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Preface

Bringing together the Lévy-process financial world and the theory of unit-linked life insurance contracts, this thesis shall contribute to the ongoing research on the interface between financial and actuarial mathematics. It concludes my doctorate research which was carried out at the Department of Number Theory and Probability Theory at the University of Ulm in the period from October 2003 to July 2006. My work was supervised by Professors Ulrich Stadtmüller and Rüdiger Kiesel, both University of Ulm.

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Chapter 1

Introduction and summary

1.1 Historical overview and existing literature

Unlike traditional life insurance the benefits and possibly the premiums of an unit-linked life insurance are random and linked to the development of some specified reference portfolio, a mutual fund or simply a stock index. Insurance companies offering these kind of products often collaborate with mutual funds and offer the policy-holder a huge variety of investment opportunities including funds that concentrate on specific countries, areas or industry sectors or even mixtures of several specific funds. Such contracts are usually equipped with a minimal capital guarantee and allow the policyholder to decide how his premiums are to be invested. Hence, they transfer parts of the investment risk to the policyholder while, simultaneously, providing insurance, a limited exposure to the downside risk and a high level of investment transparency. Moreover, the policy-holder benefits directly from the upside potential of his investment and the flexibility of the contracts allowing to adjust the riskiness of the investment depending on the remaining time of the policy. In recent years these specific life insurance products, especially those with capital guarantees, have become very popular in most countries with a fully developed insurance industry. They seem to have appeared for the first time around 1950 in the Netherlands and around 1954 in the United States of America. Later, in 1957, they were introduced to the United Kingdom. Unit-linked life insurance is also known as equity-linked or equity-based life insurance. In the United States it is called variable life insurance.

These kinds of insurance products combine financial and insurance risk and require techniques from both financial and actuarial mathematics for their proper treatment. Møller (2002) reviews valuation and hedging of life and non-life insurance products on the interface of those two areas and especially describes exhaustively the history of unit-linked life insurance products providing an overview of related literature.

The first who analyzed unit-linked insurance products using modern financial mathematics were Brennan and Schwarz (1976, 1979a,b) and Boyle and Schwartz (1977) who recognized that the payoff of an unit-linked with guarantee life insurance equals the payoff of an European call option plus the guaranteed amount. Furthermore, referring to a law of large numbers argument these authors replaced the uncertainty of insured lives by their expected values and did not pay attention to mortality risk. For this reason they could treat unit-linked with guarantee life insurance contracts as contingent claims with financial uncertainty in a complete financial market, in which they could apply the standard valuation and hedging techniques that had been developed shortly before by Black and Scholes (1973) and Merton (1973). The idea of replacing the uncertain course of mortality by its expected development was later also used by Delbaen (1986), Bacinello and Ortu (1993a) and Aase and Persson (1994) to price unit-linked life insurance contracts involving the martingale based financial techniques of Harrison and Pliska (1981) and Harrison and Kreps (1979). Moreover, Aase and Persson (1994) additionally worked with continuous and not with discrete time survival probabilities, but all authors mentioned above used constant interest rates. Bacinello and Ortu (1993b), Nielsen and Sandmann (1995) and Bacinello and Persson (2002) added stochastic interest rates to the existing models.

Møller (1998) had the new idea not to average away mortality treating simultaneously the uncertain development of the financial market and the insurance portfolio modeled together in a product probability space as two independent components and shows that the uncertainty of the insurance portfolio even makes a complete financial market incomplete, in which riskless hedging is impossible. Moreover, for a Black-Scholes financial market and a portfolio of either unit-linked pure endowment or unit-linked term insurance contracts, risk-minimizing hedging strategies and the associated hedging risk are derived. More precisely, a complete financial market which turned incomplete by mortality is considered and the risk-minimizing hedging theory of Föllmer and Sondermann (1986) is applied. Hedging strategies deduced from this approach react to the financial uncertainty as well as to the actual number of survivors in the insurance portfolio. The associated hedging risk is a measure for the mortality uncertainty which can not be hedged away. Since the theory of Föllmer and Sondermann (1986) requires contingent claims payable at the end of the considered time horizon, Møller (1998) assumes that there is only a single premium paid at issuing date and that the insurance benefits are deferred, while earning interest, and cashed out at the end of the considered time horizon. From a practical point of view not admitting intermediate premium payments is not very restrictive for the pure endowment. But for most other life insurance contracts this is a very unsatisfying assumption, because their benefits are usually due immediately upon occurrence of some insurance event. Annuity payments are

even paid on a regular basis. For this reason Møller (2001a) extended the risk-minimizing hedging theory for contingent claims with fixed maturity of Föllmer and Sondermann (1986) to payment streams, in particular, insurance payment streams admitting intermediate premium and benefit payments. However, this theory is only applicable in martingale financial markets implying that the hedging risk is only interpretable under the hedger's subjective measure if it is already a martingale measure. Møller (2001a) additionally contains the risk-minimizing hedging strategies and the associated hedging risk for a martingale Black-Scholes financial market and for general unit-linked life insurance contracts, modeled in a multi-state Markov model with intermediate payments. Analogously to the already mentioned articles (Møller (1998, 2001a)), the paper Møller (2001b) treats risk-minimizing hedging of equity-linked life insurance contracts in a discrete time set-up using exemplarily a Cox-Ross-Rubinstein financial market model.

More recently, indifference pricing techniques, involving the financial variance and standard deviation principles, have been proposed as an alternative way of valuation and hedging of unit-linked life insurance contracts by Møller (2001c, 2003a,b). Moreover, the risk-minimizing hedging theory for payment streams of Møller (2001a) is applied in Dahl and Møller (2006) for the case of traditional life insurance with stochastic interest rates and a mortality intensity of all policy-holders that is affected by some stochastic process.

Unit-linked life insurance requires for its treatment a financial market in which the *unit* is modeled. Almost all of the articles mentioned above assume a Black-Scholes kind of financial market, in which asset prices are driven by a Brownian motion featuring a normal distribution of asset returns and continuous sample paths. Recent research, however, favors more general Lévy-process financial market models, in which the Brownian motion is replaced by a more general Lévy process resulting in models with more flexible asset return distributions and discontinuous sample paths. Among others, Chan (1999) is certainly a seminal work introducing general Lévy processes to financial market modeling, but even Merton (1976) proposed a jump-diffusion process to model asset returns. According to Chapter 1 of Cont and Tankov (2004) Lévy-process models with jumps naturally meet the following observations of real financial markets, which are impossible or difficult to realize in diffusion based models.

1. Asset prices often do not follow continuous trajectories and incorporate large, sudden movements in prices.

2. Losses are concentrated in a few large downward moves represented by discontinuities in the price behavior.

3. Asset return distributions are heavy tailed.

4. Asset return distributions may be asymmetric.

5. Financial markets are incomplete and riskless hedging is impossible.
6. Perfect hedges do not exist and options are risky investments.
7. Various hedging strategies may perform differently.
8. Volatility smiles in equity and foreign exchange option markets: The implied volatility may be U-shaped (smile) as a function of the strike price of an option.
9. Volatility skews in equity and foreign exchange option markets: The implied volatility may be decreasing as a function of the strike price of an option.

As a consequence, research in general Lévy-process financial markets has gained a major role in modern financial mathematics over the last decade. Recently, Jaimungal and Young (2005) discussed the pricing of equity-linked pure endowments in a finite variation Lévy-process financial market using indifference pricing, more precisely, the principle of equivalent utility.

1.2 Summary, contributions and outline of this thesis

The main aim of this thesis is the development of locally risk-minimizing hedging strategies for unit-linked life insurance contracts whose *unit* is modeled in a general Lévy-process financial market. It therefore merges the quite advanced and in recent years developed theory of Lévy-process financial markets with the theory of unit-linked life insurances and provides moreover a quadratic hedging framework for insurance payment streams exposed to pure financial and pure insurance risk. Up to our knowledge, locally risk-minimizing hedging of unit-linked life insurance contracts in general Lévy-process financial markets has not been analyzed so far. In summary this thesis is part of the ongoing research on the interface between financial and actuarial mathematics and generalizes the earlier work of Møller (1998, 2001a) in the sense that the complete Black-Scholes financial market is replaced by an incomplete financial market in case of a more general geometric Lévy-driven model.

As Møller (1998, 2001a) we assume stochastic independence between the financial market and the insurance model and consider them combined in a common product probability space. This picks up the idea to model the uncertain development of the stock price and the insured lives simultaneously not averaging away mortality, that is, the uncertain course of insured lives is not replaced by their expected development. Furthermore, modeling the insurance part of our model with the general multi-state Markov chain model of Hoem (1969) yields very flexible results which can be adapted to various

imaginable types of life insurance. In examples, however, we put special emphasis on certainly the most important versions such as pure endowments, term insurances or life annuities.

We first derive the locally risk-minimizing hedging strategy and the associated hedging risk for a portfolio of unit-linked pure endowment and a portfolio of unit-linked term insurance contracts assuming that they are sold against a single premium at issuing date and that their insurance benefits are deferred, while earning interest, and cashed out at the end of the considered time horizon. With this simplifying assumption the corresponding claims fit into the theory of Föllmer and Sondermann (1986) and Schweizer (1991). Møller (2001a) provides an extension of the theory of Föllmer and Sondermann (1986) to risk-minimizing hedging of payment streams in martingale financial markets. Moreover, the theory of Föllmer and Sondermann (1986) has also been generalized to semimartingale financial markets by Schweizer (1991) introducing the concept of locally risk-minimizing hedging and the Föllmer-Schweizer measure. In a separate chapter we show that locally risk-minimizing hedging also extends to hedging of payment streams if the Föllmer-Schweizer measure is used. This delivers an interpretation of the hedging risk under an investor's subjective probability measure and not only under some risk-neutral martingale measure. The extension of locally risk-minimizing hedging to payment streams is new and has not been considered before. Finally we derive, in a general Lévy-process financial market, locally risk-minimizing hedging strategies for general unit-linked life insurance contracts admitting intermediate premiums and benefits.

Since we use an incomplete financial market from the beginning we get two hedging risk components reflecting the hedging risk's origin. We call them pure financial and pure insurance risk of hedging. Moreover, the total hedging risk of an unit-linked life insurance claim is shown to separate exactly in those two terms and, in accordance with standard actuarial theory, the pure insurance risk is demonstrated to diversify by raising the number of insured individuals within the portfolio. This is however, as expected, not the case with the pure financial risk. The difference from Møller (1998, 2001a) is that we additionally model in compliance with reality the pure financial risk. Because the Black-Scholes model describes a complete financial market, the total hedging risk in Møller (1998, 2001a) corresponds to the pure insurance risk of hedging in our case. Let us give a detailed outline of the thesis in the following.

Basic concepts

In Chapter 2 the essential technical concepts required in this thesis are reviewed and the models used in the following chapters are introduced. Beginning with the definition of a Lévy process we subsequently discuss the Lévy measure, the famous Lévy-Itô-decomposition, being formulated in The-

orem 2.2.4, and the distributional properties of Lévy processes such as the Lévy-Khinchin formula and infinite divisibility. Moreover, in Theorem 2.2.6 we classify the paths behavior of Lévy processes distinguishing between finite and infinite activity or variation, respectively. Subsection 2.2.2 contains a discussion of jump-diffusions, variance gamma, normal inverse Gaussian and generalized hyperbolic Lévy processes, since they are commonly used for financial market modeling. In Section 2.3 the Lévy-process financial market model of Chan (1999) is introduced and the risk-neutral Föllmer-Schweizer measure, which is required for locally risk-minimizing hedging, is treated. We explain in detail the effects of the associated change of measure on the back-driving Lévy process possibly becoming an additive process. Furthermore, in Lemmata 2.3.1 and 2.3.3, we discuss the stock price process before and after this change of measure, respectively, and show its square-integrability in both cases using the result of the appendix. This result enables us to weaken significantly an assumption of Chan (1999). In our case it suffices now to demand only the existence of the Lévy measure's third moment unlike the existence of its exponential moments. The multi-state Markov life insurance model of Hoem (1969), which admits to model quite general forms of life insurance, is summarized in Section 2.4 and historically famous but nowadays still frequently used laws of mortality are presented in Subsection 2.4.3 to give examples for the abstract hazard rate of the insurance model. Following this, in Subsection 2.4.4, we use the central limit theorem to review a conventional model of insurance risk diversification for the case of traditional life insurance and in Subsection 2.4.5 we explain, applying the strong law of large numbers, the risk-neutrality of an insurance company with respect to mortality. Finally, Section 2.5 is devoted to the construction of a common product probability space combining the Lévy-process financial market and the insurance model under the reasonable assumption of stochastic independence. Moreover, in the risk-neutral counterpart of this space the risk-minimizing hedging theory of Föllmer and Sondermann (1986) is reviewed including a discussion of its relation to local risk-minimization of Schweizer (1991) and the fundamental Galtchouk-Kunita-Watanabe decomposition in Theorem 2.5.3.

Arbitrage-free price process

In Chapter 3 we discuss the arbitrage-free price process of a square-integrable contingent claim which we represent, using the Markov property of the stock price, as a function of the time, the stock value at that time and the due date of the claim. The underlying financial market is thereby the common risk-neutral probability space of Chapter 2, in which the process modeling the stock is an additive process, that is, a càdlàg, stochastically continuous process with independent increments. The main emphasis of the chapter is the derivation of the Galtchouk-Kunita-Watanabe decomposition of the

arbitrage-free price process which is formulated in Theorem 3.3.4 and which has the same structure as it was worked out for a Lévy process. It is certainly the most important element of the locally risk-minimizing hedging theory and it is the basis for the derivation of the Galtchouk-Kunita-Watanabe decomposition of the unit-linked life insurance intrinsic values in the following chapters. Roughly speaking, the decomposition is a projection result in the Hilbertspace of square-integrable random variables. We prove Theorem 3.3.4 applying directly the orthogonality of the involved continuous and purely discontinuous martingales, an approach we have not observed before. Moreover, the theorem formulated for additive processes seems to be new. To distinguish our contribution from existing results, we provide an overview of related literature at the end of the chapter. A necessary result of decomposing the arbitrage-free price process is its Feynman-Kac type representation. In our case this is a partial integro-differential equation which is formulated and proved in Theorem 3.2.4. It follows from the martingale property of the discounted arbitrage-free price process and can be used to compute it numerically. At the beginning the applied decomposition procedure is completely analogous to the Black-Scholes setting, in which it already reduces with the Feynman-Kac type representation to the right Galtchouk-Kunita-Watanabe decomposition. In our more general set-up several further steps are necessary, which are demonstrated in Lemmata 3.3.2 and 3.3.3.

Hedging of unit-linked pure endowment and term insurance

Chapter 4 is devoted to the derivation of locally risk-minimizing hedging strategies for a portfolio of either unit-linked pure endowment or unit-linked term insurance contracts. As mentioned before, we thereby restrict ourselves for the moment to the case in which there are only payments at the beginning and at the end of the considered time horizon. This allows to apply the locally risk-minimizing hedging theory of Schweizer (1991), which has been developed for contingent claims with fixed maturity in a semimartingale financial market and which delivers an interpretation of the hedging risk under the subjective probability measure. The chapter serves therefore as an intermediate step towards the general treatment of unit-linked life insurances admitting intermediate premiums and benefits. It is based on Riesner (2006) and complements Møller (1998) by replacing the complete Black-Scholes financial market through a more general and incomplete Lévy-process financial market. We start with a general mathematical discussion of unit-linked life insurance benefits and derive the present value of an entire portfolio of the two types of life insurances mentioned above. After various further calculations, these reduce to the desired Galtchouk-Kunita-Watanabe decompositions being respectively formulated in Corollaries 4.3.1 and 4.3.6. These calculations include, among others, Itô's integration by parts for general semimartingales and, additionally for the term insurance, Fubini's the-

orem for stochastic integrals. The locally risk-minimizing hedging strategies are subsequently provided in Corollaries 4.3.2 and 4.3.7, respectively. For the general Lévy-process financial market all mentioned corollaries are new results. We discuss and compare them in detail to the results of Møller (1998). In our case the risk-minimal investment in the stock is a weighted sum of the Black-Scholes delta and a term involving the jumps of the price process. Additionally we derive the two components of the hedging risk, the pure financial and the pure insurance risk. Finally we justify why local risk-minimization should particularly be used for insurance claim hedging.

Payment stream hedging for semimartingales

Chapter 5 partly contributes to Riesner (2005) and addresses locally risk minimizing hedging of payment streams in general semimartingale financial markets. For martingale financial markets there is the risk-minimizing hedging theory for payment streams of Møller (2001a). In semimartingale financial markets one usually chooses an arbitrary risk-neutral martingale measure, however, it remains the problem to interpret the hedging risk under the subjective believes. Schweizer (1991) developed therefore locally risk minimizing hedging of contingent claims with fixed maturity for semimartingale financial markets, which delivers an interpretation of the hedging risk under the subjective probability measure provided the risk-neutral Föllmer-Schweizer measure is applied. We show in this chapter, using Schweizer (1991) as basis, that risk-minimizing hedging of payment streams has in the same sense a local version as hedging of contingent claims with fixed maturity: Using the theory of Møller (2001a) one derives a risk-minimizing hedging strategy under the Föllmer-Schweizer measure which is then locally risk-minimizing with respect to the subjective probability measure. After having discussed a lot of technical machinery this main result is summarized in Theorem 5.4.2. It is based on the same optimality equation as in the classical case which is formulated in Theorem 5.3.7. Furthermore, in Proposition 5.2.3 we relate attainable contingent claims to attainable payment streams in a semimartingale financial market. The extension of locally risk-minimizing hedging to payment streams seems to be considered in this thesis for the first time.

Hedging of general unit-linked life insurance contracts

The topic of Chapter 6 is the derivation of locally risk-minimizing hedging strategies for general unit-linked life insurance contracts having intermediate premium and benefit payments. The multi-state Markov life insurance model of Hoem (1969) is thereby used to describe the model's insurance part. It admits state-wise annuity payments as well as payments induced by a state transition. Those payments are interpreted as difference of benefit

and premium payments. The Galtchouk-Kunita-Watanabe decomposition of the contracts' intrinsic value is stated in Theorem 6.3.1 and is proved applying the techniques already known from Chapter 4. The corresponding locally risk-minimizing hedging strategy as well as the hedging risk are subsequently given in Corollary 6.3.2. For a general Lévy-process financial market these results are new. At the end of the chapter we apply this general model to unit-linked life annuities and to unit-linked term insurances considering first a single contract and subsequently a portfolio of these insurances. The chapter contributes to Riesner (2005) and generalizes Møller (2001a) by replacing the complete Black-Scholes financial market through a more general and incomplete Lévy-process financial market.

Jump-diffusion stock model

To propose one possible concrete Lévy-process financial market, we discuss the asymmetric double exponential jump-diffusion model of Kou (2002) in Chapter 7. We justify using this particular model and review the asymmetric double exponential distribution. This is followed by a discussion of the Lévy process behind the model's jump-diffusion, which we imbed in our financial market framework. Special emphasis is thereby put on the jump-size distribution and the corresponding Lévy measure. Kou (2002) uses a specific risk-neutral probability measure based on utility driven thoughts and shows that with this change of measure one stays, imposing certain assumptions on the market parameters like interest rate or drift, in the class of asymmetric double exponential jump-diffusions, whose distributional properties are essential for the derivation of the model's European call option pricing formula. Since we consider local risk-minimization, we are mainly interested in the implications of the Föllmer-Schweizer measure on the model and discuss them explicitly in Theorem 7.4.2 as well as in Lemmata 7.4.3 and 7.4.4. In Lemma 7.4.5 it is then shown that under this measure the jump-sizes are generally not asymmetric double exponential distributed resulting subsequently in alternative, the distributional class preserving conditions on the market parameters, which we additionally compare to those of Kou (2002). Section 7.5 concludes the chapter with the formula for the arbitrage-free price process of an European call, an unit-linked with guarantee pure endowment and an unit-linked with guarantee term insurance. Moreover the spatial derivative of the arbitrage-free price process is calculated. Both the price itself and its derivative are essential ingredients of the deduced hedging strategies. This jump-diffusion model has not been considered in connection with unit-linked life insurance contracts so far. Moreover, its investigation under the Föllmer-Schweizer change of measure has not been done before and the conditions for this change of measure not to lead outside the class of asymmetric double exponential jump-diffusions are new.

Statistics for jump-diffusions

Studying the asymmetric double exponential jump-diffusion of Kou (2002), we realized that satisfying parameter estimation methods are almost not available for jump-diffusions. Therefore we decided to address this topic in Chapter 8. Although we use thereby exemplarily the mentioned process, our treatment is essentially also valid for other jump-diffusions. For those processes the probability density is not expressible in closed form, however, their characteristic function is explicitly known. For that reason we propose in Section 8.3 to base the estimation on the cumulant generating function which we compare in the squared distance with the complex logarithm of the empirical characteristic function. This leads to a system of two non-linear regression models which arise from the real and imaginary part of these functions and which are even linear in special cases. Additionally we give a determining rule for the compact support on which these regression problems are considered. They may be solved using standard least-square routines. The theoretical foundation of our estimation approach is provided in Section 8.4. This section contains the Glivenko-Cantelli type theorem for the empirical characteristic function of Csörgő and Totik (1983) which we motivate and which gives us strong consistency. Thereafter we apply the invariance principle of Csörgő (1981) to the empirical characteristic function resulting in a complex valued Gaussian process in the distributional limit. With the help of this result we finally state in Theorem 8.4.7 the asymptotic finite-dimensional distributions of the complex valued residuals of our regression problem and compute their covariance structure.

Appendix

The appendix contains a result from Stochastic Analysis, which is based on Gronwall's inequality. It provides conditions under which the solution of a certain stochastic integral equation with respect to a Lévy process is a square-integrable process. We use it in Lemmata 2.3.1 and 2.3.3 to show that the stock process defined in (2.14) is square-integrable under the historical as well as under the Föllmer-Schweizer measure. An essential sufficient condition is the existence of only the third moment of the Lévy measure which we suppose in (2.12). We could therefore weaken one assumption of Chan (1999) demanding the existence of exponential moments of the Lévy measure. This opens the model to a broader class of distributions, especially to those with heavier tails. We designed and proved Theorem A.0.2 tailor-made to our situation, especially including a drift component. Other versions of the result, without a drift component, are contained (without proofs) for example in Jacod, Méléard and Protter (2000) or Ma, Protter and Zhang (2001).

Chapter 2

Basic concepts

In this chapter we provide the fundamentals required in this thesis. In doing so we begin with an introduction to the theory of Lévy processes and review those properties of a Lévy process which are beneficial for a Lévy-process financial market. Moreover, we briefly discuss specific Lévy processes that are in common use for financial market modeling, such as jump-diffusions, variance gamma, normal inverse Gaussian or generalized hyperbolic Lévy processes. Subsequently, we introduce the general Lévy-process financial market model of Chan (1999) together with changing the historical measure to the Föllmer-Schweizer measure. The local risk-minimizing hedging theory of Schweizer (1991), encountered in the chapters below, is based on this measure. Thereafter, we consider a multi-state Markov model for life insurance modeling which will frequently be used in this thesis. In our presentation of the latter model we follow Møller (2001a). Finally we describe how the financial market and the insurance model are combined to a common model being a product probability space. This model is necessary, since we treat insurance contracts for which the insurance benefit depends on a risky asset modeled in the financial market. For the product space we also review both its incompleteness and the (local) risk-minimizing hedging theory of Föllmer and Sondermann (1986) and Schweizer (1991).

2.1 Preliminaries

This little section at the beginning provides assumptions made throughout the thesis and states frequently used and common notations. For a detailed treatment of basic tools such as martingales, semimartingales and stochastic integration we refer to e.g. Jacod and Shiryaev (2003) or Protter (2004). Throughout this thesis we only consider insurance contracts and related financial markets, so it suffices to treat only the case of a finite time horizon $T \in [0, \infty)$.

Definition 2.1.1. *A filtered probability space* $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ *satisfies*

the usual conditions if $(\Omega, \mathbb{F}, \mathbb{P})$ *is complete, all the null sets of* $\mathbb{F}$ *are contained in* $\mathcal{F}_0$*, and* $(\mathcal{F}_t)_{0\le t\le T}$ *is a right-continuous filtration, that is:*

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathbb{F} \quad \text{are } \sigma\text{-algebras for} \quad 0 \le s \le t \le T,$$

and

$$\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t, \quad \forall \quad 0 \le s \le T.$$

In particular, we assume $\mathbb{F} = \mathcal{F}_T$*.*

For a filtration, $\mathcal{F}_t$ is interpreted as the information known in the model at time t which increases over time, i.e. a filtration describes how information is revealed in the model. All filtered probability spaces mentioned in this thesis are assumed to satisfy the usual conditions.

A stochastic process $X = (X_t)_{0\le t\le T}$ on some filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ is said to be càdlàg (continu à droite, limite à gauche) if its paths are right-continuous admitting left limits, i.e.

$$\lim_{s\to t, s>t} X_s = X_t \quad \text{and} \quad \exists \lim_{s\to t, s<t} X_s =: X_{s-}.$$

For such a stochastic process we define its jump at time $0 \le t \le T$ by

$$\Delta X_t := X_t - X_{t-}, \quad \text{where we set} \quad X_{0-} := X_0.$$

We further say X is adapted if X_t is $\mathcal{F}_t$-measurable for each $0 \le t \le T$. Occasionally we need the following measurability concepts.

Definition 2.1.2 (Optional σ-algebra)**.** *The optional* σ*-algebra is the* σ*-algebra* $\mathcal{O}$ *generated on* $\Omega \times [0,T]$ *by all adapted càdlàg processes.* $X : \Omega \times [0,T] \to \mathbb{R}$ *is called an optional process if it is measurable with respect to* $\mathcal{O}$*.*

Definition 2.1.3 (Predictable σ-algebra)**.** *The predictable* σ*-algebra is the* σ*-algebra* $\mathcal{P}$ *generated on* $\Omega \times [0,T]$ *by all adapted left-continuous processes.* $X : \Omega \times [0,T] \to \mathbb{R}$ *is called a predictable process if it is measurable with respect to* $\mathcal{P}$*.*

We always denote the expectation operator with respect to the canonical measure $\mathbb{P}$ by $\mathbb{E}[\cdot]$, whereas the expectation with respect to any other measure $\mathbb{Q}$ is written as $\mathbb{E}^{\mathbb{Q}}[\cdot]$. In the context of this thesis we generally use $\mathbb{P}$ for the so called real-world or historical measure and respectively $\mathbb{Q}$ for the so called risk-neutral measure.

Definition 2.1.4. *Let* $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ *be some filtered probability space and let* $\mathbb{Q}$ *be a second probability measure on it. Then* $\mathbb{P}$ *is said to be locally equivalent with respect to* $\mathbb{Q}$ *(*$\mathbb{P} \overset{loc}{\sim} \mathbb{Q}$*) if*

$$\mathbb{P}_t(A) = 0 \iff \mathbb{Q}_t(A) = 0, \quad \forall\, A \in \mathcal{F}_t,\ 0 \le t \le T.$$

$\mathbb{P}_t$ *and* $\mathbb{Q}_t$ *denote the restrictions of* $\mathbb{P}$ *and* $\mathbb{Q}$ *to the* σ*-algebra* $\mathcal{F}_t$*, respectively.*

We remaind the reader that a given positive martingale $Z = (Z_t)_{0\le t\le T}$ with $Z_0 = 1$ defines a probability measure $\mathbb{Q} \overset{loc}{\sim} \mathbb{P}$ by the Radon-Nikodým derivative $\frac{d\,\mathbb{Q}_t}{d\,\mathbb{P}_t} = Z_t$.

For two random objects Y_i, $i = 1, 2$, in most cases $Y_1 = Y_2$ (or "$\le$" etc.) stands for "$Y_1 = Y_2$ holds a.s. (almost surely)" (or $Y_1 \le Y_2$ a.s., etc.). Using the common notations we denote the quadratic (co)variation of two semimartingales X_t and Y_t by $[X, Y]_t$ and the predictable quadratic (co)variation of two (locally) square-integrable martingales X_t and Y_t by $\langle X, Y\rangle_t$.

2.2 Lévy processes

Lévy processes are named after the French mathematician Paul Lévy (1886 – 1971). Together with Andrei N. Kolmogorov (1903 – 1987) Lévy is considered as one of the founding fathers of the modern theory of stochastic processes. Taylor (1975) writes

> *"At that time there was no mathematical theory of probability — only a collection of small computational problems. Now it is a fully-fledged branch of mathematics using techniques from all branches of modern analysis and making its own contribution of ideas, problems, results and useful machinery to be applied elsewhere. If there is one person who has influenced the establishment and growth of probability theory more than any other, that person must be Paul Lévy."*

2.2.1 Definition and basic properties

In this subsection we define Lévy processes and discuss some of their most important properties. As basis we consider the filtered probability space $(\Omega, \mathbb{G}, (\mathcal{G}_t)_{0\le t\le T}, \mathbb{P})$. The next definition is adapted from Protter (2004), Chapter I, and Cont and Tankov (2004), Chapter 3.

Definition 2.2.1. *An adapted (càdlàg) stochastic process $L = (L_t)_{0\le t\le T}$ with values in $\mathbb{R}$ such that $L_0 = 0$ is called a Lévy process if it possesses the following properties:*

1. *Independent increments:*

$$L_t - L_s \quad \text{is independent of} \quad \mathcal{G}_s, \quad 0 \le s < t \le T.$$

2. *Stationary increments:*

$$L_{t+h} - L_t \overset{d}{=} L_h, \quad 0 \le t < t + h \le T,$$

that is, the distribution of $L_{t+h} - L_t$ does not depend on t.

3. Stochastic continuity:

$$\forall\, \varepsilon > 0,\ \lim_{h \to 0} \mathbb{P}(|L_{t+h} - L_t| \geq \varepsilon) = 0,$$

that is, given a fixed time t, the probability of seeing a jump of L at t is zero.

The "càdlàg" assumption in the definition of a Lévy process is not necessary and can be made without loss of generality, since every Lévy process has an unique modification that is càdlàg (cf. e.g. Protter (2004), Chapter I, Theorem 30). In Definition II.4.1 of Jacod and Shiryaev (2003) processes satisfying 1. and 2. of Definition 2.2.1 are called processes with stationary independent increments (PIIS). A straightforward choice of $\mathcal{G}_t$ could be the history of L up to and including time t, that is, the completed σ-algebra $\sigma(L_s, 0 \leq s \leq t)$. For the rest of this thesis we always assume $\mathcal{G}_t$ to be of this form.

Perhaps the most significant property of Lévy processes is the so called Lévy-Itô decomposition which reveals the building blocks of Lévy processes and at the same time a lot more structure than the definition does. In order to formulate this decomposition properly the discontinuities of a Lévy process have to be considered. Since a Lévy process has càdlàg paths, the only discontinuities occurring are at most countably many jump discontinuities (cf. Riesner (2003)). The jump counter of a Lévy process L is for $0 \leq t \leq T$ defined as

$$N_t(B) := \sum_{0 < s \leq t} \mathbb{1}\left(\Delta L_s \neq 0\right) \mathbb{1}\left(\Delta L_s \in B\right)$$

for any Borel-measurable set $B \in \mathbb{R}$. By the càdlàg paths of L it is clear that N_t is almost surely finite whenever $0 \notin \bar{B}$ (the closure of B). Protter (2004), Chapter I Theorem 35, shows that $N_t(\cdot)$ and $\mathbb{E}[N_1(\cdot)]$ are σ-finite measures on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel[1]-σ-algebra over $\mathbb{R}$.

Definition 2.2.2. *Given a Lévy process L, the measure $\nu(\cdot)$ defined on $(\mathbb{R}, \mathcal{B})$ by*

$$\nu(B) = \mathbb{E}[N_1(B)] \quad \textit{for any set} \quad B \in \mathcal{B}$$

is called the Lévy measure of L. $\nu(B)$ is the expected number, per unit time, of jumps whose size belongs to B.

Moreover for a fixed Borel-measurable set $B \in \mathbb{R}$ such that $0 \notin \bar{B}$, it holds that $(N_t(B))_{0 \leq t \leq T}$ is a Poisson process[2] (Protter (2004), Chapter I). That is, it is a counting process without explosion which is a Lévy process (cf. additionally Lemma 2.1 in Cont and Tankov (2004)). Therefore it is

[1] Félix Edouard Justin Émile Borel, French mathematician (1871 – 1956)

[2] named after the French mathematician Siméon Denis Poisson (1781 – 1840)

obvious that $N_t(B)$ has intensity $t\nu(B)$ and that $(N_t(B) - t\nu(B))_{0\le t\le T}$ is a martingale. More generally, given a Lévy process L, one considers a Poisson random measure $N(dt, dx)$ on $[0,T]\times\mathbb{R}$ (cf. e.g. Jacod and Shiryaev (2003), Definition II.1.20, Theorem II.4.8 or (more elementary) Cont and Tankov (2004), Chapter 2.6). For any Borel-measurable set $A \in [0,T]\times\mathbb{R}$ one sets

$$\iint_A N(dt,dx) := \sum_s \mathbb{1}\left(\Delta L_s \neq 0\right)\mathbb{1}\left((s,\Delta L_s)\in A\right),$$

that is, using the Dirac-measure δ_a at point a,

$$N(dt,dx) := \sum_s \mathbb{1}\left(\Delta L_s \neq 0\right)\delta_{(s,\Delta L_s)}(dt,dx).$$

The measure $N(dt,dx)$ and the jump counter $N_t(\cdot)$ are connected through

$$N_t(B) = \iint_{[0,t]\times B} N(ds,dx) =: \int_0^t\int_B N(ds,dx) \quad \text{for any set} \quad B\in\mathcal{B}.$$

Lemma 2.2.3. *Let $f:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ be a family of stochastic processes such that f is measurable with respect to $\mathcal{P}\otimes\mathcal{B}$. For any Borel-measurable set $A\in[0,T]\times\mathbb{R}$ such that*

$$\mathbb{E}[\iint_A |f(t,x)|\,N(dt,dx)] < \infty \tag{2.1}$$

it holds

$$\mathbb{E}[\iint_A f(t,x)N(dt,dx)] = \iint_A \mathbb{E}[f(t,x)]\,dt\nu(dx),$$

where $dt\nu(dx)$ denotes the product of the usual Lebesgue measure and the Lévy measure.

Proof. By the standard monotone class argument it suffices to consider

$$f(\omega,t,x) = \mathbb{1}_A(t,x)\,\mathbb{1}_{B\times(s_1,s_2]\times C}(\omega,t,x)$$

where $A=(t_1,t_2]\times\tilde{A}$, $0\le t_1<t_2\le T$, $0\le s_1<s_2\le T$, $\tilde{A},C\in\mathcal{B}$ and $B\in\mathcal{G}_{s_1}$ such that (2.1) is satisfied. According to Theorem I.2.2 of Jacod and Shiryaev (2003) sets of the form $B\times(s_1,s_2]$ constitute a generating system of $\mathcal{P}$. Without loss of generality we assume $s_1\le t_1$ and $t_2\le s_2$. By the independent increments of L we get that the random variables $\mathbb{1}_B$ and

$N(A \cap ((s_1, s_2] \times C))$ are independent. Hence,

$$\begin{aligned}
&\mathbb{E}[\iint f(t,x)N(dt,dx)] \\
&= \mathbb{E}[\mathbb{1}_B \sum_s \mathbb{1}\left(\Delta L_s \neq 0\right) \mathbb{1}\left((s, \Delta L_s) \in A \cap ((s_1, s_2] \times C)\right)] \\
&= \mathbb{P}(B)\, \mathbb{E}[\sum_{t_1 \leq s \leq t_2} \mathbb{1}\left(\Delta L_s \neq 0\right) \mathbb{1}\left(\Delta L_s \in \tilde{A} \cap C\right)] \\
&= \mathbb{P}(B)\, \mathbb{E}[N_{t_2}(\tilde{A} \cap C) - N_{t_1}(\tilde{A} \cap C)] \\
&= \mathbb{P}(B)(t_2 - t_1)\nu(\tilde{A} \cap C) = \iint \mathbb{E}[f(t,x)]\, dt\nu(dx). \quad \square
\end{aligned}$$

The measure $dt\nu(dx)$ is called intensity measure of $N(dt, dx)$ or its compensator. We define the compensated measure $M(dt, dx) := N(dt, dx) - dt\nu(dx)$. Let now $B \in \mathcal{B}$ and $f(t, x)$ be a family of stochastic processes as in the previous Lemma such that (2.1) is satisfied. Then the observation that

$$\int_s^t \int_B f(u,x)N(du,dx) \quad \text{is independent of} \quad \mathcal{G}_s$$

for $0 \leq s < t \leq T$ immediately implies that integrals of the form

$$\int_0^t \int_B f(s,x)M(ds,dx), \quad 0 \leq t \leq T,$$

are martingales. With these observations on the jump behavior of a Lévy process we state the Lévy-Itô[3]-decomposition.

Theorem 2.2.4 (Lévy-Itô-decomposition)**.** *Let L be a Lévy process and $\nu(\cdot)$ its Lévy measure. Then it holds*

$$\int_{\{|x| \leq 1\}} x^2 \nu(dx) < \infty \quad \textit{and} \quad \int_{\{|x|>1\}} \nu(dx) < \infty.$$

Further, there exists a Brownian[4] motion with variance $c^2 t$, which is a Wiener process[5] (standard Brownian motion) $W = (W_t)_{0 \leq t \leq T}$ times a constant $c \in (0, \infty)$, and a constant $\gamma = \mathbb{E}\left[L_1 - L_1^l\right] (\in \mathbb{R})$ such that for any $0 \leq t \leq T$

$$L_t = \gamma t + cW_t + L_t^l + \tilde{M}_t,$$

where

$$L_t^l = \int_0^t \int_{\{|x|>1\}} xN(ds,dx) \quad \textit{and} \quad \tilde{M}_t = \int_0^t \int_{\{|x| \leq 1\}} xM(ds,dx).$$

The processes W, L^l and $\tilde{M}$ are independent.

[3] Kiyoshi Itô, Japanese mathematician (*1915)

[4] named after the Scottish botanist Robert Brown (1773 – 1858)

[5] named after the American mathematician Norbert Wiener (1894 – 1964)

Proof. The original proof of Lévy, which has been completed by Itô, can be found in Lévy (1934) and Itô (1942). The proof is also done in Protter (2004), Chapter I.4 and in Riesner (2003), where the proof of Protter (2004) has been elaborated in detail. An outline is also contained in Cont and Tankov (2004). □

Let us briefly comment on this decomposition following the treatment in Cont and Tankov (2004). Observe that by the càdlàg paths of L, the process

$$L_t^l = \sum_{0<s\leq t} \Delta L_s \mathbb{1}(|\Delta L_s| > 1)$$

is almost surely a finite sum and hence a compound Poisson process, since it is a Lévy process with piecewise constant sample paths (cf. Cont and Tankov (2004), Proposition 3.3). There is nothing special about the threshold of jump magnitude being equal to 1. Observe that for every $0 < \varepsilon < 1$ the integral

$$\int_0^t \int_{\varepsilon\leq|x|\leq 1} xN(ds, dx) = \sum_{0<s\leq t} \Delta L_s \mathbb{1}(\varepsilon \leq |\Delta L_s| \leq 1)$$

is as well a well-defined compound Poisson process. However, as $\varepsilon \to 0$ this integral does not necessarily converge, since there can be infinitely many small jumps. Replacing this integral by its compensated version $\tilde{M}_t$, which is a martingale, yields the desired convergence (compare $\int_{\mathbb{R}} \min(1, x^2)\, \nu(dx) < \infty$). $\tilde{M}_t$ could be interpreted as an infinite superposition of independent compensated Poisson processes. The Lévy-Itô-decomposition implies therefore that every Lévy process is a combination of a Brownian motion with drift and a possibly infinite sum of independent compound Poisson processes. Since every càdlàg function may be approximated by a piecewise constant function, every Lévy process can be approximated with arbitrary precision by a jump-diffusion process, that is, by the sum of a Brownian motion with drift and a compound Poisson process. The decomposition of a Lévy process into a martingale $cW_t + \tilde{M}_t$ and a process of finite variation $\gamma t + L_t^l$ shows further that every Lévy process is a semimartingale (cf. Definition I.4.21 of Jacod and Shiryaev (2003)). In applications one often makes the additional assumption

$$\int_{\{|x|>1\}} |x|\, \nu(dx) < \infty \tag{2.2}$$

implying $\mathbb{E}[L_1^l] < \infty$. This allows to simplify the Lévy-Itô-decomposition of a Lévy process L in the following way

$$L_t = cW_t + M_t + at, \quad 0 \leq t \leq T,$$

where

$$M_t = \int_0^t \int_{\mathbb{R}} xM(ds, dx) \quad \text{and} \quad a = \mathbb{E}[L_1] = \gamma + \int_{\{|x|>1\}} x\,\nu(dx).$$

In particular, $\mathbb{E}[L_t] < \infty$ for all t if and only if (2.2) holds. Under this additional assumption the process L is a special semimartingale, since at is predictable, which implies the almost sure uniqueness of this decomposition. (cf. Corollary I.3.16 of Jacod and Shiryaev (2003)). This decomposition of a Lévy process L into a martingale $cW_t + M_t$ and an increasing, predictable process at is called the Doob[6]-Meyer[7]-decomposition (cf. Kallenberg (2002), Chapter 25) of the process L. The following Lemma states an useful observation about the moments of the Lévy measure.

Lemma 2.2.5. *The Lévy measure $\nu(dx)$ of some Lévy process L satisfies*

$$\int_{\{|x|>1\}} |x|^n \,\nu(dx) < \infty \quad \Longleftrightarrow \quad \mathbb{E}\left[|L_t|^n\right] < \infty \quad \textit{all} \quad t,$$

for some fixed $n \geq 1$. In particular, in this case one has that

$$\Big| \int_{\mathbb{R}} x^m \nu(dx) \Big| < \infty, \quad 2 \leq m \leq n.$$

Proof. The first part follows from Theorem 25.3 in Sato (1999). Moreover,

$$\Big| \int_{\{|x|>1\}} x^m \nu(dx) \Big| \leq \int_{\{|x|>1\}} |x|^m \,\nu(dx) \leq \int_{\{|x|>1\}} |x|^n \,\nu(dx)$$

and that

$$\Big| \int_{\{|x|\leq 1\}} x^n \nu(dx) \Big| \leq \int_{\{|x|\leq 1\}} |x|^n \,\nu(dx) \leq \int_{\{|x|\leq 1\}} |x|^2 \,\nu(dx). \quad \square$$

The martingale M_t is hence a square-integrable martingale if and only if

$$\int_{\{|x|>1\}} |x|^2 \,\nu(dx) < \infty.$$

Observe additionally, $\mathbb{E}[M_t^2] = \mathbb{E}[[M, M]_t] = t \int_{\mathbb{R}} x^2 \nu(dx)$ (cf. Protter (2004), Chapter II, Corollary 3 of Theorem 27, and Theorem 28). The next theorem summarizes and classifies the path behavior of a Lévy process. The triplet $(\gamma, c^2, \nu(dx))$ is called the characteristics of L (cf. Chapter II.2 of Jacod and Shiryaev (2003)).

Theorem 2.2.6 (Path properties)**.** *Let L be a Lévy process and $(\gamma, c^2, \nu(dx))$ its characteristics. Then it holds:*

[6] Joseph Leo Doob, American mathematician (1910 – 2004)
[7] Paul-André Meyer, French mathematician (1934 – 2003)

1. $\nu \equiv 0$ *if and only if almost all paths of* L *are continuous.*

2. *(a)* (Finite activity) *If* $\nu(\mathbb{R}) < \infty$, *then almost all paths of* L *have only finitely many jumps on any compact interval.*

   (b) (Infinite activity) *If* $\nu(\mathbb{R}) = \infty$, *then almost all paths of* L *have infinitely many jumps on any compact interval.*

3. *(a)* (Finite variation) *If* $c = 0$ *and* $\int_{\{|x|\leq 1\}} |x|\, \nu(dx) < \infty$, *then almost all paths of* L *are of finite variation on any compact interval.*

   (b) (Infinite variation) *If* $c > 0$ *or* $\int_{\{|x|\leq 1\}} |x|\, \nu(dx) = \infty$, *then almost all paths of* L *are of infinite variation on any compact interval.*

Proof. 1. $\nu(B) = 0$ implies $N([0,t] \times B) = 0$ almost surely for all $0 \leq t \leq T$ and for all $B \in \mathcal{B}$. Hence by the Lévy-Itô-decomposition almost all paths are continuous. If almost all paths of L are continuous, then $\Delta L_t = 0$ for all t, which implies $\nu(dx) = 0$.

2. Let $\nu(\mathbb{R}) < \infty$, then for any $0 \leq t_1 < t_2 \leq T$ one has that

$$N([t_1, t_2] \times \mathbb{R}) = \sum_{t_1 \leq s \leq t_2} \mathbb{1}\left(\Delta L_s \neq 0\right) \mathbb{1}\left(\Delta L_s \in \mathbb{R}\right) < \infty \quad \text{a.s.}$$

This shows that L has almost surely only finitely many jumps on $[t_1, t_2]$.

Let $\nu(\mathbb{R}) = \infty$ and consider the sequence ε_n given by

$$\varepsilon_1 = \sup\left\{r : \int_{\{|x|\geq r\}} \nu(dx) \geq 1\right\}$$
$$\varepsilon_n = \sup\left\{r : r < \varepsilon_{n-1}, \int_{\{|x|\geq r\}} \nu(dx) \geq n\right\}, \quad n \geq 2.$$

We define for $0 \leq t_1 < t_2 \leq T$ the following sequence of independent random variables

$$Y_n = \int_{t_1}^{t_2} \int_{\{\varepsilon_{n+1} \leq |x| < \varepsilon_n\}} N(dt, dx), \quad n \geq 1.$$

The variables Y_n are Poisson distributed with intensity $\lambda_n \geq (t_2 - t_1)$ for all n and the total number of jumps of L in the interval $[t_1, t_2]$ is equal to $\sum_{n=1}^{\infty} Y_n$. Now it is not very hard to show that

$$\mathbb{P}(Y_n \geq 1) = e^{-\lambda_n} \sum_{k=1}^{\infty} \frac{\lambda_n^k}{k!} = 1 - \frac{1}{e^{\lambda_n}} \geq 1 - \frac{1}{e^{t_2 - t_1}} \quad \text{all} \quad n.$$

Hence, $\sum_{n=1}^{\infty} \mathbb{P}(Y_n \geq 1) = \infty$, which implies, using the Borel-Cantelli[8]-Lemma,

$$\mathbb{P}\left(Y_n \geq 1, \quad \text{i. o.}\right) = 1.$$

[8] Francesco Paolo Cantelli, Italian mathematician (1875 – 1966)

Therefore, $\sum_{n=1}^{\infty} Y_n = \infty$ almost surely.

3. Let $c = 0$ and $\int_{\{|x|\le 1\}} |x|\,\nu(dx) < \infty$, then the Lévy-Itô-decomposition implies that

$$L_t = \gamma t + L_t^l + \tilde{M}_t = \Big(\gamma - \int_{\{|x|\le 1\}} x\nu(dx)\Big)t + L_t^l + \int_0^t\int_{\{|x|\le 1\}} xN(ds,dx).$$

The first two terms are clearly of finite variation. It remains to consider the third term having total variation over $[0,T]$ given by

$$\mathrm{TV}_{[0,T]}\Big(\int_0^{\cdot}\int_{\{|x|\le 1\}} xN(ds,dx)\Big) = \int_0^T\int_{\{|x|\le 1\}} |x|\,N(ds,dx).$$

We have equality here, since the total variation of any càdlàg function is greater or equal to the sum of its jumps. Therefore,

$$\mathbb{E}\big[\,\mathrm{TV}_{[0,T]}\Big(\int_0^{\cdot}\int_{\{|x|\le 1\}} xN(ds,dx)\Big)\big] = T\int_{\{|x|\le 1\}} |x|\,\nu(dx),$$

which implies that the total variation of L is finite almost surely.

For the converse statement we observe that for every $n \ge 1$ and every $0 \le t_1 < t_2 \le T$ it holds,

$$\begin{aligned}\mathrm{TV}_{[t_1,t_2]}(L) &\ge \int_{t_1}^{t_2}\int_{\{\frac{1}{n}\le|x|<1\}} |x|\,N(ds,dx)\\ &= (t_2-t_1)\int_{\{\frac{1}{n}\le|x|<1\}} |x|\,\nu(dx) + \int_{t_1}^{t_2}\int_{\{\frac{1}{n}\le|x|<1\}} |x|\,M(ds,dx).\end{aligned}$$

Let now $Y_n := \int_{t_1}^{t_2}\int_{\{\frac{1}{n}\le|x|<1\}} |x|\,M(ds,dx)$ and $X_n := Y_{n+1} - Y_n$. Then one has that $\mathbb{E}[X_n] = 0$ all n and

$$\sum_{n=1}^{\infty}\mathbb{V}\mathrm{ar}[X_n] = \sum_{n=1}^{\infty}\mathbb{E}\left[X_n^2\right] = (t_2-t_1)\sum_{n=1}^{\infty}\int_{\{\frac{1}{n+1}\le|x|<\frac{1}{n}\}} x^2\nu(dx) < \infty.$$

Kolmogorov's three series theorem implies then that $\sum_{n=1}^{\infty} X_n$ converges to a finite limit almost surely. Therefore

$$\int_{\{|x|\le 1\}} |x|\,\nu(dx) = \infty \Rightarrow \mathrm{TV}_{[t_1,t_2]}(L) = \infty \quad \text{a.s.}$$

If $c > 0$ then L contains a Brownian motion component which has almost surely paths of infinite variation. □

The technique involving Kolmogorov's three series theorem that we applied in the previous proof is also used to prove that $\tilde{M}_t$ exists in the Lévy-Itô-decomposition. Let us now briefly turn to Lévy processes with almost surely nondecreasing paths, which play an important role in financial market modeling, too.

Definition 2.2.7. *(Subordinator) A Lévy process L is called a subordinator if almost surely all its sample paths are nondecreasing.*

Corollary 2.2.8. *A Lévy process L with characteristic triplet $(\gamma, c^2, \nu(dx))$ is a subordinator if and only if $c = 0$, $\nu((-\infty, 0]) = 0$, $\int_0^1 x\,\nu(dx) < \infty$ and $\gamma - \int_0^1 x\,\nu(dx) \geq 0$.*

Another immediate consequence of the Lévy-Ito-decomposition is the Lévy-Khinchin[9] formula for the characteristic function of a Lévy process.

Theorem 2.2.9 (Lévy-Khinchin)**.** *Let L be a Lévy process and $(\gamma, c^2, \nu(dx))$ its characteristics. Then it holds*

$$\phi_t(u) = \mathbb{E}\left[e^{iuL_t}\right] = e^{t\psi(u)}, \quad u \in \mathbb{R},$$

where

$$\psi(u) = -\frac{1}{2}c^2u^2 + i\gamma u + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\,\mathbb{1}_{\{|x|\leq 1\}}\right)\nu(dx).$$

Proof. Kallenberg (2002), Corollary 15.8. □

Under assumption (2.2) the so-called characteristic exponent simplifies to

$$\psi(u) = -\frac{1}{2}c^2u^2 + iau + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right)\nu(dx).$$

According to the Lévy-Khinchin formula the characteristics of a Lévy process uniquely determine its characteristic function and hence its distribution. Further by the property

$$\phi_t(u) = (\phi_1(u))^t, \tag{2.3}$$

we see that the distribution of L_1 determines the marginal distributions of the whole Lévy process L. There is now a significant relationship between Lévy processes and infinitely divisible distributions. At any fixed time t we may write

$$L_t = \sum_{k=1}^{n} \left(L_{t\frac{k}{n}} - L_{t\frac{k-1}{n}}\right),$$

that is, L_t represented as a sum of n i.i.d. (independent and identically distributed) random variables, which holds because of the stationary and independent increments. We cite the following definition from Bingham and Kiesel (2004).

Definition 2.2.10. *A random variable X, or its distribution function F, is infinitely divisible if for each $n = 1, 2, \ldots$ there is a distribution function F_n with F as its n-fold convolution power:*

$$F = F_n * F_n * \cdots * F_n \quad \textit{(n factors)}$$

[9] Aleksandr Yakovlevich Khinchin, Russian mathematician (1894 – 1959)

that is, X has the same distribution as $\sum_{k=1}^{n} X_{nk}$ with X_{ni} $(i = 1, \ldots, n)$ independent with common distribution F_n.

Theorem 2.2.11 (Lévy processes and infinitely divisibility)**.** *Let X be any random variable. Then X is infinitely divisible if and only if $X \stackrel{d}{=} L_1$ (equal in distribution) for some Lévy process L.*

Proof. The "if" part is shown above. For the converse statement we refer to Kallenberg (2002), Theorem 15.2. □

Thus, given an infinitely divisible distribution it is fairly easy to construct the corresponding Lévy process.

2.2.2 Common examples

Our aim is it now to discuss some examples of Lévy processes frequently used in financial market modeling. Classified by their jump activity (cf. Theorem 2.2.6), we distinguish between two main categories: finite and infinite activity Lévy-processes. Again we work on the filtered probability space $(\Omega, \mathbb{G}, (\mathcal{G}_t)_{0 \le t \le T}, \mathbb{P})$. For a related discussion we refer to Chapter 4 in Cont and Tankov (2004).

2.2.3 Finite activity Lévy-processes: jump-diffusions

The most simple examples and at the same time the most well-known finite activity Lévy-processes are clearly the Brownian motion possessing almost sure continuous sample paths and suitable normally distributed increments, and the Poisson process. This is also reflected in the Lévy-Itô-decomposition. We recall that in the Black[10]-Scholes[11] model the log-asset-price is modeled using a diffusion, that is,

$$L_t = \beta t + cW_t, \quad 0 \le t \le T,$$

for $\beta \in \mathbb{R}$, $c > 0$ and $(W_t)_{0 \le t \le T}$ a standard Brownian motion. Another important example is the compound Poisson process which we write as

$$L_t = \sum_{i=1}^{N_t} Y_i, \quad 0 \le t \le T,$$

where the jump sizes Y_i are i.i.d. with distribution having a density $f(x)$ with respect to the Lebesgue measure and where $(N_t)_{0 \le t \le T}$ is a Poisson process with intensity λt, independent from Y_i, for all i. As previously mentioned compound Poisson processes are completely characterized as Lévy processes

[10] Fischer Sheffey Black, American economist (1938 – 1995)
[11] Myron S. Scholes, Canadian economist (*1941) and Nobel prize winner in 1997

with almost sure piecewise constant sample paths (cf. Cont and Tankov (2004), Proposition 3.3). For a compound Poisson process the Lévy measure is particularly simple.

Proposition 2.2.12. *Let L be a compound Poisson process. Then its Lévy measure is given by*

$$\nu(dx) = \lambda f(x)\,dx.$$

Proof. Let $B \in \mathcal{B}$. Then, conditionally on the trajectory of the Poisson process N_t, the measure $N([0,1]\times B)$ is a sum of N_1 i.i.d. Bernoulli[12] random variables taking value 1 with probability $\int_B f(x)\,dx$. Hence,

$$\begin{aligned}\nu(B) &= \mathbb{E}\left[N([0,1]\times B)\right] = \mathbb{E}\left[\mathbb{E}\left[N([0,1]\times B) \mid \mathcal{G}_1\right]\right] \\ &= \mathbb{E}[N_1 \int_B f(x)\,dx] = \lambda \int_B f(x)\,dx. \quad \square\end{aligned}$$

In particular, this shows once again that each compound Poisson process is a finite activity Lévy process. Together with the Brownian motion compound Poisson processes constitute so called jump-diffusions. A Lévy process of jump-diffusion type has the following form

$$L_t = \beta t + cW_t + \sum_{i=1}^{N_t} Y_i, \quad 0 \le t \le T,$$

where $\beta \in \mathbb{R}$, $c > 0$, $(W_t)_{0\le t\le T}$ is a standard Brownian motion and $\sum_{i=1}^{N_t} Y_i$, $0 \le t \le T$, is a compound Poisson process. Opposed to the Black-Scholes model using a jump-diffusion to describe log-prices in a financial market implies that the "normal" evolution of log-prices is given by a diffusion process interrupted by jumps at random occurrence. This jumps might represent rare events due to unexpected new market information resulting in, for example, large drawdowns or large quotation gains. To make the definition of a jump-diffusion model complete one needs to specify the jump-size distribution. The challenge here is to find the "right" tail behavior to reproduce extremal events, since the tail behavior of the process depends significantly on the tail behavior of the jump-size distribution (cf. Lemma 2.2.5). There are essentially two jump-size distributions considered in the context of financial markets. They are especially attractive since at least for European style options both lead to an almost closed option pricing formula in the sense of being a quickly converging series. First, the Merton[13] model (cf. Merton (1976)) assumes that the jump-sizes Y_i of L_t follow a normal distribution: $Y_i \sim \mathcal{N}(\mu, \delta^2)$. This implies that the probability density of L_t satisfies

$$p_t(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k \exp\left(-\frac{(x-\beta t-k\mu)^2}{2(\sigma^2 t+k\delta^2)}\right)}{k!\sqrt{2\pi(\sigma^2 t + k\delta^2)}}$$

[12] named after the Swiss mathematician Jakob Bernoulli (1654 – 1705)

[13] Robert Carhart Merton (*1944), Nobel prize winner in 1997

and that its Lévy measure is given by

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\delta^2}\right) dx.$$

In Chapter 7 we discuss in detail the second common jump-diffusion model known as the Kou model (cf. Kou (2002)). Here the jump-size distribution is assumed to follow an asymmetric double exponential distribution with Lebesgue density

$$f(x) = p\eta_1 e^{-\eta_1 x} \mathbb{1}_{[0,\infty)}(x) + q\eta_2 e^{\eta_2 x} \mathbb{1}_{(-\infty,0)}(x), \quad \eta_1 > 1, \eta_2 > 0,$$

where $p, q \geq 0$ and $p + q = 1$. The asymmetric tail behavior seems to be an advantage compared to the Merton model, since this asymmetry is observable in financial markets implied by both over- and underreaction of market participants to new market information. In this case the probability density of L_t is not available in closed form.

2.2.4 Infinite activity Lévy-processes

The most frequently used infinite activity Lévy processes are generated through Brownian subordination with an independent increasing Lévy process. This means a Brownian motion with a possible drift is evaluated at a different, new and stochastic time scale which is given by an independent subordinator. This time change is interpreted in financial terms as business time. Those models stay in the class of Lévy processes, that is, if $\beta t + cW_t$ is a diffusion and $(\tau(t))_{0\leq t\leq T}$ is an independent subordinator, then the process $\beta\tau(t) + cW_{\tau(t)}$ is again a Lévy process (cf. Cont and Tankov (2004), Theorem 4.2). Although generated through a Brownian motion, the Lévy-Itô-decomposition of this kind of Lévy processes does not necessarily contain a Brownian motion leading to purely discontinuous Lévy processes. Modeling asset prices with purely discontinuous but infinite activity Lévy processes is justified in the literature (e.g. Carr, Geman, Madan and Yor (2002)) by the argument that the jump structure with an infinite number of arbitrarily small jumps is already rich enough to give a nontrivial small time behavior.

