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Stable reduction of three-point covers

Dissertation zur Erlangung des Doktorgrades Dr. rer. nat
der Fakultät für Mathematik und Wirtschaftswissenschaften
der Universität Ulm

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Ulm, 2015

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Tag der Promotion: 06.07.2015

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Introduction

Let R be a complete discrete valuation ring with fraction field K and residue field k. We assume that the characteristic of K is 0 and the characteristic of k is $p > 0$. Additionally, we assume that k is algebraically closed. Further, we assume that all curves are absolutely irreducible, smooth, and projective.

In this thesis we study the stable reduction of a certain class of three-point covers. The three-point covers in this thesis are described as follows.

$f_K : Y_K \to X_K = \mathbb{P}^1_K$ is the Galois closure of a three-point cover $\varphi_K : Z_K = \mathbb{P}^1_K \to X_K = \mathbb{P}^1_K$, i.e. φ_K is a finite cover and is branched only in $x_1 = 0, x_2 = 1, x_3 = \infty \in X_K$. We additionally always assume that φ_K is ramified in exactly three points. We denote the stable reduction of f_K by $\bar{f} : \bar{Y} \to \bar{X}$.

The main result in this thesis is a complete description of the stable reduction for $\deg \varphi_K < 2p$ (Theorem 4.2.2). In the description of the stable reduction there occur covers which are the restriction of $\bar{f}$ to an irreducible component in $\bar{Y}$ above an irreducible component in $\bar{X}$ which are separable. We call these covers tail covers.

An important part of this thesis is the explicit description of the tail covers in general. We define tail covers (Definition 2.1.1) as objects of their own, i.e. as objects which are not connected to stable reduction. We generalize the tail covers which occur in the stable reduction of f_K for $\deg \varphi_K < 2p$ and analyze the ramification behaviour (Theorem 2.3.11). We prove that these tail covers (of arbitrary degree) occur in the stable reduction of three-point covers (Corollary 5.2.9).

Moreover, if $\deg \varphi_K < 2p$ and f_K has bad reduction, then we prove a geometric interpretation of Osserman's characterization of good reduction ([Oss08, Theorem 2.4]) in terms of the degree of the occurring tail covers (Proposition 5.3.2).

Stable reduction of three-point covers

Let R be a complete discrete valuation ring with fraction field K and residue field k. We assume that the characteristic of K is 0 and the characteristic of k is $p > 0$. Additionally, we assume that k is algebraically closed. Further, we assume that all curves are absolutely irreducible, smooth and projective.

Let $f_K : Y_K \to X_K = \mathbb{P}^1_K$ be a Galois three-point cover over K, i.e. f_K is a finite cover over K which is branched in exactly three points $x_1 = 0, x_2 = 1$ and $x_3 = \infty \in X_K$. We denote the Galois group of f_K by G. Replacing K by a finite extension, there exist semistable models $\tilde{\mathcal{Y}}$ over R such that the ramification points of f_K specialize to pairwise distinct smooth points on the special fibre (see Section 1.6). Let $\mathcal{Y}$ be minimal among all the semistable models satisfying this property. Define $\mathcal{X} = \mathcal{Y}/G$. Then $\mathcal{X}$ is a semistable model of $X_K = \mathbb{P}^1_K$ over R. The map $f_R : \mathcal{Y} \to \mathcal{X}$ is called *stable model of* f_K and its restriction to the special fibre (denoted by $\bar{f} : \bar{Y} \to \bar{X}$) is called the *stable reduction of* f_K. We say that f_K has *good reduction* if $\bar{f}$ is separable. Otherwise, f_K is said to have *bad reduction*.

The following setup holds throughout the whole thesis.

$f_K : Y_K \to X_K = \mathbb{P}^1_K$ is the Galois closure of a three-point cover $\varphi_K : Z_K = \mathbb{P}^1_K \to X_K = \mathbb{P}^1_K$ of degree $d > 1$. We always assume that φ_K is ramified in exactly three points $z_1, z_2, z_3 \in Z_K$ with ramification indices e_1, e_2, e_3, respectively. We call the datum $\mathcal{C} = (d; e_1, e_2, e_3)$ the *ramification type of* φ_K.

What is special about this setup? It is well known that there exists a bijection between (abstract) ramification types (Definition 1.2.1 and Notation 1.2.3) $\mathcal{C} = (d; e_1, e_2, e_3)$ and three-point covers $\varphi_K : Z = \mathbb{P}^1_K \to X = \mathbb{P}^1_K$ having this ramification type (Lemma 1.2.6). The cover φ_K can even be defined over $\mathbb{Q}$ and we emphasize that φ_K is uniquely determined by the ramification indices! Hence, all the information about φ_K (and hence about f_K) is contained in the ramification type $\mathcal{C}$ of φ_K. Osserman used this fact in [Oss08] and characterized the existence of three-point covers in characteristic $p > 0$ which only depends on $\mathcal{C} = (d; e_1, e_2, e_3)$ ([Oss08, Theorem 2.4] or Theorem 4.1.7). Osserman's result can be interpreted as a result on good reduction. The existence of a three-point cover with ramification type $\mathcal{C}$ in characteristic $p > 0$ is equivalent to the unique three-point cover with ramification type $\mathcal{C}$ in characteristic 0 having good reduction. Thereby, he characterized all three-point covers with good reduction. But what is the structure of the stable reduction of three-point covers with bad reduction?

The general structure of the stable reduction of three-point covers with bad reduction is unknown. However, for $d < 2p$ there exist several structure results which were proven by Wewers ([Wew03]). We briefly recall these results.

We remark that Wewers' results even hold in a more general context, which he calls mild reduction (Definition 3.2.1).

- Wewers proved in [Wew03] that if f_K has bad reduction, then $\bar{X}$ is a comb ([Wew03, Theorem 2.6] or Theorem 3.2.6). Moreover, there exists a unique component $\bar{X}_0 \subset \bar{X}$ such that the restriction of $\bar{f}$ to an irreducible component above $\bar{X}_0$ is inseparable. The restriction of $\bar{f}$ to an irreducible component which lies above any other component of $\bar{X}$ is separable. We call such covers tail covers. There are two types of tail covers which are determined by the number of branch points. We refer to Section 3.2 for more details.
- Wewers also proved a lifting result for $d < 2p$ ([Wew03, Corollary 4.6] or Theorem 3.3.4). This result says that for every three-point cover with $d < 2p$ there exists a unique set of tail covers and a unique so-called deformation datum such that the tail covers and the deformation datum satisfy certain compatibility conditions. Every set of tail covers, together with a deformation datum satisfying these conditions can be "lifted" to characteristic 0, meaning that it arises from the stable reduction of a three-point cover. We refer to Section 3.3 for more information.

Note that Wewers' results do not say how many of these tail covers exist in the stable reduction of three-point covers. Moreover, his results do neither describe the tail covers nor the deformation datum in the stable reduction.

In this thesis we complete Wewers' results in some sense for the three-point covers in our setup. We provide the missing information about the tail covers and the deformation data. Our main result is the full description of the tail covers and deformation data of the stable reduction for $d < 2p$ (Theorem 4.2.2).

Theorem 1 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $d < 2p$. Let $\varphi_K : Z_K = \mathbb{P}^1_K \to X_K = \mathbb{P}^1_K$ be the unique three-point cover with ramification type $\mathcal{C}$ and $f_K : Y \to X$ its Galois closure. Define $d_i = \max\{e_i, p\}$. Then the following holds.*

(a) f_K has bad reduction if and only if $\sum_{i=1}^{3}(d_i - p) \leq d - p$.

(b) Assuming that f_K has bad reduction we determine the deformation datum, together with the set of tail covers.

Part (a) is essentially a reformulation of the result of Osserman in the case that $d < 2p$ (Theorem 4.1.1). Note however, that Osserman's result is much more general, as it applies to arbitrary d. We refer to Section 4.2 for the precise statement of (b).

With Theorem 1 (a) we prove that the existence of a hypergeometric deformation datum associated with the type $\mathcal{C}$ is equivalent to bad reduction (Corollary 4.1.3). This result is new in the setup of this thesis. However, a similar result has been proven by Bouw and Wewers in [BW04] for a different class of three-point covers.

Further, we regard tail covers as independent objects (not necessarily connected to stable reduction). We generalize the tail covers which occur in the stable reduction of three-point covers for $d < 2p$ in a natural way, i.e. we analyze the tail covers which are defined as the Galois closure of the maps

(i)
$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^e + \bar{z}^{ap}, & \text{where } (a-1)p < e < ap, \\ \frac{\bar{z}^e}{\bar{z}^{e-ap}-1}, & \text{where } ap < e < (a+1)p, \end{cases} \quad \text{or}$$

(ii)
$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^d}{\bar{z}^{d-ap}}, \quad \text{where } ap < d < (a+1)p.$$

We prove that the tail covers defined as the Galois closures of (i) occur in the stable reduction of three-point covers for arbitrarily large a and e (Corollary 5.2.9). We prove many results on the structure of wild inertia groups (for $a < p$), which may prove useful in future research, e.g. we prove that the prime-to-p part of a wild inertia group permutes ap numbers if $a \geq 2$ (see Theorem 2.3.11). We hope that these results help to find a three-point cover with $d \geq 2p$ with mild reduction, or to prove that mild reduction can only occur if $d < 2p$.

With the additional assumption that there exists only one lower jump (in the filtration of the higher ramification groups) we compute the ramification invariant of the tail covers of (i) and (ii) (Section 2.4). This enables us to show that there are many three-point covers with nonmild reduction (Corollary 5.4.3). Moreover, we show that in some sense the tail covers of (i) are unique for any $a > 0$, whereas the tail cover of (ii) is not for $a \geq 2$ (Section 2.6). This is interesting, because of the uniqueness of the three-point covers (in our setup). The uniqueness of the three-point cover strongly suggests that the tail covers should be unique, as well. Since this is not the case for the tail cover of (ii), it is unlikely that this tail cover appears in the stable reduction of any three-point cover with $d \geq 2p$.

The results we prove for tail covers in the case that $d \geq 2p$ are sadly not as complete as one could hope. For instance, we don't know the order of a wild inertia group. We also don't know the number of lower jumps (in the filtration of higher ramification groups). However, we believe that there is a unique lower jump $\neq 0$.

Another main result is a correspondence between Osserman's result on good reduction ([Oss08, Theorem 2.4] or Theorem 4.1.7) and the degree of tail covers for $d < 2p$ (Proposition 5.3.2).

Proposition *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $d < 2p$. Let $\varphi_K : Z_K = \mathbb{P}^1_K \to X_K = \mathbb{P}^1_K$ be the unique three-point cover with ramification type $\mathcal{C}$ and $f_K : Y \to X$ its Galois closure. Define $d_i = \max\{e_i, p\}$. Suppose that f_K has bad reduction. Then the following holds.*

(a) There exists a unique set $S \subset \{1, 2, 3\}$ for which Osserman's criterion fails, i.e. there exists a unique set $S \subset \{1, 2, 3\}$ for which Definition 4.1.6 (ii) and (iii) are satisfied but not the final inequality.

(b) Let S be the unique set of (a). Then $d_j > e_j$ if and only if $j \notin S$.[1]

This result yields a beautiful interpretation of Osserman's technical result in terms of the stable reduction. We further analyze this correspondence for several three-point covers with $d \geq 2p$ and find that the correspondence always holds (at least in these examples). We hope that this correspondence can be generalized in some sense.

[1] In Theorem 4.2.2 we prove that if $p \nmid e_j$, then the degree of the tail cover is $d_j = \max\{e_j, p\}$.

Outline

The thesis consists of five chapters.

In Chapter 1 we recall the most basic definitions and results which are used throughout the whole thesis.

In Chapter 2 we study the tail covers. In Section 2.1 we define tail covers in general. In Section 2.2 we recall several known results about tail covers. In Section 2.3 we construct the tail covers of interest for this thesis (Lemma 2.3.1). As special cases the tail covers of Section 2.2 are included. We prove several structure results about the wild inertia groups (Theorem 2.3.11). In Section 2.4 we compute the ramification invariant of the tail covers of Section 2.3 under the additional assumption that there exists exactly one lower jump $\neq 0$ (Lemma 2.4.1). In Section 2.5 we compute the ramification type of the tail covers for $d < 2p$. In Section 2.6 we discuss uniqueness of the tail covers from Section 2.3.

In Chapter 3 we recall the details on the stable reduction of three-point covers. In Section 3.1 we define special deformation data, hypergeometric deformation data and special G-deformation data. We analyze the existence of hypergeometric deformation data in more detail. We recall a general result due to Bouw and Wewers (Proposition 3.1.8) which yields a criterion for the existence of a hypergeometric deformation datum and construct one which only depends on ramification types of three-point covers (Proposition 3.1.9). Hence, we link abstract ramification types with hypergeometric deformation data. In Section 3.2 we recall Wewers' structure theorem (Theorem 3.2.6) on the stable reduction of three-point covers. This section provides the link between stable reduction and tail covers. In Section 3.3 we recall Wewers' lifting result (Theorem 3.3.4). This lifting result is the main tool for the description of the stable reduction. We end this chapter with an elaborate example which demonstrates the use of this result.

In Chapter 4 we characterize the stable reduction of three-point covers with $d < 2p$. In Section 4.1 we prove the convenient simplification of Osserman's good reduction result (see Theorem 1 (a) in the introduction or Theorem 4.1.1). Using this simplification we prove a connection between the existence of hypergeometric deformation data and bad reduction (Corollary 4.1.3). In Section 4.2 we prove the characterization of the stable reduction for $d < 2p$ (Theorem 4.2.2 (a),(b)) save for one three-point cover, which we discuss separately in Section 4.3 (Theorem 4.2.2 (c)).

In Chapter 5 we mainly analyze three-point covers with $d \geq 2p$. In Section 5.1 we provide several combinatorial basics. In Section 5.2 we use the combinatorial results from Section 5.1 to compute several irreducible components of several three-point covers with $d \geq 2p$. In Section 5.3 we prove a correspondence between Osserman's result and the degree of tail covers for $d < 2p$ (Proposition 5.3.2) and discuss whether this correspondence could be generalized with several examples. In Section 5.4 we prove that many three-point covers do not have mild reduction.

1 Preliminaries

In this chapter we define all the basic objects which occur in this thesis, e.g. three-point covers, higher ramification groups, induced curves, stable reduction and so on. This chapter is a collection of basic results and most of the included results are well known.

In Section 1.1 we define decomposition and inertia groups and recall some basic properties of Galois covers. In Section 1.2 we define the objects of interest, namely three-point covers. In Section 1.3 we discuss higher ramification groups. In Section 1.4 we recall some basic properties about Sylow p-subgroups of symmetric groups. This section is mainly included to fix some notation used in later chapters. We then proceed in Section 1.5 with the definition of a curve induced by a group extension. Finally in Section 1.6 we give the most basic overview about stable reduction.

Throughout this whole chapter we assume that all curves are smooth, projective and absolutely irreducible unless explicitly stated otherwise.

1.1 Galois covers

Throughout this section let R be a complete discrete valuation ring, K its fraction field and k its residue field. Moreover, L/K is a finite Galois extension, R_L the discrete valuation ring of L and ℓ the residue field. We assume that ℓ/k is separable and denote the Galois group of L/K by G.

In this section we recall some basic but crucial results about Galois covers. The main source for this section is [Ser79, Chapter I §7].

Proposition 1.1.1 *The group G acts transitively on the set of prime ideals $\mathfrak{P} \subset R_L$ which lie above a given prime ideal $\mathfrak{p} \subset R$.*

Proof: The proof can be found in [Ser79, Chapter I, Proposition 19]. □

To see the importance of Proposition 1.1.1 we need to introduce decomposition groups and their connection to ramification points. The following definition can be found in [Ser79, Chapter 1, §7].

Definition 1.1.2 Let $\mathfrak{P} \subset R_L$ be a prime ideal. The subgroup

$$D(\mathfrak{P}) = \{s \in G \mid s(\mathfrak{P}) = \mathfrak{P}\}$$

is called *the decomposition group of* $\mathfrak{P}$.

Proposition 1.1.1 now has the following consequence.

Corollary 1.1.3 *Let $\mathfrak{P}_1, \mathfrak{P}_2 \subset R_L$ be two prime ideals with $\mathfrak{p} = \mathfrak{P}_1 \cap R = \mathfrak{P}_2$. Then the decomposition groups $D(\mathfrak{P}_1)$ and $D(\mathfrak{P}_2)$ are conjugate.*

Proof: Trivial. □

We now define the inertia group. The following definition can be found in [Ser79, Chapter 1, §7].

Definition 1.1.4 Let $\mathfrak{P} \subset R_L$ be a prime ideal $\neq 0$ and $s \in D(\mathfrak{P})$, and $\mathfrak{p} = \mathfrak{P} \cap R$. Let $a \in \ell = R_L/\mathfrak{P}$. Define $s(a + \mathfrak{P}) = s(a) + \mathfrak{P}$. This defines a $k = R/\mathfrak{p}$ automorphism of ℓ. The described passage to the quotient defines a homomorphism

$$\varepsilon : D(\mathfrak{P}) \to \mathrm{Gal}(\ell/k).$$

The kernel of ε is called the *inertia group* of $\mathfrak{P}$ and denoted by $I_{\mathfrak{P}}$.

