\frac{1}{2}-\frac{6}{2(6+\alpha)}}\|\bar{\gamma}(t)-\gamma_\infty\|_{C^{6+\alpha}}^{\frac{6}{6+\alpha}}\\ &\leq C(\tilde{r},\delta,\gamma_0)\|\bar{\gamma}(t)-\gamma_\infty\|_{L^2(\mathrm{d}s_{\gamma_\infty})}^{\beta}\end{aligned}$$

for some $\beta\in(0,1)$. With (3.5.22) and (3.5.11), we obtain

$$\|\bar{\gamma}(t)-\gamma_\infty\|_{W^{6,2}}\leq C\|\bar{\gamma}(0)-\gamma_\infty\|_{C^2}^{\frac{\vartheta\beta}{2}}<C\varepsilon^{\frac{\vartheta\beta}{2}}. \tag{3.5.23}$$

*Step 8:* $T=T'=\infty$. Choosing $\varepsilon<\big(\frac{\min\{\delta,\tilde{r}\}}{2C}\big)^{\frac{2}{\vartheta\beta}}$, we obtain from (3.5.23)

$$\|\bar{\gamma}(t)-\gamma_\infty\|_{W^{6,2}}<\frac{\min\{\delta,\tilde{r}\}}{2}$$

on $(0,T)$. This gives a contradiction to the maximality of $T$ if $T<T'$, see (3.5.13). Thus, we obtain $T=T'$ and (3.5.13) holds up to $T'$. Assume that $T'<\infty$. Since (3.5.14) holds up to $t=T'$, we can restart (3.3.16) with initial datum determined by $\bar{\gamma}(T')$. This

contradiction to the maximality of $T'$ yields $T' = \infty$ and

$$\|\bar{\gamma}(t) - \gamma_\infty\|_{W^{6,2}} < \min\{\delta, \tilde{r}\} \qquad \text{for all } t \in (0, \infty). \tag{3.5.24}$$

*Step 9: Convergence.* With (3.5.24) and (3.5.20), we obtain $\|\partial_t \bar{\gamma}\|_\star \in L^1(\mathbb{R}_+)$. From

$$\|\bar{\gamma}(t) - \bar{\gamma}(t')\|_{L^2}^2 \leq \|\bar{\gamma}(t) - \bar{\gamma}(t')\|_{H_{\gamma_\infty}} \|\bar{\gamma}(t) - \bar{\gamma}(t')\|_{H_{\gamma_\infty}^{-1}} \leq C\delta \int_{t'}^{t} \|\partial_t \bar{\gamma}\|_\star \, \mathrm{d}\tau \to 0$$

for $t', t \to \infty$, it follows that $\lim_{t\to\infty} \bar{\gamma}(t)$ exists in $L^2(\mathbb{S}^1; \mathbb{R}^2)$ and thus equals $\gamma_\infty$. By a subsequence argument, we obtain $\|\bar{\gamma}(t) - \gamma_\infty\|_{C^k} \to 0$ as $t \to \infty$ for any $k \in \mathbb{N}$, i.e. the claimed smooth convergence follows. □

### 3.5.4. Blow-up in infinite time

Without penalizing the length, solutions do not converge in general. A simple example for arbitrary rotation index is given by the situation described in the following lemma. For the sake of completeness, we state it now, even though the proof relies on an argument that is presented in the following section.

**Lemma 3.5.7** ([99, Lemma 6.9])**.** *If $A_0 = 0$ and $\lambda = 0$, the length $\mathcal{L}(\gamma)$ of the global solution $\gamma$ from Theorem 3.4.6 becomes unbounded as time goes to infinity.*

*Proof.* Assume there is $M \in \mathbb{R}$ such that $\mathcal{L}(\gamma(t)) < M$ for all $t \geq 0$. Then, proceeding as in the proof of Theorem 3.4.6 yields subconvergence. As in Section 3.5.3, this improves to full convergence to a stationary solution $\gamma_\infty$. This stationary solution satisfies (3.6.2) for a constant $c_1 \in \mathbb{R}$, i.e. we have

$$0 = 2c_1 A_0 = \mathcal{E}(\gamma_\infty) - \lambda \mathcal{L}(\gamma_\infty) = \mathcal{E}(\gamma_\infty).$$

This implies $\kappa_\infty \equiv 0$, which contradicts the closedness of $\gamma_\infty$. □

One might think that for $A_0 \neq 0$ and $\lambda = 0$, the length of a curve can be controlled by a bound on the elastic energy. We expect that this is not the case. We have in mind a curve shaped like a dumbbell (see Figure 3.2), where we can make the bar longer and thinner, allowing us to increase the length without increasing the elastic energy or the area (cf. also [23]). This means that also for $A_0 \neq 0$, in general, we do not expect the length to remain bounded, which would imply convergence for $\lambda = 0$.
We also note that for convex curves with $A_0 \neq 0$ and bounded elastic energy, it is possible to control the length. This has already been proven in [63]. However, since we are working with a higher order flow, convexity is generally not preserved, see [16].

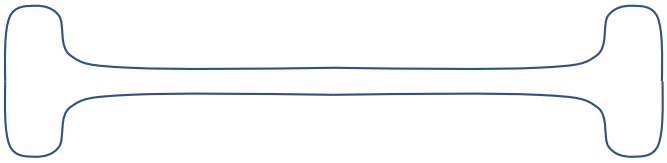


Figure 3.2: Curve shaped like a dumbbell.

## 3.6. Stationary solutions

The convergence result outlined in Section 3.5.3 raises interest in gaining a deeper understanding of stationary solutions — particularly with regard to the fact that, for certain initial data, the limit cannot be an elastica.

### 3.6.1. The defect in the elastica equation

From (3.0.5), we see that $\gamma$ is a stationary solution if and only if there exists a constant $c_1 \in \mathbb{R}$ such that

$$\nabla_s^2 \kappa + \frac{1}{2}|\kappa|^2 \kappa - \lambda \kappa = c_1 \nu. \tag{3.6.1}$$

If $c_1 = 0$ and $\lambda > 0$, we obtain the classical elastica equation (1.2.10), for which the solutions are classified. So in a certain sense, the constant $c_1$ represents the deviation from the classical case. Remember that the only closed elasticae are multifold coverings of the circle and of the figure eight elastica, see [124, Lemma 5.4].
If $A_0 \neq 0$, the defect $c_1$ in (3.6.1) can be measured in terms of the unpenalized elastic energy and the length of the curve.

**Lemma 3.6.1** ([99, Lemma 6.7])**.** *Let $\gamma$ be a stationary solution and $c_1 \in \mathbb{R}$ such that $\gamma$ satisfies* (3.6.1)*. Then*

$$2c_1 A_0 = \mathcal{E}(\gamma) - \lambda \mathcal{L}(\gamma), \tag{3.6.2}$$

*where $\mathcal{E}$ is the unpenalized elastic energy.*

*Proof.* We assume that $\gamma \colon \mathbb{S}^1 \to \mathbb{R}^2$ satisfies (3.6.1) on $\mathbb{S}^1$. This implies

$$\int_{\mathbb{S}^1} \langle \nabla_s^2 \kappa, \gamma \rangle \,\mathrm{d}s + \frac{1}{2} \int_{\mathbb{S}^1} |\kappa|^2 \langle \kappa, \gamma \rangle \,\mathrm{d}s - \lambda \int_{\mathbb{S}^1} \langle \kappa, \gamma \rangle \,\mathrm{d}s = c_1 \int_{\mathbb{S}^1} \langle \gamma, \nu \rangle \,\mathrm{d}s. \tag{3.6.3}$$

The first integral can be written as

$$\begin{aligned} \int_{\mathbb{S}^1} \langle \nabla_s^2 \kappa, \gamma \rangle \,\mathrm{d}s &= -\int_{\mathbb{S}^1} \langle \partial_s \nabla_s \kappa, \partial_s \gamma \rangle \langle \gamma, \partial_s \gamma \rangle \,\mathrm{d}s = \int_{\mathbb{S}^1} \langle \nabla_s \kappa, \kappa \rangle \langle \gamma, \partial_s \gamma \rangle \,\mathrm{d}s \\ &= -\frac{1}{2} \int_{\mathbb{S}^1} |\kappa|^2 \,\mathrm{d}s - \frac{1}{2} \int_{\mathbb{S}^1} |\kappa|^2 \langle \kappa, \gamma \rangle \,\mathrm{d}s. \end{aligned}$$

Putting this into (3.6.3) and with another integration by parts in the third integral of (3.6.3) yields (3.6.2). □

**Remark 3.6.2** (Equipartition of energy, [99, Remark 7])**.** If $A_0 = 0$ and $\gamma$ is a stationary solution, Lemma 3.6.1 implies that

$$\mathcal{E}(\gamma) = \lambda \mathcal{L}(\gamma),$$

So, the two parts of the penalized elastic energy (3.0.1) are equally weighted.

For a suitable choice of the parameters $\lambda$, $A_0$, and $\omega$, stationary solutions satisfy (3.6.1) with $c_1 \neq 0$.

**Lemma 3.6.3.** *Let $\gamma$ be a stationary solution with rotation index $\omega$ and let $c_1 \in \mathbb{R}$ be given such that $\gamma$ satisfies* (3.6.1)*. If $2\lambda A_0 \neq \pi\omega$, then $c_1 \neq 0$.*

*Proof.* If $\omega \neq 0$ and $\gamma$ satisfies (3.6.1) with $c_1 = 0$, then $\gamma$ parametrizes an $\omega$-fold covered circle with radius $r = \frac{1}{\sqrt{2\lambda}}$ (see for example [124, Lemma 5.4]). This implies $\mathcal{A}(\gamma) = \frac{\omega\pi}{2\lambda}$. If $\omega = 0$ and $\gamma$ satisfies (3.6.1) with $c_1 = 0$, then $\gamma$ parametrizes a (multifold covered) figure eight elastica with $\mathcal{A}(\gamma) = 0 = \frac{\omega\pi}{2\lambda}$. □

Note that $c_1 \neq 0$ does not necessarily imply that $\gamma$ does not describe an elastica. This becomes apparent with the following simple example. Take $A_0$ and $\omega$ having the same sign and consider a curve which describes an $\omega$-fold covered circle with radius $r = \sqrt{A_0(\pi\omega)^{-1}}$. Then this curve satisfies (3.6.1) with $c_1 \neq 0$ in general.

### 3.6.2. Existence of non-elastica stationary solutions

The dynamic approach with the $H_\gamma^{-1}$-gradient flow provides an elegant proof of the existence of solutions to (3.6.1) that are no elastica. We emphasize that this has been proven before by giving concrete examples in [142], [155] or [159]. For more details on this, see also Section 3.6.4. With our approach, there is no need for an explicit parametrization using Jacobi elliptic functions (as in [142], [155] and [159]), but simply an appropriate choice of the initial datum.

**Corollary 3.6.4** (of Theorem 3.0.1, Existence of non-elastica stationary solutions, [99, Corollary 6.8])**.** *There exists $c_1 \in \mathbb{R}$ and solutions to* (3.6.1) *that do not describe a multifold covered circle or figure eight elastica.*

We consider a concrete example. Let $\lambda > 0$, $\omega = 0$ and $A_0 \neq 0$. Clearly, for this choice of parameters, there exists an admissible initial datum, see for example Figure 3.1. By Theorem 3.5.6, the solution to (1.6.1) with this initial datum converges to a stationary solution. Since the rotation index and the enclosed signed area are preserved along the flow, this stationary solution satisfies $\omega = 0$ and $A_0 \neq 0$. The only closed elastica with $\omega = 0$

is the figure eight elastica with $\mathcal{A} = 0$. Thus, the stationary solution does not describe an elastica. The same arguments apply for $\mathcal{A} = 0$ and arbitrary $\omega \neq 0$. Figure 1.3B gives an exemplary initial datum.

### 3.6.3. Symmetries

We already know that stationary solutions exist that are not elasticae. Our goal is to better understand their structure. Let us first examine their symmetries provided by phase plane methods.

Working with the scalar curvature $k = \langle \kappa, \nu \rangle$, $\gamma$ is a stationary solution if and only if there exists $c_1 \in \mathbb{R}$ such that

$$\partial_s^2 k + \frac{1}{2}k^3 - \lambda k = c_1. \tag{3.6.4}$$

Compare this to (3.6.1) and note that $\langle \nabla_s^2 \kappa, \nu \rangle = \partial_s^2 k$. Moreover, multiplying (3.6.4) with $\partial_s k$, we see that for a stationary solution $\gamma$, there is $c_2 \in \mathbb{R}$ such that

$$\frac{1}{2}(\partial_s k)^2 + \frac{1}{8}k^4 - \frac{\lambda}{2}k^2 - c_1 k = c_2. \tag{3.6.5}$$

**Remark 3.6.5.** Let the stationary solution $\gamma$ be reparametrized by arclength, such that $\partial_s = |\partial_x \gamma|^{-1} \partial_x = \partial_x$. As in [8, Section 2.2], we can interpret the task of describing stationary solutions as the problem of determining the motion of a particle in a fourth-order potential:
If $k$ gives the displacement of the particle and $k''$ stands for the second derivative in time, the acceleration of a particle is given by $k'' = F(k)$, where $F$ stands for the forces acting on the particle. With the conservative force field

$$F(k) := -\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{8}k^4 - \frac{\lambda}{2}k^2 - c_1 k\right)$$

coming from the fourth order potential $V = \frac{1}{8}k^4 - \frac{\lambda}{2}k^2 - c_1 k$ and with $k'' = \partial_s^2 k$, we obtain (3.6.4). The total energy of the particle is preserved, i.e. with the kinetic energy $\frac{1}{2}(k')^2$, we have

$$\frac{1}{2}(k')^2 + V(k) \equiv c_2,$$

which corresponds with $k' = \partial_s k$ to (3.6.5).

In the following, the aim is to better understand the structure of (3.6.5), or, to put it differently, to gain a clearer understanding of the potential $V$ from Remark 3.6.5.

**Lemma 3.6.6.** *For $c_1 \in \mathbb{R}$ given, we consider the function*

$$V\colon \mathbb{R} \to \mathbb{R}, \quad V(k) = \frac{1}{8}k^4 - \frac{\lambda}{2}k^2 - c_1 k.$$

*(i) If $27c_1^2 > 8\lambda^3$, $V$ possesses one critical point, which corresponds to the global minimum.*

*(ii) If $27c_1^2 < 8\lambda^3$, there are three critical points of $V$, at which we have two local minima and one local maximum.*

*(iii) If $27c_1^2 = 8\lambda^3$ there exist two critical points of $V$. At one of them we have the global minimum, the other corresponds to a saddle point.*

*Proof.* We consider the derivative

$$\frac{\mathrm{d}}{\mathrm{d}k}V(k) = \frac{1}{2}k^3 - \lambda k - c_1 \tag{3.6.6}$$

with the not necessarily distinct roots $k_1, k_2, k_3 \in \mathbb{C}$. Since (3.6.6) is a third order polynomial with real coefficients, at least one of the roots is real. We follow the theory in [84, Section 10.3] to prove the claim without computing the roots explicitly. The discriminant of (3.6.6) is given by

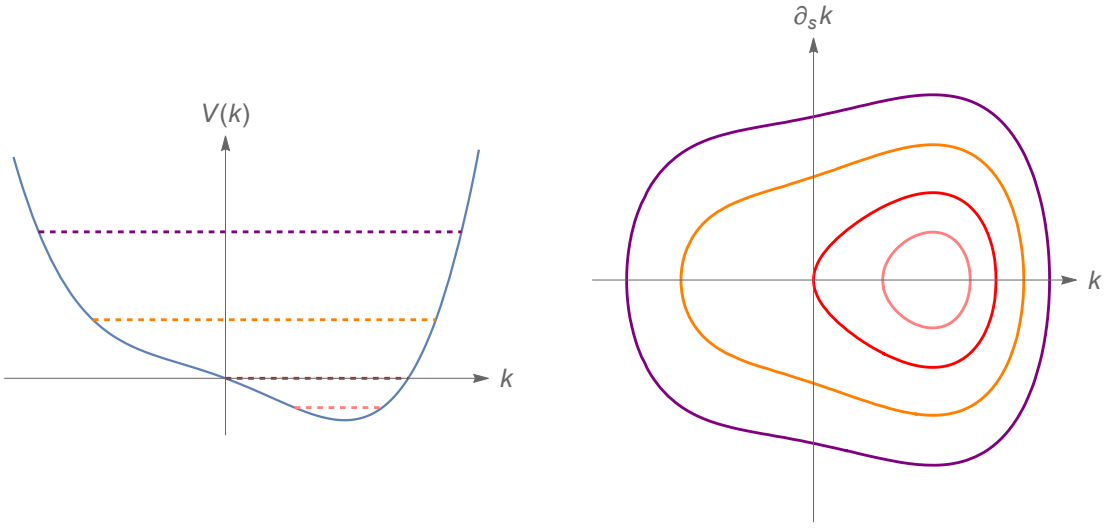
$$\frac{1}{16}(k_1 - k_2)^2(k_1 - k_3)^2(k_2 - k_3)^2 =: \Delta.$$