2.2.5 Variance gamma Lévy-process

As a first example we consider the variance gamma process following the exposition in Madan (2001). A variance gamma process is a Lévy process of infinite activity but of finite variation. It was first studied in Madan and Seneta (1990) for the symmetric case (where $\beta = 0$ in (2.5) below). The general case of an asymmetric variance gamma processes was then developed in Madan and Milne (1991) and Madan, Carr and Chang (1998). Besides the volatility, those processes feature only two additional parameters allowing

to control skewness and kurtosis of the distribution. As subordinator one considers a gamma process with mean rate equal to 1, that is, a Lévy process $\tau = (\tau_t)_{0\le t\le T}$ whose increments $\tau_{t+h} - \tau_t$ over non-overlapping intervals of length h follow for all $0 \le t < t+h \le T$ a gamma distribution with Lebesgue density

$$p_h(x) = \frac{1}{\Gamma\left(\frac{h}{v}\right) v^{\frac{h}{v}}} x^{\frac{h}{v}-1} e^{-\frac{x}{v}} \mathbb{1}_{(0,\infty)}(x), \quad v > 0, \tag{2.4}$$

where $\Gamma(x)$ denotes the gamma function. With this definition one immediately gets that $\mathbb{E}[\tau_t] = t$ and $\mathbb{V}\mathrm{ar}[\tau_t] = vt$. Therefore the average random time change in t units of calender time is t whereas its variance is proportional to t. The Lévy measure and the characteristic exponent of a gamma process are given by (cf. Cont and Tankov (2004), Table 4.4)

$$\nu(dx) = \frac{e^{-\frac{x}{v}}}{vx} \mathbb{1}_{(0,\infty)}(x)\, dx \quad \text{and} \quad \psi_\tau(u) = -\frac{1}{v}\log\left(1 - iuv\right),$$

respectively. Given now a gamma process, a standard Brownian motion $(W_t)_{0\le t\le T}$, some $c > 0$ and $\beta \in \mathbb{R}$, the variance gamma process is defined by

$$L_t = \beta\tau(t) + cW_{\tau(t)}, \quad 0 \le t \le T, \tag{2.5}$$

which is a Brownian motion with drift evaluated at gamma time. The characteristic exponent of the variance gamma process is given by

$$\psi(u) = -\frac{1}{v}\log\left(1 - iu\beta v + \frac{1}{2}c^2 v u^2\right),$$

since

$$\mathbb{E}\left[e^{iuL_t}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iuL_t} \mid \mathcal{G}_t\right]\right] = \mathbb{E}\left[e^{\tau(t)\left(-\frac{1}{2}c^2u^2 + i\beta u\right)}\right] = e^{t\psi_\tau\left(\frac{1}{2}c^2iu^2 + \beta u\right)}.$$

From this representation it immediately follows that a variance gamma process is a purely discontinuous process and further, applying l'Hospital's rule[14], that

$$\lim_{v\to 0} \psi(u) = -\frac{1}{2}c^2u^2 + i\beta u.$$

Hence, letting the variance of the time change tend to zero results again in a diffusion setting. The Lévy measure of a variance gamma process (cf. Madan, Carr and Chang (1998), eq. 14) has the following form

$$\nu(dx) = \frac{1}{v\,|x|} \exp\left(\frac{\beta}{c^2}x - \frac{\sqrt{\frac{2}{v} + \frac{\beta^2}{c^2}}}{c}\,|x|\right) dx.$$

[14] Guillaume François Antoine, Marquis de l'Hospital, French mathematician (1661 – 1704)

A formula for the Lévy measure of a subordinated Lévy process is provided for example in Theorem 4.2 of Cont and Tankov (2004). The form of its Lévy measure implies that a variance gamma Lévy-process has paths of finite variation, since $\int_{\{|x|\leq 1\}} |x|\, \nu(dx) < \infty$. Further the asymptotic behavior $\nu(dx) \sim \frac{1}{|x|}\, dx$, $(x \to 0)$, yields the infinite activity of the process. Another remarkable fact is that the variance gamma process may be written as a difference of two independent gamma processes implying as well the almost sure finite variation of its sample paths. From the moments of the variance gamma process,

$$\begin{aligned}
\mathbb{E}[L_t] &= \beta t \\
\mathbb{E}[(L_t - \mathbb{E}[L_t])^2] &= (\beta^2 v + c^2)t \\
\mathbb{E}[(L_t - \mathbb{E}[L_t])^3] &= (2\beta^3 v^2 + 3c^2\beta v)t \\
\mathbb{E}[(L_t - \mathbb{E}[L_t])^4] &= (3c^4 v + 12c^2\beta^2 v^2 + 6\beta^4 v^3)t \\
&\quad + (3c^4 + 6c^2\beta^2 v + 3\beta^4 v^2)t^2,
\end{aligned}$$

one derives that β is the parameter generating skewness while kurtosis is primarily influenced by v. The Lebesgue probability density of a variance gamma process L_t is derived in Madan, Carr and Chang (1998) (pp. 87 and 98) by making use of the mixture representation in (2.5). It has the form

$$p_t(x) = \frac{2}{v^{\frac{t}{v}}\sqrt{2\pi}c\Gamma\left(\frac{t}{v}\right)} e^{\frac{\beta}{c^2}x} \left(\frac{x^2}{2\frac{c^2}{v} + \beta^2}\right)^{\frac{t}{2v}-\frac{1}{4}} \mathrm{K}_{\frac{t}{v}-\frac{1}{2}}\left(\frac{1}{c^2}\sqrt{x^2\left(\frac{2c^2}{v} + \beta^2\right)}\right),$$

where $\mathrm{K}_\lambda(\cdot)$ denotes the modified Bessel function[15] of the second kind of order λ (cf. Section 9.6 in Abramowitz and Stegun (1964)). Sometimes the modified Bessel function of the second kind is also called Bessel function of the third kind or MacDonald function.

The variance gamma distribution is a subclass of the so-called CGMY -distributions. CGMY-distributions are infinitely divisible and were introduced in Carr, Geman, Madan and Yor (2002) as an extension of the variance gamma distribution to model log returns in financial markets.

2.2.6 Normal inverse Gaussian Lévy-process

A second three parameter Lévy process is the normal inverse Gaussian process[16]. It is of infinite activity and of infinite variation and was introduced to finance by Barndorff-Nielsen (1998). Here, we partially follow Cont and Tankov (2004), Sections 4.4.2/4.4.3. The treatment of a normal inverse Gaussian process is completely analogous to the variance Gamma process. It is defined as in (2.5) with the exception that $\tau = (\tau_t)_{0\leq t\leq T}$ is an inverse

[15] named after the German mathematician Friedrich Wilhelm Bessel (1784 – 1846)
[16] named after the German mathematician Carl Friedrich Gauss (1777 – 1855)

Gaussian subordinator, that is, a Lévy process whose increments $\tau_{t+h} - \tau_t$ over non-overlapping intervals of length h follow for all $0 \le t \le T$ an inverse Gaussian distribution with density

$$p_h(x) = \frac{h}{\sqrt{2\pi v}} x^{-\frac{3}{2}} \exp\Big(-\frac{1}{2vx}(x-h)^2 \Big) \mathbb{1}_{(0,\infty)}(x), \quad v > 0. \tag{2.6}$$

Note that $\mathbb{E}[\tau_t] = t$ and $\mathbb{V}\mathrm{ar}[\tau_t] = vt$ like for the Gamma subordinator. An inverse Gaussian Lévy process has the following Lévy measure and the following characteristic exponent

$$\nu(dx) = \frac{1}{\sqrt{2\pi v}} x^{-3/2} e^{-\frac{x}{2v}} \mathbb{1}_{(0,\infty)}(x)\, dx \quad \text{and} \quad \psi_\tau(u) = \frac{1}{v} - \frac{1}{v}\sqrt{1 - 2ivu}.$$

The normal inverse Gaussian process is purely discontinuous with characteristic exponent

$$\psi(u) = \psi_\tau\left(\frac{1}{2}c^2 iu^2 + \beta u\right) = \frac{1}{v} - \frac{1}{v}\sqrt{1 + u^2c^2v - 2i\beta uv}$$

and with moments

$$\begin{aligned}
\mathbb{E}[L_t] &= \beta t \\
\mathbb{E}[(L_t - \mathbb{E}[L_t])^2] &= (\beta^2 v + c^2)t \\
\mathbb{E}[(L_t - \mathbb{E}[L_t])^3] &= (3\beta^3 v^2 + 3c^2\beta v)t \\
\mathbb{E}[(L_t - \mathbb{E}[L_t])^4] &= (3c^4 v + 15\beta^4 v^3 + 18c^2\beta^2 v^2)t \\
&\quad + (3c^4 + 6c^2\beta^2 v + 3\beta^4 v^2)t^2.
\end{aligned}$$

Therefore skewness is again influenced by the parameter β and kurtosis is again primarily controlled by v. The Lévy measure of a normal inverse Gaussian process is

$$\nu(dx) = \frac{\sqrt{\beta^2 + \frac{c^2}{v}}}{\pi c\sqrt{v}} \frac{\exp\left(\frac{\beta}{c^2}x\right)}{|x|} \mathrm{K}_1\left(\frac{\sqrt{\beta^2 + \frac{c^2}{v}}}{c^2}\,|x|\right) dx$$

and its marginal Lebesgue probability densities are given by

$$p_t(x) = \frac{t}{\pi} e^{\left(\frac{t}{v} + \frac{\beta}{c^2}x\right)} \sqrt{\frac{\beta^2}{vc^2} + \frac{1}{v^2}} \frac{\mathrm{K}_1\left(\frac{\sqrt{\beta^2 + \frac{c^2}{v}}}{c^2}\sqrt{x^2 + \frac{t^2c^2}{v}}\right)}{\sqrt{x^2 + \frac{t^2c^2}{v}}}. \tag{2.7}$$

According to Abramowitz and Stegun (1964), formula 9.6.9, for $\lambda > 0$ one has the following asymptotic behavior:

$$\mathrm{K}_\lambda(z) \sim \frac{1}{2}\Gamma(\lambda)\left(\frac{2}{z}\right)^\lambda, \quad \text{in particular} \quad \mathrm{K}_1(z) \sim \frac{1}{z}, \quad (z \to 0). \tag{2.8}$$

Therefore $\nu(dx) \sim \frac{1}{x^2}\, dx$, $(x \to 0)$. Hence, $\int_{\{|x| \le 1\}} |x|\, \nu(dx) = \infty$ and $\nu(\mathbb{R}) = \infty$ which immediately implies that the process has paths of infinite variation and of infinite activity. We will meet again the normal inverse Gaussian distribution in the context of generalized hyperbolic distributions below.

2.2.7 Generalized hyperbolic distributions and Lévy processes

At the end of the Lévy process section let us briefly have a look at generalized hyperbolic distributions following the detailed account in Eberlein (2001). Barndorff-Nielsen (1977) introduced the class of generalized hyperbolic distributions ($\mathcal{GH}$) to model the particle size distribution of wind-blown sand. It is a broad class of distributions and features as special case the normal inverse Gaussian distribution and as limiting case the variance Gamma distribution. The Lebesgue probability density of this class is given by

$$f_{\mathcal{GH}}(x;\lambda,\alpha,\beta,\delta,\mu) = \frac{(\alpha^2-\beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}\,\mathrm{K}_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\left(\delta^2+(x-\mu)^2\right)^{\frac{1}{2}(\lambda-\frac{1}{2})} \times \mathrm{K}_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)e^{\beta(x-\mu)},$$

where the five parameters mean and satisfy: $\alpha > 0$ the shape, β with $0 \leq |\beta| < \alpha$ the skewness and $\mu \in \mathbb{R}$ the location. $\delta > 0$ is a scaling parameter, which corresponds to the standard deviation of a normal distribution. Eventually, the parameter $\lambda \in \mathbb{R}$ characterizes certain subclasses. The characteristic function of the $\mathcal{GH}$-law has the 'explicit' form

$$\phi_{\mathcal{GH}}(u) = e^{i\mu u}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\frac{\lambda}{2}}\frac{\mathrm{K}_{\lambda}\left(\delta\sqrt{\lambda^2-(\beta+iu)^2}\right)}{\mathrm{K}_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}. \tag{2.9}$$

From this representation the moments of the distribution can be derived. However, they are not very explicit and so we refer the interested reader to Eberlein (2001).

A random variable X follows a $\mathcal{GH}$-distribution if

$$\begin{aligned} X \mid Y = y &\sim \mathcal{N}(\mu+\beta y, y) \\ Y &\sim \mathcal{GIG}(\lambda,\delta,\gamma), \end{aligned} \tag{2.10}$$

where $\mathcal{N}(\mu+\beta y, y)$ denotes the normal distribution with mean $\mu+\beta y$ and variance y. $\mathcal{GIG}(\lambda,\delta,\gamma)$ denotes the generalized inverse Gaussian distribution with parameters λ, δ and γ. Thus X follows a mean-variance mixture of normal distributions, where the variance is sampled from a $\mathcal{GIG}$-law. The Lebesgue density of the $\mathcal{GIG}$-law is equal to

$$f_{\mathcal{GIG}}(x;\lambda,\delta,\gamma) = \left(\frac{\gamma}{\delta}\right)^{\lambda}\frac{1}{2\,\mathrm{K}_{\lambda}(\delta\gamma)}x^{\lambda-1}\exp\Big(-\frac{1}{2}\left(\frac{\delta^2}{x}+\gamma^2 x\right)\Big)\,\mathbb{1}_{(0,\infty)}(x),$$

where $\gamma > 0$ and the remaining parameters are as introduced above. The mixture representation (2.10) yields also a formula for the density of a generalized hyperbolic distribution (cf. Eberlein (2001))

$$f_{\mathcal{GH}}(x;\lambda,\alpha,\beta,\delta,\mu) = \int_0^{\infty} f_{\mathcal{N}(\mu+\beta y,y)}(x) f_{\mathcal{GIG}}(y;\lambda,\delta,\sqrt{\alpha^2-\beta^2})\,dy. \tag{2.11}$$

Barndorff-Nielsen and Halgreen (1977) showed that generalized inverse Gaussian laws are infinitely divisible and Kelker (1971) proved that infinite divisibility is transferred by mean-variance mixtures of normal distributions. Hence generalized hyperbolic distributions are infinitely divisible and generate Lévy processes as described in Theorem 2.2.11 and relation (2.3). In general, however, only the increments of length one of Lévy processes that are constructed in the above manner follow a generalized hyperbolic distribution. The class of $\mathcal{GH}$-distributions is universally not closed under convolution, which is an immediate consequence of the characteristic function (2.9). The only exception is the case where $\lambda = -\frac{1}{2}$ (cf. Normal inverse Gaussian distributions below). In Eberlein (2001) it is discussed that generalized hyperbolic Lévy-processes are purely discontinuous, of infinite variation and of infinite activity. Moreover the Lévy measure is stated, but the expression is rather complicated as it involves integrals of Bessel functions of the fist and second kind.

Generalized hyperbolic Lévy-processes may also be constructed via subordination. As subordinator one uses a generalized inverse Gaussian process $\tau = (\tau_t)_{0\le t\le T}$, where one specifies increments of length one to be $\mathcal{GIG}$-distributed via Theorem 2.2.11 and further derives the distribution of τ_t via relation (2.3). Note that the $\mathcal{GIG}$-law is also not closed under convolution but has positive mass only, so τ has almost surely only positive increments. The generalized hyperbolic Lévy-process is henceforth defined as

$$L_t = \mu t + \beta\tau(t) + W_{\tau(t)}, \quad 0 \le t \le T,$$

where $W = (W_t)_{0\le t\le T}$ is a standard Brownian motion independent of τ (compare the mixture representation (2.10)). If the parameters of the $\mathcal{GIG}$-law are λ, δ and $\gamma = \sqrt{\alpha^2-\beta^2}$, the $\mathcal{GH}$-law of the corresponding Lévy process $(L_t)_{0\le t\le T}$ has parameters $(\lambda, \alpha, \beta, \delta, \mu)$.

Subclass: Hyperbolic distributions

The so-called hyperbolic distributions ($\mathcal{H}$), which were first introduced to finance by Eberlein and Keller (1995), are the subclass corresponding to $\lambda = 1$. Accordingly it is a four parameter family of distributions with density

$$f_{\mathcal{H}}(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta\,\mathrm{K}_1\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \exp\Big(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\Big),$$

since $\mathrm{K}_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$ (cf. Watson (1944), p. 80). The logarithm of $f_{\mathcal{H}}$ is a hyperbola generated by the term $-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)$. For this reason this distribution-family is called "hyperbolic". Compared to a generalized hyperbolic density, the numerical effort to evaluate a hyperbolic

density over some time gird reduces considerably, since the Bessel function, as part of the normalizing factor, has to be evaluated only once.

Subclass: Normal inverse Gaussian distributions

Another important subclass of $\mathcal{GH}$-distributions are the normal inverse Gaussian ($\mathcal{NIG}$) distributions, which result from the choice $\lambda = -\frac{1}{2}$. Hence, using $\mathrm{K}_{-\nu}(z) = \mathrm{K}_{\nu}(z)$ (cf. Abramowitz and Stegun (1964), 9.6.6) and again $\mathrm{K}_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$,

$$f_{\mathcal{NIG}}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi}\exp\Big(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\Big)\frac{\mathrm{K}_1\Big(\alpha\sqrt{\delta^2+(x-\mu)^2}\Big)}{\sqrt{\delta^2+(x-\mu)^2}}.$$

The main properties of $\mathcal{NIG}$-distributions are their scaling property and their closeness under convolution. That is, for independent random variables $X \sim \mathcal{NIG}(\alpha,\beta,\delta_1,\mu_1)$ and $Y \sim \mathcal{NIG}(\alpha,\beta,\delta_2,\mu_2)$ one has

$$cX \sim \mathcal{NIG}(\frac{\alpha}{c},\frac{\beta}{c},c\delta,c\mu), \quad c>0$$
$$X+Y \sim \mathcal{NIG}(\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2).$$

The scaling property follows easily by the substitution rule and the density $f_{\mathcal{NIG}}(\cdot)$ and the closure of convolution follows directly from the product of the corresponding characteristic functions ((2.9) with $\lambda = -\frac{1}{2}$). For this reason the normal inverse Gaussian Lévy-process possesses only $\mathcal{NIG}$-distributed increments.

The $\mathcal{GIG}$-law reduces for the parameter choice $\lambda = -\frac{1}{2}$ to the inverse Gaussian distribution with density

$$f_{\mathcal{IG}}(x;\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}}x^{-\frac{3}{2}}\exp\Big(-\frac{\gamma^2}{2x}\Big(x-\frac{\delta}{\gamma}\Big)^2\Big)\mathbb{1}_{(0,\infty)}(x).$$

This explains again the name of the normal inverse Gaussian distribution, which is a mean-variance mixture of normal distributions, where the mixing distribution is inverse Gaussian. Observe that for the explicit construction in (2.6) one has $\mu = 0$ and one chooses $\delta = \frac{h}{\sqrt{v}}$ and $\gamma = \frac{1}{\sqrt{v}}$ for some $v > 0$. To derive the marginal densities in (2.7) one applies the scaling property and uses $\gamma = \sqrt{\alpha^2-\beta^2}$.

Limiting class: Variance gamma distributions

Finally we consider the case when $\lambda > 0$ and when we let $\delta \to 0$. Using (2.8) one gets that

$$\lim_{\delta\to 0} f_{\mathcal{GIG}}(x;\lambda,\delta,\gamma) = \Big(\frac{\gamma^2}{2}\Big)^{\lambda}\frac{1}{\Gamma(\lambda)}x^{\lambda-1}\exp\Big(-\frac{\gamma^2}{2}x\Big)\mathbb{1}_{(0,\infty)}(x),$$

which is the gamma distribution with parameters λ and γ. The generalized hyperbolic Lévy-process reduces hence for $\lambda > 0$ and for $\delta \to 0$ to the variance gamma Lévy-process which immediately follows by (2.11) with $\mu = 0$. Observe that in (2.4) one chooses $\lambda = \frac{h}{v}$ and $\gamma = \sqrt{\frac{2}{v}}$ for some $v > 0$.

2.3 Financial market

Now that we know the most significant properties of a Lévy process and after having seen many examples of Lévy processes we consider a Lévy-process financial market that will be used in this thesis. In particular, in the proceeding chapters our aim is to derive risk-minimizing hedging strategies for unit-linked life insurance contracts in such a financial market.

2.3.1 Lévy-process model

The Lévy-process financial market model that we will work with is introduced in Chan (1999). On the filtered probability space $(\Omega_1, \mathbb{G}, (\mathcal{G}_t)_{0\le t\le T}, \mathbb{P}_1)$ we consider a Lévy process L and assume throughout that the Lévy measure $\nu(dx)$ of L satisfies

$$\int_{\{|x|>1\}} |x|^3 \, \nu(dx) < \infty. \tag{2.12}$$

Recall (2.2) and Lemma 2.2.5 for the consequences arising out of condition (2.12), especially for the Lévy-Itô-decomposition. We suppose further a financial market with only two traded assets: a stock with price $S = (S_t)_{0\le t\le T}$ and a bond with price $B = (B_t)_{0\le t\le T}$. S could refer to a single stock or to an index of assets in some financial market. One might think of a mutual fund or of a (weighted) sum of assets of a company, which could be for an insurance company the actuarial reserve of an insurance-contract cluster. The processes S and B at any time $t \in [0, T]$ are assumed to be given by the stochastic differential equation

$$\begin{aligned} dS_t &= b_t S_{t-}\, dt + \sigma_t S_{t-}\, dL_t \\ &= (b_t + a\sigma_t) S_{t-}\, dt + \sigma_t S_{t-}\, d(cW_t + M_t), \end{aligned} \tag{2.13}$$

where the initial value S_0 is deterministic and satisfies $S_0 > 0$, or the differential equation

$$dB_t = r_t B_t\, dt, \quad B_0 = 1, \quad \text{i.e.} \quad B_t = \exp\Big(\int_0^t r_s\, ds \Big).$$

The drift b_t, the volatility σ_t and the risk-free interest rate r_t are supposed to be continuous and deterministic functions on $[0, T]$ and $\sigma_t > 0$ for any $t \in [0, T]$. The coefficient functions $f(t, x) = b_t x$ and $g(t, x) = \sigma_t x$ are obviously process-Lipschitz. Therefore the solution of (2.13) is unique (cf.

Protter (2004), Chapter V, Section 3, Theorem 6 and 7). Using the stochastic exponential (cf. Protter (2004), Chapter II.8) the stock price has the following explicit representation:

$$\begin{aligned} S_t &= S_0 \exp\Big(\int_0^t \sigma_s dL_s + \int_0^t \big(b_s - \frac{c^2\sigma_s^2}{2}\big)\, ds\Big) \\ &\quad \times \prod_{0<s\le t} (1+\sigma_s \Delta L_s) \exp(-\sigma_s \Delta L_s) \\ &= S_0 \exp\Big(\int_0^t c\sigma_s dW_s + \int_0^t \sigma_s\, dM_s + \int_0^t \big(a\sigma_s + b_s - \frac{c^2\sigma_s^2}{2}\big)\, ds\Big) \\ &\quad \times \prod_{0<s\le t} (1+\sigma_s \Delta M_s) \exp(-\sigma_s \Delta M_s). \end{aligned} \tag{2.14}$$

In Riesner (2003) this representation is developed in detail. In Cont and Tankov (2004), Chapter 8.4, the stochastic exponential specifically for Lévy processes is treated. In order to have $S_t > 0$ for all t, we furthermore assume that $\Delta L_t = \Delta M_t > -1/\sigma_t$ a.s. for all t. $\hat{S}_t = B_t^{-1} S_t$ denotes the discounted stock price.

Lemma 2.3.1. *$\hat{S}$ satisfies the following stochastic differential equation*

$$\begin{aligned} d\hat{S}_t &= (b_t - r_t)\hat{S}_{t-}\, dt + \sigma_t \hat{S}_{t-}\, dL_t \\ &= (b_t + a\sigma_t - r_t)\, \hat{S}_{t-}\, dt + \sigma_t \hat{S}_{t-}\, d(cW_t + M_t). \end{aligned}$$

Further, $\hat{S}$ is square-integrable, i.e. $\mathbb{E}\big[\sup_{0\le t\le T} \hat{S}_t^2\big] < \infty$.

Proof. For the first claim we apply integration by parts (cf. Protter (2004), Chapter II, Corollary 2 of Theorem 22) to $S_t B_t^{-1}$ and note that $[S, B^{-1}]_t = S_0 B_0^{-1}$ for all t, since B is continuous and of finite variation (cf. Protter (2004), Chapter II, Theorem 26 and 28). Hence,

$$d\hat{S}_t = S_{t-}\, dB_t^{-1} + B_t^{-1}\, dS_t = -\hat{S}_{t-} r_t\, dt + B_t^{-1}\, dS_t.$$

Using (2.13) yields the claim. The square integrability follows from Theorem A.0.2. One identifies the Lévy process with $cW_t + M_t$ and one sets $f(t,x) = (a\sigma_t + b_t - r_t)x$ and $g(t,x) = \sigma_t x$. Note that $\kappa t = \big\langle cW + M, cW + M\big\rangle_t = \big(c^2 + \int_{\mathbb{R}} x^2 \nu(dx)\big)\, t$. □

The previous lemma states the unique decomposition of the special semimartingale $\hat{S}_t$, which we can rewrite in integral form as

$$\hat{S}_t = S_0 + \int_0^t \sigma_s \hat{S}_{s-}\, d(cW_s + M_s) + \int_0^t (a\sigma_s + b_s - r_s)\hat{S}_s\, ds,$$

where $\int_0^t \sigma_s \hat{S}_{s-}\, d(cW_s + M_s)$ is a $\mathbb{P}_1$-square-integrable local martingale, in fact it is a martingale (Protter (2004), Chapter I, Theorem 51), and $\int_0^t (a\sigma_s + b_s - r_s)\hat{S}_s\, ds$ is a continuous adapted, and hence predictable, process.

Some authors (e.g. Eberlein (2001)) model the stock price using the ordinary instead of the stochastic exponential of a Lévy process and motivate this in considering distributional properties of logarithmic stock returns or justify it by the Black-Scholes setting, where the stock price is a geometric Brownian motion, that is, the ordinary exponential of a Lévy process. It can be seen that $S_t = e^{L_t}$ does not in general satisfy the stochastic differential equation (2.13). However, Goll and Kallsen (2000) (Lemma 5.8) show that the ordinary exponential of a Lévy process is the stochastic exponential of another certain Lévy process and provide a formula that transforms the corresponding Lévy processes. Hence the two models are not completely different. For the derivation of hedging strategies we are more interested in the infinitesimal time behavior of the asset price as opposed to the explicit structure of the process itself. For this reason we work with the stochastic differential equation (2.13) and the so implied stochastic exponential.

2.3.2 Föllmer-Schweizer measure

Risk-neutral valuation requires a locally $\mathbb{P}_1$-equivalent measure such that $\hat{S}$ is a martingale under this measure. We call such a measure *equivalent martingale measure*. In Chan (1999) it is shown that there are arbitrary many equivalent martingale measures in a general Lévy-process financial market. Such markets are well-known to be free of arbitrage, but riskless hedging is not possible, that is, those markets are by far not complete. For the theory of arbitrage-free and complete markets we refer to the classical paper of Harrison and Pliska (1981) or to the fundamental paper of Delbaen and Schachermayer (1998). A review of this theory is also contained in Bingham and Kiesel (2004) or in Cont and Tankov (2004). In this thesis the incompleteness of general Lévy markets becomes apparent to some extent.

For the incomplete Lévy market one such equivalent martingale measure is the so-called Föllmer-Schweizer measure needed for the hedging theory applied in the chapters below and denoted by $\mathbb{Q}_1$. In Chan (1999) this change of measure is discussed in detail: In analogy to the equivalent change of measure in the Black-Scholes model, the density process of $\mathbb{Q}_1$ with respect to $\mathbb{P}_1$ is here given by

$$Z_t = 1 + \int_0^t cG_s Z_{s-}\, dW_s + \int_0^t \int_{\mathbb{R}} G_s x Z_{s-}\, M(ds, dx), \quad 0 \le t \le T, \quad (2.15)$$

where, with $v = \int_{\mathbb{R}} x^2 \nu(dx)$, the Girsanov[17] parameter is

$$G_t = \frac{r_t - b_t - a\sigma_t}{\sigma_t(c^2 + v)}. \quad (2.16)$$

The market price of risk in the Black-Scholes world corresponds to $-G_t$. The explicit representation of Z yields that if $G_t \Delta L_t > -1$ for all $t \in [0, T]$

[17] named after the Russian mathematician Igor Vladimirovich Girsanov (†1967)

then $\mathbb{Q}_1$ is a true measure and not merely a signed measure. This holds if $\sigma_t^{-1} G_t < 1$ for all t, since the jumps of L are assumed to be bounded from below by $-1/\sigma_t$ for all t. If L has also unbounded jumps from above, it suffices to have additionally $G_t > 0$ for all $t \in [0, T]$ (cf. Riesner (2003)). The Lévy process L under $\mathbb{Q}_1$ is given by (cf. Chan (1999))

$$L_t^{\mathbb{Q}} = cW_t^{\mathbb{Q}} + M_t^{\mathbb{Q}} + A_t^{\mathbb{Q}},$$

where the predictable part $A_t^{\mathbb{Q}}$ has the following representation

$$A_t^{\mathbb{Q}} = at + \int_0^t c^2 G_s \, ds + \int_0^t \int_{\mathbb{R}} G_s x^2 \, \nu(dx) ds = \int_0^t \frac{r_s - b_s}{\sigma_s} \, ds.$$

Further, the process

$$cW_t^{\mathbb{Q}} = cW_t - \int_0^t c^2 G_s \, ds$$

is a standard Brownian motion under $\mathbb{Q}_1$ with variance $c^2 t$, and $M_t^{\mathbb{Q}}$ is a $\mathbb{Q}_1$-martingale given by

$$M_t^{\mathbb{Q}} = M_t - \int_0^t \int_{\mathbb{R}} G_s x^2 \, \nu(dx) ds = \int_0^t \int_{\mathbb{R}} x \, M^{\mathbb{Q}}(ds, dx).$$

$M^{\mathbb{Q}}(dt, dx)$ denotes the compensated jump measure of L under $\mathbb{Q}_1$, namely

$$M^{\mathbb{Q}}(dt, dx) = N^{\mathbb{Q}}(dt, dx) - \nu_t^{\mathbb{Q}}(dx) dt,$$

and

$$\nu_t^{\mathbb{Q}}(dx) = (1 + G_t x) \, \nu(dx).$$

$N^{\mathbb{Q}}(dt, dx)$ is the jump measure of the process $L^{\mathbb{Q}}$; it is defined like the measure $N(dt, dx)$ in Section 2.2.1. It is important to note that the set of possible paths of the process is invariant under an equivalent change of measure. However, the likelihood of seeing one specific paths is changed and of course the statistical properties of the paths, such as the intensity of jumps or the independence of increments, are not preserved. It is henceforth not surprising that one does not necessarily get again a Lévy process after an equivalent change of measure. Jacod and Shiryaev (2003), Theorem II.4.15, show that a semimartingale has independent increments if and only if its characteristics admit a version that is deterministic. This is the case with L under $\mathbb{Q}_1$, since G_t is deterministic. However, in Corollary II.4.19 they also show that increments of a semimartingale are stationary if and only if they are linear in time. This is not necessarily true in our case and only holds if G does not depend on t. For completeness we consider the following definition and refer the reader for a further discussion to Cont and Tankov (2004), Chapter 14.

Definition 2.3.2. *An adapted (càdlàg) stochastic process $X = (X_t)_{0\le t\le T}$ with values in $\mathbb{R}$ such that $X_0 = 0$ is called an additive process if it possesses independent increments and if it is stochastically continuous.*

The process $M_t^{\mathbb{Q}}$ is a $\mathbb{Q}_1$-martingale, since its increments are independent and its expectation is zero. It is moreover $\mathbb{Q}_1$-square-integrable, since $\mathbb{E}^{\mathbb{Q}_1}[(M_t^{\mathbb{Q}})^2] = \int_0^t\int_{\mathbb{R}} x^2\nu_s^{\mathbb{Q}}(dx)ds < \infty$. This follows immediately from the continuity of G_t and condition (2.12) implying $\left|\int_{\mathbb{R}} x^3\nu(dx)\right| < \infty$.

Lemma 2.3.3. *Under $\mathbb{Q}_1$ $\hat{S}$ satisfies the following stochastic differential equation*

$$d\hat{S}_t = \sigma_t\hat{S}_{t-}\, d(cW_t^{\mathbb{Q}} + M_t^{\mathbb{Q}}),$$

where $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}$ are as defined above. Further $\hat{S}$ is a $\mathbb{Q}_1$-square-integrable martingale, that is, $\mathbb{E}^{\mathbb{Q}_1}[\sup_{t\in[0,T]}\hat{S}_t^2] < \infty$.

Proof. From Lemma 2.3.1 we know the stochastic differential equation for $\hat{S}$ under $\mathbb{P}_1$. Hence under $\mathbb{Q}_1$ $\hat{S}$ satisfies

$$\begin{aligned} d\hat{S}_t &= (b_t - r_t)\hat{S}_{t-}\, dt + \sigma_t\hat{S}_{t-}\, dL_t^{\mathbb{Q}} \\ &= (b_t - r_t)\hat{S}_{t-}\, dt + \sigma_t\hat{S}_{t-}\, d(cW_t^{\mathbb{Q}} + M_t^{\mathbb{Q}}) + \sigma_t\hat{S}_{t-}\left(\frac{r_t - b_t}{\sigma_t}\right) dt \\ &= \sigma_t S_{t-}\, d(cW_t^{\mathbb{Q}} + M_t^{\mathbb{Q}}). \end{aligned}$$

$\hat{S}$ is a $\mathbb{Q}_1$-square-integrable martingale by Theorem A.0.2. Compare the proof of Lemma 2.3.1. □

2.3.3 Other equivalent changes of measure

Chan (1999) provides a general formula for the density process $Z = (Z_t)_{0\le t\le T}$ of any to $\mathbb{P}_1$ locally equivalent measure $\tilde{\mathbb{Q}}$. It is for $0 \le t \le T$ described as the stochastic differential equation

$$dZ_t = Z_{t-}G_t\, d(cW_t) + \int_{\mathbb{R}} Z_{t-}(H(t,x) - 1)\, M(dt, dx), \quad Z_0 = 1, \tag{2.17}$$

where G_t is a predictable process and $H(t,x) \ge 0$ is a predictable process for fixed x and a Borel-measurable function for fixed t. G_t and $H(t,x)$ are called the Girsanov parameters and have to satisfy certain conditions such that Z is a martingale. If they satisfy additionally

$$c^2\sigma_t G_t + a\sigma_t + b_t - r_t + \int_{\mathbb{R}} \sigma_t x(H(t,x) - 1)\nu(dx) = 0 \tag{2.18}$$

for $0 \le t \le T$, then $\hat{S}$ is a martingale under $\tilde{\mathbb{Q}}$. Using Girsanov's theorem for semimartingales (Jacod and Shiryaev (2003), Theorem III.3.24) Chan (1999) develops a special Girsanov theorem for Lévy processes telling how

to transform a given Lévy process under a given change of measure. The Girsanov parameters and Girsanov's theorems are introduced and discussed in Chapter III of Jacod and Shiryaev (2003). The Föllmer-Schweizer measure is an example of this general theory with $H(t,x) = 1+G_t x$ and G_t defined as above. Another important equivalent martingale measure for a Lévy process driven asset is the so-called Esscher measure, which we do not need here. The interested reader might consider the exposition in Chan (1999) or the more practically oriented treatment in Cont and Tankov (2004).

2.4 Life insurance

Besides the financial market we require a life insurance model that evolves independently of the financial market. We denote the life insurance model by the filtered probability space $(\Omega_2, \mathbb{H}, (\mathcal{H}_t)_{0\le t\le T}, \mathbb{P}_2)$ and use it to describe the stochastic life states of an insured person, which are relevant for the considered insurance contract. Further, we treat the stochastic evolution of a whole portfolio of identical insurance contracts and review some concepts of classical actuarial mathematics. For a basic introduction to actuarial mathematics, such as the fundamental actuarial principle of equivalence, we refer to Bowers, Gerber, Hickman, Jones and Nesbitt (1997) or to Gerber and Cox (1997). The precise mathematical description of specific unit-linked life insurances, such as unit-linked pure endowments or unit-linked term insurances, follows in the subsequent chapters.

2.4.1 Multi-state Markov model

One possible model to describe $(\Omega_2, \mathbb{H}, (\mathcal{H}_t)_{0\le t\le T}, \mathbb{P}_2)$ is the classical multi-state Markov[18] model of Hoem (1969); see also Møller (2001a) and Norberg (1999). One considers the set $\mathcal{J} = \{0, 1, \dots, J\}$ of possible states of a policy where usually one assumes 0 to be its initial state. $\mathcal{J} = \{\text{active, disabled, dead}\}$ describes exemplarily three possible states of an insured individual. A càdlàg Markov process $Z = (Z_t)_{0\le t\le T}$ with values in $\mathcal{J}$ and initial distribution $(1, 0, \dots, 0)$ is now used to indicate the state of the policy at time t. One further defines $\mathcal{H}_t$ to be the completed σ-algebra $\sigma(Z_s,\, 0 \le s \le t)$, i.e. $(\mathcal{H}_t)_{0\le t\le T}$ is the natural completed filtration of Z. In order to count the number of transitions from state j to state k in the time interval $(0,t]$ a multivariate counting process $(N^{jk})_{j\neq k}$ is defined by

$$N_t^{jk} = \#\{s \mid s \in (0,t], Z_{s-} = j, Z_s = k\} \quad j,k \in \mathcal{J}, \quad j \neq k, \quad 0 \le t \le T.$$

Moreover the processes $I_t^j = \mathbb{1}_{\{Z_t=j\}}$, $j \in \mathcal{J}$, are introduced indicating whether the policy is in state j at time t or not. The Markov chain Z is

[18] Andrei Andreyevich Markov, Russian mathematician (1856 - 1922)

further assumed to posses transition rates λ^{jk} given by

$$\lambda_t^{jk} = I_{t-}^j \mu_t^{jk}, \quad 0 \leq t \leq T,$$

where the intensities of transition μ_t^{jk} are supposed to exist and to be deterministic, continuous functions. The transition rates λ^{jk} compensate the counting processes N^{jk} resulting in square-integrable, mutually orthogonal and zero-mean martingales

$$M_t^{jk} = N_t^{jk} - \int_0^t \lambda_u^{jk}\, du, \quad 0 \leq t \leq T. \tag{2.19}$$

The orthogonality follows from the fact that the N^{jk} do not have any simultaneous jumps (cf. Kallenberg (2002), Lemma 15.6). The predictable quadratic variation process $\left\langle M^{jk}, M^{jk} \right\rangle_t$ is hence given by

$$\left\langle M^{jk}, M^{jk} \right\rangle_t = \int_0^t \lambda_u^{jk}\, du = \int_0^t I_u^j \mu_u^{jk}\, du, \quad 0 \leq t \leq T, \tag{2.20}$$

and for predictable processes F_t^{jk}, $j, k \in \mathcal{J}$, $j \neq k$, $0 \leq t \leq T$, such that

$$\mathbb{E}[\int_0^T F_s^{jk2}\, \mathbb{1}_{s-}^j\, \mu_s^{jk}\, ds] < \infty,$$

integrals of the form

$$\int_0^t F_s^{jk}\, dM_s^{jk}, \quad 0 \leq t \leq T,$$

are again square-integrable, zero-mean martingales (Protter (2004), Chapter IV, Lemma before Theorem 28). Given the intensities of transition the transition probabilities $p_{jk}(t,u) = \mathbb{P}_2(Z_u = k \mid Z_t = j)$ of Z are determined by Kolmogorov's backward differential equations

$$\frac{d}{dt} p_{jk}(t,u) = \sum_{l:l \neq j} \mu_t^{jl} \left(p_{jk}(t,u) - p_{lk}(t,u) \right), \quad 0 \leq t \leq u \leq T, \tag{2.21}$$

subject to the conditions $p_{jk}(t,t) = \mathbb{1}_{\{j=k\}}$. Contrarily, given the transition probabilities $p_{jk}(t,u) = \mathbb{P}_2(Z_u = k \mid Z_t = j)$ of Z one has for $j \neq k$ that

$$\mu_t^{jk} = \lim_{h \downarrow 0} \frac{p_{jk}(t, t+h)}{h}.$$

2.4.2 Typical portfolio of insured lives

Another way to describe the probability space $(\Omega_2, \mathbb{H}, (\mathcal{H}_t)_{0\le t\le T}, \mathbb{P}_2)$ is to consider a portfolio of independent and identically distributed lives as it is used in Møller (1998). Applying the usual actuarial notation, we start with a group of l_x individuals selected from a cohort of equal age x. The remaining lifetimes of those individuals are assumed to be a sequence of i.i.d. non-negative random variables $T_1, \dots, T_{l_x}$ defined on $(\Omega_2, \mathbb{H}, \mathbb{P}_2)$. The distribution of T_1 is thereby assumed to exhibit a continuous density f with respect to the Lebesgue measure. Its hazard rate function μ_{x+t}, $t \ge 0$, is then given by $\mu_{x+t} = \frac{f(t)}{\int_t^\infty f(s)\,ds}$, where the denominator is strictly positive, and zero otherwise. By the following argument involving the mean value theorem (presupposing the necessary technical conditions) the hazard rate function may be interpreted as the probability that an individual dies in a small period of time. Let $t, h > 0$, h small, then

$$\frac{\mathbb{P}_2(T_1 \le t+h) - \mathbb{P}_2(T_1 \le t)}{\mathbb{P}_2(T_1 > t)} \approx \frac{f(t)}{\int_t^\infty f(s)\,ds} h = \mu_{x+t} h.$$

The observation $\mu_{x+t} = -\frac{d}{dt} \log\big(\int_t^\infty f(s)\,ds\big)$, for $\int_t^\infty f(s)\,ds > 0$, leads to the survival function given by

$$ {}_tp_x = \mathbb{P}_2\left(T_1 > t\right) = \exp\Big(-\int_0^t \mu_{x+\tau}\,d\tau\Big) \tag{2.22}$$

and the number of deaths in the group is recorded by the counting process $N^I = (N_t^I)_{0\le t\le T}$, where

$$N_t^I = \sum_{i=1}^{l_x} \mathbb{1}(T_i \le t), \quad 0 \le t \le T.$$

$\mathcal{H}_t$ in this model is assumed to be the completed σ-algebra $\sigma\{\mathbb{1}(T_i \le u);\, 0 \le u \le t,\, i = 1, \dots, l_x\}$. The process N^I is compensated by the stochastic intensity process (cf. e.g. Jacod and Shiryaev (2003), Proposition II.3.32)

$$\lambda_t \equiv (l_x - N_t^I)\mu_{x+t}, \quad 0 \le t \le T,$$

which is the product of the number of individuals surviving until time t and the hazard rate, leading to the compensated process $M^I = (M_t^I)_{0\le t\le T}$, given by

$$M_t^I = N_t^I - \int_0^t \lambda_u\,du, \quad 0 \le t \le T.$$

It is a square-integrable, zero-mean martingale with predictable quadratic variation process

$$\big\langle M^I, M^I \big\rangle_t = \int_0^t \lambda_u\,du, \quad 0 \le t \le T.$$

This portfolio model is, however, just a special application of the multi-state Markov model discussed in the previous subsection, where the state space is modeled as $\mathcal{J} = \{0, 1, \ldots, l_x\}$ and state j corresponds to exactly j policy holders having died. The transition rates of the Markov process Z are then $\lambda_t^{jk} = I_t^j \mathbb{1}_{\{k=j+1\}}(n-j)\mu_{x+t}$ for $j, k = 0, \ldots, l_x - 1$. Note that the intensities of transition are given by

$$\mu_t^{jk} = \begin{cases} (n-j)\mu_{x+t} & k = j+1, \\ -(n-j)\mu_{x+t} & k = j, \\ 0 & \text{else.} \end{cases}$$

Moreover, the transition probabilities are

$$p_{jk}(t,u) = \begin{cases} 0 & k < j, \\ \exp\Big(-\displaystyle\int_t^u (n-j)\mu_{x+\tau}\, d\tau\Big) & k = j, \end{cases}$$

and for $k > j$ Kolmogorov's backward differential equations determine p_{jk} by

$$\frac{d}{dt} p_{jk}(t,u) = (n-j)\mu_{x+t}\left(p_{jk}(t,u) - p_{j+1,k}(t,u)\right).$$

2.4.3 Analytical laws of mortality

Previously we considered the hazard rate function of the remaining lifetimes of individuals in an insurance portfolio. To make this more precise, we discuss now some analytical choices of hazard rate functions, also called laws of mortality (cf. Bowers, Gerber, Hickman, Jones and Nesbitt (1997)). If an analytical law is applied to generate mortality tables, they are commonly used and their simple analytical form has the advantage to allow for further laborious mathematical computations. However, it might be questionable if one simple law describes accurately the mortality for a whole group of individuals spanning a whole range of different ages. Further, with the increasing speed and storage capacity of computers it is nowadays possible to apply much more advanced laws involving more parameters to generate mortality tables.

One of the first laws of mortality

$$\mu_x = (\omega - x)^{-1}, \quad 0 \le x < \omega,$$

where ω is the theoretical maximal age for individuals, was proposed in 1729 by De Moivre[19]. Another law was introduced by Gompertz[20] in 1825:

$$\mu_x = \beta\gamma^x, \quad \beta > 0,\ \gamma > 1,\ x \ge 0.$$

[19] Abraham de Moivre, French mathematician (1667 - 1754)
[20] Benjamin Gompertz, English mathematician (1779 - 1865)

A further famous choice is the Gompertz-Makeham hazard rate function. It complements the Gompertz law, which captures the hazard of aging, by a constant interpreted as capturing the accident hazard. It is given by

$$\mu_x = \alpha + \beta\gamma^x, \quad \beta > 0,\, \alpha \geq -\beta,\, \gamma > 1,\, x \geq 0.$$

Finally, in 1939, Weibull[21] proposed to apply

$$\mu_x = kx^n, \quad k > 0,\, n > 0,\, x \geq 0.$$

2.4.4 Conventional life-insurance risk diversification

The purpose of this subsection is to demonstrate how life insurance companies conventionally diversify mortality risk within an insurance portfolio. Therefore we consider the classical life insurance with deterministic premium and benefit payments. A similar consideration is contained in Bowers, Gerber, Hickman, Jones and Nesbitt (1997). We assume the portfolio model of Subsection 2.4.2 and further that some insurance company issues at time $t = 0$ to each of the l_x individuals an identical contract (same insurance period T, same insured sum K) against a single premium P. Exemplarily, we think of a term insurance paying the insured sum to the heirs, in case the insured person dies before T. The present value of the insurer's total liability is then described by

$$H = \sum_{i=1}^{l_x} H_i, \quad \text{where} \quad H_i = \mathbb{1}(T_i \leq T)e^{-rT_i}K$$

for some riskless rate of interest r, and with $H_1, \ldots, H_{l_x}$ forming a sequence of i.i.d. random variables. We suppose further that the insurance company chooses the premium P such that

$$\mathbb{P}_2(l_x P \geq H) \geq \alpha,$$

for some risk-level α being close to 1 and presuppose $H_1 \in \mathrm{L}^2(\mathbb{P}_2)$. Let $\mu = \mathbb{E}[H_1]$ and $\sigma^2 = \mathbb{V}\mathrm{ar}[H_1]$. For l_x sufficiently large the central limit theorem implies then that

$$\mathbb{P}_2\left(\frac{H - l_x\mu}{\sigma\sqrt{l_x}} \leq \frac{l_xP - l_x\mu}{\sigma\sqrt{l_x}}\right) \approx \Phi\left(\frac{l_xP - l_x\mu}{\sigma\sqrt{l_x}}\right)$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution function. Hence P is approximately chosen such that

$$P = \mu + z_\alpha \frac{\sigma}{\sqrt{l_x}},$$

[21] Ernst Hjalmar Waloddi Weibull, Swedish engineer and mathematician (1887 - 1979)

where z_α is the α-quantile of the standard normal distribution. This shows that the risk premium charged, i.e. $z_\alpha \frac{\sigma}{\sqrt{l_x}}$, becomes smaller if the number of insured persons in the portfolio is increased. To account for that risk premium life insurance companies usually include a safety loading in their mortality tables.

2.4.5 Risk-neutrality

We follow Aase and Persson (1994) and Møller (1998, 2001a) in assuming risk-neutrality of an insurance company towards mortality. This is a common assumption justified both by the reasoning in the previous section and by a law of large numbers argument stating that the mortality probabilities in a large portfolio of stochastically independent insured individuals develop as expected. That is, using the portfolio model from Section 2.4.2 of i.i.d. remaining lifetimes $T_1, \ldots, T_{l_x}$, the relative number of survivors until time $t \in [0, T]$

$$\frac{1}{l_x} \sum_{i=1}^{l_x} \mathbb{1}(T_i > t) \to_t {}_t p_x \quad \text{a.s. for} \quad l_x \to \infty.$$

Roughly speaking, the actual number of survivors $\sum_{i=1}^{l_x} \mathbb{1}(T_i > t)$ equals the expected number $l_{xt}p_x$, if the portfolio size is sufficiently large. This forms also the basis for the actuarial fundamental principle of equivalence saying that the expected present values of benefit and premium payments are equal.

2.5 Combined financial and life insurance model

2.5.1 Combined model

A unit-linked or equity-linked life insurance is a certain kind of insurance linking the insurance benefit to the market value of some specified reference portfolio which can consist of stocks, bonds or shares of a mutual fund. To model such contracts properly, we need to combine a financial market model for a reference portfolio and an insurance model for the stochastic of the insured lives. The foundation of our subsequent model is therefore a filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ modeled as a product space of two stochastically independent probability spaces. The first, $(\Omega_1, \mathbb{G}, (\mathcal{G}_t)_{0 \le t \le T}, \mathbb{P}_1)$, describes a Lévy process financial market as introduced in Section 2.3 and the second, $(\Omega_2, \mathbb{H}, (\mathcal{H}_t)_{0 \le t \le T}, \mathbb{P}_2)$, describes an insurance model as discussed in Section 2.4. To be more precise:

$$\begin{aligned} \Omega &= \Omega_1 \times \Omega_2, \\ \mathcal{F}_t &= \mathcal{G}_t \vee \mathcal{H}_t, \quad 0 \le t \le T, \\ \mathbb{P}(A) &= (\mathbb{P}_1 \otimes \mathbb{P}_2)(A), \quad \forall\, A \in \mathbb{F} = \mathcal{G}_T \vee \mathcal{H}_T. \end{aligned}$$

For risk-neutral valuation we also need to decide which equivalent martingale measure to use in the combined model. In contrast to our setting Møller (1998) considers a standard Black-Scholes financial market, which is complete, and chooses its unique equivalent martingale measure. Because of the uncertainty of the insured lives in the second probability space his combined model becomes also incomplete. In fact Møller (1998) shows that in the combined model there are, even with a complete financial market, arbitrary many equivalent martingale measures to choose from. In our case the financial market is already incomplete and since we assume risk neutrality of the insurer towards mortality, we obtain an equivalent martingale measure in the combined model building the product measure of the Föllmer Schweizer measure $\mathbb{Q}_1$ and the mortality law $\mathbb{P}_2$. We denote that risk-neutral product measure by $\mathbb{Q}$.

2.5.2 Hedging theory of Föllmer and Sondermann

We already know now that the above introduced combined model, represented by $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{Q})$, is incomplete. Therefore we briefly review the concept of risk -minimizing hedging of Föllmer and Sondermann (1986) who extended the hedging theory for complete markets to the case of incomplete markets. For a detailed treatment of this theory in the insurance context we refer to Møller (1998). In the beginning we summarize the most important facts about hedging in financial markets in a single definition. Let $\mathrm{L}^2(\mathbb{Q}_{\hat{S}})$ be the space of square-integrable predictable processes $(\xi_t)_{0\le t\le T}$ satisfying

$$\mathbb{E}^{\mathbb{Q}}[\int_0^T \xi_s^2 \, d\langle \hat{S}, \hat{S}\rangle_s] < \infty.$$

With this assumption we have that for $\xi \in \mathrm{L}^2(\mathbb{Q}_{\hat{S}})$ integrals of the form $\int_0^T \xi_s \, d\hat{S}_s$ are square integrable martingales (cf. Protter (2004), Chapter IV, Lemma before Theorem 28).

Definition 2.5.1. *1. A random variable $H \in \mathrm{L}^2(\mathbb{Q})$ is a contingent claim with maturity T if it is $\mathcal{F}_T$-measurable and if it describes a payoff at time T.*

2. A pair of stochastic processes $\varphi = (\xi, \eta)$ is called a (trading or portfolio) strategy if

(a) $\xi = (\xi_t)_{0\le t\le T} \in \mathrm{L}^2(\mathbb{Q}_{\hat{S}})$,

(b) $\eta = (\eta_t)_{0\le t\le T}$ is adapted,

(c) the (discounted) portfolio process $\hat{V}(\varphi)$, given by $\hat{V}_t(\varphi) = \xi_t \hat{S}_t + \eta_t$ for $0 \le t \le T$, satisfies $\hat{V}_t(\varphi) \in \mathrm{L}^2(\mathbb{Q})$ and is càdlàg for all $0 \le t \le T$.

3. *For a contingent claim H a portfolio strategy is called H-admissible if $H = \hat{V}_T(\varphi)$ ($\mathbb{Q}$-a.s.).*

4. *The (cumulative) cost process $C(\varphi)$ associated with a strategy φ is*

$$C_t(\varphi) = \hat{V}_t(\varphi) - \int_0^t \xi_s \, d\hat{S}_s, \quad 0 \le t \le T.$$

5. *The intrinsic risk process $R(\varphi)$ associated with a strategy φ is*

$$R_t(\varphi) := \mathbb{E}^{\mathbb{Q}}[\big(C_T(\varphi) - C_t(\varphi)\big)^2 \mid \mathcal{F}_t], \quad 0 \le t \le T.$$

6. *The intrinsic value process $(V_t^*)_{0 \le t \le T}$ of a contingent claim H is*

$$V_t^* = \mathbb{E}^{\mathbb{Q}}[H \mid \mathcal{F}_t], \quad 0 \le t \le T.$$

The first component of a portfolio strategy, ξ_t, is usually interpreted as the number of stocks and the second component, η_t, as the number of bonds held at time $t \in [0, T]$ within the portfolio. The undiscounted portfolio value is naturally given by $V_t = \hat{V}_t B_t = \xi_t S_t + \eta_t B_t$. In the following we concentrate on H-admissible strategies φ for a contingent claim H. In this case a hedger is able to generate H, but maybe only at some costs. The (cumulative) cost process $C(\varphi)$ represents the value of the portfolio less accumulated gains originated through trading in the stock S. If that cost process is not $\mathbb{Q}$-a.s. constant, a hedger has to invest additional capital after having made the initial investment $V_0(\varphi) = C_0(\varphi)$. Its risk is thereby measured by the squared value of future costs stemming from trading. It is given by the intrinsic risk process. Contrarily, if the cost process is $\mathbb{Q}$-a.s. constant Harrison and Pliska (1981) called the corresponding strategy self-financing. They also proved that in a complete market every contingent claim H is attainable, that is,

$$H = V_0(\varphi) + \int_0^T \xi_s d\hat{S}_s \quad \mathbb{Q}\text{-a.s.}$$

for a self-financing trading strategy φ. Clearly in this case we have no risk involved and $R_t(\varphi) = 0$, $\mathbb{Q}$-a.s. Extending this theory Föllmer and Sondermann (1986) call a trading strategy mean-self-financing, if the cost process is a martingale. We have already seen that we face an incomplete market, where not every contingent claim is self-financed attainable. So for hedging purposes the question arises of how to invest in the stock and the bond so that the involved risk is minimal in some sense. Föllmer and Sondermann (1986) define a trading strategy to be risk-minimizing as follows (cf. also Schweizer (1991)).

Definition 2.5.2. *Let $\varphi = (\xi, \eta)$ be a trading strategy and $t \in [0, T)$. An admissible continuation of φ from t on is a trading strategy $\tilde{\varphi} = (\tilde{\xi}, \tilde{\eta})$ satisfying*

$$\tilde{\xi}_s = \xi_s \quad \text{for} \quad s \leq t, \qquad\qquad \tilde{\eta}_s = \eta_s \quad \text{for} \quad s < t,$$

and

$$\hat{V}_T(\tilde{\varphi}) = \hat{V}_T(\varphi) \quad \mathbb{Q}\text{-a.s.}$$

The trading strategy φ is called risk-minimizing if for any $t \in [0, T)$ and for any admissible continuation $\tilde{\varphi}$ of φ from t on we have

$$R_t(\tilde{\varphi}) \geq R_t(\varphi) \quad \mathbb{Q}\text{-a.s.}$$

Föllmer and Sondermann (1986) furthermore show that a risk-minimizing strategy is found with the Galtchouk-Kunita-Watanabe decomposition of the intrinsic value process of a contingent claim H. We adapt the following theorem from Cont and Tankov (2004).

Theorem 2.5.3 (Galtchouk-Kunita-Watanabe decomposition)**.** *Let $(\hat{S}_t)_{0 \leq t \leq T}$ be a square-integrable martingale with respect to $\mathbb{Q}$. For any random variable $H \in \mathrm{L}^2(\mathbb{Q})$ which is measurable with respect to $\mathcal{F}_T$ there exists a stochastic process $\xi^H \in \mathrm{L}^2(\mathbb{Q}_{\hat{S}})$ and a square integrable $\mathcal{F}_T$-random variable K_T^H such that*

$$H = \mathbb{E}^{\mathbb{Q}}[H] + \int_0^T \xi_t^H \, d\hat{S}_t + K_T^H.$$

Further the zero-mean square-integrable martingale $K_t^H = \mathbb{E}^{\mathbb{Q}}[K_T \mid \mathcal{F}_t]$, $0 \leq t \leq T$, is orthogonal to all stochastic integrals with respect to $\hat{S}$. That is, $K_t^H \cdot \int_0^t \xi_s \, d\hat{S}_s$, $0 \leq t \leq T$, is again a martingale for any $\xi \in \mathrm{L}^2(\mathbb{Q}_{\hat{S}})$ or equivalently if $\left\langle K^H, \hat{S} \right\rangle_t = 0$ for all $0 \leq t \leq T$.

Proof. For the original result we refer to Kunita and Watanabe (1967) and Galtchouck (1976). A compact and recent treatment is Ansel and Stricker (1993). □

Using this decomposition V_t^* is now expressed as

$$V_t^* = \mathbb{E}^{\mathbb{Q}}[H] + \int_0^t \xi_s^H \, d\hat{S}_s + K_t^H, \quad 0 \leq t \leq T, \tag{2.23}$$

where $K^H = (K_t^H)_{0 \leq t \leq T}$ is a zero-mean square-integrable martingale, orthogonal to the process $\hat{S}$ and $\xi^H \in \mathrm{L}^2(\mathbb{Q}_{\hat{S}})$. Using the orthogonality Föllmer and Sondermann (1986) prove the following theorem.

Theorem 2.5.4. *(Föllmer and Sondermann) Assume $\hat{S}$ is a square-integrable martingale. For every contingent claim H exists an unique H-admissible*

risk-minimizing trading strategy φ^H. It is mean self-financing and its components are given by

$$(\xi_t, \eta_t) = (\xi_t^H, V_t^* - \xi_t^H \hat{S}_t), \quad 0 \leq t \leq T,$$

where ξ^H is the integrand in (2.23). Further, the associated risk process is given by

$$R_t(\varphi) = \mathbb{E}^{\mathbb{Q}}\left[(K_T^H - K_t^H)^2 \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T,$$

where K^H is taken from (2.23).

Proof. Föllmer and Sondermann (1986). □

2.5.3 Local risk-minimization of Schweizer

Föllmer and Sondermann (1986) considered merely the case where the historical, subjective measure $\mathbb{P}$ is already a martingale measure for the discounted stock price. Opposed to the risk-neutral word, only in this case it makes sense to talk about hedging-risk. So in the previous section we are at first glance a little bit unprecise, if we do not presuppose $\hat{S}$ to be a martingale under $\mathbb{P}$. However, we work under the Föllmer-Schweizer measure $\mathbb{Q}$ which has the following justification. Schweizer (1991) shows that the approach of Föllmer and Sondermann (1986) fails for the case, where $\hat{S}$ is a semimartingale and not a martingale under $\mathbb{P}$. It is then not possible to obtain a risk-minimizing hedging strategy reducing the involved risk uniformly over time. For a counterexample we refer the reader to Schweizer (2001). The reason is that the Galtchouk-Kunita-Watanabe projection technique fails for semimartingales because one is unable to control their finite-variation drift part. Its influence on

$$R_t^{\mathbb{P}}(\varphi) = \mathbb{E}\left[\big(C_T(\varphi) - C_t(\varphi)\big)^2 \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T,$$

is not evident. Instead Schweizer (1991) proposes to consider a class of smaller variations of trading strategies and finds the so-called locally risk-minimizing strategies (under $\mathbb{P}$). The idea is to change the subjective probability measure to the Föllmer-Schweizer measure which transforms the semimartingale $\hat{S}$ into a martingale. Under this measure one finds then a risk-minimizing strategy according to Föllmer and Sondermann (1986) which is then in turn locally risk-minimizing under $\mathbb{P}$. The perhaps most significant feature of the Föllmer-Schweizer measure is that it transforms only the semimartingale $\hat{S}$ into a martingale while preserving orthogonality relations. That is, orthogonal martingales under $\mathbb{P}$ are also orthogonal martingales under $\mathbb{Q}$ and vice versa. For completeness we remark that the extension of the Galtchouk-Kunita-Watanabe decomposition to the semimartingale case is referred to as the Föllmer-Schweizer decomposition. The orthogonality holds then with respect to the martingale part of the semimartingale. In

other words, if one derives the Galtchouk-Kunita-Watanabe decomposition under some equivalent martingale measure and writes it down under the original measure $\mathbb{P}$, it is called Föllmer-Schweizer decomposition. For further details about local risk-minimization we refer to Schweizer (1991) or for its extension towards hedging of general payment streams to Chapter 5 of this thesis.