Decomposition groups and inertia groups arise in two ways in this thesis. We briefly elaborate this. But first, we introduce the following notation.

Notation 1.1.5 Let $f : Y \to X$ be a Galois cover defined over K and $y \in Y$ a point (closed or generic). There exists a prime ideal $\mathfrak{P} \subset K[Y]$ in the coordinate ring corresponding to y. Localizing we obtain the situation from the introduction of this section and define the decomposition group and Galois group as in Definition 1.1.2 and 1.1.4. We denote the decomposition group $D(\mathfrak{P})$ (resp. the inertia group $I_{\mathfrak{P}}$) by D_y (resp. I_y).

Note that the decomposition group of a prime ideal is the stabilizer of this prime ideal (under the action of G on the set of prime ideals). Hence the decomposition group of a point $y \in Y$ describes the multiplicity of this point. We now prove that decomposition groups and inertia groups coincide under certain circumstances.

Lemma 1.1.6 *Let $f : Y \to X$ be a Galois cover defined over K. Assume that k is algebraically closed. Let $y \in Y$ be a closed point and $I_y \subset G$ its inertia group and D_y its decomposition group. Then*

$$D_y = I_y.$$

Proof: This follows immediately from the definition of ε and the fact that k is algebraically closed. □

We now sketch the appearance of decomposition and inertia groups. Part one of this explanation is derived from [BW12, Example 2.1]. Part two of this explanation uses terms and definitions of the stable reduction which are discussed in Section 1.6. However, the reader who is familiar with stable reduction may profit from this explanation at this point.

- Let $f : Y \to X$ be a Galois cover between curves defined over k. We assume that k is algebraically closed and that $\operatorname{char} k = p > 0$. Let $y \in Y$ be a closed point, $x = f(y) \in X$ its image and I_y its inertia group. Since Y and X are smooth curves, the complete local rings $\hat{\mathcal{O}}_{Y,y}$ and $\hat{\mathcal{O}}_{X,x}$ are discrete valuation rings which are isomorphic to $k[[z]]$. Let $L := \operatorname{Frac}(\hat{\mathcal{O}}_{Y,y})$ and $K := \operatorname{Frac}(\hat{\mathcal{O}}_{X,x})$ be the fraction fields. Then L/K is a totally ramified I_y-Galois extension. In particular, f is ramified in y with ramification index $|I_y|$.
- Let $f : Y \to X$ be a Galois three-point cover (see Definition 1.2.1) defined over K and assume that k is algebraically closed. Let $\bar{f} : \bar{Y} \to \bar{X}$ be the stable reduction of f and $\eta \in \bar{Y}$ the generic point of an irreducible component. Then the inertia group of this irreducible component is defined as the inertia group of η.

In this thesis we often switch between non-Galois covers and Galois covers which are induced by the former. The non-Galois covers in this thesis are always covers between genus-0 curves. It is often necessary to interpret the Galois groups as subgroups of symmetric groups. We make this precise as follows.

Remark 1.1.7 Let K be any field.

(a) Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a map of degree $d > 1$ between curves over K. Let $K(Y)$ be the Galois closure of the field extension $K(Z)/K(X)$ and Y the curve with function field $K(Y)$. Let $f : Y \to X$ be the induced map and $G = \operatorname{Gal}(f)$ (we often say that *f is the Galois closure of φ*). Moreover, let $x \in X$ be a point which is not a branch point. We consider x as base point. We choose a numbering of the d leaves above x and regard G as subgroup of S_d via the permutation representation of G on these leaves.

(b) Let $f : Y \to X = \mathbb{P}^1$ be a finite G-Galois cover over K and suppose that there exists a subgroup $H \subset G$ of index $d > 1$ such that $Y/H \simeq \mathbb{P}^1$ and f is the Galois closure of $Y/H \to X$. By (a) we regard $G \subset S_d \simeq S(\{1, \dots, d\})$ as a transitive group on d leaves. By the definition of quotient schemes, we have that $K(Y/H) = K(Y)^H$. We obtain the field diagram of Figure 1.1.1.

$$K(X) \hookrightarrow K(Y)^H \hookrightarrow K(Y)$$

Figure 1.1.1: Field diagram.

Since $K(Y)/K(X)$ is a finite and separable field extension (f is Galois and finite), $K(Y)^H/K(X)$ is finite and separable as well. Hence there exists a primitive element $\alpha \in K(Y)^H$ and an irreducible polynomial $u := \min_{K(X)}(\alpha)$ of degree d such that $K(Y)^H = K(X)(\alpha)$. By Galois theory, G acts on the d roots of u. Numbering the roots of u yields a permutation representation of G. Since $\alpha \in K(Y)^H$, H actually fixes α and hence H is a one-point stabilizer of G (i.e. $H = H_{i_0} = G \cap S(\{1, \ldots, i_0 - 1, i_0 + 1, \ldots, d\})$). Note that the permutation representation of G on the leaves is just a renumbering of the permutation representation on the roots of u. We conclude that H also fixes one leaf. Conversely, let $H_j := G \cap S(\{1, \ldots, j-1, j+1, \ldots, d\})$ be a one-point stabilizer. Then $K(Y)^{H_j}/K(X)$ is a field extension of degree d and the Galois closure is $K(Y)$. Since G is transitive on the d leaves, there exists an element $\mu \in G$ with $\mu(i_0) = j$. We conclude that $K(Y)^{H_j}$ and $K(Y)^H$ are isomorphic. It follows that Y/H_j is a curve of genus 0.

1.2 Three-point covers

In this section we define three-point covers and the ramification type of three-point covers. These are the central objects in this thesis.

We now define three-point covers. Note that since the notation used in this thesis differs from the standard notation (see Schneps' definition of Belyi morphism in [Sch94, Definition 1, Page 50]), we explain it directly after Definition 1.2.1.

Definition 1.2.1 Let K be any field.

(a) A *three-point cover* $f_K : Z_K \to X_K := \mathbb{P}^1_K$ is a finite cover defined over K between smooth and geometrically connected K-curves, branched exactly in $x_1 = 0, x_2 = 1, x_3 = \infty \in X_K$. If the field of definition K is clear from the context, we omit the subscript K to indicate that the curves X_K and Y_K and the map f_K are defined over K.

(b) Let $\varphi : Z \to X = \mathbb{P}^1$ be a three-point cover. If $Z \simeq \mathbb{P}^1$, then φ is called *genus-0 Belyi morphism*.

(c) Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism. Let L be the Galois closure of $K(Z)/K(X)$ and Y the projective curve with function field L. Let $f : Y \to X = \mathbb{P}^1$ be the unique map such that f factors through φ. Then f is called the *Galois closure of* φ.

(d) Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism of degree d, $f : Y \to X$ its Galois closure, and let $z_{i,1}, \ldots, z_{i,r_i} \in Z$ be all the points above $x_i \in X$, where $x_i \in \{0, 1, \infty\}$. Denote the ramification index of $z_{i,j}$ by $1 \leq e_{i,j} \leq d$. For $i = 1, 2, 3$ we define $C_i := (e_{i,1}, \ldots, e_{i,r_i})$. The tuple $\mathcal{C} := (d; C_1, C_2, C_3)$ is called the *ramification type of* φ *(and of* f*)*. We sometimes say that φ (or f) has type $\mathcal{C}$ or that φ (or f) is a three-point cover of type $\mathcal{C}$.

(e) An *abstract (genus-0 three-point) ramification type* is a tuple $\mathcal{C} = (d; C_1, C_2, C_3)$ with a positive integer d satisfying the following conditions.

(i) Every $C_i = (e_{i,1}, \ldots, e_{i,r_i})$ is a tuple of integers $1 \leq e_{i,j} \leq d$ such that $\sum_{j=1}^{r_i} e_{i,j} = d$, i.e. the C_i form a partition of d.

(ii) The C_i satisfy

$$2d - 2 = \sum_{i=1}^{3} \sum_{j=1}^{r_i} (e_{i,j} - 1). \tag{1.2.1}$$

(iii) For every $i \in \{1, 2, 3\}$ there exists at least one $1 \leq j \leq r_i$ such that $e_{i,j} \neq 1$.

The reason we use the term "genus-0 Belyi morphism" is the following. In this thesis we describe the stable reduction of a certain class of Galois three-point covers. One property of this class is that every Galois three-point cover in it is the Galois closure of a three-point cover between genus-0 curves. The term "three-point cover between genus-0 curves" however is lengthy and cumbersome. However, the terms "genus-0 three-point cover" and "Galois three-point cover" are too similar and prone to confusion. To avoid this problem we introduced the term "genus-0 Belyi morphism". Further, we always denote a Galois three-point cover by Latin letters and the corresponding genus-0 Belyi morphism by Greek letters (e.g. f and φ of Definition 1.2.1 (c)).

Remark 1.2.2 (a) Let $f : Y \to \mathbb{P}^1$ be the Galois closure of a genus-0 Belyi morphism $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ as in Definition 1.2.1 (c). Then f is also a three-point cover.

(b) Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism with ramification type $\mathcal{C} = (d; C_1, C_2, C_3)$. The C_i are partitions of d that is $\sum_{j=1}^{r_i} e_{i,j} = d$. Hence the C_i define conjugacy classes in S_d.

(c) The conditions on abstract ramification types in Definition 1.2.1 (e) imply the following. Condition (i) says that the ramification indices are a partition of d. Condition (ii) is Riemann-Hurwitz in genus 0 (Theorem 1.3.5) and condition (iii) guarantees that there are exactly three branch points.

To work conveniently with ramification types, we introduce the following short notation for a special case of three-point covers. The following notation is based on [BZ13, Definition 2.1].

Notation 1.2.3 (a) Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism and $\mathcal{C} = (d; C_1, C_2, C_3)$ its ramification type. Assume that φ is only ramified in exactly three points $z_1, z_2, z_3 \in Z$. Denote the ramification indices by e_1, e_2, e_3 respectively. In this case we simply write $\mathcal{C} = (d; e_1, e_2, e_3)$ instead of $\mathcal{C} = (d; C_1, C_2, C_3)$. We call this case the *single-cycle case*.

(b) Similarly, if $\mathcal{C} = (d; C_1, C_2, C_3)$ is an abstract ramification type and in every C_i there is exactly one $e_{i,j} \neq 1$, then we simply write $\mathcal{C} = (d; e_1, e_2, e_3)$, where e_i is the unique $e_{i,j} \neq 1$.

Remark 1.2.4 (i) In the single-cycle case, the formula

$$2d - 2 = \sum_{i=1}^{3} \sum_{j=1}^{r_i} (e_{i,j} - 1)$$

of (1.2.1) simplifies to

$$2d + 1 = e_1 + e_2 + e_3.$$

(ii) In this thesis we only study the single-cycle case.

Let $f : Y \to X = \mathbb{P}^1$ be any three-point cover as in Definition 1.2.1 (a). Recall that $\mathrm{Aut}_K(\mathbb{P}^1)$ consists of linear fractional maps which are uniquely determined by the images of three distinct points. Using a linear fractional map on X we can assume that the three branch points $x_1 = 0, x_2 = 1, x_3 = \infty$ are any three distinct points in $X = \mathbb{P}^1$. Thus the Definition 1.2.1 (a) of three-point covers is equivalent to the definition that f is ramified at three distinct points. But since it is easier to work with the normalized form of three-point covers, we stick to the normalization of Definition 1.2.1.

Further, we can normalize genus-0 Belyi morphisms with exactly three ramification points (single-cycle case) in a convenient way. Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism with exactly three ramification points $z_1, z_2, z_3 \in Z$ and let $\mathcal{C} = (d; e_1, e_2, e_3)$ be its ramification type. Using a linear fractional map on the coordinate of Z we may assume that $\{z_1, z_2, z_3\} = \{0, 1, \infty\} \subset Z$. After renumbering we can assume that $z_1 = 0, z_2 = 1, z_3 = \infty$ and that $\varphi(z_i) = x_i$. In particular, we can normalize φ in such a way that the ramification index above 0 is the smallest, the one above 1 is possibly larger and the ramification index above ∞ is the largest that is, we may assume that $e_1 \leq e_2 \leq e_3$.

Convention 1.2.5 (a) Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type. Unless explicitly told otherwise we assume that $e_1 \leq e_2 \leq e_3 \leq d$.

(b) Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism with three single ramification points and $\mathcal{C} = (d; e_1, e_2, e_3)$ its ramification type. We always assume that φ is normalized as follows:

- $\varphi(0) = 0$ with ramification index e_1,
- $\varphi(1) = 1$ with ramification index e_2 and
- $\varphi(\infty) = \infty$ with ramification index e_3.

(c) Every mention of Galois three-point cover means a Galois three-point cover, which is the Galois closure of a genus-0 Belyi morphism.

(d) All genus-0 Belyi morphisms and ramification types of any three-point cover are in the single-cycle case.

To see why ramification types are important we recall the following result.

Lemma 1.2.6 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type. Then there exists a genus-0 Belyi morphism $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ defined over $\mathbb{Q}$ such that φ has ramification type $\mathcal{C}$. Moreover, φ is unique up to normalization.*

In particular: if we normalize φ as in Convention 1.2.5 (b), then φ is unique.

Proof: This is a well-known result. See [LO08, Lemma 2.1]. □

Lemma 1.2.6 shows that in the single-cycle case (which we always assume), ramification types and genus-0 Belyi morphism are essentially the same i.e. given an abstract ramification type $\mathcal{C}$ (in the single-cycle case) there exists a genus-0 Belyi morphism with ramification type $\mathcal{C}$ and it is unique. Thus, all the information about the genus-0 Belyi morphism and its Galois closure are contained in $\mathcal{C}$. Hence we can use purely combinatorial results on abstract ramification types and obtain results about genus-0 Belyi morphisms and their respective Galois closures. In Section 4.1 we discuss this important property in more detail. Note however that Lemma 1.2.6 is only true in the single-cycle case. The essential conditions here are "genus 0" and exactly three ramification points. Both conditions are encoded in the notation of the abstract ramification type in Lemma 1.2.6 (see Remark 1.2.2 (c) and Notation 1.2.3).

Example 1.2.7 Let $\mathcal{C} = (7; 4, 5, 6)$ and $p = 5$. Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the unique genus-0 Belyi morphism of type $\mathcal{C}$. We claim that φ is given by

$$x = \varphi(z) = \frac{z^4\left(-10z^3 + 42z^2 - 63z + 35\right)}{7z - 3}.$$

To verify this, we look at the derivative. One computes

$$\varphi'(z) = \frac{-420z^3(z-1)^4}{(7z-3)^2}.$$

This shows that φ is indeed only ramified in $z = 0$ with ramification index 4, in $z = 1$ with ramification index 5 and comparing the degree of the numerator and denominator of φ one sees that the ramification index in $z = \infty$ is indeed 6. Thus φ is the genus-0 Belyi morphism of type $\mathcal{C}$.

1.3 Higher ramification groups

Throughout this section let K be a complete discretely valued field with residue field k. Let L/K denote a finite Galois extension, $R_L \subset L$ the complete discrete valuation ring of L and G the Galois group of L/K. We denote the discrete valuation of R_L by v_L.

In this section we recall the definition and most important properties of higher ramification groups. We end this section with a small description of the similarities and differences of wild and tame ramification. This section is heavily based on [Ser79, Chapter IV].

Definition 1.3.1 The *i-th ramification group* $G_i \subset G$ is defined by

$$G_i = \{s \in G \mid v_L(s(a) - a) \geq i + 1 \text{ for all } a \in R_L\}.$$

We summarize the most important properties of ramification groups.

Proposition 1.3.2 *The higher ramification groups have the following properties.*

(a) $G_{-1} = G$ *and* G_0 *is the inertia subgroup of* G.

(b) $G_i \subset G$ *is a normal subgroup for all* $i \geq -1$.

(c) $G_i = \{1\}$ *for* i *large enough. In particular, there is an* h *such that* $G_h \supsetneq G_{h+1}$.

(d) Let $H \subset G$ *be a subgroup. Then* $H_i = G_i \cap H$.

(e) *If $\mathfrak{D}_{L/K}$ denotes the different of L/K, then*

$$v_L(\mathfrak{D}_{L/K}) = \sum_{i=0}^{\infty}(|G_i| - 1).$$

(f) *G_0/G_1 is a cyclic group of order prime to p.*

(g) *G_1 is a p-group.*

(h) *The inertia group G_0 has the following property:*

$$G_0 \simeq \mathcal{P} \rtimes H,$$

where $\mathcal{P}$ is a p-group and $H \simeq \mathbb{Z}/m\mathbb{Z}$ is a cyclic group with $p \nmid m$.

Proof: The proof of (a), (b) and (c) is in [Ser79, Chapter IV, Proposition 1].

For (d) see [Ser79, Chapter IV, Proposition 2].

The proof of (e) is in [Ser79, Chapter IV, Proposition 4].

Property (f), (g), (h) are proven in Corollary 1, 3, and 4 to [Ser79, Chapter IV, Proposition 7]. □

Definition 1.3.3 The integers $i \in \mathbb{Z}$ with $G_i \neq G_{i+1}$ are called *(lower) jumps*.

We now recall the formula of Riemann-Hurwitz. To write it down conveniently, we introduce the following short notation.

