Universität Ulm



ulm university universität
uulm

Invariant Sets and Long Time Behaviour of Operator Semigroups

Vorgelegt von
Jochen Glück

aus Wertingen im Jahr 2016.

Dissertation zur Erlangung des Doktorgrades Dr. rer. nat. der Fakultät für Mathematik und Wirtschaftswissenschaften der Universität Ulm

Amtierender Dekan:
Prof. Dr. Werner Smolny

Gutachter:
Prof. Dr. Wolfgang Arendt
Prof. Dr. Rico Zacher
Prof. Dr. Jürgen Voigt

Tag der Promotion:
02. Dezember 2016

Contents

Introduction **v**

Preliminaries **xxi**

I Asymptotic Behaviour of Contraction Semigroups **1**

1 Geometry of the Unit Ball in Banach Spaces **3**
1.1 Extremely Non-Hilbert Spaces 3
1.2 Projectively Non-Hilbert Spaces 7
1.3 Super Projectively Non-Hilbert Spaces 11
1.4 Notes & Open Problems 14

2 Spectrum and Asymptotics of Contraction Semigroups **17**
2.1 The Point Spectrum on Extremely Non-Hilbert Spaces 18
2.2 The Point Spectrum on Projectively Non-Hilbert Spaces . . . 21
2.3 Semigroups of Kernel Operators 26
2.4 The Spectrum on Super Projectively Non-Hilbert Spaces . . . 30
2.5 Notes & Open Problems 32

3 An Application to Delay Differential Equations **37**
3.1 Basics 38
3.2 Delay Differential Equations in 1 Dimension 42
3.3 Delay Differential Equations in d Dimensions 44
3.4 Notes & Open Problems 46

II Asymptotic Behaviour of Positive Semigroups **47**

4 Geometry of Banach Space Cones **49**
4.1 Centred Cones 51
4.2 Projectively Non-Euclidean and Projectively Lattice Ordered Banach Spaces 56
4.3 Notes & Open Problems 67

5 Spectrum and Asymptotics of Positive Semigroups **69**
5.1 The Point Spectrum 69
5.2 Asymptotic Behaviour 77
5.3 Notes & Open Problems 78

III Eventual Positivity **81**

6 Eventual and Asymptotic Positivity – Spectral Properties **83**
6.1 Some Basic Notions 84
6.2 The Spectral Bound 86
6.3 The Peripheral Point Spectrum 92
6.4 Notes & Open Problems 94

7 Asymptotic Positivity – Criteria and Characterisation **99**
7.1 Positive Projections and Spectral Projections 99
7.2 The Resolvent 101
7.3 The Semigroup 104
7.4 Notes & Open Problems 111

8 Applications of Asymptotic Positivity **113**
8.1 The Finite-Dimensional Case 114
8.2 The Dirichlet-to-Neumann Operator in High Dimensions . . 119
8.3 The bi-Laplace Operator with Dirichlet Boundary Conditions 121
8.4 Delay Differential Equations 123
8.5 Network Flows with Non-Positive Mass Transition 128
8.6 A Diffusion Equation with Non-Autonomous Boundary Conditions 131
8.7 Notes & Open Problems 134

9 Individual Eventual Strong Positivity – Criteria and Characterisation **137**
9.1 Strongly Positive Spectral Projections 138
9.2 The Resolvent 139
9.3 The Semigroup 148
9.4 Notes & Open Problems 153

10 Criteria for Uniform Eventual Strong Positivity **155**
10.1 The Resolvent 155
10.2 The Semigroup 157
10.3 Counterexamples 159
10.4 Notes & Open Problems 163

11 Applications of Eventual Strong Positivity **165**
11.1 The Finite-Dimensional Case 166
11.2 The Dirichlet-to-Neumann Operator in Dimension 2 168
11.3 The Dirichlet-to-Neumann Operator on Finite Graphs 170
11.4 The bi-Laplace Operator with Dirichlet Boundary Conditions 174
11.5 Squares of Laplace Operators 175

11.6 Delay Differential Equations 178
11.7 Laplace Operators with Non-Local Boundary Conditions . . 181
11.8 Notes & Open Problems 186

Appendices **189**

A Some Aspects of Spectral Theory **191**
A.1 Spectral Projections 191
A.2 Eigenvalue Multiplicities and Generalised Eigenspaces 193
A.3 Isolated Singularities of the Resolvent 196
A.4 Pseudo-Resolvents 199

B Ordered Banach Spaces **205**
B.1 Basics 205
B.2 Duality 209
B.3 Automatic Continuity and Extension Theorems 211
B.4 Banach Lattices and Hilbert Lattices 213

C Complexifications **215**
C.1 Complexifications of Real Banach Spaces 216
C.2 Standard Constructions for Complexificiations of Real Banach Spaces 219
C.3 Spectral Theoretic Aspects 222
C.4 Complexifications of Banach Lattices 222
C.5 Standard Constructions for Complexifications of Banach Lattices 226
C.6 Complex Hilbert Lattices 230

Bibliography **233**

Nomenclature **251**

Index **257**

Zusammenfassung in deutscher Sprache **263**

Introduction

Motivation

In this thesis we study one-parameter semigroups of linear operators and their behaviour for large times. As a beginning, let us review some basic ideas which motivate our analysis and which are present throughout the thesis.

Operator semigroups In analysis, the time evolution of a linear autonomous system is usually described by a semigroup of linear operators. If E is a Banach space, usually thought of as the space of all states of our system, then an operator semigroup on E is a family of bounded linear operators $(T(t))_{t\geq 0}$ on E which fulfil the semigroup law $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$; for many applications it is natural to assume in addition that $T(0) = I_E$. If $x_0 \in E$ is a given state, then the mapping $t \mapsto T(t)x_0$ describes the time evolution of the system in case that its initial state at the time 0 equals x_0.

This concept is very general and thus operator semigroups occur in abundance in many branches of mathematics. Parabolic partial differential equations, various kinds of transport processes, delay differential equations and time-continuous Markov processes are only a few examples which fit into this framework. Semigroups occurring in applications usually satisfy some kind of continuity condition which makes sure that the underlying evolution equation is well-posed in a certain sense. One important example of such a condition is strong continuity: a semigroup $(T(t))_{t\geq 0}$ on a Banach space E which fulfils $T(0) = I_E$ is called *strongly continuous* or a C_0*-semigroup* if the mapping

$$[0,\infty) \to E, \qquad t \mapsto T(t)x$$

is continuous with respect to the norm on E for every $x \in E$. Such semigroups have been extensively studied and their basic theory is well-known. One important property is that every C_0-semigroup $(T(t))_{t\geq 0}$ has a so-called *generator* A which is a, typically unbounded, linear operator on E and which relates the semigroup to a certain evolution equation. We will always denote a C_0-semigroup $(T(t))_{t\geq 0}$ by $(T(t))_{t\geq 0} =: (e^{tA})_{t\geq 0}$ where A is the generator of the semigroup.

We should point out that there are other important notions of continuity for operator semigroups which are, for instance, widely used in stochastics. For many other applications, however, C_0-semigroups are the right notion, and we entirely focus on them throughout this thesis. We use many basic properties of such semigroups tacitly, and give references whenever we make

use of results which we consider more advanced or specialised. Standard references for the theory of C_0-semigroups are for example [Paz83], [EN00] and [EN06].

Positivity of semigroups A special role in many applications is played by so-called *positive* C_0-semigroups. Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a Banach space E on which a sensible notion of "positivity" is defined; for example, E could be some kind of function space or, from a more abstract point of view, we simply assume E to be an ordered Banach space. The semigroup $(e^{tA})_{t\geq 0}$ is called *positive* if $e^{tA}f \geq 0$ for all $t \in [0,\infty)$ whenever $0 \leq f \in E$. The role of positive semigroups in applications is, in a sense, twofold. On the one hand, positivity occurs as a natural requirement in many models from physics or biology. In such models, the physically or biologically relevant states are often only the positive ones. In order to yield a plausible description of reality it is a necessary condition on a semigroup $(e^{tA})_{t\geq 0}$ that physically relevant initial data leads to physically relevant trajectories; thus the semigroup should be positive. On the other hand, positivity of semigroups is often useful as an abstract mathematical tool in the analysis of the long term behaviour of the system; we discuss this in more detail now.

Long time behaviour of semigroups One of the most natural questions that arise in the study of any kind of dynamical system is the analysis of its long time behaviour. For example, the trajectories $(e^{tA}x)_{t\geq 0}$ of a C_0-semigroup $(e^{tA})_{t\geq 0}$ could converge to 0 as $t \to \infty$ or they could tend to a non-trivial equilibrium. Instead, they could also behave approximately periodic, tend in norm to ∞, or show instead some kind of completely irregular behaviour. Moreover, convergence of the semigroup can occur with respect to different topologies. One efficient tool in the analysis of the long time behaviour (or "asymptotics") of C_0-semigroups is the spectral theory of their generators. Although there are other important and powerful approaches, too (see e.g. the monograph [Eme07] for an overview), we focus on spectral theoretic methods.

Let us elaborate in a bit more detail on this spectral theoretic approach. Whenever we want to study the asymptotic behaviour of a C_0-semigroup $(e^{tA})_{t\geq 0}$, the spectral theoretic approach consists of two steps:

(i) Gather information about the spectrum $\sigma(A)$ of the generator A, in particular about those spectral values with maximal real part (the so-called *peripheral spectrum*).

(ii) Use theorems which relate spectral properties of A to the asymptotic behaviour of the semigroup $(e^{tA})_{t\geq 0}$.

For many applications, both steps are far from being trivial, and positivity of the semigroup can come in very handy in both steps.

Let us discuss this in more detail, starting with step (i): to compute the spectrum of an operator on an infinite-dimensional Banach space is often very hard, not to say impossible. In some special cases, i.e. for self-adjoint operators on Hilbert spaces, one has very useful a priori information on the spectrum (such as $\sigma(A) \subseteq \mathbb{R}$), but for more general operators such information in not available. Even if the generator A has compact resolvent, a situation which occurs frequently in applications and which implies that the spectrum consists entirely of eigenvalues, it can still be hard to compute those eigenvalues. One useful approach is to use structural properties of A – e.g. one could try to decompose A into an "easy" infinite-dimensional and a complicated finite-dimensional part – in order to describe the spectrum of A by so-called characteristic equations; see e.g. [Nag97] for a more detailed explanation of such an approach. Yet, even if this is possible, one often still has to solve complicated finite-dimensional spectral problems or non-trivial equations in the complex plane. Positivity of the semigroup, say e.g. with respect to a Banach lattice cone, can often help to facilitate those tasks be providing additional a priori information on the spectrum.

Indeed, the fact that positivity has a profound impact on the spectrum of a positive operator was first observed at the beginning of the 20th century by Perron [Per07a], [Per07b] and Frobenius [Fro08], [Fro09], [Fro12] who studied finite-dimensional positive matrices. This is why the spectral theory of positive operators and semigroups is often referred to as *Perron–Frobenius theory*. Pioneering work in the infinite-dimensional setting was done in the middle of the century by Kreĭn und Rutman in their seminal work [KR48] where they considered the spectrum of positive operators on ordered Banach spaces. Two decades later, much progress was made for the special case of positive operators on Banach lattices; see [Sch74, Section V.4 and V.5] for an overview. In the 1980s many spectral results for positive operators were transferred to generators of positive semigroups, and it was observed that these results could be efficiently used for the asymptotic analysis of such semigroups; we refer to the monograph [AGG$^+$86] for an overview of the state of the art at that time.

Let us now give some examples of information about the spectrum of the generator A that can be obtained by using positivity properties of the semigroup $(e^{tA})_{t\geq 0}$. It is, for instance, known that the spectral bound $\mathrm{s}(A)$ of the generator A of a positive C_0-semigroup is, unless it equals $-\infty$, always contained in the spectrum $\sigma(A)$; see [AGG$^+$86, Corollary C-III.1.4]. Another useful property of positive semigroups is that the peripheral spectrum $\sigma_{\mathrm{per}}(A) := \sigma(A) \cap (\mathrm{s}(A) + i\mathbb{R})$ is, under appropriate technical assumptions, a *cyclic* subset of the complex plane, meaning that $\alpha + in\beta \in \sigma_{\mathrm{per}}(A)$ for all $n \in \mathbb{Z}$

whenever $\alpha+i\beta \in \sigma_{\mathrm{per}}(A)$ (where $\alpha, \beta \in \mathbb{R}$). If, for some reason, it is known that $\sigma_{\mathrm{per}}(A)$ is bounded (which is for example the case of the semigroup is eventually norm-continuous), it thus follows automatically that $\sigma_{\mathrm{per}}(A)$ consists only of $\mathrm{s}(A)$.

Step (ii) in the above scheme for the asymptotic analysis of C_0-semigroups is a topic on its own, and there are many important theorems which show how information about the spectrum $\sigma(A)$ can be used to obtain information about the long term behaviour of the semigroup $(e^{tA})_{t\geq 0}$. Unfortunately, the situation for C_0-semigroups is much more complicated than it is, for example, for (the powers of) single operators. For most relations which one might expect between the spectrum of A and the asymptotic behaviour of $(e^{tA})_{t\geq 0}$ there exist more or less obscure counterexamples showing that such a relation is not in general true. This is the reason why most theorems in this branch of semigroup theory impose some kind of regularity condition on the semigroup which enables us to infer information about the behaviour of the semigroup from the shape of $\sigma(A)$. Typical examples for such theorems are spectral mapping theorems [EN00, Section IV.3], the Jacobs–deLeeuw–Glicksberg decomposition (see e.g. [EN00, Section V.2], [Kre85, Section 2.4] and [EFHN15, Section 16.3]), which is in fact applicable in a much wider range of situations than only for C_0-semigroups, and certain results which are related to Tauberian theorems for the vector-valued Laplace transform [ABHN11, Part II], most prominently the famous ABLP theorem [ABHN11, Theorems 5.5.5 and 5.5.6].

Sometimes one can also employ positivity in this second step of the analysis. One popular example for this is a theorem of Weis asserting that, for a positive C_0-semigroup $(e^{tA})_{t\geq 0}$ on an L^p-space ($1 \leq p < \infty$), the spectral bound $\mathrm{s}(A)$ of the generator coincides with the growth bound $\omega_0(A)$ of the semigroup (see e.g. [ABHN11, Theorem 5.3.6] or [vN96, Theorem 3.5.3]). Thus, positivity of the semigroup yields the equality $\mathrm{s}(A) = \omega_0(A)$, a property which is in general (i.e. without using positivity) only satisfied if some version of the spectral mapping theorem holds (see e.g. [EN00, Section IV.3]).

Contributions of this Thesis

In this thesis we shed new light on the relation between positivity and asymptotics of semigroups from several directions. More specifically, we discuss three topics, each of them presented in one of the three parts of the thesis, which are all motivated by the above discussion.

The geometry of unit balls Geometrically speaking, a positive semigroup is simply a semigroup which leaves invariant a certain closed convex set, namely

the positive cone. The most striking spectral results of positive C_0-semigroups can certainly be obtained on Banach lattices. We can thus ask what is so special about Banach lattice cones and then consider other convex sets K which have similar properties. If a semigroup $(e^{tA})_{t\geq 0}$ leaves K invariant, it is natural to expect spectral result for A which resemble the corresponding results for positive semigroups.

In finite dimensions, there is a simple characterisation of lattices cones: in fact, a closed, pointed and generating cone in $\mathbb{R}^d$ (see Definition B.1.1 for a detailed explanation of those notions) is a lattice cone if and only if it is spanned by exactly d extreme rays (this follows from [AT07, Theorem 3.21]). This suggests that lattice cones are, in a sense, particularly non-smooth and far from being "rotund" in any way.

Let us now consider other convex sets which have similar properties: for example, for $1 < p < \infty$, the unit ball in a finite-dimensional ℓ^p-space is certainly smooth but, if $p \neq 2$, it is not particularly "rotund". A semigroup which leaves the unit ball invariant is of course simply a *contraction semigroup* and hence, we would expect contraction semigroups on such spaces to have special spectral properties in case that $p \neq 2$. This idea is the leitmotif for the first part of the thesis.

Of course, we actually go much further and discuss semigroups on infinite-dimensional L^p-spaces as well as on several other types of Banach spaces. In order to turn our leitmotif into rigorous spectral results, some work is required: we borrow some ideas about contractive operators on finite-dimensional spaces from a paper of Lyubich [Lyu70], reformulate them in an infinite-dimensional setting and combine them with several deep results from the geometric theory of Banach spaces. The work pays off and we are rewarded by several surprising results about the spectrum and the long term behaviour of contraction semigroups. We can, for instance, prove the following beautiful theorem on L^p-spaces:

Theorem. *Let $(e^{tA})_{t\geq 0}$ be a contractive C_0-semigroup in $L^p(\Omega,\mu;\mathbb{R})$ for a measure space (Ω,μ) and $p \in (1,\infty)\setminus\{2\}$.*

If the set $\sigma(A)\cap i\mathbb{R}$ is bounded, then it consists at most of 0 and thus, e^{tA} is strongly convergent as $t\to\infty$.

Here, the spectrum of A is understood to be the spectrum of a complex extension of A; see Appendix C.3 for details. The above theorem is a consequence of Theorem 2.4.1 and Example 1.3.4(a) below. We point out that the assumption that $\sigma(A)\cap i\mathbb{R}$ be bounded is automatically fulfilled if the semigroup $(e^{tA})_{t\geq 0}$ is eventually norm-continuous. In particular, the theorem can be applied if the semigroup (or rather its extension to $L^p(\Omega,\mu;\mathbb{C})$) is analytic; here, it is important to note that we do *not* assume that the semigroup is

contractive on a complex sector, because otherwise the result is trivial (and also valid in the case $p = 2$).

If we employ additional compactness assumptions on the semigroup, then we can also prove convergence results on L^1 and on spaces of continuous functions; see Sections 1.2 and 2.2 for details. One such result will be employed in Chapter 3 where we analyse the long term behaviour of certain delay differential equations.

Another interesting phenomenon occurs if we turn to the special case of sequence spaces. In [Dav05] Davies considered positive contraction semigroups $(e^{tA})_{t\geq 0}$ on the space $\ell^p(\mathbb{N};\mathbb{R})$ for $p \in [1,\infty)$ and proved that the point spectrum of A always fulfils $\sigma_{\text{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$. Using our techniques one can show that the positivity assumption is not needed for this result in case that $p \neq 2$, i.e. we can prove the following theorem:

Theorem. *Let $p \in [1,\infty) \setminus \{2\}$ and let $(e^{tA})_{t\geq 0}$ be a contractive C_0-semigroup on $\ell^p(I;\mathbb{R})$ (where I is an arbitrary index set). Then $\sigma_{\text{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.*

This is shown in Corollary 2.1.5. The same result remains true if we replace $\ell^p(I;\mathbb{R})$ with the space $c(\mathbb{N};\mathbb{R})$ of real-valued convergent sequences or with any closed subspace of the latter, see Theorem 2.1.1, Example 1.1.5(c) and Remark 1.1.2.

We note that Davies' result has an analogue for positive contraction semigroups on L^p-spaces over arbitrary measure spaces in case that the semigroup contains a so-called *kernel operator* [Gre82, Korollar 3.11]. Again, it turns out that this result remains true without the positivity assumption if $p \neq 2$, see Theorem 2.3.4. For a detailed discussion of the relation between Greiner's and Davies' results as well as their interplay with our theory we refer to the notes at the end of Chapter 2.

The geometry of positive cones In the second part of the thesis we move back to the original motivation discussed at the beginning of the introduction. We have mentioned above that a positive semigroup on a Banach lattice satisfies quite strong spectral properties. In order to study the asymptotic behaviour of the semigroup, we are mainly interested in the question whether the (point) spectrum of the generator A fulfils the condition $\sigma_{\text{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$. On Banach lattices, good sufficient conditions for this property are known and thus the question arises whether similar conditions can be found in more general ordered Banach spaces E.

Simple counterexamples such as the classical ice cream cone in three dimensions show that this cannot be true without an additional geometric assumption on the positive cone E_+. A very natural assumption is to require the cone E_+ to be a lattice cone, but not necessarily normal, which means that

E is lattice ordered, but not a Banach lattice. For positive *compact* operators on such spaces, interesting results have already been obtained by Kreĭn and Rutman [KR48, Section 8] in the middle of the last century, and it is not difficult to use such results on single operators in order to infer results for C_0-semigroups. Of course, the assumption that E be lattice ordered is still rather strong; one way to get rid of it is to impose some kind of strengthened positivity assumptions on the operator or the semigroup. To give only one example, one can assume that the cone E_+ has non-empty interior and that the semigroup maps every non-zero positive vector into the interior of E_+. It was also observed by Kreĭn and Rutman [KR48] that such a condition yields interesting results without any geometric assumption on the cone; we refer to the introduction of Chapter 4 for a more detailed discussion and for further references.

Yet, we do not intend to require any strengthened positivity assumptions from our semigroups; instead, we want to stick to the first mentioned approach and derive convergence results from geometric properties of the positive cone. An important step in this direction was made by Veĭtsblit and Lyubich who considered very general conditions on the cone, but restricted themselves to the finite-dimensional case in [VL85]; one result in the infinite-dimensional setting is due to Bartoszek [Bar92] who considered positive operators under the assumption that the underlying Banach space fulfils the Riesz decomposition property.

Inspired by [VL85] we consider a rather general geometric condition, but in infinite dimensions and under very mild compactness assumptions (namely in the weak topology). Our main result in Part II is Theorem 5.1.1; the theorem is a bit too technical to state it here explicitly, but it can be used to prove that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$ for generators of sufficiently well-behaved positive semigroups $(e^{tA})_{t\geq 0}$, and it can be applied on a large class of ordered Banach spaces, such as: lattice ordered spaces; more generally, spaces with the Riesz decomposition property; Sobolev spaces with sufficiently high order of differentiability; spaces of k-times continuously differentiable functions, and several others. Moreover, we present some consequences of this theorem for the long term behaviour of positive semigroups.

Eventual positivity In the third and most extensive part of the thesis we approach the relation between positivity and the long-term behaviour of semigroups from a completely different angle. Throughout the preceding discussion we always understood "long-term behaviour" as some kind of convergence or divergence of the semigroup as time tends to infinity. Now, let us change the perspective and assume that we are given a C_0-semigroup $(e^{tA})_{t\geq 0}$ on, say, a Banach lattice E, which is not positive. A completely dif-

ferent aspect of the long-term behaviour of the semigroup is the question whether it actually *becomes* positive for large enough time.

Such an *eventually positive* behaviour, which might at first glance appear somewhat peculiar, has actually played an increasingly prominent role in the literature within the last two decades. There are two areas in analysis in which eventual positivity has become quite important: the first one is finite-dimensional matrix theory, where eventually positive powers of matrices as well as eventually positive semigroups have been studied in great detail. It seems that the initial motivation for the development of this theory were inverse spectral problems combined with the desire to prove Perron–Frobenius type theorems for a larger class of matrices than the positive ones. A lot of research in this subject has been done during the last 20 years; we refer to the notes at the end of Chapter 6 for more details and for some references to the extensive literature.

The second area where eventual positivity has played a role is the study of certain higher order elliptic differential operators such a the bi-Laplace operator with Dirichlet boundary conditions. The literature focused mainly (but not exclusively) on the resolvent of such operators and the question whether it satisfies certain eventual positivity properties close to the spectral bound of the operator. Eventual positivity of the semigroups generated by those operators was also studied in several papers. We refer to Sections 8.3 and 11.4 as well as to the notes at the end of Chapters 8 and 11 for more information and several references. Moreover, those questions are closely related to so-called *anti-maximum principles* for differential operators which have been extensively studied for several decades now; more details about this can be found in Remarks 9.2.4 and 9.2.8.

Only very recently, eventual positivity was observed for the semigroup generated by another differential operator, namely the Dirichlet-to-Neumann operator on the unit circle in two dimensions [Dan14].

Despite the intensive study of the phenomena mentioned above, until very recently only little effort has been made to develop a general and abstract theory of eventual positivity in infinite dimensions. In the third part of this thesis we present such a theory for C_0-semigroups on Banach lattices in its current stage of development. Moreover, we give numerous applications to demonstrate the usefulness of our results as well as the prevalence of the phenomenon "eventual positivity" in many concrete models. The theory we present is based on joint work with Daniel Daners and James B. Kennedy. Most of the material can either be found in the two recent papers [DGK16b] and [DGK16a] or is part of still ongoing research projects; see the introductions to the single chapters for details.

We first show that several spectral theoretic results which are well-known for positive semigroups remain valid for eventually positive semigroups. For

instance, we prove the following result (for undefined notation or terminology we refer to the preliminaries of the thesis):

Theorem. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach lattice E and assume that, for every $0 \leq f \in E$, there exists a $t_0 \geq 0$ such that $e^{tA}f \geq 0$ for all $t \geq t_0$. Then the following assertions hold:*

(a) *If the spectrum $\sigma(A)$ is non-empty, then it contains the spectral bound* $\mathrm{s}(A)$.

(b) *If* $\mathrm{s}(A)$ *is, in addition, a pole of the resolvent of A, then it is an eigenvalue of A and of the adjoint A' and both eigenspaces* $\ker(\mathrm{s}(A)-A)$ *and* $\ker(\mathrm{s}(A)-A')$ *contain a positive, non-zero vector.*

This result can be found in Theorem 6.2.5 and Proposition 6.2.7.

Afterwards we discuss a slightly weaker form of eventual positivity which we name *asymptotic positivity*. We define this notion for semigroups as well as for resolvents of operators, and it is our major goal to characterise this property by means of spectral conditions. After we have accomplished this we will demonstrate the usefulness of our results by proving asymptotic positivity for a wealth of examples from several areas of analysis: we will apply our theory to finite-dimensional matrix semigroups, we will consider several elliptic differential operator as well as the Dirichlet-to-Neumann operator, we are going to study a large class of one-dimensional delay differential equations and we will consider a particular example of a network flow with non-positive mass transition in the vertices.

A related concept which seems to be, both from an applied and from a theoretical point of view, even more interesting than asymptotic positivity is what we call *eventual strong positivity*. Here we fix a vector $u \geq 0$ in a Banach lattice E; for every other vector $f \in E$ we write $f \gg_u 0$ if $f \geq \varepsilon u$ for some $\varepsilon > 0$. We then consider the concept of a semigroup $(e^{tA})_{t\geq 0}$ which is *individually eventually strongly positive with respect to u*, meaning that for every $0 < f \in E$ there exists a $t_0 \geq 0$ such that $e^{tA}f \gg_u 0$ for all $t \geq t_0$. Characterising this property turns out to be more challenging than characterising asymptotic positivity, but under appropriate conditions we can prove that eventual strong positivity is related to certain spectral properties of the generator A, to the behaviour of the resolvent of A close to the spectral bound, to so-called *anti-maximum principles*, as well as to the convergence of the semigroup as $t \to \infty$. We will also study whether the above-mentioned individual eventual strong positivity property is in fact a *uniform* property, meaning that t_0 can be chosen independently of f. We shall see in several counterexamples (see Section 10.3) that this is not in general the case. However, if we consider self-adjoint semigroups on L^2-spaces, then it turns out that the individual and the uniform property are, under appropriate conditions, equivalent; this is the last theoretical highlight of the thesis, which we prove in Theorem 10.2.1.

We illustrate our results about eventual strong positivity by a large variety of examples: we shall see that some of our examples for asymptotic positivity even exhibit eventual strong positivity if we slightly strengthen our conditions on the parameters, and we will also discuss several further examples, among them two different Laplace operators with non-local boundary conditions and the square of a Robin Laplace operator.

How the Thesis is Organised

After the introduction the reader can find a preliminary chapter where we introduce most of our notation and where we recall some important notions from operator theory. As explained above, the main part of the thesis is divided into three parts. The first of them, consisting of Chapters 1–3, deals with the spectrum and the convergence of contractive semigroups on Banach spaces whose unit balls have special geometric properties. The second part, consisting of Chapters 4 and 5, deals with the spectrum and the long-time behaviour of positive semigroups on ordered Banach spaces provided that the positive cone satisfies certain geometric conditions. The third and longest part of the thesis, consisting of Chapters 6–11, presents a theory of eventually positive semigroups and gives a large variety of examples for such semigroups. Finally, the thesis concludes with several appendices in which we collect some important results from the literature that we use throughout the thesis. Let us now give a description of the single chapters:

Chapter 1 This chapter treats the geometric foundations for the first part of the thesis. We discuss several classes of real Banach spaces which behave, in a sense, rather different from Hilbert spaces. We study certain permanence properties of those classes and we show that they contain many important examples of Banach spaces.

Chapter 2 In this chapter we study spectral properties of contractive C_0-semigroups on Banach spaces that belong to one of the classes discussed in Chapter 1. We mainly focus on theorems which ensure that the (point) spectrum of the semigroup generator intersects the imaginary axis at most in 0, since such results yield interesting convergence theorems as corollaries. We vary the assumptions on the semigroup as well as on the Banach space several times throughout the chapter, thereby demonstrating that there is some kind of trade-off between the geometric assumptions on the underlying Banach space and the regularity assumptions of the semigroup.

Chapter 3 This is the last chapter of Part I. It is devoted to the study of semigroups generated by finite-dimensional delay differential equations. We give some general results about the long time behaviour of such semigroups, but our major goal is to demonstrate how the results from Chapter 2 can be applied in this situation.

Chapter 4 This is the first of two chapters in Part II of the thesis. We first study a certain class of positive cones in real Banach spaces which we call *centred* cones. These are loosely related to what is sometimes called *ice cream cones* in the literature. Afterwards, we discuss two classes of ordered Banach spaces: the first class includes a lot of interesting spaces which are not lattice ordered but have a somewhat related property; we call them *projectively lattice ordered spaces*. The second class is even more general and contains all Banach spaces whose positive cone is, in a sense, far from being Euclidean shaped; we call them *projectively non-Euclidean spaces*.

Chapter 5 In this second chapter of Part II we study positive semigroups on the previously discussed classes of ordered Banach spaces. We prove a result on the peripheral point spectrum of their generator and derive convergence theorems for such semigroups as a consequence. We consider this chapter mainly as a theoretical study with the objective to understand the geometric background of some results in Perron–Frobenius theory. Therefore no concrete applications of our results are discussed.

Chapter 6 This chapter is the starting point for Part III of the thesis. We give precise definitions of what we mean by an *eventually positive* and by an *asymptotically positive* C_0-semigroup and show that several results from Perron–Frobenius theory remain true for those semigroups.

Chapter 7 The main objective of this chapter is a characterisation of asymptotically positive behaviour of C_0-semigroups. Yet, we first give sufficient conditions for certain spectral projections of a linear operator A to be positive. Afterwards, we consider the behaviour of the resolvent $\mathcal{R}(\cdot, A)$ in a right neighbourhood of a spectral value λ_0 of A and characterise whether the resolvent $\mathcal{R}(\lambda, A)$ behaves, in a certain sense, asymptotically positive as λ approaches λ_0. Finally, we consider C_0-semigroups and give, under appropriate regularity assumptions, a characterisation of whether the semigroup behaves asymptotically positive. We shall see that this question is closely related to the behaviour of the resolvent $\mathcal{R}(\cdot, A)$ close to the spectral bound $\mathrm{s}(A)$ as well as to the question whether the spectral projection associated with $\mathrm{s}(A)$ is positive.

Chapter 8 This chapter is devoted to applications of our theory developed in Chapter 7. After a discussion of the finite-dimensional case, which contributes to the theory of eventually and asymptotically positive matrices, we discuss several examples of differential equations. Here, we have three different goals in mind: for some examples, like for the Dirichlet-to-Neumann operator and the bi-Laplace operator, the observation of eventually positive behaviour is a few years older than the theory presented in this thesis. Here, it is our goal to demonstrate that our theory allows a unified treatment of such phenomena (in fact, we treat *asymptotic* positivity for those operator in Chapter 8; for *eventual* positivity of this operators, which plays a more prominent role in the literature, we refer to Chapter 11). For other examples, like the delay differential equations in Section 8.4, it seems that eventual or asymptotic positivity has previously not been studied in the literature; here, we give rather general sufficient conditions for asymptotic positivity, which shows that such behaviour is far from being exotic. Finally, we study a specific network flow with non-positive mass transitions and a one-dimensional diffusion equation with non-autonomous boundary conditions at the end of the chapter. Those latter examples are not very general in nature, and the setting in which we present those examples is deliberately kept simple; it is our intention here to give only an introduction to two further settings where asymptotic positivity may occur and to stimulate further research.

Chapter 9 In this chapter we slightly change the setting; instead of considering the very general but rather weak notion of asymptotic positivity, we characterise semigroups which indeed become positive (and not only approximately positive) for large enough times. Some spectral properties of such semigroups have already been discussed in Chapter 6, but it turns out that things become much more subtle if we look for *characterisations* of such semigroups. Indeed, to obtain good results it seems necessary to further strengthen the notion of eventual positivity: we thus introduce the concept of *eventual strong positivity* for resolvents as well as for semigroups. The chapter is organised similarly to Chapter 7: we first consider strongly positive spectral projections, then we study resolvents and finally C_0-semigroups.

Chapter 10 In the definition of individually eventually strongly positive semigroups on page xiii, we allowed the time t_0 to depend on the initial value f; this explains the word "individual" in the notion. If we can actually choose t_0 to be independent of f, we shall therefore call the semigroup *uniformly eventually strongly positive* with respect to u. Taking a closer look at Chapter 7 the reader will find out that such a distinction also played a (small) role in the theory of asymptotically positive semigroups; yet, one of our results in

Chapter 7 (Theorem 7.3.6) shows that such a distinction is inessential for a large class of examples. It might now come as a surprise that the situation is completely different for eventually strongly positive semigroups; in two counterexamples in Chapter 10 we show that, even under very strong regularity assumptions, one cannot expect individual eventual strong positivity to imply uniform eventual strong positivity. On the other hand, we will give sufficient conditions for the individual property to imply the uniform property for self-adjoint semigroups on L^2-spaces.

Chapter 11 This final chapter of the thesis is devoted to present an assortment of examples for eventually strongly positive semigroups. After a brief visit to the finite-dimensional case in the first section of the chapter, we again consider a large variety of differential operators. In several occasions, we revisit operators from Chapter 8 which were proven there to generate asymptotically positive semigroups; we then show that, under additional assumptions, those semigroups are even (individually or uniformly) eventually strongly positive. Moreover, we consider some new classes of examples, namely the Dirichlet-to-Neumann operator on graphs, squares of Dirichlet and Robin Laplace operators on spaces of continuous functions, and two examples of a one-dimensional Laplace operator with non-local boundary conditions.

Appendix A In this appendix we collect a few results from spectral theory which we use throughout the thesis. Most of the material in this appendix can be found somewhere in the literature, often in well-known monographs about functional analysis, but we find it convenient to provide a common source of reference for all those results. As a rule, we give proofs only for results which we could not find in the exact version we need in the literature; otherwise we restrict ourselves to providing references for each result. The same applies to the other appendices, too.

Appendix B In this appendix, we give a brief review of the basic theory of ordered Banach spaces. In particular, we present some results in the setting of ordered Banach spaces which are often found in the literature within the much more general (though, for the unfamiliar reader, less comprehensible) setting of ordered topological vector spaces. Finally, we recall a few notions about so-called *Hilbert lattices*.

Appendix C Here we deal with complexifications of real Banach spaces. These constructions can be considered to be a kind of folklore within analysis; they are used very frequently, in particular when spectral theory plays a role,

and they are certainly considered to be basic equipment within the toolbox of every mathematician working in functional analysis. Yet, it is hard to find a comprehensive treatment of them in the literature. In this appendix we treat only the very basics of complexifications. Moreover, we pay special attention to the topic of complex Banach lattices, which are special complexfications of real Banach lattices. Although most textbooks and monographs on Banach lattices contain sections where complex Banach lattices are defined and briefly studied, we feel that the prevalence of complex Banach lattices in infinite-dimensional Perron–Frobenius theory, and in particular in Part III this thesis, justifies a thorough review of some of their most important properties.

Main sources for the text The results in Part I of the thesis are due to the author and many of them can be found in [Glü16d]; yet we have improved some of the theorems and added some new material which has not been published, yet. The results in the second part of the thesis are due to the author and have not been published, yet. Most of the results in Part III are due to Daniel Daners, James B. Kennedy and the author; many of them are contained in one of the papers [DGK16b] and [DGK16a]. Though, several theorems and examples in this part of the thesis have not been published, yet. Some of them are the result of a further collaboration with Daniel Daners, while others are based on further joint work with him and James B. Kennedy. We refer to the introductions of the single chapters for more details.

Notes & Open Problems At the end of each chapter we include a section called *Notes & Open Problems*. We use these sections for a couple of purposes: we discuss the history of some subjects in somewhat more detail and we mention results which are related to the main text, though not closely enough to be stated explicitly there. We also provide many additional references in the notes. As a rule, we quote a reference in the main text in any of the following four cases: if we employ a result from this reference in a proof; if we use an idea or inspiration from the reference; if we compare our results to the results from the reference; and if we want to give a brief overview of the literature when introducing a new topic. Any further references, which are intended to provide additional information (historical or recent), are included in the notes. Finally, we also use the notes to pose several open problems.

Acknowledgements

During my doctoral studies, lasting more than three years at the Institute of Applied Analysis at Ulm University, I had not only a very enjoyable time, but also the possibility to learn, discuss and discover a lot of interesting things,

not solely in mathematics. This great experience was the foundation for the research which I present in this thesis, as well as for many other mathematical projects. None of this would have been possible without the support of a lot of people. It is my pleasure to thank those people for all they have done for me:

My supervisor Wolfgang Arendt always guided me with his experience, his knowledge and his advise and continuously encouraged me and my research. It was also a pleasure to work with him in teaching on several occasions. Moritz Gerlach was never lost for a discussion, be it mathematical in nature or not; besides teaching me a lot of useful things in his field of expertise, he never failed to make my day with one of his shrewd remarks. I also greatly enjoyed, and still enjoy, our ongoing collaboration about the asymptotic behaviour of positive semigroups. Stephan Fackler generously shared his impressive mathematical knowledge with me and helped me to understand a lot of things which had previously been a closed book to me. I constantly benefited from his advise as well as his prudent attitude. It was, and is still, a pleasure to work with James Kennedy on several mathematical projects, some of which play an important role in this thesis. Moreover, James' superior insight into all matters of language, his cheerful attitude as well as his fondness for all kind of discussions were a constant source of delight for me. In several collaborations, some of them essential for this thesis and some of them still going on, Daniel Daners has shared his mathematical experience with me; on many occasions he made me consider things from a different, and sometimes more applied point of view. I am also very grateful to him for giving me the great opportunity of a two-weeks research stay at the University of Sydney. Manuel Bernhard, who I got to know almost at the very beginning of my undergraduate studies, had certainly a profound impact on how I look at and do mathematics. He taught me to see things from a different, often more general perspective and gave me an impression of the vastness of mathematical theories which are waiting to be learned. Manfred Sauter was, and is, always in mood for a good discussion, may it concern mathematical research, teaching, or any other topic which just happened to find its way into our lunch break. Rico Zacher opened some closed mathematical books for me – in some of his lectures as well as in some discussions at the blackboard; it was also a pleasure to work with him in teaching on several occasions. In plenty of discussions with Marie-Luise Hein I gained new insights into some mathematical areas which had been strangers to me, before.

I am also very grateful to my other colleagues, and former colleagues, at the Institute of Applied Analysis, among them Dominik Dier, Marcel Kreuter, Stefan Kunkel and Khalid Akhlil, for many lively discussions as well as for contributing to the great atmosphere at our institute. Moreover, I am indebted to all participants of several reading courses which I attended during my

studies, among them Raphael Reinauer, Julian Rüth, Adrian Spener and Christian Steck; besides being a very pleasant use of time, those reading courses have significantly improved my understanding of many aspects of mathematics and its applications. During my doctoral studies I have very often benefited from discussions with a lot of other mathematicians which were often more experienced than me; my thanks for this go in particular, though not solely, to Anna Dall'Acqua, Tom ter Elst, Markus Haase, Markus Kunze, Delio Mugnolo and Rainer Nagel.

I am, moreover, very grateful to my family, in particular to my mother, for their support, their advise and their encouragement during my studies. Last but not least, I want to express my deepest gratitude to Mandy; this thesis would certainly not have come into existence without her patience, her encouragement and her ongoing support.

I gratefully acknowledge financial support by a doctoral scholarship within the scope of the "Landesgraduiertenförderung Baden–Württemberg" which gave me the opportunity to do the research on which this thesis is based. Moreover, I gratefully acknowledge the support by a scholarship of the "Studienstiftung des deutschen Volkes" during my studies in the Bachelor's and Master's program.

Preliminaries

Some elementary notations By $\mathbb{N} := \{1, 2, 3, ...\}$ we denote the set of *strictly* positive integers, and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. By $i\mathbb{R} \subseteq \mathbb{C}$ we denote the imaginary axis and by $\mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ we denote the complex unit circle. If S is a subset of a metric space (M, d) and $x \in M$, then we denote by $\operatorname{dist}(x, S) := \inf_{y \in S} d(x, y)$ the *distance* of x to S. For every non-empty set S we denote by $\mathbb{1}_S : S \to \mathbb{R}$ the constant function with value 1. If the choice of S is clear from the context, then we shall sometimes use the abbreviation $\mathbb{1} := \mathbb{1}_S$. Whenever V is a vector space and $S \subseteq V$, we denote by $\operatorname{span} S \subseteq V$ the *linear span* (or *linear hull*) of S.

A subset $S \subseteq \mathbb{C}$ of the complex numbers is called *cyclic* or, more precisely, *imaginary additively cyclic*, if $\alpha + i\beta \in S$ (for $\alpha, \beta \in \mathbb{R}$) implies that $\alpha + in\beta \in S$ for all integers $n \in \mathbb{Z}$. We note in passing that there is also a different meaning of a *cyclic* subset of $\mathbb{C}$ which is important in the spectral theory of *single* positive operators; see e.g. [Sch74, p. 324]. The definition above is, however, better adapted to the study of (generators of) C_0-semigroups and thus more relevant for this thesis. The other notion of cyclicity that we have just mentioned will only occur on a few occasions in the notes at the end of some chapters.

Banach spaces Let E, F be (real or complex) Banach spaces. The norm on E is usually denoted by $\|\cdot\|$; if it is, for some reason, necessary to stress the space E on which our norm is defined, then we shall write $\|\cdot\|_E$, instead. We denote by $\mathcal{L}(E; F)$ to space of all *bounded* linear operators from E to F; this space is, unless stated otherwise, assumed to be endowed with the operator norm, which is also denoted by $\|\cdot\|$. We use the abbreviation $\mathcal{L}(E) := \mathcal{L}(E; E)$. By $I_E \in \mathcal{L}(E)$ we denote the *identity operator* on E. An operator $T \in \mathcal{L}(E; F)$ is called *contractive* if $\|T\| \leq 1$. The *rank* of an operator $T \in \mathcal{L}(E; F)$ is defined to be the dimension of its range TE; here, and everywhere else in the thesis, we consider the *dimension* to be either an integer or ∞, i.e. we do not distinguish between different infinite cardinalities. By $\mathcal{K}(E; F) \subseteq \mathcal{L}(E; F)$ we denote the subspace of *compact* linear operators from E to F, and we abbreviate $\mathcal{K}(E) := \mathcal{K}(E; E)$. The *dual space* of E is denoted by E'. We denote the *duality* of E and E' by $\langle \cdot, \cdot \rangle$, i.e. we write $x'(x) =: \langle x', x \rangle$ for all $x \in E$ and all $x' \in E'$. If $T \in \mathcal{L}(E; F)$, then we denote the *adjoint* operator of T by $T' \in \mathcal{L}(F', E')$. The domain of a possibly unbounded linear operator A on E is always denoted by $D(A)$; unless stated otherwise, the domain of A is always assumed to be endowed with the *graph norm* $\|\cdot\|_A$, given by $\|x\|_A := \|Ax\| + \|x\|$ for all $x \in D(A)$. If $A : E \supseteq D(A) \to E$ is a linear operator, then we define $D(A^\infty) := \bigcap_{n \in \mathbb{N}} D(A^n)$. If A is, moreover, densely defined, then we denote its *adjoint* by $A' : E' \supseteq D(A') \to E'$. Let

H be a real or complex Hilbert space; we denote the *inner product* on H by $(\cdot|\cdot)$. If $A : H \supseteq D(A) \to H$ is a densely defined linear operator, then $A^* : H \supseteq D(A^*) \to H$ denotes the *Hilbert space adjoint* of A.

If E is a real Banach space, then we use the symbol $E_{\mathbb{C}}$ to denote a *complexification* of E and if $A : E \supseteq D(A) \to E$, then we denote by $A_{\mathbb{C}} : E_{\mathbb{C}} \supseteq D(A_{\mathbb{C}}) \to E_{\mathbb{C}}$ the *complex extension* of A to $E_{\mathbb{C}}$. We refer to Appendix C for a detailed discussion of complexifications.

Let E be a real or complex Banach space. For any measure space (Ω, μ) and any $p \in [1, \infty]$ we denote the E-valued L^p-space on (Ω, μ) by $L^p(\Omega, \mu; E)$ (note that the σ-algebra on Ω is encoded in μ). If the measure μ is clear from the context, then we write simply $L^p(\Omega; E) := L^p(\Omega, \mu; E)$. In particular, we always assume open subsets Ω of $\mathbb{R}^d$ to be endowed with the Lebesgue measure, while we assume the boundary $\partial\Omega$ of such set (if it is sufficiently smooth) to be endowed with the surface measure. If I is a non-empty set and $p \in [1, \infty]$, then we define $\ell^p(I; E) := L^p(I, \#; E)$, where # denotes the counting measure on I.

If L is a locally compact Hausdorff space, then we denote by $C_0(L; E)$ the space of all E-valued continuous functions on L which vanish at infinity; this space is, of course, always assumed to be endowed with the supremum norm $\|\cdot\|_\infty$ given by $\|f\|_\infty := \sup_{\omega \in L} \|f(\omega)\|_E$. If K is a compact Hausdorff space, then we write $C(K; E) := C_0(K; E)$ for short. If I is a non-empty set, then we define $c_0(I; E) := C_0(I; E)$ where I is endowed with the discrete topology. We write $c(\mathbb{N}; E)$ for the space of all E-valued convergent sequences which are indexed over $\mathbb{N}$ (endowed with the supremum norm), and the notation $c(\mathbb{N}_0; E)$ is used in a similar way. If $\Omega \subseteq \mathbb{R}^d$ is a bounded and open set, E is a Banach space and $k \in \mathbb{N}$, then we denote by $C^k(\overline{\Omega}; E)$ the space of all E-valued functions on $\overline{\Omega}$ which are k-times continuously differentiable in Ω and whose derivatives $D^\alpha f|_\Omega$ have, for $|\alpha| \leq k$, a (necessarily unique) continuous extension to $\overline{\Omega}$ which we denote by $D^\alpha f$; we endow this space with the norm $\|f\| := \max_{|\alpha| \leq k} \|D^\alpha f\|_\infty$ which renders it a Banach space. By $C^\infty(\overline{\Omega}; E)$ we denote the vector space $\bigcap_{k \in \mathbb{N}} C^k(\overline{\Omega}; E)$. Note that $C^k(\overline{\Omega}; E)$ and $C^\infty(\overline{\Omega}; E)$ depend by definition on Ω and not only on $\overline{\Omega}$.

Spectral theory Here we introduce the notation that we use for the spectral theory of operators on complex Banach spaces. The corresponding notions for operators on real Banach spaces are defined by means of complexifications; if, for instance, A is a linear operator on a real Banach space, then the spectrum of A is defined to be the spectrum of a complex extension of A. We refer to Section C.3 in the appendix for a detailed discussion of this.

Let E be a complex Banach space, let $A : E \supseteq D(A) \to E$ be a linear operator and let $T \in \mathcal{L}(E)$. By $\sigma(A)$, $\sigma_{\mathrm{pnt}}(A)$ and $\sigma_{\mathrm{appr}}(A)$ we denote the *spectrum*, the

point spectrum and the *approximate point spectrum* of A, respectively. The *resolvent set* of A is denoted by $\rho(A)$. Recall that if $\rho(A)$ is non-empty, then A is automatically closed. For each $\lambda \in \rho(A)$, we denote by $\mathcal{R}(\lambda, A) := (\lambda - A)^{-1} \in \mathcal{L}(E)$ the *resolvent* of A at λ. We refer to Section A.4 in the appendix for a discussion of the related concept of a *pseudo-resolvent*. We say that A has *compact resolvent* if the resolvent set $\rho(A)$ is non-empty and if the operator $\mathcal{R}(\lambda, A) \in \mathcal{L}(E)$ is compact for one (equivalently: all) $\lambda \in \rho(A)$. The number

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \in [-\infty, \infty]$$

is called the *spectral bounded* of A and the number

$$s_0(A) := \inf\{c \in \mathbb{R} : c > s(A), \sup_{\operatorname{Re}\lambda > c} \|\mathcal{R}(\lambda, A)\| < \infty\} \in [-\infty, \infty]$$

is called the *abscissa of uniform boundedness of the resolvent* or the *pseudo-spectral bound* of A. By

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\} \in [0, \infty)$$

we denote the *spectral radius* of T. If $s(A) \in \mathbb{R}$, then the *peripheral spectrum* of A is defined to be the set $\sigma_{\text{per}}(A) := (s(A) + i\mathbb{R}) \cap \sigma(A)$ and the *peripheral point spectrum* of A is defined to be the set $\sigma_{\text{per,pnt}}(A) := \sigma_{\text{per}}(A) \cap \sigma_{\text{pnt}}(A)$. We point out that for *bounded* operators T it is sometimes more appropriate to define the peripheral spectrum as the set $\{\lambda \in \sigma(T) : |\lambda| = r(T)\}$. However, since we are mainly concerned with (generators of) C_0-semigroups, we will never use this latter definition in the main text; it will sometimes occur in the notes at the end of a chapter, but it should always be clear from the context what we mean by the notion "peripheral spectrum". A number $\lambda \in \mathbb{C}$ is called a *dominant spectral value* of A if $\lambda \in \sigma(A)$ and if $\operatorname{Re}\mu < \operatorname{Re}\lambda$ for all other spectral values μ of A. Note that if λ is a dominant spectral value of A, then $\operatorname{Re}\lambda = s(A)$ (but the converse implication is, in general, false).

For some more notions from spectral theory which we need, we refer the reader to Appendix A. There we discuss *spectral projections* in Section A.1, eigenvalue multiplicities in Section A.2 and isolated singularities of the resolvent (in particular, poles of the resolvent) in Section A.3.

C_0-semigroups Let E be a real or complex Banach space. We denote a C_0-semigroup on E with generator A always by $(e^{tA})_{t\geq 0}$. The number

$$\omega_0(A) := \inf\{\omega \in \mathbb{R} : \exists M \geq 1 \ \forall t \geq 0 \ \|e^{tA}\| \leq Me^{t\omega}\} \in [-\infty, \infty)$$

is called the *growth bound* or the *type* of the C_0-semigroup $(e^{tA})_{t\geq 0}$. Sometimes we call a C_0-semigroup simply a *semigroup*. By $\omega_{\text{ess}}(A)$ we denote the *essential*

growth bound of $(e^{tA})_{t\geq 0}$, i.e. the growth bound of the induced semigroup in the Calkin algebra $\mathcal{L}(E)/\mathcal{K}(E)$; see [EN00, Definition IV.2.9] for details. A C_0-semigroup $(e^{tA})_{t\geq 0}$ is called *contractive* if each operator e^{tA} is contractive, i.e. if $\|e^{tA}\| \leq 1$ for all $t \geq 0$.

Ordered Banach spaces and Banach lattices For the precise definition of an *ordered Banach space* and for many related notions we refer to Appendix B; here we only recall the most basic notions. Let E be an ordered Banach space with positive cone E_+. Let $f, g \in E$; we call f *positive* if $f \in E_+$ and we denote this by $f \geq 0$. We write $f \leq g$ (or $g \geq f$) to say that $g - f \geq 0$. By the notation $f < g$ (or $g > f$) we mean that $f \leq g$, but $f \neq g$. The set $[f, g] := \{x \in E : f \leq x \leq g\}$ is called an *order interval* in E. A linear functional $\varphi : E \to \mathbb{R}$ is called *positive* if $\varphi(x) \geq 0$ for all $x \in E_+$. Let $x' \in E'$; we write $x' \geq 0$ to denote that x' is positive, and we write $x' > 0$ if $x' \geq 0$, but $x' \neq 0$. By

$$(E')_+ := \{x' \in E' : \ x' \text{ is positive}\} \subseteq E'$$

we denote the *dual cone* of E_+; sometimes we also use that notation $E'_+ := (E')_+$, instead. Let F be another ordered Banach space with positive cone F_+. A linear mapping $T : E \to F$ is called *positive* if $TE_+ \subseteq F_+$. If $T \in \mathcal{L}(E; F)$, then we write $T \geq 0$ to denote that T is positive and we write $T > 0$ to say that $T \geq 0$, but $T \neq 0$. A C_0-semigroup $(e^{tA})_{t\geq 0}$ is called *positive* if each operator e^{tA} $(t \geq 0)$ is positive.

Let E, F be real Banach lattices; all the above notions and notations in particular apply to this case. Moreover, we use the following conventions. Let $u \in E_+$. By

$$E_u := \{f \in E : \ \exists c \geq 0 \, |f| \leq cu\}$$

we denote the *principal ideal* in E generated by u. If not stated otherwise, we always assume the principal ideal E_u to be endowed with the *gauge norm* $\|\cdot\|_u$ given by

$$\|x\|_u := \inf\{c \geq 0 : \ |x| \leq cu\}$$

for all $x \in E_u$. It can be proved that $(E_u, \|\cdot\|_u)$ is not only a Banach lattice, but even an AM-space with unit u (see [Sch74, Definition II.7.1 and the corollary to Proposition II.7.2]); hence, $(E_u, \|\cdot\|_u)$ is isometrically Banach lattice isomorphic to $\mathcal{C}(K; \mathbb{R})$ for some compact Hausdorff space K, and the corresponding isomorphism maps u to $\mathbb{1}_K$ [Sch74, Theorem II.7.4]. We call u a *quasi-interior point* of E_+ if E_u is dense in E (with respect to $\|\cdot\|_E$); this is denoted by $u \gg 0$. Let $T \in \mathcal{L}(E; F)$. We call T *strongly positive* if $Tf \gg 0$ whenever $0 < f \in E$. A functional $x' \in E'$ is called *strictly positive* if it is

strongly positive as an operator from E to $\mathbb{R}$, i.e. if $\langle x', x\rangle > 0$ for all $x > 0$. It is important to note the following ambiguity caused by this definition: let $x' \in E'_+$; then the notation $x' \gg 0$ could either mean that x' is a quasi-interior point of E'_+, or that x' is strongly positive as an operator from E to $\mathbb{R}$, i.e. that the functional x' is strictly positive; however, those two conditions are *not* equivalent (if x' is a quasi-interior point of E'_+, then it is a strictly positive functional, but the converse implication is not in general true). For this reason, we never use the notation $x' \gg 0$ for functionals. Let $u \in F_+$. If $g \in F$, then we write $g \gg_u 0$ if there exists an $\varepsilon > 0$ such that $g \geq \varepsilon u$; in this case we say that g is *strongly positive with respect to* u. An operator $T \in \mathcal{L}(E,F)$ is called *strongly positive with respect to* u if $Tf \gg_u 0$ for all $0 < f \in E$; we denote this by $T \gg_u 0$.

For the convenience of the reader we also sum up a bit of notation for complex Banach lattices; a more detailed treatment of them can be found in the Appendices C.4 and C.5. A *complex Banach lattice* E is by definition the *complexification* of a real Banach lattice, which we shall denote by $E_{\mathbb{R}}$ and which we call the *real part* of E; a vector $f \in E$ is called *real* if $f \in E_{\mathbb{R}}$. We use the notation $E_+ := (E_{\mathbb{R}})_+$. If f and g are two elements of E and if we write down any property of f or g (or both) which contains some kind of order relation, then we mean by this that the occurring vectors are contained in $E_{\mathbb{R}}$ and that the relation holds in $E_{\mathbb{R}}$. For example if we write $g > f$ for two vectors $f, g \in E$, this means that $f, g \in E_{\mathbb{R}}$ and that $g > f$ in the real Banach lattice $E_{\mathbb{R}}$; similarly, if $u \in E_+$ and $f \in E_{\mathbb{R}}$, then the notation $f \gg_u 0$ means that $f \in E_{\mathbb{R}}$ and $f \gg_u 0$ in $E_{\mathbb{R}}$. If $u \in E_+$, then the *principal ideal* generated by u in E is, again, defined to be the set

$$E_u := \{f \in E : \exists c > 0 \, |f| \leq cu\}.$$

We have $E_u := (E_{\mathbb{R}})_u + i(E_{\mathbb{R}})_u$; as in the real case we endow E_u with the *gauge norm* $\|\cdot\|_u$ given by

$$\|z\|_u := \inf\{c \geq 0 : \, |z| \leq cu\}$$

for all $z \in E_u$, which makes E_u a complexification of the real Banach lattice $(E_u, \|\cdot\|_u)$. Just as in the real case, there exists a compact Hausdorff space K and an isometric isomorphism of complex Banach lattices from E_u to $\mathcal{C}(K;\mathbb{C})$ which maps u to $\mathbb{1}_K$, see Proposition C.5.8. Note that E_u is dense in E if and only if $(E_{\mathbb{R}})_u$ is dense in $E_{\mathbb{R}}$. Hence, we can use the notion of a *quasi-interior point* of $E_+ = (E_{\mathbb{R}})_+$ in the complex case without caring whether we currently argue in the complex space E or in the real space $E_{\mathbb{R}}$. For an operator $T \in \mathcal{L}(E;F)$ between to Banach lattices E and F we write $T \geq 0$ if E maps $E_{\mathbb{R}}$ to $F_{\mathbb{R}}$ and its restriction to $E_{\mathbb{R}}$ fulfils $T|_{E_{\mathbb{R}}} \geq 0$; the notations $T > 0$, $T \gg 0$ and $T \gg_u 0$ for some fixed $u \in F_+$ are defined similarly. Those remarks apply, in particular, to bounded linear functionals on complex Banach lattices.

While ordered Banach spaces are discussed in more detail in Appendix B, we need on very few occasions the more general concept of an *ordered vector space* and of a *vector lattice* (or *Riesz space*). For those topics we refer the reader to the literature, e.g. to [Per67], [SW99, Chapter V], [LZ71], [Zaa83] and [AT07].

Ultra powers In the first part of the thesis we shall need ultra powers of Banach spaces. Detailed information about this concept can, for example, be found in the survey article [Hei80]; here, we only recall a few basics in order to fix the notation. Let E be a real or complex Banach space and let $\mathcal{U}$ be an ultra filter on a set $I \neq \emptyset$. Let $c_{0,\mathcal{U}}(I;E)$ denote the closed subspace of $\ell^\infty(I;E)$ of all sequences which converges to 0 along $\mathcal{U}$. Then the quotient space

$$E_{\mathcal{U}} := \ell^\infty(I;E)/c_{0,\mathcal{U}}(I;E)$$

is called an *ultra power* of E. For each family $x = (x_i) \in \ell^\infty(I;E)$ we denote by $x_{\mathcal{U}}$ the equivalence class of x in $E_{\mathcal{U}}$; we always have $\|x_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|$. For each $x \in E$ we denote by $x_{\mathcal{U}} \in E_{\mathcal{U}}$ the equivalence class of the constant family $(x)_{i\in I} \in \ell^\infty(\mathbb{N};E)$; we have $\|x_{\mathcal{U}}\| = \|x\|$, so the mapping $E \to E_{\mathcal{U}}$, $x \mapsto x_{\mathcal{U}}$ is an isometric linear operator which we call the *canonical embedding* of E into $E_{\mathcal{U}}$. Each operator $T \in \mathcal{L}(E)$ induces an operator $T_{\mathcal{U}} \in \mathcal{L}(E_{\mathcal{U}})$ given by $T_{\mathcal{U}} x_{\mathcal{U}} = (Tx_i)_{\mathcal{U}}$ for each $x_{\mathcal{U}} = (x_i)_{\mathcal{U}} \in E_{\mathcal{U}}$. The mapping $\mathcal{L}(E) \to \mathcal{L}(E_{\mathcal{U}})$, $T \mapsto T_{\mathcal{U}}$ is an isometric unital Banach algebra homomorphism.

If E is a real or complex Banach lattice, then $c_{0,\mathcal{U}}(I;E)$ is an ideal in the Banach lattice $\ell^\infty(I;E)$. Hence, the ultra power $E_{\mathcal{U}}$ is a Banach lattice, too, and the canonical embedding $E \hookrightarrow E_{\mathcal{U}}$ is a Banach lattice homomorphism. Moreover, an operator $T \in \mathcal{L}(E)$ is positive if and only if the lifted operator $T_{\mathcal{U}} \in \mathcal{L}(E_{\mathcal{U}})$ is positive.

Part I

Asymptotic Behaviour of Contraction Semigroups

1 Geometry of the Unit Ball in Banach Spaces

In this chapter we study three different geometric properties of real Banach spaces E which describe in some manner how far the space E is from being a Hilbert space. Those properties are chosen to be very well-adapted to the spectral analysis of contraction semigroups in Chapter 2. The first notion, studied in Section 1.1, is that of an *extremely non-Hilbert space*; this is, as the name suggests, a rather strong property, and we will see that some classical sequence spaces fulfil this property. The second class of Banach spaces, which we name *projectively non-Hilbert spaces*, is much larger and contains, for example, all L^p-spaces for $p \neq 2$ as well as many spaces of continuous functions; we consider those spaces in Section 1.2. The third notion, which we introduce in Section 1.3, is that of a *super-projectively non-Hilbert space*; a real Banach space E is said to have this properties if every ultra power of E is projectively non-Hilbert. Typical examples are, again, all L^p-spaces for $p \neq 2$ and certain spaces of continuous functions.

The material in Sections 1.1 and 1.2 is mostly taken from [Glü16d, Sections 2.1 and 3.1]. The material presented in Section 1.3 was inspired by [Glü16d, Section 4], see in particular [Glü16d, Remark 4.8(ii)], but it is not explicitly contained in this reference.

1.1 Extremely Non-Hilbert Spaces

In this section we introduce and analyse an, as we shall see, rather small class of Banach spaces which we call *extremely non-Hilbert spaces*. Let us start with the major definition.

Definition 1.1.1 (Extremely non-Hilbert space). A real Banach space E is called *extremely non-Hilbert* if it does not contain a two-dimensional subspace which is isometrically isomorphic to a Hilbert space.

The following observation is obvious but, due to its importance and for later reference, we state it in a separate remark.

Remark 1.1.2. If a real Banach space E is extremely non-Hilbert, then every closed subspace of E is extremely non-Hilbert, too.

Some examples of extremely non-Hilbert spaces are constituted by certain ℓ^p-sequence spaces. Note, however, the distinction between the two cases $p \in 2\mathbb{N}$ and $p \notin 2\mathbb{N}$ in the following examples.

Examples 1.1.3. Let $1 \leq p < \infty$.

(a) If $I \neq \emptyset$ is an arbitrary index set and $p \notin 2\mathbb{N}$, then the space $\ell^p(I;\mathbb{R})$ is extremely non-Hilbert.

(b) If $p \in 2\mathbb{N}$, then the space $\ell^p(\mathbb{N};\mathbb{R})$ is not extremely non-Hilbert.

(c) The spaces $L^p([0,1];\mathbb{R})$ and $\mathcal{C}([0,1];\mathbb{R})$ are not extremely non-Hilbert.

Proof. (a) If I is countable, this result can be found in [DJP98, Corollary 1.8]. By Remark 1.1.2 this also implies the assertion if I is finite.

So let I be uncountable and let $V \subseteq \ell^p(I;\mathbb{R})$ be a two-dimensional subspace which is spanned by two vectors f and g. There is an at most countable subset $I' \subseteq I$ such that the support of f and g is contained in I'. Hence, V is contained in the subspace $\ell^p(I';\mathbb{R})$ of $\ell^p(I;\mathbb{R})$; since $\ell^p(I';\mathbb{R})$ is extremely non-Hilbert, V cannot be isometrically isomorphic to a Hilbert space.

(b) This is obvious for $p = 2$, and for $p \in \{4,6,\ldots\}$ one can prove that there exist even finite-dimensional ℓ^p-spaces which isometrically contain a two-dimensional Hilbert space, see [Mil88, p. 283–294] (where, however, the necessary dimension increases with p).

(c) It is sufficient to show that $L^p([0,2\pi];\mathbb{R})$ and $\mathcal{C}([0,2\pi];\mathbb{R})$ are not extremely non-Hilbert. First, let $E = L^p([0,2\pi];\mathbb{R})$ and let $V \subseteq E$ be the span of the functions cos and sin. Let $\alpha,\beta \in \mathbb{R}$; we show that $\|\alpha\cos+\beta\sin\|_p = (\alpha^2+\beta^2)^{\frac{1}{2}}\|\cos\|_p$.

To this end, choose $\theta \in [0,2\pi)$ such that $|\alpha+i\beta|e^{i\theta} = \alpha - i\beta$; then we have $|\alpha+i\beta|\operatorname{Re} e^{i(\theta+\cdot)} = \alpha\cos+\beta\sin$. We thus obtain for every $p \in [1,\infty)$

$$\begin{aligned}\|\alpha\cos+\beta\sin\|_p^p &= |\alpha+i\beta|^p \int_{[0,2\pi]} |\operatorname{Re} e^{i(\theta+x)}|^p \, dx \\ &= |\alpha+i\beta|^p \int_{[0,2\pi]} |\operatorname{Re} e^{ix}|^p \, dx = (\alpha^2+\beta^2)^{\frac{p}{2}}\|\cos\|_p^p.\end{aligned}$$

Hence, $\|\alpha\cos+\beta\sin\|_p = (\alpha^2+\beta^2)^{\frac{1}{2}}\|\cos\|_p$ for all $\alpha,\beta \in \mathbb{R}$, as claimed. This shows that the linear bijection $\mathbb{R}^2 \to V$, $(\alpha,\beta) \mapsto \frac{1}{\|\cos\|_p}(\alpha\cos+\beta\sin)$ is an isometry between $(\mathbb{R}^2,\|\cdot\|_2)$ and $(V,\|\cdot\|_p)$. Hence, $V \subseteq L^p([0,2\pi];\mathbb{R})$ is a two-dimensional Hilbert space.

Now, let $E = \mathcal{C}([0,2\pi];\mathbb{R})$ and again, let $V \subseteq E$ be the span of cos and sin. Let $\alpha,\beta \in \mathbb{R}$; we show that $\|\alpha\cos+\beta\sin\|_\infty = (\alpha^2+\beta^2)^{\frac{1}{2}}$. To this end, choose $\theta \in [0,2\pi)$ as above, such that $\alpha\cos+\beta\sin = |\alpha+i\beta|\operatorname{Re} e^{i(\theta+\cdot)}$. Then we have

$$\begin{aligned}\|\alpha\cos+\beta\sin\|_\infty &= |\alpha+i\beta| \sup_{x\in[0,2\pi]} |\operatorname{Re} e^{i(\theta+x)}| \\ &= |\alpha+i\beta| \sup_{x\in[0,2\pi]} |\operatorname{Re} e^{ix}| = (\alpha^2+\beta^2)^{\frac{1}{2}}.\end{aligned}$$

As above, this proves that the linear bijection $\mathbb{R}^2 \to V$, $(\alpha, \beta) \mapsto \alpha \cos + \beta \sin$ is an isometry between $(\mathbb{R}^2, \|\cdot\|_2)$ and $(V, \|\cdot\|_\infty)$. Hence, $V \subseteq \mathcal{C}([0, 2\pi]; \mathbb{R})$ is a two-dimensional Hilbert space. □

Example 1.1.3(c) is, on the complex unit circle instead of a real interval, also discussed in [Glü16d, Example 2.11]. This example is also related to [Gol73, Remark 5]. Moreover, it is worthwhile pointing out that $L^p([0,1])$ actually contains an isometric copy of the infinite-dimensional Hilbert space $\ell^2(\mathbb{N}; \mathbb{R})$; this can be shown by probabilistic methods, see [JL01, p. 16].

At the beginning of this section we claimed that the class of extremely non-Hilbert spaces is rather small. This heuristic assertion is backed up by Example 1.1.3(c) which shows that many important function spaces on $[0,1]$ are not extremely non-Hilbert. Moreover, we can conclude from Examples 1.1.3 that the notion "extremely non-Hilbert space" is not well-behaved with respect to duality:

Remark 1.1.4. The dual space or a pre-dual space of an extremely non-Hilbert space need not be extremely non-Hilbert, in general. Indeed, the space $\ell^{\frac{4}{3}}(\mathbb{N}; \mathbb{R})$ is extremely non-Hilbert according to Example 1.1.3(a) while its dual and pre-dual space $\ell^4(\mathbb{N}; \mathbb{R})$ is not, according to Example 1.1.3(b).

A further example of an extremely non-Hilbert space is the space $c(\mathbb{N}; \mathbb{R})$ of convergent real-valued sequences, endowed with the supremum norm. Indeed, our next example shows that this space is contained in a more general class of extremely non-Hilbert spaces.

Examples 1.1.5. (a) Let L be a locally compact Hausdorff space. If L is at most countable, then $\mathcal{C}_0(L; \mathbb{R})$ is extremely non-Hilbert.

(b) Let K be a compact Hausdorff space. If K is a most countable, then $\mathcal{C}(K; \mathbb{R})$ is extremely non-Hilbert.

(c) The space $c := c(\mathbb{N}; \mathbb{R})$ of convergent real-values sequences is extremely non-Hilbert.

(d) Let $I \neq \emptyset$ be an arbitrary index set. Then the space $c_0(I; \mathbb{R})$ is extremely non-Hilbert.

Proof. (a) Assume for a contradiction that $V \subseteq \mathcal{C}_0(L; \mathbb{R})$ is a two-dimensional vector subspace which is a Hilbert space with respect to the norm induced by $\mathcal{C}_0(L; \mathbb{R})$. Then there exist vectors $f, g \in V$ such that $\|\cos\theta f + \sin\theta g\| = 1$ for all $\theta \in \mathbb{R}$. For all $l \in L$ we consider the closed set $A_l := \{\theta \in \mathbb{R} : |\cos\theta f(l) + \sin\theta g(l)| = 1\}$. We have $\bigcup_{l \in L} A_l = \mathbb{R}$, since for every $\theta \in \mathbb{R}$ the function $\cos\theta f + \sin\theta g$ has norm 1 and the norm is attained (by the modulus of the function) at at least one point $l \in L$. Since each set A_l is closed in $\mathbb{R}$ and

since the set L is countable, it follows from Baire's category theorem that at least one of the sets A_l, A_{l_0} say, has non-empty interior. Thus, there exists an open interval $\emptyset \neq I \subseteq \mathbb{R}$ such that $\cos\theta f(l_0) + \sin\theta g(l_0) \in \{-1, 1\}$ for all $\theta \in I$. For continuity reason, we must either have $\cos\theta f(l_0) + \sin\theta g(l_0) = 1$ or $\cos\theta f(l_0) + \sin\theta g(l_0) = -1$ for all $\theta \in I$, and we conclude from the identity theorem for analytic functions that this is even true for all $\theta \in \mathbb{R}$. This is, however, a contradiction since the functions cos and sin and the constant function with value 1 are linearly independent elements of the vector space $\mathbb{R}^{\mathbb{R}}$.

(b) This is an immediate consequence of (a).

(c) This follows from (b) since the Banach space c is isometrically isomorphic to $\mathcal{C}(\mathbb{N} \cup \{\infty\}; \mathbb{R})$ where $\mathbb{N} \cup \{\infty\}$ denotes the one point compactification of the discrete space $\mathbb{N}$.

(d) Let $V \subseteq c_0(I; \mathbb{R})$ be a two-dimensional vector subspace which is spanned by two vectors f, g. Then there exists an at most countable subset $I_0 \subseteq I$ such that f and g are zero outside of I_0. Hence, $V \subseteq c_0(I_0; \mathbb{R})$, and so we conclude from (a) that V is not a Hilbert space. □

The fact that for instance the sequence space $c_0(\mathbb{N}; \mathbb{R})$ is extremely non-Hilbert might suggest that we can, more generally, obtain extremely non-Hilbert spaces by considering vector-valued sequence spaces of the kind $c_0(\mathbb{N}; E)$ (where the space E should, of course, be assumed to be extremely non-Hilbert, as well). Yet, the following example shows that this idea does not in general work.

Example 1.1.6. There is a two-dimensional real Banach space E with the following properties:

(a) The space E is not a Hilbert space; in particular, E is extremely non-Hilbert.

(b) The space $c_0(\mathbb{N}; E)$ is not extremely non-Hilbert.

Proof. Indeed, let $E = \mathbb{R}^2$ and let the norm $\|\cdot\|$ on $\mathbb{R}^2$ be such that the unit sphere $S \subseteq \mathbb{R}^2$ is given by

$$\begin{aligned} S =& \{(\cos\theta, \sin\theta) : \theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]\} \\ &\cup \{(x_1, -\operatorname{sgn} x_1) : x_1 \in [-1, 0) \cup (0, 1]\} \\ &\cup \{(-\operatorname{sgn} x_2, x_2) : x_2 \in [-1, 0) \cup (0, 1]\}. \end{aligned}$$

The set S can be seen in Figure 1.1. Clearly, S is the boundary of a closed convex set $K \subseteq \mathbb{R}^2$ which contains 0 and which fulfils $-K = K$. Hence, there is indeed a norm on $\mathbb{R}^2$ such that K is the closed unit ball with respect to this norm and such that S is the unit sphere with respect to this norm.

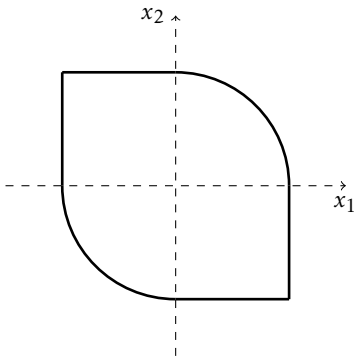

Figure 1.1: The unit sphere $S \subseteq \mathbb{R}^2$.

Let us now show the two assertions claimed above:

(a) This is obvious since S is not an ellipse.

(b) Let the set $\{1,2\}$ be endowed with the discrete topology. We show that there is even a two-dimensional Hilbert space isometrically contained in $F := \mathcal{C}(\{1,2\};E)$. To this end, let $f \in F$ be given by $f(1) = \begin{pmatrix}1\\0\end{pmatrix}$ and $f(2) = \begin{pmatrix}0\\-1\end{pmatrix}$ and let $g \in F$ be given by $g(1) = \begin{pmatrix}0\\1\end{pmatrix}$ and $g(2) = \begin{pmatrix}1\\0\end{pmatrix}$. We show that $\|\cos\theta f + \sin\theta g\| = 1$ for all $\theta \in [0,2\pi)$. So let $\theta \in [0,2\pi)$. Indeed, we have

$$\big(\cos\theta f + \sin\theta g\big)(1) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad \text{and} \quad \big(\cos\theta f + \sin\theta g\big)(2) = \begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}.$$

Hence,

$$\Big\|\big(\cos\theta f + \sin\theta g\big)(1)\Big\|_E \begin{cases} = 1 & \text{if } \theta \in [0,\frac{\pi}{2}] \cup [\pi,\frac{3\pi}{2}] \\ < 1 & \text{else} \end{cases}$$

and

$$\Big\|\big(\cos\theta f + \sin\theta g\big)(2)\Big\|_E \begin{cases} < 1 & \text{if } \theta \in (0,\frac{\pi}{2}) \cup (\pi,\frac{3\pi}{2}) \\ = 1 & \text{else.} \end{cases}$$

Since $\|\cos\theta f + \sin\theta g\|$ is the maximum of both those numbers, it follows that $\|\cos\theta f + \sin\theta g\| = 1$ for all $\theta \in [0,2\pi)$. Thus, the span of f and g is a two-dimensional Hilbert space. □

1.2 Projectively Non-Hilbert Spaces

We have seen in the previous section that many important function spaces are not extremely non-Hilbert. Yet, such spaces are, of course, much more

prevalent in analysis than those few sequence spaces which turned out to be indeed extremely non-Hilbert. We will thus show in the present section that, for instance, L^p-spaces on arbitrary measure spaces also fulfil some (weakened) geometric condition which can be employed in our spectral analysis in Chapter 2.

Recall that a closed vector subspace F of a Banach space E is called *complemented* if there exists a projection $P \in \mathcal{L}(E)$ with range F. The subspace F is called 1-*complemented* if exists a contractive projection $P \in \mathcal{L}(E)$ with range F.

Definition 1.2.1 (Projectively non-Hilbert space). A real Banach space E is called *projectively non-Hilbert* if it does not contain a 1-complemented two-dimensional subspace which is isometrically isomorphic to a Hilbert space.

The concept of a projectively non-Hilbert space goes back to a paper of Lyubich [Lyu70] (although he did not use this terminology) who found out that those spaces constitute the right setting for the spectral analysis of finite-dimensional and compact contractive operators. Lyubich himself was motivated by earlier work of Krasnosel'skiĭ [Kra68] who considered a more specific class of Banach spaces which he named *completely non-Hilbert*.

Let us explicitly note the following simple but important observation.

Remark 1.2.2. If a real Banach space E is projectively non-Hilbert and $P \in \mathcal{L}(E)$ is a contractive projection, then its range PE is projectively non-Hilbert, too.

The class of projectively non-Hilbert spaces is actually much larger than the class of extremely non-Hilbert spaces. This is illustrated by the subsequent example as well as by Examples 1.2.7 below.

Example 1.2.3. Let $p \in [1,\infty) \setminus \{2\}$ and let (Ω, μ) be an arbitrary measure space. Then $L^p(\Omega, \mu; \mathbb{R})$ is projectively non-Hilbert.

Proof. Let $V \subseteq L^p(\Omega, \mu; \mathbb{R})$ be a two-dimensional vector subspace and let P be a contractive projection on $L^p(\Omega, \mu; \mathbb{R})$ with range V. As shown in [Tza69, Theorem 6], the range of a contractive projection in $L^p(\Omega, \mu; \mathbb{R})$ is isometrically isomorphic to another L^p-space (with the same p). Hence, the Banach space V is isometrically isomorphic to $L^p(\tilde{\Omega}, \tilde{\mu}; \mathbb{R})$ for another measure space $(\tilde{\Omega}, \tilde{\mu})$. Since V is two-dimensional, it follows from the subsequent Remark 1.2.4 that V is not a Hilbert space. □

In the preceding proof we made use of the following simple observation:

Remark 1.2.4. Let $p \in [1,\infty) \setminus \{2\}$ and let (Ω, μ) be an arbitrary measure space. If $L^p(\Omega, \mu; \mathbb{R})$ has dimension ≥ 2, then it is not (isometrically) a Hilbert space.

Proof. Since $L^p(\Omega,\mu;\mathbb{R})$ has dimension at least 2, it contains two positive vectors f,g of norm 2 which are disjoint (in the sense of vector lattices). Hence, we have $2\cdot 2^p = \|f\|_p^p + \|g\|_p^p = \|f+g\|_p^p = \|f-g\|_p^p$. If $L^p(\Omega,\mu;\mathbb{R})$ were a Hilbert space, then the parallelogram identity would imply

$$2^4 = 2\|f\|_p^2 + 2\|g\|_p^2 = \|f+g\|_p^2 + \|f-g\|_p^2 = 2^{3+\frac{2}{p}},$$

which is a contradiction since $p \neq 2$. □

In Remark 1.1.4 we observed that extremely non-Hilbert spaces do not behave particularly well with respect to duality. Let us now prove that the situation is much better for projectively non-Hilbert spaces.

Proposition 1.2.5. *Let E be a real Banach space. If its dual space E' is projectively non-Hilbert, then so is E itself.*

In contrast to the above proposition, the author does currently not know whether the dual space of a projectively non-Hilbert space is always projectively non-Hilbert itself; compare however Proposition 1.3.3 and see also the notes at the end of the chapter for a brief discussion. For the proof of Proposition 1.2.5 we need the following elementary lemma:

Lemma 1.2.6. *Let E be a real or complex Banach space and let $P \in \mathcal{L}(E)$ be a projection. Then the mapping*

$$\begin{aligned} i : P'E' &\to (PE)' \\ x' &\mapsto x'|_{PE} \end{aligned}$$

is a contractive Banach space isomorphism. If P is contractive, then i is even isometric.

Proof. The mapping i is obviously linear and contractive. To show that i is injective, let $x' \in P'E'$ such that $i(x') = x'|_{PE} = 0$. For every $x \in E$ this implies

$$\langle x', x\rangle = \langle P'x', x\rangle = \langle x', Px\rangle = \langle x'|_{PE}, Px\rangle = 0.$$

Hence $x' = 0$, so i is indeed injective. To show that i is surjective, let $\tilde{x}' \in (PE)'$. Due to the Hahn-Banach theorem, $\tilde{x}'$ can be extended to a functional $x' \in E'$. For every $x \in PE$ we then have

$$\langle i(P'x'), x\rangle = \langle (P'x')|_{PE}, x\rangle = \langle P'x', x\rangle = \langle x', x\rangle = \langle \tilde{x}', x\rangle.$$

Hence, $i(P'x') = \tilde{x}'$, so $\tilde{x}'$ has a pre-image $P'x' \in P'E'$. This proves that i is surjective, and hence, i is a contractive Banach space isomorphism.

Assume now in addition that P is contractive. In order to show that i is isometric, let $x' \in P'E'$ and let $\varepsilon > 0$. There exists a vector $x \in E$ with norm 1 such that $|\langle x', x\rangle| \geq \|x'\| - \varepsilon$. The vector Px is contained in PE and we have

$$|\langle i(x'), Px\rangle| = |\langle x'|_{PE}, Px\rangle| = |\langle x', Px\rangle| = |\langle x', x\rangle| \geq \|x'\| - \varepsilon.$$

However, since P is contractive we know that $\|Px\| \leq \|x\| = 1$ and hence, $\|i(x')\| \geq \|x'\| - \varepsilon$. This shows that actually $\|i(x')\| = \|x'\|$. □

Proof of Proposition 1.2.5. Assume for a contradiction that there is a contractive projection $P \in \mathcal{L}(E)$ such that PE is a two-dimensional Hilbert space. Then the dual space $(PE)'$ is a two-dimensional Hilbert space, too. According to Lemma 1.2.6 the subspace $P'E' \subseteq E'$ is isometrically isomorphic to $(PE)'$; thus it is a two-dimensional Hilbert space. Since $P'E'$ is 1-complemented in E', this contradicts the assumption that E' is projectively non-Hilbert. □

Now we can use Proposition 1.2.5 to give some further examples of projectively non-Hilbert spaces.

Examples 1.2.7. (a) Let E be a real Banach lattice which is an AM-space (cf. [Sch74, Definition II.7.1]). Then E is projectively non-Hilbert.

(b) In particular, if K is a compact Hausdorff and L is a locally compact Hausdorff space, then $\mathcal{C}(K;\mathbb{R})$ and $\mathcal{C}_0(L;\mathbb{R})$ are projectively non-Hilbert.

(c) If (Ω, μ) is an arbitrary measure space, then $L^\infty(\Omega, \mu;\mathbb{R})$ is projectively non-Hilbert.

Proof. (a) If E is an AM-space, then its dual space E' is an AL-space [Sch74, Definition II.8.1 and Proposition II.9.1] and thus isometrically Banach lattice isomorphic to $L^1(\Omega, \mu;\mathbb{R})$ for some measure space (Ω, μ) [MN91, Theorem 2.7.1]. Thus, E' is projectively non-Hilbert according to Example 1.2.3. Proposition 1.2.5 now yields that E is projectively non-Hilbert itself.

(b) and (c) This follows immediately from (a) since $\mathcal{C}(K;\mathbb{R})$, $\mathcal{C}_0(L;\mathbb{R})$ and $L^\infty(\Omega, \mu;\mathbb{R})$ are AM-spaces. □

Let us close this section with two further, slightly more exotic, examples of extremely non-Hilbert spaces.

Examples 1.2.8. (a) Certain Orlicz sequence spaces which fulfil an appropriate technical condition are projectively non-Hilbert. This is discussed in [Ran04, the last paragraph of Remark 3.5].

(b) Let $p, q \in (1, \infty) \setminus \{2\}$. Then the vector-valued sequence space $\ell^p(\ell^q) := \ell^p(\mathbb{N};\ell^q(\mathbb{N};\mathbb{R}))$ is projectively non-Hilbert. This follows easily from [LRvG07, Theorem 5.1].

1.3 Super Projectively Non-Hilbert Spaces

We will see in our spectral analysis in Chapter 2 that the notions of extremely non-Hilbert and projectively non-Hilbert spaces are very useful if we intend to prove results on the points spectrum of (generators of) contraction semigroups. Yet, in order to obtain even results for the entire spectrum, we need to employ some kind of ultrapower technique, and to this end we now introduce the geometric notion of a *super projectively non-Hilbert space*; this notion is based on that of a projectively non-Hilbert space and it is well-adapted to the construction of ultrapowers.

Definition 1.3.1 (Super projectively non-Hilbert space). A real Banach space E is called *super projectively non-Hilbert* if for every ultra filter $\mathcal{U}$ on any index set $I \neq \emptyset$ the ultra power $E_{\mathcal{U}}$ is projectively non-Hilbert.

Just as the class of projectively non-Hilbert spaces, the class of super projectively non-Hilbert spaces is stable with respect to taking images of contractive projections:

Proposition 1.3.2. *Let E be a real Banach space and let $P \in \mathcal{L}(E)$ be a contractive projection. If E is super projectively non-Hilbert, then so is the range PE of P.*

Proof. Let $\mathcal{U}$ be an ultra filter on a set $I \neq \emptyset$; then $E_{\mathcal{U}}$ is projectively non-Hilbert and hence, the range $P_{\mathcal{U}}E_{\mathcal{U}}$ of the projection $P_{\mathcal{U}} \in \mathcal{L}(E_{\mathcal{U}})$ is projectively non-Hilbert, too. On the other hand, one easily checks that the Banach space $P_{\mathcal{U}}E_{\mathcal{U}}$ is isometrically isomorphic to the ultra power $(PE)_{\mathcal{U}}$. Thus, $(PE)_{\mathcal{U}}$ is projectively non-Hilbert, too. □

It is a natural question how the class of super projectively non-Hilbert spaces is related to the class of projectively non-Hilbert spaces. The following proposition answers this question.

Proposition 1.3.3. *If a real Banach E is super projectively non-Hilbert, then E, E' and E'' are projectively non-Hilbert.*

Proof. Suppose that E is projectively non-Hilbert. According to [Hei80, Proposition 6.7] we can find an index set $I \neq \emptyset$ and an ultra filter $\mathcal{U}$ on I with the following properties: there exists an isometric embedding $i : E'' \hookrightarrow E_{\mathcal{U}}$ and a contractive projection $P \in \mathcal{L}(E_{\mathcal{U}})$ with range $i(E'')$.

Since $E_{\mathcal{U}}$ is projectively non-Hilbert, so is $i(E'')$ and hence, so is E''. According to Proposition 1.2.5 this implies that E' and E are projectively non-Hilbert, too. □

For our purposes in Chapter 2 the most important example class of super projectively non-Hilbert spaces is constituted by reflexive L^p-spaces for $p \neq 2$;

yet, the following examples show that some further functions spaces fulfil this property, too.

Examples 1.3.4. (a) If (Ω, μ) is an arbitrary measure space and $p \in [1, \infty] \setminus \{2\}$, then $L^p(\Omega, \mu; \mathbb{R})$ is super projectively non-Hilbert.

(b) If E is a real Banach lattice which is an AM-space, then E is super projectively non-Hilbert. In particular, all spaces listed in Examples 1.2.7 are super projectively non-Hilbert.

Proof. (b) This follows from Example 1.2.7 (a) since every ultra power of an AM-space is again an AM-space.

(a) For $p < \infty$ every ultra power of an L^p-space is again an L^p-space, so the assertion follows from Example 1.2.3. For $p = \infty$ the assertion follows from (b). □

In order to obtain further examples, it would be desirable to have good sufficient criteria for a real Banach space to be super projectively non-Hilbert. Yet, it does not appear to be easy to derive such criteria. Here, we thus restrict ourselves to the negative criterion in the next proposition. In order to formulate the proposition, we introduce the following notation: for every two-dimensional real Banach space F with closed unit ball $\overline{B}$ we denote by $\delta(F)$ the number

$$\delta(F) := \sup_{x,y \in \overline{B}} \Big| \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \Big|. \tag{1.1}$$

Note that $\delta(F)$ is a measure of how far the vectors in F are from satisfying the parallelogram equality; we have $\delta(F) = 0$ if and only if F is (isometrically) a Hilbert space.

Proposition 1.3.5. *Let E be real Banach space and suppose that there exists a sequence of rank-2-projections $(P_n)_{n\in\mathbb{N}} \subseteq E$ such that*

$$\lim_{n\to\infty} \|P_n\| = 1 \qquad \textit{and} \qquad \lim_{n\to\infty} \delta(P_n E) = 0.$$

Then E is not super projectively non-Hilbert.

Proof. Let $\mathcal{U}$ be a free ultra filter on $\mathbb{N}$ and let $P_{\mathcal{U}} \in \mathcal{L}(E_{\mathcal{U}})$ be given by $P_{\mathcal{U}} x_{\mathcal{U}} = (P_n x_n)_{\mathcal{U}}$ for all $x_{\mathcal{U}} = (x_n)_{\mathcal{U}} \in E_{\mathcal{U}}$; note that $P_{\mathcal{U}}$ is well-defined since the sequence (P_n) is bounded with respect to the operator norm. Clearly, $P_{\mathcal{U}}$ is a projection and moreover, $P_{\mathcal{U}}$ is contractive since we have

$$\|P_{\mathcal{U}} x_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|P_n x_n\| \leq \lim_{\mathcal{U}} \|P_n\| \|x_n\| = \lim_{\mathcal{U}} \|P_n\| \lim_{\mathcal{U}} \|x_n\| = \|x_{\mathcal{U}}\|$$

for all $x_{\mathcal{U}} \in E_{\mathcal{U}}$.

Let us now show that the range $P_{\mathcal{U}}E_{\mathcal{U}}$ is two-dimensional: for each $n \in \mathbb{N}$ we can find, according to Riesz's lemma, two normalised vectors $x_n, y_n \in P_nE$ such that y_n has distance at last $\frac{1}{2}$ from the span of x_n. Now, let $x_{\mathcal{U}} := (x_n)_{\mathcal{U}}$ and $y_{\mathcal{U}} := (y_n)_{\mathcal{U}}$. Then $x_{\mathcal{U}}$ and $y_{\mathcal{U}}$ are both non-zero elements of the range $P_{\mathcal{U}}E_{\mathcal{U}}$, and for every scalar $\alpha \in \mathbb{R}$ we obtain

$$\|\alpha x_{\mathcal{U}} - y_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|\alpha x_n - y_n\| \geq \frac{1}{2}.$$

Hence, $x_{\mathcal{U}}$ and $y_{\mathcal{U}}$ are two linearly independent vectors in $P_{\mathcal{U}}E_{\mathcal{U}}$, so $\dim P_{\mathcal{U}}E_{\mathcal{U}} \geq 2$.

On the other hand, consider three vectors $x_{\mathcal{U}}^{(1)}, x_{\mathcal{U}}^{(2)}, x_{\mathcal{U}}^{(3)} \in P_{\mathcal{U}}E_{\mathcal{U}}$. For every $k \in \{1,2,3\}$ we can choose representatives $(x_n^{(k)})_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N};E)$ of $x_{\mathcal{U}}^{(k)}$ such that $x_n^{(k)}$ is contained in P_nE and has norm $\|x_{\mathcal{U}}^{(k)}\|$ for every $n \in \mathbb{N}$. Since P_nE has dimension 2 for every $n \in \mathbb{N}$, we can thus find scalars $\alpha_n^{(k)} \in \mathbb{R}$ (where $k \in \{1,2,3\}$ and $n \in \mathbb{N}$) with the following properties: we have $\sum_{k=1}^3 \alpha_n^{(k)} x_n^{(k)} = 0$ for each $n \in \mathbb{N}$ and, for every $n \in \mathbb{N}$, at least one of the three numbers $\alpha_n^{(1)}, \alpha_n^{(2)}, \alpha_n^{(3)}$ has modulus 1 and the other tfwo have modulus at most 1. We define $\alpha^{(k)} := \lim_{\mathcal{U}} \alpha_n^{(k)}$ for all $k \in \{1,2,3\}$. Note that at least one of those three numbers is not equal to 0 since we have

$$\sum_{k=1}^3 |\alpha^{(k)}| = \lim_{\mathcal{U}} \sum_{k=1}^3 |\alpha_n^{(k)}| \geq 1.$$

On the other hand, we have

$$\sum_{k=1}^3 \alpha^{(k)} x_{\mathcal{U}}^{(k)} = \Big(\sum_{k=1}^3 \alpha_n^{(k)} x_n^{(k)} \Big)_{\mathcal{U}} = 0,$$

so the vectors $x_{\mathcal{U}}^{(1)}, x_{\mathcal{U}}^{(2)}, x_{\mathcal{U}}^{(3)}$ are linearly dependent. This proves that actually $\dim P_{\mathcal{U}}E_{\mathcal{U}} = 2$.

Finally, we observe that $P_{\mathcal{U}}E_{\mathcal{U}}$ is (isometrically) a Hilbert space. Indeed, for all $x_{\mathcal{U}}, y_{\mathcal{U}} \in P_{\mathcal{U}}E_{\mathcal{U}}$ with $\|x_{\mathcal{U}}\|, \|y_{\mathcal{U}}\| \leq 1$ we can choose representatives $(x_n), (y_n) \in \ell^\infty(\mathbb{N};E)$ such that $x_n, y_n \in P_nE$ for all n and such that x_n and y_n have at most norm 1 for all n. Thus, we obtain

$$\begin{aligned} &\big| \|x_{\mathcal{U}} + y_{\mathcal{U}}\|^2 + \|x_{\mathcal{U}} - y_{\mathcal{U}}\|^2 - 2\|x_{\mathcal{U}}\|^2 - 2\|y_{\mathcal{U}}\|^2 \big| \\ &= \lim_{\mathcal{U}} \big| \|x_n + y_n\|^2 + \|x_n - y_n\|^2 - 2\|x_n\|^2 - 2\|y_n\|^2 \big| \leq \lim_{\mathcal{U}} \delta(P_nE) = 0. \end{aligned}$$

Hence, the parallelogram equality holds on $P_{\mathcal{U}}E_{\mathcal{U}}$ which proves that this space is in fact a Hilbert space. Thus, $E_{\mathcal{U}}$ is not projectively non-Hilbert and hence, E is not super projectively non-Hilbert. □

1.4 Notes & Open Problems

Permanence properties As observed in Remark 1.1.2, the class of extremely non-Hilbert spaces is, of course, invariant with respect to taking closed subspaces. Yet, we have noted in Remark 1.1.4 that the same class of Banach spaces is not stable with respect to taking dual or pre-dual spaces, so it is probably not very natural to expect other kinds of permanence properties from this class. On the other hand, the situation is a bit better for projectively non-Hilbert spaces; this class of Banach spaces is invariant with respect to taking pre-dual spaces as shown in Proposition 1.2.5. It is thus natural to ask whether further permanence properties are fulfilled by this class; let us formulate two special cases of this question as an open problem:

Problem 1.4.1. Let E be a real Banach space which is projectively non-Hilbert.

(a) Is it always true that the dual space E' is also projectively non-Hilbert?

(b) Let $F \subseteq E$ be a closed vector subspace. Under which conditions is it true that the quotient space E/F of E is projectively non-Hilbert, too?

In connection with question (a) above, we recall from Proposition 1.3.3 that E' (and even E'') is always projectively non-Hilbert in case that E is even *super* projectively non-Hilbert.

Example spaces Our most prominent examples of extremely non-Hilbert and projectively non-Hilbert spaces are certain L^p-spaces and spaces of continuous functions. The fact that $\ell^p(\mathbb{N};\mathbb{R})$ is extremely non-Hilbert if $p \in [1,\infty)\setminus 2\mathbb{N}$ (from which we derived Example 1.1.3(a)) is based on a thorough analysis of isometric properties of subspaces of L^p in [DJP98]. For finite-dimensional ℓ^p-spaces the same result was proved much earlier by Lyubich [Lyu70, Proposition 1] by a more direct computation. For more information about the converse fact that ℓ^p contains a two-dimensional Hilbert space if $p \in 2\mathbb{N}$ we refer for instance to [Lyu70, the proof of Proposition 2], [LV93] and [Kön04].

The fact that every L^p-space for $p \in [1,\infty) \setminus \{2\}$ is projectively non-Hilbert is based on a general description of the ranges of contractive projections in those spaces, compare the proof of Example 1.2.3. There, we referred to a paper of Tzafriri [Tza69] where such a description is given in L^p-spaces over general measures spaces. Earlier results in this direction had been obtained by Grothendieck [Gro55] and Douglas [Dou65] on L^1-spaces; in the special case of σ-finite measure spaces, Tzafriri's result had been proved by Ando [And66, Theorem 4] a few years earlier. We point out that in order to prove that every $L^p(\Omega,\mu;\mathbb{R})$-space is projectively non-Hilbert it would suffice to use Ando's result even if our underlying measure space (Ω,μ) is non-σ-finite. Indeed, if P is a contractive rank-2 projection on our L^p-space, then we

can find a σ-finite measurable subset $\tilde{\Omega} \subseteq \Omega$ such that all functions within the range of P have a representative which is supported on $\tilde{\Omega}$. Since the range of P coincides with the range of the restriction $P|_{L^p(\tilde{\Omega},\mu;\mathbb{R})}$ (and since this restriction is a contractive projection on $L^p(\tilde{\Omega},\mu;\mathbb{R})$), it follows from Ando's result that the range of P is isometrically isomorphic to some other L^p-space.