Chapter 3

Arbitrage-free price process

3.1 Introduction

In Subsection 2.5.2 we observed the importance of the Galtchouk-Kunita-Watanabe decomposition (cf. Theorem 2.5.3) for (locally) risk-minimizing hedging of contingent claims. Therefore in this chapter our focus is the derivation of that significant decomposition in our financial market keeping unit-linked life insurance in view. Since normally an insurance benefit has to be cashed out at the time a certain event occurs, we often deal in the insurance context with time-dependent payment streams, a situation being more general compared to simple contingent claim hedging. For this reason we are considering time-dependent payoff functions and call their value arbitrage-free price process. We presuppose for the whole chapter the risk-neutral combined model $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{Q})$ as discussed in Chapter 2 with all the notations and assumptions made. We remark, however, that we could perform the same analysis only for the financial market, as introduced in Section 2.3.

3.2 A Feynman-Kac type formula

We begin with the arbitrage-free price process of certain payoff functions in the financial market. As previously mentioned we consider the Lévy process stock model S (cf. (2.13)) and a time-dependent function $g : [0,T]\times[0,\infty) \to \mathbb{R}$ such that $g(t, S_t)$ is $\mathcal{G}_t$-measurable, $0 \leq t \leq T$, and such that

$$\sup_{u\in[0,T]} \mathbb{E}^{\mathbb{Q}}\left[\left(B_u^{-1} g(u, S_u)\right)^2\right] < \infty. \tag{3.1}$$

This technical assumption corresponds to the square-integrability of simple contingent claims (cf. Definition 2.5.1). Note that $g(t, S_t)$ is only influenced by the uncertainty of the financial market and is interpreted as a payoff specified at time t depending on the asset S_t. A contingent claim H from

Definition 2.5.1 is a special case of a time-dependent payoff function, where one sets $g(t, S_t) = 0$ for $t \in [0, T)$ and $g(T, S_T) = H$. Following risk-neutral valuation we obtain the next definition.

Definition 3.2.1. *The arbitrage-free price process at time $0 \leq t \leq T$ of a payoff $g(u, S_u)$ at time $0 \leq u \leq T$ is denoted by $F(t, S_t, u)$ and is given by*

$$F(t, S_t, u) = \begin{cases} \mathbb{E}^{\mathbb{Q}}[B_t B_u^{-1} g(u, S_u) \mid \mathcal{G}_t], & 0 \leq t < u \leq T, \\ B_t B_u^{-1} g(u, S_u), & 0 \leq u \leq t \leq T. \end{cases}$$

Naturally we are only interested in the case $0 \leq t < u \leq T$. Similar to the Black-Scholes differential equation, which is fulfilled by the price process in a Black-Scholes financial market, there is an useful characterization of the price process in the more general Lévy-process setting in terms of an integro-differential equation. To formulate this integro-differential equation properly we need a few observations first.

Lemma 3.2.2. *The arbitrage-free price process satisfies*

$$F(t, S_t, u) = \mathbb{E}^{\mathbb{Q}} \left[B_t B_u^{-1} g(u, S_u) \middle| S_t \right], \quad 0 \leq t < u \leq T.$$

Proof. This follows immediately from the Markov property of Lévy processes (cf. e.g. Protter (2004), Chapter I, Theorem 32), which carries over by the independent increments to the possible additive process $L^{\mathbb{Q}}$ (cf. Subsection 2.3.2 and Cont and Tankov (2004), Chapter 14). □

In the following we therefore consider for $x > 0$ and for $0 \leq t < u \leq T$ the function

$$F(t, x, u) = \mathbb{E}^{\mathbb{Q}}_{t,x} \left[B_t B_u^{-1} g(u, S_u) \right] := \mathbb{E}^{\mathbb{Q}} \left[B_t B_u^{-1} g(u, S_u) \middle| S_t = x \right],$$

and assume that $F(\cdot, \cdot, u) \in C^{1,2}([0, T] \times [0, \infty))$ for each $u \in [0, T]$. That is, it is once continuously differentiable with respect to the first and twice continuously differentiable with respect to the second variable. For notational convenience let us denote by $\dot{F}(t, \cdot, \cdot) = \frac{\partial}{\partial t} F(t, \cdot, \cdot)$ and by $F'(\cdot, x, \cdot) = \frac{\partial}{\partial x} F(\cdot, x, \cdot)$. The next Lemma states a simple fact, but is often used in the sequel, especially in the proof of the theorem below.

Lemma 3.2.3. *Given a stock price as introduced in Section 2.3. Then the continuous part of its quadratic variation is given by*

$$[S, S]^c_t = \int_0^t \sigma_s^2 S_s^2 c^2 \, ds.$$

Proof. The claim is not very hard to proof and follows by a simple calculation using the well-known rules for the quadratic variation (cf. Protter (2004), Chapter II, Section 6).

$$\begin{aligned}
[S,S]_t^c &= \Big[\int_0^{\cdot} \sigma_s S_{s-}(c\,dW_s^{\mathbb{Q}} + dM_s^{\mathbb{Q}}) + r_s S_{s-}\,ds, \\
&\qquad \int_0^{\cdot} \sigma_s S_{s-}(c\,dW_s^{\mathbb{Q}} + dM_s^{\mathbb{Q}}) + r_s S_{s-}\,ds \Big]_t^c \\
&= \int_0^t \sigma_s^2 S_{s-}^2 c^2\, d[W^{\mathbb{Q}}, W^{\mathbb{Q}}]_s + 2\int_0^t \sigma_s^2 S_{s-}^2 c\, d[W^{\mathbb{Q}}, M^{\mathbb{Q}}]_s^c \\
&\quad + 2\int_0^t \sigma_s S_{s-}^2 c r_s\, d[W^{\mathbb{Q}}, \cdot]_s^c + \int_0^t \sigma_s^2 S_{s-}^2\, d[M^{\mathbb{Q}}, M^{\mathbb{Q}}]_s^c \\
&\quad + 2\int_0^t \sigma_s S_{s-}^2 r_s\, d[M^{\mathbb{Q}}, \cdot]_s^c + \int_0^t r_s^2 S_{s-}^2\, d[\cdot,\cdot]_s^c \\
&= \int_0^t \sigma_s^2 S_s^2 c^2\, ds.
\end{aligned}$$

All terms, except of the last one, vanish because $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}$ are orthogonal. □

Theorem 3.2.4.

1. *$F(t,x,u)$ satisfies for $x \geq 0$ and $0 \leq t < u \leq T$ the following integro-differential equation:*

$$\begin{aligned}
&\dot{F}(t,x,u) + \mathcal{A}_t F(t,x,u) - r_t F(t,x,u) = 0, \\
&F(u,x,u) = g(u,x),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
\mathcal{A}_t F(t,x,u) &= \frac{1}{2} c^2 \sigma_t^2 x^2 F''(t,x,u) + r_t x F'(t,x,u) \\
&\quad + \int_{\mathbb{R}} \left\{ F(t, x+\sigma_t x y, u) - F(t,x,u) - \sigma_t x y F'(t,x,u) \right\} \nu_t^{\mathbb{Q}}(dy).
\end{aligned}$$

2. *Let $g \in C([0,T]\times[0,\infty))$ be a continuous and bounded function on its range $\mathbb{R}$. Then for $u \in [0,T]$ any bounded solution $F(\cdot,\cdot,u) \in C^{1,2}([0,T]\times[0,\infty))$ of the parabolic Cauchy problem (3.2) admits the representation*

$$F(t,x,u) = \mathbb{E}_{t,x}^{\mathbb{Q}} \left[B_t B_u^{-1} g(u, S_u) \right], \quad 0 \leq t \leq u \leq T.$$

Proof. First part: We fix $u \in [0,T]$. The initial value follows immediately. Further the process $(B_t^{-1} F(t, S_t, u))_t$ is a $(\mathcal{G}_t, \mathbb{Q})$-martingale, hence it is a special semimartingale and any decomposition into a local martingale and a

predictable process of finite variation has to satisfy that the predictable part is identically zero. To derive such a representation we apply Itô's formula to $B_t^{-1}F(t,S_t,u)$. To begin with, the product rule yields

$$d(B_t^{-1}F(t,S_t,u)) = F(t,S_{t-},u)dB_t^{-1} + B_t^{-1}dF(t,S_t,u),$$

since B is continuous and of finite variation. Using now $dB_t^{-1} = -r_tB_t^{-1}\,dt$ and applying Itô's formula (cf. e.g. Protter (2004), Chapter II, Theorem 33) to the function $F(\cdot,\cdot,u)$, we get

$$\begin{aligned}
&B_t^{-1}F(t,S_t,u)\\
&= F(0,S_0,u) - \int_0^t r_sB_s^{-1}F(s,S_s,u)\,ds\\
&\quad + \int_0^t B_s^{-1}\dot{F}(s,S_s,u)\,ds + \int_0^t B_s^{-1}F'(s,S_{s-},u)\,dS_s\\
&\quad + \frac{1}{2}\int_0^t B_s^{-1}F''(s,S_{s-},u)\,d[S,S]_s^c\\
&\quad + \sum_{0<s\le t} B_s^{-1}\left\{F(s,S_s,u) - F(s,S_{s-},u) - F'(s,S_{s-},u)\Delta S_s\right\}.
\end{aligned} \tag{3.3}$$

Recall that $N^{\mathbb{Q}}$ is the measure that counts the jumps of $L^{\mathbb{Q}}$. Using $\Delta S_t = \sigma_tS_t\Delta L_t^{\mathbb{Q}}$ we observe

$$\begin{aligned}
&\sum_{0<s\le t} B_s^{-1}\left\{F(s,S_s,u) - F(s,S_{s-},u) - F'(s,S_{s-},u)\Delta S_s\right\}\\
&= \int_0^t\int_{\mathbb{R}} B_s^{-1}\big\{F(s,S_{s-}+\sigma_sS_{s-}y,u) - F(s,S_{s-},u)\\
&\qquad\qquad - F'(s,S_{s-},u)\sigma_sS_{s-}y\big\}N^{\mathbb{Q}}(ds,dy).
\end{aligned} \tag{3.4}$$

Writing S_t as the product of $\hat{S}_t$ and B_t the product rule yields

$$dS_t = d(\hat{S}_tB_t) = \hat{S}_{t-}\,dB_t + B_t\,d\hat{S}_t = S_tr_t\,dt + B_t\,d\hat{S}_t. \tag{3.5}$$

We proceed now and plug (3.4), (3.5) and the result of Lemma 3.2.3 into equation (3.3), obtaining

$$\begin{aligned}
d(B_t^{-1}F(t,S_t,u)) &= -r_tB_t^{-1}F(t,S_t,u)\,dt + B_t^{-1}\Big[\dot{F}(t,S_t,u)\,dt\\
&\quad + F'(t,S_t,u)S_tr_t\,dt + F'(t,S_{t-},u)B_t\,d\hat{S}_t + \frac{1}{2}F''(t,S_t,u)c^2\sigma_t^2S_t^2\,dt\\
&+\int_{\mathbb{R}}\left\{F(t,S_{t-}+\sigma_tS_{t-}y,u) - F(t,S_{t-},u) - F'(t,S_{t-},u)\sigma_tS_{t-}y\right\}M^{\mathbb{Q}}(dt,dy)\\
&+\int_{\mathbb{R}}\left\{F(t,S_{t-}+\sigma_tS_{t-}y,u) - F(t,S_{t-},u) - F'(t,S_{t-},u)\sigma_tS_{t-}y\right\}\nu_t^{\mathbb{Q}}(dy)dt\Big].
\end{aligned}$$

Hence, $B_t^{-1}F(t,S_t,u)$ can be written as the sum of the local $\mathbb{Q}$-martingale

$$\begin{aligned}
&F(0,S_0,u)+\int_0^t F'(s,S_{s-},u)\,d\hat{S}_s\\
&+\int_0^t\int_{\mathbb{R}} B_s^{-1}\big\{F(s,S_{s-}+\sigma_s S_{s-}y,u)-F(s,S_{s-},u)\\
&\qquad\qquad -F'(s,S_{s-},u)\sigma_s S_{s-}y\big\}\,M^{\mathbb{Q}}(ds,dy)
\end{aligned}$$

and the predictable, finite-variation process

$$\int_0^t B_s^{-1}\big(\dot{F}(s,S_s,u)+\mathcal{A}_sF(s,S_s,u)-r_sF(s,S_s,u)\big)\,ds,$$

which has to be identically zero for any $0\le t\le T$. The continuity of F and the non-negativity of the stock price S immediately imply

$$\dot{F}(t,x,u)+\mathcal{A}_tF(t,x,u)-r_tF(t,x,u)=0,\quad 0\le t\le u,\quad \forall\, x\ge 0.$$

To prove the second part of the theorem, we fix again $u\in[0,T]$ and assume that $F(\cdot,\cdot,u)$ is a bounded solution of (3.2). Consider the bounded function

$$v(\tau,S_\tau):=B_tB_\tau^{-1}F(\tau,S_\tau,u)\quad 0\le t\le\tau\le u.$$

Differentiation yields

$$\dot{v}(\tau,S_\tau)=B_tB_\tau^{-1}\dot{F}(\tau,S_\tau,u)+B_tB_\tau^{-1}(-r_\tau)F(\tau,S_\tau,u)$$

and

$$\mathcal{A}_\tau v(\tau,S_\tau)=B_tB_\tau^{-1}\mathcal{A}_\tau F(\tau,S_\tau,u).$$

Hence,

$$\begin{aligned}
&\dot{v}(\tau,S_\tau)+\mathcal{A}_\tau v(\tau,S_\tau)\\
&=B_tB_\tau^{-1}\Big(\dot{F}(\tau,S_\tau,u)+\mathcal{A}_\tau F(\tau,S_\tau,u)-r_\tau F(\tau,S_\tau,u)\Big)\\
&\equiv 0.
\end{aligned}$$

Using this and Itô's formula, we get

$$\begin{aligned}
dv(\tau,S_\tau)&=\dot{v}(\tau,S_\tau)d\tau+v'(\tau,S_{\tau-})dS_\tau+\frac{1}{2}v''(\tau,S_{\tau-})d[S,S]^c_\tau\\
&\quad+\int_{\mathbb{R}}\big(v(\tau,S_{\tau-}+\sigma_\tau S_{\tau-}y)-v(\tau,S_{\tau-})\\
&\qquad\qquad -v'(\tau,S_{\tau-})\sigma_\tau S_{\tau-}y\big)N^{\mathbb{Q}}(d\tau,dy)\\
&=\underbrace{\big(\dot{v}(\tau,S_\tau)+\mathcal{A}_\tau v(\tau,S_\tau)\big)}_{=0}\,d\tau+v'(\tau,S_{\tau-})\sigma_\tau S_{\tau-}\left(c\,dW_\tau^{\mathbb{Q}}+dM_\tau^{\mathbb{Q}}\right)\\
&\quad+\int_{\mathbb{R}}\big(v(\tau,S_{\tau-}+\sigma_\tau S_{\tau-}y)-v(\tau,S_{\tau-})\\
&\qquad\qquad -v'(\tau,S_{\tau-})\sigma_\tau S_{\tau-}y\big)M^{\mathbb{Q}}(d\tau,dy).
\end{aligned}$$

Hence, $v(\tau, S_\tau)$ is a local $\mathbb{Q}$-martingale. Since it is furthermore bounded, it is a true martingale. Thus we obtain

$$\begin{aligned} 0 &= \mathbb{E}^{\mathbb{Q}}_{t,x}[v(\tau, S_\tau) - v(t, S_t)] = \mathbb{E}^{\mathbb{Q}}_{t,x}[v(u, S_u) - v(t, S_t)] \\ &= \mathbb{E}^{\mathbb{Q}}_{t,x}[v(u, S_u)] - F(t, x, u). \end{aligned}$$

Therefore,

$$F(t, x, u) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[v(u, S_u)\right] = \mathbb{E}^{\mathbb{Q}}_{t,x}[B_t B_u^{-1} F(u, S_u, u)].$$

The claim follows, since $F(u, S_u, u) = g(u, S_u)$. □

Theorems of this type are called Feynman[1]-Kac[2] formulas and especially the first part of the theorem is one of the main tools to derive the Galtchouk-Kunita-Watanabe decomposition of the arbitrage-free price process. For this reason we worked it out tailor-made to our situation. The second part of the theorem complements our consideration. The operator $\mathcal{A}_t$ in (3.2) is the infinitesimal generator of the transition operator that corresponds to the additive process in the background (cf. Cont and Tankov (2004), Chapter 14).

Of course, there are more general versions of the theorem both in terms of the operator $\mathcal{A}_t$ and with respect to the conditions imposed on the arbitrage-free price process or on the payoff function. For example Kallenberg (2002) (Chapter 24, Theorem 24.1) presents the theorem more general with respect to the process in the background in considering merely a generator of a Feller-semigroup. Feynman-Kac formulas are additionally useful in numerical computations of the price process $F(t, S_t, u)$. This fact is particularly recognized in Cont and Tankov (2004) (Chapter 12), where the partial integro-differential equation is analyzed in detail. It is investigated what conditions the price process or the payoff function have to satisfy such that a solution exists. For this reason the partial integro-differential equation is formulated separately for different kind of payoff functions such as European or American type options. Especially they determine conditions on an European-style payoff under which $F \in C^{1,2}([0, T] \times [0, \infty))$. Karatzas and Shreve (2004) treat the case of a Brownian motion, where the measure $\nu \equiv 0$ and hence the integral term in (3.2) is not present.

At the end we have again a brief glance at

$$\begin{aligned} \mathcal{A}_t F(t, x, u) = &\frac{1}{2} c^2 \sigma_t^2 x^2 F''(t, x, u) + r_t x F'(t, x, u) \\ &+ \int_{\mathbb{R}} \left\{ F(t, x + \sigma_t x y, u) - F(t, x, u) - \sigma_t x y F'(t, x, u) \right\} \nu_t^{\mathbb{Q}}(dy). \end{aligned}$$

[1] Richard Phillips Feynman, American physicist (1918 - 1988) and Nobel prize winner 1965

[2] Mark Kac, Polish mathematician (1914 - 1984)

If the terms under the integral are separated, the integral does in general not converge. Let us, however, consider the special case where the parameters of the financial market are constant, that is, $r_t \equiv r$, $\sigma_t \equiv \sigma$ and $b_t \equiv b$. Then the measure $\nu_t^{\mathbb{Q}}(dy) = (1 + Gy)\nu(dy)$ is independent of t (cf. (2.16)). Furthermore assume that $\int_{\{|y|\leq 1\}} |y|\, \nu(dy) < \infty$ and that the process is of finite activity, that is $\nu(\mathbb{R}) < \infty$ (cf. Lemma 2.2.5, Theorem 2.2.6 and (2.12)). Then the partial integro-differential equation of Theorem 3.2.4 simplifies to

$$\dot{F}(t,x,u) + \frac{1}{2}c^2\sigma^2x^2F''(t,x,u) + \Big(r - \sigma\int_{\mathbb{R}} y\,\nu^{\mathbb{Q}}(dy)\Big)xF'(t,x,u) \\ - \big(r + \nu^{\mathbb{Q}}(\mathbb{R})\big)\,F(t,x,u) + \int_{\mathbb{R}} F(t,x+\sigma xy,u)\,\nu^{\mathbb{Q}}(dy) = 0,$$

which should considerably simplify the analysis for a numerical solution.

3.3 Galtchouk-Kunita-Watanabe decomposition

Let us now derive the Galtchouk-Kunita-Watanabe decomposition (cf. Theorem 2.5.3) of the discounted price process $B_t^{-1}F(t,S_t,u)$. In the following chapters this turns out to be vital for the Galtchouk-Kunita-Watanabe decomposition of the intrinsic value of unit-linked life insurance contracts. A reference for the theory of orthogonal (local) martingales and its relation to the predictable quadratic covariation is, for example, Section I.4, Theorem I.4.40 and Theorem II.1.33 in Jacod and Shiryaev (2003).

In the following we always assume that there exists a non-random constant c_1 such that

$$\big|F'(t,x,u)\big| \leq c_1 < \infty \quad \forall\, t,x,u \quad \mathbb{Q}\text{-a.s.} \tag{3.6}$$

Definition 3.3.1. *For $0 \leq t < u \leq T$ we define the (discounted) jump of the price process $F(t,S_t,u)$ induced by a jump of the underlying stock process S by*

$$J(t,x,u) = B_t^{-1}\left\{F(t,S_{t-} + \sigma_t S_{t-}x,u) - F(t,S_{t-},u)\right\}.$$

Recall that $\Delta S_t = \sigma_t S_t \Delta L_t^{\mathbb{Q}}$. Hence, for $\Delta L_t^{\mathbb{Q}} = x$ the induced jump of the price process is $J(t,x,u)$. The next lemma states a first decomposition of the discounted price process.

Lemma 3.3.2. *For $0 \leq t < u \leq T$ the process $B_t^{-1}F(t,S_t,u)$ admits the decomposition*

$$B_t^{-1}F(t,S_t,u) = F(0,S_0,u) + \int_0^t F'(s,S_{s-},u)\,d\hat{S}_s + \tilde{K}_t(u),$$

where

$$\tilde{K}_t(u) = \int_0^t\int_{\mathbb{R}} \Big(J(s,y,u) - F'(s,S_{s-},u)\sigma_s\hat{S}_{s-}y\Big)\,M^{\mathbb{Q}}(ds,dy).$$

Proof. This representation was developed in the first part of the proof of Theorem 3.2.4. □

Opposed to a standard Black-Scholes setting one easily verifies that this decomposition is not the Galtchouk-Kunita-Watanabe decomposition, since the process $\hat{S}$ is not orthogonal to the process $\tilde{K}$. Recall the differential of the discounted stock price $\hat{S}$, which was derived in Lemma 2.3.3:

$$d\hat{S}_t = c\sigma_t \hat{S}_{t-}\, dW_t^{\mathbb{Q}} + \sigma_t \hat{S}_{t-} \int_{\mathbb{R}} y M^{\mathbb{Q}}(dt, dy). \tag{3.7}$$

Hence, for $0 \leq t \leq T$,

$$\begin{aligned} \left\langle \hat{S}, \tilde{K}(u) \right\rangle_t = \Big\langle \int_0^{\cdot} c\sigma_s \hat{S}_{s-}\, dW_s^{\mathbb{Q}}, \tilde{K}(u) \Big\rangle_t \\ + \Big\langle \int_0^{\cdot} \sigma_s \hat{S}_{s-} \int_{\mathbb{R}} y M^{\mathbb{Q}}(ds, dy), \tilde{K}(u) \Big\rangle_t, \end{aligned}$$

where the first term on the right hand side is equal to zero, since the integral with respect to the Brownian motion is continuous and the integral with respect to $M^{\mathbb{Q}}(dt, dy)$ is purely discontinuous. It remains the second term being equal to

$$\int_0^t \int_{\mathbb{R}} \sigma_s \hat{S}_{s-} y \left(J(s, y, u) - F'(s, S_{s-}, u) \sigma_s \hat{S}_{s-} y \right) \nu_s^{\mathbb{Q}}(dy) ds,$$

which is generally not zero. A more careful consideration of the decomposition above reveals that some terms cancel out. This is explained in the next lemma.

Lemma 3.3.3. *For $0 \leq t < u \leq T$ the process $B_t^{-1} F(t, S_t, u)$ admits the decomposition*

$$B_t^{-1} F(t, S_t, u) = F(0, S_0, u) + \int_0^t F'(s, S_{s-}, u) c\sigma_s \hat{S}_{s-}\, dW_s^{\mathbb{Q}} + \tilde{\tilde{K}}_t(u),$$

where

$$\tilde{\tilde{K}}_t(u) = \int_0^t \int_{\mathbb{R}} J(s, y, u) M^{\mathbb{Q}}(ds, dy).$$

This decomposition is orthogonal in the sense that $W^{\mathbb{Q}}$ is orthogonal to the process $\tilde{\tilde{K}}$.

Proof. We plug (3.7) into the decomposition stated in Lemma 3.3.2. This yields the claimed decomposition. The orthogonality is immediate, since the Brownian motion is continuous and the process $\tilde{\tilde{K}}$ is purely discontinuous (i.e. quadratic pure-jump). □

Of course, this is again not the desired decomposition. We do not only want to invest in the continuous part of the stock process, although it is orthogonal. But in any case this decomposition shows a nice representation property of the price process in terms of the Brownian motion $W^{\mathbb{Q}}$ and the compensated jump measure $M^{\mathbb{Q}}(dt, dy)$, which are the components of the process $L^{\mathbb{Q}}$ in the background. The previous two lemmas lead to the next theorem, which states the Galtchouk-Kunita-Watanabe decomposition of $B_t^{-1}F(t, S_t, u)$.

Theorem 3.3.4. *Let $v_t^{\mathbb{Q}} = \int_{\mathbb{R}} x^2 \nu_t^{\mathbb{Q}}(dx)$ and $\kappa_t = c^2 + v_t^{\mathbb{Q}}$, $t \in [0, T]$. For $0 \leq t < u \leq T$ the Galtchouk-Kunita-Watanabe decomposition of $B_t^{-1}F(t, S_t, u)$ is given by*

$$B_t^{-1}F(t, S_t, u) = F(0, S_0, u) + \int_0^t \xi_s(u)\, d\hat{S}_s + K_t(u),$$

where

$$\xi_t(u) = \frac{c^2}{\kappa_t} F'(t, S_{t-}, u) + \frac{1}{\sigma_t \hat{S}_{t-} \kappa_t} \int_{\mathbb{R}} x J(t, x, u)\, \nu_t^{\mathbb{Q}}(dx),$$

and

$$K_t(u) = \int_0^t \zeta_s^{(1)}(u)\, dW_s^{\mathbb{Q}} + \int_0^t \int_{\mathbb{R}} \zeta_s^{(2)}(y, u)\, M^{\mathbb{Q}}(ds, dy)$$

is orthogonal to $\hat{S}$. Further the processes $\zeta^{(1)}(u)$ and $\zeta^{(2)}(y, u)$ are given by

$$\begin{aligned} \zeta_t^{(1)}(u) &= c\sigma_t \hat{S}_{t-} \left(F'(t, S_{t-}, u) - \xi_t(u) \right), \\ \zeta_t^{(2)}(y, u) &= J(t, y, u) - \sigma_t \hat{S}_{t-} \xi_t(u) y. \end{aligned}$$

Proof. The proof consists of two steps. First we fix $u \in [0, T]$ and derive that $B_t^{-1}F(t, S_t, u)$ has the claimed decomposition. Second we show that for the claimed choice of $\xi_t(u)$ this decomposition is the desired Galtchouk-Kunita-Watanabe decomposition.

We build on the decomposition that we derived in Lemma 3.3.2 and insert some predictable process $\xi_t(u)$ resulting in

$$\begin{aligned} B_t^{-1}F(t, S_t, u) = F(0, S_0, u) &+ \int_0^t \xi_s(u)\, d\hat{S}_s \\ &+ \int_0^t \left(F'(s, S_{s-}, u) - \xi_s(u) \right) d\hat{S}_s + \tilde{K}_t(u). \end{aligned}$$

The differential (3.7) of the discounted stock price $\hat{S}$ yields further

$$\begin{aligned} B_t^{-1}F(t,S_t,u) = F(0,S_0,u) &+ \int_0^t \xi_s(u)\, d\hat{S}_s \\ &+ \int_0^t c\sigma_s \hat{S}_{s-} \left(F'(s,S_{s-},u) - \xi_s(u)\right)\, dW_s^{\mathbb{Q}} \\ &+ \int_0^t \int_{\mathbb{R}} \sigma_s \hat{S}_{s-} y \left(F'(s,S_{s-},u) - \xi_s(u)\right) M^{\mathbb{Q}}(ds,dy) \\ &+ \tilde{K}_t(u). \end{aligned}$$

Since $\tilde{K}_t(u) = \int_0^t \int_{\mathbb{R}} \left(J(s,y,u) - F'(s,S_{s-},u)\sigma_s \hat{S}_{s-} y\right) M^{\mathbb{Q}}(ds,dy)$ we deduce

$$\begin{aligned} B_t^{-1}F(t,S_t,u) = F(0,S_0,u) &+ \int_0^t \xi_s(u)\, d\hat{S}_s \\ &+ \int_0^t c\sigma_s \hat{S}_{s-} \left(F'(s,S_{s-},u) - \xi_s(u)\right)\, dW_s^{\mathbb{Q}} \\ &+ \int_0^t \int_{\mathbb{R}} \left(J(s,y,u) - \sigma_s \hat{S}_{s-} \xi_s(u) y\right) M^{\mathbb{Q}}(ds,dy). \end{aligned}$$

Now we have to choose $\xi_t(u)$ such that this decomposition is orthogonal. Consider for $0 \le t < u \le T$ the predictable quadratic covariation

$$\left\langle \hat{S}, K(u) \right\rangle_t = \left\langle \hat{S}, \int_0^{\cdot} \zeta_s^{(1)}(u)\, dW_s^{\mathbb{Q}} + \int_0^{\cdot} \int_{\mathbb{R}} \zeta_s^{(2)}(y,u) M^{\mathbb{Q}}(ds,dy) \right\rangle_t. \tag{3.8}$$

We know that integrals with respect to $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}(dt,dy)$ are orthogonal. Hence with (3.7) expression (3.8) is equal to

$$\int_0^t c\sigma_s \hat{S}_{s-} \zeta_s^{(1)}(u)\, d\left\langle W^{\mathbb{Q}}, W^{\mathbb{Q}} \right\rangle_s + \int_0^t \int_{\mathbb{R}} \sigma_s \hat{S}_{s-} y \zeta_s^{(2)}(y,u)\, \nu_s^{\mathbb{Q}}(dy)\, ds.$$

Inserting the expressions for $\zeta_s^{(1)}(u)$ and $\zeta_s^{(2)}(y,u)$ yields

$$\begin{aligned} &\int_0^t c^2 \sigma_s^2 \hat{S}_{s-}^2 \left(F'(s,S_{s-},u) - \xi_s(u)\right)\, ds \\ &+ \int_0^t \int_{\mathbb{R}} \sigma_s \hat{S}_{s-} y \left(J(s,y,u) - y\sigma_s \hat{S}_{s-} \xi_s(u)\right)\, \nu_s^{\mathbb{Q}}(dy)\, ds \\ = &\int_0^t c^2 \sigma_s^2 \hat{S}_{s-}^2 F'(s,S_{s-},u)\, ds - \int_0^t \kappa_s \sigma_s^2 \hat{S}_{s-}^2 \xi_s(u)\, ds \\ &+ \int_0^t \int_{\mathbb{R}} \sigma_s \hat{S}_{s-} y J(s,y,u)\, \nu_s^{\mathbb{Q}}(dy)\, ds. \end{aligned}$$

Finally, for $\xi_s(u) = \frac{c^2}{\kappa_s} F'(s,S_{s-},u) + \frac{1}{\sigma_s \hat{S}_{s-} \kappa_s} \int_{\mathbb{R}} x J(s,x,u)\, \nu_s^{\mathbb{Q}}(dx)$ the expression is equal to zero for any $0 \le t < u \le T$, which yields the desired

orthogonal decomposition. It remains to show that the integrals with respect to $\hat{S}$, $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}(dt,dy)$ are really square-integrable. Since it is continuous in t and (2.12) holds we have that

$$\sup_{t\in[0,T]}\left|\int_{\mathbb{R}} x^2\nu_t^{\mathbb{Q}}(dx)\right| \leq \int_{\mathbb{R}} x^2\nu(dx) + \sup_{t\in[0,T]}|G_t|\left|\int_{\mathbb{R}} x^3\nu(dx)\right| < \infty.$$

The mean-value theorem yields $J(t,x,u) = F'(t,x_0,u)\sigma_t\hat{S}_{t-}x$ for some intermediate value $x_0 = x_0(S_{t-},\sigma_t,x)$, and hence

$$\xi_t(u) = \frac{c^2}{\kappa_t}F'(t,S_{t-},u) + \frac{1}{\kappa_t}\int_{\mathbb{R}} F'(t,x_0,u)x^2\nu_t^{\mathbb{Q}}(dx). \tag{3.9}$$

We conclude by (3.6) that

$$\sup_{t\in[0,T]}|\xi_t(u)| \leq \sup_{t\in[0,T]}\Big(\frac{c_1}{\kappa_t}\big(c^2 + \Big|\int_{\mathbb{R}} x^2\nu_t^{\mathbb{Q}}(dx)\Big|\big)\Big) < \infty \quad \forall\, u \quad \mathbb{Q}\text{-a.s.} \tag{3.10}$$

This non-random boundedness of ξ implies that it is an element of $\mathrm{L}^2(\mathbb{Q}_{\hat{S}})$ (cf. the beginning of Subsection 2.5.2 for its definition) and so $\int_0^t \xi_s(u)\,d\hat{S}_s$ is a square-integrable martingale. Likewise we infer therefore that there exist non-random constants $c_2, c_3 > 0$ such that

$$\sup_{t\in[0,T]}\big|\zeta_t^{(1)}(u)\big| \leq c_2 \sup_{t\in[0,T]}\hat{S}_t \quad \forall\, u \quad \mathbb{Q}\text{-a.s.} \tag{3.11}$$

$$\sup_{t\in[0,T]}\big|\zeta_t^{(2)}(y,u)\big| \leq c_3\,|y| \sup_{t\in[0,T]}\hat{S}_t \quad \forall\, y,u \quad \mathbb{Q}\text{-a.s.} \tag{3.12}$$

The $\mathbb{Q}$-square integrability of $\hat{S}_t$ (cf. Lemma 2.3.3) together with (3.11) implies now that $\int_0^t \zeta_s^{(1)}(u)\,dW_s^{\mathbb{Q}}$ is a square-integrable martingale (cf. Protter (2004), Chapter IV, Lemma before Theorem 28). Further with (3.12) we have that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[\Big(\int_0^T\int_{\mathbb{R}} \zeta_s^{(2)}(y,u)\,M^{\mathbb{Q}}(ds,dy)\Big)^2] \\
&= \int_0^T\int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\big(\zeta_s^{(2)}(y,u)\big)^2]\,\nu_s^{\mathbb{Q}}(dy)\,ds \\
&\leq T c_3^2\,\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]}\hat{S}_t^2] \sup_{s\in[0,T]}\Big|\int_{\mathbb{R}} y^2\nu_s^{\mathbb{Q}}(dy)\Big| < \infty,
\end{aligned}$$

so $\int_0^t\int_{\mathbb{R}} \zeta_s^{(2)}(y,u)\,M^{\mathbb{Q}}(ds,dy)$ is also a square-integrable martingale. This finally yields that $K(u)$ is a square-integrable martingale. □

At the end of the chapter we give, to the best of our knowledge, a short overview of existing literature dealing with the Galtchouk-Kunita-Watanabe decomposition in the Lévy-process context. Jacod, Méléard and

Protter (2000) investigate the so-called Clark-Ocone-Haussmann formula in a general Markov-process setting. Imposing certain conditions this formula reduces to the Galtchouk-Kunita-Watanabe decomposition. Since our setting is simpler, we are able to state a rather elementary and independent proof. In addition, our theorem is much more explicit. For certain functionals of Lévy processes the Clark-Ocone-Haussmann formula is also addressed in the papers of Lokka (2004) and Benth, Di Nunno, Løkka, Øksendal and Proske (2003) in terms of Malliavin Calculus. In particular, Malliavin derivatives are used to prove and to express the decomposition, which is by far less explicit as our expression. Cont and Tankov (2004) derive in Section 10.4 a to our formulation similar explicit representation of the integrand with respect to the discounted stock. However, they neither explicitly state the Galtchouk-Kunita-Watanabe decomposition nor the process $K_t(u)$. Furthermore, they only treat European style options and formulate the theorem only for the case where the stock price has constant coefficients and is driven by a Lévy process. We consider a much more general stock process, which might be under $\mathbb{Q}$ merely driven by an additive process and which has non-constant coefficients. Moreover, we prove the theorem differently from Cont and Tankov (2004). To be more precise, they derive the integrand with respect to the discounted stock by minimizing the variance of a certain hedging error applying standard calculus techniques whereas we directly make use of the predictable quadratic covariation and the structure of the underlying Hilbertspace.

Chapter 4

Hedging of unit-linked pure endowment and term insurance

4.1 Introduction

The following is based on Riesner (2006). We begin this chapter discussing unit-linked insurance in general and also how to model unit-linked pure endowment or term insurance contracts. Thereafter we derive hedging strategies for a portfolio of either one of those contracts in an incomplete Lévy-process financial market. Presupposing the financial market introduced in Section 2.3 and assuming the insurance portfolio discussed in Subsection 2.4.2 we work on the risk-neutral combined financial and insurance model $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{Q})$ of Section 2.5. In particular, we make use of the theory developed in Chapter 3.

We study the model of Møller (1998) in detail and demonstrate how his results are affected when the complete Black-Scholes market is replaced by an incomplete financial market in case of a more general geometric Lévy-driven model as introduced by Chan (1999). In Møller (1998) the uncertainty of both the insured lives and the stock price is modeled simultaneously as a product probability space and, unlike the actuarial principle of equivalence, mortality is not averaged away. Consequently the uncertain development of insured lives is not replaced by the expected development, and the insurance claims depend on both sources of uncertainty. However, in a complete financial market there is no pure financial hedging risk and so with our replacement through an incomplete financial market, an insurance company issuing unit-linked life insurance contracts now additionally faces pure financial risk. The intrinsic risk of the company arises then no longer only from the insurance portfolio but also from the financial market. We derive this risk and the hedging strategies with the (local) risk-minimization

theory of Föllmer and Sondermann (1986) and Schweizer (1991) (cf. Subsections 2.5.2 and 2.5.3). This requires contingent claims payable at the end of the considered time horizon only. Thus for the moment we deal with the simple situation, in which insurance contracts do not generate intermediate payment times. For contracts having insurance benefits before time T, for example if the insured person dies, we assume that the benefits are deferred earning interest with the risk-free interest rate and are cashed out at the end of the considered time horizon. We also suppose the premiums to be paid as a single premium at the beginning. An extension, allowing intermediate payment times, is treated in Møller (2001a), however, only for a financial market in which the stock is a martingale under the historical measure. A generalization admitting the stock also to be a semimartingale under the historical measure is developed in Chapters 5 and 6 of this thesis. For the pure endowment we discuss in detail our results and show that the financial risk is, as expected, not diversifiable by raising the number of insured persons within the portfolio. In accordance with standard actuarial theory (cf. Subsection 2.4.4) this is however the case with risk originated by mortality. Additionally we derive the contract's intrinsic value process which is a natural candidate for the reserve that an insurance company should hold back and which provides simultaneously an idea of a fair premium. For the results of the unit-linked term insurance the same reasoning applies. In considering the pure endowment and the term insurance separately we also treat the case of the popular endowment insurance being simply the sum of them.

4.2 Unit-linked life insurance contracts

Let us start with a general introduction in the theory of unit-linked life insurance contracts. As briefly mentioned earlier, unit-linked or equity-linked life insurance contracts link the insurance benefit and possibly the premiums to the market value of some specified reference portfolio most likely consisting of stocks, bonds or shares of a mutual fund represented in our model by the stock S. In Aase and Persson (1994) unit-linkage is discussed in detail. Therefore we only summarize common types of unit-linked payoffs not taking into account for the moment the due date of the insurance benefit. We denote the payoff by $g(S)$ for some suitable function $g : [0, \infty) \to \mathbb{R}$.

With a "pure unit-linked" insurance contract the amount to be paid is given by the value of S at expiration, that is, $g(s) = s$. In this case all financial risk transfers to the policy-holder. Opposed to this, an "unit-linked with guarantee" contract is equipped with a guarantee that assures the policy-holder a minimum amount if the reference portfolio falls below a certain level at due date. For a prefixed guarantee $k > 0$ the payoff

has then the form $g(s) = \max(s, k)$. For some insurance products it is also imaginable having a guarantee that is not fixed. For example it could be the case that the guarantee depends on time and is $k(t) = k \exp\left(\int_0^t r_u \, du\right)$ for some interest rate r_t and some prefixed amount $k > 0$. That is, the amount of guarantee increases by earning interest over the insurance period.

Of course there are many other possible choices for the payoff, whereas from a practical point of view one is restricted by being able to evaluate the associated price process $F(t, S_t, u)$ (cf. Chapter 3). For instance, the function $g(s) = \max(s, k)$ is the sum of an European call option $(s - k)^+$ and a constant k for which price process evaluation techniques exist.

In this chapter we consider the insurance portfolio model that we introduced in Subsection 2.4.2 and assume that to each of the l_x individuals an unit-linked insurance contract is issued at time 0 against a single premium at issuing date. Those unit-linked insurance contracts are supposed to be either of pure endowment or of term insurance type and hence specify benefit payments being linked to the development of the financial market and contingent on the remaining lifetime of the policyholder.

4.2.1 Unit-linked pure endowment

In case the issued unit-linked contracts are unit-linked pure endowment contracts we suppose the insurance benefit to be a contingent claim $g(S_T)$ for some suitable function g such that $g(S_T) \in \mathrm{L}^2(\mathbb{Q})$. Recall that with a pure endowment the sum insured is to be paid at the end of the term T if the insured person is still alive. Therefore the present value of each insured person's insurance benefit is given by

$$H_i = \mathbb{1}(T_i > T) B_T^{-1} g(S_T), \quad \text{which is} \quad \mathcal{F}_T\text{-measurable}$$

and at the same time the individual obligation of the insurance company at time 0. Therefore the entire portfolio generates the discounted claim

$$H = B_T^{-1} g(S_T) \sum_{i=1}^{l_x} \mathbb{1}(T_i > T) = B_T^{-1} g(S_T)(l_x - N_T^I), \tag{4.1}$$

where $(l_x - N_T^I)$ represents the number of insured persons surviving the insurance period.

4.2.2 Unit-linked term insurance

Now let us assume that the issued contracts are unit-linked term insurance contracts. A term insurance is in some sense the opposite of a pure endowment. Its benefit is due immediately upon death before time T. Hence, payments can occur at any time during $[0, T]$ and the contract function $g(t, S_t)$ is allowed to be time-dependent. Such claims do not fit into our framework

of hedging contingent claims (cf. Subsection 2.5.2), since they are generally due before time T. As in Møller (1998) we therefore simply assume that all payments are deferred to the term of the contract and are accumulated with the risk-free rate of interest r_t. Hence if $T_i < T$ the insurance company has to pay $B_T B_{T_i}^{-1} g(T_i, S_{T_i})$ at time T, and this is a contingent claim fitting in our framework, provided $\sup_{u\in[0,T]} \mathbb{E}^{\mathbb{Q}}\left[(B_u^{-1} g(u, S_u))^2\right] < \infty$. The present value of the insurer's liabilities generated by l_x term insurances with deferred and accumulated payments is then given by

$$\begin{aligned} H &= B_T^{-1} \sum_{i=1}^{l_x} B_T B_{T_i}^{-1} g(T_i, S_{T_i})\, \mathbb{1}(T_i \le T) \\ &= \sum_{i=1}^{l_x} \int_0^T B_u^{-1} g(u, S_u)\, d\,\mathbb{1}(T_i \le u) \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

which is of course equal to

$$H = \int_0^T B_u^{-1} g(u, S_u)\, dN_u^I.$$

4.3 Risk-minimizing hedging strategy

The hedging strategies that we derive in this section are risk-minimizing with respect to the risk-neutral measure $\mathbb{Q}$. Whereas for a hedger, it is more important to know the risk in terms of his subjective beliefs represented by the historical, subjective measure $\mathbb{P}$. Since we work under the Föllmer-Schweizer measure the risk-minimizing strategies under $\mathbb{Q}$ are locally risk-minimizing with respect to $\mathbb{P}$. Compare the related discussion in Subsection 2.5.3.

4.3.1 Unit-linked pure endowment

Consider the claim developed in (4.1):

$$H = B_T^{-1} g(S_T)(l_x - N_T^I).$$

The intrinsic-value process $(V_t^*)_{0\le t\le T}$ of H is given by

$$V_t^* = \mathbb{E}^{\mathbb{Q}}[H \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[(l_x - N_T^I) \mid \mathcal{H}_t] B_t^{-1}\, \mathbb{E}^{\mathbb{Q}}[B_t B_T^{-1} g(S_T) \mid \mathcal{G}_t],$$

applying the independence of the financial market and the insurance portfolio. The first factor of V_t^* is just the conditional expected number of insured persons surviving until time T given the number of insured persons at time t. This is a binomial experiment with survival probability as success probability. Hence

$$\mathbb{E}^{\mathbb{Q}}[(l_x - N_T^I) \mid \mathcal{F}_t] = (l_x - N_t^I)_{T-t}p_{x+t}.$$

The second factor is exactly the arbitrage-free price process for $u = T$ (cf. Definition 3.2.1). Hence

$$V_t^* = (l_x - N_t^I)_{T-t} p_{x+t} B_t^{-1} F(t, S_t). \tag{4.2}$$

Since the contract function $g(S_T)$ is time-independent, in what follows we omit the "third" argument u of $F(t, S_t, u)$ since it is constant and equal to T. V^* may be interpreted as the market value associated with the entire portfolio of pure endowments and further, in accordance with the actuarial principle of equivalence, the initial value $V_0^* = l_{xT} p_x F(0, S_0)$ is a natural candidate for the single premium of the entire portfolio (cf. Møller (1998)). However, both values are considered under the risk-neutral pricing rule $\mathbb{Q}$. Hence, the process V^* could be seen as the prospective reserve of the insurance portfolio under the measure $\mathbb{Q}$.

To find the risk-minimizing hedging strategy we need to develop the Galtchouk-Kunita-Watanabe decomposition of V^* (cf. Subsection 2.5.2). Applying Itô's integration by parts (cf. Protter (2004), Chapter II, Theorem 22, Corollary 2) to (4.2) yields

$$\begin{aligned} V_t^* &= \int_0^t B_s^{-1} F(s, S_{s-})\, d\big((l_x - N_s^I)_{T-s} p_{x+s}\big) \\ &\quad + \int_0^t (l_x - N_{s-}^I)_{T-s} p_{x+s}\, d\big(B_s^{-1} F(s, S_s)\big) \\ &\quad + \big[B_\cdot^{-1} F(\cdot, S_\cdot), (l_x - N_\cdot^I)_{T-\cdot} p_{x+\cdot}\big]_t. \end{aligned}$$

For the quadratic covariation one has:

$$\begin{aligned} &\big[B_\cdot^{-1} F(\cdot, S_\cdot), (l_x - N_\cdot^I)_{T-\cdot} p_{x+\cdot}\big]_t \\ &= V_0^* + \big[B_\cdot^{-1} F(\cdot, S_\cdot), (l_x - N_\cdot^I)_{T-\cdot} p_{x+\cdot}\big]_t^c \\ &\quad + \sum_{0<s\le t} \Delta\big(B_s^{-1} F(s, S_s)\big) \Delta\big((l_x - N_s^I)_{T-s} p_{x+s}\big) \end{aligned}$$

Obviously, $(l_x - N_t^I)_{T-t} p_{x+t}$ is a process of finite variation, hence it is a quadratic pure-jump process and the continuous part of the quadratic covariation is identically 0. Further for fixed $0 \le s \le T$

$$\mathbb{Q}\Big(\big|\Delta\big(B_s^{-1} F(s, S_s)\big) \Delta\big((l_x - N_s^I)_{T-s} p_{x+s}\big)\big| > 0 \Big) = 0, \tag{4.3}$$

since at least $B_s^{-1} F(s, S_s)$ is continuous in probability and the processes (i.e. S and N^I) are independent by assumption. So the sum over the jump terms vanishes $\mathbb{Q}$-a.s., because we face at most countably many jump discontinuities in considering only càdlàg processes. The product rule (for Lebesgue-Stieltjes integrals) yields

$$\begin{aligned} d\big((l_x - N_t^I)_{T-t} p_{x+t}\big) &= (l_x - N_t^I) d({}_{T-t}p_{x+t}) + {}_{T-t}p_{x+t} d(l_x - N_t^I) \\ &= (l_x - N_t^I)_{T-t} p_{x+t} \mu_{x+t}\, dt - {}_{T-t}p_{x+t} dN_t^I \\ &= -({}_{T-t}p_{x+t}\, dM_t^I). \end{aligned}$$

Recall the expression for ${}_tp_x$ in (2.22). Putting things together, we get

$$\begin{aligned} V_t^* = V_0^* &+ \int_0^t (l_x - N^I_{s-})_{T-s}p_{x+s}\, d\big(B_s^{-1}F(s,S_s)\big) \\ &+ \int_0^t -B_s^{-1}F(s,S_{s-})_{T-s}p_{x+s}\, dM_s^I, \quad \mathbb{Q}\text{-a.s.} \end{aligned} \tag{4.4}$$

Although our stock process is not continuous, hence Itô's formula is quite different from the case of a continuous stock process, the integral with respect to M^I is exactly the same as obtained by Møller (1998) and we can use Theorem 3.3.4 to get:

Corollary 4.3.1. *For the pure endowment in (4.1) the process V^* has the following Galtchouk-Kunita-Watanabe decomposition:*

$$V_t^* = V_0^* + \int_0^t (l_x - N^I_{s-})_{T-s}p_{x+s}\xi_s\, d\hat{S}_s + K_t^H, \quad 0 \le t \le T,$$

$$\begin{aligned} \textit{where} \quad K_t^H = &\int_0^t (l_x - N^I_{s-})_{T-s}p_{x+s}\zeta_s^{(1)}\, dW_s^{\mathbb{Q}} \\ &+ \int_0^t\int_{\mathbb{R}} (l_x - N^I_{s-})_{T-s}p_{x+s}\zeta_s^{(2)}(y)\, M^{\mathbb{Q}}(ds,dy) \\ &+ \int_0^t -B_s^{-1}F(s,S_{s-})_{T-s}p_{x+s}\, dM_s^I \end{aligned}$$

is orthogonal to $\hat{S}$ and ξ, $\zeta^{(1)}$ and $\zeta^{(2)}$ are taken from Theorem 3.3.4.

Proof. The decomposition of V^* follows from (4.4) and Theorem 3.3.4. Note that it is indeed the desired Galtchouk-Kunita-Watanabe decomposition. The integrals with respect to $\hat{S}$, $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}(dt,dy)$ are still square-integrable, since $(l_x - N^I_{t-})_{T-t}p_{x+t}$ is $\mathbb{Q}$-a.s. bounded by the non-random constant l_x. Further the orthogonality between $\hat{S}$ and the integrals with respect to $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}$ is kept. For both claims compare the proof of Theorem 3.3.4, especially the relations (3.10), (3.11) and (3.12). The integral $\int_0^t B_s^{-1}F(s,S_{s-})_{T-s}p_{x+s}\, dM_s^I$ is orthogonal to $\hat{S}$ by the independence of $\hat{S}$ and M^I, i.e. the process $(cW_t^{\mathbb{Q}} + M_t^{\mathbb{Q}})M_t^I$ is clearly a $\mathbb{Q}$-martingale. It is a square-integrable martingale, since by Doob's inequality (cf.Protter (2004), Chapter I, Theorem 20) it holds that

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\Big(\int_0^t B_s^{-1}F(s,S_{s-})_{T-s}p_{x+s}\, dM_s^I\Big)^2\Big] \\ &\le 4\sup_{t\in[0,T]}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^t \Big(B_s^{-1}F(s,S_{s-})_{T-s}p_{x+s}\Big)^2 d\big\langle M^I, M^I\big\rangle_s\Big], \end{aligned}$$

and further we know that $d\langle M^I, M^I\rangle_s = \lambda_s\, ds$. Since λ_s, B_s^{-1} and ${}_{T-s}p_{x+s}$ can be bounded by a non-random constant, it remains to consider the expectation $\mathbb{E}^{\mathbb{Q}}[\int_0^t F^2(s, S_{s-})\, ds]$ for which we have that

$$\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{Q}}[\int_0^t F^2(s, S_{s-})\, ds] \leq \text{const.}\ \mathbb{E}^{\mathbb{Q}}[g^2(S_T)] < \infty.$$

This follows by Jensen's inequality and the square-integrability was presupposed (cf. (3.1)). □

Using Theorem 2.5.4 it is now easy to derive a risk-minimizing strategy for the claim of an unit linked pure endowment insurance.

Corollary 4.3.2. *The unique admissible risk-minimizing strategy φ^* (under $\mathbb{Q}$) for the pure endowment in (4.1) is for $0 \leq t \leq T$ given by*

$$\begin{aligned}\xi_t^* &= (l_x - N_{t-}^I)_{T-t}p_{x+t}\xi_t,\\ \eta_t^* &= (l_x - N_t^I)_{T-t}p_{x+t}B_t^{-1}F(t, S_t) - \xi_t^*\hat{S}_t.\end{aligned}$$

For the associated intrinsic risk process $R(\varphi^*)$ one has, for $0 \leq t \leq s \leq T$ and

$$\rho_t(s) = \mathbb{E}^{\mathbb{Q}}[{\zeta_s^{(1)}}^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\big(\zeta_s^{(2)}(y)\big)^2 \mid \mathcal{F}_t]\nu_s^{\mathbb{Q}}(dy),$$

$$\begin{aligned}R_t(\varphi^*) = (l_x - N_t^I)_{T-t}p_{x+t}\int_t^T \Big({}_{T-s}p_{x+s}(1 - {}_{s-t}p_{x+t})\\ + (l_x - N_t^I)_{T-t}p_{x+t}\Big)\rho_t(s)\, ds\\ + (l_x - N_t^I)_{T-t}p_{x+t}\int_t^T {}_{T-s}p_{x+s}\mu_{x+s}B_s^{-2}\,\mathbb{E}^{\mathbb{Q}}[F^2(s, S_s) \mid \mathcal{F}_t]\, ds.\end{aligned}$$

Proof. Only the second part needs justification. According to Theorem 2.5.4 the risk process is given by $R_t(\varphi^*) = \mathbb{E}^{\mathbb{Q}}\left[(K_T^H - K_t^H)^2 \mid \mathcal{F}_t\right]$. Using the orthogonality of the integrals with respect to $W^{\mathbb{Q}}$, $M^{\mathbb{Q}}(ds, dy)$ and M^I we get

$$\begin{aligned}R_t(\varphi^*) = \mathbb{E}^{\mathbb{Q}}[\int_t^T \Big({}_{T-s}p_{x+s}(l_x - N_s^I)\zeta_s^{(1)}\Big)^2 d\langle W^{\mathbb{Q}}, W^{\mathbb{Q}}\rangle_s \mid \mathcal{F}_t]\\ + \mathbb{E}^{\mathbb{Q}}[\int_t^T\int_{\mathbb{R}} \Big({}_{T-s}p_{x+s}(l_x - N_s^I)\zeta_s^{(2)}(y)\Big)^2 \nu_s^{\mathbb{Q}}(dy)ds \mid \mathcal{F}_t]\\ + \mathbb{E}^{\mathbb{Q}}[\int_t^T \big({}_{T-s}p_{x+s}B_s^{-1}F(s, S_{s-})\big)^2 d\langle M^I, M^I\rangle_s \mid \mathcal{F}_t].\end{aligned}$$

Independence and Tonelli's theorem yields

$$\begin{aligned}R_t(\varphi^*) = \int_t^T {}_{T-s}p_{x+s}^2\,\mathbb{E}^{\mathbb{Q}}\left[(l_x - N_s^I)^2 \mid \mathcal{F}_t\right]\rho_t(s)\, ds\\ + \mathbb{E}^{\mathbb{Q}}\left[\int_t^T {}_{T-s}p_{x+s}^2 B_s^{-2}F^2(s, S_{s-})\, d\lambda_s \mid \mathcal{F}_t\right].\end{aligned}$$

Further, $d\lambda_s = (l_x - N_s^I)\mu_{x+s}\,ds$ and given the past $\mathcal{F}_t$ we already observed that $(l_x - N_s^I)$ follows a binomial distribution for $s \in [t,T]$. So

$$\mathbb{E}^{\mathbb{Q}}\left[(l_x - N_s^I) \mid \mathcal{F}_t\right] = (l_x - N_t^I)_{s-t}p_{x+t}.$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[(l_x - N_s^I)^2 \mid \mathcal{F}_t\right] = (l_x - N_t^I)_{s-t}p_{x+t}(1 - {}_{s-t}p_{x+t}) + (l_x - N_t^I)^2{}_{s-t}p_{x+t}^2.$$

A simple computation shows ${}_{T-s}p_{x+s}{}_{s-t}p_{x+t} = {}_{T-t}p_{x+t}$. Hence

$$\begin{aligned} R_t(\varphi^*) = (l_x - N_t^I)_{T-t}p_{x+t} \int_t^T \big({}_{T-s}p_{x+s}(1 - {}_{s-t}p_{x+t}) \\ + (l_x - N_t^I)_{T-t}p_{x+t}\big)\rho_t(s)\,ds \\ + (l_x - N_t^I)_{T-t}p_{x+t} \int_t^T {}_{T-s}p_{x+s}\mu_{x+s}B_s^{-2}\,\mathbb{E}^{\mathbb{Q}}[F^2(s,S_s) \mid \mathcal{F}_t]\,ds. \square \end{aligned}$$

At this point we discuss our results in comparison to the results of Møller (1998), which are obviously a special case of ours. Møller (1998) obtained $(l_x - N_{t-}^I)_{T-t}p_{x+t}F'(t,S_t)$ as the risk-minimal investment in the stock. We get a similar result: The optimal investment in the stock ξ_t^* depends also on the expected number of insured persons surviving from $t-$ to T. However, if we allow additionally the stock process to have jumps, we invest not only the Black-Scholes Δ-hedge $F'(t,S_t)$ but

$$\begin{aligned} \xi_t &= \frac{c^2}{\kappa_t}F'(t,S_{t-}) + \frac{1}{\sigma_t \hat{S}_{t-}\kappa_t}\int_{\mathbb{R}} xJ(t,x)\nu_t^{\mathbb{Q}}(dx) \\ &= \frac{c^2}{\kappa_t}F'(t,S_{t-}) + \frac{1}{\kappa_t}\int_{\mathbb{R}} F'(t,x_0)x^2\nu_t^{\mathbb{Q}}(dx), \end{aligned}$$

for some intermediate value x_0 resulting from the mean-value theorem. The difference from a classical Black-Scholes setting is apparent. If our Lévy process was merely a Brownian motion the measure $\nu_t^{\mathbb{Q}}(dx)$ would be equal to 0 and $\kappa_t \equiv c^2$: we would have $\xi_t = F'(t,S_{t-})$ as expected. In the presence of jumps the (locally) risk-minimizing investment in the stock is nearly a weighted sum of $F'(t,S_{t-})$ and of the jump $J(t,x)$. Note that c^2t is the variance of the Brownian part $cW_t^{\mathbb{Q}}$ and that $\int_0^t \kappa_s\,ds = c^2t + \int_0^t\int_{\mathbb{R}} x^2\nu_s^{\mathbb{Q}}(dx)ds$ is the variance of L_t under $\mathbb{Q}$. Further the insurance company is able to reduce its total portfolio risk to the intrinsic risk, determined by the process K^H. Møller (1998) interprets K^H as the insurer's loss, in our case:

$$\begin{aligned} dK_t^H = (l_x - N_{t-}^I)_{T-t}p_{x+t}\zeta_t^{(1)}\,dW_t^{\mathbb{Q}} \\ + \int_{\mathbb{R}} (l_x - N_{t-}^I)_{T-t}p_{x+t}\zeta_t^{(2)}(y)\,M^{\mathbb{Q}}(dt,dy) \\ + \big(-B_t^{-1}F(t,S_{t-})_{T-t}p_{x+t}\,dM_t^I\big). \end{aligned}$$

The integral driven by $dM_t^I = dN_t^I - \lambda_t dt$ can be interpreted as the driver of pure insurance risk and is exactly the cost process obtained by Møller (1998). A death will produce an immediate gain for the insurer due to the downward adjustment of the expected number of survivors, whereas no death will cause a small loss. We interpret the remaining terms of K^H as the driver for financial risk, i.e. the integrals with respect to $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}$. Those do not appear in the complete financial market of Møller (1998). For the incomplete financial market this shows the influence on the insurer's risk of an investment in the possible discontinuous stock. It is of interest that both the discontinuous (i.e. $M^{\mathbb{Q}}$) and the continuous (i.e. $W^{\mathbb{Q}}$) part of the stock price influence K^H. Additionally, $\zeta^{(1)}$ and $\zeta^{(2)}$ depend on F' as well as on $J(t,x)$. Hence, the more general Lévy-process setting yields an increase of risk also influenced by the continuous part of the stock process, as one might perhaps not expect. However, if we considered an incomplete financial market with a continuous price process, there would be also an additional term for financial risk. Note that the financial risk driver is adjusted according to the expected number of insured persons surviving from $t-$ to T in the sense that a death results in a downward adjustment of the loss.