For the polynomial in (3.6.6) its value is computed to be

$$\Delta = \frac{1}{4}(8\lambda^3 - 27c_1^2).$$

If $\Delta > 0$, the three roots $k_1, k_2, k_3$ are distinct and real. Since $V$ is a fourth order polynomial with positive coefficient for the fourth order term, we conclude that $V$ possesses two local minima and one local maximum. In the case that $\Delta < 0$, (3.6.6) has one real root and two nonreal complex conjugate roots. It follows that there is only one local extremum of $V$ which is a local minimum. Moreover, if $\Delta = 0$, then (3.6.6) has a multiple root and all its roots are real. From the structure of $V$ we deduce, that there is one root with multiplicity one, which corresponds to a local minimum, and the other two roots coincide and correspond to a saddle point. □

Figure 3.3A and Figure 3.4A show $V$ for exemplary values of $\lambda$ and $c_1$ in the case (i) and (ii). Note that a change of the sign of $c_1$ causes a reflection at the vertical axis.
From the structure of $V$ we can now derive the following (see also [35, §12]). Given $\lambda > 0$ and $c_1 \in \mathbb{R}$, the set $\{k \in \mathbb{R} : V(k) \leq c_2\}$, consists of at most two bounded intervals $I_1 = [k_{\min,1}, k_{\max,1}]$ and $I_2 = [k_{\min,2}, k_{\max,2}]$. At the endpoints of these intervals (if they




(A) (B)

Figure 3.3: Example for the case $27c_1^2 > 8\lambda^3$. Parameters $\lambda = \frac{1}{2}$ and $c_1 = 1$. Potential $V$ in (A) and phase curves for $c_2 \in \{-1, 0, 1, 2, 5\}$ in (B).

are not empty), $V$ equals $c_2$. Assume that $c_2$ is a regular value in the sense that $c_2$ does not equal the value of $V$ at any of the critical points. Then the equation

$$\frac{(\partial_s k)^2}{2} + V(k) = c_2, \quad k \in [k_{\min,i}, k_{\max,i}] \tag{3.6.7}$$

defines a smooth curve diffeomorphic to a circle in the $(k, \partial_s k)$-plane. Since (3.6.7) is even in $\partial_s k$, this curve is symmetric to the $k$-axis. See Figure 3.3B and Figure 3.4B for concrete examples.

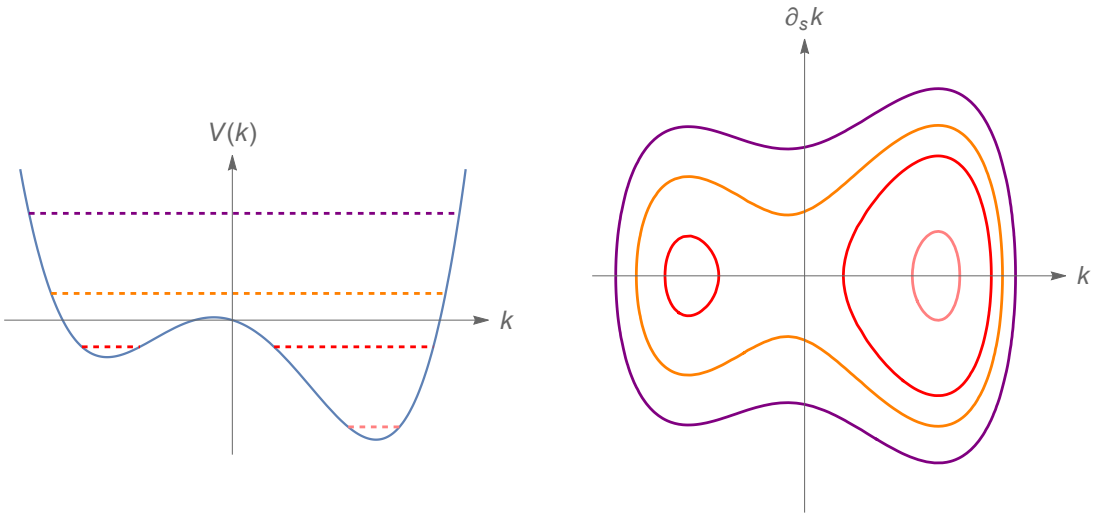
In the following section, we distinguish the cases

(I) $I_1 = [a, b]$ for $a < b$ and $I_2 = \emptyset$, i.e. there exist two real solutions $a, b \in \mathbb{R}$ of $V(k) = c_2$, (see Proposition 3.6.14) and

(II) $I_1 = [a, b]$ and $I_2 = [c, d]$ for $a < b < c \leq d$, i.e. there exist more than two real solutions $a, b, c, d \in \mathbb{R}$ of $V(k) = c_2$ (see Proposition 3.6.15).

**Remark 3.6.7.** If $c_2$ equals the global minimum of $V$, $k$ is constant (which is an admissible stationary solution for a certain choice of the parameters $\omega$ and $A_0$). If $c_1$ is as in case (ii) of Lemma 3.6.6 and $c_2$ equals the local maximum of $V$, the curve in the phase plane contradicts the closedness of $\gamma$. For this, see [8, Section 3.3 and Figure 6]. The same is the case if $c_1$ is as in case (iii) of Lemma 3.6.6 and $c_2$ equals the critical value of $V$ at which there is no local extremum. So these values of $c_1$ and $c_2$ can be excluded when considering closed curves.

Before we proceed, let us outline the properties of the solutions to (3.6.7), see [35, §12] and [158].




(A) (B)

Figure 3.4: Example for the case $27c_1^2 < 8\lambda^3$. Parameters $\lambda = \frac{6}{5}$ and $c_1 = \frac{1}{4}$. Potential $V$ in (A) and phase curves for $c_2 \in \{-1, -\frac{1}{4}, \frac{1}{4}, 1\}$ in (B).

**Lemma 3.6.8** ([158, Lemma 4])**.** *Let $\gamma$ be a stationary solution with length $L_\infty$. Then, after reparametrizing by arclength and a shift in the argument,*

*(i) $k$ is periodic with period $T = \frac{L_\infty}{n}$ and $k\left(\frac{T}{2} + s\right) = k\left(\frac{T}{2} - s\right)$ for $s \in \left[0, \frac{T}{2}\right]$,*

*(ii) $k(0) = k(T) = \max_{[0,L_\infty]} k =: k_{\max}$ and $k(\frac{T}{2}) = \min_{[0,L_\infty]} k =: k_{\min}$,*

*(iii) $\partial_s k(0) = \partial_s k(\frac{T}{2}) = 0$ and either $\partial_s k < 0$ or $\partial_s k \equiv 0$ on $\left(0, \frac{T}{2}\right)$,*

*(iv) $\int_0^{\frac{T}{2}} k \, \mathrm{d}s = \frac{\pi\omega}{n}$*

*for some $n \in \mathbb{N}$.*

**Remark 3.6.9.** We note the following about the value $n \in \mathbb{N}$ in Lemma 3.6.8.

(i) If $k$ is not constant, $n$ counts the number of minima of $k$ in $[0, L_\infty]$.

(ii) For $\omega = 1$, we have $n \geq 2$. This follows from [6, Symmetry Theorem].

(iii) If $n \geq 2$, the stationary solution $\gamma$ is $n$-fold rotationally symmetric.

### 3.6.4. Explicite parametrization

From here on, we consider only the case where $c_1$ is nonzero. For the case where $c_1$ is zero, we know that the stationary solutions are elasticae, which means that they describe either an $\omega$-fold covered circle or a multifold covered figure eight.

Let us first summarize what is already known from previous works. For $c_1 \neq 0$, (3.6.1) is the equation which determines the equilibrium shapes of cylindrical fluid membranes

exposed to a pressure difference $c_1$ and experiencing a tensile stress $\lambda$, see [142] or [155]. In this context, (3.6.1) is the Euler–Lagrange equation corresponding to the problem of minimizing the elastic energy

$$\frac{1}{2}\int_\gamma |\kappa|^2 \,\mathrm{d}s + \lambda\mathcal{L}(\gamma) + c_1\mathcal{A}(\gamma) \tag{3.6.8}$$

with penalized area ($c_1 > 0$ given) and penalized length ($\lambda > 0$ given). In [142], a complete classification of the critical points corresponding to closed curves is presented. Similarly, [155] explicitly provides solutions to (3.6.1) in a format comparable to [142]. The authors state a condition under which a solution describes a closed curve and investigate when the curve is simple. We also want to mention that for $\lambda = 0$ in (3.6.8), we obtain the Tadjbakhsh–Odeh functional, see [150] and [159]. The corresponding Euler–Lagrange equation is again of the form (3.6.4), see [159, (11)]. In [159], the authors classify the solutions for $\omega = 1$ and give an explicit condition for the existence of nonelastica solutions. By [42, Lemma 5.4], minimizing (3.6.8) is equivalent to minimize $\mathcal{E}_\lambda$ under the constraint of fixed enclosed area. The latter problem is studied in [158] and [159] for $\omega = 1$. The authors of [126] extend the problem to $\omega \in \mathbb{Z}$. They study an auxiliary problem and show all the solutions in terms of the curvature. According to [126, End of Section 2], the authors plan to investigate the existence and non-existence of solutions for given enclosed area and given rotation index (and free $\lambda$) in a forthcoming article. This would bring us closer to understanding our stationary solutions for a given initial datum. However, to our knowledge, this article has not yet been published. For some further references, see also the citations within the mentioned articles.

The variety of different references already indicates the complexity of classifying the stationary solutions. In this thesis, we limit our focus to proposing an approach for finding stationary solutions that are not an elastica. We intend to see a few concrete examples. Additionally, we aim to partially demonstrate how the results found in the literature fit together. Our approach is primarily based on [126] at the beginning and later on [155]. First, we need to introduce Jacobi elliptic functions.

#### Jacobi elliptic functions

For a detailed introduction to elliptic integrals and Jacobi elliptic functions, we refer to [1], [25] or [100]. We only summarize what we need for the following. Throughout this paragraph, let $p \in \mathbb{R}$ with $|p| < 1$.

**Definition 3.6.10** (Amplitude function)**.** The *amplitude function* $\mathrm{am}(\cdot, p)\colon \mathbb{R} \to \mathbb{R}$ is defined to be the inverse function of

$$\mathbb{R} \to \mathbb{R}, \quad z \mapsto \int_0^z \frac{1}{1 - p^2 \sin^2\varphi}\,\mathrm{d}\varphi.$$

**Definition 3.6.11** (Elliptic integrals)**.** We define the *incomplete elliptic integral of first kind* as

$$F(x,p) := \int_0^x \frac{1}{1-p^2\sin^2\varphi}\,\mathrm{d}\varphi,$$

such that $F(\cdot,p) = \mathrm{am}^{-1}(\cdot,p)$. The *incomplete elliptic integral of third kind* is defined as

$$\Pi(x,\alpha^2,p) = \int_0^x \frac{1}{(1-\alpha^2\sin^2\varphi)\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi,$$

where $\alpha^2 \in \mathbb{R}$ such that $1 \neq \alpha^2 \neq p^2$. Moreover, we define the *complete elliptic integral of first kind* as

$$K(p) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi \tag{3.6.9}$$

and the *complete elliptic integral of third kind* as

$$\Pi(\alpha^2,p) = \int_0^{\frac{\pi}{2}} \frac{1}{(1-\alpha^2\sin^2\varphi)\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi.$$

**Definition 3.6.12** (Elliptic functions)**.** The *Jacobi elliptic functions* are given by

$$\begin{aligned}
&\mathrm{cn}(\cdot,p)\colon \mathbb{R}\to\mathbb{R}, \quad \mathrm{cn}(x,p) = \cos(\mathrm{am}(x,p)),\\
&\mathrm{sn}(\cdot,p)\colon \mathbb{R}\to\mathbb{R}, \quad \mathrm{sn}(x,p) = \sin(\mathrm{am}(x,p)),\\
&\mathrm{dn}(\cdot,p)\colon \mathbb{R}\to\mathbb{R}, \quad \mathrm{dn}(x,p) = \sqrt{1-p^2\sin^2(\mathrm{am}(x,p))}.
\end{aligned}$$

We show the graphs of the Jacobi elliptic functions cn, sn and dn for $p^2 = \frac{1}{2}$ plotted on $[0, 4K(p)]$ in Figure 3.5.

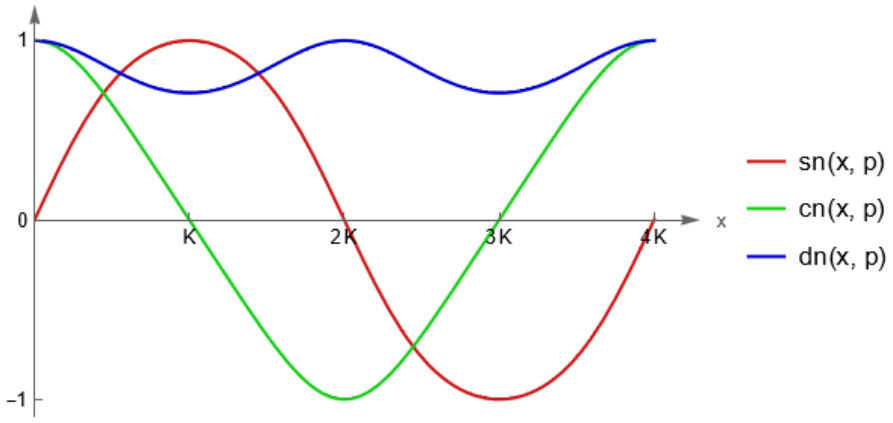



Figure 3.5: Jacobi elliptic functions for $p^2 = \frac{1}{2}$.

The periodicity of the Jacobi elliptic functions, which can also be seen in this figure, is formalized in the following lemma.

**Lemma 3.6.13** (Periodicity, [1, Section 16.2])**.** *Let $l \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then*

$$\mathrm{am}(lK(p), p) = l\frac{\pi}{2}$$

*and therefore*

$$\begin{aligned}\mathrm{cn}(x + 4lK(p), p) &= \mathrm{cn}(x, p),\\ \mathrm{sn}(x + 4lK(p), p) &= \mathrm{sn}(x, p),\\ \mathrm{dn}(x + 2lK(p), p) &= \mathrm{dn}(x, p).\end{aligned}$$

#### Explicit expressions for the curvature

In this subsection, we consider a given stationary solution $\gamma\colon [0, L_\infty] \to \mathbb{R}^2$ reparametrized by arclength with nonconstant curvature. There are $c_1 \in \mathbb{R}$ such that $\gamma$ satisfies (3.6.4) and $c_2 \in \mathbb{R}$ such that $\gamma$ satisfies (3.6.5). We work with $V\colon \mathbb{R} \to \mathbb{R}$ defined as in Lemma 3.6.6. Combining the results of [126] and [155], we obtain the following explicit expressions for the curvature $k\colon [0, L_\infty] \to \mathbb{R}$. The first proposition belongs to Case (I) of Section 3.6.3.

**Proposition 3.6.14** (cf. [155])**.** *If $c_1, c_2$ are such that there exist exactly two real solutions $a, b \in \mathbb{R}$, $a > b$, of $V(k) = c_2$, then the curvature of the stationary solution is represented by*

$$k_1(s) = \frac{(Aa + Bb) - (Aa - Bb)\mathrm{cn}\left(\frac{1}{4}\sqrt{AB}s, p\right)}{B + A + (B - A)\mathrm{cn}\left(\frac{1}{4}\sqrt{AB}s, p\right)} \tag{3.6.10}$$

*up to a shift in the argument. Here, we set*

$$A = \sqrt{4(\Im(c))^2 + (3b + a)^2} \quad \text{and} \quad B = \sqrt{4(\Im(c))^2 + (b + 3a)^2},$$

*where $c = \Re(c) + i\Im(c)$ is one of the imaginary solutions of $V(k) = c_2$. The elliptic modulus $p$ is given by*

$$p = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{4(\Im(c))^2 + (3a + b)(a + 3b)}{AB}}.$$

The next proposition covers Case (II) of Section 3.6.3. A curvature of the form $k_2$ could correspond to the red line on the right hand side in Figure 3.4A, while a curvature of the form $k_3$ could correspond to the left red line in Figure 3.4A.