Notation 1.3.4 Let $f : Y \to X$ be a finite, separable morphism of smooth curves defined over k and $y \in Y$ be a closed point. Then y corresponds to a discrete valuation ring $\mathcal{O}_{Y,y}$ and $x := f(y)$ corresponds to a valuation ring $\mathcal{O}_{X,x}$. Then we define $\mathfrak{D}(y)$ to be the different of $\mathcal{O}_{Y,y}$ over $\mathcal{O}_{X,x}$.

Theorem 1.3.5 (Riemann-Hurwitz) *Let $f : Y \to X$ be a finite Galois morphism of curves. Then*

$$2g(Y) - 2 = \deg(f) \cdot (2g(X) - 2) + \sum_{y \in Y} v_L\left(\mathfrak{D}(y)\right).$$

Proof: See [Har77, Chapter IV, Corollary 2.4] combined with [Har77, Chapter IV, Proposition 2.2 (b)] and [Ser79, Chapter III §7, Proposition 14]. □

We now introduce the upper numbering of higher ramification groups and recall the Theorem of Hasse-Arf. We refer to [Ser79, Chapter IV, §3] for more details.

Definition 1.3.6 (a) For $u \in \mathbb{R}$ with $u \geq -1$ we define $G_u := G_{\lceil u \rceil}$ and

$$\varphi(u) := \int_0^u \frac{dt}{[G_0 : G_t]}.$$

The *(upper) numbering* is then defined by

$$G^{\varphi(u)} := G_u.$$

(b) The numbers σ_i such that $G^{\sigma_i} \neq G^{\sigma_i + \varepsilon}$ for any $\varepsilon > 0$ are called *(upper) jumps*. If there exists only one upper jump $\sigma \neq 0$, then σ is called the *ramification invariant*.

We now recall the Theorem of Hasse-Arf. For a proof see [Ser79, Chapter V, §7].

Theorem 1.3.7 (Hasse-Arf) *If G is an abelian group and if v is an upper jump, then v is an integer.*

Now we prove a general result on the ramification invariant. Part of the idea of the proof is a generalization of [BW12, Exercise 2.13].

Lemma 1.3.8 *Let $1 \leq b \leq p-1$ and L/K be a Galois field extension with Galois group $G = (\mathbb{Z}/p\mathbb{Z})^b \rtimes_\chi \mathbb{Z}/m\mathbb{Z}$, where $\chi : \mathbb{Z}/m\mathbb{Z} \to \mathrm{Aut}((\mathbb{Z}/p\mathbb{Z})^b)$ is defined by the conjugation action of $\mathbb{Z}/m\mathbb{Z}$ on $(\mathbb{Z}/p\mathbb{Z})^b$. Suppose that the action of $\mathbb{Z}/m\mathbb{Z}$ on $(\mathbb{Z}/p\mathbb{Z})^b$ is irreducible and let $m_1 := |\ker \chi|$ and $m_2 = m/m_1$. Then the following holds.*

(a) There exists a unique lower jump $h \neq 0$ of the higher ramification groups of G.

(b) The integers h and m satisfy

$$\gcd(h, m) = m_1.$$

In particular, if χ is injective, then $\gcd(h, m) = 1$.

(c) The ramification invariant is $\sigma = h/m$.

Proof: Assertion (a) is well known and follows immediately from the irreducibility of the action of $\mathbb{Z}/m\mathbb{Z}$ on $(\mathbb{Z}/p\mathbb{Z})^b$ using Proposition 1.3.2 (b).

We now prove (b). By assumption $\mathbb{Z}/m_1\mathbb{Z}$ acts trivially on $(\mathbb{Z}/p\mathbb{Z})^b$. In particular, the subgroup

$$\tilde{G} := (\mathbb{Z}/p\mathbb{Z})^b \rtimes_\chi \mathbb{Z}/m_1\mathbb{Z} = (\mathbb{Z}/p\mathbb{Z})^b \times \mathbb{Z}/m_1\mathbb{Z} \subset G$$

is an abelian group. The upper jump of $\tilde{G}$ coincides with the upper jump of G and is given by

$$\varphi(h) = \frac{1}{p^b m_1} \cdot h p^b = \frac{h}{m_1}.$$

By Hasse-Arf (Theorem 1.3.7) $\frac{h}{m_1} \in \mathbb{Z}$ and hence $m_1 \mid h$.

We now prove that $\gcd(h, m_2) = 1$. By assumption $\mathbb{Z}/m_2\mathbb{Z}$ acts irreducibly on $(\mathbb{Z}/p\mathbb{Z})^b$. We assume (for contradiction) that $\gcd(h, m_2) =: n \neq 1$.

Let $\tau \in G$ be an element of order m_2 and $\gamma_0 \in (\mathbb{Z}/p\mathbb{Z})^b$ an element of order p. We define

$$\gamma_i := \tau^i \gamma_0 \tau^{-i} \quad \text{for } i = 1, \ldots, m_2 - 1.$$

Moreover, we define

$$L_i := L^{\langle \gamma_0, \ldots, \gamma_{i-1}, \gamma_{i+1} \ldots, \gamma_{m_2-1} \rangle} \quad \text{for } i = 0, \ldots, m_2 - 1 \text{ and}$$
$$M := L^{\langle \gamma_0, \ldots, \gamma_{m_2-1} \rangle}.$$

We obtain the following field of Figure 1.3.1.

The field extensions L_i/M are cyclic of order p and M/K is cyclic of order m_2. By Artin-Schreier and Kummer theory, there exist $y_i \in L_i$ and $x \in M$ with

$$\begin{aligned} \tau(x) &= \xi_{m_2} x, \\ y_0^p - y_0 &= x^{-h}(1 + x c_1 + \ldots) \quad \text{and} \\ \tau(y_i) &= y_{i+1}, \quad \text{for } i = 0, \ldots, m_2 - 2. \end{aligned}$$

where ξ_{m_2} denotes a primitive m_2-th root of unity. Restricting to the subgroup of order n and accordingly replacing τ by an appropriate power we may assume that $\mathrm{ord}(\tau) = n$. Note that τ need no longer act irreducibly on $(\mathbb{Z}/p\mathbb{Z})^b$ but we will not use this in the following. Renumbering the y_i we may still assume that $\tau(y_i) = y_{i+1}$ for $i = 0, \ldots, n-1$.

Note that the injectivity of the action of $\mathbb{Z}/m_2\mathbb{Z}$ on $(\mathbb{Z}/p\mathbb{Z})^b$ implies that the orbit of y_0 under $\langle \tau \rangle$ has length n. Define

$$y := y_0 + \ldots + y_{n-1}.$$

Since the orbit of y_0 has length n, we obtain

$$\begin{aligned} \tau(y^p - y) &= y^p - y \\ &= x^{-h}(1 + \xi_n^{-h} + \xi_n^{-h \cdot 2} + \ldots + \xi_n^{-h(n-1)}) + \ldots. \end{aligned}$$

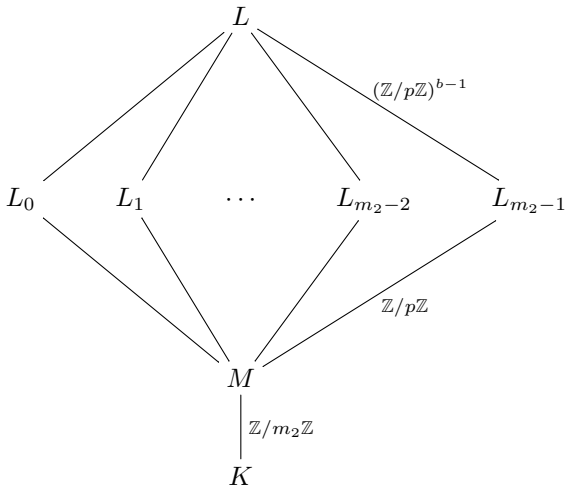

Figure 1.3.1: Diagram of the field extensions.

Since $\gcd(h, m_2) = n$, we have that $\xi_n^{-h} = 1$. In particular,

$$1 + \xi_n^{-h} + \xi_n^{-h\cdot 2} + \ldots + \xi_n^{-h(n-1)} = n \neq 0 \in k.$$

Since τ fixes $y^p - y$, but does not fix the right side, this yields a contradiction.

This proves that $\gcd(h, m_2) = 1$ and hence

$$\gcd(h, m) = m_1 \cdot \gcd(h, m_2) = m_1.$$

We now prove (c). Since there is only one lower jump $\neq 0$, we compute

$$\sigma = \varphi(h) = \frac{1}{|G_0|} \cdot h|G_1| = \frac{1}{p^b m} \cdot hp^b = \frac{h}{m}.$$

□

Remark 1.3.9 Results similar to Lemma 1.3.8 (b) in a more general setup can be derived from [Pri06, Lemma 2.6].

We now give a small summary of the differences between tame and wild ramification.

Lemma 1.3.10 *Let K be a field and $p = \operatorname{char} K$. Let $f : Y \to X$ be a Galois cover defined over K and $y \in Y$ a point. Denote the ramification index of y by e.*

(a) If $p \nmid e$ that is, if y is a tame ramification point, then $v_L(\mathfrak{D}(y)) = e - 1$.

(b) If $p \mid e$, i.e. y is a wild ramification point, then $v_L(\mathfrak{D}(y)) > e - 1$.

Now suppose that f is the Galois closure of a cover between genus-0 curves $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ of degree $d > 1$. Denote the Galois group of f by G. Regard G as subgroup of S_d (Remark 1.1.7).

(c) Let $z_1, \ldots, z_r \in Z$ denote all points above $\varphi(z)$ and e_i the ramification index of z_i. If all e_i are prime to the characteristic of the ground field, then there exists a permutation in G with cycle type $(e_1, \ldots, e_r)$.

Proof: Assertions (a) and (b) are well known and do not require a proof.

Assertion (c) is obvious. □

We point out that Lemma 1.3.10 (c) is false if there is a wild ramification point above $\varphi(z)$.

1.4 Sylow p-subgroups of symmetric groups

In this section we give the group-theoretical background on Sylow p-subgroups of symmetric groups. We begin with Sylow's Theorem.

Theorem 1.4.1 *Let G be a group of order $p^\alpha m$, where p is a prime not dividing m. Then*

(a) Sylow p-subgroups of G exist.

(b) If P is a Sylow p-subgroup of G and Q is any p-subgroup of G, then there exists $g \in G$ such that $Q \subset gPg^{-1}$, i.e. Q is contained in some conjugate of P. In particular, any two Sylow p-subgroups of G are conjugate in G.

(c) The number of Sylow p-subgroups of G (notation: n_p) satisfies

$$n_p \equiv 1 \mod p.$$

Further, n_p is the index in G of the normalizer $N_G(P)$ for any Sylow p-subgroup P.

Proof: The result is well known. See [DF04, Sec. 4.5. Theorem 18]. □

The Sylow p-subgroups of S_d arise frequently in this thesis.

Example 1.4.2 Let $1 \le a \le p-1$ be such that $ap \le d < (a+1)p$. Note that $v((ap)!) = a$, i.e. $p^a \mid |S_d|$ and $p^{a+1} \nmid |S_d|$. In particular, Sylow p-subgroups of S_d have p^a elements. Define

$$\rho_i := \big((i-1)p+1,\ (i-1)p+2,\ \cdots,ip\big) \in S_d = S(\{1,\dots,d\}), \qquad i = 1,\dots,a.$$

Note that the ρ_i are pairwise disjoint. Hence the group $P_a := \langle \rho_1,\dots,\rho_a\rangle$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^a$.

Now $\big(\mathbb{Z}/p\mathbb{Z}\big)^a$ has p^a elements. This shows that P_a is a Sylow p-subgroup of S_d. Since all Sylow p-subgroups are conjugate, it follows that all Sylow p-subgroups of S_d can be generated by a disjoint p-cycles.

Note that the Sylow p-subgroups of S_d wit $d < p^2$ are still rather "nice" and all elements have order 1 or p. For $d \ge p^2$ the Sylow p-subgroups of S_d are not abelian any longer and there are elements with order $> p$. This makes working with them much harder.

We recall a known result on the normalizer of a group generated by a p-cycle in S_p.

Lemma 1.4.3 *Let $\rho = (1\ 2\ \dots\ p)$. Then there exists $\theta \in S_p$ of order $p-1$ such that $N_{S_p}(\langle\rho\rangle) = \langle\rho\rangle \rtimes \langle\theta\rangle$ and such that θ fixes 1. Moreover, one can choose θ to be a $(p-1)$-cycle.*

Proof: For a proof of the first assertion see [HOPS, Lemma 4.13].

The second assertion follows from the proof of [HOPS, Lemma 4.13]. □

Proposition 1.4.4 *Let $a < p$ and $P_a \subset S_{ap}$ be a Sylow p-subgroup as in Example 1.4.2. Then the normalizer $N_{S_{ap}}(P_a)$ has order $p^a(p-1)^a a!$ and contains an element ξ of order $a(p-1)$ which is the disjoint product of an a-cycle ξ_1 and an $a(p-1)$-cycle ξ_2 and acts transitively on the set of generators $\{\rho_i\}$ by conjugation.*

Proof: Recall that in S_p there are $p!$ p-cycles. Since nontrivial powers of a p-cycle generate the same subgroup, there are $(p-2)!$ different Sylow p-subgroups in S_p. With the same reasoning, and the fact that there are $a!$ possibilities to order the p-cycles, we compute that there are

$$\begin{aligned} n_{p^a} &= \frac{1}{a!}\prod_{i=1}^{a}\frac{ip\cdot(ip-1)\cdots\big(ip-(p-1)\big)}{p(p-1)} \\ &= \frac{(ap)!}{p^a(p-1)^a a!} \end{aligned}$$

Sylow p-subgroups in S_{ap}. Because the index of the normalizer in S_{ap} equals the number of Sylow p-subgroups in S_{ap} (Theorem 1.4.1 (c)) the assertion on the order of the normalizer follows.

We now prove the existence of an element of order $a(p-1)$ by explicitly constructing it.

Note that $N_{S_{ap}}(P_a) \cap S(\{1,\dots,p\}) = N_{S_p}(\langle\rho_1\rangle)$ has order $p(p-1)$ (see Lemma 1.4.3). This subgroup is generated by ρ_1 and a $(p-1)$-cycle τ_1 with $\tau_1\rho_1\tau_1^{-1} = \rho_1^\alpha$, where $\alpha \in \mathbb{F}_p^*$ is a primitive $(p-1)$-th root of unity. This is the explicit description of the element θ from Lemma 1.4.3 given in the proof of [HOPS, Lemma 4.13]. We may further assume that $\tau_1(1) = 1$. Define

$$\xi_1 := (1,\ p+1,\ 2p+1,\ \cdots,(a-1)p+1)$$
$$\xi_2 := (2,\ p+2,\ \cdots,\ (a-1)p+2,\ \tau_1(2),\ p+\tau_1(2),\ \cdots,(a-1)p+\tau_1^{p-2}(2))$$

and

$$\xi := \xi_1\xi_2.$$

We see that ξ is the product of an a-cycle and a disjoint $a(p-1)$-cycle. In particular, ξ has order $a(p-1)$. Note that for $i < a$

$$\xi^i\rho_1\xi^{-i} = \rho_{i+1}.$$

In particular,

$$\xi\rho_i\xi^{-1} = \rho_{i+1} \tag{1.4.1}$$

for $i < a$.

For $i = a$ we note that $\xi\left((a-1)p+\ell\right) = \tau_1(\ell)$ and recall that $\tau_1\rho_1\tau_1^{-1} = \rho_1^\alpha$. In particular, we get

$$\xi\rho_a\xi^{-1} = (1\ \tau_1(2)\ \tau_1(3)\ \cdots\ \tau_1(p)) = \rho_1^\alpha.$$

This shows that

$$\xi^a\rho_1\xi^{-a} = \rho_1^\alpha,$$

and it follows that $\xi P_a\xi^{-1} = P_a$ and thus $\xi \in N_{S_{ap}}(P_a)$. □

1.5 The induced curve

In this section we define the induced curve which plays an important role in the description of the stable reduction of three-point covers.

The induced curve is a reducible curve with a group action on it. To define the group action we prove an auxiliary result.

Lemma 1.5.1 *Let $\tilde{G}$ be a finite group and $S \subset \tilde{G}$ be a subgroup. Let $T = \{g_1 = e,\dots,g_r\} \subset \tilde{G}$ be a set of representatives for the left cosets of S in $\tilde{G}$. Let $g \in \tilde{G}$ be any element. Then the following holds.*

For every $i \in \{1,\dots,r\}$ there exists a unique $j \in \{1,\dots,r\}$ and a unique $h \in S$ such that

$$g \cdot g_i = g_j \cdot h.$$

Proof: It is trivial that $\tilde{G} = \coprod_{i=1}^r g_iS$.

Let $i \in \{1,\dots,r\}$ and $g \in \tilde{G}$. Then trivially $g \cdot g_i \in \tilde{G}$. In particular, as the union of left cosets of $\tilde{G}$ is disjoint, there exists a unique $j \in \{1,\dots,r\}$ such that $g \cdot g_i \in g_jS$. Hence there exists $h \in S$ such that $g \cdot g_i = g_j \cdot h$.

The uniqueness of h is trivial. □

We can now define the induced curve.