Let us consider the initial intrinsic risk R_0 more detailed.

$$\begin{aligned} R_0(\varphi^*) = l_{xT} p_x \int_0^T \big({}_{T-s}p_{x+s}(1 - {}_s p_x) + l_{xT} p_x\big)\rho_0(s)\, ds \\ + l_{xT} p_x \int_0^T {}_{T-s}p_{x+s}\mu_{x+s} B_s^{-2}\, \mathbb{E}^{\mathbb{Q}}[F^2(s, S_s)]\, ds. \end{aligned}$$

Møller (1998) suggests taking the initial intrinsic risk R_0 as a measure associated with the non-hedgeable part of the claims and he examines the ratio $\sqrt{R_0}/l_x$. This is in analogy to the classical actuarial theory, where the relative portfolio risk decreases as the size l_x of the portfolio increases (cf. Subsection 2.4.4). By classical theory we mean the case of deterministic premiums and deterministic insurance benefits. The difference of the classical theory from our setting is that the unit-linked contracts are all linked to the same asset and hence are not independent. Considering consequentially the same ratio in our case, we see for the insurance risk part of R_0 that one obtains, in accordance with Møller (1998),

$$\frac{1}{l_x^2}\left(l_{xT} p_x \int_0^T {}_{T-s}p_{x+s}\mu_{x+s} B_s^{-2}\, \mathbb{E}^{\mathbb{Q}}[F^2(s, S_s)]\, ds\right) \to 0 \text{ as } l_x \to \infty.$$

The financial risk part of R_0,

$$l_{xT} p_x \int_0^T ({}_{T-s}p_{x+s}(1 - {}_s p_x) + l_{xT} p_x)\rho_0(s) ds,$$

when divided by l_x^2 apparently does not tend to 0 if l_x is increased. This only holds true for the first part of it. However, the second part of the ratio

is independent of l_x. This means there is some risk originating from trading in the market that can not be eliminated by an increase of the number of insured individuals. In Møller (1998) this effect is not present. Of course, the same results are obtained at any time $t \in [0,T]$ if the ratio $\sqrt{R_t}/(l_x - N_t^I)$ is considered instead. Finally let us consider examples for the same choice of a contract function as in Møller (1998), to see how his results are affected.

Example 4.3.3. We take $g(s) = s$. Thus we consider a pure-unit linked insurance promising the insured person, if then still alive, the value of the stock at maturity date. Not surprisingly, this yields $F(t,S_t) = \mathbb{E}^{\mathbb{Q}}[B_t B_T^{-1} S_T \mid \mathcal{F}_t] = S_t$. Hence, the intrinsic value process of the contract is $V_t^* = (l_x - N_t^I)_{T-t}p_{x+t}\hat{S}_t$ and in particular a fair premium could be $V_0^* = l_{xT}p_x S_0$. A simple calculation of the variables of Theorem 3.3.4 yields

$$\xi \equiv 1, \qquad \zeta^{(1)} \equiv \zeta^{(2)} \equiv 0.$$

Observe that $\zeta^{(1)}$ and $\zeta^{(2)}$ influence the financial risk part of the company's intrinsic risk. In case of a pure-unit linked contract the insurance company does not bear the investment risk, hence it is clear that they have to be 0. It is also quite obvious that the best hedge for this kind of contract is to invest the whole capital in the stock. This is reflected by ξ. Corollary 4.3.2 gives the risk-minimizing hedging strategy

$$(\xi_t^*, \eta_t^*) = \left((l_x - N_{t-}^I)_{T-t}p_{x+t}, -\Delta N_t^I \, {}_{T-t}p_{x+t}\hat{S}_t\right)$$

and the intrinsic risk

$$R_t(\varphi^*) = (l_x - N_t^I)_{T-t}p_{x+t} \int_t^T {}_{T-s}p_{x+s}\mu_{x+s}\, \mathbb{E}^{\mathbb{Q}}[\hat{S}_s^2 \mid F_t]\, ds.$$

At any time t the insurance company holds exactly the expected number of stocks to be cashed out at time T and each death adjusts the reserve V^* by the amount $-\Delta N_t^I \, {}_{T-t}p_{x+t}\hat{S}_t$.

Example 4.3.4. We take $g(s) = k$, $k > 0$. This choice of g represents the case of deterministic benefits. Then $F(t,S_t) = B_t B_T^{-1} k$ and $V_t^* = (l_x - N_t^I)_{T-t}p_{x+t}B_T^{-1}k$. Further

$$\xi \equiv \zeta^{(1)} \equiv \zeta^{(2)} \equiv 0.$$

Hence,

$$(\xi_t^*, \eta_t^*) = \left(0, (l_x - N_t^I)_{T-t}p_{x+t}kB_T^{-1}\right).$$

The intrinsic risk can be computed explicitly, as

$$R_t(\varphi^*) = (l_x - N_t^I)_{T-t}p_{x+t}(1 - {}_{T-t}p_{x+t})k^2 B_T^{-2}.$$

The previous two examples do not involve financial risk for the insurer. Therefore the obtained hedging strategies and the intrinsic risks are exactly the same as in Møller (1998).

Example 4.3.5. The perhaps most interesting case is the choice $g(s) = \max(s, k)$ for some $k > 0$ representing an unit-linked with guarantee contract. In case of a standard Black-Scholes financial market (cf. Møller (1998)) we have an explicit expression for the price process $F(t, S_t)$, since $g(s) = k + (s - k)^+$. However this is not the case in a more general Lévy-process setting. For a concrete choice of a Lévy process one is forced to apply, for example, Monte-Carlo simulation techniques or numerical methods to solve the partial integro-differential equation in Theorem 3.2.4. In Chapter 7 we treat the computation of $F(t, S_t)$ for an asymmetric double exponential jump-diffusion process.

4.3.2 Unit-linked term insurance

To complete this chapter we include the term insurance, although we have to make the additional assumption that in case a death occurs the insurance benefit is deferred, while earning interest, and cashed out at the end of the considered time horizon only. Of course, from a practical point of view this is not satisfactory and as mentioned in the introduction we will come back to this in Chapter 6 allowing insurance contracts to have intermediate payments. In that chapter the term insurance will only be briefly mentioned as a special case and we will refer to the current section instead. Moreover, the decomposition technique that is applied here is a special case of the general proof of Theorem 6.3.1 in Chapter 6. Recall that the present value of the insurer's liabilities for a portfolio of unit-linked term insurances is given by

$$H = \int_0^T B_u^{-1} g(u, S_u)\, dN_u^I, \tag{4.5}$$

where

$$\sup_{u\in[0,T]} \mathbb{E}[(B_u^{-1} g(u, S_u))^2] < \infty.$$

Again we need to compute the Galtchouk-Kunita-Watanabe decomposition of the intrinsic value process $V_t^* = \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t]$. For the first steps we follow the ideas of Møller (1998) and use the measurability to see that

$$V_t^* = \int_0^t B_u^{-1} g(u, S_u)\, dN_u^I + \mathbb{E}^{\mathbb{Q}}[\int_t^T B_u^{-1} g(u, S_u)\, dN_u^I \mid \mathcal{F}_t].$$

With the compensator λ of N^I we get

$$\begin{aligned} V_t^* &= \int_0^t B_u^{-1} g(u, S_u)\, dN_u^I + \mathbb{E}^{\mathbb{Q}}[\int_t^T B_u^{-1} g(u, S_u)(l_x - N_u^I)\mu_{x+u}\, du \mid \mathcal{F}_t] \\ &= \int_0^t B_u^{-1} g(u, S_u)\, dN_u^I \\ &\quad + \int_t^T B_t^{-1} F(t, S_t, u)(l_x - N_t^I)_{u-t}p_{x+t}\mu_{x+u}\, du, \end{aligned} \tag{4.6}$$

where $F(t, S_t, u)$ is defined as in Definition 3.2.1. We compare now

$$B_t^{-1} F(t, S_t, u)(l_x - N_t^I)_{u-t}p_{x+t}$$

with equation (4.2), which is V_t^* of the pure endowment insurance, and note they are almost equal. We only have to substitute u for T. Hence, we plug decomposition (4.4) with u instead of T into (4.6) yielding

$$\begin{aligned} V_t^* = \int_0^t B_u^{-1} g(u, S_u)\, dN_u^I + \int_t^T \Big(& F(0, S_0, u) l_{xu} p_x \\ & + \int_0^t (l_x - N_{s-}^I)_{u-s}p_{x+s}\, d\big(B_s^{-1} F(s, S_s, u)\big) \\ & - \int_0^t B_s^{-1} F(s, S_{s-}, u)_{u-s}p_{x+s}\, dM_s^I \Big)\mu_{x+u}\, du. \end{aligned}$$

Using Theorem 3.3.4 we get

$$\begin{aligned} V_t^* = \int_0^t B_u^{-1} g(u, S_u)\, dN_u^I + \int_t^T \Big\{ & F(0, S_0, u) l_{xu} p_x \mu_{x+u} \\ & + \int_0^t \alpha_s^u\, dM_s^I + \int_0^t \beta_s^u\, d\hat{S}_s + \int_0^t \gamma_s^u\, dW_s^{\mathbb{Q}} \\ & + \int_0^t \int_{\mathbb{R}} \delta_s^u(y) M^{\mathbb{Q}}(ds, dy) \Big\}\, du, \end{aligned}$$

where

$$\begin{aligned} \alpha_t^u &:= -B_t^{-1} F(t, S_{t-}, u)_{u-t}p_{x+t}\mu_{x+u} \\ \beta_t^u &:= (l_x - N_{t-}^I)_{u-t}p_{x+t}\mu_{x+u}\xi_t(u) \\ \gamma_t^u &:= (l_x - N_{t-}^I)_{u-t}p_{x+t}\mu_{x+u}\zeta_t^{(1)}(u) \\ \delta_t^u(y) &:= (l_x - N_{t-}^I)_{u-t}p_{x+t}\mu_{x+u}\zeta_t^{(2)}(y, u). \end{aligned}$$

Further we change limits of integration to get

$$\begin{aligned}
V_t^* = &\int_0^t B_u^{-1} g(u, S_u)\, dN_u^I + \int_0^T \Big\{ F(0, S_0, u) l_{xu} p_x \mu_{x+u} \\
&+ \int_0^t \mathbb{1}_{\{s \le u\}}\, \alpha_s^u\, dM_s^I + \int_0^t \mathbb{1}_{\{s \le u\}}\, \beta_s^u\, d\hat{S}_s + \int_0^t \mathbb{1}_{\{s \le u\}}\, \gamma_s^u\, dW_s^{\mathbb{Q}} \\
&+ \int_0^{\mathrm{t}} \int_{\mathbb{R}} \mathbb{1}_{\{s \le u\}}\, \delta_s^u(y) M^{\mathbb{Q}}(ds, dy) \Big\}\, du \\
&- \int_0^t \Big\{ F(0, S_0, u) l_{xu} p_x \mu_{x+u} + \int_0^u \alpha_s^u\, dM_s^I \\
&+ \int_0^u \beta_s^u\, d\hat{S}_s + \int_0^u \gamma_s^u\, dW_s^{\mathbb{Q}} + \quad \int_0^u \int_{\mathbb{R}} \delta_s^u(y) M^{\mathbb{Q}}(ds, dy) \Big\}\, du.
\end{aligned}$$

The last integral (with respect to du) includes in curely brackets again the expression $B_u^{-1} F(u, S_u, u)(l_x - N_u^I)_0 p_{x+u}$ written in terms of the decomposition (4.4) for the special case $t = u$. Note that ${}_0p_{x+u} = 1$. Hence,

$$\begin{aligned}
V_t^* = &\int_0^t B_u^{-1} g(u, S_u)\, dN_u^I - \int_0^t B_u^{-1} F(u, S_u, u)(l_x - N_u^I) \mu_{x+u}\, du \\
&+ \int_0^T F(0, S_0, u) l_{xu} p_x \mu_{x+u}\, du + \int_0^T \int_0^t \mathbb{1}_{\{s \le u\}}\, \alpha_s^u\, dM_s^I\, du \\
&+ \int_0^T \int_0^t \mathbb{1}_{\{s \le u\}}\, \beta_s^u\, d\hat{S}_s\, du + \int_0^T \int_0^t \mathbb{1}_{\{s \le u\}}\, \gamma_s^u\, dW_s^{\mathbb{Q}}\, du \\
&+ \int_0^T \int_0^{\mathrm{t}} \int_{\mathbb{R}} \mathbb{1}_{\{s \le u\}}\, \delta_s^u(y) M^{\mathbb{Q}}(ds, dy)\, du.
\end{aligned}$$

We simplify this further noting that $\displaystyle\int_0^T F(0, S_0, u) l_{xu} p_x \mu_{x+u}\, du = V_0^*$ and using $F(u, S_u, u) = g(u, S_u)$. Thus,

$$\begin{aligned}
V_t^* = V_0^* &+ \int_0^t B_s^{-1} g(s, S_s)\, dM_s^I + \int_0^T \int_0^t \mathbb{1}_{\{s \le u\}}\, \alpha_s^u\, dM_s^I\, du \\
&+ \int_0^T \int_0^t \mathbb{1}_{\{s \le u\}}\, \beta_s^u\, d\hat{S}_s\, du + \int_0^T \int_0^t \mathbb{1}_{\{s \le u\}}\, \gamma_s^u\, dW_s^{\mathbb{Q}}\, du \\
&+ \int_0^T \int_0^{\mathrm{t}} \int_{\mathbb{R}} \mathbb{1}_{\{s \le u\}}\, \delta_s^u(y) M^{\mathbb{Q}}(ds, dy)\, du.
\end{aligned}$$

The function $(\omega, t, u) \mapsto \alpha_t^u(\omega)$ is now $\mathcal{O} \otimes \mathcal{B}([0, T])$-measurable and by (3.1) we have

$$\int_0^T \int_0^t \mathbb{1}_{\{s \le u\}}\, |\alpha_s^u|\, dM_s^I\, du < \infty \quad \mathbb{Q}\text{-a.s.}$$

Since M^I is of finite variation we apply the standard Fubini theorem and obtain that

$$\begin{aligned}&\int_0^t B_s^{-1} g(s,S_s)\,dM_s^I + \int_0^T\int_0^t \mathbb{1}_{\{s\le u\}}\,\alpha_s^u\,dM_s^I\,du\\ &= \int_0^t \Big(B_s^{-1} g(s,S_s) + \int_s^T \alpha_s^u\,du\Big)\,dM_s^I \quad \mathbb{Q}\text{-a.s.}\end{aligned}$$

The function $(\omega,t,u)\mapsto \beta_t^u(\omega)$ is $\mathcal{P}\otimes\mathcal{B}([0,T])$-measurable and uniformly bounded by a non-random constant (cf. (3.10) in the proof of Theorem 3.3.4). Fubini's theorem for stochastic integrals (cf. Protter (2004), Chapter IV, Theorem 64) yields then:

$$\int_0^T\int_0^t \mathbb{1}_{\{s\le u\}}\,\beta_s^u\,d\hat{S}_s\,du = \int_0^t \Big(\int_s^T \beta_s^u\,du\Big)\,d\hat{S}_s.$$

Also $(\omega,t,u)\mapsto \gamma_t^u(\omega)$ is $\mathcal{P}\otimes\mathcal{B}([0,T])$-measurable and further there is a non-random constant $\tilde{c}_2$ such that

$$\sup_{t\in[0,T]} |\gamma_t^u| \le \tilde{c}_2 \sup_{t\in[0,T]} \hat{S}_t, \quad \forall\, u \quad \mathbb{Q}\text{-a.s.},$$

since (3.11) holds and $(l_x - N_{t-}^I)_{u-t}p_{x+t}\mu_{x+u}$ is non-randomly bounded. Hence for all $0\le s\le T$ we have that

$$\Big(\int_s^T (\gamma_s^u)^2\,du\Big)^{\frac{1}{2}} \le \tilde{\tilde{c}}_2 \sup_{t\in[0,T]} \hat{S}_t.$$

The square-integrability of $\hat{S}$ implies now that the integral of $\big(\int_s^T \gamma_s^u\,du\big)$ with respect to $W^{\mathbb{Q}}$ exists (cf. Protter (2004), Chapter IV, Lemma before Theorem 28). Therefore we are allowed to apply the second version of the Fubini Theorem for stochastic integrals (cf. Protter (2004), Chapter IV, Theorem 65) yielding

$$\int_0^T\int_0^t \mathbb{1}_{\{s\le u\}}\,\gamma_s^u\,dW_s^{\mathbb{Q}}\,du = \int_0^t \Big(\int_s^T \gamma_s^u\,du\Big)\,dW_s^{\mathbb{Q}}.$$

Analogously, the function $(\omega,t,u,y)\mapsto \delta_t^u(y)(\omega)$ is $\mathcal{O}\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R})$-measurable, and by (3.12) we have that

$$\sup_{t\in[0,T]} |\delta_t^u(y)| \le \tilde{c}_3\,|y| \sup_{t\in[0,T]} \hat{S}_t, \quad \forall\, y,u \quad \mathbb{Q}\text{-a.s.},$$

for a non-random constant $\tilde{c}_3$. This implies for all $0\le s\le T$ that

$$\Big(\int_s^T (\delta_s^u(y))^2\,du\Big)^{\frac{1}{2}} \le \tilde{\tilde{c}}_3\,|y| \sup_{t\in[0,T]} \hat{S}_t.$$

Hence,

$$\mathbb{E}^{\mathbb{Q}}[\Big(\int_0^T\int_{\mathbb{R}}\Big(\int_s^T \delta_s^u(y)\,du\Big)M^{\mathbb{Q}}(ds,dy)\Big)^2] < \infty$$

by the square-integrability of $\hat{S}$ (cf. proof of Theorem 3.3.4). So we may apply again Fubini's theorem for stochastic integrals:

$$\begin{aligned}&\int_0^T\int_0^{\mathrm{t}}\int_{\mathbb{R}} \mathbb{1}_{\{s\le u\}}\,\delta_s^u(y)M^{\mathbb{Q}}(ds,dy)\,du\\ &=\int_0^{\mathrm{t}}\int_{\mathbb{R}}\Big(\int_s^T \delta_s^u(y)\,du\Big)M^{\mathbb{Q}}(ds,dy)\quad \mathbb{Q}\text{-a.s.}\end{aligned}$$

V^* admits therefore $\mathbb{Q}$-a.s. for $0\le t\le T$ the following decomposition.

$$\begin{aligned}V_t^* = V_0^* &+ \int_0^t\Big(\int_s^T \beta_s^u\,du\Big)\,d\hat{S}_s + \int_0^t\Big(\int_s^T \gamma_s^u\,du\Big)\,dW_s^{\mathbb{Q}}\\ &+\int_0^{\mathrm{t}}\int_{\mathbb{R}}\Big(\int_s^T \delta_s^u(y)\,du\Big)M^{\mathbb{Q}}(ds,dy)\\ &+\int_0^t\Big(B_s^{-1}g(s,S_s)+\int_s^T \alpha_s^u\,du\Big)\,dM_s^I.\end{aligned}$$

Corollary 4.3.6. *For the term insurance in (4.5) the process V^* has the following Galtchouk-Kunita-Watanabe decomposition:*

$$V_t^* = V_0^* + \int_0^t (l_x - N_{s-}^I)\xi_s\,d\hat{S}_s + K_t^H,\quad 0\le t\le T,$$

where $\xi_t = \displaystyle\int_t^T {}_{u-t}p_{x+t}\mu_{x+u}\xi_t(u)\,du$, *and*

$$\begin{aligned}K_t^H = &\int_0^t (l_x - N_{s-}^I)\zeta_s^{(1)}\,dW_s^{\mathbb{Q}} + \int_0^{\mathrm{t}}\int_{\mathbb{R}}(l_x - N_{s-}^I)\zeta_s^{(2)}(y)\,M^{\mathbb{Q}}(ds,dy)\\ &+\int_0^t \vartheta_s\,dM_s^I\end{aligned}$$

is orthogonal to $\hat{S}$ with

$$\begin{aligned}\zeta_t^{(1)} &= \int_t^T {}_{u-t}p_{x+t}\mu_{x+u}\zeta_t^{(1)}(u)\,du,\\ \zeta_t^{(2)}(y) &= \int_t^T {}_{u-t}p_{x+t}\mu_{x+u}\zeta_t^{(2)}(y,u)\,du,\\ \vartheta_t &= B_t^{-1}g(t,S_t) - \int_t^T B_t^{-1}F(t,S_{t-},u)_{u-t}p_{x+t}\mu_{x+u}\,du\end{aligned}$$

$\xi(u)$, $\zeta^{(1)}(u)$ and $\zeta^{(2)}(y,u)$ are taken from Theorem 3.3.4.

Proof. To prove that the derived decomposition is indeed the Galtchouk-Kunita-Watanabe decomposition one follows the same lines as in the proof of Corollary 4.3.1. The boundedness and orthogonality results carry over to the term insurance without difficulties. □

Corollary 4.3.7. *The unique admissible risk-minimizing strategy φ^* (under $\mathbb{Q}$) for the term insurance in (4.5) is for $0 \le t \le T$ given by*

$$\begin{aligned}\xi_t^* &= (l_x - N_{t-}^I) \int_t^T {}_{u-t}p_{x+t}\mu_{x+u}\xi_t(u)\,du,\\ \eta_t^* &= \int_0^t B_u^{-1} g(u,S_u)\,dN_u^I + (l_x - N_t^I)\int_t^T B_t^{-1}F(t,S_t,u)\,{}_{u-t}p_{x+t}\mu_{x+u}\,du\\ &\quad - \xi_t^*\hat{S}_t.\end{aligned}$$

For the associated intrinsic risk process $R(\varphi^*)$ one has, for $0 \le t \le s \le T$ and

$$\rho_t(s) = \mathbb{E}^{\mathbb{Q}}[{\zeta_s^{(1)}}^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\big(\zeta_s^{(2)}(y)\big)^2 \mid \mathcal{F}_t]\nu_s^{\mathbb{Q}}(dy),$$

that

$$\begin{aligned}R_t(\varphi^*) &= (l_x - N_t^I)\int_t^T {}_{s-t}p_{x+t}\big(1 - {}_{s-t}p_{x+t} + (l_x - N_t^I)\,{}_{s-t}p_{x+t}\big)\rho_t(s)\,ds\\ &\quad + (l_x - N_t^I)\int_t^T \mathbb{E}^{\mathbb{Q}}[\vartheta_s^2 \mid \mathcal{F}_t]\,{}_{s-t}p_{x+t}\mu_{x+s}\,ds.\end{aligned}$$

Proof. As for the pure endowment this is just an application of Theorem 2.5.4. The risk process is derived analogously. □

We only briefly comment on these results, since the same reasoning as for the pure endowment applies. The strategy φ^* in this case is chosen such that the portfolio value

$$\hat{V}_t(\varphi^*) = V_t^* = \int_0^t B_u^{-1} g(u,S_u)\,dN_u^I + \mathbb{E}^{\mathbb{Q}}[\int_t^T B_u^{-1} g(u,S_u)\,dN_u^I \mid \mathcal{F}_t],$$

that is, at any time t it is equal to the sum of payments for deaths already occurred and expected future payments of possible future deaths. Compared to the pure endowment, ξ_t^* is additionally integrated over all possible times a death could occur from time t on. Other than that it posses the same properties as the risk-minimal stock investment of the pure endowment. The insurer reduces again his total risk to the intrinsic risk driven by the insurer's loss,

$$dK_t^H = (l_x - N_{t-}^I)\zeta_t^{(1)}\,dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} (l_x - N_{t-}^I)\zeta_t^{(2)}(y)\,M^{\mathbb{Q}}(dt,dy) + \vartheta_t\,dM_t^I,$$

where the integrals with respect to $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}$ represent again the financial risk driver. Additionally, as for ξ^* and in contrast to the pure endowment insurance, $\zeta^{(1)}(u)$ and $\zeta^{(2)}(y,u)$ are integrated over all possible times a death could occur. Moreover, each time a death occurs the investment in the stock and hence the financial risk is reduced according to the number of insured persons. The integral with respect to M_t^I is the pure insurance risk driver and is equal to the cost process obtained by Møller (1998). Considering the process ϑ_t we constitute that a death results in the insurer setting aside the payment $B_t^{-1}g(t,S_t)$ to be cashed out at T and in reducing simultaneously his reservers by the amount $\int_t^T B_t^{-1}F(t,S_{t-},u)_{u-t}p_{x+t}\mu_{x+u}\,du$. Of course, the examination of the ratio $R_t/(l_x - N_t^I)$, $t \in [0,T]$, yields the same result as for the pure endowment insurance: the insurance risk can be reduced by raising l_x, whereas the financial risk is independent of l_x.

4.4 Why use local risk-minimization?

One might ask why we use locally risk-minimizing hedging in the context of insurance claims instead of some other hedging theory.

Insurance claims may arise arbitrarily in the interval $[0,T]$ therefore it is quite reasonable to apply local risk-minimization for insurance claim hedging, since it guarantees the optimal portfolio value $V_t(\varphi^*)$ being exactly equal to the intrinsic value V_t^* at any time $0 \le t \le T$. This is the best approximate we have on the insurance claim at time t corresponding simultaneously to the prospective reserve of the insurance claim under $\mathbb{Q}$. For other hedging approaches this equality is usually not true in the open interval $(0,T)$ and is only required at time T, when some contingent claim is due. Therefore the solvency of the insurer is enhanced with local risk-minimization compared to other hedging approaches. For example consider a term insurance, where a death and hence a claim can occur at any random time during the interval $[0,T]$ and not only at T. Moreover, in case of adverse development within the insurance portfolio, a difference of the optimal portfolio value from the intrinsic value could be relevant to the solvency as well (see also Møller (1998)).

Let us briefly describe mean-variance hedging as one possible alternative belonging, as well as local risk-minimization, to the theory of quadratic hedging approaches. Here an initial investment and a hedging strategy is chosen such that it is self-financing on the interval $[0,T)$ and such that the variance between the portfolio value at time T and a contingent claim (initial intrinsic risk) is minimal under the historical measure, which is not an easy task. In particular, finding the so-called variance optimal martingale measure needs a lot of technical machinery and seems to be quite advanced for models with jumps (cf. Cont and Tankov (2004), Chapter 10.4). For the special case having the original measure already as martingale mea-

sure for $\hat{S}$, the ξ-component is identical for the mean-variance and the local risk-minimizing hedging strategy. However, the η-components differ by the self-financing property of mean-variance hedging strategies. The major disadvantage for our purposes is that during the insurance period for a mean-variance hedging strategy the portfolio value V_t will in general not equal the conditional expected present value of the claim V_t^*. It just minimizes the initial intrinsic risk.

Chapter 5

Payment stream hedging for semimartingales

5.1 Introduction

In the previous chapter we did not have a locally risk-minimizing hedging theory for insurance payment streams forcing us to defer payments to the end of the considered time horizon. Therefore in this chapter we justify that the risk-minimizing hedging theory for payment streams of Møller (2001a) can be extended to locally risk-minimizing hedging of payment streams. The starting point for all those considerations is the fundamental hedging theory of Föllmer and Sondermann (1986) for incomplete financial markets. Considering the stock price being already a martingale under the historical measure, Föllmer and Sondermann (1986) developed a risk-minimizing hedging theory for square-integrable contingent claims with fixed maturity. Later, generalizing this theory to semimartingale financial markets, Schweizer (1991) introduced the notion of local risk-minimization. Still considering the martingale financial market, Møller (2001a) enlarged the theory of Föllmer and Sondermann (1986) to risk-minimizing hedging of general payment streams, where the hedger's liabilities are given by a square-integrable payment process and not, as before, by a contingent claim with fixed maturity. Such payment processes occur, for instance, with unit-linked life insurances admitting both premium and benefit payments to occur at any time in the considered time interval. Møller (2001a) explicitly poses the open question of locally risk-minimizing hedging of payment streams. This includes, in particular, the choice of the equivalent risk-neutral martingale measure such that the hedging risk is interpretable under the historical measure. In this chapter we are now able to answer this question in treating the remaining semimartingale case. The chapter is self-contained and to a large extend independent from the previous chapters of this thesis. Moreover, it is the basis for one part of Riesner (2005).

5.2 The model

Aligning our setup to Schweizer (1991), we investigate local risk-minimization to be applicable to general payment streams. Throughout the chapter we work on some filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness. Especially $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathbb{F}$. $T \in \mathbb{R}$ is fixed and denotes the finite time horizon. Further we consider a financial market with time horizon T consisting of one risky asset (stock) with discounted price process X and a riskless investment alternative (bond) assuming its value to be constant and equal to one. $X = (X_t)_{0\leq t\leq T}$ is defined to be a semimartingale with decomposition

$$X = X_0 + M + A,$$

such that the following assumptions (cf. Schweizer (1991)) hold:

$(X1)$ $M = (M_t)_{0\leq t\leq T}$ is a square-integrable martingale with $M_0 = 0$ and $A = (A_t)_{0\leq t\leq T}$ is a predictable process of finite variation $|A|$ with $A_0 = 0$.

$(X2)$ For $\mathbb{P}$ almost all ω, the measure induced by $\langle M\rangle_\cdot(\omega)$ has the whole interval $[0, T]$ as its support.

$(X3)$ A is continuous.

$(X4)$ A is absolutely continuous with respect to $\langle M\rangle$ with a density α satisfying $\mathbb{E}_M[|\alpha| \log^+ |\alpha|] < \infty$.

$(X5)$ X is continuous at T $\mathbb{P}$-a.s.

Assumption $(X1)$ implies the existence of the predictable variation process $\langle M\rangle$ of M with respect to $\mathbb{P}$. $\mathbb{P}_M$ denotes the measure $\mathbb{P} \times \langle M\rangle$ on the product space $\bar{\Omega} := \Omega \times [0, T]$ with the σ-algebra $\mathcal{P}$ of predictable sets.

Definition 5.2.1. *A trading strategy $\varphi = (\xi, \eta)$ is a process consisting of the predictable process $\xi = (\xi_t)_{0\leq t\leq T}$ such that*

$$\mathbb{E}[\int_0^T \xi_u^2 \, d\langle M\rangle_u + \Big(\int_0^T |\xi_u| \, d\,|A|_u\Big)^2] < \infty, \tag{5.1}$$

i.e. $\xi \in \mathrm{L}^2(\mathbb{P}_M)$ and $\int_0^T |\xi_u| \, d\,|A|_u \in \mathrm{L}^2(\mathbb{P})$, and consisting of the adapted process $\eta = (\eta_t)_{0\leq t\leq T}$ such that the portfolio value $V_t(\varphi) := \xi_t X_t + \eta_t$ is càdlàg and satisfies $V_t(\varphi) \in \mathrm{L}^2(\mathbb{P})$, $0 \leq t \leq T$.

As usual, at time t, ξ_t denotes the amount of shares held in the portfolio and η_t describes the investment in the bond. The predictability assumption on ξ reflects the fact that we have to make the investment decision before actually knowing the value of the stock at some specific time.

Definition 5.2.2. *A payment stream or process is an adapted, càdlàg square-integrable process $Y = (Y_t)_{0\le t\le T}$ and the cost process $C(\varphi)$ is defined by*

$$C_t(\varphi) := V_t(\varphi) - \int_0^t \xi_u \, dX_u + Y_t, \quad 0 \le t \le T,$$

where ξ satisfies the conditions of Definition 5.2.1.

The payment process Y may be interpreted as a hedger's discounted liabilities towards a buyer of the contract to be hedged, that is, it represents contractual payments. More specifically, from the hedger's point of view, $Y_t - Y_s$, $0 \le s < t \le T$, is taken to be the total discounted outgoings less income during the interval $(s, t]$. If Y is negative, the hedger receives money from the buyer and if it is positive the buyer receives contractual payments from the hedger. The introduction of such a process to describe more general payment streams instead of contingent claims (i.e. $H \in \mathrm{L}^2$, $\mathcal{F}_T$-adapted and describing a payment at time T) is due to Møller (2001a). Note that the payment process $Y_t = \mathbb{1}_{\{t\ge T\}} H$, $0 \le t \le T$, is a contingent claim for an appropriate choice of H. Furthermore, the cost process $C(\varphi)$, being clearly càdlàg and square-integrable, represents the total costs of the hedger that are generated by contractual payments and additional investments in the portfolio being not covered by trading gains. Especially the initial costs $C_0(\varphi) = V_0(\varphi) + Y_0$ reflect the sum of the initial investment and the first contractual payment. This immediately delivers an interpretation of the portfolio value $V_t(\varphi)$: It is the value of the portfolio held at time t after the payments Y_t have been made. Since we are primarily concerned to hedge the payment process Y, it is apparent to concentrate on so called 0-admissible strategies φ, i.e. $V_T(\varphi) = 0$ $\mathbb{P}$-a.s. In other words, $V_T(\varphi)$ is the value of the portfolio after settling all liabilities. Such a strategy always exists. Just choose $\xi \equiv \eta \equiv 0$ resulting in $C_t(\varphi) = Y_t$, $0 \le t \le T$. Probably this is not the best way of dealing with the considered problem of hedging the payment process Y and so, as Møller (2001a), we additionally consider the following risk process:

$$R_t(\varphi) = \mathbb{E}[(C_T(\varphi) - C_t(\varphi))^2 \mid \mathcal{F}_t], \quad 0 \le t \le T,$$

given by its càdlàg version. Møller (2001a) calls a strategy Y-self-financing if $C_t(\varphi) \equiv C_0(\varphi)$ $\mathbb{P}$-a.s. This is in analogy to the original framework of Föllmer and Sondermann (1986) to which Y-self-financing is only equivalent, if $Y_t \equiv Y_0$ $\mathbb{P}$-a.s. Y-self-financing means that all fluctuations of the value process are either trading gains or payments described by Y. It is possible that for a Y-self-financing strategy φ: $\mathbb{P}(V_T(\varphi) < 0 \mid V_0(\varphi) > 0) > 0$, so Møller (2001a) calls a payment process Y attainable, if there exists a strategy φ which is Y-self-financing and 0-admissible. To link the new and the classical theory we consider the following proposition, which is a combination of a

proposition in Schweizer (1991) and a proposition in Møller (2001a). As in Møller (2001a) we denote entities with a bar if they are related to the classical theory.

Proposition 5.2.3. *Let Y be a payment process. The following statements are equivalent:*

(a) $H = Y_T$ is (classically) attainable, i.e. it exists a (classically) self-financing strategy $\bar{\varphi} = (\bar{\xi}, \bar{\eta})$ such that $\bar{C}_t(\bar{\varphi}) = \bar{V}_t(\bar{\varphi}) - \int_0^t \bar{\xi}_u \, dX_u \equiv C_0(\varphi)$ $\mathbb{P}$-a.s. and $\bar{V}_T(\bar{\varphi}) = H$.

(b) There exists a 0-admissible trading strategy φ with $R_t(\varphi) = 0$, $0 \le t \le T$, $\mathbb{P}$-a.s.

(c) Y is attainable.

Proof. **(c)⇒(b):** There exists a Y-self-financing strategy φ with $V_T(\varphi) = 0$. Especially, this implies $C_t(\varphi) \equiv C_0(\varphi)$ $\mathbb{P}$-a.s.

(b)⇒(a): Since $R_t(\varphi) = 0$, we have $C_t(\varphi) = C_0(\varphi)$, $0 \le t \le T$, $\mathbb{P}$-a.s. Given φ define $\bar{\varphi} = (\bar{\xi}, \bar{\eta})$ with $\bar{\xi}_t := \xi_t$ and $\bar{\eta}_t := \eta_t + Y_t$, $0 \le t \le T$. Then on the one hand we get

$$\bar{V}_T(\bar{\varphi}) = \bar{\xi}_T X_T + \bar{\eta}_T = V_T(\varphi) + Y_T = Y_T$$

and on the other hand for $0 \le t \le T$

$$\begin{aligned} \bar{C}_t(\bar{\varphi}) &= \bar{V}_t(\bar{\varphi}) - \int_0^t \bar{\xi}_u \, dX_u \\ &= V_t(\varphi) - \int_0^t \xi_u \, dX_u + Y_t = C_t(\varphi) \equiv C_0(\varphi) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

(a)⇒(c): Define $\varphi = (\xi, \eta)$ by $\xi_t := \bar{\xi}_t$ and $\eta_t := \bar{\eta}_t - Y_t$, $0 \le t \le T$. Then $V_T(\varphi) = \xi_T X_T + \eta_T = \bar{V}_T(\bar{\varphi}) - Y_T = 0$, and also

$$\begin{aligned} C_t(\varphi) &= V_t(\varphi) - \int_0^t \xi_u \, dX_u + Y_t \\ &= \bar{V}_t(\bar{\varphi}) - \int_0^t \bar{\xi}_u \, dX_u \equiv C_0(\varphi) \quad \mathbb{P}\text{-a.s.} \quad \square \end{aligned}$$

In general not every payment process is attainable. Møller (2001a) calls thus, in analogy to Föllmer and Sondermann (1986), a trading strategy Y-mean-self-financing, if its cost process $C_t(\varphi)$ is a martingale. For hedging general payment processes in a semimartingale market, we consider now the same optimization problem as in the classical case. The following definition is taken from Schweizer (1991).

Definition 5.2.4. *Let $\varphi = (\xi, \eta)$ be a trading strategy and $t \in [0,T]$.*

(a) An admissible continuation of φ from t on is a trading strategy $\tilde{\varphi} = (\tilde{\xi}, \tilde{\eta})$ satisfying

$$\tilde{\xi}_s = \xi_s \quad \text{for } s \le t, \quad \tilde{\eta}_s = \eta_s \quad \text{for } s < t,$$

and

$$V_T(\varphi) = V_T(\tilde{\varphi}) \quad \mathbb{P}\text{-a.s.}$$

(b) An admissible variation of φ from t on is a trading strategy $\Delta = (\delta, \varepsilon)$ such that $\varphi + \Delta$ is an admissible continuation of φ from t on.

(c) φ is called risk-minimizing, if for any $t \in [0,T]$ and for any admissible continuation $\tilde{\varphi}$ of φ from t on one has

$$R_t(\tilde{\varphi}) \ge R_t(\varphi) \quad \mathbb{P}\text{-a.s.}$$

or if, equivalently,

$$R_t(\varphi + \Delta) - R_t(\varphi) \ge 0 \quad \mathbb{P}\text{-a.s.}$$

for every admissible variation Δ of φ from t on.

It is now possible to restate the following Lemma, which is for the classical case formulated in Schweizer (1991).

Lemma 5.2.5. *Let Y be a payment process, $\varphi = (\xi, \eta)$ be a 0-admissible trading strategy and $t_0 \in [0,T]$. Then there exists a trading strategy $\hat{\varphi}$ satisfying*

(a) $V_T(\hat{\varphi}) = 0$

(b) $C_t(\hat{\varphi}) = \mathbb{E}[C_T(\hat{\varphi}) \mid \mathcal{F}_t]$, $\mathbb{P}$-a.s., for $t_0 \le t \le T$.

(c) $R_t(\hat{\varphi}) \le R_t(\varphi)$, $\mathbb{P}$-a.s., for $t_0 \le t \le T$.

If we choose $t_0 := 0$, then $\hat{\varphi}$ is Y-mean-self-financing. If φ is risk-minimizing, then $C_t(\varphi)$ is a martingale.

Proof. Let $\hat{\xi} := \xi$ and

$$\begin{aligned}\hat{\eta}_t &:= \begin{cases} \eta_t, & 0 \le t < t_0, \\ \mathbb{E}[Y_T - \int_0^T \xi_u \, dX_u \mid \mathcal{F}_t] + \int_0^t \xi_u \, dX_u - Y_t - \xi_t X_t, & t_0 \le t \le T, \end{cases} \\ &= \begin{cases} \eta_t, & 0 \le t < t_0, \\ \mathbb{E}[Y_T - Y_t - \int_t^T \xi_u \, dX_u \mid \mathcal{F}_t] - \xi_t X_t, & t_0 \le t \le T. \end{cases}\end{aligned}$$

We choose the càdlàg version. This implies that

$$V_t(\hat{\varphi}) = \begin{cases} V_t(\varphi), & 0 \le t < t_0, \\ \mathbb{E}[Y_T - Y_t - \int_t^T \xi_u \, dX_u \mid \mathcal{F}_t], & t_0 \le t \le T, \end{cases}$$

is càdlàg and square-integrable. $\hat{\varphi}$ is therefore a trading strategy. Immediately we have that $V_T(\hat{\varphi}) = 0$ and $C_T(\varphi) = Y_T - \int_0^T \xi_u \, dX_u = Y_T - \int_0^T \hat{\xi}_u \, dX_u = C_T(\hat{\varphi})$. Further,

$$\begin{aligned}
\mathbb{E}[C_T(\hat{\varphi}) \mid \mathcal{F}_t] &= \mathbb{E}[C_T(\hat{\varphi}) - C_t(\hat{\varphi}) \mid \mathcal{F}_t] + C_t(\hat{\varphi}) \\
&= C_t(\hat{\varphi}) + \mathbb{E}[Y_T - \int_0^T \hat{\xi}_u \, dX_u - V_t(\hat{\varphi}) \\
&\qquad + \int_0^t \hat{\xi}_u \, dX_u - Y_t \mid \mathcal{F}_t] \\
&= C_t(\hat{\varphi}) + \mathbb{E}[Y_T - Y_t - \int_t^T \hat{\xi}_u \, dX_u \mid \mathcal{F}_t] - V_t(\hat{\varphi}) \\
&= C_t(\hat{\varphi}), \quad t_0 \le t \le T.
\end{aligned}$$

Using the established results, one deduces

$$\begin{aligned}
R_t(\hat{\varphi}) &= \mathbb{E}[(C_T(\hat{\varphi}) - C_t(\hat{\varphi}))^2 \mid \mathcal{F}_t] \\
&= \mathbb{E}[(C_T(\varphi) - C_t(\varphi) + C_t(\varphi) - C_t(\hat{\varphi}))^2 \mid \mathcal{F}_t] \\
&= R_t(\varphi) + (C_t(\varphi) - C_t(\hat{\varphi}))^2 + 2(C_t(\hat{\varphi}) - C_t(\varphi))(C_t(\varphi) - C_t(\hat{\varphi})) \\
&= R_t(\varphi) - (C_t(\varphi) - C_t(\hat{\varphi}))^2 \\
&\le R_t(\varphi), \quad t_0 \le t \le T \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

To prove that $C(\varphi)$ is a martingale if φ is risk-minimizing, we note that $\hat{\varphi}$ is an admissible continuation of φ and that by the above for $0 \le t \le T$

$$R_t(\varphi) = R_t(\hat{\varphi}) + (C_t(\varphi) - \mathbb{E}[C_T(\varphi) \mid \mathcal{F}_t])^2 \ge R_t(\hat{\varphi}) \ge R_t(\varphi).$$

Therefore, $C_t(\varphi) = \mathbb{E}[C_T(\varphi) \mid \mathcal{F}_t]$, $0 \le t \le T$. □

If X is already a square-integrable $\mathbb{P}$-martingale, the question of finding a risk-minimizing trading strategy for general payment processes is completely solved in Møller (2001a). In fact it is enough to have a locally square-integrable local martingale X. The central idea is to utilize the well-known Galtchouk-Kunita-Watanabe decomposition (cf. Theorem 2.5.3). For completeness we review the result of Møller (2001a) and include its proof. We assume for the moment that X is a square-integrable martingale. For a square-integrable process Y one gets for $0 \le t \le T$

$$V_t^* := \mathbb{E}[Y_T \mid \mathcal{F}_t] = V_0^* + \int_0^t \xi_u^Y \, dX_u + K_t^Y, \quad \mathbb{P}\text{-a.s.}, \tag{5.2}$$

where $\xi^Y \in \mathrm{L}^2(\mathbb{P}_X)$ and $K^Y = (K_t^Y)_{0 \le t \le T}$ is a square-integrable martingale orthogonal to X with $K_0^Y = 0$, $\mathbb{P}$-a.s. As usual we call two martingales orthogonal if their product is a martingale, or equivalently for square-integrable martingales if their predictable quadratic covariation is identically zero.

Proposition 5.2.6 (Møller)**.** *Assume that X is a square-integrable martingale. For every payment process Y there exists an unique 0-admissible risk-minimizing trading strategy $\varphi = (\xi, \eta)$ for Y given by*

$$(\xi_t, \eta_t) = \left(\xi_t^Y, V_t^* - Y_t - \xi_t^Y X_t\right), \quad 0 \le t \le T.$$

The associated risk-process is given by $R_t(\varphi) = \mathbb{E}[\left(K_T^Y - K_t^Y\right)^2 \mid \mathcal{F}_t]$.

Proof. First one proves existence. Using (5.2) we get

$$\begin{aligned} Y_T = V_T^* &= V_0^* + \int_0^T \xi_u^Y \, dX_u + K_T^Y \\ &= V_t^* + \int_t^T \xi_u^Y \, dX_u + \left(K_T^Y - K_t^Y\right). \end{aligned}$$

Let now $\tilde{\varphi}$ be a 0-admissible trading strategy. So, since $\tilde{\varphi}$ is 0-admissible,

$$\begin{aligned} C_T(\tilde{\varphi}) - C_t(\tilde{\varphi}) &= V_T(\tilde{\varphi}) - \int_0^T \tilde{\xi}_u \, dX_u + Y_T \\ &\quad - \left(V_t(\tilde{\varphi}) - \int_0^t \tilde{\xi}_u \, dX_u + Y_t\right) \\ &= (V_t^* - Y_t - V_t(\tilde{\varphi})) + \left(K_T^Y - K_t^Y\right) \\ &\quad + \int_t^T \left(\xi_u^Y - \tilde{\xi}_u\right) dX_u. \end{aligned} \tag{5.3}$$

Using the orthogonality of X and K and the measurability of the first term in (5.3) we get

$$\begin{aligned} R_t(\tilde{\varphi}) = \mathbb{E}[\left(K_T^Y - K_t^Y\right)^2 \mid \mathcal{F}_t] + (V_t^* - Y_t - V_t(\tilde{\varphi}))^2 \\ + \mathbb{E}[\int_t^T \left(\xi_u^Y - \tilde{\xi}_u\right)^2 d\langle X \rangle_u \mid \mathcal{F}_t]. \end{aligned}$$

Choosing now $\xi_t = \xi_t^Y$ and $\eta_t = V_t^* - Y_t - \xi_t X_t$, $0 \le t \le T$, yields that the last two terms vanish and that $\varphi = (\xi, \eta)$ is a risk-minimizing strategy, since the first term does not depend on the strategy. For the uniqueness note that if $\hat{\varphi}$ is risk-minimizing and 0-admissible, one must have $\hat{\xi} = \xi^Y$, since R_0 is minimized. Further by Lemma 5.2.5 $\hat{\varphi}$ is Y-mean-self-financing, hence $C(\hat{\varphi})$ is a martingale and so by (5.3) $V_t(\hat{\varphi}) = V_t^* - Y_t$, i.e. $\hat{\eta}_t = V_t^* - Y_t - \hat{\xi}_t X_t$, $0 \le t \le T$. □

The difference from the classical theory is that a risk-minimizing strategy φ is chosen such that at time t, $0 \le t \le T$, the value of the portfolio is equal to the conditional expected value of the hedger's balance, i.e. $V_t(\varphi) = \mathbb{E}[Y_T - Y_t \mid \mathcal{F}_t]$.

For contingent claim hedging in the semimartingale case we already mentioned in Chapter 2 that things are different. According to Schweizer (1991) it is impossible to find a risk-minimizing hedging strategy that minimizes the process $R_t(\varphi)$ uniformly over time and over all admissible continuations of a trading strategy. For a detailed counterexample we refer to Schweizer (2001). The technical reason is the finite-variation part of the semimartingale X and its influence on the process $R(\varphi)$. The Kunita-Watanabe projection technique fails in this case. Since contingent claim hedging is just a special case ($Y_t = \mathbb{1}_{\{t \geq T\}} H$) of our more general setting, the projection technique must of course fail here, too. To solve this problem Schweizer (1991) introduced locally risk-minimizing hedging strategies and showed that they are the right ones to consider in the classical semimartingale case.

5.3 Local risk-minimization

We investigate now if local risk-minimization can be extended from contingent claim hedging to hedging of general payment processes. By intuition this might be quite reasonable, however, in technical terms it is somewhat sophisticated. In particular, the influence of the payment process Y on the concept is not so obvious. For this reason we decided to go into a little bit more detail and work through the theory of Schweizer (1991) while considering the expanded set-up. The following definitions are taken from Schweizer (1991).

Definition 5.3.1. *A trading strategy $\Delta = (\delta, \varepsilon)$ is called a small perturbation if δ and $\int_0^T |\delta|_u \, d\,|A|_u$ are bounded and further $\delta_T = \varepsilon_T = 0$. For a subinterval $(s,t]$ of $[0,T]$ and a small perturbation Δ, the small perturbation*

$$\Delta|_{(s,t]} := (\delta|_{(s,t]}, \varepsilon|_{[s,t)})$$

is given by

$$\delta|_{(s,t]}(\omega, u) := \delta_u(\omega)\, \mathbb{1}_{(s,t]}(u), \quad \varepsilon|_{[s,t)}(\omega, u) := \varepsilon_u(\omega)\, \mathbb{1}_{[s,t)}(u).$$

Note that $\delta|_{(s,t]}$ is predictable and $\varepsilon|_{[s,t)}$ is adapted and obviously, for any small perturbation Δ and for any 0-admissible strategy φ it holds: Δ and $\varphi + \Delta$ are 0-admissible. The boundedness condition for a small perturbation can be interpreted as not allowing unlimited systematic trading gains. Recall therefore that A is the finite variation predictable part of X. The idea of Schweizer (1991) is to consider local variations of a trading strategy. To perform this we need a partition of the interval $[0,T]$. Let this partition be given by

$$\tau = (t_i)_{0 \leq i \leq N} \quad \text{where} \quad 0 = t_0 < t_1 < \cdots < t_N = T,$$

with mesh size $|\tau| := \max_{1 \leq i \leq N}(t_i - t_{i-1})$. A sequence of partitions $(\tau_n)_{n \in \mathbb{N}}$ is called increasing, if $\tau_n \subset \tau_{n+1}$ for all n and 0-convergent if $\lim_{n \to \infty} |\tau_n| = 0$.

Definition 5.3.2. *Given a trading strategy φ, a small perturbation Δ and a partition τ of $[0,T]$, consider the following quotient:*

$$r^{\tau}[\varphi,\Delta](\omega,t) := \sum_{t_i\in\tau} \frac{R_{t_i}(\varphi+\Delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\varphi)}{\mathbb{E}[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}|\mathcal{F}_{t_i}]}(\omega)\,\mathbb{1}_{(t_i,t_{i+1}]}(t).$$

A strategy φ is called locally risk-minimizing, if

$$\liminf_{n\to\infty} r^{\tau_n}[\varphi,\Delta] \geq 0, \quad \mathbb{P}_M\text{-}a.e.,$$

for every small perturbation Δ and every increasing 0-convergent sequence (τ_n) of partitions of $[0,T]$.

The assumption $(X2)$ avoids a division by 0 in the above definition, i.e. the measure induced by $\langle M\rangle_{\cdot}$ is not constant over some subinterval of $[0,T]$. We generalize now a Lemma of Schweizer (1991).

Lemma 5.3.3. *Let Y be a payment process presupposing X to satisfy assumptions $(X1)$ and $(X2)$. Furthermore let φ be a 0-admissible trading strategy. If φ is locally risk-minimizing, then it is Y-mean-self-financing.*

Proof. As in the proof of Lemma 5.2.5 we define for $0\leq t\leq T$, $\hat{\xi}_t := \xi_t$ and

$$\hat{\eta}_t := \mathbb{E}[Y_T - Y_t - \int_t^T \xi_u\,dX_u \mid \mathcal{F}_t] - \xi_t X_t.$$

Let now $\Delta := \hat{\varphi}-\varphi$, then Δ is a small perturbation ($\Delta=(\delta,\varepsilon)=(\hat{\xi}-\xi,\hat{\eta}-\eta)=(0,\hat{\eta}-\eta)$ with $\varepsilon_T=0$). Let τ_n be the n-th dyadic partition of $[0,T]$ and denote by $d' := (d+2^{-n}T)\wedge T$ the successor in τ_n of $d\in\tau_n$. Then

$$V_d(\varphi+\Delta|_{(d,d']}) = \xi_d X_d + \eta_d + \varepsilon_d\,\mathbb{1}_{[d,d')}(d) = V_d(\varphi)+\epsilon_d = V_d(\hat{\varphi}),$$

and $V_T(\varphi+\Delta|_{(d,d']}) = V_T(\hat{\varphi}) = 0$. Further,

$$\begin{aligned} &C_T(\varphi+\Delta|_{(d,d']}) - C_d(\varphi+\Delta|_{(d,d']}) \\ &= Y_T - \int_0^T \xi_u\,dX_u - \big(V_d(\varphi+\Delta|_{(d,d']}) - \int_0^d \xi_u\,dX_u + Y_d\big) \\ &= C_T(\hat{\varphi}) - C_d(\hat{\varphi}), \quad d\in\tau_n, \text{ for all } n\in\mathbb{N}. \end{aligned}$$

Using these identities, one considers the influence of Δ on the risk process:

$$\begin{aligned} &R_d(\varphi+\Delta|_{(d,d']}) - R_d(\varphi) \\ &= \mathbb{E}[\big(C_T(\varphi+\Delta|_{(d,d']}) - C_d(\varphi+\Delta|_{(d,d']})\big)^2 \mid \mathcal{F}_d] - R_d(\varphi) \\ &= \mathbb{E}[(C_T(\hat{\varphi}) - C_d(\hat{\varphi}))^2 \mid \mathcal{F}_d] - R_d(\varphi) \\ &= R_d(\hat{\varphi}) - R_d(\varphi). \end{aligned}$$

In the proof of Lemma 5.2.5 we saw that

$$R_d(\hat{\varphi}) = R_d(\varphi) - (C_d(\varphi) - \mathbb{E}[C_T(\varphi) \mid \mathcal{F}_d])^2.$$

Hence,

$$r^{\tau_n}[\varphi, \Delta] = -\sum_{d \in \tau_n} \frac{(C_d(\varphi) - \mathbb{E}[C_T(\varphi) \mid \mathcal{F}_d])^2}{\mathbb{E}[\langle M \rangle_{d'} - \langle M \rangle_d \mid \mathcal{F}_d]} \mathbb{1}_{(d,d']}.$$

Since r^τ does not depend on the specific structure of $C(\varphi)$, the proof is now exactly the same as in the classical case: Assume there exists a dyadic rational d_0 and a set B of positive probability such that

$$C_{d_0}(\varphi)(\omega) \neq \mathbb{E}[C_T(\varphi) \mid \mathcal{F}_{d_0}](\omega) \quad \forall\, \omega \in B.$$

The right-continuity of $C_\cdot(\varphi)$ and $\mathbb{E}[C_T(\varphi) \mid \mathcal{F}_\cdot]$ implies now that $\forall\, \omega \in B$ there exists $\gamma(\omega) > 0$ and $\beta(\omega)$ such that

$$|C_d(\varphi) - \mathbb{E}[C_T(\varphi) \mid \mathcal{F}_d]|\,(\omega) \geq \gamma(\omega) > 0,$$

for every dyadic rational $d \in [d_0, d_0 + \beta(\omega)]$ yielding immediately

$$\liminf_{n \to \infty} r^{\tau_n}[\varphi, \Delta](\omega, t) < 0$$

for all $\omega \in B$ and for any $t \in (d_0, d_0 + \beta(\omega))$. This contradicts

$$\liminf_{n \to \infty} r^{\tau_n}[\varphi, \Delta](\omega, t) \geq 0 \quad \mathbb{P}_M\text{-a.e.},$$

since $\langle M \rangle_\cdot$ is strictly increasing $\mathbb{P}$-a.s. So we must have

$$C_d(\varphi) = \mathbb{E}[C_T(\varphi) \mid \mathcal{F}_d] \quad \forall \text{ dyadic rationals } d,\ \mathbb{P}\text{-a.s.}$$

The claim follows then by right-continuity. □

In the classical case it suffices to vary only the ξ component to find a locally risk-minimizing strategy. The same is true for the more general setting.

Lemma 5.3.4. *Let Y be a payment process and φ be a strategy which is 0-admissible and Y-mean-self-financing. Then φ is uniquely determined by its ξ component.*

Proof. $C(\varphi)$ is a martingale with terminal value $C_T(\varphi) = Y_T - \int_0^T \xi_u \, dX_u$, $\mathbb{P}$-a.s. Hence η_t, $0 \leq t \leq T$, is determined by the following identity:

$$\mathbb{E}[C_T(\varphi) \mid \mathcal{F}_t] = C_t(\varphi) = \xi_t X_t + \eta_t - \int_0^t \xi_u \, dX_u + Y_t, \quad \mathbb{P}\text{-a.s.} \quad \square$$

The previous Lemma tells us in addition that choosing

$$\eta_t = \mathbb{E}[C_T(\varphi) \mid \mathcal{F}_t] - \xi_t X_t + \int_0^t \xi_u \, dX_u - Y_t, \quad 0 \leq t < T,$$

and $\eta_T = 0$ yields a Y-mean-self-financing strategy. Considering now a payment process Y and a 0-admissible, Y-mean-self-financing strategy φ, which is determined by its ξ-component, we write $C(\xi) := C(\varphi)$ and $R(\xi) := R(\varphi)$. Letting moreover $\Delta = (\delta, \varepsilon)$ be a small perturbation and τ a partition of $[0, T]$ we compare for $t_i \in \tau$ the strategy $\varphi + \Delta|_{(t_i, t_{i+1}]}$, which is 0-admissible but not necessarily Y-mean-self-financing, with the 0-admissible Y-mean-self-financing trading strategy associated to $\xi + \delta|_{(t_i, t_{i+1}]}$.