In order to obtain further examples of projectively non-Hilbert spaces, it would be interesting to understand the structure of contractive projections in other classes of Banach spaces. One interesting result in this context asserts, for instance, that the Hardy spaces $H_p(\mathbb{T})$, for $p \in [1,\infty) \setminus \{2\}$, do not admit any contractive projections whose range is finite-dimensional but not one-dimensional [LRR05, Theorem 3.1]; unfortunately, those spaces do, nevertheless, not constitute examples of projectively non-Hilbert spaces since they are defined over the complex scalar field and, as we shall see in Example 2.1.7(b) below, all the applications that we have in mind break down if we do not restrict ourselves to real Banach spaces.

An overview of many interesting results about contractive projections in Banach spaces can be found in [Ran01]; yet, it seems that even until today many questions still remain open. Let us discuss a few specific examples of them which are important in our context. In [Ran04, last paragraph of Remark 3.5] the question was raised under what conditions an Orlicz sequence space is projectively non-Hilbert. Moreover, a result by Goldstein [Gol73, the theorem on p. 390] (compare also [FGJ76] where a slight inaccuracy in Goldstein's paper was corrected), which is related to our results in Section 2.1 below, also suggests that Orlicz sequence spaces should have some kind of geometric property related to the notions discussed in this chapter. Thus, the following question is particularly interesting in our setting:

Problem 1.4.2. Under which conditions is an Orlicz sequence space projectively non-Hilbert? Under which conditions is it even extremely non-Hilbert?

An answer to the first question was conjectured, but not proved, in [Ran04, last paragraph of Remark 3.5].

A look at Example 1.2.8(b) immediately raises the question whether more general L^q-valued L^p-spaces are projectively non-Hilbert, too. Let us state this question explicitly:

Problem 1.4.3. Let (Ω_1,μ_1) and (Ω_2,μ_2) be measure spaces and let $p,q \in (1,\infty) \setminus \{2\}$. Consider the L^q-valued L^p-space $E := L^p(\Omega_1,\mu_1;L^q(\Omega_2,\mu_2;\mathbb{R}))$. Is E always projectively non-Hilbert?

We point out that a description of the range of *positive* contractive projections on $L^p(L^q)$-spaces is known [HR07, Theorem 2.8]; however, in order to

prove that Problem 1.4.3 has an affirmative answer we would need a description of the range of arbitrary contractive projections on such spaces.

Note that if the answer to the above problem is "yes", then it follows that the space $E := L^p([0,1]; L^q([0,1]; \mathbb{R}))$ is even super projectively non-Hilbert. Indeed, it is known that every ultra power $E_{\mathcal{U}}$ of E is isometric to a band B in another $L^p(L^q)$-space F, see e.g. [HLR91, Corollary 8.8]. As B is a band in the order complete Banach lattice F, there exists a band projection $P \in \mathcal{L}(F)$ with range B [MN91, Theorem 1.2.9]; since every band projection on a Banach lattice is contractive, so is P. Now assume that the answer to Problem 1.4.3 is positive; then F is projectively non-Hilbert and hence, so are B and $E_{\mathcal{U}}$.

2

Spectrum and Asymptotics of Contraction Semigroups

Now we employ the geometric notions introduced in the previous chapter for the study of contraction semigroups. The main thrust of our argumentation is, in all four sections of this chapter, the same: we consider a real Banach space E which has a certain geometric property discussed in Chapter 1, and a contractive C_0-semigroup $(e^{tA})_{t\geq 0}$. Under appropriate conditions on the semigroup (which depend on the strength of the geometric assumption on E) we then derive that the (point) spectrum of E intersects the imaginary axis $i\mathbb{R}$ at most in $\{0\}$. Afterwards we use general theorems which relate the spectral properties of the generator to the long-term behaviour of the semigroup in order to obtain convergence theorems for $(e^{tA})_{t\geq 0}$.

In the first section we consider semigroups on extremely non-Hilbert spaces. Since this geometric assumption is rather strong, we obtain results on the point spectrum of the semigroup generator under very general assumptions on the semigroup; the price for this generality is that only a few Banach spaces fulfil the assumption of being extremely non-Hilbert, compare Section 1.1. In Section 2.2 we widen the range of admissible Banach spaces by considering projectively non-Hilbert spaces, at the cost of an a priori condition on the point spectrum of the semigroup. In Section 2.3 we consider the special case of L^p-spaces; we drop the a priori condition on the point spectrum again and replace it with additional assumptions on the semigroup, one of them requiring that at least one semigroup operator e^{tA} be a *regular kernel operator* or, more generally, a so-called *AM-compact* operator.

In each of the first three sections we need some kind of compactness condition to turn our results on the point spectrum into convergence results. This is different in the last section of the chapter, where we derive results on the spectrum rather than the point spectrum. By a special case of the ABLP theorem, this yields strong convergence of the semigroup without any compactness assumption. On the other hand, our results in this section are only valid on super-projectively non-Hilbert spaces, and only under a few additional assumptions.

The material presented in Sections 2.1 and 2.2 stems from [Glü16d]. The material in Section 2.4 was inspired by [Glü16d, Section 4]; however, the main result of Section 2.4, Theorem 2.4.1, needs only weaker assumptions on the semigroup than the main result in [Glü16d, Section 4] and we give a rather different proof here. Both, the theorem in its full generality and its proof presented below, have not been published yet. The material in Section 2.3 is

also new and has not been published, yet; we point out that the content of this section was strongly inspired by results about the convergence of positive semigroups which have been obtained as part of an ongoing collaboration with Moritz Gerlach.

2.1 The Point Spectrum on Extremely Non-Hilbert Spaces

If $(e^{tA})_{t\geq 0}$ is a C_0-semigroup on a real Banach space E and the generator A has non-zero eigenvalues on the imaginary axis, then the orbits of the corresponding eigenvectors behave, in some sense, periodic. If, moreover, the underlying Banach space E is extremely non-Hilbert, this periodic behaviour conflicts with the geometry of the unit ball in E. This is the major idea on which the following theorem, the main result of this section, is based.

Theorem 2.1.1. *Let E be a real Banach space which is extremely non-Hilbert and let $(e^{tA})_{t\geq 0}$ be a contrative C_0-semigroup on E. Then $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.*

Proof. Let $E_{\mathbb{C}}$ be a complexification of E and let $A_{\mathbb{C}}$ be the complex extension of A to $E_{\mathbb{C}}$. Assume for a contradiction that $i\beta \in \sigma_{\mathrm{pnt}}(A)$ for some $\beta \in \mathbb{R} \setminus \{0\}$ and that $z = x + iy \in E_{\mathbb{C}}$ (with $x, y \in E$) is a corresponding eigenvector. Then x and y are linearly independent (over $\mathbb{C}$) according to Proposition C.3.2(c); in particular, $x \neq 0$, so we may assume that $\|x\| = 1$. Let us compute the orbit of x under the semigroup: for every $t \in [0, \infty)$ we have

$$e^{tA}x = \operatorname{Re} e^{tA_{\mathbb{C}}}z = \operatorname{Re} e^{it\beta}z = \cos(\beta t)x - \sin(\beta t)y.$$

Hence, the orbit of x is periodic. Since the semigroup is contractive and $\|x\| = 1$, this implies that we have $\|e^{tA}x\| = 1$ for all $t \geq 0$. Thus, $\|\cos(\beta t)x - \sin(\beta t)y\| = 1$ for all $t \geq 0$, which shows that the span of x and y over $\mathbb{R}$ is isometrically isomorphic to the two-dimensional Hilbert space $(\mathbb{R}^2, \|\cdot\|_2)$. This is a contradiction since E is assumed to be extremely non-Hilbert. □

It follows from Examples 1.1.5(c) and 1.1.3(a) (and that fact that a closed subspace of an extremely non-Hilbert space is extremely non-Hilbert itself) that the Banach spaces $(\mathbb{R}^d, \|\cdot\|_\infty)$ and $(\mathbb{R}^d, \|\cdot\|_1)$ are both extremely non-Hilbert. Let us briefly discuss the meaning of Theorem 2.1.1 on these spaces:

Remark 2.1.2. Let $A \in \mathbb{R}^{d\times d}$.

(a) The semigroup $(e^{tA})_{t\geq 0}$ is contractive on $(\mathbb{R}^d, \|\cdot\|_\infty)$ if and only if we have $A_{kk} \leq -\sum_{j\in\{1,\dots,d\}\setminus\{k\}} |A_{kj}|$ for all $k \in \{1, \dots, d\}$.

In this case we can also use Gershgorin's circle theorem [HJ90, Theorem 6.1.1] instead of Theorem 2.1.1 to conclude that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.

(b) The semigroup $(e^{tA})_{t\geq 0}$ is contractive on $(\mathbb{R}^d, \|\cdot\|_1)$ if and only if we have $A_{kk} \leq -\sum_{j\in\{1,\dots,d\}\setminus\{k\}} |A_{jk}|$ for all $k \in \{1,\dots,d\}$.

Again, we can use either Gershgorin's circle theorem [HJ90, Corollary 6.1.3] or Theorem 2.1.1 to conclude that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.

Proof. (a) If the entries of A satisfy the estimates in (a), then one can readily check that A is dissipative with respect to the norm $\|\cdot\|_\infty$ and hence, $(e^{tA})_{t\geq 0}$ is contractive. On the other hand, if $(e^{tA})_{t\geq 0}$ is contractive, then A is strictly dissipative. Now, let $k \in \{1,\dots,d\}$; define $x \in \mathbb{R}^d$ by $x_k = 1$ and $x_j = \operatorname{sgn} A_{kj}$ for all $j \in \{1,\dots,d\} \setminus \{k\}$. Then the k-th canonical unit vector e_k, considered as an element of $(\mathbb{R}^d, \|\cdot\|_1)$, is a functional on $\mathbb{R}^d$ with norm 1 which fulfils $\langle e_k, x\rangle = 1 = \|x\|_\infty$. Therefore we have

$$0 \geq \langle e_k, Ax\rangle = A_{kk} + \sum_{j\in\{1,\dots,d\}\setminus\{k\}} |A_{kj}|,$$

which proves the assertion.

(b) The semigroup $(e^{tA})_{t\geq 0}$ is contractive in $(\mathbb{R}^d, \|\cdot\|_1)$ if and only if the semigroup generated by the transposed of A is contractive in $(\mathbb{R}^d, \|\cdot\|_\infty)$, so the assertion follows from (a). □

We would like, for example, to apply Theorem 2.1.1 to ℓ^p-spaces; yet, we have noted in Example 1.1.3(b) that these spaces are not extremely non-Hilbert in case that $p \in 2\mathbb{N}$, so we cannot apply Theorem 2.1.1 directly. This is not surprising for $p = 2$ (compare Example 2.1.7(a) below), but for $p \in 2\mathbb{N} \setminus \{2\}$ we would like to have a similar result as Theorem 2.1.1. We can indeed achieve this by observing that, for every $p \in 2\mathbb{N} \setminus \{2\}$, the dual space of ℓ^p is extremely non-Hilbert since the number p' which is conjugate to p is contained in the interval $(1, 2)$. In situations like this we can apply the following corollary.

Corollary 2.1.3. *Let E be a reflexive real Banach space and suppose that E' is extremely non-Hilbert. If $(e^{tA})_{t\geq 0}$ is a contractive C_0-semigroup on E, then $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.*

Proof. Since E is reflexive, the adjoint operator A' generates a C_0-semigroup on E' which is given by $((e^{tA})')_{t\geq 0}$ and which is thus contractive. Hence, it follows from Theorem 2.1.1 that $\sigma_{\mathrm{pnt}}(A') \cap i\mathbb{R} \subseteq \{0\}$. On the other hand, since the semigroup generated by A is bounded, we have $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \sigma_{\mathrm{pnt}}(A') \cap i\mathbb{R}$ [AB88, Lemma 2.3], which proves the assertion. □

In order to turn the spectral results above into convergence result, we need some kind of compactness assumption. Recall that a C_0-semigroup $(e^{tA})_{t\geq 0}$ on a real or complex Banach space E is said to have *relatively compact orbits* if the set $\{e^{tA}x : t \geq 0\}$ is relatively compact in E (with respect to the norm

topology) for every $x \in E$. In the literature, semigroups with relatively compact orbits are sometimes called *almost periodic* or, more precisely, *strongly almost periodic*. Note that if $(e^{tA})_{t\geq 0}$ is bounded and A (respectively, any complex extension of it) has compact resolvent, then $(e^{tA})_{t\geq 0}$ has relatively compact orbits [EN00, Corollary V.2.15]. For contractive semigroups with relatively compact orbits we obtain the following convergence result:

Corollary 2.1.4. *Suppose that the assumptions of Theorem 2.1.1 or of Corollary 2.1.3 are fulfilled. If $(e^{tA})_{t\geq 0}$ has relatively compact orbits, then e^{tA} converges strongly as $t \to \infty$.*

Proof. This follows from [ABHN11, Theorem 5.4.6] or from [EN00, Theorem V.2.14]. □

In case that the semigroup $(e^{tA})_{t\geq 0}$ is eventually compact, one can show even operator norm convergence instead of only strong convergence. Actually, this even holds on projectively non-Hilbert spaces, so we skip this to the next section, see Corollary 2.2.5 (a).

Let us now state the above results explicitly for ℓ^p-spaces.

Corollary 2.1.5. *Let $I \neq \emptyset$ be an arbitrary index set and let $p \in [1,\infty) \setminus \{2\}$. If $(e^{tA})_{t\geq 0}$ is a contractive C_0-semigroup on $\ell^p(I;\mathbb{R})$, then $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.*

If the semigroup has, in addition, has relatively compact orbits, then e^{tA} converges strongly as $t \to \infty$.

Proof. If p is not an even integer, then $\ell^p(I;\mathbb{R})$ is extremely non-Hilbert according to Example 1.1.3(a). On the other hand, if p is an even integer, then the conjugate index p' (given by $\frac{1}{p} + \frac{1}{p'} = 1$) is not an even integer, and hence the dual space $\ell^p(I;\mathbb{R})' \simeq \ell^{p'}(I;\mathbb{R})$ is projectively non-Hilbert.

Thus, the assertions follow from Theorem 2.1.1 and Corollaries 2.1.3 and 2.1.4. □

As already mentioned in the introduction of the thesis, there is an interesting connection between the above corollary and the spectral theory of positive semigroups:

Remark 2.1.6. In [Dav05, Theorem 9], Davies proved that if $(e^{tA})_{t\geq 0}$ is a contractive and *positive* C_0-semigroup on $\ell^p(\mathbb{N};\mathbb{R})$, where $p \in [1,\infty)$, then the point spectrum of A intersects the imaginary axis at most in 0. Our Corollary 2.1.5 above shows that the positivity assumption is not needed for this result, unless $p = 2$.

On the other hand, we should point out that, if the semigroup $(e^{tA})_{t\geq 0}$ is actually assumed to be positive, Davies' result can be considered as a special case of more general convergence theorems for positive semigroups; see the notes at the end of this chapter for more details.

We close this section with two examples, the first of them illustrating the special role of the condition $p \neq 2$ in Corollary 2.1.5 while the second one shows that it is essential in all our results that the underlying Banach space is real.

Examples 2.1.7. (a) To understand what role the geometry of the unit ball played for the results of this section, it is very instructive to have a look at the two-dimensional spaces $(\mathbb{R}^2, \|\cdot\|_p)$ where $p \in [1, \infty]$. The matrix

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

generates a rotation semigroup on $\mathbb{R}^2$, and this semigroup is clearly contractive in case that $p = 2$. Yet, we cannot apply Theorem 2.1.1 since $(\mathbb{R}^2, \|\cdot\|_2)$ is a Hilbert space and, indeed, the generator A has spectrum $\sigma(A) = \{-i, -i\}$.

If, on the other hand, $p \neq 2$, then $(\mathbb{R}^2, \|\cdot\|_p)$ is extremely non-Hilbert; this follows from Example 1.1.3(a) if $p \notin 2\mathbb{N} \cup \{\infty\}$, but its is even true for all $p \in \{4, 6, ...\} \cup \{\infty\}$ since the space under consideration is two-dimensional. The spectrum of A is of course still given by $\{-i, i\}$. This time we cannot apply Theorem 2.1.1 because the rotation semigroup $(e^{tA})_{t\geq 0}$ is not contractive with respect to the $\|\cdot\|_p$-norm due to the low symmetry of the unit ball of this norm.

(b) For all results in this section (as well as in the subsequent sections) it is essential that the Banach space under consideration is real. Indeed, endow $\mathbb{C}^d$ with an arbitrary norm and let $A \in \mathbb{C}^{d\times d}$ be the diagonal matrix whose diagonal entries are all equal to i. Then $(e^{tA})_{t\geq 0}$ is obviously contractive, but we have $\sigma(A) = \{i\}$.

On the other hand, we refer to [Lyu70, Theorem $3^{(c)}$ and the subsequent discussion] for some observations which are still true in the complex case.

2.2 The Point Spectrum on Projectively Non-Hilbert Spaces

One of the great disadvantages of our results in Section 2.1 is that we always needed to assume the underlying Banach space, or its dual, to be extremely non-Hilbert, a property which is only fulfilled by a few spaces. In order to widen the applicability of our results we now want to show that similar assertions for the spectrum of A are true on the much larger class of projectively non-Hilbert spaces, provided that the semigroup fulfils two additional

assumptions. The results in this section were inspired by two papers of Krasnosel'skiĭ [Kra68] and Lyubich [Lyu70] where the spectrum of contractive compact operators on certain types of Banach spaces was studied. We will use several techniques from [Lyu70] and adapt them to our setting; see below for detailed references.

The first additional assumption that we need is some a priori information about the spectrum; more precisely, we assume that the set $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ is already known to be bounded; this is, for example, always the case if the semigroup $(e^{tA})_{t\geq 0}$ is analytic or, more generally, eventually norm continuous; see [EN00, Theorem II.4.18].

The second assumption that we need is a weak compactness condition on the semigroup orbits. Recall that a C_0-semigroup $(e^{tA})_{t\geq 0}$ on a real or complex Banach space E is said to have *relatively weakly compact orbits* if the set $\{e^{tA}x : t \geq 0\}$ is relatively compact with respect to the weak topology in E for every $x \in E$. In the literature, semigroups which have relatively weakly compact orbits are sometimes called *weakly almost periodic*. Note that a bounded C_0-semigroup on a reflexive Banach space automatically has relatively weakly compact orbits. Moreover, a C_0-semigroup which has relatively compact orbits has also relatively weakly compact orbits.

Let us now state the main result of this section.

Theorem 2.2.1. *Let E be a real Banach space which is projectively non-Hilbert. Let $(e^{tA})_{t\geq 0}$ be a contractive C_0-semigroup on E which has relatively weakly compact orbits. If $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ is bounded, then we actually have $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.*

Note that the condition on $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ to be bounded cannot be dropped in the above theorem as can be seen by considering the rotation semigroup on $L^p(\mathbb{T};\mathbb{R})$, where $p \in [1,\infty) \setminus \{2\}$: this semigroup is contractive and has relatively weakly compact orbits, but the point spectrum of its generator equals $i\mathbb{Z}$ (or a multiple of this set, depending on the speed of rotation).

For the proof of Theorem 2.2.1 we need a few auxiliary results. The first one, which is stated in the following lemma, is based on an idea of Lyubich from [Lyu70, the proof of Theorem 1]. In this paper, Lyubich studied projectively non-Hilbert spaces in finite dimensions, as well as contractive operators on them; he also briefly considered the case of compact operator in infinite dimensions. Although we are mainly interested in C_0-semigroups, we point out that one can also use our theorems to derive results for single operators; see the notes at the end of this chapter for details.

Lemma 2.2.2. *Let E be a projectively non-Hilbert space and let $A \in \mathcal{L}(E)$ be such that the semigroup $(e^{tA})_{t\geq 0}$ is contractive. If $E_{\mathbb{C}}$ is a complexification of E and $A_{\mathbb{C}} \in \mathcal{L}(E_{\mathbb{C}})$ is the complex extension of A to $E_{\mathbb{C}}$, then we cannot have $E_{\mathbb{C}} = \ker(i - A_{\mathbb{C}}) \oplus \ker(-i - A_{\mathbb{C}})$.*

Proof. Assume for a contradiction that $E_{\mathbb{C}} = \ker(i - A_{\mathbb{C}}) \oplus \ker(-i - A_{\mathbb{C}})$. This implies that the orbit of every $z \in E_{\mathbb{C}}$ under $(e^{tA_{\mathbb{C}}})_{t\geq 0}$ is periodic; in particular, $(e^{tA})_{t\geq 0}$ is isometric on E. Now, we define a binary operation $\circ : \mathbb{C} \times E_{\mathbb{C}} \to E_{\mathbb{C}}$ by

$$\lambda \circ (z_1 + z_2) = \lambda z_1 + \overline{\lambda} z_2$$

for all $\lambda \in \mathbb{C}$ and all $z_1 \in \ker(i - A_{\mathbb{C}})$, $z_2 \in \ker(-i - A_{\mathbb{C}})$. One readily checks that $(E_{\mathbb{C}}, +, \circ)$ is a complex vector space. Moreover, for all $r, t \geq 0$ and all $z_1 \in \ker(i - A_{\mathbb{C}})$, $z_2 \in \ker(-i - A_{\mathbb{C}})$ we have

$$(re^{it}) \circ z = re^{it} z_1 + re^{-it} z_2 = re^{tA_{\mathbb{C}}} z_1 + re^{tA_{\mathbb{C}}} z_2 = re^{tA_{\mathbb{C}}} z. \tag{2.1}$$

In particular it follows from (2.1) that $re^{it} \circ x = re^{tA} x \in E$ for all $r, t \geq 0$ and all $x \in E$, i.e. E is a *complex* vector subspace of $(E_{\mathbb{C}}, +, \circ)$.

Moreover, (2.1) implies that $\|re^{it} \circ x\| = \|re^{tA} x\| = |re^{it}| \|x\|$ for all $r, t \geq 0$ and all $x \in E$. Hence, the complex vector space $(E, +, \circ)$ is a normed vector space (and thus a Banach space) with respect to the norm $\|\cdot\|$. Now, choose an element $x_0 \in E$ with norm $\|x_0\| = 1$. By the Hahn-Banach theorem there is a contractive $\mathbb{C}$-linear (with respect to $\circ$) projection $P : E \to E$ with range $PE = \mathbb{C} \circ x_0$. Since we have $\alpha \circ x = \alpha x$ for all $\alpha \in \mathbb{R}$ and all $x \in E$ (i.e. for real scalars, the multiplication $\circ$ coincides with the original multiplication on E), the projection P is also $\mathbb{R}$-linear with respect to the original multiplication. Moreover, the vector space $(\mathbb{C} \circ x_0, +, \circ)$ is two-dimensional over the real field and, again since the restriction of $\circ$ to the real field coincides with the original multiplication on E, it follows that $\mathbb{C} \circ x_0$ is a two-dimensional vector subspace of the original space E which admits a contractive projection P.

On the other hand, $\mathbb{C} \circ x_0$ is a two-dimensional real Hilbert space since the mapping $\mathbb{C} \to \mathbb{C} \circ x_0$, $\lambda \mapsto \lambda \circ x_0$ is an isometric isomorphism between the two-dimensional real Banach spaces $\mathbb{C}$ (with the modulus as its norm) and $\mathbb{C} \circ x_0$. This contradicts the assumption that E be projectively non-Hilbert. □

The second auxiliary result which we need, Corollary 2.2.4 below, follows from the following elementary lemma.

Lemma 2.2.3. *Let E be a complex vector space and let $T : E \to E$ be a linear mapping. Moreover, let p by a complex polynomial of degree $n \in \mathbb{N}$ with n distinct roots $\lambda_1, ..., \lambda_n \in \mathbb{C}$. Then we have* $\ker p(T) = \oplus_{k=1}^{n} \ker(\lambda_k - T)$.

Proof. Since the kernel $\ker p(T)$ does not change if we multiply $p(T)$ by a complex number $\neq 0$, we may assume that the leading coefficient of p equals 1. Thus, we have $p(z) = \prod_{k=1}^{n} (z - \lambda_k)$ for all $z \in \mathbb{C}$.

Since the values $\lambda_1, ..., \lambda_n$ are all distinct from each other, it is easy to prove by induction over n that the sum $\oplus_{k=1}^{n} \ker(\lambda_k - T)$ is indeed direct.

The inclusion "$\supseteq$" is obvious. To prove the converse inclusion, let $x \in \ker p(T)$ and let V be the finite-dimensional space $V := \operatorname{span}\{x, Tx, ..., T^{n-1}x\}$. Since $p(T)x = 0$, we have $T^n x \in V$ and thus, the space V is invariant with respect to T. Moreover, we have $p(T|_V) = p(T)|_V = 0$. Hence, it follows from the spectral mapping theorem for polynomials that $\sigma(T|_V) \subseteq \{\lambda_1, ..., \lambda_n\}$.

Let us show that all eigenvalues of $T|_V$ are semi-simple. To this end, let $\lambda \in \sigma(T|_V) \subseteq \{\lambda_1, ..., \lambda_n\}$. According to Proposition A.2.3(b) we only have to show that there exists no generalised eigenvector of rank 2 for the eigenvalue λ of $T|_V$. So, let $v \in \ker\left((T|_V - \lambda)^2\right)$ and define a complex polynomial q by

$$q(z) = (z - \lambda) + \prod_{\mu \in \{\lambda_1, ..., \lambda_n\} \setminus \{\lambda\}} (z - \mu)$$

for all $z \in \mathbb{C}$. Since $p(T|_V)v = 0$, we obtain

$$q(T|_V)(T|_V - \lambda)v = (T|_V - \lambda)^2 v + p(T|_V)v = 0.$$

Moreover, the spectral mapping theorem yields that $\sigma(q(T|_V)) = q(\sigma(T|_V)) \subseteq \{q(\lambda_1), ..., q(\lambda_n)\}$. Since 0 is not contained in this set, the operator $q(T|_V) : V \to V$ is bijective, and we conclude that $(T|_V - \lambda)v = 0$. Hence, the eigenvalue λ of $T|_V$ has no generalised eigenvectors of rank 2 and thus, it is semi-simple.

Since all eigenvectors of $T|_V$ are semi-simple, it follows that V possesses a basis which consists of eigenvectors of $T|_V$. Thus, we have $x \in V = \oplus_{k=1}^n \ker(\lambda_k - T|_V) \subseteq \oplus_{k=1}^n \ker(\lambda_k - T)$. □

If in the situation of the above lemma the polynomial p has only $m < n$ distinct roots $\lambda_1, ..., \lambda_m$ (some of them with multiplicity ≥ 2), then we do not in general have $\ker p(T) = \oplus_{k=1}^m \ker(\lambda_k - T)$. To see this, simply choose $E = \mathbb{C}^2$, $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ and $p(z) = z^2$.

Corollary 2.2.4. *Let E be a complex vector space and let $T : E \to E$ be a linear operator. For every $\mu \in \mathbb{C} \setminus \{0\}$ we have* $\ker(\mu^2 - T^2) = \ker(\mu - T) \oplus \ker(-\mu - T)$.

Proof. Apply Lemma 2.2.3 to the polynomial $p(z) = \mu^2 - z^2$. □

Now we are ready to give the proof of Theorem 2.2.1. We have already mentioned above that one major ingredient of the proof, Lemma 2.2.2, is adapted from a paper of Lyubich [Lyu70]; another essential step in the following proof is the use of the mean ergodic theorem in order to obtain a contractive projection onto the subspace F; this idea was also employed in [Lyu70, the proof of Theorem 1]. For the convenience of the reader we recall that an operator $T \in \mathcal{L}(E)$ on a real or complex Banach space E is called *mean ergodic* if the sequence $(\frac{1}{n}\sum_{k=0}^{n-1} T^k)_{n\in\mathbb{N}}$ converges with respect to the strong operator topology. In this case, the limit operator $P \in \mathcal{L}(E)$ is a projection and it range coincides with the fixed space $\ker(1 - T)$ of T.

Proof of Theorem 2.2.1. Let $E_{\mathbb{C}}$ be a complexification of E and let $A_{\mathbb{C}}$ be the complex extension of A to $E_{\mathbb{C}}$. Assume for a contradiction that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \nsubseteq \{0\}$. Replacing A with cA for some $c > 0$ we may assume that $\pm i \in \sigma_{\mathrm{pnt}}(A)$ and that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq i \cdot (-3, 3)$ (here we used that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ is bounded).

Define $F_{\mathbb{C}} := \ker(-1 - e^{\pi A_{\mathbb{C}}})$, $F := F_{\mathbb{C}} \cap E$, $G_i := \ker(i - e^{\frac{\pi}{2} A_{\mathbb{C}}})$ and $G_{-i} := \ker(-i - e^{\frac{\pi}{2} A_{\mathbb{C}}})$. We are going to show that the spaces G_i, G_{-i}, $F_{\mathbb{C}}$ and F fulfil the following properties:

(a) $F_{\mathbb{C}} = G_i \oplus G_{-i}$.

(b) The real Banach space F is projectively non-Hilbert and $F_{\mathbb{C}}$ is a complexification of F.

(c) $G_i = \ker(i - A_{\mathbb{C}})$ and $G_{-i} = \ker(-i - A_{\mathbb{C}})$.

(a) This follows immediately from Corollary 2.2.4.

(b) Since the operator $e^{\pi A_{\mathbb{C}}}$ is real, we clearly have $F_{\mathbb{C}} = F + iF$; hence, $F_{\mathbb{C}}$ is a complexification of F. To see that F is projectively non-Hilbert, first note that the semigroup $(e^{tA_{\mathbb{C}}})_{t \geq 0}$ has relatively weakly compact orbits since the same is true for $(e^{tA})_{t \geq 0}$. Thus, the operator $-e^{\pi A_{\mathbb{C}}}$ is mean ergodic [EFHN15, Remark 16.17(4)]. Let $P_{\mathbb{C}} \in \mathcal{L}(E_{\mathbb{C}})$ denote the corresponding mean ergodic projection. Then the range of P coincides with the fixed space of $-e^{\pi A_{\mathbb{C}}}$, i.e. with $F_{\mathbb{C}}$. Since $P_{\mathbb{C}}$ is a real operator we have $PE = E \cap P_{\mathbb{C}} E_{\mathbb{C}} = F$, where $P := P_{\mathbb{C}}|_E$ denotes the restriction of $P_{\mathbb{C}}$ to E. Since P is the strong limit of the operator sequence $(\frac{1}{n} \sum_{k=0}^{n-1} (-e^{\pi A})^k)_{n \in \mathbb{N}}$ we conclude that P is contractive and hence, $F = PE$ is indeed projectively non-Hilbert.

(c) According to [EN00, Corollary IV.3.8] we have

$$\ker(e^{\mu t} - e^{tA_{\mathbb{C}}}) = \overline{\operatorname{span}}_{n \in \mathbb{Z}} \ker(\mu + \frac{2\pi i n}{t} - A_{\mathbb{C}})$$

for all $\mu \in \mathbb{C}$ and all $t \in (0, \infty)$. Plugging in $t = \frac{\pi}{2}$ and $\mu = \pm i$, we thus obtain

$$\begin{aligned} G_i &= \ker(e^{i\frac{\pi}{2}}) = \overline{\operatorname{span}}_{n \in \mathbb{Z}} \ker(i(4n+1) - A_{\mathbb{C}}) = \ker(i - A_{\mathbb{C}}), \\ G_{-i} &= \ker(e^{-i\frac{\pi}{2}}) = \overline{\operatorname{span}}_{n \in \mathbb{Z}} \ker(i(4n-1) - A_{\mathbb{C}}) = \ker(-i - A_{\mathbb{C}}); \end{aligned}$$

for the last equality in both lines we used that $\sigma_{\mathrm{pnt}}(A_{\mathbb{C}}) \subseteq i \cdot (-3, 3)$. Hence, (c) holds, too.

Let us now use (a), (b) and (c) to derive a contradiction. Since $F_{\mathbb{C}} = G_i \oplus G_{-i} = \ker(i - A_{\mathbb{C}}) \oplus \ker(-i - A_{\mathbb{C}})$, we have $F_{\mathbb{C}} \subseteq D(A_{\mathbb{C}})$; moreover, $A_{\mathbb{C}}$ leaves $F_{\mathbb{C}}$ invariant and we have $A_{\mathbb{C}}|_{F_{\mathbb{C}}} \in \mathcal{L}(F_{\mathbb{C}})$. Hence, $A|_F \in \mathcal{L}(F)$.

The semigroup $(e^{tA|_F})_{t \geq 0} = (e^{tA}|_F)_{t \geq 0}$ is, of course, contractive on the projectively non-Hilbert space F. Since $A_{\mathbb{C}}|_{F_{\mathbb{C}}}$ is the complex extension of $A|_F$ to the complexification $F_{\mathbb{C}}$ of F and since $F_{\mathbb{C}} = \ker(i - A_{\mathbb{C}}|_{F_{\mathbb{C}}}) \oplus \ker(-i - A_{\mathbb{C}}|_{F_{\mathbb{C}}})$, this contradicts Lemma 2.2.2. □

Now, we want to use Theorem 2.2.1 to derive convergence theorems for contractive semigroups. Depending on what kind of compactness assumption we impose on the semigroup, we obtain convergence with respect to the operator norm or with respect to the strong topology.

Recall that a C_0-semigroup $(e^{tA})_{t\geq 0}$ on a Banach space E is called *quasi-compact* if $\operatorname{dist}(e^{tA}, \mathcal{K}(E)) \to 0$ as $t \to \infty$; here the distance is computed in the Banach space $\mathcal{L}(E)$. Note that a C_0-semigroup $(e^{tA})_{t\geq 0}$ is quasi-compact if and only if the essential growth bound $\omega_{\mathrm{ess}}(A)$ is strictly negative, see [EN00, Proposition V.3.5].

Corollary 2.2.5. *Let E be a real Banach space which is projectively non-Hilbert and let $(e^{tA})_{t\geq 0}$ be a contractive C_0-semigroup on E.*

(a) *If $(e^{tA})_{t\geq 0}$ is eventually compact, then e^{tA} converges with respect to the operator norm topology as $t \to \infty$.*

(b) *More generally, if $(e^{tA})_{t\geq 0}$ is quasi-compact, then e^{tA} converges with respect to the operator norm topology as $t \to \infty$.*

(c) *If $(e^{tA})_{t\geq 0}$ has relatively compact orbits and $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ is bounded, then e^{tA} converges strongly as $t \to \infty$.*

Proof. (b) Let $A_{\mathbb{C}}$ be the complex extension of A to a complexification $E_{\mathbb{C}}$ of E. According to [EN00, Theorem V.3.7(i)] the set $\sigma(A) \cap i\mathbb{R}$ is finite or empty and all its elements (if any exist) are poles of the resolvent $\mathcal{R}(\cdot, A_{\mathbb{C}})$. Moreover, they are actually first order pole according to Propositions A.2.4(b) and A.3.2(d). Hence, it follows from [EN00, Theorem V.3.7(ii)] that $(e^{tA_{\mathbb{C}}})_{t\geq 0}$ has relatively compact orbits; in particular, the orbits are relatively weakly compact.

Theorem 2.2.1 now implies that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$. By applying [EN00, Corollary V.3.2(iii)], it follows that $e^{tA_{\mathbb{C}}}$ (and hence, e^{tA}) converges with respect to the operator norm as $t \to \infty$.

(a) This is a special case of (b).

(c) It follows from Theorem 2.2.1 that actually $\sigma_{\mathrm{pnt}}(A_{\mathbb{C}}) \subseteq \{0\}$. Hence, the assertion follows from [EN00, Theorem V.2.14] or [ABHN11, Theorem 5.4.6]. □

We stress once again that the condition in Corollary 2.2.5 (c) that $\sigma(A) \cap i\mathbb{R}$ be bounded is automatically fulfilled if $(e^{tA})_{t\geq 0}$ is eventually norm-continuous.

2.3 Semigroups of Kernel Operators

The price we had to pay for allowing projectively instead of extremely non-Hilbert spaces in Section 2.2 was that we had to require the set $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ a priori to be bounded (we do not consider the second assumption, relative

weak compactness of the semigroup orbits, to be particularly restrictive). This resembles to some extend a situation in Perron–Frobenius theory: there one knows, under rather general assumptions, that peripheral spectrum of the generator of a positive semigroup is cyclic; in order derive triviality of the peripheral spectrum on thus has to know for some reason that it is bounded (see [AGG+86, Section C-III.2] for some details).

However, there is another considerable result in the theory of positive semigroups where the situation is different: a theorem of Greiner [Gre82, Korollar 3.11] states that a contractive and positive C_0-semigroup on an L^p-space is strongly convergent, provided that it has a fixed point which is strictly positive almost everywhere and provided that one of the operators e^{tA} is a so-called *kernel operator*. The major point here is that we obtain convergence (and thus, in particular, triviality of $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$) without any boundedness assumptions on the spectrum, merely since one of the semigroup operators is a kernel operator.

It is our goal in this section to prove that a result quite similar to Greiner's theorem holds for contractive, non-positive semigroups on L^p-spaces if $p \in [1, \infty) \setminus \{2\}$. On the other hand, if one sticks to the positivity condition, then Greiner's result can be generalised in several other respects; see the notes for some more details. The reader might have noticed that the preceding discussion resembles, to some extend, the discussion in Remark 2.1.6 where we compared our results on ℓ^p-spaces with a result of Davies. This resemblance is not a coincidence, as we explain in the notes at the end of this chapter.

At the time of submission of this thesis, the material presented in this section has not been published in a research article (or in a preprint), yet. However, we point out once again that many of the ideas presented below, in particular the use of the Jacobs–deLeeuw–Glicksberg decomposition in this context and the utilisation of so-called *AM-compact* operators have been strongly inspired by an ongoing collaboration with Moritz Gerlach which aims, among other goals, to a generalisation of Greiner's above mentioned theorem. Moreover, the approach used below was, to some extend, inspired by a paper of Keicher [Kei06, in particular the proof of Corollary 3.8].

We first need to recall a few notions from Banach lattice theory. Let E be a real Banach lattice. An operator $T \in \mathcal{L}(E)$ is called *regular* if T can be written as the difference of two positive operators, i.e. if there exist operators $0 \leq T_1, T_2 \in \mathcal{L}(E)$ such that $T = T_1 - T_2$. We denote the vector subspace of $\mathcal{L}(E)$ consisting of all regular operators by $\mathcal{L}^{\mathrm{reg}}(E)$. When ordered by the cone of positive operators, $\mathcal{L}^{\mathrm{reg}}(E)$ clearly becomes an ordered vector space with generating cone; if we assume, in addition, that E is order complete, then $\mathcal{L}^{\mathrm{reg}}(E)$ can be shown to be an order complete vector lattice (and even a Banach lattice with respect to an appropriate norm), see [Sch74, Proposition IV.1.4]. This fact is used in part (i) of the following definition.

Definition 2.3.1 (Kernel operator and AM-compact operator). Let E be an order complete real Banach lattice and let $T \in \mathcal{L}(E)$.

(i) The operator T is called a *regular kernel operator* if T is regular and, moreover, contained in the band generated by the finite rank operators in $\mathcal{L}^{\mathrm{reg}}(E)$.

(ii) The operator T is called *AM-compact* if it maps every order interval in E to a relatively compact set.

We refer for instance to [Sch74, Section IV.1] for more details about regular operators, to [Sch74, Section IV.9], [MN91, Section 3.3 and Corollary 3.7.3] and [Ger14, Sections 4.1 and 4.2] for more information about kernel operators and to [MN91, Section 3.7] for details about AM-compact operators.

As pointed out above, we are particularly interested in kernel operators on L^p-spaces. Their description in the following example shows that this class of operators is indeed a very important (and well-known) one.

Example 2.3.2. Let (Ω, μ) be a σ-finite measure space and let $p \in [1, \infty)$. An operator $T \in \mathcal{L}(L^p(\Omega, \mu; \mathbb{R}))$ is a regular kernel operator if and only if there exists a measurable function $k : \Omega \times \Omega \to \mathbb{R}$ with the following properties:

(a) For each $f \in L^p(\Omega, \mu; \mathbb{R})$ the function $k(\cdot, \omega) f(\cdot)$ is measurable and contained in $L^1(\Omega, \mu; \mathbb{R})$ for almost all $\omega \in \Omega$ and the function

$$\int_\Omega |k(\omega, \cdot) f(\omega)| \, d\mu(\omega)$$

is contained in $L^p(\Omega, \mu; \mathbb{R})$.

(b) We have $Tf = \int_\Omega k(\omega, \cdot) f(\omega) \, d\mu(\omega)$ for every $f \in L^p(\Omega, \mu; \mathbb{R})$.

Proof. See e.g. the last assertion in Proposition IV.9.8 and the beginning of the examples in Section IV.8 of [Sch74]. □

Whenever we have to deal with regular kernel operators in proofs, it turns out that their definition is not very handy. In fact, it is another, more general property of regular kernel operators which, on the one hand, is very easy to use in proofs and, on the other hand, implies some interesting properties of them; namely, every regular kernel operator is AM-compact, at least on sufficiently regular Banach lattices. We make this precise in the following proposition.

Proposition 2.3.3. *Let E be a real Banach lattice with order continuous norm (e.g. $E = L^p(\Omega, \mu; \mathbb{R})$ for some measure space (Ω, μ) and $p \in [1, \infty)$). Then every regular kernel operator on E is AM-compact.*

Proof. This result can for instance be found in [MN91, Corollary 3.7.3]. □

Now we present the main result of this section.

Theorem 2.3.4. *Let (Ω, μ) be an arbitrary measure space and let $p \in [1, \infty) \setminus \{2\}$. Suppose that $(e^{tA})_{t\geq 0}$ is a contractive C_0-semigroup on $E = L^p(\Omega, \mu; \mathbb{R})$ which has relatively weakly compact orbits. If $e^{t_0 A}$ is AM-compact for some $t_0 \geq 0$, then $\sigma_{\text{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.*

Recall once again that the assumption on $(e^{tA})_{t\geq 0}$ to have relatively weakly compact orbits is automatically fulfilled if $p \neq 1$.

For the proof we recall that an element $a > 0$ of a Banach lattice E is called an *atom* if the principal ideal E_a is one-dimensional. The Banach lattice E is called *atomic* if there is no non-zero vector in E which is disjoint to all atoms (equivalently: the band generated by all atoms is the entire space E). If $1 \leq p < \infty$ and the space $L^p(\Omega, \mu; \mathbb{R})$ over some measure space (Ω, μ) is an atomic Banach lattice, then $L^p(\Omega, \mu; \mathbb{R})$ is isometrically Banach lattice isomorphic to $\ell^p(I; \mathbb{R})$ for some index set I. This is easy to see: choose I to be the set of all atoms of norm 1 and use for instance [Sch74, Exercise 7(b) in Chapter II].

Proof of Theorem 2.3.4. By means of the Jacobs-deLeeuw-Glicksberg decomposition (see for instance [Kre85, Section 2.4]) we can find a contractive projection $P \in \mathcal{L}(E)$ which commutes with all operators e^{tA} such that $(e^{tA}|_{PE})_{t\geq 0} = (e^{tA|_{PE}})_{PE}$ extends to a contractive C_0-group $(e^{tA|_{PE}})_{t\in\mathbb{R}}$ on PE. Moreover, the point spectrum of the restriction $A|_{PE}$ fulfils $\sigma_{\text{pnt}}(A|_{PE}) = \sigma_{\text{pnt}}(A) \cap i\mathbb{R}$.

Since the projection P is contractive, it follows from [BL74, Theorem 4.1] that there exists a closed sublattice F of E and a bijective isometry $U : PE \to F$. Since F is a sublattice of E, the norm on F is p-additive, so F is itself (isometrically Banach lattice isomorphic to) an L^p-space over some measure space.

Now, consider the C_0-group $(e^{tB})_{t\in\mathbb{R}}$ induced on F by $(e^{tA|_{PE}})_{t\in\mathbb{R}}$ via U, (i.e. $e^{tB} = Ue^{tA|_{PE}}U^{-1}$ for all $t \in \mathbb{R}$). The mapping $U^{-1} : F \to PE \subseteq E$ is an isometry from the L^p-space F to the L^p-space E and thus disjointness preserving since $p \neq 2$ [Fac14, Lemma 4.2.2]; hence, U^{-1} is order bounded according to [Are83, Theorem 2.5]. Now, let $0 \leq f \in F$ and consider the order interval $[0, f]_F$ in F. Then we have

$$[0, f]_F = e^{-t_0 B} e^{t_0 B} [0, f]_F = e^{-t_0 B} U e^{t_0 A} U^{-1} [0, f]_F.$$

Since U^{-1} is order bounded, we can find vectors g_1, g_2 in E such that $U^{-1}[0, f]_F$ is contained in the order interval $[g_1, g_2]_E$ in E. Since $e^{t_0 A}$ is AM-compact, it follows that $e^{t_0 A}[g_1, g_2]_E$ is relatively compact in E and thus, $e^{t_0 A} U^{-1}[0, f]_F \subseteq$

$e^{t_0 A}[g_1, g_2]_E \cap PE$ is relatively compact in PE. Hence, the set $[0, f]_F$ is relatively compact (and thus compact) in F.

We have thus proved that all order intervals in the Banach lattice F are compact. According to [AS87, Satz 1.1] this implies that the Banach lattice F is atomic. Since F is also an L^p-space, it follows that F is in fact isometrically Banach lattice isomorphic to $\ell^p(I;\mathbb{R})$ for some index set I. Thus, we conclude from Corollary 2.1.5 that $\sigma_{\text{pnt}}(B) \cap i\mathbb{R} \subseteq \{0\}$. However, we have $\sigma_{\text{pnt}}(A) \cap i\mathbb{R} = \sigma_{\text{pnt}}(A|_{PE}) \cap i\mathbb{R} = \sigma_{\text{pnt}}(B) \cap i\mathbb{R}$, so the assertion is proved. □

Let us briefly comment on the relation of the above theorem to our results in Section 2.1:

Remark 2.3.5. We derived Theorem 2.3.4 from the first assertion in Corollary 2.1.5. On the other hand, at least in the case $p \neq 1$, the first assertion of Corollary 2.1.5 is a special case of Theorem 2.3.4.

Indeed, let $I \neq \emptyset$ be an arbitrary index set, let $p \in (1,\infty) \setminus \{2\}$ and let $(e^{tA})_{t\geq 0}$ be a contractive C_0-semigroup on $\ell^p(I;\mathbb{R})$; note that this semigroup has relatively weakly compact orbits since $\ell^p(I;\mathbb{R})$ is reflexive. As every order interval in $\ell^p(I;\mathbb{R})$ is compact, it follows that every bounded linear operator on $\ell^p(I;\mathbb{R})$ is AM-compact. Hence, Theorem 2.3.4 implies that $\sigma_{\text{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.

If the orbits of our semigroup are even relatively compact with respect to the norm topology on $L^p(\Omega,\mu;\mathbb{R})$ (which is, for bounded semigroups, for instance fulfilled if the generator has compact resolvent), then we even obtain convergence in the situation of Theorem 2.3.4:

Corollary 2.3.6. *Let (Ω,μ) be an arbitrary measure space and let $p \in [1,\infty) \setminus \{2\}$. Suppose that $(e^{tA})_{t\geq 0}$ is a contractive C_0-semigroup on $E = L^p(\Omega,\mu;\mathbb{R})$ which has relatively compact orbits.*

If $e^{t_0 A}$ is AM-compact for some $t_0 \geq 0$, then e^{tA} converges strongly as $t \to \infty$.

Proof. This follows from Theorem 2.3.4 and e.g. from [EN00, Theorem V.2.14] or [ABHN11, Theorem 5.4.6]. □

This corollary is, of course, a generalisation of the second assertion in Corollary 2.1.5.

2.4 The Spectrum on Super Projectively Non-Hilbert Spaces

In each of the previous sections in this chapter we first proved a result on the point spectrum of the semigroup generator A; we then imposed additional compactness assumptions on the semigroup in order to obtain convergence

results. If we want to avoid such compactness assumptions and prove convergence nevertheless, then we need results on the spectrum rather than the point spectrum. Since by now we have only proved theorems about the point spectrum, a typical way to obtain information about the spectrum is to employ ultra power techniques. This is why we assume the underlying Banach space in this section to be super projectively non-Hilbert.

According to Examples 1.3.4 (a) the following theorem can, for instance, be applied to $L^p(\Omega,\mu;\mathbb{R})$ for an arbitrary measure space (Ω,μ) and $p\in(1,\infty)\setminus\{2\}$. In fact, those are the only spaces of which the author currently knows that they fulfil the assumptions of the theorem. Yet, we state the result in a more general version in case that further spaces can be identified as super projectively non-Hilbert in the future.

Theorem 2.4.1. *Let E be a real Banach space which is super reflexive and super projectively non-Hilbert. Let $(e^{tA})_{t\geq 0}$ be a contractive C_0-semigroup on E and assume that $\sigma(A)\cap i\mathbb{R}$ is bounded.*

Then we actually have $\sigma(A)\cap i\mathbb{R}\subseteq\{0\}$. In particular, e^{tA} converges strongly as $t\to\infty$.

Proof. Assume for a contradiction that $i\beta\in\sigma(A)$ for some $0\neq\beta\in\mathbb{R}$. Let $E_{\mathbb{C}}$ be a complexification of E and let $A_{\mathbb{C}}$ be the complex extension of A to $E_{\mathbb{C}}$. We fix a free ultra filter $\mathcal{U}$ on $\mathbb{N}$ and we note that the ultra power $(E_{\mathbb{C}})_{\mathcal{U}}$ is a complexification of the ultra power $E_{\mathcal{U}}$.

If we define $\mathcal{R}_{\mathcal{U}}(\lambda):=\mathcal{R}(\lambda,A_{\mathbb{C}})_{\mathcal{U}}$ for every $\lambda\in\rho(A)$, then $\mathcal{R}_{\mathcal{U}}(\cdot)$ is clearly a pseudo-resolvent on $(E_{\mathbb{C}})_{\mathcal{U}}$. Since this space is reflexive and since

$$\sup_{r\in(0,\infty)} r\|\mathcal{R}_{\mathcal{U}}(r)\|<\infty,$$

we can now find a bounded linear projection P on $(E_{\mathbb{C}})_{\mathcal{U}}$ with the properties described in Theorem A.4.5. It follows from those properties that P is the weak operator limit of $r\mathcal{R}_{\mathcal{U}}(r)$ as $r\to\infty$, so P is real and acts contractively on $E_{\mathcal{U}}$. In particular, $PE_{\mathcal{U}}$ is projectively non-Hilbert.

Let F denote the range of P and note that F is a complexification of $PE_{\mathcal{U}}$. Finally, the restriction of $\mathcal{R}_{\mathcal{U}}(\cdot)$ to F is the resolvent of a densely defined real linear operator $B:F\supseteq D(B)\to F$ with $\rho(B)\supseteq\rho(A_{\mathbb{C}})$. Since $r\mathcal{R}_{\mathcal{U}}(r)$ acts contractively on $E_{\mathcal{U}}$ (and thus on $PE_{\mathcal{U}}$), it follows that B generates a bounded C_0-semigroup on F which acts contractively on the projectively non-Hilbert space $PE_{\mathcal{U}}$. Hence, $\sigma_{\mathrm{pnt}}(B)\cap i\mathbb{R}\subseteq\{0\}$ according to Theorem 2.2.1.

On the other hand, it follows from Proposition A.4.7 that $\frac{1}{\lambda-i\beta}$ is an eigenvalue of $\mathcal{R}_{\mathcal{U}}(\lambda)$ for each $\lambda\in\rho(A_{\mathbb{C}})$. Hence, $i\beta$ is an eigenvalue of B, which is a contradiction.

We have thus proved that $\sigma(A)\cap i\mathbb{R}\subseteq\{0\}$. It now follows from [EN00, Exercise V.2.25 (4)(ii)] that e^{tA} converges strongly as $t\to\infty$. □

Let us briefly discuss the assumption on $\sigma(A) \cap i\mathbb{R}$ to be bounded in Theorem 2.4.1. Again, the rotation semigroup on $L^p(\mathbb{T};\mathbb{R})$ shows that this condition cannot in general be dropped. We point out once again that $\sigma(A) \cap i\mathbb{R}$ is automatically bounded if the semigroup $(e^{tA})_{t\geq 0}$ is eventually norm continuous, see [EN00, Theorem II.4.18]. More generally, $\sigma(A) \cap i\mathbb{R}$ is always bounded for semigroups which are *norm continuous at* ∞; see the notes at the end of this chapter for a brief discussion of this.

We close the section by restating Theorem 2.4.1 explicitly for the case of L^p-spaces.

Corollary 2.4.2. *Let (Ω, μ) be a measure space and let $p \in (1,\infty) \setminus \{2\}$. Let $(e^{tA})_{t\geq 0}$ be a contractive C_0-semigroup on $L^p(\Omega,\mu;\mathbb{R})$ and assume that $\sigma(A) \cap i\mathbb{R}$ is bounded.*

Then we actually have $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$. In particular, e^{tA} converges strongly as $t \to \infty$.

2.5 Notes & Open Problems

Asymptotically contractive semigroups Throughout the chapter we only considered contractive semigroups. However, it turns out that for most of our results a somewhat weaker conditions suffices. For example, let us call a C_0-semigroup $(e^{tA})_{t\geq 0}$ on a Banach space E *weakly asymptotically contractive* if $\limsup_{t\to\infty} |\langle x', e^{tA}x\rangle| \leq 1$ for all $x \in E$, $x' \in E'$ with $\|x\|, \|x'\| \leq 1$. It then turns out that our results in Sections 2.1 and 2.2 remain true for weakly asymptotically contractive instead of contractive semigroups; see [Glü16d, Sections 2 and 3] for details. On the other hand, the condition of weak asymptotic contractivity is not sufficient in Theorem 2.4.1, see [Glü16d, Example 4.9] for a counterexample. In case that the semigroup in Theorem 2.4.1 is eventually norm continuous we can, however, replace the contractivity assumption with the *uniform asymptotic contractivity* condition $\limsup_{t\to\infty} \|e^{tA}\| \leq 1$; cf. [Glü16d, Theorem 4.5].

We have chosen not to include those asymptotic contractivity assumptions in the main text since they complicate the presentation of some of the proofs (though only slightly) and since it is not clear how to check such a condition for a given semigroup (while contractivity can, for example, be checked by proving that the semigroup generator is dissipative). On the other hand, one possible motivation to consider such asymptotically contractive semigroups is provided by Part III of the thesis. There we consider what we call *eventually* and *asymptotically positive* semigroups; their natural analogue in the present part of the thesis would certainly be asymptotically contractive semigroups.

Regularity assumptions on the semigroup In some of our spectral results we required an additional regularity assumption. For instance, in Theorems 2.2.1 and 2.3.4 we assumed that the orbits of the semigroup be relatively compact with respect to the weak operator topology. Although this is not a particularly strong assumption (it is e.g. fulfilled for all bounded C_0-semigroups on reflexive spaces), it is a natural question whether this condition is really needed; let us formulate this problem explicitly:

Problem 2.5.1. Do Theorems 2.2.1 and 2.3.4 remain true without the assumption on the semigroup to have relatively weakly compact orbits?

We should point out that this question is mostly of theoretical interest. In fact, if we want to use the spectral theoretical assertions of those theorems to derive convergence results for the semigroup, we need even stronger compactness assumptions on the semigroup anyway (compare Corollaries 2.2.5 and 2.3.6).

We note in passing that the above question is also connected to the proof of the assertion $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$ in Theorem 2.4.1. In this theorem we assumed the underlying Banach space E to be super reflexive. This assumption, which implies that the ultra power $E_{\mathcal{U}}$ in the proof of the theorem is reflexive, was needed in order to employ Theorems 2.2.1 and A.4.5. So, if the answer to Problem 2.5.1 was positive, the application of Theorem 2.2.1 would no longer require E to be super reflexive. Yet, we would still need super reflexivity of E in order to use Theorem A.4.5 in the proof.

On the other hand, if we assume the semigroup $(e^{tA})_{t\geq 0}$ to be eventually norm continuous, then the assertion $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$ from Theorem A.4.5 remains true at least on L^1-spaces. The proof of this fact is based on a completely different approach and uses order theoretical properties of L^1-spaces; see [Glü16d, Remark 4.8(i)] for details. Hence, there seems to be some hope for the following problem to have a positive answer:

Problem 2.5.2. Does the assertion $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$ in Theorem 2.4.1 remain true without the assumption that E be super reflexive?

Of course, even if the answer to this question is "yes", one still needs some additional regularity assumption on the semigroup in order to conclude strong convergence from that fact that $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$. An assumption which is sufficient for this conclusion is, for instance, total ergodicity of the semigroup (this follows from [ABHN11, Theorem 5.5.5(a)]).

Kernel Operators and atomic Banach lattices The typical argument to derive triviality of the peripheral spectrum of a positive C_0-semigroup $(e^{tA})_{t\geq 0}$ on a Banach lattice works as follows: under rather weak assumptions it is

known that the peripheral spectrum $\sigma_{\mathrm{per}}(A)$ is cyclic; see [AGG$^+$86, Theorem C-III.2.10]. If we know for some reason that $\sigma_{\mathrm{per}}(A)$ is bounded, than it follows that $\sigma_{\mathrm{per}}(A) = \{\mathrm{s}(A)\}$. This argument is, to some extend, resembled by our Theorems 2.2.1 and 2.4.1 where we replaced positivity with contractivity, but where we also need to know *a priori* that $\sigma(A) \cap i\mathbb{R}$ (or $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$) is bounded in order to conclude that it is in fact trivial.

On the other hand, as pointed out at the beginning of Section 2.3, there are other results in the asymptotic theory of positive semigroups which do not need priori information about the spectrum. Instead, such theorems usefully require that the semigroup contains, in some sense, a kernel operator; from this assumption it can often be derived that $\sigma_{\mathrm{per,pnt}}(A) \subseteq \{\mathrm{s}(A)\}$. Apparently this was first observed by Greiner [Gre82, Korollar 3.11] who deduced such a result from a 0-2-law for positive semigroups. Later on, this result was rediscovered and varied in several respects. For instance, one can show that on an atomic Banach lattice with order continuous norm every positive operator is a kernel operator, see for instance [Ger14, Lemma 4.1.5]. Hence, one would expect similar results as Greiner's theorem to hold, under appropriate conditions, on atomic Banach lattices. This is indeed true and, during the last decade, it was in the focus of several papers by Davies [Dav05], Keicher [Kei06] and Wolf [Wol08]. Moreover, Greiner's theorem for semigroups containing a kernel operator was revisited in a paper by Arendt [Are08] and generalised in several respects by Gerlach [Ger13]. Very recently, it was proved by Gerlach and the author that similar results are true even without any continuity assumptions on the semigroup (a paper with this result is in preparation).

Our theorems in Sections 2.1 and 2.3 correspond to such results, where we replaced positivity of the semigroup with contractivity; in particular, Theorem 2.1.1 corresponds to results in atomic Banach lattices (since our most prominent examples of extremely non-Hilbert spaces are sequence spaces; compare also Remark 2.1.6) while Theorem 2.3.4 corresponds to those results which deal with semigroups containing a kernel operator on more general spaces.

Semigroups which are norm continuous at ∞ In order to apply Theorem 2.4.1 one needs to know that the set $\sigma(A) \cap i\mathbb{R}$ is bounded, so one might ask for simple sufficient conditions for this. As pointed out on several occasions in this chapter, a typical example of such a condition is eventual norm continuity of the semigroup. A slightly more general condition is *norm continuity at* ∞. This notion was introduced by Martinez and Mazon in [MM96, Definition 1.1], and they proved in [MM96, Theorem 1.9] that for a C_0-semigroup $(e^{tA})_{t\geq 0}$ which is norm continuous at ∞ there exists an $\varepsilon > 0$ such that the set $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq \mathrm{s}(A) - \varepsilon\}$ is bounded. In particular, the set $\sigma(A) \cap i\mathbb{R}$ is

bounded in case that $s(A) \leq 0$.

For concepts related to norm continuity at ∞ for also refer to the papers [NP00] and [Bla01].

Results for single operators All our results in this chapter are formulated for C_0-semigroups. However, many of them have analogues for contractive single operators. Actually, those operators were in the focus of Krasnosel'skiĭ's paper [Kra68] and Lyubich's paper [Lyu70] on which some of our techniques in this chapter are based. However, it seems that proofs run actually more smoothly for C_0-semigroups than for single operators. Moreover, one can use our theorems about C_0-semigroups to derive rather general results about the spectrum of single operators; see [Glü16d, Sections 2.3, 3.3 and 4.3] for details. Here, we only point out that, for a contractive operator T on real Banach space E which is, say, projectively non-Hilbert, one cannot expect that 1 is the only eigenvalue on the unit circle (consider the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$); yet, under appropriate assumptions, one can prove that all eigenvalues of T on the unit circle are roots of unity. For, say, compact operators T this implies that the powers of T behave asymptotically periodic. For related results was also refer to [LVG03].

Non-linear operators We note that, under sufficiently strong assumptions, one can even prove asymptotic results for iterates of *non-linear* contractive mappings which are similar to the results described in the previous paragraph. We refer the interested reader for instance to [Sin90] and [LvG09].

Isometries on L^p-spaces In the proof of Theorem 2.3.4 we used the fact that an isometry between two L^p-spaces is automatically disjointness perserving if $p \in [1, \infty) \setminus \{2\}$. We quoted this from [Fac14, Lemma 4.2.2], but we note that the result is essentially due to Lamperti [Lam58] who, however, considered only σ-finite measure spaces in this paper.

3

An Application to Delay Differential Equations

Now we want to give an application of our results from Section 2.2. Although the results from this section require rather strong compactness assumptions to obtain convergence, they have the advantage to be applicable on spaces of continuous functions.

One interesting class of equations which generate C_0-semigroups on spaces of continuous functions are certain delay differential equations, and those will be our topic in this chapter. Throughout, we restrict ourselves to finite-dimensional delay differential equations (which yield semigroups on infinite-dimensional phase spaces, though). We note in passing that it is also possible to model the phase space of such equations as an L^p-space over a time-interval (as is e.g. done in the monograph [BP05]), but here we stick to the approach which is presented for instance in [EN00, Section VI.6] where the phase space is assumed to be the space of continuous functions over a time interval.

In Section 3.1 we introduce the setting and we make some general observations about the long-term behaviour of the corresponding semigroups; these observations are mainly based on well-known regularity properties of the semigroups. Moreover, it is known that, if the semigroup generated by a delay differential equation is positive, one can give a good characterisation of whether the semigroup converges to 0 as $t \to \infty$. In Theorem 3.1.2 we briefly demonstrate that positivity is also very useful to characterise whether the trajectories of the semigroup converge to a (possibly non-zero) equilibrium. Afterwards, in Sections 3.2 and 3.3, we drop the assumption of positivity and assume that the semigroup is contractive, instead. This is not always the case, but we give some sufficient conditions to ensure that it is. In Section 3.2 we demonstrate, though, that this situation is not particularly interesting for *one*-dimensional delay differential equations. In fact if, in such a case, the corresponding semigroup is contractive but does not converge to 0, then it is automatically positive. This is no longer true in the higher-dimensional case which we study in Section 3.3. Here it can happen that the semigroup is contractive, but not positive, and we use our theory from Section 2.2 to show that the semigroup is automatically convergent despite its non-positivity in this case.

The results that we present in this chapter have not been published, or submitted for publication, yet.

3.1 Basics

Let us consider the time evolution of a real-valued vector $y(t) \in \mathbb{R}^d$ where the change rate of $y(t)$ depends on its values on the past time interval $[t-r, t]$ for some $r > 0$ and is given by the formula

$$\dot{y}(t) = Bu(t) + \Phi(y(t+\cdot)); \tag{3.1}$$

here, $B \in \mathbb{R}^{d\times d}$ is a fixed matrix and Φ is a fixed bounded linear operator from $\mathcal{C}([-r,0];\mathbb{R}^d)$ to $\mathbb{R}^d$. For such a *delay differential equation* in order to have a unique solution one would intuitively expect that we have to prescribe the values $y(t)$ for $t \in [-r, 0]$.

This is indeed true and it motivates the formulation of the delay differential equation (3.1) as an abstract Cauchy problem on the Banach space $E = \mathcal{C}([-r,0];\mathbb{R}^d)$. Here $\mathbb{R}^d$ can, for the moment, be endowed with any norm. The delay differential equation (3.1) can be rewritten as the abstract Cauchy problem

$$\begin{cases} \dot{f}(t) = Af(t) & \text{for } t \in [0,\infty) \\ f(0) = f_0 \end{cases}$$

on E, where $f(t)(s) = y(t+s)$ for all $s \in [-r, 0]$ and all $t \geq 0$ and where the operator $A : E \supseteq D(A) \to E$ is given by

$$\begin{aligned} D(A) &= \{f \in \mathcal{C}^1([-r,0];\mathbb{R}^d) : f'(0) = Bf(0) + \Phi f\}, \\ Af &= f'. \end{aligned} \tag{3.2}$$

For a derivation of this reformulation and for a precise statement of the relation between this abstract Cauchy problem and (3.1) we refer to [EN00, Section VI.6]. There it is also shown that the operator A generates a C_0-semigroup $(e^{tA})_{t\geq 0}$ on E. Moreover, in this reference the delay differential equation (3.1) is not only treated for values $y(t) \in \mathbb{R}^d$ (or $\mathbb{C}^d$), but, more generally, for values y in an arbitrary Banach space. Yet, it is not our intention here to achieve the greatest possible generality, but rather to demonstrate how our results from the previous chapter can, in principle, be applied. Hence, we restrict ourselves to the case where $y(t) \in \mathbb{R}^d$.

To understand the long-time behaviour of the solution of (3.1) we have to analyse the long-time behaviour of the C_0-semigroup $(e^{tA})_{t\geq 0}$. Since this semigroup is eventually compact according to [EN00, Proposition VI.6.9], the asymptotic behaviour of $(e^{tA})_{t\geq 0}$ is closely related to the spectrum of (the complex extension of) A. More precisely, we have the following characterisation which is true for every eventually compact C_0-semigroup, and which is a version of [EN00, Corollary V.3.3]; compare also [DvGVLW95, Section IV.2]

Proposition 3.1.1. *Let $(e^{tA})_{t\geq 0}$ be an eventually compact C_0-semigroup on a real or complex Banach space E. Then the following assertion are equivalent.*

(i) *e^{tA} converges with respect to the operator norm as $t \to \infty$.*

(ii) *e^{tA} converges strongly as $t \to \infty$.*

(iii) *The semigroup $(e^{tA})_{t\geq 0}$ is bounded and $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$.*

(iv) *We have $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{0\}$; moreover, 0 is either contained in the resolvent set of A, or it is a first order pole of the resolvent of (a complex extension of) A.*

Proof. We may assume throughout the proof that E is a complex Banach space; otherwise, we replace E with a complexification of E.

The implication "(i) $\Rightarrow$ (ii)" is obvious. Due to [EN00, Corollary V.3.2(i)], the spectrum of A consists of poles of the resolvent of $\mathcal{R}(\cdot, A)$ and hence of eigenvalues of A (cf. Proposition A.3.2(a)). Thus, the implication "(ii) $\Rightarrow$ (iii)" is clear, too.

"(iii) $\Rightarrow$ (iv)" If (iii) is fulfilled, then we clearly have $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{0\}$. Assume now that $0 \in \sigma(A)$. It follows from Proposition A.2.4(b) that 0 is a semi-simple eigenvalue of A and thus, the pole order of $\mathcal{R}(\cdot, A)$ at 0 equals 1 according to Proposition A.3.2(d).

"(iv) $\Rightarrow$ (i)" This follows from [EN00, Corollary V.3.2(iii)]. □

We would like to apply the above Proposition to the operator A on $E = \mathcal{C}([-r, 0]; \mathbb{R}^d)$ which is defined by (3.2). It is therefore of major interest to described the spectrum $\sigma(A)$ and this can be done by a certain characteristic equation. To this end, let $\lambda \in \mathbb{C}$ and $y \in \mathbb{C}$. We define $e_\lambda \in \mathcal{C}([-r, 0]; \mathbb{C})$ by $e_\lambda(s) := e^{\lambda s}$ and $e_\lambda \otimes y \in \mathcal{C}([-r, 0]; \mathbb{C})$ by $(e_\lambda \otimes y)(s) := e^{\lambda s} y$. Finally, let the matrix $\Phi_\lambda \in \mathbb{C}^{d\times d}$ be given by $\Phi_\lambda y := \Phi(e_\lambda \otimes y)$. Then the spectrum of A can be described by the condition

$$\lambda \in \sigma(A) \quad \Leftrightarrow \quad \lambda \in \sigma(B + \Phi_\lambda); \tag{3.3}$$

see [EN00, Proposition VI.6.7] for a proof. The condition $\lambda \in \sigma(B + \Phi_\lambda)$ is of course equivalent to the so-called *characteristic equation* $\det(\lambda - B - \Phi_\lambda) = 0$. The problem here is that it can be very difficult to determined whether a complex number λ fulfils this equation. Even if $d = 1$, this is equivalent to the equation $\lambda = B + \Phi(e_\lambda)$ which is, in general, not easy to solve. To get a good impression of the difficulties which can occur, we refer the reader to Example 8.4.3(a) in Section 8.4 and Example 11.6.3 in Section 11.6 where we prove by explicit estimates that $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{0\}$ for some concrete choices of Φ in the case $d = 1$. On the other hand, we will see in Theorem 8.4.2(a) that it is

possible to give rather abstract sufficient conditions for this property; yet, the aforementioned examples show that those conditions are far from optimal.

To obtain more generic results about the asymptotic behaviour of $(e^{tA})_{t\geq 0}$, one is therefore lead to the application of structure theoretical results which yield information about $\sigma(A)$ without considering the characteristic equation $\det(\lambda - B - \Phi_\lambda) = 0$. Such structure theoretical results can, for example, be theorems from Perron-Frobenius theory in case that the semigroup $(e^{tA})_{t\geq 0}$ is positive (note that E is a Banach lattice in case that the norm on $\mathbb{R}^d$ is chosen to be a Banach lattice norm). In the literature the spectral theory of positive semigroups is often used to give criteria for the operator e^{tA} to converge to 0 as $t \to \infty$; see e.g. [KN84], [KN88] and [EN00, Section VI.6(c)]. In the subsequent Theorems 3.1.2 and 3.1.3 we briefly demonstrate that Perron-Frobenius theory can also be used to prove convergence of the semigroup to a non-zero equilibrium.

Theorem 3.1.2. *Let $\mathbb{R}^d$ be endowed with a Banach lattice norm and endow $E = \mathcal{C}([-r,0];\mathbb{R}^d)$ with the canonical order (such that E becomes a Banach lattice). Assume that $\Phi \in \mathcal{L}(E;\mathbb{R}^d)$ is a positive operator and that $B \in \mathbb{R}^{d\times d}$ generates a positive C_0-semigroup on $\mathbb{R}^d$ (equivalently: that all off-diagonal entries of B are ≥ 0).*

For the operator A on E defined by (3.2)*, the following assertions are equivalent:*

(i) *e^{tA} converges strongly (equivalently: with respect to the operator norm) as $t \to \infty$.*

(ii) *$(e^{tA})_{t\geq 0}$ is bounded.*

Proof. Since $(e^{tA})_{t\geq 0}$ is eventually compact (see [EN00, Proposition VI.6.9]), it follows from Proposition 3.1.1 that strong convergence of e^{tA} is equivalent to convergence with respect to the operator norm.

"(i) $\Rightarrow$ (ii)" This follows from the uniform boundedness principle.

"(ii) $\Rightarrow$ (i)" According to Proposition 3.1.1(iii) we only have to show that $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cap \{0\}$ and since $(e^{tA})_{t\geq 0}$ is bounded, we clearly have $\operatorname{Re}\lambda \leq 0$ for all $\lambda \in \sigma(A)$. Assume now that $\sigma(A) \cap i\mathbb{R}$ is non-empty. Then $s(A) = 0$ and the resolvent $\mathcal{R}(\cdot, A)$ fulfils, due to the boundedness $(e^{tA})_{t\geq 0}$, the Abel-type boundedness condition $\sup_{\lambda\in(0,\infty)} \|\lambda\mathcal{R}(\lambda, A)\| < \infty$.

Moreover, it follows from the assumptions on B and Φ that the semigroup $(e^{tA})_{t\geq 0}$ is positive [EN00, Theorem VI.6.11]. Hence, we conclude from the Perron-Frobenius type theorem in [AGG+86, Theorem C-II-2.10] that the set $\sigma(A) \cap i\mathbb{R}$ is cyclic. On the other hand, the set $\sigma(A) \cap i\mathbb{R}$ is bounded since $(e^{tA})_{t\geq 0}$ is eventually compact [EN00, Corollary V.3.2(ii)]. Hence, we conclude that $\sigma(A) \cap i\mathbb{R}$ is contained in $\{0\}$. □

The situation is even simpler in case that $d = 1$:

Theorem 3.1.3. *Let $d = 1$ and let $E = C([-r, 0]; \mathbb{R})$ be endowed with the canonical order. Assume that $\Phi \in E'$ is a positive functional and let $B \in \mathbb{R}$.*

For the operator A on E defined by (3.2)*, the following assertions are equivalent:*

(i) *e^{tA} converges strongly (equivalently: with respect to the operator norm) as $t \to \infty$.*

(ii) *$(e^{tA})_{t\geq 0}$ is bounded.*

(iii) $s(A) \leq 0$.

(iv) $B + \langle \Phi, \mathbb{1}_{[-1,0]} \rangle \leq 0$.

Proof. "(i) $\Leftrightarrow$ (ii)" This was shown in Theorem 3.1.2.

"(ii) $\Rightarrow$ (iii)" This is obvious.

"(iii) $\Rightarrow$ (i)" If $s(A) < 0$, then the semigroup converges to 0 with respect to the operator norm (since it is eventually compact and thus eventually norm continuous). Now assume that $s(A) = 0$. Since the semigroup $(e^{tA})_{t\geq 0}$ is positive according to [EN00, Theorem VI.6.11], it follows that the spectral bound 0 is contained in $\sigma(A)$ [AGG$^+$86, Corollary C-III-1.4]. Again since the semigroup is eventually compact, we conclude that 0 is a pole of the resolvent [EN00, Corollary V.3.2(i)]. It thus follows from Perron–Frobenius spectral theory, see [AGG$^+$86, Corollary C-III-2.13] that $\sigma(A)$ intersects the imaginary axis only in 0. According to Proposition 3.1.1(iv) it thus suffices to show that 0 is a first order pole of the resolvent.

It readily follows from the definition of A that $\ker A$ consists of the constant functions. Moreover, there exists a positive functional $\varphi \in \ker A'$ according to [AGG$^+$86, Theorems B-III.1.6 and B-IV.1.1]. Hence, we conclude from Proposition 7.1.3 below that 0 is indeed a first order pole of the resolvent.

"(iii) $\Leftrightarrow$ (iv)" It follows readily from the definition of A that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $\lambda = B + \langle \Phi, e^{\lambda \cdot} \rangle$. Since the semigroup $(e^{tA})_{t\geq 0}$ is eventually compact and positive, $s(A)$ is either $-\infty$ or an eigenvalue of A. Hence, $s(A) > 0$ if and only A has an eigenvalue in $(0, \infty)$, and this is in turn fulfilled if and only if the mapping

$$\alpha : \mathbb{R} \to \mathbb{R}, \quad \alpha(\lambda) = B + \langle \Phi, e^{\lambda \cdot} \rangle - \lambda$$

has a zero on $(0, \infty)$. Since α is strictly decreasing on $\mathbb{R}$ and tends to $-\infty$ as $\lambda \to \infty$, this condition is fulfilled if and only if $\alpha(0) > 0$, which is equivalent to $B + \langle \Phi, \mathbb{1}_{[-1,0]} \rangle > 0$. □

In general it might, of course, still be difficult to show that $(e^{tA})_{t\geq 0}$ is bounded; however, if the semigroup is even contractive, then one can try to prove this by employing the Lumer-Phillips theorem. On the other hand, if $\mathbb{R}^d$ is even endowed with the ∞-norm $\|\cdot\|_\infty$, then the Banach space $E =$

$\mathcal{C}([-r,0];\mathbb{R}^d)$ is isometrically isomorphic to the space $\mathcal{C}([-r,0]\times\{1,...,d\};\mathbb{R})$, which is projectively non-Hilbert according to Example 1.2.7(b). Hence, if the semigroup $(e^{tA})_{t\geq 0}$ is contractive, then we can obtain convergence results even without any positivity assumptions. We will demonstrate this in the next two sections. In Section 3.2 we first consider the one-dimensional case $d = 1$, where we will encounter an interesting relation between contractivity and positivity of the semigroup $(e^{tA})_{t\geq 0}$. In Section 3.3 we analyse the general case $d \in \mathbb{N}$ under an additional assumption on the operator Φ.

3.2 Delay Differential Equations in 1 Dimension

We consider the situation from the previous section and assume, in addition, that $d = 1$. In this case we have $E = \mathcal{C}([-r,0];\mathbb{R})$ and $\Phi \in E'$, i.e. Φ can be considered as a regular Borel measure in $[-r,0]$. We may assume throughout this section that $B = 0$ because it does not change the delay differential equation (3.1) or the definition of the operator A in (3.2) if we add B times the Dirac measure δ_0 to Φ and then replace B with 0. We then have the following result.

Theorem 3.2.1. *Let $E = \mathcal{C}([-r,0];\mathbb{R})$, let $\Phi \in E'$ and define $A : E \supseteq D(A) \to E$ by*

$$\begin{aligned} D(A) &= \{f \in \mathcal{C}^1([-r,0];\mathbb{R}) : f'(0) = \langle \Phi, f\rangle\} \\ Af &= f'. \end{aligned}$$

The following assertions are true:

(a) *If the semigroup $(e^{tA})_{t\geq 0}$ is contractive, then e^{tA} converges with respect to the operator norm as $t \to \infty$.*

(b) *If the semigroup $(e^{tA})_{t\geq 0}$ is contractive and the limit operator $P := \lim_{t\to\infty} e^{tA}$ is non-zero, then the semigroup $(e^{tA})_{t\geq 0}$ is positive and P is a rank-1-operator.*

(c) *Consider Φ as a regular Borel measure on $[-r,0]$. If $\Phi(\{0\}) \leq -|\Phi|([-r,0))$. Then the semigroup $(e^{tA})_{t\geq 0}$ is contractive.*

Proof. (a) It follows from Example 1.2.7(b) that E is projectively non-Hilbert. Since the semigroup $(e^{tA})_{t\geq 0}$ is eventually compact and contractive, it follows from Corollary 2.2.5(a) that e^{tA} converges with respect to the operator norm as $t \to \infty$.

(b) The range PE of P consists of fixed points of the semigroup $(e^{tA})_{t\geq 0}$. If $PE \neq \{0\}$, we conclude that there exists an eigenvector v of A for the eigenvalue 0. Hence, $v \in \mathcal{C}^1([-r,0];\mathbb{R})$, $v'(0) = \langle \Phi, v\rangle$ and $v' = Av = 0$, from which we

conclude that $v = c\mathbb{1}$ for some $c \in \mathbb{R} \setminus \{0\}$. From $\langle \Phi, v \rangle = v'(0) = 0$ we thus obtain $\langle \Phi, \mathbb{1} \rangle = 0$.

Moreover, since $(e^{tA})_{t\geq 0}$ is contractive, the generator A is strictly dissipative, i.e. whenever $f \in D(A)$ and $\varphi \in E'$ such that $\|\varphi\| = 1$ and $\langle \varphi, f \rangle = \|f\|$, then $\langle \varphi, Af \rangle \leq 0$.

We now use this information to show that $(e^{tA})_{t\geq 0}$ is positive. To do so, it suffices to prove that A is dispersive in the sense of [AGG+86, p. 249]. To this end, let $f \in D(A)$ with $f^+ \neq 0$ and let $s_0 \in [-r, 0]$ be such that $f(s)$ is maximal on $[-r, 0]$ for $s = s_0$. Consider the Dirac delta functional $\delta_{s_0} \in E'$, which is given by the point evaluation at s_0, has norm 1 and fulfils $\langle \delta_{s_0}, f \rangle = \|f^+\|$. In order to show that A is dispersive, we have to prove that $\langle \delta_{s_0}, Af \rangle \leq 0$, which is equivalent to $f'(s_0) \leq 0$.

In order to do so, choose $\alpha \in [0, \infty)$ sufficiently large such that the function $g := f + \alpha\mathbb{1}$ is positive. We have $g \in D(A)$ since $\mathbb{1} \in D(A)$. Moreover, $\langle \delta_{s_0}, g \rangle = \|g\|$ since g is positive and attains its maximum at s_0. Using that A is strictly dissipative, this implies that

$$\langle \delta_{s_0}, Af \rangle = \langle \delta_{s_0}, Ag \rangle \leq 0.$$

Hence, we have shown that A is dispersive and thus, we conclude that the semigroup $(e^{tA})_{t\geq 0}$ is positive.

(c) It suffices to show that A is dissipative. To this end, let $f \in D(A)\setminus\{0\}$ and choose $s_0 \in [-r, 0]$ such that $|f(s)|$ is maximal on $[-r, 0]$ at $s = s_0$. Let $\alpha \in \{-1, 1\}$ denote the signum of $f(s_0)$ and let $\delta_{s_0} \in E'$ be the Dirac delta functional which is given by the point evaluation at s_0. Then $\alpha\delta_{s_0}$ has norm 1 and we have $\langle \alpha\delta_{s_0}, f \rangle = \|f\|$. We have to show that $\langle \alpha\delta_{s_0}, Af \rangle \leq 0$, and to this end, we may assume that $f(s_0) \geq 0$, since otherwise we can replace f with $-f$ and α with $-\alpha$. Thus, $\alpha = 1$ and we have to show that $f'(s_0) \leq 0$. To do so, we distinguish three cases:

First case: $s_0 \in (-r, 0)$. Since f attains its maximum at s_0, we conclude in this case that $f'(s_0) = 0$.

Second case: $s_0 = -r$. Again, since f attains its maximum at $s_0 = -r$, we conclude that f cannot be increasing at s_0, so $f'(s_0) \leq 0$.

Third case: $s_0 = 0$. In this case we use that $f \in D(A)$ and thus obtain

$$\begin{aligned} f'(0) = \langle \Phi, f \rangle &\leq f(0)\Phi(\{0\}) + \int_{[-r,0)} |f(s)|\, d|\Phi(s)| \\ &\leq f(0)\Big(\Phi(\{0\}) + |\Phi([-r, 0])|\Big) \leq 0. \end{aligned}$$

Hence, A is indeed dissipative and the semigroup $(e^{tA})_{t\geq 0}$ is therefore contractive. □

3.3 Delay Differential Equations in d Dimensions

Now we consider again the situation from Section 3.1, but this time for the general case $d \in \mathbb{N}$. We then have $E = \mathcal{C}([-r,0];\mathbb{R}^d)$ and $\Phi \in \mathcal{L}(E;\mathbb{R}^d)$. Our main theorem in this section reads as follows:

Theorem 3.3.1. *Endow $\mathbb{R}^d$ with the ∞-norm $\|\cdot\|_\infty$ and let $B \in \mathbb{R}^{d\times d}$. Let $E = \mathcal{C}([-r,0];\mathbb{R}^d)$, let $\Phi \in \mathcal{L}(E;\mathbb{R}^d)$ and let A be the operator on E given by (3.2). The the following assertions hold.*

(a) *If the semigroup $(e^{tA})_{t\geq 0}$ on E is contractive, then e^{tA} converges with respect to the operator norm as $t \to \infty$.*

(b) *Assume that there are functionals $\Phi_1,...,\Phi_d \in \mathcal{C}([-r,0];\mathbb{R})'$ such that $\Phi f = (\langle \Phi_1, f_1\rangle,...,\langle \Phi_d, f_d\rangle)$ for all $f \in E$ and consider each functional Φ_k as a regular Borel measure on $[-r,0]$.*

If we have $\Phi_k(\{0\}) \leq -|\Phi_k|([-r,0))$ for every $k \in \{1,...,d\}$ and if the semigroup $(e^{tB})_{t\geq 0}$ is contractive on $(\mathbb{R}^d, \|\cdot\|_\infty)$, then $(e^{tA}))_{t\geq 0}$ is contractive on E.

The condition that $\Phi f = (\langle \Phi_1, f_1\rangle,...,\langle \Phi_d, f_d\rangle)$ for appropriate functionals Φ_k and all $f \in E$ which we assumed in Theorem 3.3.1(b) simply means that the k-th component of Φf only depends on the k-the component of f. This means for the delay differential equation (3.1) that information from the past of the k-th component of $y(t)$ only influences the change rates of the k-th component of $y(t)$, but not the other components. The only interaction between the different components of y is then given by the term $By(t)$, i.e. there is no delayed interaction between the components of y.

To check the condition that $(e^{tB})_{t\geq 0}$ be contractive in $(\mathbb{R}^d, \|\cdot\|_\infty)$ we recall the following criterion from Remark 2.1.2 (a): the semigroup $(e^{tB})_{t\geq 0}$ is contractive in $(\mathbb{R}^d, \|\cdot\|_\infty)$ if and only if we have $B_{kk} \leq -\sum_{j\in\{1,...,d\}\setminus\{k\}} |B_{kj}|$ for all $k \in \{1,...,d\}$.

Proof of Theorem 3.3.1. Throughout the proof, we identify the Banach space $\mathcal{C}([-r,0];\mathbb{R}^d)$ with the Banach space $\mathcal{C}([-r,0]\times\{1,...,d\};\mathbb{R})$ to which it is isometrically isomorphic.

(a) The space E is projectively non-Hilbert according to Example 1.2.7(b). Since the semigroup $(e^{tA})_{t\geq 0}$ is contractive and eventually compact, it thus follows from Corollary 2.2.5(a) that e^{tA} converges with respect to the operator norm as $t \to \infty$.

(b) The proof is very similar to the proof of Theorem 3.2.1(c); we therefore only sketch the major steps:

We have to show that A is dissipative, and to do so, we use that B is strictly dissipative. Let $f \in D(A)$. We can consider f as an element of $\mathcal{C}([-r,0]\times\{1,...,d\};\mathbb{R})$ or as an element $f = (f_1,...,f_d) \in \mathcal{C}([-r,0];\mathbb{R}^d)$, where we have $f(s,k) = f_k(s)$ for each $k \in \{1,...,d\}$ and each $s \in [-r,0]$.

Choose $(s_0, k_0) \in [-r, 0] \times \{1, ..., d\}$ such that $|f(s,k)|$ becomes maximal at $(s,k) = (s_0, k_0)$; without loss of generality, we may assume that $f(s_0, k_0) \geq 0$, and we then have to show that $f'_{k_0}(s_0) \leq 0$. If $s_0 < 0$, this follows from the maximality of f at (s_0, k_0), so let $s_0 = 0$. Since $f \in D(A)$, we then have

$$f'_{k_0}(0) = (Bf(0))_{k_0} + \langle \Phi_{k_0}, f \rangle.$$

Here we used the special form of Φ that we assume in (b). Using the assumption on Φ_{k_0}, one shows as in the proof of Theorem 3.2.1(c) that $\langle \Phi_{k_0}, f \rangle \leq 0$. On the other hand, $|f(0,k)|$ is maximal on $\{1, ..., d\}$ at $k = k_0$ and we have $f(0, k_0) \geq 0$. Since B is strictly dissipative, we must thus have $(Bf(0))_{k_0} \leq 0$. Therefore, we conclude that $f'_{k_0}(0) \leq 0$.

We have shown that A is dissipative, so the semigroup $(e^{tA})_{t \geq 0}$ is contractive. □

Let us give only one explicit example to demonstrate that Theorem 3.3.1 can sometimes be very useful to obtain convergence of the semigroup $(e^{tA})_{t \geq 0}$.

Example 3.3.2. Let $d = 2$ and $r = 1$ and define $B = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$. Moreover, let

$$\Phi(f) = \begin{pmatrix} -f_1(0) + f_1(-1) \\ -f_2(0) + \int_{-1}^{0} f_2(s)\, ds \end{pmatrix}$$

for every $f \in E = \mathcal{C}([-1, 0]; \mathbb{R}^2)$.

If A is given by (3.2), then the C_0-semigroup $(e^{tA})_{t \geq 0}$ describes the solutions of the delay differential equation

$$\frac{d}{dt} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} y_1(t-1) \\ \int_{t-1}^{t} y_2(s)\, ds \end{pmatrix},$$

where $y : [-1, \infty) \to \mathbb{R}^2$.

One easily checks that the assumptions of Theorem 3.3.1(b) are fulfilled and thus, the semigroup $(e^{tA})_{t \geq 0}$ is contractive. Hence, it follows from Theorem 3.3.1(a) that e^{tA} converges with respect to the operator norm as $t \to \infty$.

To see this without the theory developed in the previous chapters, we could try to use the characterisation (3.3) in order to prove that $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$: it follows from (3.3) that a complex number $\lambda \neq 0$ is a spectral value of A if and only if

$$\det \begin{pmatrix} e^{-\lambda} - 2 - \lambda & -1 \\ -2 & \frac{1 - e^{-\lambda}}{\lambda} - 3 - \lambda \end{pmatrix} = 0.$$

Proving by elementary methods that this equation does not have a solution $\lambda \in i\mathbb{R} \setminus \{0\}$ would not seem to be a particularly inviting task.

3.4 Notes & Open Problems

Delay differential equations A brief introduction to the treatment of delay differential equations by means of semigroup methods can, for instance, be found in [EN00, Section VI.6]. For detailed information about delay differential equations we refer e.g. to the monograph [DvGVLW95]. The latter reference contains, in particular, many interesting results about the long term behaviour of delay differential equations.

Delay differential equations and positivity An exposition of the role of positivity for semigroups generated by delay differential equations can, for instance, be found in [KN84], [AGG+86, Section B-IV.3], [KN88] or [EN00, Section VI.6(c)]; there, a particular emphasis is put on how positivity can be used to decide whether the semigroup converges to 0. For more information, as well as for some remarks on the history of the subject, we refer to those references and to the references therein.

Delay differential equations on L^p-spaces Throughout this chapter we have modelled delay differential equations on a phase space consisting of continuous functions. It is worth mentioning again that there are similar approaches which use an L^p-space as phase space; for detailed information we refer, e.g., to the monograph [BP05].

Contractivity and positivity In Proposition 3.2.1(b) we proved that, under certain circumstances, contractivity of the solution semigroup to a one-dimensional delay differential equation implies positivity. We did this by showing that dissipativity of the generator implies dispersivity. Let us point out that we can also deduce the same result from a more general principle: indeed, in the situation of Proposition 3.2.1(b) one can first show, as we did in the proof, that $\mathbb{1} \in \ker A$. This implies that $\mathbb{1}$ is a fixed vector of e^{tA} for each $t \geq 0$. Now, we can employ a general result which asserts that a contractive operator T on a $C(K;\mathbb{R})$-space which maps $\mathbb{1}_K$ to $\mathbb{1}_K$ is automatically positive (see e.g. [AGG+86, Lemma B-III.2.1]). Hence, e^{tA} is positive for all $t \geq 0$.

Part II

Asymptotic Behaviour of Positive Semigroups

4 Geometry of Banach Space Cones

Our analysis of the spectral and asymptotic behaviour of contraction semigroups in the first part of this thesis was mainly motivated by spectral results of Perron–Frobenius type for positive semigroups on Banach lattices. We replaced the Banach lattice cones with unit balls and the positivity assumption on the semigroup with contractivity. Our results mainly depended on geometric properties of the unit ball, in combination with some regularity properties of the semigroup. Now we want to make a step back towards the original question about the spectrum and the long-term behaviour of positive semigroups. Yet, in contrast to the theory on Banach lattices discussed in the introduction of the thesis, we now consider positive semigroups on more general ordered Banach spaces.

To reach a deeper understanding of what we are going to do, it is necessary to first discuss some important concepts in Perron–Frobenius theory in a bit more detail: many theorems in Perron–Frobenius theory which deal with the so-called *peripheral spectrum* belong to one of two large classes of results. Let us consider a positive operator T with spectral radius $\mathrm{r}(T) = 1$ and a positive C_0-semigroup $(e^{tA})_{t\geq 0}$ whose generator has spectral bound $\mathrm{s}(A) = 0$, on an ordered Banach space E with, say, generating cone. The first class of Perron–Frobenius results essentially deals with the outstanding role of $\mathrm{r}(T)$ among all spectral values on the unit circle (or of $\mathrm{s}(A)$ among all spectral values on the imaginary axis). For example it is proved, under appropriate assumptions, that $\mathrm{r}(T)$ and $\mathrm{s}(A)$ are themselves spectral values, and that, if they are poles of the resolvent, they admit a positive eigenvector and their pole order dominates the order of all other poles on the unit circle (respectively, on the imaginary axis). Such results can, of course, be proved by many different methods (especially in finite dimensions), but from a purely functional analytic point of view, most of them can be considered as consequences of some kind of resolvent estimate. It is characteristic for such results that they are true under rather weak assumptions on the ordered Banach space; typically, it is sufficient if the cone E_+ is normal (see e.g. [Sch60, Proposition 1], [GVW81, Theorem 3.3], [SW99, Section 2.2 in the Appendix] or [ABHN11, Theorem 5.3.1]); if the cone is not normal, this can often be compensated by some kind of compactness assumption on T and $(e^{tA})_{t\geq 0}$ (see e.g. [KR48, Theorem 6.1] or [SW99, Section 2.4 of the Appendix]). If, however, the cone E_+ is not normal and T (or $(e^{tA})_{t\geq 0}$) fulfils no additional compactness assumptions, then Perron–Frobenius results of this type may fail to hold as e.g. shown in [Bon58, Example (iv) in Section 2] (compare also [ABHN11, Example 5.3.2]).

The second class of results about the peripheral spectrum is a bit more delicate. Those result are essentially structural results which prove that the spectrum of T or A fulfils automatically some kind of restriction in its shape. Such a result could for instance assert that all spectral values of T on the unit circle are automatically roots of unity, or that those spectral values form a group, or that such spectral values, except for 1, do not exist at all. Similarly, such a result could state that the peripheral spectrum of A is cyclic (see the preliminaries for a definition of this notion) or that $\mathrm{s}(A)$ is a dominant spectral value of A. In order to prove such results, one needs either stronger positivity assumptions on T and $(e^{tA})_{t\geq 0}$ (for example that the positive cone has non-empty interior and that T maps that cone into its interior), or one needs some kind of geometric assumption on the cone (or both). Typical examples of results which require some a strengthened version of positivity can be found in [KR48, Theorem 6.3], [Sch60, Proposition 5] or [Els70, Satz 4.4]. Results which employ geometric properties of the cone instead are particularly delicate since they require some kind of technique to transform the geometric information about the cone into information about the spectrum. Inspired by the geometric approach in Part I we focus on this latter class of results. Most known theorems of this types are finite-dimensional in nature (as e.g. in [Pul71], [VL85], [Veĭ85] or [TS94]) or require the underlying space to be lattice ordered (see e.g. [KR48, Section 8], [Sch74, Theorem V.4.9] and [AGG$^+$86, Section C-III.2]). A notable exception is [Bar92] where infinite-dimensional ordered Banach spaces with the Riesz decomposition property are considered. Here, we go much further; we use the ideas presented by Veĭtsblit and Lyubich in [VL85] (see also [Veĭ85]) in the finite-dimensional setting and adapt them to prove a result about the triviality of the peripheral point spectrum on a large class of infinite-dimensional ordered Banach spaces (see Theorem 5.1.1).

Before we proceed, we want to point out once again that one cannot expect any non-trivial structural properties of the spectrum of positive operators or semigroups without additional assumptions. This was for instance demonstrated in [Sch63, Section 1]. Here, we want to give a simple three-dimensional example to illustrate what can, for instance, go wrong if the positive cone of the underlying Banach space is, in a sense, too "Euclidean". Let $K := \{z \in \mathbb{R}^3 : z_1 \geq \sqrt{z_2^2 + z_3^2}\}$ be the classical ice cream cone in $\mathbb{R}^3$ and consider the C_0-semigroup $(e^{tA})_{t\geq 0}$ on $\mathbb{R}^3$ which is given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad e^{tA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \quad \text{for all } t \geq 0.$$

For each $t \geq 0$ the operator e^{tA} describes a rotation of angle t about the first

coordinate axis in $\mathbb{R}^3$. Hence we have $e^{tA}K \subseteq K$ for all $t \geq 0$, meaning that the semigroup $(e^{tA})_{t\geq 0}$ is positive on the ordered Banach space $(\mathbb{R}^3, K)$. Yet, the spectrum of A is given by $\sigma(A) = \{-i, i, 0\}$. Hence, the peripheral spectrum of A is not cyclic in any sense, nor is it reduced to 0. This demonstrates that, if we intend to prove triviality of the peripheral (point) spectrum for the generator of positive semigroups, we need to impose additional geometric assumptions on the cone. Moreover, the above example suggests that such conditions should require the cone not to be "too Euclidean" in a sense. This is the reason why we begin Part II of the thesis with the present chapter about the geometry of Banach space cones. The basic notions and properties for ordered Banach spaces, which we use throughout this part of the thesis, can be found in Appendix B.

As a foundation we first consider what we call *centred cones* in Section 4.1. Loosely speaking, such cones are obtained by fixing a vector subspace H of co-dimension 1 in a real Banach space E, by fixing furthermore a vector $x \in E$, not located in H, and by inflating the unit ball of H as we move away from H in the direction of x.

In Section 4.2 we then come to the heart of the matter and introduce two classes of ordered Banach spaces whose properties are well-adapted to our spectral analysis of positive semigroups in Chapter 5. The first notion is that of a *projectively non-Euclidean* ordered Banach space; the cone in such a space is, as we shall see, not rotund enough to allow a positive semigroup to exhibit a certain kind of periodic behaviour. The second, somewhat smaller class of ordered Banach spaces that we consider is that of *projectively lattice ordered* Banach spaces. We will see that many important function spaces, despite not being lattice ordered, belong to this class.