Definition 1.5.2 Let $\tilde{G}$ be a finite group and $S \subset \tilde{G}$ a subgroup. Let $T = \{g_1 = e,\dots,g_r\}$ be a set of representatives for the left cosets $\tilde{G}/S$. Moreover, let Y be a curve over a field k together with a faithful action of S. The *induced curve of Y* is the set

$$\mathfrak{Z} := \mathrm{Ind}_S^{\tilde{G}} Y := \coprod_{i=1}^r Y^{(i)} = \{(i,y) \mid i \in \{1,\dots,r\}, y \in Y\}$$

with the following group action $\tilde{G} \to \mathrm{Aut}(\mathfrak{Z}), \quad g \mapsto \rho_g$:

Let $g \in \tilde{G}, (i,y) \in \mathfrak{Z}$. Then by Lemma 1.5.1 there exists a unique $j \in \{1,\dots,r\}$ and $h \in S$ with

$$g \cdot g_i = g_j \cdot h.$$

We define $g(i,y) = (j, h(y))$. This defines a map

$$\rho_g : \mathfrak{Z} \to \mathfrak{Z}.$$

We now prove that the action described in Definition 1.5.2 indeed is a group action.

Lemma 1.5.3 *In the notation of Definition 1.5.2 the map $\tilde{G} \to \mathrm{Aut}(\mathfrak{Z}), \quad g \mapsto \rho_g$ is a group homomorphism. In particular, this defines a group action of $\tilde{G}$ on $\mathfrak{Z}$.*

Proof: Let $f_1, f_2 \in \tilde{G}$ and $(i,y) \in \mathfrak{Z}$. By Lemma 1.5.1 there exist unique $j_1, j_2 \in \{1,\dots,r\}$ and $h_1, h_2 \in S$ such that $f_1 g_i = g_{j_1} h_1$ and $f_2 g_{j_1} = g_{j_2} h_2$.

Then by definition

$$\begin{aligned}(\rho_{f_2} \circ \rho_{f_1})(i,y) &= \rho_{f_2}\Big(\big(j_1, h_1(y)\big)\Big) \\ &= \big(j_2, h_2 h_1(y)\big).\end{aligned}$$

Moreover, $f_2 f_1 g_i = f_2(f_1 g_i) = f_2 g_{j_1} h_1 = (f_2 g_{j_1}) h_1 = g_{j_2} h_2 h_1$. Therefore

$$\rho_{f_2 f_1}(i,y) = \big(j_2, h_2 h_1(y)\big).$$

We have shown that for every $(i,y) \in \mathfrak{Z}$ and for arbitrary $f_1, f_2 \in \tilde{G}$ we have

$$(\rho_{f_2} \circ \rho_{f_1})(i,y) = \rho_{f_2 f_1}(i,y).$$

This shows that $\tilde{G} \to \mathrm{Aut}(\mathfrak{Z}), \quad g \mapsto \rho_g$ defines a group homomorphism. □

Let $g : Y \to X = \mathbb{P}^1_k$ be a Galois cover with $S := \mathrm{Gal}(g)$. Suppose that there exists a subgroup $H \subset S$ of index $\ell > 1$ such that Y/H has genus 0 and g is the Galois closure of $\varphi : Y/H \to X$. Let $\tilde{G}$ be a finite group with $S \subset \tilde{G}$. Let $\mathfrak{Z} = \mathrm{Ind}_S^{\tilde{G}} Y$ be the induced curve of Y and suppose that there exists a subgroup $\tilde{H} \subset \tilde{G}$ of index $d > 1$ such that $\tilde{H} \cap S = H$. Note that the map $\mathfrak{Z}/\tilde{H} \to X$ has degree d.

Recall that we denote by $Y^{(1)}$ the copy of Y in $\mathfrak{Z}$ on which the subgroup S of $\tilde{G}$ acts via the given action. It follows that $Y^{(1)}/\tilde{H} \cap S = Y^{(1)}/H \to X$ equals φ.

We number the leaves of $\mathfrak{Z}/\tilde{H} \to X$ (over a base point) with $1,\dots,d$ such that the leaves of $Y^{(1)}/H \to X$ are numbered as $1,\dots,\ell$. Since $Y^{(1)}/H$ is a component of $\mathfrak{Z}/\tilde{H}$, we obtain an embedding $S \subset S(\{1,\dots,\ell\})$ in $\tilde{G} \subset S(\{1,\dots,\ell,\ell+1,\dots,d\})$ via the natural embedding of sets. Note that $\tilde{G}$ is transitive on $\{1,\dots,d\}$ and S is transitive on $\{1,\dots,\ell\} \subset \{1,\dots,d\}$.

There exists a unique map $f : \mathfrak{Z} \to X = \mathbb{P}^1_k$ which commutes with the action of $\tilde{G}$ on $\mathfrak{Z}$, i.e. f is $\tilde{G}$-equivariant. It can be defined as follows:

$$f : \mathfrak{Z} \to X = \mathbb{P}^1_k, \quad (i,y) \mapsto g(y). \tag{1.5.1}$$

With these notations, we have the following result.

Proposition 1.5.4 *Assume that $d < 2\ell$ and let $H_{i_0} := \tilde{G} \cap S(\{1,\dots,i_0-1,i_0+1,\dots,d\})$ be the stabilizer of i_0 in $\tilde{G}$. Let $f : \mathfrak{Z} \to X = \mathbb{P}^1_k$ be the unique $\tilde{G}$-equivariant map. Then the following holds.*

There are $1 + (d-\ell)$ irreducible components in $\mathfrak{Z}/H_{i_0}$ above X. One induced map has degree ℓ whereas $(d-\ell)$ induced maps have degree 1.

Proof: Since $\{g_1 = e, \dots, g_r\}$ is a set of representatives of $\tilde{G}/S$ (Definition 1.5.2), we have that $G = \coprod_{i=1}^r g_i S$. The transitivity of $\tilde{G}$ on $\{1,\dots,d\}$ implies that for all $j \in \{1,\dots,d\}$ there exists a $g \in \tilde{G}$ with $g(i_0) = j$.

Moreover, there exists a unique $i \in \{1, \dots, r\}$ with $g \in g_i S$ and hence there exists a unique element $\hat{g}_j \in S$ with $g = g_i \hat{g}_j$. Note that for $1 \leq j \leq \ell$ the transitivity of S on $\{1, \dots, \ell\}$ implies that $g_i = g_1 = e$. Since $\hat{g}_j \in S$ we have $S = \hat{g}_j S \hat{g}_j^{-1}$ and hence

$$\begin{aligned}\left[g_i S g_i^{-1} : g_i S g_i^{-1} \cap H_{i_0}\right] &= \left[(g_i \hat{g}_j) S (g_i \hat{g}_j)^{-1} : (g_i \hat{g}_j) S (g_i \hat{g}_j)^{-1} \cap H_{i_0}\right] \\ &= \left[S : S \cap (g_i \hat{g}_j) H_{i_0} (g_i \hat{g}_j)^{-1}\right].\end{aligned}$$

Note that by definition of g, the group $g H_{i_0} g^{-1}$ fixes j and thus $S \cap g H_{i_0} g^{-1}$ is the subgroup of S which fixes j. Now S acts transitively on $\{1, \dots, \ell\}$ and fixes $\{\ell + 1, \dots, d\}$. Thus

$$\left[S : S \cap g H_{i_0} g^{-1}\right] = \begin{cases} \ell & \text{, if } 1 \leq j \leq \ell \\ 1 & \text{, if } j > \ell. \end{cases} \tag{1.5.2}$$

Now let $W \subset \mathfrak{Z}/H_{i_0}$ be any irreducible component and $\xi \in W$ a generic point. There exists an $i \in \{1, \dots, r\}$ and $y \in Y$ with $(i, y) \mapsto \xi$. Since every $Y^{(i)} \subset \mathfrak{Z}$ is closed, so is $\bar{\xi} = W \subset Y^{(i)}$. Thus we get the diagram of Figure 1.5.1.

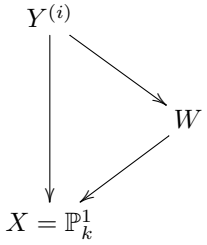

Figure 1.5.1: Diagram of situation.

Recall that the degree of a cover is the degree of the corresponding field extension. Further, note that $D(Y^{(i)}) = g_i S g_i^{-1}$. This shows that the degree of $Y^{(i)} \to X$ is $|g_i S g_i|$, the degree of $Y^{(i)} \to W$ is $|g_i S g_i^{-1} \cap H_{i_0}|$ and the degree of $W \to X$ is $[g_i S g_i : g_i S g_i^{-1} \cap H_{i_0}]$.

By equation (1.5.2), we see that there are two possible degrees for irreducible components above X, namely ℓ and 1.

Note that $H_{i_0} \subset \tilde{G}$ is a subgroup of index d (since $\tilde{G}$ is transitive on $\{1, \dots, d\}$). In particular, the map $\mathfrak{Z}/H_{i_0} \to X$ has degree d. By assumption $d < 2\ell$, so there can be at most one irreducible component over X with degree ℓ. But since f (the unique $\tilde{G}$-equivariant cover) restricted to $Y^{(1)}$ is equal to g, it is trivial that there is at least one component over $\bar{X}$ with degree ℓ. Moreover, there are $d - \ell$ different j such that

$$\left[S : S \cap (g_i \hat{g}_j) H_{i_0} (g_i \hat{g}_j)^{-1}\right] = \left[S : S \cap (g_i \hat{g}_j) H_{i_0} (g_i \hat{g}_j)^{-1}\right] = 1.$$

Thus there is one irreducible component of degree ℓ and $d - \ell$ components of degree 1 above $\bar{X}$. □

Remark 1.5.5 In Example 3.3.6 and in the proof of Theorem 4.2.2 we use Proposition 1.5.4 to compute the number of irreducible components. In one case the map $g : Y \to X = \mathbb{P}^1_k$ is not Galois, but inseparable. The assertion of Proposition 1.5.4 stays true nonetheless.

1.6 Stable reduction

In this section we recall the definition of stable reduction. This whole section is strongly inspired by [BW12, Chapter 3]. Throughout this section let R be a complete discrete valuation ring, K its fraction field of characteristic 0 and k its residue field of characteristic $p > 0$. We assume that k is algebraically closed.

The following can be found in [BW12, Chapter 3] with slightly different notation.

Let Y be a smooth projective curve defined over K and $G \subset \mathrm{Aut}_K(Y)$ be a finite group. The map $f : Y \to X := Y/G$ is a G-Galois cover between smooth projective curves. Denote by $\{y_1, \ldots, y_r\}$ the set of ramification points of f.

Definition 1.6.1 [BW12, Definition 3.1]

(a) An *R-model* $\mathcal{Y}$ of Y is a normal, flat and proper R-scheme such that $\mathcal{Y} \otimes_R K \simeq Y$.

(b) A model $\mathcal{Y}$ of Y is called *semistable* if the special fiber $\bar{Y} := \mathcal{Y} \otimes_R k$ is semistable that is the special fibre has only ordinary double points as singularities.

(c) A semistable model $\mathcal{Y}$ of Y is call *G-semistable* if

  (i) The action of G on Y extends to $\mathcal{Y}$ and

  (ii) the ramification points $y_1, \ldots, y_r$ specialize to pairwise distinct smooth points $\bar{y}_1, \ldots, \bar{y}_r \in \bar{Y}$.

The following result ist stated in [BW12, Lemma 3.2].

Lemma 1.6.2 *Let $\mathcal{Y}$ be a G-semistable R-model of Y. Let $\mathcal{X} := \mathcal{Y}/G$ denote the quotient scheme, $f_R : \mathcal{Y} \to \mathcal{X}$ the canonical map and $\bar{f} : \bar{Y} \to \bar{X}$ the restriction of f_R to the special fiber. Then the following holds.*

(a) The R-scheme $\mathcal{X}$ is a semistable R-model of $X = Y/G$.

(b) The branch points $x_1, \ldots, x_{r'}$ of f specialize to pairwise distinct smooth points $\bar{x}_1, \ldots, \bar{x}_{r'} \in \bar{X}$.

(c) If $\bar{y} \in \bar{Y}$ is a singular point, then $\bar{x} := \bar{f}(\bar{y}) \in \bar{X}$ is singular as well.

The following result is due to Deligne and Mumford. We use the formulation of [BW12, Lemma 3.3].

Theorem 1.6.3 *Assume that $2g(Y) + r - 2 > 0$. Then after replacing K by a finite separable extension $\tilde{K}/K$, there exists a minimal G-semistable $\tilde{R}$-model $\mathcal{Y}$ of Y, where $\tilde{R}$ denotes the integers of $\tilde{K}$. It is unique up to unique isomorphism.*

Proof: The uniqueness follows from the minimality. The existence from [DM69, Corollary 2.7]. □

The following definition is taken from [BW12, Definition 3.4].

Definition 1.6.4 The minimal G-semistable model $\mathcal{Y}$ from Theorem 1.6.3 is called the *stably marked model of Y*. The resulting morphism $f_{\tilde{R}} : \mathcal{Y} \to \mathcal{X}$ is called the *stable model* of the G-cover f and its restriction to the special fiber $\bar{f} : \bar{Y} \to \bar{X}$ is called the *stable reduction of f*. We say that f has *good reduction* if $\bar{f}$ is separable, otherwise we say that f has *bad reduction*.

Remark 1.6.5 In this thesis we work with three-point covers. In particular, the Galois three-point covers $Y \to X = \mathbb{P}^1$ are branched in $x_1 = 0, x_2 = 1, x_3 = \infty$ (cf. Definition 1.2.1 and Convention 1.2.5). It follows that in the notation of Lemma 1.6.2 $r' = 3$ and thus $r \geq 3$. Thus by Theorem 1.6.3 the stable reduction always exist.

Example 1.6.6 Let $\mathcal{C} = (7; 4, 5, 6)$ and $p = 5$. Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the unique genus-0 Belyi morphism of type $\mathcal{C}$ and $f : Y \to X$ the Galois closure. We verify bad reduction with explicit calculation.

Recall that by Example 1.2.7 φ is given by

$$x = \varphi(z) = \frac{z^4\left(-10z^3 + 42z^2 - 63z + 35\right)}{7z - 3}.$$

We easily compute

$$\varphi(z) \equiv \frac{z^4\left(2z^2 - 3z\right)}{2z - 3} \equiv z^5 \mod 5.$$

From this it follows that the Galois three-point cover f has bad reduction to characteristic 5.

2 Tail covers

Throughout this whole chapter let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$. We assume that k is algebraically closed. We assume that all curves are smooth, projective and absolutely irreducible unless explicitly stated otherwise.

Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $d < 2p$ (Notation 1.2.3 (a)). Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ over K be the genus-0 Belyi morphism with ramification type $\mathcal{C}$ and $f : Y \to X$ its Galois closure (see Lemma 1.2.6 for the uniqueness of φ and that φ can be defined over $\mathbb{Q}$). Suppose that f has bad reduction at p and denote the stable reduction of f by $\bar{f} : \bar{Y} \to \bar{X}$. What is the stable reduction of f and why are tail covers important? In this Chapter we do not answer these questions, but we state the fact that tail covers appear in the stable reduction of three-point covers. The explicit connection connection between tail covers and the stable reduction is discussed in Section 3.2.

To avoid confusion with the following notation, we briefly explain it (see Definition 2.1.1). Tail covers in general are Galois covers between curves $Y \to \mathbb{P}^1_k$ defined over k with at most two branch points such that the map is wildly ramified at exactly one of these branch point. Thus we could simply use a notation similar to $g : Y \to X = \mathbb{P}^1_k$ to denote a tail cover. In Section 3.2 we recall that tail covers arise in the stable reduction of three-point covers. To stress this fact, we use the notation $\bar{g} : \bar{Y} \to \bar{X}$ for tail covers analogous to the notation of the stable reduction. In Section 3.2 we will see that many tail covers occur in the stable reduction of three-point covers. To distinguish the different tail covers occurring in $\bar{f}$ we use indices e.g. $\bar{g}_j : \bar{Y}_j \to \bar{X}_j$.

In Section 2.1 we define tail covers in general. In Section 2.2 we recall several known results about tail covers. In Section 2.3 we generalize the tail covers from Section 2.2 to arbitrary degree. In this section we also prove many results about the structure of wild inertia groups of wild ramification points of these tail covers (Theorem 2.3.11). This section however, is not as complete as one could hope. For example we do not know the number of jumps of the higher ramification groups of a wild inertia group. We also do not know the cardinality of an inertia group. In Section 2.4 we compute the ramification invariants of the tail covers defined in the previous section. Sadly, the computation depends on the assumption that the inertia group has only one nontrivial lower jump. We believe that this is true, but we haven't been able to prove it yet. In Section 2.5 we compute the so called ramification type of tail covers for $d < 2p$. This is needed for the computation of certain deformation data (see Chapter 3 and 4 for more details). We finish this chapter with results about the uniqueness of the tail covers from Section 2.3 as mere covers.

2.1 Definition of tail covers

In this section we define tail covers in general.

Definition 2.1.1 (a) A *new tail cover* $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ is a Galois cover defined between smooth curves over k such that $\bar{g}$ is only ramified above $\bar{x} = \bar{\infty}$ and unramified elsewhere.