Lemma 5.3.5. *Under these assumptions, one has*

$$R_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) = R_{t_i}(\xi + \delta|_{(t_i, t_{i+1}]}) + \big(\varepsilon_{t_i} + \mathbb{E}[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \mid \mathcal{F}_{t_i}]\big)^2,$$

and hence,

$$r^\tau[\varphi, \Delta] = r^\tau[\xi, \delta] + \sum_{t_i \in \tau} \frac{(\varepsilon_{t_i} + \mathbb{E}[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \mid \mathcal{F}_{t_i}])^2}{\mathbb{E}[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]}, \tag{5.4}$$

with

$$r^\tau[\xi, \delta] := \sum_{t_i \in \tau} \frac{R_{t_i}(\xi + \delta|_{(t_i, t_{i+1}]}) - R_{t_i}(\xi)}{\mathbb{E}[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]}.$$

Proof. $\delta_T = \varepsilon_T = 0$, since Δ is a small perturbation, and so

$$\begin{aligned} C_T(\varphi + \Delta|_{(t_i, t_{i+1}]}) &= -\int_0^T \xi_u \, dX_u - \int_{t_i}^{t_{i+1}} \delta_u \, dX_u + Y_T \\ &= C_T(\varphi) - \int_{t_i}^{t_{i+1}} \delta_u \, dX_u = C_T(\xi + \delta|_{(t_i, t_{i+1}]}). \end{aligned}$$

For $t_i \in \tau$ we have that

$$\begin{aligned} C_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) &= V_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) - \int_0^{t_i} \xi_u \, dX_u + Y_{t_i} \\ &= \xi_{t_i} X_{t_i} + \eta_{t_i} - \int_0^{t_i} \xi_u \, dX_u + Y_{t_i} + \varepsilon_{t_i} \\ &= C_{t_i}(\varphi) + \varepsilon_{t_i} \end{aligned} \tag{5.5}$$

and since φ is Y-mean-self-financing,

$$\begin{aligned} C_{t_i}(\xi + \delta|_{(t_i, t_{i+1}]}) &= \mathbb{E}[C_T(\xi + \delta|_{(t_i, t_{i+1}]}) \mid \mathcal{F}_{t_i}] \\ &= C_{t_i}(\varphi) - \mathbb{E}[\int_{t_i}^{t_{i+1}} \delta_u \, dX_u \mid \mathcal{F}_{t_i}] \\ &= C_{t_i}(\varphi) - \mathbb{E}[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \mid \mathcal{F}_{t_i}]. \end{aligned} \tag{5.6}$$

(5.5) and (5.6) imply

$$C_{t_i}(\varphi + \Delta|_{(t_i,t_{i+1}]}) = C_{t_i}(\xi + \Delta|_{(t_i,t_{i+1}]}) + \varepsilon_{t_i} + \mathbb{E}[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \mid \mathcal{F}_{t_i}],$$

and hence

$$\begin{aligned} &C_T(\varphi + \Delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\varphi + \Delta|_{(t_i,t_{i+1}]}) \\ &= C_T(\xi + \delta|_{(t_i,t_{i+1}]}) - C_{t_i}(\xi + \delta|_{(t_i,t_{i+1}]}) - \big(\varepsilon_{t_i} + \mathbb{E}[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \mid \mathcal{F}_{t_i}]\big). \end{aligned}$$

The martingale property of $C(\xi + \delta|_{(t_i,t_{i+1}]})$ yields then

$$R_{t_i}(\varphi + \Delta|_{(t_i,t_{i+1}]}) = R_{t_i}(\xi + \delta|_{(t_i,t_{i+1}]}) + \big(\varepsilon_{t_i} + \mathbb{E}[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \mid \mathcal{F}_{t_i}]\big)^2. \quad \square$$

We now adapt the following Lemma of Schweizer (1991).

Lemma 5.3.6. *Assume X satisfies (X1)-(X5). Let Y be a payment process and $\varphi = (\xi, \eta)$ a 0-admissible trading strategy. Then the following statements are equivalent:*

(i) φ is locally risk-minimizing.

(ii) φ is Y-mean-self-financing, and $\liminf_{n\to\infty} r^{\tau_n}[\xi, \delta] \geq 0$ $\mathbb{P}_M$-a.e. for every δ satisfying Definition 5.3.1 and for every 0-convergent sequence (τ_n) of partitions of $[0, T]$.

(iii) φ is Y-mean-self-financing, and the martingale $C(\varphi)$ is orthogonal to M.

Proof. If (i) or (ii) holds, then $C_t(\varphi)$ is a square-integrable martingale. This follows, since φ is Y-mean-self-financing in both cases. Furthermore the second term on the right hand side of (5.4) (the sum) is not influenced by the payment process Y. Hence the equivalence of (i) and (ii) can be derived from Lemma 2.2 in Schweizer (1991) and Proposition 3.1 in Schweizer (1990). The equivalence of (i) and (iii) follows from Theorem 3.2 in Schweizer (1990). $\square$

Remark

(a) The orthogonality results of Schweizer (1991) and Schweizer (1990) suppose the cost process $C_t(\varphi)$ to be a square-integrable martingale only. For this reason, adding the square-integrable payment process Y_t does not disrupt the classical theory.

(b) Note that (ii) $\Rightarrow$ (i) immediately follows from (5.4) and Lemma 5.3.3.

(c) For (i) $\Rightarrow$ (ii) it suffices to show that

$$\sum_{t_i \in \tau_n} \frac{(\varepsilon_{t_i} + \mathbb{E}[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \mid \mathcal{F}_{t_i}])^2}{\mathbb{E}[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]}$$

converges to 0 as $\lim_{n\to\infty} |\tau_n| = 0$ for any increasing 0-convergent sequence of partitions τ_n. For a special case this is done in the proof of Lemma 2.2 in Schweizer (1991) and in general in the proof of Proposition 3.1 in Schweizer (1990).

One now proceeds to characterize a locally risk-minimizing strategy by the Kunita-Watanabe decomposition of $C_T(\varphi)$ with respect to $\mathbb{P}$ and M. This yields an optimality equation for the ξ component and the problem of finding a locally risk-minimizing strategy is hence reduced to solve this stochastic optimality equation.

Theorem 5.3.7. *Assume that X satisfies (X1)-(X5). Let Y be a payment process and let $\varphi = (\xi, \eta)$ be a 0-admissible trading strategy. Then φ is locally risk-minimizing if and only if φ is Y-mean-self-financing and ξ satisfies the optimality equation:*

$$\mu^{Y;\mathbb{P}} - \xi - \mu^{\xi,A;\mathbb{P}} = 0 \quad \mathbb{P}_M\text{-a.e.} \tag{5.7}$$

Proof. We consider the Galtchouk-Kunita-Watanabe decomposition (cf. Theorem 2.5.3) of $V_t^* := \mathbb{E}[Y_T \mid \mathcal{F}_t]$, which is for $0 \le t \le T$ given by

$$V_t^* = \mathbb{E}[Y_T \mid \mathcal{F}_t] = V_0^* + \int_0^t \mu_u^{Y;\mathbb{P}} \, dM_u + K_t^{Y;\mathbb{P}}, \quad \mathbb{P}\text{-a.s.}, \tag{5.8}$$

as well as the Galtchouk-Kunita-Watanabe decomposition

$$\int_0^T \xi_u \, dA_u = \mathbb{E}[\int_0^T \xi_u \, dA_u] + \int_0^T \mu_u^{\xi,A;\mathbb{P}} \, dM_u + K_T^{\xi,A;\mathbb{P}} \quad \mathbb{P}\text{-a.s.}$$

Note that $\int_0^T \xi_u \, dA_u$ is square-integrable, since φ is a trading strategy. The two decompositions together yield

$$\begin{aligned} C_T(\varphi) &= Y_T - \int_0^T \xi_u \, dX_u \\ &= V_0^* + \int_0^T \mu_u^{Y;\mathbb{P}} \, dM_u + K_T^{Y;\mathbb{P}} - \int_0^T \xi_u \, d(M_u + A_u) \\ &= \underbrace{V_0^* - \mathbb{E}[\int_0^T \xi_u \, dA_u]}_{=C_0(\varphi)} + \int_0^T \left(\mu_u^{Y;\mathbb{P}} - \xi_u - \mu_u^{\xi,A;\mathbb{P}}\right) dM_u \\ &\quad + K_T^{Y;\mathbb{P}} - K_T^{\xi,A;\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Lemma 5.3.6 yields then the result. $\square$

5.4 Finding a locally risk-minimizing strategy

In this section it is shown that an unique solution to the optimality equation exists. First one adds some more structure on the considered probability space (cf. Schweizer (1991)).

Definition 5.4.1. *Two stochastic processes M and N are said to form a $\mathbb{P}$-basis of $\mathrm{L}^2(\mathbb{P})$ if both M and N are square-integrable martingales under $\mathbb{P}$ and $M - M_0$ and $N - N_0$ are $\mathbb{P}$-orthogonal (as martingales). Further, if every $H \in \mathrm{L}^2(\mathbb{P})$ has an unique representation*

$$H = \mathbb{E}[H] + \int_0^T \mu_u^{H;\mathbb{P}}\, dM_u + \int_0^T \nu_u^{H;\mathbb{P}}\, dN_u \quad \mathbb{P}\text{-a.s.}$$

for two predictable processes $\mu^{H;\mathbb{P}} \in \mathrm{L}^2(\mathbb{P}_M)$ and $\nu^{H;\mathbb{P}} \in \mathrm{L}^2(\mathbb{P}_N)$.

The following assumptions are added to the model.

($P1$) There exists a process $N = (N_t)_{0\le t\le T}$ such that M and N form a $\mathbb{P}$-basis of $\mathrm{L}^2(\mathbb{P})$.

($P2$) There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that X and N form a $\mathbb{Q}$-basis of $\mathrm{L}^2(\mathbb{Q})$.

Because of ($P2$) the considered financial market must be free of arbitrage. Schweizer (1991) provides moreover a formula for the Radon-Nikodým density of the measure $\mathbb{Q}$, called the Föllmer-Schweizer measure, and shows that it is essentially unique. This formula provides an insight into the performed Girsanov change of measure removing the drift A from X. It gives furthermore an idea of how to construct the measure $\mathbb{Q}$.

Consider now a payment process $Y = (Y_t)_{0\le t\le T}$ which is also square-integrable with respect to $\mathbb{Q}$. Under assumption ($P2$) we apply Proposition 5.2.6 to the hedging problem and find an unique hedging strategy $\varphi^{Y;\mathbb{Q}} = (\xi^{Y;\mathbb{Q}}, \eta^{Y;\mathbb{Q}})$ which is risk-minimizing with respect to $\mathbb{Q}$ and satisfies $V_T(\varphi^{Y;\mathbb{Q}}) = 0$. $\xi^{Y;\mathbb{Q}}$ is given by the Kunita-Watanabe decomposition

$$Y_T = \mathbb{E}^{\mathbb{Q}}[Y_T] + \int_0^T \xi_u^{Y;\mathbb{Q}}\, dX_u + \int_0^T \nu_u^{Y;\mathbb{Q}}\, dN_u \quad \mathbb{Q}\text{-a.s.} \tag{5.9}$$

and

$$\eta_t^{Y;\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[Y_T \mid \mathcal{F}_t] - Y_t - \xi_t^{Y;\mathbb{Q}} X_t, \quad \mathbb{Q}\text{-a.s.}, \quad 0 \le t \le T.$$

The idea is now to use $\varphi^{Y;\mathbb{Q}}$ as candidate for a locally risk-minimizing strategy under $\mathbb{P}$.

Theorem 5.4.2. *Assume that X satisfies ($X1$)-($X5$) and that ($P1$) and ($P2$) hold. Let Y be a payment process and square-integrable with respect to $\mathbb{Q}$. Suppose additionally that $\nu^{Y;\mathbb{Q}} \in \mathrm{L}^2(\mathbb{P}_N)$ and that $\xi^{Y;\mathbb{Q}}$ satisfies (5.1). Then $\varphi^{Y;\mathbb{Q}}$ is locally risk-minimizing with respect to $\mathbb{P}$.*

Proof. According to Theorem 5.3.7 first one has to show that $\xi^{Y;\mathbb{Q}}$ is a solution to the optimality equation (5.7). Since $\xi^{Y;\mathbb{Q}}$ satisfies the conditions of a trading strategy and because of assumption $(P1)$, we may write

$$\int_0^T \xi_u^{Y;\mathbb{Q}}\, dA_u = \mathbb{E}[\int_0^T \xi_u^{Y;\mathbb{Q}}\, dA_u] + \int_0^T \mu_u^{\xi^{Y;\mathbb{Q}},A;\mathbb{P}}\, dM_u + \int_0^T \nu_u^{\xi^{Y;\mathbb{Q}},A;\mathbb{P}}\, dN_u.$$

Using this result, and

$$\mathbb{E}[Y_T] = \mathbb{E}^{\mathbb{Q}}[Y_T] + \mathbb{E}[\int_0^T \xi_u^{Y;\mathbb{Q}}\, dA_u],$$

which holds because of $(P2)$, one rewrites (5.9) as follows:

$$\begin{aligned} Y_T &= \mathbb{E}^{\mathbb{Q}}[Y_T] + \int_0^T \xi_u^{Y;\mathbb{Q}}\, dM_u + \int_0^T \xi_u^{Y;\mathbb{Q}}\, dA_u + \int_0^T \nu_u^{Y;\mathbb{Q}}\, dN_u \\ &= \mathbb{E}[Y_T] + \int_0^T \left(\xi_u^{Y;\mathbb{Q}} + \mu_u^{\xi^{Y;\mathbb{Q}},A;\mathbb{P}}\right) dM_u \\ &\quad + \int_0^T \left(\nu_u^{Y;\mathbb{Q}} + \nu_u^{\xi^{Y;\mathbb{Q}},A;\mathbb{P}}\right) dN_u. \end{aligned}$$

We compare this representation of Y_T with (5.8) and refer to the uniqueness of the Galtchouk-Kunita-Watanabe decomposition to conclude that the optimality equation (5.7) is satisfied:

$$\mu^{Y;\mathbb{P}} = \xi^{Y;\mathbb{Q}} + \mu^{\xi^{Y;\mathbb{Q}},A;\mathbb{P}} \quad \mathbb{P}_M\text{-a.e.}$$

It remains now to show that $\varphi^{Y;\mathbb{Q}}$ is Y-mean-self-financing with respect to $\mathbb{P}$. (5.9) yields

$$C_T(\varphi^{Y;\mathbb{Q}}) = \underbrace{V_T(\varphi^{Y;\mathbb{Q}})}_{=0} - \int_0^T \xi_u^{Y;\mathbb{Q}}\, dX_u + Y_T = \mathbb{E}^{\mathbb{Q}}[Y_T] + \int_0^T \nu_u^{Y,\mathbb{Q}}\, dN_u.$$

Further,

$$\begin{aligned} C_t(\varphi^{Y;\mathbb{Q}}) &= V_t(\varphi^{Y;\mathbb{Q}}) - \int_0^t \xi_u^{Y;\mathbb{Q}}\, dX_u + Y_t \\ &= \xi_t^{Y;\mathbb{Q}} X_t + \eta_t^{Y;\mathbb{Q}} - \int_0^t \xi_u^{Y;\mathbb{Q}}\, dX_u + Y_t. \end{aligned}$$

Recall that $\eta_t^{Y;\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[Y_T \mid \mathcal{F}_t] - Y_t - \xi_t^{Y;\mathbb{Q}} X_t$. Using $(P2)$ we get

$$\mathbb{E}^{\mathbb{Q}}[Y_T \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[Y_T] + \int_0^t \xi_u^{Y;\mathbb{Q}}\, dX_u + \int_0^t \nu_u^{Y;\mathbb{Q}}\, dN_u.$$

Hence,

$$C_t(\varphi^{Y;\mathbb{Q}}) = \mathbb{E}^{\mathbb{Q}}[Y_T] + \int_0^t \nu_u^{Y;\mathbb{Q}}\, dN_u, \quad 0 \le t \le T,$$

The claim follows, since N is a martingale under $\mathbb{P}$ and $\nu^{Y;\mathbb{Q}} \in \mathrm{L}^2(\mathbb{P}_N)$. □

Summarizing this chapter we remark that the local risk-minimization theory for the classical case transfers to the case of general payment processes and is applied in the same manner. First, one removes the drift of the semimartingale X performing an equivalent change of measure to the Föllmer-Schweizer measure $\mathbb{Q}$ and second, one applies the theory of Møller (2001a) to find a risk-minimizing hedging strategy under $\mathbb{Q}$. This strategy is then locally risk-minimizing with respect to the original measure $\mathbb{P}$.

Chapter 6

Hedging of general unit-linked life insurance contracts

6.1 Introduction

The following chapter contributed to one part of Riesner (2005) and generalizes the considerations of Chapter 4 with respect to two aspects. First, in Chapter 4 only insurance contracts that do not generate intermediate payments were treated. We restricted ourselves to the case in which only a single premium payment at the time of issue occurs and in which benefit payments are always deferred to the end of the considered time horizon. Benefits only cashed out in this way are no restriction for the pure endowment but certainly for the term insurance for which it is simply not reality. The reason for this constraint is the local risk-minimization theory of Schweizer (1991) which can only be applied for contingent claims with fixed maturity. Using Chapter 5, where we justified locally risk-minimizing hedging to work also for general payment streams, we allow here the insurance contracts to have both intermediate premium and benefit payments. The second aspect of generalization is the insurance model which is now based on the multi-state Markov model of Hoem (1969) which we discussed in Subsection 2.4.1 and which allows to model quite general forms of life insurance. To be more specific, we consider life insurance contracts driven by a Markov jump process and having unit-linked benefits (and possibly premiums) based on our Lévy-process stock model of Section 2.3. Those benefits are usually due immediately upon occurrence of some insurance event at random times. In life insurance such an event could be for instance a disability or the death of the policy-holder. Similarly, premiums are paid according to a predefined premium scheme for a fixed time or as long as the policy-holder is alive.

For those general contracts we derive locally risk-minimizing hedging

strategies and compute additionally the involved hedging risk, which has now a local interpretation under the historical measure. This is an important figure, since, as in reality, we face an incomplete financial market in which riskless hedging is not possible. The model and the results presented here complement Møller (2001a) who derives risk-minimizing hedging strategies for unit-linked life insurance payment streams in the same multi-state Markov model but for a Black-Scholes martingale financial market. Moreover, in analogy to Chapter 4 we show for the generalized model that the involved hedging risk can be separated in two components: financial risk and insurance risk. In Møller (2001a) the pure financial risk is not present, since for a complete financial market, turned incomplete by the mortality law, only the insurance risk appears. We conclude the chapter with examples of unit-linked life annuity and term insurance contracts both in a single insurance and in a portfolio model consideration. In particular, we apply the general theory in detail and reduce it to the specific insurance models.

Let us briefly recall the necessary technicalities. As in Chapter 4 the Lévy-process financial market, $(\Omega_1, \mathbb{G}, (\mathcal{G}_t)_{0\le t\le T}, \mathbb{P}_1)$, of Section 2.3 including the change of measure is supposed and as mentioned for the insurance part of our model, $(\Omega_2, \mathbb{H}, (\mathcal{H}_t)_{0\le t\le T}, \mathbb{P}_2)$, the multi-state Markov model discussed in Subsection 2.4.1 is assumed. Using those two models we consider the risk-neutral combined financial and insurance model $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{Q})$ of Subsection 2.5.1 recalling that $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ is supposed to be the product of the two independent probability spaces $(\Omega_1, \mathbb{G}, (\mathcal{G}_t)_{0\le t\le T}, \mathbb{P}_1)$ and $(\Omega_2, \mathbb{H}, (\mathcal{H}_t)_{0\le t\le T}, \mathbb{P}_2)$. We also remind the reader that the measure $\mathbb{Q}$ denotes the product measure of the Föllmer-Schweizer measure $\mathbb{Q}_1$ and the measure $\mathbb{P}_2$ which is assumed to be risk-neutral. Moreover, all the theory developed in Chapters 3 and 5 is a prerequisite for this chapter.

6.2 General unit-linked benefit and premium payments

The multi-state Markov insurance model admits quite general forms of benefit and premium payments (cf. Møller (2001a)). First, it might be the case that a transition from state j to state k at time t immediately induces a payment $g_t^{jk} = g^{jk}(t, S_t)$ and second, it is possible that depending on the policy sojourning in state j the insurance company continuously pays the rate $g_t^j = g^j(t, S_t)$ at time t. Payments of the first type usually occur with general life insurances whereas state-wise life annuities typically generate payments of the second type. Being in state j, the policy shall additionally admit lump-sum annuity payments $\tilde{g}_t^j = \tilde{g}^j(t, S_t)$ at fixed deterministic times $t \in \Gamma = \{\tau_1, \ldots, \tau_n\}$ for some $n \ge 1$. The amount payable at time $t \in [0, T]$, depending on the policy being in state j, is therefore equal to $G_t^j = G^j(t, S_t) = g_t^j + \tilde{g}_t^j \mathbb{1}_{\{t\in\Gamma\}}$ and up to time t the total state-wise annuity

payment is equal to $\int_0^t G^j(u,S_u)\,du = \int_0^t g^j(u,S_u)\,du + \sum_{\tau\in\Gamma,\tau\le t}\tilde{g}^j(\tau,S_\tau)$. Here and for the rest of this chapter we treat $\mathbb{1}_{\{u\in\Gamma\}}\,du$ as discrete counting measure on the set Γ with jump hight one. This enables us to discuss the continuous and the discrete case simultaneously. Moreover, we assume benefit and premium payments represented as difference in the functions g^{jk}, g^j and $\tilde{g}^j$, where negative values are premium payments. For technical reasons $(t,s)\mapsto g^{jk}(t,s)$, $(t,s)\mapsto g^j(t,s)$ and $(t,s)\mapsto \tilde{g}^j(t,s)\,\mathbb{1}_{\{t\in\Gamma\}}$ are supposed to be measurable for $j,k\in\mathcal{J}$, and we need that

$$\sup_{u\in[0,T]}\mathbb{E}^{\mathbb{Q}}\left[\left(B_u^{-1}g(u,S_u)\right)^2\right]<\infty,\quad \text{for all } g^{jk},g^j,\tilde{g}^j\,\mathbb{1}_{\{\cdot\in\Gamma\}}. \tag{6.1}$$

That condition is set in accordance with (3.1) and guarantees that each well-defined stochastic integral of $B_u^{-1}g(u,S_u)$ with respect to a square-integrable martingale is a square-integrable martingale. Following now Definition 3.2.1 we consider the arbitrage-free price process of those insurance claims given for $0\le t\le u\le T$ by

$$\begin{aligned}F^{jk}(t,S_t,u)&:=\mathbb{E}^{\mathbb{Q}}[B_tB_u^{-1}g^{jk}(u,S_u)\mid\mathcal{G}_t],\\ F^{j}(t,S_t,u)&:=\mathbb{E}^{\mathbb{Q}}[B_tB_u^{-1}G^{j}(u,S_u)\mid\mathcal{G}_t].\end{aligned}$$

Obviously we have

$$F^j(t,S_t,u)=\mathbb{E}^{\mathbb{Q}}[B_tB_u^{-1}g_u^j\mid\mathcal{G}_t]+\mathbb{E}^{\mathbb{Q}}[B_tB_u^{-1}\tilde{g}_u^j\,\mathbb{1}_{\{u\in\Gamma\}}\mid\mathcal{G}_t].$$

Recall that $F^{\cdot}(t,S_t,u)$, $0\le t\le u\le T$, is the price at time t of an insurance claim due at time u. The superscript " $\cdot$ " either stands for the state j or for a transition jk from state j to state k. In order to perform the following calculations properly we again suppose that the functions $F^{\cdot}(\cdot,\cdot,u)\in C^{1,2}([0,T]\times[0,\infty))$ for each $u\in[0,T]$, that is, once continuously differentiable in the first and twice continuously differentiable in the second variable. For notational convenience we denote by $F^{\cdot\prime}(t,x,u)=\frac{d}{dx}F^{\cdot}(t,x,u)$. Lemma 3.2.2 and the remarks thereafter apply similarly. In analogy to (3.6) we further assume $F^{\cdot\prime}(t,x,u)$ to be uniformly bounded, that is, there exists a non-random constant $c_1>0$ such that

$$\left|F^{\cdot\prime}(t,x,u)\right|\le c_1<\infty\quad\forall\,t,x,u\quad\text{and for all}\quad F^j,F^{jk}\quad\mathbb{Q}\text{-a.s.} \tag{6.2}$$

Having defined the price processes we introduce now the payment process $(Y_t)_{0\le t\le T}$ (cf. Definition 5.2.2 and Møller (2001a)) which is described by general unit-linked insurance contracts and for which we will derive a hedging strategy. It is given by

$$Y_t=Y_0+\sum_{j\in\mathcal{J}}\int_0^t B_u^{-1}\Big(I_u^jG_u^j\,du+\sum_{k:k\neq j}g_u^{jk}\,dN_u^{jk}\Big),\quad 0\le t\le T.$$

Here it is important that S and N^{jk} jump simultaneously only with zero probability. Hence the integrals $\int_0^t B_u^{-1} g_u^{jk} dN_u^{jk}$ are $\mathbb{Q}$-a.s. well-defined. Considering the process Y the notion unit-linked becomes finally apparent. The process describes payments depending on the current state of the policy and on the market value of the stock S_t.

6.3 Locally risk-minimizing hedging strategies

To find now the locally risk-minimizing hedging strategy (under $\mathbb{P}$) for the payment process Y, we have to derive a risk-minimizing strategy for it under the measure $\mathbb{Q}$ (cf. Chapter 5). As we already know, the intrinsic value process $V_t^* = \mathbb{E}^{\mathbb{Q}}[Y_T \mid \mathcal{F}_t]$, $0 \le t \le T$, of Y under $\mathbb{Q}$ and its Galtchouk-Kunita-Watanabe decomposition play a crucial role. We proceed as in Møller (2001a):

$$\begin{aligned}
V_t^* &= Y_t + \mathbb{E}^{\mathbb{Q}}[Y_T - Y_t \mid \mathcal{F}_t] \\
&= Y_t + \sum_{j \in \mathcal{J}} \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T B_u^{-1} \Big(I_u^j G_u^j \, du + \sum_{k:k \neq j} g_u^{jk} \, dN_u^{jk} \Big) \mid \mathcal{F}_t\Big] \\
&= Y_t + \sum_{j \in \mathcal{J}} \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T B_u^{-1} \Big(I_u^j G_u^j \, du + \sum_{k:k \neq j} g_u^{jk} I_u^j \mu_u^{jk} \, du \Big) \mid \mathcal{F}_t\Big] \\
&= Y_t + B_t^{-1} \sum_{j \in \mathcal{J}} \int_t^T p_{Z_t,j}(t,u) \Big(F^j(t, S_t, u) + \sum_{k:k \neq j} \mu_u^{jk} F^{jk}(t, S_t, u) \Big) du.
\end{aligned}$$

In the third row we used that $\int B_u^{-1} g_u^{jk} \, d(N_u^{jk} - I_u^j \mu_u^{jk} \, du)$ are martingales (cf. (2.19)). Furthermore the independence of the financial market and the insurance portfolio allows to factorize the expectations and to calculate

$$\mathbb{E}^{\mathbb{Q}}[I_u^j \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}_2}[I_u^j \mid \mathcal{H}_t] = p_{Z_t,j}(t,u), \quad 0 \le t \le u \le T.$$

In order to abbreviate the expression of V^* the following auxiliary processes V_t^i, $0 \le t \le T$ and $i \in \mathcal{J}$, are introduced:

$$V^i(t, S_t) = \sum_{j \in \mathcal{J}} \Big[\int_t^T p_{ij}(t,u) \Big(F^j(t, S_t, u) + \sum_{k:k \neq j} \mu_u^{jk} F^{jk}(t, S_t, u) \Big) \, du \Big].$$

As previously mentioned, in Møller (2001a) the original measure $\mathbb{P}$ is already a martingale measure for the stock admitting to evaluate the processes V^i directly under $\mathbb{P}$. They reflect then the current market value of future benefits less premiums at time t conditional on the policy being in state i at time t and on the value of the stock being S_t. In this case the processes V^i do actually represent the classical state-wise prospective reserve which is the expected value of discounted benefits less premiums under the physical

measure $\mathbb{P}$. In our more general setting we have to interpret this under the risk-neutral measure $\mathbb{Q}$ calling V^i the state-wise $\mathbb{Q}$-prospective reserve. In conclusion we express the processes V^i as

$$V^i(t,s) = \mathbb{E}^{\mathbb{Q}}[B_t(Y_T - Y_t) \mid Z_t = i, S_t = s]$$

and finally write

$$V_t^* = Y_t + \sum_{i \in \mathcal{J}} I_t^i B_t^{-1} V^i(t, S_t). \tag{6.3}$$

Theorem 6.3.1. *The Galtchouk-Kunita-Watanabe decomposition of V^* (under $\mathbb{Q}$) is given by*

$$V_t^* = V_0^* + \int_0^t \Big(\sum_{i \in \mathcal{J}} I_{\tau-}^i \xi_\tau^i \Big)\, d\hat{S}_\tau + K_t, \quad 0 \le t \le T,$$

where

$$\xi_t^i = \sum_{j \in \mathcal{J}} \int_t^T \Big(p_{ij}(t,u)\big(\xi_t^j(u) + \sum_{k:k \neq j} \mu_u^{jk} \xi_t^{jk}(u)\big)\Big)\, du$$

and

$$\begin{aligned} K_t = &\int_0^t \Big(\sum_{i \in \mathcal{J}} I_{\tau-}^i \zeta_\tau^{(1)i} \Big)\, dW_\tau^{\mathbb{Q}} + \int_0^{\mathrm{t}} \int_{\mathbb{R}} \Big(\sum_{i \in \mathcal{J}} I_{\tau-}^i \zeta_\tau^{(2)i}(y) \Big)\, M^{\mathbb{Q}}(d\tau, dy) \\ &+ \sum_{i,k: i \neq k} \int_0^t \vartheta_\tau^{ik}\, dM_\tau^{ik} \end{aligned}$$

is orthogonal to $\hat{S}$ with

$$\begin{aligned} \zeta_t^{(1)i} &= \sum_{j \in \mathcal{J}} \int_t^T \Big(p_{ij}(t,u)\big(\zeta_t^{(1)j}(u) + \sum_{k:k \neq j} \mu_u^{jk} \zeta_t^{(1)jk}(u)\big)\Big)\, du, \\ \zeta_t^{(2)i}(y) &= \sum_{j \in \mathcal{J}} \int_t^T \Big(p_{ij}(t,u)\big(\zeta_t^{(2)j}(y,u) + \sum_{k:k \neq j} \mu_u^{jk} \zeta_t^{(2)jk}(y,u)\big)\Big)\, du, \\ \vartheta_t^{ik} &= B_t^{-1} \Big(g_t^{ik} + V^k(t, S_t) - V^i(t, S_t) \Big). \end{aligned}$$

$\xi_t^{\cdot}(u)$, $\zeta_t^{(1)\cdot}(u)$ and $\zeta_t^{(2)\cdot}(y,u)$ are taken from Theorem 3.3.4.

Proof. The first step is to decompose the process $B_t^{-1} V^i(t, S_t)$, $i \in \mathcal{J}$. As in Møller (2001a) we let

$$Y_t^{i,u} = \sum_{j \in \mathcal{J}} p_{ij}(t,u) B_t^{-1} \Big(F^j(t, S_t, u) + \sum_{k:k \neq j} \mu_u^{jk} F^{jk}(t, S_t, u) \Big)$$

for all $i \in \mathcal{J}$ and $0 \le t \le u \le T$ and apply Itô's integration-by-parts formula (cf. Protter (2004), Chapter II, Theorem 22, Corollary 2) yielding

$$\begin{aligned} dY_t^{i,u} = &\sum_{j\in\mathcal{J}} B_t^{-1}\Big(F^j(t,S_{t-},u) + \sum_{k:k\neq j}\mu_u^{jk}F^{jk}(t,S_{t-},u)\Big)dp_{ij}(t,u) \\ &+ \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Big[d(B_t^{-1}F^j(t,S_t,u)) + \sum_{k:k\neq j}\mu_u^{jk}d(B_t^{-1}F^{jk}(t,S_t,u))\Big]. \end{aligned}$$

The quadratic covariation term is identically equal to 0, since $p_{ij}(\cdot,u)$ is of finite variation and continuous for each u. This follows immediately from (2.21) stating that it is differentiable in the first variable with a bounded derivative on $[0,T]$. Substitution of (2.21) for $dp_{ij}(t,u)$ yields

$$\begin{aligned} dY_t^{i,u} &= \sum_{j\in\mathcal{J}}\sum_{k:k\neq i}\mu_t^{ik}\big(p_{ij}(t,u) - p_{kj}(t,u)\big) \\ &\qquad B_t^{-1}\Big(F^j(t,S_t,u) + \sum_{k:k\neq j}\mu_u^{jk}F^{jk}(t,S_t,u)\Big)\,dt \\ &\qquad + \sum_{j\in\mathcal{J}} p_{ij}(t,u)\big[\cdots\big] \\ &= \sum_{k:k\neq i}\mu_t^{ik}\Big(Y_t^{i,u} - Y_t^{k,u}\Big)\,dt + \sum_{j\in\mathcal{J}} p_{ij}(t,u)\big[\cdots\big]. \end{aligned}$$

Now we apply Theorem 3.3.4 to the differentials $d(B_t^{-1}F^{\cdot}(t,S_t,u))$, which yields

$$\begin{aligned} dY_t^{i,u} &= \sum_{k:k\neq i}\mu_t^{ik}\Big(Y_t^{i,u} - Y_t^{k,u}\Big)\,dt \\ &\quad + \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Bigg[\Big(\xi_t^j(u)\,d\hat{S}_t + \zeta_t^{(1)j}(u)\,dW_t^{\mathbb{Q}} \\ &\qquad\qquad + \int_{\mathbb{R}}\zeta_t^{(2)j}(y,u)\,M^{\mathbb{Q}}(dt,dy)\Big) \\ &\quad + \sum_{k:k\neq j}\mu_u^{jk}\Big(\xi_t^{jk}(u)\,d\hat{S}_t + \zeta_t^{(1)jk}(u)\,dW_t^{\mathbb{Q}} \\ &\qquad\qquad + \int_{\mathbb{R}}\zeta_t^{(2)jk}(y,u)\,M^{\mathbb{Q}}(dt,dy)\Big)\Bigg] \\ &= \alpha_t^{i,u}\,dt + \beta_t^{i,u}\,d\hat{S}_t + \gamma_t^{i,u}\,dW_t^{\mathbb{Q}} + \int_{\mathbb{R}}\delta_t^{i,u}(y)M^{\mathbb{Q}}(dt,dy), \end{aligned}$$

where

$$\begin{aligned} \alpha_t^{i,u} &:= \sum_{k:k\neq i}\mu_t^{ik}\left(Y_t^{i,u} - Y_t^{k,u}\right), \\ \beta_t^{i,u} &:= \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Big(\xi_t^j(u) + \sum_{k:k\neq j}\mu_u^{jk}\xi_t^{jk}(u)\Big), \end{aligned}$$

$$\gamma_t^{i,u} := \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Big(\zeta_t^{(1)j}(u) + \sum_{k:k\neq j} \mu_u^{jk}\zeta_t^{(1)jk}(u)\Big),$$
$$\delta_t^{i,u}(y) := \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Big(\zeta_t^{(2)j}(y,u) + \sum_{k:k\neq j} \mu_u^{jk}\zeta_t^{(2)jk}(y,u)\Big).$$

This implies

$$Y_t^{i,u} = Y_0^{i,u} + \int_0^t \alpha_\tau^{i,u}\,d\tau + \int_0^t \beta_\tau^{i,u}\,d\hat{S}_\tau + \int_0^t \gamma_\tau^{i,u}\,dW_\tau^{\mathbb{Q}} + \int_0^t\int_{\mathbb{R}} \delta_\tau^{i,u}(y)\,M^{\mathbb{Q}}(d\tau,dy).$$

Note now that $B_t^{-1}V^i(t,S_t) = \int_t^T Y_t^{i,u}\,du$. Hence (cf. Møller (2001a)),

$$\begin{aligned} B_t^{-1}V^i(t,S_t) = {} & \int_0^T Y_0^{i,u}\,du + \int_0^T\int_0^t \mathbb{1}_{\{\tau\le u\}}\,\alpha_\tau^{i,u}\,d\tau du \\ & + \int_0^T\int_0^t \mathbb{1}_{\{\tau\le u\}}\,\beta_\tau^{i,u}\,d\hat{S}_\tau\,du + \int_0^T\int_0^t \mathbb{1}_{\{\tau\le u\}}\,\gamma_\tau^{i,u}\,dW_\tau^{\mathbb{Q}}\,du \\ & + \int_0^T\int_0^t\int_{\mathbb{R}} \mathbb{1}_{\{\tau\le u\}}\,\delta_\tau^{i,u}(y)M^{\mathbb{Q}}(d\tau,dy)du - \int_0^t Y_u^{i,u}\,du. \end{aligned}$$

Let us evaluate each of the six summands on the right hand sight separately realizing first that

$$\int_0^T Y_0^{i,u}\,du = V^i(0,S_0).$$

Furthermore, as in Møller (2001a), it holds that

$$\begin{aligned} & \int_0^t Y_u^{i,u}\,du \\ = {} & \int_0^t \sum_{j\in\mathcal{J}} \underbrace{p_{ij}(u,u)}_{=\mathbb{1}_{\{i=j\}}} B_u^{-1}\Big(F^j(u,S_u,u) + \sum_{k:k\neq j}\mu_u^{jk}F^{jk}(u,S_u,u)\Big)\,du \\ = {} & \int_0^t B_u^{-1}\Big(G_u^i + \sum_{k:k\neq i}\mu_u^{ik}g_u^{ik}\Big)\,du, \end{aligned}$$

and

$$\begin{aligned} \int_0^T\int_0^t \mathbb{1}_{\{\tau\le u\}}\,\alpha_\tau^{i,u}\,d\tau\,du &= \int_0^t\int_\tau^T \alpha_\tau^{i,u}\,du\,d\tau \\ &= \int_0^t \sum_{k:k\neq i}\mu_\tau^{ik}\int_\tau^T \Big(Y_\tau^{i,u} - Y_\tau^{k,u}\Big)\,du\,d\tau \\ &= \int_0^t \Big(\sum_{k:k\neq i}\mu_\tau^{ik}B_\tau^{-1}\big(V^i(\tau,S_\tau) - V^k(\tau,S_\tau)\big)\Big)\,d\tau. \end{aligned}$$

The application of Fubini's theorem is allowed here $\mathbb{Q}$-a.s., since the function $(\omega, t, u) \mapsto \alpha_t^{i,u}(\omega)$, $i \in \mathcal{J}$, is $\mathcal{O} \otimes \mathcal{B}([0,T])$-measurable, and since by (6.1) we have uniformly for all t, u that $|F^{\cdot}(t, S_t, u)| < \infty$ $\mathbb{Q}$-a.s. This implies together with the continuity of $\mu_t^{\cdot}$ that

$$\int_0^T \int_0^t \mathbb{1}_{\{\tau \le u\}} \left|\alpha_\tau^{i,u}\right| \, d\tau \, du < \infty \quad \mathbb{Q}\text{-a.s.}$$

Additionally, the function $(\omega, t, u) \mapsto \beta_t^{i,u}(\omega)$, $i \in \mathcal{J}$, is $\mathcal{P} \otimes \mathcal{B}([0,T])$-measurable and uniformly bounded by a non-random constant (cf. (3.10) in the proof of Theorem 3.3.4). Fubini's theorem for stochastic integrals (cf. Protter (2004), Chapter IV, Theorem 64) yields then:

$$\int_0^T \int_0^t \mathbb{1}_{\{\tau \le u\}} \, \beta_\tau^{i,u} \, d\hat{S}_\tau \, du = \int_0^t \Big(\int_\tau^T \beta_\tau^{i,u} \, du \Big) \, d\hat{S}_\tau = \int_0^t \xi_\tau^i \, d\hat{S}_\tau.$$

Also $(\omega, t, u) \mapsto \gamma_t^{i,u}(\omega)$, $i \in \mathcal{J}$, is $\mathcal{P} \otimes \mathcal{B}([0,T])$-measurable and further there is a non-random constant $\tilde{c}_2$ such that

$$\sup_{t \in [0,T]} \left|\gamma_t^{i,u}\right| \le \tilde{c}_2 \sup_{t \in [0,T]} \hat{S}_t, \quad \forall \, u \quad \mathbb{Q}\text{-a.s.},$$

since (3.11) holds and the remaining terms are non-randomly bounded. Hence for all $0 \le \tau \le T$ we have that

$$\Big(\int_\tau^T (\gamma_\tau^{i,u})^2 \, du \Big)^{\frac{1}{2}} \le \tilde{\tilde{c}}_2 \sup_{t \in [0,T]} \hat{S}_t.$$

The square-integrability of $\hat{S}$ implies now that the integral of $\big(\int_s^T \gamma_s^{i,u} \, du \big)$ with respect to $W^{\mathbb{Q}}$ exists (cf. Protter (2004), Chapter IV, Lemma before Theorem 28). Therefore we may apply the second version of Fubini's Theorem for stochastic integrals (cf. Protter (2004), Chapter IV, Theorem 65) yielding

$$\int_0^T \int_0^t \mathbb{1}_{\{\tau \le u\}} \, \gamma_\tau^{i,u} \, dW_\tau^{\mathbb{Q}} \, du = \int_0^t \Big(\int_\tau^T \gamma_\tau^{i,u} \, du \Big) \, dW_\tau^{\mathbb{Q}} = \int_0^t \zeta_\tau^{(1)i} \, dW_\tau^{\mathbb{Q}}.$$

Analogously, the function $(\omega, t, u, y) \mapsto \delta_t^{i,u}(y)$, $i \in \mathcal{J}$, is $\mathcal{O} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R})$-measurable. So, in similarity to the considerations above and to the calculations before Corollary 4.3.6 in the discussion of the term insurance we are allowed to apply again Fubini's theorem for stochastic integrals:

$$\begin{aligned} &\int_0^T \int_0^t \int_{\mathbb{R}} \mathbb{1}_{\{\tau \le u\}} \, \delta_\tau^{i,u}(y) M^{\mathbb{Q}}(d\tau, dy) \, du \\ &= \int_0^t \int_{\mathbb{R}} \Big(\int_\tau^T \delta_\tau^{i,u}(y) \, du \Big) \, M^{\mathbb{Q}}(d\tau, dy) = \int_0^t \int_{\mathbb{R}} \zeta_\tau^{(2)i}(y) \, M^{\mathbb{Q}}(d\tau, dy) \, \mathbb{Q}\text{-a.s.} \end{aligned}$$

The computation so far yields the desired decomposition of $B_t^{-1}V^i(t,S_t)$.

$$\begin{aligned} B_t^{-1}V^i(t,S_t) = V^i(0,S_0) &+ \int_0^t \xi_\tau^i \, d\hat{S}_\tau + \int_0^t \zeta_\tau^{(1)i} \, dW_\tau^{\mathbb{Q}} \\ &+ \int_0^{\mathrm{t}} \int_{\mathbb{R}} \zeta_\tau^{(2)i}(y) \, M^{\mathbb{Q}}(d\tau, dy) \\ &- \int_0^t \Big(B_\tau^{-1} G_\tau^i + \sum_{k:k\neq i} \mu_\tau^{ik} \vartheta_\tau^{ik} \Big) \, d\tau, \quad \mathbb{Q}\text{-a.s.} \end{aligned}$$

Using (6.3) and integration by parts we now obtain the decomposition of V^*:

$$\begin{aligned} V_t^* &= Y_0 + \int_0^t dY_\tau + \sum_{i\in\mathcal{J}} \int_0^t d\left(I_\tau^i B_\tau^{-1} V^i(\tau, S_\tau) \right) \\ &= Y_0 + \int_0^t B_\tau^{-1} \sum_{i\in\mathcal{J}} \Big(I_\tau^i G_\tau^i \, d\tau + \sum_{k:k\neq i} g_\tau^{ik} \, dN_\tau^{ik} \Big) \\ &\quad + \sum_{i\in\mathcal{J}} \Big(\int_0^t I_{\tau-}^i \, d\left(B_\tau^{-1} V^i(\tau, S_\tau) \right) + \int_0^t B_\tau^{-1} V^i(\tau, S_{\tau-}) \, dI_\tau^i \\ &\quad + \left[I_\cdot^i, B_\cdot^{-1} V^i(\cdot, S_\cdot) \right]_t \Big). \end{aligned}$$

Since I_t^i is of finite variation, it is a quadratic pure-jump semimartingale (cf. Protter (2004), Chapter II, Theorem 26). So the continuous part of the quadratic covariation above is identically 0, and we get

$$\left[I_\cdot^i, B_\cdot^{-1} V^i(\cdot, S_\cdot) \right]_t = I_0^i B_0^{-1} V^i(0, S_0) + \sum_{0<\tau\leq t} \Delta I_\tau^i \Delta \left(B_\tau^{-1} V^i(\tau, S_\tau) \right).$$

This term simplifies considerably because of the initial state of the Markov chain Z_t implying $I_0^i = \mathbb{1}_{\{i=0\}}$ and because of the sum of jumps being

$$\sum_{0<\tau\leq t} \Delta I_\tau^i \Delta \left(B_\tau^{-1} V^i(\tau, S_\tau) \right) = 0 \quad \mathbb{Q}\text{-a.s.}, \quad 0 \leq t \leq T.$$

This holds, since at least $V^i(t,S_t)$ is continuous in probability and the processes I^i and S are independent by assumption. Moreover, we only treat càdlàg processes and hence face only countably many jump discontinuities which yields the claim by the σ-additivity of $\mathbb{Q}$ (cf. the consideration in (4.3)). Additionally observe that

$$dI_t^i = \sum_{k:k\neq i} \left(dN_t^{ki} - dN_t^{ik} \right).$$

Hence, $\mathbb{Q}$-a.s.,

$$\begin{aligned}
V_t^* &= V_0^* + \sum_{i\in\mathcal{J}} \Big(\int_0^t I^i_{\tau-}\xi^i_\tau \, d\hat{S}_\tau + \int_0^t I^i_{\tau-}\zeta^{(1)i}_\tau \, dW^{\mathbb{Q}}_\tau \\
&\quad + \int_0^t \int_{\mathbb{R}} I^i_{\tau-}\zeta^{(2)i}_\tau(y) \, M^{\mathbb{Q}}(d\tau, dy)\Big) \\
&\quad + \sum_{i,k:i\neq k} \int_0^t B_\tau^{-1}\Big(V^k(\tau, S_{\tau-}) - V^i(\tau, S_{\tau-})\Big) \, dN^{ik}_\tau \\
&\quad + \int_0^t B_\tau^{-1} \sum_{i\in\mathcal{J}} \Big(I^i_\tau G^i_\tau \, d\tau + \sum_{k:k\neq i} g^{ik}_\tau \, dN^{ik}_\tau \Big) \\
&\quad - \sum_{i\in\mathcal{J}} \int_0^t I^i_\tau \Big(B_\tau^{-1} G^i_\tau + \sum_{k:k\neq i} \mu^{ik}_\tau \vartheta^{ik}_\tau \Big) \, d\tau \\
&= V_0^* + \int_0^t \Big(\sum_{i\in\mathcal{J}} I^i_{\tau-}\xi^i_\tau \Big) \, d\hat{S}_\tau + K_t.
\end{aligned}$$

In the last step we used that $dN^{ik}_t - I^i_t \mu^{ik}_t \, dt = dM^{ik}_t$. By the same lines as in the proof of Corollary 4.3.1 one shows now that this is indeed the desired Galtchouk-Kunita-Watanabe decomposition. Observe therefore that $\big(\sum_{i\in\mathcal{J}} I^i_{t-}\xi^i_t\big)$ is $\mathbb{Q}$-a.s. bounded by a non-random constant (cf. proof of Theorem 3.3.4 and (3.10)) and so the integral with respect to $\hat{S}_t$ is a square-integrable martingale. Also by the boundedness of $(\sum_{i\in\mathcal{J}} I^i_{t-}\zeta^{(1)i}_t)$ and of $(\sum_{i\in\mathcal{J}} I^i_{t-}\zeta^{(2)i}_t(y))$ (in the sense of (3.11) and (3.12), respectively) the integrals with respect to $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}(\cdot,\cdot)$ are square-integrable martingales. The square-integrability of the ϑ^{ik}, which follows from (6.1), implies that the integrals with respect to the M^{ik} are also square-integrable martingales (cf. additionally (2.20)). The orthogonality of K_t and $\hat{S}$ is a direct consequence on the one hand of the orthogonal decomposition of $B_t^{-1}F^{\cdot}(t, S_t, u)$ and on the other hand of the independence of S and M^{ik}. □

Corollary 6.3.2. *The unique* 0*-admissible locally risk minimizing strategy* φ *(under* $\mathbb{P}$*) for the payment process* Y *of an unit-linked life insurance contract is given by*

$$(\xi_t, \eta_t) = \Big(\sum_{i\in\mathcal{J}} I^i_{t-}\xi^i_t, \sum_{i\in\mathcal{J}} I^i_t B_t^{-1} V^i(t, S_t) - \sum_{i\in\mathcal{J}} I^i_{t-}\xi^i_t \, \hat{S}_t \Big), \quad 0 \le t \le T.$$

The intrinsic risk process (under $\mathbb{Q}$) is for $0 \le t \le T$ given by

$$R_t(\varphi) = \sum_{i,j\in\mathcal{J}} I_t^i \int_t^T \Big(\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)j}\big)^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)j}(y)\big)^2 \mid \mathcal{F}_t]\nu_\tau^{\mathbb{Q}}(dy)\Big) \\ p_{ij}(t,\tau)\,d\tau \\ + \sum_{i\in\mathcal{J}} I_t^i \int_t^T \sum_{j,k:j\neq k} \mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{jk}\big)^2 \mid \mathcal{F}_t] p_{ij}(t,\tau)\mu_\tau^{jk}\,d\tau,$$

where ξ^i, $\zeta^{(1)j}$, $\zeta^{(2)j}$ and ϑ^{jk} are taken from Theorem 6.3.1.

Proof. The hedging strategy is an immediate consequence of Proposition 5.2.6 and Theorem 5.4.2. Further $R_t(\varphi) = \mathbb{E}^{\mathbb{Q}}[(K_T - K_t)^2|\mathcal{F}_t]$ with K_t as defined in Theorem 6.3.1. To simplify this expression we use the pairwise orthogonality of the integrals with respect to $W^{\mathbb{Q}}$, $M^{\mathbb{Q}}(\cdot,\cdot)$ and M^{ik}. Note that in our case $\int_0^t \cdots W_\tau^{\mathbb{Q}}$ is a continuous, $\int_0^t\int_{\mathbb{R}} \cdots M^{\mathbb{Q}}(d\tau, dy)$ is a purely discontinuous martingale and that the M^{ik} are independent of $W^{\mathbb{Q}}$ and $M^{\mathbb{Q}}$ by assumption. For $0 \le t \le T$ one gets

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}[\Big(\int_t^T \sum_{j\in\mathcal{J}} I_{\tau-}^j \zeta_\tau^{(1)j}\, dW_\tau^{\mathbb{Q}}\Big)^2 \mid \mathcal{F}_t] \\ &= \sum_{j\in\mathcal{J}} \int_t^T \mathbb{E}^{\mathbb{Q}}[I_\tau^j \mid \mathcal{F}_t]\, \mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)j}\big)^2 \mid \mathcal{F}_t]\, d\tau, \\ &= \sum_{i,j\in\mathcal{J}} I_t^i \int_t^T \mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)j}\big)^2 \mid \mathcal{F}_t] p_{ij}(t,\tau)\, d\tau. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}[\Big(\int_t^T \int_{\mathbb{R}} \sum_{j\in\mathcal{J}} I_{\tau-}^j \zeta_\tau^{(2)j}(y)\, M^{\mathbb{Q}}(d\tau, dy)\Big)^2 \mid \mathcal{F}_t] \\ &= \sum_{i,j\in\mathcal{J}} I_t^i \int_t^T \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)j}(y)\big)^2 \mid \mathcal{F}_t] p_{ij}(t,\tau)\nu_\tau^{\mathbb{Q}}(dy)\, d\tau. \end{aligned}$$

Finally, (2.20) and the mutual orthogonality of the M^{ik} itself yields

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}[\Big(\sum_{j,k:j\neq k} \int_t^T \vartheta_\tau^{jk}\, dM_\tau^{jk}\Big)^2 \mid \mathcal{F}_t] \\ &= \mathbb{E}^{\mathbb{Q}}[\int_t^T \sum_{j,k:j\neq k} \big(\vartheta_\tau^{jk}\big)^2 \lambda_\tau^{jk}\, d\tau \mid \mathcal{F}_t] \\ &= \sum_{i\in\mathcal{J}} I_t^i \int_t^T \sum_{j,k:j\neq k} \mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{jk}\big)^2 \mid \mathcal{F}_t] p_{ij}(t,\tau)\mu_\tau^{jk}\, d\tau. \quad \square \end{aligned}$$

By Proposition 5.2.6 we know that the portfolio value of the locally risk-minimizing strategy satisfies

$$V_t(\varphi) = \mathbb{E}^{\mathbb{Q}}[Y_T - Y_t \mid \mathcal{F}_t] = \sum_{i \in \mathcal{J}} I_t^i B_t^{-1} V^i(t, S_t).$$

If we worked under the physical measure $\mathbb{P}$, $V_t(\varphi)$ would exactly correspond to the prospective reserve of a classical life insurance. Thus we call here $V_t(\varphi)$ the $\mathbb{Q}$-prospective reserve indicating the evaluation under the measure $\mathbb{Q}$. In the following we comment additionally on the strategy and the hedging risk. For a related discussion we also refer to Subsection 4.3.1. The locally risk-minimal investment in the stock at time t depending on the state i of the policy is

$$\xi_t^i = \sum_{j \in \mathcal{J}} \int_t^T \Big(p_{ij}(t,u) \big(\xi_t^i(u) + \sum_{k : k \neq j} \mu_u^{jk} \xi_t^{jk}(u) \big) \Big) \, du,$$

where

$$\xi_t^{\cdot}(u) = \frac{c^2}{\kappa_t} F^{\cdot\prime}(t, S_{t-}, u) + \frac{1}{\sigma_t \hat{S}_{t-} \kappa_t} \int_{\mathbb{R}} x J^{\cdot}(t,x,u) \nu_t^{\mathbb{Q}}(dx).$$

The difference from a classical Black-Scholes setting like in Møller (2001a) is apparent. If our Lévy process was merely a Brownian motion the measure $\nu_t^{\mathbb{Q}}(dx)$ would be equal to 0 and $\kappa_t \equiv c^2$: we would only invest $F^{\cdot\prime}(t, S_{t-}, u)$, as expected. In the presence of the underlying having jumps the locally risk-minimizing investment in the stock is nearly a weighted sum of $F^{\cdot\prime}(t, S_{t-}, u)$ and the jump $J^{\cdot}(t,x,u) = B_t^{-1}\{F^{\cdot}(t, S_{t-} + \sigma_t S_{t-} x, u) - F^{\cdot}(t, S_{t-}, u)\}$. Note that $c^2 t$ is the variance of the Brownian part $cW_t^{\mathbb{Q}}$ and that $\int_0^t \kappa_s \, ds = c^2 t + \int_0^t \int_{\mathbb{R}} x^2 \nu_s^{\mathbb{Q}}(dx) ds$ is the variance of L_t (under $\mathbb{Q}$). For the jump part the "weight" is not so obvious but, as seen before, the mean-value theorem yields $J^{\cdot}(t,x,u) = F^{\cdot\prime}(t, x_0, u) \sigma_t \hat{S}_{t-} x$ for some suitable intermediate value x_0 making the weighting more clear. Formula (6.4) shows that the insurer's intrinsic risk $R_t(\varphi)$ of the locally risk-minimizing strategy has two components (dependent on the policy being in state i):

(a) financial risk driven by trading in the market (only present in an incomplete financial market)

$$\sum_{j \in \mathcal{J}} \int_t^T \Big(\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)j}\big)^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)j}(y)\big)^2 \mid \mathcal{F}_t] \nu_\tau^{\mathbb{Q}}(dy) \Big) p_{ij}(t,\tau) \, d\tau.$$

(b) insurance risk driven by mortality

$$\int_t^T \sum_{j,k : j \neq k} \mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{jk}\big)^2 \mid \mathcal{F}_t] p_{ij}(t,\tau) \mu_\tau^{jk} \, d\tau.$$

Note that the pure financial risk appears only in an incomplete financial market and that Møller (2001a), considering merely a complete financial market, neglects this significant risk figure. The insurance risk is driven by the sum-at-risk, $\vartheta_t^{jk} = B_t^{-1}\big(g_t^{jk} + V^k(t,S_t) - V^j(t,S_t)\big)$, and results from the uncertainty of the insured lives. It is equal to the total intrinsic risk in Møller (2001a). Each time a transition of the policy from state j to state k takes place, the payment g_t^{jk} has to be cashed out contributing immediately to the insurer's loss. At the same time the state-wise $\mathbb{Q}$-prospective reserve $V^j(t,S_t)$ has to be adjusted to the new state k and is then given by $V^k(t,S_t)$. If $V^k(t,S_t) - V^j(t,S_t)$ is negative, it is used to cover the payment g_t^{jk}.

6.4 Examples

Without specifying the type of insurance contract, there are two simple choices of contract functions at which we briefly have a look at.

If the contractual payments g^{jk} and G^j are deterministic with respect to the financial market, nothing is invested in the stock. Theorem 3.3.4 immediately implies here that $\xi^{\cdot} \equiv 0$ and that no financial risk is present, i.e. $\zeta^{(1)\cdot} \equiv \zeta^{(2)\cdot} \equiv 0$. In this case the insurance risk becomes the risk of a classical life insurance. For a related result compare Example 4.3.4.

If the contract functions are given by $g_t^{jk} = G_t^j = S_t$, for all j,k and for all t, the insured takes over the whole financial risk and consequently it is not present for the insurer. From Theorem 3.3.4 one immediately derives that $\xi^{\cdot} \equiv 1$ and $\zeta^{(1)\cdot} \equiv \zeta^{(2)\cdot} \equiv 0$. This is again similarly discussed in Example 4.3.3.

As an application of the advanced and general theory presented so far, we will model now the unit-linked life annuity and the unit-linked term insurance both as a single contract and as a portfolio of n identical contracts. This illustrates the theory and helps to understand the foregoing discussion. For a standard Black-Scholes market the term insurance examples are also treated in Møller (2001a). Since there are some differences from the incomplete general Lévy-process financial market, we reconsider them for completeness of this thesis.