None of our results presented in this chapter has been published, or submitted for publication, yet.

4.1 Centred Cones

In the introduction to this chapter we have seen that the classical ice cream cone in three dimensions is a prototypical example of how a cone should *not* be shaped if we intend to show triviality of the peripheral (point) spectrum for generators of positive semigroups. More generally, we can also consider the classical ice cream cone

$$K := \{z \in \mathbb{R}^d : z_1 \geq \sqrt{z_2^2 + \ldots + z_d^2}\}$$

in d dimensions and make similar observations. In order to know what exactly we are dealing with, we use the present section, among other purposes, to

obtain a better understanding of the geometry of the classical ice cream cone K.

Clearly, this cone is, in some manner, determined by the vector $e_1 := (1,0,0,...) \in \mathbb{R}^d$, the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^d$ and the hyperplane $H := \{z \in \mathbb{R}^d : z_1 = 0\}$ or, from another point of view, the functional $e_1' \in (\mathbb{R}^d)'$ given by $\langle e_1', z\rangle = z_1$ for all $z \in \mathbb{R}^d$, whose kernel is H.

In fact, there are several possibilities to write down the cone K by means of the data e_1, e_1' and $\|\cdot\|_2$. For example, we have

$$K = \{y + re_1 : y \in \ker e_1',\ r \geq \|y\|_2\} \tag{4.1}$$

$$= \{z \in \mathbb{R}^d : \langle e_1', z\rangle \geq \|Qz\|_2\}, \tag{4.2}$$

where $Q \in \mathcal{L}(\mathbb{R}^d)$ denotes the projection onto $\ker e_1'$ along e_1. On the other hand, we can also find a representation of K which is rather similar to (4.2) but avoids use of the projection Q as well as the use of the vector e_1, namely

$$K = \{z \in \mathbb{R}^d : \langle e_1', z\rangle \geq \frac{1}{\sqrt{2}}\|z\|_2\}. \tag{4.3}$$

The representation (4.3) motivates the definition of what is sometimes called an *ice cream cone* in the more general context of ordered Banach spaces in the literature: given a number $\eta \in (0,1)$ and a normalised functional $x' \in E'$ on some real Banach space E, one defines

$$\{z \in E : \langle x', z\rangle \geq \eta\|z\|\}$$

to be the *ice cream cone* with parameters x' and η in E; see e.g. [AT07, Section 2.6]. Here, the parameter η is related to the aperture angle of the cone. Such cones can be shown to have various interesting properties, see again [AT07, Section 2.6] for some details.

Yet, despite the intriguing elegance of this concept, we prefer a slightly different approach here which is based on the formula (4.1) rather than (4.3). The advantage of the first formula is that it makes explicit use of the vector e_1 which we consider to be the "center" (or more precisely: to span the "central axis") of the cone. This is why we call the cones in the following definition *centred cones*.

Definition 4.1.1 (Centred cone). Let E be a real Banach space and let $x \in E$, $x' \in E'$ such that $\langle x', x\rangle = 1$. The set

$$E_{+,x,x'} := \{y + rx : y \in \ker x',\ r \geq \|y\|\}$$

is called the *centred cone* with parameters x and x'.

The idea to consider such cones is not new; for instance, cones of this type were used by Schaefer in [Sch63, Section 1] to construct a counterexample in the spectral theory of positive operators. In finite dimensions, a very similar constructions, which the authors called *topheavy cones*, was studied in a bit more detail by Fielder and Haynsworth in [FH73]. See also [Lyu95], [Lyu98] and [AT07, the last part of Section 2.6]; in the latter reference, a connection between centred cones and the general ice cream cones described above is given. Here we are going to prove a few results about centred cones, the most interesting of them being probably the duality result in Theorem 4.1.5.

The following note on a possible rescaling of x and x' is interesting:

Remark 4.1.2. Let x and x' be as in Definition 4.1.1. If $\alpha \in (0,\infty)$ is a scalar, then the vectors αx and $\frac{x'}{\alpha}$ still fulfil $\langle \frac{x'}{\alpha}, \alpha x\rangle = 1$. However, the centered cone $E_{+,\alpha x,\frac{x'}{\alpha}}$ is now given by

$$E_{+,\alpha x,\frac{x'}{\alpha}} = \{y + r\alpha x : y \in \ker \frac{x'}{\alpha},\ r \geq \|y\|\} = \{y + rx : y \in \ker x',\ r \geq \alpha\|y\|\}.$$

Thus, we can control the aperture angle of the centred cone $E_{+,x,x'}$ by multiplying x with α and x' with $\frac{1}{\alpha}$, respectively, where a larger parameter α corresponds to a smaller aperture angle of the cone.

In the next proposition we show a representation formula for centred cones which is similar to (4.2).

Proposition 4.1.3. *Let E be a real Banach space and let $x \in E$, $x' \in E'$ be such that $\langle x', x\rangle = 1$. Let $Q \in \mathcal{L}(E)$ denote the projection onto* $\ker x'$ *along* $\operatorname{span} x$*, i.e. $Qz = z - \langle x', z\rangle x$ for all $z \in E$. Then the centred cone with parameters x and x' is given by*

$$E_{+,x,x'} = \{z \in E : \langle x', z\rangle \geq \|Qz\|\}. \tag{4.4}$$

Proof. This follows from the fact that, for every $z \in E$, we have $z = y + xr$ for uniquely determined elements $y \in \ker x'$ and $r \in \mathbb{R}$ which are given by $y = Qz$ and $r = \langle x', z\rangle$. □

It turns out that formula (4.4) is very handy when dealing with centred cones. Let us now prove some general properties of such cones.

Proposition 4.1.4. *Let E be a real Banach space and let $x \in E$, $x' \in E'$ be such that $\langle x', x\rangle = 1$. Let $Q \in \mathcal{L}(E)$ denote the projection onto* $\ker x'$ *along* $\operatorname{span} x$*, i.e. $Qz = z - \langle x', z\rangle x$ for all $z \in E$.*

(a) *The centred cone $E_{+,x,x'}$ is indeed a cone, it is closed and it is pointed. Thus, $(E, E_{+,x,x'})$ is an ordered Banach space.*

(b) *We have* $\langle x', z\rangle \geq 0$ *for* $z \in E_{+,x,x'}$ *and* $\langle x', z\rangle > 0$ *for* $z \in E_{+,x,x'} \setminus \{0\}$.

(c) *The interior of* $E_{+,x,x'}$ *contains the point* x *and is given by* $\{z \in E : \langle x', z\rangle > \|Qz\|\}$. *Hence, the boundary of* $E_{+,x,x'}$ *is given by* $\{z \in E : \langle x', z\rangle = \|Qz\|\}$

(d) *The positive cone* $E_{+,x,x'}$ *in the ordered Banach space* E *is generating and normal. The vector* x *is an order unit in* E.

Proof. Throughout the proof we use the representation (4.4) of $E_{+,x,x'}$.

(b) If $z \in E_{+,x,x'}$, then we have $\langle x', z\rangle \geq \|Qz\| \geq 0$. If $z \in E_+$ and $\langle x', z\rangle = 0$, then we have

$$0 = \langle x', z\rangle \geq \|Qz\| \geq 0,$$

so $\langle x', z\rangle = 0$ and $Qz = 0$. This shows that $z = \langle x', z\rangle x + Qz = 0$.

(a) It readily follows from the representation (4.4) that $E_{+,x,x'}$ is a closed cone. If z and $-z$ are contained in $E_{+,x,x'}$, then we have $\langle x', -z\rangle \geq 0$. Hence, z is an element of $E_{+,x,x'}$ which fulfils $\langle x', z\rangle \leq 0$, so $z = 0$ according to assertion (b). This shows that the cone $E_{+,x,x'}$ is pointed.

(c) The set $\{z \in E : \langle x', z\rangle > \|Qz\|\}$ is an open subset of $E_{+,x,x'}$ and it is therefore contained in the interior of $E_{+,x,x'}$. Moreover, we have $\langle x', x\rangle = 1 > 0 = \|Qx\|$, so x is contained in this set and thus, x is an interior point of $E_{+,x,x'}$.

Now suppose that $z \in E_{+,x,x'}$ and $\langle x', z\rangle = \|Qz\|$. Let $\varepsilon > 0$. If we define $z_\varepsilon := z - \varepsilon x$, then $\|z_\varepsilon - z\| \leq \varepsilon\|x\|$ and we have

$$\langle x', z_\varepsilon\rangle = \langle x', z\rangle - \varepsilon = \|Qz\| - \varepsilon = \|Qz_\varepsilon\| - \varepsilon < \|Qz_\varepsilon\|.$$

Hence, z_ε is not contained in $E_{+,x,x'}$. This shows that z is a boundary point of $E_{+,x,x'}$.

(d) Since x is an interior point of $E_{+,x,x'}$, it is also an order unit according to Proposition B.1.4(b), and thus $E_{+,x,x'}$ is generating in E (see Proposition B.1.3).

To show that the cone $E_{+,x,x'}$ is normal, let $0 \leq z_1 \leq z_2$, i.e. $\langle x', z_1\rangle \geq \|Qz_1\|$ and $\langle x', z_2 - z_1\rangle \geq \|Q(z_2 - z_1)\|$. Then we have $\langle x', z_1\rangle \geq 0$ and $\langle x', z_2\rangle \geq 0$. Moreover,

$$\langle x', z_2\rangle - \langle x', z_1\rangle \geq \|Qz_1\| - \|Qz_2\|,$$

and thus

$$\|Qz_1\| + |\langle x', z_1\rangle| = \|Qz_1\| + \langle x', z_1\rangle \leq \|Qz_2\| + \langle x', z_2\rangle = \|Qz_2\| + |\langle x', z_2\rangle|.$$

Since $z \mapsto \|Qz\| + |\langle x', z\rangle|$ defines an equivalent norm on E, this proves that the cone $E_{+,x,x'}$ is normal. □

We close this section with a theorem which yields very precise information on how centred cones behave with respect to duality. Let E be a real Banach space and let $x \in E$, $x' \in E'$ with $\langle x', x\rangle = 1$. Since $(E, E_{+,x,x'})$ is an ordered Banach space with generating cone, the dual Banach space E' is also an ordered Banach space with respect to the dual cone $(E_{+,x,x'})' = \{z' \in E' : \langle z', z\rangle \geq 0 \text{ for all } z \in E_{+,x,x'}\}$; see Section B.2 for details. On the other hand, if we consider E as subset of E'' by means of evaluation, then $x \in E''$ and we can thus consider the centred cone $(E')_{+,x',x}$ in E' with parameters x' and x. The following theorem describes the relation between $(E_{+,x,x'})'$ and $(E')_{+,x',x}$. In finite dimensions, a related (though not identical) result can be found in [FH73, Theorem 2].

Theorem 4.1.5. *Let E be a real Banach space and let $x \in E$, $x' \in E'$ be such that $\langle x', x\rangle = 1$. Let $Q \in \mathcal{L}(E)$ denote the projection onto* $\ker x'$ *along* $\operatorname{span} x$*, i.e. $Qz = z - \langle x', z\rangle x$ for all $z \in E$.*

(a) *The centred cone $(E')_{+,x',x}$ in E' is always a subset of the dual cone $(E_{+,x,x'})'$.*

(b) *We have $(E')_{+,x',x} = (E_{+,x,x'})'$ if and only if Q is contractive.*

Proof. We first note that the projection in E' onto $\ker x$ along $\operatorname{span} x'$ is given by Q' and that we have $Q'z' = z' - \langle x, z'\rangle x'$ for all $z' \in E'$

(a) Let $z' \in (E')_{+,x',x}$ and let $z \in E_{+,x,x'}$. We have to show that $\langle z', z\rangle \geq 0$, and indeed we have

$$\begin{aligned}\langle z', z\rangle &= \left\langle Q'z' - \langle x, z'\rangle x', Qz - \langle x', z\rangle x\right\rangle \\ &= \langle Q'z', Qz\rangle + \langle x, z'\rangle\langle x', z\rangle \geq -\|Q'z'\|\|Qz\| + \langle x, z'\rangle\langle x', z\rangle \geq 0,\end{aligned}$$

where we used that $\|Q'z'\| \leq \langle x, z'\rangle$ and $\|Qz\| \leq \langle x', z\rangle$.

(b) "$\Rightarrow$" Suppose that Q is contractive. According to assertion (a) we only have to show that $(E_{+,x,x'})' \subseteq E_{+,x',x}$, so let $z' \in (E_{+,x,x'})'$. This means that we have $\langle z', z\rangle \geq 0$ for all $z \in E_{+,x,x'}$. We have to show that $\|Q'z'\| \leq \langle z', x\rangle$.

Let $\varepsilon > 0$ and choose $y \in E$ such that $\|y\| = 1$ and $\langle Q'z', y\rangle \geq \|Q'z'\| - \varepsilon$. If we define $z := x - Qy$, then we obtain

$$\langle z', x\rangle - \|Q'z'\| \geq \langle z', x'\rangle - \langle Q'z', y\rangle - \varepsilon = \langle z', z\rangle - \varepsilon.$$

We show that z is contained in $E_{+,x,x'}$. Indeed, we have $Qz = -Qy$ and, using that Q is contractive, we thus obtain

$$\|Qz\| = \|-Qy\| \leq \|y\| = 1 = \langle x', x\rangle = \langle x', z\rangle.$$

Hence, $z \in E_{+,x,x'}$. This shows that $\langle z', z\rangle \geq 0$ and thus, $\langle z', x\rangle - \|Q'z'\| \geq -\varepsilon$. As $\varepsilon > 0$ we arbitrary we conclude that $\langle z', x\rangle \geq \|Q'z'\|$, as claimed.

"⇐" Assume that $(E_{+,x,x'})' = E_{+,x',x}$. It suffices to prove that $\|Q'\| \leq 1$, so let $y' \in E'$ with $\|y'\| = 1$. We define $s := 1 - \langle y', x\rangle \in \mathbb{R}$ and $z' := y' + sx' \in E'$.

We show that z' is an element of the dual cone $(E_{+,x,x'})'$: for every $z \in E_{+,x,x'}$ we have

$$\langle x', z\rangle \geq \|Qz\| \geq -\langle y', Qz\rangle = -\Big\langle y', z - \langle x', z\rangle x\Big\rangle = \langle x', z\rangle\langle y', x\rangle - \langle y', z\rangle.$$

This shows that $\langle y', z\rangle \geq \langle x', z\rangle(\langle y', x\rangle - 1)$ and hence,

$$\langle z', z\rangle = \langle y', z\rangle + s\langle x', z\rangle \geq \langle x', z\rangle\Big(\langle y', x\rangle - 1 + s\Big) = 0.$$

This proves that $z' \in (E_{+,x,x'})'$. It thus follows from the assumption that we also have $z' \in E_{+,x',x}$, i.e. $\|Q'z'\| \leq \langle z', x\rangle$. Using that $Q'y' = Q'z'$, we thus obtain

$$\|Q'y'\| = \|Q'z'\| \leq \langle z', x\rangle = \langle y', x\rangle + s = 1.$$

This proves that Q' is contractive and hence, so is Q. □

4.2 Projectively Non-Euclidean and Projectively Lattice Ordered Banach Spaces

Now we are going to consider certain classes of ordered Banach spaces which turn out to provide a good setting for proving a Perron–Frobenius result for positive C_0-semigroups. We have already argued above that the positive cones in such Banach spaces should behave in some manner oppositely to classical ice creams cones. Let us recall that the three-dimensional classical ice cream cone K is the centred cone $\mathbb{R}^d_{+,e_1,e_1'}$, where $\mathbb{R}^d$ is endowed with the Euclidean norm and where $e_1 \in \mathbb{R}^d$ is the first canonical unit vector and where $e_1' \in (\mathbb{R}^d)'$ is given by $\langle e_1', z\rangle = z_1$ for all $z \in \mathbb{R}^d$. We have

$$K = \mathbb{R}^d_{+,e_1,e_1'} = \{z \in \mathbb{R}^d : z_1 \geq \Big(\sum_{k=2}^{d} z_k^2\Big)^{1/2}\}.$$

Recall from Proposition B.1.9(b) that if E is an ordered Banach space and $P \in \mathcal{L}(E)$ is a positive projection, then PE together with the cone $(PE)_+ := E_+ \cap PE = PE_+$ is an ordered Banach space, too. We now introduce the following notions which will be very important throughout the rest of Part II.

Definition 4.2.1 (Euclidean cones and projectively non-Euclidean cones). Let E be an ordered Banach space.

(i) If E has dimension $d < \infty$, then the positive cone E_+ of E is called *Euclidean* of the ordered Banach space E is isomorphic to the ordered Banach space $(\mathbb{R}^d, \mathbb{R}^d_{+,e_1,e_1'})$ where $\mathbb{R}^d$ is endowed with the Euclidean norm.

(ii) The positive cone E_+ in E is called *projectively non-Euclidean* if for every positive rank-3 projection $P \in \mathcal{L}(E)$ the cone $(PE)_+$ in the ordered Banach space PE is not Euclidean.

The ordered Banach space E is called *projectively non-Euclidean* if the positive cone E_+ in E is projectively non-Euclidean.

Projectively non-Euclidean cones were introduced by Veĭtsblit and Lyubich in [VL85] (though they did not use this terminology) in order to analyse the peripheral spectrum of positive operators on finite-dimensional ordered Banach spaces.

For later reference we explicitly note the following remark:

Remark 4.2.2. If E is a finite-dimensional ordered Banach space and the positive cone E_+ in E is Euclidean, then the positive cone E'_+ in the dual space E' is Euclidean, too.

Proof. This follows readily from Theorem 4.1.5(b). □

Let us now introduce a second class of ordered Banach spaces which is a bit more specific than projectively non-Euclidean spaces but contains, as we shall see, still a wealth of function spaces.

Definition 4.2.3 (Projectively lattice ordered Banach space). An ordered Banach space E is called *projectively lattice ordered* if for every positive finite-rank projection $P \in \mathcal{L}(E)$ the ordered Banach space $(PE, (PE)_+)$ is lattice ordered.

We will see below that this class a spaces is surprisingly large; on the other hand, every such space is projectively non-Euclidean as the following Proposition shows.

Proposition 4.2.4. *If an ordered Banach space E is projectively lattice ordered, then it is projectively non-Euclidean.*

Proof. Suppose that E is projectively lattice ordered and let $P \in \mathcal{L}(E)$ be a positive projection of rank 3. Then the ordered Banach space $(PE, (PE)_+)$ is lattice ordered. Let $\mathbb{R}^3$ be endowed with the Euclidean norm; since the ordered Banach space $(\mathbb{R}^3, \mathbb{R}^3_{+,e_1,e_1})$ is not lattice ordered (this follows e.g. from [Sch74, Corollary 1 to Theorem II.3.9]), it can therefore not be isomorphic to $(PE, (PE)_+)$. □

Some simple example classes of projectively lattice ordered spaces are listed in the subsequent examples. Recall that an ordered Banach space E is said to have the *Riesz decomposition property* if $[0,x] + [0,y] = [0, x+y]$ for all $x, y \in E_+$. The Riesz decomposition property is slightly weaker than being

lattice ordered; indeed, every lattice ordered Banach space has the Riesz decomposition property [AT07, Corollary 1.55], and every ordered Banach space with a generating and normal cone which has the Riesz decomposition property and is reflexive, is automatically lattice ordered [AT07, Corollary 2.48]. An important example of a space which has the Riesz decomposition property, but is not lattice ordered, is the space $\mathcal{C}^1([0,1];\mathbb{R})$, see [AT07, Example 1.58]. We note in passing that the Riesz decomposition property can be characterised by a certain *interpolation property*, see [AT07, Theorem 1.54] for details. Lattice ordered Banach spaces and ordered Banach spaces with the Riesz decomposition property are simple, but natural examples for spaces which are projectively lattice ordered:

Examples 4.2.5. Let E be an ordered Banach space.

(a) If E is lattice ordered, then E is also projectively lattice ordered.

(b) More generally, if the cone in E is generating and E has the Riesz decomposition property, then E is projectively lattice ordered.

Proof. (a) One easily checks that if E is lattice ordered, then, for every positive projection $P \in \mathcal{L}(E)$, the space PE is also lattice ordered by the cone $(PE)_+$.

(b) If E has generating cone and the Riesz decomposition property and if $P \in \mathcal{L}(E)$ is a positive projection, then it is easy to check that the ordered Banach space $(PE, (PE)_+)$ has also the Riesz decomposition property and that the positive cone $(PE)_+$ is generating in PE. If PE is, moreover, finite-dimensional, then it follows that PE is lattice ordered since every finite-dimensional ordered Banach space with generating cone and the Riesz decomposition property is lattice ordered [AT07, Theorem 3.9]. □

We point out that an ordered Banach space which is lattice ordered need not be a Banach lattice, in general; simple examples for this situation are constituted by first order Sobolev spaces. According to the above example, such spaces, despite not necessarily being Banach lattices, are examples of projectively lattice-ordered spaces. We will see below, in Example 4.2.9(b) that certain higher order Sobolev spaces are also projectively lattice ordered (though not lattice ordered according to [AN09, Example (d) in Section 2.3]).

Next we give a result about how projectively lattice ordered and projectively non-Euclidean spaces behave with respect to duality; this proposition and Lemma 4.2.7 below, which is used in the proof of the proposition, resemble in some way Proposition 1.2.5 and Lemma 1.2.6.

Proposition 4.2.6. *Let E be an ordered Banach space with generating cone.*

(a) *If the ordered Banach space E' is projectively lattice ordered, then so is E.*

(b) *If the ordered Banach space E' is projectively non-Euclidean, then so is E.*

For the proof we need the following lemma:

Lemma 4.2.7. *Let E be an ordered Banach space with generating cone and let $P \in \mathcal{L}(E)$ be a positive projection. Then the mapping*

$$\begin{aligned} i : P'E' &\to (PE)' \\ x' &\mapsto x'|_{PE} \end{aligned}$$

is an isomorphism between the ordered Banach spaces $P'E'$ and $(PE)'$.

Proof. It follows from Lemma 1.2.6 that i is a Banach space isomorphism and obviously, i is positive. Now, let $x' \in P'E'$ and assume that $i(x') = x'|_{PE}$ is positive. For every $x \in E_+$ we have

$$\langle x', x\rangle = \langle x', Px\rangle \geq 0$$

since $Px \in (PE)_+$. Thus, $x' \geq 0$, which shows that the inverse mapping of i is also positive. □

Proof of Proposition 4.2.6. (a) Assume that E' is projectively lattice ordered and that $P \in \mathcal{L}(E)$ is a positive projection of finite rank. Then $P' \in \mathcal{L}(E')$ is also a positive projection of finite rank, and the ordered Banach space $P'E'$ is isomorphic to the ordered Banach space $(PE)'$ according to Lemma 4.2.7. We know that $P'E'$ is lattice ordered and thus, so is $(PE)'$. This in turn implies that PE itself is lattice ordered, see e.g. [AT07, Theorem 3.9].

(b) Assume that E' is projectively non-Euclidean and let $P \in \mathcal{L}(E)$ be a positive projection of rank 3. Then $P' \in \mathcal{L}(E')$ is also a positive projection of rank 3 and the ordered Banach spaces $(PE)'$ and $P'E'$ are isomorphic according to Lemma 4.2.7. If the cone in the ordered Banach space PE was Euclidean, then, according to Remark 4.2.2, the same would be true for its dual space $(PE)'$ and thus for $P'E'$. However, the cone in ordered Banach space $P'E'$ cannot be Euclidean since E' is projectively non-Euclidean. □

Next we prove a theorem which will allow us to identify several classes of ordered Banach spaces as projectively lattice-ordered. The terminology used in the theorem is explained in Definition B.1.8.

Theorem 4.2.8. *Let F be an ordered Banach space and let E be a real Banach lattice. Suppose that there exists a dense embedding of ordered Banach spaces $j : F \to E$ which is, in addition, dominating. Then F is projectively lattice ordered.*

Proof. Let $P \in \mathcal{L}(F)$ be a positive projection of finite rank and consider the operator $jPj^{-1} : j(F) \to j(F) \subseteq E$. Since $j(F)$ is a dense and dominating subspace of E and since we have $jPj^{-1}x \geq 0$ for all $0 \leq x \in E$, it follows from

Corollary B.3.5 that there exists a positive linear operator $\tilde{P} : E \to E$ which extends jPj^{-1}. We have $\tilde{P} \in \mathcal{L}(E)$ since $\tilde{P}$ is positive and since the positive cone E_+ is generating in E; cf. Theorem B.3.2. Note that the range $(jPj^{-1})j(F)$ of jPj^{-1} is finite-dimensional and thus closed in E. Since $j(F)$ is dense in E, it follows that the range of jPj^{-1} coincides with the range of $\tilde{P}$.

Of course we have $(jPj^{-1})^2 = jPj^{-1}$, and again since $j(F)$ is dense in E, it follows that $\tilde{P}$ is a projection. Since E is a Banach lattice, the range $\tilde{P}E$ of $\tilde{P}$ is thus lattice ordered and, since the range of $\tilde{P}$ coincides with the range of jPj^{-1}, it follows that this latter range is lattice ordered, too. Hence, the range PF of P is a lattice ordered subspace of F. □

Theorem 4.2.8 immediately yields several examples for projectively lattice ordered Banach spaces:

Examples 4.2.9. The following ordered Banach spaces E are projectively lattice-ordered; in particular they are projectively non-Euclidean. Moreover, all of them contain an order unit and thus have a generating cone.

(a) The space $E = \mathcal{C}^k(\overline{\Omega})$ where $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ is a bonded domain, $k \in \mathbb{N}_0$ and $\mathcal{C}^k(\overline{\Omega})$ is endowed with the cone $\mathcal{C}^k(\overline{\Omega})_+ := C(\overline{\Omega})_+ \cap \mathcal{C}^k(\overline{\Omega})$ and the usual norm.

(b) The Sobolev space $E = W^{k,p}(\Omega;\mathbb{R})$ where $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and where $k \in \mathbb{N}$ and $p \in [1,\infty]$ such that $kp > d$ (here, we endow E with the cone $E_+ := E \cap L^p(\Omega;\mathbb{R})_+$).

(c) Let $\emptyset \neq I \subseteq \mathbb{R}$ be a bounded, open interval and let $\Omega \subseteq \mathbb{C}$ be a bounded open set which contains I. Denote by $E := \mathcal{H}_I(\overline{\Omega})$ the set of all continuous functions $\overline{\Omega} \to \mathbb{C}$ which are analytic on Ω and which take only real values on I. When endowed with the supremum norm, $\mathcal{H}_I(\overline{\Omega})$ is a real Banach space.

Let $\mathcal{H}_I(\overline{\Omega})_+ := \{f \in \mathcal{H}_I(\overline{\Omega}) : f(x) \geq 0 \text{ for all } x \in I\}$. Then $(\mathcal{H}_I(\overline{\Omega}), \mathcal{H}_I(\overline{\Omega})_+)$ is an ordered Banach space which is projectively lattice ordered and which contains an order unit.

Proof. (a) Obviously, $\mathcal{C}^k(\overline{\Omega};\mathbb{R})$ is an ordered Banach space with an order unit. The canonical embedding $\mathcal{C}^k(\overline{\Omega};\mathbb{R}) \hookrightarrow \mathcal{C}(\overline{\Omega};\mathbb{R})$ is a dense embedding of ordered Banach spaces and it is dominating since $\mathbb{1}_{\overline{\Omega}} \in \mathcal{C}^k(\overline{\Omega};\mathbb{R})$. Hence, it follows from Theorem 4.2.8 that $\mathcal{C}^k(\overline{\Omega};\mathbb{R})$ is projectively lattice ordered.

(b) Since we have $W^{k,p}(\Omega;\mathbb{R}) \hookrightarrow \mathcal{C}(\overline{\Omega};\mathbb{R})$ and since $\mathbb{1}_\Omega \in W^{k,p}(\Omega;\mathbb{R})$, the ordered Banach space $W^{k,p}(\Omega;\mathbb{R})$ contains an order unit. For the same reason, the dense embedding $W^{k,p}(\Omega;\mathbb{R}) \hookrightarrow \mathcal{C}(\overline{\Omega};\mathbb{R})$ of ordered Banach spaces is dominating. Thus, Theorem 4.2.8 implies that the ordered Banach space $W^{k,p}(\Omega;\mathbb{R})$ is projectively lattice ordered.

(c) Clearly, $\mathcal{H}_I(\overline{\Omega})$ is a real Banach space when endowed with the supremum norm. The set $\mathcal{H}_I(\overline{\Omega})_+$ is obviously a closed cone and it follows from the Identity theorem for holomorphic functions that this cone is pointed, i.e. it renders $\mathcal{H}_I(\overline{\Omega})$ an ordered Banach space. The vector $\mathbb{1}_\Omega$ is an order unit in $\mathcal{H}_I(\overline{\Omega})_+$.

The continuous linear map $j : \mathcal{H}_I(\overline{\Omega}) \mapsto \mathcal{C}(\overline{I};\mathbb{R})$ is an embedding of ordered Banach spaces; it is dominating since $j(\mathbb{1}_\Omega) = \mathbb{1}_{\overline{I}}$ and it is dense by the Weierstraß approximation theorem. Hence, it follows from Theorem 4.2.8 that $\mathcal{H}_I(\overline{\Omega})$ is projectively lattice ordered. □

It is known that the space $\mathcal{C}^k([0,1];\mathbb{R})$ (where $k \in \mathbb{N}$) has the Riesz decomposition property if $k = 1$ [AT07, Example 1.58], but it can be shown as in [NR74, Example 1] that $\mathcal{C}^k([0,1];\mathbb{R})$ does not have the Riesz decomposition property if $k \geq 3$. Hence, for $k \geq 3$, Example 4.2.9(a) yields a simple class of space which are projectively lattice ordered, but do not have the Riesz decomposition property.

Besides that, note again that a Sobolev space $W^{k,p}(\Omega;\mathbb{R})$ is lattice ordered in case that $k = 1$. In particular, $W^{k,p}(\Omega;\mathbb{R})$ is projectively lattice ordered if $k = 1$. On the other hand, if $k \geq 2$, then $W^{k,p}(\Omega;\mathbb{R})$ is no longer lattice ordered [AN09, Example (d) in Section 2.3]; yet, it follows from Example 4.2.9(b) that the same space is projectively lattice ordered for sufficiently large k or p. It is therefore interesting to ask what happens if k is larger than 1, but still too small for $W^{k,p}(\Omega;\mathbb{R})$ to embed into $\mathcal{C}(\overline{\Omega};\mathbb{R})$. In this case, it does not even seem to be clear whether the positive cone $W^{k,p}(\Omega;\mathbb{R})_+$ is generating in $W^{k,p}(\Omega;\mathbb{R})$ (see however [AN09, Example (d) in Section 2.3] for an interesting observation in the limit case $kp = d$).

Let us now present a simple class of finite-dimensional ordered Banach spaces which are projectively non-Euclidean but, in general, not lattice ordered (and hence, as they are finite-dimensional, not projectively lattice ordered): let E be an ordered Banach space with $\dim E < \infty$; the positive cone E_+ in E is called *polyhedral* if there exist finitely many functionals $\varphi_1', ..., \varphi_n' \in E'$ such that $E_+ = \bigcap_{k=1}^n \varphi_k^{-1}([0,\infty))$ (intuitively speaking, this means that E is spanned by finitely many "extreme rays"; compare [AT07, Theorem 3.37]). We have the following simple result (see also [Veĭ85, Theorem 3]):

Proposition 4.2.10. *Let E be a finite-dimensional ordered Banach space with a polyhedral cone. Then E is projectively non-Euclidean.*

Proof. Let $P \in \mathcal{L}(E)$ be a positive rank-3 projection. Then the cone $(PE)_+ := PE \cap E_+$ is also polyhedral and hence, it is not Euclidean. □

In order to find further examples of projectively non-Euclidean spaces we use the following proposition:

Proposition 4.2.11. *Let E be a real Banach space and let $x \in E$, $x' \in E'$ be such that $\langle x', x\rangle = 1$. Endow E with the centred cone $E_+ := E_{+,x,x'}$ and suppose that E contains a three-dimensional subspace F such that the positive cone $F_+ := F \cap E_+$ in F is Euclidean.*

Then there exist vectors $a, b \in \ker x' \setminus \{0\}$ and $q \in \ker x'$ which fulfil the equality $\|q + \cos(t)a + \sin(t)b\| = 1$ for all $t \in \mathbb{R}$

Proof. Since $\ker x'$ has co-dimension 1 and since F has dimension 3, it follows that $V := F \cap \ker x'$ has dimension 2 or 3. Moreover, if $\dim V$ were equal to 3, then F would be a subspace of $\ker x'$, so $F_+ := F \cap E_+ = \{0\}$. This cannot be true since (F, F_+) is isomorphic to $(\mathbb{R}^3, \mathbb{R}^3_{+,e_1,e_1})$ (where $\mathbb{R}^3$ is endowed with the Euclidean norm) which implies that F_+ is a generating cone in F. Hence, we proved that $\dim V = 2$.

Moreover, the plane V intersects E_+ (and thus F_+) only in 0. Now, let $j : (F, F_+) \to (\mathbb{R}^3, \mathbb{R}^3_{+,e_1,e_1'})$ be an isomorphism of ordered Banach spaces. Then $j(V)$ is a plane in $\mathbb{R}^3$ which intersects the cone $\mathbb{R}^3_{+,e_1,e_1'}$ only in 0. Hence, we can find a vector $\tilde{e}$ in the interior of $\mathbb{R}^3_{+,e_1,e_1'}$ such that the plane $\tilde{e} + j(V)$ intersects the boundary of the cone $\mathbb{R}^3_{+,e_1,e_1'}$ in an ellipse with barycenter $\tilde{e}$, i.e. there are two vectors $\tilde{a}, \tilde{b} \in j(V) \setminus \{0\}$ such that $\tilde{e} + \cos(t)\tilde{a} + \sin(t)\tilde{b}$ is contained in the boundary of the cone $\mathbb{R}^3_{+,e_1,e_1'}$ for all $t \in \mathbb{R}$.

Now, define $\hat{e} := j^{-1}(\tilde{e})$, $\hat{a} := j^{-1}(\tilde{a}) \in V$ and $\hat{b} := j^{-1}(\tilde{b}) \in V$. Then, for every $t \in \mathbb{R}$, the vector $\hat{e} + \cos(t)\hat{a} + \sin(t)\hat{b}$ is contained in the boundary of F_+ in F and thus in the boundary of E_+ in E. According to Proposition 4.1.4(c) this shows that

$$\|Q\hat{e} + \cos(t)\hat{a} + \cos(t)\hat{b}\| = \|Q(\hat{e} + \cos(t)\hat{a} + \cos(t)\hat{b})\| = \langle x', \hat{e}\rangle$$

for all $t \in \mathbb{R}$, where $Q \in \mathcal{L}(E)$ denotes the projection onto $\ker x'$ along the span of x. Note that $\alpha := \langle x', \hat{e}\rangle > 0$ since $\hat{e} \in E_+ \setminus \{0\}$. Thus, we only have to define $q := \frac{Q\hat{e}}{\alpha}$, $a := \frac{\hat{a}}{\alpha}$ and $b := \frac{\hat{b}}{\alpha}$ to obtain the assertion. □

We can use the above result to prove that certain centred cones in finite-dimensional ℓ^p-spaces are projectively non-Euclidean; this slightly generalises a result of Veĭtsblit and Lyubich from [VL85, Theorem 3]; we note, however, that our proof is rather similar to theirs.

Example 4.2.12. Endow $\mathbb{R}^d$ with the p-norm $\|\cdot\|_p$ for some $p \in (1, \infty)$ and let $x \in \mathbb{R}^d$, $x' \in (\mathbb{R}^d)'$ such that $\langle x', x\rangle = 1$. Endow $\mathbb{R}^d$ with the Euclidean norm and the centred cone $\mathbb{R}^d_{+,x,x'}$.

(a) If $p \in (1, 2)$, then $\mathbb{R}^d$ does not contain a subspace F such that the ordered Banach space $(F, F \cap \mathbb{R}^d_{+,x,x'})$ is isomorphic to $(\mathbb{R}^3, \mathbb{R}^3_{+,e_1,e_1})$.

In particular, the ordered Banach space $(\mathbb{R}^d, \mathbb{R}^d_{+,x,x'})$ is projectively non-Euclidean.

(b) If $p \neq 2$ and if the projection Q onto $\ker x'$ along $\operatorname{span} x$ is contractive, then the ordered Banach space $(\mathbb{R}^d, \mathbb{R}^d_{+,x,x'})$ is projectively non-Euclidean.

Proof. (a) Let $p \notin (1,2)$ and assume that the assertion is false. Then, according to Proposition 4.2.11, we can find vectors $a, b \in \mathbb{R}^d \setminus \{0\}$ and $q \in \mathbb{R}^d$ such that

$$\sum_{k=1}^{d} |q_k + \cos(t)a_k + \sin(t)b_k|^p = 1$$

for all $t \in \mathbb{R}$. Let $M \subseteq \{1,...,d\}$ be the set of all indices k for which we have $a_k = b_k = 0$ and let $K := \{1,...,d\} \setminus M$ be its complementary set; note that K is non-empty since $a \neq 0$. We have

$$\sum_{k\in K} |q_k + \cos(t)a_k + \sin(t)b_k|^p = 1 - q$$

for all $t \in \mathbb{R}$, where $q := \sum_{k\in M} |q_k|^p$. Note that $1 - q > 0$; indeed, if we assume $1 - q = 0$ then we obtain $q_k + \cos(t)a_k + \sin(t)a_k = 0$ for all $k \in K$ and all $t \in \mathbb{R}$. This is a contradiction, since for $k \in K$ not all coefficients q_k, a_k, b_k are equal to 0, but cos, sin and the constant function with value 1 are linearly independent as functions from $\mathbb{R}$ to $\mathbb{R}$. Hence, $1 - q > 0$ as claimed.

Now, let $z \in \mathbb{R}$ and define $t := 2\arctan z$. Then we have $\cos t = \frac{1-z^2}{1+z^2}$ and $\sin t = \frac{2z}{1+z^2}$. Plugging this into the above equation, we obtain after a short computation

$$\sum_{k\in K} \left|\alpha_k z^2 + \beta_k z + \gamma_k\right|^p = (1-q)\left(1+z^2\right)^p$$

for all $z \in \mathbb{R}$, where the coefficients $\alpha_k, \beta_k, \gamma_k$ are defined by

$$\begin{pmatrix} \alpha_k \\ \beta_k \\ \gamma_k \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \\ q_k \end{pmatrix}.$$

for all $k \in K$. For no index $k \in K$ the vector (a_k, b_k, q_k) is a multiple of $(0,0,1)$ (this is ensured by the choice of K). Hence, for no index $k \in K$, the vector $(\alpha_k, \beta_k, \gamma_k)$ is a multiple of $(1,0,1)$.

Replacing $\alpha_k, \beta_k, \gamma_k$ with $-\alpha_k, -\beta_k, -\gamma_k$ for all $k \in K$ for which it is necessary, we may assume that the leading coefficient of the polynomial $\alpha_k z^2 + \beta_k z + \gamma_k$ is > 0 for all $k \in K$ (note however that some of the numbers α_k and β_k can be zero, so β_k or even γ_k might be the leading coefficient for some indices k). We point out that replacing some of the tuples $(\alpha_k, \beta_k, \gamma_k)$ with $(-\alpha_k, -\beta_k, -\gamma_k)$

does not change the fact that none of those tuples is a multiple of $(1,0,1)$ for any $k \in K$.

For every $k \in K$ we define the polynomial $p_k : \mathbb{C} \to \mathbb{C}$ by $p_k(z) = \alpha_k z^2 + \beta_k z + \gamma_k$. Let $s_0 \in \mathbb{R}$ be an upper bound of the real parts of all complex roots of the polynomials q_k ($k \in K$). Then we have $q_k(z) > 0$ for all $z \in (s_0, \infty)$ and all $k \in K$, and hence we obtain

$$\frac{1}{1-q} \sum_{k \in K} q_k(z)^p = \left(1 + z^2\right)^p \tag{4.5}$$

for all $z \in (s_0, \infty)$. We will now derive a contradiction by extending equation (4.5) to an appropriate region of the complex plane. First note that the number $1 + z^2$ is contained in the sliced plane $\mathbb{C} \setminus (-\infty, 0]$ whenever $\operatorname{Re} z > 0$. Hence, the right side of equation (4.5) extends to an analytic function on the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ if we use the principal part of the logarithm to define the p-th power. On the other hand, it is easy to see that this analytic function has singularities at i and $-i$, meaning that it cannot be analytically extended to any neighbourhood of i or $-i$.

Now, let $K' \subseteq K$ be the set of all $k \in K$ for which the polynomial q_k has degree 2 and two distinct roots in the complex plane. For every $k \in K \setminus K'$ the function $q_k^p : (s_0, \infty) \to \mathbb{R}$ can be extended to an analytic function $q_k^p : \mathbb{C} \setminus [s_0, \infty) \to \mathbb{C}$. Indeed, if q_k has degree ≤ 1, then q_k maps the sliced plane $\mathbb{C} \setminus (-\infty, s_0]$ to the sliced plane $\mathbb{C} \setminus (-\infty, 0]$; hence, the p-th power of $q_k(z)$ can be defined for every $z \in \mathbb{C} \setminus (-\infty, s_0]$ by means of the principal branch of logarithm. If, on the other hand, q_k has degree 2, then we have $q_k(z) = \alpha_k (z - z_k)^2$ for a fixed $z_k \in (-\infty, s_0]$ and all $z \in (s_0, \infty)$. Since the mapping $z \mapsto z - z_k$ maps the sliced plane $\mathbb{C} \setminus (-\infty, s_0]$ to the sliced plane $\mathbb{C} \setminus (-\infty, 0]$, we can, again, define $q_k(z)^p$ for every $z \in \mathbb{C} \setminus (-\infty, s_0]$ by means of the principal branch of the logarithm. Hence, the partial sum $\sum_{k \in K \setminus K'} q_k(z)^p$ on the left-hand side of (4.5) extends analytically to a function on $\mathbb{C} \setminus (-\infty, s_0]$. Since the right side of (4.5) has a singularity at i while the sum $\sum_{k \in K \setminus K'} q_k(z)^p$ does not, it follows that K' is non-empty.

For each $k \in K'$ we denote by $z_k, \overline{z_k} \in \mathbb{C}$ the two distinct roots of q_k; then we have $q_k(z) = \alpha_k (z - z_k)(z - \overline{z_k})$ for all $z \in \mathbb{C}$. Hence, q_k maps the half plan $\mathbb{H}_K := \{z \in \mathbb{C} : \operatorname{Re} z > \operatorname{Re} z_k\}$ to the sliced plane $\mathbb{C} \setminus (-\infty, 0]$, so $z \mapsto q_k(z)^p$ extends from (s_0, ∞) to an analytic function on $\mathbb{H}_k$. On the other hand, this function has singularities at z_k and $\overline{z_k}$. Now we choose $k_1 \in K'$ such that $\operatorname{Re} z_{k_1}$ is maximal among the real parts of all z_k ($k \in K'$). Then the entire left side of (4.5) extends to an analytic function on $\mathbb{H}_{k_1} \setminus (-\infty, s_0]$, but it has singularities at z_{k_1} and $\overline{z_{k_1}}$ (note that several of the polynomials q_k, $k \in K'$, might have z_{k_1} and $\overline{z_{k_1}}$ as roots, but the singularities of these q_k^p cannot cancel out since $\alpha_k > 0$ for all $k \in K'$).

Using what we have already observed about the right side of (4.5), it follows that $\operatorname{Re} z_{k_1}$ cannot be strictly positive (since the right side of (4.5) is analytic on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$) and that $\operatorname{Re} z_{k_1}$ cannot be strictly negative (since the right side of (4.5) has singularities at i and $-i$). Thus, $\operatorname{Re} z_{k_1} = 0$. Yet, since none of the tuples $(\alpha_k, \beta_k, \gamma_k)$ is a multiple of $(1, 0, 1)$, it follows that none of the roots z_k ($k \in K'$) is equal to i or $-i$. This implies that the left side of (4.5) extends to an analytic function on a small neighbourhood of i and $-i$. This is a contradiction since the right side of (4.5) has singularities at i and $-i$.

(b) For $p \in (1, 2)$, this follows from (a). Now, let $p \in (2, \infty)$. Denote by $p' \in (1, 2)$ the dual index of p, meaning that $\frac{1}{p} + \frac{1}{p'} = 1$. Since the projection Q is contractive, we have $(\mathbb{R}^d_{+,x,x'})' = (\mathbb{R}^d)'_{+,x',x}$ according to Theorem 4.1.5(b). Since the norm on $(\mathbb{R}^d)'$ is the p'-norm, we conclude from (a) that $((\mathbb{R}^d)', (\mathbb{R}^d)'_{+,x',x})$ is projectively non-Euclidean and thus, so is $(\mathbb{R}^d, \mathbb{R}^d_{+,x,x'})$ according to Proposition 4.2.6(b). □

Unfortunately, we do currently not know an example where we can apply Proposition 4.2.11 to a centred cone in an infinite-dimensional space. Yet, Examples 4.2.9 still provide several interesting classes of infinite-dimensional spaces which are projectively lattice ordered and thus, in particular, projectively non-Euclidean.

We close this section with an example for a class of ordered Banach spaces which are *not* projectively non-Euclidean. First, we need the following simple observation from linear algebra:

Remark 4.2.13. Consider the self-adjoint 2×2-matrix

$$A = \begin{pmatrix} a & c \\ \overline{c} & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

where $a, d \in \mathbb{R}$ and $c \in \mathbb{C}$. The matrix A is positively semi-definite if and only if the inequalities

$$ad - c\overline{c} \geq 0 \qquad \text{and} \qquad a + d \geq 0$$

hold.

Proof. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ denote the eigenvalues of A. We have $\lambda_1, \lambda_2 \geq 0$ if and only if $\lambda_1\lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \geq 0$. On the other hand, we have $\lambda_1\lambda_2 = \det A = ad - c\overline{c}$ and $\lambda_1 + \lambda_2 = \operatorname{tr}(A) = a + d$. □

Now we can show that the set E of all self-adjoint operators on a Hilbert space of dimension at least two, ordered by its usual cone, is not projectively non-Euclidean:

Examples 4.2.14. (a) Let E be the three-dimensional real Banach space of all symmetric matrices in $\mathbb{R}^{2\times 2}$ (endowed with any norm) and let $E_+ \subseteq E$ be the cone of all those matrices in E which are positively semi-definite. Then the cone E_+ is Euclidean.

(b) Let E be the four-dimensional real Banach space of all self-adjoint matrices in $\mathbb{C}^2$ (endowed with any norm) and let $E_+ \subseteq E$ be the cone of all those matrices in E which are positively semi-definite. Then the ordered Banach space (E, E_+) is not projectively non-Euclidean.

(c) Let H be a complex Hilbert space with $2 \leq \dim H \leq \infty$ and let $E \subseteq \mathcal{L}(H)$ be the real Banach space of all self-adjoint bounded operators on H (endowed with the operator norm). Let $E_+ := \{A \in E : \sigma(A) \subseteq [0,\infty)\}$. Then the ordered Banach space (E, E_+) is not projectively non-Hilbert.

Proof. (a) We know from Remark 4.2.13 that

$$E_+ = \{\begin{pmatrix} a & c \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} : ad - c^2 \geq 0,\ a + d \geq 0\}.$$

Now consider the linear bijection

$$E \to \mathbb{R}^3,$$
$$\begin{pmatrix} a & c \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ c \end{pmatrix} := \begin{pmatrix} \frac{1}{2}(a+d) \\ \frac{1}{2}(a-d) \\ c \end{pmatrix}$$

and note that we have $ad - c^2 \geq 0$ and $a + d \geq 0$ if and only if $x^2 \geq y^2 + c^2$ and $x \geq 0$. Hence, the ordered Banach space (E, E_+) is isomorphic (via the above mapping) to the ordered Banach space $(\mathbb{R}^3, \mathbb{R}^3_{+,e_1,e_1'})$ where $\mathbb{R}^3$ is endowed with the Euclidean norm. This proves that the cone E_+ in E is Euclidean.

(b) The linear mapping

$$\begin{pmatrix} a & c \\ \overline{c} & d \end{pmatrix} \mapsto \begin{pmatrix} a & \operatorname{Re} c \\ \operatorname{Re} c & d \end{pmatrix}$$

is a projection from E onto the space considered in (a), and it follows from Remark 4.2.13 that this projection is positive. This proves the assertion.

(c) Fix a two-dimensional vector subspace $V \subseteq H$ and let $Q \in \mathcal{L}(H)$ be the orthogonal projection with range V. Denote by $F \subseteq \mathcal{L}(V)$ the space of all self-adjoint operators in $\mathcal{L}(V)$ and let $F_+ := \{B \in F : \sigma(B) \subseteq [0,\infty)\}$. Then

$$P : E \to E, \qquad A \mapsto QAQ$$

is a positive linear projection. Moreover, its range PE, when endowed with the cone $(PE)_+ := E_+ \cap PE = P(E_+)$, is, as an ordered Banach space, isomorphic

to the ordered Banach space (F, F_+) (via the mapping $A \mapsto A|_V$). Since the latter space is not projectively non-Euclidean according to (b), it follows that $(PE, (PE)_+)$ is not projectively non-Euclidean, either; hence, nor is (E, E_+) □

The assertion in Example 4.2.14(a) can also be found in [vGK08, Section 4]; moreover, the same reference [vGK08, Section 4] contains several other examples of three dimenional spaces with Euclidean cones.

4.3 Notes & Open Problems

Cones and ordered vector spaces The theory of cones and ordered vector spaces, as well as the study of positive operators on them, is quite an old topic in analysis. Of particular importance are so-called *vector lattices* (sometimes also named *Riesz spaces*), i.e. ordered vector spaces in which any two elements have a supremum. Those spaces have a very deep and rich theory; see for example the classical monographs [LZ71] and [Zaa83] as well as [Zaa97]. On the other hand, a large part of the literature about ordered vector spaces focuses on the interplay between order theoretic and topological properties; this interplay constitutes the subject of *ordered topological vector spaces*; we refer the interested reader for example to [Per67], [SW99, Section V] and to the more recent monograph [AT07] for an overview. An important special case of ordered topological vector spaces are ordered Banach spaces, which we considered in this chapter. A brief introduction to them can be found in Appendix B. An even more specialised class of spaces is given by the class of *Banach lattices*; those are ordered Banach spaces which are lattice ordered and whose norm is, in a certain sense, compatible with the order. Due to their prevalence in applications, as well as for intrinsic reasons, an extensive theory for such spaces (and for positive operators on them) has been developed. We refer for example to the monographs [Sch74], [AB85, Chapter 4], [MN91] for more information.

Lattice ordered Banach spaces By a *lattice ordered Banach space* we mean an ordered Banach space which is a vector lattice with respect to its canonical order. It is worthwhile pointing out again that a lattice ordered Banach space is not the same as a *Banach lattice*; in fact, a Banach lattice is defined to be a lattice ordered Banach space E which fulfils the additional condition that $\|x\| \leq \|y\|$ whenever $x, y \in E$ and $|x| \leq |y|$ (where the modulus $|x|$ is defined by $|x| := x \vee (-x)$). It is not difficult to show that a lattice ordered Banach space E is a Banach lattice with respect to some equivalent norm if and only if the positive cone E_+ in E is *normal* (see Definition B.1.5). Prototypical examples of lattice ordered Banach spaces which do not have a normal cone (and which can therefore not be renormed as Banach lattices) are first order Sobolev spaces

(compare [AN09, Example (c) in Section 2.3]). Example 4.2.5(a) shows that every lattice ordered Banach space, be it a Banach lattice or not, is projectively lattice ordered.

Positive semigroups and Euclidean cones We have demonstrated at the beginning of the chapter that positive C_0-semigroups on finite-dimensional spaces with a Euclidean cone are not particularly well-suited to prove certain types of Perron–Frobenius theorems. Yet, it is worthwhile pointing out that such semigroups have quite an interesting structure theory, see for example [SW91a]; see also [LS75] and [SW91b] for related results about single positive operators on $\mathbb{R}^n$ endowed with an Euclidean cone.

Centred cones and the geometry of unit balls We have seen in Proposition 4.2.4 that every projectively lattice ordered Banach space is also projectively non-Euclidean; hence, Example 4.2.9 yields several classes of projectively non-Euclidean spaces. But conversely, our stock of examples for projectively non-Euclidean spaces which are not projectively lattice ordered is very small; in fact we only found two finite-dimensional classes of examples in Proposition 4.2.10 and in Example 4.2.12. On the other hand, it is very tempting to assume that a centred cone in a projectively (or even extremely) non-Hilbert Banach space is automatically projectively non-Euclidean; compare also Proposition 4.2.11. Let us pose this as an open problem:

Problem 4.3.1. Let K be a centred cone in a real Banach space E.

(a) If E is projectively non-Hilbert, under which additional conditions does it follow that K is projectively non-Euclidean?

(b) If E is extremely non-Hilbert, under which additional conditions does it follows that K is projectively non-Euclidean?

Finite-dimensional C^*-algebras Let $\mathcal{A}$ be a finite-dimensional C^*-algebra and let $\mathcal{A}_{\mathrm{sa}}$ denote the space of self-adjoint elements of $\mathcal{A}$. Then $\mathcal{A}_{\mathrm{sa}}$ is an ordered Banach space with respect to its canonical order. One can show that this space is not projectively non-Euclidean whenever $\mathcal{A}$ is not commutative: indeed, this follows from Example 4.2.14(c) and from the special structure of finite-dimensional C^*-algebras which is, for example, described in [Mur90, Theorem 6.3.8]. Alternatively, one can also conclude this from Theorem 5.1.1 below and from a "negative" Perron–Frobenius theorem on non-commutative C^*-algebras in [Glü16b, Theorem 2.1].

5

Spectrum and Asymptotics of Positive Semigroups

Now we use the concepts discussed in Chapter 4 to study the spectral properties of positive C_0-semigroups on ordered Banach spaces. The first section deals with the point spectrum of the semigroup generator while we derive convergence theorems for the semigroup in Section 5.2. We should point out that we consider this part of the thesis mainly a theoretical study with the objective to comprehend how geometric properties of the cone in an ordered Banach space influences the validity of Perron–Frobenius type spectral results. Despite the generality of our results (with respect to the ordered Banach spaces on which they are applicable), we do not have any specific applications in mind.

In fact, the ordered Banach spaces which are most important in many (not all) applications are Banach lattices, and for those spaces a much more general theory is available (see e.g. [AGG+86, Section C-III]). On the other hand, one class of ordered Banach spaces which is extremely important in applications, in particular in physics, is constituted by spaces of self-adjoint elements of C^*-algebras. While different types of Perron–Frobenius results are available for such spaces (see the notes for details), we have indicated in the notes at the end of Chapter 4 that such spaces do not fall within the scope of our results; moreover, it has recently been demonstrated by the author in [Glü16b, Theorem 2.1] that Perron–Frobenius type theorems about triviality of the peripheral spectrum which exceed a certain generality can never hold on any non-commutative C^*-algebra.

Our results presented in this chapter have not yet been published nor submitted for publication. They were, as well as the results in the previous chapter, inspired by the paper [VL85] by Veĭtsblit and Lyubich; see also [Veĭ85].

5.1 The Point Spectrum

Let E be an ordered Banach space. Inspired by previous work of Krasnosel'skiĭ [Kra68], Veĭtsblit and Lyubich [VL85] have analysed the peripheral spectrum of positive operators on E in case that E is projectively non-Euclidean and finite-dimensional. We are now going to allow infinite-dimensional spaces, too, and, as always in this thesis, we are interested in C_0-semigroups rather than in single operators. As in Chapter 2, dealing with C_0-semigroups instead of single operators makes things a bit easier to handle. On the other hand,

the fact that the space E might be infinite-dimensional makes it difficult, if not impossible, to adapt the arguments of Veĭtsblit and Lyubich. They essentially employ a result about contractions on projectively non-Hilbert spaces from [Lyu70] and combine it with a few geometric arguments, an approach which does not seem to work in the infinite-dimensional setting. Instead we perform a careful analysis of the action of the semigroup (and its dual) on certain two- and three-dimensional subspaces in order to infer a similar result.

Recall that a C_0-semigroup $(e^{tA})_{t\geq 0}$ on E is called *positive* if e^{tA} is a positive operator for every $t \geq 0$. Our main theorem in this section now reads as follows:

Theorem 5.1.1. *Let E be an ordered Banach space whose cone is generating and projectively non-Euclidean and let $(e^{tA})_{t\geq 0}$ be a positive C_0-semigroup on E. Assume furthermore that $(e^{tA})_{t\geq 0}$ has weakly pre-compact orbits, that the set $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ is bounded and that at least one of the following assumptions holds:*

(a) *For each $\lambda \in \sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ the eigenspace $\ker(\lambda - A)$ is finite-dimensional.*

(b) *There exists an order unit in E.*

Then $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq \{0\}$.

In the proof of Theorem 5.1.1 we employ the following auxiliary result:

Lemma 5.1.2. *Let $(e^{tA})_{t\geq 0}$ be a bounded C_0-semigroup on a complex Banach space E, let $t > 0$ and $\gamma \in \mathbb{R}$. Assume that the set $\{\beta \in \mathbb{R} : e^{it\beta} = e^{it\gamma} \text{ and } i\beta \in \sigma_{\mathrm{pnt}}(A)\} \subseteq \gamma + \frac{2\pi}{t}\mathbb{Z}$ is finite and consists of n pairwise distinct numbers $\beta_1, ..., \beta_n$. Then we have*

$$\ker(e^{it\gamma} - e^{tA}) = \oplus_{k=1}^{n} \ker(i\beta_k - A).$$

Proof. According to [EN00, Corollary IV.3.8(ii)], $\ker(e^{it\gamma} - e^{tA})$ equals the closure of $F := \oplus_{k=1}^{n} \ker(i\beta_k - A)$; hence, we only have to show that F is closed. To this end, it suffices to prove that for each $k \in \{1, ..., n\}$ the projection $P_k : F \to \ker(i\beta_k - A)$ along the other eigenspaces in the direct sum is continuous (since this implies that F is complete).

Define $M := \sup_{\mathrm{Re}\,\lambda>0} \|\mathrm{Re}\,\lambda\mathcal{R}(\lambda, A)\|$; as the semigroup $(e^{tA})_{t\geq 0}$ is bounded, we have $M < \infty$ according to the Hille–Yosida theorem. Now, let $f_1, ..., f_n$ be vectors in $\ker(i\beta_1 - A)$, ..., $\ker(i\beta_n - A)$, respectively, and let $k \in \{1, ..., n\}$. Then we have

$$\|f_k\| = \lim_{r\downarrow 0} \|r\mathcal{R}(i\beta_k + r)(f_1 + ... + f_n)\| \leq M\|f_1 + ... + f_n\|.$$

This proves that P_k is indeed bounded. □

An argument very similar to the above proof was, for example, used in the proof of [MNN71, Theorem 1]. Now we come to the proof of Theorem 5.1.1.

Proof of Theorem 5.1.1. Throughout the proof, let $E_{\mathbb{C}}$ be a complexification of the real Banach space E and let $A_{\mathbb{C}}$ denote the complex extension of A to $E_{\mathbb{C}}$. We assume for a contradiction that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \nsubseteq \{0\}$. After replacing A with cA for some appropriate $c > 0$ we may assume that $\pm i \in \sigma_{\mathrm{pnt}}(A)$ and that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq i \cdot (-2, 2)$.

Note that the operators $e^{\pi A}$, $-e^{\pi A}$ and $e^{2\pi A}$ all have weakly pre-compact orbits and thus, those three operators are mean ergodic (see [EFHN15, Remark 16.17(4)]). Let $R, Q, P \in \mathcal{L}(E)$ denote their mean ergodic projections, respectively. All three projections R, Q and P commute; moreover, R and P are both positive and a brief computation shows that $P = R + Q$. The ranges of their complexifications $R_{\mathbb{C}}$, $Q_{\mathbb{C}}$ and $P_{\mathbb{C}}$ are given by

$$R_{\mathbb{C}}E_{\mathbb{C}} = \ker A_{\mathbb{C}}, \tag{5.1}$$

$$Q_{\mathbb{C}}E_{\mathbb{C}} = \ker(i - A_{\mathbb{C}}) \oplus \ker(-i - A_{\mathbb{C}}), \tag{5.2}$$

$$P_{\mathbb{C}}E_{\mathbb{C}} = \ker A_{\mathbb{C}} \oplus \ker(i - A_{\mathbb{C}}) \oplus \ker(-i - A_{\mathbb{C}}). \tag{5.3}$$

Indeed, these equalities follow from Lemma 5.1.2 and from the fact that $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R} \subseteq i \cdot (-2, 2)$: for equality (5.1) one only needs to observe that $R_{\mathbb{C}}E_{\mathbb{C}}$ is the fixed space of $e^{\pi A_{\mathbb{C}}}$, for equality (5.2) one uses that $Q_{\mathbb{C}}E_{\mathbb{C}}$ is the fixed space of $-e^{\pi A_{\mathbb{C}}}$ and for equality (5.3) we uses that $P_{\mathbb{C}}E_{\mathbb{C}}$ is the fixed space of $e^{2\pi A_{\mathbb{C}}}$.

We now consider the range of P in more detail: let $F := PE$ and let $F_{\mathbb{C}} := P_{\mathbb{C}}E_{\mathbb{C}}$. Clearly, F is an ordered Banach space with respect to the cone $F_+ := F \cap E_+ = PE_+$, and $F_{\mathbb{C}}$ is a complexification of F. The semigroup $(e^{tA}|_F)_{t\geq 0}$ on F is positive; moreover, it is periodic and operator norm continuous according to (5.3). Due to the periodicity, it extends to a positive, norm continuous operator group on F.

It is clear from (5.1) that our semigroup $(e^{tA})_{t\geq 0}$ acts as the identity on the range of R. Moreover it rotates all vectors in the range of Q in some sense; more precisely, the following observation is true:

(∗) *For every $x \in QE$ there exists exactly one vector $y \in QE$ such that $x + iy \in \ker(i - A_{\mathbb{C}})$, and for this vector y we have $e^{tA}x = x\cos t - y \sin t$ for all $t \geq 0$. In particular, we have $e^{\frac{3}{2}\pi}x = y$ and $e^{\pi A}x = -x$ and thus, no vector in $QE \setminus \{0\}$ is positive.*

To prove (∗), let $x \in QE \subseteq Q_{\mathbb{C}}E_{\mathbb{C}}$. According to (5.2) we can thus decompose x as $x = z_1 + z_2$, where $z_1 \in \ker(i - A_{\mathbb{C}})$ and $z_2 \in \ker(-i - A_{\mathbb{C}})$. Since $z_1, z_2 \in Q_{\mathbb{C}}E_{\mathbb{C}} = QE + iQE$, we have $z_1 = x_1 + iy_2$ and $z_2 = x_2 + iy_2$ for some $x_1, x_2, y_1, y_2 \in QE$. We have $x = x_1 + x_2$ and, as z_2 is an eigenvector of $A_{\mathbb{C}}$ for the eigenvalue $-i$

and $A_{\mathbb{C}}$ is real, the conjugate vector $\overline{z_2} = x_2 - iy_2$ is an eigenvector of $A_{\mathbb{C}}$ for the eigenvalue i. Hence, if we choose $y := y_1 - y_2$, then $x + iy = z_1 + \overline{z_2}$ is indeed an eigenvector of $A_{\mathbb{C}}$ for the eigenvalue i.

On the other hand, if $\hat{y} \in QE$ is another vector with this property, then $y - \hat{y} \in QE$ would also be an eigenvector of $A_{\mathbb{C}}$ for the eigenvalue i, which is a contradiction since $A_{\mathbb{C}}$ maps real vectors to real vectors.

Now, let $y \in QE$ such that $x + iy \in \ker(i - A_{\mathbb{C}})$. Then we obtain for each $t \geq 0$

$$e^{tA}x = \operatorname{Re}\left(e^{tA_{\mathbb{C}}}(x + iy)\right) = \operatorname{Re}\left(e^{it}(x + iy)\right) = x\cos t - y\sin t.$$

This clearly implies that $e^{\frac{3}{2}\pi}x = y$ and $e^{\pi A}x = -x$. Finally, assume that $x \in QE$ is positive, i.e. contained in E_+. Then we have $0 \leq e^{\pi A}x = -x \leq 0$, so $x = 0$. This proves $(*)$.

It is now our intention to construct a three-dimensional subspace U of F which admits a positive projection $F \to U$ and whose positive cone U_+ is Euclidean. To this end, we construct three vectors $u_1, u_2, u_3 \in F$ and three functionals $u_1', u_2', u_3' \in F'$ with the following properties:

(i) The vector u_1 is positive, contained in RE and an order unit of (F, F_+); the vectors u_2 and u_3 are contained in QE.

(ii) The functional u_1' is positive and contained in the range of $(R|_F)'$; the functionals u_2' and u_3' are contained in the range of $(Q|_F)'$.

(iii) We have $\langle u_k', u_j \rangle = \delta_{kj}$ for all $k, j \in \{1, 2, 3\}$.

(iv) The operator $S := \sum_{k=1}^{3} u_k \otimes u_k'$ is a positive projection on F with range $U := \operatorname{span}_{\mathbb{R}}\{u_1, u_2, u_3\}$; moreover, S commutes with $e^{tA}|_F$ for all $t \geq 0$.

To construct those vectors, first note that there exists an order unit z in (F, F_+). Indeed, this is the point where we need assumption (a) or (b): if (a) is fulfilled, then it follows from the description of F in (5.3) that F is finite-dimensional. Since F_+ is generating in F, it thus has non-empty interior and therefore, there exists an order unit z in (F, F_+) (see Proposition B.1.4). If, instead, (b) is fulfilled and $\tilde{z}$ is an order unit in (E, E_+), then $z := P\tilde{z}$ is clearly an order unit in (F, F_+).

Now, define $u_1 := Rz \geq 0$. We show that u_1 is still an order unit in (F, F_+): let $\tilde{z} \in F$. Then we have $\varepsilon\tilde{z} \leq u_1$ for some $\varepsilon > 0$. We have $0 \leq e^{\pi A}u_1 = e^{\pi A}(Ru_1 + Qu_1) = Ru_1 - Qu_1$, so we conclude that $2Ru_1 = u_1 + Ru_1 - Qu_1 \geq u_1 \geq \varepsilon\tilde{z}$. Hence, $Ru_1 \geq \frac{\varepsilon}{2}\tilde{z}$, which shows that u_1 is indeed an order unit in (F, F_+).

Since i is an eigenvalue of A by assumption, it follows from (5.2) that $Q \neq 0$ and thus, $Q|_F$ is non-zero since the range of Q is contained in F. Hence, we can find vector $0 \neq u \in QE$. Since QE intersects RE only in 0, it follows that

u_1 and u are linearly independent. Now, consider the two-dimensional real Banach space W which is spanned by u_1 and u; let $W_+ := W \cap E_+ = W \cap F_+$. Since u_1 is an order unit in F, it follows that W_+ is generating in W. Thus, W_+ is a closed, pointed and generating cone in the two-dimensional vector space W and hence, W_+ is spanned by two extreme rays. Again since u_1 and u are linearly independent, it follows that there exists a scalar $\alpha \in \mathbb{R} \setminus \{0\}$ such that $u_1 + \alpha u$ spans an extreme ray of W_+. Now, we note that the operator $e^{\pi A}|_W$ is positive on (W, W_+) and it maps u_1 to u_1 and u to $-u$ according to $(*)$. This shows that $u_1 - \alpha u$ spans the other extreme ray of W_+.

We now define $u_2 := \alpha u$. To construct the first two functionals u_1', u_2', we need a bit of preparation. First, we define auxiliary vectors $\hat{u}_1, \hat{u}_2 \in W_+$ which are given by $\hat{u}_1 := u_1 + u_2 \in W_+$ and $\hat{u}_2 := u_1 - u_2 \in W_+$ and which span the two extreme rays of W_+. Note that $e^{\pi A}\hat{u}_1 = \hat{u}_2$ and $e^{\pi A}\hat{u}_2 = \hat{u}_1$.

We endow the dual space W' of W with the dual cone $W_+' := \{w' \in W' | \langle w', w\rangle \geq 0 \text{ for all } w \in W_+\}$. Let $\hat{w}_1' \in W_+'$ be a vector which spans one of the two extreme rays of W_+'. Then $\hat{w}_1'$ annihilates one of the two vectors $\hat{u}_1$ and $\hat{u}_2$, say $\hat{u}_2$. On the other hand we then have $\langle \hat{w}_1', \hat{u}_1\rangle > 0$, and after rescaling $\hat{w}_1'$ with a positive factor, we may assume that $\langle \hat{w}_1', \hat{u}_1\rangle = 1$. We now use that W_+ contains u_1 which is an order unit of (F, F_+): this implies that $\hat{w}_1'$ can be extended to a positive linear functional $\hat{u}_1' \in F_+'$; see Corollary B.3.4. We have $\langle \hat{u}_1', \hat{u}_1\rangle = 1$ and $\langle \hat{u}_1', \hat{u}_2\rangle = 0$.

We define a second auxiliary functional $\hat{u}_2'$ by $\hat{u}_2' := (e^{\pi A}|_F)'\hat{u}_1' \in F'$. We are going to show that $\hat{u}_2'$ is positive, that $\langle \hat{u}_2', \hat{u}_2\rangle = 1$ and $\langle \hat{u}_2', \hat{u}_1\rangle = 0$; note that those properties show that the restriction $\hat{w}_2' := \hat{u}_2'|_W$ of the functional $\hat{u}_2'$ to W spans the second extreme ray of W_+'.

We have $\langle \hat{u}_2', v\rangle = \langle \hat{u}_1', e^{\pi A}v\rangle \geq 0$ for all $v \in F_+$. Hence, $\hat{u}_2'$ is a positive functional on F. Moreover, we have

$$\langle \hat{u}_2', \hat{u}_2\rangle = \langle \hat{u}_1', e^{\pi A}\hat{u}_2\rangle = \langle \hat{u}_1', \hat{u}_1\rangle = 1$$

and, on the other hand,

$$\langle \hat{u}_2', \hat{u}_1\rangle = \langle \hat{u}_1', e^{\pi A}\hat{u}_1\rangle = \langle \hat{u}_1', \hat{u}_2\rangle = 0.$$

Thus, all the properties of $\hat{u}_2'$ that we claimed above are indeed true.

We now define the two functionals u_1' and u_2': let $u_1' := 2(R|_F)'\hat{u}_1'$ and let $u_2' := 2(Q|_F)'\hat{u}_1'$. Note that u_1' is positive since $\hat{u}_1'$ and $(R|_F)'$ are positive. Moreover, we have $(e^{tA}|_F)'u_1' = u_1'$ for all $t \geq 0$, i.e. u_1' is a fixed vector of the adjoint semigroup $(e^{tA}|_F)_{t\geq 0}'$. Furthermore, we obtain

$$u_1' + u_2' = 2(R|_F + Q|_F)'\hat{u}_1' = 2(P|_F)'\hat{u}_1' = 2\hat{u}_1' \qquad \text{and}$$
$$u_1' - u_2' = 2(R|_F - Q|_F)'\hat{u}_1' = 2(e^{\pi A}|_F R|_F + e^{\pi A}|_F Q|_F)'\hat{u}_1' = 2(R|_F + Q|_F)'\hat{u}_2' = 2\hat{u}_2'.$$

We can use this observation to prove that u_1, u_2, u_1' and u_2' behave indeed as we would like them to: first recall that we have already proved above that $\langle \hat{u}_k', \hat{u}_j \rangle = \delta_{kj}$ for $k, j \in \{1, 2\}$. Using the equations

$$\begin{aligned} u_1 + u_2 &= \hat{u}_1 \qquad \text{and} \qquad u_1' + u_2' = 2\hat{u}_1', \\ u_1 - u_2 &= \hat{u}_2 \qquad \text{and} \qquad u_1' - u_2' = 2\hat{u}_2', \end{aligned}$$

one readily obtains

$$\begin{aligned} u_1 &= \frac{1}{2}(\hat{u}_1 + \hat{u}_2) \qquad \text{and} \qquad u_1' = \hat{u}_1' + \hat{u}_2', \\ u_2 &= \frac{1}{2}(\hat{u}_1 - \hat{u}_2) \qquad \text{and} \qquad u_2' = \hat{u}_1' - \hat{u}_2'. \end{aligned} \tag{5.4}$$

From (5.4) it now easily follows that $\langle u_k', u_j \rangle = \delta_{kj}$ for $k, j \in \{1, 2\}$.

Employing ideas from [MP07], we can now prove that the projection $\tilde{S} := u_1 \otimes u_1' + u_2 \otimes u_2' \in \mathcal{L}(F)$ is positive, which is an important preliminary step to show that the projection S in assertion (iv) is positive. It follows from (5.4) that $\tilde{S} = \hat{u}_1 \otimes \hat{u}_1' + \hat{u}_2 \otimes \hat{u}_2'$. We show that $\tilde{S}' \in \mathcal{L}(F')$ is positive. Let $x' \in F_+'$. Then the restriction $x'|_W$ of x' to W is positive, too, i.e. it is contained in W_+'. Hence, since W_+' is the convex hull of the two extreme rays spanned by $\hat{u}_1'|_W$ and $\hat{u}_2'|_W$, we can find two scalars $\alpha_1, \alpha_2 \in [0, \infty)$ such that $x'|_W = \alpha_1 \hat{u}_1'|_W + \alpha_2 \hat{u}_2'|_W$. Thus, we have

$$\tilde{S}'x' = \langle x', \hat{u}_1 \rangle \hat{u}_1' + \langle x', \hat{u}_2 \rangle \hat{u}_2' = \alpha_1 \hat{u}_1' + \alpha_2 \hat{u}_2' \in F_+'.$$

Thus, $\tilde{S}'$ is positive, and hence, so is $\tilde{S}$.

Now, it is time to introduce the missing vectors u_3 and u_3': let $u_3 := e^{\frac{3}{2}\pi A} u_2$ and $u_3' = (e^{\frac{1}{2}\pi A}|_F)' u_2'$. Note that $u_3 \in QE$ since $u_2 \in QE$ and since $e^{\frac{3}{2}\pi A}$ leaves QE invariant. Moreover, $(e^{\frac{1}{2}\pi A}|_F)'$ commutes with $(Q|_F)'$ and since u_2' is contained in the range of $(Q|_F)'$, it follows that $u_3' \in (Q|_F)'F$, too.

Let us now show that property (iii) holds: we already know that $\langle u_k', u_j \rangle = \delta_{kj}$ for all $k, j \in \{1, 2\}$. Moreover, we have the equalities

$$\begin{aligned} \langle u_1', u_3 \rangle &= \langle (e^{\frac{3}{2}\pi A}|_F)' u_1', u_2 \rangle = \langle u_1', u_2 \rangle = 0, \\ \langle u_3', u_1 \rangle &= \langle u_2', e^{\frac{1}{2}\pi A} u_1 \rangle = \langle u_2', u_1 \rangle = 0, \\ \langle u_3', u_3 \rangle &= \langle u_2', e^{\frac{1}{2}\pi A} e^{\frac{3}{2}\pi A} u_2 \rangle = \langle u_2', u_2 \rangle = 1. \end{aligned}$$

We also obtain $\langle u_3', u_2 \rangle = \langle u_2', e^{\frac{1}{2}\pi A} u_2 \rangle = -\langle u_2', u_3 \rangle$ since $e^{\frac{1}{2}\pi A} u_2 = e^{\frac{5}{2}\pi A} u_2 = -e^{\frac{3}{2}\pi A} u_2 = -u_3$. Hence, in order to prove property (iii), it remains to show that $\langle u_2', u_3 \rangle = 0$.

To this end, we take a look at the observation $(*)$ and see that $u_3 = e^{\frac{3}{2}\pi A} u_2$ is the unique element of QE for which we have $u_2 + iu_3 \in \ker(i - A_{\mathbb{C}})$. Hence,

we obtain for every time $t \geq 0$

$$\begin{aligned} 0 \leq \langle 2\hat{u}_2', e^{tA}\hat{u}_2\rangle &= \langle u_1' - u_2', e^{tA}(u_1 - u_2)\rangle = 1 + \langle u_2', e^{tA}u_2\rangle \\ &= 1 + \Big(\langle u_2', u_2\rangle \cos t - \langle u_2', u_3\rangle \sin t\Big) = 1 + \mathrm{Re}\Big(e^{it}(1 + i\langle u_2', u_3\rangle)\Big) \end{aligned}$$

We can now choose $t \in [0, \infty)$ such that the complex number $e^{it}(1 + i\langle u_2', u_3\rangle)$ equals minus the absolute value of $1 + i\langle u_2', u_3\rangle$, and for this t we obtain

$$1 \geq -\mathrm{Re}\Big(e^{it}(1 + i\langle u_2', u_3\rangle)\Big) = |1 + i\langle u_2', u_3\rangle|.$$

This proves that $\langle u_2', u_3\rangle = 0$, as claimed. Hence, property (iii) holds.

Properties (i) and (ii) have already been proved in the process, but we still have to prove (iv). It follows from (iii) that S is a projection on F with range U.

Next we show that S commutes with $e^{tA}|_F$ for all $t \geq 0$; so let $t \geq 0$. First note that we have $e^{tA}|_F(u_1 \otimes u_1') = u_1 \otimes u_1' = (u_1 \otimes u_1')e^{tA}|_F$. Hence, it suffices to show that $e^{tA}|_F$ commutes with $u_2 \otimes u_2' + u_3 \otimes u_3'$. To this end, first observe that

$$e^{tA}Q = \cos(t)Q + \sin(t)e^{\frac{1}{2}\pi A}Q = \cos(t)Q - \sin(t)e^{\frac{3}{2}\pi A}Q$$

according to $(*)$. Using that $Q|_F$ commutes with $e^{sA}|_F$ for every $s \geq 0$, this also implies that

$$\begin{aligned} (e^{tA}|_F)'(Q|_F)' &= \cos(t)(Q|_F)' + \sin(t)(e^{\frac{1}{2}\pi A}|_F)'(Q|_F)' \\ &= \cos(t)(Q|_F)' - \sin(t)(e^{\frac{3}{2}\pi A}|_F)'(Q|_F)'. \end{aligned}$$

Recall that we have $u_3 = e^{\frac{3}{2}\pi A}u_2$ and thus $u_2 = e^{\frac{1}{2}\pi A}u_3$ as well as $u_3' = (e^{\frac{1}{2}\pi A}|_F)'u_2'$ and thus $u_2' = (e^{\frac{3}{2}\pi A}|_F)'u_3'$. Hence, we obtain the formulas

$$e^{tA}u_2 = \cos(t)u_2 - \sin(t)u_3 \quad \text{and} \quad e^{tA}u_3 = \cos(t)u_3 + \sin(t)u_2, \tag{5.5}$$

$$(e^{tA}|_F)'u_2' = \cos(t)u_2' + \sin(t)u_3' \quad \text{and} \quad (e^{tA}|_F)'u_3' = \cos(t)u_3' - \sin(t)u_2' \tag{5.6}$$

for all $t \geq 0$. Using these formulas, one can now check by a brief computation that we indeed have $e^{tA}|_F(u_2 \otimes u_2' + u_3 \otimes u_3') = (u_2 \otimes u_2' + u_3 \otimes u_3')e^{tA}|_F$.

Now we can finally show that S is positive. We already know that $\tilde{S}$ is positive. To show that $S = \tilde{S} + u_3 \otimes u_3'$ is positive, too, first note that we have $(e^{\pi A}|_F)'u_3' = -u_3'$ since u_3' is contained in the range of $(Q|_F)'$. Now, let $x \in F_+$. Then we have $\langle u_3', e^{\pi A}x\rangle = -\langle u_3', x\rangle$. Since the mapping $t \mapsto \langle u_3', e^{tA}x\rangle$ is continuous from $[0, \infty)$ to $\mathbb{R}$, it follows that we have $\langle u_3', e^{tA}x\rangle = 0$ for some $t \in [0, \pi]$.

As our semigroup is 2π-periodic on F, we can find a time $s \geq 0$ such that $e^{sA}|_F e^{tA}|_F = I_F$. Using that $e^{tA}|_F$ commutes with S we thus obtain

$$Sx = e^{sA}|_F e^{tA}|_F Sx = e^{sA}|_F Se^{tA}|_F x$$

$$= e^{sA}|_F \tilde{S} e^{tA}|_F x + e^{sA}|_F \langle u_3', e^{tA}x\rangle u_3 = e^{sA}|_F \tilde{S} e^{tA}|_F x \geq 0$$

since $\tilde{S}$ is positive.

Thus, we have proved all assertions (i)–(iv); in particular, we have constructed a positive projection $S \in \mathcal{L}(F)$ with range U. Note that all the vectors u_1, u_2, u_3 are linearly independent due to (iii). This shows that U is three-dimensional. Since E is projectively non-Euclidean, so is F and hence, the ordered Banach space (U, U_+) cannot by isomorphic to the ordered Banach space $(\mathbb{R}^3, \mathbb{R}^3_{+,e_1,e_1})$ (where $\mathbb{R}^3$ is endowed with the Euclidean norm).

However, since $u_1 + u_2$ and $u_1 - u_2$ span, respectively, the two extreme rays of the positive cone in $\operatorname{span}_{\mathbb{R}}\{u_1, u_2\}$, it follows that $\alpha_1 u_1 + \alpha_2 u_2$ (where $\alpha_1, \alpha_2 \in \mathbb{R}$) is positive if and only if $\alpha_1 \geq |\alpha_2|$. Moreover, we know that $(e^{tA}|_U)_{t\geq 0}$ is a positive periodic semigroup on U. Hence, for every $u \in U$ we have $u \geq 0$ if and only if $e^{tA}|_U \geq 0$ for some or, equivalently, all $t \geq 0$. It thus follows from formula (5.5) that $\sum_{k=1}^3 \alpha_k u_k$ (where $\alpha_k \in \mathbb{R}$) is positive if and only if $\alpha_1 \geq \sqrt{\alpha_2^2 + \alpha_3^2}$. Hence, (U, U_+) *is* in fact isomorphic to $(\mathbb{R}^3, \mathbb{R}^3_{+,e_1,e_1'})$. This is a contradiction. □

Note that Theorem 5.1.1 can, in particular, be applied on projectively lattice-ordered spaces since those are projectively non-Euclidean according to Proposition 4.2.4. On the other hand one would expect that the proof of Theorem 4.2.4 becomes a bit simpler for those spaces. In the following remark we discuss this in more detail:

Remark 5.1.3. Assume that the assumptions of Theorem 4.2.4 are fulfilled and that, in addition, E is projectively lattice ordered.

(a) If the first one of the assumptions (a) and (b) in Theorem 5.1.1 is fulfilled, then the proof of the theorem becomes significantly simpler:

First one constructs the subspace $F \subseteq E$ exactly as it is done in the above proof. Then, however, one observes that F is finite-dimensional due to assumption (a); since F is the range of a positive projection, this implies that F is lattice ordered and hence, as an ordered Banach space, isomorphic to $\mathbb{R}^d$ with its usual lattice order. Thus, one can now apply standard results from Perron–Frobenius theory on Banach lattices (or even on finite-dimensional spaces) to conclude that the point spectrum of $A|_F$ intersects $i\mathbb{R}$ at most in 0.

In the special case of lattice ordered Banach spaces, the fact that certain finite-dimensionality assumptions on the eigenspaces (or compactness assumptions on the operator) yield interesting spectral and asymptotic results for positive operators was already observed by Kreĭn and Rutman in [KR48, Section 8].

(b) Now assume instead that assumption (b) of the theorem is fulfilled. Then the space F in the proof can still be infinite-dimensional, and thus it is not necessarily lattice ordered (and even if it is, it is not clear whether it is a Banach lattice with respect to some equivalent norm). It thus seems that the assumption on E to be projectively lattice ordered does not yield any simplification for the proof in this case.

Finally, note that the case where E is projectively lattice ordered and contains an order unit is particularly interesting since, as the Examples 4.2.9 show, there are many function spaces which have those two properties.

5.2 Asymptotic Behaviour

We now employ Theorem 5.1.1 in order to obtain convergence results for positive C_0-semigroups. As pointed out earlier, one can find many results in the literature where the long-term behaviour of operator semigroups is analysed with the help of positivity. Many of those results either impose some strengthened form of positivity assumption on the semigroup, or they focus on the important special case of Banach lattices; see the introduction to Chapter 4 for some references and see also the notes at the end of the present chapter for some more details. On the other hand, a particular example for an analysis of the long-term behaviour on more general spaces, namely on ordered Banach spaces with the Riesz decomposition property, is due to Bartoszek and can be found in the paper [Bar92]. It follows from Example 4.2.5(b) that every ordered Banach space with the Riesz decomposition property is projectively lattice ordered and thus, in particular, projectively non-Euclidean. Hence, our results below apply to a much wider range of spaces; we should, however, point out that the main result in [Bar92] is somewhat different from our results below, concerning the further assumptions as well as the conclusions.

Theorem 5.2.1. *Let E be an ordered Banach space whose cone is generating and projectively non-Hilbert. Let $(e^{tA})_{t\geq 0}$ be a positive and bounded C_0-semigroup on E.*

(a) *If A has compact resolvent and $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ is bounded, then e^{tA} converges strongly as $t \to \infty$.*

(b) *If $(e^{tA})_{t\geq 0}$ is eventually compact or, more generally, quasi-compact, then e^{tA} converges with respect to the operator norm as $t \to \infty$.*

Proof. (a) This follows from [EN00, Theorem V.3.7] and Theorem 5.1.1 since assumption (a) of this theorem is fulfilled.

(b) This follows from [EN00, Corollary V.3.2] and, again, from Theorem 5.1.1 since assumption (a) of the theorem is fulfilled. □

In the above theorem we only considered cases in which all eigenspaces of the generator are automatically finite-dimensional. In the most important case where the Banach space E is projectively lattice ordered, this assumption allows for a much simpler proof of the spectral result in Theorem 5.1.1, as explained in Remark 5.1.3(a). Hence, at least if the space E is projectively lattice ordered, the above convergence theorem is not particularly deep.

The situation is different in the following convergence result: there we assume that E contains an ordered unit, but the eigenspaces of the generator A are allowed to be infinite-dimensional. Even if E is projectively lattice ordered Theorem 5.1.1 (on which the following convergence result is based) is much deeper in this case, as explained in Remark 5.1.3(b).

Theorem 5.2.2. *Let E be an ordered Banach space which is projectively non-Euclidean and which contains an order unit. Let $(e^{tA})_{t\geq 0}$ be a positive and bounded C_0-semigroup on E.*

If the semigroup $(e^{tA})_{t\geq 0}$ has relatively compact orbits and if $\sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ is bounded, then e^{tA} converges strongly as $t \to \infty$.

Proof. This follows from, for instance, [EN00, Theorem V.2.14] and from Theorem 5.1.1 since assumption (b) of the theorem is fulfilled. □

We point out once again that the assumptions on E to be projectively non-Euclidean and to contain an order unit are fulfilled by a couple of interesting spaces, see Examples 4.2.9.

5.3 Notes & Open Problems

Perron–Frobenius theory on Banach lattices As pointed out in the introduction to Chapter 4, there are several types of Perron–Frobenius results which require different kinds of conditions on the underlying Banach space and on the operator or semigroup under consideration. A particularly rich theory is available on Banach lattices; we refer to [Kri69], [Sch74, Sections V.4 and V.5], [MN91, Chapter 4], [Gro95] and [AGG$^+$86, Sections B-III and C-III] for an overview. Some recent contributions to the theory, which focus mainly on symmetry properties (in particular, cyclicity) of the peripheral spectrum, can be found in [Glü16c] and [Glü16a]. A particularly beautiful result is available for positive elements of so-called *Banach lattice algebras*, see [Sch71, Section 4].

Perron–Frobenius theory in the setting of C^*-algebras Another important class of ordered Banach spaces where many Perron–Frobenius like theorems have been proved are C^*-algebras and related spaces. See for example [EHK78], [AHK79], [Gro83], [AGG+86, Section D-III] and [Łuc10] for some results in this context. On the other hand, we note that our results from this chapter cannot be applied to positive semigroups on C^*-algebras as indicated in Example 4.2.14(c) and as also explained in the notes at the end of Chapter 4. Moreover, we recall again that a certain class of Perron-Frobenius like results can *never* be true on any non-commutative C^*-algebra as was recently proved in [Glü16b, Theorem 2.1].

Perron–Frobenius theory on ordered Banach algebras During the last two decades an extensive spectral theory for positive elements of ordered Banach algebras has been developed. We refer for instance to [Muz12] and to the references therein for an overview.

Geometric Perron–Frobenius theory In particular in the finite-dimensional setting there exist many different approaches to Perron–Frobenius theory (see for instance the survey article [Mac00] for an overview). In our context it is worthwhile to point out one approach which one could describe as *geometric Perron–Frobenius theory*. An early and important contribution to this theory is a paper by Pullman [Pul71] where the so-called *core* of a positive linear operator was introduced. For a detailed overview of large parts of the theory we refer to the survey article by Tam [Tam01].

Existence of subspaces with Euclidean cones The notion of a projectively non-Euclidean ordered Banach space corresponds to the notion of a projectively non-Hilbert space from Part I of the thesis. Thus, it would be quite natural to introduce an analogue for the notion "extremely non-Hilbert space", too. For example, we could call an ordered Banach space E *extremely non-Euclidean* if there is no three-dimensional subspace $F \subseteq E$ for which the cone $F_+ := F \cap E_+$ is Euclidean in F. On such spaces we could then try to prove similar spectral results as in Theorem 5.1.1, but without a priori assumptions on the spectrum (compare Theorem 2.1.1). Yet, we chose not to pursue this in detail since the discussion after Example 4.2.12 indicates that it might be difficult to find interesting examples for such spaces (while projectively lattice ordered spaces, which exist in abundance, provide a natural source of examples for projectively non-Euclidean spaces).

Eventual and asymptotic positivity It turns out that our main result in this chapter, Theorem 5.1.1, can also be proved under slightly weaker assumptions.

In fact, one does not really need to assume the semigroup $(e^{tA})_{t\geq 0}$ to be positive, but some kind of "asymptotic positivity" suffices. This could be interesting in view of Part III of the thesis where we study eventually and asymptotically positive semigroup (though, only on Banach lattices). Yet, such a generalisation of Theorem 5.1.1 does not really contribute to the goal of Part II of the thesis (which is to study the relation of Perron–Frobenius theory and the geometry of positive cones). Moreover, the proof of Theorem 5.1.1 is long enough anyway.

Invariant convex sets Looking at the results presented in Parts I and II of the thesis, it is natural to assume that some kind of generalisation should be possible. In fact, if $(e^{tA})_{t\geq 0}$ is a C_0-semigroup on a Banach space E which leaves a certain closed, convex set $K \subseteq E$ invariant, then we would expect that this has consequences for the spectrum of A in case that K is, in some sense, large enough (we cannot expect non-trivial results if K consists for instance only of 0) and if its behaves in some sense "non-Euclidean". In finite dimensions this idea was also suggested by Lyubich [Lyu97, Section 3] and some preliminary results were presented by him in this reference. On the other hand, despite the generality and elegance of this approach, we find it somewhat doubtful whether a generalisation to infinite dimensions, though it would certainly not be trivial, would yield any significant new insights. Hence, we shall not pursue this here.

Part III

Eventual Positivity

6 Eventual and Asymptotic Positivity – Spectral Properties

The third part of this thesis is devoted to semigroups which are not positive but become, in some sense, positive for large times. In contrast to Part II we put particular emphasis on applications now (see, in particular, Chapters 8 and 11). Therefore, we restrict our attention to those ordered Banach spaces which are most important for the applications we have in mind, namely to Banach lattices. In contrast to the preceding to parts of the thesis, it now appears that the easiest way to present our results is to work on *complex* Banach lattices from the beginning; we refer to Sections C.4 and C.5 in the appendix for an introduction to the theory of complex Banach lattices.

Two of the most basic notions for this part of the thesis are already defined in the first section of this chapter: eventually positive semigroups and asymptotically positive semigroups. The rest of this chapter is devoted to proving that some important Perron–Frobenius type spectral results carry over from positive semigroups to the case of eventually positive semigroups. In Section 6.2 we deal with the spectral bound of the semigroup generator: we give sufficient conditions for the spectral bound to be in the spectrum, or to be even an eigenvalue which admits a positive eigenvector. Moreover, we prove that the spectral bound of the generator of an eventually positive semigroup coincides with the growth bound of the semigroup on a couple of important spaces. In Section 6.3 we deal not only with the spectral bound, but with the entire peripheral (point) spectrum of the semigroup generator; we give, for instance, a sufficient condition for the peripheral point spectrum to be cyclic.

While some of the Perron–Frobenius type results in this chapter are valid under rather weak assumptions on the semigroup, we shall see in the subsequent chapters that, under stronger assumptions on the semigroup, Perron–Frobenius results provide the appropriate setting to even *characterise* (some versions of) eventual positivity. Hence, eventually positive semigroups do not only allow for some generalisations of Perron–Frobenius theory, but, under some assumptions, they provide the most general setting in which such generalisations can be obtained.

Most of the material in this chapter can be found in the papers [DGK16b] and [DGK16a]. An exception is Theorem 6.3.2 which has not been published, or submitted for publication, yet.

6.1 Some Basic Notions

One of the most essential concepts in Part III of the thesis, and a leitmotif for the definition of many similar notions throughout the following chapters, is that of an *eventually positive semigroup*. In fact, we have to distinguish two different versions of this notion, an *individual* one and a *uniform* one:

Definition 6.1.1 (Eventually positive semigroups). Let E be a complex Banach lattice and let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on E.

(i) The semigroup $(e^{tA})_{t\geq 0}$ is called *individually eventually positive* if for every $0 \leq f \in E$ there is a time $t_0 \geq 0$ such that $e^{tA}f \geq 0$ for all $t \geq t_0$.

(ii) The semigroup $(e^{tA})_{t\geq 0}$ is called *uniformly eventually positive* if there is a time $t_0 \geq 0$ such that $e^{tA} \geq 0$ for all $t \geq t_0$.

In finite dimensions, it is easy to see that both the individual and the uniform version of eventual positivity coincide. Yet, we will see later on, in Section 10.3, that the distinction between both of them is essential on infinite-dimensional Banach lattices.

At first glance, the concept of eventual positivity might appear to be a bit obscure; yet, we will see in numerous examples (see Chapters 8 and 11) that this and related phenomena occur frequently in the analysis of concrete differential equations. For the moment, however, we restrict ourselves to demonstrating the concept by a simple but quite instructive finite-dimensional example.

Example 6.1.2. Let E be a finite-dimensional complex Banach lattice with dimension at least 3. Then there exists a real linear operator $A \in \mathcal{L}(E)$ such that $s(A) = 0$ is a dominant spectral value of A and such that the semigroup $(e^{tA})_{t\geq 0}$ is bounded and individually (equivalently: uniformly) eventually positive, but not positive.

Due to [Sch74, Corollary 1 to Theorem II.3.9] it suffices to show that this is true on the three-dimensional space $\mathbb{C}^3$ (whose real part $\mathbb{R}^3$ is endowed with any Banach lattice norm). Now, consider the basis (u_1, u_2, u_3) of $\mathbb{C}^3$ which is given by

$$u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}.$$

Let $A \in \mathcal{L}(\mathbb{C}^3)$ be the operator whose representation matrix with respect to the basis (u_1, u_2, u_3) is given by

$$\begin{pmatrix}0 & 0 & 0\\0 & -1 & -1\\0 & 1 & -1\end{pmatrix}.$$

Then, for every $x \in \mathbb{R}^3$, the orbit $(e^{tA}x)_{t\geq 0}$ spirals towards the axis spanned by u_1. It is thus easy to conclude from geometric considerations (or, alternatively, from an explicit computation) that the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually positive, but not positive. Moreover, the further properties of A claimed above are clearly fulfilled.

We refer for instance to [NT08, Example 3.5] for a numerical example of a matrix $A \in \mathbb{R}^{4\times 4}$ which generates an eventually positive semigroup (note however that the terminology in [NT08] is slightly different from ours).

In the case $\dim E = 2$, it can be shown that no eventually positive real semigroup exists which is not already positive; see Proposition 8.1.7.

The second concept we introduce in this section is that of an *asymptotically positive* semigroup; this notion is somewhat weaker than that of an eventually positive semigroup. For the definition we need the distance $\operatorname{dist}(\cdot, E_+)$ from the positive cone in a complex Banach lattice. For several properties of this distance function, which we used tacitly throughout Part III of the thesis, we refer to Proposition B.1.7 in the appendix.

Definition 6.1.3 (Asymptotically positive semigroups). Let E be a complex Banach lattice and let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on E. Assume that $\mathrm{s}(A) > -\infty$ and that the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded.

(i) The semigroup $(e^{tA})_{t\geq 0}$ is called *individually asymptotically positive* if $\operatorname{dist}(e^{t(A-\mathrm{s}(A))}f, E_+) \to 0$ as $t \to \infty$ for every $f \geq 0$.

(ii) The semigroup $(e^{tA})_{t\geq 0}$ is called *uniformly asymptotically positive* if for every $\varepsilon > 0$ there exists a time $t_0 \geq 0$ such that $\operatorname{dist}(e^{t(A-\mathrm{s}(A))}f, E_+) \leq \varepsilon\|f\|$ for all $t \geq t_0$ and all $0 \leq f \in E$.

Again, we shall see that the distinction between individual and uniform behaviour is really necessary, see Example 7.3.5 below.