(b) A *primitive tail cover* $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ is a Galois cover defined between smooth curves over k such that $\bar{g}$ is wildly ramified above $\bar{x} = \bar{\infty}$ and tamely ramified above $\bar{x} = \bar{0}$ and unramified elsewhere.

(c) A *tail cover* $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ is either a new tail cover or a primitive tail cover.

(d) Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be a tail cover and denote its Galois group by G. We say that $\bar{g}$ is a *genus-0 (primitive)* (resp. *genus-0 (new)*)-tail cover if there exists a subgroup $H \subset G$ of index $[G : H] = d > 1$ such that $\bar{Y}/H$ is isomorphic to $\mathbb{P}^1_k$ and $\bar{g}$ is the Galois closure of $\bar{Y}/H \to \bar{X}$.

Remark 2.1.2 (a) The Riemann-Hurwitz formula (Theorem 1.3.5) implies that a new tail cover is necessarily wildly ramified above $\bar{\infty}$.

(b) Let $\bar{Z} := \mathbb{P}^1_k$ and $\bar{\varphi} : \bar{Z} \to \bar{X} = \mathbb{P}^1_k$ be a cover of degree d which is wildly ramified above $\bar{x} = \bar{\infty}$, possibly tamely ramified above $\bar{x} = \bar{0}$ and unramified elsewhere. Let $k(\bar{Y})$ be the Galois closure of the field extension $k(\bar{Z})/k(\bar{X})$ and $\bar{Y}$ the curve with function field $k(\bar{Y})$. Let $\bar{g} : \bar{Y} \to \bar{X}$ be the corresponding Galois cover such that $\bar{g}$ factors through $\bar{\varphi}$. Then $\bar{g}$ is a genus-0 tail cover of degree d. If $\bar{\varphi}$ is branched in only one point (resp. in two points), then $\bar{g}$ is a new tail (resp. primitive tail) cover.

(c) If we want to emphasize the Galois group G of a tail cover, we often say G-tail cover.

(d) All tail covers in this thesis are genus-0 tail covers.

In Section 2.6 we also discuss the uniqueness of tail covers. To do so we require a proper definition for isomorphism.

Definition 2.1.3 Let $\bar{g}_i : \bar{Y}_i \to \bar{X}_i = \mathbb{P}^1_k$ for $i = 1, 2$ be two tail covers with Galois group G. We say that $\bar{g}_1$ and $\bar{g}_2$ are *isomorphic as mere covers*, if there exists an isomorphism $\bar{Y}_1 \xrightarrow{\sim} \bar{Y}_2$ and an isomorphism $\bar{X}_1 \xrightarrow{\sim} \bar{X}_2$ with $\bar{0}_1 \mapsto \bar{0}_2$ and $\bar{\infty}_1 \mapsto \bar{\infty}_2$ such that the following diagram is commutative.

$$\begin{array}{ccc} \bar{Y}_1 & \xrightarrow{\sim} & \bar{Y}_2 \\ {\scriptstyle \bar{g}_1}\downarrow & & \downarrow{\scriptstyle \bar{g}_2} \\ \bar{X}_1 & \xrightarrow{\sim} & \bar{X}_2 \end{array}$$

If we can choose $\bar{Y}_1 \xrightarrow{\sim} \bar{Y}_2$ as G-equivariant, then we say that $\bar{g}_1$ and $\bar{g}_2$ are *isomorphic (as G-cover).*

2.2 Some known results

In this section we briefly recall some known results about tail covers. All the following results require $d < 2p$ (see Definition 2.1.1 (d)).

First we discuss new tail covers and then proceed with primitive tail covers.

Note that $d < 2p$ implies that the inertia group of a wild ramification point has order pm (see Section 1.4). The following results are due to Bouw ([Bou11]). Note that in [Bou11] Bouw assumes $p > 2$. This restriction is not needed for Proposition 2.2.1 (a) and Lemma 2.2.2 (a).

Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be a genus-0 new tail cover of degree $d < 2p$ and let $\bar{\varphi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X}$ be the corresponding map between genus-0 curves such that $\bar{g}$ is the Galois closure of $\bar{\varphi}$. Let $\bar{y} \in \bar{Y}$ be a wild ramification point and $I_{\bar{y}}$ its inertia group. We have $|I_{\bar{y}}| = pm$ where $\gcd(p, m) = 1$. Choose generators $\theta, \tau \in I_{\bar{y}}$ with $\operatorname{ord} \tau = m$ and $\operatorname{ord} \theta = p$. Since $\deg(\bar{\varphi}) = d$, we can identify G with a subgroup of S_d. This allows us to assume that $\theta = (1 \ \dots \ p)$. Since τ normalizes θ, it follows that $\langle \tau \rangle$ acts on $\{1, \dots, p\}$. It follows that we may write $\tau = \tau_1 \tau_2$ with $\tau_1 \in S(\{1, \dots, p\}), \tau_2 \in S(\{p+1, \dots, 2p\})$. Let $n_1, \dots, n_t$ be the lengths of the cycles of τ_2.

Proposition 2.2.1 *(a) The inverse image $\bar{\varphi}^{-1}(\bar{\infty}) \subset \bar{Z}$ consists of $t + 1$ points with ramification indices $p, n_1, \dots, n_t$ respectively.*

(b) The ramification invariant satisfies

$$\sigma = \frac{d + t - 1}{p - 1}.$$

(c) There is no genus-0 new tail cover with $d = p$.

(d) Let $p + 1 \leq d < 2p$. There exists a unique degree-d cover $\bar{\varphi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ with $\sigma = d/(p-1)$. The Galois group of the Galois closure of $\bar{\varphi}$ is

$$\begin{cases} \mathrm{PSL}_2(p) & \text{, if } d = p + 1 \text{ and } p \geq 5 \\ \mathrm{PSL}_2(8) & \text{, if } p = 7 \text{ and } d = p + 2 \\ A_d & \text{otherwise.} \end{cases}$$

The cover $\bar{\varphi}$ is given by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^d}{\bar{z}^{d-p}}.$$

Proof: Part (a) and (b) are shown in [Bou11, Proposition 1.3] using $g(\bar{Z}) = 0$. Part (c) is shown in the proof of [Bou11, Corollary 1.4 (a)]. Part (d) is [Bou11, Proposition 2.1]. □

Note that t can be interpreted as the number of points above $\bar{x} = \bar{\infty}$ "aside" from the wild ramification point. In particular, if $t = d - p$, then $\bar{\varphi}$ is ramified in exactly one point above $\bar{x} = \bar{\infty}$.

We now recall several already known results about primitive tail covers.

Lemma 2.2.2 *(a) Let $p \leq d < 2p$ and $\bar{\varphi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be a degree-d cover which is wildly branched above $\bar{\infty}$, tamely branched above $\bar{0}$, and unbranched elsewhere. Let $\bar{g} : \bar{Y} \to \bar{X}$ be the Galois closure of φ, $\bar{y} \in \bar{Y}$ a wild ramification point and σ the ramification invariant of $I_{\bar{y}}$. Finally, let $m_1, \ldots, m_r$ with $\sum_{i=1}^r m_r = d$ be the ramification indices of $\bar{\varphi}$ above $\bar{0}$ and define t as in Proposition 2.2.1. Then*

$$\sigma = \frac{r + t - 1}{p - 1}.$$

(b) Let $2 \leq e < p$ be an integer. There exists a genus-0 primitive tail cover $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ with ramification index e above $\bar{x} = \bar{0}$ and p above $\bar{x} = \bar{\infty}$. Moreover, $\bar{g}$ is unique up to isomorphism. The primitive tail cover is given by Galois closure of the cover

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \bar{z}^e + \bar{z}^p.$$

Proof: Part (a) is [Bou11, Lemma 3.2] using $g(\bar{Z}) = 0$.

The formula for the primitive tail cover as well as the uniqueness as mere covers follow from the proof of [BO11, Lemma 5.1]. The uniqueness up to isomorphism in the sense of Definition 2.1.3 follows from [BO11, Lemma 5.3 (c)]. □

There also exist several results concerning the Galois group of the tail cover from Lemma 2.2.2 (b). We will not use these results. See [BO11, Chapter 5] for more details.

Remark 2.2.3 (i) If $t = 1$, then the ramification invariant in Proposition 2.2.1 simplifies to

$$\sigma = \frac{d}{p - 1}.$$

In Section 2.4 we generalize this formula for new tail covers which are given as the Galois closure of

$$(\bar{x}, \bar{z}) \mapsto \frac{1 + \bar{z}^d}{\bar{z}^{d-ap}},$$

where $ap < d < (a+1)p \leq p^2$, with the additional assumption that there exists exactly one lower jump $\neq 0$.

(ii) Note that the ramification invariant in Lemma 2.2.2 (a) has two interesting special cases. If $\bar{\varphi}$ is fully ramified above $\bar{\infty}$, then $d = p$ and $t = 0$. If additionally $\bar{\varphi}$ is only ramified in one single point above $\bar{0}$ with ramification index e, then we have $r = 1 + d - e = 1 + p - 1$. In particular,

$$\sigma = \frac{p - e}{p - 1}.$$

If $\bar{\varphi}$ is fully ramified above $\bar{0}$, then $r = 1$ and $d = e$. If additionally $\bar{\varphi}$ is only ramified in one single point above $\bar{\infty}$, then $t = e - p$ and hence

$$\sigma = \frac{e - p}{p - 1}.$$

The primitive tail covers which appear in the stable reduction of three-point covers with $d < 2p$ all satisfy the above assumptions and have the described ramification invariants. We refer to Chapter 4 for more details.

2.3 General candidates for tail covers and properties of inertia groups

In this section we generalize the tail covers from the previous section. All of the following results are new.

We now define the tail covers of interest for this thesis.

Lemma 2.3.1 *Let a, e, d be positive integers with $e, d > 1$ and $0 < |e - ap| < p$ and $ap < d < (a+1)p$. Let $\bar{\varphi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be defined by*

(i)

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^e + \bar{z}^{ap}, & \text{if } e < ap \\ \frac{\bar{z}^e}{\bar{z}^{e-ap}-1}, & \text{if } e > ap \end{cases} \text{ or}$$

(ii)

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^d}{\bar{z}^{d-ap}}.$$

Let $\bar{g} : \bar{Y} \to \bar{X}$ be the Galois closure of $\bar{\varphi}$. Then the following holds

(a) In (i) $\bar{\varphi}$ is ramified in exactly two points $\bar{z} = \bar{0}$ and $\bar{z} = \bar{\infty}$ and unramified else. $\bar{\varphi}$ is ramified in $\bar{z} = \bar{0}$ with ramification index e (tame) and in $\bar{\infty}$ with ramification index ap (wild). We have $\bar{\varphi}(\bar{0}) = \bar{0}$ and $\bar{\varphi}(\bar{\infty}) = \bar{\infty}$.

(b) In (ii) $\bar{\varphi}$ is only branched in $\bar{x} = \bar{\infty}$. Moreover, $\bar{\varphi}$ is ramified in $\bar{z} = \bar{0}$ with ramification index $d - ap$ and wildly ramified in $\bar{z} = \bar{\infty}$ with ramification index ap and else unramified.

(c) In case (i) $\bar{g}$ is a genus-0 primitive tail cover and in case (ii) $\bar{g}$ is a genus-0 new tail cover[2].

Proof: We begin with (a).

One computes that

$$\bar{\varphi}'(\bar{z}) = \begin{cases} e\bar{z}^{e-1} & , \text{ if } e < ap \\ -e\frac{\bar{z}^{e-1}}{(\bar{z}^{e-ap}-1)^2} & , \text{ if } e > ap. \end{cases}$$

This shows that $\bar{\varphi}$ is only ramified above $\bar{x} = \bar{0}$ and $\bar{x} = \bar{\infty}$. Moreover, we see that $\bar{\varphi}$ is tamely ramified only in $\bar{z} = 0$ with ramification index e. We immediately see, that $\bar{\varphi}$ is ramified in $\bar{\infty}$ with ramification index ap and else unramified.

This proves the assertion (a).

We now prove (b).

One computes that

$$\bar{\varphi}'(z) = -\frac{d}{\bar{z}^{d-ap+1}}.$$

This shows that $\bar{\varphi}$ is only ramified above $\bar{x} = \bar{\infty}$. One immediately sees that $\bar{\varphi}$ is ramified in $\bar{\infty}$ with ramification index ap. That $\bar{\varphi}$ is also ramified in $\bar{0}$ with ramification index $d - ap$ is obvious. Since $d = ap + (d - ap)$ there can be no other points above $\bar{x} = \bar{\infty}$. This proves (b).

Assertion (c) is a direct consequence of (a) and (b). □

Remark 2.3.2 Note that for $a = 1$ the tail covers defined in Lemma 2.3.1 are the tail covers from Section 2.2. In Chapter 5 we prove that all the primitive tail covers from Lemma 2.3.1 occur in the stable reduction of three-point covers (see Corollary 5.2.9).

We now prove several results about the structure of the inertia group of a wild ramification point of a tail cover defined as in Lemma 2.3.1. We use the following notation.

[2]The reason why we introduce the letter d (which obviously also satisfies $0 < |d - ap| < p$) in the definition of the new tail cover is the following. Throughout this whole thesis e always denotes a ramification index. If we used e instead of d for the new tail cover, this could easily lead to confusion.

Notation 2.3.3 Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1$ be a tail cover, $\bar{y} \in \bar{Y}$ a wild ramification point and $I_{\bar{y}}$ its inertia group. We denote the Sylow p-subgroup of $I_{\bar{y}}$ by $I^{(p)}$.

Remark 2.3.4 In the situation of Lemma 2.3.1 assume that $a < p$. We note the following. Recall that we can regard the Galois group G of $\bar{g}$ as subgroup of S_d where d is the degree of $\bar{\varphi}$ (Remark 1.1.7) (Note that if $\bar{g}$ is one of the two primitive tail covers as defined in Lemma 2.3.1 (i), then $d = \max\{e, ap\}$. If however $\bar{g}$ is the new tail cover in Lemma 2.3.1 (ii), then the degree of $\bar{\varphi}$ is d by definition). Since $ap < d < (a+1)p \leq p^2$, by Section 1.4 we know that all the Sylow p-subgroups of S_d are conjugate to the subgroup

$$P_a := \langle \rho_1, \ldots, \rho_a \rangle, \quad \text{where} \quad \rho_i := ((i-1)p+1,\ (i-1)p+2,\ \ldots\ , ip) \quad \text{for } i = 1, \ldots, a. \tag{2.3.1}$$

This has the following consequence.

If $\bar{y} \in \bar{Y}$ denotes a wild ramification point and $I_{\bar{y}}$ its inertia group, then $I^{(p)}$ is contained in a Sylow p-subgroup of S_d which is conjugate to P_a. Note that P_a is elementary abelian. In particular, we have that

$$I^{(p)} \simeq \left(\mathbb{Z}/p\mathbb{Z}\right)^b,$$

where $b \leq a$ and consequently we have that

$$I_{\bar{y}} \simeq \left(\mathbb{Z}/p\mathbb{Z}\right)^b \rtimes \mathbb{Z}/m\mathbb{Z}$$

with a positive integer m which is relatively prime to p.

In the following we analyze the inertia groups of wild ramification groups of the tail covers of Lemma 2.3.1. To do so we assume that $a < p$ for the rest of this chapter. Hence we can assume that the Sylow p-subgroup is an elementary abelian p-group as in Remark 2.3.4. For better readability, we repeat this assumption in every result where it is explicitly needed. Moreover, we restrict to the case $d = ap + 1$ for the new tail cover of Lemma 2.3.1 (ii) unless explicitly said otherwise. We also repeat this condition wherever it is needed.

Remark 2.3.5 The following results use the fact that there is exactly one single ramification point above $\bar{x} = \bar{\infty} \in \bar{X}$. For the primitive tail covers of Lemma 2.3.1 this holds by Lemma 2.3.1 (a). For the new tail cover of Lemma 2.3.1 this holds only if $d = ap + 1$.

Lemma 2.3.6 *Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be a tail cover as in Lemma 2.3.1, with $a < p$ and $d = ap + 1$ in (ii). Let $\bar{y} \in \bar{Y}$ be a wild ramification point and $I_{\bar{y}} = I^{(p)} \rtimes \langle \tau \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^b \rtimes \mathbb{Z}/m\mathbb{Z}$ its inertia group. Then the following holds.*

(a) $I_{\bar{y}} \subset S_{ap}$.

(b) $|\bigcup_{\gamma \in I^{(p)}} \operatorname{supp}(\gamma)| = ap$.