6.4.1 Single unit-linked life annuity insurance

We discuss a life annuity insurance issued to a single person against a single premium P at time 0. This contract immediately starts to pay the insured person an annuity conditional on the person being alive. Furthermore we assume that this payment is of unit-linked with guarantee type. To be more precise, conditional on being alive the insured receives at time $t \geq 0$ a guaranteed deterministic annuity $\gamma e^{\delta t} > 0$ increasing continuously by compounding interest with some rate $\delta > 0$. However, if the value of some

reference portfolio S_t exceeds this minimal payment the insured gets the amount S_t. For this contract the state space of the multi-state Markov model is $\mathcal{J} = \{0, 1\}$, where 0 represents the state *policy holder alive* and 1 the state *policy holder dead*. Let the policy holder be of age x at time 0 with remaining lifetime T_x after time 0. The policy holder dying at time T_x implies $N_t^{01} = \mathbb{1}(T_x \leq t)$ and moreover the intensity μ^{01} of the only possible transition from state 0 to state 1 is the hazard-rate function μ of T_x. Note that naturally there are no transitions form state 1 to state 0. The transition probabilities can be determined by Kolmogorov's backward equations (2.21):

$$\frac{d}{dt} p_{00}(t, u) = \mu_t^{01} p_{00}(t, u).$$

Hence,

$$p_{00}(t, u) = \exp\Big(- \int_t^u \mu_\tau \, d\tau \Big).$$

The probability $p_{00}(t, u)$ is the survival probability until time u given the policy holder is alive at time t, in actuarial notation this is written as ${}_{u-t}p_{x+t}$. In contrast

$$p_{01}(t, u) = {}_{u-t}\, q_{x+t} = 1 - p_{00}(t, u)$$

is therefore the probability that the person at age $x + t$ dies before time u. The contract functions are all equal to zero except of

$$\begin{aligned} G_t^0 = G^0(t, S_t) &= g^0(t, S_t) - P\, \mathbb{1}(t = 0), \quad \text{where} \\ g^0(t, S_t) &= \max(S_t, \gamma e^{\delta t}). \end{aligned}$$

This results in only one price process $F^0(t, S_t, u)$, which has to be evaluated depending on the particular Lévy-process model of the financial market (cf. Chapter 7). In case the Lévy process is a Brownian motion the Black-Scholes formula can be applied. The locally risk-minimal investment in the stock is now given by (cf. Corollary 6.3.2)

$$\xi_t = I_{t-}^0 \xi_t^0 + I_{t-}^1 \xi_t^1 = I_{t-}^0 \xi_t^0 = \mathbb{1}(T_x \geq t) \xi_t^0$$

since $\xi^1 \equiv 0$, that is, there are no more payments after the policyholder's death. By Theorem 6.3.1 we have now, since all other terms are zero,

$$\begin{aligned} \xi_t^0 &= \sum_{j=0}^{1} \int_t^T \Big(p_{0j}(t, u) \big(\xi_t^j(u) + \sum_{k : k \neq j} \mu_u^{jk} \xi_t^{jk}(u) \big) \Big) \, du \\ &= \int_t^T \xi_t^0(u) p_{00}(t, u) \, du, \end{aligned} \tag{6.4}$$

where (cf. Theorem 3.3.4)

$$\xi_t^0(u) = \frac{c^2}{\kappa_t} F^{0\prime}(t, S_{t-}, u) + \frac{1}{\sigma_t \hat{S}_{t-} \kappa_t} \int_{\mathbb{R}} x J^0(t, x, u) \nu_t^{\mathbb{Q}}(dx).$$

Similarly,

$$\begin{aligned}\eta_t &= \mathbb{1}(T_x > t)B_t^{-1}V^0(t,S_t) + \mathbb{1}(T_x \le t)B_t^{-1}V^1(t,S_t) - \xi_t\hat{S}_t \\ &= \mathbb{1}(T_x > t)B_t^{-1}V^0(t,S_t) - \xi_t\hat{S}_t,\end{aligned}$$

since $V^1(t,S_t) = 0$ and where

$$\begin{aligned}V^0(t,S_t) &= \sum_{j=0}^{1}[\int_t^T p_{0j}(t,u)\big(F^j(t,S_t,u) + \sum_{k:k\neq j}\mu_u^{jk}F^{jk}(t,S_t,u)\big)\,du] \\ &= \int_t^T F^0(t,S_t,u)p_{00}(t,u)\,du.\end{aligned}$$

$V^0(t,S_t)$ is the expected total amount to be cashed out to the policy-holder in the interval $[t,T]$, if the policy-holder is still alive. The intrinsic hedging risk (cf. Corollary 6.3.2) is given by

$$\begin{aligned}R_t(\varphi) = &\sum_{i,j\in\{0,1\}} I_t^i \int_t^T \Big(\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)j}\big)^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}}\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)j}(y)\big)^2 \mid \mathcal{F}_t]\nu_\tau^{\mathbb{Q}}(dy)\Big) \\ &\qquad p_{ij}(t,\tau)\,d\tau \\ &+ \sum_{i=0}^{1} I_t^i \int_t^T \sum_{j,k:j\neq k}\mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{jk}\big)^2 \mid \mathcal{F}_t]p_{ij}(t,\tau)\mu_\tau^{jk}\,d\tau, \\ = &\,\mathbb{1}(T_x > t)\int_t^T \Big(\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)0}\big)^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}}\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)0}(y)\big)^2 \mid \mathcal{F}_t]\nu_\tau^{\mathbb{Q}}(dy)\Big) \\ &\qquad p_{00}(t,\tau)\,d\tau \\ &+ \mathbb{1}(T_x > t)\int_t^T \mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{01}\big)^2 \mid \mathcal{F}_t]p_{00}(t,\tau)\mu_\tau^{01}\,d\tau,\end{aligned}$$

since $\zeta^{(1)1} \equiv \zeta^{(2)1}(y) \equiv \mu^{10} \equiv p_{10} \equiv 0$. In analogy to (6.4) one obtains

$$\zeta_t^{(1)0} = \int_t^T \zeta_t^{(1)0}(u)p_{00}(t,u)\,du \quad \text{and} \quad \zeta_t^{(2)0}(y) = \int_t^T \zeta_t^{(2)0}(y,u)p_{00}(t,u)\,du,$$

where

$$\begin{aligned}\zeta_t^{(1)0}(u) &= c\sigma_t\hat{S}_{t-}\left(F^{0\prime}(t,S_{t-},u) - \xi_t^0(u)\right), \\ \zeta_t^{(2)0}(y,u) &= J^0(t,y,u) - \sigma_t\hat{S}_{t-}\xi_t^0(u)y.\end{aligned}$$

It remains to consider the sum at risk

$$\begin{aligned}\vartheta_t^{01} &= B_t^{-1}\left(g_t^{01} + V^1(t,S_t) - V^0(t,S_t)\right) \\ &= -B_t^{-1}V^0(t,S_t).\end{aligned}$$

This shows the amount by which the insurance company can reduce its reserve if the insured person dies at time t. Recall Definition 5.2.2 of the initial hedging costs. Moreover we have that

$$C_0(\varphi) = V_0(\varphi) + Y_0 = V^0(0, S_0) - P.$$

It is reasonable to demand $C_0(\varphi) = 0$, hence an idea of a minimal premium is

$$P = \int_0^T F^0(0, S_0, u) p_{00}(0, u)\, du.$$

We note that the hedging strategy is generally not self-financing and that there are additional costs of hedging during the interval $(0, T]$.

6.4.2 Single unit-linked term insurance

In this example we assume the same set-up as we did for the single life annuity except that now we consider a single unit-linked life term insurance with guarantee. The contract is again issued at time 0 against the single premium P and specifies the insurance benefit payable immediately upon death of the policy holder if occurred before time T. Again we have two contract functions different from zero:

$$g^{01}(t, S_t) = \max(S_t, \gamma e^{\delta t}) \quad \text{and} \quad G_t^0 = \tilde{g}_t^0\, \mathbb{1}_{\{t=0\}} = -P\, \mathbb{1}_{\{t=0\}}\,.$$

Thus we only have one price process $F^{01}(t, S_t, u)$ and the locally risk-minimizing strategy (cf. Corollary 6.3.2) for the stock is given by

$$\xi_t = I_{t-}^0 \xi_t^0 + I_{t-}^1 \xi_t^1 = I_{t-}^0 \xi_t^0 = \mathbb{1}(T_x \geq t) \xi_t^0,$$

where

$$\begin{aligned} \xi_t^0 &= \sum_{j=0}^{1} \int_t^T \Big(p_{0j}(t, u) \big(\xi_t^j(u) + \sum_{k:k\neq j} \mu_u^{jk} \xi_t^{jk}(u) \big) \Big)\, du \\ &= \int_t^T \xi_t^{01}(u) p_{00}(t, u) \mu_u\, du, \end{aligned}$$

with

$$\xi_t^{01}(u) = \frac{c^2}{\kappa_t} F^{01\prime}(t, S_{t-}, u) + \frac{1}{\sigma_t \hat{S}_{t-} \kappa_t} \int_{\mathbb{R}} x J^{01}(t, x, u) \nu_t^{\mathbb{Q}}(dx).$$

As for the single life annuity the locally risk-minimal investment in the bond is

$$\eta_t = \mathbb{1}(T_x > t) B_t^{-1} V^0(t, S_t) - \xi_t \hat{S}_t,$$

where

$$V^0(t,S_t) = \sum_{j=0}^{1}[\int_t^T p_{0j}(t,u)\big(F^j(t,S_t,u) + \sum_{k:k\neq j}\mu_u^{jk}F^{jk}(t,S_t,u)\big)\,du]$$
$$= \int_t^T F^{01}(t,S_t,u)p_{00}(t,u)\mu_u\,du.$$

Note that $p_{00}(t,u)\mu_u$ is the probability that the person survives from t to u and dies immediately at u. The intrinsic risk is (cf. Corollary 6.3.2)

$$R_t(\varphi) = \sum_{i,j\in\{0,1\}} I_t^i \int_t^T \Big(\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)j}\big)^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}}\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)j}(y)\big)^2 \mid \mathcal{F}_t]\nu_\tau^{\mathbb{Q}}(dy)\Big)$$
$$p_{ij}(t,\tau)\,d\tau$$
$$+\sum_{i=0}^{1} I_t^i \int_t^T \sum_{j,k:j\neq k}\mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{jk}\big)^2 \mid \mathcal{F}_t]p_{ij}(t,\tau)\mu_\tau^{jk}\,d\tau,$$
$$= \mathbb{1}(T_x > t)\int_t^T \Big(\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)0}\big)^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}}\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)0}(y)\big)^2 \mid \mathcal{F}_t]\nu_\tau^{\mathbb{Q}}(dy)\Big)$$
$$p_{00}(t,\tau)\,d\tau$$
$$+\,\mathbb{1}(T_x > t)\int_t^T \mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{01}\big)^2 \mid \mathcal{F}_t]p_{00}(t,\tau)\mu_\tau^{01}\,d\tau,$$

where

$$\zeta_t^{(1)0} = \int_t^T \zeta_t^{(1)01}(u)p_{00}(t,u)\mu_u\,du,$$
$$\zeta_t^{(2)0}(y) = \int_t^T \zeta_t^{(2)01}(y,u)p_{00}(t,u)\mu_u\,du,$$

with

$$\zeta_t^{(1)01}(u) = c\sigma_t\hat{S}_{t-}\left(F^{01\prime}(t,S_{t-},u) - \xi_t^{01}(u)\right),$$
$$\zeta_t^{(2)01}(y,u) = J^{01}(t,y,u) - \sigma_t\hat{S}_{t-}\xi_t^{01}(u)y.$$

In this case the sum at risk is given by

$$\vartheta_t^{01} = B_t^{-1}\left(g_t^{01} - V^0(t,S_t)\right),$$

which clearly shows the origin of the insurance risk. If the person dies at time t the company's effort is the payment g_t^{01} less reserve $V^0(t,S_t)$. Further, the initial costs are

$$C_0(\varphi) = V_0(\varphi) - P = \int_0^T F^{01}(0,S_0,u)p_{00}(0,u)\mu_u\,du - P.$$

An idea of a minimal premium could therefore be

$$P = \int_0^T F^{01}(0,S_0,u)p_{00}(0,u)\mu_u\,du.$$

6.4.3 Portfolio of n unit-linked life annuity contracts

Starting form the single unit-linked life annuity with guarantee (cf. Subsection 6.4.1), we consider now a portfolio of n identical such contracts issued to n policy holders with i.i.d. remaining lifetimes and common hazard-rate function μ. In Subsection 2.4.2, to which we refer, we already discussed the embedding of a portfolio of independent and identically distributed lives in the multi-state Markov model. Recall that in this case the state space is $\mathcal{J} = \{0, 1, \dots, n\}$, where state j corresponds to exactly j policy holders having died, and that the transition probabilities satisfy $p_{jk}(t,u) = 0$ for $k < j$. Since each policy holder receives the same annuity, the non-zero state-dependent payment functions are given by

$$\begin{aligned} G^j(t,S_t) &= g^j(t,S_t) - nP\,\mathbb{1}(t=0), \quad \text{where} \\ g^j(t,S_t) &= (n-j)\max(S_t, \gamma e^{\delta t}). \end{aligned}$$

This immediately results in the price process $F^j(t,S_t,u) = (n-j)F(t,S_t,u)$, where $F(t,S_t,u)$ denotes the price of the payment: $\max(S_t, \gamma e^{\delta t})$. The locally risk-minimizing hedging strategy for the stock investment (cf. Corollary 6.3.2) is then

$$\begin{aligned} \xi_t &= \sum_{i=0}^{n} I^i_{t-}\xi^i_t = \xi^{Z_{t-}}_t \\ &= \sum_{j=Z_{t-}}^{n} \int_t^T \Big(p_{Z_{t-}j}(t,u)\big(\xi^j_t(u) + \sum_{k:k\neq j} \mu^{jk}_u \xi^{jk}_t(u)\big)\Big)\,du \\ &= \sum_{j=Z_{t-}}^{n} \int_t^T p_{Z_{t-}j}(t,u)\xi^j_t(u)\,du = \sum_{j=Z_{t-}}^{n} \int_t^T p_{Z_{t-}j}(t,u)(n-j)\xi_t(u)\,du, \end{aligned}$$

where

$$\xi_t(u) = \frac{c^2}{\kappa_t} F'(t,S_{t-},u) + \frac{1}{\sigma_t \hat{S}_{t-}\kappa_t} \int_{\mathbb{R}} xJ(t,x,u)\nu^{\mathbb{Q}}_t(dx).$$

One observes now that $(n - Z_u)$ given Z_t $(u > t)$ follows a binomial distribution with parameters $((n-Z_t), p(t,u))$, where $p(t,u) = {}_{u-t}p_{x+t} = \exp\big(-\int_t^u \mu_\tau\,d\tau\big)$. Hence,

$$\sum_{j=Z_t}^{n} p_{Z_t,j}(t,u)(n-j) = \mathbb{E}^{\mathbb{Q}}[(n-Z_u) \mid Z_t] = (n-Z_t)p(t,u),$$

and

$$\xi_t = (n - Z_{t-}) \int_t^T \xi_t(u)p(t,u)\,du.$$

That is, the investment is adjusted according to the number of persons in the portfolio at time t. For the bond we get

$$\eta_t = \sum_{i\in\mathcal{J}} I_t^i B_t^{-1} V^i(t,S_t) - \xi_t \hat{S}_t = B_t^{-1} V^{Z_t}(t,S_t) - \xi_t \hat{S}_t,$$

where

$$\begin{aligned} V^{Z_t}(t,S_t) &= \sum_{j=Z_t}^{n} [\int_t^T p_{Z_t j}(t,u)\big(F^j(t,S_t,u) + \sum_{k:k\neq j} \mu_u^{jk} F^{jk}(t,S_t,u)\big)\, du] \\ &= \sum_{j=Z_t}^{n} \int_t^T p_{Z_t j}(t,u)(n-j)F(t,S_t,u)\, du \\ &= (n-Z_t)\int_t^T F(t,S_t,u)p(t,u)\, du, \end{aligned} \tag{6.5}$$

which is the single unit-linked life annuity's $\mathbb{Q}$-prospective reserve times the number of insured persons at time t. The computation of the hedging risk (cf. Corollary 6.3.2) yields that

$$\begin{aligned} R_t(\varphi) = \sum_{j=Z_t}^{n} \int_t^T \Big(& \mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)j}\big)^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)j}(y)\big)^2 \mid \mathcal{F}_t]\nu_\tau^{\mathbb{Q}}(dy)\Big) \\ & p_{Z_t j}(t,\tau)\, d\tau \\ + \sum_{j=Z_t}^{n} \int_t^T & \mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{j,j+1}\big)^2 \mid \mathcal{F}_t] p_{Z_t j}(t,\tau)\mu_\tau^{j,j+1}\, d\tau, \end{aligned}$$

since, we recall,

$$\mu_t^{jk} = \begin{cases} (n-j)\mu_t & k = j+1, \\ -(n-j)\mu_t & k = j, \\ 0 & \text{else.} \end{cases}$$

Let us consider $\sum_{j=Z_t}^n \big(\zeta_\tau^{(1)j}\big)^2$ more detailed (cf. Theorem 6.3.1).

$$\begin{aligned} \sum_{j=Z_t}^{n} \big(\zeta_\tau^{(1)j}\big)^2 &= \sum_{j=Z_t}^{n} \Big(\sum_{l=0}^{n} \int_\tau^T \Big(p_{jl}(\tau,u)\big(\zeta_\tau^{(1)l}(u) + \sum_{k:k\neq l} \mu_u^{lk}\zeta_\tau^{(1)lk}(u)\big)\Big)\, du\Big)^2 \\ &= \sum_{j=Z_t}^{n} \Big(\sum_{l=j}^{n} \int_\tau^T p_{jl}(\tau,u)(n-l)\zeta_\tau^{(1)}(u)\, du\Big)^2 \\ &= \sum_{j=0}^{n-Z_t} \Big(\sum_{l=Z_t}^{n-j} \int_\tau^T p_{Z_t+j,l+j}(\tau,u)(n-j-l)\zeta_\tau^{(1)}(u)\, du\Big)^2 \\ &= \sum_{j=Z_t}^{n} (n-j)^2 \Big(\int_\tau^T p(\tau,u)\zeta_\tau^{(1)}(u)\, du\Big)^2, \end{aligned}$$

where $\zeta_\tau^{(1)}(u) = c\sigma_t \hat{S}_{\tau-} \left(F'(\tau, S_{\tau-}, u) - \xi_\tau(u)\right)$. Analogously we derive

$$\sum_{j=Z_t}^{n} \left(\zeta_\tau^{(2)j}\right)^2 = \sum_{j=Z_t}^{n} (n-j)^2 \Big(\int_\tau^T p(\tau, u) \zeta_\tau^{(2)}(y, u)\, du \Big)^2,$$

where $\zeta_\tau^{(2)}(y, u) = J(\tau, y, u) - \sigma_\tau \hat{S}_{\tau-} \xi_\tau(u) y$. Let now

$$\begin{aligned} \rho_t(\tau) = \mathbb{E}^{\mathbb{Q}}[\Big(\int_\tau^T p(\tau, u) \zeta_\tau^{(1)}(u)\, du \Big)^2 \mid \mathcal{F}_t] \\ + \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\Big(\int_\tau^T p(\tau, u) \zeta_\tau^{(2)}(y, u)\, du \Big)^2 \mid \mathcal{F}_t] \nu_\tau^{\mathbb{Q}}(dy), \end{aligned}$$

then

$$\begin{aligned} R_t(\varphi) = \sum_{j=Z_t}^{n} \int_t^T \rho_t(\tau)(n-j)^2 p_{Z_t j}(t, \tau)\, d\tau \\ + \sum_{j=Z_t}^{n} \int_t^T \mathbb{E}^{\mathbb{Q}}[\left(\vartheta_\tau^{j,j+1}\right)^2 \mid \mathcal{F}_t](n-j) p_{Z_t j}(t, \tau) \mu_\tau\, d\tau. \end{aligned}$$

This simplifies further, using

$$\begin{aligned} \sum_{j=Z_t}^{n} (n-j)^2 p_{Z_t j}(t, \tau) &= \mathbb{E}^{\mathbb{Q}}[(n - Z_\tau)^2 \mid Z_t] \\ &= (n - Z_t) p(t, \tau)(1 - p(t, \tau)) + (n - Z_t)^2 p^2(t, \tau), \end{aligned}$$

and

$$\vartheta_\tau^{j,j+1} = B_\tau^{-1} \left(g_\tau^{j,j+1} + V^{j+1}(\tau, S_\tau) - V^j(\tau, S_\tau) \right),$$

for which we have with (6.5) and $j = 0, \ldots, n - Z_t$ that

$$\begin{aligned} \vartheta_\tau^{Z_t+j, Z_t+j+1} &= B_\tau^{-1} \big((n - Z_t - j - 1) - (n - Z_t - j) \big) \\ &\qquad \int_\tau^T F(\tau, S_\tau, u) p(\tau, u)\, du \\ &= -B_\tau^{-1} \int_\tau^T F(\tau, S_\tau, u) p(\tau, u)\, du. \end{aligned}$$

This shows exactly the amount by which the insurance company can reduce its reserve if a person in the portfolio dies. Hence,

$$\begin{aligned} R_t(\varphi) = (n - Z_t) \int_t^T p(t, \tau) \big(1 - p(t, \tau) + (n - Z_t) p(t, \tau) \big) \rho_t(\tau)\, d\tau \\ + (n - Z_t) \int_t^T \mathbb{E}^{\mathbb{Q}}[\Big(B_\tau^{-1} \int_\tau^T F(\tau, S_\tau, u) p(\tau, u)\, du \Big)^2 \mid \mathcal{F}_t] \\ p(t, \tau) \mu_\tau\, d\tau. \end{aligned}$$

For the premium we obtain again that

$$C_0(\varphi) = V_0(\varphi) + Y_0 = V^0(0, S_0) - nP = n \int_0^T F(0, S_0, u)p(0, u)\, du - nP.$$

Note that all the portfolio results reduce to the single life annuity of Subsection 6.4.1 if $n = 1$.

6.4.4 Portfolio of n unit-linked term insurance contracts

To conclude the chapter, let us finally consider the same insurance portfolio as in Subsection 6.4.3, however, for n identical unit-linked term insurance contracts being already discussed for the single policy case in Subsection 6.4.2. Since the computations here are very similar to the computations in the previous examples, we only briefly summarize the results. If an insured individual dies, we observe a transition from state j to state $j+1$ and the company has to cash out

$$g^{jk}(t, S_t) = \mathbb{1}_{\{k=j+1\}} \max(S_t, \gamma e^{\delta t}),$$

which results in the price process $F^{jk}(t, S_t, u) = \mathbb{1}_{\{k=j+1\}} F(t, S_t, u)$. The premium is due at issuing date, thus

$$G_t^0 = \tilde{g}_t^0\, \mathbb{1}_{\{t=0\}} = -nP\, \mathbb{1}_{\{t=0\}}\,.$$

The locally risk-minimal stock investment is (cf. Corollary 6.3.2)

$$\begin{aligned}
\xi_t &= \sum_{i=0}^{n} I_{t-}^i \xi_t^i = \xi_t^{Z_{t-}} = \sum_{j=Z_{t-}}^{n} \int_t^T \Big(p_{Z_{t-}j}(t, u) \sum_{k:k\neq j} \mu_u^{jk} \xi_t^{jk}(u) \Big)\, du \\
&= \sum_{j=Z_{t-}}^{n} \int_t^T p_{Z_{t-}j}(t, u)(n-j)\mu_u \xi_t(u)\, du \\
&= (n - Z_{t-}) \int_t^T \xi_t(u) p(t, u) \mu_u\, du
\end{aligned}$$

with

$$\xi_t(u) = \frac{c^2}{\kappa_t} F'(t, S_{t-}, u) + \frac{1}{\sigma_t \hat{S}_{t-} \kappa_t} \int_{\mathbb{R}} x J(t, x, u) \nu_t^{\mathbb{Q}}(dx).$$

Analogously to (6.5) one derives

$$\eta_t = \sum_{i\in\mathcal{J}} I_t^i B_t^{-1} V^i(t, S_t) - \xi_t \hat{S}_t = B_t^{-1} V^{Z_t}(t, S_t) - \xi_t \hat{S}_t$$

with

$$V^{Z_t}(t, S_t) = (n - Z_t) \int_t^T F(t, S_t, u) p(t, u) \mu_u\, du.$$

Also the hedging risk is

$$R_t(\varphi) = \sum_{j=Z_t}^{n} \int_t^T \Big(\mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(1)j}\big)^2 \mid \mathcal{F}_t] + \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\big(\zeta_\tau^{(2)j}(y)\big)^2 \mid \mathcal{F}_t] \nu_\tau^{\mathbb{Q}}(dy) \Big) \\ p_{Z_t j}(t,\tau)\, d\tau \\ + \sum_{j=Z_t}^{n} \int_t^T \mathbb{E}^{\mathbb{Q}}[\big(\vartheta_\tau^{j,j+1}\big)^2 \mid \mathcal{F}_t] p_{Z_t j}(t,\tau) \mu_\tau^{j,j+1}\, d\tau,$$

where

$$\sum_{j=Z_t}^{n} \big(\zeta_\tau^{(1)j}\big)^2 = \sum_{j=Z_t}^{n} (n-j)^2 \Big(\int_\tau^T \zeta_\tau^{(1)}(u) p(\tau,u) \mu_u\, du \Big)^2,$$

and

$$\sum_{j=Z_t}^{n} \big(\zeta_\tau^{(2)j}\big)^2 = \sum_{j=Z_t}^{n} (n-j)^2 \Big(\int_\tau^T \zeta_\tau^{(2)}(y,u) p(\tau,u) \mu_u\, du \Big)^2.$$

Again we have here, $\zeta_\tau^{(1)}(u) = c\sigma_t \hat{S}_{\tau-} \left(F'(\tau, S_{\tau-}, u) - \xi_\tau(u)\right)$ and $\zeta_\tau^{(2)}(y,u) = J(\tau,y,u) - \sigma_\tau \hat{S}_{\tau-} \xi_\tau(u) y$. Furthermore we observe that

$$\vartheta_\tau^{j,j+1} = B_\tau^{-1} \Big(\max(S_\tau, \gamma e^{\delta\tau}) + V^{j+1}(\tau, S_\tau) - V^j(\tau, S_\tau) \Big) = \vartheta_\tau,$$

since for $j = 0, \ldots, n - Z_t$ it holds that

$$V^{Z_t+j+1}(\tau, S_\tau) - V^{Z_t+j}(\tau, S_\tau) = -\int_\tau^T F(\tau, S_\tau, u) p(\tau,u) \mu_u\, du.$$

This has the following interpretation. If an insured person dies at time τ, the insurance company reduces its $\mathbb{Q}$-prospective reserve by the amount $\int_\tau^T F(\tau, S_\tau, u) p(\tau,u) \mu_u\, du$ to cover the payment $\max(S_\tau, \gamma e^{\delta\tau})$. Let again

$$\rho_t(\tau) = \mathbb{E}^{\mathbb{Q}}[\Big(\int_\tau^T \zeta_\tau^{(1)}(u) p(\tau,u) \mu_u\, du \Big)^2 \mid \mathcal{F}_t] \\ + \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[\Big(\int_\tau^T \zeta_\tau^{(2)}(y,u) p(\tau,u) \mu_u\, du \Big)^2 \mid \mathcal{F}_t] \nu_\tau^{\mathbb{Q}}(dy),$$

which yields

$$R_t(\varphi) = (n - Z_t) \int_t^T p(t,\tau) \big(1 - p(t,\tau) + (n - Z_t) p(t,\tau)\big) \rho_t(\tau)\, d\tau \\ + (n - Z_t) \int_t^T \mathbb{E}^{\mathbb{Q}}[\vartheta_\tau^2 \mid \mathcal{F}_t] p(t,\tau) \mu_\tau\, d\tau.$$

An idea of a minimal premium may again derived from the initial hedging costs:

$$C_0(\varphi) = V^0(0, S_0) - nP = n \int_0^T F(0, S_0, u) p(0, u) \mu_u \, du - nP.$$

In Subsection 4.3.2 we already considered a portfolio of unit-linked term insurance contracts, however, with the constraint that the insurance benefits are deferred and cashed out at the end T of the insurance period. In Chapter 4 we had to make this unrealistic additional assumption, since we were not able to treat intermediate payment times. Compared to our result here, allowing intermediate payment times, we recall that in Corollary 4.3.7 we obtained almost the same result. Only the bond investment, η_t, differs by the amount $\int_0^t B_\tau g(\tau, S_\tau) \, d \sum_{j=0}^{n-1} N_\tau^{j,j+1}$ which represents in Chapter 4 the payments already set aside to be cashed out at time T.

Chapter 7

Jump-diffusion stock model

7.1 Introduction

In the previous chapters we derived hedging strategies for unit-linked life insurance contracts modeling the *unit* in a Lévy-process financial market. However, so far we have not specified the Lévy process behind the financial market and so, given a certain payoff $g(u, S_u)$ for a contract, the risk-neutral price process $F(t, S_t, u)$ of Definition 3.2.1 and its derivative $F'(t, S_t, u)$ were an abstract ingredient for the hedging strategy and the involved hedging risk. Of course, depending on the *unit*, one has to decide which Lévy process fits best. In case the Lévy process of the financial market is a Brownian motion one has the Black-Scholes option pricing formula to express explicitly the price process of unit-linked with guarantee life insurance contracts. For almost all other Lévy processes, besides possibly very few jump-diffusions, or more general payoff structures one usually has no explicit formula. In particular for infinite activity Lévy-processes (cf. Chapter 2) one has to rely at an early stage on numerical methods to compute the price process. A useful tool for numerical evaluations in those cases is the Feynman-Kac partial integro-differential equation of Theorem 3.2.4. In Chapter 12 of Cont and Tankov (2004) various methods to solve this equation numerically are discussed. For finite activity Lévy-processes things are a little bit better. The two most common jump-diffusion models, the Merton and the Kou model, feature an *almost closed* formula for the price process of common options; it is a quickly converging series.

We decided to consider the jump-diffusion model for option pricing proposed in Kou (2002). The model assumes that the logarithm of the asset price follows a jump-diffusion, where the jump sizes are asymmetric double exponential distributed. The jumps may be interpreted as rare events due to unexpected new market information. According to Kou (2002) this model features several advantages compared to a Black-Scholes financial market. First, from empirical investigations it is widely known that asset return dis-

tributions incorporate asymmetric leptokurtic features. This means they may be skewed to the left and may have a higher peak and two asymmetric heavier tails than those of a normal distribution. This follows by the fact that markets tend to have both overreaction and underreaction to various good or bad news. Further, in an option pricing framework one usually observes a volatility smile effect (i.e. the implied volatility curve is a convex function of the strike price). Both the leptokurtic features and the volatility smile effect can be explained by the asymmetric double exponential jump-diffusion model. Second, under a specific change of measure, it is possible to derive in this model closed-form solutions to a number of option pricing problems such as European call and put options as well as path dependent options like American or barrier options (cf. Kou (2002), Kou and Wang (2003), and Kou and Wang (2004)).

The decisive point, however, is the change of the historical probability measure to an equivalent risk-neutral probability under which the price process of an option is calculated. In Kou (2002) the risk-neutral measure is chosen considering a rational expectations equilibrium framework in which an investor with a power utility function tries to maximize his utility. Under some restrictive assumptions on the model parameters, like interest rate and drift, the logarithm of the asset price process stays in the class of asymmetric double exponential jump-diffusion processes. The derivation of the option pricing formula relies heavily on the distributional properties of this class (cf. Section 7.2) and it remains unclear what happens if another risk-neutral probability is chosen perhaps leading outside the class of asymmetric double exponential jump-diffusions. Moreover, it is an open question to find a risk-minimizing hedging strategy under an *arbitrary* chosen risk-neutral measure. This is a significant issue, since riskless hedging is not possible. We know that a Lévy-process financial market has in general an arbitrary number of equivalent risk-neutral measures and we are mainly interested in the derivation of locally risk-minimizing hedging strategies using the Föllmer-Schweizer measure as risk-neutral probability. For this reason we will investigate the behavior of an asymmetric double exponential jump-diffusion process under an equivalent change of measure to the Föllmer-Schweizer measure and show that in this case the process stays only in his distribution class, if the corresponding Girsanov parameter is equal to one or if it is equal to zero representing the case where no change of measure is necessary. It seems that only in those two special cases one has an analogous closed-form solution for the price process. The reason is the effect of the Girsanov parameter (i.e. the market price of risk) on the jump-size distribution. Since this effect does not depend on a specific distribution, our treatment is exemplarily for similar cases.

Before discussing the change of measure, we review some fundamental and needed results of Kou (2002). We imbed the model in our Lévy-process framework of Subsection 2.2.1 and in the financial market of Section 2.3.

Finally, we derive exemplarily the price process for an unit-linked pure endowment and term insurance with guarantee. Moreover, we compute the spatial derivative of the price process.

7.2 The asymmetric double exponential distribution

Let us begin discussing the asymmetric double exponential distribution and some of its main properties. In the following we work on some probability space $(\Omega, \mathbb{G}, \mathbb{P})$. Most of what is presented here can be traced back to Kou (2002).

Definition 7.2.1. *A random variable Y is said to follow an asymmetric double exponential distribution if it has a density with respect to the Lebesgue measure of the form*

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbb{1}_{[0,\infty)}(y) + q\eta_2 e^{\eta_2 y} \mathbb{1}_{(-\infty,0)}(y), \quad \eta_1 > 1,\, \eta_2 > 0,$$

where $p, q \geq 0$ and $p + q = 1$.

If Y is an asymmetric double exponential distributed random variable, then one has

$$Y \stackrel{d}{=} \begin{cases} \xi^+, & \text{with probability} \quad p \\ -\xi^-, & \text{with probability} \quad q, \end{cases}$$

where ξ^+ and ξ^- are exponential random variables with means $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$, respectively. Furthermore it is not very hard to compute the following moments of Y:

$$\begin{aligned} \mathbb{E}[Y] &= \frac{p}{\eta_1} - \frac{q}{\eta_2}, \\ \mathbb{V}\mathrm{ar}[Y] &= \frac{2p}{\eta_1^2} + \frac{2q}{\eta_2^2} - \left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)^2, \\ \mathbb{E}[e^Y] &= p\frac{\eta_1}{\eta_1 - 1} + q\frac{\eta_2}{\eta_2 + 1}. \end{aligned} \tag{7.1}$$

The condition $\eta_1 > 1$ in Definition 7.2.1 implies that $\mathbb{E}[e^Y] < \infty$. As described in the introduction we will consider jump-diffusions whose jumps follow an asymmetric double exponential distribution. It will turn out to be useful for it to know the distribution of the sum of asymmetric double exponential random variables on the one hand and on the other hand the distribution of the sum of a normal and a gamma distributed random variable, i.e. the sum of a normal random variable and a sum of exponential random variables.

Lemma 7.2.2. *Let ξ^+ and ξ^- be independent exponential distributed random variables with parameters η_1 and η_2, respectively. Then*

(a) $\mathbb{P}(\xi^+ > \xi^-) = \frac{\eta_2}{\eta_1+\eta_2}$ *and* $\mathbb{P}(\xi^+ < \xi^-) = \frac{\eta_1}{\eta_1+\eta_2}$.

(b) $\mathbb{P}(\xi^+ - \xi^- \le x \mid \xi^+ > \xi^-) = \mathbb{P}(\xi^+ \le x)$ *for* $x \ge 0$.

(c) $\mathbb{P}(\xi^+ - \xi^- \le x \mid \xi^+ < \xi^-) = \mathbb{P}(-\xi^- \le x)$ *for* $x \le 0$.

Proof. By symmetry it suffices to consider for the first claim

$$\mathbb{P}(\xi^+ > \xi^-) = \iint_{\{u \ge v \ge 0\}} \eta_1 e^{-\eta_1 u} \eta_2 e^{-\eta_2 v}\, d(u,v) = \frac{\eta_2}{\eta_1+\eta_2}.$$

For the second part we look for $x \ge 0$ at

$$\begin{aligned}\mathbb{P}(\xi^- < \xi^+ \le x + \xi^-) &= \iint_{\{0 \le v < u \le v+x\}} \eta_1 e^{-\eta_1 u} \eta_2 e^{-\eta_2 v}\, d(u,v) \\ &= \frac{\eta_2}{\eta_1+\eta_2}\left(1 - e^{-\eta_1 x}\right),\end{aligned}$$

and use part (a) to obtain

$$\mathbb{P}(\xi^+ - \xi^- \le x \mid \xi^+ > \xi^-) = \frac{\mathbb{P}(\xi^- < \xi^+ \le x + \xi^-)}{\mathbb{P}(\xi^+ > \xi^-)} = \mathbb{P}(\xi^+ \le x).$$

Let now $x \le 0$. Then

$$\begin{aligned}\mathbb{P}(\xi^+ - \xi^- \le x, \xi^+ < \xi^-) &= \mathbb{P}(\xi^- \ge -x + \xi^+, \xi^- > \xi^+) \\ &= \mathbb{P}(\xi^- \ge \xi^+ - x) \\ &= \iint_{\{v \ge u - x\}} \eta_1 e^{-\eta_1 u}\, \mathbb{1}_{[0,\infty)}(u) \eta_2 e^{-\eta_2 v}\, d(u,v) \\ &= \frac{\eta_1}{\eta_1+\eta_2} e^{\eta_2 x},\end{aligned}$$

and using again part (a) yields

$$\mathbb{P}(\xi^+ - \xi^- \le x \mid \xi^+ < \xi^-) = \frac{\mathbb{P}(\xi^- \ge -x + \xi^+, \xi^- > \xi^+)}{\mathbb{P}(\xi^+ < \xi^-)} = \mathbb{P}(-\xi^- \le x).$$

□

Applying the previous lemma one shows that the sum of i.i.d. asymmetric double exponential random variables is equal in distribution to a randomly mixed gamma random variable.

Proposition 7.2.3. *Let* Y_i, $i = 1, \dots, n$, $n \ge 1$, *be i.i.d. asymmetric double exponential random variables. Then*

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} \begin{cases} \sum_{i=1}^{k} \xi_i^+, & \text{with probability} \quad P_{n,k},\ k = 1, \dots, n, \\ -\sum_{i=1}^{k} \xi_i^-, & \text{with probability} \quad Q_{n,k},\ k = 1, \dots, n, \end{cases}$$

where ξ_i^+ and ξ_i^- are i.i.d. exponential random variables with rates η_1 and η_2, respectively. For $1 \le k \le n-1$ the probabilities are given by

$$P_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-k} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i} p^i q^{n-i},$$

and

$$Q_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-k} p^{n-i} q^i.$$

Moreover, $P_{n,n} = p^n$ and $Q_{n,n} = q^n$.

Proof. The claim follows by a random walk argument using Lemma 7.2.2, cf. Kou (2002). □

Proposition 7.2.4. *Let X be a normal distributed random variable with mean 0 and variance σ^2. Let further ξ_i, $i = 1, \dots, n$, $n \ge 1$, be i.i.d. exponential random variables with rate $\eta > 0$. Then for the random variables $X + \sum_{i=1}^n \xi_i$ and $X - \sum_{i=1}^n \xi_i$ holds:*

(a) The density functions are given by

$$f_{X+\sum_{i=1}^n \xi_i}(t) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} e^{-t\eta} \operatorname{Hh}_{n-1}\Big(-\frac{t}{\sigma} + \sigma\eta\Big),$$
$$f_{X-\sum_{i=1}^n \xi_i}(t) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} e^{t\eta} \operatorname{Hh}_{n-1}\Big(\frac{t}{\sigma} + \sigma\eta\Big),$$

where the function $\operatorname{Hh}_n(x)$ for $n \ge 0$ is defined as follows:

$$\operatorname{Hh}_n(x) := \int_x^\infty \operatorname{Hh}_{n-1}(y)\, dy = \frac{1}{n!} \int_x^\infty (t-x)^n e^{-t^2/2}\, dt,$$
$$\text{and} \quad \operatorname{Hh}_{-1}(x) := e^{-x^2/2}.$$

(b) The tail probabilities are given by

$$\mathbb{P}\Big(X + \sum_{i=1}^n \xi_i \ge x\Big) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} \operatorname{I}_{n-1}\Big(x; -\eta, -\frac{1}{\sigma}, -\sigma\eta\Big),$$
$$\mathbb{P}\Big(X - \sum_{i=1}^n \xi_i \ge x\Big) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} \operatorname{I}_{n-1}\Big(x; \eta, \frac{1}{\sigma}, -\sigma\eta\Big),$$

where

$$\operatorname{I}_n(c; \alpha, \beta, \delta) := \int_c^\infty e^{\alpha x} \operatorname{Hh}_n(\beta x - \delta)\, dx, \quad n \ge 0.$$

Proof. We include the proof of Kou (2002), since it is not very long. Recall $\sum_{i=1}^n \xi_i$ follows a gamma distribution with parameters n and η, hence its density is

$$f_{\sum_{i=1}^n \xi_i}(x) = \frac{\eta^n}{(n-1)!} x^{n-1} e^{-\eta x} \mathbb{1}_{(0,\infty)}(x).$$

Therefore,

$$\begin{aligned} f_{X+\sum_{i=1}^n \xi_i}(t) &= \int_{-\infty}^{\infty} f_{\sum_{i=1}^n \xi_i}(t-x) f_X(x)\, dx \\ &= e^{-t\eta}\eta^n \int_{-\infty}^{t} \frac{e^{x\eta}(t-x)^{n-1}}{(n-1)!} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\, dx \\ &= e^{-t\eta}\eta^n e^{(\sigma\eta)^2/2} \int_{-\infty}^{t} \frac{(t-x)^{n-1}}{(n-1)!} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\sigma^2\eta)^2}{2\sigma^2}}\, dx. \end{aligned}$$

Substituting now $y = (x - \sigma^2\eta)/\sigma$ yields

$$\begin{aligned} f_{X+\sum_{i=1}^n \xi_i}(t) &= e^{-t\eta} e^{(\sigma\eta)^2/2} \sigma^{n-1}\eta^n \int_{-\infty}^{\frac{t}{\sigma}-\sigma\eta} \frac{(t/\sigma - y - \sigma\eta)^{n-1}}{(n-1)!} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy \\ &= \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} e^{-t\eta}\, \mathrm{Hh}_{n-1}\Big(-\frac{t}{\sigma} + \sigma\eta\Big), \end{aligned}$$

where the last step follows from the substitution $y = -u$. The other density is derived analogously noting that

$$\mathbb{P}\Big(-\sum_{i=1}^n \xi_i \le x\Big) = \int_{-\infty}^{x} f_{\sum_{i=1}^n \xi_i}(-u)\, du.$$

Hence,

$$f_{X-\sum_{i=1}^n \xi_i}(t) = e^{t\eta}\eta^n e^{(\sigma\eta)^2/2} \int_{t}^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x+\sigma^2\eta)^2}{2\sigma^2}}\, dx.$$

Substituting $y = (x + \sigma^2\eta)/\sigma$ yields the claimed result. □

For a detailed treatment of the functions $\mathrm{Hh}_n(\cdot)$ and $\mathrm{I}_n(\cdot)$ we refer to Kou (2002) who also refers to Abramowitz and Stegun (1964). Those functions can be evaluated using the cumulative standard normal distribution function $\Phi(\cdot)$ and the following results. Obviously,

$$\mathrm{Hh}_0(x) = \sqrt{2\pi}\Phi(-x),$$

and furthermore for all $n \ge 1$ one has that

$$n\,\mathrm{Hh}_n(x) = \mathrm{Hh}_{n-2}(x) - x\,\mathrm{Hh}_{n-1}(x).$$

Depending on the parameters, the function $\mathrm{I}_n(\cdot)$ can be computed through the following relations. For $n \geq 0$ it holds that

$$\mathrm{I}_n(c;\alpha,\beta,\delta) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}\mathrm{Hh}_i(\beta c-\delta) + \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\Phi\Big(-\beta c+\delta+\frac{\alpha}{\beta}\Big),$$

for $\alpha \neq 0$, $\beta > 0$ and that

$$\mathrm{I}_n(c;\alpha,\beta,\delta) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}\mathrm{Hh}_i(\beta c-\delta) - \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\Phi\Big(\beta c-\delta-\frac{\alpha}{\beta}\Big),$$

for $\alpha < 0$, $\beta < 0$. Finally for $n \geq 0$ one uses that

$$\mathrm{I}_n(c;\alpha,\beta,\delta) = \begin{cases} \frac{1}{\beta}\mathrm{Hh}_{n+1}(\beta c-\delta) & \text{for} \quad \alpha = 0,\ \beta > 0, \\ \infty & \text{for} \quad \alpha \geq 0,\ \beta \leq 0, \\ \mathrm{Hh}_n(-\delta)e^{\alpha c} & \text{for} \quad \alpha < 0,\ \beta = 0. \end{cases}$$

The next proposition is vital for the evaluation of European call options to be considered below. In particular the derivation of the pricing formula depends on these results.

Proposition 7.2.5. *Let X be a normal random variable with mean 0 and variance σ^2 and let ξ_i, $i = 1,\ldots,n$, $n \in \mathbb{N}$, be i.i.d. exponential random variables with rate η. Then, for any real numbers α, β, $K \in \mathbb{R}$, $K/\alpha > 0$ and*

$$h = \log\Big(\frac{K}{\alpha}\Big) - \beta,$$

one gets for $n \geq 1$ the following expectations. For $\eta > 1$,

$$\mathbb{E}[\big(\alpha e^{\beta+X+\sum_{i=1}^n \xi_i} - K\big)^+] = \alpha e^{\beta}\frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}}e^{(\sigma\eta)^2/2}\,\mathrm{I}_{n-1}\Big(h;1-\eta,-\frac{1}{\sigma},-\sigma\eta\Big) - K\frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}}e^{(\sigma\eta)^2/2}\,\mathrm{I}_{n-1}\Big(h;-\eta,-\frac{1}{\sigma},-\sigma\eta\Big)$$

and for $\eta > 0$,

$$\mathbb{E}[\big(\alpha e^{\beta+X-\sum_{i=1}^n \xi_i} - K\big)^+] = \alpha e^{\beta}\frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}}e^{(\sigma\eta)^2/2}\,\mathrm{I}_{n-1}\Big(h;1+\eta,\frac{1}{\sigma},-\sigma\eta\Big) - K\frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}}e^{(\sigma\eta)^2/2}\,\mathrm{I}_{n-1}\Big(h;\eta,\frac{1}{\sigma},-\sigma\eta\Big).$$

Proof. Observe that

$$\begin{aligned}
&\mathbb{E}[\big(\alpha e^{\beta+X+\sum_{i=1}^n \xi_i} - K\big)^+] \\
&= \alpha e^{\beta} \int_h^\infty e^t f_{X+\sum_{i=1}^n \xi_i}(t)\, dt - K\,\mathbb{P}\left(X + \sum_{i=1}^n \xi_i \geq h\right) \\
&= \alpha e^{\beta} \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} \int_h^\infty e^t e^{-t\eta}\, \mathrm{Hh}_{n-1}\Big(-\frac{t}{\sigma} + \sigma\eta\Big)\, dt \\
&\quad - K\,\mathbb{P}\left(X + \sum_{i=1}^n \xi_i \geq h\right)
\end{aligned}$$

and that this expectation only exists if $\eta > 1$. Moreover for all $\eta > 0$ we have that

$$\begin{aligned}
&\mathbb{E}[\big(\alpha e^{\beta+(X-\sum_{i=1}^n \xi_i)} - K\big)^+] \\
&= \alpha e^{\beta} \int_h^\infty e^t f_{X-\sum_{i=1}^n \xi_i}(t)\, dt - K\,\mathbb{P}\left(X - \sum_{i=1}^n \xi_i \geq h\right) \\
&= \alpha e^{\beta} \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} \int_h^\infty e^t e^{t\eta}\, \mathrm{Hh}_{n-1}\Big(\frac{t}{\sigma} + \sigma\eta\Big)\, dt \\
&\quad - K\,\mathbb{P}\left(X - \sum_{i=1}^n \xi_i \geq h\right).
\end{aligned}$$

The claim follows then from Proposition 7.2.4. □

7.3 The model

In this section we discuss the model of Kou (2002) in the context of Subsection 2.2.1 (Basic properties of Lévy processes) and Section 2.3 (Financial market). We work on a filtered probability space $(\Omega, \mathbb{G}, (\mathcal{G}_t)_{0\leq t\leq T}, \mathbb{P})$ for some finite number $T > 0$. As always in this thesis the probability space is supposed to satisfy the usual conditions and we assume that $\mathcal{G}_0 = \{\emptyset, \Omega\}$, $\mathcal{G}_T = \mathbb{G}$.

7.3.1 The asymmetric double exponential jump-diffusion

We start with a Lévy process, $L = (L_t)_{0\leq t\leq T}$, which is defined by

$$L_t = cW_t + \sum_{i=1}^{N_t} (V_i - 1), \quad 0 \leq t \leq T, \tag{7.2}$$

where $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion, $c > 0$ is deterministic, $(N_t)_{0\leq t\leq T}$ is a Poisson process with rate λt and V_i is a sequence of i.i.d. nonnegative random variables such that $Y = \log(V)$ follows an asymmetric

double exponential distribution. Moreover, (W_t), (N_t) and (Y_i) are supposed to be stochastically independent. As before, we let the filtration $(\mathcal{G}_t)_{0\le t\le T}$ be the natural, completed filtration of $(L_t)_{0\le t\le T}$.

Lemma 7.3.1. *The Lévy measure of the process L has the form:*

$$\nu(dx) = \lambda f_{V-1}(x)\,dx,$$

where

$$f_{V-1}(x) = p\eta_1(1+x)^{-(1+\eta_1)}\,\mathbb{1}_{[0,\infty)}(x) + q\eta_2(1+x)^{-(1-\eta_2)}\,\mathbb{1}_{(-1,0)}(x),$$

with $\eta_1 > 1$, $\eta_2 > 0$, $p, q \ge 0$ and $p + q = 1$. $f_{V-1}(x)$ denotes the Lebesgue density of the jump size distribution of the process L.

Proof. The first claim follows from Proposition 2.2.12. Observe further that the distribution of $V - 1$ is given by

$$F_{V-1}(x) = \mathbb{P}(V - 1 \le x) = F_Y(\log(x+1)), \quad x > -1.$$

Let therefore $x > -1$. Then for the Lebesgue density $f_Y(x)$ of Y one has that

$$\begin{aligned} f_{V-1}(x) &= f_Y(\log(x+1))\frac{1}{1+x} \\ &= \frac{p\eta_1}{1+x}e^{-\eta_1\log(1+x)}\,\mathbb{1}_{[0,\infty)}(\log(1+x)) \\ &\quad + \frac{q\eta_2}{1+x}e^{\eta_2\log(1+x)}\,\mathbb{1}_{(-\infty,0)}(\log(1+x)), \end{aligned}$$

which yields the claim. □

Theorem 7.3.2. *The decomposition*

$$L_t = cW_t + M_t + at, \quad 0 \le t \le T,$$

where

$$M_t = \sum_{i=1}^{N_t}(V_i - 1) - \mathbb{E}[V_1 - 1]\lambda t,$$

and $a = \mathbb{E}[V_1 - 1]\lambda$ is the Lévy-Itô-decomposition of L. Furthermore, $M = (M_t)_{0\le t\le T}$ is a pure-jump martingale and if $\eta_1 > 2$, then M is a square-integrable martingale. $A = (A_t)_{0\le t\le T} = (at)_{0\le t\le T}$ is predictable and of finite variation implying L_t to be a special semi-martingale.

Proof. The decomposition follows from Theorem 2.2.4 and condition (2.2). That condition, being

$$\int_{\{|x|>1\}} |x|\ \nu(dx) < \infty,$$

is satisfied here, since the form of the Lévy measure (cf. Lemma 7.3.1) implies that $(1+x)^{-(1+\eta_1)}$ is the only relevant part to consider. For this part it holds now that $x(1+x)^{-(1+\eta_1)}$ behaves asymptotically as $x^{-(1+\varepsilon)}$ for some $\varepsilon > 0$ and $x \to \infty$, since we generally have that $\eta_1 > 1$. Moreover,

$$a = \mathbb{E}[L_1] = \mathbb{E}[\sum_{i=1}^{N_1}(V_i - 1)] = \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=1}^{n}(V_i - 1)]\, \mathbb{P}(N_1 = n).$$

Using the i.i.d. assumption of the V_i and the fact that N_1 follows a Poisson distribution we get that

$$a = \sum_{n=1}^{\infty} n\, \mathbb{E}[V_1 - 1] \frac{\lambda^n}{n!} e^{-\lambda} = \mathbb{E}[V_1 - 1] \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} e^{-\lambda} = \mathbb{E}[V_1 - 1]\lambda.$$

The function $A_t = at$ is obviously of finite variation and predictable, since it is monotonically increasing and continuous. Finally, if $\eta_1 > 2$ one has that

$$\int_{\{|x|>1\}} x^2\, \nu(dx) = \lambda p \eta_1 \int_1^{\infty} \frac{x^2}{(1+x)^{1+\eta_1}}\, dx < \infty,$$

since $x^2(1+x)^{-(1+\eta_1)}$ also behaves asymptotically as $x^{-(1+\varepsilon)}$ for some $\varepsilon > 0$ and $x \to \infty$. Hence in this case M is square-integrable (cf. Lemma 2.2.5). □

We remark that the predictable part A_t also could have been computed using the Lévy measure, that is,

$$a = \mathbb{E}[L_1] = \int_{\mathbb{R}} x\, \nu(dx) = \lambda \int_{\mathbb{R}} x f_{V-1}(x)\, dx.$$

This shows again the necessity demanding $\eta_1 > 1$, since $\mathbb{E}[V] = \mathbb{E}[e^Y] = p\frac{\eta_1}{\eta_1 - 1} + q\frac{\eta_2}{\eta_2 + 1}$.

7.3.2 Financial market

Let us now consider the financial market of Section 2.3 with the Lévy process L of (7.2). Using the stochastic differential equation (2.13) we describe the stock $S = (S_t)_{0 \le t \le T}$ via

$$dS_t = bS_{t-}\, dt + cS_{t-}\, dW_t + S_{t-} d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big), \tag{7.3}$$

where $S_0 > 0$ is deterministic and b, modeling the drift of the stock-price movement, is supposed to be a deterministic real constant. Also recall that for the risk-free interest rate r the riskless investment in this financial market is given by

$$B_t = e^{rt}, \quad 0 \le t \le T.$$

Lemma 7.3.3. *The process S_t has the following explicit form:*

$$S_t = S_0 \exp\Big(cW_t + \big(b - \frac{c^2}{2}\big)t + \sum_{i=1}^{N_t} Y_i\Big), \quad 0 \le t \le T.$$

By convention we set $\prod_{i=1}^{0} V_i = 1$.

Proof. The explicit formula for the stock price is generally formulated in (2.14) and reduces in our case to

$$S_t = S_0 \exp\Big(L_t + \big(b - \frac{c^2}{2}\big)t\Big) \prod_{0<s\le t} (1 + \Delta L_s) \exp(-\Delta L_s).$$

Having now $\Delta L_s = \Delta\Big(\sum_{i=1}^{N_s}(V_i - 1)\Big) = (V_{N_s} - 1)\Delta N_s$, we get that

$$\begin{aligned} S_t &= S_0 \exp\Big(cW_t + \sum_{k=1}^{N_t}(V_k - 1) + \big(b - \frac{c^2}{2}\big)t\Big) \prod_{i=1}^{N_t} V_i \exp\Big(-\sum_{j=1}^{N_t}(V_j - 1)\Big) \\ &= S_0 \exp\Big(cW_t + \big(b - \frac{c^2}{2}\big)t\Big) \prod_{i=1}^{N_t} V_i. \end{aligned}$$

The claim follows, since $Y_i = \log(V_i)$. □

To get an impression of the stock price behavior Figure 7.1 exemplarily displays four sample paths for a fixed set of parameters. From the previous lemma we see that the process S is the exponential function of the asymmetric double exponential jump-diffusion process $\theta t + cW_t + \sum_{i=1}^{N_t} Y_i$, where $\theta = b - \frac{c^2}{2}$. However, it is defined in (7.3) as the stochastic exponential of the Lévy process $\tilde{L}_t = bt + L_t$, i.e. $S_t = \mathcal{E}(\tilde{L})_t$. The construction of S_t illustrates therefore Lemma 5.8 in Goll and Kallsen (2000) which shows that $\mathcal{E}(\tilde{L})_t = e^{X_t}$, where

$$\begin{aligned} X_t &= \tilde{L}_t - \frac{c^2 t}{2} + \sum_{0<s\le t} \big(\log(1 + \Delta\tilde{L}_s) - \Delta\tilde{L}_s\big) \\ &= \theta t + cW_t + \sum_{i=1}^{N_t} Y_i = \log \mathcal{E}(\tilde{L})_t. \end{aligned}$$

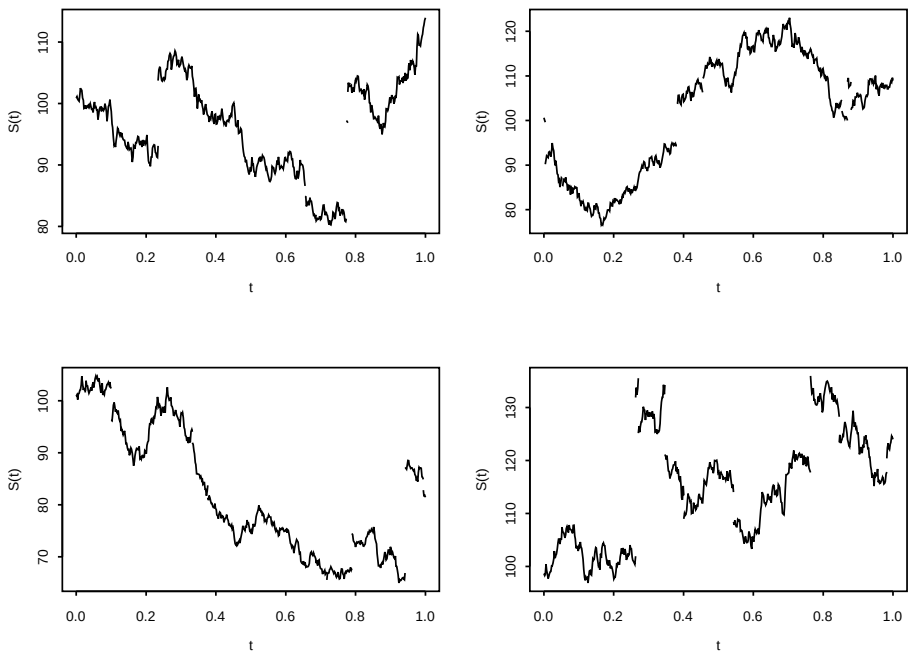


Figure 7.1: Stock price sample paths. Parameters: $S_0 = 100$, $c = 0.2$, $b = 0.05$, $\lambda = 5$, $\eta_1 = 17$, $\eta_2 = 15$, $p = 0.5$.

Moreover, Goll and Kallsen (2000) state a formula for the Lévy measure of X which we denote by $\nu_X(\cdot)$. According to this formula one has for any $-1 < a < b \leq \infty$ that

$$\begin{aligned}\nu_X((a,b)) &= \int \mathbb{1}_{(a,b)}(\log(1+x))\nu(dx) \\ &= \int \mathbb{1}_{(\log(1+a),\log(1+b))}(u)\lambda f_{V-1}(e^u - 1)e^u\, du \\ &= \int \mathbb{1}_{(\log(1+a),\log(1+b))}(u)\lambda f_Y(u)\, du\end{aligned}$$

Of course, in our case we already know the Lévy measure of the jump-diffusion process X. It is given by $\nu_X(dx) = \lambda f_Y(x)\, dx$ (cf. Proposition 2.2.12). As always for the financial market we require additionally the condition

$$\int_{\{|x|>1\}} |x|^3 \, \nu(dx) < \infty$$

to be satisfied. It is mainly formulated in (2.12) to guarantee the square-integrability of the martingale M both under the historical and under the

Föllmer-Schweizer measure. The following lemma provides a sufficient condition.

Lemma 7.3.4. *If $\eta_1 > 3$, then condition (2.12) is satisfied.*

Proof. We use the same argument as before. For $|x| > 1$ it holds that $\nu(dx) = \lambda p \eta_1 (1+x)^{-(1+\eta_1)}\, dx$. If $\eta_1 > 3$, then $x^3(1+x)^{-(1+\eta_1)}$ behaves asymptotically as $x^{-(1+\varepsilon)}$ for $\varepsilon > 0$ and for $x \to \infty$ which yields the claim. □

7.4 Risk-neutral measure

Here we work again on $(\Omega, \mathbb{G}, (\mathcal{G}_t)_{0\le t\le T}, \mathbb{P})$ of the previous section.

7.4.1 Föllmer-Schweizer measure

As before in this thesis, $\mathbb{Q}$ denotes the Föllmer-Schweizer measure and Z_T the Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. From (2.15) we know that the density process $Z_t = \mathbb{E}[Z_T \mid \mathcal{F}_t]$, $0 \le t \le T$, satisfies

$$dZ_t = G Z_{t-} \left(c\, dW_t + dM_t\right),$$

where the Girsanov parameter G (cf. (2.16)) has the form

$$G = \frac{r - (b+a)}{c^2 + v} \quad \text{with} \quad v = \int_{\mathbb{R}} x^2 \nu(dx).$$

Lemma 7.4.1. *The explicit representation of Z_t is given by*

$$Z_t = \exp\Big(cGW_t - \frac{c^2G^2}{2}t\Big) e^{-\lambda G\, \mathbb{E}[V_1 - 1]t} \prod_{i=1}^{N_t} (1 + G(V_i - 1)), \quad 0 \le t \le T.$$

Z is positive for $0 < G \le 1$.

Proof. The formula for the stochastic exponential (cf. Protter (2004), Chapter II.8) together with Theorem 7.3.2 and $\Delta M_s = (V_{N_s} - 1)\Delta N_s$ yields

$$\begin{aligned} Z_t &= \exp\Big(G(cW_t + M_t) - \frac{c^2G^2}{2}t\Big) \prod_{0<s\le t} (1 + G\Delta M_s) \exp(-G\Delta M_s) \\ &= \exp\Big(cGW_t - \frac{c^2G^2}{2}t + G\sum_{k=1}^{N_t}(V_k - 1) - \lambda G\, \mathbb{E}[V_1 - 1]t\Big) \\ &\quad \times \prod_{i=1}^{N_t} (1 + G(V_i - 1)) \exp\Big(-G\big(\sum_{j=1}^{N_t}(V_j - 1)\big)\Big), \end{aligned}$$

from which the claim follows. Furthermore, since the random variables V_i are concentrated on $[0, \infty)$, we have that Z is strictly positive if $0 < G \le 1$. □

Note that if $G = 0$, then $Z \equiv 1$ and no change of measure is performed. Moreover, it is quite easy to check that the process Z is indeed a density. Using the independence, for every $0 \leq t \leq T$ one has that

$$\begin{aligned}
\mathbb{E}[Z_t] &= \mathbb{E}[\exp\Big(cGW_t - \frac{c^2G^2}{2}t\Big)]e^{-\lambda G\,\mathbb{E}[V_1-1]t}\,\mathbb{E}[\prod_{i=1}^{N_t}\big(1+G(V_i-1)\big)] \\
&= e^{-\lambda G\,\mathbb{E}[V_1-1]t}\sum_{k=0}^{\infty}\mathbb{E}[\prod_{i=1}^{k}(1+G(V_i-1))]e^{-\lambda t}\frac{(\lambda t)^k}{k!} \\
&= e^{-\lambda t(1+G\,\mathbb{E}[V_1-1])}\sum_{k=0}^{\infty}\left(1+G\,\mathbb{E}[V_1-1]\right)^k\frac{(\lambda t)^k}{k!} \\
&= 1.
\end{aligned}$$

The general theory of Subsection 2.3.2 yields now the following theorem, which describes the Lévy process L under the measure $\mathbb{Q}$.