It is easy to give examples of asymptotically positive semigroups which are not eventually positive. In two dimensions, we can for instance consider the semigroup $(e^{tA})_{t\geq 0}$ where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & i-1 \end{pmatrix}, \qquad e^{tA} = \begin{pmatrix} 1 & 0 \\ 0 & e^{t(i-1)} \end{pmatrix} \quad \text{for all } t \geq 0.$$

This also shows that asymptotic positivity can occur for semigroups which are not real. On the other hand, if the semigroup (equivalently: A) is real, then asymptotic positivity in two dimensions already implies positivity of the semigroup, see Proposition 8.1.6. Yet, there exist real semigroups in three dimensions which are asymptotically positive, but not eventually positive. An

example is the semigroup $(e^{tA})_{t\geq 0}$ given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad e^{tA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t}\cos t & -e^{-t}\sin t \\ 0 & e^{-t}\sin t & e^{-t}\cos t \end{pmatrix} \quad \text{for all } t \geq 0.$$

We refer the reader to Section 8.1 for more information about asymptotically positive semigroups in finite dimensions, and to the rest of Chapter 8 for several examples of asymptotically positive semigroups in infinite dimensions.

6.2 The Spectral Bound

In this section we prove various results for the spectral bound of the generator A of an individually eventually positive semigroup. To this end, we start by considering the representation of the resolvent $\mathcal{R}(\cdot, A)$ as the Laplace transform of the semigroup: it is well-known that, for any C_0-semigroup $(e^{tA})_{t\geq 0}$ on a complex Banach space E, the resolvent $\mathcal{R}(\lambda, A)$ can be represented by means of a Laplace integral whenever $\operatorname{Re}\lambda > \omega_0(A)$; see e.g. [EN00, Theorem II.1.10]. Moreover, if E is a Banach lattice and the semigroup $(e^{tA})_{t\geq 0}$ is positive, such a representation of $\mathcal{R}(\lambda, A)$ even holds for $\operatorname{Re}\lambda > \mathrm{s}(A)$, where the Laplace integral has to be understood as an improper Riemann integral; see [AGG+86, Theorem C-III.1.2] or [ABHN11, Theorem 5.3.1]. The first result in this section shows that the same is true if the semigroup $(e^{tA})_{t\geq 0}$ is only assumed to be individually eventually positive.

Proposition 6.2.1. *Let $(e^{tA})_{t\geq 0}$ be an individually eventually positive C_0-semigroup on a complex Banach lattice E. For all $f \in E$ and all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \mathrm{s}(A)$ we then have*

$$\mathcal{R}(\lambda, A)f = \int_0^\infty e^{-t\lambda} e^{tA} f \, d\lambda,$$

where the integral is understood as an improper Riemann integral.

Proof. It suffices to show the assertion for $f \geq 0$. By assumption there exists $t_0 \geq 0$ such that $e^{tA}f \geq 0$ for all $t \geq t_0$. Now, consider the functions $u, v : [0,\infty) \to E$ which are given by

$$u(t) = e^{tA}f \quad \text{and} \quad v(t) = e^{(t_0+t)A}f$$

for all $t \geq 0$. Let $\operatorname{abs}(u)$ and $\operatorname{abs}(v)$ denote the abscissas of convergence of the Laplace transforms $\hat{u}$ and $\hat{v}$ of u and v; see [ABHN11, Section 1.4] for details. We have $\operatorname{abs}(u) \leq \omega_0(A)$ and from the formula

$$\int_0^T e^{-t\lambda} u(t)\, dt = \int_0^{t_0} e^{-t\lambda} u(t)\, dt + e^{-t_0\lambda} \int_0^{T-t_0} e^{-t\lambda} v(t)\, dt,$$

which is valid for all $T \geq t_0$ and all $\lambda \in \mathbb{C}$, we can conclude that $\operatorname{abs}(u) = \operatorname{abs}(v)$. According to [ABHN11, Theorem 1.5.1], both Laplace transforms $\hat{u}$ and $\hat{v}$ are analytic on the half plane $\{\lambda \in \mathbb{C} | \operatorname{Re}\lambda > \operatorname{abs}(v)\}$. Moreover, $\hat{u}(\lambda)$ coincides with $\mathcal{R}(\lambda, A)f$ whenever $\operatorname{Re}\lambda > \omega_0(A)$; if we can show that $\mathrm{s}(A) \geq \operatorname{abs}(v)$, then $\hat{u}(\lambda)$ exists and coincides with $\mathcal{R}(\lambda, A)$ (due to the Identity theorem for analytic functions) whenever $\operatorname{Re}\lambda > \mathrm{s}(A)$, which proves the assertion.

So assume for a contradiction that $\mathrm{s}(A) < \operatorname{abs}(v)$. Then $\operatorname{abs}(v) > -\infty$ and $\operatorname{abs}(v) \in \rho(A)$. As $v(t)$ is positive for all $t \geq 0$, it follows from [ABHN11, Theorem 1.5.3] that $\hat{v}$ has a singularity at $\operatorname{abs}(v)$, meaning that $\hat{v}$ cannot be extended analytically to any open neighbourhood of $\operatorname{abs}(v)$. However, we have

$$\mathcal{R}(\lambda, A)f = \hat{u}(\lambda) = \int_0^{t_0} e^{-t\lambda} u(t)\, dt + e^{-t_0\lambda}\hat{v}(\lambda)$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \operatorname{abs}(v)$, which shows that $\mathcal{R}(\cdot, A)f$ has a singularity at $\operatorname{abs}(v)$, too. This is a contradiction since $\operatorname{abs}(v) \in \rho(A)$. □

A moment of reflection reveals that the above proof actually shows that [ABHN11, Theorem 1.5.3] is not only true for positive functions, but for eventually positive functions, too.

Before we proceed we prove an auxiliary result about eventually positive operator families. Recall that if T is a positive operator on a complex Banach lattice E, then we have $|Tf| \leq T|f|$ for all $f \in E$. The following lemma contains a similar observation for eventually positive operator families, at least if $f \in E_{\mathbb{R}}$.

Lemma 6.2.2. *Let E be a complex Banach lattice and let $(T_j)_{j\in J} \subseteq \mathcal{L}(E)$ be a net of operators on E which is individually eventually positive in the following sense: for every $f \in E_+$ there exists $j_0 \in J$ such that $T_j f \geq 0$ for all $j \geq j_0$.*

Then, for every f in the real part $E_{\mathbb{R}}$ of E, there exists $j_1 \in J$ such that $|T_j f| \leq T_j|f|$ for all $j \geq j_1$.

Proof. We can choose $j_1 \in J$ such that $T_j(|f| - f) = 2T_j f^- \geq 0$ and $T_j(|f| + f) = 2T_j f^+ \geq 0$ for all $j \geq j_1$. For all those j we thus obtain $T_j|f| \geq T_j f$ and $T_j|f| \geq -T_j f$, and therefore $T_j|f| \geq T_j f \vee (-T_j f) = |T_j f|$. □

If $(e^{tA})_{t\geq 0}$ is a positive C_0-semigroup on a Banach lattice E, the Laplace representation formula of the resolvent immediately yields the estimate $|\mathcal{R}(\lambda, A)f| \leq \mathcal{R}(\operatorname{Re}\lambda, A)|f|$ for all $f \in E$ whenever $\operatorname{Re}\lambda > \mathrm{s}(A)$; this estimate has many applications in the spectral theory of positive semigroups. In the next lemma we show, by means of Proposition 6.2.1, a similar estimate for individually eventually positive semigroups; since the positivity occurs only eventually, the estimate contains a bounded error term.

Lemma 6.2.3. *Let $(e^{tA})_{t\geq 0}$ be an individually eventually positive C_0-semigroup on a complex Banach lattice E. Then for every $f \in E_{\mathbb{R}}$ there exists a map $r_f : (\mathrm{s}(A), \infty) \to E$ which fulfils the following properties:*

(a) *We have $|\mathcal{R}(\lambda, A)f| \leq \mathcal{R}(\operatorname{Re}\lambda, A)|f| + r_f(\operatorname{Re}\lambda)$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \mathrm{s}(A)$.*

(b) *If $\mathrm{s}(A) > -\infty$, then r_f is bounded on $(\mathrm{s}(A), \infty)$.*

(c) *If $\mathrm{s}(A) = -\infty$, then r_f is bounded on (c, ∞) for every $c \in \mathbb{R}$.*

Proof. Let $f \in E_{\mathbb{R}}$. According to Lemma 6.2.2 there exists $t_0 \geq 0$ such that $|e^{tA}f| \leq e^{tA}|f|$ for all $t \geq t_0$. For all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \mathrm{s}(A)$ and all $T \geq t_0$ we thus obtain

$$\begin{aligned}\Big|\int_0^T e^{-t\lambda}e^{tA}f\,dt\Big| &\leq \int_0^T e^{-t\operatorname{Re}\lambda}|e^{tA}f|\,dt \\ &\leq \int_0^T e^{-t\operatorname{Re}\lambda}e^{tA}|f|\,dt + \int_0^{t_0} e^{-t\operatorname{Re}\lambda}(|e^{tA}f| - e^{tA}|f|)\,dt.\end{aligned}$$

Proposition 6.2.1 asserts that the expressions on the left and on the right in the above estimate converge as $T \to \infty$ and that we obtain

$$|\mathcal{R}(\lambda, A)f| \leq \mathcal{R}(\operatorname{Re}\lambda, A)|f| + \int_0^{t_0} e^{-t\operatorname{Re}\lambda}(|e^{tA}f| - e^{tA}|f|)\,dt$$

whenever $\operatorname{Re}\lambda > \mathrm{s}(A)$. Defining the latter integral as $r_f(\operatorname{Re}\lambda)$, we obtain the assertion. □

Let us give a couple of applications of Lemma 6.2.3. Recall that for an operator $A : E \supseteq D(A) \to E$ on a complex Banach space E the *abscissa of uniform boundedness of the resolvent* (or the *pseudo-spectral bound*) $\mathrm{s}_0(A)$ of A is defined as

$$\mathrm{s}_0(A) := \inf\{c \in \mathbb{R} : c > \mathrm{s}(A),\ \sup_{\operatorname{Re}\lambda > c} \|\mathcal{R}(\lambda, A)\| < \infty\} \in [-\infty, \infty].$$

Obviously, we always have $\mathrm{s}(A) \leq \mathrm{s}_0(A)$; for positive semigroups we even have $\mathrm{s}(A) = \mathrm{s}_0(A)$, a fact which follows readily from that estimate $|\mathcal{R}(\lambda, A)f| \leq \mathcal{R}(\operatorname{Re}\lambda, A)|f|$ which is true for all f and whenever $\operatorname{Re}\lambda \geq \mathrm{s}(A)$ (compare [AGG+86, Corollary C-III.1.3] and also [AGG+86, formula (2.13) in the proof of Proposition C-III.2.7]). Using Lemma 6.2.3 we now obtain the same result for individually eventually positive semigroups:

Corollary 6.2.4. *Let $(e^{tA})_{t\geq 0}$ by an individually eventually positive C_0-semigroup on a complex Banach lattice E. Then we have $\mathrm{s}(A) = \mathrm{s}_0(A)$.*

Proof. Let $c > \mathrm{s}(A)$ and $f \in E_{\mathbb{R}}$. With r_f as in Lemma 6.2.3 we obtain

$$\sup_{\operatorname{Re}\lambda>c} \|\mathcal{R}(\lambda, A)f\| \leq \sup_{\operatorname{Re}\lambda>c} \|\mathcal{R}(\operatorname{Re}\lambda, A)|f|\| + \sup_{\operatorname{Re}\lambda>c} \|r_f(\operatorname{Re}\lambda)\| < \infty.$$

Hence, the assertion follows from the uniform boundedness principle. □

A second application of the estimate in Lemma 6.2.3 is the following result which is well-known for positive semigroups, see [AGG+86, Corollary C-III.1.4].

Theorem 6.2.5. *Let $(e^{tA})_{t\geq 0}$ be an individually eventually positive C_0-semigroup on a complex Banach lattice E. If $\mathrm{s}(A) > -\infty$, then $\mathrm{s}(A) \in \sigma(A)$.*

Proof. Let $\mathrm{s}(A) > -\infty$, i.e. $\sigma(A) \neq \emptyset$. Choose a sequence a complex numbers (λ_n) such that $\mathrm{s}(A) < \operatorname{Re}\lambda_n$ for all indices n and such that $\operatorname{dist}(\lambda_n, \sigma(A)) \to 0$. We then have $\operatorname{Re}\lambda_n \to \mathrm{s}(A)$ and $\|\mathcal{R}(\lambda_n, A)\| \to \infty$.

By means of the uniform boundedness principle we can find a vector $f \in E_{\mathbb{R}}$ and a subsequence (λ_{n_k}) of (λ_n) such that $\|\mathcal{R}(\lambda_{n_k}, A)f\| \to \infty$. Due to Lemma 6.2.3, this implies $\|\mathcal{R}(\operatorname{Re}\lambda_{n_k}, A)|f|\| \to \infty$. Since $\operatorname{Re}\lambda_{n_k} \to \mathrm{s}(A)$, we conclude that $\mathrm{s}(A) \in \sigma(A)$. □

For a general C_0-semigroup $(e^{tA})_{t\geq 0}$ one always has the (trivial) estimates $\mathrm{s}(A) \leq \mathrm{s}_0(A) \leq \omega_0(A)$. For positive semigroups and, according to Corollary 6.2.4, also for individually eventually positive semigroups we even know that $\mathrm{s}(A) = \mathrm{s}_0(A)$. Moreover, on certain classes of Banach lattices, one can even prove the equality $\mathrm{s}(A) = \omega_0(A)$ whenever the semigroup $(e^{tA})_{t\geq 0}$ is positive (see e.g. [AGG+86, Theorem C-IV.1.1(a)] and [ABHN11, Theorem 5.3.6]. Let us now show that at least on three important classes of Banach lattices, the same is true for individually eventually strongly positive semigroups:

Theorem 6.2.6. *Let $(e^{tA})_{t\geq 0}$ be an individually eventually positive C_0-semigroup on a complex Banach lattice E and assume that one of the following assertions is fulfilled:*

(a) *E is a Hilbert space.*

(b) *E is an AL-space.*

(c) *E is an AM-space with unit and A is real.*

Then $\mathrm{s}(A) = \omega_0(A)$.

Proof. (a) This follows readily from Corollary 6.2.4 and the Gearhart–Prüss theorem [ABHN11, Theorem 5.2.1].

For any of the remaining two assertions it suffices to prove that the inequality $s(A) < 0$ implies the inequality $\omega_0(A) \leq 0$, so assume for the rest of the proof that $s(A) < 0$.

(b) Let $f \in E_+$ and choose $t_0 \geq 0$ such that $e^{tA} f \geq 0$ for all $t \geq t_0$. Due to the additivity of the norm on the positive cone we have

$$\int_0^T \|e^{tA} f\| \, dt \leq \int_0^{t_0} \|e^{tA} f\| \, dt + \Big\| \int_0^{t_0} e^{tA} f \, dt \Big\| + \Big\| \int_0^T e^{tA} f \, dt \Big\|$$

for all $T \geq t_0$. Since $s(A) < 0$, it follows from Proposition 6.2.1 that the latter integral converges to a vector in E as $T \to \infty$. Hence, we have $\int_0^T \|e^{tA} f\| \, dt < \infty$ for all $f \in E_+$, and hence for all $f \in E$. It thus follows from the Datko–Pazy theorem, see [EN00, Theorem 5.1.8], that the semigroup e^{tA} converges to 0 with respect to the operator norm as $t \to \infty$. In particular, $\omega_0(A) < 0$.

(c) We may assume that $E = \mathcal{C}(K; \mathbb{C})$ for some compact Hausdorff space K. For every $f \in E$ Proposition 6.2.1 shows that the abscissa of convergence for the Laplace transform of the orbit $(e^{tA} f)_{t \geq 0}$ is at most $s(A)$. Hence, it follows from [ABHN11, Proposition 1.4.5(a)] that, whenever $\operatorname{Re} \lambda > s(A)$, the operator $\int_0^T e^{-\lambda t} e^{tA} \, dt$ converges with respect to the operator norm as $T \to \infty$ (where the integral is understood in the strong sense, of course). In particular, for $\lambda = 0$ we obtain $M := \sup_{T \geq 0} \| \int_0^T e^{tA} \, dt \| < \infty$.

Since the domain $D(A)$ is dense in E and since A is real, we can find a vector $0 \ll_{\mathbb{1}_K} u \in D(A)$. For every $t \geq 0$ we have

$$\|e^{tA} u - u\| = \| \int_0^t e^{sA} Au \, ds \| \leq M \|Au\|;$$

hence, the orbit $(e^{tA} u)_{t \geq 0}$ is bounded in E.

Now, let $f \in E_+$. We have $0 \leq f \leq cu$ for some constant $c \geq 0$. For all sufficiently large $t \geq 0$ we thus obtain $0 \leq e^{tA} f \leq c e^{tA} u$. This shows that the orbit $(e^{tA} f)_{t \geq 0}$ is bounded for all $f \in E_+$ and hence, for all $f \in E$. Thus, the semigroup $(e^{tA})_{t \geq 0}$ is bounded due to the uniform boundedness principle; in particular, $\omega_0(A) \leq 0$. □

If the C_0-semigroup $(e^{tA})_{t \geq 0}$ is even positive, the assertions of the above theorem remain valid on L^p-spaces, at least over σ-finite measure spaces; see e.g. [ABHN11, Theorem 5.3.6] or [vN96, Theorem 3.5.3]. It is an interesting questions whether this remains true for individually eventually positive semigroups; see the notes at the end of this chapter for a more detailed discussion.

Let $(e^{tA})_{t \geq 0}$ be an individually eventually positive semigroup. According to Theorem 6.2.5, the spectral bound $s(A)$ is then a spectral value, unless it equals $-\infty$. If the $s(A)$ is, moreover, a pole of the resolvent (which is, e.g., always the case if the resolvent $\mathcal{R}(\cdot, A)$ is compact), then it is, of course, also

an eigenvalue of A. The following proposition now shows that it even admits a positive eigenvector and a positive eigenfunctional. For related results about positive semigroups, see for instance [AGG+86, Corollary B-III.1.4, Theorem B-III.1.6, Remark B-III.2.15(a), Theorem C-III.1.1(c) and (d) and Proposition C-III.3.5].

Proposition 6.2.7. *Let $(e^{tA})_{t\geq 0}$ be an individually eventually positive C_0-semigroup on a complex Banach lattice E and suppose that $\mathrm{s}(A) > -\infty$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$. Then $\mathrm{s}(A)$ is an eigenvalue of A which admits a positive eigenvector; moreover, $\mathrm{s}(A) = \mathrm{s}(A')$ is also an eigenvalue of A' which admits a positive eigenvector.*

Proof. We may assume that $\mathrm{s}(A) = 0$. Let $m \in \mathbb{N}$ be the order of 0 as a pole of $\mathcal{R}(\cdot, A)$ and let

$$\mathcal{R}(\lambda, A) = \sum_{k=-m}^{\infty} Q_k \lambda^k$$

denote the Laurent series expansion of the resolvent $\mathcal{R}(\cdot, A)$ about 0. Then Q_{-m} is non-zero and its range is contained in $\ker A$; see Proposition A.3.2(a). Since the resolvent of the adjoint operator A' is given by

$$\mathcal{R}(\lambda, A') = \sum_{k=-m}^{\infty} Q_k' \lambda^k$$

for λ close to 0, we conclude, moreover, that the range of the non-zero operator Q_{-m}' is contained in $\ker A'$.

We have $Q_{-m} = \lim_{\lambda \downarrow 0} \lambda^m \mathcal{R}(\lambda, A)$, and we intend to show that Q_{-m} is positive. So let $f \in E_+$. Choose $t_0 \geq 0$ such that $e^{tA} f \geq 0$ for all $t \geq t_0$. According to Proposition 6.2.1 we have

$$\operatorname{dist}(\mathcal{R}(\lambda, A) f, E_+) = \lim_{T \to \infty} \operatorname{dist}\Big(\int_0^T e^{-\lambda t} e^{tA} f \, dt, E_+ \Big) \leq \| \int_0^{t_0} e^{-\lambda t} e^{tA} f \, dt \|$$

for every $\lambda > 0$. Hence, we obtain

$$\operatorname{dist}(Q_{-m} f, E_+) = \lim_{\lambda \downarrow 0} \lambda^m \operatorname{dist}(\mathcal{R}(\lambda, A) f, E_+) = 0,$$

so $Q_{-m} f \geq 0$. This shows that Q_{-m} is positive, as claimed. Since Q_{-m} is non-zero, its range contains a non-zero positive vector; as the range of Q_{-m} is contained in $\ker A$, it follows that A admits a positive eigenvector for the eigenvalue 0. Moreover, it follows that the adjoint operator Q_{-m}' is also positive (and non-zero, of course); hence, its range contains a non-zero, positive functional. Since the range of Q_{-m}' is contained in $\ker A'$, we conclude that A' admits also a positive eigenvector for the eigenvalue 0. □

6.3 The Peripheral Point Spectrum

After being mainly concerned with the spectral bound of an individually eventually positive semigroup in the previous section, we now deal with the eigenvalues located on the line $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \mathrm{s}(A)\}$, i.e. with the peripheral point spectrum. We start with a result about the order of poles in the peripheral spectrum; compare [AGG+86, Corollary C-III.1.5].

Proposition 6.3.1. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach lattice E which is individually eventually positive and assume that $\mathrm{s}(A) > -\infty$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$*

If another number $\lambda \in \sigma_{\mathrm{per}}(A)$ is also a pole of the resolvent, then the pole order of λ is at most as large as the pole order of $\mathrm{s}(A)$.

Proof. We may assume that $\mathrm{s}(A) = 0$. Let $i\beta$, where $\beta \in \mathbb{R}$, be a pole of the resolvent $\mathcal{R}(\cdot, A)$ and let $m \in \mathbb{N}$ denote the order of 0 as a pole of $\mathcal{R}(\cdot, A)$. It suffices to show that

$$\sup_{\alpha\in(0,1]} \alpha^m \|\mathcal{R}(\alpha + i\beta, A)\| < \infty. \tag{6.1}$$

Let $f \in E_{\mathbb{R}}$ and let $r_f : (0,\infty) \to E$ be given as in Lemma 6.2.3. We then have

$$\sup_{\alpha\in(0,1]} \alpha^m \|\mathcal{R}(\alpha + i\beta, A)f\| \leq \sup_{\alpha\in(0,1]} \alpha^m \|\mathcal{R}(\alpha, A)|f|\| + \sup_{\alpha\in(0,1]} \alpha^m \|r_f(\alpha)\|.$$

The latter summand is finite since r_f is norm bounded on $(0,\infty)$ and the first summand is finite since 0 is a pole of $\mathcal{R}(\cdot, A)$ of order m. Hence, the uniform boundedness principle implies the estimate (6.1), thus proving the assertion. □

Our second result in this section deals with a certain symmetry property of the peripheral point spectrum of the semigroup generator. We recall from the preliminaries that a subset $S \subseteq \mathbb{C}$ of the complex numbers is called *cyclic* if we have $\alpha + in\beta \in S$ for all integers $n \in \mathbb{Z}$ whenever $\alpha + i\beta \in S$ (where $\alpha, \beta \in \mathbb{R}$). We now prove that the peripheral point spectrum of an individually asymptotically positive semigroup is cyclic provided that a certain regularity assumption is fulfilled. We refer to [Gre82, Theorem 2.5] and [AGG+86, Corollary C-III.4.3] for similar results under slightly weaker regularity assumptions in case that the semigroup is positive; see also the discussion in the notes.

Theorem 6.3.2. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach lattice E, assume that $\mathrm{s}(A) > -\infty$ and that the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ has weakly pre-compact orbits (and is thus, in particular, bounded).*

If $(e^{tA})_{t\geq 0}$ is individually asymptotically positive, then the peripheral point spectrum $\sigma_{\mathrm{per,pnt}}(A)$ is cyclic.

For the proof we need the following simple lemma:

Lemma 6.3.3. *Let $(e^{tA})_{t\geq 0}$ be a bounded C_0-semigroup with $\mathrm{s}(A) = 0$ on a complex Banach lattice E. If $(e^{tA})_{t\geq 0}$ is individually asymptotically positive, then*

$$\operatorname{dist}(\langle x', e^{tA}x\rangle, [0,\infty)) \to 0 \quad \text{as } t \to \infty$$

for all $x \in E_+$ and all $x' \in E'_+$.

Proof. Let $x \in E_+$ and $x' \in E'_+$. We obtain

$$\begin{aligned}\operatorname{dist}(\langle x', e^{tA}x\rangle, [0,\infty)) &\leq \inf_{z\in E_+} |\langle x', e^{tA}x\rangle - \langle x', z\rangle| \\ &\leq \inf_{z\in E_+} \|x'\|\|e^{tA}x - z\| = \|x'\|\operatorname{dist}(e^{tA}x, E_+) \to \infty\end{aligned}$$

as $t \to \infty$, thus proving the assertion. □

Proof of Theorem 6.3.2. We may assume that A has spectral bound 0 and that its peripheral point spectrum $\sigma_{\mathrm{per,pnt}}(A) = \sigma_{\mathrm{pnt}}(A) \cap i\mathbb{R}$ is non-empty. Our proof is based on employing the Jacobs–deLeeuw–Glicksberg decomposition of the semigroup $(e^{tA})_{t\geq 0}$.

Let $\mathcal{T} := \{e^{tA} | t \geq 0\}$ and let $\mathcal{S}$ denote the closure of $\mathcal{T}$ in $\mathcal{L}(E)$ with respect to the weak operator topology. Since our semigroup has weakly pre-compact orbits, it follows that $\mathcal{S}$ is compact with respect to the weak operator topology, see [EN00, Lemma V.2.7]. Hence, $\mathcal{S}$ is a semitopological compact semigroup with respect to operator multiplication. Now, let

$$\mathcal{K} := \bigcap_{T\in\mathcal{S}} T\mathcal{S}$$

denote the so-called *Sushkevich kernel* of $\mathcal{S}$. It is easy to see, but important for us to note, that actually

$$\mathcal{K} = \bigcap_{T\in\mathcal{S}} \overline{T\mathcal{T}}^{\mathrm{w}} \subseteq \bigcap_{T\in\mathcal{T}} \overline{T\mathcal{T}}^{\mathrm{w}} = \bigcap_{t\in[0,\infty)} \overline{\{e^{(t+s)A} | s \in [0,\infty)\}}^{\mathrm{w}}; \tag{6.2}$$

here, $\overline{\mathcal{A}}^{\mathrm{w}}$ denotes the closure of a set $\mathcal{A} \subseteq \mathcal{L}(E)$ with respect to the weak operator topology.

It is known that $\mathcal{K}$ is a compact group with respect to operator multiplication and the weak operator topology; moreover, it is an ideal in the semigroup $\mathcal{S}$. The identity element P of $\mathcal{K}$ is a projection in $\mathcal{L}(E)$; its range PE is invariant with respect to the semigroup $(e^{tA})_{t\geq 0}$ (since P commutes with the semigroup), and the restriction of the semigroup to PE extends to a C_0-group on PE; for every $t > 0$, the operator $(e^{tA}|_{PE})^{-1}$ can be written as $R_t|_{PE}$ for some $R_t \in \mathcal{K}$. Finally, it is also known that PE is the closed linear span of eigenvectors of

A belonging to purely imaginary eigenvalues (or to the eigenvalue 0). In particular, we have $\sigma_{\mathrm{per,pnt}}(A) = \sigma_{\mathrm{per,pnt}}(A|_{PE})$. All those results can be found e.g. in [Kre85, Section 2.4] and in [EN00, Section V.2].

Let us now show that each operator $K \in \mathcal{K}$ is positive. Fix $K \in \mathcal{K}$, let $x \in E_+$, $x' \in E'_+$ and $\varepsilon > 0$. According to Lemma 6.3.3 we have

$$\operatorname{dist}(\langle x', e^{tA}x\rangle, [0,\infty)) < \varepsilon$$

for all sufficiently large t, $t \geq t_0$ say. It follows from (6.2) that we can find a net of times $(t_j) \subseteq [0,\infty)$ such that $e^{(t_0+t_j)A}$ converges to K with respect to the weak operator topology. Hence,

$$\operatorname{dist}(\langle x', Kx\rangle, [0,\infty)) = \lim_j \operatorname{dist}(\langle x', e^{(t_0+t_j)A}x\rangle, [0,\infty)) \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\langle x', Kx\rangle \geq 0$, so K is indeed positive.

In particular, the projection P is positive, so it follows from Proposition C.5.5 that PE is itself a complex Banach lattice with respect to some equivalent norm. Moreover, we have $e^{tA}|_{PE} = (e^{tA}P)|_{PE}$ for each $t \geq 0$, and since $e^{tA}P$ is contained in $\mathcal{K}$, it follows that $e^{tA}P$, and thus $e^{tA}|_{PE}$, is positive. Finally, recall that for every $t > 0$ the inverse operator $(e^{tA}|_{PE})^{-1}$ is given by $R_t|_{PE}$ for some $R_t \in \mathcal{K}$; hence, the inverse of $e^{tA}|_{PE}$ is positive to, so we conclude that the C_0-semigroup $(e^{tA}|_{PE})_{t\geq 0}$ extends in fact to a C_0-group of lattice homomorphism on the complex Banach lattice PE.

According to [AGG+86, Theorem C-III.4.2] this implies that the point spectrum of its generator $A|_{PE}$ is cyclic and thus, so is the peripheral point spectrum of A. □

6.4 Notes & Open Problems

Eventually positive matrices In finite dimensions, the idea to consider matrices for which some powers, or eventually all powers, are positive and to prove Perron–Frobenius type results for those matrices is not particularly new. For instance, in [Bra61] the reader can find results about matrices which have a positive power; quite recently such matrices were again considered in [TCDF15]. In [Sen81, pp. 48–54] matrices A are studied for which some polynomial $p(A)$ is positive. Moreover, several early papers related to eventual positivity were motivated by inverse spectral problems as e.g. [Fri78] and [ZT99].

Yet, during the last two decades, the literature concerned with eventually positive matrices literally exploded; here we can give only some examples for the countless references one can find: in [TRH01], [JT04], [Nou06], [ES08] and [ES09] eventually positive matrices were employed in the study of Perron–Frobenius type properties of matrices with some non-positive entries. Further

structure result for eventually positive matrices can for instance be found in [CNM02], [CNM04] and [HTW15]. The relation of eventual positivity to matrix roots is studied in [MPT14] and an algorithm to test whether a given matrix is eventually positive is described in [Hog09]. Relations between eventually positive matrices and sign patterns are, for example, studied in [BCD+09], [EHT09] and [CEH+12]. Eventually positive matrix semigroups rather than eventually positive matrices occur for instance in [NT08], [OTvdD09], [EHT09, Theorem 2.9] and [Eri15]. We point out that those references are far from being complete; the interested reader can find many more articles on these topics in the references of the articles quoted above.

Laplace representation of the resolvent In Proposition 6.2.1 we proved that the resolvent of an individually eventually positive C_0-semigroup can be represented as Laplace integral whenever $\operatorname{Re}\lambda > \mathrm{s}(A)$. The proof presented here is based on results about vector-valued Laplace transforms on ordered Banach spaces. For positive semigroups an alternative proof of this result can be found in [AGG+86, Theorem C-III.1.2].

Equality of the spectral and the growth bound The fact that the growth bound and the spectral bound of *positive* semigroups coincide on certain classes of Banach lattices was first observed on the three types of Banach lattices which we considered in Theorem 6.2.6. On AL-spaces and on Hilbert lattices the proofs presented for Theorem 6.2.6 are the same as for positive semigroups, compare [AGG+86, Theorem C-IV.1.1(a)]. On AM-spaces several proofs for the equality $\mathrm{s}(A) = \omega_0(A)$ for positive semigroups are known: on AM-spaces with unit, a proof can be found in [Der80, Satz 3.3]; on general AM-spaces one can prove the result by a different method, see [AGG+86, Theorem B-IV.1.4 and Theorem C-IV.1.1(a)]. The argument for individually eventually positive semigroups given in Theorem 6.2.6 is different from those two proofs and was suggested by Arendt. Yet, it leaves the following question open:

Problem 6.4.1. Let $(e^{tA})_{t\geq 0}$ be an individually eventually positive C_0-semigroup on a complex Banach lattice E.

(a) Suppose that $E = \mathcal{C}_0(L;\mathbb{C})$ for some locally compact Hausdorff space L; does it follow that $\mathrm{s}(A) = \omega_0(A)$?

(b) Suppose, more generally, that E is an AM-space; does it follows that $\mathrm{s}(A) = \omega_0(A)$?

Another question, no less interesting, concernes the situation on L^p-spaces. Let (Ω, μ) be σ-finite measure space, let $1 \leq p < \infty$ and let $(e^{tA})_{t\geq 0}$ be a positive

C_0-semigroup on $E := L^p(\Omega, \mu; \mathbb{C})$. It was proved by Weis in [Wei95] that this already implies $s(A) = \omega_0(A)$. Some years later, Weis published a different (and somewhat simpler) proof for the same result in [Wei98]. It seems to be unclear whether the techniques used in [Wei95] and [Wei98] can be adapted to the case of eventually positive semigroups; hence, the following question arises:

Problem 6.4.2. Let $p \in [1, \infty)$ and let (Ω, μ) be a measure space (assume it to be σ-finite if necessary). If $(e^{tA})_{t\geq 0}$ is an individually eventually positive C_0-semigroup on $L^p(\Omega, \mu; \mathbb{C})$, does it follow that $s(A) = \omega_0(A)$?

Cyclicity of the peripheral spectrum The cyclicity of the peripheral (point) spectrum is an interesting and delicate topic in the theory of positive operators and semigroups. Motivated by the classical theorem of Frobenius, such symmetry properties of the spectrum were studied by many authors. Striking results were achieved by Krieger [Kri69, Folgerung 2.2.1(b)] and Lotz [Lot68, Theorems 4.7, 4.9 and 4.10] who proved independently that the peripheral spectrum of a positive operator is cyclic in case that the operator is Abel bounded (note, however, that the notion "cyclic" has to be defined differently from the notion considered in this thesis if it is supposed to fit the single operator case rather than the case of C_0-semigroups). Until today it is an open problem whether to peripheral spectrum of *every* positive operator on a Banach lattice is cyclic; we refer to [Glü16c] and [Glü16a] for an overview of this topic as well as for some recent partial results. For the generator of positive C_0-semigroups, cyclcitiy of the peripheral spectrum was proved under similar assumptions as in the single operator case by Greiner [Gre81, Theorem 2.4] after a partial result had been obtained by Derndinger [Der80, Theorem 3.7]; compare also [AGG$^+$86, Section C-III.2]. We point out again that some sufficient conditions for the peripheral *point* spectrum of positive semigroup to be cyclic can be found in [Gre82, Theorem 2.5] and [AGG$^+$86, Corollary C-III.4.3]. Note however that there exist examples of positive semigroups for which the peripheral point spectrum is not cyclic (cf. [AGG$^+$86, Examples B-III.2.13 and C-III.4.4] and [Glü16c, Example 8.2]).

The question whether similar results hold for operators and semigroups which are only eventually (or asymptotically) positive is not easy since the classical proofs do not carry over to this case. Using somewhat different techniques the author recently proved that a power bounded operator T with spectral radius 1 has cyclic peripheral spectrum if it is uniformly asymptotically positive. A paper containing this result, along with several other theorems, is in preparation; we chose not to include those results here since we intend to focus on the theory of C_0-semigroups in this thesis. Yet, it seems

that the proof of the above mentioned result cannot simply be adjusted to work for C_0-semigroups. Therefore, the following question remains open:

Problem 6.4.3. Let $(e^{tA})_{t\geq 0}$ be a bounded C_0-semigroup on a complex Banach lattice E with $\mathrm{s}(A) = 0$ and assume that this semigroup $(e^{tA})_{t\geq 0}$ is uniformly asymptotically positive. Does it follow that the peripheral spectrum of A is cyclic?

7 Asymptotic Positivity – Criteria and Characterisation

In this chapter we focus on asymptotic positivity and, in particular, on characterisations of this property. Actually, we do not only consider asymptotically positive semigroups, but also what we are going to call *asymptotically positive resolvents*. We start our analysis in Section 7.1 by considering spectral projections associated to isolated spectral values of linear operators; we give sufficient conditions for such spectral projections to be positive in terms of the associated eigenvectors and eigenfunctionals. In Section 7.2 we characterise asymptotically positive behaviour of the resolvent $\mathcal{R}(\cdot, A)$ of an operator A in a right neighbourhood of an isolated spectral value λ_0 in terms of positivity properties of the spectral projection associated with λ_0. Finally, in Section 7.3, we characterise asymptotic positivity of a C_0-semigroup $(e^{tA})_{t\geq 0}$, under appropriate assumptions, via the positivity of the spectral projection associated with the spectral bound $\mathrm{s}(A)$.

Hence, the question whether the spectral projection P associated with the spectral bound $\mathrm{s}(A)$ of a semigroup generator A is positive relates the existence of positive eigenvectors and eigenfunctionals to the asymptotic positivity of the resolvent $\mathcal{R}(\cdot, A)$ close to $\mathrm{s}(A)$ as well as to the asymptotic positivity of the semigroup $(e^{tA})_{t\geq 0}$.

Most results in this chapter have been published in [DGK16a]; see also Proposition 2.3 in [DGK16b] which was the blueprint for Proposition 7.3.1 and Corollary 7.3.2 below.

7.1 Positive Projections and Spectral Projections

We will see later on that positive spectral projections play an important role in our characterisations of asymptotically positive resolvents and semigroups (see Theorems 7.2.2, 7.3.3 and 7.3.6). Hence, it will be useful to have a sufficient criterion for a spectral projection to be positive. We begin with a simple observation about the positivity of rank-1 projections in general:

Proposition 7.1.1. *Let E be a complex Banach lattice and let $P \in \mathcal{L}(E)$ be a projection. If PE is one-dimensional and PE and $P'E'$ both contains positive, non-zero elements, then P is positive.*

Proof. The operator P is given by $Pf = \langle \varphi_0, f\rangle f_0$ for some $f_0 \in PE \setminus \{0\}$, some $\varphi_0 \in P'E' \setminus \{0\}$ and all $f \in E$. Since PE is one-dimensional, so is $P'E'$ (this follows for example from Lemma 1.2.6). Hence, PE and $P'E'$ are spanned by

vectors $f_1 > 0$ and $\varphi_1 > 0$, respectively, and therefore we have $f_0 = \alpha f_1$ and $\varphi_0 = \beta\varphi_1$ for two appropriate scalars $\alpha, \beta \neq 0$.

This implies that $Pf = \alpha\beta\langle\varphi_1, f\rangle f_1$ for all $f \in E$. To conclude that $P \geq 0$ it suffices to shows that $\alpha\beta \geq 0$. We have $f_1 = Pf_1 = \alpha\beta\langle\varphi_1, f_1\rangle f_1$, so $1 = \alpha\beta\langle\varphi_1, f_1\rangle$. Since $\langle\varphi_1, f_1\rangle \geq 0$, this shows that indeed $\alpha\beta \geq 0$. □

If P is a spectral projection associated to an isolated eigenvalue of an operator A, which is the most important case for our purposes, then Proposition 7.1.1 takes the following form:

Corollary 7.1.2. *Let E be a complex Banach lattice and let $A : E \supseteq D(A) \to E$ be a densely defined linear operator. Let $\lambda_0 \in \sigma(A)$ be a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. Assume that $\ker(\lambda_0 - A)$ is one-dimensional and that $\ker(\lambda_0 - A)$ and $\ker(\lambda_0 - A')$ both contain a positive, non-zero element. Then the spectral projection P associated to λ_0 is positive.*

Proof. Since λ_0 is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$, the eigenspace $\ker(\lambda_0 - A)$ coincides with the range PE, see Proposition A.3.2(d). Hence, PE is one-dimensional and contains a positive, non-zero vector. On the other hand, P' is the spectral projection associated to the first order pole λ_0 of $\mathcal{R}(\cdot, A')$, so again by Proposition A.3.2(d) the range $P'E'$ coincides with the eigenspace $\ker(\lambda_0 - A')$; hence, $P'E'$ contains a positive, non-zero vector. The assertion thus follows from Proposition 7.1.1. □

There exist good abstract criteria to ensure that a spectral value of an operator A is isolated and even a pole of the resolvent. If, for example, A has compact resolvent, then *every* spectral value of A is isolated and, moreover, a pole of the resolvent. Yet, we do not know in general whether a given pole is of order 1; in such cases, the following result can be useful.

Proposition 7.1.3. *Let E be a complex Banach lattice, let $A : E \supseteq D(A) \to E$ be a densely defined linear operator and let $\lambda_0 \in \sigma(A)$ be a pole of the resolvent $\mathcal{R}(\cdot, A)$. Assume that $\ker(\lambda_0 - A)$ is one-dimensional, that $\ker(\lambda_0 - A)$ and $\ker(\lambda_0 - A)'$ both contain a positive, non-zero element and that at least one of the following two conditions is fulfilled:*

(a) *$\ker(\lambda_0 - A)$ contains a quasi-interior point of E_+.*

(b) *$\ker(\lambda_0 - A')$ contains a strictly positive functional.*

Then λ_0 is a first order pole of $\mathcal{R}(\cdot, A)$ and an algebraically simple eigenvalue of A; moreover, the associated spectral projection P is positive.

Proof. We may assume that $\lambda_0 = 0$. It suffices to show that 0 is a semi-simple eigenvalue of A; then 0 is algebraically simple since it is geometrically simple

by assumption and it is a first order pole of the resolvent according to Proposition A.3.2(d). Moreover, Proposition 7.1.1 then implies that P is positive.

To show that 0 is semi-simple, we only have to show that A has no generalised eigenvector of rank 2 for the eigenvalue 0, see Proposition A.2.3(b). So assume for a contradiction that there exists an $f \in \ker(A^2) \setminus \ker A$. Then we have

$$\langle \varphi, g \rangle = 0 \tag{7.1}$$

for all $g \in \ker A$ and all $\varphi \in \ker A'$. Indeed, if g and φ are such vectors, then we have $g = \alpha A f$ for some scalar α since $Af \in \ker A \setminus \{0\}$ and since $\ker A$ is one-dimensional. Hence, $\langle \varphi, g \rangle = \alpha \langle \varphi, Af \rangle = \alpha \langle A'\varphi, f \rangle = 0$, as claimed.

By assumption, condition (a) or (b) is fulfilled. If (a) holds, then we can find a functional $0 < \varphi \in \ker A'$ and a vector $0 < g \in \ker A$ which is a quasi-interior point of E_+. We then have $\langle \varphi, g \rangle > 0$; however, this contradicts (7.1).

If, on the other hand, (b) holds, then we can find a vector $0 < g \in \ker A$ and a strictly positive functional $\varphi \in \ker A'$. Again we have $\langle \varphi, g \rangle > 0$, in contradiction to (7.1). □

The proof of the above proposition uses techniques which are quite standard in Perron–Frobenius theory. For related methods we refer, for example, to [AGG+86, Remark B-III.2.15(a)] or to [Gro95, Theorem 4.12(ii)].

7.2 The Resolvent

Now we turn to the analysis of asymptotically positive resolvents. Note that if $\lambda_0 \in \mathbb{R}$ is a spectral value of an operator A and if $(\lambda_0, \lambda_0 + \delta]$ is contained in the resolvent set $\rho(A)$, then the resolvent of A fulfils $\|\mathcal{R}(\lambda, A)\| \geq \frac{1}{\lambda - \lambda_0}$ for $\lambda > \lambda_0$; in particular, the resolvent explodes as λ approaches λ_0 from above. On the other hand, in many applications a so-called *Abe-type growth condition* is fulfilled, by what we mean that $\limsup_{\lambda \downarrow \lambda_0} (\lambda - \lambda_0) \|\mathcal{R}(\lambda, A)\| < \infty$. Hence, the number $(\lambda - \lambda_0)\|\mathcal{R}(\lambda, A)\|$ remains bounded, but it does not approach 0 as $\lambda \downarrow \lambda_0$. In such a case, consider a vector $f \geq 0$ and suppose for a moment that the distance of $(\lambda - \lambda_0)\mathcal{R}(\lambda, A) f$ to the positive cone tends to 0 as $\lambda \downarrow \lambda_0$. This means that the distance of $\mathcal{R}(\lambda, A) f$ to the positive cone is, for λ close to λ_0 and positive f, small compared to the norm of $\mathcal{R}(\lambda, A)$; we might thus say that the resolvent behaves *asymptotically positive* at 0. This is the motivation for the subsequent definition.

On the other hand, we should note that the distance of $\mathcal{R}(\lambda, A) f$ to the positive cone, while it is small compared to the operator norm $\|\mathcal{R}(\lambda, A)\|$, might not necessarily be small compared to the norm of the *vector* $\mathcal{R}(\lambda, A) f$; this indicates that there are also other possibilities to define an "asymptotically

positive resolvent"; see the notes at the end of this chapter for a more detailed discussion.

Definition 7.2.1 (Asymptotically positive resolvent). Let E be a complex Banach lattice and let $A : E \supseteq D(A) \to E$ be a linear operator. Let $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ such that we have $(\lambda_0, \lambda_0 + \delta] \subseteq \rho(A)$ for some $\delta > 0$ and assume that the resolvent of A fulfils the growth condition

$$\limsup_{\lambda \downarrow \lambda_0} (\lambda - \lambda_0) \|\mathcal{R}(\lambda, A)\| < \infty.$$

(i) The resolvent $\mathcal{R}(\cdot, A)$ is called *individually asymptotically positive at* λ_0 if we have $(\lambda - \lambda_0) \operatorname{dist}(\mathcal{R}(\lambda, A) f, E_+) \to 0$ as $\lambda \downarrow \lambda_0$ for every $0 \leq f \in E$.

(ii) The resolvent $\mathcal{R}(\cdot, A)$ is called *uniformly asymptotically positive at* λ_0 if for every $\varepsilon > 0$ there exists a $\lambda_1 > \lambda_0$ such that $(\lambda_0, \lambda_1] \subseteq \rho(A)$ and such that $(\lambda - \lambda_0) \operatorname{dist}(\mathcal{R}(\lambda, A) f, E_+) \leq \varepsilon \|f\|$ for all $0 \leq f \in E$ and all $\lambda \in (\lambda_0, \lambda_1]$.

We recall once again that some properties of the distance $\operatorname{dist}(\cdot, E_+)$ to the positive cone are collected in Proposition B.1.7; we will use these properties frequently (and tacitly) in the sequel.

Note that if $\lambda_0 \in \sigma(A)$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$, then the growth condition $\limsup_{\lambda \downarrow \lambda_0} (\lambda - \lambda_0) \|\mathcal{R}(\lambda, A)\| < \infty$ is fulfilled if and only if the pole order of $\mathcal{R}(\cdot, A)$ at λ_0 equals 1.

We should point out that one can also introduce a slightly stronger notion of asymptotic positivity for resolvent, which is called *asymptotic positivity of bounded type* in [DGK16a, Definition 7.3]. Yet, this notion does not lead to particularly different concepts, and it does not require new methods to be handled in proofs. Therefore, we will not discuss this notion in detail, here.

When dealing with semigroups in the next section, as well as in the applications in Chapter 8, we will see that case $\lambda_0 = \mathrm{s}(A)$ is particularly important in Definition 7.2.1 (and in this case, the assumption $(\lambda_0, \lambda_0 + \delta] \subseteq \rho(A)$ is automatically fulfilled). However, the general theory presented here works for other choices of λ_0, too.

Our main result in this section is the following characterisation of asymptotically positive resolvents.

Theorem 7.2.2. *Let E be a complex Banach lattice, let $A : E \supseteq D(A) \to E$ be a linear operator and let $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ be a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. Then the following assertions are equivalent:*

(i) *The spectral projection P associated with λ_0 is positive.*

(ii) *The resolvent $\mathcal{R}(\cdot, A)$ is individually asymptotically positive at λ_0.*

(iii) *The resolvent $\mathcal{R}(\cdot, A)$ is uniformly asymptotically positive at λ_0.*

Proof. We may assume that $\lambda_0 = 0$.

"(i) $\Rightarrow$ (iii)" Assume that P is positive and let $\varepsilon > 0$. Since 0 is an isolated spectral value of A, we can find a number $\lambda_2 > 0$ such that the interval $(0, \lambda_2]$ is contained in $\rho(A)$. Let $F := (I-P)E$. Since 0 is not a spectral value of $A|_F$, the restriction of $\mathcal{R}(\cdot, A)$ to F is uniformly bounded on $(0, \lambda_2]$. Thus, $M := \sup_{\lambda \in (0,\lambda_2]} \|\mathcal{R}(\cdot, A)(I-P)\| < \infty$.

Now, let $\varepsilon > 0$ and choose $\lambda_1 \in (0, \lambda_2]$ such that $\lambda_1 \leq \frac{\varepsilon}{M}$. Since 0 is a first order pole of the resolvent we have $PE = \ker A$. Hence we obtain for all $\lambda \in (0, \lambda_1]$ and all $f \in E_+$

$$\begin{aligned}\lambda \operatorname{dist}(\mathcal{R}(\lambda, A)f, E_+) &\leq \lambda \operatorname{dist}(\mathcal{R}(\lambda, A)Pf, E) + \lambda \|\mathcal{R}(\lambda, A)(I-P)f\| \\ &\leq \operatorname{dist}(Pf, E_+) + \lambda M \|f\| \leq \varepsilon \|f\|.\end{aligned}$$

Thus, $\mathcal{R}(\cdot, A)$ is indeed uniformly asymptotically positive at $\lambda_0 = 0$.

"(iii) $\Rightarrow$ (ii)" This implication is obvious.

"(ii)" $\Rightarrow$ (i)" Assume that the resolvent $\mathcal{R}(\cdot, A)$ is individually asymptotically positive at $\lambda_0 = 0$ and let $f \in E_+$. Since 0 is a first order pole of the resolvent, we have $\lim_{\lambda \downarrow 0} \lambda \mathcal{R}(\lambda, A) = P$ with respect the operator norm. Hence,

$$\operatorname{dist}(Pf, E_+) = \lim_{\lambda \downarrow 0} \lambda \operatorname{dist}(\mathcal{R}(\lambda, A)f, E_+) = 0.$$

This shows that $Pf \geq 0$, so P is indeed positive. □

Finally, we consider the special case where the underlying Banach lattice is an L^2-space or, using a more abstract language, a Hilbert lattice. On such spaces we obtain a particularly simple sufficient condition for asymptotic positivity of the resolvent:

Theorem 7.2.3. *Let H be a complex Hilbert lattice, let A be a densely-defined, self-adjoint operator and let $\lambda_0 \in \mathbb{R}$ be an isolated spectral value of A. Moreover, assume that* $\ker(\lambda_0 - A)$ *is one-dimensional and contains a non-zero positive vector.*

Then λ_0 is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$ and the resolvent is uniformly asymptotically positive at λ_0.

Proof. We may assume that $\lambda_0 = 0$. As A is self-adjoint and 0 is an isolated spectral value of A, it follows that 0 is in fact a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. Since the Hilbert space adjoint $A^* = A$ of A has a positive non-zero eigenvector for the eigenvalue λ_0, we conclude from Corollary C.6.4 that the Banach space adjoint A' of A has also a positive non-zero eigenvector for the eigenvalue λ_0. Hence, Corollary 7.1.2 implies that the spectral projection P associated to the spectral value 0 is positive. According to Theorem 7.2.2 this implies that $\mathcal{R}(\cdot, A)$ is uniformly asymptotically positive at $\lambda_0 = 0$. □

7.3 The Semigroup

In this section we are going to prove, under appropriate assumptions, characterisations of individually and uniformly asymptotically positive semigroups. As a preparation, we need the following simple observation about complex numbers of modulus one:

Proposition 7.3.1. *Let $\lambda_1,...,\lambda_m \in \mathbb{C}$ have modulus 1. Then there exists a sequence of integers $0 \leq k_n \to \infty$ such that $\lambda_1^{k_n},...,\lambda_m^{k_n} \to 1$ as $n \to \infty$.*

Proof. Let $\mathbb{T} \subseteq \mathbb{C}$ denote the complex unit circle and consider the element $\lambda := (\lambda_1,...,\lambda_m)$ of the compact topological group $\mathbb{T}^m$. The mapping $\varphi : \mathbb{T}^m \to \mathbb{T}^m$, $\mu \mapsto \lambda \cdot \mu$ is a so-called *group rotation* on $\mathbb{T}^m$ (see [EFHN15, Example 2.9]). Hence, it follows from topological dynamics that each point in $\mathbb{T}^m$ is *recurrent* for φ (see [Fur81, Theorem 1.2] or [EFHN15, Proposition 3.12(d)]) and hence even *infinitely recurrent* [EFHN15, Exercise 6 in Chapter 3]. In particular, the point $1 := (1,...,1) \in \mathbb{T}^m$ is infinitely recurrent, meaning that there exists a sequence of integers $0 \leq k_n \to \infty$ such that $\lambda^{k_n} = \varphi^{k_n}(1) \to 1$. This proves the assertion. □

Proposition 7.3.1 yields the following result about C_0-semigroups and about powers of single operators.

Corollary 7.3.2. *Let E be a complex Banach space, let $T \in \mathcal{L}(E)$ be a power bounded operator with spectral radius $\mathrm{r}(T) = 1$ and let $(e^{tA})_{t\geq 0}$ be a bounded C_0-semigroup on E such that $\mathrm{s}(A) = 0$.*

(a) *Suppose that $\sigma(T) \cap \mathbb{T}$ consists of poles of the resolvent $\mathcal{R}(\cdot, T)$ and let P_{per} denote the spectral projection associated with $\sigma(T) \cap \mathbb{T}$. Then there exists a sequence of integers $0 \leq k_n \to \infty$ such that $T^{k_n} f \to f$ for each $f \in P_{\mathrm{per}}E$.*

(b) *Suppose that $\sigma_{\mathrm{per}}(A) = \sigma(A) \cap i\mathbb{R}$ is non-empty, finite and consists of poles of the resolvent $\mathcal{R}(\cdot, A)$; let P_{per} denote the corresponding spectral projection. Then there exists a sequence of times $0 \leq t_n \to \infty$ such that $e^{t_n A} f \to f$ for all $f \in P_{\mathrm{per}}E$.*

Proof. (a) It follows from the assumptions that $\sigma(T) \cap \mathbb{T}$ is non-empty, finite and isolated from the rest of the spectrum. In particular, the spectral projection P_{per} corresponding to $\sigma(T) \cap \mathbb{T}$ is defined. Since every $\lambda \in \sigma(A) \cap \mathbb{T}$ is a pole of the resolvent $\mathcal{R}(\cdot, T)$ and since T is power bounded, it follows from Propositions A.2.4(a) and A.3.2(d) that each such λ is in fact a simple pole of the resolvent. Thus, the range of the spectral projection corresponding to λ coincides with the eigenspace $\ker(\lambda - T)$.

Let $\sigma(T) \cap \mathbb{T} = \{\lambda_1,...,\lambda_m\}$; then it follows that $P_{\mathrm{per}}E = \oplus_{j=1}^m \ker(\lambda_j - T)$. Now choose a sequence of positive integers $0 \leq k_n \to \infty$ as in Proposition 7.3.1.

Every $f \in P_{\mathrm{per}}E$ can be decomposed as $f = f_1 + ... + f_m$, where $f_j \in \ker(\lambda_j - T)$ for each $j \in \{1, ..., m\}$. We thus obtain

$$T^{k_n} f = \sum_{j=1}^{m} \lambda_j^{k_n} f_j \overset{n\to\infty}{\to} \sum_{j=1}^{m} f_j = f.$$

This proves (a).

(b) Let $\sigma_{\mathrm{per}}(A) = \{i\beta_1, ..., i\beta_m\}$, where $\beta_1, ..., \beta_m \in \mathbb{R}$. It follows from Propositions A.2.4(b) and A.3.2(d) that each $i\beta_j$ is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$ and thus, the range of the spectral projection corresponding to $i\beta_j$ coincides with the eigenspace $\ker(i\beta_j - A)$. Hence, $P_{\mathrm{per}}E = \oplus_{j=1}^{m} \ker(i\beta_j - A)$.

Let $\lambda_j = e^{i\beta_j}$ for each $j \in \{1, ..., m\}$ and choose a sequence $0 \leq t_n := k_n \to \infty$ as in Proposition 7.3.1. Every $f \in P_{\mathrm{per}}E$ can be decomposed as $f = f_1 + ... + f_m$ where $f_j \in \ker(i\beta_j - A)$ for each $j \in \{1, ..., m\}$. Hence we obtain

$$e^{t_n A} f = \sum_{j=1}^{m} \lambda_j^{t_n} f_j \overset{n\to\infty}{\to} \sum_{j=1}^{m} f_j = f.$$

This proves (b). □

Assertion (b) of the above corollary will be very useful in the proof of the next theorem. Assertion (a) will be used in the analysis of the finite-dimensional case, cf. the proof of Theorem 8.1.3.

We recall that *asymptotically positive semigroups* were defined in Definition 6.1.3. We are now ready to prove a characterisation of such semigroups. Our first theorem deals with *individual* asymptotic positivity; afterwards we discuss the assumptions of the theorem by means of several examples before we prove a similar result about *uniform asymptotic positivity*. While we have seen in Theorem 7.2.2 above that those notions coincide for resolvents, we will demonstrate in Example 7.3.5 that it is actually necessary to distinguish between the individual and the uniform behaviour for semigroups.

Theorem 7.3.3. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach lattice E, suppose that $\mathrm{s}(A) > -\infty$ and that the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded. Assume moreover that $\sigma_{\mathrm{per}}(A)$ is non-empty, finite and consists of poles of the resolvent. Then the following assertions are equivalent:*

(i) *$\mathrm{s}(A)$ is a dominant spectral value of A and the associated spectral projection P is positive.*

(ii) *The semigroup $(e^{tA})_{t\geq 0}$ is individually asymptotically positive.*

(iii) *The rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ converges strongly to a positive operator as $t \to \infty$.*

Proof. We may assume throughout the proof that $s(A) = 0$.

"(i) $\Rightarrow$ (iii)" Since 0 is a dominant spectral value of A, it follows that the restriction of A to the space $(I-P)E$ has no spectral values on the imaginary axis. As $(e^{tA})_{t\geq 0}$ is bounded, it follows from [vN96, Corollary 5.2.6] that $(e^{tA})_{t\to 0}$ converges strongly to 0 on $(I-P)E$. On the other hand we have $PE = \ker A$ since 0 is a first order pole of $\mathcal{R}(\cdot, A)$ according to Propositions A.2.4(b) and A.3.2(d). Hence, PE consists of fixed vectors of the semigroup and we thus obtain for each $f \in E_+$

$$e^{tA}f = Pf + e^{tA}(I-P)f \to Pf$$

as $t \to \infty$. This shows that the semigroup converges strongly to the positive operator P as $t \to \infty$.

"(iii) $\Rightarrow$ (ii)" Assume that e^{tA} converges strongly to an operator $Q \geq 0$ as $t \to \infty$. For every $f \in E_+$ we then obtain

$$\operatorname{dist}(e^{tA}f, E_+) \leq \operatorname{dist}(Qf) + \|e^{tA}f - Qf\| = \|e^{tA}f - Qf\| \to 0$$

as $t \to \infty$. This proves that $(e^{tA})_{t\geq 0}$ is individually asymptotically positive.

"(ii) $\Rightarrow$ (i)" Assume that $(e^{tA})_{t\geq 0}$ is individually asymptotically positive and let P_{per} denote the spectral projection associated to the peripheral spectrum $\sigma_{\mathrm{per}}(A)$. According to Corollary 7.3.2(b) we can find a sequence of times $0 \leq t_n \to \infty$ such that $e^{t_n A}f \to f$ for all $f \in P_{\mathrm{per}}E$. For every $t \geq 0$ and every $f \in E_+$ we conclude that

$$\begin{aligned}\operatorname{dist}(e^{tA}P_{\mathrm{per}}f, E_+) &= \lim_{n\to\infty} \operatorname{dist}(e^{(t+t_n)A}P_{\mathrm{per}}f) \\ &\leq \lim_{n\to\infty} \operatorname{dist}(e^{(t+t_n)A}f) + \lim_{n\to\infty} \|e^{(t+t_n)A}(I-P_{\mathrm{per}})f\| = 0;\end{aligned}$$

here we used that $(e^{tA})_{t\geq 0}$ converges strongly to 0 on $(I-P_{\mathrm{per}})E$, a fact which follows from [vN96, Corollary 5.2.6]. Hence, $e^{tA}P_{\mathrm{per}} \geq 0$ for all $t \geq 0$. Using $t = 0$, we see that this in particular implies $P_{\mathrm{per}} \geq 0$. Hence it follows from Proposition C.5.5 that, with respect to an equivalent norm, the range $P_{\mathrm{per}}E$ is itself a complex Banach lattice with positive cone $(P_{\mathrm{per}}E)_+ := E_+ \cap P_{\mathrm{per}}E = P_{\mathrm{per}}E_+$. For each $f \in (P_{\mathrm{per}}E)_+$ we have $e^{tA}f = e^{tA}P_{\mathrm{per}}f \geq 0$ for every time $t \geq 0$. Thus, the restricted semigroup $(e^{tA}|_{P_{\mathrm{per}}E})_{t\geq 0}$ is positive on the complex Banach lattice $P_{\mathrm{per}}E$.

Note that $\sigma(A|_{P_{\mathrm{per}}E}) = \sigma_{\mathrm{per}}(A)$, and since the latter set is non-empty by assumption, we conclude that $s(A|_{P_{\mathrm{per}}E}) = s(A) = 0$; thus, $\sigma_{\mathrm{per}}(A|_{P_{\mathrm{per}}E}) = \sigma(A|_{P_{\mathrm{per}}E}) = \sigma_{\mathrm{per}}(A)$. It follows from Perron–Frobenius theory that the spectral bound of $A|_{P_{\mathrm{per}}E}$ is contained in its spectrum [AGG+86, Corollary C-III.1.4] and, since the semigroup $(e^{tA}|_{P_{\mathrm{per}}E})_{t\geq 0}$ is bounded, that the peripheral spectrum of $A|_{P_{\mathrm{per}}E}$ is cyclic [AGG+86, Theorem C-III.2.10]. Since the peripheral spectrum of $A|_{P_{\mathrm{per}}E}$ is also bounded, we conclude that it consists of 0 only. This

proves that $s(A) = 0$ is a dominant spectral value of 0. In particular, we have $P = P_{\text{per}} \geq 0$. □

It is natural to ask whether the technical assumptions in Theorem 7.3.3 are really needed. Let us demonstrate by two counterexamples that some of them cannot be dropped:

Examples 7.3.4. (a) In Theorem 7.3.3 the assumption that $\sigma_{\text{per}}(A)$ be non-empty cannot be dropped.

Indeed, let $B \in \mathbb{R}^{2\times 2}$ be a matrix with spectrum $\sigma(B) = \{-i, i\}$ and set $A_n := nB - \frac{1}{n}$ for each $n \in \mathbb{N}$. Endow $\mathbb{C}^2$ with the Euclidean norm, set $E := \ell^2(\mathbb{N};\mathbb{C}^2)$ and let $A : E \supseteq D(A) \to E$ be the matrix multiplication operator with symbol (A_n), meaning that

$$D(A) = \{(x_n) \in E |\, (Ax_n) \in E\}, \qquad A(x_n) = (A_n x_n).$$

Then $\sigma(A) = \{\pm ni - \frac{1}{n} |\, n \in \mathbb{N}\}$, so the spectral bound of A equals 0 and the peripheral spectrum of A is empty.

It is easily seen that A generates a bounded C_0-semigroup on E which converges strongly to 0.

Hence, $(e^{tA})_{t\geq 0}$ is individually asymptotically positive for trivial reasons. Yet, the conclusion of Theorem 7.3.3 that $s(A)$ is a dominant spectral value does not hold since $s(A)$ is not a spectral value of A at all.

(b) The assumption that $\sigma_{\text{per}}(A)$ consist of poles of the resolvent cannot be dropped in Theorem 7.3.3.

To construct a counterexample we proceed similarly as in (a). This time, let $B \in \mathbb{R}^{3\times 3}$ be such that for each $t \geq 0$ the operator e^{tB} is the rotation of angle t about the line in the direction $(1,1,1) \in \mathbb{R}^3$. Let P denote the spectral projection for the eigenvalue 0 of B. Then $I-P$ is the projection along the span of $(1,1,1)$ onto its orthogonal complement. We set $A_n := B - \frac{1}{n}(I-P)$ for each $n \in \mathbb{N}$.

Endow $\mathbb{C}^3$ with the euclidean norm and set $E = \ell^2(\mathbb{N};\mathbb{C}^3)$. Let $A : E \supseteq D(A) \to E$ be the multiplication operator with symbol (A_n). Then A generates a bounded C_0-semigroup $(e^{tA})_{t\geq 0}$ which is individually asymptotically positive. The spectrum of A is given by

$$\sigma(A) = \{0, \pm i\} \cup \bigcup_{n\in\mathbb{N}} \{\pm i - \frac{1}{n}\},$$

so $s(A) = 0$ is an isolated point in the spectrum and it can readily be checked to be even a first order pole of the resolvent. However, the

spectral values i and $-i$ are not isolated in the spectrum; in particular, they cannot be poles of the resolvent. Hence the only assumption of Theorem 7.3.3 which fails is that the peripheral spectrum of A consist of poles of the resolvent. Yet, despite the semigroup being individually asymptotically positive, assertion (i) from Theorem 7.3.3 does not hold since $s(A)$ is not a dominant spectral value.

One might wonder whether, under the assumptions of Theorem 7.3.3, individual asymptotic positivity of the semigroup does already imply uniform asymptotic positivity. Our answer to this question is twofold: first we show in the following Example 7.3.5 that the answer is "no", in general. Afterwards, in Theorem 7.3.6, we show that the answer is "yes" under an additional assumption on the semigroup.

Example 7.3.5. There are semigroups, fulfilling the assumptions of Theorem 7.3.3, which are individually asymptotically positive but not uniformly asymptotically positive.

The counterexample we are going to construct is rather similar to the semigroup in Example 7.3.4(b). Let $B \in \mathbb{R}^{3\times 3}$ be such that for each $t \in \mathbb{R}$ the operator e^{tB} is the rotation of angle t about the line in the direction $(1,1,1) \in \mathbb{R}^3$. Let P denote the spectral projection for the eigenvalue 0 of B. Thus, $I-P$ is the projection along the span of $(1,1,1)$ onto its orthogonal complement. We now set $A_n := nB - \frac{1}{n}(I-P)$ for each $n \in \mathbb{N}$.

Endow $\mathbb{C}^3$ with the Euclidean norm, let $E := \ell^2(\mathbb{N};\mathbb{C}^3)$ and denote by $A : E \supseteq D(A) \to E$ the matrix multiplication operator with symbol (A_n). Then it can be readily checked that A generates a bounded C_0-semigroup $(e^{tA})_{t\geq 0}$ which is individually, but not uniformly asymptotically positive. Moreover, the spectrum of A is given by

$$\sigma(A) = \{0\} \cup \{\pm ni - \frac{1}{n} |\ n \in \mathbb{N}\};$$

hence, $s(A) = 0$ is a dominant spectral value of A and one can easily see that it is also a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. Thus, all assumptions of Theorem 7.3.3 are fulfilled and we conclude that individual asymptotic positivity of the semigroup does imply uniform asymptotic positivity under the assumptions of Theorem 7.3.3.

Yet, it is worthwhile recalling that the situation is somewhat different for resolvents. Since, according to Theorem 7.3.3, the spectral projection of A associated to 0 is positive, we can conclude from Theorem 7.2.2 that the resolvent $\mathcal{R}(\cdot, A)$ is not only individually, but even uniformly asymptotically positive at $s(A) = 0$.

To obtain a characterisation for uniformly asymptotically positive semigroups, we have to make an additional assumption on the long time behaviour

of the semigroup; this additional assumption is automatically fulfilled if the semigroup is sufficiently regular, see Remark 7.3.7 below. Our characterisation of uniform asymptotic positivity reads as follows:

Theorem 7.3.6. *Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach lattice E, suppose that $\mathrm{s}(A) > -\infty$ and that the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded. Assume moreover that $\sigma_{\mathrm{per}}(A)$ is non-empty, finite and consists of poles of the resolvent and let P_{per} be the spectral projection associated with $\sigma_{\mathrm{per}}(A)$. If the restriction of the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ to $(I-P_{\mathrm{per}})E$ converges to 0 with respect to the operator norm as $t \to \infty$, then the following assertions are equivalent:*

(i) *$\mathrm{s}(A)$ is a dominant spectral value of A and the associated spectral projection P is positive.*

(ii) *The semigroup $(e^{tA})_{t\geq 0}$ is individually asymptotically positive.*

(iii) *The semigroup $(e^{tA})_{t\geq 0}$ is uniformly asymptotically positive.*

(iv) *The rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ converges with respect to the operator norm to a positive operator as $t \to \infty$.*

Proof. We may assume that $\mathrm{s}(A) = 0$.

"(i) $\Rightarrow$ (iv)" Suppose that 0 is a dominant spectral value of A and that $P \geq 0$. Then we have $P_{\mathrm{per}} = P$. Moreover, since the semigroup $(e^{tA})_{t\geq 0}$ is bounded, 0 is a simple pole of the resolvent and hence the range of P coincides with the kernel of A. Thus, $e^{tA}P = P$ for all $t \geq 0$. By assumption, $e^{tA}(I-P)$ converges to 0 with respect to the operator norm as $t \to \infty$, so we conclude that $e^{tA} = P + e^{tA}(I-P)$ converges to the positive operator P with respect to the operator norm as $t \to \infty$.

"(iv) $\Rightarrow$ (iii)" Assume that e^{tA} converges to an operator $Q \geq 0$ with respect to the operator norm as $t \to \infty$ and let $\varepsilon > 0$. For all sufficiently large times t, $t \geq t_0$ say, we then have $\|e^{tA} - Q\| \leq \varepsilon$. Thus, we obtain for all $t \geq t_0$ and all $f \in E_+$ the estimate

$$\operatorname{dist}(e^{tA}f, E_+) \leq \operatorname{dist}(Qf, E_+) + \|e^{tA}f - Qf\| \leq \|e^{tA} - Q\|\|f\| \leq \varepsilon\|f\|.$$

This shows that $(e^{tA})_{t\geq 0}$ is uniformly asymptotically positive, as claimed.

"(iii) $\Rightarrow$ (ii)" and "(ii) $\Rightarrow$ (i)" The first of these implications is obvious and the second has already been proved in Theorem 7.3.3. □

In case that the semigroup $(e^{tA})_{t\geq 0}$ is eventually norm continuous some of the assumptions of Theorem 7.3.6 are automatically fulfilled or easier to check. Let us sum this up in the following remark:

Remark 7.3.7. Let $(e^{tA})_{t\geq 0}$ be an eventually norm continuous C_0-semigroup on a complex Banach space E and assume that $\mathrm{s}(A) > -\infty$. Then the following assertions hold:

(a) The peripheral spectrum $\sigma_{\mathrm{per}}(A)$ is non-empty and compact.

(b) If $\sigma_{\mathrm{per}}(A)$ consists of pole of the resolvent, then it is finite. If P_{per} denotes the corresponding spectral projection, then the restriction of the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ to the space $(I-P_{\mathrm{per}})E$ converges to 0 with respect to the operator norm.

(c) If $\sigma_{\mathrm{per}}(A)$ consists of poles of the resovlent, then the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded if and only if all these pole are of order 1.

Proof. We may assume that $\mathrm{s}(A) = 0$.

(a) For eventually norm continuous semigroups the set $\{\lambda \in \sigma(A) | \operatorname{Re}\lambda \geq s\}$ is compact for every $s \in \mathbb{R}$, see [EN00, Theorem II.4.18]. This easily implies the assertion.

(b) Since $\sigma_{\mathrm{per}}(A)$ is compact and consists of isolated points, it must be finite. Since set $\{\lambda \in \sigma(A) | \operatorname{Re}\lambda \geq s\}$ is compact for every $s \in \mathbb{R}$, the spectral bound of the restriction of A to $(I-P_{\mathrm{per}})E$ is strictly negative. As the semigroup $(e^{tA}|_{(I-P_{\mathrm{per}})E})_{t\geq 0}$ is eventually norm continuous, we conclude that its growth bound is strictly negative, too [EN00, Corollary IV.3.11]. This proves the assertion.

(c) If $(e^{tA})_{t\geq 0}$ is bounded, then all poles of the resolvent on $i\mathbb{R}$ are of order one according to Propositions A.2.4(b) and A.3.2(d).

On the other hand, we conclude from (b) that the peripheral spectrum $\sigma_{\mathrm{per}}(A)$ is finite and that the semigroup $(e^{tA})_{t\geq 0}$ operator norm converges to 0 on the space $(I-P_{\mathrm{per}})E$. In particular, $(e^{tA})_{t\geq 0}$ is bounded on $(I-P_{\mathrm{per}})E$. Now, if $\sigma_{\mathrm{per}}(A) = \{i\beta_1, ..., i\beta_m\}$ consists of first order poles of the resolvent, then we have $P_{\mathrm{per}}E = \oplus_{j=1}^{m} \ker(i\beta_j - A)$ and hence, $(e^{tA})_{t\geq 0}$ is also bounded on $P_{\mathrm{per}}E$. □

We close the section with a result on the special case of self-adjoint semigroups on L^2-spaces. For this specal case, we obtain the following sufficient condition for uniform asymptotic positivity as a consequence of Theorem 7.3.6:

Theorem 7.3.8. *Let H be a complex Hilbert lattice and let A be a densely-defined, self-adjoint operator such that $\mathrm{s}(A)$ is not ∞ and an isolated point of the spectrum $\sigma(A)$. Moreover, assume that $\ker(\mathrm{s}(A) - A)$ is one-dimensional and contains a non-zero positive vector.*

Then the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded and the semigroup $(e^{tA})_{t\geq 0}$ is uniformly asymptotically positive.

Proof. First note that A generates a C_0-semigroup since $s(A) < \infty$ and since A is self-adjoint; for the same reason, the semigroup $(e^{t(A-s(A))})_{t\geq 0}$ is bounded. Since $s(A)$ is an isolated spectral value, and again since A is self-adjoint, we conclude that $s(A)$ is even a first order pole of the resolvent.

Since the Hilbert space adjoint $A^* = A$ of A contains a non-zero positive eigenvector for the eigenvalue $s(A)$, it follows from Corollary C.6.4 that the Banach space adjoint A' of A has also a non-zero positive eigenvector for the eigenvalue $s(A)$. Hence, the spectral projection P corresponding to $s(A)$ is positive according to Corollary 7.1.2. Since the semigroup $(e^{tA})_{t\geq 0}$ is analytic, it fulfils the assumptions of Theorem 7.3.6 according to Remark 7.3.7. Theorem 7.3.6 thus implies that the semigroup is uniformly asymptotically positive. □

7.4 Notes & Open Problems

On the definition of asymptotic positivity When trying to find an appropriate definition of the notion *asymptotic positivity*, the following question arises immediately: the positivity property we are interested in should occur "asymptotically" compared to what? Our definition of an individually asymptotically positive semigroup is based on the idea that, for a positive initial value f, the distance $\operatorname{dist}(e^{tA}f, E_+)$ becomes, for large time, small compared to the operator norm $\|e^{tA}\|$. However, this raises the following question:

Problem 7.4.1. It can happen that for some $f \geq 0$ the vector $e^{tA}f$ tends to 0 as $t \to \infty$, while the operator norm $\|e^{tA}\|$ does not. So, is our definition of *asymptotic positivity* appropriate, or should we rather call a C_0-semigroup individually asymptotically positive if, for every $f > 0$, the quotient $\operatorname{dist}(e^{tA}f, E_+)/\|e^{tA}f\|$ tends to 0 as $t \to \infty$?

A similar question arises, of course, for the definition of individually asymptotically positive resolvents. The Theorems 7.2.2 and 7.3.3 suggest that our definition of asymptotic positivity is the right one if we seek to prove characterisations for positive spectral projections. Yet, it is possible that the alternative definition suggested above yields an interesting theory, too.

Let us also pose a second question which is loosely related to the first one:

Problem 7.4.2. Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on a complex Banach lattice E with $s(A) > -\infty$. How should asymptotic positivity of $(e^{tA})_{t\geq 0}$ be defined if the rescaled semigroup $(e^{tA})_{t\geq 0}$ is not bounded?

This question can be considered on several levels of generality: if, for instance, $\omega_0(A) = s(A)$, then it follows that $\|e^{tA}\|$ can grow only sub-exponentially faster then $e^{s(A)}$. For general semigroups it may, however, happen that even

$\omega_0(A) > \mathrm{s}(A)$, which means that the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ grows even exponentially fast. In both cases it does not seem to be clear what would be an appropriate notion of asymptotic positivity.

More general characterisation theorems All our characterisation results for asymptotically positive resolvents and semigroups impose rather strong conditions on the peripheral spectrum of the semigroup generator; namely, it has to be finite and it has to consist of poles of the resolvent. This raises the following question:

Problem 7.4.3. How can asymptotic positivity of a semigroup $(e^{tA})_{t\geq 0}$ be characterised if $\sigma_{\mathrm{per}}(A)$ is infinite or even unbounded?

Asymptotically positive operators While we dealt with characterisations of asymptotically positive C_0-semigroups in this chapter, on cane also consider single operators with asymptotically positive powers. In this context it is interesting to note that, if $A : E \supseteq D(A) \to E$ is a linear operator on a complex Banach lattice E, asymptotic positivity of the resolvent $\mathcal{R}(\cdot, A)$ at the spectral bound $\mathrm{s}(A)$ can, under appropriate assumptions, be characterised by asymptotic positivity of the powers of $\mathcal{R}(\lambda, A)$ for any fixed $\lambda \in (\mathrm{s}(A), \infty)$; see [DGK16b, Theorem 7.6] for details.

Topological dynamics and the Jacobs-de-Leeuw-Glicksberg decomposition We briefly note that Corollary 7.3.2, whose proof relies on arguments from topological dynamics, is related to the famous Jacobs-de-Leeuw-Glicksberg decomposition of a semigroup (see e.g. [EN00, Section V.2], [Kre85, Section 2.4] and [EFHN15, Section 16.3] for details about this decomposition). Since, however, a thorough discussion of this relation would lead to far away from the main thrust of the thesis, we leave the details to the interested reader.

8 Applications of Asymptotic Positivity

While eventual positivity of certain differential operators has been studied for several years in the literature, it seems that asymptotic positivity has played a less prominent role. Nevertheless, we shall see in the present chapter that this phenomenon also occurs in many concrete applications. We are going to demonstrate the applicability of our results from Chapter 7 by presenting a large variety of examples for asymptotically positive behaviour. We begin in Section 8.1 where we show that interesting results for finite-dimensional matrix semigroups can be derived with the help of our general theory. In Section 8.2 we consider the semigroup generated by the Dirichlet-to-Neumann operator on the boundary of a smooth domain $\Omega \subseteq \mathbb{R}^d$ and give sufficient conditions for this semigroup to be asymptotically positive in terms of the eigenfunctions corresponding to the spectral bound. In Section 8.3 we consider the bi-Laplace operator on domains close to the unit ball and prove that its resolvent behaves asymptotically positive at its spectral bound; we are not going to consider the semigroup generated by the bi-Laplace operator here, since for the semigroup much stronger results will be available in Chapter 11 after we have presented a general theory for what we are going to call *eventually strongly positive semigroups*. In Section 8.4 we consider delay differential equations and prove that they generate asymptotically positive semigroups under rather general assumptions. A network flow with non-positive mass transition in the vertices of the network is considered in Section 8.5. Again, we will see that the corresponding semigroup behaves asymptotically positive. In the final Section 8.6 we consider a diffusion process on an interval under the influence of non-autonomous boundary conditions. We indicate how such a model can actually be treated by semigroup methods and we prove asymptotic positivity of the corresponding semigroup under appropriate assumptions.

Some of our examples, namely the finite-dimensional case, the Dirichlet-to-Neumann operator, the bi-Laplace operator and delay differential equations, will be revisited in Chapter 11 below; there we will see that under some additional assumptions we can even prove much stronger forms of eventually positive behaviour.

The results presented in Sections 8.2, 8.3 and 8.5, as well as many of the results from Section 8.1, have been published in [DGK16a, Section 9]. An exception is Proposition 8.1.7 which is contained in [DGK16b, Proposition 6.2]; further exceptions are Theorem 8.1.3 and Corollary 8.1.4 which have not been stated explicitly in [DGK16a, Section 9], though large parts of their

proofs are implicitly contained in the proof of [DGK16a, Theorem 9.1]. A special case of the results presented in Section 8.4 (namely the one discussed in Example 8.4.3(a)) has also been published in [DGK16a, Section 9]. The other, more general results from Section 8.4, as well as the results discussed in Section 8.6, have not been published or submitted for publication, yet.

8.1 The Finite-Dimensional Case

Let us begin by applying our general theory from Chapter 7 to the special case of finite-dimensional Banach lattices. On such Banach lattices, the phenomenon of eventual positivity has been observed and studied for a bit longer in the literature than the infinite-dimensional case; see the notes at the end of Chapter 6 for more details and some historical remarks.

We point out that our definition, as well as our characterisations, of asymptotically positive semigroups do not assume the semigroup generator A to be real. Hence, our results below are also a contribution to the Perron–Frobenius theory of matrices with some complex entries which was, from various perspectives, also studied in [Rum03], [NV12] and [TCDF15].

Recall that a finite-dimensional real Banach lattice is always Banach lattice isomorph to $\mathbb{R}^d$, endowed with its standard cone and some lattice norm [Sch74, Corollary 1 to Theorem II.3.9]; hence, a finite-dimensional complex Banach lattice is always isomorphic to $\mathbb{C}^d$. This is why we restrict ourselves to considerations on $\mathbb{C}^d$ throughout this section. Moreover, we identify each operator $A \in \mathcal{L}(\mathbb{C}^d)$ with its representation matrix with respect to the canonical basis. Similar to the definition $E_+ := (E_{\mathbb{R}})_+$ for general complex Banach lattices E, we use the convention $\mathbb{C}^d_+ := \mathbb{R}^d_+$.

Let $A \in \mathbb{C}^d$. It is well-known that the semigroup $(e^{tA})_{t\geq 0}$ is positive if and only if A is real and all off-diagonal entries of A are ≥ 0, i.e. there exists of good description of positivity of the semigroup in terms of the generator A. Similarly, it is known that certain eventual positivity properties of the semigroup $(e^{tA})_{t\geq 0}$ can be described by what one might call eventual positivity properties of the powers $(A + cI)^n$ (for some appropriate choice $c \in \mathbb{R}$); see e.g. [NT08, Theorem 3.3] and also Section 11.1 below. It is thus natural to expect that we can obtain a characterisation for asymptotic positivity of $(e^{tA})_{t\geq 0}$ in terms of the powers of (a shift of) A. To this end, we define asymptotic positivity for powers of single matrices, first:

Definition 8.1.1 (Asymptotically positive matrices). Suppose that $T \in \mathbb{C}^{d\times d}$ has positive spectral radius $r(T) > 0$ and that $\frac{T}{\mathrm{r}(T)}$ is power-bounded.

(i) The matrix T is called *individually asymptotically positive* if we have $\operatorname{dist}((\frac{T}{\mathrm{r}(T)})^n f, \mathbb{C}^d_+) \to 0$ as $n \to \infty$ for every $0 \leq f \in \mathbb{C}^d$.

(ii) The matrix T is called *uniformly asymptotically positive* if for every $\varepsilon > 0$ there exists $n_0 \geq 0$ such that we have $\operatorname{dist}((\frac{T}{\mathrm{r}(T)})^n f, \mathbb{C}^d_+) \leq \varepsilon \|f\|$ for all $n \geq n_0$ and every $0 \leq f \in \mathbb{C}^d$.

Of course, eventual positivity can also be defined and studied for powers of single operators on infinite-dimensional Banach lattices; see the notes at the end of Chapter 7 for more details.

Since the underlying space is finite-dimensional in this section, it turns out that individual and uniform asymptotic positivity of operators are the same (this is also true for C_0-semigroups: see the discussion after the following proposition):

Proposition 8.1.2. *Suppose that $T \in \mathbb{C}^{d\times d}$ has positive spectral radius and that $\frac{T}{\mathrm{r}(T)}$ is power-bounded. Then T is individually asymptotically positive if and only if T is uniformly asymptotically positive.*

Proof. "$\Leftarrow$" This implication is obvious.

"$\Rightarrow$" We may assume that $r(T) = 1$ and that $\mathbb{C}^d$ is endowed with the $\|\cdot\|_1$-norm. Assume that T is individually asymptotically positive and let $\varepsilon > 0$. If $e_1, \ldots, e_d \in \mathbb{C}^d$ denote the canonical unit vectors, then we have

$$\operatorname{dist}(T^n e_j, \mathbb{C}^d_+) \leq \varepsilon$$

for all $j \in \{1, \ldots, d\}$ and all sufficiently large $n \in \mathbb{N}$, $n \geq n_0$ say. Thus we obtain for each $n \geq n_0$ and each $f \in \mathbb{C}^d_+$

$$\operatorname{dist}(T^n f, \mathbb{C}^d_+) \leq \sum_{j=1}^{d} f_j \operatorname{dist}(e_j, \mathbb{C}^d_+) \leq \sum_{j=1}^{d} f_j \varepsilon = \varepsilon \|f\|_1.$$

Hence, T is indeed uniformly asymptotically positive. □

According to the above proposition it is justified to simply speak of *asymptotically positive matrices* instead of individually or uniformly asymptotically positive matrices. Similarly, if $A \in \mathbb{C}^{d\times d}$ is a matrix such that the semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded, then it follows from Theorem 7.3.6 that the semigroup $(e^{tA})_{t\geq 0}$ is individually asymptotically positive if and only if it is uniformly asymptotically positive. Therefore we may simply speak of *asymptotically positive semigroups* in finite dimensions.

Before we give our characterisation of asymptotically positive matrix semigroup, we prove a characterisation and several properties of asymptotically positive matrices:

Theorem 8.1.3. *Suppose that $T \in \mathbb{C}^{d\times d}$ has positive spectral radius and that $\frac{T}{\mathrm{r}(T)}$ is power-bounded. Then the following assertions are equivalent:*

(i) *The matrix T is asymptotically positive.*

(ii) *If P_{per} denotes the spectral projection associated to $\{\lambda \in \sigma(T) | \, |\lambda| = \mathrm{r}(T)\}$, then TP_{per} is positive.*

If the equivalent conditions (i) *and* (ii) *are fulfilled, then P_{per} is positive, too; moreover, $\mathrm{r}(T) \in \sigma(T)$ and the spectral projection P associated to $\mathrm{r}(T)$ is also positive.*

Proof. We may assume that $\mathrm{r}(T) = 1$.

"(i) $\Rightarrow$ (ii)" Assume that T is asymptotically positive. According to Corollary 7.3.2(b) we can find a sequence of integers $0 \leq k_n \to \infty$ such that $T^{k_n} f \to f$ for each $f \in P_{\mathrm{per}}\mathbb{C}^d$. Thus we obtain for every $k \in \mathbb{N}_0$ and every $f \in \mathbb{C}^d_+$

$$\begin{aligned}\operatorname{dist}(T^k P_{\mathrm{per}} f, E_+) &= \lim_{n\to\infty} \operatorname{dist}(T^{k+k_n} P_{\mathrm{per}} f, E_+) \\ &\leq \lim_{n\to\infty} \operatorname{dist}(T^{k+k_n} f, E_+) + \lim_{n\to\infty} \|T^{k+k_n}(I_E - P_{\mathrm{per}})f\| = 0.\end{aligned}$$

Thus, $T^k P_{\mathrm{per}} \geq 0$ for each $k \in \mathbb{N}_0$. In particular we obtain $P_{\mathrm{per}} \geq 0$ and $TP_{\mathrm{per}} \geq 0$.

"(ii) $\Rightarrow$ (i)" Assume that $TP_{\mathrm{per}} \geq 0$; then we have $T^n P_{\mathrm{per}} = (TP_{\mathrm{per}})^n \geq 0$ for each $n \in \mathbb{N}$.

Let $\varepsilon > 0$. We clearly have $T^n(I - P_{\mathrm{per}}) \to 0$ as $n \to \infty$, so we obtain $\|T^n(I - P_{\mathrm{per}})\| \leq \varepsilon$ for all sufficiently large n, $n \geq n_0$ say. We thus obtain for every $n \geq n_0$ and every $f \in \mathbb{C}^d_+$

$$\operatorname{dist}(T^n f, E_+) \leq \operatorname{dist}(T^n P_{\mathrm{per}} f, E_+) + \|T^n(I - P_{\mathrm{per}})f\| \leq \varepsilon \|f\|.$$

Hence, T is asymptotically positive.

Now assume that the equivalent assertions (i) and (ii) are fulfilled. We have already seen in the proof of the implication "(i) $\Rightarrow$ (ii)" that P_{per} is then positive. Moreover, this implies that the range $P_{\mathrm{per}}\mathbb{C}^d$ of P_{per} is also a complex Banach lattice with respect to an equivalent norm and with positive cone $(P_{\mathrm{per}}\mathbb{C}^d)_+ := \mathbb{C}^d_+ \cap P_{\mathrm{per}}\mathbb{C}^d = P_{\mathrm{per}}\mathbb{C}^d_+$; see Proposition C.5.5. Since $TP_{\mathrm{per}} \geq 0$, it follows that the restriction $T|_{P_{\mathrm{per}}}$ is a positive operator on the complex Banach lattice $P_{\mathrm{per}}\mathbb{C}^d$. Hence, the spectral radius of $T|_{P_{\mathrm{per}}}$ is contained in its spectrum. On the other hand we have $\mathrm{r}(T|_{P_{\mathrm{per}}}) = \mathrm{r}(T) = 1$ and $\sigma(T|_{P_{\mathrm{per}}}) \subseteq \sigma(T)$. This shows that $1 = \mathrm{r}(T) \in \sigma(T)$, as claimed.

It remains to show that the spectral projection P associated to the spectral value $\mathrm{r}(T) = 1$ is positive. Note that 1 is a first order pole of the resolvent $\mathcal{R}(\cdot, T)$ since T is power bounded; see Propositions A.2.4(a) and A.3.2(d). Hence, we have $P = \lim_{\lambda \downarrow 1}(\lambda - 1)\mathcal{R}(\lambda, T)$. Let $f \in \mathbb{C}^d_+$. Using the Neumann series representation of the resolvent we obtain for $\lambda > 1$

$$\operatorname{dist}((\lambda - 1)\mathcal{R}(\lambda, T)f, \mathbb{C}^d_+) \leq (\lambda - 1) \sum_{n=0}^{\infty} \frac{\operatorname{dist}(T^n f, \mathbb{C}^d_+)}{\lambda^{n+1}} \overset{\lambda \downarrow 1}{\to} 0;$$

here we used that $(\lambda-1)\sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}=1$ for each $\lambda>1$ and that $\operatorname{dist}(T^n f,\mathbb{C}_+^d)\to 0$ as $n\to\infty$. Hence, $\operatorname{dist}(Pf,\mathbb{C}_+^d)=0$, proving that P is positive. □

As a consequence of Theorem 8.1.3 we obtain the following result:

Corollary 8.1.4. *Let $T\in\mathbb{C}^{d\times d}$ be a matrix with spectral radius $\mathrm{r}(T)>0$. Assume that $\mathrm{r}(T)$ is a first order pole of the resolvent $\mathcal{R}(\cdot,T)$, let P denote the corresponding spectral projection and assume that $|\lambda|<\mathrm{r}(T)$ for all other spectral values $\lambda\in\sigma(T)\setminus\{\mathrm{r}(T)\}$.*

Then $\frac{T}{\mathrm{r}(T)}$ is power bounded. Moreover, T is asymptotically positive if and only if $P\geq 0$.

Proof. We may assume that $\mathrm{r}(T)=1$.

The powers of T clearly converge to 0 on $(I-P)\mathbb{C}^d$. On the other hand, the range $P\mathbb{C}^d$ coincides with the fixed space of T since $\mathrm{r}(T)$ is a first order pole of the resolvent. Hence, the powers of T are bounded on $P\mathbb{C}^d$. This shows that T is power bounded.

Note that P is the spectral projection associated to $\sigma(T)\cap\mathbb{T}$ since the latter set consists only of $\{\mathrm{r}(T)\}$. Hence, it follows from Theorem 8.1.3 that T is asymptotically positive if and only if $TP\geq 0$. Since, however, the range of P consists of fixed points of T, we have $TP=P$. This proves the asserted equivalence. □

Now we can finally give a characterisation of asymptotically positive matrix semigroups, our main result in this section:

Theorem 8.1.5. *For every matrix $A\in\mathbb{C}^{d\times d}$ the following assertions are equivalent:*

(i) *The rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded and the semigroup $(e^{tA})_{t\geq 0}$ is asymptotically positive.*

(ii) *There is a number $c>0$ such that $A+c$ has strictly positive spectral radius, $\frac{A+c}{\mathrm{r}(A+c)}$ is power-bounded and $A+c$ is asymptotically positive.*

(iii) *For all sufficiently large numbers $c>0$ the matrix $A+c$ has strictly positive spectral radius, $\frac{A+c}{\mathrm{r}(A+c)}$ is power-bounded and $A+c$ is asymptotically positive.*

Proof. "(i) ⇒ (iii)" Assume that the rescaled semigroup is bounded and that $(e^{tA})_{t\geq 0}$ is asymptotically positive. Then $\mathrm{s}(A)$ is a dominant spectral value of A and the corresponding spectral projection P is positive according to Theorem 7.3.3. Let $\overline{\mathbb{D}}$ denote the closed unit disk and $\mathbb{T}$ the unit circle in $\mathbb{C}$. If we choose $c>0$ sufficiently large, then the closed disk $(\mathrm{s}(A)+c)\overline{\mathbb{D}}$ contains the set $\sigma(A)+c$ and the circle $(\mathrm{s}(A)+c)\mathbb{T}$ intersects $\sigma(A)+c$ only in $\mathrm{s}(A)+c>0$. Hence $\mathrm{s}(A+c)=\mathrm{s}(A)+c$ equals the spectral radius $\mathrm{r}(A+c)$ of $A+c$ and every other spectral value of $A+c$ has modulus strictly smaller than $\mathrm{s}(A+c)$.

Since the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded, $\mathrm{s}(A)$ is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. Hence, $\mathrm{r}(A+c) = \mathrm{s}(A+c)$ is a first order pole of the resolvent $\mathcal{R}(\cdot, A+c)$; the corresponding spectral projection is also given by $P \geq 0$. Thus, according to Corollary 8.1.4, $\frac{A+c}{\mathrm{r}(A+c)}$ is power bounded and $A+c$ is asymptotically positive.

"(iii) $\Rightarrow$ (ii)" This implication is obvious.

"(ii) $\Rightarrow$ (i)" If $\frac{A+c}{\mathrm{r}(A+c)}$ is power bounded and $A+c$ is asymptotically positive, then, according to Theorem 8.1.3, the spectral radius $\mathrm{r}(A+c)$ is a spectral value of $A+c$ and the corresponding spectral projection P is positive. It follows from $\mathrm{r}(A+c) \in \sigma(A+c)$ that the spectral bound $\mathrm{s}(A+c)$ equals $\mathrm{r}(A+c)$ and is a dominant spectral value of $A+c$. Hence, $\mathrm{s}(A)$ is a dominant spectral value of A and the corresponding spectral projection is also given by $P \geq 0$.

Note that $\mathrm{r}(A+c) = \mathrm{s}(A+c)$ is a first order pole of the resolvent $\mathcal{R}(\cdot, A+c)$ since $\frac{A+c}{\mathrm{r}(A+c)}$ is power bounded; this follows from Propositions A.2.4(a) and A.3.2(d). Thus, $\mathrm{s}(A)$ is a first order pole of $\mathcal{R}(\cdot, A)$ and hence, the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded according to Remark 7.3.7(c). Theorem 7.3.6 therefore implies that the semigroup $(e^{tA})_{t\geq 0}$ is asymptotically positive. □

Let us finally take a closer look at the two-dimensional case. In this case we can show that an asymptotically positive semigroup is automatically positive, at least if the generator is real:

Proposition 8.1.6. *Let $A \in \mathbb{R}^{2\times 2}$ and suppose that $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded. If the semigroup $(e^{tA})_{t\geq 0}$ is asymptotically positive, then it is positive.*

Proof. We distinguish two cases. In the first cast, A has only one eigenvalue $\lambda \in \mathbb{C}$. Since A is real, it then follows that $\lambda \in \mathbb{R}$ and thus $\lambda = \mathrm{s}(A)$. Moreover, the eigenvalue λ is semi-simple since $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded by assumption. Hence, $A = \lambda I$ and it follows that $(e^{tA})_{t\geq 0} = (e^{t\lambda} I)_{t\geq 0}$ is positive.

In the second case, A has two different eigenvalues $\lambda_1 \neq \lambda_2$. According to Theorem 7.3.3 we have $\mathrm{s}(A) \in \sigma(A)$, so $\lambda_2 = \mathrm{s}(A)$, say; moreover, the corresponding spectral projection P is positive, again according to Theorem 7.3.3. Since A is real, it follows that λ_1 is real, too and hence, $\lambda_1 < \lambda_2$. For every $t \geq 0$ we thus obtain

$$e^{tA} = e^{t\lambda_2} P + e^{t\lambda_1}(I-P) = (e^{t\lambda_2} - e^{t\lambda_1})P + e^{t\lambda_1} I \geq 0.$$

This proves the assertion. □

Note that the conclusion of the above proposition does not in general hold if the matrix A is allowed to have complex entries. For instance, the matrix $A = \begin{pmatrix} 0 & 0 \\ 0 & i-1 \end{pmatrix}$, which we have already briefly considered in Section 6.1,

has spectral bound 0 and generates a bounded and asymptotically positive semigroup $(e^{tA})_{t\geq 0}$; yet, this semigroup is not positive.