Proof: We prove (a). Note that the assertion is trivial if $\bar{\varphi}$ is given by $\bar{x} = \bar{z}^e + \bar{z}^{ap}$. Therefore, assume that $\bar{\varphi}$ is given by $\bar{x} = \bar{z}^e/(\bar{z}^{e-ap} - 1)$ or $\bar{x} = (1 + \bar{z}^{ap+1})/\bar{z}$. In Remark 2.3.4 we noted that $I^{(p)}$ is contained in a Sylow p-subgroup conjugate to P_a (see (2.3.1) for the definition of P_a). Thus we may assume that $I^{(p)} \subset P_a$. In particular, we may assume that

$$I^{(p)} \subset S_{ap} = S(\{1, \ldots, ap\}).$$

Let $\tau \in I_{\bar{y}}$ be an element of order m. Suppose that there exists an $i_0 \in \{ap+1, \ldots, e\}$ with $\tau(i_0) \neq i_0$. Let $H_{i_0} = G \cap S(\{1, \ldots, ap, \ldots, i_0 - 1, i_0 + 1, \ldots, e\})$ denote the one-point stabilizer of i_0 in G. Then $\tau \notin H_{i_0}$ and in particular $|I_{\bar{y}} \cap H_{i_0}| = p^b x$ with $x \mid m$. Note that this implies that

$$\left[I_{\bar{y}} : I_{\bar{y}} \cap H_{i_0}\right] = \frac{|I_{\bar{y}}|}{|I_{\bar{y}} \cap H_{i_0}|} = \frac{p^b m}{p^b x} = \frac{m}{x}.$$

Since $x \mid m$ we see that $1 < m/x \leq m$. Moreover, $p \nmid m/x$ as well. This implies that $\bar{\varphi}$ has a ramification point with ramification index $m/x > 1$ above $\bar{\infty}$, which it does not (Lemma 2.3.1). This is obviously a contradiction. Hence $I_{\bar{y}} \subset S_{ap}$.

Now we prove (b). The assertion is trivial for $a = 1$. Thus let $a \geq 2$. Similar to the proof of (a) we may assume that $I^{(p)} \subset P_a$. Hence it is obvious that $|\bigcup_{\gamma \in I^{(p)}} \operatorname{supp}(\gamma)| \leq ap$ and it follows immediately that $p \mid |\bigcup_{\gamma \in I^{(p)}} \operatorname{supp}(\gamma)|$.

Now suppose that $|\bigcup_{\gamma\in I^{(p)}} \mathrm{supp}(\gamma)| < ap$. Then $|\bigcup_{\gamma\in I^{(p)}} \mathrm{supp}(\gamma)| = rp$ for some $r < a$. For simplicity we may assume that

$$\bigcup_{\gamma\in I^{(p)}} \mathrm{supp}(\gamma) = \{1, 2, \ldots, rp\}. \tag{2.3.2}$$

Now let $rp < i_0 \leq ap$ and let $H_{i_0} = G \cap S(\{1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, ap\}$ be the one-point stabilizer of i_0 in G. Recall that $I_{\bar{y}}$ is a transitive group on $\{1, \ldots, ap\}$. By equation (2.3.2) this implies that $I^{(p)} \subset H_{i_0}$ but $\tau \notin H_{i_0}$ (because $I_{\bar{y}}$ is transitive on $\{1, \ldots, ap\}$). We conclude that

$$[I_{\bar{y}} : I_{\bar{y}} \cap H_{i_0}] = \frac{|I_{\bar{y}}|}{|I_{\bar{y}} \cap H_{i_0}|} = \frac{p^b m}{p^b x} = \frac{m}{x}$$

for some $1 \leq x < m$ with $x \mid m$. Now $\gcd(m, p) = 1$ and hence $\frac{m}{x} \neq ap$. This is a contradiction. □

Lemma 2.3.6 has a surprisingly strong implication on inertia groups of a wild ramification point. We make this precise as follows.

Example 2.3.7 Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be one of the tail covers from Lemma 2.3.1 with $a < p$ and $d = ap + 1$ for the new tail cover. Let $\bar{y} \in \bar{Y}$ be a wild ramification point and $I_{\bar{y}}$ its inertia group. One question is: what is the possible inertia group of a wild ramification point $\bar{y} \in \bar{Y}$? We may assume that

$$I^{(p)} \subset P_a = \langle \rho_1, \ldots, \rho_a \rangle.$$

If $b = 1$, then Lemma 2.3.6 implies that

$$I^{(p)} = \langle \gamma \rangle$$

with

$$\gamma = \prod_{i=1}^{a} \rho_i^{n_i}$$

with $n_i \in \mathbb{F}_p^*$ for all $1 \leq i \leq a$, i.e. the power of every ρ_i is nontrivial!

For general $b < a$, Lemma 2.3.6 implies that

$$I^{(p)} = \langle \gamma_1, \ldots, \gamma_b \rangle,$$

and

$$\bigcup_{i=1}^{b} \mathrm{supp}(\gamma_i) = \{1, \ldots, ap\}.$$

In particular, for every $j \in \{1, \ldots, a\}$ there exists at least one $i \in \{1, \ldots, b\}$ such that

$$\gamma_i = \rho_j^{n_j} \prod_{r\neq j} \rho_r^{n_r}$$

and $n_j \in \mathbb{F}_p^*$, i.e. the power of ρ_j is nontrivial. This demonstrates that Lemma 2.3.6 greatly decreases the number of possible inertia groups, or rather the p-part of inertia groups.

We now prove a structure result for the prime-to-p part $\langle \tau \rangle$ for an inertia group of a tail cover as in Lemma 2.3.1. This again reduces the possibilities for wild inertia groups. We emphasize that the result (part (a)) is only true for $a \geq 2$. Part (b) however is obviously true for $a = 1$, as well.

Proposition 2.3.8 *Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be a tail cover as in Lemma 2.3.1, with $a < p$ and $d = ap + 1$ in (ii). Additionally assume that $a \geq 2$. Moreover, let $\bar{y} \in \bar{Y}$ be a wild ramification point and $I_{\bar{y}} = I^{(p)} \rtimes \langle \tau \rangle \simeq \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z}$ its inertia group. Then the following holds.*

(a) $|\mathrm{supp}(\tau)| = ap$.

(b) $a \mid m$.

Proof: Similar to the proof of Lemma 2.3.6 we may assume that

$$I^{(p)} \subset P_a.$$

Define $B_i := \operatorname{supp} \rho_i$, where the ρ_i are as in (2.3.1), and $B := \{1, \dots, ap\}$. Let $j \in B$ be an arbitrary number. There exists a unique $j_0 \in \{1, \dots, a\}$ with $j \in B_{j_0}$. Since $I^{(p)} \subset P_a$ it follows that the orbit of j_0 under $I^{(p)}$ is B_{j_0}.

Note that $I_{\bar{y}} \subset S_{ap} = S(\{1, \dots, ap\})$ is a transitive subgroup on $\{1, \dots, ap\}$ and $I^{(p)} \subset I_{\bar{y}}$ is a normal subgroup. By [DF04, Section 4.1, Exercise 9], $I_{\bar{y}}$ acts transitively on the set of orbits of $I^{(p)}$.

Let $\tau \in I_{\bar{y}}$ be an element of order m. The transitivity of $I_{\bar{y}}$ on the orbits of $I^{(p)}$ implies that $\langle \tau \rangle$ acts transitively on the orbits of $I^{(p)}$ on B, i.e. $\langle \tau \rangle$ acts transitively on the set of blocks. Since $a \geq 2$ it follows that τ acts on ap numbers nontrivially and hence

$$|\operatorname{supp} \tau| = ap.$$

This proves (a).

That $a \mid m$ follows from the fact that φ has a ramification point with ramification index ap. □

We now prove a result on the structure of $\langle \tau \rangle \subset I_{\bar{y}}$. Recall that the elements ξ_1 and ξ_2 from the proof of Proposition 1.4.4 are defined as

$$\xi_1 := (1,\ p+1,\ 2p+1,\ \cdots, (a-1)p+1)$$
$$\xi_2 := (2,\ p+2,\ \cdots,\ (a-1)p+2,\ \tau_1(2),\ p+\tau_1(2),\ \cdots, (a-1)p+\tau_1^{p-2}(2)),$$

where $\tau_1 \in N_{S_p}(\langle \rho_1 \rangle)$ with $\tau_1 \rho_1 \tau_1^{-1} = \rho_1^{\alpha}$.

Lemma 2.3.9 *Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be a tail cover as in Lemma 2.3.1, with $a < p$ and $d = ap + 1$ in (ii). Let $\bar{y} \in \bar{Y}$ be a wild ramification point and $I_{\bar{y}} = I^{(p)} \rtimes \langle \tau \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^b \rtimes \mathbb{Z}/m\mathbb{Z}$ its inertia group. Let ρ_i be defined as in (2.3.1). Then the following holds.*

(a) Changing the ρ_i by powers, if necessary, we may assume that

$$\tau \rho_i \tau^{-1} = \begin{cases} \rho_{i+1} & , \text{ if } i < a \\ \rho_1^{\alpha}, & \text{if } i = a \end{cases}$$

for some $\alpha \in \mathbb{F}_p^*$.

(b) τ is conjugate to the product of ξ_1 with a power of ξ_2 inside S_{ap}.

(c) $m = a \cdot \operatorname{ord}(\alpha)$.

Proof: We first prove the assertions for $a = 1$.

Let $a = 1$. Then assertion (a) is trivial. Moreover, $I_{\bar{y}} \simeq \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z} \subset S_p$. By Lemma 1.4.3, τ is the power of a $(p-1)$-cycle and hence (b) is satisfied. Assertion (c) is obvious in this case.

Now let $a > 1$.

We use the notation and definitions introduced in the proof of Proposition 2.3.8.

Recall that τ acts transitively on the a blocks $B_i \subset B = \{1, \dots, ap\}$. It follows that τ^a fixes all blocks.

In particular $\tau^a \in N_{S_{ap}}(\langle \rho_i \rangle)$ for all i and hence $\operatorname{ord}(\tau^a) \mid (p-1)$.

Since τ acts transitively on the a blocks, it follows that the a elements

$$\tau^{i-1} \rho_1 \tau^{-(i-1)}, \quad \text{for } 0 \leq i \leq a-1$$

generate P_a as well. Hence, redefining $\rho_i := \tau^{i-1} \rho_1 \tau^{-(i-1)}$ for $0 \leq i \leq a-1$, we obtain

$$\tau \rho_i \tau^{-1} = \begin{cases} \rho_{i+1}, & \text{for } i < a \\ \rho_1^{\alpha}, & \text{for } i = a \end{cases} \tag{2.3.3}$$

for some $\alpha \in \mathbb{F}_p^*$, because τ^a fixes all blocks. This proves (a).

Additionally, the fact that τ^a fixes all blocks implies that

$$\tau^a = \prod_{i=1}^{a} \tau_i^{n_i}$$

where $\tau_i \in N_{S(\mathrm{supp}(\rho_i))}(\langle \rho_i \rangle)$ are elements of order $p-1$ (see Lemma 1.4.3). There are two possible cases.

1. Case: There exists an i_0 with $n_i = 0$

Then $\tau^a \rho_{i_0} \tau^{-a} = \rho_{i_0}$. Since $\tau^{i_0-1}\rho_1\tau^{-(i_0-1)} = \rho_{i_0}$ we see that

$$\begin{aligned}
\rho_{i_0} &= \tau^a \left(\tau^{i_0-1}\rho_1\tau^{-(i_0-1)}\right)\tau^{-a} \\
&= \tau^{i_0-1}\left(\tau^a\rho_1\tau^{-a}\right)\tau^{-(i_0-1)} \\
&= \tau^{i_0-1}\rho_1^\alpha\tau^{-(i_0-1)} \\
&= \left(\tau^{i_0-1}\rho_1\tau^{-(i_0-1)}\right)^\alpha \\
&= \rho_{i_0}^\alpha
\end{aligned}$$

and hence $\alpha = 1$. This shows that the length of the orbit of every $t \in \{1, \ldots, ap\}$ is exactly a. Since $|\mathrm{supp}(\tau)| = ap$ it follows that τ is the product of p disjoint a-cycles. In particular, τ is conjugate to $\xi_1\xi_2^{p-1}$.

2. Case: $n_i \neq 0$ for all i

Then $\alpha \neq 1$ and $|\mathrm{supp}(\tau^a)| = a(p-1)$. Recall that by Proposition 2.3.8 (a) $|\mathrm{supp}(\tau)| = ap$. The transitivity of the action of τ on the blocks B_i now implies that $\tau = \sigma_1\sigma_2$ where σ_1 is an a-cycle and $\sigma_2 \in S(\{1, \ldots, ap\} \setminus \mathrm{supp}(\sigma_1))$. We may assume that σ_1 permutes the smallest numbers in each block that is, we may assume that

$$\sigma_1 = \xi_1.$$

Let $t \in \mathrm{supp}(\sigma_2)$. Since $\tau^a\rho_1\tau^{-a} = \rho_1^\alpha$, it follows that the orbit of t has $a \cdot \mathrm{ord}_p\,\alpha$ elements. It follows that τ is conjugate to $\xi_1\xi_2^{(p-1)/\,\mathrm{ord}_p\,\alpha}$.

In both cases $m = a \cdot \mathrm{ord}(\alpha)$ and hence (b) and (c) are proven. □

We note the following

Lemma 2.3.10 *Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be a tail cover as in Lemma 2.3.1, with $a < p$ and $d = ap + 1$ in (ii). Let $\bar{y} \in \bar{Y}$ be a wild ramification point and $I_{\bar{y}} = I^{(p)} \rtimes \langle\tau\rangle \simeq (\mathbb{Z}/p\mathbb{Z})^b \rtimes_\chi \mathbb{Z}/m\mathbb{Z} \subset S_{ap}$ be its inertia group. Let $m_1 := |\ker\chi|$ and $m_2 = m/m_1$. Then $m_1 \mid a$.*

Proof: We may assume that $I^{(p)}$ is contained in P_a. Let τ be an element of order m. By Lemma 2.3.9 (a) we may assume that

$$\tau\rho_i\tau^{-1} = \begin{cases} \rho_{i+1} & , \text{ if } i < a \\ \rho_1^\alpha & , \text{ if } i = a. \end{cases}$$

The conjugation action of τ on the ρ_i defines a linear map on P_a considered as $\mathbb{F}_p$-vector space. The matrix representation of τ with respect to the basis $(\rho_1, \ldots, \rho_a)$ is given by

$$M(\tau) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \alpha \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $M(\tau)$ is $t^a - \alpha$. Let β be a root of $t^a - \alpha$ with $\mathrm{ord}(\beta) = m$. From this it follows easily that all roots of $t^a - \alpha$ are of the form $\beta^{1+i\,\mathrm{ord}\,\alpha}$ for $0 \leq i \leq a-1$.

Recall that subgroups of finite cyclic groups are uniquely determined by their order. This implies that τ^{m_2} generates $\ker\chi$, because τ^{m_2} has order m_1. In particular, $M(\tau)^{m_2}$ has an eigenvalue 1.

Note that since $t^a - \alpha$ is a polynomial with only simple roots and $p \nmid a$, $M(\tau)$ is diagonalizable over the algebraic closure of $\mathbb{F}_p$. Since the characteristic polynomial of any square matrix is invariant under base change, the eigenvalues of $M(\tau)^{m_2}$ are given by $\beta^{(1+i\,\mathrm{ord}\,\alpha)\cdot m_2}$.

Note that $M(\tau)^{m_2}$ has eigenvalue 1 if and only if there exists an i such that $\mathrm{ord}(\beta^{1+i\,\mathrm{ord}\,\alpha}) \mid m_2$. We compute

$$\mathrm{ord}(\beta^{1+i\,\mathrm{ord}\,\alpha}) = \frac{m}{\gcd(m, \mathrm{ord}_p(\alpha)\cdot i + 1)}.$$

Recall that $m = m_1 m_2$ and hence

$$\mathrm{ord}(\beta^{1+i\,\mathrm{ord}\,\alpha}) = \frac{m}{\gcd(m, \mathrm{ord}_p(\alpha)\cdot i + 1)} \quad \Big| \quad m_2$$

if and only if $m_1 \mid \gcd(m, \mathrm{ord}_p(\alpha)\cdot i + 1)$. Since $m_1 \mid m$ this is equivalent to $m_1 \mid (\mathrm{ord}_p(\alpha)\cdot i + 1)$. Now $m = a\cdot \mathrm{ord}_p\,\alpha$ and hence $m_1 \mid (\mathrm{ord}_p(\alpha)\cdot i + 1)$ if and only if there exists an integer x such that $m_1 \cdot x = \frac{m}{a} i + 1$. This equation is equivalent to

$$a = \frac{m\cdot i}{m_1 x - 1}.$$

We see that

$$a = \frac{m\cdot i}{m_1 x - 1} = \frac{m_1 m_2 \cdot i}{m_1 x - 1} \equiv 0 \mod m_1.$$

This implies that $m_1 \mid a$. □

We summarize the results for better reference.

Theorem 2.3.11 *Let a, e, d be positive integers with $e, d > 1$ and $0 < |e - ap| < p$ and $ap < d < (a+1)p$. Let $\bar\varphi : \bar Z = \mathbb{P}^1_k \to \bar X = \mathbb{P}^1_k$ be defined by*

(i)

$$(\bar x, \bar z) \mapsto \bar x = \begin{cases} \bar z^e + \bar z^{ap}, & \text{if } e < ap \\ \frac{\bar z^e}{\bar z^{e-ap}-1}, & \text{if } e > ap \end{cases} \quad \text{or}$$

(ii)

$$(\bar x, \bar z) \mapsto \bar x = \frac{1 + \bar z^d}{\bar z^{d-ap}}.$$

Let $\bar g : \bar Y \to \bar X$ be the Galois closure of $\bar\varphi$. Then the following holds.