Theorem 7.4.2. *Under the measure $\mathbb{Q}$, L (denoted by $L^{\mathbb{Q}}$) is again a Lévy process and has the following Lévy-Itô-decomposition*

$$L_t^{\mathbb{Q}} = cW_t^{\mathbb{Q}} + M_t^{\mathbb{Q}} + A_t^{\mathbb{Q}},$$

where $cW_t^{\mathbb{Q}} = cW_t - c^2Gt$ is a Brownian motion under $\mathbb{Q}$. For the martingale $M^{\mathbb{Q}}$ one has that

$$\begin{aligned}
M_t^{\mathbb{Q}} &= \sum_{i=1}^{N_t}(V_i-1) - t\int_{\mathbb{R}} x\,\nu(dx) - t\int_{\mathbb{R}} Gx^2\nu(dx) \\
&= \sum_{i=1}^{N_t}(V_i-1) - t\int_{\mathbb{R}} x\,\nu^{\mathbb{Q}}(dx).
\end{aligned}$$

and the Lévy measure $\nu^{\mathbb{Q}}(dx) = (1+Gx)\,\nu(dx)$. $M^{\mathbb{Q}}$ is a square-integrable, quadratic pure-jump martingale. Furthermore, $A_t^{\mathbb{Q}} = (r-b)t$ is predictable and of finite variation.

Although the previous theorem follows from the general theory we consider the next lemma which complements the theorem's assertion and explains further this specific change of measure.

Lemma 7.4.3. *Under the measure $\mathbb{Q}$ we have that*

(a) $cW_t^{\mathbb{Q}} = cW_t - c^2Gt$, $0 \leq t \leq T$, is a Brownian motion with variance c^2t.

(b) $\sum_{i=1}^{N_t}(V_i-1)$, $0 \leq t \leq T$, is a compound Poisson process with rate

$$\mathbb{E}^{\mathbb{Q}}[N_t] = (1+G\,\mathbb{E}[V_1-1])\lambda t =: \lambda^{\mathbb{Q}}t,$$

and

$$M_t^{\mathbb{Q}} = \sum_{i=1}^{N_t} (V_i - 1) - \mathbb{E}^{\mathbb{Q}}[V_1 - 1]\lambda^{\mathbb{Q}} t.$$

(c) the Lévy measure of L is given by

$$\nu^{\mathbb{Q}}(dx) = \lambda^{\mathbb{Q}} f_{V-1}^{\mathbb{Q}}(x)\, dx,$$

where

$$f_{V-1}^{\mathbb{Q}}(x) = \frac{\lambda}{\lambda^{\mathbb{Q}}}(1 + Gx) f_{V-1}(x).$$

Proof. (a) This is a standard argument and is repeated for completeness only. Girsanov's theorem for local martingales (cf. Jacod and Shiryaev (2003), Theorem III.3.11) states that

$$cW_t^{\mathbb{Q}} := cW_t - \int_0^t \frac{1}{Z_{s-}} d\big\langle cW, Z\big\rangle_s, \quad 0 \le t \le T,$$

is a local $\mathbb{Q}$-martingale. Recalling the definition of Z we compute (under $\mathbb{P}$)

$$\big\langle cW, Z\big\rangle_t = \Big\langle cW, G \int_0^{\cdot} Z_{s-} d(cW_s + M_s)\Big\rangle_t = c^2 G \int_0^t Z_s\, ds.$$

We used that $\big\langle cW, cW\big\rangle_t = c^2 t$ and $\big\langle cW, M\big\rangle_t = 0$, since M is quadratic pure-jump and W is continuous. The associativity of stochastic integrals yields then $cW_t^{\mathbb{Q}} = cW_t - c^2 G t$. Further, $W^{\mathbb{Q}}$ is $\mathbb{Q}$-a.s. continuous and

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[(cW_t^{\mathbb{Q}})^2] &= \mathbb{E}^{\mathbb{Q}}[(cW_t - c^2 G t)^2] = c^2\, \mathbb{E}[(W_t - cGt)^2 Z_t] \\
&= c^2 \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (x - cGt)^2 e^{-\frac{(x-cGt)^2}{2t}}\, dx \\
&= c^2 t = \big\langle cW^{\mathbb{Q}}, cW^{\mathbb{Q}}\big\rangle_t.
\end{aligned}$$

Lévy's theorem (cf. Protter (2004), Chapter II, Theorem 38) yields then that $W_t^{\mathbb{Q}}$ is a standard $\mathbb{Q}$-Brownian motion.

(b) Let $0 \le t \le T$. First we compute the expectation of N_t under $\mathbb{Q}$.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[N_t] &= \mathbb{E}[N_t Z_t] = e^{-\lambda t(1 + G\,\mathbb{E}[V_1 - 1])} \sum_{k=0}^{\infty} k\, (1 + G\,\mathbb{E}[V_1 - 1])^k\, \frac{(\lambda t)^k}{k!} \\
&= (1 + G\,\mathbb{E}[V_1 - 1])\lambda t =: \lambda^{\mathbb{Q}} t.
\end{aligned}$$

Let now $H_t := \sum_{i=1}^{N_t}(V_i - 1)$ and $g_j := 1 + G(V_j - 1))$, then

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[e^{iuH_t}] = \mathbb{E}[e^{iuH_t} Z_t] &= \mathbb{E}[e^{iuH_t} e^{(\lambda-\lambda^{\mathbb{Q}})t} \prod_{j=1}^{N_t} g_j] \\
&= e^{(\lambda-\lambda^{\mathbb{Q}})t}\, \mathbb{E}[\prod_{j=1}^{N_t} e^{iu(V_j-1)} g_j] \\
&= e^{-\lambda^{\mathbb{Q}}t} \sum_{k=0}^{\infty} \Big(\mathbb{E}[e^{iu(V_1-1)} g_1]\Big)^k \frac{(\lambda t)^k}{k!} \\
&= e^{-\lambda^{\mathbb{Q}}t} \exp\Big(t \int_{\mathbb{R}} e^{iux} \lambda(1+Gx) f_{V-1}(x)\, dx\Big) \\
&= \exp\Big(t \int_{\mathbb{R}} \big(e^{iux} - 1\big)\, \nu^{\mathbb{Q}}(dx)\Big).
\end{aligned}$$

This shows the characteristic function of H_t is that of a compound Poisson process.

(c) The Lévy measure of L under $\mathbb{Q}$ is given by (cf. Proposition 2.2.12)

$$\nu^{\mathbb{Q}}(dx) = \lambda^{\mathbb{Q}} f^{\mathbb{Q}}_{V-1}(x)\, dx.$$

Together with $\nu^{\mathbb{Q}}(dx) = (1+Gx)\nu(dx) = \lambda(1+Gx) f_{V-1}(x)\, dx$ we have

$$f^{\mathbb{Q}}_{V-1}(x) = \frac{\lambda}{\lambda^{\mathbb{Q}}}(1+Gx) f_{V-1}(x). \qquad \square$$

Before proceeding with the next lemma, let us remark that $\int_{\mathbb{R}} \nu^{\mathbb{Q}}(dx) = \lambda^{\mathbb{Q}}$, which is the expected result, and moreover that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[V_1 - 1]\lambda^{\mathbb{Q}} t = t \int_{\mathbb{R}} x \nu^{\mathbb{Q}}(dx) \\
&= \left(\frac{p}{\eta_1 - 1} - \frac{q}{\eta_2 + 1} + G\left(\frac{2p}{(\eta_1-1)(\eta_1-2)} + \frac{2q}{(\eta_2+1)(\eta_2+2)}\right)\right) \lambda t.
\end{aligned}$$

The next lemma shows that under the measure $\mathbb{Q}$ the stock price is the exponential function of a jump-diffusion process like under $\mathbb{P}$.

Lemma 7.4.4. *The explicit representation of S_t under $\mathbb{Q}$ is given by*

$$S_t = S_0 \exp\Big(cW_t^{\mathbb{Q}} + \sum_{i=1}^{N_t} Y_i + \big(r - \frac{c^2}{2} - \mathbb{E}^{\mathbb{Q}}[V_1 - 1]\lambda^{\mathbb{Q}}\big)t\Big), \quad 0 \le t \le T.$$

Proof. We apply the stochastic differential equation (7.3) with $L^{\mathbb{Q}}$. In anal-

ogy to the proof of Lemma 7.3.3 we have, using Theorem 7.4.2, that

$$\begin{aligned} S_t &= S_0 \exp\Big(L_t^{\mathbb{Q}} + \big(b - \frac{c^2}{2}\big)t\Big) \prod_{0<s\le t} \big(1 + \Delta L_s^{\mathbb{Q}}\big) \exp\big(-\Delta L_s^{\mathbb{Q}}\big) \\ &= S_0 \exp\Big(cW_t^{\mathbb{Q}} + M_t^{\mathbb{Q}} + \big(r - \frac{c^2}{2}\big)t\Big) \prod_{0<s\le t} \big(1 + \Delta M_s^{\mathbb{Q}}\big) \exp\big(-\Delta M_s^{\mathbb{Q}}\big) \\ &= S_0 \exp\Big(cW_t^{\mathbb{Q}} + \big(r - \frac{c^2}{2} - \mathbb{E}^{\mathbb{Q}}[V_1 - 1]\lambda^{\mathbb{Q}}\big)t\Big) \prod_{i=1}^{N_t} V_i. \quad \square \end{aligned}$$

Recall that the Föllmer-Schweizer measure is chosen such that the discounted stock price $\hat{S}_t := B_t^{-1} S_t$ is a martingale under this measure. In our case this is relatively simple to check. One just uses the independent and stationary increments and immediately gets that

$$\mathbb{E}^{\mathbb{Q}}\Big[\frac{\hat{S}_t}{\hat{S}_s} \mid \mathcal{F}_s\Big] = 1, \quad 0 \le s \le t \le T.$$

In the introduction we already mentioned that for the option pricing formula of Kou (2002) it is fundamental to stay in the class of asymmetric double exponential jump diffusions after the change of measure. According to Lemma 7.4.4 the stock price under $\mathbb{Q}$ is the exponential function of a jump-diffusion process. It is now the question, if this is again an asymmetric double exponential jump-diffusion. This can be answered if one knows the jump-size distribution, that is, the distribution of Y_i under $\mathbb{Q}$.

Lemma 7.4.5. *The $\mathbb{Q}$-distribution of the random variables Y_i is described by the following Lebesgue density:*

$$\begin{aligned} f_Y^{\mathbb{Q}}(y) = p\frac{\lambda}{\lambda^{\mathbb{Q}}}\Big((1-G)\eta_1 e^{-\eta_1 y} + G\frac{\eta_1}{\eta_1 - 1}(\eta_1 - 1)e^{-(\eta_1 - 1)y}\Big)\mathbb{1}_{[0,\infty)}(y) \\ + q\frac{\lambda}{\lambda^{\mathbb{Q}}}\Big((1-G)\eta_2 e^{\eta_2 y} + G\frac{\eta_2}{\eta_2 + 1}(\eta_2 + 1)e^{(\eta_2 + 1)y}\Big)\mathbb{1}_{(-\infty,0)}(y). \end{aligned}$$

Proof. The cumulative distribution function of Y under $\mathbb{Q}$ is given by

$$F_Y^{\mathbb{Q}}(y) = \mathbb{Q}(Y \le y) = \mathbb{Q}(V - 1 \le e^y - 1) = F_{V-1}^{\mathbb{Q}}(e^y - 1).$$

Hence,

$$\begin{aligned} f_Y^{\mathbb{Q}}(y) &= f_{V-1}^{\mathbb{Q}}(e^y - 1)e^y = \frac{\lambda}{\lambda^{\mathbb{Q}}}(1 + G(e^y - 1))e^y f_{V-1}(e^y - 1) \\ &= \frac{\lambda}{\lambda^{\mathbb{Q}}}(1 + G(e^y - 1))e^y\Big(p\eta_1 e^{-(1+\eta_1)y}\,\mathbb{1}_{[0,\infty)}(y) \\ &\quad + q\eta_2 e^{(\eta_2 - 1)y}\,\mathbb{1}_{(-\infty,0)}(y)\Big). \end{aligned}$$

Simplifying this expression yields the claimed density. $\square$

Recall that we require $0 \leq G \leq 1$. Unfortunately for $G \in (0,1)$ the density of Y under $\mathbb{Q}$ is not convenient for further explicit computations. Both the upward and the downward jump follow a certain linear combination of the densities of two different exponential distributions. This shows that the Föllmer-Schweizer change of measure does generally not result anymore in an asymmetric double exponential jump-diffusion model. In these cases we can therefore not expect to obtain a similar *closed-form* option pricing formula as Kou (2002). This has several reasons. First, for an asymmetric double exponential random variable, the sum of upward and the sum of downward jumps follow a gamma distribution, respectively. This is the option pricing formula's basis relying on the distribution of the sum of a normal random variable and a gamma distributed random variable (cf. Proposition 7.2.4). Under $\mathbb{Q}$, however, we neither generally know the distribution of the sum of upward nor the sum of downward jumps. Second, a further key feature is the memoryless property of the exponential distribution (cf. Lemma 7.2.2) which is used to split the sum of asymmetric double exponential random variables with a certain probability and in distribution into a sum of upward and a sum of downward jumps. Such a memoryless property does in our case neither hold for the upward nor for the downward jump distribution. It seems that to compute expectations under $\mathbb{Q}$ for $G \in (0,1)$, involving the stock S, one is forced to apply Monte-Carlo simulation techniques using Lemma 7.4.4. Having an absolutely continuous distribution with respect to the Lebesgue measure Lemma 7.4.5 together with the inversion method can be applied to simulate Y.

There remain two special cases for which one stays in the class of asymmetric double exponential jump-diffusions. If $G = 0$, then one does not perform a change of measure or, in other words, the historical measure is already the risk-neutral measure. If $G = 1$, then we get the following Corollary.

Corollary 7.4.6. *Let $G = 1$. Then Y follows under $\mathbb{Q}$ an asymmetric double exponential distribution with density*

$$f_Y^{\mathbb{Q}}(y) = p^{\mathbb{Q}}(\eta_1 - 1)e^{-(\eta_1 - 1)y}\,\mathbb{1}_{[0,\infty)}(y) + q^{\mathbb{Q}}(\eta_2 + 1)e^{(\eta_2 + 1)y}\,\mathbb{1}_{(-\infty,0)}(y),$$

where

$$p^{\mathbb{Q}} := p\frac{\lambda}{\lambda^{\mathbb{Q}}}\frac{\eta_1}{\eta_1 - 1} \quad \text{and} \quad q^{\mathbb{Q}} := q\frac{\lambda}{\lambda^{\mathbb{Q}}}\frac{\eta_2}{\eta_2 + 1}.$$

Let us discuss this a little bit further assuming from now on $G = 1$. Observing $\lambda^{\mathbb{Q}} = \lambda\,\mathbb{E}[V]$, one immediately gets that

$$p^{\mathbb{Q}} + q^{\mathbb{Q}} = \frac{\lambda}{\lambda^{\mathbb{Q}}}\,\mathbb{E}[V] = 1.$$

Moreover, one has that (cf. (7.1))

$$\mathbb{E}^{\mathbb{Q}}[V] = p^{\mathbb{Q}}\frac{\eta_1 - 1}{\eta_1 - 2} + q^{\mathbb{Q}}\frac{\eta_2 + 1}{\eta_2 + 2},$$

and that, under $\mathbb{Q}$,

$$Y \stackrel{d}{=} \begin{cases} \xi^+, & \text{with probability} \quad p^{\mathbb{Q}} \\ -\xi^-, & \text{with probability} \quad q^{\mathbb{Q}}, \end{cases}$$

where ξ^+ and ξ^- are exponential random variables with parameters $\eta_1 - 1$ and $\eta_2 + 1$, respectively. For practical purposes the assumption $G = 1$ is very restrictive. It implies constraints on the parameters of the financial market such as interest rate r, drift b, volatility $(c^2 + v)$ and jump sizes influenced by a. In particular, this assumes a market price of risk equal to -1. From a theoretical point of view, this special case offers an interesting insight in this change of measure. The subjective probabilities of an upward or a downward jump p and q change to the risk-neutral probabilities $p^{\mathbb{Q}}$ and $q^{\mathbb{Q}}$. Note here that $\lambda/\lambda^{\mathbb{Q}}$, $\eta_1/(\eta_1 - 1)$ and $\eta_2/(\eta_2 + 1)$ are the proportions of the subjective parameters $(\lambda, \eta_1, \eta_2)$ to the risk-neutral parameters $(\lambda^{\mathbb{Q}}, \eta_1 - 1, \eta_2 + 1)$, respectively.

7.4.2 Kou's risk-neutral measure

Finally we consider the change of measure in Kou (2002) and compare the resulting jump-size distribution with the jump-size distribution derived under the Föllmer-Schweizer measure. We already mentioned that in Kou (2002) the choice of the risk-neutral measure is based on a rational expectations equilibrium framework, where the investor uses the power utility function $U(c,t) = e^{-\theta t}\frac{c^\gamma}{\gamma}$ for some $0 < \gamma < 1$. According to Kou (2002), this results in the following density:

$$Z_t^{\text{Kou}} = \exp\Big(c(\gamma-1)W_t - \frac{1}{2}c^2(\gamma-1)^2 t - \lambda t\left(\mathbb{E}[V^{\gamma-1}] - 1\right)\Big)\prod_{i=1}^{N_t} V_i^{\gamma-1}.$$

In equation (2.17) of Subsection 2.3.3 we briefly considered the general stochastic differential equation describing the density of any equivalent martingale measure in a Lévy-process financial market. If the Girsanov parameters are time-independent, then its explicit expression is given by

$$Z_t^{\text{Kou}} = \exp\Big(cGW_t - \frac{1}{2}c^2G^2t - t\int_{\mathbb{R}}\left(H(x) - 1\right)\,\nu(dx)\Big)\prod_{i=1}^{N_t} H(V_i - 1)$$

for some $G \in \mathbb{R}$ and some Borel-measurable function $H : \mathbb{R} \to [0,\infty)$, provided that $H(0) = 1$. It is an equivalent martingale measure only if the Girsanov parameters satisfy the necessary martingale condition (2.18) for the stock:

$$c^2G + b - r + \int_{\mathbb{R}} xH(x)\nu(dx) = 0. \tag{7.4}$$

We recall that the choice $H(x) = 1 + Gx$ with $G = \frac{r-(b+a)}{c^2+v}$ belongs to the Föllmer-Schweizer measure. Comparing now the general form of the density with the density of Kou (2002) yields that $G^{\text{Kou}} = \gamma - 1$ and $H^{\text{Kou}}(x) = (1+x)^{\gamma-1}$ for a γ such that (7.4) is satisfied. Note that $\lambda(\mathbb{E}[V^{\gamma-1}] - 1)$ arises from the following equation

$$\int_{\mathbb{R}} (H^{\text{Kou}}(x) - 1)\ \nu(dx) = \int_{\mathbb{R}} \left((1+x)^{\gamma-1} - 1\right) \lambda f_{V-1}(x)\, dx.$$

In analogy to Lemma 7.4.3 we derive that

$$f_{V-1}^{\mathbb{Q},\text{Kou}}(x) = \frac{\lambda}{\lambda^{\mathbb{Q}}} H^{\text{Kou}}(x) f_{V-1}(x).$$

Hence,

$$\begin{aligned} f_Y^{\mathbb{Q},\text{Kou}}(y) &= f_{V-1}^{\mathbb{Q},\text{Kou}}(e^y - 1)e^y \\ &= p\frac{\lambda}{\lambda^{\mathbb{Q}}} \frac{\eta_1}{\eta_1 + 1 - \gamma} (\eta_1 + 1 - \gamma) e^{-(\eta_1+1-\gamma)y}\, \mathbb{1}_{[0,\infty)}(y) \\ &\quad + q\frac{\lambda}{\lambda^{\mathbb{Q}}} \frac{\eta_2}{\eta_2 - 1 + \gamma} (\eta_2 - 1 + \gamma) e^{(\eta_2-1+\gamma)y}\, \mathbb{1}_{(-\infty,0)}(y). \end{aligned} \tag{7.5}$$

This restates the result of Kou (2002) that for any γ one stays in the class of asymmetric double exponential jump-diffusions. However, recall that $0 < \gamma < 1$ has to be chosen such that equation (7.4) is satisfied, which is not an easy task. In Kou (2002) this condition is formulated differently and it is presupposed to be satisfied by γ. However, finding or computing γ remains an open question. One might argue that the specification of γ is a problem in the same way as it is for our choice $G = 1$ for the Föllmer-Schweizer measure. In any case, if the parameters η_1 and η_2 are not too small the density (7.5) is *similar* to the density in Corollary 7.4.6.

7.5 Arbitrage-free price process

We work now on the probability space $(\Omega, \mathbb{G}, (\mathcal{G}_t)_{0\le t\le T}, \mathbb{Q})$, where $\mathbb{Q}$ is the Föllmer-Schweizer measure with $G = 1$. With this assumption the stock price S is the exponential function of an asymmetric double exponential jump-diffusion.

7.5.1 Valuation of an European Call

Let us consider an European call option which is due at time u, $0 \le u \le T$, with payoff $(S_u - K)^+$ for some deterministic strike price $K > 0$. For this option we will formulate the risk-neutral price formula of Kou (2002) in the next theorem. We derive it in detail, since the formula is quite advanced and in our case the jump-size distribution is slightly different from the one in Kou (2002). Note that the involved series are quickly convergent.

Theorem 7.5.1. *At time $0 \le t < u \le T$, the arbitrage-free price process*

$$F(t, S_t, u) = e^{-r(u-t)}\, \mathbb{E}^{\mathbb{Q}}[(S_u - K)^+ \mid \mathcal{G}_t]$$

of $(S_u - K)^+$ is given by

$$\begin{aligned} F(t, S_t, u) = \pi_0(t,u)\big(S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)}\Phi(d_+(t,u)) - Ke^{-r(u-t)}\Phi(d_-(t,u))\big) \\ + \sum_{n=1}^{\infty}\pi_n(t,u)\sum_{k=1}^{n} P_{n,k}^{\mathbb{Q}}\big((\eta_1 - 1)c\sqrt{u-t}\big)^k \\ \Big(S_t e^{-\left(\lambda^{\mathbb{Q}}\mu + \frac{c^2}{2}\right)(u-t)} a_1(t,u) - Ke^{-r(u-t)} b_1(t,u)\Big) \\ + \sum_{n=1}^{\infty}\pi_n(t,u)\sum_{k=1}^{n} Q_{n,k}^{\mathbb{Q}}\left((\eta_2 + 1)c\sqrt{u-t}\right)^k \\ \Big(S_t e^{-\left(\lambda^{\mathbb{Q}}\mu + \frac{c^2}{2}\right)(u-t)} a_2(t,u) - Ke^{-r(u-t)} b_2(t,u)\Big), \end{aligned}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\, dy$ is the cumulative probability function of a standard normal distribution,

$$d_{\pm}(t,u) = \frac{\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{c^2}{2} - \lambda^{\mathbb{Q}}\mu\right)(u-t)}{c\sqrt{u-t}}, \quad \mu = \mathbb{E}^{\mathbb{Q}}[V] - 1,$$

and

$$\pi_n(t,u) = \mathbb{Q}(N_{u-t} = n) = \frac{\left(\lambda^{\mathbb{Q}}(u-t)\right)^n}{n!} e^{-\lambda^{\mathbb{Q}}(u-t)}.$$

Furthermore,

$$\begin{aligned} a_1(t,u) &= f_1(t,u)\, \mathrm{I}_{k-1}\left(h(t,u); 2 - \eta_1, \frac{-1}{c\sqrt{u-t}}, (1-\eta_1)c\sqrt{u-t}\right), \\ b_1(t,u) &= f_1(t,u)\, \mathrm{I}_{k-1}\left(h(t,u); 1 - \eta_1, \frac{-1}{c\sqrt{u-t}}, (1-\eta_1)c\sqrt{u-t}\right), \\ a_2(t,u) &= f_2(t,u)\, \mathrm{I}_{k-1}\left(h(t,u); 2 + \eta_2, \frac{1}{c\sqrt{u-t}}, -(1+\eta_2)c\sqrt{u-t}\right), \\ b_2(t,u) &= f_2(t,u)\, \mathrm{I}_{k-1}\left(h(t,u); 1 + \eta_2, \frac{1}{c\sqrt{u-t}}, -(1+\eta_2)c\sqrt{u-t}\right) \end{aligned}$$

with

$$\begin{aligned} f_1(t,u) &= \frac{e^{(c(1-\eta_1))^2(u-t)/2}}{c\sqrt{2\pi(u-t)}}, \\ f_2(t,u) &= \frac{e^{(c(1+\eta_2))^2(u-t)/2}}{c\sqrt{2\pi(u-t)}}, \end{aligned}$$

and

$$h(t,u) = \log\Big(\frac{K}{S_t}\Big) + \Big(\lambda^{\mathbb{Q}}\mu - r + \frac{c^2}{2}\Big)(u-t).$$

The probability weights are for $1 \le k \le n-1$ *as follows:*

$$P^{\mathbb{Q}}_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i}\left(\frac{\eta_1 - 1}{\eta_1+\eta_2}\right)^{i-k}\left(\frac{\eta_2+1}{\eta_1+\eta_2}\right)^{n-i} p^{\mathbb{Q}^i} q^{\mathbb{Q}^{n-i}}$$

and

$$Q^{\mathbb{Q}}_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i}\left(\frac{\eta_1 - 1}{\eta_1+\eta_2}\right)^{n-i}\left(\frac{\eta_2+1}{\eta_1+\eta_2}\right)^{i-k} p^{\mathbb{Q}^{n-i}} q^{\mathbb{Q}^i}.$$

For $k = n$ *one has that*

$$P^{\mathbb{Q}}_{n,n} = p^{\mathbb{Q}^n} \quad \textit{and} \quad Q^{\mathbb{Q}}_{n,n} = q^{\mathbb{Q}^n}.$$

Proof. Let $0 \le t < u \le T$. Using the stationary and independent increments we get that

$$\begin{aligned} F(t,S_t,u) &= e^{-r(u-t)}\,\mathbb{E}^{\mathbb{Q}}[\Big(S_t\frac{S_u}{S_t} - K\Big)^+ \mid \mathcal{G}_t] \\ &= e^{-r(t-u)}\,\mathbb{E}^{\mathbb{Q}}[\Big(S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} e^{X_{u-t}(N_{u-t})} - K\Big)^+ \mid \mathcal{G}_t], \end{aligned}$$

where

$$X_{u-t}(n) = \big(r - \frac{c^2}{2}\big)(u-t) + cW^{\mathbb{Q}}_{u-t} + \sum_{i=1}^{n} Y_i. \tag{7.6}$$

Applying Proposition 7.2.3 and the Poisson distribution of N_{u-t} with rate $\lambda^{\mathbb{Q}}(u-t)$ we obtain that

$$\begin{aligned} &F(t,S_t,u) \\ &= e^{-r(u-t)}\sum_{n=0}^{\infty}\pi_n(t,u)\,\mathbb{E}^{\mathbb{Q}}[\Big(\ S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} e^{X_{u-t}(n)} - K\Big)^+ \mid \mathcal{G}_t] \\ &= \pi_0(t,u)e^{-r(u-t)}\,\mathbb{E}^{\mathbb{Q}}[\Big(S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} e^{X_{u-t}(0)} - K\Big)^+ \mid \mathcal{G}_t] \\ &\quad + e^{-r(u-t)}\sum_{n=1}^{\infty}\pi_n(t,u)\Big(\sum_{k=1}^{n} P^{\mathbb{Q}}_{n,k}\,\mathbb{E}^{\mathbb{Q}}[\Big(S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} e^{X^{+}_{u-t}(k)} - K\Big)^+ \mid \mathcal{G}_t] \\ &\qquad\qquad + \sum_{k=1}^{n} Q^{\mathbb{Q}}_{n,k}\,\mathbb{E}^{\mathbb{Q}}[\Big(S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} e^{X^{-}_{u-t}(k)} - K\Big)^+ \mid \mathcal{G}_t]\Big), \end{aligned}$$

where

$$X^{\pm}_{u-t}(k) = \big(r - \frac{c^2}{2}\big)(u-t) + cW^{\mathbb{Q}}_{u-t} \pm \sum_{i=1}^{k}\xi^{\pm}_i.$$

ξ^+ and ξ^- are exponential random variables with rate $\eta_1 - 1$ and $\eta_2 + 1$, respectively. We evaluate now each of the three expectations above separately. Letting $\alpha = S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)}$ and $\beta = \left(r - \frac{c^2}{2}\right)(u-t)$ we apply Proposition 7.2.5 which yields

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[\Big(S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} e^{X^+_{u-t}(k)} - K\Big)^+ \mid \mathcal{G}_t] \\
&= \big((\eta_1 - 1)c\sqrt{u-t}\big)^k \left(e^{r(u-t)} S_t e^{-\left(\lambda^{\mathbb{Q}}\mu + \frac{c^2}{2}\right)(u-t)} a_1(t,u) - K b_1(t,u)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[\Big(S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} e^{X^-_{u-t}(k)} - K\Big)^+ \mid \mathcal{G}_t] \\
&= \big((1 + \eta_2)c\sqrt{u-t}\big)^k \left(e^{r(u-t)} S_t e^{-\left(\lambda^{\mathbb{Q}}\mu + \frac{c^2}{2}\right)(u-t)} a_2(t,u) - K b_2(t,u)\right),
\end{aligned}$$

where

$$h(t,u) = \log\left(\frac{K}{\alpha}\right) - \beta = \log\left(\frac{K}{S_t}\right) + \left(\lambda^{\mathbb{Q}}\mu - r + \frac{c^2}{2}\right)(u-t).$$

Finally, the Black-Scholes formula implies

$$\begin{aligned}
&e^{-r(u-t)}\, \mathbb{E}^{\mathbb{Q}}[\Big(S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} e^{X_{u-t}(0)} - K\Big)^+ \mid \mathcal{G}_t] \\
&= S_t e^{-\lambda^{\mathbb{Q}}\mu(u-t)} \Phi(d_+(t,u)) - K e^{-r(u-t)} \Phi(d_-(t,u)).
\end{aligned}$$

This completes the proof. □

Corollary 7.5.2. *Let $0 \le t < u \le T$ and let*

$$F^{BS}(t, S_t, u) = S_t \Phi(d^{BS}_+(t,u)) - K e^{-r(u-t)} \Phi(d^{BS}_-(t,u)),$$

with

$$d^{BS}_\pm(t,u) = \frac{\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{c^2}{2}\right)(u-t)}{c\sqrt{u-t}}$$

the Black-Scholes option pricing formula. Then one has that

(a) $F(t, S_t, u) = F^{BS}(t, S_t, u)$ for $\lambda = 0$.

(b) $F(t, S_t, u) \to F^{BS}(t, S_t, u)$ for ($\eta_1 \to \infty$ and $\eta_2 \to \infty$).

Proof. The first part is obvious. Since the process X in (7.6) converges in distribution to a Brownian motion with drift if both η_1 and η_2 tend to infinity, the second claim follows. For details we refer to Kou (2002). □

7.5.2 Valuation of unit-linked pure endowment and term insurance with guarantee

Let us start with an unit-linked pure endowment with guarantee, for which the insurance company pays to the insured, if alive at time T, the amount $g(S_T) = \max(S_T, K)$ for some deterministic capital guarantee $K > 0$. As in Chapter 4 we set $u = T$ in this case and suppress the explicit dependence on the due date u of the insurance claim.

Corollary 7.5.3. *At time $0 \leq t \leq T$, the arbitrage-free price process of an unit-linked pure-endowment with guarantee $K > 0$ is given by*

$$F^{pe}(t, S_t) = F(t, S_t) + Ke^{-r(T-t)},$$

where $F(t, S_t)$ is taken from Theorem 7.5.1.

Proof. The claim follows immediately from Theorem 7.5.1 once noticed that $\max(S_T, K) = (S_T - K)^+ + K$. □

In case of an unit-linked term insurance with guarantee the contract function is time dependent and additionally we assume that the capital guarantee $K > 0$ earns interest with some risk-free interest rate $\delta > 0$. Hence, the contract function is $g(u, S_u) = \max(S_u, Ke^{\delta u})$ for $0 \leq u \leq T$.

Corollary 7.5.4. *At time $0 \leq t < u \leq T$, the arbitrage-free price process of an unit-linked term-insurance with guarantee $Ke^{\delta u}$ is computable via the formula:*

$$F^{ti}(t, S_t, u) = F(t, S_t, u) + Ke^{(\delta - r)u + rt},$$

where $F(t, S_t, u)$ is taken from Theorem 7.5.1 with the following adaptations

$$d_{\pm}(t, u) = \frac{\log\left(\frac{S_t}{Ke^{\delta u}}\right) + \left(r \pm \frac{c^2}{2} - \lambda^{\mathbb{Q}}\mu\right)(u - t)}{c\sqrt{u - t}},$$

$$h(t, u) = \log\left(\frac{Ke^{\delta u}}{S_t}\right) + \left(\lambda^{\mathbb{Q}}\mu - r + \frac{c^2}{2}\right)(u - t).$$

Proof. Analogously to Corollary 7.5.3 the claim follows from Theorem 7.5.1. □

7.5.3 Spatial derivative

In the previous chapters we saw that in a Lévy-process financial market the derivative of the arbitrage-free price process with respect to its space variable is required. This derivative occurs with the Galtchouk-Kunita-Watanabe decomposition (cf. Theorem 3.3.4) of the arbitrage-free price process and corresponds to the well-known delta in a Black-Scholes financial market. We recall from Chapter 3 that for $x > 0$ and $0 \leq t < u \leq T$ we have that

$$F'(t, x, u) = \frac{\partial}{\partial x} e^{-r(u-t)}\, \mathbb{E}^{\mathbb{Q}}[g(u, S_u) \mid S_t = x].$$

Corollary 7.5.5. *For $0 \le t < u \le T$ it holds*

$$\begin{aligned}
F'(t,S_t,u) &= \pi_0(t,u)e^{-\lambda^{\mathbb{Q}}\mu(u-t)}\Phi(d_+(t,u)) \\
&\quad + \sum_{n=1}^{\infty}\pi_n(t,u)\sum_{k=1}^{n}P_{n,k}^{\mathbb{Q}}\big((\eta_1-1)c\sqrt{u-t}\big)^k \\
&\quad \Big\{e^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)}a_1(t,u) \\
&\quad + \Big(e^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)+(2-\eta_1)h(t,u)} - \frac{K}{S_t}e^{-r(u-t)+(1-\eta_1)h(t,u)}\Big) \\
&\quad f_1(t,u)\,\mathrm{Hh}_{k-1}\Big(\frac{-h(t,u)}{c\sqrt{u-t}} + (\eta_1-1)c\sqrt{u-t}\Big)\Big\} \\
&\quad + \sum_{n=1}^{\infty}\pi_n(t,u)\sum_{k=1}^{n}Q_{n,k}^{\mathbb{Q}}\left((\eta_2+1)c\sqrt{u-t}\right)^k \\
&\quad \Big\{e^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)}a_2(t,u) \\
&\quad + \Big(e^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)+(2+\eta_2)h(t,u)} - \frac{K}{S_t}e^{-r(u-t)+(1+\eta_2)h(t,u)}\Big) \\
&\quad f_2(t,u)\,\mathrm{Hh}_{k-1}\Big(\frac{h(t,u)}{c\sqrt{u-t}} + (1+\eta_2)c\sqrt{u-t}\Big)\Big\}
\end{aligned}$$

Proof. We consider the formula from Theorem 7.5.1 and compute first

$$\begin{aligned}
&\frac{\partial}{\partial x}\pi_0(t,u)\big(xe^{-\lambda^{\mathbb{Q}}\mu(u-t)}\Phi(d_+(t,u)) - Ke^{-r(u-t)}\Phi(d_-(t,u))\big)\Big|_{x=S_t} \\
&= \pi_0(t,u)\Big(e^{-\lambda^{\mathbb{Q}}\mu(u-t)}\Phi(d_+(t,u)) \\
&\quad + \big(xe^{-\lambda^{\mathbb{Q}}\mu(u-t)}\varphi(d_+(t,u)) - Ke^{-r(u-t)}\varphi(d_-(t,u))\big)\frac{\partial}{\partial x}d_+(t,u)\Big|_{x=S_t}\Big),
\end{aligned}$$

where $\varphi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}$ denotes the density of the standard normal distribution. Moreover, the derivatives of $d_+(t,u)$ and $d_-(t,u)$ coincide, since $d_-(t,u) = d_+(t,u) - c\sqrt{u-t}$. It holds now that

$$d_+(t,u)c\sqrt{u-t} = \log\Big(\frac{S_t}{K}\Big) + \Big(r + \frac{c^2}{2} - \lambda^{\mathbb{Q}}\mu\Big)(u-t),$$

and hence that

$$\begin{aligned}
&S_te^{-\lambda^{\mathbb{Q}}\mu(u-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_+^2(t,u)} - Ke^{-r(u-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(d_+(t,u)-c\sqrt{u-t}\right)^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_+^2(t,u)}\Big(S_te^{-\lambda^{\mathbb{Q}}\mu(u-t)} - Ke^{-r(u-t)}e^{\left(d_+(t,u)c\sqrt{u-t}-\frac{c^2}{2}(u-t)\right)}\Big) \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_+^2(t,u)}\Big(S_te^{-\lambda^{\mathbb{Q}}\mu(u-t)} - Ke^{-r(u-t)}\frac{S_t}{K}e^{r(u-t)}e^{-\lambda^{\mathbb{Q}}\mu(u-t)}\Big) \\
&= 0.
\end{aligned}$$

This yields the first term of the derivative and we proceed in computing the derivative of the second summand of $F(t, S_t, u)$.

$$\begin{aligned}
&\frac{\partial}{\partial x}\Big(xe^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)}a_1(t,u) - Ke^{-r(u-t)}b_1(t,u)\Big)\Big|_{x=S_t} \\
&= e^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)}a_1(t,u) + xe^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)}\frac{\partial}{\partial x}a_1(t,u)\Big|_{x=S_t} \\
&\quad - Ke^{-r(u-t)}\frac{\partial}{\partial x}b_1(t,u)\Big|_{x=S_t}.
\end{aligned}$$

For $x = S_t$ one obtains that

$$\begin{aligned}
\frac{\partial}{\partial x}a_1(t,u) = f_1(t,u)\frac{\partial}{\partial x}\,\mathrm{I}_{k-1}\left(h(t,u); 2-\eta_1, \frac{-1}{c\sqrt{u-t}}, (1-\eta_1)c\sqrt{u-t}\right) \\
\frac{\partial}{\partial x}h(t,u).
\end{aligned}$$

Since

$$\frac{\partial}{\partial y}\,\mathrm{I}_n(y;\alpha,\beta,\delta) = -\frac{\partial}{\partial y}\int_{\infty}^{y} e^{\alpha x}\,\mathrm{Hh}_n(\beta x-\delta)\,dx = -e^{\alpha y}\,\mathrm{Hh}_n(\beta y-\delta)$$

and

$$\frac{\partial}{\partial x}h(t,u)\Big|_{x=S_t} = -\frac{1}{S_t}.$$

we get for $x = S_t$ that

$$\frac{\partial}{\partial x}a_1(t,u) = \frac{f_1(t,u)}{S_t}e^{(2-\eta_1)h(t,u)}\,\mathrm{Hh}_{k-1}\left(\frac{-h(t,u)}{c\sqrt{u-t}} + (\eta_1-1)c\sqrt{u-t}\right).$$

Analogously for $x = S_t$ we have that

$$\frac{\partial}{\partial x}b_1(t,u) = \frac{f_1(t,u)}{S_t}e^{(1-\eta_1)h(t,u)}\,\mathrm{Hh}_{k-1}\left(\frac{-h(t,u)}{c\sqrt{u-t}} + (\eta_1-1)c\sqrt{u-t}\right).$$

It remains to consider the derivative of the last summand of $F(t, S_t, u)$.

$$\begin{aligned}
&\frac{\partial}{\partial x}\Big(xe^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)}a_2(t,u) - Ke^{-r(u-t)}b_2(t,u)\Big)\Big|_{x=S_t} \\
&= e^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)}a_2(t,u) + xe^{-\left(\lambda^{\mathbb{Q}}\mu+\frac{c^2}{2}\right)(u-t)}\frac{\partial}{\partial x}a_2(t,u)\Big|_{x=S_t} \\
&\quad - Ke^{-r(u-t)}\frac{\partial}{\partial x}b_2(t,u)\Big|_{x=S_t}.
\end{aligned}$$

Performing for $x = S_t$ similar computations as for $a_1(t,x)$ and $b_1(t,x)$ we get that

$$\begin{aligned}
\frac{\partial}{\partial x}a_2(t,u) &= \frac{f_2(t,u)}{S_t}e^{(2+\eta_2)h(t,u)}\,\mathrm{Hh}_{k-1}\left(\frac{h(t,u)}{c\sqrt{u-t}} + (1+\eta_2)c\sqrt{u-t}\right) \\
\frac{\partial}{\partial x}b_2(t,u) &= \frac{f_2(t,u)}{S_t}e^{(1+\eta_2)h(t,u)}\,\mathrm{Hh}_{k-1}\left(\frac{h(t,u)}{c\sqrt{u-t}} + (1+\eta_2)c\sqrt{u-t}\right).
\end{aligned}$$

This finally yields the result. □

Chapter 8

Statistics for jump-diffusions

8.1 Introduction

In this chapter we consider parameter estimation for the asymmetric double exponential jump-diffusion stock model treated in the previous chapter. However, the presented theory, including the estimation method, is to a large extent independent of a concrete jump-size distribution and hence applicable to various types of jump-diffusions, for which it seems that satisfying parameter estimation methods are almost not available.

The particular difficulty of statistical inference for jump-diffusions having only finitely many observations is to discriminate the diffusion and the jump part of the process. It often remains unobserved, whether a discrete realization stems from a large movement of the diffusion or from a jump. Aït-Sahalia (2004) shows that maximum likelihood estimation is theoretically able to perfectly disentangle Brownian noise from jumps provided one samples frequently enough or, in other words, at least in the limit of infinitely frequent sampling. Usually jump-diffusions do not exhibit a transition probability density in closed form. Instead their density is only expressible as a series of improper convolution integrals coming from the summation of the related random variables. This function may be non-concave and may have several local maxima in a several dimensions spanning parameter space. Thus, it is far from trivial to apply numerical methods for maximum likelihood estimation in this case. Another disadvantage is that the transition density may be unbounded, which is discussed in Honoré (1998) in connection with pitfalls of maximum-likelihood estimation for the Merton jump-diffusion model.

We propose to rely the estimation on the characteristic and the cumulant generating function which are both explicitly known for jump-diffusions and which carry the same amount of information like the probability density. Because of the stationary and independent increments, the logarithmic stock returns constitute an i.i.d. sample of the jump-diffusion. Comparing

this sample's empirical with the theoretical cumulant generating function in their squared distance leads to a system of nonlinear regression problems, whose components arise from the real and imaginary part of these functions. The regression models can be solved using standard numerical routines and in case the expected jump sizes are known, they even reduce to linear regression problems. We demonstrate our estimation method with examples of two special cases, whereas we postpone the more involved numerical treatment of the general case to future research. To justify our approach, we discuss strong consistency of the empirical characteristic function and show that in our case an invariance principle is applicable leading to a complex-valued Gaussian process in the distributional limit. Using the latter we derive the asymptotic finite-dimensional distributions of the empirical characteristic function and formulate, moreover, a theorem for the asymptotic finite-dimensional distributions of the complex residuals of our nonlinear regression problem including their covariance structure.

Belomestny and Reiß (2006a,b) also base the calibration of exponential jump-diffusion models on the cumulant generating function and estimate the Lévy measure of the process in a nonparametric way applying Fourier inversion. However, they use option data to draw inference on the underlying risk-neutral price process of the stock, that is, they do not calibrate the model under the historical measure. Hence, their approach can neither be applied to asset models for which option data is not available nor to situations in which a particular risk-neutral measure such as the Föllmer-Schweizer measure for hedging purposes is chosen.

8.2 Model of stock price and returns

Suppose we are given a filtered probability space $(\Omega, \mathbb{G}, (\mathcal{G}_t)_{0\leq t\leq T}, \mathbb{P})$ for some finite number $T > 0$ which supports the following stochastic process:

$$S_t = S_0 \exp\Big(cW_t + \big(b - \frac{c^2}{2}\big)t + \sum_{j=1}^{N_t} Y_j\Big), \quad 0 \leq t \leq T, \tag{8.1}$$

for some $S_0 > 0$. Like in Chapter 7, $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion, $c > 0$ is deterministic, $(N_t)_{0\leq t\leq T}$ is a Poisson process with rate λt and Y_i is a sequence of independent and asymmetric double exponential distributed random variables (cf. Section 7.2 and Lemma 7.3.3). Recall Definition 7.2.1 in which the probability density of Y_1 is given by

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbb{1}_{[0,\infty)}(y) + q\eta_2 e^{\eta_2 y} \mathbb{1}_{(-\infty,0)}(y), \quad \eta_1 > 0,\, \eta_2 > 0, \tag{8.2}$$

where $p, q \geq 0$ and $p + q = 1$. Only for technical reasons we additionally required $\eta_1 > 1$ or $\eta_1 > 3$ in Chapter 7. We assume now that we observe S_t

at $n > 0$ points in time, say $0 \leq t_0 < t_1 < \ldots < t_n \leq T$, and set $S_k := S_{t_k}$, $0 \leq k \leq n$, for notational convenience.

Given these discrete observations their relative return is usually defined by

$$R_k = \frac{S_k - S_{k-1}}{S_{k-1}} = \frac{S_k}{S_{k-1}} - 1, \quad k = 1, \ldots, n.$$

Because of the exponential form of the asset price it is more convenient to apply the logarithmic return, commonly known as log-return, which is of the form:

$$X_k = \log(1 + R_k) = \log \frac{S_k}{S_{k-1}}, \quad k = 1, \ldots, n.$$

Of course, considering X_k instead of R_k has to be justified. A simple justification is the following. The order of R_k is typically less than 10^{-2} and in this case a simple Taylor series expansion of $\log(1 + R_k)$ immediately yields

$$|X_k - R_k| \leq \frac{R_k^2}{2} + \frac{|R_k|^3}{3} + \frac{R_k^4}{4} \ldots,$$

which shows that this difference is numerically negligible. Moreover, the independent and stationary increments of the Brownian motion and of the compound Poisson process in (8.1) imply on the one hand that the X_k, $k = 1, \ldots, n$, are independent and on the other hand that, without loss of generality,

$$X_k \stackrel{d}{=} L_1 := cW_1 + \left(b - \frac{c^2}{2}\right) + \sum_{j=1}^{N_1} Y_j, \quad k = 1, \ldots, n. \tag{8.3}$$

If $t_k \neq k$ we rescale the relevant parameters by $t_k - t_{k-1} = \Delta$. The random variable L_1 is a sum of a normal distributed random variable and a Poisson random sum having a certain jump-size distribution. It follows immediately from the convolution of the corresponding densities that the probability density of L_1 is only expressible as a series involving terms of improper integration, which are not expressible in closed form (cf. Proposition 7.2.4). This makes it almost impossible to apply maximum-likelihood estimation, since the local behavior of the density in the six dimensional parameter space

$$\Theta = \mathbb{R} \times (0, \infty) \times (0, \infty) \times (1, \infty) \times (0, \infty) \times [0, 1]$$

of the parameter $\vec{\theta} = (b, c, \lambda, \eta_1, \eta_2, p)$ is not known. We do know, however, the characteristic function of L_1, which is described by the Lévy-Khinchin formula discussed in Theorem 2.2.9 and which stores the same amount of information like the density. We have that

$$\phi(u) := \phi_1(u) = \mathbb{E}\left[e^{iuL_1}\right] = e^{\psi(u)}, \quad u \in \mathbb{R},$$

where

$$\psi(u) = -\frac{1}{2}c^2u^2 + iau + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right) \nu(dx)$$

is the characteristic exponent, also called the cumulant generating function. Note that condition (2.2) is satisfied here.

Lemma 8.2.1. *The cumulant generating function of L_1 has the following explicit form:*

$$\begin{aligned}\psi(u, \vec{\theta}) = -u^2\Big(\frac{c^2}{2} + \lambda\Big(\frac{p}{\eta_1^2 + u^2} + \frac{q}{\eta_2^2 + u^2}\Big)\Big) \\ + iu\Big(b - \frac{c^2}{2} + \lambda\Big(\frac{p\eta_1}{\eta_1^2 + u^2} - \frac{q\eta_2}{\eta_2^2 + u^2}\Big)\Big).\end{aligned}$$

Proof. We first observe that

$$a = \mathbb{E}[L_1] = \Big(b - \frac{c^2}{2}\Big) + \lambda\,\mathbb{E}[Y_1]$$

(cf. the discussion after Theorem 2.2.4). Moreover, from Proposition 2.2.12 we know that $\nu(dx) = \lambda f_Y(x)\,dx$ yielding

$$\begin{aligned}\int_{\mathbb{R}} (e^{iux} - 1)\lambda f_Y(x)\,dx &= \lambda p\eta_1 \int_0^\infty e^{(iu-\eta_1)x}\,dx + \lambda q\eta_2 \int_{-\infty}^0 e^{(iu+\eta_2)x}\,dx - \lambda \\ &= \lambda\Big(\frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1\Big),\end{aligned}$$

and

$$\int_{\mathbb{R}} iux\lambda f_Y(x)\,dx = iu\lambda\,\mathbb{E}[Y_1].$$

Hence,

$$\psi(u) = -\frac{1}{2}c^2u^2 + iu\Big(b - \frac{c^2}{2}\Big) + \lambda\Big(\frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1\Big).$$

We use now $p + q = 1$ to obtain that

$$\frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1 = \frac{ipu}{\eta_1 - iu} - \frac{iqu}{\eta_2 + iu}.$$

Further for the complex numbers it holds that

$$\frac{ipu}{\eta_1 - iu} = -\frac{u^2p}{\eta_1^2 + u^2} + i\frac{p\eta_1 u}{\eta_1^2 + u^2},$$

and

$$-\frac{iqu}{\eta_2 + iu} = -\frac{u^2q}{\eta_2^2 + u^2} - i\frac{q\eta_2 u}{\eta_2^2 + u^2}.$$

Finally the claim follows. □

For notational convenience we let

$$\psi_1(u,\vec{\theta}) := -u^2\Big(\frac{c^2}{2} + \lambda\Big(\frac{p}{\eta_1^2+u^2} + \frac{q}{\eta_2^2+u^2}\Big)\Big)$$

be the real and

$$\psi_2(u,\vec{\theta}) := u\Big(b - \frac{c^2}{2} + \lambda\Big(\frac{p\eta_1}{\eta_1^2+u^2} - \frac{q\eta_2}{\eta_2^2+u^2}\Big)\Big)$$

be the imaginary part of $\psi(u,\vec{\theta})$.

8.3 Estimation method

Our aim is to estimate $\vec{\theta} \in \Theta$ for the random variable L_1 of the previous section. As already mentioned, maximum likelihood estimation is hard to apply, so we concentrate on the characteristic function instead. For a given random sample $(X_1,\dots,X_n)$ of i.i.d. observations of L_1 we first define the empirical characteristic function.

Definition 8.3.1. *The empirical characteristic function $\hat{\phi}_n(u)$ of i.i.d. random variables $(X_1,\dots,X_n)$ is defined by*

$$\hat{\phi}_n(u) = \int_{-\infty}^{\infty} e^{iux}\, d\hat{F}_n(x), \tag{8.4}$$

where

$$\hat{F}_n(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}(X_k \le x), \quad x \in \mathbb{R},$$

is the empirical distribution function of $(X_1,\dots,X_n)$.

Equation (8.4) immediately implies that

$$\hat{\phi}_n(u) = \frac{1}{n}\sum_{k=1}^{n} e^{iuX_k} = \frac{1}{n}\sum_{k=1}^{n}\cos(uX_k) + i\frac{1}{n}\sum_{k=1}^{n}\sin(uX_k).$$

To get an estimate for $\vec{\theta}$, our idea is to compare the empirical with the theoretical characteristic function. Since the cumulant generating function ψ already carries all significant information, we consider this function instead of the characteristic function which leads to a system of nonlinear regression models.

Let $\mathbb{C}^* := \mathbb{C} \setminus \{x+iy : x \in (-\infty,0], y = 0\}$ be the complex plain excluding the negative real line and let $\operatorname{Log} : \mathbb{C}^* \to \mathbb{C}$ be the complex logarithm. Recall that for any $z \in \mathbb{C}^*$ with representation $z = |z|\, e^{i \operatorname{Arg} z}$ one has that $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$, where $\operatorname{Arg} z \in (-\pi,\pi)$. Using this little review of the complex logarithm we define the empirical cumulant generating function.

Definition 8.3.2. *The empirical cumulant generating function* $\hat{\psi}_n(u)$ *of i.i.d. random variables* $(X_1, \dots, X_n)$ *is defined by*

$$\begin{aligned}\hat{\psi}_n(u) &= \operatorname{Log} \hat{\phi}_n(u) \\ &= \log\left|\hat{\phi}_n(u)\right| + i \operatorname{Arg} \hat{\phi}_n(u).\end{aligned}$$

Furthermore, we let $\hat{\psi}_{n,1}(u) = \log\left|\hat{\phi}_n(u)\right|$ *and* $\hat{\psi}_{n,2}(u) = \operatorname{Arg} \hat{\phi}_n(u)$.

Let us express $\hat{\psi}_{n,1}(u)$ and $\hat{\psi}_{n,2}(u)$ explicitly:

$$\begin{aligned}\hat{\psi}_{n,1}(u) &= \log \sqrt{\frac{1}{n^2}\Big(\sum_{k=1}^{n} \cos(uX_k)\Big)^2 + \frac{1}{n^2}\Big(\sum_{k=1}^{n} \sin(uX_k)\Big)^2} \\ &= -\log n + \frac{1}{2}\log\left(\Big(\sum_{k=1}^{n} \cos(uX_k)\Big)^2 + \Big(\sum_{k=1}^{n} \sin(uX_k)\Big)^2\right)\end{aligned}$$

and

$$\hat{\psi}_{n,2}(u) = \begin{cases} \arctan\Big(\frac{\sum_{k=1}^{n} \sin(uX_k)}{\sum_{k=1}^{n} \cos(uX_k)}\Big), & \text{if} \quad \sum_{k=1}^{n} \cos(uX_k) > 0 \\ \arctan\Big(\frac{\sum_{k=1}^{n} \sin(uX_k)}{\sum_{k=1}^{n} \cos(uX_k)}\Big) + \pi, & \text{if} \quad \sum_{k=1}^{n} \cos(uX_k) < 0. \end{cases}$$

Following our idea we now compare $\psi(u, \vec{\theta})$ with $\hat{\psi}_n(u)$ in their squared distance, that is,

$$\Big|\psi(u, \vec{\theta}) - \hat{\psi}_n(u)\Big|^2 = \Big(\psi_1(u, \vec{\theta}) - \hat{\psi}_{n,1}(u)\Big)^2 + \Big(\psi_2(u, \vec{\theta}) - \hat{\psi}_{n,2}(u)\Big)^2,$$

which should be as small as possible for an estimated $\hat{\vec{\theta}}$, for each $u \in \mathbb{R}$. Figure 8.1 shows plots of the real part $\hat{\psi}_{n,1}(u)$ (left column) and the imaginary part $\hat{\psi}_{n,2}(u)$ (right column) of the empirical cumulant generating function for an i.i.d. sample of L_1 random variables. The dashed lines represent the corresponding real and imaginary parts, $\psi_1(u, \vec{\theta})$ and $\psi_2(u, \vec{\theta})$, of the true cumulant generating function, respectively. The first row of Figure 8.1 clearly indicates that for large absolute values of u the empirical function starts to blur. For that reason we suggest to reduce its support to some compact interval $[-U, U]$ on which those functions are considered and on which the parameter estimation is based. For the support reduction we propose to choose U such that the slope of $\hat{\psi}_{n,1}(u)$ does not change its sign on $[-U, U]$. The second row of Figure 8.1 pictures the result for the reduced support.

Now we discretize this support by an equidistant partition $-U \le u_1 < \dots < u_m \le U$ leading generally to a system of two nonlinear regression models of the form:

$$\hat{\psi}_{n,l}(u_j) = \psi_l(u_j, \vec{\theta}) + \varepsilon_{n,l}(u_j), \quad j = 1, \dots, m,\ l = 1, 2, \tag{8.5}$$

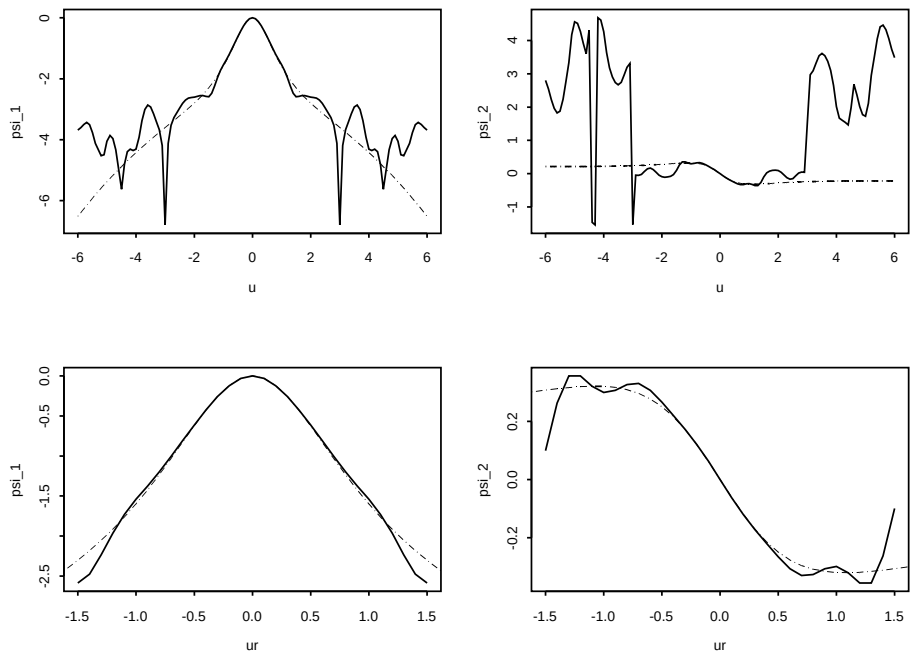


Figure 8.1: Empirical vs. theoretical cumulant generating function. Left column: Real part. Right column: Imaginary part. Dashed line: Theoretical functions. Parameters: Sample size $n = 1000$, $c^2 = 0.2$, $b = 0.08$, $\lambda = 3$, $\eta_1 = \eta_2 = 1$, $p = 0.4$.

for some residuals $\varepsilon_{n,l}(u)$, $l = 1, 2$. Let us consider these regression models a little bit more detailed starting with two special cases.

Special case 1

Assuming that the asymmetric double exponential distribution underlying L_1 satisfies $\eta_1 = \eta_2 = \rho$, where $\rho > 0$ is known, leads in fact to a system of two linear regression models, which are:

$$\hat{\psi}_{n,1}(u_j) = c^2\Big(\frac{-u_j^2}{2}\Big) + \lambda\Big(\frac{-u_j^2}{\rho^2 + u_j^2}\Big) + \varepsilon_{n,1}(u_j), \quad j = 1, \ldots, m$$

and

$$\hat{\psi}_{n,2}(u_j) = \Big(b - \frac{c^2}{2}\Big)u_j + \lambda(2p-1)\frac{u_j\rho}{\rho^2 + u_j^2} + \varepsilon_{n,2}(u_j), \quad j = 1, \ldots, m.$$

The first linear regression yields estimates for c^2 and λ denoted by $\hat{c}^2$ and $\hat{\lambda}$. The second linear model produces subsequently estimates for $e_1 := b - \frac{c^2}{2}$

and $e_2 := \lambda(2p-1)$ denoted by $\hat{e}_k$, $k = 1, 2$ which imply the following estimates for b and p:

$$\hat{b} = \hat{e}_1 + \frac{\hat{c}^2}{2} \qquad \text{and} \qquad \hat{p} = \frac{1}{2}\Big(\frac{\hat{e}_2}{\hat{\lambda}} + 1\Big).$$

These regression problems are easily solved using a standard statistics package such as S-Plus. Let us illustrate this case with an example, which demonstrates that this estimation approach really separates c^2 and λ, which is the basic difficulty of estimating jump-diffusions having only discrete observations and no further indication distinguishing between the continuities and the discontinuities of the paths.