Finally, we briefly consider eventually positive semigroups: since, in finite dimensions, a C_0-semigroup is individually eventually positive if and only if it is uniformly eventually positive, we shall call such semigroups simply *eventually positive*. Suppose that $(e^{tA})_{t\geq 0}$ is an eventually positive semigroup on $\mathbb{C}^2$ with real generator A. If we assume that the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded, then the semigroup $(e^{tA})_{t\geq 0}$ is obviously asymptotically positive and hence Proposition 8.1.6 implies that the semigroup is actually positive. If, however, the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is not bounded, then the notion is asymptotic positivity is not even well-defined for our semigroup, so we cannot apply Proposition 8.1.6. Yet, as similar result as in this proposition holds; this is our final result in this section.

Proposition 8.1.7. *Let $A \in \mathbb{R}^{2\times 2}$. If the semigroup $(e^{tA})_{t\geq 0}$ is eventually positive, then it is even positive.*

Proof. We may assume that $\mathrm{s}(A) = 0$ and, according to Theorem 6.2.5, we then have $\mathrm{s}(A) = 0 \in \sigma(A)$. If the eigenvalue 0 of A is semi-simple, then the semigroup $(e^{tA})_{t\geq 0}$ is bounded and hence asymptotically positive; the assertion thus follows from Proposition 8.1.6. If, on the other hand, the eigenvalue 0 of A is not semi-simple, then $A^2 = 0$. Hence, $0 \leq e^{tA} = I + tA$ for sufficiently large $t \geq 0$. This shows that $A \geq 0$ and thus the semigroup $(e^{tA})_{t\geq 0}$ is positive in this case, too. □

8.2 The Dirichlet-to-Neumann Operator in High Dimensions

In this Section we consider the semigroup generated by the so-called Dirichlet-to-Neumann operator on the boundary of a bounded domain Ω in $\mathbb{R}^d$. In the case where $d = 2$ and Ω is the uni disk, an analysis of eventual positivity properties for this semigroup was performed in [Dan14]; this was one of the motivations to develop the general theory presented in Part III of this thesis.

In this section we deal with the *asymptotic* positivity of the semigroup and consider domains in arbitrary dimensions; in the case $d = 2$, *eventual* positivity of the semigroup is studied in Section 11.2.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain; for simplicity, we assume that Ω has $\mathcal{C}^\infty$-boundary. Let $\Delta_D : L^2(\Omega;\mathbb{C}) \supseteq D(\Delta_D) \to L^2(\Omega;\mathbb{C})$ denote the Dirichlet Laplacian and fix a number $\lambda \in \mathbb{R} \setminus \sigma(\Delta_D)$. For any given boundary data

$g \in L^2(\partial\Omega;\mathbb{C})$ we solve, whenever possible, the equation

$$\begin{cases} \Delta f = \lambda f & \text{in } \Omega, \\ f = g & \text{on } \partial\Omega \end{cases}$$

for a function $f \in H^1(\Omega;\mathbb{C})$; then we map f to its normal derivative $\frac{\partial}{\partial \nu} f$ on $\partial\Omega$, in case that this normal derivative exists and is contained in $L^2(\partial\Omega;\mathbb{C})$. The mapping $g \mapsto \frac{\partial}{\partial \nu} f$ yields a linear operator $\mathcal{D}_\lambda : L^2(\partial\Omega;\mathbb{C}) \supseteq D(\mathcal{D}_\lambda) \to L^2(\partial\Omega;\mathbb{C})$ which is called the *Dirichlet-to-Neumann operator*. For a precise definition of this operator we refer to [AM12] or [AtEKS14]. The operator $-\mathcal{D}_\lambda$ is densely defined, self-adjoint, has finite spectral bound and compact resolvent; in particular, it generates a C_0-semigroup on $L^2(\partial\Omega;\mathbb{C})$. In [Dan14, Section 4] the special case of the Dirichlet-to-Neumann operator on a two-dimensional disk was analysed and it was shown that, for some parameters λ, the semigroup generated by $-\mathcal{D}_\lambda$ exhibits eventually positive behaviour. Using our theory developed in Section 7.3 we can now give a sufficient criterion for the Dirichlet-to-Neumann semigroup to be asymptotically positive:

Theorem 8.2.1. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^∞-boundary and let $\lambda \in \mathbb{R} \setminus \sigma(\Delta_D)$. Consider the Robin problem*

$$(*) \qquad \begin{cases} \Delta f = \lambda f & \text{in } \Omega, \\ \frac{\partial}{\partial \nu} f = \mathrm{s}(-\mathcal{D}_\lambda) f & \text{on } \partial\Omega. \end{cases}$$

for $f \in H^1(\Omega;\mathbb{C})$.

(a) *If the Robin problem $(*)$ has a solution which is unique up to scalars and which has non-zero positive trace on $\partial\Omega$, then the semigroup $(e^{-t\mathcal{D}_\lambda})_{t\geq 0}$ is uniformly asymptotically positive.*

(b) *If the semigroup $(e^{-t\mathcal{D}_\lambda})_{t\geq 0}$ is individually (equivalently: uniformly) asymptotically positive, then the Robin problem $(*)$ has at least one solution with non-zero positive trace on $\partial\Omega$.*

Proof. (a) First note that each eigenvector of $-\mathcal{D}_\lambda$ for the eigenvalue $\mathrm{s}(-\mathcal{D}_\lambda)$ is the trace of a solution of the Robin problem $(*)$. Hence, if this solution is unique up to scalars, then the eigenspace $\ker\big(\mathrm{s}(-\mathcal{D}_\lambda) + \mathcal{D}_\lambda\big)$ is one-dimensional. Thus, assertion (a) follows from Theorem 7.3.8.

(b) It follows from Theorem 7.3.6 and Remark 7.3.7 that the semigroup $(e^{-t\mathcal{D}_\lambda})_{tge0}$ is individually asymptotically positive if and only if it is uniformly asymptotically positive. Moreover, the range of the spectral projection associated with $\mathrm{s}(-\mathcal{D}_\lambda)$ consists of eigenvalues, so assertion (b) is a consequence of Theorem 7.3.3(i). □

Much more about the Dirichlet-to-Neumann operator can be said in two dimensions, i.e. if $\Omega \subseteq \mathbb{R}^2$; for this we refer to Section 11.2 below.

8.3 The bi-Laplace Operator with Dirichlet Boundary Conditions

Eventual positivity properties of the resolvent as well as the semigroup associated to the bi-Laplace operator with Dirichlet-Boundary conditions have been studied quite extensively for the last two decades; see e.g. [GS96], [GS97] and [FGG08]. For more information, we also refer to the notes at the end of this chapter, to Section 11.4 below and to the notes at the end of Chapter 11.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^\infty$-boundary. Let $p \in (1,\infty)$. We consider the bi-Laplace operator $A_p : L^p(\Omega;\mathbb{C}) \supseteq D(A_p) \to L^p(\Omega;\mathbb{C})$ which is given by

$$\begin{aligned} D(A_p) &= W^{4,p}(\Omega;\mathbb{C}) \cap W_0^{2,p}(\Omega;\mathbb{C}), \\ A_p f &= -\Delta^2 f. \end{aligned} \tag{8.1}$$

As pointed out above, eventual positivity properties of the resolvent of this operator have been the topic of many research papers, see the notes for some details. We will discuss eventually positive behaviour for the resolvent of A_p, as well as for the semigroup generated by A_p, in Section 11.4. In the current section we focus on asymptotic positivity.

Let us first sum up a few properties of the operator A_p.

Proposition 8.3.1. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^\infty$-boundary and let $p \in (1,\infty)$. The operator A_p is a closed, densely defined linear operator on $L^p(\Omega;\mathbb{C})$; it has compact resolvent and generates an analytic C_0-semigroup on $L^p(\Omega;\mathbb{C})$. If $p = 2$, then the operator $A_p = A_2$ is self-adjoint.*

Moreover, the (point) spectrum $\sigma(A_p) = \sigma_{\mathrm{pnt}}(A_p)$ and the eigenspaces of A_p are independent of p and the resolvent operators are consistent on the L^p-scale.

Proof. Obviously, A_p is densely defined. It follows from [GGS10, Corollary 2.21] that $0 \in \rho(A_p)$; in particular, A_p is closed and, since $D(A_p)$ embeds compactly into $L^p(\Omega;\mathbb{C})$, A_p has compact resolvent. It is proved in [Tan97, Theorem 5.6 on p. 189] that A_p generates an analytic semigroup on $L^p(\Omega;\mathbb{C})$. The operator A_2 is clearly symmetric and, as it generates an analytic C_0-semigroup, it is self-adjoint.

Now, let $1 < p_1 < p_2 < \infty$. For each $\lambda \in \mathbb{C}$ we have $\ker(\lambda - A_{p_2}) \subseteq \ker(\lambda - A_{p_1})$ as $D(A_{p_2}) \subseteq D(A_{p_1})$. If, on the other hand, v is contained in $\ker(\lambda - A_{p_1})$, then [GGS10, Corollary 2.21] together with a bootstrapping argument shows that $v \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{C})$ and hence, $v \in \ker(\lambda - A_{p_2})$. This shows that the point spectrum of A_p does not depend on p, and neither do the corresponding eigenspaces. Since each operator A_p has compact resolvent, its spectrum consists of eigenvalues, so we also conclude that $\sigma(A_p)$ is p-independent.

Finally, assume that $1 < p_1 < p_2 < \infty$ and let $\lambda \in \rho(A_{p_1}) = \rho(A_{p_2})$. For each $f \in L^{p_2}(\Omega;\mathbb{C})$, the function $w = \mathcal{R}(\lambda, A_{p_2})f$ is contained in $D(A_{p_2}) \subseteq D(A_{p_1})$ and we have $(\lambda - A_{p_1})w = (\lambda - A_{p_2})w = f$. This shows that $\mathcal{R}(\lambda, A_{p_1})$ and $\mathcal{R}(\lambda, A_{p_2})$ coincide on $L^{p_2}(\Omega;\mathbb{C})$. □

The following result on the first eigenfunction of A_p is due to Grunau and Sweers [GS98, Theorem 5.2].

Theorem 8.3.2. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain and let $p \in (1,\infty)$. Let u be the quasi-interior point of $L^p(\Omega;\mathbb{C})_+$ given by $u(\omega) = \operatorname{dist}(\omega, \partial\Omega)^2$ for all $\omega \in \Omega$.*

If Ω is sufficiently close to the unit ball in $\mathbb{R}^d$ in the sense of [GS98, Theorem 5.2], (where $m = 2$), then $\ker(\mathrm{s}(A_p) - A_p)$ is spanned by a function $v \gg_u 0$.

Combining this theorem with Proposition 9.1.1 below we obtain the following result about the spectral bound of A_p and the associated spectral projection:

Corollary 8.3.3. *Let $p \in (1,\infty)$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^∞-boundary such that the conclusion of Theorem 8.3.2 holds. Then $\mathrm{s}(A_p)$ is a dominant spectral value of A_p and a first order pole of the resolvent $\mathcal{R}(\cdot, A_p)$; the corresponding spectral projection P satisfies $P \gg_u 0$.*

Proof. As $\sigma(A_p) = \sigma(A_2)$ and A_2 is self-adjoint, it follows that $\mathrm{s}(A_p)$ is a dominant spectral value. Since A_p has compact resolvent, we also know that $\mathrm{s}(A_p)$ is a pole of $\mathcal{R}(\cdot, A_p)$. By assumption, $\ker(\mathrm{s}(A_p) - A_p)$ is one-dimensional and contains a function $v \gg_u 0$. Since the eigenspaces of A_p do not depend on p, it follows that we also have $v \in \ker(\mathrm{s}(A_p) - A_2)$. Using that A_2 is self-adjoint, we obtain $v \in \ker(\mathrm{s}(A_p) - A_2^*)$ and, as v is real, it follows from the discussion before Corollary C.6.4 that $v \in \ker(\mathrm{s}(A_p) - A_2')$. Moreover, it follows from [GGS10, Corollary 2.21] together with a bootstrapping argument that $v \in C^\infty(\overline{\Omega};\mathbb{C})$. Hence, $v \in \ker(\mathrm{s}(A_p) - A_p')$.

Proposition 9.1.1 below now implies that $P \gg_u 0$ and that $\mathrm{s}(A_p)$ is a first order pole of the resolvent $\mathcal{R}(\cdot, A_p)$. □

The assertion of the above corollary is stronger than needed in this section; we chose to prove the stronger version above because we will need this version in Section 11.4. For the current section we only need that $\mathrm{s}(A_p)$ is a dominant spectral value and a first order pole of the resolvent and that $P \geq 0$. This weaker assertion follows from Theorem 8.3.2 and Proposition 7.1.3 without employing Proposition 9.1.1

Combining Corollary 8.3.3 with Theorem 7.2.2, we now immediately obtain the following result on asymptotic positivity of the resolvent $\mathcal{R}(\cdot, A_p)$:

Proposition 8.3.4. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^\infty$-boundary and let $p \in (1, \infty)$. Let u be as in Theorem 8.3.2.*

If Ω as such that the conclusion of Theorem 8.3.2 holds, then the resolvent $\mathcal{R}(\cdot, A_p)$ is uniformly asymptotically positive at $\mathrm{s}(A_p)$.

Of course, we could also prove a similar result for the semigroup $(e^{tA_p})_{t\geq 0}$; however, we do not state this explicitly here as we are going to prove of much stronger result for the semigroup in Section 11.4. In the same section we will also prove a stronger result for the resolvent in case that $p > \frac{n}{2}$.

8.4 Delay Differential Equations

Delay differential equations have already been considered in Chapter 3. Here we show that, under quite general assumptions, asymptotic positivity occurs for the semigroup associated with such equations. Since our theory of asymptotic positivity was developed on complex Banach lattices we always consider the case of complex scalars now.

Let us briefly recall the most important facts about delay differential equations in one dimension: we consider the time evolution of a complex value $y(t) \in \mathbb{C}$ whose change rate is given by

$$\dot{y}(t) = \langle \Phi, y(t + \cdot) \rangle,$$

where $\Phi \in \mathcal{C}([-r, 0]; \mathbb{C})'$ and where $r > 0$ is a fixed number. This equation is closely related to the abstract Cauchy problem $\dot{f} = Af$ on the Banach space $E = \mathcal{C}([-r, 0]; \mathbb{C})$, where A is given by

$$\begin{aligned} D(A) &= \{f \in \mathcal{C}^1([-r, 0]; \mathbb{C}) : f'(0) = \langle \Phi, f \rangle\}, \\ Af &= f'. \end{aligned} \tag{8.2}$$

This operator generates a C_0-semigroup $(e^{tA})_{t\geq 0}$ on E whose orbits describe in some sense the solutions of the delay differential equation considered above, see [EN00, Section VI.6(a)].

If Φ maps constant functions to 0, then $0 \in \sigma(A)$, and in this case we can prove the following theorem about asymptotic positivity of the resolvent $\mathcal{R}(\cdot, A)$ at 0. In order to state the theorem, let us agree on the following terminology: if S_1, S_2 are two subsets of the real numbers, then we say that S_1 is *located on the left* of S_2 if $s_1 \leq s_2$ for all $s_1 \in S_1$ and all $s_2 \in S_2$. Moreover, recall that the *support* of a positive Borel measure μ on a topological space X is defined to be the set

$$\{x \in X : \mu(U) > 0 \text{ for every open neighbourhood } U \text{ of } x\}.$$

It is easy to see that this set is always closed. If K is a compact Hausdorff space and $\varphi \in \mathcal{C}(K;\mathbb{C})'$ is a positive functional, then we define the *support* of φ to be the support of the positive regular Borel measure associated to φ.

Theorem 8.4.1. *Let $E = \mathcal{C}([-r,0];\mathbb{C})$ for some $r > 0$, let $\Phi \in E'$ be real and let A be the operator on E defined in* (8.2)*. Suppose that $\langle \Phi, \mathbb{1}_{[-r,0]} \rangle = 0$ and assume that the support of Φ^+ is located on the left of the support of Φ^-. Then the following assertions hold:*

(a) *A has compact resolvent and 0 is an eigenvalue of A with eigenvector $\mathbb{1} := \mathbb{1}_{[-r,0]}$.*

(b) *The eigenvalue 0 is algebraically simple; in particular, it is a first order pole of $\mathcal{R}(\cdot, A)$.*

(c) *The resolvent $\mathcal{R}(\cdot, A)$ is uniformly asymptotically positive at 0.*

Proof. (a) Since $\langle \Phi, \mathbb{1} \rangle = 0$, the function $\mathbb{1}$ is an eigenvector of A for the eigenvalue 0. Moreover, the embedding $D(A) \hookrightarrow E$ is compact by the Arzelà–Ascoli theorem and, as A has non-empty resolvent set, it follows that A has compact resolvent. In particular, all spectral values of A are poles of the resolvent $\mathcal{R}(\cdot, A)$.

(b) Clearly, the eigenvalue 0 is geometrically simple. Hence, it suffices to show that it is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. For each $\lambda \in \mathbb{C}$, let $e_\lambda \in E$ be given by $e_\lambda(s) := e^{\lambda s}$ for every $s \in [-r, 0]$. Moreover, for each $\lambda \in \mathbb{C}$, define $H_\lambda \in \mathcal{L}(E)$ by $(H_\lambda f)(s) := \int_s^0 e^{\lambda(s-\tau)} f(\tau)\, d\tau$ for all $f \in E$ and all $s \in [-r, 0]$. According to [EN00, Proposition VI.6.7], the spectrum of A is given by $\sigma(A) = \{\lambda \in \mathbb{C} |\ \lambda = \langle \Phi, e_\lambda \rangle\}$ and the resolvent of A is given by

$$\mathcal{R}(\lambda, A) f = \frac{f(0) + \langle \Phi, H_\lambda f \rangle}{\lambda - \langle \Phi, e_\lambda \rangle} e_\lambda + H_\lambda f$$

for each $\lambda \in \rho(A)$ and each $f \in E$. It suffices to show that $\limsup_{\lambda \downarrow 0} \lambda \|\mathcal{R}(\lambda, A)\| < \infty$, so let $\lambda > 0$. Note that we have $\|\Phi^+\| = \|\Phi^-\|$ since $\langle \Phi, \mathbb{1} \rangle = 0$. Using that the support of Φ^+ is located on the left of the support of Φ^- and that e_λ is increasing, we conclude that $\langle \Phi, e_\lambda \rangle \leq 0$. Hence, $(0, \infty) \subseteq \rho(A)$ and $|\frac{1}{\lambda - \langle \Phi, e_\lambda \rangle}| = \frac{1}{\lambda - \langle \Phi, e_\lambda \rangle} \leq \frac{1}{\lambda}$ for each $\lambda > 0$. We thus obtain for each such λ

$$\|\mathcal{R}(\lambda, A)\| \leq \frac{1 + \|\Phi\| \|H_\lambda\|}{\lambda - \langle \Phi, e_\lambda \rangle} \|e_\lambda\|_\infty + \|H_\lambda\| \leq \frac{1 + \|\Phi\|}{\lambda} + 1;$$

here we used that the operator $H_\lambda \in \mathcal{L}(E)$ has norm $\|H_\lambda\| \leq 1$ whenever $\lambda > 0$. This proves that $\limsup_{\lambda \downarrow 0} \lambda \|\mathcal{R}(\lambda, A)\| \leq 1 + \|\Phi\| < \infty$, so 0 is indeed a first order pole of the resolvent $\mathcal{R}(\cdot, A)$.

(c) For every $\lambda > 0$ and every $f \in E_+$ we have $-\langle \Phi, e_\lambda \rangle \geq 0$, $f(0) \geq 0$ and $H_\lambda f \geq 0$. Hence,

$$\lambda \operatorname{dist}(\mathcal{R}(\lambda, A), E_+) \leq \frac{\lambda}{\lambda - \langle \Phi, e_\lambda \rangle} \operatorname{dist}(\langle \Phi, H_\lambda f \rangle e_\lambda, E_+) \leq \langle \Phi, H_\lambda f \rangle^-.$$

It is easy to see that $H_\lambda f$ converges to $H_0 f$ with respect to $\|\cdot\|_\infty$ as $\lambda \downarrow 0$, so $\langle \Phi, H_\lambda f \rangle^- \to \langle \Phi, H_0 f \rangle^-$ as $\lambda \downarrow 0$. However, the function $H_0 f$ is decreasing, so the properties of Φ imply that $\langle \Phi, H_0 f \rangle \geq 0$. Thus, $\langle \Phi, H_\lambda f \rangle^- \to 0$ as $\lambda \downarrow 0$, so we have proved that $\mathcal{R}(\cdot, A)$ is indeed asymptotically positive at 0. □

If we want the semigroup $(e^{tA})_{t \geq 0}$, and not only the resolvent, to be asymptotically positive, we need a criterion which implies that the 0 is the spectral bound of A and, moreover, a dominant spectral value; such a criterion, together with information about asymptotic positivity of the semigroup, is given in the following theorem.

Theorem 8.4.2. *Assume that the assumptions of Theorem 8.4.1 are fulfilled. Then, in addition to the assertions* (a)–(c) *from Theorem 8.4.1, the following properties hold:*

(a) *If $r \leq \frac{\pi}{2\|\Phi\|}$, then the spectral bound* $\mathrm{s}(A)$ *equals* 0 *and is a dominant spectral value of A.*

(b) *If the spectral bound* $\mathrm{s}(A)$ *equals* 0 *and is a dominant spectral value, then the semigroup $(e^{tA})_{t \geq 0}$ is uniformly asymptotically positive.*

(c) *The semigroup $(e^{tA})_{t \geq 0}$ is not positive, unless the support of Φ^- equals* $\{0\}$.

Proof. (a) We know from Theorem 8.4.1 that 0 is a spectral value. If $\Phi = 0$ (in which case we interpret the number $\frac{\pi}{2\|\Phi\|}$ as ∞), a short computation shows that 0 is the only eigenvalue, and thus the only spectral value of A. Hence, we assume for the rest of the proof that $\Phi \neq 0$.

Let $\lambda = \alpha + i\beta \in \mathbb{C}$ be a spectral value of A (where $\alpha, \beta \in \mathbb{R}$) and assume that $\alpha \geq 0$; we have to show that $\lambda = 0$. Since A is real, all spectral values of A occur in complex conjugate pairs, so we may also assume that $\beta \geq 0$.

Since λ is a spectral value, we have $\lambda = \langle \Phi, e^{\lambda \cdot} \rangle$; see e.g. [EN00, Proposition VI.6.7]. Splitting this equation into its real and imaginary part, we obtain

$$\alpha = \langle \Phi, e^{\alpha \cdot} \cos(\beta \cdot) \rangle, \tag{8.3}$$

$$\beta = \langle \Phi, e^{\alpha \cdot} \sin(\beta \cdot) \rangle. \tag{8.4}$$

Since $\alpha \geq 0$, equation (8.4) yields that $\beta \leq \langle |\Phi|, \mathbb{1} \rangle = \|\Phi\|$. It thus follows from the assumption that $r \leq \frac{\pi}{2\beta}$. This shows that the function $\cos(\beta \cdot)$ has no zeros on the interval $(-r, 0]$; in particular, $\cos(\beta x) \geq 0$ for all $x \in [-r, 0]$.

Now, assume for a contradiction that $\lambda \neq 0$. This implies that the function $e^{\alpha \cdot} \cos(\beta \cdot)$, which is ≥ 0 on $[-r, 0]$, is also strictly increasing on this interval. Indeed, if $\beta = 0$, then $\alpha > 0$, so the claim is true. If, on the other hand, $\beta \neq 0$, then $e^{\alpha \cdot}$ is non-decreasing and $\cos(\beta \cdot)$ is strictly increasing on $[-r, 0]$ (since it has not zeros on $(-r, 0]$); hence, in this case the claim holds, too.

Yet, our assumption on Φ now implies that $\langle \Phi, e^{\alpha \cdot} \cos(\beta \cdot) \rangle < 0$, and equations (8.3) thus shows that $\alpha < 0$. This is a contradiction, so $\lambda = 0$.

(b) Assume that $s(A)$ equals 0 and is a dominant spectral value of A. Since 0 is a first order pole of the resolvent according to Theorem 8.4.1(b) and since the semigroup $(e^{tA})_{t \geq 0}$ is eventually compact, it follows that the semigroup is bounded (in fact, it is even convergent with respect to the operator norm, see Proposition 3.1.1).

It follows from Theorems 8.4.1 and 7.2.2 that the spectral projection P associated with the first order pole $s(A) = 0$ is positive. Hence we conclude from Remark 7.3.7 and Theorem 7.3.6 that the semigroup $(e^{tA})_{t \geq 0}$ is uniformly asymptotically positive.

(c) This follows from [AGG$^+$86, Example B-II.1.22]; actually, the result is only proved for the case $r = 1$ in this reference, but it is, of course, also true for arbitrary $r \in (0, \infty)$. □

One might ask whether, under the conditions of Theorem 8.4.1, the spectral bound of A is always equals to 0 and even a dominant spectral value. We will see in the subsequent example that this is not true, in general. On the other hand, we will also see that it can sometimes happen that $s(A)$ is equal to 0 and a dominant spectral value even if the estimate $r \leq \frac{\pi}{2\|\Phi\|}$ from Theorem 8.4.2(a) is not fulfilled.

Example 8.4.3. Let $r > 1$, $c > 0$ and $E = \mathcal{C}([-r, 0]; \mathbb{C})$, let $\Phi \in E'$ be given by $\langle \Phi, f \rangle = cf(-r) - cf(-r+1)$ for all $f \in E$ and let the operator A on E be defined by (8.2). Then all assumptions of Theorem 8.4.1 are fulfilled. Moreover, the following assertions hold:

(a) Let $r = 2$ and $c = 1$. Then the estimate $r \leq \frac{\pi}{2\|\Phi\|}$ is not fulfilled.

Nevertheless, the spectral bound of A equals 0 and is a dominant spectral value.

(b) The numbers $r > 1$ and $c > 0$ can be chosen such that $1 + i \in \sigma(A)$; hence, we have $s(A) > 0$ in this case.

Proof. Obviously, the assumptions of Theorem 8.4.1 are fulfilled.

(a) Let $r = 2$ and $c = 1$. Then we have $\|\Phi\| = 2$ and thus, $\frac{\pi}{2\|\Phi\|} = \frac{\pi}{4} < 1 < r$.

Note that 0 is a spectral value of A according to Theorem 8.4.1(a). To show that $s(A)$ equals 0 and is a dominant spectral value, let $\lambda = \alpha + i\beta \in \mathbb{C}$ be a

spectral value of A with $\alpha \in [0,\infty)$ and $\beta \in \mathbb{R}$; we have to show that $\lambda = 0$. Since A is real, all spectral values of A occur in complex conjugate pairs, so we may assume that $\beta \geq 0$. Since $\lambda \in \sigma(A)$, we have $\lambda = \langle \Phi, e^{\lambda \cdot} \rangle = e^{-2\lambda} - e^{-\lambda}$ (this follows from [EN00, Proposition VI.6.7]). Splitting the equation for λ into real and imaginary part, we obtain

$$\alpha = e^{-2\alpha}\cos(2\beta) - e^{-\alpha}\cos(\beta) \tag{8.5}$$

$$\beta = -e^{-2\alpha}\sin(2\beta) + e^{-\alpha}\sin(\beta). \tag{8.6}$$

We first show that $\beta = 0$, and to do so we use the following observation:

(∗) *Let $\delta > 0$ with $2\delta \in [\pi, \frac{3}{2}\pi]$ and suppose that we know for some reason that $\beta \in [0, \delta]$. Then we also have $\beta \in [0, 1 - \sin(2\delta)]$.*

Indeed, if $2\delta \in [\pi, \frac{3}{2}\pi]$ and $\beta \leq \delta$, then $\sin(2\beta) \geq \sin(2\delta)$ since the sine function is positive on $[0,\pi]$ and negative and decreasing on $[\pi, \frac{3}{2}\pi]$. Hence, (8.6) implies that

$$\beta \leq -e^{-2\alpha}\sin(2\delta) + e^{-\alpha}\sin(\beta) \leq -\sin(2\delta) + 1,$$

where we used that $-\sin(2\delta) \geq 0$ and $e^{-2\alpha} \leq 1$. Thus, (∗) holds.

Now we argue as follows: it is immediate from (8.6) that $\beta \leq 2$. If we set $\delta = 2$, then $2\delta \in [\pi, \frac{3}{2}\pi]$. Thus, we may apply (∗) to obtain $\beta \leq 1 - \sin(4)$. Now define $\delta = 1 - \sin(4)$; again, we have $2\delta \in [\pi, \frac{3}{2}\pi]$, so (∗) now yields $\beta \leq 1 - \sin\Big(2(1 - \sin 4)\Big)$. The latter number is smaller then $\frac{\pi}{2}$, so we have $\sin(2\beta) \geq 0$ and $\sin(\beta) \geq 0$. Hence, equation (8.6) now yields $\beta \leq \sin\beta$. Thus $\beta = 0$, as claimed.

Now we can show that we have $\alpha = 0$, too. Indeed, equation (8.5) yields $\alpha = e^{-2\alpha} - e^{-\alpha} \leq 0$, so $\alpha = 0$. This proves $\lambda = 0$, so the spectral bound $s(A)$ is indeed equal to 0 and a dominant spectral value.

(b) Let $\lambda = 1 + i$ and let $\theta \in [0, 2\pi)$ be the argument of the complex number $1 - e^{\lambda}$. We define $r = \theta + \frac{7}{4}\pi > 1$ and $c = \frac{\sqrt{2}e^r}{|1-e^{\lambda}|} > 0$. Let us show that we have $\lambda \in \sigma(A)$ for these choices of r and c.

According to [EN00, Proposition VI.6.7] we have to prove that the characteristic equation $\lambda = \langle \Phi, e^{\lambda \cdot} \rangle$ is fulfilled. Since

$$\begin{aligned}\langle \Phi, e^{\lambda \cdot} \rangle = ce^{-\lambda r}(1 - e^{\lambda}) = \frac{\sqrt{2}e^r}{|1 - e^{\lambda}|} e^{-(1+i)r}|1 - e^{\lambda}|e^{i\theta} \\ = \sqrt{2}e^{-ir}e^{i\theta} = \sqrt{2}e^{-i\frac{7}{4}\pi} = \lambda,\end{aligned}$$

this is indeed fulfilled. □

8.5 Network Flows with Non-Positive Mass Transition

Transport equations on networks have been intensively studied by means of semigroup methods during the last decade; we refer, for instance, to [KS05], [DKS09] and [DKFNR10] for some contributions to the theory.

In all the models considered in the literature, it seems to be a natural assumption that the mass which flows through the network is distributed with positive weights in the vertices of the network, so the associated semigroup becomes positive. In this section we demonstrate by means of a concrete example that, if we allow some of the weights in the vertices to be negative, asymptotic positivity can occur.

It is certainly justified to ask whether such model as particularly relevant for applications; in fact, already the above use of the word "mass" suggests that positivity of the transition weights, and thus of the semigroup, is a very natural assumption for such a model. Yet, we consider our example with non-positive mass transition as some kind of "toy example" in order to study how asymptotic positivity can occur instead of positivity if one changes certain assumptions in a known model.

Let us consider a directed graph G consisting of two vertices v_{-1} and v_0 and three edges e_1, e_2, e_3, where e_1 points from v_{-1} to v_0, where e_2 is a loop from v_0 to v_0 and where e_3 is another loop from v_0 to v_0. We assume that e_1 and e_2 both have length 1 while the third edge e_3 has a number $L > 0$ as its length. The graph G is shown in Figure 8.1.

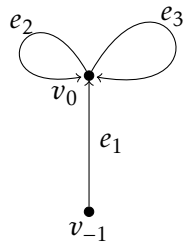

Figure 8.1: The graph G.

The edges e_1 and e_2 are modelled as the real interval $[0,1]$ where 0 is supposed to be the starting point and 1 is supposed to be the end point of the edge. Similarly, the edge e_3 is modelled as the interval $[0,L]$, where again 0 is the starting point and L is the end point. Assume that an initial mass distribution f on the graph G is given; f is modelled as an element $f = (f_1, f_2, f_3)$ of the space $E = L^1([0,1];\mathbb{C}) \oplus L^1([0,1];\mathbb{C}) \oplus L^1([0,L];\mathbb{C})$, where f_k describes the mass distribution on the k-th edge. Here, the space E is endowed with the norm $\|(f_1, f_2, f_3)\| = \|f_1\|_1 + \|f_2\|_1 + \|f_3\|_1$.

We want to model a transport process on the graph G which moves the mass distribution f according to the following laws:

- On each edge, the mass is shifted with velocity 1 from the starting point of the edge to its end point.
- When some mass from e_1 arrives at v_0, then two thirds of this mass are diverted to e_3. The remaining third of the mass is diverted to e_2, but with a flipped sign.
- When some mass from e_2 arrives at v_0, then half of it is diverted to e_3 and half of it re-enters e_2 at its starting point. Similarly, half of the mass arriving at v_0 from e_3 is diverted to e_2, the other mass re-enters e_3 at its starting point.
- No mass enters the system at the starting point of e_1, i.e. at v_{-1}.

Note that the mass diversion in the vertex v_0 is subject to a slightly finer condition than is usually considered in the literature: there, it is assumed in most cases that the mass arriving at a vertex is summed up, and then diverted to the outgoing edges according to some prescribed weight. In our model, we assume that how the mass is diverted depends on the edge it arrives from.

The transport process specified above can be described by the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t) \\ u(0) = f \end{cases}$$

on the space E, where the operator A is given by

$$\begin{aligned} D(A) = \{(f_1, f_2, f_3) \in W^{1,1}((0,1);\mathbb{C}) \oplus W^{1,1}((0,1);\mathbb{C}) \oplus W^{1,1}(0,L);\mathbb{C}) : \\ f_2(0) = \frac{1}{2} f_2(1) + \frac{1}{2} f_3(L) - \frac{1}{3} f_1(1), \\ f_3(0) = \frac{1}{2} f_2(1) + \frac{1}{2} f_3(L) + \frac{2}{3} f_1(1), \quad f_1(0) = 0\} \\ A(f_1, f_2, f_3) = -(f_1', f_2', f_3'). \end{aligned} \tag{8.7}$$

Now we prove that the above Cauchy problem is well-posed; moreover, if the length L of e_3 is irrational, then the semigroup generated by A is individually asymptotically positive:

Theorem 8.5.1. *Let $L > 0$, let $E = L^1([0,1];\mathbb{C}) \oplus L^1([0,1];\mathbb{C}) \oplus L^1([0,L];\mathbb{C})$ and let $A : E \supseteq D(A) \to E$ be given by* (8.7)*. Then A is a closed and densely defined operator, and the following assertions hold:*

(a) *A complex number $\lambda \in \mathbb{C}$ is a spectral value of A if and only if it is an eigenvalue of A if and only if* 0 *is an eigenvalue of the matrix*

$$S(\lambda) := \begin{pmatrix} e^{-\lambda} - 2 & e^{-\lambda L} \\ e^{-\lambda} & e^{-\lambda L} - 2 \end{pmatrix}.$$

Moreover, A has compact resolvent and for every eigenvalue of A the corresponding eigenspace is one-dimensional.

(b) *The operator A is dissipative and generates a contractive C_0-semigroup on E.*

(c) *The kernels of A and A' are given by* $\ker A = \operatorname{span}\{(0, \mathbb{1}_{[0,1]}, \mathbb{1}_{[0,L]})\}$ *and* $\ker A' = \operatorname{span}\{(\frac{1}{3}\mathbb{1}_{[0,1]}, \mathbb{1}_{[0,1]}, \mathbb{1}_{[0,L]})\}$.

(d) *The spectral bound* $\mathrm{s}(A)$ *equals* 0*; if L is irrational, then* $\mathrm{s}(A)$ *is, moreover, a dominant spectral value of A.*

(e) *The semigroup $(e^{tA})_{t\geq 0}$ is not positive. However, if L is irrational, then the semigroup is individually asymptotically positive.*

Proof. Clearly, A is closed and densely defined.

(a) A simple computation shows that the matrix $S(\lambda)$ is singular if and only if $\ker(\lambda - A)$ is non-zero, and that in this case the eigenspace $\ker(\lambda - A)$ is one-dimensional. Another straightforward, though somewhat tedious, computation proves that $\lambda - A$ is surjective if the matrix $S(\lambda)$ is invertible. One readily checks that the function $\lambda \mapsto \det S(\lambda)$ is not constantly 0. Hence, the resolvent set $\rho(A)$ is non-empty, and since $D(A)$ embeds compactly into E, it follows that A has compact resolvent.

(b) Since A has compact resolvent, its spectrum is countable, so we have $\lambda \in \rho(A)$ for some $\lambda > 0$. Note that A is a real operator. Using the boundary conditions in the definition of $D(A)$ one can readily check that the restriction of A to the real part $E_{\mathbb{R}}$ of E is dissipative. Hence, $A|_{E_{\mathbb{R}}}$ generates a contractive C_0-semigroup on $E_{\mathbb{R}}$, and thus A generates a bounded C_0-semigroup on E. Moreover, as E is an L^1-space, it follows e.g. from [Fen98, Proposition 2.1.1] that even the complex C_0-semigroup $(e^{tA})_{t\geq 0}$ is contractive; in particular, A is dissipative.

(c) The first assertion is obvious and the fact that $(\frac{1}{3}\mathbb{1}_{[0,1]}, \mathbb{1}_{[0,1]}, \mathbb{1}_{[0,L]}) \in \ker A'$ follows by a short computation from the definition of the adjoint A' and from the boundary conditions satisfied by functions on $D(A)$. Since A has compact resolvent, 0 is a pole of $\mathcal{R}(\cdot, A)$ and the associated spectral projection has finite rank. Thus, it follows from Proposition A.3.3(d) that $\dim\ker A = \dim\ker A'$. This proves the assertion.

(d) Since A generates a contractive C_0-semigroup we clearly have $\mathrm{s}(A) \leq 0$ and it follows from (c) that actually $\mathrm{s}(A) = 0$.

Now, assume that L is irrational. Let us show that there is no other spectral value except 0 on the imaginary axis. Indeed, if we assume that $i\beta \in \sigma(A)$ for some $\beta \in \mathbb{R} \setminus \{0\}$, then

$$0 = \det S(i\beta) = -2(e^{-i\beta} + e^{-i\beta L}) + 4$$

according to (a). Taking the real part of this equation we obtain $\cos(\beta) + \cos(\beta L) = 2$, so $\cos(\beta) = \cos(\beta L) = 1$. We conclude that β and βL are both contained in $\pi(\mathbb{Z} + \frac{1}{2})$. This is, however, a contradiction since L was assumed to be irrational.

(e) The semigroup $(e^{tA})_{t\geq 0}$ is contractive; hence, in order to show that it is not positive, it suffices to show that the generator A is not dispersive; cf. [AGG+86, Theorem C-II.1.2]. To this end, consider a vector $f = (f_1, f_2, f_3) \in D(A)$ with the properties $f_1, f_3 \leq 0$, $f_2 \geq 0$ and

$$\begin{aligned} f_1(0) = 0, \quad f_2(0) = \frac{1}{3}, \quad f_3(0) = -\frac{2}{3}, \\ \text{and} \quad f_1(1) = -1, \quad f_2(1) = 0, \quad f_3(L) = 0; \end{aligned}$$

clearly, such a vector $0 \leq f \in D(A)$ exists. Let $g \in E'$ be given by $g = \operatorname{sgn} f^+$. If A was dispersive, we would have $\langle g, Af\rangle \leq 0$. However, we obtain $\langle g, Af\rangle = f_2(0) = \frac{1}{3} > 0$. Hence, A is not dispersive, so the semigroup $(e^{tA})_{t\geq 0}$ is not positive.

Now, assume again that L is irrational. The spectral bound $\mathrm{s}(A) = 0$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$ since the resolvent of A is compact. The description of $\ker A$ and $\ker A'$ in (c) allows us to employ Proposition 7.1.3; hence, 0 is a first order pole of $\mathcal{R}(\cdot, A)$ and the spectral projection P corresponding to 0 is positive. Since 0 is also a dominant spectral value, we conclude from Theorem 7.3.3 that the semigroup $(e^{tA})_{t\geq 0}$ is individually asymptotically positive. □

8.6 A Diffusion Equation with Non-Autonomous Boundary Conditions

Suppose that a function $f_b : [0, \infty) \to \mathbb{C}$ is given which is supported on $[0, r]$ for some $r > 0$. We want to consider the diffusion equation

$$\begin{cases} \dot{f}(t,x) & = \Delta f(t,x) \qquad \text{for } x \in (0,1) \\ -f'(t,0) & = \beta_1 f_b(t) \\ f'(t,1) & = -\beta_2 f_b(t) \end{cases} \tag{8.8}$$

on $L^1((0,1);\mathbb{C})$, where $\beta_1, \beta_2 \geq 0$ are constants with $\beta_1 + \beta_2 \leq 1$ and where, of course, an initial function $f(0,\cdot) \in L^1((0,1);\mathbb{C})$ is given. Hence, the outer

normal derivatives of the solution are prescribed to be multiples of f_b, where the factor $-\beta_2$ at the boundary point 1 is negative!

Since the boundary conditions are non-homogeneous, the problem (8.8) can, at first glance, not be analysed by semigroup methods. However, we can use a similar approach as in [EN00, Section VI.7] to rewrite the above problem in a semigroup setting nevertheless. To this end, we let the left shift semigroup $(e^{t\frac{d}{ds}})_{t\geq 0}$ on $L^1((0,r);\mathbb{C})$ act on the given boundary function f_b and we transmit the value $(e^{t\frac{d}{ds}}f_b)(0) = f_b(t)$ in an appropriate way to the boundary values of the wanted solution of (8.8) on $L^1((0,1);\mathbb{C})$.

More precisely, we reformulate the diffusion equation (8.7) as the abstract Cauchy problem

$$\frac{d}{dt}(f, f_b) = A(f, f_b)$$

on the space $E := L^1((0,1);\mathbb{C}) \times L^1((0,r);\mathbb{C})$ where the operator A is given by

$$\begin{aligned} D(A) = \{(f, f_b) \in W^{2,1}((0,1);\mathbb{C}) \oplus W^{1,1}((0,r);\mathbb{C}) : \\ -f'(0) = \beta_1 f_b(0), \quad f'(1) = -\beta_2 f_b(0), \quad f_b(r) = 0\} \qquad (8.9) \\ A(f, f_b) = (f'', f_b'). \end{aligned}$$

Let us prove that the abstract Cauchy problem associated to the operator A is well-posed and that, under an appropriate condition on β_1 and β_2, the C_0-semigroup $(e^{tA})_{t\geq 0}$ is asymptotically positive. To the best of our knowledge, asymptotic positivity properties of such a semigroup have not been studied before; we mention, however, the loosely related paper [Chu15] where it is studied whether a positive forcing term results in eventually positive behaviour of a harmonic oscillator.

Theorem 8.6.1. *Let $E = L^1((0,1);\mathbb{C}) \oplus L^1((0,r);\mathbb{C})$ be endowed with the norm given by $\|(f, f_b)\| = \|f\|_1 + \|f_b\|_1$, let $\beta_1, \beta_2 \geq 0$ such that $\beta_1 + \beta_2 \leq 1$ and let $A : E \supseteq D(A) \to E$ be given by* (8.9)*. Then the following assertions hold:*

(a) *The spectrum of A coincides with the spectrum of the Neumann–Laplace operator Δ_N on $L^1((0,1);\mathbb{C})$, i.e. $\sigma(A) = \{-k^2\pi^2 | \, k \in \mathbb{N}_0\}$.*

(b) *The operator A is dissipative and generates a contractive C_0-semigroup on E. Moreover, A has compact resolvent.*

(c) *The kernel of A is spanned by $(\mathbb{1}_{(0,1)}, 0)$ and the kernel of the adjoint A' is spanned by the functional $(\mathbb{1}_{(0,1)}, (\beta_1 - \beta_2)\mathbb{1}_{(0,r)}) \in L^\infty((0,1);\mathbb{C}) \otimes L^\infty((0,r);\mathbb{C})$.*

(d) *The semigroup $(e^{tA})_{t\geq 0}$ is individually asymptotically positive if and only if $\beta_1 \geq \beta_2$.*

(e) *The semigroup $(e^{tA})_{t\geq 0}$ is positive if and only if $\beta_2 = 0$.*

According to assertions (d) and (e) in the above theorem, the semigroup $(e^{tA})_{t\geq 0}$ is individually asymptotically positive but not positive if we have $0 < \beta_2 \leq \beta_1$.

Proof of Theorem 8.6.1. (a) Let $\Delta_N : L^1((0,1);\mathbb{C}) \supseteq D(\Delta_N) \to L^1((0,1);\mathbb{C})$ be the Neumann-Laplace operator, i.e.

$$D(\Delta_N) = \{u \in W^{2,1}((0,1);\mathbb{C}) |\, f'(0) = f'(1) = 0\}, \quad \Delta_N u = u''.$$

It is well-known that $\sigma(\Delta_N) = \{-k^2\pi^2 |\, k \in \mathbb{N}_0\}$. Let $\lambda \in \mathbb{C}$. If $\lambda - \Delta_N$ is not bijective, then $\lambda - \Delta_N$ is in fact not even injective, so we have $(\lambda - \Delta_N)f = 0$ for some $0 \neq f \in D(\Delta_N)$. It now immediately follows that $(f, 0) \in D(A)$ is an eigenvector of A for the eigenvalue λ. We have thus shown that $\sigma(\Delta_N) \subseteq \sigma(A)$. On the other hand, assume now that $\lambda - \Delta_N$ is bijective. If λ was an eigenvalue of A with eigenvector (f, f_b), then we could immediately verify that $f_b = 0$, and thus, $f \neq 0$ would be contained in $D(\Delta_N)$ and we would therefore obtain $(\lambda - \Delta_N)f = 0$; this cannot be true. Hence, $\lambda - A$ is injective. Moreover, using that $\lambda - \Delta_N$ is bijective, it is easy to see that the boundary value problem

$$\begin{cases} (\lambda - \Delta)u &= f \\ \frac{\partial}{\partial \nu} u = g \end{cases}$$

has a solution $u \in W^{2,1}((0,1);\mathbb{C})$ for each $f \in L^1((0,1);\mathbb{C})$ and each $g \in \mathbb{C}^2$. From this fact we can immediately derive that $\lambda - A$ is surjective. Thus, $\rho(\Delta_N) \subseteq \rho(A)$.

(b) The operator A is clearly real. We first show that the restriction of A to the real part $E_{\mathbb{R}}$ of E is dissipative. So let $(f, f_b) \in D(A)$ be a real-valued function and let $(g, g_b) \in E'_{\mathbb{R}} = L^\infty((0,1);\mathbb{R}) \oplus L^\infty((0,r);\mathbb{R})$ be given by $(g, g_b) = (\operatorname{sgn} f, \operatorname{sgn} f_b)$. We have to show that $\langle g, Af\rangle \leq 0$. One easily computes that

$$\begin{aligned} \langle (g, g_b), A(f, f_b)\rangle &\leq |f'(1)| + |f'(0)| + |f_b(r)| - |f_b(0)| \\ &= \beta_2 |f_b(0)| + \beta_1 |f_b(0)| - |f_b(0)| \leq 0, \end{aligned}$$

as $\beta_1 + \beta_2 \leq 1$. Hence, $A|_{E_{\mathbb{R}}}$ is indeed dissipative. It now follows from the description of the spectrum $\sigma(A)$ in (a) that $A|_{E_{\mathbb{R}}}$ generates a contractive C_0-semigroup on $E_{\mathbb{R}}$ and thus, A generates a bounded C_0-semigroup $(e^{tA})_{t\geq 0}$ on E. Moreover, since E is an L^1-space, it follows from [Fen98, Proposition 2.1.1] that the semigroup $(e^{tA})_{t\geq 0}$ is even contractive; thus, A is dissipative.

Since the resolvent set $\rho(A)$ is non-empty and the embedding $D(A) \hookrightarrow E$ is compact, we also conclude that A has compact resolvent.

(c) Obviously, the kernel of A is spanned by $(\mathbb{1}_{(0,1)}, 0)$. Moreover, it follows easily from the definition of the adjoint A' and from the boundary conditions

satisfied by functions in $D(A)$ that $(\mathbb{1}_{(0,1)}, (\beta_1 - \beta_2)\mathbb{1}_{(0,r)})$ is an eigenvector of A' for the eigenvalue 0. Since A has compact resolvent, the number 0 is a pole of $\mathcal{R}(\cdot, A)$ and the corresponding spectral projection has finite rank. Hence, we have $\dim \ker A = \dim \ker A'$ according to Proposition A.3.3(d). This proves the assertion.

(d) The spectral bound $s(A) = 0$ is a dominant spectral value and, as A has compact resolvent, a pole of $\mathcal{R}(\cdot, A)$. In fact, it is even a first order pole since the semigroup $(e^{tA})_{t\geq 0}$ is bounded.

Now assume that $\beta_1 \geq \beta_2$. Then $\ker A$ and $\ker A'$ both contain positive, non-zero vectors according to (c); since $\ker A$ is one-dimensional we can conclude from Corollary 7.1.2 that the spectral projection P associated with 0 is positive. Hence, the individual asymptotic positivity of $(e^{tA})_{t\geq 0}$ follows from the characterisation in Theorem 7.3.3.

On the other hand, assume now that the semigroup $(e^{tA})_{t\geq 0}$ is individually asymptotically positive. Then it follows again from Theorem 7.3.3 that the spectral projection P associated with $s(A) = 0$ is positive. The spectral value 0 is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$ since the resolvent is compact and since the semigroup generated by A is bounded; hence, 0 is also a first order pole of $\mathcal{R}(\cdot, A')$. The corresponding spectral projection is given by P'. Since 0 is a geometrically simple eigenvalue of A', it is also an algebraically simple eigenvalue of A', so the range of P' is one-dimensional. As P' is positive, its range contains a non-zero, positive vector and thus, all real elements of $P'E'$ are either positive or negative. As $(\mathbb{1}_{(0,1)}, (\beta_1 - \beta_2)\mathbb{1}_{(0,r)}) \in \ker A' = P'E'$, we conclude that $\beta_1 \geq \beta_2$.

(e) If $\beta_2 = 0$ one can readily show that A is dispersive and hence $(e^{tA})_{t\geq 0}$ is positive (cf. [AGG+86, Theorem C-II.1.2]). Assume now that $(e^{tA})_{t\geq 0}$ is positive. Since the semigroup is also contractive, it follows that A is dispersive. Choose a vector $(f, f_b) \in D(A)$ such that $f_b \leq 0$, $f_b(0) = -1$ as well as $f(x) < 0$ for all $x \in [0, \frac{1}{2})$, $f(x) > 0$ for all $x \in (\frac{1}{2}, 1]$ and $f'(\frac{1}{2}) = f(\frac{1}{2}) = 0$. It is easy to see that such a vector exists in $D(A)$. Let $(g, g_b) = (\operatorname{sgn} f^+, \operatorname{sgn} f_b^+) = (\mathbb{1}_{(\frac{1}{2},1)}, 0) \in E'$. Since A is dispersive we obtain

$$0 \geq \langle (g, g_b), A(f, f_b) \rangle = f'(1) - f'(\frac{1}{2}) = f'(1) = \beta_2 \geq 0.$$

Hence, $\beta_2 = 0$. □

8.7 Notes & Open Problems

The finite-dimensional case Some remarks on the history of the finite-dimensional theory can be found in the notes at the end of Chapter 6. Here we restrict ourselves to noting that the concepts of *eventual positivity* and *eventual strong positivity* seem to be somewhat more prevalent in the literature than

the notion of *asymptotic positivity*. For example, we could not find a result close to Theorem 8.1.5 in the literature. Hence, our theory seems to add some new aspects even to the finite-dimensional case.

The bi-Laplace operator One reason why the bi-Laplace operator with Dirichlet boundary conditions is, for example in two dimensions, of some importance, is the fact that it appears in a mathematical model for clamped plates. The theory of eventually and asymptotically positive resolvents for this operator has quite a long history (although the notion "eventual positivity" was used only seldom in the literature). In fact, it was conjectured by Boggio and Hadamard at the beginning of the 20th century that on every convex domain and for every $f \geq 0$ the so-called *clamped plate equation*

$$-Au = f$$

(where A denotes the bi-Laplace operator with Dirichlet boundary conditions) has a positive solution u; see e.g. the introduction of [GS96] for some details. We point out that, if the equation is realised on an appropriate function space, this means that the resolvent $\mathcal{R}(\cdot, A)$ is positive in 0 and from this one can conclude eventual positivity properties of the resolvent at $s(A) < 0$ (compare Proposition 10.1.1 below). The above conjecture, as natural as it might appear, has later been found out to be wrong on many important domains; we refer again to the introduction of [GS96] for several references. On the other hand, of the underlying domain is a ball, then the conjecture is indeed true; this follows from an explicit solution formula for the clamped plate equation on balls which is due to Boggio [Bog05]. Those observations have inspired a lot of research concerned with the question on which domain the resolvent of the bi-Laplace operator exhibits some kind of eventual positivity property as well as with the related question whether the first eigenfunction can be chosen to be positive; see e.g. the references quoted in Section 8.3 as well as [Swe01], [DS04], [DS05], [Dal05] and [LW15]. Our results in Section 8.2 can be seen as a contribution to this theory. Note, however, that concrete properties of the underlying domain are related to our results only by means of Theorem 8.3.2 which was proved by Grunau and Sweers in [GS98]. Hence, we only applied our abstract theory; the hard analysis related to the concrete equation is due to Grunau and Sweers. Moreover, it is worthwhile pointing out again that we focused on *asymptotic* positivity in Section 8.2 while most of the results quoted above deal with *eventual* positivity. We treat eventual positivity properties of the bi-Laplace operator in Section 11.4 below.

Let us briefly quote some results about the bi-Laplace operator which are related to the above discussion: in [GS97] the reader can find some results about positivity properties of biharmonic equations with inhomogeneous

(Dirichlet) boundary conditions and in [RG12] conditions on f are given to ensure positivity of the solution to $\Delta^2 u = f$ on more general domains. Moreover, it is worthwhile pointing out that there are several results in the literature which show that, on rather general domains, the negative part of the corresponding Green function is, in a sense, small compared to the positive part; see for example [DMS05], [GR09], [GR10] and [GRS11]. We also note that similar questions as for the bi-Laplacian can be asked for higher order elliptic operators and indeed, many results in the papers quoted above also deal with such higher order problems. For some negative results concerning this case we refer, for instance, to the recent paper [Swe16] and the references therein.

Delay differential equations We note that in Example 8.4.3(a) it is actually possible to explicitly compute an eigenvector of the adjoint A' for the eigenvalue 0, see [DGK16a, Section 9]. Using this one can readily determine the spectral projection P of A associated to 0.

Diffusion equations with non-autonomous boundary conditions We note that the model discussed in Section 8.6, as simple as it is, suggests some possible generalisations. First of all we could, of course, consider a diffusion process in higher dimensions; moreover, one would probably be interested in a vector-valued boundary function f_b such that the inhomogeneous boundary term does not only depend on one (time-dependent) parameter $f_b(t)$. Yet, for such vector-valued functions, it does not seem to be clear which conditions can still ensure asymptotic positivity of the semigroup.

Beside this, it would certainly be worthwhile allowing for boundary functions f_b which do not have compact support, meaning that the inhomogeneous boundary term does not disappear after a finite period of time. This, however, causes considerable problems in the analysis presented above. In fact, if we replace the "boundary space" $L^1((0,r);\mathbb{C})$ with, say, the space $L^1((0,\infty);\mathbb{C})$, then we can no longer expect the peripheral spectrum of our semigroup generator A to be finite or to consist of poles of the resolvent. Hence, we can no longer apply our characterisation results in this situation. This indicates that it is worthwhile seeking for generalisations of our theorems which do not require such strong assumptions on the peripheral spectrum of A; compare Problem 7.4.3 above.

9

Individual Eventual Strong Positivity – Criteria and Characterisation

Let us now slightly change our topic and consider a form of eventual positivity which is somewhat stronger than the asymptotic positivity properties from the previous chapters. Again, we are aiming mainly for characterisations, but it turns out that the notion of eventual positivity introduced in Definition 6.1.1 is not particularly well-suited for this task; compare Example 9.2.1 below. Instead, we will consider a slightly stronger notion which we call *eventual strong positivity*; see Definitions 9.2.2 and 9.3.1. We will see that it is possible to find good characterisations for this property.

The present chapter is organised analogously to Chapter 7. First, we consider spectral projections which are now, in an appropriate sense, strongly positive instead of only positive. We characterise this property by the existence of certain positive eigenvectors and eigenfunctionals. Then, in Section 9.2, we deal with eventual strong positivity of resolvents and characterise this property by means of a certain spectral projections. We shall see that there is also an interesting related property of resolvents which we call *eventual strong negativity* and which is closely related to what is known as *anti-maximum principles* in the literature. In the final Section 9.3 we characterise eventual strong positivity of C_0-semigroups, again by means of the spectral projection associated to the spectral bound of the generator A.

Hence, we have a similar situation as in Chapter 7: the existence of positive eigenvectors and eigenfunctionals for $\mathrm{s}(A)$, the eventually strongly positive behaviour of the resolvent close to $\mathrm{s}(A)$ and the eventually strongly positive behaviour of the semigroup are all related via the spectral projection P associated to $\mathrm{s}(A)$.

We should, however, point out that all the properties considered in this section are in fact *individual* eventual positivity properties; this means that, for instance, the time t_0 from which on the semigroup trajectory $(e^{tA}f)_{t\geq 0}$ for a positive initial value f becomes positive depends, in general, on f. The counterpart of this property, where t_0 can be chosen independently of f, is called *uniform eventual strong positivity* and is studied in Chapter 10.

Most of the results presented Sections 9.1 and 9.3, as well as Example 9.2.1 and Theorem 9.2.6 have been published in [DGK16a]; moreover, many of those results have predecessors on spaces of continuous functions which were published in [DGK16b]. On the other hand, Theorem 9.2.5, Proposition 9.2.9,

Corollary 9.3.6 and the auxiliary results to prove them have not been published, nor submitted for publication, yet. Theorem 9.2.5 is part of an ongoing research project with Daniel Daners; the other results just mentioned are based on joint work with Daniel Daners and James B. Kennedy.

9.1 Strongly Positive Spectral Projections

In this section we are, again, interested in positive projections and, more specifically, positive spectral projections. Yet, we consider a somewhat stronger form of positivity here. Let E be a complex Banach lattice and let $u \in E_+$. Recall from the preliminaries that a vector $f \in E$ is called *strongly positive with respect to* u if $f \in E_+$ and if there exists a number $\varepsilon > 0$ such that $f \geq \varepsilon u$; in this case we write $f \gg_u 0$. Also recall that an operator $T \in \mathcal{L}(E)$ is called *strongly positive with respect to* u if $Tf \gg_u 0$ for all $0 < f \in E$. Of course, an operator T which is strongly positive with respect to u is also positive.

In what follows we are only interested in the case where u is a quasi-interior point of E_+. Let us now characterise when a given spectral projection P, associated with a pole of the resolvent of an operator A, is strongly positive with respect to a fixed quasi-interior point u of E_+.

Proposition 9.1.1. *Let E be a complex Banach lattice and let $A : E \supseteq D(A) \to E$ be a real and densely defined linear operator. Let $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ be a pole of the resolvent $\mathcal{R}(\cdot, A)$ (and thus an eigenvalue of A) and let P be the spectral projection associated to λ_0. For every quasi-interior point u of E_+ the following assertions are equivalent:*

(i) $P \gg_u 0$.

(ii) *The eigenvalue λ_0 of A is geometrically simple,* $\ker(\lambda_0 - A)$ *contains a vector $v \gg_u 0$ and* $\ker(\lambda_0 - A')$ *contains a strictly positive functional.*

(iii) *The eigenvalue λ_0 of A is algebraically simple,* $\ker(\lambda_0 - A)$ *contains a vector $v \gg_u 0$ and the range of $\lambda_0 - A$ intersects the positive cone E_+ only in* 0.

If the equivalent assertions (i)–(iii) *are fulfilled, then λ_0 is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$ and λ_0 is the only eigenvalue of A which has a positive eigenvector.*

Proof. "(i) $\Rightarrow$ (ii)" If (i) is true, then the range PE is one-dimensional. Indeed, as P is positive, its range PE is again a Banach lattice with respect to some equivalent norm, see Proposition C.5.5. Since u is a quasi-interior point of E_+, one readily checks that Pu is a quasi-interior point of the positive cone $(PE)_+ := E_+ \cap PE = PE_+$. Moreover, we actually have $Pf \gg_{Pu} 0$ for every $0 < f \in E$. Hence, every $0 < f \in PE$ fulfils $f \gg_{Pu} 0$ and is thus a quasi-interior

point of $(PE)_+$. Yet, a Banach lattice whose positive cone contains only quasi-interior points except 0 has to be one-dimensional, see [Lot68, Lemma 5.1].

As the range PE is one-dimensional, it follows that λ_0 is in fact an algebraically simple eigenvalue of A; in particular, it is geometrically simple. We have $\ker(\lambda_0 - A) = PE$ and thus, $\ker(\lambda_0 - A)$ contains a vector $v \gg_u 0$ (namely $v := Pf$ for any $0 < f \in E$). On the other hand, let $0 < \varphi \in E'$. For every $0 < f \in E$ the vector Pf is a quasi-interior point of E_+ and thus we have $\langle P'\varphi', f\rangle = \langle \varphi, Pf\rangle > 0$. Hence, $P'\varphi'$ is a strictly positive functional. According to Proposition A.3.3(d), λ_0 is also an algebraically simple eigenvalue of the adjoint A'. Moreover, λ_0 is clearly an isolated spectral value of A' and P' is the corresponding spectral projection. Hence, we conclude that $P'E' = \ker(\lambda_0 - A')$, which proves assertion (ii).

"(ii) $\Rightarrow$ (iii)" If (ii) is fulfilled, then it follows from Proposition 7.1.3 that λ_0 is in fact an algebraically simple eigenvalue of A. Let $\varphi \in \ker(\lambda_0 - A)$ be strictly positive. If $g \in E_+ \cap (\lambda_0 - A)D(A)$, then $g = (\lambda_0 - A)f$ for some $f \in D(A)$ and hence we have $\langle \varphi, g\rangle = \langle (\lambda_0 - A')\varphi, f\rangle = 0$. Thus, $g = 0$.

"(iii) $\Rightarrow$ (i)" If (iii) holds, then the range PE of P is one-dimensional and spanned by a vector $v \gg_u 0$. Let $0 < f \in E$; we then have $Pf = \alpha v$ for some $\alpha \in \mathbb{C}$. Since λ_0 and A are real, so is P; hence, $\alpha \in \mathbb{R}$. Assume for a contradiction that $\alpha \leq 0$. Then we have $0 < f - \alpha v = f - Pf \in \ker P$. Yet, as λ_0 is an algebraically simple eigenvalue and therefore a first order pole of the resolvent, we conclude that $\ker P$ coincides with the range of $\lambda_0 - A$ according to Proposition A.3.2(c). Thus, $f - \alpha v$ is a non-zero positive vector in the range of $\lambda_0 - A$, which contradicts (iii). Hence $\alpha > 0$, which implies that $Pf \gg_u 0$.

Now assume that the equivalent assertions (i)–(iii) are fulfilled. Since λ_0 is an algebraically simple eigenvalue according to assertion (iii) of A, it is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. Assume that $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$ is another eigenvalue of A which admits an eigenvector $w > 0$. Then we have $0 < w = \frac{1}{\lambda_0 - \lambda}(\lambda_0 - A)w$, and the latter vector is contained in the range of $\lambda_0 - A$; this contradicts assertion (iii). □

9.2 The Resolvent

It is our major objective in this chapter to study eventually positive resolvents and semigroups. However, it turns out that the notion of an eventually positive semigroup from Definition 6.1.1 is not particularly well-suited for a characterisation. Let $A : E \supseteq D(A) \to E$ be a linear operator on a complex Banach lattice E and let $\lambda_0 \in \sigma(A) \cap \mathbb{R} \cup \{-\infty\}$. We say that the resolvent $\mathcal{R}(\cdot, A)$ is *individually eventually positive* if there exists $\lambda_2 > \lambda_0$ with the following property: we have $(\lambda_0, \lambda_2] \subseteq \rho(A)$ and for all $0 \leq f \in E$ there exists $\lambda_1 \in (\lambda_0, \lambda_2]$ such that $\mathcal{R}(\lambda, A)f \geq 0$ for all $\lambda \in (\lambda_0, \lambda_1]$. Similarly, we say that the resolvent is

uniformly eventually positive if there exists $\lambda_1 > \lambda_0$ with the following property: we have $(\lambda_0, \lambda_1] \subseteq \rho(A)$ and $\mathcal{R}(\lambda, A) \geq 0$ for all $\lambda \in (\lambda_0, \lambda_1]$.

The next example demonstrates that individual eventual positivity of a semigroup $(e^{tA})_{t\geq 0}$ cannot, in general, be characterised be individual eventual positivity of the resolent $\mathcal{R}(\cdot, A)$ at $\mathrm{s}(A)$ (even of $\mathrm{s}(A)$ is a dominant spectral value of A and the underlying space is finite-dimensional):

Example 9.2.1. There exists a finite-dimensional complex Banach lattice E and a real operator $A \in \mathcal{L}(E)$ with the following properties:

(a) The spectral bound $\mathrm{s}(A)$ equals 0 and is a dominant spectral value of A.

(b) The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually positive.

(c) The resolvent $\mathcal{R}(\cdot, A)$ is not individually eventually positive at $\mathrm{s}(A)$.

To construct such an example, let E_0 be a finite-dimensional complex Banach lattice of dimension $\dim E_0 \geq 3$. Then it is easy to see that there exists an eventually positive C_0-semigroup $(e^{tA_1})_{t\geq 0}$ on E_1 which is not positive, which is bounded and which fulfils $\mathrm{s}(A_1) = 0$; see Example 6.1.2. Theorems 6.2.5 and 7.3.3 imply that $\mathrm{s}(A_1) = 0$ is a dominant spectral value of A_1.

Now, consider the set $U := \{\lambda \in (0, \infty) : \mathcal{R}(\lambda, A_1) \ngeq 0\}$. Then U is open in $(0, \infty)$ and, moreover, U is non-empty since the semigroup $(e^{tA_1})_{t\geq 0}$ is not positive. Choose $\lambda_1 \in U$ and define an operator A on $E := E_1 \times \mathbb{C}$ by $A = (A_1 - \lambda_1) \oplus 0_{\mathbb{C}}$.

Obviously, $\mathrm{s}(A)$ equals 0 and is a dominant spectral value of A. Moreover, the semigroup $(e^{tA})_{t\geq 0} = (e^{-t\lambda_1} e^{tA_1} \oplus I_{\mathbb{C}})_{t\geq 0}$ is individually eventually positive. Yet, we have $\mathcal{R}(\lambda, A) = \mathcal{R}(\lambda_1 + \lambda, A_1) \oplus \frac{1}{\lambda} I_{\mathbb{C}}$ for each $\lambda \in (0, \infty)$, and this operator is, of course, not positive for λ close to 0 since U is open.

Another example where the semigroup is individually eventually positive, while the resolvent is not, can be found in [DGK16b, Example 8.2]; however, the Banach lattice in that example is infinite-dimensional and we have $\mathrm{s}(A) = -\infty$.

Since we have now seen that eventually positive semigroups and resolvents are presumably not easy to characterise, we focus on a slightly stronger notion of eventual positivity, namely eventual positivity with respect to a given quasi-interior point u of E_+. The precise definitions are, for resolvents, as follows:

Definition 9.2.2 (Eventually strongly positive resolvents). Let E be a complex Banach lattice, let $A : E \supseteq D(A) \to E$ be a linear operator and let $\lambda_0 \in [-\infty, \infty)$ be a spectral value of A or $-\infty$. Let u be a quasi-interior point of E_+.

(i) The resolvent $\mathcal{R}(\cdot, A)$ is called *individually eventually strongly positive with respect to u at λ_0* if there exists $\lambda_2 > \lambda_0$ with the following properties: $(\lambda_0, \lambda_2] \subseteq \rho(A)$ and for every $0 < f \in E$ there is a number $\lambda_1 \in (\lambda_0, \lambda_2]$ such that $\mathcal{R}(\lambda, A)f \gg_u 0$ for all $\lambda \in (\lambda_0, \lambda_1]$.

(ii) The resolvent $\mathcal{R}(\cdot, A)$ is called *uniformly eventually strongly positive with respect to u at λ_0* if there exists $\lambda_1 > \lambda_0$ with the following properties: $(\lambda_0, \lambda_1] \subseteq \rho(A)$ and $\mathcal{R}(\lambda, A) \gg_u 0$ for all $\lambda \in (\lambda_0, \lambda_1]$.

Instead of considering the behaviour of the resolvent in a right neighbourhood of a given spectral value λ_0, we can also study its behaviour in a left neighbourhood. If the resolvent is positive in a right neighbourhood of λ_0, it seems natural to presume that it might behave, in a sense, negatively on the left of λ_0. This motivates the following definition:

Definition 9.2.3 (Eventually strongly negative resolvents). Let E be a complex Banach lattice, let $A : E \supseteq: D(A) \to E$ be a linear operator and let $\lambda_0 \in (-\infty, \infty]$ be a spectral value of A or ∞. Let u be a quasi-interior point of E_+.

(i) The resolvent $\mathcal{R}(\cdot, A)$ is called *individually eventually strongly negative with respect to u at λ_0* if there exists $\lambda_2 < \lambda_0$ with the following properties: $[\lambda_2, \lambda_0) \subseteq \rho(A)$ and for every $0 < f \in E$ there is a number $\lambda_1 \in [\lambda_2, \lambda_0)$ such that $-\mathcal{R}(\lambda, A)f \gg_u 0$ for all $\lambda \in [\lambda_1, \lambda_0)$.

(ii) The resolvent $\mathcal{R}(\cdot, A)$ is called *uniformly eventually strongly negative with respect to u at λ_0* if there exists $\lambda_1 < \lambda_0$ with the following properties: $[\lambda_1, \lambda_0) \subseteq \rho(A)$ and $-\mathcal{R}(\lambda, A) \gg_u 0$ for all $\lambda \in [\lambda_1, \lambda_0)$.

We point out that we could have also defined the notion "asymptotically negative resolvent" in Section 7.2; yet, such a notion would not have lead to many new insights and it would, presumably, not have been particularly significant in applications. The notion of an eventually strongly negative resolvent is, by contrast, rather interesting from an applied point of view since it is closely related to what is known as *anti-maximum principles* in the literature:

Remark 9.2.4. Consider for example the Dirichlet Laplace operator Δ_D on $L^p(\Omega; \mathbb{C})$ where $\Omega \subseteq \mathbb{R}^d$ is a sufficiently smooth domain and where $p \in (1, \infty)$. It is well-known that, for any $\lambda > s(A)$ and any $0 < f \in L^p(\Omega; \mathbb{C})$, the unique solution $v \in D(\Delta_D)$ of the problem

$$(\lambda - \Delta_D)v = f \tag{9.1}$$

fulfils $v \gg 0$. This is a consequence of the maximum principle and it is equivalent to the fact that the semigroup $(e^{t\Delta_D})_{t\geq 0}$ on $L^p(\Omega;\mathbb{C})$ is positive and irreducible.

Assume now that $p > d$. Then, on the other hand, Clément and Peletier proved in [CP79, Section 3] that for every $0 < f \in L^p(\Omega;\mathbb{C})$ there exists $\lambda_1 < s(\Delta_D)$ such that $(\lambda_1, s(\Delta_D)) \subseteq \rho(\Delta_D)$ and such that, for any $\lambda \in (\lambda_1, s(\Delta_D))$, the solution v of the problem (9.1) fulfils $-v \gg 0$ (in fact, they proved such a result for a more general class of elliptic operators, see [CP79] for details). They named this result an *anti-maximum principle*; in order to highlight the dependence of λ_1 on f, we might in fact call it an *individual* anti-maximum principle. Adapting our terminology from Definition 9.2.3 we would rather refer to this property of Δ_D by saying that the resolvent $\mathcal{R}(\cdot, \Delta_D)$ is *individually eventually strongly negative* at $s(\Delta_D)$ (note, however, that we considered individual eventual strong negativity *with respect to a fixed quasi-interior point* u).

Anti-maximum principles were in the focus of many research papers; here we refer only to a few examples: in [Bir95] versions of the anti-maximum principle for general domains where given; in [Tak96] anti-maximum (and maximum) principles were discussed in a rather abstract setting of ordered Banach spaces; in [GS01] anti-maximum principles for polyharmonic operators were studied.

We will see in Theorem 9.2.6 below that anti-maximum principles (in our terms: eventual strong negativity of the resolvent) can be studied within our general framework of eventual positivity. Yet, before we come to this theorem, we first prove that operators with eventually strongly positive (or negative) resolvents satisfy a certain Perron–Frobenius type property:

Theorem 9.2.5. *Let E be a complex Banach lattice and let $A : E \supseteq D(A) \to E$ be a densely defined and real linear operator. Suppose that $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$ and let P be the corresponding spectral projection. Let u be a quasi-interior point of E_+.*

If $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to u at λ_0 or if $\mathcal{R}(\cdot, A)$ is individually eventually strongly negative with respect to u at λ_0, then we have $P \gg_u 0$.

The subsequent proof of Theorem 9.2.5 has been inspired by the proof of [AGG+86, Proposition B-III-3.5].

Proof of Theorem 9.2.5. We may assume that $\lambda_0 = 0$. Suppose that $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to u at 0; we intend to use the implication "(ii) $\Rightarrow$ (i)" in Proposition 9.1.1. Denote by $m \geq 1$ the

order of 0 as a pole of the resolvent $\mathcal{R}(\cdot, A)$ and let

$$\mathcal{R}(\lambda, A) = \sum_{k=-m}^{\infty} Q_k \lambda^k \tag{9.2}$$

be the Laurent series expansion of $\mathcal{R}(\cdot, A)$ for λ close to 0. Then $0 \neq Q_{-m} = \lim_{\lambda \downarrow 0} \lambda^m \mathcal{R}(\lambda, A)$; hence, Q_{-m} is non-zero and positive.

We show that $\ker A$ contains a non-zero positive element, and each such element v fulfils $v \gg_u 0$; in particular, each such v is a quasi-interior point of E_+. Indeed, since the non-zero positive operator Q_{-m} maps into $\ker A$ according to Proposition A.3.2 (a), we conclude that $\ker A$ contains a non-zero positive vector. On the other hand, since $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to the quasi-interior point u, it follows that every $0 < v \in \ker A$ fulfils $v = \lambda \mathcal{R}(\lambda, A)v \gg_u 0$, where $\lambda > 0$ is chosen sufficiently small.

Let us show next that the range of Q'_{-m} contains a strictly positive functional φ. By taking adjoints, the series representation (9.2) yields the Laurent series expansion of $\mathcal{R}(\lambda, A')$ about 0. Hence, the range of Q'_{-m} is a subset of $\ker A'$ and it contains a functional $\varphi > 0$ since $Q'_{-m} > 0$. For every $0 < f \in E$ we can find $\lambda > 0$ such that $\mathcal{R}(\lambda, A)f \gg_u 0$; in particular, $\mathcal{R}(\lambda, A)f$ is a quasi-interior point of E_+. Since φ is contained in the fixed space of $\lambda \mathcal{R}(\lambda, A') = \lambda \mathcal{R}(\lambda, A)'$ we obtain

$$\langle \varphi, f \rangle = \langle \varphi, \lambda \mathcal{R}(\lambda, A) f \rangle > 0,$$

as claimed.

To conclude from Proposition 9.1.1 that $P \gg_u 0$, it remains to show that $\ker A$ is one-dimensional. To this end, we first show that $E_{\mathbb{R}} \cap \ker A$ is a sublattice of the real part $E_{\mathbb{R}}$ of E. Let $f \in E_{\mathbb{R}} \cap \ker A$. According to Lemma 6.2.2 we obtain for all sufficiently small $\lambda > 0$

$$|f| = |\lambda \mathcal{R}(\lambda, A) f| \leq \lambda \mathcal{R}(\lambda, A)|f|.$$

Now, let $\varphi \in \ker A'$ be a strictly positive functional; for all such λ as above we thus conclude that

$$\langle \varphi, \lambda \mathcal{R}(\lambda, A)|f| - |f| \rangle = \langle \lambda \mathcal{R}(\lambda, A')\varphi, |f| \rangle - \langle \varphi, |f| \rangle = 0.$$

Hence, $\lambda \mathcal{R}(\lambda, A)|f| - |f| = 0$, so $|f|$ is indeed contained in $\ker A$. Hence, $E_{\mathbb{R}} \cap \ker A$ is a sublattice of $E_{\mathbb{R}}$, as claimed.

We have already observed above that every non-zero positive element of $\ker A$ is a quasi-interior point of E_+. Hence, every non-zero positive element of the sublattice $E_{\mathbb{R}} \cap \ker A$ of $E_{\mathbb{R}}$ is also a quasi-interior point of the cone $(E_{\mathbb{R}} \cap \ker A)_+ := (E_{\mathbb{R}})_+ \cap \ker A$; see [Sch74, Corollary 2 to Theorem II.6.3].

According to [Lot68, Lemma 5.1], this implies that $E_{\mathbb{R}} \cap A$ is one-dimensional over the real field. As A is real, we conclude that $\ker A = E_{\mathbb{R}} \cap \ker A + iE_{\mathbb{R}} \cap \ker A$ is one-dimensional over the complex field. Now, Proposition 9.1.1 finally yields that $P \gg_u 0$.

If we assume instead that the resolvent $\mathcal{R}(\cdot, A)$ is individually eventually strongly negative with respect to u at 0, then the resolvent $\mathcal{R}(\cdot, -A)$ of $-A$ is individually eventually strongly positive with respect to u at 0. Since the spectral projection P of A associated to 0 coincides with the spectral projection of $-A$ associated to 0, we conclude from what we have already shown that $P \gg_u 0$. □

It is natural to ask whether the converse implication in Theorem 9.2.5 does also hold, i.e. whether the condition $P \gg_u 0$ implies that the resolvent $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to u at λ_0. Example 9.3.7 below shows that this is not in general true. However, the converse implication is indeed true if we impose an additional domination assumption on the domain of the generator A; moreover, under the same additional assumption, we can prove that individual eventual strong positivity is equivalent to individual eventual strong negativity:

Theorem 9.2.6. *Let E be a complex Banach lattice and let $A : E \supseteq D(A) \to E$ be a densely defined and real linear operator. Suppose that $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$ and let P be the corresponding spectral projection. Moreover, let $u \in E_+$ and assume that $D(A) \subseteq E_u$. Then u is a quasi-interior point of E_+ and the following assertions are equivalent:*

(i) $P \gg_u 0$.

(ii) *The resolvent $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to u at λ_0.*

(iii) *The resolvent $\mathcal{R}(\cdot, A)$ is individually eventually strongly negative with respect to u at λ_0.*

Certainly, some comments on the domination condition $D(A) \subseteq E_u$ in the above theorem are in order.

Remark 9.2.7. Assume that the assumptions of Theorem 9.2.6 are fulfilled.

(a) As pointed out above, the condition $D(A) \subseteq E_u$ cannot be dropped in Theorem 9.2.6; a counterexample can be found in Example 9.3.7 below.

(b) If we endow, as usual, the space $D(A)$ with the graph norm and the space E_u with the gauge norm with respect to u, then the canonical embedding $D(A) \hookrightarrow E_u$ is continuous. This follows from the closed graph theorem and will be used several times in the sequel.

(c) In case that $E = \mathcal{C}(K;\mathbb{C})$ for some compact Hausdorff space K, the condition $D(A) \subseteq E_u$ is automatically fulfilled: indeed, as u is a quasi interior point of E_+, it follows that $u \gg_{\mathbb{1}_K} 0$ and hence, $E_u = E$. On the other hand, if the Banach space E is, in some sense, sufficiently regular (e.g. an L^p-space for some $p \in [1,\infty)$), then the condition $D(A) \subseteq E_u$ is quite strong (though it is still fulfilled in many applications) as Proposition 9.2.9 below suggests.

(d) If the equivalent conditions of Theorem 9.2.6 are fulfilled, then it follows from Proposition 9.1.1 that A has an eigenvector v for the eigenvalue λ_0 which is unique up to scalar multiplication and which can be chosen such that $v \gg_u 0$. On the other hand, since $v \in D(A) \subseteq E_u$, it follows that $u \gg_v 0$. This shows that the two vectors u and v dominate each other in some sense. In particular, we have $E_u = E_v$.

Hence, we might always think of the vector u in Theorem 9.2.6 as some kind of substitute for the eigenvector v (note that, in concrete applications, we may not know v exactly and, moreover, v might be rather complicated while u can often be chosen to be quite simple; compare the concrete applications in Chapter 11).

(e) One might wonder how to check the domination condition $D(A) \subseteq E_u$ in applications. A prototypical situation is as follows: E is an L^p-space ($1 < p < \infty$) over some bounded domain $\Omega \subseteq \mathbb{R}^d$, A is a differential operator defined on some Sobolev space $D(A)$ and u is the constant function $\mathbb{1} := \mathbb{1}_\Omega$. In this case, the principal ideal E_u is equal to $L^\infty(\Omega;\mathbb{C})$. Hence, the inclusion $D(A) \subseteq E_u$ is fulfilled if an appropriate Sobolev embedding theorem holds.

Before we prove Theorem 9.2.6, we make a few further comments on the relation of assertion (iii) in the theorem to anti-maximum principles:

Remark 9.2.8. Here we continue the discussion from Remark 9.2.4: Theorem 9.2.6 shows that certain spectral properties of the operator A are sufficient to conclude that A fulfils an individual anti-maximum principle in case that the domination condition $D(A) \subseteq E_u$ is fulfilled.

This explains on an abstract level why the condition $p > d$ is needed in the result of Clément and Peletier quoted in Remark 9.2.4: in this case, we know from Sobolev embedding theorems that the domain $D(\Delta_D)$ of the Dirichlet Laplace operator, which is given by $D(\Delta_D) = W^{2,p}(\Omega;\mathbb{C}) \cap W_0^{1,p}(\Omega;\mathbb{C})$, embeds into $\mathcal{C}^1(\overline{\Omega};\mathbb{C})$. Hence, every function in $D(\Delta_D)$ is 0 on $\partial\Omega$ in the classical sense and we conclude that $D(\Delta_D)$ is contained in the principal ideal $L^p(\Omega;\mathbb{C})_u$, where $u(x) := \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$. On the other hand, the first eigenvalue of Δ_D admits an eigenvector $v \gg_u 0$ (this follows for instance

from [Dav89, Lemma 4.6.1]) as well as a strictly positive eigenfunctional. Thus, we can conclude the validity of the individual anti-maximum principle from Theorem 9.2.6.

If, on the other hand, $p \leq d$, then we cannot conclude that $D(\Delta_D)$ is contained in $L^p(\Omega;\mathbb{C})_u$, so the theorem is not applicable; in fact, the anti-maximum principle turns out to fail in this case as shown in [Swe97].

Let us now prove Theorem 9.2.6. It is worthwhile pointing out that some ideas that we employ in the proof are related to the abstract approach to anti-maximum principles in [Tak96].

Proof of Theorem 9.2.6. We may assume that $\lambda_0 = 0$.

"(ii) $\Rightarrow$ (i)" According to Theorem 9.2.5 this implication is true even under weaker assumptions.

"(i) $\Rightarrow$ (ii)" Assume that (i) is true; according to Proposition 9.1.1, 0 is then a first order pole of the resolvent, so we have $\lim_{\lambda\downarrow 0} \lambda\mathcal{R}(\lambda,A) = P$ is $\mathcal{L}(E)$. Now, fix an arbitrary element $\mu \in \rho(A)$. We have $\mathcal{R}(\mu,A) \in \mathcal{L}(E;D(A))$. Using the resolvent identity we therefore obtain

$$\lambda\mathcal{R}(\lambda,A) = \lambda\mathcal{R}(\mu,A) + (\mu-\lambda)\mathcal{R}(\mu,A)\,\lambda\mathcal{R}(\lambda,A) \to \mu\mathcal{R}(\mu,A)P = P$$

in $\mathcal{L}(E;D(A))$ as $\lambda \downarrow 0$. Moreover, it follows from the closed graph theorem that the canonical embedding $D(A) \hookrightarrow E_u$ is continuous; hence, we also have $\lambda\mathcal{R}(\lambda,A) \to P$ in $\mathcal{L}(E;E_u)$ as $\lambda \downarrow 0$. Now, let $0 < f \in E$. Since A is real, so is $\lambda\mathcal{R}(\lambda,A)f$ for all $\lambda \in \mathbb{R} \cap \rho(A)$. Since we have $\lambda\mathcal{R}(\lambda,A)f \to Pf \gg_u 0$ in E_u as $\lambda \downarrow 0$, it follows that $\lambda\mathcal{R}(\lambda,A)f \gg_u 0$ for all sufficiently large $\lambda > 0$.

"(i) $\Leftrightarrow$ (iii)" Note that P is also the spectral projection of $-A$ associated to the spectral value $-\lambda_0 = 0$. Hence, according to what we have just shown, we have $P \gg_u 0$ if and only if $\mathcal{R}(\cdot,-A)$ is individually eventually strongly positive with respect to u at 0, and this is in turn true if and only if $\mathcal{R}(\cdot,A)$ is individually eventually strongly negative with respect to u at 0. □

While the domination condition $D(A) \subseteq E_u$ is, for instance, always fulfilled on $\mathcal{C}(K;\mathbb{C})$-spaces (see Remark 9.2.7(c)), it is a rather strong condition on many other important Banach lattices, namely on Banach lattices with *order continuous norm*. We refer to [Sch74, Definition II.5.12] or [MN91, Definition 2.4.1] for a definition of those spaces and more information about them (and we point out that a complex Banach lattice is said to have order continuous norm if the same is true for its real part); here we only recall that all L^p-spaces for $1 \leq p < \infty$ and, for instance, the space $c_0(\mathbb{N};\mathbb{C})$ of sequences which converge to 0 have order continuous norm. On such Banach lattices the following proposition holds:

Proposition 9.2.9. *Let E be a complex Banach lattice with order continuous norm, let $A : E \supseteq D(A) \to E$ be a linear operator with non-empty resolvent set and let $u \in E_+$ such that $D(A) \subseteq E_u$. Then all spectral values of A are poles of the resolvent $\mathcal{R}(\cdot, A)$; moreover, the generalised eigenspace corresponding to each spectral value of A is finite-dimensional.*

The proof is based on the following observation from Banach lattice theory:

Lemma 9.2.10. *Let E be a real or complex Banach lattice with order continuous norm and let $T_1, T_2, T_3 \in \mathcal{L}(E)$. If $u \in E_+$ and $T_k E \subseteq E_u$ for all $k \in \{1, 2, 3\}$, then $T_3 T_2 T_1$ is compact.*

Proof. First assume that the scalar field is real; as usual, we endow E_u with the gauge norm with respect to u.

We have $T_k \in \mathcal{L}(E; E_u)$ for all $k \in \{1, 2, 3\}$ due to the closed graph theorem. In particular, the restriction $T_3|_{E_u}$ is a bounded linear operator from E_u to E_u. Thus T_3 maps the unit ball of E_u to an order interval in E. Since E has order continuous norm, every order interval in E is weakly compact [MN91, Theorem 2.4.2], so $T_3|_{E_u}$ is a weakly compact operator from E_u to E. The space E_u is an AM-space and thus, it has the Dunford–Pettis property, meaning that every weakly compact operator from E_u into any Banach space F is a Dunford–Pettis operator (see [MN91, Proposition 3.7.9]); in particular, the operator $T_3|_{E_u} : E_u \to E$ is a Dunford–Pettis operator, i.e. we have $\|T_3 x_n\|_E \to 0$ for every sequence $(x_n) \subseteq E_u$ which converges weakly to 0 in E_u.

As T_1 is continuous from E to E_u, it maps the unit ball of E into an order interval of E; thus, T_1 is weakly compact as an operator from E to E. Since T_2 is continuous from E to E_u we know that T is also weakly continuous from E to E_u. Thus, $T_2 T_1$ is weakly compact as an operator from E to E_u.

Now, let $(x_n) \subseteq E$ be a bounded sequence. By the Eberlein-Šmulian theorem [DS58, Theorem V.6.1], $(T_2 T_1 x_n)$ has a subsequence $(T_2 T_1 x_{n_k})$ which weakly converges in E_u to a vector $y \in E_u$. Since $T_3|_{E_u}$ is a Dunford–Pettis operator from E_u to E, it follows that $T_3(T_2 T_1 x_{n_k} - y)$ norm converges to 0 in E. Hence, $(T_3 T_2 T_1 x_{n_k})$ is norm convergent to $T_3 y$ in E. This proves that $T_3 T_2 T_1$ is compact.

Now assume that the scalar field is complex. For each $k \in \{1, 2, 3\}$ we can write $T_k = R_k + i S_k$, where $R, S \in \mathcal{L}(E)$ are real operators (see Proposition C.1.7). Let $k \in \{1, 2, 3\}$ and $f \in E_{\mathbb{R}}$. There exists $c > 0$ such that $|T_k f| \leq cu$. Hence, $|R_k f| = |\mathrm{Re}(T_k f)| \leq |T_k f| \leq cu$ and $|S_k f| = |\mathrm{Im}(T_k f)| \leq |T_k f| \leq cu$. This shows that $R_k E_{\mathbb{R}} \subseteq (E_u)_{\mathbb{R}} = (E_{\mathbb{R}})_u$ and, similarly, $S_k E_{\mathbb{R}} \subseteq (E_u)_{\mathbb{R}} = (E_{\mathbb{R}})_u$. Hence, if we multiply three (possibly identical) operators out of the set $\{R_k|_{E_{\mathbb{R}}} : k \in \{1, 2, 3\}\} \cup \{S_k|_{E_{\mathbb{R}}} : k \in \{1, 2, 3\}\}$, we obtain a compact operator from $E_{\mathbb{R}}$ to $E_{\mathbb{R}}$; and thus, if we multiply three (possibly identical) operators out of the set $\{R_k : k \in \{1, 2, 3\}\} \cup \{S_k : k \in \{1, 2, 3\}\}$, then we obtain a compact operator from

E to E. As the operator $T_3T_2T_1$ is a linear combination of such products, it is compact, too. □

For the proof of Proposition 9.2.9 we are actually interested in the following corollary which is an immediate consequence of Lemma 9.2.10

Corollary 9.2.11. *Let E be a real or complex Banach lattice with order continuous norm and let $T \in \mathcal{L}(E)$. If $u \in E_+$ and $TE \subseteq E_u$, then T^3 is compact.*

From the above corollary we can now conclude the assertion of Proposition 9.2.9:

Proof of Proposition 9.2.9. Fix $\lambda \in \rho(A)$. Then we have $\mathcal{R}(\lambda, A)E = D(A) \subseteq E_u$, so $\mathcal{R}(\lambda, A)^3$ is compact according to Corollary 9.2.11. Theorem A.3.5(b) in the appendix thus implies the assertion. □

9.3 The Semigroup

Let us now turn, again, from resolvents to semigroups. As we did for resolvents we are going to consider eventual strong positivity with respect to a fixed quasi-interior point u of the Banach lattice cone E_+.

Definition 9.3.1 (Eventually strongly positive semigroups). Let E be a complex Banach lattice, let $(e^{tA})_{t\geq 0}$ be C_0-semigroup on E and let u be a quasi-interior point of E_+.

(i) The semigroup $(e^{tA})_{t\geq 0}$ is called *individually eventually strongly positive with respect to u* if for every $0 < f \in E$ there exists a time $t_0 \geq 0$ such that $e^{tA}f \gg_u 0$ for all $t \geq t_0$.

(ii) The semigroup $(e^{tA})_{t\geq 0}$ is called *uniformly eventually strongly positive with respect to u* if there exists a time $t_0 > 0$ such that $e^{tA} \gg_u 0$ for all $t \geq t_0$.

Let us formulate right away the main theorem of this section, namely a characterisation of semigroups which are individually eventually strongly positive with respect to a given quasi-interior point u. While we needed the domination condition $D(A) \subseteq E_u$ to characterise eventual strong positivity of resolvents in Theorem 9.2.6 (which is equivalent to $\mathcal{R}(\lambda, A)E \subseteq E_u$ for some $\lambda \in \rho(A) \cap \mathbb{R}$), we now assume that $e^{t_0}E \subseteq E_u$ for some $t_0 \geq 0$; in the prototypical special case $E = L^p(\Omega, \mu; \mathbb{C})$ (for a finite measure space (Ω, μ) and some $p \in [1, \infty)$) and $u = \mathbb{1}_\Omega$, this can be interpreted as some kind of "smoothing condition"; compare also Corollary 9.3.6 below. We point out that, for a large class of semigroups, the condition $e^{t_0A} \subseteq E_u$ turns out to be weaker than the condition $D(A) \subseteq E_u$, see Remark 9.3.4 below.

Theorem 9.3.2. *Let E be a complex Banach lattice and let $(e^{tA})_{t\geq 0}$ be a real C_0-semigroup on E with $\mathrm{s}(A) > -\infty$. Suppose that the peripheral spectrum $\sigma_{\mathrm{per}}(A)$ is finite and consists of pole of the resolvent $\mathcal{R}(\cdot, A)$. Let u be a quasi-interior point of E_+ and assume that there exists a time $t_0 \geq 0$ for which we have $e^{t_0 A}E \subseteq E_u$. Then the following assertions are equivalent:*

(i) *The rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded, $\mathrm{s}(A)$ is a dominant spectral value of A and the spectral projection P associated to $\mathrm{s}(A)$ fulfils $P \gg_u 0$.*

(ii) *The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to u.*

(iii) *$e^{t(A-\mathrm{s}(A))}$ converges strongly to an operator $Q \gg_u 0$ as $t \to \infty$.*

The comments that we made in Remark 9.2.7 concerning the condition $D(A) \subseteq E_u$ apply, of course, in a similar way to the condition $e^{t_0 A}E \subseteq E_u$; see, however, also Remark 9.3.4 below.

For the proof of Theorem 9.3.2 we need the following lemma:

Lemma 9.3.3. *Let E be a complex Banach lattice and let u be a quasi-interior point of E_+. Let $(J, \leq)$ be a non-empty, totally ordered set and let $(T_j)_{j\in J} \subseteq \mathcal{L}(E)$ be a family of operators whose fixed space F is defined as*

$$F := \{f \in E : T_j f = f \text{ for all } j \in J\}.$$

Assume that $(T_j)_{j\in J}$ is eventually strongly positive with respect to u in the sense that for every $0 < f \in E$ there exists $j_0 \in J$ such that $T_j f \gg_u 0$ for all $j \geq j_0$.

(a) *Suppose that the family $(T_j|_{E_u})_{j\leq j_0}$ is bounded in $\mathcal{L}(E_u; E)$ for every $j_0 \in J$ and assume that F contains a non-zero positive element. Then the entire family $(T_j|_{E_u})_{j\in J}$ is bounded in $\mathcal{L}(E_u; E)$.*

(b) *Let $0 < P \in \mathcal{L}(E)$ be a projection whose range PE is contained in F and suppose that every operator T_j $(j \in J)$ leaves $\ker P$ invariant. Then $P \gg_u 0$.*

Proof. (a) It suffices to show that the orbit $(T_j f)_{j\in J}$ is bounded in E for every $0 < f \in E_u$. Due to our assumption, F contains a vector $v > 0$. For some $j \in J$ we have $0 \ll_u T_j v = v$, so $v \geq \varepsilon u$ for some $c > 0$. Since f is contained in E_u, we can thus find a constant $c > 0$ such that $0 \leq f \leq cv$. As $cv - f \geq 0$, we conclude for all sufficiently large j, $j \geq j_0$ say, that $T_j f \geq 0$ and $T_j(cv - f) \geq 0$; hence, $0 \leq T_j f \leq T_j cv = cv$ for all such j. This shows that the orbit $(T_j f)_{j\geq j_0}$ is bounded in E_u and thus in E. Since $(T_j f)_{j\leq j_0}$ is bounded in E be assumption, we conclude that the entire orbit $(T_j f)_{j\in J}$ is bounded in E.

(b) Let $f > 0$. If $Pf \neq 0$ we have $Pf = T_j Pf \gg_u 0$ for some sufficiently large index $j \in J$; thus, it suffices to show that $Pf \neq 0$. Since P is non-zero and E_u

is dense in E, we can find a vector $0 < g \in E_u$ such that $Pg > 0$. We can find $j \in J$ such that $T_j f \gg_u 0$ and thus, we have $cT_j f - g \geq 0$ for a sufficiently large number $c > 0$. Hence, $cPT_j f \geq Pf > 0$ and, in particular, $PT_j f \neq 0$. As T_j leaves the kernel of P invariant by assumption, we conclude that $Pf \neq 0$. □

Let us now prove Theorem 9.3.2:

Proof of Theorem 9.3.2. We may assume that $s(A) = 0$.

"(i) ⇒ (iii)" If (i) is fulfilled it follows from [vN96, Corollary 5.2.6] that e^{tA} converges strongly to 0 on the kernel of P. On the other hand, we have $PE = \ker A$ since the boundedness of the semigroup implies that 0 is a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. Thus, $e^{tA}P = P$ for all $t \geq 0$. Therefore, e^{tA} converges strongly to the operator $P \gg_u 0$ as $t \to \infty$.

"(iii) ⇒ (ii)" Assume that (iii) holds. Then Qf is a fixed vector of the semigroup for every $f \in E$. Hence, $e^{tA}Q = Q$ for all $t \geq 0$. Since $e^{t_0 A}E \subseteq E_u$, the closed graph theorem implies that $e^{t_0 A} \in \mathcal{L}(E; E_u)$. Now, let $0 < f \in E$. Since our semigroup is real, so is $e^{tA}f$ for every $t \geq 0$. We have

$$e^{tA}f = e^{t_0 A}e^{(t-t_0)A}f \to e^{t_0 A}Qf = Qf \gg_u 0$$

in E_u as $t \to \infty$. Hence, $e^{tA}f \gg_u 0$ for all sufficiently large t.