**Proposition 3.6.15** (cf. [155]). *If $c_1, c_2$ are such that there exist real solutions $a, b, c, d \in \mathbb{R}$, $a > b$ and $a \neq c \geq d \neq b$, of $V(k) = c_2$, then the curvature of the stationary solution is represented either by*

$$k_2(s) = d + \frac{(a-d)(b-d)}{(b-d) + (a-b)\mathrm{sn}^2\Big(\frac{1}{4}\sqrt{(c-a)(d-b)}s, p\Big)} \tag{3.6.11}$$

*(in this case $d \leq c < b < a$), or by*

$$k_3(s) = c - \frac{(c-a)(c-b)}{(c-b) - (a-b)\mathrm{sn}^2\Big(\frac{1}{4}\sqrt{(c-a)(d-b)}s, p\Big)} \tag{3.6.12}$$

*(in this case $b < a < d \leq c$), up to a shift in the argument. Here, we set*

$$p = \sqrt{\frac{(c-d)(a-b)}{(c-a)(d-b)}}.$$

Note that if $c = d$ in Proposition 3.6.15, then $k_2$ and $k_3$ coincide. We are slightly anticipating here, but we also point out that this is only interesting in the case where $V$ has a local minimum at $c = d$. If $V$ has a local maximum at $c = d$, the curvature $k_2 = k_3$ does not describe a closed curve, as explained in [8, Section 3.3].

To prove the preceding two propositions, not many new ideas are required. Rather, our task is to bring together the already existing results. To understand what is happening, we partially reproduce the arguments of these established results. The briefly summarized proofs of Proposition 3.6.14 and Proposition 3.6.15 are provided on page 175 and page 179, respectively.

What is the idea? Due to Lemma 3.6.8, we know that after a shift in the argument, the curvature $k\colon [0, L_\infty] \to \mathbb{R}$ is either constant or its restriction to $[0, \frac{L_\infty}{2n}]$ is a solution of

$$\begin{cases} \partial_s^2 k + \frac{1}{2}k^3 - \lambda k = c_1 & \text{on } [0, \frac{L_\infty}{2n}], \\ \partial_s k(0) = \partial_s k(\frac{L_\infty}{2n}) = 0, & \\ \partial_s k < 0 & \text{on } (0, \frac{L_\infty}{2n}) \end{cases} \tag{3.6.13}$$

for some $n \in \mathbb{N}$. Here, the differential operator $\partial_s = \partial_x$ is just the ordinary derivative. Without loss of generality, we assume that $k$ is monotone decreasing on $(0, \frac{L_\infty}{2n})$. If not, we achieve this through an additional shift in the argument.

Problem (3.6.13) is studied in [126]. As in this article, we intend to find a representation of the solutions to (3.6.13). So, the following steps are similar to what is done in [126].

For a given smooth solution $k\colon [0, \frac{L_\infty}{2n}] \to \mathbb{R}$ of (3.6.13), we define

$$a := \max_{[0,\frac{L_\infty}{2n}]} k = k(0) \quad \text{and} \quad b := \min_{[0,\frac{L_\infty}{2n}]} k = k(\tfrac{L_\infty}{2n}).$$

Since $\partial_s k < 0$ on $(0, \frac{L_\infty}{2n})$, we know that $a \neq b$. By integration, we obtain

$$(\partial_s k)^2 = -\frac{1}{4}k^4 + \lambda k^2 + 2c_1 k - \Big( -\frac{1}{4}a^4 + \lambda a + 2c_1 a\Big) \tag{3.6.14}$$

Moreover, there exists $\delta \in \mathbb{R}$ such that

$$(\partial_s k)^2 = -\frac{1}{4}(k-a)(k-b)\bigg(\Big(\frac{a+b}{2} + k\Big)^2 + 4\delta\bigg). \tag{3.6.15}$$

This is because of the conditions in (3.6.13), which tell us that the right hand side of (3.6.14) can be written as $(k-a)(k-b)$ times a second order polynomial in $k$. And since no third order terms appear at the right hand side of (3.6.14), a short computation shows that this second order polynomial is of the form $-\frac{1}{4}((\frac{a+b}{2} + k)^2 + 4\delta)$ for some $\delta \in \mathbb{R}$. We now define

$$P := \frac{3a+b}{4} \quad \text{and} \quad Q := \frac{a+3b}{4}. \tag{3.6.16}$$

With this definition, $P + Q = a + b$ and $2(P - Q) = a - b$. Moreover, we set

$$\hat{k} := \frac{1}{2}\left(k + \frac{P+Q}{2}\right). \tag{3.6.17}$$

Then $\hat{k}(0) = P$ and $\hat{k}(\frac{L_\infty}{2n}) = Q$ and $\hat{k}$ solves

$$(\partial_s \hat{k})^2 = (P - \hat{k})(\hat{k} - Q)(\hat{k}^2 + \delta). \tag{3.6.18}$$

This is an approach which is also found, for example, in [116]. From now on, we have to distinguish different cases. The case $\delta > 0$ leads to Proposition 3.6.14 and the case $\delta \leq 0$ leads to Proposition 3.6.15.

First, let us assume that

$$\boxed{\delta > 0 \ \text{ and } \ P + Q \leq 0.}$$

In this case, we consider $\tilde{k}$ given as

$$\tilde{k} := (\hat{k} - Q)^{-1} \quad \text{ on } [0, \tfrac{L_\infty}{2n}). \tag{3.6.19}$$

Note that $\hat{k}$ is strictly monotone decreasing and attains values between $P$ and $Q$, such

that $\tilde{k}$ is well defined. Equation (3.6.18) transforms to

$$\begin{aligned}\partial_s\tilde{k} &= -(\hat{k}-Q)^{-2}\partial_s\hat{k} = (\hat{k}-Q)^{-2}\sqrt{(P-\hat{k})(\hat{k}-Q)(\hat{k}^2+\delta)} \\ &= \tilde{k}^2\sqrt{\left(P-Q-\tfrac{1}{\tilde{k}}\right)\left(\tfrac{1}{\tilde{k}}\right)\left(\tfrac{1}{\tilde{k}^2}+\tfrac{2Q}{\tilde{k}}+Q^2+\delta\right)} \\ &= \sqrt{(P-Q)\left(\tilde{k}-\tfrac{1}{P-Q}\right)(Q^2+\delta)\left(\tilde{k}^2+\tfrac{2Q}{Q^2+\delta}\tilde{k}+\tfrac{1}{Q^2+\delta}\right)}. \end{aligned} \tag{3.6.20}$$

Moreover, since $P > Q$ we know that

$$\tilde{k}(0) = (P-Q)^{-1} > 0 \quad \text{and} \quad \tilde{k}(s) \to \infty \ \text{ for } \ s \to \tfrac{L_\infty}{2n}.$$

Since $\tan^2(\cdot)$ attains values between 0 and $\infty$ on $[0, \frac{\pi}{2})$, we make the ansatz to write $\tilde{k}$ on $[0, \frac{L_\infty}{2n})$ as

$$\tilde{k}(s) = (P-Q)^{-1}\left(1 + \frac{\sqrt{P^2+\delta}}{\sqrt{Q^2+\delta}}\tan^2\left(\frac{\varphi(s)}{2}\right)\right) \tag{3.6.21}$$

for some function $\varphi(s)$ with $\varphi(0) = 0$, $\varphi(\frac{L_\infty}{2n}) = \pi$ and attaining values in $[0, \pi)$. For this change of variable from $\tilde{k}$ to $\varphi$, see also [116, p. 160]. Note that we are in the case $\delta > 0$, such that $\sqrt{Q^2+\delta} \neq 0$ and $\sqrt{P^2+\delta} \neq 0$. So the coefficient in front of the tangent is well defined and different from zero. One the one hand, differentiating (3.6.21) yields

$$\partial_s\tilde{k} = \frac{1}{P-Q}\frac{\sqrt{P^2+\delta}}{\sqrt{Q^2+\delta}} \cdot 2\tan\tfrac{\varphi}{2}\cos^{-2}\left(\tfrac{\varphi}{2}\right) \cdot \tfrac{1}{2}\partial_s\varphi. \tag{3.6.22}$$

On the other hand, a rather long computation shows that putting (3.6.20) and (3.6.21) together, yields

$$\begin{aligned}\partial_s\tilde{k} &= \sqrt{P-Q}\sqrt{Q^2+\delta}\sqrt{\frac{\sqrt{P^2+\delta}}{\sqrt{Q^2+\delta}}\frac{1}{P-Q}\tan^2\tfrac{\varphi}{2}} \\ &\quad \cdot \left(\frac{1}{(P-Q)^2} + \frac{2}{(P-Q)^2}\frac{\sqrt{P^2+\delta}}{\sqrt{Q^2+\delta}}\tan^2\tfrac{\varphi}{2} + \frac{1}{(P-Q)^2}\frac{P^2+\delta}{Q^2+\delta}\tan^4\tfrac{\varphi}{2}\right. \\ &\qquad \left. + \frac{2Q}{Q^2+\delta}\frac{1}{P-Q} + \frac{2Q}{Q^2+\delta}\frac{1}{P-Q}\frac{\sqrt{P^2+\delta}}{\sqrt{Q^2+\delta}}\tan^2\tfrac{\varphi}{2} + \frac{1}{Q^2+\delta}\right)^{\frac{1}{2}} \\ &= \left(\frac{P^2+\delta}{Q^2+\delta}\right)^{\frac{1}{4}}\tan\tfrac{\varphi}{2}\frac{\sqrt{P^2+\delta}}{P-Q}\left(1+\tan^2\tfrac{\varphi}{2}\right) \\ &\quad \cdot \sqrt{1+\sin^2\tfrac{\varphi}{2}\cos^2\tfrac{\varphi}{2}\left(\frac{2\delta+2QP}{\sqrt{P^2+\delta}\sqrt{Q^2+\delta}}-2\right)} \end{aligned} \tag{3.6.23}$$

Let us further define

$$p := \sqrt{\frac{1}{2}\left(1 - \frac{\delta + QP}{\sqrt{P^2+\delta}\sqrt{Q^2+\delta}}\right)}. \tag{3.6.24}$$

Note that this is well defined since

$$\delta + QP \leq \sqrt{Q^2P^2 + 2\delta QP + \delta^2} \leq \sqrt{Q^2P^2 + P^2\delta + Q^2\delta + \delta^2} = \sqrt{P^2+\delta}\sqrt{Q^2+\delta}.$$

Comparing (3.6.22) and (3.6.23) and using trigonometric identities then yields

$$\begin{aligned}\partial_s\varphi &= (P^2+\delta)^{\frac{1}{4}}(Q^2+\delta)^{\frac{1}{4}}\left(\cos^2\tfrac{\varphi}{2} + \sin^2\tfrac{\varphi}{2}\right)\sqrt{1 + \sin^2\tfrac{\varphi}{2}\cos^2\tfrac{\varphi}{2}\left(\frac{2\delta+2QP}{\sqrt{P^2+\delta}\sqrt{Q^2+\delta}} - 2\right)} \\ &= (P^2+\delta)^{\frac{1}{4}}(Q^2+\delta)^{\frac{1}{4}}\sqrt{1-p^2\sin^2\varphi}\,.\end{aligned}$$

Separation of variables and using that $\varphi(0) = 0$ leads to

$$(P^2+\delta)^{\frac{1}{4}}(Q^2+\delta)^{\frac{1}{4}}s = \int_0^{\varphi(s)} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi. \tag{3.6.25}$$

With $\varphi(\frac{L_\infty}{2n}) = \pi$ and (3.6.9), we obtain

$$(P^2+\delta)^{\frac{1}{4}}(Q^2+\delta)^{\frac{1}{4}}\frac{L_\infty}{2n} = \int_0^{\pi} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi = 2\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi = 2K(p), \tag{3.6.26}$$

which turns (3.6.25) into

$$\frac{4n}{L_\infty}K(p)s = \int_0^{\varphi(s)} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi.$$

Comparing this to Definition 3.6.10 and Definition 3.6.12, we see that

$$\cos\varphi(s) = \mathrm{cn}\left(\tfrac{4n}{L_\infty}K(p)s, p\right).$$

Using trigonometric identities, this turns our ansatz (3.6.21) into

$$\begin{aligned}\tilde{k}(s) &= (P-Q)^{-1}\left(1 + \frac{\sqrt{P^2+\delta}}{\sqrt{Q^2+\delta}}\,\frac{1-\cos\varphi(s)}{1+\cos\varphi(s)}\right) \\ &= \frac{\sqrt{Q^2+\delta} + \sqrt{P^2+\delta} + \left(\sqrt{Q^2+\delta} - \sqrt{P^2+\delta}\right)\cos\varphi(s)}{(P-Q)\sqrt{Q^2+\delta}(1+\cos\varphi(s))}\end{aligned}$$

$$= \frac{\sqrt{Q^2+\delta} + \sqrt{P^2+\delta} + \left(\sqrt{Q^2+\delta} - \sqrt{P^2+\delta}\right) \operatorname{cn}\left(\frac{4n}{L_\infty} K(p)s, p\right)}{(P-Q)\sqrt{Q^2+\delta}\left(1 + \operatorname{cn}\left(\frac{4n}{L_\infty} K(p)s, p\right)\right)}.$$

With (3.6.19) and (3.6.17) we arrive at

$$\begin{aligned} k(s) &= 2\left(\frac{1}{\bar{k}} + Q\right) - \frac{P+Q}{2} \\ &= 2\frac{P\sqrt{Q^2+\delta} + Q\sqrt{P^2+\delta} + \left(P\sqrt{Q^2+\delta} - Q\sqrt{P^2+\delta}\right) \operatorname{cn}\left(\frac{4n}{L_\infty} K(p)s, p\right)}{\sqrt{P^2+\delta} + \sqrt{Q^2+\delta} + \left(\sqrt{Q^2+\delta} - \sqrt{P^2+\delta}\right) \operatorname{cn}\left(\frac{4n}{L_\infty} K(p)s, p\right)} - \frac{P+Q}{2} \end{aligned} \tag{3.6.27}$$

on $\left[0, \frac{L_\infty}{2n}\right)$. Clearly, $k(s) \to 2Q - \frac{P+Q}{2} = b$ for $s \to \frac{L_\infty}{2n}$.