Example 8.3.3. We assume $\eta_1 = \eta_2 = 1$. The descriptive statistics in Tables 8.1 and 8.2 result from 500 independent estimations each based on an i.i.d. sample of L_1 with the following parameters: $b = 0.08$, $c^2 = 0.2$, $\lambda = 2$ and $p = 0.4$. We distinguish between two sample sizes being $n = 1000$ and $n = 500$. Moreover the u_j were chosen such that $|u_j - u_{j-1}| = \frac{1}{10}$.

$n = 1000$	$\hat{b}$	$\hat{c}^2$	$\hat{\lambda}$	$\hat{p}$
Mean	0.128	0.278	1.840	0.379
Median	0.125	0.263	1.858	0.382
Std. dev.	0.132	0.121	0.267	0.092
1st. Qu.	0.037	0.197	1.680	0.330
3rd. Qu.	0.208	0.350	2.026	0.436
Minimum	−0.260	−0.094	0.842	−0.091
Maximum	0.663	0.749	2.567	0.670

Table 8.1: Descriptive statistics of 500 independent parameter estimates in the special case $\eta_1 = \eta_2 = 1$. Sample size: $n = 1000$.

$n = 500$	$\hat{b}$	$\hat{c}^2$	$\hat{\lambda}$	$\hat{p}$
Mean	0.154	0.310	1.796	0.368
Median	0.146	0.288	1.810	0.380
Std. dev.	0.200	0.173	0.328	0.147
1st. Qu.	0.023	0.198	1.586	0.308
3rd. Qu.	0.267	0.404	2.003	0.450
Minimum	−0.354	−0.232	0.768	−0.477
Maximum	0.851	1.061	2.773	0.775

Table 8.2: Descriptive statistics of 500 independent parameter estimates in the special case $\eta_1 = \eta_2 = 1$. Sample size: $n = 500$.

Special case 2

Generalizing the previous case we suppose now $\eta_1 = \eta_2$ and get a nonlinear regression problem that is partially linear:

$$\hat{\psi}_{n,1}(u_j) = c^2\Big(\frac{-u_j^2}{2}\Big) + \lambda\Big(\frac{-u_j^2}{\eta_1^2 + u_j^2}\Big) + \varepsilon_{n,1}(u_j), \quad j = 1, \ldots, m$$

and

$$\hat{\psi}_{n,2}(u_j) = \Big(b - \frac{c^2}{2}\Big)u_j + \lambda(2p-1)\Big(\frac{\eta_1 u_j}{\eta_1^2 + u_j^2}\Big) + \varepsilon_{n,2}(u_j), \quad j = 1, \ldots, m.$$

In this case we apply nonlinear regression to the first model and estimate c^2, λ and η_1. In S-Plus this is done using the procedure `nls()` with the option `plinear` for 'partial linear', which accounts for the linearity in c^2 and λ. Once η_1 is estimated, we consider

$$\hat{\psi}_{n,2}(u_j) = \Big(b - \frac{c^2}{2}\Big)u_j + \lambda(2p-1)\Big(\frac{\hat{\eta}_1 u_j}{\hat{\eta}_1^2 + u_j^2}\Big) + \tilde{\tilde{\varepsilon}}_{n,2}(u_j), \quad j = 1, \ldots, m.$$

To this model simple linear regression is applied like in special case 1. It turned out that the following equivalent transformed system of regression models stabilizes the numerical estimation procedure, provided $u_j \neq 0$, $j = 1, \ldots, m$.

$$\frac{\hat{\psi}_{n,1}(u_j)}{u_j^2} = c^2\Big(\frac{-1}{2}\Big) + \lambda\Big(\frac{-1}{\eta_1^2 + u_j^2}\Big) + \tilde{\varepsilon}_{n,1}(u_j), \quad j = 1, \ldots, m$$

and

$$\frac{\hat{\psi}_{n,2}(u_j)}{u_j} = \Big(b - \frac{c^2}{2}\Big) + \lambda(2p-1)\Big(\frac{\eta_1}{\eta_1^2 + u_j^2}\Big) + \tilde{\varepsilon}_{n,2}(u_j), \quad j = 1, \ldots, m.$$

We consider again an example and point out that the parameter estimation does not take much computational time, in fact it only takes a few seconds, since it relies on implemented standard solutions for least-square problems.

Example 8.3.4. We assume $\eta_1 = \eta_2$. The descriptive statistics in Tables 8.3 and 8.4 result from 500 independent estimations each based on an i.i.d. sample of L_1 with the following parameters: $b = 0.1$, $c^2 = 0.2$, $\lambda = 2$, $\eta_1 = 2$ and $p = 0.4$. We distinguish between two sample sizes being $n = 1000$ and $n = 500$. Moreover the u_j were chosen such that $|u_j - u_{j-1}| = \frac{1}{10}$ and the initial value for the nonlinear least-square procedure determining η_1^2 was set arbitrarily to 8 for both sample sizes.

$n = 1000$	$\hat{b}$	$\hat{c}^2$	$\hat{\lambda}$	η_1	$\hat{p}$
Mean	0.174	0.195	2.139	2.010	0.293
Median	0.149	0.175	2.034	1.987	0.348
Std. dev.	0.216	0.122	0.913	0.343	0.275
1st. Qu.	0.032	0.113	1.531	1.792	0.196
3rd. Qu.	0.283	0.272	2.607	2.238	0.460
Minimum	−0.342	−0.174	0.364	1.067	−1.950
Maximum	1.239	0.596	7.190	3.246	0.789

Table 8.3: Descriptive statistics of 500 independent parameter estimates in the special case $\eta_1 = \eta_2$. Sample size: $n = 1000$.

$n = 500$	$\hat{b}$	$\hat{c}^2$	$\hat{\lambda}$	η_1	$\hat{p}$
Mean	0.174	0.212	2.078	1.946	0.277
Median	0.163	0.199	1.809	1.911	0.341
Std. dev.	0.249	0.163	1.308	0.452	0.326
1st. Qu.	0.012	0.105	1.238	1.631	0.155
3rd. Qu.	0.300	0.310	2.548	2.154	0.464
Minimum	−0.765	−0.415	0.115	0.806	−1.601
Maximum	1.776	0.773	12.128	3.832	1.442

Table 8.4: Descriptive statistics of 500 independent parameter estimates in the special case $\eta_1 = \eta_2$. Sample size: $n = 500$.

General case

In general we get the following system of nonlinear regression models.

$$\frac{\hat{\psi}_{n,1}(u_j)}{u_j^2} = c^2\Big(\frac{-1}{2}\Big) + e_1\Big(\frac{-1}{\eta_1^2 + u_j^2}\Big) + e_2\Big(\frac{-1}{\eta_2^2 + u_j^2}\Big) + \tilde{\varepsilon}_{n,1}(u_j),$$

and

$$\frac{\hat{\psi}_{n,2}(u_j)}{u_j} = \Big(b - \frac{c^2}{2}\Big) + \lambda p\Big(\frac{\eta_1}{\eta_1^2 + u_j^2}\Big) - \lambda(1-p)\Big(\frac{\eta_2}{\eta_2^2 + u_j^2}\Big) + \tilde{\varepsilon}_{n,2}(u_j),$$

where $j = 1, \ldots, m$, $e_1 = \lambda p$ and $e_2 = \lambda(1-p)$. Nonlinear regression of the first model yields $\hat{c}^2$, $\hat{\eta}_1$, $\hat{\eta}_2$,

$$\hat{\lambda} = \hat{e}_1 + \hat{e}_2 \qquad \text{and} \qquad \hat{p} = \frac{\hat{e}_1}{\hat{e}_1 + \hat{e}_2}.$$

These estimates can then be used to consider the second regression model as a linear regression problem. Note that if η_1 and η_2 are known, both regressions are linear.

8.4 Mathematics behind the estimation

In this section our aim is to derive, at least asymptotically, some statements about the distribution and the covariance of the residuals in (8.5). To achieve this we first investigate the asymptotic behavior of the empirical characteristic function for an increasing sample size. For a sequence of i.i.d. random variables $X_1, X_2, X_3, \ldots$ with common distribution function $F(x)$ we denote by $\hat{F}_n(x)$ the empirical distribution function of the first n variables. $\phi(u)$, $\hat{\phi}_n(u)$, $\psi(u)$ and $\hat{\psi}_n(u)$ are analogously the true and the empirical characteristic or cumulant generating functions, respectively. For notational convenience we suppress the explicit dependence on the parameter vector $\vec{\theta}$ in this subsection.

Let us recall the Glivenko-Cantelli theorem which states that the empirical distribution function converges uniformly almost surely to the true distribution function, that is,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\hat{F}_n(x) - F(x)\right| = 0 \quad \text{a.s.}$$

This result is not self-evident for the empirical characteristic function, since one generally has an improper Riemann-Stieltjes integral. However, the properties of a distribution function imply that for a given $\varepsilon > 0$ there exists a $M > 0$ such that $F(-M) < \varepsilon/6$ and $1 - F(M) < \varepsilon/6$. Furthermore, the Glivenko-Cantelli Theorem yields that for sufficiently large random n one has $\hat{F}_n(-M) < \varepsilon/6$, $1 - \hat{F}_n(M) < \varepsilon/6$ and $||\hat{F}_n - F||_\infty < \varepsilon/6$ a.s., where $||f||_\infty := \sup_{x\in\mathbb{R}} |f(x)|$. Hence for any compact set $K \in \mathbb{R}$ and sufficiently large n it holds that

$$\begin{aligned}
\sup_{u\in K} \left|\hat{\phi}_n(u) - \phi(u)\right| &= \sup_{u\in K} \left|\int_{-\infty}^{\infty} e^{iux}\, d\big(\hat{F}_n(x) - F(x)\big)\right| \\
&\le \int_{-\infty}^{-M} d\hat{F}_n(x) + \int_{-\infty}^{-M} dF(x) + \int_{M}^{\infty} d\hat{F}_n(x) + \int_{M}^{\infty} dF(x) \\
&\quad + 2\left|\left|\hat{F}_n - F\right|\right|_\infty + \sup_{u\in K} |u| \int_{-M}^{M} \left|\hat{F}_n(x) - F(x)\right| dx \\
&\le \varepsilon + \sup_{u\in K} |u|\, 2M \left|\left|\hat{F}_n - F\right|\right|_\infty \quad \text{a.s.}
\end{aligned}$$

for any compact set $K \subset \mathbb{R}$. Choosing $K = [-T_n, T_n]$, the law of iterated logarithm for the empirical process implies $T_n = o((n/\log\log n)^{1/2})$ (cf. Csörgő (1981)). More general one has the following theorem:

Theorem 8.4.1 (Csörgő and Totik)**.** *Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. random variables. If*

$$\lim_{n\to\infty} \frac{\log T_n}{n} = 0,$$

then the empirical characteristic function satisfies:

$$\lim_{n\to\infty} \sup_{|u|\leq T_n} \left|\hat{\phi}_n(u) - \phi(u)\right| = 0 \quad a.s.$$

Proof. Csörgő and Totik (1983). □

The theorem's rate $T_n = \exp(o(n))$ is best possible for almost sure convergence. Note that Figure 8.1 partly reflects this behavior. Since the complex logarithm is continuous, the almost sure convergence of Theorem 8.4.1 carries forward to the cumulant generating function and gives us strong consistency. In the next lemma we investigate the moments of the empirical characteristic function.

Lemma 8.4.2. *For given i.i.d. random variables $(X_1, \ldots, X_n)$ the empirical characteristic function has the following properties.*

1. $\mathbb{E}[\hat{\phi}_n(u)] = \phi(u)$, $\forall\, u \in \mathbb{R}$.
2. $\mathbb{C}\mathrm{ov}\left[\hat{\phi}_n(u), \overline{\hat{\phi}_n(v)}\right] = \frac{1}{n}\big(\phi(u-v) - \phi(u)\phi(-v)\big)$, $\forall\, u, v \in \mathbb{R}$.

As usual, $\overline{z}$ denotes the conjugate, $x - iy$, for a given complex number $z = x + iy$, $x, y \in \mathbb{R}$.

Proof. Let $u \in \mathbb{R}$. By the i.i.d. assumption one has that

$$\mathbb{E}[\hat{\phi}_n(u)] = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[e^{iuX_k}] = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[e^{iuX_1}] = \phi(u).$$

Now let $u, v \in \mathbb{R}$. The covariance is then given by

$$\begin{aligned}
\mathbb{C}\mathrm{ov}\left[\hat{\phi}_n(u), \overline{\hat{\phi}_n(v)}\right] &= \mathbb{E}\left[\hat{\phi}_n(u)\overline{\hat{\phi}_n(v)}\right] - \phi(u)\overline{\phi(v)} \\
&= \mathbb{E}\Big[\frac{1}{n^2}\sum_{k=1}^{n} e^{i(u-v)X_k} + \frac{1}{n^2}\sum_{\substack{j,k=1 \\ j\neq k}}^{n} e^{iuX_k}e^{-ivX_j}\Big] \\
&\quad - \phi(u)\overline{\phi(v)} \\
&= \frac{1}{n}\phi(u-v) + \Big(1 - \frac{1}{n}\Big)\phi(u)\overline{\phi(v)} - \phi(u)\overline{\phi(v)} \\
&= \frac{1}{n}\big(\phi(u-v) - \phi(u)\overline{\phi(v)}\big).
\end{aligned}$$

Since $\overline{\phi(v)} = \phi(-v)$, the claim follows. □

Note that the previous Lemma implies that

$$\mathbb{V}\mathrm{ar}\left[\hat{\phi}_n(u)\right] = \frac{1}{n}\big(1 - |\phi(u)|^2\big), \quad \forall\, u \in \mathbb{R}.$$

For the empirical characteristic function it is even possible to formulate an invariance principle to derive a limiting process for the convergence in distribution. Therefore we need the following process.

Definition 8.4.3. *A stochastic process $B = B(t)$, $0 \le t \le 1$, is called a Brownian bridge if $B(t) = W(t) - tW(1)$ for a standard Wiener process $W = W(t)$, $0 \le t \le 1$.*

Definition 8.4.3 yields that a Brownian bridge B is a Gaussian process with $\mathbb{E}[B(t)] = 0$ and $\mathbb{E}[B(t)B(s)] = \min(t,s) - ts$ for all $0 \le s,t \le 1$. Donsker's invariance principle (cf. Chapter 4 in Csörgő and Révész (1981)) states now for $\beta_n(x) = \sqrt{n}\big(\hat{F}_n(x) - F(x)\big)$ and a Brownian bridge B that

$$\beta_n(x) \xrightarrow{d} B(F(x)), \quad n \to \infty.$$

This has been transferred to the characteristic function by Csörgő (1981). However, the transformation is again not as easy as it might seem at first glance. It uses the following sequence of stochastic processes,

$$Y_n(u) = \sqrt{n}\big(\hat{\phi}_n(u) - \phi(u)\big) = \int_{-\infty}^{\infty} e^{iux}\, d\beta_n(x), \quad u \in \mathbb{R},$$

and is formulated in a very general theorem by Csörgő (1981) which we reduce to our situation.

Theorem 8.4.4 (Csörgő)**.** *Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. random variables and let $-\infty < T_1 < T_2 < \infty$. If $\mathbb{E}[|X|^\alpha] < \infty$ for arbitrary large α, then there exists for each n a Brownian bridge B_n such that*

$$\sup_{T_1 \le u \le T_2} \left| Y_n(u) - \int_{-\infty}^{\infty} e^{iux}\, dB_n(F(x)) \right| = O(n^{-1/2} \log n) \quad a.s.$$

In particular, there exists a Brownian bridge B such that for all $u \in \mathbb{R}$

$$Y_n(u) \xrightarrow{d} Y(u) = \int_{-\infty}^{\infty} e^{iux}\, dB(F(x)), \quad n \to \infty.$$

$Y(u)$ is a complex valued Gaussian process satisfying $\mathbb{E}[Y(u)] = 0$ and $\mathbb{E}[Y(u)\overline{Y(v)}] = \phi(u-v) - \phi(u)\phi(-v)$ for all $u, v \in \mathbb{R}$. Here, $\xrightarrow{d}$ denotes convergence in distribution.

Proof. The proof is given in Csörgő (1981). □

For completeness let us note that

$$\begin{aligned} Y(u) &= -iu \int_{-\infty}^{\infty} e^{iux} B(F(x))\, dx = \int_0^1 e^{iuF^{-1}(x)}\, dB(x) \\ &= \int_0^1 e^{iuF^{-1}(x)}\, dW(x) - W(1) \int_0^1 e^{iuF^{-1}(x)}\, dx, \end{aligned}$$

where $F^{-1}(y) = \sup\{x \mid F(x) \le y\}$, $0 \le y \le 1$. Since $e^{iux}B(F(x))$ is almost everywhere continuous, $Y(u)$ exists as an improper Riemann-Stieltjes integral whenever $F(-x)$ and $1 - F(x)$ decrease fast enough for $x \to \infty$, which is guaranteed by the existence of all moments. Moreover, $Y(u)$ exists also as a stochastic integral, since $|\,\mathbb{E}[\big(\int_0^1 e^{iuF^{-1}(x)}\,dW(x)\big)^2]| \le 1$ a.s. for each $u \in \mathbb{R}$. Hence, for each $u \in \mathbb{R}$, the stochastic process $\int_0^t e^{iuF^{-1}(x)}\,dW(x)$, $0 \le t \le 1$, is a martingale and $\mathbb{E}[Y(u)] = 0$. For $u, v \in \mathbb{R}$ the covariance is computed using the following observation:

$$\begin{aligned}\mathbb{C}\mathrm{ov}[Y(u), Y(-v)] &= \mathbb{E}[Y(u)Y(-v)] \\ &= [Y(u), Y(-v)]_1 \\ &= \int_0^1 e^{i(u-v)F^{-1}(x)}\,d[W,W]_x - \phi(u)\phi(-v)[W,W]_1 \\ &= \phi(u-v) - \phi(u)\phi(-v).\end{aligned}$$

This implies the covariance structure for the real and imaginary part of the process. We denote by $\operatorname{Re} z = \frac{z+\overline{z}}{2}$ the real part and by $\operatorname{Im} z = \frac{z-\overline{z}}{2i}$ the imaginary part of a complex number z. These relations yield for all $u, v \in \mathbb{R}$:

$$\begin{aligned}\mathbb{C}\mathrm{ov}[\operatorname{Re} Y(u), \operatorname{Re} Y(v)] &= \frac{1}{2}\big(\operatorname{Re}\phi(u+v) + \operatorname{Re}\phi(u-v)\big) - \operatorname{Re}\phi(u)\operatorname{Re}\phi(v), \\ \mathbb{C}\mathrm{ov}[\operatorname{Re} Y(u), \operatorname{Im} Y(v)] &= \frac{1}{2}\big(\operatorname{Im}\phi(u-v) - \operatorname{Im}\phi(u+v)\big) + \operatorname{Re}\phi(u)\operatorname{Im}\phi(v), \\ \mathbb{C}\mathrm{ov}[\operatorname{Im} Y(u), \operatorname{Im} Y(v)] &= \frac{1}{2}\big(\operatorname{Re}\phi(u+v) - \operatorname{Re}\phi(u-v)\big) + \operatorname{Im}\phi(u)\operatorname{Im}\phi(v).\end{aligned}$$

The next lemma demonstrates that Theorem 8.4.4 is applicable in our case.

Lemma 8.4.5. *The random variable L_1 in (8.3) satisfies the condition $\mathbb{E}[|L_1|^\alpha] < \infty$ for arbitrary large α.*

Proof. According to Lemma 2.2.5 it suffices to have $\int_{\{|x|>1\}} |x|^\alpha\,\nu(dx) < \infty$, which is certainly satisfied for $\nu(dx) = \lambda f_Y(x)\,dx$, where $f_Y(x)$ is the density of the asymmetric double exponential distribution in (8.2). □

An immediate consequence of Theorem 8.4.4 is the following asymptotic statement of the finite-dimensional distributions of the empirical characteristic function.

Corollary 8.4.6. *Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. random variables and let $u_1, \ldots, u_m \in \mathbb{R}$. Then the empirical characteristic function satisfies:*

$$\begin{pmatrix} \sqrt{n}\big(\hat{\phi}_n(u_1) - \phi(u_1)\big) \\ \vdots \\ \sqrt{n}\big(\hat{\phi}_n(u_m) - \phi(u_m)\big) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Y(u_1) \\ \vdots \\ Y(u_m) \end{pmatrix}, \quad n \to \infty,$$

where $Y(u)$ is the limit process from Theorem 8.4.4 with the covariance structure described above.

Let us come back to our system of nonlinear regression models in (8.5). Using the theory developed so far, we are now able to derive an asymptotic statement about the finite-dimensional distributions of the residuals $\varepsilon_{n,l}(u)$, $l = 1, 2$.

Theorem 8.4.7. *Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. random variables and let $u_1, \ldots, u_m \in \mathbb{R}$ such that $\phi(u_j) \in \mathbb{C}^*$ for all $j = 1, \ldots, m$. Then it holds:*

$$\begin{pmatrix} \sqrt{n}\big(\varepsilon_{n,1}(u_1) + i\varepsilon_{n,2}(u_1)\big) \\ \vdots \\ \sqrt{n}\big(\varepsilon_{n,1}(u_m) + i\varepsilon_{n,2}(u_m)\big) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \tilde{Y}(u_1) \\ \vdots \\ \tilde{Y}(u_m) \end{pmatrix}, \quad n \to \infty,$$

where

$$\tilde{Y}(u) = \frac{Y(u)}{\phi(u)}$$

for the limit process $Y(u)$ from Theorem 8.4.4. Moreover, $\tilde{Y}(u)$ has the following covariance structure:

$$\mathbb{E}[\tilde{Y}(u)\tilde{Y}(-v)] = \frac{\phi(u-v)}{\phi(u)\phi(-v)} - 1, \quad \forall\, u, v \in \mathbb{R}.$$

Proof. There is a $R > 0$ such that the open balls $B_R(\phi(u_j)) = \{z \in \mathbb{C} : |z - \phi(u_j)| < R\} \in \mathbb{C}^*$ for all $j = 1, \ldots, m$. Furthermore, according to Theorem 8.4.1 we find a n_0 such that for $0 \le \tau \le 1$ the straight lines

$$z_{n,j}(\tau) = \phi(u_j) + \tau\big(\hat{\phi}_n(u_j) - \phi(u_j)\big) \in B_R(\phi(u_j)) \quad \text{a.s.}$$

for all $n \ge n_0$ and $j = 1, \ldots, m$. We have therefore for each $j = 1, \ldots, m$ almost surely that

$$\begin{aligned} \varepsilon_{n,1}(u_j) + i\varepsilon_{n,2}(u_j) &= \big(\hat{\psi}_{n,1}(u_j) - \psi_1(u_j)\big) + i\big(\hat{\psi}_{n,2}(u_j) - \psi_2(u_j)\big) \\ &= \operatorname{Log}\hat{\phi}_n(u_j) - \operatorname{Log}\phi(u_j) \\ &= \int_{\phi(u_j)}^{\hat{\phi}_n(u_j)} \frac{1}{z}\, dz = \int_0^1 \frac{z'_{n,j}(\tau)}{z_{n,j}(\tau)}\, d\tau \\ &= \big(\hat{\phi}_n(u_j) - \phi(u_j)\big) \int_0^1 \frac{1}{z_{n,j}(\tau)}\, d\tau, \quad \forall\, n \ge n_0. \end{aligned}$$

Again by Theorem 8.4.1 it exists a $\tilde{n}_0 \ge n_0$ such that

$$|z_{n,j}(\tau)| \ge |\phi(u_j)| - \left|\hat{\phi}_n(u_j) - \phi(u_j)\right| \ge \frac{|\phi(u_j)|}{2}, \quad \text{a.s.} \quad \forall\, n \ge \tilde{n}_0.$$

Hence,

$$\frac{1}{|z_{n,j}(\tau)|} \le \frac{2}{|\phi(u_j)|}, \quad \text{a.s.} \quad \forall\, n \ge \tilde{n}_0,$$

so by dominated convergence we get that

$$\int_0^1 \frac{1}{z_{n,j}(\tau)}\,d\tau \to \frac{1}{\phi(u_j)}, \quad \text{a.s.} \quad n \to \infty.$$

Slutsky's Lemma and Corollary 8.4.6 yield then the claim. □

The previous theorem implies in particular that the residuals of the regression problem are not independent.

Let us finally consider again the special case 1 of Section 8.3. In this case we assume for a given ρ that $\eta_1 = \eta_2 = \rho$ for the distribution of the random variable L_1, which is defined in (8.3). This implies a linear regression with the following matrix representation:

$$\vec{\hat{\psi}}_n = A\vec{\theta} + \vec{\varepsilon}_n,$$

where

$$\vec{\hat{\psi}}_n = \begin{pmatrix} \hat{\psi}_n(u_1) \\ \vdots \\ \hat{\psi}_n(u_m) \end{pmatrix}, \quad A = \begin{pmatrix} \frac{-u_1^2}{2} & \frac{-u_1^2}{\rho^2+u_1^2} & iu_1 & i\frac{u_1\rho}{\rho^2+u_1^2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-u_m^2}{2} & \frac{-u_m^2}{\rho^2+u_m^2} & iu_m & i\frac{u_m\rho}{\rho^2+u_m^2} \end{pmatrix}, \quad \vec{\theta} = \begin{pmatrix} c^2 \\ \lambda \\ e_1 \\ e_2 \end{pmatrix}$$

and

$$\vec{\varepsilon}_n = \begin{pmatrix} \varepsilon_{n,1}(u_1) + i\varepsilon_{n,2}(u_1) \\ \vdots \\ \varepsilon_{n,1}(u_m) + i\varepsilon_{n,2}(u_m) \end{pmatrix}.$$

Recall that $u_1, \ldots, u_m \in \mathbb{R}$ and that $m \geq 1$ is fixed. The least-square estimate $\hat{\vec{\theta}}_n$ for $\vec{\theta} \in \mathbb{R}^4$ is given by

$$\hat{\vec{\theta}}_n = \left(A^H A\right)^{-1} \mathrm{Re}\left(A^H \vec{\hat{\psi}}_n\right),$$

where A^H denotes the transpose of the conjugate matrix $\overline{A}$ of A. The covariance of $\sqrt{n}\hat{\vec{\theta}}_n$ is therefore asymptotically determined by

$$\lim_{n\to\infty} \mathbb{C}\mathrm{ov}\left[\sqrt{n}\hat{\vec{\theta}}_n\right] = \left(A^H A\right)^{-1} \lim_{n\to\infty} n\,\mathbb{C}\mathrm{ov}[\mathrm{Re}(A^H\vec{\varepsilon}_n)]\left(A^H A\right)^{-1},$$

where

$$\begin{aligned} \mathbb{C}\mathrm{ov}[\mathrm{Re}(A^H\vec{\varepsilon}_n)] &= \mathbb{E}[\Big(\frac{A^H\vec{\varepsilon}_n + \overline{A^H\vec{\varepsilon}_n}}{2}\Big)\Big(\frac{A^H\vec{\varepsilon}_n + \overline{A^H\vec{\varepsilon}_n}}{2}\Big)^H] \\ &= \frac{1}{2}\,\mathrm{Re}\left(A^H\,\mathbb{E}[\vec{\varepsilon}_n\vec{\varepsilon}_n^H]A\right) + \frac{1}{2}\,\mathrm{Re}\left(A^H\,\mathbb{E}[\vec{\varepsilon}_n\overline{\vec{\varepsilon}_n}^H]\overline{A}\right). \end{aligned}$$

Hence, because of Theorem 8.4.7,

$$\lim_{n\to\infty} n\,\mathbb{C}\mathrm{ov}[\mathrm{Re}(A^H\vec{\varepsilon}_n)] = C$$

for some matrix $C \in \mathbb{R}^{4\times 4}$, which shows that

$$\lim_{n\to\infty} \mathbb{C}\mathrm{ov}\left[\sqrt{n}\hat{\vec{\theta}}_n\right] = \left(A^H A\right)^{-1} C \left(A^H A\right)^{-1}.$$

8.5 Future research

There are still some open questions to be answered in connection with the suggested estimation method. We postpone them to future research raising here, however, the most important issues from our point of view.

1. Estimation of special case 1 applying the complex linear regression that we discussed at the end of Section 8.4. The results should then be compared to the results in Section 8.3, where we estimated the parameters using separately the two linear models corresponding to the real and imaginary part.

2. Numerical treatment of the general case of Section 8.3, where the problem is to separate η_1 and η_2. A nonlinear least-square routine has to be applied and the question of how to choose the initial values for the routine has to be answered. An ad-hoc application of the S-Plus routine `nls()` often terminates irregularly because of singularities or too many iteration steps.

3. Improvement of the estimations in special case 2, where the standard deviation of some estimators is still not satisfying.

4. Describing in analogy to the linear case the estimator's covariance structure in the nonlinear case.

5. Optimization of the support for the function $\psi_1(u, \vec{\theta})$ of Section 8.3. Considering this function suggests to discriminate large values of u to estimate c^2 and possibly λ from small values of u to estimate η_1 and η_2.

6. Finding an optimal set of supporting points for the regression models. This includes their number as well as their location. Note that the residuals at different supporting points are dependent.

7. Application of the estimation method to real financial data to get parameter estimates for an asymmetric double exponential jump-diffusion stock model.

Appendix A

Supplementary Material

Lemma A.0.1 (Gronwall's inequality[1]). *Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be non-negative and locally bounded. If there exist $C, h \geq 0$ such that for all $t \in [a, b]$*

$$f(t) \leq C + h \int_a^t f(s)\, ds,$$

then we have that

$$f(t) \leq C\, e^{h(t-a)}$$

for all $t \in [a, b]$.

Proof. This follows from Theorem 6.1.4 in Applebaum (2004). □

Theorem A.0.2. *Suppose that $x \in \mathbb{R}$ and that the process $X = (X_t)_{0 \leq t \leq T}$ is given by the following stochastic integral equation*

$$X_t = x + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_{s-}) dL_s, \quad 0 \leq t \leq T,$$

where, on some filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, $L = (L_t)_{0 \leq t \leq T}$ is a square-integrable martingale and a Lévy process with $\langle L, L \rangle_t = \kappa t$ for some $\kappa > 0$. Further let f, g be continuous functions on $[0, T] \times \mathbb{R}$ that are continuously differentiable in the spatial variable x. Additionally assume that there exists a constant $K > 0$ such that

$$\sup_{0 \leq t \leq T} (|f(t, 0)| + |g(t, 0)|) \leq K$$
$$\left| \frac{\partial}{\partial x} f(t, x) \right| + \left| \frac{\partial}{\partial x} g(t, x) \right| \leq K.$$

[1] named after the Swedish mathematician Thomas Hakon Grönwall (1877 - 1932)

Then for any $0 \leq t \leq T$ there exists a constant $C = C(K, T, \kappa) > 0$, such that

$$\mathbb{E}[\sup_{0\leq s\leq t} X_s^2] \leq C\left(1 + x^2\right).$$

Especially, if $f(t,x) = 0$ for all (t,x) then X is a martingale.

Proof. By the mean-value-theorem we have that $|f(t,x)| \leq K(1+|x|)$ for all (t,x). The same holds for $|g(t,x)|$. Further the integral $\int_0^t g(s, X_{s-})dL_s$ is a local martingale, since X_{s-} is a left continuous process and hence locally bounded (Protter (2004), Chapter IV, Theorem 29 and remark before Theorem 15). That is, there exists a reducing sequence of stopping times $(\tau_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} \tau_n = \infty$ such that the stopped process $\int_0^{t\wedge\tau_n} g(s, X_{s-})dL_s$ is a martingale for each $n \geq 1$. We observe now that

$$\begin{aligned}\mathbb{E}[\sup_{0\leq s\leq t} X_s^2] \leq 4x^2 &+ \mathbb{E}[\sup_{0\leq s\leq t} 4\Big(\int_0^s f(u, X_u)\, du\Big)^2] \\ &+ \mathbb{E}[\sup_{0\leq s\leq t} 2\Big(\int_0^s g(u, X_{u-})dL_u\Big)^2].\end{aligned}$$

By the Cauchy-Schwarz-inequality and the above estimate for f we have that

$$\begin{aligned}\mathbb{E}[\sup_{0\leq s\leq t} 4\Big(\int_0^s f(u, X_u)\, du\Big)^2] &\leq 4T\,\mathbb{E}[\int_0^t f^2(u, X_u)\, du] \\ &\leq 8TK^2\Big(T + \int_0^t \mathbb{E}\left[X_u^2\right]\, du\Big) \\ &\leq 8TK^2\Big(T + \int_0^t \mathbb{E}[\sup_{0\leq s\leq u} X_s^2]\, du\Big).\end{aligned}$$

Next,

$$\mathbb{E}[\sup_{0\leq s\leq t} 2\Big(\int_0^s g(u, X_{u-})dL_u\Big)^2] = \mathbb{E}[\lim_{n\to\infty} \sup_{0\leq s\leq t\wedge\tau_n} 2\Big(\int_0^s g(u, X_{u-})dL_u\Big)^2].$$

Using now Fatou's lemma as well as Doob's maximal quadratic inequality we get that

$$\begin{aligned}\mathbb{E}[\sup_{0\leq s\leq t} 2\Big(\int_0^s g(u, X_{u-})dL_u\Big)^2] &\leq 8 \lim_{n\to\infty} \mathbb{E}[\Big(\int_0^{t\wedge\tau_n} g(u, X_{u-})dL_u\Big)^2] \\ &= 8 \lim_{n\to\infty} \mathbb{E}[\int_0^{t\wedge\tau_n} g^2(u, X_{u-})d\langle L, L\rangle_u] \\ &= 8\kappa \int_0^t \mathbb{E}[g^2(u, X_u)]\, du.\end{aligned}$$

Note that the interchange of integration in both cases is justified by Tonelli's theorem. Moreover, using the above estimate for g yields

$$\begin{aligned} 8\kappa \int_0^t \mathbb{E}[g^2(u, X_u)]\, du &\leq 16\kappa K^2 \Big(T + \int_0^t \mathbb{E}\left[X_u^2\right]\, du \Big) \\ &\leq 16\kappa K^2 \Big(T + \int_0^t \mathbb{E}[\sup_{0\leq s\leq u} X_s^2]\, du \Big). \end{aligned}$$

Altogether we obtain that

$$\mathbb{E}[\sup_{0\leq s\leq t} X_s^2] \leq 4x^2 + 8TK^2(T + 2\kappa) + h(T, K, \kappa) \int_0^t \mathbb{E}[\sup_{0\leq s\leq u} X_s^2]\, du,$$

where

$$h(T, K, \kappa) = 8K(T + 2\kappa).$$

Lemma A.0.1 yields now

$$\mathbb{E}[\sup_{0\leq s\leq t} X_s^2] \leq \left(4x^2 + 8TK^2\left(T + 2\kappa\right)\right) e^{h(T,K,\kappa)T} \leq C(1 + x^2).$$

If $f(t, x) = 0$ for all (t, x) then X is a local martingale and by Theorem I.51 in Protter (2004) a martingale. □

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Zusammenfassung

Im Mittelpunkt dieser Arbeit steht die Entwicklung von lokal risikominimalen Hedgingstrategien für fondsgebundene Lebensversicherungsverträge, wobei der Fonds in einem allgemeinen Lévy-Prozess Finanzmarktmodell modelliert wird. Dadurch soll die in den letzten Jahren entwickelte und ziemlich weit vorangeschrittene Theorie der Lévy-Prozess Finanzmärkte mit der Theorie für fondsgebundene Lebensversicherungen verbunden und außerdem ein Rahmen bereitgestellt werden, der quadratisches Hedgen von Zahlungsströmen ermöglicht, die einem Investitionsrisiko und einem Versicherungsrisiko ausgesetzt sind. Unserem Wissen zur Folge, wurde lokal risikominimales Hedgen von fondsgebundenen Lebensversicherungsverträgen in allgemeinen Lévy-Prozess Finanzmärkten bisher noch nicht analysiert. Diese Arbeit liefert daher einen Beitrag zur laufenden Forschung an der Schnittstelle zwischen Finanz- und Versicherungsmathematik und verallgemeinert die früheren Arbeiten von Møller (1998, 2001a), indem der vollständige Black-Scholes Finanzmarkt durch einen unvollständigen Finanzmarkt, am Beispiel eines allgemeineren geometrischen Lévy-Prozess Modells, ersetzt wird.

Wie Møller (1998, 2001a) nehmen wir stochastische Unabhängigkeit zwischen dem Finanzmarkt und dem Versicherungsmodell an und betrachten beide zusammengefasst in einem gemeinsamen Produktwahrscheinlichkeitsraum. Dies greift die Idee auf, die Unsicherheit des Finanzmarktes und die der versicherten Leben gleichzeitig zu modellieren und nicht die Sterblichkeit durch ihren erwarteten Verlauf zu ersetzen. Zudem liefert die Verwendung des Markovkettenmodells von Hoem (1969) zur Beschreibung der Lebensversicherungen sehr flexible Resultate, die an die verschiedensten vorstellbaren Lebensversicherungsarten angepasst werden können. In Beispielen liegt jedoch unser Schwerpunkt auf den sicherlich wichtigsten Varianten wie die Erlebensfallversicherung, die Todesfallversicherung und die Leibrentenversicherung.

Wir beginnen mit der Herleitung von lokal risikominimalen Hedgingstrategien und dem damit verbundenen Hedgingrisiko für ein Portfolio von fondsgebundenen Erlebensfallversicherungen und für ein Portfolio von fondsgebundenen Todesfallversicherungen, wobei wir zunächst annehmen, dass diese gegen eine Einmalprämie zum Ausstellungsdatum verkauft und dass ihre Versicherungsleistungen bis zum Ende des betrachteten Zeithorizonts

verzinslich aufgeschoben und erst dann ausbezahlt werden. Mit dieser vereinfachenden Annahme passen die entsprechenden Zahlungsansprüche in die lokal risikominimale Hedgingtheorie von Schweizer (1991). Die risikominimale Hedgingtheorie von Föllmer und Sondermann (1986) für Zahlungsansprüche mit festem Auszahlungszeitpunkt wurde von Møller (2001a) für allgemeine Zahlungsströme erweitert. Beide Theorien gelten jedoch in Martingalfinanzmärkten, in denen das Hedgingrisiko unter dem jeweils betrachteten Martingalmaß interpretiert werden kann. Schweizer (1991) verallgemeinerte die Theorie von Föllmer und Sondermann (1986) für Semimartingalfinanzmärkte, indem er lokal risikominimales Hedgen und das risikoneutrale Föllmer-Schweizer Maß eingeführt hat. Eine unter diesem Maß risikominimale Hedgingstrategie ist lokal risikominimal bezüglich dem ursprünglichen Maß und kann somit nicht nur unter einem möglicherweise beliebigen, risikoneutralen Martingalmaß hinsichtlich ihres Risikos interpretiert werden. Aus diesem Grund untersuchen wir, ob sich lokal risikominimales Hedgen auch auf allgemeine Zahlungsströme anwenden lässt. Diese Erweiterung auf Zahlungsströme ist neu und wurde zuvor noch nicht betrachtet. Schließlich leiten wir in einem allgemeinen Lévy-Prozess Finanzmarkt lokal risikominimale Hedgingstrategien für allgemeine fondsgebundene Lebensversicherungsverträge her, welche innerhalb ihrer Laufzeit Versicherungsleistungen und Prämienzahlungen zulassen.

Da wir von Anfang an einen unvollständigen Finanzmarkt benutzen, erhalten wir zwei Bestandteile des Hedgingrisikos, welche dessen Ursprung widerspiegeln. Wir bezeichnen sie als das reine Finanz- und das reine Versicherungsrisiko. Das gesamte Hedgingrisiko eines fondsgebundenen Lebensversicherungsanspruchs teilt sich genau in diese beiden Größen auf, wobei das reine Versicherungsrisiko durch eine Erhöhung der Anzahl der versicherten Personen innerhalb des Portfolios diversifizierbar ist. Dies gilt jedoch nicht für das reine Finanzrisiko, was man ohnehin in der Realität nicht erwarten würde. Der Unterschied zu Møller (1998, 2001a) ist, dass wir in Übereinstimmung zur Realität zusätzlich das reine Finanzrisiko modellieren. Da das Black-Scholes Modell einen vollständigen Finanzmarkt beschreibt, stimmt das gesamte Hedgingrisiko von Møller (1998, 2001a) mit dem reinen Versicherungsrisiko in unserem Fall überein. Im Folgenden geben wir eine detaillierte Inhaltsübersicht der Dissertation.

Grundlegende Konzepte

In Kapitel 2 werden die in dieser Arbeit benötigten wesentlichen technischen Grundlagen besprochen und es werden die in den folgenden Kapiteln häufig verwendeten Modelle eingeführt. Nach der Definition eines Lévy-Prozesses diskutieren wir das Lévy-Maß, die berühmte Lévy-Itô-Zerlegung, welche in Theorem 2.2.4 formuliert wird, und die Verteilungseigenschaften von Lévy Prozessen, zu denen die Lévy-Khinchin Formel und die un-

endliche Teilbarkeit gehören. Zusätzlich klassifizieren wir in Theorem 2.2.6 das Pfadverhalten von Lévy-Prozessen, indem wir jeweils zwischen endlicher und unendlicher Aktivität oder Variation unterscheiden. Der Unterabschnitt 2.2.2 beinhaltet eine Diskussion von üblicherweise zur Modellierung von Finanzmärkten verwendeten Lévy-Prozessen. Im Anschluss daran führen wir im Abschnitt 2.3 das Lévy-Prozess Finanzmarktmodell von Chan (1999) ein und behandeln das für lokal risikominimales Hedgen benötigte Föllmer-Schweizer Maß. Wir erklären ausführlich die Auswirkungen des damit zusammenhängenden Maßwechsels auf den Lévy-Prozess des Modells, welcher zu einem additiven Prozess wird. Außerdem diskutieren wir jeweils in den Lemmata 2.3.1 und 2.3.3 den Aktienpreisprozess vor und nach diesem Maßwechsel und zeigen in beiden Fällen seine quadratische Integrierbarkeit, indem wir das Resultat aus dem Anhang dieser Arbeit verwenden. Dieses Resultat ermöglicht es uns, die von Chan (1999) vorausgesetzte Existenz exponentieller Momente des Lévy-Maßes wesentlich abzuschwächen, indem wir nur noch die Existenz des dritten Moments benötigen. Im darauffolgenden Abschnitt 2.4 fassen wir das auf einer zeitstetigen Markovkette basierende Lebensversicherungsmodell von Hoem (1969) zusammen, welches die Modellierung von ziemlich allgemeinen Lebensversicherungsverträgen zulässt. Im Unterabschnitt 2.4.3 präsentieren wir dann historische und zugleich noch heute häufig benutzte Sterblichkeitsgesetze, um Beispiele für die Hasardrate des Versicherungsmodells zu geben. Anschließend verwenden wir den zentralen Grenzwertsatz, um im Unterabschnitt 2.4.4 ein weit verbreitetes Diversifikationsmodell für das Versicherungsrisiko am Beispiel der gewöhnlichen Lebensversicherung zu besprechen. Weiterhin zeigt eine Anwendung des starken Gesetzes der großen Zahlen im Unterabschnitt 2.4.5 die Risikoneutralität eines Versicherungsunternehmens in Bezug auf die Sterblichkeit. Schließlich ist Abschnitt 2.5 der Konstruktion eines gemeinsamen Produktwahrscheinlichkeitsraums gewidmet, welcher das Lévy-Prozess Finanzmarktmodell mit dem Versicherungsmodell unter der begründeten Annahme von stochastischer Unabhängigkeit zusammenfasst. Zudem wiederholden wir im risikoneutralen Gegenstück dieses Raumes die risikominimale Hedgingtheorie von Föllmer und Sondermann (1986), einschließlich der für sie grundlegenden Galtchouk-Kunita-Watanabe Zerlegung in Theorem 2.5.3 und einer Diskussion ihres Zusammenhangs zum lokal risikominimalen Hedgen von Schweizer (1991).

Arbitrage-freier Preisprozess

In Kapitel 3 diskutieren wir den arbitragefreien Preisprozess eines quadratisch integrierbaren Zahlungsanspruchs, welchen wir mit Hilfe der Markoveigenschaft des Aktienprozesses als eine Funktion der Zeit, des Aktienwerts zu dieser Zeit und dem Fälligkeitsdatum des Anspruchs darstellen. Dabei liegt der gemeinsame risikoneutrale Finanzmarkt aus Kapitel 2 zu Grunde.

In diesem wird die Aktie durch einen additiven Prozess, das heißt, einem càdlàg, stochastisch stetigen Prozess mit unabhängigen Zuwächsen, modelliert. Der Schwerpunkt dieses Kapitels ist die in Theorem 3.3.4 hergeleitete Galtchouk-Kunita-Watanabe Zerlegung des arbitragefreien Preisprozesses. Sie hat dieselbe Struktur, wie wenn sie für einen Lévy-Prozess ausgearbeitet worden wäre und ist sicherlich der wichtigste Teil der lokal risikominimalen Hedgingtheorie. Darüber hinaus ist sie die Grundlage der Galtchouk-Kunita-Watanabe Zerlegung des inneren Werts der fondsgebundenen Lebensversicherungen in den folgenden Kapiteln. Grob umschrieben ist diese Zerlegung ein Projektionsresultat im Hilbertraum der quadratisch integrierbaren Zufallsvariablen. Wir beweisen Theorem 3.3.4, indem wir die Orthogonalität der beteiligten stetigen und rein unstetigen Martingale direkt verwenden. Diese Herangehensweise haben wir zuvor noch nicht beobachtet. Darüber hinaus scheint die Formulierung des Theorems für additive Prozesse neu zu sein. Um unseren Beitrag von bereits existierenden Resultaten abzugrenzen, stellen wir am Ende des Kapitels einen Überblick über damit zusammenhängende Literatur zur Verfügung. In Theorem 3.2.4 betrachten wir die Feynman-Kac partielle Differenzial- und Integralgleichung, welche vom arbitragefreien Preisprozess notwendigerweise erfüllt und zum Beweis der Galtchouk-Kunita-Watanabe Zerlegung benötigt wird. Diese Gleichung kann zur numerischen Berechnung des arbitragefreien Preisprozesses verwendet werden und folgt aus der Martingaleigenschaft des diskontierten arbitragefreien Preisprozesses. Die angewendete Zerlegungsmethode ist am Anfang sehr ähnlich zur der für Black-Scholes Finanzmärkte bekannten Herangehensweise. Allerdings erhält man in diesem Fall durch die entsprechende Feynman-Kac Darstellung bereits die richtige Galtchouk-Kunita-Watanabe Zerlegung. In unserem Fall sind dagegen weitere Schritte erforderlich, welche in den Lemmata 3.3.2 und 3.3.3 genauer ausgeführt werden.

Hedging von fondsgebundener Erlebens- und Todesfallversicherung

Kapitel 4 basiert auf Riesner (2006) und ist der Entwicklung von lokal risikominimalen Hedgingstrategien für ein Portfolio von entweder fondsgebundenen Erlebens- oder Todesfallversicherungsverträgen gewidmet. Wie bereits erwähnt, schränken wir uns dabei zunächst auf den Fall von Zahlungen nur zu Beginn und zum Ende des betrachteten Zeitintervalls ein. Diese Einschränkung ermöglicht es uns, die für Zahlungsansprüche mit festem Auszahlungszeitpunkt entwickelte lokal risikominimale Hedgingtheorie von Schweizer (1991) anzuwenden. Das Kapitel dient als Zwischenschritt bevor wir Prämienzahlungen und Versicherungsleistungen auch innerhalb der Versicherungsperiode betrachten und verallgemeinert darüber hinaus Møller (1998), indem der vollständige Black-Scholes Finanzmarkt durch einen allgemeineren und unvollständigen Lévy-Prozess Finanzmarkt ersetzt wird. Wir beginnen das Kapitel mit einer allgemeinen mathematischen Diskussion

fondsgebundener Lebensversicherungsverträge und bestimmen die Barwerte des gesamten Portfolios der oben erwähnten Versicherungsarten. Verschiedene weitere Umformungen führen uns dann jeweils in den Korollaren 4.3.1 und 4.3.6 zur gewünschten Galtchouk-Kunita-Watanabe Zerlegung der zu diesen Barwerten gehörenden inneren Werte der betrachteten Versicherungsportfolios. Diese Umformungen beinhalten unter anderem die Anwendung von Itôs partieller Integrationsformel für allgemeine Semimartingale und, zusätzlich bei der Todesfallversicherung, von Fubinis Theorem für stochastische Integrale. Die lokal risikominimalen Hedgingstrategien werden anschließend jeweils in den Korollaren 4.3.2 und 4.3.7 wiedergegeben. Alle erwähnten Korollare sind für einen allgemeinen Lévy-Prozess Finanzmarkt neue Resultate, welche wir ausführlich diskutieren und mit denen von Møller (1998) vergleichen. In unserem allgemeineren Fall ist der risikominimale Investmentanteil der risikobehafteten Anlage eine gewichtete Summe des Black-Scholes Delta and einem weiteren, aus den Sprüngen des Preisprozesses resultierenden Term. Anschließend betrachten wir das Hedgingrisiko und leiten seine zwei Bestandteile, das reine Finanz- und das reine Versicherungsrisiko, her. Am Ende des Kapitels begründen wir, warum lokale Risikominimierung speziell für Versicherungszahlungsansprüche verwendet werden sollte.

Hedging von allgemeinen Zahlungsströmen für Semimartingale

In Kapitel 5 wird lokal risikominimales Hedgen von Zahlungsströmen in einem allgemeinen Semimartingalfinanzmarkt untersucht. Für Martingalfinanzmärkte existiert die risikominimale Hedgingtheorie für Zahlungsströme von Møller (2001a). Diese liefert eine Interpretation des Hedgingrisikos unter dem jeweils betrachteten Martingalmaß. Ist das subjektive Maß jedoch kein Martingalmaß, so stellt sich die Frage wie man das Hedgingrisiko auch unter diesem Maß und nicht nur unter einem, möglicherweise beliebig gewählten, risikoneutralen Martingalmaß interpretieren kann. Schweizer (1991) hat die lokal risikominimale Hedgingtheorie für Zahlungsansprüche mit festem Auszahlungszeitpunkt für Semimartingalfinanzmärkte entwickelt. Diese ermöglicht eine lokale Interpretation des Hedgingrisikos unter dem subjektiven Wahrscheinlichkeitsmaß, vorausgesetzt das Föllmer-Schweizer Maß wird als risikoneutrales Maß verwendet. Wir benützen nun Schweizer (1991), um in diesem Kapitel zu zeigen, dass das risikominimale Hedgen von Zahlungsströmen im selben Sinne eine lokale Version besitzt wie das Hedgen von Zahlungsansprüchen mit festem Auszahlungszeitpunkt. Man benutzt die Theorie von Møller (2001a) und leitet unter dem Föllmer-Schweizer Maß eine risikominimale Hedgingstrategie her. Diese ist dann lokal risikominimal in Bezug auf das subjektive Wahrscheinlichkeitsmaß. Nach einigen technischen Details, fassen wir dieses Hauptresultat in Theorem 5.4.2 zusammen. Es basiert letztendlich auf derselben Optimalitätsgleichung wie im klassischen Fall, welche wir in Theorem 5.3.7 formulieren. In Proposition 5.2.3 verglei-

chen wir außerdem den Begriff von erreichbaren Zahlungsströmen mit dem klassischen Begriff von erreichbaren Zahlungsansprüchen. Die Erweiterung von lokal risikominimalem Hedgen auf Zahlungsströme ist ein neues Resultat. Ein Teil von Riesner (2005) ist diesem Kapitel entnommen.

Hedging von allgemeinen fondsgebunden Lebensversicherungsverträgen

Das Thema von Kapitel 6 ist die Herleitung von lokal risikominimalen Hedgingstrategien für allgemeine fondsgebundene Lebensversicherungsverträge, welche innerhalb ihrer Laufzeit sowohl Prämien- als auch Leistungszahlungen zulassen. Zur Modellierung der Lebensversicherungsverträge verwenden wir das zeitstetige Markovkettenmodell von Hoem (1969). Es ermöglicht sowohl zustandsabhängige als auch durch einen Zustandsübergang ausgelöste Zahlungen, welche wir als Differenz von Prämien- und Versicherungsleistungen interpretieren. Das Kapitel trägt zu Riesner (2005) bei und verallgemeinert Møller (2001a), indem der vollständige Black-Scholes Finanzmarkt durch einen allgemeineren und unvollständigen Lévy-Prozess Finanzmarkt ersetzt wird. Die Galtchouk-Kunita-Watanabe Zerlegung des inneren Werts dieser allgemeinen Versicherungen ist als Hauptresultat in Theorem 6.3.1 wiedergegeben und wird mit aus Kapitel 4 bereits bekannten Techniken bewiesen. Die daraus anschließend abgeleitete lokal risikominimale Hedgingstrategie sowie das damit zusammenhängende Hedgingrisiko sind in Korollar 6.3.2 enthalten. Für einen allgemeinen Lévy-Prozess Finanzmarkt sind diese Resultate neu. Am Ende des Kapitels wenden wir das allgemeine Modell auf die fondsgebundene Leibrenten- und die fondsgebundene Todesfallversicherung an, wobei wir zunächst jeweils einen einzelnen Vertrag und im Anschluss daran ein Portfolio dieser Verträge betrachten.

Modellierung der Aktie durch eine Sprungdiffusion

Um einen möglichen, konkreten Lévy-Prozess Finanzmarkt vorzuschlagen, diskutieren wir in Kapitel 7 das asymmetrisch doppelt exponentiale Sprungdiffusionsmodell von Kou (2002). Am Anfang des Kapitels begründen wir warum wir dieses Modell für geeignet halten und besprechen die asymmetrisch doppelte Exponentialverteilung. Danach diskutieren wir den Lévy-Prozess, der der Sprungdiffusion zu Grunde liegt und binden das Modell in unseren Finanzmarkt ein. Dabei legen wir besonderen Wert auf die Sprunghöhenverteilung und das damit zusammenhängende Lévy-Maß. Als risikoneutrales Wahrscheinlichkeitsmaß verwendet Kou (2002) ein spezielles, auf Nutzenüberlegungen basierendes Maß und zeigt, dass man mit dem entsprechenden Maßwechsel, unter gewissen Annahmen an die Marktparameter wie Zins- und Driftrate, innerhalb der Klasse der asymmetrisch doppelt exponentialen Sprungdiffusionen bleibt. Die Verteilungseigenschaften dieser Klasse sind für das Aufstellen der Preisformel des Modells für europäische

Calloptionen wesentlich. Da wir lokal risikominimales Hedgen betrachten, sind wir an den Auswirkungen des Föllmer-Schweizer Maßes auf das Modell interessiert. Diese diskutieren wir ausführlich in Theorem 7.4.2 sowie in den Lemmata 7.4.3 und 7.4.4. Im Anschluss daran zeigen wir in Lemma 7.4.5, dass im Allgemeinen die Sprunghöhen unter diesem Maß keiner asymmetrisch doppelten Exponentialverteilung mehr folgen und leiten daraus entsprechende Bedingungen an die Marktparameter ab, unter denen man auch nach diesem Maßwechsel innerhalb der gewünschten Verteilungsklasse bleibt. Diese Bedingungen vergleichen wir anschließend mit denen von Kou (2002). Abschnitt 7.5 beendet das Kapitel mit den Formeln für den arbitragefreien Preisprozess einer europäischen Calloption, einer fondsgebundenen Erlebensfallversicherung und einer fondsgebundenen Todesfallversicherung, wobei beide eine Mindestzahlung garantieren. Außerdem berechnen wir die räumliche Ableitung des entsprechenden arbitragefreien Preisprozesses. Sowohl der Preisprozess als auch seine räumliche Ableitung sind Bestandteile der in dieser Arbeit aufgestellten Hedgingstrategien. Das Sprungdiffusionsmodell von Kou (2002) wurde bisher noch nicht im Zusammenhang mit fondsgebundenen Lebensversicherungen und dem Föllmer-Schweizer Maßwechsel betrachtet. Außerdem sind die Bedingungen neu, unter denen dieser Maßwechsel nicht aus der Klasse der asymmetrisch doppelt exponentialen Sprungdiffusionen herausführt.

Statistik für Sprungdiffusionen

Während dem Studium der asymmetrisch doppelt exponentialen Sprungdiffusion von Kou (2002) fiel uns auf, dass es keine zufriedenstellenden Parameterschätzmethoden für Sprungdiffusionen gibt. Aus diesem Grund behandeln wir dieses Thema in Kapitel 8. Obwohl wir den genannten Prozess dabei beispielhaft verwenden, ist unsere Betrachtung im wesentlichen auch für andere Sprungdiffusionen gültig. Da die Wahrscheinlichkeitsdichte dieser Prozesse keine geschlossene Form besitzt, ihre charakteristische Funktion aber explizit bekannt ist, basieren wir in Abschnitt 8.3 die Schätzung auf die kumulantenerzeugende Funktion. Diese vergleichen wir mit dem komplexen Logarithmus der empirischen charakteristischen Funktion im Hinblick auf ihren quadratischen Abstand und erhalten somit ein System aus zwei nichtlinearen Regressionsmodellen, welche aus dem Real- und Imaginärteil dieser Funktionen hervorgehen. Zusätzlich geben wir eine Regel zur Bestimmung des kompakten Trägers an, auf dem diese Regressionsprobleme betrachtet werden. Diese können dann mit Hilfe von Standardverfahren zur Lösung von Kleinstquadratproblemen gelöst werden und sind sogar in einigen Spezialfällen linear. Die theoretischen Grundlagen unserer Schätzmethode stellen wir in Abschnitt 8.4 bereit. Dieser Abschnitt enthält das „Glivenko-Cantelli Theorem“ für die charakteristische Funktion von Csörgő und Totik (1983), welches wir motivieren und aus welchem wir starke Konsistenz erhalten. Da-

nach wenden wir das Invarianzprinzip von Csörgő (1981) auf die empirische charakteristische Funktion an und bekommen einen komplexwertigen Gaußschen Prozess als Grenzwert in Verteilung. Dieses Resultat führt uns dann asymptotisch in Theorem 8.4.7 zu den endlichdimensionalen Verteilungen der komplexwertigen Residuen unseres Regressionsproblems einschließlich ihrer Kovarianzstruktur.

Anhang

Der Anhang enthält ein Resultat aus der stochastischen Analysis, welches auf der Ungleichung von Gronwall basiert. Es liefert Bedingungen, unter denen die Lösung einer bestimmten stochastischen Integralgleichung bezüglich eines Lévy-Prozesses ein quadratisch integrierbarer Prozess ist. Wir benutzen es in den Lemmata 2.3.1 und 2.3.3, um zu zeigen, dass der in (2.14) definierte Aktienprozess sowohl unter dem historischen als auch unter dem Föllmer-Schweizer Maß quadratisch integrierbar ist. Mit diesem Ergebnis genügt uns die Existenz des dritten Moments des Lévy-Maßes, welche wir in (2.12) voraussetzen. Wir konnten deswegen die von Chan (1999) geforderte Existenz exponentieller Momente abschwächen und dadurch das Modell für eine größere Klasse von Verteilungen, insbesondere für diejenigen mit dickeren Tails, öffnen. Wir haben Theorem A.0.2 speziell an unsere Erfordernisse angepasst und bewiesen. Andere Versionen des Resultats sind zum Beispiel in Jacod, Méléard and Protter (2000) oder Ma, Protter and Zhang (2001) enthalten, allerdings ohne explizite Beweisführung und ohne eine Driftkomponente, welche wir hinzugefügt haben.