"(ii) ⇒ (i)" Assume (ii). According to Proposition 6.2.7, 0 is an eigenvalue of A with an eigenvector $v > 0$. Thus, we can apply Lemma 9.3.3(a) to the operator family $(T_j)_{j\in J} := (e^{tA})_{t\in[0,\infty)}$ and conclude that $(e^{tA}|_{E_u})_{t\in[0,\infty)}$ is bounded in $\mathcal{L}(E_u; E)$. Since $e^{t_0 A} \in \mathcal{L}(E; E_u)$ due to the closed graph theorem, we conclude that the operator family $(e^{tA})_{t\geq t_0}$ is bounded in $\mathcal{L}(E)$ and this readily implies that the entire semigroup is bounded in $\mathcal{L}(E)$.

Theorem 7.3.3 now yields that $s(A) = 0$ is a dominant spectral value of A and that the associated spectral projection P is positive (and non-zero as $0 \in \sigma(A)$). Since the semigroup $(e^{tA})_{t\geq 0}$ is bounded, 0 is a first order pole of the resolvent and hence, the range of P coincides with $\ker A$ and thus with the fixed space of the semigroup. Therefore, we can finally employ Lemma 9.3.3(b) to conclude that $P \gg_u 0$. □

Let us briefly comment on the question when the condition $e^{t_0 A}E \subseteq E_u$ is fulfilled for some $t_0 \geq 0$. Of course, it is automatically fulfilled if $E = \mathcal{C}(K; \mathbb{C})$ for some compact Hausdorff space K. We can also give more abstract conditions which are sufficient for $e^{t_0}E \subseteq E_u$ to hold for at least one $t_0 \geq 0$. Recall that a C_0-semigroup $(e^{tA})_{t\geq 0}$ on a Banach space E is called *eventually differentiable* if there exists $t_0 \geq 0$ such that mapping $(t_0, \infty) \to \mathcal{L}(E)$, $t \mapsto e^{tA}$ is differentiable; it is called *immediately differentiable* if we can choose $t_0 = 0$ (compare [EN00, Definition II.4.13 and Exercise II.4.16(1)]).

Remark 9.3.4. Let E be a complex Banach lattice and let u be a quasi-interior point of E_+. Let $(e^{tA})_{t\geq 0}$ be a C_0-semigroup on E.

(a) If the semigroup $(e^{tA})_{t\geq 0}$ is eventually differentiable and if there is an integer $n \in \mathbb{N}$ such that $D(A^n) \subseteq E_u$, then we have $e^{tA}E \subseteq E_u$ for one (hence all) sufficiently large $t > 0$.

(b) If the semigroup $(e^{tA})_{t\geq 0}$ is immediately differentiable (which is in particular fulfilled if $(e^{tA})_{t\geq 0}$ is analytic) and if we have $D(A^\infty) \subseteq E_u$, then $e^{tA}E \subseteq E_u$ for all $t > 0$.

Proof. (a) Since the semigroup is eventually differentiable, we have $e^{t_1 A}E \subseteq D(A)$ for some $t_1 \geq 0$. This readily implies that $e^{nt_1 A}E \subseteq D(A^n) \subseteq E_u$.

(b) If the semigroup is immediately differentiable, then we have $e^{tA}E \subseteq D(A^\infty) \subseteq E_u$ for all $t > 0$. □

Let us give several corollaries of Theorem 9.3.2. In the first corollary we show that a weakened version of assertion (i) in the theorem is still equivalent to the other assertions in case that the semigroup is eventually norm continuous.

Corollary 9.3.5. *Let E be a complex Banach lattice and let $(e^{tA})_{t\geq 0}$ be a real, eventually norm-continuous C_0-semigroup on E with $\mathrm{s}(A) > -\infty$. Suppose that the peripheral spectrum $\sigma_{\mathrm{per}}(A)$ consists of poles of the resolvent $\mathcal{R}(\cdot, A)$. Let u be a quasi-interior point of E_+ and assume that there exists a time $t_0 \geq 0$ for which we have $e^{t_0 A}E \subseteq E_u$. Then the following assertions are equivalent:*

(i) $\mathrm{s}(A)$ *is a dominant spectral value of A and the spectral projection P associated to* $\mathrm{s}(A)$ *fulfils $P \gg_u 0$.*

(ii) *The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to u.*

(iii) *$e^{t(A-\mathrm{s}(A))}$ converges strongly to an operator $Q \gg_u 0$ as $t \to \infty$.*

Proof. Since the semigroup is eventually norm continuous, the peripheral spectrum of A is automatically bounded, see [EN00, Theorem II.4.18]. Moreover, since $\sigma_{\mathrm{per}}(A)$ is assumed to consist of poles of the resolvent, and thus of isolated spectral values, it follows that $\sigma_{\mathrm{per}}(A)$ is in fact finite. Hence, the assumptions of Theorem 9.3.2 are fulfilled and, according to the theorem we only have to show that if assertion (i) is fulfilled, then the rescaled semigroup $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is automatically bounded.

So assume that (i) holds. Then 0 is a first order pole of the resolvent according to Proposition 9.1.1. Since the semigroup is eventually norm continuous, Remark 7.3.7(c) thus asserts that $(e^{t(A-\mathrm{s}(A))})_{t\geq 0}$ is bounded. □

Our second corollary is an analogue of Corollary 9.2.9 for semigroups instead of resolvents. As in Corollary 9.2.9 we consider the important case where the Banach lattice E has order continuous norm; in this case we can use the "smoothing condition" $e^{t_0 A}E \subseteq E_u$ together with Corollary 9.2.11 to show that many assumptions of Theorem 9.3.2, respectively of Corollary 9.3.5, are automatically fulfilled.

Corollary 9.3.6. *Let E be a complex Banach lattice with order continuous norm and let $(e^{tA})_{t\geq 0}$ be a real C_0-semigroup on E with $\mathrm{s}(A) > -\infty$. Let u be a quasi-interior point of E_+ and assume that there exists a time $t_0 \geq 0$ for which we have $e^{t_0 A}E \subseteq E_u$. Then the semigroup is eventually compact (in particular, all spectral values of A are poles of the resolvent) and the following assertions are equivalent:*

(i) $\mathrm{s}(A)$ *is a dominant spectral value of A and the spectral projection P associated to $\mathrm{s}(A)$ fulfils $P \gg_u 0$.*

(ii) *The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to u.*

(iii) *$e^{t(A-\mathrm{s}(A))}$ converges strongly to an operator $Q \gg_u 0$ as $t \to \infty$.*

Proof. Since E has order continuous norm, the condition $e^{t_0 A}E \subseteq E_u$ implies that $e^{3t_0 A}$ is compact, see Corollary 9.2.11. Thus, our semigroup is eventually compact. This implies in particular that the semigroup is eventually norm continuous and that all spectral values of A are poles of the resolvent, see [EN00, Lemma II.4.22 and Corollary V.3.2(i)]. Hence, the assumptions of Corollary 9.3.5 are fulfilled, which implies the assertion. □

We close the section with a counterexample showing that that the domination condition $D(A) \subseteq E_u$ in Theorem 9.2.6 and the smoothing condition $e^{t_0 A}E \subseteq E_u$ in Theorem 9.3.2 cannot in general be dropped.

Example 9.3.7. Let $p \in [1, \infty)$ and let $E = L^p((-1,1);\mathbb{C})$. There is a bounded real operator $A \in \mathcal{L}(E)$ with the following properties:

(a) The semigroup $(e^{tA})_{t\geq 0}$ is bounded, 0 is a dominant spectral value of A and a first order pole of the resolvent and the associated spectral projection P fulfils $P \gg_{\mathbb{1}} 0$, where $\mathbb{1} := \mathbb{1}_{(-1,1)}$.

(b) The semigroup $(e^{tA})_{t\geq 0}$ is not individually eventually positive; in particular, it is not individually eventually strongly positive with respect to $\mathbb{1}$.

(c) Similarly, the resolvent $\mathcal{R}(\cdot, A)$ is not individually eventually positive at 0; in particular, it is not individually eventually strongly positive with respect to $\mathbb{1}$ at 0.

(d) The smoothing condition $e^{t_0 A}E \subseteq E_{\mathbb{1}}$ is not fulfilled for any $t_0 \geq 0$, nor is the domination condition $D(A) \subseteq E_{\mathbb{1}}$.

Proof. To construct A, let $\varphi \in E'$ be given by $\langle \varphi, f \rangle = \int_{(-1,1)} f(\omega)\, d\omega$ for all $f \in E$ and consider the decomposition $E = \operatorname{span} \mathbb{1} \oplus F$, where $F := \ker \varphi$. We denote by $S \in \mathcal{L}(F)$ the *reflection operator* which is given by $(Sf)(\omega) = f(-\omega)$ for all $f \in F$ and almost all $\omega \in (-1,1)$. Using that $S^2 = I_F$, we obtain $\sigma(A) = \{-1, 1\}$. Now, let us define $A \in \mathcal{L}(E)$ by

$$A = 0_{\operatorname{span} \mathbb{1}} \oplus (-S - 2I_F).$$

Then A is a real operator, we have $\sigma(A) = \{-3, -1, 0\}$, the number 0 is a first order pole of the resolvent and the corresponding spectral projection P is given by $Pf = \frac{1}{2}\langle \varphi, f \rangle \mathbb{1}$. Moreover, the semigroup $(e^{tA})_{t \geq 0}$ is clearly bounded. Thus, assertion (a) is fulfilled.

Let us now show that assertions (b) and (c) are fulfilled, to. Using again that $S^2 = I_F$ one easily computes that the semigroup $(e^{tA})_{t \geq 0}$ and the resolvent $\mathcal{R}(\cdot, A)$ are given by

$$e^{tA} = I_{\operatorname{span} \mathbb{1}} \oplus e^{-2t}\Big(\cosh(t) I_F - \sinh(t) S\Big), \tag{9.3}$$

$$\mathcal{R}(\lambda, A) = \frac{1}{\lambda} I_{\operatorname{span} \mathbb{1}} \oplus \frac{1}{(\lambda+2)^2 - 1}\Big((\lambda + 2) I_F - S\Big) \tag{9.4}$$

for all $t \geq 0$ and all $\lambda \in \mathbb{C} \setminus \{-3, -1, 0\}$.

Now, let $f \in E$ be given by $f(\omega) = (1 - \omega)^{-\frac{1}{2p}}$ for all $\omega \in (-1, 1)$. The function f is bounded outside any neighbourhood of 1, but it is unbounded in any neighbourhood of 1. Decomposing f as $f = Pf + (I_E - P)f$ and using formulas (9.3) and (9.4) we can thus see that $\mathcal{R}(\lambda, A)f$ is not positive for any $\lambda \in (0, \infty)$ and that $e^{tA}f$ is not positive for any $t \in (0, \infty)$. This proves (b) and (c).

Finally, observe that $D(A) = E \not\subseteq L^\infty((-1,1); \mathbb{C}) = E_{\mathbb{1}}$ and, since $e^{t_0 A}$ is bijective from E to E, $e^{t_0 A}E = E \not\subseteq L^\infty((-1,1); \mathbb{C}) = E_{\mathbb{1}}$ for any $t_0 \geq 0$. Of course, we could alternatively conclude from Theorem 9.2.6 that $D(A) \not\subseteq E_{\mathbb{1}}$ and from Theorem 9.3.2 that $e^{t_0 A}E \not\subseteq E_{\mathbb{1}}$ for any $t_0 \geq 0$ since all other assumptions of these theorems are fulfilled by our example, but the conclusions of those theorems are not. □

9.4 Notes & Open Problems

Eventual strong positivity Throughout the chapter we always considered strong positivity with respect to a given quasi-interior point $u \in E_+$. It would, of course, also be interesting to consider strong positivity in a more general sense; we could, for instance, call a C_0-semigroup $(e^{tA})_{t \geq 0}$ on a complex

Banach lattice *individually eventually strongly positive* if, for every $f > 0$, there exists a time $t_0 > 0$ such that $e^{tA}f$ is a quasi-interior point of E_+ for every $t \geq t_0$. It is natural to ask for characterisations of such semigroups:

Problem 9.4.1. How can individual eventual strong positivity of a semigroup (as defined in the previous paragraph) be characterised (under sufficiently strong regularity assumptions, say)?

This seems to be a difficult question since, if we do not fix a vector $u \in E_+$, no domination or smoothing condition such as $D(A) \subseteq E_u$ or $e^{t_0 A}E \subseteq E_u$ can be formulated; yet, we have seen in Example 9.3.7 that the characterisations in Theorems 9.2.6 and 9.3.2 fail without such a condition.

Perron–Frobenius results without a domination or smoothing condition Note that the implication "(ii) $\Rightarrow$ (i)" in Theorem 9.2.6 also holds without the domination condition $D(A) \subseteq E_u$ (this is the content of Theorem 9.2.5) while the converse implication is in general false without this assumption (see Example 9.3.7). The observation that we do not need a domination condition for the first mentioned implication was not known yet when the paper [DGK16a] was written; it was obtained later as part of an ongoing collaboration with Daniel Daners. It would be interesting to know whether a similar result holds for semigroups, i.e. whether the implication "(ii) $\Rightarrow$ (i)" in Theorem 9.3.2 (or, say, in Corollary 9.3.5) remains true without the smoothing assumption $e^{t_0 A}E \subseteq E_u$.

10 Criteria for Uniform Eventual Strong Positivity

While, in the preceding chapter, we focussed on *individually* eventually strongly positive behaviour of resolvents and semigroups, we are now going to study *uniform* eventual strong positivity. Section 10.1 contains some simple criteria for a resolvent to be uniformly eventually strongly positive. For semigroups, however, it seems quite difficult to give good criteria for uniform eventual strong positivity. We study this problem in Section 10.2; actually, we merely consider the special case of self-adjoint semigroups on L^2-spaces since in this situation it turns out that individual eventual strong positivity is, under appropriate assumptions, equivalent to uniform eventual strong positivity. In the third section we provide two counterexamples which show that, in more general situations, it is necessary to distinguish between individual and uniform eventual behaviour even under very strong regularity assumptions.

The results presented in Section 10.1 have, in the special case of $\mathcal{C}(K;\mathbb{C})$-spaces, been published in [DGK16b, Section 4]. The only result in Section 10.2, Theorem 10.2.1, has not been published or submitted for publication, yet. It is based on joint work with Daniel Daners and part of an ongoing collaboration with him. The counterexamples in Section 10.3 have both been published in [DGK16b, Section 5].

10.1 The Resolvent

The following sufficient condition for uniform eventual strong positivity of a resolvent is based on the Taylor series expansion of $\mathcal{R}(\cdot, A)$ around a point $\lambda_1 \in \rho(A)$ where it is already known that the resolvent is strongly positive. Due to the Taylor series expansion, the strong positivity extends to an interval on the left of λ_1 until a spectral value λ_0 gets in the way.

Proposition 10.1.1. *Let E be a complex Banach lattice, let u be a quasi-interior point of E_+ and let $A : E \supseteq D(A) \to E$ be a real linear operator. Let $\lambda_0 \in [-\infty, \infty)$ be a spectral value of A or $-\infty$ and suppose there exists a number $\lambda_1 > \lambda_0$ such that $(\lambda_0, \lambda_1] \subseteq \rho(A)$, $\mathcal{R}(\lambda_1, A) \geq 0$ and $\mathcal{R}(\lambda_1, A)^n \gg_u 0$ for some integer $n \in \mathbb{N}_0$.*

Then the resolvent $\mathcal{R}(\cdot, A)$ is uniformly eventually strongly positive with respect to u at λ_0; more precisely, we have $\mathcal{R}(\lambda, A) \gg_u$ for all $\lambda \in (\lambda_0, \lambda_1)$.

Proof. Let $I := \{\lambda \in (\lambda_0, \lambda_1) | \mathcal{R}(\lambda_1, A) \gg_u 0\}$. We have to show that $I = (\lambda_0, \lambda_1)$.

Let us first show that $(\lambda_1 - \varepsilon, \lambda_1) \subseteq I$ for some $\varepsilon > 0$. To this end, choose $\varepsilon > 0$ so small that the open ball with radius ε and center λ_1 is contained in the resolvent set $\rho(A)$. Then we have

$$\mathcal{R}(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_1 - \lambda)^n \mathcal{R}(\lambda_1, A)^{n+1} \gg_u 0$$

for all $\lambda \in (\lambda_1 - \varepsilon, \lambda_1)$. Hence, $\emptyset \neq (\lambda_1 - \varepsilon, \lambda_1) \subseteq I$, as claimed.

Now, let $J \subseteq I$ be the largest subinterval of I which contains $(\lambda_1 - \varepsilon, \lambda_1)$. It suffices to show that $\inf J = \lambda_0$, so assume for a contradiction that $\inf J > \lambda_0$. Choose a $\delta > 0$ such that the open ball around $\inf J$ with radius δ is contained in the resolvent set $\rho(A)$ and such that $\inf J + \delta \leq \lambda_1$. Then the number $\inf J + \frac{\delta}{4}$ is contained in J and the ball of radius $\frac{\delta}{2}$ with center $\inf J + \frac{\delta}{4}$ is contained in the resolvent set $\rho(A)$. Using again the series expansion of the resolvent, this time about $\inf J + \frac{\delta}{4}$, we obtain for all $\lambda \in (\inf J - \frac{\delta}{4}, \inf J]$

$$\mathcal{R}(\lambda, A) = \sum_{n=0}^{\infty} (\inf J + \frac{\delta}{4} - \lambda)^n \mathcal{R}(\inf J + \frac{\delta}{4}, A)^{n+1} \gg_u 0$$

Thus, $(\inf J - \frac{\delta}{4}, \inf J] \subseteq I$ and hence, $(\inf J - \frac{\delta}{4}, \lambda_1) \subseteq J$, which is a contradiction. □

A simple but interesting consequence of Proposition 10.1.1 is the following result about squares of operators:

Corollary 10.1.2. *Let E be a complex Banach lattice, let u be a quasi-interior point of E_+ and let $A : E \supseteq D(A) \to E$ be a linear operator. Let $\lambda_0 \in [-\infty, 0)$ be a spectral value of A or $-\infty$ and assume that $(\lambda_0, 0] \subseteq \rho(A)$. Moreover, assume that there is a closed linear operator $B : E \supseteq D(B) \to E$ such that $A = -B^2$.*

If $\mathcal{R}(0, B) \gg_u 0$, then the resolvent $\mathcal{R}(\cdot, A)$ is uniformly eventually strongly positive with respect to u at λ_0.

Proof. We have $\mathcal{R}(0, A) = (B^2)^{-1} = \left(-B^{-1}\right)^2 = \mathcal{R}(0, B)^2 \gg_u 0$. Hence, it follows from Proposition 10.1.1 that $\mathcal{R}(\lambda, A) \gg_u 0$ for all $\lambda \in (\lambda_0, 0]$. □

Corollary 10.1.2 has nice applications if B is a second order elliptic differential operator with spectral bound $s(B) < 0$. For such operators it follows, under rather general conditions, from the strong maximum principle that the resolvent $\mathcal{R}(0, B)$ is, in some sense, strongly positive. Hence we can conclude from Corollary 10.1.2 that the resolvent of $-B^2$ is eventually strongly positive at its spectral bound. This can sometimes be used to derive (at least individual) eventual strong positivity of the semigroup $(e^{-tB^2})_{t \geq 0}$; on the other hand, this semigroup cannot be positive since the differential operator $-B^2$ has order > 2 (see e.g. [MO86, Theorem 3.6], [Miy86, Section 2] and [ABR90, Proposition 2.2]). We refer to Section 11.5 for some concrete examples of this.

10.2 The Semigroup

We have seen in the preceding section that it is easy to obtain uniform eventual strong positivity of a resolvent $\mathcal{R}(\cdot, A)$ if it is known that $\mathcal{R}(\lambda, A)$ is strongly positive at at least *one* point $\lambda \in \rho(A) \cap \mathbb{R}$. On the other hand, it seems to be quite difficult to give criteria for uniform eventual strong positivity of $\mathcal{R}(\cdot, A)$ at a point λ_0 in terms of spectral properties. A similar difficulty occurs when we look for criteria for uniform eventual strong positivity of C_0-semigroups. One special case where it is possible to give such a criterion is that of self-adjoint semigroups on L^2-spaces:

Theorem 10.2.1. *Let $H \neq \{0\}$ be a complex Hilbert lattice and let $(e^{tA})_{t\geq 0}$ be a real C_0-semigroup on H with self-adjoint generator A. Let u be a quasi-interior point of H_+ and assume that we have $D(A^\infty) \subseteq H_u$. Then the semigroup is immediately compact (in particular, all spectral values of A are poles of the resolvent) and the following assertions are equivalent:*

(i) *The spectral projection P associated with $\mathrm{s}(A)$ fulfils $P \gg_u 0$.*

(ii) *$\ker(\mathrm{s}(A) - A)$ is one-dimensional and contains a vector $v \gg_u 0$.*

(iii) *The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to u.*

(iv) *The semigroup $(e^{tA})_{t\geq 0}$ is uniformly eventually strongly positive with respect to u.*

(v) *$e^{t(A-\mathrm{s}(A))}$ converges strongly to an operator $Q \gg_u 0$ as $t \to \infty$.*

Proof. We may assume that $\mathrm{s}(A) = 0$. Since the semigroup $(e^{tA})_{t\geq 0}$ is analytic, it follows from $D(A^\infty) \subseteq H_u$ that $e^{tA}H \subseteq H_u$ for all $t > 0$, compare Remark 9.3.4(b). Moreover, since the generator A is self-adjoint, it has non-empty spectrum. Finally, the Banach lattice H is, of course, reflexive and hence, it has order continuous norm (this follows for example from [MN91, Theorem 2.4.2(v)]). Thus, our semigroup is immediately compact due to Corollary 9.2.11 and hence, all spectral values of A are poles of $\mathcal{R}(\cdot, A)$. Moreover, all assumptions of Corollary 9.3.6 are fulfilled.

Since $\sigma(A) \subseteq \mathbb{R}$, the spectral bound $\mathrm{s}(A)$ is of course a dominant spectral value and thus, it follows from Corollary 9.3.6 that assertions (i), (iii) and (v) of our theorem are equivalent.

The implication "(i) $\Rightarrow$ (ii)" follows from Proposition 9.1.1. If, conversely, asssertion (ii) holds, then it follows from the self-adjointness of A and from Corollary C.6.4 that $\ker A'$ contains a strictly positive functional. Hence, we can employ again Proposition 9.1.1 to conclude that (i) is true.

The implication "(iv) ⇒ (iii)" is obvious. Hence it only remains to show, say, "(i) ⇒ (iv)", and this is, of course, the most interesting part here. Assume that $P \gg_u 0$

We note that, for every $t > 0$, the operator e^{tA} is a Hilbert-Schmidt operator on H. Indeed, since $e^{tA}H_{\mathbb{R}} \subseteq (H_u)_{\mathbb{R}} = (H_{\mathbb{R}})_u$, the operator $e^{tA}|_{H_{\mathbb{R}}}$ is a so-called *majorizing* operator on the real Hilbert lattice $H_{\mathbb{R}}$, see [Sch74, Definition IV.3.1]. Hence, $e^{tA}|_{H_{\mathbb{R}}}$ is a Hilbert–Schmidt operator on the real Hilbert space $H_{\mathbb{R}}$ according to [Sch74, Theorem IV.6.9]; this readily implies that e^{tA} is a Hilbert–Schmidt operator on the complex Hilbert space H.

Since the semigroup $(e^{tA})_{t\geq 0}$ is immediately compact, it follows that A has compact resolvent [EN00, Theorem II.4.29]; in particular, H is separable. As A is self-adjoint, there exists on orthonormal basis $(e_n)_{n\in\mathbb{N}_0}$ of eigenvectors of A, and we may choose each vector e_n to be real. Let $0 = -\lambda_0 \geq -\lambda_1 \geq \lambda_2 \geq \ldots$ be the corresponding eigenvalues. Note that we have $\lambda_0 \neq \lambda_1$ since the eigenspace $\ker A$ is one-dimensional according to assertion (ii). Due to the same assertion we may assume that $e_0 \gg_u 0$ after replacing e_0 with $-e_0$ if necessary; so $e_0 \geq cu$ for some fixed constant $c > 0$. For every $t \geq 0$ and every $n \in \mathbb{N}_0$, e_n is an eigenvector of e^{tA} for the eigenvalue $e^{-\lambda_n t}$.

Now, let $0 < f \in H$. For every $t \geq 0$ we obtain

$$e^{tA}f = \sum_{n=0}^{\infty}(f|e_n)e^{-t\lambda_n}e_n \geq c^2(f|u)u + \sum_{n=1}^{\infty}(f|e_n)e^{-t\lambda_n}e_n,$$

where the series converge in H. Due to the closed graph theorem we have $e^A \in \mathcal{L}(H, H_u)$; let $M > 0$ denote the operator norm of e^A as an operator from H to H_u. We then have $\|e_n\|_u = e^{\lambda_n}\|e^A e_n\|_u \leq e^{\lambda_n}M\|e_n\| = Me^{\lambda_n}$ for every $n \in \mathbb{N}_0$. Thus, we obtain for $t > 0$

$$\sum_{n=1}^{\infty}\|(f|e_n)e^{-t\lambda_n}e_n\|_u \leq \sum_{n=1}^{\infty}(f||e_n|)e^{(-t+1)\lambda_n} \leq (f|u)M^2\sum_{n=1}^{\infty}e^{(-t+2)\lambda_n}.$$

For every $n \in \mathbb{N}$, the number $e^{(-t+2)\lambda_n}$ decreases to 0 as $t \to \infty$ since $\lambda_n > 0$. Now we use that e^A is a Hilbert–Schmidt operator: this implies that the eigenvalues of e^A are square summable, so $\sum_{n=1}^{\infty}e^{-2\lambda_n} < \infty$. The dominated convergence theorem now implies that $\sum_{n=1}^{\infty}e^{(-t+2)\lambda_n}$ converges to 0 as $t \to \infty$. In particular, the series is smaller then $\frac{c^2}{2M^2}$ for all sufficiently large t, $t \geq t_0$ say. For all those $t \geq t_0$ and all $0 < f \in H$ the series $\sum_{n=1}^{\infty}(f|e_n)e^{-t\lambda_n}e_n$ converges absolutely in H_u and we have $\|\sum_{n=1}^{\infty}(f|e_n)e^{-t\lambda_n}e_n\|_u \leq \frac{c^2}{2M^2}$. Thus,

$$\begin{aligned} e^{tA}f &\geq c^2(f|u)u - \|\sum_{n=1}^{\infty}(f|e_n)e^{-t\lambda_n}\|_u u \\ &\geq c^2(f|u)u - (f|u)M^2\frac{c^2}{2M^2} = \frac{c^2}{2}(f|u)u \gg_u 0. \end{aligned}$$

Hence, the semigroup $(e^{tA})_{t\geq 0}$ is indeed uniformly eventually strongly positive with respect to u. □

It does not seem to be completely clear what causes the uniform eventual strong positivity in Theorem 10.2.1 on an abstract level. A better understanding of this would be important to obtain criteria for uniform eventual strong positivity in more general situations.

10.3 Counterexamples

In Theorem 10.2.1 we showed, under very special assumptions, that individual eventual strong positivity of a semigroup is equivalent to the uniform property. On the other hand, we could not prove such an equivalence in the more general setting of Theorem 9.3.2; similarly, the characterisation theorem for resolvents, Theorem 9.2.6, did not contain any assertions about uniform eventual strong positivity. It is thus natural to ask for counterexamples which show that individual eventual strong positivity with respect to a quasi-interior point u does not in general imply uniform eventual strong positivity with respect to u.

We are going to give two such counterexamples in this section. The first counterexample is rather simple, but the generator in this example is bounded, so we cannot expect the semigroup to have any compactness properties. The second example is technically a bit more involved; in this example, the semigroup generator has compact resolvent and the semigroup itself is analytic and thus immediately compact. Both examples are constructed on $\mathcal{C}(K;\mathbb{C})$-spaces and we choose $u := \mathbb{1} := \mathbb{1}_K$. Thus, the domination and smoothing conditions $D(A) \subseteq E_u$ and $e^{t_0 A}E \subseteq E_u$ are automatically fulfilled in those examples.

For the convenience of the reader we recall that the resolvent of a linear operator $A : E \supseteq D(A) \to E$ on a complex Banach lattice E is called *uniformly eventually positive* at a number $\lambda_0 \in \sigma(A) \cap \mathbb{R} \cup \{-\infty\}$ if there exists a $\lambda_1 > \lambda_0$ with the following properties: $(\lambda_0, \lambda_1] \in \rho(A)$ and $\mathcal{R}(\lambda, A) \geq 0$ for all $\lambda \in (\lambda_0, \lambda_1]$.

Example 10.3.1. There exists a real C_0-semigroup $(e^{tA})_{t\geq 0}$ on $E = \mathcal{C}([-1,1];\mathbb{C})$ with bounded generator $A \in \mathcal{L}(E)$ which has the following properties:

(a) The semigroup $(e^{tA})_{t\geq 0}$ and its generator A fulfil all assumptions of the Theorems 9.3.2 and 9.2.6 (with $u = \mathbb{1}$).

(b) The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$, but not uniformly eventually positive (in particular, it is not uniformly eventually strongly positive with respect to $\mathbb{1}$).

(c) The resolvent $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to $\mathbb{1}$ at $\mathrm{s}(A)$, but not uniformly positive at $\mathrm{s}(A)$ (in particular, it is not uniformly eventually strongly positive with respect to $\mathbb{1}$ at $\mathrm{s}(A)$).

Proof. The construction of A is very similar to our construction in Example 9.3.7, except for the underlying Banach lattice being a different one. We set $E := \mathcal{C}([-1,1];\mathbb{C})$.

Let $\varphi \in E'$ be given by $\langle \varphi, f\rangle = \int_{-1}^{1} f(\omega)\, d\omega$ and consider the decomposition $E = \operatorname{span}\mathbb{1} \oplus F$, where $F = \ker\varphi$. As in Example 9.3.7 we let $S \in \mathcal{L}(F)$ be the reflection operator given by $(Sf)(\omega) = f(-\omega)$ for each $f \in F$ and each $\omega \in [-1,1]$ and, again, we let

$$A = 0_{\operatorname{span}\mathbb{1}} \oplus (-S - 2I_F) \in \mathcal{L}(E).$$

The operator A is real and we have $\sigma(A) = \{-3,-1,0\}$, 0 is a first order pole of the resolvent and the corresponding spectral projection P is given by $Pf = \frac{1}{2}\langle \varphi, f\rangle \mathbb{1}$ for all $f \in E$; clearly, $P \gg_{\mathbb{1}} 0$. One can easily see that the semigroup $(e^{tA})_{t\geq 0}$ is bounded. Moreover, we have $e^{tA}E \subseteq E = E_{\mathbb{1}}$ and $D(A) = E = E_{\mathbb{1}}$ for all $t \geq 0$ and all $\lambda \in \rho(A)$.

So all assumptions of Theorems 9.3.2 and 9.2.6 are fulfilled and, since $P \gg_{\mathbb{1}} 0$, the first of the equivalent assertions in these theorems also hold. Hence, we conclude from these theorems that the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$ and that the resolvent $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to $\mathbb{1}$ at $\mathrm{s}(A) = 0$.

Using that $S^2 = I_F$ we can easily check that the semigroup $(e^{tA})_{t\geq 0}$ and the resolvent $\mathcal{R}(\cdot, A)$ are given by the formulas (9.3) and (9.4) in Example 9.3.7.

Now we show that the semigroup $(e^{tA})_{t\geq 0}$ is not uniformly eventually positive. For each $\varepsilon > 0$ we choose a function $0 < f_\varepsilon \in E$ which fulfils $\|f_\varepsilon\|_\infty = 1$, $\langle \varphi, f_\varepsilon\rangle = \varepsilon$, $f_\varepsilon(1) = 1$ and $f_\varepsilon(-1) = 0$. We can decompose f_ε as $f_\varepsilon = \frac{\varepsilon}{2}\mathbb{1} + (f_\varepsilon - \frac{\varepsilon}{2}\mathbb{1}) \in \langle \mathbb{1}\rangle \oplus F$. Using this and formula (9.3), a straightforward computation shows that

$$(e^{tA} f_\varepsilon)(-1) = \frac{\varepsilon}{2}\Big(1 - e^{-2t}\cosh(t) + e^{-2t}\sinh(t)\Big) - e^{-2t}\sinh(t).$$

for all $t \geq 0$. Thus, for each $t > 0$ we can choose $\varepsilon > 0$ sufficiently small such that $e^{tA} f_\varepsilon \not\geq 0$. Actually, this even shows that e^{tA} is not a positive operator for *any* $t > 0$.

By a similar computation, one can also show that for any $\lambda > 0$ there exists a $\varepsilon > 0$ such that $(\mathcal{R}(\lambda, A) f_\varepsilon)(-1) < 0$. Hence, the resolvent $\mathcal{R}(\cdot, A)$ is not uniformly eventually positive. □

Our second example, although being technically a bit more involved, is based on the same idea as Example 10.3.1 above. Our main point is that

the resolvent of the generator is now compact; moreover, the semigroup $(e^{tA})_{t\geq 0}$ is analytic and thus immediately compact (this follows from [EN00, Theorem II.4.29]).

Example 10.3.2. Let $E := c := c(\mathbb{Z};\mathbb{C})$ be the space of all complex valued sequences $x = (x_n)_{n\in\mathbb{Z}}$, such that the limits $\lim_{n\to\infty} x_n$ and $\lim_{n\to-\infty} x_n$ both exist and coincide. We endow E with the $\|\cdot\|_\infty$-norm and we denote the constant sequence with value 1 in E by $\mathbb{1}$. Note that $E \simeq \mathcal{C}(K;\mathbb{C})$, where K denotes the one-point compactification of the discrete space $\mathbb{Z}$.

There exists a real, analytic C_0-semigroup $(e^{tA})_{t\geq 0}$ on E such that A has compact resolvent and such that the following properties hold:

(a) The semigroup $(e^{tA})_{t\geq 0}$ and its generator A fulfils all assumptions of Theorem 9.3.2.

(b) The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$, but not uniformly eventually positive (in particular, it is not uniformly eventually strongly positive with respect to $\mathbb{1}$).

Proof. The main idea of the construction is the same as in Example 10.3.1, although the details are a bit more involved, now. We decompose $E = c$ as

$$c = \operatorname{span}\mathbb{1} \oplus c_0 = \operatorname{span}\mathbb{1} \oplus c_0^{\mathrm{s}} \oplus c_0^{\mathrm{a}},$$

where $c_0 := \{x \in c | \lim_{n\to\pm\infty} x_n = 0\}$ and where c_0^{s} and c_0^{a} are the *symmetric* and the *anti-symmetric* part of c_0, given by

$$\begin{aligned} c_0^{\mathrm{s}} &:= \{x \in c_0 | x_n = x_{-n} \text{ for all } n \in \mathbb{Z}\},\\ c_0^{\mathrm{a}} &:= \{x \in c_0 | x_n = -x_{-n} \text{ for all } n \in \mathbb{Z}\}. \end{aligned}$$

If $S \in \mathcal{L}(c_0)$ denotes the *reflection operator*, given by $(Sx)_n = x_{-n}$ for all $x \in c_0$ and all $n \in \mathbb{Z}$, then c_0^{s} is the eigenspace of S for the eigenvalue 1 and c_0^{a} is the eigenspace of S for the eigenvalue -1. Note that for each $x \in c_0$ the decomposition in its symmetric and anti-symmetric part is given by

$$x = \frac{1}{2}(x + Sx) + \frac{1}{2}(x - Sx) \in c_0^{\mathrm{s}} \oplus c_0^{\mathrm{a}}.$$

We now choose symmetric sequences $\alpha = (\alpha_n)_{n\in\mathbb{Z}}, \beta = (\beta_n)_{n\in\mathbb{Z}}$ of real numbers which fulfil $0 < \alpha_n < \beta_n$ for all $n \in \mathbb{Z}$ and $\alpha_n, \beta_n \to \infty$ as $n \to \pm\infty$. It is easy to see that we can find another symmetric sequence $g = (g_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z};\mathbb{C})$ which fulfils $g_n > 0$ for all $n \in \mathbb{Z}$, as well as $\langle \mathbb{1}, g\rangle = 1$ and

$$2g_n + e^{-n\beta_n} - e^{-n\alpha_n} < 0 \tag{10.1}$$

for all $n \in \mathbb{Z} \setminus \{0\}$. Consider the Banach space isomorphism $B \in \mathcal{L}(c)$ and its inverse $B^{-1} \in \mathcal{L}(c)$ which are given by

$$B(c\mathbb{1} + x) = \Big(c + \langle g, x\rangle\Big)\mathbb{1} + x, \quad B^{-1}(c\mathbb{1} + x) = \Big(c - \langle g, x\rangle\Big)\mathbb{1} + x$$

for all $x \in c_0$ and all $c \in \mathbb{C}$. Now, let M_β be the multiplication operator with symbol β on c_0^{s} and let M_α be the multiplication operator with symbol α on c_0^{a}. Then $-M_\beta$ and $-M_\alpha$ are closed operators with compact resolvents and generate analytic semigroups on c_0^{s} and c_0^{a}, respectively. Now, consider the C_0-semigroup $(I_{\operatorname{span}\mathbb{1}} \oplus e^{-tM_\beta} \oplus e^{-tM_\alpha})_{t\geq 0}$ on $c = \operatorname{span}\mathbb{1} \oplus c_0^{\mathrm{s}} \oplus c_0^{\mathrm{a}}$. By means of the Banach space isomorphism $B \in \mathcal{L}(c)$, we apply a similarity transformation to this semigroup and thus obtain a C_0-semigroup $(e^{tA})_{t\geq 0}$ on c given by

$$e^{tA} := B^{-1}\Big(I_{\operatorname{span}\mathbb{1}} \oplus e^{-tM_\beta} \oplus e^{-tM_\alpha}\Big)B.$$

This semigroup has the generator $A = -B^{-1}\Big(0_{\operatorname{span}\mathbb{1}} \oplus M_\beta \oplus M_\alpha\Big)B$. Clearly, A has compact resolvent, 0 is a dominant spectral value of A and one can easily check that the spectral projection P is given by $P(c\mathbb{1} + x) = (c + \langle g, x\rangle)\mathbb{1}$ for all $c \in \mathbb{C}$ and all $x \in c_0$. Hence, $P \gg_{\mathbb{1}} 0$. Moreover, the semigroup generated by A is bounded.

We also have $D(A) \subseteq c = c_{\mathbb{1}}$ and $e^{tA}c \subseteq c = c_{\mathbb{1}}$ for all $t \geq 0$. Hence, all assumptions of Theorem 9.3.2 are fulfilled and assertion (i) of the theorem holds, too. We thus conclude that the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$.

Let us now show that the semigroup $(e^{tA})_{t\geq 0}$ is not uniformly eventually positive. Using the formulas for B and B^{-1} given above, we can compute the orbit $(e^{tA}x)_{t\geq 0}$ for every $x \in c_0$: for each such x and each $t \geq 0$ we obtain after a short computation

$$e^{tA}x = \frac{1}{2}\Big(\langle g, 2x - e^{-tM_\beta}(x + Sx)\rangle\mathbb{1} + e^{-tM_\beta}(x + Sx) + e^{-tM_\alpha}(x - Sx)\Big).$$

For $x \geq 0$, this yields

$$e^{tA}x \leq \frac{1}{2}\Big(2\langle g, x\rangle\mathbb{1} + e^{-tM_\beta}(x + Sx) + e^{-tM_\alpha}(x - Sx)\Big)$$

for all $t \geq 0$. Now, fix $t_0 > 0$ and choose an integer $n \geq t_0$. We define $x := \mathbb{1}_{\{n\}}$ and thus, we obtain for the $-n$-the component of $e^{tA}x$ the estimate

$$(e^{tAx})_{-n} \leq \frac{1}{2}\Big(2g_n + e^{-t\beta_n} - e^{-t\alpha_n}\Big).$$

According to the estimate (10.1), this number is strictly negative if we choose $t = n$. Hence, for each $t_0 \geq 0$ we can find a time $t := n \geq t_0$ such that e^{tA} is not a positive operator. This proves, as claimed, that the semigroup $(e^{tA})_{t\geq 0}$ is not uniformly eventually positive. □

10.4 Notes & Open Problems

Uniform anti-maximum principles In Section 9.2 we have already pointed out that individual eventual strong negativity of resolvents corresponds to so-called *individual anti-maximum principles*. On the other hand, *uniform* eventual strong negativity of the resolvent corresponds to uniform anti-maximum principles. Unfortunately, we do at the moment not have an analogue of Theorem 9.2.6 for uniform eventual strong positivity available; even in the setting of self-adjoint operators on Hilbert lattices the technique presented in Theorem 10.2.1 to prove uniform eventual strong positivity for semigroups does not seem to work for resolvents. Moreover, the results from Section 10.1 cannot be used to derive uniform anti-maximum principles, either, since they only work on the right side of a given spectral value λ_0.

On the other hand, we should point out that one can find some results about uniform anti-maximum principles in the literature, see e.g. [Tak96, Section 5], [CS00] and [CS01]; the first reference contains some abstract theorems, while the latter two references focus on elliptic differential operators. It is an interesting task for future research to find more results which help to understand uniform anti-maximum principles on an abstract level.

11 Applications of Eventual Strong Positivity

In this final chapter we give a large variety of examples for semigroups which behave eventually strongly positive. In Section 11.1 we revisit the finite-dimensional case and show that our general theory does not only yield known theorems as special cases, but it is even able to strengthen some results known from the finite-dimensional literature. In Section 11.2 we revisit the Dirichlet-to-Neumann operator, but this time in two dimensions. We demonstrate that one can even obtain results about eventual strong positivity instead of only asymptotic positivity in this setting. In Section 11.3 we make a brief detour to the Dirichlet-to-Neumann operator on graphs, though only in a simple special case. Our results in this section are somewhat different from our other applications in that we show that, under some circumstances, eventual positivity does *not* occur for the corresponding semigroup. The question whether it *does* occur under different assumptions seems to be difficult to answer, given the current state of the theory. In Section 11.4 we come back to the bi-Laplace operator with Dirichlet boundary conditions. We show under rather weak assumptions that the resolvent as well as the generated semigroup behaves eventually strongly positive. This does not only resemble some results from the literature but, again, it even strengthens some of them, merely by the use of our abstract theory. Another type of bi-Laplace operator, namely the square of the Dirichlet– and the Robin–Laplace operator, are considered in Section 11.5. In contrast to the preceding sections we do not work on L^p-spaces now, but on spaces of continuous functions instead. This demonstrate that our theory is well-adapted to applications on such spaces, too. Delay differential equations are revisited in Section 11.6. While we have already given sufficient conditions for the generated semigroup to be asymptotically positive in Chapter 8, we now show that the semigroup is even eventually strongly positive under slightly stronger assumptions. In the final Section 11.7 we consider two examples of a (one-dimensional) Laplace operator with non-local boundary conditions. While it is well-known that the Laplace operator with several kinds of local boundary conditions generates a positive semigroup, the situation seems to be much more involved for non-local boundary conditions. We give two concrete examples where the generated semigroup is not positive, but eventually strongly positive.

Most of the results presented in this chapter have been published in [DGK16b, Section 6] or [DGK16a, Section 6], some of them only in weaker versions since Theorem 10.2.1 was not yet available when those papers were

written. The results in Section 11.3 have not been published yet and are part of an ongoing research project in collaboration with Daniel Daners and James B. Kennedy. Only a special case of the results from Section 11.6 was published in [DGK16b], namely the special case presented in Example 11.6.3.

11.1 The Finite-Dimensional Case

As in Chapter 8 we start our grand tour of applications by considering the finite-dimensional case. We have already explained in Section 8.1 that every finite-dimensional complex Banach lattice is isomorphic to $\mathbb{C}^n$, so we assume for the rest of the section that $\mathbb{R}^d$ is endowed with some Banach lattice norm and that $\mathbb{C}^d$ is endowed with the Banach lattice compexification of this norm.

Let $\mathbb{1} \in \mathbb{R}^d$ denote the vector whose entries are all equal to 1. We have $f \gg 0$ for a vector $f \in \mathbb{R}^d$ if and only if $f \gg_{\mathbb{1}} 0$; hence, we have $A \gg 0$ for a matrix $A \in \mathbb{R}^{d\times d}$ if and only if $A \gg_{\mathbb{1}} 0$. Obviously, $f \gg 0$ (respectively, $A \gg 0$) if and only if all entries of f (respectively, of A) are > 0.

It is easy to see that for every $A \in \mathbb{R}^{d\times d}$ the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$ if and only if it is uniformly eventually strongly positive with respect to $\mathbb{1}$. In this case we shall simply say that $(e^{tA})_{t\geq 0}$ is *eventually strongly positive*.

It is again our intention to describe eventual positivity properties of the semigroup by considering the powers of (a shift of) its generator. We therefore start with the following definition:

Definition 11.1.1 (Eventually strongly positive matrix). A matrix $A \in \mathbb{R}^{d\times d}$ is called *eventually strongly positive* if there is an integer $n_0 \in \mathbb{N}$ such that $A^n \gg 0$ for all $n \geq n_0$.

Let us proceed right away by giving the main result of this section; a similar result has been obtained by Noutsos and Tsatsomeros in [NT08, Theorem 3.3] by using only finite-dimensional theory; in fact, their theorem is stated in a slightly different way, but it is not difficult to see that it is, with a small exception, equivalent to our result below. The only major difference is that, in our theorem, it suffices to assume that the eigenvalue $\mathrm{s}(A)$ is geometrically (instead of algebraically) simple in assertion (ii) and that we then obtain algebraic simplicity for free.

Theorem 11.1.2. *For every matrix $A \in \mathbb{R}^{d\times d}$ the following assertions are equivalent:*

(i) *The semigroup $(e^{tA})_{t\geq 0}$ is eventually strongly positive.*

(ii) *The spectral bounded* $\mathrm{s}(A)$ *is a dominant and geometrically simple eigenvalue of* A*; moreover,* $\ker(\mathrm{s}(A)-A)$ *and* $\ker(\mathrm{s}(A)-A^T)$ *both contain a vectors which are* $\gg 0$.

(iii) *There exists a number* $c \in \mathbb{R}$ *such that the matrix* $A+c$ *is eventually strongly positive.*

(iv) *For all sufficiently large* $c \in \mathbb{R}$ *the matrix* $A+c$ *is eventually strongly positive.*

If the equivalent assertions (i)–(iv) *are fulfilled, then* $\mathrm{s}(A)$ *is even an algebraically simple eigenvalue of* A *and, moreover, the only eigenvalue of* A *which admits a positive eigenvector.*

Proof. "(i) $\Leftrightarrow$ (ii)" This follows from the equivalence "(i) $\Leftrightarrow$ (ii)" in Theorem 9.3.5 and the equivalence "(i) $\Leftrightarrow$ (ii)" in Proposition 9.1.1. Moreover, the same proposition yields that $\mathrm{s}(A)$ is even an algebraically simple eigenvalue of A, and moreover the only eigenvalue with a positive eigenvector, if the equivalent conditions of the current theorem are fulfilled.

"(ii) $\Rightarrow$ (iv)" If assertion (ii) holds, then the spectral projection P associated with $\mathrm{s}(A)$ is strongly positive according to Proposition 9.1.1. Since $\mathrm{s}(A)$ is a dominant spectral value of A, we conclude for all sufficiently large $c \in \mathbb{R}$ that the spectral bound of the matrix $A+c$ is strictly positive, that it is even equal to the spectral radius of $A+c$ and, moreover, that it is the only eigenvalue of $A+c$ with maximal modulus. For such $c \in \mathbb{R}$, let $T := \frac{A+c}{\mathrm{r}(A+c)} = \frac{A+c}{\mathrm{s}(A+c)}$.

The matrix T has spectral radius 1, and 1 is the only eigenvalue of T on the unit circle; the associated spectral projection is still given by $P \gg 0$. In particular, 1 is a first order pole the resolvent $\mathcal{R}(\cdot, T)$ according to Proposition 9.1.1, so the range of P coincides with the fixed space of T. Hence, T^n converges to $P \gg 0$ is $n \to \infty$, so we conclude that $T^n \gg 0$ for all sufficiently large n. Thus, $(A+c)^n \gg 0$ for all sufficiently large n.

"(iv) $\Rightarrow$ (iii)" This is obvious.

"(iii) $\Rightarrow$ (i)" Assume that the matrix $B := A+c$ is eventually strongly positive for some $c \in \mathbb{R}$. We have $e^{tA} = e^{-tc}e^{tB}$ for all $t \geq 0$, so it suffices to show that the semigroup $(e^{tB})_{t\geq 0}$ is eventually strongly positive.

Choose $n \in \mathbb{N}_0$ such that $B^n \gg 0$ for all $n \geq n_0$. For $t > 0$ we have

$$t^{-n_0}e^{tB} \geq B^{n_0} + \sum_{n=0}^{n_0-1} t^{k-n_0}B^k \to B^{n_0} \gg 0$$

as $t \to \infty$. Hence, $t^{-n_0}e^{tB} \gg 0$ for all sufficiently large $t > 0$ and this proves that $e^{tB} \gg 0$ for all sufficiently large $t > 0$, too. □

In Proposition 8.1.6 we showed that a real, asymptotically positive semigroup in two dimensions is automatically positive and in Proposition 8.1.7

such a result was proved for eventually positive semigroups. In the following proposition we prove a similar result, namely that an eventually strongly positive semigroup in two dimensions is automatically strongly positive:

Proposition 11.1.3. *Let $A \in \mathbb{R}^{2\times 2}$. The semigroup $(e^{tA})_{t\geq 0}$ is eventually strongly positive if and only if it is* strongly positive, *meaning that $e^{tA} \gg 0$ for all $t > 0$.*

Proof. "$\Leftarrow$" This implication is obvious.

"$\Rightarrow$" Assume that $(e^{tA})_{t\geq 0}$ is eventually strongly positive. According to Theorem 11.1.2, $\lambda_0 := \mathrm{s}(A)$ is an algebraically simple eigenvalue of A. Moreover, the corresponding spectral projection P is strongly positive according to Theorem 9.3.2.

Since $\mathbb{R}^2$ is two-dimensional, the second eigenvalue λ_1 of A is also algebraically simple; moreover, it is real since A is so, and of course we have $\lambda_1 < \lambda_0$. The spectral projection corresponding to λ_1 is $I-P$, and its range coincides with the eigenspace $\ker(\lambda_1 - A)$. Hence, we have

$$e^{tA} = e^{t\lambda_0}P + e^{t\lambda_1}(I-P) = e^{t\lambda_1}I + (e^{t\lambda_0} - e^{t\lambda_1})P \gg 0$$

for all $t > 0$ since $\lambda_0 > \lambda_1$. □

11.2 The Dirichlet-to-Neumann Operator in Dimension 2

Now we revisit the Dirichlet-to-Neumann operator $\mathcal{D}_\lambda$ discussed in Section 8.2, but this time in the two-dimensional case $\Omega \subseteq \mathbb{R}^2$. Considering the two-dimensional case has the advantage that the boundary of Ω, if assumed to be sufficiently smooth, is a one-dimensional compact manifold. Since it is known that the domain of the Dirichlet-to-Neumann operator is a first order Sobolev space over $\partial\Omega$ (see [BtE15, Theorem 5.2]), it thus follows that it embeds into $L^\infty(\partial\Omega;\mathbb{C})$ which gives us exactly the kind of domination condition which we need to prove eventual strong positivity instead of asymptotic positivity. Our main result reads as follows.

Theorem 11.2.1. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^\infty$-boundary and let $\lambda \in \mathbb{R} \setminus \sigma(\Delta_D)$. Then the following assertions are equivalent:*

(i) *The Dirichlet-to-Neumann semigroup $(e^{-t\mathcal{D}_\lambda})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}_{\partial\Omega}$.*

(ii) *The Dirichlet-to-Neumann semigroup $(e^{-t\mathcal{D}_\lambda})_{t\geq 0}$ is uniformly eventually strongly positive with respect to $\mathbb{1}_{\partial\Omega}$.*

(iii) *The spectral bound $\mathrm{s}(-\mathcal{D}_\lambda)$ is a geometrically simple eigenvalue of $-\mathcal{D}_\lambda$ and the corresponding eigenspace contains a function which is $\gg_{\mathbb{1}_{\partial\Omega}} 0$.*

(iv) *The resolvent* $\mathcal{R}(\cdot, -\mathcal{D}_\lambda)$ *is individually eventually strongly positive with respect to* $\mathbb{1}_{\partial\Omega}$ *at* $\mathrm{s}(-\mathcal{D}_\lambda)$.

(v) *The resolvent* $\mathcal{R}(\cdot, -\mathcal{D}_\lambda)$ *is individually eventually strongly negative with respect to* $\mathbb{1}_{\partial\Omega}$ *at* $\mathrm{s}(-\mathcal{D}_\lambda)$.

We note in passing that the eigenvector of $-\mathcal{D}_\lambda$ belonging to $\mathrm{s}(-\mathcal{D}_\lambda)$ can be described by a corresponding Robin problem as was done in Theorem 8.2.1.

Proof of Theorem 11.2.1. We have $D(\mathcal{D}_\lambda) = H^1(\partial\Omega;\mathbb{C})$ according to [BtE15, Theorem 5.2]. Since Ω is a bounded two-dimensional domain with $\mathcal{C}^\infty$-boundary, its boundary is a one-dimensional smooth and compact manifold. Thus, we have

$$D(\mathcal{D}_\lambda) = H^1(\partial\Omega;\mathbb{C}) \subseteq L^\infty(\partial\Omega;\mathbb{C}) = L^2(\partial\Omega;\mathbb{C})_{\mathbb{1}_{\partial\Omega}}.$$

Since the above domination condition is fulfilled and since the Dirichlet-to-Neumann operator is self-adjoint, we note the all assumptions of Theorem 10.2.1 hold. Hence, this theorem immediately implies the equivalence of assertions (i), (ii) and (iii). Moreover, the same theorem yields that $\mathrm{s}(A)$ is a pole of the resolvent and that the assertions (i)–(iii) are equivalent to the condition $P \gg_{\mathbb{1}_{\partial\Omega}} 0$; this latter condition is in turn equivalent to assertions (iv) and (v) according to Theorem 9.2.6. □

We point out that the above theorem is far from giving a complete answer to the question whether the semigroup generated by the Dirichlet-to-Neumann operator is eventually strongly positive. In fact, the major achievement in Theorem 11.2.1 is to show that the problem can be completely reduced to a spectral theoretic question. Yet, it does not seem to be clear at all under which conditions on Ω the spectral condition (iii) in the theorem is fulfilled.

One special case in which this is understood quite properly is that of the unit disk in $\mathbb{R}^2$. This example as studied by Daners in [Dan14] where he proved by means of concrete computations that condition (iii) is fulfilled for certain choices of the parameter λ. Since the abstract characterisation in Theorem 10.2 was not known by then, he used more explicit properties of Fourier series on the unit circle in order to conclude that the semigroup is uniformly eventually positive. In this context it is worthwhile recalling that, in the proof of Theorem 10.2, we employed the spectral theorem for compact self-adjoint operators and the series expansion of vectors along an orthonormal basis of the underlying Hilbert space. This can, of course, be considered as some kind of abstract Fourier series argument.

11.3 The Dirichlet-to-Neumann Operator on Finite Graphs

As explained in the preceding section it seems, in general, to be difficult to find out for a given domain $\Omega \subseteq \mathbb{R}^2$ whether the spectral condition (iii) in Theorem 11.2.1 is fulfilled. Seeking for a better understanding of eventual positivity phenomena related to Dirichlet-to-Neumann operators, it is therefore natural to study similar operators in different settings. Thus, we use the current section for a brief look at the Dirichlet-to-Neumann operator on finite graphs and its eventual positivity properties. Our results are rather elementary; we study only graphs without interior vertices (i.e. without vertices where Kirchhoff conditions are imposed on the graph) and our main result actually states that, for many choices of the paramater λ, eventual positivity does *not* occur. The question whether it occurs for different choices of λ seems to require some kind of perturbation theory for eventually positive semigroup which is not yet available at the current state of the theory. We point out, however, that the development of such a theory is part of an ongoing project with Daniel Daners; moreover, the results presented in this section are part of a collaboration with Daniel Daners and James B. Kennedy where we also intend to generalise those results below to graphs with interior vertices.

For our analysis we only need some elementary properties of metric graphs; for a more detailed discussion of analysis on metric graphs (often referred to as "quantum graphs") we refer to the extensive literature on the topic, e.g. to the monograph [Mug14] and to the survey article [Kuc08]). Throughout the section, assume that the following data is given:

- A undirected, connected graph $G = (V, E)$ with vertex set V and a finite set of edges E; we assume that V has at least two elements.

  Moreover, we assume that each two vertices are connected by at most one edge and that there are no loops in G, i.e. that no vertex is connected to itself by a single edge.

- Despite the fact that G is undirected, assume that for each edge $e \in E$ one of the vertices connected by e is called the *left endpoint* of e and is denoted by e_l, while the other vertex is called the *right endpoint* of e and is denoted by e_r.

  The assignment of e_l and e_r is arbitrary and is only needed to assign each edge to a real interval later on.

- A function $L : E \to (0, \infty)$; for every $e \in E$ we interpret $L_e := L(e)$ as the length of the edge e.

Whenever two vertices v and w are connected by a (necessarily unique) edge e, we use the notation $L_{v,w} := L_e$. We define $L_{v,w} := \infty$ whenever there is no edge between v and w. If we define $\frac{1}{\infty} := 0$, then the matrix $(\frac{1}{L_{v,w}})_{v,w\in V} \in \mathbb{R}^{V\times V}$ is called the *weighted adjacency matrix* of the graph G.

Throughout the section we assume $\mathbb{C}^V$ to be endowed with the Euclidean norm which renders it a complex Hilbert lattice.

The Dirichlet-to-Neumann operator on G is an operator on $\mathbb{C}^V = L^2(V;\mathbb{C})$ which is defined in a similar fashion as the Dirichlet-to-Neumann operator on (the boundary of) domains in $\mathbb{R}^d$.

Definition 11.3.1 (Dirichlet-to-Neumann operator on graphs). Let $f \in \mathbb{C}^V$ and let $\lambda \in \mathbb{R} \setminus \{-(\frac{\pi k}{L_e})^2 : k \in \mathbb{N}\, e \in E\}$. For every $e \in E$ let $u_{e,f}$ be the function in $H^2((0,L_e);\mathbb{C})$ solving the equation

$$\begin{cases} u'' & = \lambda u \quad \text{in } (0,L_e) \\ u(0) & = f(e_{\mathrm{l}}) \\ u(L_e) & = f(e_{\mathrm{r}}) \end{cases}$$

(which exists and is uniquely determined due to the choice of λ). We then define $g_f \in \mathbb{C}^V$ by

$$g_f(v) = \sum_{e\in E:\, e_{\mathrm{l}}=v} -u'_{e,f}(0) + \sum_{e\in E:\, e_{\mathrm{r}}=v} u'_{e,f}(L_e),$$

i.e. for every edge e which meets v we compute the outer normal derivative of $u_{e,f}$ at v and sum up all those derivatives.

Then the linear operator $\mathcal{D}_\lambda : \mathbb{C}^V \to \mathbb{C}^V$, $f \mapsto g_f$ is called the *Dirichlet-to-Neumann operator* on the graph G with spectral parameter λ.

In our simple setting, it is not difficult to compute the Dirichlet-to-Neumann operator explicitly:

Proposition 11.3.2. *Let $\lambda \in \mathbb{R} \setminus \{-(\frac{\pi k}{L_e})^2 : k \in \mathbb{N},\, e \in E\}$. Then the representation matrix of the Dirichlet-to-Neumann operator $\mathcal{D}_\lambda$ with respect to the canonical basis of $\mathbb{C}^V$ is given by the following formulas:*

(a) *If $\lambda = 0$, then*

$$(\mathcal{D}_\lambda)_{v,w} = \begin{cases} \frac{-1}{L_{v,w}} & \text{if } v \neq w \text{ and } L_{v,w} \neq \infty \\ 0 & \text{if } v \neq w \text{ and } L_{v,w} = \infty \\ \sum_{u\in V:\, L_{v,u}\neq\infty} \frac{1}{L_{v,u}} & \text{if } v = w. \end{cases}$$

(b) *If* $\lambda < 0$ *and* $\mu := \sqrt{-\lambda}$, *then*

$$(\mathcal{D}_\lambda)_{v,w} = \begin{cases} \frac{-\mu}{\sin(\mu L_{v,w})} & \text{if } v \neq w \text{ and } L_{v,w} \neq \infty \\ 0 & \text{if } v \neq w \text{ and } L_{v,w} = \infty \\ \sum_{u \in V:\, L_{v,u} \neq \infty} \frac{\mu \cos(\mu L_{v,u})}{\sin(\mu L_{v,u})} & \text{if } v = w. \end{cases}$$

(c) *If* $\lambda > 0$ *and* $\mu := \sqrt{\lambda}$, *then*

$$(\mathcal{D}_\lambda)_{v,w} = \begin{cases} \frac{-\mu}{\sinh(\mu L_{v,w})} & \text{if } v \neq w \text{ and } L_{v,w} \neq \infty \\ 0 & \text{if } v \neq w \text{ and } L_{v,w} = \infty \\ \sum_{u \in V:\, L_{v,u} \neq \infty} \frac{\mu \cosh(\mu L_{v,u})}{\sinh(\mu L_{v,u})} & \text{if } v = w. \end{cases}$$

Proof. The proof is a straightforward computation, so we omit the details. □

Let us now state the main result of this section which makes a positive assertion about certain *positivity* properties of the semigroup $(e^{-\mathcal{D}_\lambda})_{t \geq 0}$ and a partially negative assertion about *eventual positivity* properties.

Theorem 11.3.3. *Let* $\sigma := \{-(\frac{\pi k}{L_e})^2 : k \in \mathbb{N}\ e \in E\}$. *The semigroup generated by minus the Dirichlet-to-Neumann operator has the following properties:*

(a) *If* $\lambda > \max \sigma$, *then the semigroup* $(e^{-t\mathcal{D}_\lambda})_{t \geq 0}$ *is strongly positive, meaning that* $e^{-t\mathcal{D}_\lambda} \gg 0$ *for all* $t > 0$.

(b) *Let* $\lambda_0 \in \mathbb{R}$. *Then there exists a number* $\lambda \in \mathbb{R} \setminus \sigma$, $\lambda \leq \lambda_0$ *such that the semigroup* $(e^{-t\mathcal{D}_\lambda})_{t \geq 0}$ *is strongly positive.*

(c) *Let* $\lambda_0 \in \mathbb{R}$. *Then there exists a number* $\lambda \in \mathbb{R} \setminus \sigma$, $\lambda \leq \lambda_0$ *such that the semigroup* $(e^{-t\mathcal{D}_\lambda})_{t \geq 0}$ *is not eventually positive.*

Proof. (a) If $\lambda > \max \sigma$, then it follows from Proposition 11.3.2 that $(-\mathcal{D}_\lambda)_{v,w} > 0$ if $v \neq w$ and $L_{v,w} \neq \infty$, and that $(-\mathcal{D}_\lambda)_{v,w} = 0$ if $v \neq w$ and $L_{v,w} = \infty$. Hence, all off-diagonal entries of $-\mathcal{D}_\lambda$ are ≥ 0, so $-\mathcal{D}_\lambda$ generates a positive semigroup. Choose $c \geq 0$ such that $-\mathcal{D}_\lambda + c$ is positive. Since the graph G is connected, it follows that $-\mathcal{D}_\lambda + c$ is irreducible, so $e^{t(-\mathcal{D}_\lambda + c)} \gg 0$ for all $t > 0$ and hence, $e^{-t\mathcal{D}_\lambda} \gg 0$ for all $t > 0$.

To show the remaining two assertions, we first make the following observation: if $\lambda < 0$, then the Dirichlet-to-Neumann operator is given by Proposition 11.3.2(b). Then signs of the off-diagonal entries of $-\mathcal{D}_\lambda$ are determined by the values $\sin(\mu L_e)$, where $\mu = \sqrt{-\lambda}$ and $e \in E$. We now show the following claim:

Claim: For every $\mu_0 \geq 0$ *there exist* $\mu_1, \mu_2 \geq \mu_0$ *such that* $\sin(\mu_1 L_e) > 0$ *and* $\sin(\mu_2 L_e) < 0$ *for all* $e \in E$.

To prove the claim, first note that if we choose two fixed numbers $\tilde{\mu}_1 > 0$ and $\tilde{\mu}_2 < 0$ which are sufficiently close to 0, then $\sin(\tilde{\mu}_1 L_e) > 0$ and $\sin(\tilde{\mu}_2 L_e) < 0$ for all $e \in E$. Now, consider the complex numbers $\lambda_e := \exp(iL_e)$ for $e \in E$. Those numbers are contained in the complex unit circle $\mathbb{T}$, and the vector $\lambda := (\lambda_e)_{e\in E}$ is an element of the compact group $\mathbb{T}^E$. It follows from Proposition 7.3.1 that we can find a sequence of integers $1 \leq n_k \to \infty$ such that $\lambda^{n_k} \to (1, ..., 1) \in \mathbb{T}^E$. Hence, we obtain for all $e \in E$

$$\sin((\tilde{\mu}_1 + n_k)L_e) = \operatorname{Im}\Big(\exp(i\tilde{\mu}_1 L_e)\lambda_e^{n_k}\Big) \to \operatorname{Im}\Big(\exp(i\tilde{\mu}_1 L_e)\Big) = \sin(\tilde{\mu}_1 L_e) > 0$$

for all $e \in E$ and, similarly,

$$\sin((\tilde{\mu}_2 + n_k)L_e) \to \sin(\tilde{\mu}_2 L_e) < 0$$

for all $e \in E$. Hence, we only have to choose $\mu_1 := \tilde{\mu}_1 + n_k$ and $\mu_2 := \tilde{\mu}_2 + n_k$ for some sufficiently large index k and thus obtain the claim.

Now we can prove the remaining two assertions:

(b) We may assume that $\lambda_0 \leq 0$. Let $\mu_0 := \sqrt{-\lambda_0}$, choose $\mu_1 \geq \mu_0$ as in the above claim and set $\lambda := -\mu_1^2$. Since $\sin(\mu_1 L_e) > 0$ for all $e \in E$, it follows that all off-diagonal entries of $-\mathcal{D}_\lambda$ are ≥ 0 and that $(-\mathcal{D}_\lambda)_{v,w}$ is even > 0 whenever $v \neq w$ and $L_{v,w} < \infty$. As in the proof of assertion (a) we conclude that $e^{-t\mathcal{D}_\lambda} \gg 0$ for all $t > 0$.

(c) Again, we may assume that $\lambda_0 \leq 0$ and we set $\mu_0 := \sqrt{-\lambda_0}$. Choose $\mu_2 \geq \mu_0$ as in the above claim and set $\lambda := -\mu_2^2$. Since $\sin(\mu_2 L_e) < 0$ for all $e \in E$, it follows that all off-diagonals entries of $\mathcal{D}_\lambda$ are ≥ 0 and that $(\mathcal{D}_\lambda)_{v,w}$ is even > 0 whenever $v \neq w$ and $L_{v,w} < \infty$. With a similar argument as in the proof of the previous assertion we now conclude that the semigroup generated by *plus* the operator $\mathcal{D}_\lambda$ is strongly positive.

Let $\eta_{\max}$ and $\eta_{\min}$ denote the largest and the smallest eigenvalue of $\mathcal{D}_\lambda$, respectively. We conclude from Theorem 11.1.2 that $\eta_{\max}$ is an algebraically simple eigenvalue of $\mathcal{D}_\lambda$ and that it is the only eigenvalue of $\mathcal{D}_\lambda$ which admits a positive eigenvector.

Now, assume for a contradiction that the semigroup generated by $-\mathcal{D}_\lambda$ is eventually positive. Then it follows from Theorem 6.2.5 and Proposition 6.2.7 that the spectral bound of $-\mathcal{D}_\lambda$, which coincides with $-\eta_{\min}$, admits of a positive eigenvector. Thus, the eigenvalue $\eta_{\min}$ of $\mathcal{D}_\lambda$ admits a positive eigenvector. However, we have observed before that $\eta_{\max}$ is the only eigenvalue of $\mathcal{D}_\lambda$ with this property. Hence, $\eta_{\max} = \eta_{\min}$, and we thus conclude $\eta_{\max}$ is the only eigenvalue of $\mathcal{D}_\lambda$. Yet, we have already seen that the eigenvalue $\eta_{\max}$ of $\mathcal{D}$ is algebraically simple, so the underlying space $\mathbb{C}^V$ is one-dimensional. This is a contradiction since we assumed that our graph has at least two vertices. □

11.4 The bi-Laplace Operator with Dirichlet Boundary Conditions

In this section we revisit the bi-Laplace operator with Dirichlet boundary conditions which we have already considered in Section 8.3 and which is given by (8.1). But this time, we focus on eventual strong positivity instead of asymptotic positivity.

The following result strengthens the assertion of Proposition 8.3.4 considerably in case that $p > \frac{n}{2}$. It is a generalisation of a result contained in [GS98, Proposition 5.3] where a similar assertion was proved under the additional assumption that $p > 2$ (note, however, that [GS98, Proposition 5.3] also contains information about higher order polyharmonic operators).

Proposition 11.4.1. *Let $1 < p < \infty$, let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^\infty$-boundary and let $u \in L^p(\Omega;\mathbb{C})$ be given by $u(\omega) = \operatorname{dist}(\omega, \partial\Omega)^2$ for all $\omega \in \Omega$. If the conclusion of Theorem 8.3.2 holds and if $p > \frac{d}{2}$, then the resolvent $\mathcal{R}(\cdot, A_p)$ is individually eventually strongly positive with respect to u at $\mathrm{s}(A_p)$ and individually eventually strongly negative with respect to u at $\mathrm{s}(A_p)$.*

Proof. According to Corollary 8.3.3, the spectral bound $\mathrm{s}(A_p)$ is a first order pole of the resolvent and the corresponding spectral projection P fulfils $P \gg_u 0$. Due to Theorem 9.2.6 it thus suffices to prove that the domination condition $D(A_p) \subseteq L^p(\Omega;\mathbb{C})_u$ is fulfilled.

Since $p > \frac{n}{2}$ we have $D(A_p) \subseteq W^{4,p}(\Omega;\mathbb{C}) \hookrightarrow \mathcal{C}^2(\overline{\Omega};\mathbb{C})$. By the definition of $D(A_p)$ we thus obtain $D(A_p) \subseteq \mathcal{C}^2(\overline{\Omega};\mathbb{C}) \cap W_0^{2,p}(\Omega;\mathbb{C})$, and it follows that the trace and the gradient of every $f \in D(A_p)$ are 0 on $\partial\Omega$ in the classical sense. Thus, $D(A_p) \subseteq L^p(\Omega;\mathbb{C})_u$. □

Let us now also give a result about the semigroup generated by A_p. It seems that results in the literature about eventual positivity of bi-harmonic heat equation focus mainly on the case $\Omega = \mathbb{R}^n$; see e.g. [FGG08] and [GG08]. On the other hand, it is not difficult to prove quite strong results on bounded domains which are sufficiently close to the unit ball; this is due to the fact that our general theory directly relates eventual positivity properties of the elliptic problem to eventual positivity properties of the parabolic problem.

It is interesting that we do not need the above assumption $p > \frac{n}{2}$ if we consider the semigroup instead of the resolvent. This is due to the fact that our characterisation of eventually strongly positive semigroups in Theorem 9.3.2 only needs the assumption $e^{t_0 A}E \subseteq E_u$ which is, in many cases, weaker than the assumption $D(A) \subseteq E_u$ needed in our result about resolvents in Theorem 9.2.6 (compare Remark 9.3.4).

Theorem 11.4.2. *Let $1 < p < \infty$, let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^\infty$-boundary and let $u \in L^p(\Omega;\mathbb{C})$ be given by $u(\omega) = \operatorname{dist}(\omega, \partial\Omega)^2$ for all $\omega \in \Omega$. If the conclusion of Theorem 8.3.2 holds, then the C_0-semigroup $(e^{tA_p})_{t\geq 0}$ on $L^p(\Omega;\mathbb{C})$ is individually eventually strongly positive with respect to u.*

Proof. It follows from Corollary 8.3.3 that $\mathrm{s}(A_p)$ is a dominant spectral value of A_p and that the corresponding spectral projection P fulfils $P \gg_u 0$. Hence, the assertion follows from Corollary 9.3.5 if we can prove that $e^{t_0 A_p} L^p(\Omega;\mathbb{C}) \subseteq L^p(\Omega;\mathbb{C})_u$ for some $t_0 \geq 0$. Since the semigroup $(e^{tA_p})_{t\geq 0}$ is analytic, this condition is fulfilled, according to Remark 9.3.4(b), if we have $D(A_p^\infty) \subseteq L^p(\Omega;\mathbb{C})_u$.

The latter domination condition is indeed fulfilled: we conclude from [GGS10, Corollary 2.21] that $D(A_p^n) \subseteq W^{4n,p}(\Omega;\mathbb{C})$ for each $n \in \mathbb{N}$. Hence, for sufficiently large n, we obtain $D(A_p^n) \subseteq \mathcal{C}^2(\overline{\Omega};\mathbb{C})$ and, by the definition of $D(A_p)$, thus $D(A_p^n) \subseteq \mathcal{C}^2(\overline{\Omega};\mathbb{C}) \cap W_0^{2,p}(\Omega;\mathbb{C})$. Hence, the trace and the normal derivative of every $f \in D(A_p^n)$ are 0 on $\partial\Omega$ in the classical sense, so we have indeed $D(A_p^n) \subseteq L^p(\Omega;\mathbb{C})_u$ if we choose $n \in \mathbb{N}$ sufficiently large. □

In the case $p = 2$ we can, due to Theorem 10.2.1, even prove uniform eventual strong positivity of the semigroup:

Theorem 11.4.3. *Let $p = 2$, let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^\infty$-boundary and let $u \in L^2(\Omega;\mathbb{C})$ be given by $u(\omega) = \operatorname{dist}(\omega, \partial\Omega)^2$ for all $\omega \in \Omega$. If the conclusion of Theorem 8.3.2 holds, then the C_0-semigroup $(e^{tA_2})_{t\geq 0}$ on $L^2(\Omega;\mathbb{C})$ is uniformly eventually strongly positive with respect to u.*

Proof. The proof is the same as the proof of Theorem 11.4.2, except for the fact that we can now employ the stronger Theorem 10.2.1 since the underlying space is a Hilbert lattice and since A_2 is self-adjoint. □

11.5 Squares of Laplace Operators

In this section we study eventual positivity for another class of bi-Laplace operators. We consider a Laplace operator Δ with Dirichlet or Robin boundary conditions and analyse eventual positivity properties of its negative square $A := -\Delta^2$. This operator is, of course, also some kind of bi-Laplace operators, but with very different boundary conditions than those considered in Section 11.4. Moreover, we now study the operator on a space of continuous functions rather than on an L^p-space which shows that our theory is also well-adapted to such spaces.

Our strategy is as follows: we use that the Laplace operator Δ, endowed with appropriate boundary conditions, has strictly negative spectral bound and that its resolvent in 0 is, in some sense, strongly positive. We then employ

Corollary 10.1.2 to derive information about the resolvent of the negative square $A = -\Delta^2$ and from this we finally obtain, by means of our general theory, eventual positivity properties of the semigroup generated by A. Compare also the discussion after Corollary 10.1.2.

Before we discuss our two examples, we need to recall a few facts from the theory of analytic semigroups. We use the notions of an *analytic C_0-semigroup* and of a *bounded analytic C_0-semigroup* in the usual sense, see e.g. [EN00, Definition II.4.5]. We shall need the following simple observation: if a C_0-semigroup $(e^{tA})_{t\geq 0}$ on a complex Banach space E extends to an analytic C_0-semigroup of angle $\frac{\pi}{2}$ and if $\sigma(A) \subseteq (-\infty, 0)$, then $(e^{tA})_{t\geq 0}$ actually extends to a *bounded* analytic semigroup of angle $\frac{\pi}{2}$. This is not difficult to see; we refer to [DGK16b, Lemma 6.5] for details. By using a generation theorem for bounded analytic semigroups as e.g. given in [EN00, Theorem II.4.6] we can thus conclude that $-A^2$ generates a bounded analytic C_0-semigroup of angle $\frac{\pi}{2}$, too.

Let us now give our first example, the square of the Dirichlet-Laplace operator:

Theorem 11.5.1. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^2$-boundary and define the Dirichlet Laplace operator Δ_D on the Banach lattice $\mathcal{C}_0(\Omega;\mathbb{C})$ by*

$$D(\Delta_D) := \{f \in \mathcal{C}_0(\Omega;\mathbb{C}) : \Delta f \in \mathcal{C}_0(\Omega;\mathbb{C})\}, \quad \Delta_D f := \Delta f,$$

where Δf is understood in the sense of distributions. Define $u \in \mathcal{C}_0(\Omega;\mathbb{C})$ by $u(x) := \operatorname{dist}(x, \partial\Omega)$ for every $x \in \Omega$. Then the operator $A := -\Delta_D^2$ has the following properties:

(a) *The operator A generates a bounded analytic C_0-semigroup of angle $\pi/2$ on $\mathcal{C}_0(\Omega;\mathbb{C})$ and we have $\emptyset \neq \sigma(A) \subseteq (-\infty, 0)$. Moreover, A has compact resolvent.*

(b) *The resolvent $\mathcal{R}(\cdot, A)$ is uniformly eventually strongly positive with respect to u at $\mathrm{s}(A)$ and individually eventually strongly negative with respect to u at $\mathrm{s}(A)$.*

(c) *The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to u.*

(d) *The semigroup $(e^{tA})_{t\geq 0}$ is not positive.*

Proof. (a) The operator Δ_D generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$ on $\mathcal{C}_0(\Omega;\mathbb{C})$ according to [AB99, Theorem 2.3]. Since $\sigma(\Delta_D) \subseteq (-\infty, 0)$, it thus follows from the discussion preceding to this theorem that $A = -\Delta_D^2$ generates an bounded analytic C_0-semigroup of angle $\frac{\pi}{2}$ on $\mathcal{C}_0(\Omega;\mathbb{C})$. Moreover, we have $\emptyset \neq \sigma(A) \subseteq (-\infty, 0)$ since $\emptyset \neq \sigma(\Delta_D) \subseteq (-\infty, 0)$.

We note that we have $D(A) \subseteq W^{2,p}(\Omega;\mathbb{C}) \cap \mathcal{C}_0(\Omega;\mathbb{C})$ for every $p > n$ by classical regularity theory. Hence, $D(A) \subseteq \mathcal{C}^1(\overline{\Omega};\mathbb{C})$, which implies that the embedding $D(A) \hookrightarrow \mathcal{C}_0(\Omega;\mathbb{C})$ is compact, so we conclude that A has compact resolvent.

(b) We first show that $\mathcal{R}(0,\Delta_D) \gg_u 0$: let $f \in D(\Delta_D)$ and assume that $g := -\Delta_D f > 0$. Since we have $f \in \mathcal{C}^1(\overline{\Omega};\mathbb{C})$, it follows from a Sobolev space version of the maximum principle (see e.g. [Bon67] or [Ama83, Theorem 6.1]) that the outer normal derivative is strictly negative everywhere on $\partial\Omega$. Hence, $f \gg_u 0$.

Now, as we have shown $\mathcal{R}(0,\Delta_D) \gg_u 0$, it follows from Corollary 10.1.2 that $\mathcal{R}(\cdot,A)$ is uniformly eventually strongly positive with respect to u at $\mathrm{s}(A)$. Moreover, the mapping u is $\mathcal{C}^2$ in a neighbourhood of $\partial\Omega$ (see [KP81, Theorem 3]) and the outer normal derivative of u is strictly negative everywhere on $\partial\Omega$; hence, we conclude that the domination condition $D(A) \subseteq \mathcal{C}_0(\Omega;\mathbb{C})_u$ is fulfilled. Thus, Theorem 9.2.6 shows that the resolvent $\mathcal{R}(\cdot,A)$ is also individually eventually strongly negative with respect to u at $\mathrm{s}(A)$.

(c) We first conclude from Theorem 9.2.5 and from assertion (b) above that the spectral projection P associated to $\mathrm{s}(A)$ fulfils $P \gg_u 0$. Since the semigroup $(e^{tA})_{t\geq 0}$ is analytic, the domination condition $D(A) \subseteq \mathcal{C}_0(\Omega;\mathbb{C})_u$ implies that $e^{tA}\mathcal{C}_0(\Omega;\mathbb{C}) \subseteq \mathcal{C}_0(\Omega;\mathbb{C})_u$ for every $t > 0$, see Remark 9.3.4(b). We can thus apply Theorem 9.3.2 and conclude that the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to u.

(d) Since A acts as a fourth-order differential operator on the test functions on Ω, this follows e.g. from [ABR90, Proposition 2.2]. □

In our second theorem in this section we consider the square of the Robin Laplace operator. Again, we intend to work on a space of continuous functions and hence, the appropriate space for these boundary conditions is $\mathcal{C}(\overline{\Omega};\mathbb{C})$.

Theorem 11.5.2. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^2$-boundary and let $\beta \in \mathcal{C}(\partial\Omega;\mathbb{C})$ a function which fulfils $\beta \gg_{\mathbb{1}_{\partial\Omega}} 0$. Let Δ_β denote the realisation of the Laplace operator, subject to the Robin boundary conditions*

$$\frac{\partial}{\partial \nu} u + \beta u = 0,$$

on the Banach lattice $\mathcal{C}(\overline{\Omega};\mathbb{C})$. Then the operator $A := -\Delta_\beta^2$ has the following properties:

(a) *The operator A generates a bounded analytic C_0-semigroup of angle $\pi/2$ on $\mathcal{C}(\overline{\Omega};\mathbb{C})$ and we have $\emptyset \neq \sigma(A) \subseteq (-\infty, 0)$. Moreover, A has compact resolvent.*

(b) *The resolvent $\mathcal{R}(\cdot,A)$ is uniformly eventually strongly positive with respect to $\mathbb{1} := \mathbb{1}_{\overline{\Omega}}$ at $\mathrm{s}(A)$ and individually eventually strongly negative with respect to $\mathbb{1}$ at $\mathrm{s}(A)$.*

(c) *The semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$.*

(d) *The semigroup $(e^{tA})_{t\geq 0}$ is not positive.*

Proof. (a) Since $\beta \gg_{\mathbb{1}_{\overline{\Omega}}} 0$, we have $\emptyset \neq \sigma(\Delta_\beta) \subseteq (-\infty, 0)$ and thus, $\emptyset \neq \sigma(A) \subseteq (-\infty, 0)$. Moreover, it is shown in [War06, Theorem 3.3] that Δ_β generates an immediately compact, analytic C_0-semigroup of angle $\frac{\pi}{2}$ on $\mathcal{C}(\overline{\Omega};\mathbb{C})$. Hence, it follows from the discussion preceding to Theorem 11.5.1 that A generates a bounded analytic C_0-semigroup of angle $\frac{\pi}{2}$ on $\mathcal{C}(\overline{\Omega};\mathbb{C})$. Moreover, since the C_0-semigroup $(e^{t\Delta_\beta})_{t\geq 0}$ is compact, it follows that Δ_β has compact resolvent [EN00, Theorem II.4.29] and thus, so has A.

(b) It is shown in [Ama83, Theorem 6.1] that $\mathcal{R}(0, \Delta_\beta) \gg_{\mathbb{1}} 0$. Hence, it follows from Corollary 10.1.2 that the resolvent $\mathcal{R}(\cdot, A)$ is uniformly eventually strongly positive with respect to $\mathbb{1}$ at $s(A)$. Theorem 9.2.6 thus implies that the resolvent is also individually eventually strongly negative with respect to $\mathbb{1}$ at $s(A)$.

(c) Theorem 9.2.5 and assertion (b) above imply that the spectral projection P associated with $s(A)$ fulfils $P \gg_{\mathbb{1}} 0$. Theorem 9.3.2 thus yields that the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$.

(d) Here we can argue as in the proof of Theorem 11.5.1: as A acts as a fourth-order differential operator on the test functions, it follows from [ABR90, Proposition 2.2] that the semigroup $(e^{tA})_{t\geq 0}$ is not positive. □

11.6 Delay Differential Equations

In this section we revisit delay differential equations which have already been in the focus of Section 8.4. However, instead of asymptotic positivity we now concentrate on eventual strong positivity.

As before, we let $r > 0$, $E = \mathcal{C}([-r, 0];\mathbb{C})$ and $\Phi \in E'$. We define

$$\begin{aligned} D(A) &= \{f \in \mathcal{C}^1([-r, 0];\mathbb{C} : f'(0) = \langle \Phi, f\rangle\}, \\ Af &= f', \end{aligned} \tag{11.1}$$

and we want to analyse the eventual positivity properties of the resolvent $\mathcal{R}(\cdot, A)$ and of the semigroup $(e^{tA})_{t\geq 0}$. In Thereom 8.4.1 we gave a sufficient criterion for the asymptotic positivity of the resolvent at 0; let us now show what condition we need in addition to ensure $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to $\mathbb{1} := \mathbb{1}_{[-r,0]}$ at 0.

Theorem 11.6.1. *Assume that the assumptions of Theorem 8.4.1 are fulfilled. Then, of course, the assertions of Theorem 8.4.1 are fulfilled. Moreover, the following assertions are equivalent to each other:*

(i) *The resolvent* $\mathcal{R}(\cdot, A)$ *is individually eventually strongly positive with respect to* $\mathbb{1}$ *at* 0.

(ii) *The resolvent* $\mathcal{R}(\cdot, A)$ *is individually eventually strongly negative with respect to* $\mathbb{1}$ *at* 0.

(iii) *The support of* Φ^- *contains the number* 0.

Proof. The equivalence "(i) $\Leftrightarrow$ (ii)" follows readily from Theorem 9.2.6. It thus remains to show the equivalence of (i) and (iii).

For each $\lambda \in \mathbb{C}$, let $e_\lambda \in E$ and $H_\lambda \in \mathcal{L}(E)$ be defined as in the proof of Theorem 8.4.1, i.e. $e_\lambda(s) = e^{\lambda s}$ and $(H_\lambda f)(s) := \int_s^0 e^{\lambda(s-\tau)} f(\tau)\, d\tau$ for all $f \in E$ and all $s \in [-r, 0]$. Recall that we have $\sigma(A) = \{\lambda \in \mathbb{C} |\ \lambda = \langle \Phi, e_\lambda \rangle\}$ and that the resolvent of A is given by

$$\mathcal{R}(\lambda, A) f = \frac{f(0) + \langle \Phi, H_\lambda f \rangle}{\lambda - \langle \Phi, e_\lambda \rangle} e_\lambda + H_\lambda f$$

for each $\lambda \in \rho(A)$ and each $f \in E$.

"(iii) $\Rightarrow$ (i)" Let $0 < f \in H$ and $\lambda > 0$. Then $f(0) \geq 0$ and $H_\lambda f \geq 0$; moreover, since e_λ is an increasing function, the condition on Φ from Theorem 8.4.1 implies $\langle \Phi, e_\lambda \rangle \leq 0$.

We have $\langle \Phi, H_\lambda f \rangle \to \langle \Phi, H_0 f \rangle$ as $\lambda \downarrow 0$. Clearly, $H_0 f$ is a positive, non-zero and decreasing function which fulfils $(H_0 f)(0) = 0$. It thus follows from assertion (iii) that $\langle \Phi, H_0 f \rangle > 0$. Hence, $\langle \Phi, H_\lambda f \rangle > 0$ for all sufficiently small $\lambda > 0$. For all those λ we thus have $\mathcal{R}(\lambda, A) f \geq \frac{\langle \Phi, H_\lambda f \rangle}{\lambda - \langle \Phi, e_\lambda \rangle} e_\lambda \gg_{\mathbb{1}} 0$.

"(i) $\Rightarrow$ (iii)" Assume that $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to $\mathbb{1}$ at 0, and let P denote the spectral projection associated to the spectra value 0. Then we have $P \gg_{\mathbb{1}} 0$ according to Theorem 9.2.5. Since 0 is a first order pole of $\mathcal{R}(\cdot, A)$, we thus obtain for every $f > 0$

$$\begin{aligned} 0 \ll_{\mathbb{1}} Pf &= \lim_{\lambda \downarrow 0} \lambda \mathcal{R}(\lambda, A) f \\ &\leq \lim_{\lambda \downarrow 0} \Big[(f(0) + \langle \Phi, H_\lambda f \rangle) e_\lambda + \lambda H_\lambda f \Big] = (f(0) + \langle \Phi, H_0 f \rangle) \mathbb{1}. \end{aligned}$$

Hence, $f(0) + \langle \Phi, H_0 f \rangle > 0$ for every $f > 0$. Now, assume that 0 is not contained in the support of Φ^-. Since the support of Φ^- is closed, there exists $\delta > 0$ such that $(-\delta, 0]$ is contained in $[-r, 0]$ and does not intersect the support of Φ^-; hence, $(-\delta, 0]$ does not intersect the support of Φ^+, either. Now choose a function $0 < f \in E$ whose support is contained in $(-\delta, 0)$. Then $H_0 f$ is constant on $[-r, -\delta]$; this implies that $\langle \Phi, H_0 f \rangle = 0$ since $\langle \Phi, \mathbb{1} \rangle = 0$. As we also have $f(0) = 0$, this contradicts the fact that $f(0) + \langle \Phi, H_0 f \rangle > 0$. □

As in Theorem 8.4.2 we now consider the semigroup generated by A. Moreover, we recall assertion (a) of Theorem 8.4.2 in the following result for the convenience of the reader.

Theorem 11.6.2. *Assume that the assumptions of Theorem 8.4.1 are fulfilled. Then the following assertions hold:*

(a) *If $r \leq \frac{\pi}{2\|\Phi\|}$, then the spectral bound* $\mathrm{s}(A)$ *equals* 0 *and is a dominant spectral value of A.*

(b) *Suppose that the spectral bound* $\mathrm{s}(A)$ *equals* 0 *and is a dominant spectral value. Then the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$ if and only if the support of Φ^- contains* 0.

Proof. (a) This has already been shown in Theorem 8.4.2(a).

(b) Suppose that $\mathrm{s}(A)$ equals 0 and is a dominant spectral value of A. Then it follows from Theorem 9.2.6 and Corollary 9.3.5 that the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$ if and only if the resolvent $\mathcal{R}(\cdot, A)$ is individually eventually strongly positive with respect to $\mathbb{1}$ at 0. The asserted equivalence therefore follows from Theorem 11.6.1. □

Again, the condition $r \leq \frac{\pi}{2\|\Phi\|}$ is by no means necessary for $\mathrm{s}(A)$ to be equal to 0 and a dominant spectral value. We close this section with an example where the above estimate is not fulfilled but where the semigroup is nevertheless individually eventually strongly positive with respect to $\mathbb{1}$:

Example 11.6.3. Let $r = 2$, $E = \mathcal{C}([-2, 0]; \mathbb{C})$ and let $\Phi \in E'$ be given by $\langle \Phi, f \rangle = \int_{-2}^{-1} f(s)\, ds - \int_{-1}^{0} f(s)\, ds$ for all $f \in E$. Let the operator A on E be defined by (11.1).

Then the estimate $r \leq \frac{\pi}{2\|\Phi\|}$ is not fulfilled. Yet, the spectral bound $\mathrm{s}(A)$ equals 0 and is a dominant spectral value of A. Hence, the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$, but not positive.

Proof. We have $\|\Phi\| = 2$ and thus, $\frac{\pi}{2\|\Phi\|} = \frac{\pi}{4} < 1 < r$, so the estimate $r \leq \frac{\pi}{2\|\Phi\|}$ is indeed not fulfilled.

Since 0 is an eigenvalue of A with eigenvector $\mathbb{1}$, we have to show that A has no other eigenvalue in the right half plane. To this end, assume that $\lambda = \alpha + i\beta \in \sigma(A)$, where $\alpha \in [0, \infty)$ and $\beta \in \mathbb{R}$. We have to show that $\lambda = 0$. Recall that we have $\lambda = \langle \Phi, e^{\lambda \cdot} \rangle$ since λ is an eigenvalue of A. A short computation, using the definition of Φ, yields that this is equivalent to the equation $-\lambda^2 = (1 - e^{-\lambda})^2$, and this equation is is turn satisfied if and only if one of the following to equations if fulfilled:

$$i\lambda = 1 - e^{-\lambda}, \tag{11.2}$$

$$-i\lambda = 1 - e^{-\lambda}. \tag{11.3}$$

Since λ fulfils (11.3) if and only if $\overline{\lambda}$ fulfils (11.2), it suffices to show that $\lambda = 0$ if λ fulfils (11.2). So assume that λ fulfils (11.2); by splitting this equation

into its real and imaginary part we obtain

$$-\beta = 1 - e^{-\alpha}\cos(\beta), \tag{11.4}$$

$$\alpha = e^{-\alpha}\sin(\beta). \tag{11.5}$$

Since $\alpha \geq 0$, we conclude from (11.4) that $\beta \in [-2, 0]$. However, (11.5) shows us that $\sin(\beta) \geq 0$, so actually $\beta = 0$. Employing again equation (11.5) we conclude that $\alpha = 0$, too.

According to Theorem 11.6.2(b) it now follows that the semigroup $(e^{tA})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$. Moreover, the semigroup is not positive since the support of Φ^- is not equal to $\{0\}$; cf. Theorem 8.4.2(c). □

11.7 Laplace Operators with Non-Local Boundary Conditions

Let us close the thesis with two examples of a somewhat surprising phenomenon. In many important cases it is known that the Laplace operator with local boundary conditions, when realised on an appropriate function space, generates a positive semigroup. This is essentially a consequence of the maximum principle. Yet, the situation changes considerably if we impose non-local boundary conditions on the operator, instead. Indeed, it can happen in such a situation that the generated semigroup is no longer positive, but still eventually positive. In this section we demonstrate this by two examples, the first of them being subject to symmetric boundary conditions while the boundary conditions in the second example are non-symmetric.

We start with some general observations. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain; since we are only interested in the basic principles, and since later on we are going to restrict ourselves to the one-dimensional case anyway, we assume for simplicity that Ω has $\mathcal{C}^\infty$-boundary. Let $B \in \mathcal{L}(L^2(\partial\Omega;\mathbb{C}))$ and consider the sesquilinear form

$$a : H^1(\Omega;\mathbb{C}) \times H^1(\Omega;\mathbb{C}) \to \mathbb{C}, \quad a(u,v) = \int_\Omega \nabla u \overline{\nabla v}\, dx - (B\gamma(u)|\gamma(v)), \tag{11.6}$$

where $\gamma(u), \gamma(v) \in L^2(\partial\Omega;\mathbb{C})$ denote the traces of u and v on the boundary of Ω. We note in passing that we chose the sign of the term $-(B\gamma(u)|\gamma(v))$ differently than in [DGK16a, Section 6]. This has some advantages for the formulation of the following proposition, where we collect a few properties of the operator on $L^2(\Omega;\mathbb{C})$ associated to a. For more information about such operators we refer to [GMN14].

Proposition 11.7.1. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^∞-boundary, let $B \in \mathcal{L}(L^2(\partial\Omega;\mathbb{C}))$ and let $-\Delta_B : L^2(\Omega;\mathbb{C}) \supseteq D(-\Delta_B) \to L^2(\Omega;\mathbb{C})$ be the operator associated with the form a in* (11.6)*. Then Δ_B has the following properties:*

(a) *We have*

$$D(\Delta_B) \supseteq H^2_B(\Omega;\mathbb{C}) := \{u \in H^2(\Omega;\mathbb{C}) : \frac{\partial}{\partial \nu} u = B\gamma(u)\},$$
$$\Delta_B u = \Delta u \qquad \text{for all } u \in H^2_B(\Omega;\mathbb{C}),$$

where $\frac{\partial u}{\partial \nu} \in L^2(\partial\Omega;\mathbb{C})$ denotes the outer normal derivative of u and where $\gamma(u) \in L^2(\partial\Omega;\mathbb{C})$ denotes trace of u on $\partial\Omega$.

(b) *The operator Δ_B generates an analytic C_0-semigroup on $L^2(\Omega;\mathbb{C})$ and has compact resolvent.*

(c) *If B is real, then so is Δ_B; if B is self-adjoint, then so is Δ_B.*

(d) *The semigroup $(e^{t\Delta_B})_{t\geq 0}$ is positive if and only if the semigroup $(e^{tB})_{t\geq 0}$ on $L^2(\partial\Omega;\mathbb{C})$ is positive (if and only if $B + c$ is positive for some $c \in [0,\infty)$).*

Proof. (a) This follows from the divergence theorem.

(b) We clearly have $\operatorname{Re} a(u,u) + c\|u\|^2_{L^2} \geq \|u\|^2_{H^1}$ for some constant $c > 0$ and all $u \in H^1(\Omega;\mathbb{C})$, i.e. a is $H^1(\Omega;\mathbb{C})$-elliptic. Hence, Δ_B generates an analytic semigroup.

Since $D(\Delta_B) \hookrightarrow H^1(\Omega;\mathbb{C}) \hookrightarrow L^2(\Omega;\mathbb{C})$ and since the latter embedding is compact, it follows that Δ_B has compact resolvent.

(c) If B is self-adjoint, then the form a is symmetric and hence Δ_B is self-adjoint. Moreover, if B is real, then so is a and hence, so is A; see Proposition C.6.5 for details.