Let us now assume that for smooth solution $k\colon [0, \frac{L_\infty}{2n}] \to \mathbb{R}$ of (3.6.13), it holds that

$$\boxed{\delta > 0 \ \text{ and } \ P + Q > 0.}$$

So, we remain in the case that $\delta > 0$, but the sum of the maximum and the minimum of the curvature is now positive. Intuitively, one might think of reversing the orientation of the curve described by $k$, which changes the sign of the curvature and, consequently, the sign of the sum of the maximum and minimum. This is, visually speaking, the basic idea. We consider

$$\bar{k}(s) := -k\left(\tfrac{L_\infty}{2n} - s\right), \quad s \in \left[0, \tfrac{L_\infty}{2n}\right].$$

Then $\bar{k}$ is monotone decreasing and attains values between

$$\bar{a} = -\min_{[0, \frac{L_\infty}{2n}]} k = -b \quad \text{and} \quad \bar{b} = -\max_{[0, \frac{L_\infty}{2n}]} k = -a.$$

For $c_1$ replaced by $-c_1$, $\bar{k}$ solves (3.6.13) and (3.6.19). And with a short computation, one checks that $\bar{k}$ solves

$$(\partial_s \bar{k})^2 = -\frac{1}{4}(\bar{k} - \bar{a})(\bar{k} - \bar{b})\left(\left(\frac{\bar{a}+\bar{b}}{2} + \bar{k}\right)^2 + 4\delta\right)$$

with the same $\delta > 0$ as for which $k$ solves (3.6.15). So defining

$$\bar{P} := \frac{3\bar{a} + \bar{b}}{4} = -Q \quad \text{and} \quad \bar{Q} := \frac{\bar{a} + 3\bar{b}}{4} = -P,$$

we obtain the representation

$$\bar{k}(s) = 2\frac{\bar{P}\sqrt{\bar{Q}^2+\delta} + \bar{Q}\sqrt{\bar{P}^2+\delta} + \left(\bar{P}\sqrt{\bar{Q}^2+\delta} - \bar{Q}\sqrt{\bar{P}^2+\delta}\right)\operatorname{cn}\left(\frac{4n}{L_\infty}K(p)s,p\right)}{\sqrt{\bar{P}^2+\delta}+\sqrt{\bar{Q}^2+\delta}+\left(\sqrt{\bar{Q}^2+\delta}-\sqrt{\bar{P}^2+\delta}\right)\operatorname{cn}\left(\frac{4n}{L_\infty}K(p)s,p\right)} - \frac{\bar{P}+\bar{Q}}{2},$$

where $p$ is given as in (3.6.24). This yields

$$\begin{aligned} k(s) &= -\left(-k\left(\tfrac{L_\infty}{2n} - \left(\tfrac{L_\infty}{2n} - s\right)\right)\right) = -\bar{k}\left(\tfrac{L_\infty}{2n} - s\right) \\ &= 2\frac{P\sqrt{Q^2+\delta} + Q\sqrt{P^2+\delta} + \left(P\sqrt{Q^2+\delta} - Q\sqrt{P^2+\delta}\right)\operatorname{cn}\left(\frac{4n}{L_\infty}K(p)s,p\right)}{\sqrt{P^2+\delta}+\sqrt{Q^2+\delta}+\left(\sqrt{Q^2+\delta}-\sqrt{P^2+\delta}\right)\operatorname{cn}\left(\frac{4n}{L_\infty}K(p)s,p\right)} - \frac{P+Q}{2}, \end{aligned} \tag{3.6.28}$$

since

$$\operatorname{cn}\left(\tfrac{4n}{L_\infty}K(p)\left(\tfrac{L_\infty}{2n}-s\right),p\right) = \operatorname{cn}\left(2K(p) - \tfrac{4n}{L_\infty}K(p)s,p\right) = -\operatorname{cn}\left(\tfrac{4n}{L_\infty}K(p)s,p\right).$$

Comparing (3.6.27) and (3.6.28), we summarize that in the case

$$\boxed{\delta > 0,}$$

the solution $k\colon [0,\frac{L_\infty}{2n}] \to \mathbb{R}$ of (3.6.13) is represented by (3.6.27) with $p$ defined as in (3.6.24). Moreover, it holds that

$$K(p) = \frac{L_\infty}{4n}((P^2+\delta)(Q^2+\delta))^{\frac{1}{4}},$$

see (3.6.26). We note that this result can be found in [126, Lemma 4.1].

To be able to compare the representation that we found to the formulas in [155] and [49] and to come to the statement of Proposition 3.6.14, we now write (3.6.27) in a slightly different form. For this, let us consider the implication of $\delta > 0$. Specifically, it means that the right hand side of (3.6.15), when viewed as a polynomial in $k$, is only zero at $k = a$ and $k = b < a$. Therefore, we write (3.6.15) as

$$-\frac{1}{4}(k-a)(k-b)\left(\left(\frac{a+b}{2}+k\right)^2 + 4\delta\right) = -\frac{1}{4}(k-a)(k-b)(k-c)(k-d) \tag{3.6.29}$$

where $c, d \in \mathbb{C}\setminus\mathbb{R}$ and $d = c^*$ where $c^* = \Re(c) - i\Im(c)$ is the complex conjugate of $c$. Comparing the coefficients in (3.6.29) yields

$$\delta = \frac{(\Im(c))^2}{4}.$$

Next, we define

$$A := \sqrt{4(\Im(c))^2 + (3b+a)^2} \quad \text{and} \quad B := \sqrt{4(\Im(c))^2 + (b+3a)^2}. \tag{3.6.30}$$

A short computation shows that

$$A = 4\sqrt{Q^2+\delta} \quad \text{and} \quad B = 4\sqrt{P^2+\delta}.$$

With this, we see that $p$ defined in (3.6.24) can be written as

$$p = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{4(\Im(c))^2 + (3a+b)(a+3b)}{AB}}$$

Moreover, a quick look reveals that (3.6.26) is expressed in terms of $A$ and $B$ as

$$\frac{1}{4}\sqrt{AB} = \frac{4n}{L_\infty}K(p).$$

With (3.6.16) and (3.6.30), we rewrite (3.6.27) as

$$\begin{aligned} k(s) &= 2\frac{PA + BQ + (PA - BQ)\mathrm{cn}\left(\frac{1}{4}\sqrt{AB}s, p\right)}{B + A - (B-A)\mathrm{cn}\left(\frac{1}{4}\sqrt{AB}s, p\right)} - \frac{P+Q}{2} \\ &= \frac{(Aa+Bb) + (Aa-Bb)\mathrm{cn}\left(\frac{1}{4}\sqrt{AB}s, p\right)}{B + A - (B-A)\mathrm{cn}\left(\frac{1}{4}\sqrt{AB}s, p\right)} \end{aligned} \tag{3.6.31}$$

for $s \in [0, \frac{L_\infty}{2n}]$. Having in mind Lemma 3.6.8, we set

$$k(s) = k\left(\frac{L_\infty}{n} - s\right) \tag{3.6.32}$$

for $s \in [\frac{L_\infty}{2n}, \frac{L_\infty}{n}]$. Since $\frac{1}{4}\sqrt{AB} = \frac{4n}{L_\infty}K(p)$ and

$$\mathrm{cn}\left(\tfrac{4n}{L_\infty}K(p)\left(\tfrac{L_\infty}{n} - s\right), p\right) = \mathrm{cn}\left(-\tfrac{4n}{L_\infty}K(p)s, p\right) = \mathrm{cn}\left(\tfrac{4n}{L_\infty}K(p)s, p\right)$$

due to Lemma 3.6.13, (3.6.32) is given by the right hand side of (3.6.31). For $s \in [0, L_\infty]$, we consider the $\frac{L_\infty}{n}$-periodic extension of $k$ given on $[0, \frac{L_\infty}{n}]$ as in (3.6.31) and in (3.6.32). Shifting the argument by

$$\frac{2K(p)}{\frac{1}{4}\sqrt{AB}} = \frac{8K(p)}{\sqrt{AB}} = \frac{L_\infty}{2n}$$

which changes the sign of cn, we define

$$k_1(s) := k\left(s + \frac{8K(p)}{\sqrt{AB}}\right) = \frac{(Aa+Bb) - (Aa-Bb)\mathrm{cn}\left(\frac{1}{4}\sqrt{AB}s, p\right)}{B + A + (B-A)\mathrm{cn}\left(\frac{1}{4}\sqrt{AB}s, p\right)}$$

This is exactly the representation given in Proposition 3.6.14. Let us summarize what we have done.

*Proof of Proposition 3.6.14.* Let $\gamma\colon [0, L_\infty] \to \mathbb{R}^2$ be a stationary solution, parametrized by arclength, such that $\gamma$ satisfies (3.6.5) for $c_1, c_2 \in \mathbb{R}$. Then, by Lemma 3.6.8, after a shift in the argument, the curvature $k$ satisfies (3.6.13). Since we assume that $c_1, c_2$ are such that $V(k) = c_2$ has exactly two real solutions, $k$ satisfies (3.6.15) for some $\delta > 0$. It follows that $k$ is represented by [126, (4.1)]. Relating $\delta$ to the imaginary solutions of $V(k) = c_2$, we derive the representation in Proposition 3.6.14, which is also found in [155, Theorem 1]. □

Now, we consider the case that $\delta \leq 0$ in (3.6.15). First, let us assume that

$$\boxed{\delta \leq 0 \text{ and } P + Q > 0.}$$

Since $P > Q$, this implies that $P > 0$. If we assume that $Q < 0$, then $\hat{k}(s^*) = 0$ for some $s^* \in (0, \frac{L_\infty}{2n})$. By (3.6.18), $(\partial_s \hat{k}(s^*))^2 = -PQ\delta \leq 0$, a contradiction. Thus, $0 \leq Q < P$. Moreover, if we assume that $\sqrt{-\delta} \geq P$, we obtain a contradiction to (3.6.18), which is

$$0 \leq (\partial_s \hat{k})^2 = (P - \hat{k})(\hat{k} - Q)(\hat{k} - \sqrt{-\delta})(\hat{k} + \sqrt{-\delta}).$$

Assuming $Q < \sqrt{-\delta} < P$ yields the existence of $s^* \in (0, \frac{L_\infty}{2n})$ such that $\hat{k}(s^*) = \sqrt{-\delta}$. This implies $(\partial_s k(s^*))^2 = 0$, a contradiction. Thus, we know that $0 \leq \sqrt{-\delta} \leq Q$. We further assume that $\sqrt{-\delta} \neq Q$ and clarify later why. In summary, we consider

$$0 \leq \sqrt{-\delta} < Q < P. \tag{3.6.33}$$

In the current case $\delta \leq 0$ and $P + Q > 0$, we define

$$\tilde{k}(s) := (\hat{k}(s) + \sqrt{-\delta})^{-1} \text{ on } [0, \tfrac{L_\infty}{2n}).$$

Since $\hat{k}$ decreases, $\tilde{k}$ increases and

$$\begin{aligned}\partial_s \tilde{k} &= -(\hat{k} + \sqrt{-\delta})^{-2} \partial_s \hat{k} = (\hat{k} + \sqrt{-\delta})^{-2} \sqrt{(P - \hat{k})(\hat{k} - Q)(\hat{k}^2 + \delta)} \\ &= \tilde{k}^2 \sqrt{\left(P - \frac{1}{\tilde{k}} + \sqrt{-\delta}\right)\left(\frac{1}{\tilde{k}} - \sqrt{-\delta} - Q\right)\left(\frac{1}{\tilde{k}^2} - \frac{2\sqrt{-\delta}}{\tilde{k}} + (\sqrt{-\delta})^2 + \delta\right)}\end{aligned}$$

$$= \sqrt{(P+\sqrt{-\delta})(Q+\sqrt{-\delta})}\sqrt{\left(\tilde{k}-\frac{1}{P+\sqrt{-\delta}}\right)\left(\frac{1}{Q+\sqrt{-\delta}}-\tilde{k}\right)\left(1-2\sqrt{-\delta}\tilde{k}\right)}. \tag{3.6.34}$$

Since $\tilde{k}$ is monotone increasing and attains values between $\frac{1}{P+\sqrt{-\delta}}$ and $\frac{1}{Q+\sqrt{-\delta}}$, we make the ansatz

$$\tilde{k}(s) = \frac{1}{P+\sqrt{-\delta}} + \left(\frac{1}{Q+\sqrt{-\delta}} - \frac{1}{P+\sqrt{-\delta}}\right)\sin^2\varphi(s) \tag{3.6.35}$$

for some smooth function $\varphi$ with $\varphi(0) = 0$ and $\varphi(\frac{L_\infty}{2n}) = \frac{\pi}{2}$. On the one hand, differentiating this ansatz yields

$$\partial_s\tilde{k} = 2\left(\frac{1}{Q+\sqrt{-\delta}} - \frac{1}{P+\sqrt{-\delta}}\right)\sin\varphi\cos\varphi\partial_s\varphi. \tag{3.6.36}$$

On the other hand, combining the ansatz and (3.6.34), we obtain

$$\begin{aligned}\partial_s\tilde{k} = \sqrt{(P+\sqrt{-\delta})(Q+\sqrt{-\delta})}\Bigg(&\left(\frac{1}{Q+\sqrt{-\delta}} - \frac{1}{P+\sqrt{-\delta}}\right)\sin^2\varphi \\ &\cdot\left(\frac{1}{P+\sqrt{-\delta}} - \frac{1}{Q+\sqrt{-\delta}} + \left(\frac{1}{Q+\sqrt{-\delta}} - \frac{1}{P+\sqrt{-\delta}}\right)\sin^2\varphi\right) \\ &\cdot\left(\frac{2\sqrt{-\delta}}{P+\sqrt{-\delta}} + \left(\frac{2\sqrt{-\delta}}{Q+\sqrt{-\delta}} - \frac{2\sqrt{-\delta}}{P+\sqrt{-\delta}}\right)\sin^2\varphi - 1\right)\Bigg)^{\frac{1}{2}} \\ = \sqrt{(P+\sqrt{-\delta})(Q+\sqrt{-\delta})}&\sin\varphi\cos\varphi\left(\frac{1}{Q+\sqrt{-\delta}} - \frac{1}{P+\sqrt{-\delta}}\right)\sqrt{\frac{P-\sqrt{-\delta}}{P+\sqrt{-\delta}}} \\ &\cdot\sqrt{1-p^2\sin^2\varphi},\end{aligned} \tag{3.6.37}$$

where we set

$$p := \sqrt{\frac{2\sqrt{-\delta}(P-Q)}{(Q+\sqrt{-\delta})(P-\sqrt{-\delta})}}.$$

Due to (3.6.33), $p$ is well defined. Comparing (3.6.36) and (3.6.37) yields

$$\partial_s\varphi = \frac{1}{2}\sqrt{Q+\sqrt{-\delta}}\sqrt{P-\sqrt{-\delta}}\sqrt{1-p^2\sin^2\varphi}.$$

Separation of variables and using that $\varphi(0) = 0$ leads to

$$\frac{1}{2}\sqrt{Q+\sqrt{-\delta}}\sqrt{P-\sqrt{-\delta}}\;s = \int_0^{\varphi(s)}\frac{1}{\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi. \tag{3.6.38}$$

With $\varphi(\frac{L_\infty}{2n}) = \frac{\pi}{2}$, we obtain

$$\frac{L_\infty}{4n}\sqrt{Q+\sqrt{-\delta}}\sqrt{P-\sqrt{-\delta}} = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi = K(p).$$

Comparing (3.6.38) to Definition 3.6.10 and Definition 3.6.12, we see that

$$\cos\varphi(s) = \mathrm{cn}\Big(\frac{2n}{L_\infty}K(p)s, p\Big).$$

With this, (3.6.35) becomes

$$\tilde{k}(s) = \frac{1}{P+\sqrt{-\delta}} + \left(\frac{1}{Q+\sqrt{-\delta}} - \frac{1}{P+\sqrt{-\delta}}\right)\left(1-\mathrm{cn}^2\Big(\frac{2n}{L_\infty}K(p)s,p\Big)\right),$$

which brings us to

$$\begin{aligned} k(s) &= 2\hat{k}(s) - \frac{P+Q}{2} = 2\left(\frac{1}{\tilde{k}(s)} - \sqrt{-\delta}\right) - \frac{P+Q}{2} \\ &= \frac{2}{\frac{1}{Q+\sqrt{-\delta}} + \left(\frac{1}{P+\sqrt{-\delta}} - \frac{1}{Q+\sqrt{-\delta}}\right)\mathrm{cn}^2\Big(\frac{2n}{L_\infty}K(p)s,p\Big)} - 2\sqrt{-\delta} - \frac{P+Q}{2} \\ &= \frac{2(Q+\sqrt{-\delta})(P+\sqrt{-\delta})}{P+\sqrt{-\delta}-(P-Q)\mathrm{cn}^2\Big(\frac{2n}{L_\infty}K(p)s,p\Big)} - 2\sqrt{-\delta} - \frac{P+Q}{2}. \end{aligned} \tag{3.6.39}$$

We check that $k(0) = 2P - \frac{P+Q}{2} = a$ and $k(\frac{L_\infty}{2n}) = 2Q - \frac{P+Q}{2} = b$.