(a) In (i) $\bar\varphi$ is ramified in exactly two points $\bar z = \bar 0$ and $\bar z = \bar\infty$ and unramified else. $\bar\varphi$ is ramified in $\bar z = 0$ with ramification index e (tame) and in $\bar\infty$ with ramification index ap (wild). We have $\bar\varphi(\bar 0) = \bar 0$ and $\bar\varphi(\bar\infty) = \bar\infty$.

(b) In (ii) $\bar\varphi$ is only branched in $\bar x = \bar\infty$. Moreover, $\bar\varphi$ is ramified in $\bar z = 0$ with ramification index $d - ap$ and wildly ramified in $\bar z = \bar\infty$ with ramification index ap and else unramified.

Now suppose that $d = ap + 1$ and $a < p$. Let $\bar y \in \bar Y$ be a wild ramification point and $I_{\bar y} = I^{(p)} \rtimes_\chi \langle\tau\rangle \simeq \left(\mathbb{Z}/p\mathbb{Z}\right)^b \rtimes_\chi \mathbb{Z}/m\mathbb{Z}$ its inertia group. Define $m_1 := |\ker(\chi)|$ and ρ_i as in (2.3.1).Then the following results hold for the wild inertia group.

(c) $I_{\bar y} \subset S_{ap}$.

(d) $|\bigcup_{\gamma\in I^{(p)}} \mathrm{supp}(\gamma)| = ap$.

(e) $a \mid m$.

(f) $m_1 \mid a$.

(g)
- *If $a = 1$, then $|\operatorname{supp}(\tau)| = p - 1$.*
- *If $a > 1$, then $|\operatorname{supp}(\tau)| = ap$.*

(h) *τ is conjugate to the product of ξ_1 with a power of ξ_2 in S_{ap} (where ξ_1 and ξ_2 are defined as in the paragraph before Lemma 2.3.9) and we may assume that*

$$\tau\rho_i\tau^{-1} = \begin{cases} \rho_{i+1}, & \text{if } i < a \\ \rho_1^{\alpha}, & \text{if } i = a \end{cases}$$

for some $\alpha \in \mathbb{F}_p^*$.

(i) $m = a \cdot \operatorname{ord}(\alpha)$.

Proof: Assertion (a) and (b) are proven in Lemma 2.3.1. Assertion (c) and (d) are proven in Lemma 2.3.6.

Assertion (e) is proven in Proposition 2.3.8 for $a \geq 2$. For $a = 1$ this is trivial.

Assertion (f) is proven in Lemma 2.3.10.

Assertion (g) is proven in Proposition 2.3.8 for $a > 1$. For $a = 1$, note that the inertia group is a subgroup of the normalizer of $N_{S_p}(\langle\rho\rangle)$ where $I^{(p)} = \langle\rho\rangle$. Thus the assertion follows with Lemma 1.4.3.

Assertion (h) and (i) are proven in Lemma 2.3.9. □

2.4 Ramification invariants

In this section we compute the ramification invariants of the tail covers from Theorem 2.3.11 under the assumption that there exists only one nontrivial lower jump. The results in this section are new.

Lemma 2.4.1 *Let a, e be integers with $a \geq 1$, $e \geq 2$ with $0 < |ap - e| < p$.*

(a) *Let $\bar{\varphi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be defined by*

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{x} = \bar{z}^e + \bar{z}^{ap}, & \text{if } e < ap \\ \bar{x} = \frac{\bar{z}^e}{\bar{z}^{e-ap}-1}, & \text{if } e > ap. \end{cases}$$

Let $\bar{g} : \bar{Y} \to \bar{X}$ be the Galois closure of $\bar{\varphi}$ and $\bar{y} \in \bar{Y}$ a wild ramification point and $I_{\bar{y}}$ its inertia group. Assume that $I_{\bar{y}}$ has only one lower jump $\neq 0$. Finally, let $d = \max\{e, ap\}$ denote the degree of $\bar{\varphi}$. Then the ramification invariant is given by

$$\sigma = \frac{2d - e - ap}{a(p-1)} = \frac{|e - ap|}{a(p-1)}.$$

(b) *Let $ap < d < (a+1)p$ and $\bar{\varphi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be defined by*

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^d}{\bar{z}^{d-ap}}.$$

Let $\bar{g} : \bar{Y} \to \bar{X}, \bar{y} \in \bar{Y}$ and $I_{\bar{y}}$ be defined as in (a). Assume that $I_{\bar{y}}$ has only one lower jump $\neq 0$. Then the ramification invariant is given by

$$\sigma = \frac{d}{a(p-1)}.$$

Proof: The idea in the proof of (a) and (b) is identical. In a first step we factorize the genus-0 tail cover $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ through the cover $\bar{\theta} : \bar{Y} \to \bar{Z} = \mathbb{P}^1_k$, where $\bar{\theta}$ is such that Diagram 2.4.1 is commutative, i.e. $\bar{g} = \bar{\varphi} \circ \bar{\theta}$.

Note that by Galois theory the map $\bar{\theta}$ is a Galois cover as well. We then use the Riemann-Hurwitz formula (Theorem 1.3.5) on $\bar{g}$ and on $\bar{\theta}$. Note that $\bar{\theta}$ and $\bar{g}$ map from the same curve, namely $\bar{Y}$. Hence, the left side

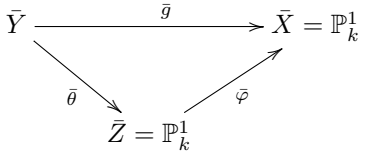

Figure 2.4.1: Factorization of $\bar{g}$.

of the Riemann-Hurwitz formula coincides and we can equate the right sides. This enables us to compute the ramification invariant.

In the proof of (a) and (b) we denote the Galois group of $\bar{g}$ by G and the Galois group of $\bar{\theta}$ by H. Recall that H is a one-point stabilizer of G, and thus $|G| = d|H|$ (Remark 1.1.7).

We now prove (a).

The Riemann-Hurwitz formula (Theorem 1.3.5) implies that the map $\bar{g} : \bar{Y} \to \bar{X}$ satisfies

$$2g(\bar{Y}) - 2 = -2|G| + \frac{|G|}{e}(e-1) + \frac{|G|}{p^b m}\left(p^b m - 1 + h(p^b - 1)\right).$$

The Riemann-Hurwitz formula applied to the cover $\bar{\theta} : \bar{Y} \to \bar{Z}$ satisfies

$$\begin{aligned} 2g(\bar{Y}) - 2 = &- 2|H| + (d-e)\frac{|H|}{e}(e-1) + \frac{|H|}{p^{b-1}\frac{m}{a}}\left(p^{b-1}\frac{m}{a} - 1 + h(p^{b-1}-1)\right) \\ &+ (d-ap)\frac{|H|}{p^b m}\left(p^b m - 1 + h(p^b - 1)\right). \end{aligned}$$

Since the left side of both equations equals $2g(\bar{Y}) - 2$, the right sides are equal. Arranging the coefficient with h on one side of the equation and the other summands on the other side we obtain (we use $|G| = d|H|$)

$$\begin{aligned} & h \cdot \frac{1}{p^b m}\left(d(p^b - 1) - ap(p^{b-1} - 1) - (d - ap)(p^b - 1)\right) = \frac{h}{m}a(p-1) \\ =& 2d - \frac{d}{e}(e-1) - \frac{d}{p^b m}\left(p^b m - 1\right) - 2 + \frac{(d-e)}{e}(e-1) + \frac{a}{p^{b-1}m}\left(p^{b-1}\frac{m}{a} - 1\right) \\ &+ \frac{(d-ap)}{p^b m}\left(p^b m - 1\right) \\ =& 2d - 2 - (e-1) + \frac{1}{p^b m}\left(-d(p^b m - 1) + ap\left(p^{b-1}\frac{m}{a} - 1\right) + (d-ap)(p^b m - 1)\right) \\ =& 2d - 1 - e + \frac{1}{p^b m}\left(p^b m - ap^{b+1}m\right) \\ =& 2d - e - ap. \end{aligned}$$

This proves (a). Now we prove (b).

The Riemann-Hurwitz formula implies that the map $\bar{g} : \bar{Y} \to \bar{X}$ satisfies

$$2g(\bar{Y}) - 2 = -2|G| + \frac{|G|}{p^b m}\left(p^b m - 1 + h(p^b - 1)\right).$$

Note that by Theorem 2.3.11 (b), $\bar{\varphi}$ is ramified in two points. One of these has tame ramification with ramification index $d - ap$ and the other is a wild ramification point with ramification index ap. The Riemann-Hurwitz formula applied to the cover $\bar{\theta} : \bar{Y} \to \bar{Z}$ satisfies

$$2g(\bar{Y}) - 2 = -2|H| + \frac{|H|}{p^b \frac{m}{d-ap}}\left(p^b \frac{m}{d-ap} - 1 + h(p^b - 1)\right) + \frac{|H|}{p^{b-1}\frac{m}{a}}\left(p^{b-1}\frac{m}{a} - 1 + h(p^{b-1} - 1)\right).$$

Since the left side of both equations equals $2g(\bar{Y})-2$, the right sides are equal. Arranging the coefficient with h on one side of the equation and the other summands on the other side we obtain (we use $|G| = d|H|$)

$$\begin{aligned}
&h \cdot \frac{1}{p^b m}\left(d(p^b-1)-(d-ap)(p^b-1)-ap(p^{b-1}-1)\right) = \frac{h}{m}a(p-1)\\
=&2d-2+\frac{1}{p^b m}\left(-d(p^b m-1)+(d-ap)\left(p^b\frac{m}{d-ap}-1\right)+ap\left(p^{b-1}\frac{m}{a}-1\right)\right)\\
=&2d-2+\frac{1}{p^b m}\left(-dp^b m+d+p^b m-(d-ap)+p^b m-ap\right)\\
=&2d-2+\frac{1}{p^b m}\left(-dp^b m+2p^b m\right)\\
=&2d-2-d+2\\
=&d.
\end{aligned}$$

This yields the assertion. □

Remark 2.4.2 Note that in Lemma 2.4.1 the assumption that there exists a unique lower jump $\neq 0$ is crucial. This assumption is satisfied, if for example the action of $\langle\tau\rangle$ on $I^{(p)}$ is irreducible. We believe that this is true.

2.5 Ramification type of tail covers

In this chapter we define the ramification type of a tail cover and compute it in the case $d < 2p$. The results in this section can be proved with different methods as well. The results in this section are not new, but we have not found a suitable source.

The following definition can be found in [Wew03, Definition 2.9 ff.].

Definition 2.5.1 (a) A *pointed G-tail cover* is a G-tail cover together with a point $\bar{y} \in \bar{Y}$ above $\bar{\infty} \in \bar{X}$ and a subgroup H of the inertia group $I_{\bar{y}}$ such that $I_{\bar{y}} = I^{(p)} \rtimes H$, where $I^{(p)}$ denotes the first higher ramification group of $I_{\bar{y}}$. Additionally, $I^{(p)}$ is required to have order p.

(b) Let $(\bar{g}, \bar{y}, H)$ be a pointed G-tail cover. Let $m = m(\bar{g})$ denote the order of H and $h = h(\bar{g})$ the unique lower jump $\neq 0$ of $I_{\bar{y}}$. The pair (m, h) is called the *ramification type of $\bar{g}$*.

We now compute the ramification types of the tail covers defined in Lemma 2.3.1 for $a = 1$.

Proposition 2.5.2 *(a) Let $1 < e < 2p$ with $e \neq p$. Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be the primitive tail cover defined by the Galois closure of*

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^e + \bar{z}^p & , \text{ if } e < p \\ \frac{\bar{z}^e}{\bar{z}^{e-p}-1} & , \text{ if } e > p. \end{cases}$$

Then the ramification type of $\bar{g}$ is (h, m) with

$$h = \frac{|e-p|}{\gcd(e-p, p-1)}, \quad m = \frac{p-1}{\gcd(e-p, p-1)}.$$

(b) Let $\bar{g} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ be the new tail cover defined by the Galois closure of

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1+\bar{z}^{p+1}}{\bar{z}}.$$

Then the ramification type of $\bar{g}$ is (h, m) with

$$h = \frac{p+1}{2}, \quad m = \frac{p-1}{2}.$$

Proof: Note that Theorem 2.3.11 (f) shows that $m_1 = 1$ (for the primitive tail cover defined by $\bar{x} = \bar{z}^e + \bar{z}^p$ this follows from [BO11, Lemma 5.1 (a)] and $\sigma = \frac{p-e}{p-1}$, as well. For the new tail cover defined by $\bar{x} = \frac{1+\bar{z}^{p+1}}{\bar{z}}$ this has also been shown in [HOPS, Proposition 4.17]). By Lemma 1.3.8 this implies that $\gcd(h, m) = 1$. Thus the assertion follows from

$$\sigma = \frac{h}{m} = \frac{|e-p|}{p-1} = \begin{cases} \frac{p-e}{p-1} & , \text{ if } e < p \\ \frac{e-p}{p-1} & , \text{ if } e > p \end{cases}$$

in the case of the primitive tail covers and

$$\sigma = \frac{p+1}{p-1}$$

in the case (b) of the new tail cover. □

2.6 Uniqueness of tail covers

In Lemma 2.2.2 we recalled that primitive tail covers of degree p are unique (see Definition 2.1.3 for the definition of isomorphisms between tail covers and Lemma 2.2.2 for the uniqueness of the tail cover with degree p). In this section we prove a similar result for arbitrary $a \geq 1$ and the primitive tail covers from Lemma 2.3.1. This result is new. At the end of this section (Remark 2.6.2) we show that the new tail cover from Lemma 2.3.1 (ii) is not unique.

Proposition 2.6.1 *(a) Let $e, a \geq 1$ be integers with $(a-1)p < e < ap$. There exists a unique cover $\bar{\varphi} : \bar{Z} = \mathbb{P}^1 \to \bar{X} = \mathbb{P}^1$ (unique as mere covers) of degree ap such that $\bar{\varphi}$ is ramified in exactly two points $\bar{z}_1, \bar{z}_2 \in \bar{Z}$ and unramified else with the following properties. $\bar{\varphi}$ is tamely ramified in $\bar{z}_1$ with ramification index e and $\bar{\varphi}$ is fully ramified in $\bar{z}_2$ (wildly). The coordinates on $\bar{X}$ and $\bar{Z}$ can be normalized such that $\bar{\varphi}$ is given by*

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \bar{z}^e + \bar{z}^{ap}.$$

(b) Let $e, a \geq 1$ be integers with $ap < e < (a+1)p$. There exists a unique cover $\bar{\varphi} : \bar{Z} = \mathbb{P}^1 \to \bar{X} = \mathbb{P}^1$ (unique as mere covers) of degree e such that $\bar{\varphi}$ is ramified in exactly two points $\bar{z}_1, \bar{z}_2$ and unramified else with the following properties. $\bar{\varphi}$ is tamely ramified in $\bar{z}_1$ with ramification index e and wildly ramified in $\bar{z}_2$ with ramification index ap. The coordinates on $\bar{X}$ and $\bar{Z}$ can be normalized such that $\bar{\varphi}$ is given by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{\bar{z}^e}{\bar{z}^{e-ap} - 1}.$$

Proof: If $\bar{\varphi}$, is defined by $\bar{x} = \bar{z}^e + \bar{z}^{ap}$ or $\bar{x} = \bar{z}^e/(\bar{z}^{e-ap} - 1)$, then $\bar{\varphi}$ satisfies the asserted properties of (a) and (b) by Lemma 2.3.1. Hence we only have to show the uniqueness as mere covers.

We prove (a).

Let $\bar{\psi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be any cover with the properties of assertion (a), i.e. $\bar{\psi}$ is a cover of degree ap, tamely ramified in exactly one point $\bar{z}_1$ with ramification index e and fully ramified in another point $\bar{z}_2$ (and unramified else). Recall that the automorphisms of $\mathbb{P}^1_k$ are given by linear fractional maps, which are defined by the images of three distinct points.

We normalize the coordinate on $\bar{Z}$ such that $\bar{z}_1$ has coordinate $\bar{z} = \bar{0}$ and $\bar{z}_2$ has coordinate $\bar{z} = \bar{\infty}$. Moreover, we normalize the coordinate on $\bar{X}$ such that $\bar{\psi}(\bar{0}) = \bar{0}$ and $\bar{\psi}(\bar{\infty}) = \bar{\infty}$. Note that this normalization still leaves two degrees of freedom (we have only chosen the image of two points in $\bar{X}$ and $\bar{Z}$ so far, but we have the freedom to choose the image of three points).

Then we can write

$$\bar{x} = \bar{z}^e P(\bar{z})$$

with a polynomial P of degree $ap - e$. We further normalize $\bar{z}$ such that $\bar{\psi}(\bar{1}) = \bar{2}$ (note that in characteristic 2 this means that $P(\bar{1}) = \bar{0}$). Changing the coordinate $\bar{x}$ by a suitable multiple we may assume that P is monic.