(d) Let $\gamma : H^1(\Omega;\mathbb{C}) \to L^2(\Omega;\mathbb{C})$ denote the trace operator. Employing the Beurling–Deny criterion and assertion (c), we see that the semigroup $(e^{r\Delta_B})_{t\geq 0}$ is positive if and only if B is real and $a(u^+,u^-) \leq 0$ for all $u \in H^1(\Omega;\mathbb{R})$. This is true if and only if B is real and $(-Bv^+|v^-) \leq 0$ for all $v \in \gamma(H^1(\Omega;\mathbb{R}))$. Since the latter space is dense in $L^2(\partial\Omega;\mathbb{R})$, this condition is in turn equivalent to B being real and the condition $(-Bv^+|v^-) \leq 0$ for all $v \in L^2(\partial\Omega;\mathbb{R})$. Again by the Beurling–Deny criterion, this is equivalent to the semigroup $(e^{tB})_{t\geq 0}$ on $L^2(\partial\Omega;\mathbb{C})$ being positive.

Finally, it follows e.g. from [AGG+86, Theorem C-II.1.11] that the semigroup $(e^{tB})_{t\geq 0}$ is positive if and only if the operator $B + c$ is positive for some $c \in [0,\infty)$. □

Let us now discuss two concrete one-dimensional examples where eventual positivity occurs. A two-dimensional, though technically more involved example which is related to the modelling of Bose condensation can be found

in [DGK16a, Theorem 6.13]. Note that if $\Omega \subseteq \mathbb{R}$ is a bounded interval, then $L^2(\partial\Omega;\mathbb{C})$ may be identified with $\mathbb{C}^2$ where the first component of a vector $v \in \mathbb{C}^2$ denotes its value on the left boundary point of Ω and where its second component denotes its value on the right boundary point of Ω. Thus an operator $B \in \mathcal{L}(L^2(\Omega;\mathbb{C}))$ is given by a matrix $B \in \mathbb{C}^{2\times 2}$.

We also point out that, in the one-dimensional case, the description of the operator Δ_B from Proposition 11.7.1(a) can be improved; indeed, one can easily check that the following remark is true:

Remark 11.7.2. If, in the situation of Proposition 11.7.1, the dimension d equals 1, then we have $D(\Delta_B) = H^2_B(\Omega;\mathbb{C})$.

We now come to our first example. In this example, the matrix B is self-adjoint.

Theorem 11.7.3. *Let $\Omega = (0,1) \subseteq \mathbb{R}$, let $B = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ and let $-\Delta_B : L^2(\Omega;\mathbb{C}) \supseteq D(-\Delta_B) \to L^2(\Omega;\mathbb{C})$ be the operator associated with the form a in (11.6). Define $\mathbb{1} := \mathbb{1}_{(0,1)}$. Then Δ_B has the following properties:*

(a) *We have $\emptyset \neq \sigma(\Delta_B) \subseteq (-\infty, 0)$ and $\mathcal{R}(0,\Delta_B) \gg_{\mathbb{1}} 0$. In particular, the resolvent $\mathcal{R}(\cdot,\Delta_B)$ is uniformly eventually strongly positive with respect to $\mathbb{1}$ at $\mathrm{s}(\Delta_B)$.*

(b) *The resolvent $\mathcal{R}(\cdot,\Delta_B)$ is individually eventually strongly negative with respect to $\mathbb{1}$ at $\mathrm{s}(\Delta_B)$.*

(c) *The semigroup $(e^{t\Delta_B})_{t\geq 0}$ is uniformly eventually strongly positive with respect to $\mathbb{1}$.*

(d) *The semigroup $(e^{t\Delta_B})_{t\geq 0}$ is not positive.*

Proof. (a) Since B is self-adjoint, so is A. Moreover, we have $a(u,u) \leq 0$ for all $u \in H^1(\Omega;\mathbb{C})$ since $-B$ is positively semi-definite. Thus, $\sigma(\Delta_B) \subseteq (-\infty, 0]$. Since Δ_B has compact resolvent, it suffices to show that Δ_B is injective in order to prove that $0 \in \rho(\Delta_B)$. So let $u \in \ker \Delta_B$. Then we have $u \in D(\Delta_B) \subseteq H^1(\Omega;\mathbb{C})$ and $a(u,u) = 0$. Again since $-B$ is positively semi-definite, it follows that $\int_\Omega |u'|^2\,dx = 0$, so u is constant. Finally, since $-(B\mathbb{1}_{\{0,1\}}|\mathbb{1}_{\{0,1\}}) > 0$, it follows that $u = 0$. Hence, Δ_B is injective, so $\sigma(\Delta_B) \subseteq (\infty, 0)$, as claimed. Moreover, the spectrum of Δ_B is non-empty since Δ_B is self-adjoint.

Using Remark 11.7.2, one can check by a straightforward computation that $\mathcal{R}(0,\Delta_B)$ is given by

$$(\mathcal{R}(0,\Delta_B)f)(x) = \frac{1}{2}\int_0^x \int_y^1 f(z)\,dz\,dy + \frac{1}{2}\int_x^1 \int_0^y f(z)\,dz\,dy \quad \text{for all } x \in (0,1),$$

for every $f \in L^2((0,1);\mathbb{C})$. From this, one can easily see that $\mathcal{R}(0,\Delta_B) \gg_{\mathbb{1}} 0$. Hence, it follows from Proposition 10.1.1 that the resolvent $\mathcal{R}(\cdot,\Delta_B)$ is uniformly eventually strongly positive with respect to $\mathbb{1}$ at $s(\Delta_B)$.

(b) Note that, as Ω is one-dimensional and bounded, $D(\Delta_B) \subseteq H^1(\Omega;\mathbb{C}) \subseteq L^\infty(\Omega;\mathbb{C}) = L^2(\Omega;\mathbb{C})_{\mathbb{1}}$. Hence, assertion (b) follows from (a) and Theorem 9.2.6.

(c) It follows from assertion (a) and Theorem 9.2.5 that the spectral projection P associated to $s(\Delta_B)$ fulfils $P \gg_{\mathbb{1}} 0$.

As the semigroup is analytic, the domination condition $D(\Delta_B) \subseteq L^2(\Omega;\mathbb{C})_{\mathbb{1}}$ implies the smoothing condition $e^{t\Delta_B} L^2(\Omega;\mathbb{C}) \subseteq L^2(\Omega;\mathbb{C})_{\mathbb{1}}$ for all $t > 0$, see Remark 9.3.4(b). Since Δ_B is self-adjoint and $P \gg_{\mathbb{1}} 0$, we can conclude from Theorem 10.2.1 that the semigroup $(e^{t\Delta_B})_{t\geq 0}$ is uniformly eventually strongly positive with respect to $\mathbb{1}$.

(d) This follows from Proposition 11.7.1(d) since B has some strictly negative off-diagonal entries. □

We close this section, as well as the entire thesis, with our second example. Here, we consider a parametrised family of non-symmetric matrices B. A similar example was considered e.g. in [GM97] and [GM00] as a simple model for a thermostat (actually, in those papers the sign of the parameter β is chosen differently and the transposed matrix of B is considered – which simply interchanges the role of the two boundary points of the interval; we prefer this choice of B since it seems to facilitate some computations).

Theorem 11.7.4. *Let $\Omega = (0,\pi) \subseteq \mathbb{R}$, let $\beta \in \mathbb{R}$ and define $B = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$. Let $-\Delta_B : L^2(\Omega;\mathbb{C}) \supseteq D(-\Delta_B) \to L^2(\Omega;\mathbb{C})$ be the operator associated with the form a in* (11.6)*. Define $\mathbb{1} := \mathbb{1}_{(0,\pi)}$. Then Δ_B has the following properties:*

(a) *The semigroup $(e^{t\Delta_B})_{t\geq 0}$ is positive if and only if $\beta \geq 0$.*

(b) *All eigenvalues of A are geometrically simple.*

(c) *Let $\beta < 0$. Then $\sigma(A) \cap \mathbb{R}$ is non-empty and the largest real eigenvalue λ_1 of A fulfils $\lambda_1 < 0$; moreover, the corresponding eigenfunction can be chosen to be $\gg_{\mathbb{1}} 0$ if and only if $\beta \in (-\frac{1}{2}, 0)$.*

(d) *Let $\beta < 0$. The semigroup $(e^{t\Delta_B})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$ if and only if $\beta \in (-\frac{1}{2}, 0)$.*

Proof. (a) This follows from Proposition 11.7.1(d).

(b) According to Proposition 11.7.1(a) and Remark 11.7.2 the operator Δ_B is given by

$$D(\Delta_B) = \{u \in H^2((0,\pi);\mathbb{C}) \mid u'(0) = 0,\ u'(\pi) = \beta u(0)\},$$
$$\Delta_B u = u''.$$

Assertion (b) follows from this by a straightforward computation.

(c) Let $\beta < 0$. It is easy to check that 0 is not an eigenvalue, and hence not a spectral value, of Δ_B. Now let $\lambda \in \mathbb{C} \setminus \{0\}$ and choose $\mu \in \mathbb{C}$ such that $\mu^2 = -\lambda$. Then an easy computation shows that $\lambda \in \sigma(A)$ if and only

$$\mu \sin(\mu\pi) = -\beta, \tag{11.7}$$

and in this case the corresponding eigenfunction (which is unique up to scalar multiples) is given by by $u(x) = \cos(\mu x)$.

We show that Δ_B has no real eigenvalues > 0: assume for a contradiction that $\lambda > 0$. Then $\mu = i\eta$ for some $\eta \in \mathbb{R}$, so equation (11.7) yields $0 < -\beta = -\eta \sinh(\eta) \leq 0$, which is a contradiction.

On the other hand, assume now that $\lambda < 0$. Then $\mu \in \mathbb{R}$, and we may assume that $\mu > 0$. Since $-\beta > 0$, equation (11.7) has a solution $\mu \in (0, \infty)$, so $\sigma(\Delta_B) \cap \mathbb{R} \neq \emptyset$. Moreover, the corresponding eigenfunction, which is a multiple of $u(x) = \cos(\mu x)$, can be chosen to be $\gg_{\mathbb{1}} 0$ if and only if $\mu < \frac{1}{2}$. Now, note that the function

$$f : [0, \infty) \to \mathbb{R}, \quad \mu \mapsto \mu \sin(\mu\pi)$$

fulfils $f(0) = 0$ and $f(\frac{1}{2}) = \frac{1}{2}$, and it is strictly increasing on the interval $[0, \frac{1}{2}]$. Hence, the first solution $\mu > 0$ of (11.7), which equals $\sqrt{-\lambda_1}$, fulfils $\mu < \frac{1}{2}$ if and only if $\beta > -\frac{1}{2}$. This proves (c).

(d) First of all we note that

$$D(\Delta_B) \subseteq H^1((0,\pi);\mathbb{C}) \subseteq L^\infty((0,\pi);\mathbb{C}) = L^2((0,\pi);\mathbb{C})_{\mathbb{1}}.$$

Since the semigroup $(e^{t\Delta_B})_{t\geq 0}$ is analytic, it thus follows that $e^{t\Delta_B} L^2((0,\pi);\mathbb{C}) \subseteq L^\infty((0,\pi);\mathbb{C})_{\mathbb{1}}$ for all $t > 0$, see Remark 9.3.4(b).

Now, assume that $(e^{t\Delta_B})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$. Then it follows from Theorem 9.3.2 and Proposition 9.1.1 that $s(\Delta_B)$ is a dominant spectral value of Δ_B and that we can find a corresponding eigenfunction $u \gg_{\mathbb{1}} 0$. Hence, $s(\Delta_B) = \lambda_1$ and it follows from assertion (c) that $\beta \in (-\frac{1}{2}, 0)$.

Assume now on the other hand that $\beta \in (-\frac{1}{2}, 0)$. Then it follows from assertion (c) that the eigenvalue λ_1 admits an eigenfunction $u \gg_{\mathbb{1}}$. Moreover, interchanging A with its Hilbert space adjoint A^* is equivalent to interchanging the matrix B with B^*, and this is in turn equivalent to interchanging the role of the boundary points of the interval $(0, \pi)$. Hence, A^* has the same spectrum as A with correspondingly reflected eigenfunctions. In particular, the eigenvalue λ_1 of A^* possesses an eigenfunction $\tilde{u} \gg_{\mathbb{1}} 0$. Thus, the Banach space adjoint A' of A posses a strictly positive eigenvector for the eigenvalue λ_1 (see Corollary C.6.4), and therefore it follows from Proposition 9.1.1 that

the spectral projection P of A corresponding to the spectral value λ_1 fulfils $P \gg_{\mathbb{1}} 0$.

Finally, it is noted in [GM97, Section 3] that the equation (11.7) has only real solutions as long as β is contained in the interval $(\beta_r, 0)$ for some appropriately chosen $\beta_r < -\frac{1}{2}$ (note that the sign of β, and thus of β_r, in [GM97] is opposite to the sign our parameters). Thus, as $\beta > -\frac{1}{2}$, the spectrum of Δ_B is real and, in particular, the spectral bound $s(\Delta_B)$ equals λ_1 and is a dominant spectral value of Δ_B. Hence, it follows from Corollary 9.3.5 that the semigroup $(e^{t\Delta_B})_{t\geq 0}$ is individually eventually strongly positive with respect to $\mathbb{1}$. □

11.8 Notes & Open Problems

The finite-dimensional case For some historical remarks about the finite-dimensional theory of eventual positivity we refer, again, to the notes at the end of Chapter 6.

The Dirichlet-to-Neumann operator The reason why we need Ω to be a *two*-dimensional domain in Theorem 11.2.1 is that this implies that the Sobolev space $H^1(\partial\Omega;\mathbb{C})$, which is the domain of $\mathcal{D}_\lambda$, is contained in $L^\infty(\partial\Omega;\mathbb{C})$ since the boundary $\partial\Omega$ is a *one*-dimensional manifold. On the other hand, if one could prove in higher dimensions d that $D(\mathcal{D}_\lambda^n) \subseteq L^\infty(\partial\Omega;\mathbb{C})$ for some (d-dependent) integer n, then we could still employ Theorem 10.2.1 (due to Remark 9.3.4) to derive the equivalence of conditions (i)–(iii) in Theorem 11.2.1. Using this, one could for instance try to generalise some results on the two-dimensional disk which were obtained in [Dan14] to higher-dimensional balls.

The bi-Laplace operator We have already made some comments about eventual positivity of the resolvent of the bi-Laplace operator (with Dirichlet boundary conditions) in the notes at the end of Chapter 8. As pointed out in Section 11.4 some of the results from the literature can even be improved by using our abstract theory.

In the notes of Chapter 8 we focussed on references related to eventual positivity of the elliptic biharmonic problem. Some result about the parabolic problem can be found in the papers [FGG08] and [GG08], which have already been quoted before Theorem 11.4.2; see also [GG09] some related results.

A comparison of our results in Sections 11.4 and 11.5 shows that eventual positivity properties of bi-Laplace operators strongly depend on the boundary conditions. While, for the bi-Laplacian with Dirichlet boundary conditions, eventual positivity properties only hold on very specific domain, the negative

square of the Dirichlet (or Robin) Laplace operator exhibits eventual positivity properties on every bounded domain with sufficiently regular boundary.

Finally, we point out that for the bi-Laplace operator A in one dimension, endowed with different sets of boundary conditions, one can explicitly compute the largest number λ_0 for which the resolvent $\mathcal{R}(\lambda_0, A)$ is still positive; see [Ulm99] for details.

Delay differential equations As in the notes at the end of Chapter 8 we point out that it is possible in Example 11.6.3 to explicitly compute an eigenvector of the adjoint operator A' for the eigenvalue 0; see [DGK16b, Section 6.5]. From this one can easily derive a formula for the spectral projection of A associated to 0.

Laplace operators with non-local boundary conditions Laplace operators with non-local boundary conditions are for example studied in a rather general setting in [GM09], [GMN14] and [GMNO15]. Besides the two one-dimensional examples presented in Section 11.7 one can also prove eventual positivity results in a certain two-dimensional model; see [DGK16a, Theorem 6.13] for details.

The operator which we studied in Theorem 11.7.3 was also considered by Arendt and Warma in [AW03, Example 4.5] where it was claimed, among other things, that the generated semigroup is positive. Theorem 11.7.3 shows that this is not true, but that the semigroup exhibits a weaker form of positivity, namely it is uniformly eventually strongly positive.

Appendices

A Some Aspects of Spectral Theory

In Appendix A we collect some results from spectral theory which we use throughout the thesis. All the material in this appendix (some of it maybe in a slightly different version) can be found in the literature, mostly in textbooks and monographs about functional analysis and operator theory; see for instance [Kat95, Section III.6], [Yos95, Chapter VIII], [Tay58, Chapter 5], [HP57, Section 5.2], and also [EN00, Section IV.1 and IV.2] and [CD13]. Yet, we prefer to summarise the material in this appendix. On the one hand, this allows us to use a common source of reference throughout the main text; on the other hand, we can write all results down in exactly the version we need.

Some very elementary notions from spectral theory, which we use throughout in the thesis and in this appendix, are recalled in the preliminaries of the thesis. In Appendix A we only deal with spectral theory on *complex* Banach spaces. Most of the concepts can be carried over to operators on real Banach spaces by means of *complexifications*, see Section C.3 in Appendix C.

A.1 Spectral Projections

Spectral projections are a very common – and very useful – tool in operator theory. They are defined by means of contour integration over the resolvent of an operator. We use the notion of a *contour* in the complex plane in a rather broad sense as it is defined for instance in [Sar07, Section IX.6], i.e. we consider a contour as a formal sum of closed piecewise $\mathcal{C}^1$-curves in $\mathbb{C}$, weighted by integers.

Let $S \subseteq \mathbb{C}$ be a closed set. Recall that a subset $\sigma \subseteq S$ is called an *isolated subset of* S if σ and $S \setminus \sigma$ are both closed in $\mathbb{C}$ (or, equivalently, in S). A number $\lambda_0 \in S$ is called *isolated in* S if $\{\lambda_0\}$ is an isolated subset of S, i.e. if $S \setminus \{\lambda_0\}$ is closed.

Definition A.1.1 (Spectral projection). Let E be a complex Banach space, let $A : E \supseteq D(A) \to E$ be a linear operator and let $\sigma \subseteq \sigma(A)$ be a bounded isolated subset of $\sigma(A)$. Let γ be a contour in $\rho(A) \subseteq \mathbb{C}$ whose winding number around each point of σ equals 1 and whose winding number around each point of $\sigma(A) \setminus \sigma$ equals 0 (note that such a contour always exists). The linear operator

$$P := \frac{1}{2\pi i} \oint_\gamma \mathcal{R}(\lambda, A)\, d\lambda \ \in \ \mathcal{L}(E)$$

is called the *spectral projection* of A associated to σ; here the integration is performed in the Banach space $\mathcal{L}(E)$. If $\sigma = \{\lambda_0\}$ for some $\lambda_0 \in \sigma(A)$, then we call P the spectral projection associated to λ_0.

Note that the spectral projection is independent of the choice of the contour γ according to Cauchy's integral theorem. The reason why spectral projections are very useful in operator theory is the subsequent theorem. To formulate it, we need the following notion: let $A : E \supseteq D(A) \to E$ be a linear operator on a real or complex Banach space E; let $P \in \mathcal{L}(E)$ be projection and let $Q := I_E - P$. We say that P *reduces* A if the sum of the spaces $D(A) \cap PE$ and $D(A) \cap QE$ equals $D(A)$ and if A maps the first of those space to PE and the second to QE.

Theorem A.1.2. *Let E be a complex Banach space, let $A : E \supseteq D(A) \to E$ be a linear operator and let $\sigma \subseteq \sigma(A)$ be a bounded isolated subset of $\sigma(A)$. Let P be the spectral projection of A associated to σ, define $\tilde{\sigma} := \sigma(A) \setminus \sigma$ and $Q := I_E - P$. Then the following assertions holds:*

(a) *The projection P reduces A; moreover, we have $PE \subseteq D(A)$.*

(b) *The operator $A|_{PE} \in \mathcal{L}(PE)$ has spectrum $\sigma(A|_{PE}) = \sigma$ and the operator $A|_{QE} : QE \supseteq QE \cap D(A) \to QE$ has spectrum $\sigma(A|_{QE}) = \tilde{\sigma}$*

(c) *The operator P is the only projection in $\mathcal{L}(E)$ which fulfils properties* (a) *and* (b).

(d) *If $A \in \mathcal{L}(E)$, then Q is the spectral projection of A associated with $\tilde{\sigma}$.*

Proof. Assertions (a) and (b) follow for instance from [Kat95, Theorem III.6.17] (a brief look at the proof of this theorem shows that the projection which reduces the operator there is indeed the spectral projection). Assertion (d) follows from that fact that we have

$$I_E = \frac{1}{2\pi i} \oint_\gamma \mathcal{R}(\lambda, A)\, d\lambda$$

for every operator $A \in \mathcal{L}(E)$ if the contour γ circumvents the entire spectrum of A (see [Yos95, Section VIII.7]).

To prove assertion (c), let $\tilde{P} \in \mathcal{L}(E)$ be another projection for which (a) and (b) hold (where we replace Q with $\tilde{Q} := I_E - \tilde{P}$, of course). Let γ be a contour in $\mathbb{C}$ such as in Definition A.1.1. The projection $\tilde{P}$ reduces the resolvent $\mathcal{R}(\lambda, A)$ for every $\lambda \in \rho(A)$ and $\mathcal{R}(\cdot, A|_{\tilde{Q}E})$ is holomorphic on $\rho(A) \cup \sigma$; hence,

$$\frac{1}{2\pi i} \oint_\gamma \mathcal{R}(\lambda, A)\tilde{Q}\, d\lambda = 0.$$

On the other hand, $A|_{\tilde{P}E}$ is by assumption a bounded linear operator on $\tilde{P}E$. Hence

$$\frac{1}{2\pi i} \oint_\gamma \mathcal{R}(\lambda, A|_{\tilde{P}E})\, d\lambda = I_{\tilde{P}E}$$

and therefore,

$$\frac{1}{2\pi i}\oint_\gamma \mathcal{R}(\lambda, A)\tilde{P}\, d\lambda = \tilde{P}.$$

This prove that $P = \tilde{P} + 0 = \tilde{P}$. □

It is important to note that the roles of P and Q are *not* symmetric in the above theorem. Obviously, we cannot simply define Q to be the spectral projection associated to $\tilde{\sigma}$ since the latter set is, in general, unbounded. However, even if $\tilde{\sigma}$ is bounded, the roles of P and Q are still not symmetric! Indeed, suppose that the spectrum of A is bounded and can be split into to closed parts as $\sigma(A) = \sigma \dot{\cup} \tilde{\sigma}$. Then we can define a spectral projection P associated to σ and a spectral projection $\tilde{P}$ associated to $\tilde{\sigma}$; moreover, we define $Q := I_E - P$ as in the above theorem. If we have $A \in \mathcal{L}(E)$, then $Q = \tilde{P}$; this is the content of part (d) of the theorem. However, if A is not a bounded operator (there exist, of course, unbounded operators with bounded spectrum), then it can in general happen that $Q \neq \tilde{P}$ (more precisely, the range of $\tilde{P}$ might be smaller than the range of Q). There is also a second reason why the roles of P and Q cannot be interchanged even if $\sigma(A)$ (and thus $\tilde{\sigma}$) is bounded: according to assertion (a) we have $PE \subseteq D(A)$, but this is not in general true for QE. We also point out that the assertion $PE \subseteq D(A)$ in (a) is essential for the characterisation in assertion (c) to be true.

We also recall from the proof of Theorem A.1.2(d) that the spectral projection associated with the entire spectrum of A equals I_E in case that $A \in \mathcal{L}(E)$.

In the following simple proposition we note how several spectral projections belonging to different parts of the spectrum sum up.

Proposition A.1.3. *Let E be a complex Banach space, let $A : E \supseteq D(A) \to E$ be a linear operator and let $\sigma_1, ..., \sigma_n \subseteq \sigma(A)$ be isolated subsets of $\sigma(A)$ which are bounded and pairwise disjoint. Let $P_1, ..., P_n$ denote the spectral projections of A associated to $\sigma_1, ..., \sigma_n$, respectively, and let P denote the spectral projection of A associated to $\bigcup_{k=1}^n \sigma_k$. Then $P = \sum_{k=1}^n P_k$.*

Proof. This is clear since the spectral projection P does not depend on the choice of the contour in its definition. □

A.2 Eigenvalue Multiplicities and Generalised Eigenspaces

In this section we consider multiplicities of eigenvalues for linear operators; those notions play an important role in Part III of the thesis. Let us start right away with the major definitions. We recall that, throughout the thesis, we

always understand the *dimension* of a vector space to be either an integer or ∞, i.e. we do not distinguish between different infinite cardinalities.

Definition A.2.1 (Eigenvalue multiplicities and generalised eigenspace). Let E be a complex Banach space and let $A : E \supseteq D(A) \to E$ be a linear operator with an eigenvalue $\lambda \in \mathbb{C}$.

(i) The number $\dim \ker(\lambda - A) \in \mathbb{N} \cup \{\infty\}$ is called the *geometric multiplicity* of the eigenvalue λ.

The eigenvalue λ is called *geometrically simple* if it has geometric multiplicity 1.

(ii) The space $\bigcup_{n\in\mathbb{N}} \ker\left((\lambda - A)^n\right)$ is called the *generalised eigenspace* of A for the eigenvalue λ. Its dimension (which is contained in $\mathbb{N} \cup \{\infty\}$) is called the *algebraic multiplicity* of the eigenvalue λ.

The eigenvalue λ is called *algebraically simple* if it has algebraic multiplicity 1.

(iii) Let $n \in \mathbb{N}$. A vector $x \in E$ is called a *generalised eigenvector of rank n* for the eigenvalue λ if $x \in \ker\left((\lambda - A)^n\right) \setminus \ker\left((\lambda - A)^{n-1}\right)$.

(iv) The eigenvalue λ is called *semi-simple* if its generalised eigenspace coincides with its eigenspace $\ker(\lambda - A)$.

Note that a vector $x \in E$ is an eigenvector of A for the eigenvalue λ if and only if it is a generalised eigenvector of rank 1 for the eigenvalue λ. Of course the generalised eigenspace of an eigenvalue λ always contains the eigenspace and thus, the algebraic multiplicity is at least as large as the geometric multiplicity. In particular, an algebraically simple eigenvalue is always geometrically simple, too, and thus semi-simple.

In finite dimensions the algebraic multiplicity of an operator can be found out by means of the following remark:

Remark A.2.2. Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then the algebraic multiplicity of λ equals the order of λ as a root of the characteristic polynomial $z \mapsto \det(z - A)$.

Proof. This can easily be seen by using the Jordan normal form of A. □

Assertion (b) of the following proposition, as simple as it is, can sometimes be very useful to find out whether an eigenvalue is semi-simple.

Proposition A.2.3. *Let E be a complex Banach space and let $A : E \supseteq D(A) \to E$ be a linear operator with an eigenvalue λ.*

(a) *Suppose that we have* $\ker\big((\lambda - A)^m\big) = \ker\big((\lambda - A)^{m+1}\big)$ *for some* $m \in \mathbb{N}$. *Then* $\ker\big((\lambda - A)^m\big) = \ker\big((\lambda - A)^n\big)$ *for all* $n \geq m$ *and hence* $\ker\big((\lambda - A)^m\big)$ *is the generalised eigenspace of* λ.

(b) *The eigenvalue* λ *is semi-simple if and only if it has no generalised eigenvector of rank* 2.

Proof. (a) This can easily be seen by induction.

(b) If λ has a generalised eigenvector of rank 2, then λ is obviously not semi-simple. On the other hand, if λ has no generalised eigenvector of rank 2, then we have $\ker(\lambda - A) = \ker\big((\lambda - A)^2\big)$ and it thus follows from (a) that $\ker(\lambda - A)$ is the generalised eigenspace of λ. Hence, the eigenvalue λ is semi-simple. □

Two sufficient conditions for the semi-simplicity of an eigenvalue are given in the following proposition.

Proposition A.2.4. *Let* E *be a complex Banach space.*

(a) *If* $T \in \mathcal{L}(E)$ *is power bounded and* $\lambda \in \mathbb{C}$ *is an eigenvalue of* T *with modulus* $|\lambda| = 1$, *then* λ *is semi-simple.*

(b) *If* $A : E \supseteq D(A) \to E$ *generates a bounded* C_0*-semigroup and* $\lambda \in \mathbb{C}$ *is an eigenvalue of* A *with real part* $\operatorname{Re}\lambda = 0$, *then* λ *is semi-simple.*

Proof. (a) If λ is not semi-simple then we can find, according to Proposition A.2.3 (b), a generalised eigenvector x of rank 2 for λ. Define $y := (T - \lambda)x \neq 0$. We have $Ty = \lambda y$ and it is easy to prove via induction that $T^n x = n\lambda^{n-1} y + \lambda^n x$ for every $n \in \mathbb{N}_0$. Since $|\lambda| = 1$ and $y \neq 0$ this contradicts the power boundedness of T.

(b) Again, if λ is not semi-simple then we can find, according to Proposition A.2.3 (b), a generalised eigenvector x of rank 2 for λ. As above we define $y := (A - \lambda)x \neq 0$. We have $y \in D(A)$ and $Ay = \lambda y$; using this, one easily checks that the mapping $u : [0, \infty) \to D(A)$, $u(t) = te^{t\lambda} y + e^{t\lambda} x$ solves the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) &= Au(t), \\ u(0) &= x. \end{cases}$$

Hence we have $e^{tA} x = u(t)$ for all $t \geq 0$. Since $\lambda \in i\mathbb{R}$ and $y \neq 0$, this contradicts the boundedness of the C_0-semigroup $(e^{tA})_{t \geq 0}$. □

A.3 Isolated Singularities of the Resolvent

In this section we deal with isolated points of the spectrum of a linear operator. Since the resolvent of an operator is an analytic function, such points are isolated singularities of the resolvent. Of particular importance are spectral values which are even poles of the resolvent; those occur frequently in Part III of the thesis. We shall see in Proposition A.3.2 that such poles are always eigenvalues and that it is particularly easy to obtain certain information about the corresponding eigenvalue multiplicities.

We start by having a closer look at the Laurent series expansion of the resolvent about an isolated spectral value:

Proposition A.3.1. *Let E be a complex Banach space, let $A : E \supseteq D(A) \to E$ be a linear operator and let $\lambda_0 \in \sigma(A)$ be an isolated point of $\sigma(A)$. If*

$$\mathcal{R}(\lambda, A) = \sum_{k=-\infty}^{\infty} Q_k (\lambda - \lambda_0)^k$$

is the Laurent expansion of the analytic function $\mathcal{R}(\cdot, A) : \rho(A) \to \mathcal{L}(E)$ about λ_0, then the coefficients $Q_k \in \mathcal{L}(E)$ fulfil the following properties:

(a) *The residuum $P := Q_{-1}$ is the spectral projection of A associated to λ_0.*

(b) *If $Q_{k_0} = 0$ for an integer $k_0 \leq -1$, then $Q_k = 0$ for all $k \leq k_0$ and hence, λ_0 is a pole of $\mathcal{R}(\cdot, A)$ of order $< -k_0$.*

(c) *If $PE = Q_{-1}E$ is finite-dimensional, then λ_0 is a pole of $\mathcal{R}(\cdot, A)$ of order $\leq \dim Q_{-1}E$.*

Proof. (a) Assertion (a) is obvious. Assertion (b) follows for example from [Yos95, Theorem 1 in Section VIII.8] and assertion (c) can be found in [EN00, assertion (ii) at the end of Section IV.1.17]. □

The following result shows that every pole of the resolvent is in fact an eigenvalue; it gives us also a possibility to compute a corresponding eigenvector and it provides some information about semi-simplicity of the eigenvalue. We use this result, often tacitly, in many proofs throughout Part III of the thesis.

Proposition A.3.2. *Let E be a complex Banach space, let $A : E \supseteq D(A) \to E$ be a linear operator and let $\lambda_0 \in \sigma(A)$ be an isolated point of $\sigma(A)$ and a pole of $\mathcal{R}(\cdot, A)$ of order $k_0 \in \mathbb{N}$. If*

$$\mathcal{R}(\lambda, A) = \sum_{k=-k_0}^{\infty} Q_k (\lambda - \lambda_0)^k$$

is the Laurent expansion of $\mathcal{R}(\cdot, A)$ about λ_0 and $P := Q_{-1}$ denotes the spectral projection of A associated to λ_0, then the following assertions hold:

(a) $\{0\} \neq Q_{-k_0}E \subseteq \ker(\lambda_0 - A) \subseteq PE$. *In particular, λ_0 is an eigenvalue of A.*

(b) *We have $PE = \ker\left((\lambda_0 - A)^{k_0}\right)$, and this space is the generalised eigenspace of A for the eigenvalue λ_0.*

(c) *For all $k \geq k_0$ the range of $(\lambda_0 - A)^k$ does not depend on k and coincides with the kernel* $\ker P$.

(d) *The eigenvalue λ_0 of A is semi-simple if and only if* $\ker(\lambda_0 - A) = PE$ *if and only if λ_0 is a first order pole of $\mathcal{R}(\cdot, A)$ (i.e. if and only if $k_0 = 1$).*

Proof. Assertion (a) follows from [Yos95, Theorem 2 in Section VIII.8] and assertions (b) and (c) can be found in [Yos95, Theorem 3 in Section VIII.8]. Assertion (d) is a consequence of (b) and of [Yos95, Theorem 2 in Section VIII.8]. □

Let us also collect a few facts about isolated spectral values of the adjoint of an operator:

Proposition A.3.3. *Let E be a complex Banach space, let $A : E \supseteq D(A) \to E$ be a densely defined linear operator and let $\lambda_0 \in \sigma(A)$ be an isolated point of $\sigma(A)$ and a pole of $\mathcal{R}(\cdot, A)$. If*

$$\mathcal{R}(\lambda, A) = \sum_{k=-\infty}^{\infty} Q_k(\lambda - \lambda_0)^k$$

is the Laurent expansion of $\mathcal{R}(\cdot, A)$ about λ_0 and $P := Q_{-1}$ denotes the spectral projection of A associated to λ_0, then the following assertions hold:

(a) *The number λ_0 is also an isolated singularity of the resolvent $\mathcal{R}(\cdot, A') = \mathcal{R}(\cdot, A)'$ of the adjoint operator A'. The Laurent series expansion of $\mathcal{R}(\cdot, A')$ about λ_0 is given by*

$$\mathcal{R}(\lambda, A') = \sum_{k=-\infty}^{\infty} Q_k'(\lambda - \lambda_0)^k.$$

Moreover, the spectral projection of A' associated to λ_0 equals P'.

(b) *The number λ_0 is a pole of $\mathcal{R}(\cdot, A)$ if and only if it is a pole of $\mathcal{R}(\cdot, A)$, and in this case the pole orders coincide.*

(c) *Suppose that λ_0 is a pole of $\mathcal{R}(\cdot, A)$ (equivalently: of $\mathcal{R}(\cdot, A')$). Then the generalised eigenspaces of λ_0 for the operator A and for the adjoint operator A' have the same dimension; in particular, λ_0 is an algebraically simple eigenvalue of A if and only if it is an algebraically simple eigenvalue of A'.*

(d) *If P has finite rank, then the dimensions of the eigenspaces* $\ker(\lambda_0 - A)$ *and* $\ker(\lambda_0 - A')$ *coincide.*

Proof. (a) This is obvious since $\sigma(A) = \sigma(A')$ and $\mathcal{R}(\lambda, A') = \mathcal{R}(\lambda, A)'$ for all $\lambda \in \rho(A) = \rho(A')$.

(b) This follows readily from (a).

(c) According to Proposition A.3.2(b) the generalised eigenspaces we are interested in are given by PE and $P'E'$. It follows for instance from Lemma 1.2.6 that those spaces have the same dimension.

(d) For this result we refer, for instance, to [Kat95, Remark III.6.23]. □

Finally, we need criteria to check whether a spectral value of a given operator is a pole of the resolvent. We sum up such criteria in the next proposition and the subsequent theorem.

Proposition A.3.4. *Let $A : E \supseteq D(A) \to E$ be a linear operator on a complex Banach space E, let $\lambda_0 \in \rho(A)$ and define $T := \mathcal{R}(\lambda_0, A)$. For each $\lambda \in \mathbb{C}$ the following assertions hold:*

(a) *The number λ is an isolated point of $\sigma(A)$ if and only if $(\lambda_0 - \lambda)^{-1}$ is an isolated point of $\sigma(T)$; in this case the spectral projection of A associated to λ coincides with the spectral projection of T associated to $(\lambda_0 - \lambda)^{-1}$.*

(b) *The number λ is a pole of $\mathcal{R}(\cdot, A)$ if and only if $(\lambda_0 - \lambda)^{-1}$ is a pole of $\mathcal{R}(\cdot, A)$, and in this case the pole orders coincide.*

Proof. This proposition is taken from [EN00, Proposition IV.1.18]. □

The following theorem is a part of what is sometimes called *analytic Fredholm theory*. For a related result we refer the reader to [Ste69].

Theorem A.3.5. *Let E be a complex Banach space, let $T \in \mathcal{L}(E)$ and let $A : E \supseteq D(A) \to E$ be a linear operator.*

(a) *If T^n is compact for some $n \in \mathbb{N}$, then all spectral values of T, except possibly 0, are poles of the resolvent $\mathcal{R}(\cdot, T)$ and the corresponding generalised eigenspaces are all finite-dimensional.*

(b) *Suppose that there exist $\lambda_0 \in \rho(A)$ and $n \in \mathbb{N}$ such that $\mathcal{R}(\lambda_0, A)^n$ is compact. Then all spectral values of T are poles of the resolvent and the corresponding generalised eigenspaces are all finite-dimensional.*

Proof. (a) This is a special case of [Tay58, Theorem 5.8-F].

(b) This also follows from [Tay58, Theorem 5.8-F]. Alternatively, one can derive the assertion from (a) and from Proposition A.3.4 (since the generalised eigenspace of a pole of the resolvent is the range of the corresponding spectral projection according to Proposition A.3.2(b)). □

Note that assertion (b) of the above theorem includes in particular the well-known result that all spectral values of an operator with compact resolvent are poles of the resolvent and have a finite-dimensional generalised eigenspace.

A.4 Pseudo-Resolvents

Pseudo-resolvents are a technical tool which we need in the proof of Theorem 2.4.1. They resemble resolvents of linear operators in the sense that they also satisfy the resolvent identity and they share many analytic properties with them. In fact, it can even by shown that every pseudo-resolvent is the resolvent of a so-called *multi-valued* operator, see e.g. [Haa06, Appendix A] for details. One of the reasons why pseudo-resolvents are a useful tool in spectral theory is their stability with respect to ultra powers: an unbounded linear operator $A : E \supseteq D(A) \to E$ on a complex Banach lattice cannot in general be lifted to an ultra power $E_{\mathcal{U}}$ of E. However, the resolvent operators $\mathcal{R}(\lambda, A)$ ($\lambda \in \rho(A)$) can be lifted, and we thus obtain a mapping $\rho(A) \to \mathcal{L}(E_{\mathcal{U}})$, $\lambda \mapsto \mathcal{R}(\lambda, A)_{\mathcal{U}}$ which might no longer be a resolvent of an operator, but which is still a pseudo-resolvent.

After this motivation we give the definition of a pseudo-resolvent:

Definition A.4.1 (Pseudo-resolvent). Let E be a complex Banach space and let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be an open set. A mapping $\mathcal{R} : \Omega \to \mathcal{L}(E)$ is called a *pseudo-resolvent* if it fulfils the equation

$$\mathcal{R}(\lambda) - \mathcal{R}(\mu) = (\mu - \lambda)\mathcal{R}(\lambda)\mathcal{R}(\mu)$$

for all $\lambda, \mu \in \Omega$; this equation is known as the *resolvent equation* or the *resolvent identity*.

Note that if $\mathcal{R} : \Omega \to \mathcal{L}(E)$ is a pseudo-resolvent, then $\mathcal{R}(\lambda)$ commutes with $\mathcal{R}(\mu)$ for all $\lambda, \mu \in \Omega$. The following property shows that several operator theoretical properties of $\mathcal{R}(\lambda)$ do in fact not depend on λ:

Proposition A.4.2. *Let E be a complex Banach space, let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and let $\mathcal{R} : \Omega \to \mathcal{L}(E)$ be a pseudo-resolvent. Then the kernel* $\ker \mathcal{R}(\lambda)$ *and the range $\mathcal{R}(\lambda)E$ are independent of λ.*

Proof. See e.g. [EN00, Lemma III.4.5]. □

The following proposition characterises whether a pseudo-resolvent is in fact the resolvent of a linear operator.

Proposition A.4.3. *Let E be a complex Banach space, let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and let $\mathcal{R} : \Omega \to \mathcal{L}(E)$ be a pseudo-resolvent. Then the following assertions are equivalent:*

(i) *There is a $\lambda \in \Omega$ such that the operator $R(\lambda) : E \to E$ is injective.*

(ii) *For every $\lambda \in \Omega$ the operator $R(\lambda) : E \to E$ is injective.*

(iii) *There is a linear operator $A : E \supseteq D(A) \to E$ such that $\rho(A) \supseteq \Omega$ and such that $\mathcal{R}(\lambda, A) = \mathcal{R}(\lambda)$ for all $\lambda \in \Omega$.*

Proof. "(iii) $\Rightarrow$ (ii)" This implication is clear since the inverse of a (single-valued) linear operator is always injective.

"(ii) $\Rightarrow$ (i)" This is obvious.

"(i) $\Rightarrow$ (iii)" Let $\lambda_0 \in \Omega$ be such that $\mathcal{R}(\lambda_0)$ is injective. We define $D(A) := D(B) := \mathcal{R}(\lambda_0)E$, $B : E \supseteq D(B) \to E$, $B = \mathcal{R}(\lambda_0)^{-1}$ and $A : E \supseteq D(A) \to E$, $A = \lambda_0 - B$.

According to Proposition A.4.2 the range of $\mathcal{R}(\lambda)$ is independent of λ, so we have $D(A) = \mathcal{R}(\lambda)E$ for every $\lambda \in \Omega$. Moreover, the resolvent equations yields

$$\begin{aligned}\mathcal{R}(\lambda)(\lambda - A) &= \mathcal{R}(\lambda)(\lambda - \lambda_0) + \mathcal{R}(\lambda)B = (\lambda - \lambda_0)\mathcal{R}(\lambda)\mathcal{R}(\lambda_0)B + \mathcal{R}(\lambda)B \\ &= \big(\mathcal{R}(\lambda_0) - \mathcal{R}(\lambda)\big)B + \mathcal{R}(\lambda)B = \mathcal{R}(\lambda_0)B = I_{D(A)}\end{aligned}$$

for each $\lambda \in \Omega$; note that these equalities are indeed true since all the operators have the correct domain, namely $D(B) = D(A)$. A similar computation shows that

$$(\lambda - A)\mathcal{R}(\lambda) = I_E$$

for all $\lambda \in \Omega$. Hence, each $\lambda \in \Omega$ is contained in the resolvent set of A and fulfils $\mathcal{R}(\lambda) = \mathcal{R}(\lambda, A)$. □

We point out once again that a pseudo-resolvent can always be interpreted as the resolvent of a multi-valued operator A, see e.g. [Haa06, Appendix A]; hence, the above Proposition A.4.3 gives equivalent conditions for A to be single-valued. The following analytic properties of pseudo-resolvents can be derived from the resolvent equation:

Proposition A.4.4. *Let E be a complex Banach space, let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and let $\mathcal{R} : \Omega \to \mathcal{L}(E)$ be a pseudo-resolvent. Then the following assertion are fulfilled:*

(a) *The mapping $\mathcal{R} : \Omega \to \mathcal{L}(E)$ is analytic and its derivatives are given by $\mathcal{R}^{(n)}(\lambda) = (-1)^n n! \mathcal{R}(\lambda)^{n+1}$ for every $n \in \mathbb{N}_0$ and every $\lambda \in \Omega$.*

(b) *If $\lambda_0 \in \Omega$ and $r := \operatorname{dist}(\lambda_0, \partial\Omega)$, then we have*

$$\mathcal{R}(\lambda) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \mathcal{R}(\lambda_0)^{n+1}$$

for every $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < r$, where the series converges absolutely in $\mathcal{L}(E)$.

Proof. (a) This can be found for instance in [HP57, Theorem 5.8.2 and formula (5.8.5)].

(b) This follows readily from (a) and from the Taylor series expansion of analytic functions. □

If a pseudo-resolvent on a reflexive Banach space E satisfies a certain boundedness condition, then it can always be restricted to a complemented subspace of E where it becomes the resolvent of a linear operator. Moreover, this complemented subspace can be chosen large enough to conserve most of the information about the pseudo-resolvent. This was proved by Kato in [Kat59]; in the subsequent theorem, we state the result in a slightly different way.

Theorem A.4.5. *Let E be a reflexive complex Banach space and let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be an open set which contains the line (r_0, ∞) for some $r_0 \geq 0$. Let $\mathcal{R} : \Omega \to \mathcal{L}(E)$ be a pseudo-resolvent such that $\sup_{r\in(r_0,\infty)} \|r\mathcal{R}(r)\| < \infty$. Then there exists a projection $P \in \mathcal{L}(E)$ which has the following properties (and which is uniquely determined by property* (b)*):*

(a) *P commutes with $\mathcal{R}(\lambda, A)$ for all $\lambda \in \Omega$; hence, $\mathcal{R}(\lambda)$ leaves the kernel and the range of P invariant for every $\lambda \in \Omega$.*

(b) *For every $\lambda \in \Omega$ we have $\ker P = \ker \mathcal{R}(\lambda)$ and $PE = \overline{\mathcal{R}(\lambda)E}$.*

(c) *Let $F := PE$ be the range of P. There exists a densely defined linear operator $A : F \supseteq D(A) \to F$ such that $\rho(A) \supseteq \Omega$ and such that $\mathcal{R}(\lambda, A) = \mathcal{R}(\lambda)|_F \in \mathcal{L}(F)$ for every $\lambda \in \Omega$.*

Moreover, $r\mathcal{R}(r)$ converges to P with respect to the weak operator topology as $r \to \infty$.

Proof. If such a projection P exists, then it clearly is uniquely determined by (b). The operator family $(r(\mathcal{R}(r))_{r\in(r_0,\infty)}$ is a bounded net in $\mathcal{L}(E)$ (where the index set (r_0, ∞) is endowed with the order induced by $\mathbb{R}$). We now proceed in several steps.

Step 1: We prove that whenever a subnet $(r_j\mathcal{R}(r_j))_{j\in J}$ of $(r(\mathcal{R}(r))_{r\in(r_0,\infty)}$ converges to an operator $\tilde{P} \in \mathcal{L}(E)$ with respect to the weak operator topology, then $\tilde{P}$ is a projection which fulfils the assertions (a), (b) and (c).

We first show that $\tilde{P}\mathcal{R}(\lambda) = \mathcal{R}(\lambda)$ for all $\lambda \in \Omega$. Indeed, we have

$$r_j\mathcal{R}(r_j)\mathcal{R}(\lambda) - \mathcal{R}(\lambda) = \frac{r_j}{r_j - \lambda}\Big(\mathcal{R}(\lambda) - \mathcal{R}(r_j)\Big) - \mathcal{R}(\lambda)$$

for all $\lambda \in \Omega$ and all $j \in J$ with $r_j \neq \lambda$. The left-hand side of the above equation converges to $\tilde{P}\mathcal{R}(\lambda) - \mathcal{R}(\lambda)$ with respect to the weak operator topology. The

fraction $\frac{r_j}{r_j-\lambda}$ converges to 1 and the operator $\mathcal{R}(r_j)$ converges to 0 with respect to operator norm; hence the right-hand side of the above equation converges to 0 with respect to the operator norm. This shows that $\tilde{P}\mathcal{R}(\lambda) = \mathcal{R}(\lambda)$ for every $\lambda \in \Omega$. In particular, we have $\tilde{P} r_j \mathcal{R}(r_j) = r_j \mathcal{R}(r_j)$ for each index j and therefore $\tilde{P}^2 = \tilde{P}$, so $\tilde{P}$ is indeed a projection.

Obviously $\tilde{P}$ fulfils (a). We show that (b) holds for $\tilde{P}$, too. Since $\ker \mathcal{R}(\lambda)$ is independent of λ according to Proposition A.4.2 and since $\tilde{P}$ is the weak operator limit of $(r_j \mathcal{R}(r_j))_{j\in J}$ it follows that $\ker \mathcal{R}(\lambda) \subseteq \ker \tilde{P}$ for all $\lambda \in \Omega$. On the other hand we have $\mathcal{R}(\lambda) = \mathcal{R}(\lambda)\tilde{P}$ and thus $\ker \tilde{P} \subseteq \ker \mathcal{R}(\lambda)$ for every $\lambda \in \Omega$.

We know that $\mathcal{R}(\lambda) = \tilde{P}\mathcal{R}(\lambda)$ and hence $R(\lambda)E \subseteq \tilde{P}E$ for every $\lambda \in \Omega$; since $\tilde{P}E$ is closed we even conclude that even $\overline{\mathcal{R}(\lambda)E} \subseteq \tilde{P}E$. On the other hand, the range of $\mathcal{R}(\lambda)$ does not depend on λ according to Proposition A.4.2 and $\tilde{P}$ is the weak operator limit of $(r_j \mathcal{R}(r_j))_{j\in J}$. Therefore $\tilde{P}E$ is contained in the weak closure, and thus also in the norm closure, of $\mathcal{R}(\lambda)E$ for every $\lambda \in \Omega$.

Now we prove that assertion (c) is fulfilled for $\tilde{P}$. Indeed, $\ker \mathcal{R}(\lambda)|_F = \{0\}$ for every $\lambda \in \Omega$ since $\ker \mathcal{R}(\lambda) = \ker \tilde{P}$ intersects $F = \tilde{P}E$ only in $\{0\}$. Hence, $\mathcal{R}(\lambda)$ is injective for each $\lambda \in \Omega$ and it thus follows from Proposition A.4.3 that there is an operator $A : F \supseteq D(A) \to F$ such that $\rho(A) \supseteq \Omega$ and such that $\mathcal{R}(\lambda, A) = \mathcal{R}(\lambda)|_F$. To see that A is densely defined, fix $\lambda \in \Omega$ and note that we have $D(A) = \mathcal{R}(\lambda, A)F = \mathcal{R}(\lambda)|_F F = \mathcal{R}(\lambda)\tilde{P}E = R(\lambda)E$, and the latter space is dense in F according to (b).

Step 2: We show that there exists a uniquely determined projection $P \in \mathcal{L}(E)$ which fulfils (a), (b) and (c). Indeed, we have already observed at the beginning at the proof that such a projection is, if it exists, uniquely determined by property (b). To show existence note that the net $(r\mathcal{R}(r))_{r\in(r_0,\infty)}$ is bounded in $\mathcal{L}(E)$ and since E is reflexive, there exists a subnet $(r_j \mathcal{R}(r_j))_{j\in J}$ which converges to an operator $P \in \mathcal{L}(E)$ with respect to the weak operator topology. As shown in Step 1, P is a projection which has the required properties.

Step 3: We finally prove that $r\mathcal{R}(r)$ converges to P with respect to the weak operator topology as $r \to \infty$. To this end, let $(r_k \mathcal{R}(r_k))_{k\in K}$ be an arbitrary subnet of $(r\mathcal{R}(r))_{r\in(r_0,\infty)}$. Then there exists another subnet $(r_{k_j} \mathcal{R}(r_{k_j}))_{j\in J}$ of $(r_k \mathcal{R}(r_k))_{k\in K}$ which converges to an operator $\tilde{P} \in \mathcal{L}(E)$ with respect to the weak operator topology. It follows from Step 1 that $\tilde{P}$ is a projection which fulfils assertions (a), (b) and (c). Hence $\tilde{P} = P$. We have thus shown that every subnet of $(r\mathcal{R}(r))_{r\in(r_0,\infty)}$ has a subnet which converges weakly to P. This proves the assertion. □

The following proposition shows that, for a pseudo-resolvent $\mathcal{R} : \Omega \to \mathcal{L}(E)$, the eigenvalues of all operators $\mathcal{R}(\lambda)$ (for $\lambda \in \Omega$) are closely related.

Proposition A.4.6. *Let E be a complex Banach space, let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and let $\mathcal{R} : \Omega \to \mathcal{L}(E)$ be a pseudo-resolvent. For every $x \in E \setminus \{0\}$ and every $\alpha \in \mathbb{C}$ the following assertions are equivalent:*

(i) *There exists $\lambda \in \Omega$ such that $\lambda \neq \alpha$ and such that $\frac{1}{\lambda-\alpha}$ is an eigenvalue of $\mathcal{R}(\lambda)$ with corresponding eigenvector x.*

(ii) *We have $\alpha \notin \Omega$; moreover, for every $\lambda \in \Omega$ the number $\frac{1}{\lambda-\alpha}$ is an eigenvalue of $\mathcal{R}(\lambda)$ with corresponding eigenvector x.*

Proof. See e.g. [AGG+86, Proposition C-III.2.6(a)]. □

We close this section with a result that shows how certain spectral values of a linear operator can be lifted to eigenvalues of a pseudo-resolvent on an ultra power of the underlying Banach space.

Proposition A.4.7. *Let E be a complex Banach space, let $A : E \supseteq D(A) \to E$ be a linear operator and assume that $\alpha \in \sigma(A)$ is located on the topological boundary of the spectrum $\sigma(A)$ in $\mathbb{C}$. If $\mathcal{U}$ is a free ultra filter on $\mathbb{N}$, then*

$$\rho(A) \to \mathcal{L}(E_{\mathcal{U}}), \qquad \lambda \mapsto \mathcal{R}_{\mathcal{U}}(\lambda) := \Big(\mathcal{R}(\lambda, A)\Big)_{\mathcal{U}}$$

defines a pseudo-resolvent on the ultra power $E_{\mathcal{U}}$. Moreover, $\frac{1}{\lambda-\alpha}$ is an eigenvalue of $\mathcal{R}_{\mathcal{U}}(\lambda)$ for all $\lambda \in \rho(A)$.

Proof. Obviously, $\mathcal{R}_{\mathcal{U}}(\cdot)$ is a pseudo-resolvent on $E_{\mathcal{U}}$. Choose a sequence $(\mu_n) \subseteq \rho(A)$ which converges to α and observe that $\|\mathcal{R}(\mu_n, A)\| \to \infty$. Hence, we can find a sequence of normalised vectors $(x_n) \subseteq E$ such that even $r_n := \|\mathcal{R}(\mu_n, A)x_n\| \to \infty$. Now, let

$$z_n := \frac{1}{r_n}\mathcal{R}(\mu_n, A)x_n$$

for each index n. Then $z := (z_n)$ is a sequence of normalised vectors in E and the vector $z_{\mathcal{U}} = (z_n)_{\mathcal{U}}$ is a non-zero element of the ultra power $E_{\mathcal{U}}$.

Now let $\lambda \in \rho(A)$ be an arbitrary element. Then we obtain

$$\begin{aligned}(\lambda-\alpha)\mathcal{R}(\lambda, A)z_n &= (\lambda-\mu_n)\mathcal{R}(\lambda, A)\frac{1}{r_n}\mathcal{R}(\mu_n, A)x_n + (\mu_n-\alpha)\mathcal{R}(\lambda, A)z_n \\ &= \frac{1}{r_n}\Big(\mathcal{R}(\mu_n, A)x_n - \mathcal{R}(\lambda, A)x_n\Big) + (\mu_n-\alpha)\mathcal{R}(\lambda, A)z_n \\ &= z_n - \frac{1}{r_n}\mathcal{R}(\lambda, A)x_n + (\mu_n-\alpha)\mathcal{R}(\lambda, A)z_n\end{aligned}$$

for each $n \in \mathbb{N}$. Hence, the difference of $(\lambda-\alpha)\mathcal{R}(\lambda, A)z_n$ and z_n converges to 0 which proves that $(\lambda-\alpha)\mathcal{R}_{\mathcal{U}}(\lambda)z_{\mathcal{U}} = z_{\mathcal{U}}$, as claimed. □

B

Ordered Banach Spaces

In Appendix B we briefly recall those concepts from the theory of ordered Banach spaces which we need in Part II of the thesis. We point out that there is an extensive literature about ordered topological vector spaces; see e.g. [Per67], [SW99, Chapter V] and [AT07]. Ordered Banach spaces can be seen as a simple special of those spaces; despite (or maybe because) of this fact, it is difficult to find a comprehensive treatment of ordered Banach spaces in the literature (although some monographs about ordered topological vector spaces contain important results about ordered Banach spaces as special cases; see e.g. [AT07, Section 2.5]). Therefore, and in order to fix our terminology, we collect all the material that we need in this appendix. We quote most of our results from different sources in the literature.

B.1 Basics

The notion *ordered Banach space* is used ambiguously in the literature. Here, we agree on the following definition:

Definition B.1.1 (Ordered Banach space). An *ordered Banach space* is a pair (E, E_+) where E is a real Banach space and E_+ is a subset of E with the following properties:

(i) E_+ is closed.

(ii) E_+ is a *cone*, i.e. $E_+ \neq \emptyset$ and for all $x, y \in E_+$ and all scalars $\alpha, \beta \geq 0$ we have $\alpha x + \beta y \in E_+$.

(iii) The cone E_+ is *pointed*, i.e. we have $(-E_+) \cap E_+ = \{0\}$.

The subset $E_+ \subseteq E$ is called the *positive cone* in E. The positive cone E_+ is called *generating* if $E_+ - E_+ = E$. The positive cone E_+ is called *total* if $E_+ - E_+$ is dense in E.

What we named a *cone* above is for example called a *wedge* in [AT07]; in the same reference the notion "cone" is used to described a "pointed wedge" (i.e. a set which we would call a *pointed cone* in our terminology). Moreover, some authors (as e.g. in [AT07]) do not require the cone in an ordered Banach space to be closed; yet, this assumption is required for most of the deeper theorems about ordered Banach spaces. Anyway, the above definition seems to fit our purposes best.

Let (E, E_+) be an ordered Banach space. We shall often suppress the positive cone E_+ in the notation and merely say that E is an ordered Banach space. There is a canonical order $\leq$ on E which is given by

$$x \leq y \qquad \text{if and only if} \qquad y - x \in E_+.$$

We write $x < y$ to indicate that $x \leq y$, but $x \neq y$. An element $x \in E$ is called *positive* if $x \in E_+$ or, equivalently, if $x \geq 0$.

If the ordered Banach space E has generating cone, then every vector $x \in E$ can be written as a difference of two positive vectors y and z. It is not obvious that y and z can be chosen such that their norm is, in a sense, bounded by the norm of x. This is the content of the subsequent theorem:

Theorem B.1.2. *Let E be an ordered Banach space with generating cone. Then there exists a constant $M > 0$ with the following property: for every $x \in E$ we can find $y, z \in E_+$ such that $\|y\|, \|z\| \leq M\|x\|$ and $x = y - z$.*

Proof. See for instance [AT07, Theorem 2.37]. □

Let E be an ordered Banach space. An element u of E is called an *order unit* if for every $x \in E$ there is a real number $\varepsilon > 0$ such that $\varepsilon x \leq u$; note that an order unit is automatically positive. Not very ordered Banach space contains an ordered unit, but the space $C(K;\mathbb{R})$ of all real-valued continuous functions on a compact Hausdorff space K does, for example. Let us prove a few facts about order units in the following two propositions.

Proposition B.1.3. *Let E be an ordered Banach space and let $u \in E$. Then the following assertions are equivalent:*

(i) *The vector u is an order unit in E.*

(ii) *The positive cone E_+ is generating in E and for every $x \in E_+$ there exists $\varepsilon > 0$ such that $\varepsilon x \leq u$.*

Proof. "(i) $\Rightarrow$ (ii)" Let u be an order unit in E. If $x \in E$, then we can find $\varepsilon > 0$ such that $x \leq \frac{1}{\varepsilon}u$. Hence, $x = \frac{1}{\varepsilon}u - (\frac{1}{\varepsilon}u - x) \in E_+ - E_+$, so E_+ is indeed generating in E. The second assertion of (ii) is obvious.

"(ii) $\Rightarrow$ (i)" Suppose that (ii) holds. Let $x \in E$. Then we can find $x_1, x_2 \in E_+$ such that $x = x_1 - x_2$ and, moreover, there exists $\varepsilon > 0$ such that $\varepsilon x_1 \leq u$. Hence, $\varepsilon x = \varepsilon x_1 - \varepsilon x_2 \leq \varepsilon x_1 \leq u$. This proves that u is indeed an order unit in E. □

Proposition B.1.4. *Let E be an ordered Banach space and let $u \in E$.*

(a) *The vector u is an order unit in E if and only if there exists $\varepsilon > 0$ such that $x \leq u$ for all $x \in E$ with $\|x\| \leq \varepsilon$.*

(b) *The vector u is an order unit in E if and only if u is contained in the interior of E_+.*

(c) *The cone E_+ has non-empty interior if and only if E contains an ordered unit.*

(d) *Suppose that E is finite-dimensional. Then the cone E_+ is generating in E if and only if it has non-empty interior.*

Proof. (a) "⇒" Let u be an order unit. We first show that there exists $\varepsilon > 0$ such that $x \leq u$ for all *positive* x with $\|x\| \leq \varepsilon$: assume for a contradiction that this is not true. Then we can find a sequence of vectors $(x_n) \subseteq E_+$ such that $\|x_n\| \leq \frac{1}{n2^n}$ but $x_n \not\leq u$ for all $n \in \mathbb{N}$. We define $x := \sum_{n=1}^{\infty} n x_n \in E_+$. For every $n \in \mathbb{N}$ we then have $\frac{1}{n} x \geq x_n$; as $x_n \not\leq u$, this implies that $\frac{1}{n} x \not\leq u$ for any $n \in \mathbb{N}$. This is a contradiction since u is an order unit.

Now let $\varepsilon > 0$ such that $x \leq u$ for all $x \in E_+$ with $\|x\| \leq \varepsilon$. Recall that E_+ is generating in E since u is an order unit. Hence we can choose $M > 0$ as in Theorem B.1.2. Let $x \in E$ with norm $\|x\| \leq \frac{\varepsilon}{M}$. We can find two vectors $y, z \in E_+$ such that $x = y - z$ and such that $\|y\|, \|z\| \leq \varepsilon$. Hence, $x = y - z \leq y \leq u$.

"⇐" This implication is obvious.

(b) "⇒" Suppose that u is an order unit in E. According to assertion (a) there exists $\varepsilon > 0$ such that $x \leq u$ for all vectors $x \in E$ with norm $\|x\| \leq \varepsilon$. Now, assume that $y \in E$ and $\|u - y\| \leq \varepsilon$. Then we have $u - y \leq u$ and thus $y \geq 0$. This proves that the ball with radius ε and center u is contained in E_+, so u is indeed an interior point of E_+.

"⇐" Now assume in turn that u is an interior point of E_+ and let $\varepsilon > 0$ such that the closed ball with radius ε and center u is contained in E_+. For every $x \in E$ with $\|x\| \leq \varepsilon$ we have $\|(u - x) - u\| \leq \varepsilon$ and thus $u - x \geq 0$. Hence, $u \geq x$ for all $x \in E$ with norm $\|x\| \leq \varepsilon$, which shows that u is indeed an order unit in E.

(c) This follows immediately from (b).

(d) This result can, for instance, be found in [AT07, Lemma 3.2]. □

The first of the following notions is important when we deal with duality (compare Theorem B.2.5), and the second one is closely related to the first one (compare Proposition B.1.6 below).

Definition B.1.5 (Normal cone and monotone norm). Let E be an ordered Banach space.

(i) The positive cone E_+ is called *normal* if there is a constant $C > 0$ such that we have $\|x\| \leq C\|y\|$ for all $x, y \in E$ with $0 \leq x \leq y$.

(ii) The norm on E is called *monotone* if we have $\|x\| \leq \|y\|$ for all $x, y \in E$ with $0 \leq x \leq y$.

For the convenience of the reader we quote a non-trivial characterisation of normal cones in the following proposition. To do so, we need to recall the notion of an *order interval*: if x, y are elements of an ordered Banach space, then the set $[x,y] := \{v \in E : x \leq v \leq y\}$ is called an order interval in E. Clearly, $[x,y]$ is non-empty if and only if $x \leq y$. Note that an order interval in an ordered Banach space is always closed.

Proposition B.1.6. *Let E be an ordered Banach space. Then the following assertions are equivalent:*

(i) *The positive cone E_+ is normal.*

(ii) *There is an equivalent norm on E which is monotone.*

(iii) *Every order interval in E is norm bounded.*

Proof. These assertions can for instance be found in [AT07, Theorems 2.38 and 2.40]. □

When dealing with asymptotic positivity in Part III of the main text we frequently need the distance of a vector x in an ordered Banach space to the positive cone. In the next proposition we collect some elementary but important properties of this distance function; we omit the simple proofs of those properties.

Proposition B.1.7. *Let E be an ordered Banach space. The distance function* $\operatorname{dist}(\cdot, E_+)$ *fulfils the following properties:*

(a) $\operatorname{dist}(x+y, E_+) \leq \operatorname{dist}(x, E_+) + \operatorname{dist}(y, E_+)$ *for all $x, y \in E$.*

(b) $\operatorname{dist}(\alpha x, E_+) = \alpha \operatorname{dist}(x, E_+)$ *for all $x \in E$ and all $\alpha \in [0, \infty)$.*

(c) $\operatorname{dist}(x, E_+) \leq \|x\|$ *for all $x \in E$.*

(d) *A vector $x \in E$ fulfils* $\operatorname{dist}(x, E_+) = 0$ *if and only if $x \in E_+$.*

The same assertions remain true if we replace E with a complexification $E_{\mathbb{C}}$ of the real Banach space E (cf. Appendix C), where we define $(E_{\mathbb{C}})_+ := E_+$; in particular the assertions remain true on complex Banach lattices.

We also discuss some notions related to positive operators. Let E, F be ordered Banach spaces. A linear map $T : E \to F$ is called *positive* if $TE_+ \subseteq F_+$; if $T \in \mathcal{L}(E;F)$, then we write $T \geq 0$ to say that T is positive. A linear map $T : E \to F$ is called *bi-positive* if $TE_+ \subseteq F_+$ and $T^{-1}(F_+) \subseteq E_+$, i.e. T is bi-positive if and only if for all $x, y \in E$ the inequality $x \leq y$ is equivalent to the inequality $Tx \leq Ty$. If a linear map $T : E \to F$ is bijective, then it is bi-positive if and only if T and T^{-1} are both positive.

The following notions are needed in Part II of the thesis. Since they are possibly not completely standard we introduce them in a separate definition.

Definition B.1.8 (Special classes of positive operators). Let E, F be to ordered Banach spaces and let $T : E \to F$ be a linear map.

(i) T is called an *embedding of ordered Banach spaces* if T is continuous, injective and bi-positive.

(ii) T is called a *dense embedding of ordered Banach spaces* if T is an embedding of ordered Banach spaces and if, in addition, TE is dense in F.

(iii) T is called an *isomorphism of ordered Banach spaces* if T is continuous, bijective and bi-positive.

(iv) Assume that T is positive. The map T is called *dominating* if for every $y \in F$ there exists an $x \in E$ such that $Tx \geq y$.

If T is an isomorphism of ordered Banach spaces, then T^{-1} is, of course, an isomorphism of ordered Banach spaces, too (continuity of T^{-1} follows from the open mapping theorem; this is why we did not require it in the definition). Two ordered Banach spaces E and F are said to be *isomorphic* if there exists an isomorphism of ordered Banach spaces from E to F.

We close this section with the following simple but important result about closed subspaces of ordered Banach spaces.

Proposition B.1.9. *Let E be an ordered Banach space, let $V \subseteq E$ be a closed vector subspace and define $V_+ := V \cap E_+$.*

(a) *(V, V_+) is an ordered Banach space, too, and the canonical injection $V \hookrightarrow E$ is an embedding.*

(b) *Suppose that there exists a positive projection $P \in \mathcal{L}(E)$ with range $PE = V$. Then we have $V_+ = P(E_+)$. In particular, V_+ is generating in V if E_+ is generating in E.*

Proof. This is an easy exercise. □

B.2 Duality

An important concept in the theory of ordered Banach spaces is duality. In the previous section we defined the notion of *positivity* for a linear map between two ordered Banach spaces. If we endow the real field $\mathbb{R}$ with the cone $\mathbb{R}_+ := [0, \infty)$ and make it thus an ordered Banach space, then we obtain

the notion of a *positive linear functional*; more precisely, a linear mapping $\varphi : E \to \mathbb{R}$ (on an ordered Banach space E) is called *positive* if $\varphi(x) \geq 0$ for all $x \in E_+$. This definition in particular applies to *continuous* linear functionals $x' \in E'$; for $x' \in E'$ we write $x' \geq 0$ to say that x' is positive; we write $x' > 0$ if x' is positive and non-zero. The set of all positive $x' \in E'$ is of particular importance:

Definition B.2.1 (Dual cone). Let E be an ordered Banach space. The set $(E_+)' := E'_+ := \{x' \in E' : x' \text{ is positive}\}$ is called the *dual cone* of E_+.

Note that the dual cone E'_+ is indeed a cone, i.e. $\alpha x' + \beta y' \in E'_+$ for all $x', y' \in E'_+$ and all scalars $\alpha, \beta \geq 0$. Moreover it is easy to see that E'_+ is closed in E'. In order for the dual space E' to be an ordered Banach space with respect to the dual cone E'_+, we also need that the cone E'_+ is pointed. The following proposition describes under which circumstances this happens.

Proposition B.2.2. *Let E be an ordered Banach space. The following assertions are equivalent:*

(i) *The dual cone E'_+ is pointed, i.e. $(-E'_+) \cap E'_+ = \{0\}$.*

(ii) *The dual space E' is an ordered Banach space with respect to the dual cone E'_+.*

(iii) *The positive cone E_+ is total on E.*

Proof. The equivalence "(i) $\Leftrightarrow$ (ii)" is obvious; the equivalence "(i) $\Leftrightarrow$ (iii)" follows for example from [AT07, Theorem 2.13]. □

The next proposition shows that the question whether a vector in an ordered Banach space is positive can be answered by testing against positive functionals.

Proposition B.2.3. *Let E be an ordered Banach space. A vector $x \in E$ is positive if and only if $\langle x', x\rangle \geq 0$ for every positive functional $x' \in E'$.*

Proof. This follows from the Hahn-Banach theorem; see for example [MP07, Lemma 1] for details. □

As a corollary we obtain the following correspondence between positivity of a bounded linear operator and its adjoint:

Corollary B.2.4. *Let E, F be ordered Banach spaces. An operator $T \in \mathcal{L}(E, F)$ is positive if and only if $T'F'_+ \subseteq E'_+$.*

Proof. The implication "$\Rightarrow$" is obvious and the converse implication "$\Leftarrow$" follows from Proposition B.2.3. □

Note that if (E', E'_+) and (F', F'_+) are ordered Banach spaces (which is fulfilled if and only if E_+ and F_+ are total in E and F, respectively), then the condition $T'F'_+ \subseteq E'_+$ in the above corollary simply means that T' is positive.

The following theorem is one reason why normality of positive cones is an important concept. Although we do not need this theorem in the main text, we chose to include it here in order to demonstrate the importance of the concept "normality".

Theorem B.2.5. *Let E be an ordered Banach space. Then the following assertions are equivalent:*

(i) *The positive cone E_+ is normal.*

(ii) *The dual cone E'_+ is generating in E', i.e. $E'_+ - E'_+ = E'$.*

Proof. This theorem can, for instance, be found in [AT07, Theorem 2.40]. □

B.3 Automatic Continuity and Extension Theorems

It is an outstanding property of ordered Banach spaces that, under rather weak assumptions, positivity of linear mappings automatically implies their continuity. This is the content of the following two theorems.

Theorem B.3.1. *Let E be an ordered Banach space with generating cone. Then every positive linear functional $\varphi : E \to \mathbb{R}$ is continuous.*

Proof. See e.g. [AN09, Lemma 2.3]. □

More generally, we have:

Theorem B.3.2. *Let E, F be ordered Banach spaces and suppose that the positive cone E_+ is generating in E. Then every positive linear map $T : E \to F$ is continuous.*

Proof. This result can, for instance, be found in [AN09, Theorem 2.8]. □

We need one more important concept, namely the extension of positive linear mappings. One can find rather general theorems on this topic in the literature, even in the setting of ordered vector spaces. Here, we list only some important special cases which we use in Part II of the thesis. To state them we need the following notion: let E be an ordered Banach space. A (not necessarily closed) vector subspace $V \subseteq E$ is called *dominating* if for every $y \in E$ there exists a vector $x \in V$ with $x \geq y$. Note that if E possesses an order unit $u \in E_+$ and this order unit is contained in V, then the subspace V is dominating.

Theorem B.3.3. *Let E, F be ordered Banach spaces and suppose that F is an order complete vector lattice. Let $V \subseteq E$ be a dominating vector subspace and let $T : V \to F$ be a positive linear mapping, meaning that $Tx \geq 0$ for all $x \in V \cap E_+$. Then there exists a positive linear mapping $\tilde{T} : E \to F$ which extends T.*

Proof. This is a special case of [AT07, Theorem 1.30]. □

What we actually need are the following two corollaries of Theorem B.3.3:

Corollary B.3.4. *Let E be an ordered Banach space with generating cone and let $V \subseteq E$ be a dominating vector subspace. If $\varphi : V \to \mathbb{R}$ is linear and positive, meaning that $\varphi(x) \geq 0$ for all $x \in V \cap E_+$, then there exists a positive functional $x' \in E'$ which extends φ.*

Proof. Since the ordered Banach space $\mathbb{R}$ is an order complete vector lattice, follows from Theorems B.3.3 and B.3.1. □

Our second corollary of Theorem B.3.3 states that, if the cone in E is generating, if V is dense in E and if the space F is even a Banach lattice, then we can drop the condition of order completeness on F:

Corollary B.3.5. *Let E be an ordered Banach space with generating cone and let F be a real Banach lattice. Suppose that $V \subseteq E$ is a dominating vector subspace and let $T : V \to F$ be a positive linear mapping, meaning that $Tx \geq 0$ for all $x \in V \cap E_+$. If V is dense in E, then there exists an operator $\tilde{T} \in \mathcal{L}(E; F)$ which extends T.*

Proof. The bi-dual space F'' of a F is an order complete Banach lattice; we consider F as sublattice of F'' by means of evaluation. Then, due to Theorem B.3.3, the mapping $T : V \to F \hookrightarrow F''$ can be extended to a positive linear mapping $\tilde{T} : E \to F''$. Since the cone in E is generating, the mapping $\tilde{T}$ is continuous according to Theorem B.3.2. Moreover, $\tilde{T}$ maps the dense subset V of E into $F \subseteq F''$. Since F is closed in F'', this implies that $\tilde{T}E \subseteq F$. Hence, we have indeed $\tilde{T} \in \mathcal{L}(E; F)$. □

Note that in Corollary B.3.4 we only assumed E_+ to be generating for convenience reasons (this is the version that we need in the main text); nevertheless, if E_+ is not generating, the corollary still holds except for the fact that x' might no longer be continuous. By contrast, the assumption that E_+ be generating is essential in Corollary B.3.5; we need it to derive continuity of $\tilde{T}$, and this continuity is in turn needed to prove that the range of $\tilde{T}$ is contained in F.

B.4 Banach Lattices and Hilbert Lattices

Banach lattices are an important special case of ordered Banach spaces; we freely use their theory throughout the thesis. Even a basic introduction to this field would lead way too far here, so we refer the reader to the extensive literature about Banach lattices, instead; see for instance the monographs [Sch74], [AB85, Chapter 4], [MN91] and [Zaa97]. Most of the notation that we use is described in the preliminaries of the thesis. Moreover, we give a concise introduction to the theory of *complex* Banach lattices and Sections C.4 and C.5 of the appendix.

In the present section we restrict ourselves to a brief discussion of so-called *Hilbert lattices*, a concept which we employ on a few occasions in the main text and with which the reader might no be completely familiar. For some additional information we refer for example to [Sch74, Section IV.6]. Complex Hilbert lattices will be briefly discussed in Section C.6.

Definition B.4.1 (Hilbert lattice). A *Hilbert lattice* is a (real) Banach lattice H whose norm is induced by an inner product.

The structure theory of Hilbert lattices is not particularly rich since the following proposition says that they are in fact nothing else but L^2-spaces.

Proposition B.4.2. *For every Hilbert lattice H the following assertions hold:*

(a) *There exists a measure space (Ω, μ) and an isometric (and thus unitary) Banach lattice isomorphism $H \to L^2(\Omega, \mu; \mathbb{R})$.*

(b) *The canonical mapping $H \to H'$, $x \mapsto (\cdot|, x)$ is an isometric Banach lattice isomorphism.*

Proof. Assertion (a) can for example be found, along with some additional properties, in [Sch74, Theorem IV.6.7]. Assertion (b) is an immediate consequence of (a). □

The above proposition shows that, in all our theorems which deal with Hilbert lattices, we could simply restrict ourselves us to L^2-spaces without any loss of generality. Yet, at least in Part III of the thesis, we prefer to use L^2-spaces only in concrete examples and applications and to formulate all general theorems in the language of Hilbert lattices, instead. In fact, we find that this better fits into the main thrust of third part, where the entire theory is formulated in the setting of Banach lattices.

C

Complexifications

Complexifications of real Banach spaces are an essential tool in functional analysis; they are, for example, needed to apply spectral theory on real Banach spaces. There can be several reasons why one insists on working with real Banach spaces and using complexifications of them instead of working with complex Banach spaces right from the beginning. For example, our main reason to do this in Part I of the thesis is that our results are actually only true on real Banach spaces (cf. Example 2.1.7(b)). Another reason becomes important if we work on Banach lattices as we do throughout Part III of the thesis: the theory of Banach lattices is intrinsically real, so one has no choice but to start with real Banach lattices and complexify them in order to obtain the concept of a *complex Banach lattice*. Moreover, for many complex-valued function spaces there is a canonical object which we might consider to be the "real part" of the space, namely the subset of all real-valued functions in this space. This suggests the definition of what we might call a "real operator" on such a space, namely an operator which maps real-valued functions to real-valued functions. Such a notion cannot be defined in a canonical way on an abstract complex Banach space, but it can be defined on a complexification of a real Banach space.

In Section C.1 we define complexifications of real Banach spaces and consider some elementary properties of them. In Section C.2 we demonstrate that complexifications are quite well-behaved with respect to certain standard constructions of Banach spaces (such as quotient spaces). In Section C.3 we show how complexifications can be used to employ spectral theoretic notions on real Banach spaces. Sections C.4 and C.5 deal with the important special case of *complex Banach lattices* and in Section C.6 briefly discusses complex Hilbert lattices and sesqui-linear forms on them.

Most of the results in this appendix are very elementary and we thus omit their proofs. The major task here is not to prove or discuss deep results, but to find the right notions in order to build a general and abstract framework for complexifications which can be used in the main text of the thesis. Yet, a few of the results that we need are not completely elementary (in particular in the theory of complex Banach lattices); for those results we include a reference to the literature or we briefly indicate the major steps of the proof.

Most concepts which we introduce here are used tacitly in the main text. Whenever the reader encounters an unfamiliar result about complexifications in the thesis, the result can be looked up in this appendix.

C.1 Complexifications of Real Banach Spaces

It seems to be difficult to find a comprehensive treatment of complexifications of real Banach spaces in the literature. A paper which contains quite a wealth of information about this topic is [MST99]; the following definition is also taking from this paper (what we call a *complexification* here is called a *reasonable complexification* in [MST99]).

Definition C.1.1 (Complexification of a real Banach space). Let E be a real Banach space. A *complexification* of E is a tuple $(E_{\mathbb{C}}, \varphi)$ with the following properties:

(i) $E_{\mathbb{C}}$ is a complex Banach space and $\varphi : E \to E_{\mathbb{C}}$ is an isometric $\mathbb{R}$-linear map.

(ii) We have $\varphi(E) + i\varphi(E) = E$ and $\varphi(E) \cap i\varphi(E) = \{0\}$.

(iii) We have $\|\varphi(x) + i\varphi(y)\| = \|\varphi(x) - i\varphi(y)\|$ for all $x, y \in E$.

In [Glü16d] we used a slightly weaker definition of complexifications, but it seems that the above definition comes closest to what one would expect intuitively from such a notion.

Let $(E_{\mathbb{C}}, \varphi)$ be a complexification of a real Banach space E. By means of the mapping φ we may, of course, consider E as a subset of $E_{\mathbb{C}}$. If we do so, we simply say that $E_{\mathbb{C}}$ is a complexification of E, thereby suppressing φ in the notation. Although we shall do this frequently, it is important to always keep in mind that whatever we say about complexifications can be restated more precisely by invoking the mapping φ.

If $E_{\mathbb{C}}$ is a complexification of a real Banach space E, then every vector $z \in E_{\mathbb{C}}$ can be decomposed as $z = x + iy$, where x and y are two uniquely determined elements of E. The vector x is called the *real part* of z and is denoted by $x =: \operatorname{Re} z$; the vector y is called the *imaginary part* of z and is denoted by $y =: \operatorname{Im} z$. The vector $\overline{z} := x - iy$ is called the *complex conjugate* of z. Property (iii) in Definition C.1.1 means that $\|z\| = \|\overline{z}\|$ for every $z \in E_{\mathbb{C}}$. Note that we have $\overline{iz} = -i\overline{z}$ as well as $\operatorname{Re}(iz) = -\operatorname{Im} z$ and $\operatorname{Im}(iz) = \operatorname{Re} z$ for all $z \in E_{\mathbb{C}}$. We shall often say that E is the *real part* of $E_{\mathbb{C}}$.

Whenever one introduces a certain mathematical concept, it is a good idea to make sure that we are not talking about the empty set. This is ensured by the following proposition.

Proposition C.1.2. *Let E be a real Banach space. Then there exists a complexification $(E_{\mathbb{C}}, \varphi)$ of E.*

Proof. Such a complexification can be easily constructed by endowing an algebraic complexification of the real vector space E with the norm given in [MST99, Proposition 3]. □

The question whether the complexification of a real Banach space is uniquely determined is settled by the following proposition; its proof follows right from the definition of a complexification, so we omit the details.

Proposition C.1.3. *Let E be a real Banach space and let $(E_{\mathbb{C},1}, \varphi_1)$ and $(E_{\mathbb{C},1}, \varphi_2)$ be two complexifications of E. Then there exists exactly one isomorphism of complex Banach spaces $\psi : E_{\mathbb{C},1} \to E_{\mathbb{C},2}$ which fulfils $\psi\varphi_1 = \varphi_2$.*

If we consider E as a subspace of $E_{\mathbb{C},1}$ and $E_{\mathbb{C},2}$ by means of φ_1 and φ_2, then the equality $\psi\varphi_1 = \varphi_2$ simply means that ψ acts as the identity on E. We call the isomorphism ψ in the above proposition the *canonical isomorphism* between the complexifications $(E_{\mathbb{C},1}, \varphi_1)$ and $(E_{\mathbb{C},2}, \varphi_2)$ of E. Note that the canonical isomorphism need not be isometric, in general, but its restriction to $\varphi_1(E)$ is isometric. If ψ is the canonical isomorphism between $(E_{\mathbb{C},1}, \varphi_1)$ and $(E_{\mathbb{C},2}, \varphi_2)$, then ψ^{-1} is the canonical isomorphism between $(E_{\mathbb{C},2}, \varphi_2)$ and $(E_{\mathbb{C},1}, \varphi_1)$.

If $E_{\mathbb{C}}$ is a complexification of a real Banach space E, then the mappings $\operatorname{Re} : E_{\mathbb{C}} \to E$ and $\operatorname{Im} : E_{\mathbb{C}} \to E$ are contractive; this is the content of the following proposition which is taken from [MST99, Proposition 1].

Proposition C.1.4. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach space E. Then we have $\|\operatorname{Re} z\| \leq \|z\|$ and $\|\operatorname{Im} z\| \leq \|z\|$ for all $z \in E_{\mathbb{C}}$.*

Proof. This follows readily from property (iii) in Definition C.1.1. □

Now we want to discuss how a linear operator between two Banach spaces can be extended to complexifications of those Banach spaces; the answer is given by the following simple, but important result. Again, we omit the proof.

Proposition C.1.5. *Let $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ be complexifications of real Bananch spaces E and F, respectively. Let $A : E \supseteq D(A) \to F$ be a linear operator and define $D(A_{\mathbb{C}}) := D(A) + iD(A) \subseteq E_{\mathbb{C}}$. Then there exists a uniquely determined linear (over $\mathbb{C}$) operator $A_{\mathbb{C}} : E_{\mathbb{C}} \supseteq D(A_{\mathbb{C}}) \to F_{\mathbb{C}}$ such that $A_{\mathbb{C}}|_{D(A)} = A$.*

The operator A is closed if and only if $A_{\mathbb{C}}$ is closed and we have $A \in \mathcal{L}(E;F)$ if and only if $A_{\mathbb{C}} \in \mathcal{L}(E_{\mathbb{C}};F_{\mathbb{C}})$.

In the situation of Proposition C.1.5 the operator $A_{\mathbb{C}}$ is called the *complex extension* of the operator A to the complexifications $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ of E and F.

Here, we point out again that, by abuse of notation, we suppressed the embeddings $\varphi_E : E \to E_{\mathbb{C}}$ and $\varphi_F : F \to F_{\mathbb{C}}$ in the above proposition. If we intend to be more precise, we have have to restate, for instance, the first part of the proposition as follows:

Let $(E_{\mathbb{C}}, \varphi_E)$ and $(F_{\mathbb{C}}, \varphi_F)$ be complexifications of real Banach spaces E and F, let $A : E \supseteq D(A) \to F$ be a linear operator and define $D(A_{\mathbb{C}}) := \varphi_E(D(A)) +$

$i\varphi_E(D(A)) \subseteq E_{\mathbb{C}}$. Then there exists a uniquely determined linear operator $A_{\mathbb{C}} : E_{\mathbb{C}} \supseteq D(A_{\mathbb{C}}) \to F_{\mathbb{C}}$ such that $A_{\mathbb{C}}|_{\varphi_E(D(A))} = \varphi_F A$.

Such formulations are of course a bit cumbersome, so we avoid them whenever possible; nevertheless, whenever we suppress the embedding of a real Banach space into a complexification in our notation, we encourage the reader to recall that this is actually a bit imprecise and can be rephrased in a rigorous way.

In the context of Proposition C.1.5 we point out that the *complex extension* $A_{\mathbb{C}}$ of A is only uniquely determined since the complexifications $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ are already chosen. If we choose other complexifications, then we obtain another complex extension of A. On the other hand, all complex extensions of A are, of course, essentially the same in the following sense: let $E_{\mathbb{C},1}$ and $E_{\mathbb{C},2}$ be two complexifications of E, let $F_{\mathbb{C},1}$ and $F_{\mathbb{C},2}$ be two complexifications of F and let

$$A_{\mathbb{C},1} : E_{\mathbb{C},1} \supseteq D(A_{\mathbb{C},1}) \to F_{\mathbb{C},1} \quad \text{and} \quad A_{\mathbb{C},2} : E_{\mathbb{C},2} \supseteq D(A_{\mathbb{C},2}) \to F_{\mathbb{C},2}$$

be the complex extensions of a linear map $A : E \supseteq D(A) \to F$, respectively. Then $A_{\mathbb{C},1}$ and $A_{\mathbb{C},2}$ are intertwined by the canonical isomorphisms $E_{\mathbb{C},1} \to E_{\mathbb{C}_2}$ and $F_{\mathbb{C},1} \to F_{\mathbb{C},2}$. In particular, $A_{\mathbb{C},1}$ and $A_{\mathbb{C},2}$ are similar in case that $E = F$, $E_{\mathbb{C},1} = F_{\mathbb{C},1}$ and $E_{\mathbb{C},2} = F_{\mathbb{C},2}$.

Let us now briefly deal with the question whether a given operator between two complexifications is in fact the complex extension on an operator between the corresponding real spaces. The answer is given by the following proposition whose proof is, again, elementary.

Proposition C.1.6. *Let $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ be complexifications of real Bananch spaces E and F, respectively. Let $\tilde{A} : E_{\mathbb{C}} \supseteq D(\tilde{A}) \to F_{\mathbb{C}}$ be a linear operator. Then the following assertions are equivalent:*

(i) *There exists a linear operator $A : E \supseteq D(A) \to F$ such that $\tilde{A} = A_{\mathbb{C}}$.*

(ii) *We have $D(\tilde{A}) = D(\tilde{A}) \cap E + iD(\tilde{A}) \cap E$ and $\tilde{A}x \in F$ for all $x \in D(\tilde{A}) \cap E$.*

If the equivalent assertions (i) *and* (ii) *are fulfilled, then the operator A is uniquely determined and we have $D(A) = D(\tilde{A}) \cap E$ and $A = \tilde{A}|_{D(A)}$.*

Suppose that the assumptions of Proposition C.1.6 are fulfilled. The operator $\tilde{A}$ is called *real* if it fulfils the equivalent assertions (i) and (ii). Note that an operator $\tilde{A} \in \mathcal{L}(E;F)$ is real if and only if $\tilde{A}E \subseteq F$.

Every bounded linear operator between the complexifications of two real Banach spaces can be decomposed into a (complex) linear combination of two real operators. More precisely, we have the following result:

Proposition C.1.7. *Let $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ be complexifications of real Bananch spaces E and F and let $\tilde{T} \in \mathcal{L}(E_{\mathbb{C}}; F_{\mathbb{C}})$ Then there exist two uniquely determined real operators $\tilde{R}, \tilde{S} \in \mathcal{L}(E_{\mathbb{C}}; F_{\mathbb{C}})$ such that $\tilde{T} = \tilde{R} + i\tilde{S}$.*

Proof. The uniqueness of $\tilde{R}$ and $\tilde{S}$ is easy to see. To prove the existence, one defines operators $R, S \in \mathcal{L}(E; F)$ by

$$Rx := \frac{1}{2}(\tilde{T}x + \overline{\tilde{T}x}) \quad \text{and} \quad Sx := \frac{1}{2i}(\tilde{T}x - \overline{\tilde{T}x})$$

for all $x \in E$. We let $\tilde{R} := R_{\mathbb{C}}$ and $\tilde{S} := S_{\mathbb{C}}$. Then we have $\tilde{T}x = \tilde{R}x + i\tilde{S}x$ for all $x \in E$ and hence for all $x \in E_{\mathbb{C}}$. □

We close this section with the following two simple propositions about C_0-semigroups which are needed on several occasions in the main text. Again, we omit the straightforward proofs.

Proposition C.1.8. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach space E and let $A : E \supseteq D(A) \to E$ be a linear operator. Then A generates a C_0-semigroup on E if and only if $A_{\mathbb{C}}$ generates a C_0-semigroup on $E_{\mathbb{C}}$, and in this case we have $(e^{tA})_{\mathbb{C}} = e^{tA_{\mathbb{C}}}$ for all $t \geq 0$.*

Proposition C.1.9. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach space E and let $(e^{t\tilde{A}})_{t\geq 0}$ be a C_0-semigroup on $E_{\mathbb{C}}$. Then $\tilde{A}$ is real if and only if the operator $e^{t\tilde{A}}$ is real for every $t \geq 0$.*

If the above equivalent assertions are fulfilled and A is the unique linear operator on A with $\tilde{A} = A_{\mathbb{C}}$, then A generates a C_0-semigroup on E and we have $e^{t\tilde{A}} = (e^{tA})_{\mathbb{C}}$ for all $t \geq 0$.

C.2 Standard Constructions for Complexificiations of Real Banach Spaces

As a first standard construction for Banach spaces we consider taking the dual space. The Banach space $\mathbb{C}$ is a complexification of the real Banach space $\mathbb{R}$, so everything which was said in the previous section about bounded linear operators in particular applies to bounded linear functionals. If $E_{\mathbb{C}}$ is a complexification of a real Banach space E, then every functional $x' \in E'$ has a unique complex extensions $(x')_{\mathbb{C}} \in (E_{\mathbb{C}})'$. On the other hand, according to Proposition C.1.6, a linear functional $\tilde{x}' \in (E_{\mathbb{C}})'$ is real (i.e. the complexification of a functional $x' \in E'$) if and only if $\langle \tilde{x}', x\rangle \in \mathbb{R}$ for all $x \in E$. Moreover, every functional $\tilde{x}' \in (E_{\mathbb{C}})'$ can be decomposed into a (complex) linear combination of two real fuctionals in $(E_{\mathbb{C}})'$.

If we identify E' with a subset of $(E_{\mathbb{C}})'$ via the mapping $x' \mapsto (x')_{\mathbb{C}}$ then, using what was said above, it is not difficult to prove the following consistence

property which one might briefly (but inaccurately) sum up as $(E_{\mathbb{C}})' = (E')_{\mathbb{C}}$ and $(A_{\mathbb{C}})' = (A')_{\mathbb{C}}$:

Proposition C.2.1. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach space E and let $A : E \supseteq D(A) \to E$ be a densely defined linear operator.*

Then $(E_{\mathbb{C}})'$ is a complexification of E'; moreover, $A_{\mathbb{C}}$ is also densely defined and, with respect to the aforementioned complexification, we have $(A_{\mathbb{C}})' = (A')_{\mathbb{C}}$.

More precisely, if $(E_{\mathbb{C}}, \varphi)$ is a complexification of a real Banach space E and we define

$$\eta : E' \to (E_{\mathbb{C}})', \qquad x' \mapsto (x')_{\mathbb{C}},$$

then $((E_{\mathbb{C}})', \eta)$ is a complex extension of E'.

Let us now come to subspaces and quotient spaces of complexifications. Again, everything turns out to work as expected; the only difficulty in the proof is not to get lost in the notation.

Proposition C.2.2. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach space E and let $V \subseteq E$ be a closed vector subspace. Let $T \in \mathcal{L}(E)$ be an operator which leaves V invariant.*

(a) *The space $V_{\mathbb{C}} := V + iV$ is a closed vector subspace of $E_{\mathbb{C}}$ and a complexification of V. Moreover, $V_{\mathbb{C}}$ is invariant with respect to $T_{\mathbb{C}}$ and the restriction $(T_{\mathbb{C}})|_{V_{\mathbb{C}}}$ is the complex extension of the restriction $T|_V$.*

(b) *If we identify E/V with a subspace of $E_{\mathbb{C}}/V_{\mathbb{C}}$ via the mapping $x+V \mapsto x+V_{\mathbb{C}}$, then $E_{\mathbb{C}}/V_{\mathbb{C}}$ is a complexification of E/V. Moreover, the mapping induced by $T_{\mathbb{C}}$ on $E_{\mathbb{C}}/V_{\mathbb{C}}$ is the complex extension of the mapping induced by T on E/V.*

We now leave it to the reader to rephrase the above result in a more precise way which explicitly uses the embedding φ of a real Banach space into a given complexification.

In view of the above proposition it is natural to ask under what conditions a given subspace $\tilde{V} \subseteq E_{\mathbb{C}}$ can be written in the form $V + iV$ for some $V \subseteq E$. The answer to this question is easy. We state it, even for non-closed subspaces, in the following proposition.

Proposition C.2.3. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach space E and let $\tilde{V} \subseteq E_{\mathbb{C}}$ be a vector subspace. If we define $V := V_{\mathbb{C}} \cap E$, then the following assertions are equivalent:*

(a) $\tilde{V} = V + iV$.

(b) *There exists a subset W of E such that $\tilde{V} = W + iW$.*

(c) *$\tilde{V}$ is invariant with respect to taking real parts, i.e. we have* $\operatorname{Re} z \in \tilde{V}$ *whenever* $z \in \tilde{V}$.

(d) *$\tilde{V}$ is* conjugation invariant, *i.e. we have* $\overline{z} \in \tilde{V}$ *whenever* $z \in \tilde{V}$.

If these equivalent assertions are fulfilled, then we have $V = W$.

An important special case of subspaces of a given Banach space are ranges of continuous projections. In the subsequent proposition we collect a few results about this special case. To state the proposition we need to recall the following notion: let E be a real or complex Banach space, let $A : E \supseteq D(A) \to E$ be a linear operator and let $P \in \mathcal{L}(E)$ be a projection; define $Q := I_E - P$. We say that P *reduces* A if we have $D(A) = D(A) \cap PE + D(A) \cap QE$ and if A maps $D(A) \cap PE$ to PE and $D(A) \cap QE$ to QE. In this case, we can define the restricted operators

$$A|_{PE} : PE \supseteq D(A|_{PE}) := D(A) \cap PE \to PE, \quad x \mapsto Ax \quad \text{and}$$
$$A|_{QE} : QE \supseteq D(A|_{QE}) := D(A) \cap QE \to QE, \quad x \mapsto Ax.$$

It is not difficult to see that this construction is compatible with complexifications. More explicitly, the situation is as follows:

Proposition C.2.4. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach space E, let $A : E \supseteq D(A) \to E$ be a linear operator and let $P, Q \in \mathcal{L}(E)$ be projections such that $P + Q = I_E$.*

(a) *We have $PE = E \cap P_{\mathbb{C}}E_{\mathbb{C}}$ and $QE = E \cap Q_{\mathbb{C}}E_{\mathbb{C}}$; moreover, $P_{\mathbb{C}}E_{\mathbb{C}}$ is a complexification of PE and $Q_{\mathbb{C}}E_{\mathbb{C}}$ is a complexification of QE.*

(b) *The projection P reduces A if and only if $P_{\mathbb{C}}$ reduces $A_{\mathbb{C}}$; in this case, $(A_{\mathbb{C}})|_{P_{\mathbb{C}}E_{\mathbb{C}}}$ is the complex extension of $A|_{PE}$, and the analogous result for Q is also true.*

Finally, we formulate a result about the compatibility of complexifications with ultra powers; this is a simple consequence of Proposition C.2.2 above.

Proposition C.2.5. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach space E and let $\mathcal{U}$ be an ultra filter on an index set $I \neq \emptyset$. Let $T \in \mathcal{L}(E)$. If we identify $E_{\mathcal{U}}$ with a subspace of $(E_{\mathbb{C}})_{\mathcal{U}}$ via the mapping*

$$(x_i) + c_{0,\mathcal{U}}(I; E) \mapsto (x_i) + c_{0,\mathcal{U}}(I; E_{\mathbb{C}}),$$

then $(E_{\mathbb{C}})_{\mathcal{U}}$ is a complexification of $E_{\mathcal{U}}$. Moreover, the lifted operator $(T_{\mathbb{C}})_{\mathcal{U}}$ is the complex extension of the lifted operator $T_{\mathcal{U}}$.