Let us now assume that for the smooth solution $k\colon [0, \frac{L_\infty}{2n}] \to \mathbb{R}$ of (3.6.13), we have

$$\boxed{\delta \le 0 \ \text{ and } \ P+Q<0.}$$

With the same observations as on page 172 for the case $\delta > 0$, we see that

$$\bar{k}(s) := -k\left(\tfrac{L_\infty}{2n} - s\right), \quad s \in \left[0, \tfrac{L_\infty}{2n}\right].$$

is monotone decreasing, attains values between $-b$ and $-a$ and is thus represented by

$$\bar{k}(s) = \frac{2(\bar{Q}+\sqrt{-\delta})(\bar{P}+\sqrt{-\delta})}{\bar{P}+\sqrt{-\delta}-(\bar{P}-\bar{Q})\mathrm{cn}^2\Big(\frac{2n}{L_\infty}K(p)s,p\Big)} - 2\sqrt{-\delta} - \frac{\bar{P}+\bar{Q}}{2},$$

where

$$\bar{P} := \frac{1}{4}\Big(3\max_{[0,\frac{L_\infty}{2n}]}\bar{k} + \min_{[0,\frac{L_\infty}{2n}]}\bar{k}\Big) = \frac{-3b-a}{4} = -Q \quad \text{and} \quad \bar{Q} := -P \quad \text{with} \quad \bar{P}+\bar{Q} > 0.$$

This yields

$$\begin{aligned} k(s) &= -\left(-k\left(\tfrac{L_\infty}{2n} - \left(\tfrac{L_\infty}{2n} - s\right)\right)\right) = -\bar{k}\left(\tfrac{L_\infty}{2n} - s\right) \\ &= \frac{2(P-\sqrt{-\delta})(Q-\sqrt{-\delta})}{P-\sqrt{-\delta}-(P-Q)\mathrm{cn}^2\left(\frac{2n}{L_\infty}K(p)s,p\right)} + 2\sqrt{-\delta} - \frac{P+Q}{2}. \end{aligned} \tag{3.6.40}$$

We summarize that in the case

$$\boxed{\delta \leq 0,}$$

the solution $k\colon [0, \frac{L_\infty}{2n}] \to \mathbb{R}$ of (3.6.13) is represented by (3.6.39) with $p$ defined as in (3.6.24) if $P+Q>0$. If the sum of maximum and minimum is negative, i.e. $P+Q<0$, then $k$ is represented by (3.6.40). Moreover, it holds that

$$K(p) = \frac{L_\infty}{4n}\sqrt{Q+\sqrt{-\delta}}\sqrt{P-\sqrt{-\delta}}.$$

We note that the first part of this result can be found in [126, Lemma 5.1].

**Remark 3.6.16** ($\delta \leq 0$ and $P+Q=0$)**.** We exclude the case that $\delta \leq 0$ and $P+Q=0$. If $P+Q=a+b=0$, then $a<0<b$, since $a \neq b$. So there is $s^* \in (0, \frac{L_\infty}{2n})$ such that $k(s^*)=0$. Since $a+b=0$, this implies that $(\partial_s k(s^*))^2 = -ab\delta \leq 0$, see (3.6.14). We recall, that this contradicts (3.6.13).

Again, we rewrite (3.6.39) and (3.6.40) to be able to compare the results to [155]. We observe that $\delta<0$ implies that the right hand side of (3.6.14), when viewed as a polynomial in $k$, is zero at $k=a$, $k=b<a$ and $k=c$, $k=d\leq c$ for some $c,d \in \mathbb{R}$. A short computation reveals

$$\delta = -\frac{1}{16}(c-d)^2.$$

Moreover, by comparing the coefficients or by Vieta's formulas, we know that

$$a+b+c+d=0. \tag{3.6.41}$$

With this, we find

$$p = \sqrt{\frac{(c-d)(a-b)}{(a-c)(b-d)}} \tag{3.6.42}$$

and

$$\frac{2n}{L_\infty}K(p) = \frac{1}{4}\sqrt{(b-d)(a-c)}. \tag{3.6.43}$$

With (3.6.41), we find that if $P+Q>0$, then $d \le c < b < a$. Indeed, $P+Q>0$ implies $0 \le \sqrt{-\delta} < Q < P$, which yields $c<b$. Similarly, if $P+Q<0$, then $Q<P<-\sqrt{-\delta} \le 0$. This leads to $a<d$ and thus $b<a<d\le c$. In both cases, (3.6.42) and (3.6.43) are well defined.

Working with $a,b,c,d$, it becomes clear why we excluded $\sqrt{-\delta}=Q$ in the case $\delta \le 0$ and $P+Q>0$. If we would have $\sqrt{-\delta}=Q$, then a short computation shows using (3.6.41) shows that $b=c$. This brings us to the case that $c_2$ equals the local maximum of $V(k)$, which does not lead to a curve with finite length, see Remark 3.6.7 and [8, Figure 6].

If $P+Q>0$, i.e. $d\le c<b<a$, combining (3.6.39), and (3.6.41), (3.6.43) yields the representation

$$k_2(s) := d + \frac{(a-d)(b-d)}{(b-d)+(a-b)\mathrm{sn}^2\left(\frac{1}{4}\sqrt{(a-c)(b-d)}s,p\right)}. \tag{3.6.44}$$

If $P+Q<0$, i.e. $b<a<d\le c$, we find with (3.6.40) and (3.6.41) that

$$k_3(s) := c - \frac{(c-a)(c-b)}{(c-b)-(a-b)\mathrm{sn}^2\left(\frac{1}{4}\sqrt{(c-a)(d-b)}s,p\right)}. \tag{3.6.45}$$

Again, we set

$$k_i(s) = k_i\left(\tfrac{L_\infty}{n}-s\right) \tag{3.6.46}$$

for $s\in[\frac{L_\infty}{2n},\frac{L_\infty}{n}]$, $i=2,3$. Since $\frac{1}{4}\sqrt{(a-d)(b-d)}=\frac{2n}{L_\infty}K(p)$ and

$$\mathrm{sn}^2\left(\tfrac{2n}{L_\infty}K(p)\left(\tfrac{L_\infty}{n}-s\right),p\right)=\mathrm{sn}^2\left(-\tfrac{2n}{L_\infty}K(p)s,p\right)=\mathrm{sn}^2\left(\tfrac{2n}{L_\infty}K(p)s,p\right)$$

due to Lemma 3.6.13, (3.6.46) is given by the right hand sides of (3.6.44) and (3.6.45). For $s\in[0,L_\infty]$, we consider the $\frac{L_\infty}{n}$-periodic extension of $k_i$, $i=2,3$ given on $[0,\frac{L_\infty}{n}]$ as in (3.6.44) and in (3.6.45). This yields exactly the representation given in Proposition 3.6.15. Let us again summarize what we have done.

*Proof of Proposition 3.6.15.* Let $\gamma\colon[0,L_\infty]\to\mathbb{R}^2$ be a stationary solution, parametrized by arclength. Then $\gamma$ satisfies (3.6.5) for $c_1,c_2\in\mathbb{R}$. By Lemma 3.6.8, after a shift in the argument, the curvature $k$ satisfies (3.6.13). Since we assume that $c_1,c_2$ are such that $V(k)=c_2$ has more than two real solutions, $k$ satisfies (3.6.15) for some $\delta\le 0$. If the sum $\max k+\min k$ is positive, it follows that $k$ is represented by [126, (5.1)]. If $\max k+\min k$ is negative, we derive that $k$ is also given by [126, (5.1)] when replacing $\sqrt{-\delta}$ by $-\sqrt{-\delta}$. Relating $\delta$ to the two additional solutions $c,d\in\mathbb{R}$ of $V(k)=c_2$, we derive the representation in Proposition 3.6.15. □

**Remark 3.6.17.** It is possible to verify by a computation that $k_1$, $k_2$, and $k_3$ indeed solve (3.6.4) and (3.6.5). For this one uses that comparing the coefficients of (3.6.14) and (3.6.15) reveals

$$\lambda = \frac{3}{8}(a+b)^2 - \frac{1}{16}(3a+b)(a+3b) - \delta,$$
$$c_1 = \frac{1}{32}(a+b)(a-b)^2 + \frac{1}{2}\delta,$$

see also [126, (3.4)]. Written with the (in one case imaginary and in one case real) zero $c \in \mathbb{C}$ of $V(k) = c_2$, this is

$$\lambda = \frac{1}{4}(a^2+b^2+c^2+ab+ac+bc), \tag{3.6.47}$$
$$c_1 = -\frac{1}{8}(a+b)(a+c)(b+c), \tag{3.6.48}$$

see also [155, (17),(18)]. Observe that the right-hand sides of (3.6.47) and (3.6.48) are real even if $c$ is not real. This is evident with (3.6.41) and $d = c^*$ if $c \in \mathbb{C} \setminus \mathbb{R}$. Moreover, we have

$$c_2 = -\frac{1}{4}a^4 + \lambda a + 2c_1 a = \frac{1}{4}abc(a+b+c)$$

for the integration constant in (3.6.5).

#### Closing criterion

When looking at the explicit representations of the curvature of stationary solutions in Proposition 3.6.14 and Proposition 3.6.15, one naturally wonders for which choice of the parameters, (3.6.10), (3.6.11), and (3.6.12) describe closed curves. To derive an explicit expression for the curve $\gamma$ from the curvature $k$, two integrations are required. For the given equations, this is a difficult task. However, in [155, Section 4], a condition for closedness is presented that requires only one integration. This is made possible by exploiting the structure of equation (3.6.4). We would like to present this simple yet elegant trick here. We note that while the ideas are elaborated on in [155], the trick is already known from [8].

**Proposition 3.6.18.** *Let $0 < P_i < \infty$ be a period of $k_i$, $i = 1,2,3$, for $k_i$ given as in (3.6.10), (3.6.11), or (3.6.12). Then a curve $\gamma_i$ described by $k_i$ is smoothly closed if and only if*

$$\int_0^{P_i} k_i(s)\,\mathrm{d}s = \frac{2\pi\omega}{n} \tag{3.6.49}$$

*for some $\omega \in \mathbb{Z}$ and $n \in \mathbb{N}$.*

We would like to point out that a similar closedness condition can also be found in [9]. The authors provide an additional constraint on a periodic curvature function which forces the associated curve to close up. Using the structure of (3.6.4), we are able to require less on the coefficients $\omega$ and $n$ than in [9].

*Proof of Proposition 3.6.18.* Let $\gamma_i$ be a curve described by $k_i$ with tangent vector field $\tau_i$ and normal vector field $\nu_i$. Since $k_i$ solves

$$\partial_s^2 k_i + \frac{1}{2}k_i^3 - \lambda k_i = c_1$$

with $\lambda > 0$ and $c_1 \in \mathbb{R}$, it holds that

$$\begin{aligned}\partial_s(\partial_s k_i\,\tau_i - \frac{1}{2}k_i^2\nu_i + \lambda\nu_i - c_1\gamma_i) &= \partial_s^2 k_i\,\tau_i + \partial_s k_i\,k_i\nu_i - k_i\partial_s k_i\,\nu_i + \frac{1}{2}k_i^2 k_i\tau_i - \lambda k_i\tau_i - c_1\tau_i \\ &= \left(\partial_s^2 k_i + \frac{1}{2}k_i^3 - \lambda k_i - c_1\right)\tau_i = 0.\end{aligned}$$

Integration of this equation yields

$$\gamma_i = \frac{1}{c_1}\left(\partial_s k_i\,\tau_i - \frac{1}{2}k_i^2\nu_i + \lambda\nu_i\right) + C \tag{3.6.50}$$

for a constant $C \in \mathbb{R}^2$. By suitably translating the origin, we have $C = 0$. Further, we define the angle function $\theta_i\colon [0, \mathbb{R}) \to \mathbb{R}$ by

$$\theta_i(s) - \theta_i(0) := \int_0^s k_i(r)\,\mathrm{d}r.$$

Let us now assume that there is $n \in \mathbb{N}$ and $\omega \in \mathbb{Z}$ such that (3.6.49) holds. Then

$$\theta_i(nP_i) - \theta_i(0) = n\int_0^{P_i} k_i(s)\,\mathrm{d}s = 2\pi\omega,$$

since $k_i$ is $P_i$-periodic. Thus, we obtain

$$\nu_i(nP_i) = \begin{pmatrix} -\sin\theta_i(nP_i) \\ \cos\theta_i(nP_i)\end{pmatrix} = \begin{pmatrix} -\sin\theta_i(0) \\ \cos\theta_i(0)\end{pmatrix} = \nu_i(0), \quad \tau_i(nP_i) = \tau_i(0). \tag{3.6.51}$$

It follows by (3.6.50) that $\gamma(nP_i) = \gamma(0)$. Thus, the curve described by $k_i\colon [0, nP_i] \to \mathbb{R}$ is zeroth order closed. First order closedness follows by (3.6.51) and smooth closedness by the periodicity of $k_i$.

For the other implication, let us now consider a smoothly closed curve $\gamma_i$ with length $L$ that is described by $k_i$. For its tangent vector, it holds that $\tau_i(L) = \tau_i(0)$. By the definition of the tangent vector, this implies that there exists $\omega \in \mathbb{Z}$ such that $\theta_i(L) = \theta_i(0) + 2\pi\omega$, where $\theta_i$ is an angle function describing $\gamma_i$. Without loss of generality we assume that

$\theta_i(0) = 0$. Then (3.6.50) reveals

$$\gamma_i(0) = \frac{1}{c_1}\begin{pmatrix} \partial_s k_i(0) \\ -\frac{1}{2}k_i^2(0) + \lambda \end{pmatrix} \quad \text{and} \quad \gamma_i(L) = \frac{1}{c_1}\begin{pmatrix} \partial_s k_i(L) \\ -\frac{1}{2}k_i^2(L) + \lambda \end{pmatrix}.$$

It follows that $k_i(L) = \pm k_i(0)$ and $\partial_s k_i(L) = \partial_s k_i(0) = 0$. When deriving the representation formulas for $k_i$, we have seen, that the structure of equation (3.6.5) implies that $\partial_s k_i(s) = 0$ if and only if $s = m\frac{L}{2n}$ for $m \in \mathbb{Z}$ and some fixed $n \in \mathbb{N}$ and $\frac{L}{2n} = \frac{P_i}{2}$. This implies that

$$2\pi\omega = \theta_i(L) - \theta_i(0) = \theta_i(nP_i) = \int_0^{nPi} k_i(s)\,\mathrm{d}s = n\int_0^{P_i} k_i(s)\,\mathrm{d}s$$

and the second part of the claim follows. □

We now aim to formulate the closing condition (3.6.49) for the explicit formulas $k_1$, $k_2$, and $k_3$ for the curvature. The periods of $k_i$, $i = 1, 2, 3$, can be determined by the periods of $\mathrm{cn}(s, p)$ and $\mathrm{sn}^2(x, p)$. The integrals are then computed using a handbook of elliptic integrals. With formulas (110.08), (361.54), and (361.63) from [25], we obtain the following lemma.

**Lemma 3.6.19** (cf. [155, Theorem 1, Theorem 2, and Section 4])**.** *Let $0 < P_i < \infty$ be the least period of $k_i$, $i = 1, 2, 3$, for $k_i$ given as in* (3.6.10)*,* (3.6.11)*, and* (3.6.12)*. Then*

$$P_1 = \frac{16}{\sqrt{AB}}K(p) \quad \textit{and} \quad P_2 = P_3 = \frac{8}{\sqrt{(c-a)(d-b)}}K(p).$$

*Moreover,*

$$\begin{aligned}
\int_0^{P_1} k_1(s)\,\mathrm{d}s &= 16\frac{Aa - Bb}{\sqrt{AB}(A-B)}K(p) + 8\frac{(A+B)(b-a)}{\sqrt{AB}(A-B)}\Pi\left(-\frac{(A-B)^2}{4AB}, p\right), \\
\int_0^{P_2} k_2(s)\,\mathrm{d}s &= \frac{8}{\sqrt{(c-a)(d-b)}}\left(dK(p) + (b-d)\Pi\left(\frac{a-b}{a-d}, p\right)\right), \\
\int_0^{P_3} k_3(s)\,\mathrm{d}s &= \frac{8}{\sqrt{(c-a)(d-b)}}\left(cK(p) + (c-a)\Pi\left(\frac{a-b}{c-b}, p\right)\right).
\end{aligned}$$

With this, we have derived explicit closing conditions that are comparatively straightforward to verify.

### Examples for stationary solutions

We are now ready to find some examples of stationary solutions and gain insight into their immense variety.

First, we consider the case where the curvature of the stationary solution is given by $k_1$. We use Lemma 3.6.19 and solve both (3.6.49) and $L_\infty = nP_i$ for fixed parameters $n$ and

$\omega$ and for fixed length $L_\infty = 2\pi$ using Mathematica. This yields the closed curves shown in Figure 3.6, Figure 3.7, and Figure 3.8.




Figure 3.6: Examples of embedded stationary solutions represented by $k_1$ as in (3.6.10). For (A): $\lambda \approx 0.219$, $c_1 = 5$, $n = 2$, for (B): $\lambda \approx 0.853$, $c_1 = 18$, $n = 3$, for (C): $\lambda \approx 2.043$, $c_1 = 40$, $n = 4$, and for (D): $\lambda \approx 2.237$, $c_1 = 100$, $n = 6$. For all examples $L_\infty = 2\pi$ and $\omega = 1$. The figures are scaled differently.

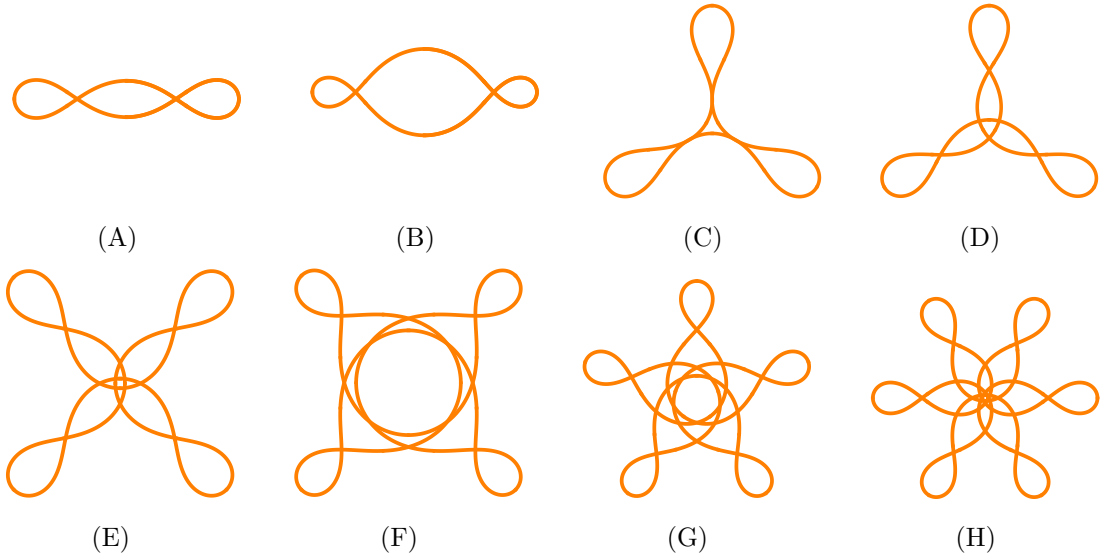



Figure 3.7: Examples of nonembedded stationary solutions represented by $k_1$ as in (3.6.10). For (A): $\lambda \approx 2.331$, $c_1 = 7$, $n = 2$, for (B): $\lambda \approx 6.891$, $c_1 = 12$, $n = 2$, for (C): $\lambda \approx 2.982$, $c_1 = 21.7$, $n = 3$, for (D): $\lambda \approx 6.175$, $c_1 = 28$, $n = 3$, for (E): $\lambda \approx 14.226$, $c_1 = 75$, $n = 4$, for (F): $\lambda \approx 43.296$, $c_1 = 180$, $n = 4$, for (G): $\lambda \approx 44.679$, $c_1 = 250$, $n = 5$, and for (H): $\lambda \approx 43.625$, $c_1 = 300$, $n = 6$. For all examples $L_\infty = 2\pi$ and $\omega = 1$. The figures are scaled differently.

Naturally, many additional questions arise immediately, such as the conditions under which the solution is embedded. To answer these questions would exceed the scope of this section. For the question of embeddedness, we refer to [155, Section 5] or [8, Section 4.4].



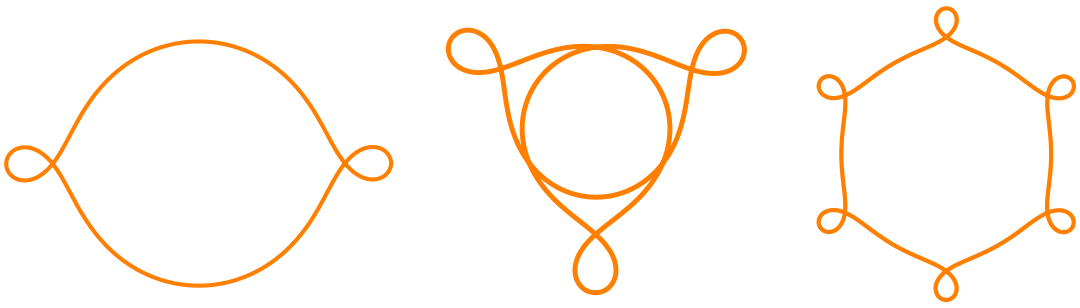
(A) (B)

Figure 3.8: Examples of stationary solutions represented by $k_1$ as in (3.6.10) with $\omega = 0$ in (A) and $\omega = 2$ in (B). For (A): $\lambda \approx 25.995$, $c_1 = 70$, $n = 4$, 4-fold covered, for (B): $\lambda \approx 6.308$, $c_1 = 508$, $n = 4$, double covered. For both examples $L_\infty = 2\pi$. The figures are scaled differently.

As explained above, to find examples of closed curves described by $k_1$, we fix the length. We observe that if a curve $\gamma$ is a stationary solution, i.e. satisfies (3.6.4) for some $c_1 \in \mathbb{R}$, then a scaling $\tilde{\gamma} = r\gamma$, $r > 0$, of $\gamma$ satisfies

$$\partial_s^2 \tilde{k} + \frac{1}{2}\tilde{k}^3 - \frac{\lambda}{r^2}\tilde{k} = \frac{c_1}{r^3}, \tag{3.6.52}$$

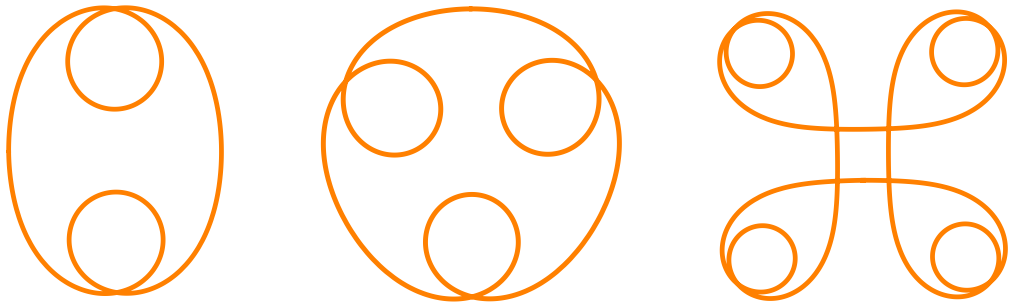
i.e. solves the equation for stationary solutions for $\tilde{\lambda} = \frac{\lambda}{r^2}$ and $\tilde{c_1} = \frac{c_1}{r^3}$. The enclosed area scales with the factor $r^2$.



(A) (B) (C)

Figure 3.9: Examples of stationary solutions represented by $k_2$ as in (3.6.11). For (A): $\lambda = \frac{3}{2}$, $c_1 = 0.50000063$, $n = 2$, $L_\infty \approx 24.844$, for (B): $\lambda = \frac{1}{3}$, $c_1 = \frac{1}{10}$, $n = 3$, $L_\infty \approx 58.643$, and for (C): $\lambda = 1$, $c_1 = \frac{1}{5}$, $n = 6$, $L_\infty \approx 55.0164$. The figures are scaled differently.

Next, we consider stationary solutions whose curvature is given by $k_2$ as in (3.6.11) or $k_3$ as in (3.6.12). This time, we take a different approach. We specify explicit values for $\lambda$ and $c_1$, and use Mathematica to search for solutions to (3.6.49) for these parameters. The examples we find have different lengths. Taking into account (3.6.52), the curves can be scaled arbitrarily. Examples for stationary solutions with curvature given by $k_2$ are presented in Figure 3.9. In Figure 3.10, examples for stationary solutions described by $k_3$ are given.



(A) (B) (C)

Figure 3.10: Examples of stationary solutions represented by $k_3$ as in (3.6.12). For (A): $\lambda = \frac{3}{2}$, $c_1 = 0.50000063$, $n = 2$, $L_\infty \approx 17.136$, for (B): $\lambda = \frac{1}{3}$, $c_1 = \frac{1}{10}$, $n = 3$, $L_\infty \approx 22.811$, and for (C): $\lambda = 1$, $c_1 = \frac{1}{5}$, $n = 6$, $L_\infty \approx 19.495$. The figures are scaled differently.

There are various ways of fixing certain parameters and comparing the corresponding stationary solutions. We have only presented a few of the many possible closed curves. For more examples of stationary solutions, see [155], [142], [126], [116], [150], or [159].

CHAPTER 4

# Outlook

In this thesis, we have investigated two variants of the classical elastic flow. There are, of course, many other variants inspired by physical and biological considerations. Based on the two problems we studied, we would like to suggest a few ideas for further research topics that could be explored.

#### Heterogeneous elastic wires with density dependent spontaneous curvature

Let us assume we are in the situation described in Chapter 2 and consider elastic curves with a heterogeneity that is described by a density function. From an application perspective, it might be interesting to choose the spontaneous curvature $c_0$, see (2.0.2), as a function dependent on the density. In three dimensions, this would, for example, describe the situation of BAR-domain proteins attaching to cell membranes, see for example [145], [149], or [89].
Mathematically, we would consider a curve $\gamma$ described by an angle function $\theta\colon [0,L] \to \mathbb{R}$ with density $\rho\colon [0,L] \to \mathbb{R}$, the sufficiently smooth spontaneous curvature

$$c_0 \colon \mathbb{R} \to \mathbb{R}$$

and the energy

$$\mathcal{E}_\mu(\theta,\rho) = \int_0^L \left(\beta(\rho)(\partial_s\theta - c_0(\rho))^2 + \mu(\partial_s\rho)^2\right) \mathrm{d}s$$

(for the details, see also Section 1.5). For the $L^2$-gradient flow under the constraints of closedness and fixed total mass, we would obtain

$$\begin{cases} \partial_t\theta = \partial_s\left(\beta(\rho)(\partial_s\theta - c_0(\rho))\right) + \lambda_{\theta 1}\sin\theta - \lambda_{\theta 2}\cos\theta, \\ \partial_t\rho = \mu\partial_s^2\rho - \frac{1}{2}\beta'(\rho)(\partial_s\theta - c_0(\rho))^2 + \beta(\rho)(\partial_s\theta - c_0(\rho))c_0'(\rho) - \lambda_\rho, \end{cases} \tag{4.0.1}$$

where the Lagrange multipliers $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$ are given as in (2.1.2) with $c_0 = c_0(\rho)$ and

$$\lambda_\rho(t) = \frac{1}{L}\int_0^L \left(-\frac{1}{2}\beta'(\rho)(\partial_s\theta - c_0)^2 + \beta(\rho)(\partial_s\theta - c_0(\rho))c_0'(\rho)\right) \mathrm{d}s.$$

Since the additional terms are of lower order, we expect the same results for local well

posedness and global existence as in Chapter 2. However, regarding the qualitative properties, it would be very interesting to study the modified flow equations (4.0.1) in more detail. For example, let us briefly look at the preservation of convexity. The evolution equation for the signed scalar curvature $k$ is given by

$$\partial_t k = \beta(\rho)\partial_s^2 k + 2\partial_s\beta(\rho)\partial_s k + (\partial_s^2\beta(\rho) + \lambda_{\theta 1}\cos\theta + \lambda_{\theta 2}\sin\theta)k - \partial_s^2(\beta(\rho)c_0(\rho)). \quad (4.0.2)$$

We observe that the structure of the last term in (4.0.2) allows for new results on the preservation of convexity, depending on the relationship between the functions $\beta$ and $c_0$.

#### Fourth order elastic flow with modulated stiffness

Another interesting project would be to study the $L^2$-gradient flow of the energy $\mathcal{E}_\mu(\gamma, \rho)$, see (1.5.1), directly, without working with an angle function. This would have the advantage of removing the restriction to planar curves. It would result in a coupled system of a fourth order equation and a second order equation. While this would be a gradient flow for the same energy, the decrease of energy would behave differently, and the flow would exhibit different properties. For example, one might think of the preservation of convexity, which we do not expect for a fourth order equation. Instead of a Lagrange multiplier for the closedness constraint, one would obtain a Lagrange multiplier to keep the length fixed (if desired). For this system, a different theory than in Chapter 2 would be needed to establish local well posedness. The author is not aware of any existing maximal regularity results in the literature, such as the one in [101] used in Section 2.3, that could be directly applied.

#### Rotationally symmetric heterogenous vesicles

A naturally arising idea is to consider not heterogeneous elastic wires, but heterogeneous elastic shells. In other words, to study biological vesicles whose geometry is influenced by a density on the surface. Put differently, this involves considering the Canham–Helfrich energy (1.1.2) with density dependent bending rigidities. Concerning discrete densities, we mention [31], where the minimization problem for axisymmetric multiphase surfaces is studied. Our suggestion would be to study continuous densities. We propose to consider axisymmetric surfaces obtained by rotating a curve contained in the $xz$-plane around the $z$-axis. One could consider this curve as heterogeneous and generating a density distribution on the surface. This would exclude certain configurations, we particularly think of the images in [13, Figure 2]. Therefore, another step could be to consider the density independent of rotational symmetry. Additionally, examining non-rotationally symmetric surfaces could be another further step. For this problem, we suggest working within the weak varifold setting from [19]. There, the existence of varifold minimizers for the Canham–Helfrich energy in the presence of multiple phases, so essentially discrete

densities, is studied. In this context, we also want to mention [130], where the authors consider not a sharp but a diffuse interface model, which allows for mixtures between phases and models concentration fields of the different phases by smooth density functions. They derive a sharp interface limit for diffuse phase separation energies of Modica–Mortola type for phase fields on varying generalized surfaces.

### Combining heterogeneity and incompressibility

To cover an even broader range of suitable modeling questions, one could also consider studying an elastic flow that accounts for both heterogeneities *and* incompressibility. One option would be to work with the system of a fourth order and a second order equation, mentioned above, and introduce an additional Lagrange multiplier, as in [129]. Alternatively, one could work with an angle function as in Chapter 2. Note that a closed curve described by the angle function $\theta\colon [0,L] \to \mathbb{R}$ encloses the signed area

$$\mathcal{A}(\theta) = \frac{1}{2}\int_0^L \int_0^s \sin(\theta(s) - \theta(\xi))\,\mathrm{d}\xi\,\mathrm{d}s.$$

This formula can be shown by a short computation using (1.2.6) and (1.5.2).

### Self similar solutions for the area preserving elastic flow

Another interesting question is whether self similar solutions to the area preserving elastic flow exist. For the free elastic flow, we observed in Example 1.4.2 that both the lemniscate of Bernoulli and multifold covered circles are self similar solutions. Additionally, for the curve diffusion flow, the lemniscate of Bernoulli also acts as a self similar solution, see [55]. Are there such solutions for the evolution equation (1.6.1)? Of course, we need to consider the special case $A_0 = 0$ and $\lambda = 0$ for this. Similar to [55, Lemma 9], we prove the following lemma.

**Lemma 4.0.1.** *Let $\lambda = 0$. Let $\gamma_0\colon \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth regular initial datum. Then the solution to* (1.6.1) *with initial datum $\gamma_0$ evolves self similarly if and only if there is a constant $K > 0$ such that*

$$\partial_s^2\left(\partial_s^2 k_0 + \frac{1}{2}k_0^3\right) = K\langle \gamma_0, \nu_0\rangle, \tag{4.0.3}$$

*where $k_0$ is the signed scalar curvature of $\gamma_0$ and $\nu_0$ denotes its normal.*

Integrating (4.0.3) yields the condition $\mathcal{A}(\gamma_0) = 0$, which must obviously hold for self similar solutions. The question is, which curves satisfy (4.0.3)?

### Incompressibility for networks of curves

In the last two decades, networks of curves and their evolution along gradient flows have gained increasing attention. See [88], [113], [141], [133], or [152], to name just a few examples. One version is to study the motion by curvature of networks, which is essentially the analogue of the curve shortening flow for networks. We do not address the exact definition of a network or the formulation of the geometric flow with the appropriate boundary conditions here. However, it is worth noting that, for example, [112, Section 5.1] provides interesting insights into the evolution of the area, which is enclosed by multiple curves forming the network. We wonder how an incompressibility condition can be incorporated into the study of flows of networks, which could certainly find its justification in physical and biological contexts. However, it is first necessary to choose a specific definition of a network. How many curves are considered, and how are they connected? From the mathematical and also from the modeling perspective, several questions need to be addressed initially. Next, it must be determined which area should be kept constant. This also depends on the chosen definition of a network.
This remains a very open and poorly defined question but could lead to an interesting project.

APPENDIX A

# Formalities

## Declaration of first submission

Ich erkläre hiermit, dass die von mir vorgelegte Dissertation bisher nicht im In- oder Ausland in dieser oder ähnlicher Form in einem anderen Promotionsverfahren vorgelegt wurde.