C.3 Spectral Theoretic Aspects

Using complexifications, we can now define the most important spectral theoretic notions for operators on real Banach spaces. We start right away with the major definition.

Definition C.3.1 (Spectrum of operators on real Banach spaces). Let E be a real Banach space and let $A : E \supseteq D(A) \to E$ be a linear operator. We define the *spectrum*, the *point spectrum* and the *approximate point spectrum* of A as

$$\sigma(A) := \sigma(A_{\mathbb{C}}), \quad \sigma_{\mathrm{pnt}}(A) := \sigma_{\mathrm{pnt}}(A_{\mathbb{C}}), \quad \sigma_{\mathrm{appr}}(A) := \sigma_{\mathrm{appr}}(A_{\mathbb{C}})$$

where $A_{\mathbb{C}}$ is the complex extension of A to any complexification $E_{\mathbb{C}}$ of E.

It follows from the discussion before Proposition C.1.6 that the above notions do not depend on the choice of the complexification of E. Let A be a linear operator on a real Banach space E and let $T \in \mathcal{L}(E)$. The *spectral bounded* $\mathrm{s}(A)$ and the *pseudo-spectral bound* $\mathrm{s}_0(A)$ of A as well as the *spectral radius* $\mathrm{r}(T)$ of T are also defined to be the corresponding values for any complex extension of A and T.

In the main text we need the following simple observations about the spectrum and the eigenvalues of an operator on a real Banach space. As usual in this appendix we omit the simple proof.

Proposition C.3.2. *Let E be a real Banach space and let $E : E \supseteq D(A) \to E$ be a linear operator. Let $A_{\mathbb{C}}$ by the complex extension of A to some complexification $E_{\mathbb{C}}$ of E. Let $\lambda \in \mathbb{C}$. Then the following assertions hold:*

(a) *We have $\lambda \in \sigma(A)$ if and only $\overline{\lambda} \in \sigma(A)$.*

(b) *We have $\lambda \in \sigma_{\mathrm{pnt}}(A)$ if and only if $\overline{\lambda} \in \sigma_{\mathrm{pnt}}(A)$.*

Moreover, if $\lambda \in \sigma_{\mathrm{pnt}}(A) = \sigma_{\mathrm{pnt}}(A_{\mathbb{C}})$ and $z \in E_{\mathbb{C}}$ is an eigenvector of $A_{\mathbb{C}}$ for the eigenvalue λ, then $\overline{z}$ is an eigenvector of $A_{\mathbb{C}}$ for the eigenvalue $\overline{\lambda}$.

(c) *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of $A_{\mathbb{C}}$ with eigenvector $z = x + iy$ (where $x, y \in E$), then x and y are linearly independent elements of $E_{\mathbb{C}}$.*

C.4 Complexifications of Banach Lattices

In this section we discuss an important special case of complexifications, namely complexifications of Banach lattices. An introduction to this topic can also be found in [Sch74, Section II.11] or [MN91, Section 2.2]. We refer also to [MW74] where an axiomatic theory of complex vector lattices is developed. Before we start with the major definitions, we need the following observation:

Remark C.4.1. Let E be a real Banach lattice and let $x, y \in E$. Then the supremum $\sup_{\theta\in[0,2\pi]}(\cos\theta\, x + \sin\theta\, y)$ exists in E.

Proof. This can, for instance, be found in [Sch74, Section II.11] (more precisely, it is shown in this reference that the supremum $\sup_{\theta\in[0,2\pi]}|\cos\theta\, x + \sin\theta\, y|$ exists, but it is easy to see that this is equivalent to the existence of the above supremum and that both suprema actually coincide). □

Now we define complexifications of real Banach lattices. The main idea is very similar to Definition C.1.1; the only difference is that we now require a stronger property of the norm on the complexification:

Definition C.4.2 (Complexification of a Banach lattice). Let E be a real Banach lattice. A *Banach lattice complexification* of E is a tuple $(E_{\mathbb{C}}, \varphi)$ with the following properties:

(i) $E_{\mathbb{C}}$ is a complex Banach space and $\varphi : E \to E_{\mathbb{C}}$ is an $\mathbb{R}$-linear map.

(ii) We have $\varphi(E) + i\varphi(E) = E$ and $\varphi(E) \cap i\varphi(E) = \{0\}$.

(iii) We have $\|\varphi(x) + i\varphi(y)\| = \|\sup_{\theta\in[0,2\pi]}(\cos\theta\, x + \sin\theta\, y)\|$ for all $x, y \in E$.

Note that in the situation of the above definition the mapping φ is automatically isometric due to property (iii). The fact that every real Banach lattice possesses a complexification follows from the following simple observation:

Remark C.4.3. Let E be a real Banach lattice, let V be a complex vector space which contains E as a real vector subspace and assume that $E \cap iE = \{0\}$ (it is easily seen that such a space V exists). Define $E_{\mathbb{C}} = E + iE \subseteq V$ and endow $E_{\mathbb{C}}$ with the norm $\|x+iy\| = \|\sup_{\theta\in[0,2\pi]}(\cos\theta\, x+\sin\theta\, y)\|$ (note that the supremum is contained in E, so the norm of the supremum is defined). Then $(E_{\mathbb{C}}, \|\cdot\|)$ is a complex Banach space and, with respect to the canonical embedding $E \hookrightarrow E_{\mathbb{C}}$, a Banach lattice complexification of the real Banach lattice E.

We call the norm defined on $E + iE$ in the preceding remark the *complexification* of the norm on E.

One can readily verify that a complexification of a real Banach lattice E in the sense of Definition C.4.2 is also a complexification of the Banach space E in the sense of Definition C.1.1 (while, on the other hand, one should note that a Banach space complexification of a real Banach lattice need, in general, not be a Banach lattice complexification). This shows that such notions as the real and the imaginary part or the complex conjugate of a vector, which are defined for Banach space complexifications in Section C.1, can also be used for Banach lattice complexifications. In particular, the notion of a real operator,

and the complex extension of operators are also defined in the setting of complexifications of Banach lattices.

Note that if $E_{\mathbb{C},1}$ and $E_{\mathbb{C},2}$ are two complexifications of a real Banach lattice E, then the canonical isomorphism between them (which exists since $E_{\mathbb{C},1}$ and $E_{\mathbb{C},2}$ are also Banach space complexifications of E) is isometric; this follows from property (iii) in Definition C.4.2.

We often use the same notational simplification as in the previous sections: if $(E_{\mathbb{C}}, \varphi)$ is a Banach lattice complexification of a real Banach lattice E, then we may consider E as a subset of $E_{\mathbb{C}}$ by means of φ and we may thus simply say that $E_{\mathbb{C}}$ is a Banach lattice complexification of E, thereby suppressing the map φ.

Let $E_{\mathbb{C}}$ be a Banach lattice complexification of a real Banach lattice E. For all $z = x + iy$ (where $x, y \in E$) we define $|z| := \sup_{\theta\in[0,2\pi]}(\cos\theta\, x + \sin\theta\, y)$ to be the *complex modulus* of z. Then $|z| \in E_+$ and, according to property (iii) in Definition C.4.2, we have $\||z|\| = \|z\|$. It is worthwhile pointing out that, for $z \in E$, this definition of $|z|$ coincides with the usual definition of $|z|$ in the real vector lattice E. As for instance pointed out in [Sch74, p. 134], the complex modulus has the following useful properties: for each $z \in E_{\mathbb{C}}$ we have $|z| = 0$ if and only if $z = 0$; we have $|\alpha z| = |\alpha||z|$ for all $\alpha \in \mathbb{C}$ and all $z \in E_{\mathbb{C}}$; and we have $|z_1 + z_2| \leq |z_1 + z_2|$ for all $z_1, z_2 \in \mathbb{C}$. Moreover, we note that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$ for all $z \in E_{\mathbb{C}}$. Finally, we point out that the following norm estimates are sometimes useful: if $z \in E_{\mathbb{C}}$, then we have $\|z\| \leq \|\operatorname{Re} z\| + \|\operatorname{Im} z\| \leq 2\|z\|$.

In Part III of the thesis we work on complexifications of real Banach lattices throughout. It is thus convenient to switch the notation and denote the real Banach lattice by $E_{\mathbb{R}}$ and its complexification simply by E. Moreover, we shall call E a *complex Banach lattice*; more precisely, we use the following definition:

Definition C.4.4 (Complex Banach lattice). A *complex Banach lattice* is a tuple $(E, E_{\mathbb{R}}, \varphi)$ where $E_{\mathbb{R}}$ is a real Banach lattice and (E, φ) is a Banach lattice complexification of $E_{\mathbb{R}}$.

By abuse of notation we shall often simply say that E is a complex Banach lattice, thereby suppressing the real Banach lattice $E_{\mathbb{R}}$ as well as the embedding φ in the notation. We call the space $E_{\mathbb{R}}$ the *real part* of E. As mentioned above, it is often most convenient to simply consider $E_{\mathbb{R}}$ as a subset of E by means of the embedding $\varphi : E_{\mathbb{R}} \to E$, and we shall use this simplification almost throughout this appendix and the main text. The set $(E_{\mathbb{R}})_+$ is thus a subset of E, too, and we use the notation $E_+ := (E_{\mathbb{R}})_+$. We write $z \geq 0$ for an element $z \in E$ to say that $z \in E_+$. Similary, we write $z_1 \leq z_2$ for two elements $z_1, z_2 \in E$ to say that both elements z_1 and z_2 are contained in $E_{\mathbb{R}}$ and that $z_1 \leq z_2$. As in the real case we write $z_1 < z_2$ to indicate that $z_1 \leq z_2$, but $z_1 \neq z_2$.

It turns out that the Banach *lattice* complexification seems to be the natural choice of a complexification for many important function spaces. Let us list a few examples explicitly:

Examples C.4.5. (a) Let (Ω, μ) be a measure space and let $p \in [1, \infty]$. Then $L^p(\Omega, \mu; \mathbb{C})$ is a complex Banach lattice with real part $L^p(\Omega, \mu; \mathbb{R})$.

(b) Let K be a compact Hausdorff space. Then $\mathcal{C}(K; \mathbb{C})$ is a complex Banach lattice with real part $\mathcal{C}(K; \mathbb{R})$.

(c) More generally, let L be a locally compact Hausdorff space. Then $\mathcal{C}_0(L; \mathbb{C})$ is a complex Banach lattice with real part $\mathcal{C}_0(L; \mathbb{R})$.

Proof. This follows, for instance, from the uniqueness result in [MW74, Korollar 2.3]. □

Let E be a complex Banach lattice. An operator $T \in \mathcal{L}(E)$ is called *positive* if $TE_+ \subseteq E_+$. Note that a positive operator is always real. An operator in $\mathcal{L}(E_\mathbb{R})$ is positive if and only if its complex extension to E is positive. An operator $T \in \mathcal{L}(E)$ is called a *lattice homomorphism* if T is real and if the restricted operator $T|_{E_\mathbb{R}} \in \mathcal{L}(E_\mathbb{R})$ is a lattice homomorphism. Clearly, a lattice homomorphism on E is positive. An operator $T \in \mathcal{L}(E)$ is called an *isomorphism of complex Banach lattices* (or a *complex Banach lattice isomorphism*) if T is real and if the restricted operator $T|_{E_\mathbb{R}} \in \mathcal{L}(E_\mathbb{R})$ is a Banach lattice isomorhism. It is easy to see that T is a complex Banach lattice isomorphism if and only if T is bijective and T and T^{-1} are both positive. Two complex Banach lattices are called *(isometrically) Banach lattice isomorphic* if there exists an (isometric) isomorphism of complex Banach lattices between them. Two complex Banach lattices are (isometrically) Banach lattice isomorphic if and only if their real parts are (isometrically) Banach lattice isomorphic.

We have the following characterisation of complex lattice homomorphisms:

Proposition C.4.6. *Let E be a complex Banach lattice and let $T \in \mathcal{L}(E_\mathbb{C})$. The following assertions are equivalent:*

(a) *T is a lattice homomorphism.*

(b) *We have $|Tz| = |z|$ for all $z \in E$.*

Proof. See for instance [Sch74, p. 136]; a few more details are given in [Lot68, Satz 1.8]. □

C.5 Standard Constructions for Complexifications of Banach Lattices

In Section C.1 we have seen that Banach space complexifications are compatible with many standard constructions. Next we will demonstrate that the same is true for Banach lattice complexifications. To state those results, it is best to return to the notation that we used at the beginning of Section C.4, i.e. we denote the real Banach lattice under consideration by E and its complexification by $E_{\mathbb{C}}$. The constructions in most of the subsequent propositions can be considered to be special cases of the constructions in Section C.2; the two major points are that the underlying real Banach space that we construct (e.g. a dual or a quotient space) is still a real Banach lattice, and that the complex Banach space which we construct is not only a Banach space complexification of the real space (which follows in most cases immediately from the results in Section C.2), but also a Banach *lattice* complexification, meaning that the norm fulfils property (iii) from Definition C.4.2.

In this section, we only state results about complexifications of spaces. The corresponding results for operators on those spaces are immediate consequences of the corresponding propositions in Section C.2. We begin again with dual spaces:

Proposition C.5.1. *Let $E_{\mathbb{C}}$ be a Banach lattice complexification of a real Banach lattice E and consider E' as a subspace of $(E_{\mathbb{C}})'$ by means of the embedding $x' \mapsto (x')_{\mathbb{C}}$. Then $(E_{\mathbb{C}})'$ is a Banach lattice complexification of the real Banach lattice E'.*

Moreover, we have $\langle |x'|, x\rangle = \sup_{|z|\leq x} |\langle x', z\rangle|$ for all $x' \in (E_{\mathbb{C}})'$ and all $x \in E_+$

Proof. This result can be found in [Sch74, Corollary 3 to Theorem IV.1.8]. □

We want to recall how an *ideal* and a *sublattice* of a complex Banach lattice are defined. The definition of an ideal relies on the following observation:

Remark C.5.2. Let $E_{\mathbb{C}}$ be a Banach lattice complexification of a real Banach lattice E, let $I \subseteq E$ be a vector subspace (not necessarily closed) and define $I_{\mathbb{C}} := I + iI$. Then the following assertions are equivalent:

(i) I is an ideal in E.

(ii) If $z_1, z_2 \in I_{\mathbb{C}}$ with $|z_1| \leq |z_2|$ and $z_2 \in I_{\mathbb{C}}$, then $z_1 \in I_{\mathbb{C}}$.

Proof. "(ii) ⇒ (i)" This implications is obvious.

"(i) ⇒ (ii)" Assume that I is an ideal in E and let $z_1, z_2 \in E_{\mathbb{C}}$ such that $z_2 \in I_{\mathbb{C}}$ and $|z_1| \leq |z_2|$. It follows that $\operatorname{Re} z_2, \operatorname{Im} z_2 \in I$ and hence, $|\operatorname{Re} z_2| + |\operatorname{Im} z_2| \in I$. On the other hand, we have $|\operatorname{Re} z_1| \leq |z_1| \leq |z_2| \leq |\operatorname{Re} z_2| + |\operatorname{Im} z_2|$ and thus $\operatorname{Re} z_1 \in I$. For the same reason we have $\operatorname{Im} z_1 \in I$, so $z_1 \in I_{\mathbb{C}}$, as claimed. □

Let $E_{\mathbb{C}}$ be a complexification of a real Banach lattice E and let $\tilde{I} \subseteq E_{\mathbb{C}}$ be a vector subspace. We call $\tilde{I}$ an *ideal* if $\tilde{I}$ is conjugation invariant and if $z_1, z_2 \in E_{\mathbb{C}}$ with $|z_1| \leq |z_2|$ and $z_2 \in \tilde{I}$ implies that $z_1 \in \tilde{I}$. According to Proposition C.2.3 and Remark C.5.2 this is fulfilled if and only if there exists an ideal I in E such that $\tilde{I} = I + iI$.

A similar equivalence as in Remark C.5.2 also holds for *sublattices*, at least under the additional assumption that the subspace we consider is closed. Recall that a vector subspace V of a real Banach lattice E is called a *sublattice* of E if $|x| \in V$ for all $x \in V$; equivalently, V is stable with respect to all finite lattice operations. We now make the following observation which indicates how we should define a (closed) sublattice of a complex Banach space.

Remark C.5.3. Let $E_{\mathbb{C}}$ be a Banach lattice complexification of a real Banach lattice E, let $V \subseteq E$ be closed vector subspace of E and define $V_{\mathbb{C}} := V + iV \subseteq E_{\mathbb{C}}$. Then $V_{\mathbb{C}}$ is a closed vector subspace of $E_{\mathbb{C}}$ and the following assertion are equivalent:

(i) V is a sublattice of E.

(ii) For all $z \in V_{\mathbb{C}}$ we have $|z| \in V_{\mathbb{C}}$.

Proof. Clearly, $V_{\mathbb{C}}$ is closed.

"(ii) $\Rightarrow$ (i)" This implication is obvious.

"(i) $\Rightarrow$ (ii)" If V is a sublattice of E, then it is a Banach lattice with respect to the norm induced by E. Hence, it follows from Remark C.4.3 that there exists an norm $\|\cdot\|_1$ on the complex vector space $V_{\mathbb{C}}$ which renders it a Banach lattice complefication of V. For each $z \in V_{\mathbb{C}}$, let $|z|_{V_{\mathbb{C}}}$ denote the modulus of z in the complex Banach lattice $V_{\mathbb{C}}$.

Now, denote by $j : V_{\mathbb{C}} \to E_{\mathbb{C}}$ the canonical embedding. Obviously, j is a lattice homomorphism between the two complex Banach lattices $V_{\mathbb{C}}$ and $E_{\mathbb{C}}$. If we denote by $|\cdot|$ the complex modulus on $E_{\mathbb{C}}$, the we obtain according to Proposition C.4.6

$$|z|_{V_{\mathbb{C}}} = j(|z|_{V_{\mathbb{C}}}) = |j(z)| = |z|$$

for all $z \in V_{\mathbb{C}}$; in particular, $|z| \in V_{\mathbb{C}}$. □

We point out that the above proof also shows that we have $\|z\|_1 = \|z\|$ for all $z \in V_{\mathbb{C}}$.

The above remark motivates the following definition: let $E_{\mathbb{C}}$ be a complexification of real Banach lattice E. A closed vector subspace $\tilde{V} \subseteq E_{\mathbb{C}}$ is called a *sublattice* of $E_{\mathbb{C}}$ if it is conjugation invariant and if we have $|z| \in \tilde{V}$ for all $z \in \tilde{V}$. According to Proposition C.2.3 and Remark C.5.3 this is fulfilled if and only if there exists a closed sublattice V of E such that $\tilde{V} = V + iV$.

Now we can state the following proposition about sublattices and quotient spaces:

Proposition C.5.4. *Let $E_{\mathbb{C}}$ be a Banach lattice complexification of a real Banach lattice E, let $V \subseteq E$ be a closed sublattice and let $I \subseteq E$ a closed ideal in E; we define $V_{\mathbb{C}} := V + iV$ and $I_{\mathbb{C}} := I + iI$. Then $V_{\mathbb{C}}$ and $I_{\mathbb{C}}$ are closed and the following assertions hold:*

(a) *The space $V_{\mathbb{C}}$ is (with respect to the norm induced by $E_{\mathbb{C}}$) a Banach lattice complexification of the real Banach lattice V; the canonical embedding $V_{\mathbb{C}} \hookrightarrow E_{\mathbb{C}}$ is a lattice homomorphism.*

(b) *If we identify E/I with a subset of $E_{\mathbb{C}}/I_{\mathbb{C}}$ by means of the mapping $x + I \mapsto x + I_{\mathbb{C}}$, then $E_{\mathbb{C}}/I_{\mathbb{C}}$ is (with respect to the quotient norm) a Banach lattice complexification of the Banach lattice E/I; the quotient map $E_{\mathbb{C}} \twoheadrightarrow E_{\mathbb{C}}/I_{\mathbb{C}}$ is a lattice homomorphism.*

Proof. (a) This follows from the same arguments used in the proof of Remark C.5.3; see in particular what we noted immediately after the proof.

(b) See e.g. [Sch74, Proposition II.11.4]. □

There is another important concept of subspaces of Banach lattices: a closed vector subspace V of a real Banach lattice is called a *lattice subspace* of E if V is a vector lattice with respect to the order induced by E (equivalently: with respect to the order induced by the cone $V_+ := E_+ \cap V$). It can be shown with the help of an automatic continuity theorem (for instance [SW99, assertion V.5.6]) that a closed lattice subspace of a real Banach lattice is always a Banach lattice with respect to an equivalent norm (a result which, surprisingly, we could not find in the literature). However, we will not discuss this in detail here, and we do not need complexifications of general lattice subspaces, either. Instead, we are interested in the following special case:

Proposition C.5.5. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach lattice E and let $P \in \mathcal{L}(E)$ be a positive projection.*

(a) *The range PE is lattice ordered with respect to the order induced by E and for every $x \in PE$ the modulus of x in PE is given by $P|x|$.*

Moreover, $x \mapsto \|x\|_{PE} := \|P|x|\|$ defines a norm on PE which is equivalent to the norm induced by E and which renders PE a real Banach lattice.

(b) *The mapping $z \mapsto \|z\|_{P_{\mathbb{C}}E_{\mathbb{C}}} := \|P|z|\|$ defines a norm on $P_{\mathbb{C}}E_{\mathbb{C}}$ which is equivalent to the norm induced by $E_{\mathbb{C}}$ and with respect to which $P_{\mathbb{C}}E_{\mathbb{C}}$ is a Banach lattice complexification of the real Banach lattice PE.*

Moreover, the complex modulus of any vector $z \in P_{\mathbb{C}}E_{\mathbb{C}}$ in the complex Banach lattice $P_{\mathbb{C}}E_{\mathbb{C}}$ is given by $|z|_{P_{\mathbb{C}}E_{\mathbb{C}}} = P|z|$.

Proof. (a) This assertion is straightforward to prove; compare [Sch74, Proposition III.11.5] for some details.

(b) It is easy to check that whenever a subset $S \subseteq PE$ has a supremum $\sup_E S$ in E, then S has also a supremum in PE and this supremum is given by $P(\sup_E S)$. Hence, the assertion follows from Remark C.4.3. □

Next, we state the following result about ultra powers of complex Banach lattices which is an immediate consequences of Proposition C.5.4(b).

Proposition C.5.6. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach lattice E and let $\mathcal{U}$ be an ultra filter on an index set $I \neq \emptyset$. If we identify $E_{\mathcal{U}}$ with a subset of $(E_{\mathbb{C}})_{\mathcal{U}}$ by means of the embedding*

$$(x_i) + c_{0,\mathcal{U}}(I;E) \mapsto (x_i) + c_{0,\mathcal{U}}(I;E_{\mathbb{C}}),$$

then $(E_{\mathbb{C}})_{\mathcal{U}}$ is a complexification of the real Banach lattice $E_{\mathcal{U}}$; the canonical embedding $E_{\mathbb{C}} \mapsto (E_{\mathbb{C}})_{\mathcal{U}}$ is lattice homomorphism.

Another important concept in a Banach lattice is the construction of a principal ideal. Let E be a real Banach lattice and let $u \in E_+$. We call the set

$$E_u := \{x \in E : \exists c \geq 0 \, |x| \leq cu\}$$

the *principal ideal* generated by u in E; it is easy to see that E_u is indeed an ideal in E. Moreover, for each $x \in E_u$ we define

$$\|x\|_u := \inf\{c \geq 0 : \ |x| \leq cu\};$$

then $\|\cdot\|_u$ can be shown to be a complete norm on E_u which is called the *gauge norm* on E_u. It is known that $(E_u, \|\cdot\|_u)$ is a real Banach lattice and, in fact, even an AM-space with unit u [Sch74, the corollary to Proposition II.7.2] (see [Sch74, Definition II.7.1] for the definition of an *AM-space with unit*). Moreover, the canonical embedding $E_u \hookrightarrow E$ is a lattice homomorphism (and thus, in particular, continuous).

Now, let $E_{\mathbb{C}}$ be a complexification of a real Banach lattice E and let $u \in E_+$. As above we define

$$(E_{\mathbb{C}})_u := \{z \in E_{\mathbb{C}} : \exists c \geq 0 \, |z| \leq cu\}$$

to be the *principal ideal* generated by u. It is not difficult to see that E_u is indeed an ideal in the complex Banach lattice $E_{\mathbb{C}}$. Finally, we also define

$$\|z\|_u := \inf\{c \geq 0 : \ |z| \leq cu\}$$

for all $z \in (E_{\mathbb{C}})_u$. The mapping $z \mapsto \|z\|_u$ is called the *gauge norm* on $(E_{\mathbb{C}})_u$. Using what we have said above about the real case, one can now immediately prove the following result:

Proposition C.5.7. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach lattice E and let $u \in E_+$. Then the gauge norm $\|\cdot\|_u$ is a complete norm on $(E_{\mathbb{C}})_u$ with respect to which $(E_{\mathbb{C}})_u$ is a complexification of the real Banach lattice $(E_u, \|\cdot\|_u)$ and the canonical embedding $(E_{\mathbb{C}})_u \hookrightarrow E_{\mathbb{C}}$ is a lattice homomorphism.*

Motivated by the above discussion we close the section with the following complex version of Kakutani's representation theorem for AM-spaces with unit:

Proposition C.5.8. *Let $E_{\mathbb{C}}$ be a complexification of a real Banach lattice E; assume moreover that E is an AM-space with unit u. Then there exists a compact Hausdorff space and an isometric isomorphism of complex Banach lattices $E_{\mathbb{C}} \to \mathcal{C}(K;\mathbb{C})$ which maps u to $\mathbb{1}$.*

Proof. Since, for every compact Hausdorff space K, the space $\mathcal{C}(K;\mathbb{C})$ is a complexification of the real Banach lattice $\mathcal{C}(K;\mathbb{R})$, the assertion follows readily from Kakutani's representation theorem for AM-space with unit in the real case, which can e.g. be found in [Sch74, Theorem II.7.4] and [MN91, Theorem 2.1.3]. □

C.6 Complex Hilbert Lattices

In Part III of the main text we use the concept of a *complex Hilbert lattice* on a few occasions. Here we sum up a bit of information about them. We are going to need what has been said in Section B.4 about real Hilbert lattices. Let us begin with the definition:

Definition C.6.1 (Complex Hilbert lattice). By a *complex Hilbert lattice* we mean a complex Banach lattice whose real part is a Hilbert lattice.

Hence, a complex Hilbert lattice $H_{\mathbb{C}}$ is simply a complexification of a real Hilbert lattice H. We first point out that the inner product on H can be extended to $H_{\mathbb{C}}$ in a way compatible with the norm on $H_{\mathbb{C}}$:

Proposition C.6.2. *Let H be a real Hilbert lattice and let $H_{\mathbb{C}}$ denote a Banach lattice complexification of H. Let $(\cdot|\cdot)$ denote the inner product on H and define*

$$(z_1|z_2) := (\operatorname{Re} z_1|\operatorname{Re} z_2) + (\operatorname{Im} z_1|\operatorname{Im} z_2) - i(\operatorname{Re} z_1|\operatorname{Im} z_2) + i(\operatorname{Im} z_1|\operatorname{Re} z_2)$$

for all $z_1, z_2 \in H_{\mathbb{C}}$. Then $(\cdot|\cdot)$ is an inner product on the complex vector space $H_{\mathbb{C}}$ and we have $\|z\|^2 = (z|z)$ for all $z \in H_{\mathbb{C}}$; in particular, the complex Banach space $H_{\mathbb{C}}$ is a Hilbert space.

Proof. It is straightforward to check that $(\cdot|\cdot)$ defines indeed an inner product on $H_{\mathbb{C}}$. According to Proposition B.4.2(a) there exists a measure space (Ω, μ) and an isometric Banach lattice isomorphism $\phi : H \to L^2(\Omega, \mu; \mathbb{R})$. Since $L^2(\Omega, \mu; \mathbb{C})$ is a complexification of the real Banach lattice $L^2(\Omega, \mu; \mathbb{R})$, it follows that ϕ extends to an isometric isomorphism of complex Banach lattices $\phi_{\mathbb{C}} : H_{\mathbb{C}} \to L^2(\Omega, \mu; \mathbb{C})$. Since ϕ is isometric, it is unitary, i.e. we have

$$(\phi x_1 | \phi x_2) = (x_1 | x_2)$$

for all $x_1, x_2 \in H$ and from this one readily concludes that the same formula holds for $\phi_{\mathbb{C}}$ and for all $z_1, z_2 \in H_{\mathbb{C}}$. Since $\phi_{\mathbb{C}}$ is isometric, this implies that $\|z\|^2 = (z|z)$ for all $z \in H_{\mathbb{C}}$, as claimed. □

We will always assume that any complex Hilbert lattice $H_{\mathbb{C}}$ is endowed with the inner product described in the Proposition C.6.2 (which is, according to the proposition, automatically compatible with the norm on $H_{\mathbb{C}}$)). The above proof also shows the representation result in assertion (a) of the following proposition.

Proposition C.6.3. *Let $H_{\mathbb{C}}$ by a complex Hilbert lattice with real part H. Then the following assertions hold:*

(a) *There exists a measure space (Ω, μ) and an isometric (hence, unitary) isomorphism of complex Banach lattices $H_{\mathbb{C}} \to L^2(\Omega, \mu; \mathbb{C})$.*

(b) *Let $z \in H_{\mathbb{C}}$ and define $z' \in (H_{\mathbb{C}})'$ by $\langle z', w \rangle := (w|z)$ for all $w \in H_{\mathbb{C}}$. Then z' is a positive element of the complex Banach lattice $(H_{\mathbb{C}})'$ if and only if z is a positive element of the complex Banach lattice $H_{\mathbb{C}}$.*

Moreover, z' is a strictly positive functional on $H_{\mathbb{C}}$ if and only if z is a quasi-interior point of H_+.

Proof. As pointed out above, assertion (a) is a consequence of the proof of Proposition C.6.2. Assertion (b) can be concluded either from (a) or from Proposition B.4.2(b). □

Let $H_{\mathbb{C}}$ by a complex Hilbert lattice with real part H and consider the conjugate linear mapping $H_{\mathbb{C}} \to (H_{\mathbb{C}})'$, $z \mapsto z' := (\cdot|z)$. For every densely defined linear operator $A : H_{\mathbb{C}} \supseteq D(A) \to H_{\mathbb{C}}$ we have $z \in D(A^*)$ if and only if $z' \in D(A')$, and in this case $A'z' = (A^*z)'$. This observation, together with Proposition C.6.3(b) yields the following corollary:

Corollary C.6.4. *Let $H_{\mathbb{C}}$ be a complex Hilbert lattice with real part H, let $A : H_{\mathbb{C}} \supseteq D(A) \to H_{\mathbb{C}}$ be a densely defined linear operator and let $\lambda_0 \in \mathbb{C}$.*

(a) *If λ_0 is an eigenvalue of A^* with a positive eigenvector, then $\overline{\lambda_0}$ is an eigenvalue of A' with a positive eigenvector.*

(b) *If λ_0 is an eigenvalue of A^* with an eigenvector which is a quasi-interior point of H_+, then $\overline{\lambda_0}$ is an eigenvalue of A' with an eigenvector which is a strictly positive functional.*

Finally, we briefly discuss the question whether a linear operator on complex Hilbert lattice which is induced by means of a form is real. Let $V_{\mathbb{C}}$ be a complex Hilbert lattice with real part V. A sesqui-linear form $a : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ is called *real* if $a(x,x) \in \mathbb{R}$ for all $x \in V$.

We have the following simple result:

Proposition C.6.5. *Let V and H be real Hilbert lattices and suppose that V is a dense subset of H such that the canonical embedding $V \hookrightarrow H$ is a lattice homomorphism. Let $V_{\mathbb{C}}$ and $H_{\mathbb{C}}$ denote Banach lattice complexifications of V and H, respectively; then we can also consider $V_{\mathbb{C}}$ as a dense subset of $H_{\mathbb{C}}$ and the canonical embedding $V_{\mathbb{C}} \hookrightarrow H_{\mathbb{C}}$ is a lattice homomorphism.*

Let $a : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ be a continuous sesqui-linear form and let $A : H_{\mathbb{C}} \supseteq D(A) \to H_{\mathbb{C}}$ be the linear operator induced by a on $H_{\mathbb{C}}$. If $\tilde{a}$ is real, then so is A.

Proof. First, let $x \in D(A) \cap H = D(A) \cap V$. For every $v \in V$ we have $(Ax|v)_{H_{\mathbb{C}}} = a(x,v) \in \mathbb{R}$. Since V is dense in H, this implies that we even have $(Ax|v)_{H_{\mathbb{C}}} \in \mathbb{R}$ for all $v \in H$ and hence, $Ax \in H$.

Thus, it only remains to show that $D(A) \cap H + iD(A) \cap H = D(A) \cap H$; according to Proposition C.2.3 this is equivalent to the condition that $\operatorname{Re} z \in D(A)$ for each $z \in D(A)$. So, let $z \in D(A)$. Then we obtain $(Az|x) = a(z,x)$ for all $x \in V_{\mathbb{C}}$ and in particular for all $x \in V$. Hence, we have

$$(\operatorname{Re}(Az)|x) + i(\operatorname{Im}(Az)|x) = a(\operatorname{Re} z, x) + ia(\operatorname{Im} z, x)$$

for all $x \in V$. Since a is real, we have $a(\operatorname{Re} z, x) \in \mathbb{R}$ and $a(\operatorname{Im} z, x) \in \mathbb{R}$ and thus, the above equation implies that $(\operatorname{Re}(Az)|x) = a(\operatorname{Re} z, x)$ for all $x \in V$; since both sides of this equation are conjugate linear with respect to x, the equation even holds for all $x \in V_{\mathbb{C}}$. This proves that $\operatorname{Re} z \in D(A)$. □

Bibliography

[AB85] C. D. Aliprantis and O. Burkinshaw. *Positive operators*, volume 119 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1985.

[AB88] W. Arendt and C. J. K. Batty. Tauberian theorems and stability of one-parameter semigroups. *Trans. Amer. Math. Soc.*, 306(2):837–852, 1988.

[AB99] Wolfgang Arendt and Philippe Bénilan. Wiener regularity and heat semigroups on spaces of continuous functions. In Joachim Escher and Gieri Simonett, editors, *Topics in nonlinear analysis*, volume 35 of *Progr. Nonlinear Differential Equations Appl.*, pages 29–49. Birkhäuser, Basel, 1999.

[ABHN11] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.

[ABR90] Wolfgang Arendt, Charles J. K. Batty, and Derek W. Robinson. Positive semigroups generated by elliptic operators on Lie groups. *J. Operator Theory*, 23(2):369–407, 1990.

[AGG+86] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[AHK79] Sergio Albeverio and Raphael Høegh-Krohn. Frobenius theory for positive maps of von Neumann algebras. *Comm. Math. Phys.*, 64(1):83–94, 1978/79.

[AM12] Wolfgang Arendt and Rafe Mazzeo. Friedlander's eigenvalue inequalities and the Dirichlet-to-Neumann semigroup. *Commun. Pure Appl. Anal.*, 11(6):2201–2212, 2012.

[Ama83] Herbert Amann. Dual semigroups and second order linear elliptic boundary value problems. *Israel J. Math.*, 45(2-3):225–254, 1983.

[AN09] Wolfgang Arendt and Robin Nittka. Equivalent complete norms and positivity. *Arch. Math. (Basel)*, 92(5):414–427, 2009.

[And66] T. Andô. Contractive projections in L_p spaces. *Pacific J. Math.*, 17:391–405, 1966.

[Are83] Wolfgang Arendt. Spectral properties of Lamperti operators. *Indiana Univ. Math. J.*, 32(2):199–215, 1983.

[Are08] W. Arendt. Positive semigroups of kernel operators. *Positivity*, 12(1):25–44, 2008.

[AS87] Wolfgang Arendt and Hans-Ulrich Schwarz. Ideale regulärer Operatoren und Kompaktheit positiver Operatoren zwischen Banachverbänden. *Math. Nachr.*, 131:7–18, 1987.

[AT07] Charalambos D. Aliprantis and Rabee Tourky. *Cones and duality*, volume 84 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007.

[AtEKS14] W. Arendt, A. F. M. ter Elst, J. B. Kennedy, and M. Sauter. The Dirichlet-to-Neumann operator via hidden compactness. *J. Funct. Anal.*, 266(3):1757–1786, 2014.

[AW03] Wolfgang Arendt and Mahamadi Warma. Dirichlet and Neumann boundary conditions: What is in between? *J. Evol. Equ.*, 3(1):119–135, 2003. Dedicated to Philippe Bénilan.

[Bar92] Wojciech Bartoszek. Riesz decomposition property implies asymptotic periodicity of positive and constrictive operators. *Proc. Amer. Math. Soc.*, 116(1):101–111, 1992.

[BCD+09] Abraham Berman, Minerva Catral, Luz M. DeAlba, Abed Elhashash, Frank J. Hall, Leslie Hogben, In-Jae Kim, D. D. Olesky, Pablo Tarazaga, Michael J. Tsatsomeros, and P. van den Driessche. Sign patterns that allow eventual positivity. *Electron. J. Linear Algebra*, 19:108–120, 2009.

[Bir95] Isabeau Birindelli. Hopf's lemma and anti-maximum principle in general domains. *J. Differential Equations*, 119(2):450–472, 1995.

[BL74] S. J. Bernau and H. E. Lacey. The range of a contractive projection on an L_p-space. *Pacific J. Math.*, 53:21–41, 1974.

[Bla01] Mark D. Blake. A spectral bound for asymptotically norm-continuous semigroups. *J. Operator Theory*, 45(1):111–130, 2001.

[Bog05] Tommaso Boggio. Sulle funzioni di green d'ordinem. *Rend. Circ. Mat. Palermo*, 20(1):97–135, 1905.

[Bon58] F. F. Bonsall. Linear operators in complete positive cones. *Proc. London Math. Soc. (3)*, 8:53–75, 1958.

[Bon67] Jean-Michel Bony. Principe du maximum dans les espaces de Sobolev. *C. R. Acad. Sci. Paris Sér. A-B*, 265:A333–A336, 1967.

[BP05] András Bátkai and Susanna Piazzera. *Semigroups for delay equations*, volume 10 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, 2005.

[Bra61] Alfred Brauer. On the characteristic roots of power-positive matrices. *Duke Math. J.*, 28:439–445, 1961.

[BtE15] J. Behrndt and A. F. M. ter Elst. Dirichlet-to-Neumann maps on bounded Lipschitz domains. *J. Differential Equations*, 259(11):5903–5926, 2015.

[CD13] Alexander P. Campbell and Daniel Daners. Linear Algebra via Complex Analysis. *Amer. Math. Monthly*, 120(10):877–892, 2013.

[CEH+12] M. Catral, C. Erickson, L. Hogben, D. D. Olesky, and P. van den Driessche. Sign patterns that allow strong eventual nonnegativity. *Electron. J. Linear Algebra*, 23:1–10, 2012.

[Chu15] Jaywan Chung. Asymptotic positivity of solutions of second order differential equations. Positivity, 2015. Online First.

[CNM02] Sarah Carnochan Naqvi and Judith J. McDonald. The combinatorial structure of eventually nonnegative matrices. *Electron. J. Linear Algebra*, 9:255–269, 2002.

[CNM04] Sarah Carnochan Naqvi and Judith J. McDonald. Eventually nonnegative matrices are similar to seminonnegative matrices. *Linear Algebra Appl.*, 381:245–258, 2004.

[CP79] Ph. Clément and L. A. Peletier. An anti-maximum principle for second-order elliptic operators. *J. Differential Equations*, 34(2):218–229, 1979.

[CS00] Ph. Clément and G. Sweers. Uniform anti-maximum principles. *J. Differential Equations*, 164(1):118–154, 2000.

[CS01] Philippe Clément and Guido Sweers. Uniform anti-maximum principle for polyharmonic boundary value problems. *Proc. Amer. Math. Soc.*, 129(2):467–474, 2001.

[Dal05] A. Dall'Acqua. *Higher Order Elliptic Problems and Positivity*. PhD thesis, Delft University of Technology, 2005.

[Dan14] Daniel Daners. Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator. *Positivity*, 18(2):235–256, 2014.

[Dav89] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[Dav05] E. B. Davies. Triviality of the peripheral point spectrum. *J. Evol. Equ.*, 5(3):407–415, 2005.

[Der80] Roland Derndinger. Über das Spektrum positiver Generatoren. *Math. Z.*, 172(3):281–293, 1980.

[DGK16a] Daniel Daners, Jochen Glück, and James B. Kennedy. Eventually and asymptotically positive semigroups on Banach lattices. *J. Differential Equations*, 261(5):2607–2649, 2016.

[DGK16b] Daniel Daners, Jochen Glück, and James B. Kennedy. Eventually positive semigroups of linear operators. *J. Math. Anal. Appl.*, 433(2):1561–1593, 2016.

[DJP98] F. Delbaen, H. Jarchow, and A. Pełczyński. Subspaces of L_p isometric to subspaces of l_p. *Positivity*, 2(4):339–367, 1998.

[DKFNR10] B. Dorn, M. Kramar Fijavž, R. Nagel, and A. Radl. The semigroup approach to transport processes in networks. *Phys. D*, 239(15):1416–1421, 2010.

[DKS09] Britta Dorn, Vera Keicher, and Eszter Sikolya. Asymptotic periodicity of recurrent flows in infinite networks. *Math. Z.*, 263(1):69–87, 2009.

[DMS05] Anna Dall'Acqua, Christian Meister, and Guido Sweers. Separating positivity and regularity for fourth order Dirichlet problems in 2d-domains. *Analysis (Munich)*, 25(3):205–261, 2005.

[Dou65] R. G. Douglas. Contractive projections on an L_1 space. *Pacific J. Math.*, 15:443–462, 1965.

[DS58] N. Dunford and J. T. Schwartz. *Linear Operators. I. General Theory*. Interscience Publishers, Inc., New York, 1958.

[DS04] A. Dall'Acqua and G. Sweers. On domains for which the clamped plate system is positivity preserving. In *Partial differential equations and inverse problems*, volume 362 of *Contemp. Math.*, pages 133–144. Amer. Math. Soc., Providence, RI, 2004.

[DS05] Anna Dall'Acqua and Guido Sweers. The clamped-plate equation for the limaçon. *Ann. Mat. Pura Appl. (4)*, 184(3):361–374, 2005.

[DvGVLW95] Odo Diekmann, Stephan A. van Gils, Sjoerd M. Verduyn Lunel, and Hans-Otto Walther. *Delay equations*, volume 110 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995. Functional, complex, and nonlinear analysis.

[EFHN15] Tanja Eisner, Bálint Farkas, Markus Haase, and Rainer Nagel. *Operator theoretic aspects of ergodic theory*, volume 272 of *Graduate Texts in Mathematics*. Springer, Cham, 2015.

[EHK78] David E. Evans and Raphael Høegh-Krohn. Spectral properties of positive maps on C^*-algebras. *J. London Math. Soc. (2)*, 17(2):345–355, 1978.

[EHT09] Elisabeth M. Ellison, Leslie Hogben, and Michael J. Tsatsomeros. Sign patterns that require eventual positivity or require eventual nonnegativity. *Electron. J. Linear Algebra*, 19:98–107, 2009.

[Els70] Ludwig Elsner. Monotonie und Randspektrum bei vollstetigen Operatoren. *Arch. Rational Mech. Anal.*, 36:356–365, 1970.

[Eme07] E. Yu. Emel'yanov. *Non-spectral asymptotic analysis of one-parameter operator semigroups*, volume 173 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2007.

[EN00] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[EN06] Klaus-Jochen Engel and Rainer Nagel. *A short course on operator semigroups*. Universitext. Springer, New York, 2006.

[Eri15] Craig Erickson. Sign patterns that require eventual exponential nonnegativity. *Electron. J. Linear Algebra*, 30:171–195, 2015.

[ES08] Abed Elhashash and Daniel B. Szyld. On general matrices having the Perron-Frobenius property. *Electron. J. Linear Algebra*, 17:389–413, 2008.

[ES09] Abed Elhashash and Daniel B. Szyld. Two characterizations of matrices with the Perron-Frobenius property. *Numer. Linear Algebra Appl.*, 16(11-12):863–869, 2009.

[Fac14] S. Fackler. *Regularity Properties of Sectorial Operators: Extrapolation, Counterexamples and Generic Classes*. PhD thesis, Universität Ulm, 2014.

[Fen98] G. Fendler. On dilations and transference for continuous one-parameter semigroups of positive contractions on L^p-spaces. *Ann. Univ. Sarav. Ser. Math.*, 9(1):iv+97, 1998.

[FGG08] Alberto Ferrero, Filippo Gazzola, and Hans-Christoph Grunau. Decay and eventual local positivity for biharmonic parabolic equations. *Discrete Contin. Dyn. Syst.*, 21(4):1129–1157, 2008.

[FGJ76] R. J. Fleming, J. A. Goldstein, and J. E. Jamison. One parameter groups of isometries on certain Banach spaces. *Pacific J. Math.*, 64(1):145–151, 1976.

[FH73] Miroslav Fiedler and Emilie Haynsworth. Cones which are topheavy with respect to a norm. *Linear and Multilinear Algebra*, 1:203–211, 1973.

[Fri78] Shmuel Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel J. Math.*, 29(1):43–60, 1978.

[Fro08] Georg Frobenius. Über Matrizen aus positiven Elementen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 471–476, 1908.

[Fro09] Georg Frobenius. Über Matrizen aus positiven Elementen. II. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 514–518, 1909.

[Fro12] Georg Frobenius. Über Matrizen aus nicht negativen Elementen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 456–477, 1912.

[Fur81] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory*. Princeton University Press, Princeton, N.J., 1981. M. B. Porter Lectures.

[Ger13] M. Gerlach. On the peripheral point spectrum and the asymptotic behavior of irreducible semigroups of Harris operators. *Positivity*, 17(3):875–898, 2013.

[Ger14] M. Gerlach. *Semigroups of Kernel Operators*. PhD thesis, Universität Ulm, 2014.

[GG08] Filippo Gazzola and Hans-Christoph Grunau. Eventual local positivity for a biharmonic heat equation in $\mathbb{R}^n$. *Discrete Contin. Dyn. Syst. Ser. S*, 1(1):83–87, 2008.

[GG09] Filippo Gazzola and Hans-Christoph Grunau. Some new properties of biharmonic heat kernels. *Nonlinear Anal.*, 70(8):2965–2973, 2009.

[GGS10] Filippo Gazzola, Hans-Christoph Grunau, and Guido Sweers. *Polyharmonic boundary value problems*, volume 1991 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010. Positivity preserving and nonlinear higher order elliptic equations in bounded domains.

[Glü16a] Jochen Glück. Growth rates and the peripheral spectrum of positive operators. 2016. Preprint. Available from arxiv.org/abs/1512.07483.

[Glü16b] Jochen Glück. A note on lattice ordered C^*-algebras and perron-Frobenius theory. 2016. Preprint. Available from arxiv.org/abs/1605.05190v1.

[Glü16c] Jochen Glück. On the peripheral spectrum of positive operators. *Positivity*, 20(2):307–336, 2016.

[Glü16d] Jochen Glück. Spectral and asymptotic properties of contractive semigroups on non-Hilbert space. *J. Operator Theory*, 76(1):3–31, 2016.

[GM97] Patrick Guidotti and Sandro Merino. Hopf bifurcation in a scalar reaction diffusion equation. *J. Differential Equations*, 140(1):209–222, 1997.

[GM00] Patrick Guidotti and Sandro Merino. Gradual loss of positivity and hidden invariant cones in a scalar heat equation. *Differential Integral Equations*, 13(10-12):1551–1568, 2000.

[GM09] Fritz Gesztesy and Marius Mitrea. Nonlocal Robin Laplacians and some remarks on a paper by Filonov on eigenvalue inequalities. *J. Differential Equations*, 247(10):2871–2896, 2009.

[GMN14] Fritz Gesztesy, Marius Mitrea, and Roger Nichols. Heat kernel bounds for elliptic partial differential operators in divergence form with Robin-type boundary conditions. *J. Anal. Math.*, 122:229–287, 2014.

[GMNO15] Fritz Gesztesy, Marius Mitrea, Roger Nichols, and El Maati Ouhabaz. Heat kernel bounds for elliptic partial differential operators in divergence form with Robin-type boundary conditions II. *Proc. Amer. Math. Soc.*, 143(4):1635–1649, 2015.

[Gol73] J. A. Goldstein. Groups of isometries on Orlicz spaces. *Pacific J. Math.*, 48:387–393, 1973.

[GR09] Hans-Christoph Grunau and Frédéric Robert. Boundedness of the negative part of biharmonic Green's functions under Dirichlet boundary conditions in general domains. *C. R. Math. Acad. Sci. Paris*, 347(3-4):163–166, 2009.

[GR10] Hans-Christoph Grunau and Frédéric Robert. Positivity and almost positivity of biharmonic Green's functions under Dirichlet boundary conditions. *Arch. Ration. Mech. Anal.*, 195(3):865–898, 2010.

[Gre81] Günther Greiner. Zur Perron-Frobenius-Theorie stark stetiger Halbgruppen. *Math. Z.*, 177(3):401–423, 1981.

[Gre82] G. Greiner. Spektrum und Asymptotik stark stetiger Halbgruppen positiver Operatoren. *Sitzungsber. Heidelb. Akad. Wiss. Math.-Natur. Kl.*, pages 55–80, 1982.

[Gro55] A. Grothendieck. Une caractérisation vectorielle-métrique des espaces L^1. *Canad. J. Math.*, 7:552–561, 1955.

[Gro83] Ulrich Groh. On the peripheral spectrum of uniformly ergodic positive operators on C^*-algebras. *J. Operator Theory*, 10(1):31–37, 1983.

[Gro95] J. J. Grobler. Spectral theory in Banach lattices. In *Operator theory in function spaces and Banach lattices*, volume 75 of *Oper. Theory Adv. Appl.*, pages 133–172. Birkhäuser, Basel, 1995.

[GRS11] Hans-Christoph Grunau, Frédéric Robert, and Guido Sweers. Optimal estimates from below for biharmonic Green functions. *Proc. Amer. Math. Soc.*, 139(6):2151–2161, 2011.

[GS96] Hans-Christoph Grunau and Guido Sweers. Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions. *Math. Nachr.*, 179:89–102, 1996.

[GS97] Hans-Christoph Grunau and Guido Sweers. Positivity properties of elliptic boundary value problems of higher order. In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 8 (Athens, 1996)*, volume 30, pages 5251–5258, 1997.

[GS98] Hans-Christoph Grunau and Guido Sweers. The maximum principle and positive principal eigenfunctions for polyharmonic equations. In *Reaction diffusion systems (Trieste, 1995)*, volume 194 of *Lecture Notes in Pure and Appl. Math.*, pages 163–182. Dekker, New York, 1998.

[GS01] Hans-Christoph Grunau and Guido Sweers. Optimal conditions for anti-maximum principles. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 30(3-4):499–513 (2002), 2001.

[GVW81] Günther Greiner, Jürgen Voigt, and Manfred Wolff. On the spectral bound of the generator of semigroups of positive operators. *J. Operator Theory*, 5(2):245–256, 1981.

[Haa06] Markus Haase. *The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[Hei80] S. Heinrich. Ultraproducts in Banach space theory. *J. Reine Angew. Math.*, 313:72–104, 1980.

[HJ90] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[HLR91] R. Haydon, M. Levy, and Y. Raynaud. *Randomly normed spaces*, volume 41 of *Travaux en Cours [Works in Progress]*. Hermann, Paris, 1991.

[Hog09] Leslie Hogben. Eventually cyclic matrices and a test for strong eventual nonnegativity. *Electron. J. Linear Algebra*, 19:129–140, 2009.

[HP57] Einar Hille and Ralph S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed.

[HR07] C. Ward Henson and Yves Raynaud. On the theory of $L_p(L_q)$-Banach lattices. *Positivity*, 11(2):201–230, 2007.

[HTW15] Leslie Hogben, Bit-Shun Tam, and Ulrica Wilson. Note on the Jordan form of an irreducible eventually nonnegative matrix. *Electron. J. Linear Algebra*, 30:279–285, 2015.

[JL01] W. B. Johnson and J. Lindenstrauss. Basic concepts in the geometry of Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 1–84. North-Holland, Amsterdam, 2001.

[JT04] Charles R. Johnson and Pablo Tarazaga. On matrices with Perron-Frobenius properties and some negative entries. *Positivity*, 8(4):327–338, 2004.

[Kat59] Tosio Kato. Remarks on pseudo-resolvents and infinitesimal generators of semi-groups. *Proc. Japan Acad.*, 35:467–468, 1959.

[Kat95] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[Kei06] V. Keicher. On the peripheral spectrum of bounded positive semigroups on atomic Banach lattices. *Arch. Math. (Basel)*, 87(4):359–367, 2006.

[KN84] W. Kerscher and R. Nagel. Asymptotic behavior of one-parameter semigroups of positive operators. *Acta Appl. Math.*, 2(3-4):297–309, 1984.

[KN88] Wolfgang Kerscher and Rainer Nagel. Positivity and stability for Cauchy problems with delay. In *Partial differential equations (Rio de Janeiro, 1986)*, volume 1324 of *Lecture Notes in Math.*, pages 216–235. Springer, Berlin, 1988.

[Kön04] H. König. Applications of spherical designs to Banach space theory. In *Orlicz centenary volume*, volume 64 of *Banach Center Publ.*, pages 127–134. Polish Acad. Sci., Warsaw, 2004.

[KP81] Steven G. Krantz and Harold R. Parks. Distance to C^k hypersurfaces. *J. Differential Equations*, 40(1):116–120, 1981.

[KR48] M. G. Kreĭn and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Uspehi Matem. Nauk (N. S.)*, 3(1(23)):3–95, 1948.

[Kra68] M. A. Krasnosel'skiĭ. A certain spectral property of linear completely continuous operators in the space of continuous functions. *Problemy Mat. Anal. Slož. Sistem*, (Vyp. 2):68–71, 1968. (Russian).

[Kre85] U. Krengel. *Ergodic theorems*, volume 6 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.

[Kri69] Hans-Jürgen Krieger. *Beiträge zur Theorie positiver Operatoren*. Akademie-Verlag, Berlin, 1969. Schriftenreihe der Institute für Mathematik. Reihe A: Reine Mathematik, Heft 6.

[KS05] Marjeta Kramar and Eszter Sikolya. Spectral properties and asymptotic periodicity of flows in networks. *Math. Z.*, 249(1):139–162, 2005.

[Kuc08] Peter Kuchment. Quantum graphs: an introduction and a brief survey. In *Analysis on graphs and its applications*, volume 77 of *Proc. Sympos. Pure Math.*, pages 291–312. Amer. Math. Soc., Providence, RI, 2008.

[Lam58] J. Lamperti. On the isometries of certain function-spaces. *Pacific J. Math.*, 8:459–466, 1958.

[Lot68] Heinrich P. Lotz. Über das Spektrum positiver Operatoren. *Math. Z.*, 108:15–32, 1968.

[LRR05] Florence Lancien, Beata Randrianantoanina, and Eric Ricard. On contractive projections in Hardy spaces. *Studia Math.*, 171(1):93–102, 2005.

[LRvG07] B. Lemmens, B. Randrianantoanina, and O. van Gaans. Second derivatives of norms and contractive complementation in vector-valued spaces. *Studia Math.*, 179(2):149–166, 2007.

[LS75] Raphael Loewy and Hans Schneider. Positive operators on the n-dimensional ice cream cone. *J. Math. Anal. Appl.*, 49:375–392, 1975.

[Łuc10] Andrzej Łuczak. The eigenvectors of semigroups of positive maps on von Neumann algebras. *Ergodic Theory Dynam. Systems*, 30(5):1495–1502, 2010.

[LV93] Yu. I. Lyubich and L. N. Vaserstein. Isometric embeddings between classical Banach spaces, cubature formulas, and spherical designs. *Geom. Dedicata*, 47(3):327–362, 1993.

[LVG03] B. Lemmens and O. Van Gaans. Iteration of linear p-norm nonexpansive maps. *Linear Algebra Appl.*, 371:265–276, 2003.

[LvG09] B. Lemmens and O. van Gaans. Dynamics of non-expansive maps on strictly convex Banach spaces. *Israel J. Math.*, 171:425–442, 2009.

[LW15] Philippe Laurençot and Christoph Walker. Sign-preserving property for some fourth-order elliptic operators in one dimension or in radial symmetry. *J. Anal. Math.*, 127:69–89, 2015.

[Lyu70] Yu. I. Lyubich. On the boundary spectrum of contractions in minkowski spaces. *Siberian Mathematical Journal*, 11(2):271–279, 1970.

[Lyu95] Yu. Lyubich. Perron-Frobenius theory for finite-dimensional spaces with a hyperbolic cone. In *Proceedings of the Workshop "Nonnegative Matrices, Applications and Generalizations" and the Eighth Haifa Matrix Theory Conference (Haifa, 1993)*, volume 220, pages 283–309, 1995.

[Lyu97] Yu. Lyubich. The boundary spectrum of linear operators in finite-dimensional spaces. In *Linear operators (Warsaw, 1994)*, volume 38 of *Banach Center Publ.*, pages 217–226. Polish Acad. Sci., Warsaw, 1997.

[Lyu98] Yuri Lyubich. Linear operators in one-dimensional extensions of Banach spaces. In *Voronezh Winter Mathematical Schools*, volume 184 of *Amer. Math. Soc. Transl. Ser. 2*, pages 145–155. Amer. Math. Soc., Providence, RI, 1998.

[LZ71] W. A. J. Luxemburg and A. C. Zaanen. *Riesz spaces. Vol. I*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971. North-Holland Mathematical Library.

[Mac00] C. R. MacCluer. The many proofs and applications of Perron's theorem. *SIAM Rev.*, 42(3):487–498 (electronic), 2000.

[Mil88] V. D. Milman. A few observations on the connections between local theory and some other fields. In *Geometric aspects of functional analysis (1986/87)*, volume 1317 of *Lecture Notes in Math.*, pages 283–289. Springer, Berlin, 1988.

[Miy86] Shizuo Miyajima. Generators of positive C_0-semigroups. In *Aspects of positivity in functional analysis (Tübingen, 1985)*, volume 122 of *North-Holland Math. Stud.*, pages 241–246. North-Holland, Amsterdam, 1986.

[MM96] J. Martinez and J. M. Mazon. C_0-semigroups norm continuous at infinity. *Semigroup Forum*, 52(2):213–224, 1996.

[MN91] P. Meyer-Nieberg. *Banach lattices*. Universitext. Springer-Verlag, Berlin, 1991.

[MNN71] A. S. Markus, L. N. Nikol'skaja, and N. K. Nikol'skiĭ. The unitary spectrum of a contraction in a Banach space. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 22:65–74, 1971. (Russian).

[MO86] Shizuo Miyajima and Noboru Okazawa. Generators of positive C_0-semigroups. *Pacific J. Math.*, 125(1):161–176, 1986.

[MP07] D. Mupasiri and M. P. Prophet. A note on the existence of shape-preserving projections. *Rocky Mountain J. Math.*, 37(2):573–585, 2007.

[MPT14] Judith J. McDonald, Pietro Paparella, and Michael J. Tsatsomeros. Matrix roots of eventually positive matrices. *Linear Algebra Appl.*, 456:122–137, 2014.

[MST99] G. A. Muñoz, Y. Sarantopoulos, and A. Tonge. Complexifications of real Banach spaces, polynomials and multilinear maps. *Studia Math.*, 134(1):1–33, 1999.

[Mug14] Delio Mugnolo. *Semigroup methods for evolution equations on networks*. Understanding Complex Systems. Springer, Cham, 2014.

[Mur90] Gerard J. Murphy. *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.

[Muz12] Kelvin Muzundu. *Spectral theory in commutatively ordered banach algebras*. PhD thesis, Stellenbosch: Stellenbosch University, 2012.

[MW74] Günter Mittelmeyer and Manfred Wolff. Über den Absolutbetrag auf komplexen Vektorverbänden. *Math. Z.*, 137:87–92, 1974.

[Nag97] Rainer Nagel. Characteristic equations for the spectrum of generators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24(4):703–717 (1998), 1997.

[Nou06] Dimitrios Noutsos. On Perron-Frobenius property of matrices having some negative entries. *Linear Algebra Appl.*, 412(2-3):132–153, 2006.

[NP00] Rainer Nagel and Jan Poland. The critical spectrum of a strongly continuous semigroup. *Adv. Math.*, 152(1):120–133, 2000.

[NR74] Alexander Nagel and Walter Rudin. Riesz decompositions. *Canad. J. Math.*, 26:753–760, 1974.

[NT08] Dimitrios Noutsos and Michael J. Tsatsomeros. Reachability and holdability of nonnegative states. *SIAM J. Matrix Anal. Appl.*, 30(2):700–712, 2008.

[NV12] Dimitrios Noutsos and Richard S. Varga. On the Perron-Frobenius theory for complex matrices. *Linear Algebra Appl.*, 437(4):1071–1088, 2012.

[OTvdD09] D. D. Olesky, M. J. Tsatsomeros, and P. van den Driessche. M_V-matrices: a generalization of M-matrices based on eventually nonnegative matrices. *Electron. J. Linear Algebra*, 18:339–351, 2009.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[Per07a] Oskar Perron. Grundlagen für eine theorie des jacobischen kettenbruchalgorithmus. *Mathematische Annalen*, 64(1):1–76, 1907.

[Per07b] Oskar Perron. Zur Theorie der Matrices. *Math. Ann.*, 64(2):248–263, 1907.

[Per67] Anthony L. Peressini. *Ordered topological vector spaces*. Harper & Row, Publishers, New York-London, 1967.

[Pul71] N. J. Pullman. A geometric approach to the theory of nonnegative matrices. *Linear Algebra and Appl.*, 4:297–312, 1971.

[Ran01] B. Randrianantoanina. Norm-one projections in Banach spaces. *Taiwanese J. Math.*, 5(1):35–95, 2001. International Conference on Mathematical Analysis and its Applications (Kaohsiung, 2000).

[Ran04] B. Randrianantoanina. Contractive projections in Orlicz sequence spaces. *Abstr. Appl. Anal.*, (2):133–145, 2004.

[RG12] Hermann Render and Marius Ghergu. Positivity properties for the clamped plate boundary problem on the ellipse and strip. *Math. Nachr.*, 285(8-9):1052–1062, 2012.

[Rum03] Siegfried M. Rump. Perron-Frobenius theory for complex matrices. *Linear Algebra Appl.*, 363:251–273, 2003. Special issue on nonnegative matrices, M-matrices and their generalizations (Oberwolfach, 2000).

[Sar07] Donald Sarason. *Complex function theory*. American Mathematical Society, Providence, RI, second edition, 2007.

[Sch60] Helmut Schaefer. Some spectral properties of positive linear operators. *Pacific J. Math.*, 10:1009–1019, 1960.

[Sch63] Helmut H. Schaefer. Spektraleigenschaften positiver linearer Operatoren. *Math. Z.*, 82:303–313, 1963.

[Sch71] Egon Scheffold. Das Spektrum von Verbandsoperatoren in Banachverbänden. *Math. Z.*, 123:177–190, 1971.

[Sch74] H. H. Schaefer. *Banach lattices and positive operators*. Springer-Verlag, New York, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 215.

[Sen81] E. Seneta. *Nonnegative matrices and Markov chains*. Springer Series in Statistics. Springer-Verlag, New York, second edition, 1981.

[Sin90] R. Sine. A nonlinear Perron-Frobenius theorem. *Proc. Amer. Math. Soc.*, 109(2):331–336, 1990.

[Ste69] Stanly Steinberg. Meromorphic families of compact operators. *Arch. Rational Mech. Anal.*, 31:372–379, 1968/1969.

[SW91a] Ronald J. Stern and Henry Wolkowicz. Exponential nonnegativity on the ice cream cone. *SIAM J. Matrix Anal. Appl.*, 12(1):160–165, 1991.

[SW91b] Ronald J. Stern and Henry Wolkowicz. Invariant ellipsoidal cones. In *Proceedings of the First Conference of the International Linear Algebra Society (Provo, UT, 1989)*, volume 150, pages 81–106, 1991.

[SW99] H. H. Schaefer and M. P. Wolff. *Topological vector spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1999.

[Swe97] Guido Sweers. L^n is sharp for the anti-maximum principle. *J. Differential Equations*, 134(1):148–153, 1997.

[Swe01] Guido Sweers. When is the first eigenfunction for the clamped plate equation of fixed sign? In *Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viña del Mar-Valparaiso, 2000)*, volume 6 of *Electron. J. Differ. Equ. Conf.*, pages 285–296. Southwest Texas State Univ., San Marcos, TX, 2001.

[Swe16] Guido Sweers. An elementary proof that the triharmonic Green function of an eccentric ellipse changes sign. *Arch. Math. (Basel)*, 107(1):59–62, 2016.

[Tak96] Peter Takáč. An abstract form of maximum and anti-maximum principles of Hopf's type. *J. Math. Anal. Appl.*, 201(2):339–364, 1996.

[Tam01] Bit-Shun Tam. A cone-theoretic approach to the spectral theory of positive linear operators: the finite-dimensional case. *Taiwanese J. Math.*, 5(2):207–277, 2001.

[Tan97] Hiroki Tanabe. *Functional analytic methods for partial differential equations*, volume 204 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1997.

[Tay58] Angus E. Taylor. *Introduction to functional analysis*. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958.

[TCDF15] Francesco Tudisco, Valerio Cardinali, and Carmine Di Fiore. On complex power nonnegative matrices. *Linear Algebra Appl.*, 471:449–468, 2015.

[TRH01] Pablo Tarazaga, Marcos Raydan, and Ana Hurman. Perron-Frobenius theorem for matrices with some negative entries. *Linear Algebra Appl.*, 328(1-3):57–68, 2001.

[TS94] Bit Shun Tam and Hans Schneider. On the core of a cone-preserving map. *Trans. Amer. Math. Soc.*, 343(2):479–524, 1994.

[Tza69] L. Tzafriri. Remarks on contractive projections in L_p-spaces. *Israel J. Math.*, 7:9–15, 1969.

[Ulm99] Michael Ulm. The interval of resolvent-positivity for the biharmonic operator. *Proc. Amer. Math. Soc.*, 127(2):481–489, 1999.

[Veĭ85] A. I. Veĭtsblit. A property of the boundary spectrum of nonnegative operators. *Ukrain. Mat. Zh.*, 37(1):114–116, 136, 1985.

[vGK08] Onno van Gaans and Anke Kalauch. Ideals and bands in pre-Riesz spaces. *Positivity*, 12(4):591–611, 2008.

[VL85] A. I. Veĭtsblit and Yu. I. Lyubich. The boundary spectrum of nonnegative operators. *Sibirsk. Mat. Zh.*, 26(6):24–28, 188, 1985.

[vN96] J. van Neerven. *The asymptotic behaviour of semigroups of linear operators*, volume 88 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1996.

[War06] Mahamadi Warma. The Robin and Wentzell-Robin Laplacians on Lipschitz domains. *Semigroup Forum*, 73(1):10–30, 2006.

[Wei95] Lutz Weis. The stability of positive semigroups on L_p spaces. *Proc. Amer. Math. Soc.*, 123(10):3089–3094, 1995.

[Wei98] Lutz Weis. A short proof for the stability theorem for positive semigroups on $L_p(\mu)$. *Proc. Amer. Math. Soc.*, 126(11):3253–3256, 1998.

[Wol08] M. P. H. Wolff. Triviality of the peripheral point spectrum of positive semigroups on atomic Banach lattices. *Positivity*, 12(1):185–192, 2008.

[Yos95] K. Yosida. *Functional analysis*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

[Zaa83] A. C. Zaanen. *Riesz spaces. II*, volume 30 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983.

[Zaa97] Adriaan C. Zaanen. *Introduction to operator theory in Riesz spaces*. Springer-Verlag, Berlin, 1997.

[ZT99] Boris G. Zaslavsky and Bit-Shun Tam. On the Jordan form of an irreducible matrix with eventually non-negative powers. *Linear Algebra Appl.*, 302/303:303–330, 1999. Special issue dedicated to Hans Schneider (Madison, WI, 1998).

Nomenclature

Elementary notation

$\mathbb{1}_S$	The constant function $S \to \mathbb{R}$ with value 1 on a given set S.
$\mathbb{1}$	The function $\mathbb{1}_S$ if the set S is clear from the context.
$\operatorname{dist}(x, S)$	Distance of an element x in a metric space to a subset S of the same metric space.
$i\mathbb{R}$	Imaginary axis in the complex plane.
$\mathbb{N}$	Strictly positive integers, $\mathbb{N} = \{1, 2, 3, ...\}$.
$\mathbb{N}_0$	Non-negative integers, $\mathbb{N}_0 = \{0, 1, 2, ...\}$.
$\operatorname{span} S$	The linear span of a set S in a vector space.
$\mathbb{T}$	Complex unit circle.

Banach spaces

$\mathcal{C}(K; E)$	Space of all continuous, E-valued functions on a compact Hausdorff space K (where E is a Banach space).
$\mathcal{C}_0(L; E)$	Space of all continuous, E-valued functions on a locally compact Hausdorff space L which vanish at ∞ (where E is a Banach space).
$c_0(I; E)$	The space $C_0(I; E)$, where I is a non-empty set endowed with the discrete topology.
$c(\mathbb{N}; E)$	The space of all convergent E-valued sequences, indexed over $\mathbb{N}$ (where E is a Banach space).
$c(\mathbb{N}_0; E)$	The space of all convergent E-valued sequences, indexed over $\mathbb{N}_0$ (where E is a Banach space).
$\mathcal{C}^k(\overline{\Omega}; E)$	Space of all E-valued functions on $\overline{\Omega}$ which are k-times continuously differentiable in Ω and whose derivatives of order $\leq k$ all admit continuous extensions to $\overline{\Omega}$ (where E is a Banach space and $\Omega \subseteq \mathbb{R}^d$ is a bounded open set).
$\mathcal{C}^\infty(\overline{\Omega}; E)$	The vector space $\bigcap_{k\in\mathbb{N}} \mathcal{C}^k(\overline{\Omega}; E)$, where E is a Banach space and $\Omega \subseteq \mathbb{R}^d$ is a bounded open set.
$\langle \cdot, \cdot \rangle$	Duality between a Banach space and its dual space.

E'	The dual space of a Banach space E.
$E_{\mathbb{C}}$	Complexification of a real Banach space E.
$H^k(\Omega,\mathbb{C})$	Abbreviation for $W^{k,2}(\Omega,\mathbb{C})$.
$H_0^k(\Omega,\mathbb{C})$	Abbreviation for $W_0^{k,2}(\Omega,\mathbb{C})$.
$(\cdot\|\cdot)$	Inner product on a Hilbert space.
$L^p(\Omega,\mu;E)$	Lebesgue space over some measure space (Ω,μ) with values in a Banach space E.
$L^p(\Omega;E)$	Lebesgue space with values in a Banach space E, if the σ-algebra and the measure on Ω are clear from the context.
$\ell^p(I;E)$	ℓ^p-sequence space over a non-empty index set I, with values in a Banach space E.
$W^{k,p}(\Omega,\mathbb{C})$	Sobolev space of complex-valued functions of differentiability order k.
$W_0^{k,p}(\Omega,\mathbb{C})$	Closure of the test functions in $W^{k,p}(\Omega,\mathbb{C})$.

Operators and operators spaces

A'	The adjoint of a densely defined linear operator A on a Banach space.
A^*	The Hilbert space adjoint of a densely defined linear operator A on a Hilbert space.
$A_{\mathbb{C}}$	For an operator $A : E \supseteq D(A) \to E$ on a real Banach space E, $A_{\mathbb{C}}$ denotes the complex extension to a complexification $E_{\mathbb{C}}$ of E.
$D(A)$	Domain of a linear operator A on a Banach space.
$D(A^\infty)$	The space $\bigcap_{n\in\mathbb{N}} D(A^n)$, where A is an operator on a Banach space.
I_E	The identity operator on a Banach space E.
$\mathcal{K}(E;F)$	The space of all compact linear operators between two Banach spaces E and F.
$\mathcal{K}(E)$	The space of all compact linear operators on a Banach space E.
$\mathcal{L}(E;F)$	The space of bounded linear operators between two Banach spaces E and F.

$\mathcal{L}(E)$ The space of bounded linear operators on a Banach space E.

T' The adjoint of a bounded linear operator T between two Banach spaces.

Spectral theory

$\rho(A)$ Resolvent set of (a complex extension of) an operator A on a Banach space.

$\mathcal{R}(\lambda, A)$ Resolvent of an operator A on a complex Banach space at a point $\lambda \in \rho(A)$; given by $\mathcal{R}(\lambda, A) := (\lambda - A)^{-1}$.

$\mathrm{r}(T)$ Spectral radius of (a complex extension of) a bounded linear operator T on a Banach space.

$\sigma(A)$ Spectrum of (a complex extension of) an operator A on a Banach space.

$\sigma_{\mathrm{pnt}}(A)$ Point spectrum of (a complex extension of) an operator A on a Banach space.

σ_{appr} Approximate point spectrum of (a complex extension of) an operator A on a Banach space.

$\sigma_{\mathrm{per}}(A)$ Peripheral spectrum of (a complex extension of) an operator A on a Banach space.

$\sigma_{\mathrm{per,pnt}}(A)$ Peripheral point spectrum of (a complex extension of) an operator A on a Banach space.

$\mathrm{s}(A)$ Spectral bound of (a complex extension of) an operator A on a Banach space.

$\mathrm{s}_0(A)$ Abscissa of uniform boundedness of the resolvent of (a complex extension of) an operator A on a Banach space; sometimes also called the pseudo-spectral bound of A.

C_0-semigroups

$(e^{tA})_{t\geq 0}$ C_0-semigroup with generator A on a Banach space.

$\omega_0(A)$ Growth bound, also called the type, of a C_0-semigroup with generator A on a Banach space.

$\omega_{\mathrm{ess}}(A)$ Essential growth bound of a C_0-semigroup with generator A on a Banach space.

Ordered Banach spaces and Banach lattices

E_+	Positive cone in an ordered Banach space E.
$(E')_+$	Dual cone of a positive cone E_+ in an ordered Banach space E; alternative notation: E'_+.
E'_+	Dual cone of a positive cone E_+ in an ordered Banach space E; alternative notation: $(E')_+$.
$E_{+,x,x'}$	Centered cone in a real Banach space E with parameters $x \in E$ and $x' \in E'$ (where $\langle x', x\rangle = 1$).
$E_{\mathbb{R}}$	Real part of a complex Banach lattice E.
$E_{\mathbb{C}}$	Complefixication of a real Banach lattice E.
E_u	Principal ideal generated by an element $u \in E_+$ in a real or complex Banach lattice E.
$\geq$	Order relation in an ordered Banach space E, defined by: $f \geq g$ iff $f - g \in E_+$.
≥ 0	Notation for linear functionals $x' \in E'$ on an ordered Banach space E; we write $x' \geq 0$ iff $\langle x', x\rangle \geq 0$ for all $0 \leq x \in E$.
≥ 0	Notation for a bounded linear operator between to ordered Banach spaces E and F; we write $T \geq 0$ iff $TE_+ \subseteq F_+$.
$>$	Order relation in an ordered Banach space E, defined by: $f > g$ iff $f \geq g$ and $f \neq g$.
> 0	Notation for linear functionals $x' \in E'$ and an ordered Banach space E; we write $x' > 0$ iff $x' \geq 0$, but $x' \neq 0$.
> 0	Notation for a bounded linear operator between two ordered Banach spaces E and F; we write $T > 0$ iff $T \geq 0$ and $T \neq 0$.
$\gg 0$	Notation in a real or complex Banach lattice E: for $f \in E$ we write $f \gg 0$ iff f is a quasi-interior point of E_+.
$\gg 0$	Notation for a bounded linear operator operator T between two real or complex Banach lattices E and F: we write $T \gg 0$ if $Tf \gg 0$ for each $0 < f \in E$.
$\gg_u 0$	Notation in a real or complex Banach lattice E: for $f \in E$ and $u \in E_+$ we write $f \gg_u 0$ iff f is real and there exists $\varepsilon > 0$ such that $f \geq \varepsilon u$.

$\gg_u 0$	Notation for a bounded linear operator T between two real or complex Banach lattices E and F: if $u \in E_+$ is fixed, then we write $T \gg_u 0$ if $Tf \gg_u 0$ for each $0 < f \in E$.
$\mathcal{L}^{\mathrm{reg}}(E)$	The space of all regular linear operators on a Banach lattice E.
$\leq$	Order relation in an ordered Banach space E, defined by: $f \leq g$ iff $g - f \in E_+$.
$<$	Order relation in an ordered Banach space E, defined by: $f < g$ iff $f \leq g$ and $f \neq g$.

Ultra powers

$c_{0,\mathcal{U}}(I;E)$	The subspace of $\ell^\infty(I;E)$ of all families of vectors which converge to 0 along a given ultra filter $\mathcal{U}$ on I (where E is a Banach space).
$E_{\mathcal{U}}$	Ultra power of a Banach space E with respect to an ultra filter $\mathcal{U}$ on an index set $I \neq \emptyset$.
$T_{\mathcal{U}}$	For an operator $T \in \mathcal{L}(E)$ this denotes the operator induced by T on the ultra power $E_{\mathcal{U}}$.
$x_{\mathcal{U}}$	For a family $x = (x_i) \in \ell^\infty(I;E)$ this denotes the equivalence class of x in the ultra power $E_{\mathcal{U}}$.
$x_{\mathcal{U}}$	For a vector x in a Banach space E this denotes the equivalence class of the constant family $(x)_{i\in I}$ in the ultra power $E_{\mathcal{U}}$.

Index

1-complemented subspace, 8

ABLP theorem, viii
abscissa of uniform boundedness of the resolvent, xxiii
adjacency matrix of a graph
    weighted –, 171
algebra
    Banach lattice –, 78
algebraic multiplicity, 194
algebraically simple eigenvalue, 194
almost periodic, 20
    strongly –, 20
    weakly –, 22
AM-compact operator, 28
AM-space, 229
analytic Fredholm theory, 198
analytic semigroup, 176
    bounded –, 176
anti-maximum principle, 141
approximate point spectrum on a real Banach space, 222
asymptotically positive matrix, 115
    individually –, 114
    uniformly –, 115
asymptotically positive matrix semigroup, 115
asymptotically positive resolvent
    individually –, 102
    uniformly –, 102
asymptotically positive semigroup
    individually –, 85
    uniformly –, 85
atom in a Banach lattice, 29
atomic Banach lattice, 29

Banach lattice algebra, 78
Banach lattice complexification, 223
Banach lattice isomorphism
    complex –, 225
bi-Laplace operator, 121, 174
bi-positive operator, 208
Bose condensation, 182
bounded analytic semigroup, 176

canonical embedding of a Banach space into an ultra power, xxvi
canonical isomorphism between two compexifications, 217
centred cone, 52
characteristic equation for the spectrum, 39
characteristic polynomial, 194
compact
    quasi-, 26
compact operator
    AM-, 28
compact orbits
    relatively –, 19
    relatively weakly –, 22
compact resolvent, xxiii
complemented subspace, 8
completely non-Hilbert space, 8
complex Banach lattice, 224
complex Banach lattice isomorphism, 225
complex conjugate vector, 216
complex extension of a linear operator, 217
complex Hilbert lattice, 230
complex modulus, 224
complexification of a Banach lattice, 223

complexification of a Banach lattice norm, 223
complexification of a real Banach space, 216
complexifications
  canonical isomorphism between two –, 217
cone, 205
  centred –, 52
  dual –, 210
  Euclidean –, 56
  generating –, 205
  ice cream, 52
  normal –, 207
  pointed –, 205
  polyhedral –, 61
  positive –, 205
  projectively non-Euclidean –, 57
  topheavy –, 53
  total –, 205
conjugate vector
  complex –, 216
conjugation invariant subspace of a complexification, 221
contractive C_0-semigroup, xxiv
contractive operator, xxi
cyclic subset of the complex numbers, xxi

decomposition property
  Riesz –, 57
delay differential equation, 38, 123, 178
dense embedding of ordered Banach spaces, 209
differentiable C_0-semigroup
  eventually –, 150
  immediately –, 150
diffusion with non-autonomous boundary conditions, 131
dimension of a vector space, xxi, 194
Dirichlet Laplace operator
  square of the, 176
Dirichlet-to-Neumann operator, 120, 168
Dirichlet-to-Neumann operator on a finite graph, 171
dominant spectral value, xxiii
dominating operator, 209
dominating subspace, 211
dual cone, 210

eigenspace
  generalised –, 194
eigenvalue
  algebraically simple –, 194
  geometrically simple –, 194
  semi-simple –, 194
eigenvector
  generalised –, 194
embedding of ordered Banach spaces, 209
  dense –, 209
essential growth bound of a C_0-semigroup, xxiv
Euclidean cone, 56
eventually differentiable C_0-semigroup, 150
eventually positive C_0-semigroup, 119
eventually positive resolvent
  individually —, 139
  uniformly –, 140, 159
eventually positive semigroup
  individually –, 84
  uniformly –, 84
eventually strongly negative resolvent
  individually –, 141
  uniformly –, 141

eventually strongly positive matrix, 166
eventually strongly positive matrix semigroup, 166
eventually strongly positive resolvent
  individually –, 141
  uniformly –, 141
eventually strongly positive semigroup
  individually –, 148
  uniformly –, 148
extension of a linear operator
  complex –, 217
extremely non-Hilbert space, 3

form
  real –, 232
Fredholm theory
  analytic –, 198

gauge norm, 229
gauge norm on a complex Banach lattice, 229
generalised eigenspace, 194
generalised eigenvector, 194
generating cone, 205
geometric multiplicity, 194
geometrically simple eigenvalue, 194
growth bound of a C_0-semigroup, xxiii
  essential –, xxiv

Hilbert lattice, 213
  complex –, 230

ice cream cone, 52
ideal
  principal –, 229
ideal in a complex Banach lattice, 227
  principal –, 229
imaginary part, 216
immediately differentiable C_0-semigroup, 150
individually asymptotically positive matrix, 114
individually asymptotically positive resolvent, 102
individually asymptotically positive semigroup, 85
individually eventually positive resolvent, 139
individually eventually positive semigroup, 84
individually eventually strongly negative resolvent, 141
individually eventually strongly positive resolvent, 141
individually eventually strongly positive semigroup, 148
interpolation property, 58
interval
  order –, 208
isolated element of a set, 191
isolated subset, 191
isomorphic complex Banach lattices, 225
isomorphic ordered Banach spaces, 209
isomorphism of complex Banach lattices, 225
isomorphism of ordered Banach spaces, 209

kernel operator
  regular –, 28

Laplace operator
  square of the –, 175
Laplace operator with non-local boundary conditions, 181
lattice homomorphism on a complex Banach lattice, 225

lattice ordered Banach space
projectively –, 57
located on the left, 123

maximum principle
anti-, 141
mean ergodic operator, 24
measure
support of a –, 123
modulus of a vector in a complex Banach lattice, 224
monotone norm, 207
multiplicity
algebraic –, 194
geometric –, 194

network flow, 128
non-autonomous boundary conditions
diffusion with –, 131
non-Euclidean cone
projectively –, 57
non-Euclidean ordered Banach space
projectively –, 57
non-Hilbert space
completely –, 8
extremely –, 3
projectively –, 8
super projectively –, 11
non-local boundary conditions
Laplace operator with –, 181
norm
gauge –, 229
monotone –, 207
norm on a complex Banach lattice
gauge –, 229
normal cone, 207

order continuous norm
Banach lattice with –, 146
order interval, 208
order unit, 206
ordered Banach space, 205
projectively lattice –, 57
projectively non-Euclidean –, 57

periodic
almost –, 20
strongly almost –, 20
weakly almost –, 22
peripheral point spectrum, xxiii
peripheral spectrum, xxiii
point spectrum on a real Banach space, 222
pointed cone, 205
polyhedral cone, 61
positive C_0-semigroup, xxiv
positive cone, 205
positive element of an ordered Banach space, 206
positive functional, 210
positive operator, 208
bi-, 208
strongly, xxiv
positive operator on a complex Banach lattice, 225
positive operator semigroup, 70
principal ideal, 229
principal ideal in a complex Banach lattice, 229
projection
reducing –, 192, 221
spectral –, 191
projectively lattice ordered Banach space, 57
projectively non-Euclidean cone, 57
projectively non-Euclidean ordered Banach space, 57
projectively non-Hilbert space, 8
super –, 11
pseudo-resolvent, 199
pseudo-spectral bound, xxiii

pseudo-spectral bound on a real Banach space, 222

quasi-compact, 26
quasi-interior point of a Banach lattice cone, xxiv

rank of a generalised eigenvector, 194
rank of an operator, xxi
real linear operator, 218
real part, 216
real part of a complex Banach lattice, 224
real part of the complexification of a Banach space, 216
real sesqui-linear form, 232
reducing projection, 192, 221
regular operator on a Banach lattice, 27
relatively compact orbits, 19
relatively weakly compact orbits, 22
resolvent equation, 199
resolvent identity, 199
Riesz decomposition property, 57
Robin Laplace operator
  square of the –, 177

semi-simple eigenvalue, 194
sequi-linear form
  real –, 232
simple eigenvalue
  algebraically –, 194
  geometrically –, 194
  semi-, 194
spectral bound on a real Banach space, 222
  pseudo-, 222
spectral projection, 191
spectral radius on a real Banach space, 222
spectrum on a real Banach space, 222
  approximate point –, 222
  point –, 222
square of a Laplace operator, 175
square of the Dirichlet Laplace operator, 176
square of the Robin Laplace operator, 177
strictly positive functional, xxiv
strongly almost periodic, 20
strongly positive matrix semigroup, 168
strongly positive operator, xxiv
strongly positive operator with respect to u, xxv
strongly positive vector with respect to u, xxv
sublattice of a complex Banach lattice, 227
subspace
  1-complemented –, 8
  complemented –, 8
  dominating –, 211
super projectively non-Hilbert space, 11
support of a functional, 124
support of a measure, 123

thermostat, 184
topheavy cone, 53
total cone, 205
transport process on a metric graph, 129
type of a C_0-semigroup, xxiii

ultra power of a Banach space, xxvi
uniform boundedness of the resolvent
  abscissa of –, xxiii
uniformly asymptotically positive matrix, 115

uniformly asymptotically positive
resolvent, 102
uniformly asymptotically positive
semigroup, 85
uniformly eventually positive
resolvent, 140, 159
uniformly eventually positive
semigroup, 84
uniformly eventually strongly
negative resolvent, 141
uniformly eventually strongly
positive resolvent, 141
uniformly eventually strongly
positive semigroup, 148
unit
order –, 206

vector space
dimension of a –, xxi, 194

weakly almost periodic, 22
weakly compact orbits
relatively –, 22
weighted adjacency matrix of a
graph, 171

Zusammenfassung in deutscher Sprache

In der vorliegenden Arbeit studieren wir den Zusammenhang zwischen dem Langzeitverhalten von C_0-Halbgruppen auf Banachräumen und invarianten konvexen Mengen. Unsere Untersuchungen sind durch die sogenannte Perron–Frobenius–Theorie positiver Halbgruppen motiviert. In dieser Theorie betrachtet man positive C_0-Halbgruppen $(e^{tA})_{t\geq 0}$ auf einem geordneten Banachraum, meist einem Banachverband, und untersucht, welche Auswirkungen die Positivität der Halbgruppe auf das Spektrum des Generators A und somit auf das Langzeitverhalten der Halbgruppe hat. Ausgehend von dieser Theorie leiten wir drei Fragestellungen ab, die im Zentrum der drei Hauptteile unserer Arbeit stehen.

In Teil I untersuchen wir Halbgruppen $(e^{tA})_{t\geq 0}$ auf reellen Banachräumen, insbesondere auf Funktionenräumen, die nicht positiv, sondern stattdessen kontraktiv sind. Es stellt sich heraus, dass diese vermeintliche schwache Voraussetzung genügt um äußerst überraschende Spektralresultate zu beweisen, falls die Einheitskugel des Banachraumes sich hinreichend von der eines Hilbertraumes unterscheidet. Diese Eigenschaft – in geeignetem Sinne "nicht–Hilbertsch" zu sein – präzisieren wir, indem wir eine Reihe unterschiedlich starker geometrischer Bedingungen an einen Banachraum definieren. Aufbauend auf einer Arbeit von Lyubich zeigen wir anschließend, dass solche Bedingungen zusammen mit der Kontraktivität der Halbgruppe $(e^{tA})_{t\geq 0}$ (und unter geeigneten Regularitätsvoraussetzungen) erhebliche Konsequenzen für das Spektrum von A haben. Aus unseren spetraltheoretischen Resultaten leiten wir schließlich Konvergenzsätze für die betrachteten Halbgruppen her.

Die geometrischen Bedingungen an die Einheitskugel, welche wir benutzen, werden von einer Reihe von wichtigen Funktionenräumen erfüllt, darunter einige klassiche Folgenräume und L^p-Räume für $p \neq 2$. Eines unserer Hauptresultate in Teil I lautet wie folgt: Auf einem reellen L^p-Raum ($p \in (1,\infty)\setminus\{2\}$) ist jede schließlich Norm-stetige, kontraktive C_0-Halbgruppe stark konvergent. Am Ende des ersten Teils der Arbeit zeigen wir, wie ein anderes unserer Resultate zur Analyse des Langzeitverhaltens von retardierten Differentialgleichungen benutzt werden kann.

In Teil II wenden wir uns wieder den positiven Halbgruppen zu, welche die ursprüngliche Motivation für diese Arbeit sind. Wir schränken uns nun aber nicht mehr auf Banachverbände ein, sondern betrachten positive C_0-Halbgruppen auf allgemeinen geordneten Banachräumen. Ähnlich wie in Teil I stellt sich heraus, dass wir nicht-triviale spektraltheoretische Resultate

nur unter zusätzlichen Voraussetzungen an die Geometrie des positiven Kegels erwarten können – dieser sollte sich in geeignetem Sinne "nicht-Euklidisch" verhalten. Wir beschäftigen uns deshalb zunächst ausführlich mit der Geometrie von positiven Kegeln. Anschließend beweisen wir, inspiriert durch eine Arbeit von Veĭtsblit und Lyubich, ein spektraltheoretisches Resultat über die Generatoren positiver Halbgruppen auf geordenten Banachräumen mit solchen "nicht-Euklidischen" Kegeln. Außerdem sprechen wir kurz einige Konsequenzen dieses Resultats für das Langzeitverhalten der Halbgruppe an.

Wir betrachten den kompletten zweiten Teil der Arbeit vornehmlich als theoretische Studie. Wir wollen einerseits den Zusammenhang zwischen Perron–Frobenius–Theorie und der Geometrie von Kegeln herausarbeiten; andererseits möchten wir die Grenzen der Perron–Frobenius–Theorie ausloten und darlegen, dass man sich für viele Resultate nicht auf verbandsgeordnete Räume beschränken muss. Aufgrund dieser Sichtweise gehen wir in Teil II nicht weiter auf potentielle Anwendungen ein.

Es sollte betont werden, dass wir unter dem „Langzeitverhalten" einer Halbgruppe in den ersten beiden Teilen der Arbeit stets die Frage verstehen, ob die Halbgruppe in einer geeigneten Topologie für $t \to \infty$ konvergiert. Die Invarianz konvexer Mengen (genauer: des positiven Kegels oder der Einheitskugel) dienten uns vornehmlich als Mittel zum Zweck um spektraltheoretische Resultate und auf diesem Wege Konvergenzaussagen beweisen zu können. Andererseits sollte man nicht vergessen, dass gerade Positivität auch eine wichtige intrinsische Bedeutung in vielen Anwendungen besitzt.

Aus diesem Grund wechseln wir in Teil III der Arbeit die Perspektive und widmen uns anstelle von bloßen Konvergenzuntersuchungen einem anderen Aspekts des Langzeitverhaltens von Halbgruppen: Wir studieren solche C_0-Halbgruppen, die womöglich nicht für kleine, wohl aber für genügend große Zeiten positiv werden; genauer gesprochen betrachten wir Halbgruppen $(e^{tA})_{t\geq 0}$ mit der Eigenschaft, dass für jeden Anfangszustand $f \geq 0$ eine Zeit t_0 existiert, von der an die Trajektorie $(e^{tA}f)_{t\geq 0}$ ebenfalls im positiven Kegel liegt. Da der Zeitpunkt t_0 hier von f abhängen kann, umschreiben wir dieses Verhalten als *individuelle schließliche Positivität* der Halbgruppe; von *gleichmäßiger schließlicher Positivität* sprechen wir hingegen, wenn wir t_0 unabhängig von f wählen können. Mehrere Gegenbeispiele zeigen, dass individuelle und gleichmäßige schließiche Positivität nicht gleichbedeutend sind.

Unser Ziel im dritten Teil der Arbeit ist es eine Theorie von schließlich positiven Halbgruppen zu entwickeln, die einerseits Auskunft über wichtige, inbesondere spektraltheoretische Eigenschaften solcher Halbgruppen gibt und die uns andererseits Mittel zur Verfügung stellt um in konkreten Anwendungen zu entscheiden, ob eine – oft nur durch ihren Generator gegebene – Halbgruppe schließlich positiv ist. Aufgrund unseres Interesses an

konkreten Anwendungen in diesem Teil der Arbeit konzentrieren wir uns hierbei auf solche geordneten Banachräume, die in Anwendungen besonders häufig auftreten, namentlich auf Banachverbände. Wir führen verschiedene Variationen der oben beschriebenen Konzepte ein, die wir alle unter dem Begriff *Schließliche Positivität* zusammenfassen. Nachdem wir zunächst beweisen, dass zahlreiche Resultate aus der Perron–Frobenius–Theorie positiver Halbgruppen für schließlich positive Halbgruppen richtig bleiben, konzentrieren wir uns im Rest des dritten Teils auf *Charakterisierungen* schließlich positiven Verhaltens. Diese Charakterisierungen beruhen einerseits auf spektraltheoretischen Eigenschaften des Halbgruppengenerators, welche sich in konkreten Anwendungen häufig nachprüfen lassen. Andererseits spannen unsere Charakterisierungen auch den Bogen zurück zu den ersten beiden Teilen der Arbeit, denn wir zeigen, dass das dort betrachtete Langzeitverhalten, also die Konvergenz der Halbgruppe, eng mit dem nun betrachteten Langzeitverhalten, also der schließlichen Positivität, zusammenhängt. Neben der schließlichen Positivität der Halbgruppe untersuchen wir zudem schließlich positives Verhalten der Resolvente des Halbgruppengenerators.

Unsere Hauptresultate in Teil III sind mehrere Theoreme, die verschiedene Arten von schließlicher Positivität einer Halbgruppe spektraltheoretisch charakterisieren. In zwei eigenen Kapiteln stellen wir zahlreiche Anwendungen dieser Resultate auf konkrete Beispiele von Differentialoperatoren und anderen Halbgruppengeneratoren vor. Wir verfolgen hierbei drei Ziele: Erstens möchten wir aufzeigen, dass einige konkrete Beispiele von Differentialoperatoren, bei denen in der Literatur bereits schließlich positives Verhalten beobachtet wurde, sich innerhalb unserer allgemeinen Theorie einheitlich behandeln lassen. Bei diesen Beispielen handelt es sich um bi-Laplace-Operatoren und um Dirichlet-zu-Neumann-Operatoren. Eine von Daners veröffentlichte Fallstudie über das schließlich positive Verhalten des letzteren lieferte die ursprüngliche Motivation für die Entwicklung einer allgemeinen Theorie schließlich positiver Halbgruppen. Zweitens ist es unser Ziel zu zeigen, dass schließliche Positivität in einigen Modellen unter sehr allgemeinen Bedingungen auftritt. Dies belegt, dass es sich hier keineswegs um ein sonderbares oder rein akademisches Phänomen handelt. Stattdessen kann schließliche Positivität als ein Verhalten gesehen werden, welches unter Umständen deutlich häufiger auftritt als Positivität der Halbgruppe. Unser drittes Ziel ist es, Anregungen für weitere Forschung auf diesem Gebiet zu liefern. Zum Beispiel zeigen wir anhand zweier konkreter Beispiele, dass schließliche Positivität bei bestimmten Laplace-Operatoren mit nicht-lokalen Randbedingungen auftritt. Dieses und weitere Beispiele legen nahe, dass weitere Untersuchungen erforderlich sind um ein noch besseres Verständnis sowohl der abstrakten Theorie als auch konkreter Beispiele zu erlangen.

Ehrenwörtliche Erklärung

Ich versichere hiermit, dass ich die Arbeit selbstständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe.

Ulm, den 27.07.2016

.....
Jochen Glück

Zahlreiche Resultate in dieser Dissertation sind bereits, teils in modifizierter Version, in den folgenden Fachartikeln erschienen:

[DGK16a] Daniel Daners, Jochen Glück, und James B. Kennedy. Eventually and asymptotically positive semigroups on Banach lattices. *J. Differential Equations*, 261(5):2607–2649, 2016.

[DGK16b] Daniel Daners, Jochen Glück, und James B. Kennedy. Eventually positive semigroups of linear operators. *J. Math. Anal. Appl.*, 433(2):1561–1593, 2016.

[Glü16d] Jochen Glück. Spectral and asymptotic properties of contractive semigroups on non-Hilbert space. *J. Operator Theory*, 76(1):3–31, 2016.

Der Lebenslauf wurde aus Datenschutzgründen in der digitalen Version entfernt.

Der Lebenslauf wurde aus Datenschutzgründen in der digitalen Version entfernt.

Der Lebenslauf wurde aus Datenschutzgründen in der digitalen Version entfernt.