I hereby declare that this thesis has not been submitted in this or in similar form in Germany or abroad in any other doctoral procedure.

---

Leonie Langer

## Declaration of co-authorship

Teile dieser Dissertation wurden bereits in folgenden Fachartikeln und Vordrucken veröffentlicht, wobei die jeweils genannten Ko-Autoren gleichermaßen an der Erstellung der jeweiligen Abhandlungen beteiligt waren. Bis auf den Autor dieser Dissertation haben keine Autoren Teile dieser Publikationen in einem Promotionsverfahren verwendet.

Parts of this thesis have been already published in the following articles and preprints, whereby the respective co-authors were equally involved in the creation of the articles. No author except for the author of this thesis has used any parts of these publications in any other thesis.

**Article 1:** *A dynamic approach to heterogeneous elastic wires.*
DOI: `https://doi.org/10.1016/j.jde.2024.02.001`
Prof. Dr. Anna Dall'Acqua, *contact:* `anna.dallacqua@uni-ulm.de`,
Leonie Langer,
Dr. Fabian Rupp, *contact:* `fabian.rupp@univie.ac.at`

**Article 2:** *Conservation, convergence, and computation for evolving heterogeneous elastic wires.*
DOI: `https://doi.org/10.1137/23M159086X`
Prof. Dr. Anna Dall'Acqua, *contact:* `anna.dallacqua@uni-ulm.de`,
Dr. Gaspard Jankowiak, *contact:* `gaspard.jankowiak@uni-graz.at`,
Leonie Langer,
Dr. Fabian Rupp, *contact:* `fabian.rupp@univie.ac.at`

**Article 3:** *Dynamics of Elastic Wires: Preserving Area without Nonlocality.*
DOI: `https://doi.org/10.1007/s00526-025-02948-0`
Leonie Langer

---

Leonie Langer

---

Prof. Dr. Anna Dall'Acqua (Academic Supervisor)

## Declaration of consent

Ich bin damit einverstanden, dass die Arbeit auch zum Zweck der Überprüfung der Einhaltung allgemein geltender wissenschaftlicher Standards benutzt wird, insbesondere auch unter Verwendung elektronischer Datenverarbeitung.

I permit the inspection of my thesis for general standards of good scientific practice, including by means of electronic data processing.

Leonie Langer

## Declaration of independent scientific work

Ich versichere hiermit, dass ich die Arbeit selbstständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich und inhaltlich übernommenen Stellen als solche kenntlich gemacht habe und die Satzung der Universität Ulm zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

I declare that I prepared this doctoral thesis independently and did not use any other sources than indicated by me and, furthermore, I cited the passages where I quoted or referred to works by other people and their contents and I comply with the Statutes of Ulm University for Safeguarding Good Scientific Practice in the applicable version.

Leonie Langer

## Curriculum vitae

*Der Lebenslauf wurde aus Gründen des Datenschutzes entfernt.*

*The curriculum vitae has been removed for privacy reasons.*

# Bibliography


[1] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. No. 55, U. S. Government Printing Office, Washington, DC, 1964, For sale by the Superintendent of Documents.

[2] Jean-Jacques Alibert, Alessandro Della Corte, Ivan Giorgio, and Antonio Battista, *Extensional elastica in large deformation as Γ-limit of a discrete 1D mechanical system*, Z. Angew. Math. Phys. **68** (2017), no. 2, Paper No. 42, 19.

[3] Holm Altenbach and Pavel A. Zhilin, *The theory of simple elastic shells*, pp. 1–12, Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.

[4] Ben Andrews, James McCoy, Glen Wheeler, and Valentina-Mira Wheeler, *Closed ideal planar curves*, Geom. Topol. **24** (2020), no. 2, 1019–1049.

[5] Sigurd B. Angenent, *The zero set of a solution of a parabolic equation.*, Journal für die reine und angewandte Mathematik (Crelles Journal) **1988** (1988), 79 – 96.

[6] Stuart S. Antman, *A note on a paper of Tadjbakhsh and Odeh*, J. Math. Anal. Appl. **21** (1968), 132–135.

[7] Jürgen Appell and Petr P. Zabrejko, *Nonlinear superposition operators*, Cambridge Tracts in Mathematics, vol. 95, Cambridge University Press, Cambridge, 1990.

[8] Guillermo Arreaga, Riccardo Capovilla, Chryssomalis Chryssomalakos, and Jemal Guven, *Area-constrained planar elastica*, Phys. Rev. E **65** (2002), 031801.

[9] Josu Arroyo, Oscar J. Garay, and José J. Mencía, *When is a periodic function the curvature of a closed plane curve?*, Amer. Math. Monthly **115** (2008), no. 5, 405–414.

[10] Charles Baker, *The mean curvature flow of submanifolds of high codimension*, arXiv **1104.4409** (2011).

[11] Christian Bär, *Elementare Differentialgeometrie*, expanded ed., Walter de Gruyter & Co., Berlin, 2010.

[12] Sören Bartels, *Finite element simulation of nonlinear bending models for thin elastic rods and plates*, Geometric partial differential equations. Part I, Handb. Numer. Anal., vol. 21, Elsevier/North-Holland, Amsterdam, [2020] ©2020, pp. 221–273.

[13] Tobias Baumgart, Samuel T. Hess, and Watt W. Webb, *Imaging coexisting fluid domains in biomembrane models coupling curvature and line tension*, Nature **425** (2003), 821–824.

[14] Jöran Bergh and Jörgen Löfström, *Interpolation spaces: An introduction*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2012.

[15] Florence Bertails, Basile Audoly, Marie-Paule Cani, Bernard Querleux, Frédéric Leroy, and Jean-Luc Lévêque, *Super-helices for predicting the dynamics of natural hair*, ACM SIGGRAPH 2006 Papers (New York, NY, USA), SIGGRAPH '06, Association for Computing Machinery, 2006, p. 1180–1187.

[16] Simon Blatt, *Loss of convexity and embeddedness for geometric evolution equations of higher order*, J. Evol. Equ. **10** (2010), no. 1, 21–27.

[17] Sahbi Boussandel, Ralph Chill, and Eva Fasangova, *Maximal regularity, the local inverse function theorem, and local well-posedness for the curve shortening flow*, Czechoslovak Mathematical Journal **62** (2012), no. 2, 335–346.

[18] Katharina Brazda, Gaspard Jankowiak, Christian Schmeiser, and Ulisse Stefanelli, *Bifurcation of elastic curves with modulated stiffness*, European Journal of Applied Mathematics (2022), 1–27.

[19] Katharina Brazda, Luca Lussardi, and Ulisse Stefanelli, *Existence of varifold minimizers for the multiphase Canham-Helfrich functional*, Calc. Var. Partial Differential Equations **59** (2020), no. 3, Paper No. 93, 26.

[20] Haim Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011.

[21] Guido Brunnett and Peter E. Crouch, *Elastic curves on the sphere*, Adv. Comput. Math. **2** (1994), no. 1, 23–40.

[22] Robert Bryant and Phillip Griffiths, *Reduction for constrained variational problems and* $\int \frac{1}{2}k^2\,ds$, Amer. J. Math. **108** (1986), no. 3, 525–570.

[23] Dorin Bucur and Antoine Henrot, *A new isoperimetric inequality for elasticae*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 11, 3355–3376.

[24] Giuseppe Buttazzo, Mariano Giaquinta, and Stefan Hildebrandt, *One-dimensional variational problems: An introduction*, Oxford lecture series in mathematics and its applications, Clarendon Press, 1998.

[25] Paul F. Byrd and Morris D. Friedman, *Handbook of elliptic integrals for engineers and scientists*, second ed., Die Grundlehren der mathematischen Wissenschaften, vol. Band 67, Springer-Verlag, New York-Heidelberg, 1971.

[26] Marcel Campen and Leif Kobbelt, *Dual strip weaving: interactive design of quad layouts using elastica strips*, ACM Trans. Graph. **33** (2014), no. 6.

[27] Peter B. Canham, *The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell*, J. Theor. Biol. **26** (1970), no. 1, 61–81.

[28] Italo Capuzzo Dolcetta, Stefano Finzi Vita, and Riccardo March, *Area-preserving curve-shortening flows: from phase separation to image processing*, Interfaces Free Bound. **4** (2002), no. 4, 325–343.

[29] Ralph Chill and Eva Fašangová, *Gradient systems, 13th international internet seminar on evolution equations*, MatFyzPress, Prague, 2010.

[30] Ralph Chill, Eva Fašangová, and Reiner Schätzle, *Willmore blowups are never compact*, Duke Math. J. **147** (2009), no. 2, 345–376.

[31] Rustum Choksi, Marco Morandotti, and Marco Veneroni, *Global minimizers for axisymmetric multiphase membranes*, ESAIM Control Optim. Calc. Var. **19** (2013), no. 4, 1014–1029.

[32] Kai-Seng Chou, *A blow-up criterion for the curve shortening flow by surface diffusion*, Hokkaido Math. J. **32** (2003), no. 1, 1–19.

[33] Kai-Seng Chou and Xi-Ping Zhu, *The curve shortening problem*, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[34] Philippe Clément and Gieri Simonett, *Maximal regularity in continuous interpolation spaces and quasilinear parabolic equations*, 2001, pp. 39 – 67.

[35] Roger Cooke and Vladimir I. Arnold, *Ordinary differential equations*, Springer Textbook, Springer Berlin Heidelberg, 1992.

[36] Anna Dall'Acqua, Gaspard Jankowiak, Leonie Langer, and Fabian Rupp, *Conservation, Convergence, and Computation for Evolving Heterogeneous Elastic Wires*, SIAM J. Math. Anal. **56** (2024), no. 4, 4494–4529.

[37] Anna Dall'Acqua, Leonie Langer, and Fabian Rupp, *A dynamic approach to heterogeneous elastic wires*, Journal of Differential Equations **392** (2024), 1–42.

[38] Anna Dall'Acqua, Tim Laux, Lin, Paola Pozzi, and Adrian Spener, *The elastic flow of curves on the sphere*, Geometric Flows **3** (2018), no. 1, 1–13.

[39] Anna Dall'Acqua, Chun-Chi Lin, and Paola Pozzi, *Evolution of open elastic curves in $\mathbb{R}^n$ subject to fixed length and natural boundary conditions*, Analysis (Berlin) **34** (2014), no. 2, 209–222.

[40] ———, *A gradient flow for open elastic curves with fixed length and clamped ends*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **17** (2017), no. 3, 1031–1066.

[41] Anna Dall'Acqua, Marius Müller, Reiner Schätzle, and Adrian Spener, *The Willmore flow of tori of revolution*, Anal. PDE **17** (2024), no. 9, 3079–3124.

[42] Anna Dall'Acqua and Alessandra Pluda, *Some minimization problems for planar networks of elastic curves*, Geom. Flows **2** (2017), no. 1, 105–124.

[43] Anna Dall'Acqua and Paola Pozzi, *A Willmore-Helfrich $L^2$-flow of curves with natural boundary conditions*, Comm. Anal. Geom. **22** (2014), no. 4, 617–669.

[44] Anna Dall'Acqua, Paola Pozzi, and Adrian Spener, *The Łojasiewicz-Simon gradient inequality for open elastic curves*, J. Differential Equations **261** (2016), no. 3, 2168–2209.

[45] Anna Dall'Acqua and Adrian Spener, *The elastic flow of curves in the hyperbolic plane*, arXiv **1710.09600** (2017).

[46] ———, *Circular solutions to the elastic flow in hyperbolic space*, RIMS Kôkyûroku (2018), no. 2082.

[47] Friederike Dittberner, *Area preserving curve shortening flow*, Mean curvature flow, De Gruyter Proc. Math., De Gruyter, Berlin, 2020, pp. 75–80.

[48] Peter A. Djondjorov, Mariana Ts. Hadzhilazova, Ivaïlo M. Mladenov, and Vassil M. Vassilev, *Explicit parameterization of Euler's elastica*, Geometry, integrability and quantization (2008), 175 – 186.

[49] Peter A. Djondjorov, Vassil M. Vassilev, and Ivaïlo M. Mladenov, *Analytic description and explicit parametrisation of the equilibrium shapes of elastic rings and tubes under uniform hydrostatic pressure*, International Journal of Mechanical Sciences **53** (2011), no. 5, 355–364.

[50] Manfredo P. do Carmo, *Riemannian geometry*, portuguese ed., Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992.

[51] ———, *Differential geometry of curves and surfaces*, 2nd revised edition ed., Dover Puclications, Mineola, New York, 2016.

[52] Patrick Dondl, Coffi Aristide Hounkpe, and Martin Jesenko, *$\Gamma$-convergence of a discrete Kirchhoff rod energy*, ESAIM Control Optim. Calc. Var. **30** (2024), Paper No. 53.

[53] Gerhard Dziuk, Ernst Kuwert, and Reiner Schätzle, *Evolution of elastic curves in $\mathbb{R}^n$: existence and computation*, SIAM J. Math. Anal. **33** (2002), no. 5, 1228–1245.

[54] Klaus Ecker, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, vol. 57, Birkhäuser Boston, Inc., Boston, MA, 2004.

[55] Maureen Edwards, Alexander Gerhardt-Bourke, James McCoy, Glen Wheeler, and Valentina-Mira Wheeler, *The shrinking figure eight and other solitons for the curve diffusion flow*, J. Elasticity **119** (2015), no. 1-2, 191–211.

[56] Samuil D. Eidelman and Nicolae V. Zhitarashu, *Parabolic boundary value problems*, Operator Theory: Advances and Applications, Birkhäuser Basel, 1998.

[57] Charles M. Elliott and Harald Garcke, *Existence results for diffusive surface motion laws*, Adv. Math. Sci. Appl. **7** (1997), no. 1, 467–490.

[58] Joachim Escher and Kazuo Ito, *Some dynamic properties of volume preserving curvature driven flows*, Math. Ann. **333** (2005), no. 1, 213–230.

[59] Joachim Escher, Uwe F. Mayer, and Gieri Simonett, *On the surface diffusion flow*, Navier-Stokes equations and related nonlinear problems (Palanga, 1997), VSP, Utrecht, 1998, pp. 69–79.

[60] Malena I. Español, Dmitry Golovaty, and J. Patrick Wilber, *Euler elastica as a $\Gamma$-limit of discrete bending energies of one-dimensional chains of atoms*, Math. Mech. Solids **23** (2018), no. 7, 1104–1116.

[61] Paul C. Fife, *Models for phase separation and their mathematics*, Electron. J. Differential Equations (2000), No. 48, 26.

[62] Patrick B. Furrer, Robert S. Manning, and John H. Maddocks, *DNA rings with multiple energy minima*, Biophysical Journal **79** (2000), no. 1, 116–136.

[63] Michael Gage, *An isoperimetric inequality with applications to curve shortening*, Duke Math. J. **50** (1983), no. 4, 1225–1229.

[64] ———, *On an area-preserving evolution equation for plane curves*, Nonlinear problems in geometry (Mobile, Ala., 1985), Contemp. Math., vol. 51, Amer. Math. Soc., Providence, RI, 1986, pp. 51–62.

[65] Michael Gage and Richard S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), no. 1, 69–96.

[66] Harald Garcke and Michael Gößwein, *Non-linear stability of double bubbles under surface diffusion*, J. Differential Equations **302** (2021), 617–661.

[67] Claus Gerhardt, *Curvature problems*, Series in geometry and topology, International Press, 2006.

[68] Yoshikazu Giga and Kazuo Ito, *On pinching of curves moved by surface diffusion*, Commun. Appl. Anal. **2** (1998), no. 3, 393–405.

[69] ———, *Loss of convexity of simple closed curves moved by surface diffusion*, Topics in nonlinear analysis, Progr. Nonlinear Differential Equations Appl., vol. 35, Birkhäuser, Basel, 1999, pp. 305–320.

[70] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer, Berlin Heidelberg, 2001.

[71] Alain Goriely and Michael Tabor, *Spontaneous helix hand reversal and tendril perversion in climbing plants*, Phys. Rev. Lett. **80** (1998), 1564–1567.

[72] Victor Geoffrey Alan Goss, *The history of the planar elastica: Insights into mechanics and scientific method*, Science & Education **18** (2008), 1057–1082.

[73] Michael Gößwein, *Surface diffusion flow of triple junction clusters in higher space dimensions*, PhD thesis (2019).

[74] Sachin Goyal, Noel C. Perkins, and Christopher L. Lee, *Nonlinear dynamics and loop formation in kirchhoff rods with implications to the mechanics of DNA and cables*, Journal of Computational Physics **209** (2005), no. 1, 371–389.

[75] Matthew A. Grayson, *Shortening embedded curves*, Ann. of Math. (2) **129** (1989), no. 1, 71–111.

[76] Oliver Gross, Ulrich Pinkall, and Moritz Wahl, *Elastic curves with variable bending stiffness*, arXiv **2404.04027** (2024).

[77] Christian Hafner and Bernd Bickel, *The design space of plane elastic curves*, ACM Trans. Graph. **40** (2021), no. 4.

[78] Ryan A. Harris and John E. Hearst, *On Polymer Dynamics*, The Journal of Chemical Physics **44** (1966), no. 7, 2595–2602.

[79] Robert Haslhofer, *Lectures on mean curvature flow*, arXiv **1406.7765** (2014).

[80] John E. Hearst and Yaoming Shi, *The elastic rod provides a model for DNA and its functions*, Mathematical Approaches to Biomolecular Structure and Dynamics, The IMA Volumes in Mathematics and its Applications, vol. 82, Springer, New York, 1996.

[81] Wolfgang Helfrich, *Elastic properties of lipid bilayers: Theory and possible experiments*, Zeitschrift für Naturforschung C **28** (1973), no. 11, 693–703.

[82] Michael Helmers, *Snapping elastic curves as a one-dimensional analogue of two-component lipid bilayers*, Math. Models Methods Appl. Sci. **21** (2011), no. 5, 1027–1042.

[83] ———, *Convergence of an approximation for rotationally symmetric two-phase lipid bilayer membranes*, Q. J. Math. **66** (2015), no. 1, 143–170.

[84] Ronald S. Irving, *Integers, polynomials, and rings*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2004, A course in algebra.

[85] Jürgen Jost, *Riemannian geometry and geometric analysis*, seventh ed., Universitext, Springer, Cham, 2017.

[86] Frank Jülicher and Reinhard Lipowsky, *Domain-induced budding of vesicles*, Phys. Rev. Lett. **70** (1993), 2964–2967.

[87] Eric Kim and Dohyun Kwon, *Area-preserving anisotropic mean curvature flow in two dimensions*, Calc. Var. Partial Differential Equations **64** (2025), no. 1, Paper No. 27, 39.

[88] David Kinderlehrer and Chun Liu, *Evolution of grain boundaries*, Math. Models Methods Appl. Sci. **11** (2001), no. 4, 713–729.

[89] Christoph Kluge, Matthias Pöhnl, and Rainer A. Böckmann, *Spontaneous local membrane curvature induced by transmembrane proteins*, Biophysical Journal **121** (2022), no. 5, 671 – 683.

[90] Petr Kmoch, Ugo Bonanni, and Nadia Magnenat-Thalmann, *Hair simulation model for real-time environments*, Proceedings of the 2009 Computer Graphics International Conference (New York, NY, USA), CGI '09, Association for Computing Machinery, 2009, p. 5–12.

[91] Sebastian Knoche and Jan Kierfeld, *The secondary buckling transition: Wrinkling of buckled spherical shells*, Eur. Phys. J. E **37** (2014), no. 62.

[92] Norihito Koiso, *On the motion of a curve towards elastica*, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 403–436.

[93] Ernst Kuwert and Reiner Schätzle, *The Willmore flow with small initial energy*, J. Differential Geom. **57** (2001), no. 3, 409–441.

[94] ———, *Gradient flow for the Willmore functional*, Comm. Anal. Geom. **10** (2002), no. 2, 307–339.

[95] ———, *The Willmore functional*, Topics in Modern Regularity Theory (Pisa) (Giuseppe Mingione, ed.), Edizioni della Normale, 2012, pp. 1–115.

[96] Olga A. Ladyženskaja, Vsevolod A. Solonnikov, and Nina N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, reprinted edition ed., American Mathematical Society, United States of America, 1988.

[97] Serge Lang, *Real and functional analysis*, third ed., Graduate Texts in Mathematics, vol. 142, Springer-Verlag, New York, 1993.

[98] Joel Langer and David A. Singer, *The total squared curvature of closed curves*, Journal of Differential Geometry **20** (1984), no. 1, 1 – 22.

[99] Leonie Langer, *Dynamics of elastic wires: preserving area without nonlocality*, Calc. Var. Partial Differential Equations **64** (2025), no. 106.

[100] Derek F. Lawden, *Elliptic functions and applications*, Applied Mathematical Sciences, vol. 80, Springer-Verlag, New York, 1989.

[101] Jeremy LeCrone, *Elliptic operators and maximal regularity on periodic little-hölder spaces*, **12**, no. 2, 295–325.

[102] John M. Lee, *Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, Springer-Verlag, New York, 1997, An introduction to curvature.

[103] ———, *Introduction to smooth manifolds*, volume 218 of Graduate Texts in Mathematics, Springer, 2013.

[104] ———, *Introduction to Riemannian manifolds*, second ed., Graduate Texts in Mathematics, vol. 176, Springer, Cham, 2018.

[105] Raph Levien, *The elastica: a mathematical history*, Tech. Report UCB/EECS-2008-103, EECS Department, University of California, Berkeley (2008).

[106] Peter Li and Shing Tung Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. **69** (1982), no. 2, 269–291.

[107] Yen-Chih Lin, Y. W. Lin, and Tzay-Ming Hong, *Crumpling wires in two dimensions*, Phys. Rev. E **78** (2008), 067101.

[108] Anders Linnér, *Some properties of the curve straightening flow in the plane*, Trans. Amer. Math. Soc. **314** (1989), no. 2, 605–618.

[109] ———, *Unified representations of nonlinear splines*, J. Approx. Theory **84** (1996), no. 3, 315–350.

[110] Alessandra Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser Verlag, Basel, 1995.

[111] Carlo Mantegazza, *Lecture notes on mean curvature flow*, Progress in Mathematics, vol. 290, Birkhäuser/Springer Basel AG, Basel, 2011.

[112] Carlo Mantegazza, Matteo Novaga, and Alessandra Pluda, *Lectures on curvature flow of networks*, Contemporary research in elliptic PDEs and related topics, Springer INdAM Ser., vol. 33, Springer, Cham, 2019, pp. 369–417.

[113] Carlo Mantegazza, Matteo Novaga, and Vincenzo Maria Tortorelli, *Motion by curvature of planar networks*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), no. 2, 235–324.

[114] Carlo Mantegazza, Alessandra Pluda, and Marco Pozzetta, *A survey of the elastic flow of curves and networks*, Milan J. Math. **89** (2021), no. 1, 59–121.

[115] Carlo Mantegazza and Marco Pozzetta, *The Łojasiewicz-Simon inequality for the elastic flow*, Calc. Var. Partial Differential Equations **60** (2021), no. 1, Paper No. 56, 17.

[116] Waichiro Matsumoto, Minoru Murai, and Shoji Yotsutani, *By which kind of sound, can one hear the shape of a drum?*, Kyoto University Research Information Repository **1315** (2003), 156 – 175.

[117] Shigeki Matsutani, *Euler's Elastica and Beyond*, Journal of Geometry and Symmetry in Physics **17** (2010), no. none, 45 – 86.

[118] James A. McCoy, Glen E. Wheeler, and Yuhan Wu, *A length-constrained ideal curve flow*, Q. J. Math. **73** (2022), no. 2, 685–699.

[119] Harvey T. McMahon and Jennifer L. Gallop, *Membrane curvature and mechanisms of dynamic cell membrane remodelling*, Nature **438** (2005), 590–596.

[120] Tatsuya Miura, *Li-Yau type inequality for curves in any codimension*, Calc. Var. Partial Differential Equations **62** (2023), no. 8, Paper No. 216, 28.

[121] Tatsuya Miura, Marius Müller, and Fabian Rupp, *Optimal thresholds for preserving embeddedness of elastic flows*, Amer. J. Math. **147** (2025), no. 1, 33–80.

[122] Tatsuya Miura and Glen Wheeler, *The free elastic flow for closed planar curves*, arXiv **2404.12619** (2024).

[123] Maria Giovanna Mora and Stefan Müller, *Derivation of the nonlinear bending-torsion theory for inextensible rods by* $\Gamma$*-convergence*, Calc. Var. Partial Differential Equations **18** (2003), no. 3, 287–305.

[124] Marius Müller and Fabian Rupp, *A Li-Yau inequality for the 1-dimensional Willmore energy*, Adv. Calc. Var. **16** (2023), no. 2, 337–362.

[125] Marius Müller and Adrian Spener, *On the convergence of the elastic flow in the hyperbolic plane*, Geom. Flows **5** (2020), no. 1, 40–77.

[126] Minoru Murai, Waichiro Matsumoto, and Shoji Yotsutani, *Representation formula for the plane closed elastic curves*, Conference Publications **2013** (2013), no. special, 565–585.

[127] Sébastien Neukirch, *Extracting dna twist rigidity from experimental supercoiling data*, Phys. Rev. Lett. **93** (2004), 198107.

[128] Matteo Novaga and Paola Pozzi, *A second order gradient flow of p-elastic planar networks*, SIAM J. Math. Anal. **52** (2020), no. 1, 682–708.

[129] Shinya Okabe, *The motion of elastic planar closed curves under the area-preserving condition*, Indiana Univ. Math. J. **56** (2007), no. 4, 1871–1912.

[130] Heiner Olbermann and Matthias Röger, *Phase separation on varying surfaces and convergence of diffuse interface approximations*, Calc. Var. Partial Differential Equations **62** (2023), no. 9, Paper No. 250, 21.

[131] Olivier Pantz and Karim Trabelsi, *Derivation of nonlinear shell models combining shear and flexure: application to biological membranes*, Math. Mech. Complex Syst. **3** (2015), no. 2, 101–138.

[132] Mark A. Peletier and Matthias Röger, *Partial localization, lipid bilayers, and the elastica functional*, Arch. Ration. Mech. Anal. **193** (2009), no. 3, 475–537.

[133] Alessandra Pluda, *Evolution of spoon-shaped networks*, Netw. Heterog. Media **11** (2016), no. 3, 509–526.

[134] Alexander Polden, *Curves and surfaces of least total curvature and fourth-order flows*, PhD thesis, Universität Tübingen (1996).

[135] Fabian Rupp, *On the Łojasiewicz-Simon gradient inequality on submanifolds*, J. Funct. Anal. **279** (2020), no. 8, 108708, 33.

[136] ———, *The volume-preserving Willmore flow*, Nonlinear Analysis **230** (2023), 113220.

[137] ———, *The Willmore flow with prescribed isoperimetric ratio*, Comm. Partial Differential Equations **49** (2024), no. 1-2, 148–184.

[138] ———, *The Willmore flow with prescribed isoperimetric ratio*, Communications in Partial Differential Equations **49** (2024), no. 1-2, 148–184.

[139] Fabian Rupp and Adrian Spener, *Existence and convergence of the length-preserving elastic flow of clamped curves*, J. Evol. Equ. **24** (2024), no. 3, Paper No. 59.

[140] Manuel Schlierf, *Spontaneous curvature effects of the Helfrich flow: singularities and convergence*, Comm. Partial Differential Equations **50** (2025), no. 3, 441–476.

[141] Oliver C. Schnürer, Abderrahim Azouani, Marc Georgi, Juliette Hell, Nihar Jangle, Amos Koeller, Tobias Marxen, Sandra Ritthaler, Mariel Sáez, Felix Schulze, and Brian Smith, *Evolution of convex lens-shaped networks under the curve shortening flow*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2265–2294.

[142] Zhang Shao-guang, *A complete classification of closed shapes for cylindrical vesicles*, **6** (1997), no. 9, 641.

[143] Leon Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525–571.

[144] Gieri Simonett, *The Willmore flow near spheres*, Differential Integral Equations **14** (2001), no. 8, 1005–1014.

[145] Mijo Simunovic, Gregory A. Voth, Andrew Callan-Jones, and Patricia Bassereau, *When physics takes over: Bar proteins and membrane curvature*, Trends in Cell Biology **25** (2015), no. 12, 780–792, Special Issue: Quantitative Cell Biology.

[146] David A. Singer, *Lectures on elastic curves and rods*, Curvature and variational modeling in physics and biophysics, AIP Conf. Proc., vol. 1002, Amer. Inst. Phys., Melville, NY, 2008, pp. 3–32.

[147] Jonas Spillmann and Matthias Harders, *Inextensible elastic rods with torsional friction based on lagrange multipliers*, Computer Animation and Virtual Worlds **21** (2010), no. 6, 561–572.

[148] Jonas Spillmann and Matthias Teschner, *CORDE: Cosserat Rod Elements for the Dynamic Simulation of One-Dimensional Elastic Objects*, Eurographics/SIGGRAPH Symposium on Computer Animation (Dimitris Metaxas and Jovan Popovic, eds.), The Eurographics Association, 2007.

[149] Tatiana B. Stanishneva-Konovalova, N.I. Derkacheva, Svetlana V. Polevova, and Olga S. Sokolova, *The role of bar domain proteins in the regulation of membrane dynamics*, Acta Naturae **8** (2016), no. 4, 60 – 69.

[150] Itadji Tadjbakhsh and Faroukh Odeh, *Equilibrium states of elastic rings*, J. Math. Anal. Appl. **18** (1967), 59–74.

[151] Jean E. Taylor and John W. Cahn, *Linking anisotropic sharp and diffuse surface motion laws via gradient flows*, J. Statist. Phys. **77** (1994), no. 1-2, 183–197.

[152] Yoshihiro Tonegawa and Neshan Wickramasekera, *The blow up method for Brakke flows: networks near triple junctions*, Arch. Ration. Mech. Anal. **221** (2016), no. 3, 1161–1222.

[153] Clifford Ambrose Truesdell, *The influence of elasticity on analysis: The classic heritage*, Bulletin of the American Mathematical Society **9** (1983), 293–310.

[154] Dong-Ho Tsai and Xiao-Liu Wang, *On length-preserving and area-preserving nonlocal flow of convex closed plane curves*, Calc. Var. Partial Differential Equations **54** (2015), no. 4, 3603–3622.

[155] Vassil M. Vassilev, Peter A. Djondjorov, and Ivaïlo M. Mladenov, *Cylindrical equilibrium shapes of fluid membranes*, Journal of Physics A: Mathematical and Theoretical **41** (2008), no. 43, 435201.

[156] Chien Ming Wang, Hong Zhang, Noel Challamel, and Wenhao H. Pan, *Hencky bar-chain/net for structural analysis*, World Scientific (Europe), 2020.

[157] Xiao-Liu Wang, *The evolution of area-preserving and length-preserving inverse curvature flows for immersed locally convex closed plane curves*, J. Funct. Anal. **284** (2023), no. 1, Paper No. 109744, 25.

[158] Kohtaro Watanabe, *Plane domains which are spectrally determined*, Ann. Global Anal. Geom. **18** (2000), no. 5, 447–475.

[159] Kohtaro Watanabe and Izumi Takagi, *Representation formula for the critical points of the Tadjbakhsh-Odeh functional and its application*, Japan J. Indust. Appl. Math. **25** (2008), no. 3, 331–372.

[160] Yingzhong Wen, *$L^2$ flow of curve straightening in the plane*, Duke Mathematical Journal **70** (1993), no. 3, 683 – 698.

[161] ———, *Curve straightening flow deforms closed plane curves with nonzero rotation number to circles*, J. Differential Equations **120** (1995), no. 1, 89–107.

[162] Emmet Finlay Whittlesey, *Analytic functions in Banach spaces*, Proceedings of the American Mathematical Society **16** (1965), no. 5, 1077–1083.

[163] Thomas James Willmore, *Note on embedded surfaces*, An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) **11B** (1965), 493–496.

[164] ———, *Riemannian geometry*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993.

[165] Yuhan Wu, *Gradient flow of the dirichlet energy for the curvature of plane curves*, Doctor of Philosophy thesis, School of Mathematics and Applied Statistics, University of Wollongong (2021), no. https://ro.uow.edu.au/theses1/1039.

[166] Jonas Zehnder, Stelian Coros, and Bernhard Thomaszewski, *Designing structurally-sound ornamental curve networks*, ACM Trans. Graph. **35** (2016), no. 4.

[167] Eberhard Zeidler, *Nonlinear functional analysis and its applications - III: Variational methods and optimization. translated by L. F. Boron*, Springer-Verlag, New York, 1985.