We compute that

$$\bar{\psi}'(\bar{z}) = \bar{z}^{e-1}(eP + \bar{z}P').$$

The assumption that $\bar{\psi}$ is only ramified in $\bar{z} = \bar{\infty}$ and $\bar{z} = \bar{0}$ implies that $eP + \bar{z}P' = c$ is constant and nonzero. Write $P = \sum_{i=0}^{ap-e} a_i \bar{z}^i$. Then the equation $eP + \bar{z}P' = c$ yields the following system of equations for the coefficients

$$(e+i)a_i = \begin{cases} c, & \text{for } i = 0 \\ 0, & \text{else.} \end{cases}$$

Hence $a_0 = \frac{c}{e} \neq 0$. Moreover, since $0 < ap - e < p$ it follows that $a_1 = a_2 = \ldots = a_{ap-e-1} = 0$. Since P is monic with $P(\bar{1}) = \bar{2}$ and has degree $ap - e$ it follows that $a_0 = 1$. $\bar{\psi}$ is ramified in $\bar{z} = \bar{\infty}$ with ramification index ap, it follows that $a_{ap-e} \neq 0$. This proves that $\bar{\psi}$ is given by

$$\bar{x} = \bar{z}^e \left(1 + \bar{z}^{ap-e}\right).$$

This proves (a).

We now prove (b).

Let $\bar{\psi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be any cover with the properties of assertion (b), i.e. $\bar{\psi}$ is a cover of degree e, tamely ramified in exactly one point $\bar{z}_1$ with ramification index e and wildly ramified in another point $\bar{z}_2$ with ramification index ap (and unramified else).

We may normalize the coordinate $\bar{z}$ of $\bar{Z} = \mathbb{P}^1_k$ such that the unique wild (resp. tame) ramification point has coordinate $\bar{z} = \bar{\infty}$ (resp. $\bar{z} = \bar{0}$). Moreover, we normalize the coordinate $\bar{x}$ of $\bar{X} = \mathbb{P}^1_k$ such that $\bar{\psi}(\bar{\infty}) = \bar{\infty}$ and $\bar{\psi}(\bar{0}) = \bar{0}$ (note that we have used 4 degrees of freedom and thus have still 2 left). Finally we normalize the coordinate on $\bar{Z}$ such that $\bar{\psi}(\bar{1}) = \bar{\infty}$. Then $\bar{\psi}$ is given by an equation

$$\bar{x} = c\frac{\bar{z}^e}{P(\bar{z})}, \tag{2.6.1}$$

where c is a nonzero constant and P is a monic polynomial of degree $e - p - ap$ with $P(\bar{1}) = \bar{0}$. Replacing the coordinates $\bar{x}$ by a multiple, if necessary, we may assume that $c = \bar{1}$.

The assumption that $\bar{\psi}$ is only ramified in exactly one point above $\bar{x} = \bar{0}$ implies that

$$\bar{\varphi}'(z) = \tilde{c}\frac{\bar{z}^{e-1}}{P^2}, \tag{2.6.2}$$

where $\tilde{c}$ is a non-zero constant. Equation (2.6.1) yields

$$\bar{\varphi}'(\bar{z}) = \frac{e\bar{z}^{e-1}P - z^e P'}{P^2}. \tag{2.6.3}$$

Comparing equation (2.6.2) with (2.6.3) yields

$$eP - \bar{z}P' = \tilde{c}. \tag{2.6.4}$$

Writing $P(\bar{z}) = \sum_{i=0}^{e-ap} a_i \bar{z}^i$ we see from equation (2.6.4) that

$$(e-i)a_i = \begin{cases} \tilde{c} & \text{for } i = 0 \\ 0 & \text{for } i = 1, \ldots, e - ap. \end{cases}$$

This, for $i = 0$, yields that $a_0 = \tilde{c}/e$. Moreover, for all $0 < i < e - ap$ this yields $a_i = 0$. Recall that P is monic with $P(\bar{1}) = \bar{0}$. In particular $a_0 = -\bar{1}$. This proves that $\bar{\psi}$ is given by

$$\bar{x} = \frac{\bar{z}^e}{\bar{z}^{e-p} - 1}.$$

□

Remark 2.6.2 The new tail cover in Lemma 2.3.1 defined by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^d}{\bar{z}^{d-ap}}, \tag{2.6.5}$$

is not unique for $a \geq 2$. The reason is the following. For simplicity let $d = ap+1$. Let $\bar{\psi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be a degree $d = ap+1$ cover, which is only ramified in one single point with ramification index ap (note that

for $d = ap + 1$ the new tail cover defined by the Galois closure of (2.6.5) satisfies this assumption, since the second ramification point $\bar{z} = 0$ has ramification index 1 (Lemma 2.3.1 (b))). Normalizing the coordinates on $\bar{Z}$ and $\bar{X}$ we may assume that the ramification point is $\bar{z} = \bar{\infty}$ and the branch point is $\bar{x} = \bar{\infty}$. Then $\bar{\psi}(\bar{\infty}) = \bar{\infty}$. Since the ramification index in $\bar{\infty}$ is ap, this implies that $\bar{\psi}$ is given by

$$\bar{x} = \frac{P(z)}{h(z)},$$

where $\deg P = ap + 1$ and $\deg h = 1$. We further normalize the coordinate on $\bar{Z}$ such that $\bar{\psi}(0) = \infty$, as well (note that we have used 3 degrees of freedom until now and have 3 left). Thus $h(z) = cz$ with a unit c. Dividing the coefficients of P by c we may assume that h is monic. This yields

$$\bar{\psi}'(\bar{z}) = \frac{P'(\bar{z})\bar{z} - P(\bar{z})}{\bar{z}^2}.$$

Write $P(\bar{z}) = \sum_{i=0}^{ap+1} a_i \bar{z}^i$. The fact that $\bar{\psi}$ is only ramified in one single point above $\bar{\infty}$ implies that the numerator of $\bar{\psi}'(\bar{z})$ has to be a nonzero constant. Using $P(\bar{z}) = \sum_{i=0}^{ap+1} a_i \bar{z}^i$ we get

$$P'(\bar{z})\bar{z} - P(\bar{z}) = \sum_{i=0}^{ap+1} a_i(i-1)z^i.$$

Now $\sum_{i=0}^{ap+1} a_i(i-1)z^i$ is constant if and only if

$$a_i(i-1) = \begin{cases} \text{const.} \neq 0, & \text{for } i = 0 \\ 0, & \text{else.} \end{cases}$$

Since this computation is done in a field of characteristic $p > 0$, this implies that $P(\bar{z})$ simplifies to

$$P(\bar{z}) = a_0 + \sum_{r=0}^{a} a_{rp+1} \bar{z}^{rp+1}.$$

In the case $a = 1$ we see that there are exactly three coefficients a_0, a_1 and a_{p+1} left to determine. Since we have 3 degrees of freedom left, we can normalize the coordinates and obtain the new tail cover defined by $\bar{x} = \frac{1+\bar{z}^{p+1}}{\bar{z}}$.

In the case $a > 1$ the same argument does not work, as there are at least 4 unknown coefficients, but we can only fix 3 of them. This shows that there are non-isomorphic new tail covers.

Note that since the new tail covers defined by $\bar{x} = \frac{1+\bar{z}^d}{\bar{z}^{d-ap}}$ are not unique, it is very unlikely that they appear in the stable reduction of three-point covers for $a > 1$. The reason is that three-point covers in the single-cycle case (Lemma 1.2.6) are unique.

3 Stable reduction of three-point covers

In Section 1.6 we discussed stable reduction in general. In the present chapter we recall several known results about the stable reduction of three-point covers where the three-point cover has mild reduction (Definition 3.2.1). In particular, we will see that the stable reduction of a three-point cover with mild reduction is described by a deformation datum and a set of tail covers which satisfy some compatibility conditions with the deformation datum (Theorem 3.3.4). Mild reduction is automatically satisfied if $d < 2p$ and this case is explicitly analyzed in Chapter 4, where we describe the stable reduction in detail (Theorem 4.2.2).

Throughout this whole chapter let $d < 2p$.

In Chapter 2 we described the tail covers which occur in the stable reduction of three-point covers. In Section 3.1 we similarly describe the deformation data. We define special deformation data, hypergeometric deformation data and special G-deformation data. In Theorem 4.2.2 we prove that hypergeometric deformation data play a very important role. Therefore, we study the existence of hypergeometric deformation data in detail. We recall a general existence result for hypergeometric deformation data due to Bouw and Wewers (Proposition 3.1.8). Using the ramification invariants of the tail covers in Lemma 2.4.1 for $a = 1$, we are able to use the existence result Proposition 3.1.8 and prove the existence of hypergeometric deformation data in dependence of an abstract ramification type $\mathcal{C} = (d; e_1, e_2, e_3)$ with the additional assumption that $\sum_{i=1}^{3}(\max\{e_i, p\} - p) \leq d - p$. In Chapter 4 we prove that $\sum_{i=1}^{3}(\max\{e_i, p\} - p) \leq d - p$ is equivalent to the three-point cover of type $\mathcal{C}$ having bad reduction (Theorem 4.1.1).

In Section 3.2 we recall a theoretical result due to Wewers on the structure of the stable reduction of three-point covers (Theorem 3.2.6). This result shows that $\bar{X}$ is a comb (where $f : Y \to X = \mathbb{P}^1$ is a three-point cover in characteristic 0 with mild reduction and $\bar{f} : \bar{Y} \to \bar{X}$ is its stable reduction), consisting of one unique central component and several other irreducible components which have only one single intersection point with the central component. The restriction of $\bar{f}$ to an irreducible component in $\bar{Y}$ which lies above any non-central component is a tail cover (Lemma 3.2.5). Hence this section shows the connection between tail covers and the stable reduction.

In Section 3.3 we recall a lifting result for three-point covers (Theorem 3.3.4). This lifting result is the main tool for the characterization of the stable reduction of three-point covers for $d < 2p$ in Chapter 4. We end this chapter with an example, where we demonstrate the use of Theorem 3.3.4.

3.1 Deformation data

In this section we recall the definition of special deformation data, hypergeometric deformation data and special G-deformation data. We also study the existence of hypergeometric deformation data in the case $d < 2p$.

The following definitions of special deformation datum and hypergeometric deformation datum can be found in [BW04, Definition 2.1] (and [Wew03, Definition 2.7]). We recall these definitions for the sake of completeness and because of the paramount importance in this work. To avoid confusion with our previous notation in Section 2.3, we use a notation which is slightly different from that of [BW04, Definition 2.1]. What is called Z_k in [BW04, Definition 2.1], we call $\bar{W}_k$. We use the notation with bars to stress the fact that the deformation datum occurs in the stable reduction of a three-point cover.

Let k be an algebraically closed field of characteristic $p > 0$.

Let H_0 be a finite group of order prime to p. Fix a character $\chi : H_0 \to \mathbb{F}_p^*$. Let $g : \bar{W}_k \to \bar{X} = \mathbb{P}^1_k$ be an H_0-Galois cover over k and ω a meromorphic differential on $\bar{W}_k$. We assume that ω is logarithmic (i.e. can be written as $\omega = \mathrm{d}u/u$ for a function u on $\bar{W}_k$) and

$$\beta^*\omega = \chi(\beta) \cdot \omega, \quad \text{for all } \beta \in H_0.$$

Let $\xi \in \bar{W}_k$ be a closed point and τ its image in $\bar{X} = \mathbb{P}^1_k$. Denote by $H_{0,\xi}$ the stabilizer of ξ in H_0. Define

$$m_\tau(\omega) := |H_{0,\xi}|, \quad h_\tau(\omega) := \mathrm{ord}_\xi(\omega) + 1, \quad \sigma_\tau(\omega) := h_\tau(\omega)/m_\tau(\omega). \tag{3.1.1}$$

Since ω is logarithmic, we have $h_\tau(\omega) \geq 0$. We say that τ is a *critical point* of the differential form ω if $(m_\tau(\omega), h_\tau(\omega)) \neq (1, 1)$. Let $(\tau_i)_{i \in \mathbb{B}}$ be the critical points of ω, indexed by a finite set $\mathbb{B}$. For $i \in \mathbb{B}$, we write

$m_i(\omega), h_i(\omega), \sigma_i(\omega)$ instead of $m_{\tau_i}(\omega), h_{\tau_i}(\omega), \sigma_{\tau_i}(\omega)$. For every i, choose a point $\xi_i \in \bar{W}_k$ above τ_i. Define $\mathbb{B}_{\text{wild}} := \{i \in \mathbb{B} \mid h_i = 0\}$.

Definition 3.1.1 A *special deformation datum of type* (H_0, χ) is a pair (g, ω), where $g : \bar{W}_k \to \bar{X} = \mathbb{P}^1_k$ is an H_0-Galois cover and ω is a logarithmic differential on $\bar{W}_k$ such that the following holds.

(i) We have
$$\beta^*\omega = \chi(\beta) \cdot \omega, \quad \text{for all } \beta \in H_0.$$

(ii) For every $i \in \mathbb{B}\backslash\mathbb{B}_{\text{wild}}$, we have that $0 < \sigma_i(\omega) < 2$ and $\sigma_i(\omega) \neq 1$.

(iii) Define $\mathbb{B}_{\text{prim}} = \{i \in \mathbb{B} \mid 0 < \sigma_i(\omega) < 1\}$. Then $|\mathbb{B}_{\text{prim}} \cup \mathbb{B}_{\text{wild}}| = 3$.

Write $\mathbb{B}_0 = \mathbb{B}_{\text{wild}} \cup \mathbb{B}_{\text{prim}}$ and $\mathbb{B}_{\text{new}} = \mathbb{B}\backslash\mathbb{B}_0$.

We call the special deformation datum (g, ω) *normalized* if $\{\tau_j \mid j \in \mathbb{B}_0\} = \{\bar{0}, \bar{1}, \bar{\infty}\}$.

We remark that our definition of special deformation datum is a reformulation of [Wew03, Definition 2.7]. It differs from [BW04, Definition 2.1] in two points, namely we restrict to $\sigma(\omega) \neq 1, 2$. We do this to use the results of [Wew03] without the need for modification.

Remark 3.1.2 There exists one three-point cover where the definition of special deformation data (Definition 3.1.1) is inappropriate (ramification type $\mathcal{C} = (2p-1; p, p, 2p-1)$). The reason is that the ramification invariant of the primitive tail cover given by $\bar{x} = \frac{\bar{z}^{2p-1}}{\bar{z}^{p-1}-1}$ is 1 (Lemma 2.4.1 (a)). We discuss this problem in detail in Section 4.3.

Convention 3.1.3 All special deformation data are assumed to be normalized. From now on we identify the set $\mathbb{B}_0$ with $\{1, 2, 3\}$ and let $\tau_1 = \bar{0}, \tau_2 = \bar{1}$ and $\tau_3 = \bar{\infty}$.

We now define special G-deformation data. Although the definition is lengthy, special G-deformation data play a very important role in this thesis. Therefore, we include all the details. This definition can be found in [Wew03, §2.2.3].

Let G be a finite group and suppose we are given the following data.

- A subgroup $G_0 \subset G$, of the form $G_0 = I_0 \rtimes H_0$, where $I_0 \subset G_0$ is cyclic of order p and $H_0 \subset G_0$ is of prime-to-p order. Let $\chi_0 : H_0 \to \mathbb{F}_p^*$ denote the character defined by the rule $\beta\alpha\beta^{-1} = \alpha^{\chi_0(\beta)}$, where $\alpha \in I_0, \beta \in H_0$.
- A normalized special deformation datum $(g_0 : \bar{W} \to \bar{X}_0 := \mathbb{P}^1_k, \omega_0)$ of type (H_0, χ_0). Let $(\tau_j)_{j\in\mathbb{B}}$ denote the set of critical points, $m_j(\omega_0), h_j(\omega_0)$ and $\sigma_j(\omega_0)$ the numbers from (3.1.1).
- For all $j \in \mathbb{B}\backslash\mathbb{B}_{\text{wild}}$ a point $\xi_j \in \bar{W}$ above $\tau_j \in \bar{X}_0 = \mathbb{P}^1_k$.
- For all $j \in \mathbb{B}\backslash\mathbb{B}_{\text{wild}}$ a subgroup $G_j \subset G$ and a connected pointed G_j-tail cover $(\bar{g}_j : \bar{Y}_j \to \bar{X}_j, \bar{y}_j, H_j)$ (see Definition 2.5.1). By abuse of notation, we shall write $(g_0, \omega_0; \bar{g}_j)$ to denote all of the data introduced above.

Definition 3.1.4 The datum $(g_0, \omega_0; \bar{g}_j)$ is called a *special G-deformation datum* if the following conditions holds

(i) The group G is generated by its subgroups G_0 and G_j.

(ii) The tail cover $\bar{g}_j : \bar{Y}_j \to \bar{X}_j$ has the ramification type $(m_j(\bar{g}_j), h_j(\bar{g}_j)) = (m_j(\omega), h_j(\omega))$ (see Definition 2.5.1). Moreover,
$$I_{\bar{y}_j} = I_0 \rtimes H_j, \quad \text{and} \quad H_j = H_0 \cap I_{\bar{y}_j}.$$

Remark 3.1.5 (a) In [Wew03] it is shown that the stable reduction of a three-point cover with mild reduction (Definition 3.2.1) gives rise to a special G-deformation datum. We do not go into detail on this topic because this thesis has another aim. For more details on this topic we refer to §1 and §2 of [Wew03].

(b) In the context of three-point covers with mild reduction we use the notation $\mathbb{B}_{\text{prim}}, \mathbb{B}_{\text{new}}$ etc. on the special G-deformation datum of (a).

We now define the hypergeometric deformation datum. This definition requires that $\operatorname{char} k > 2$ (see [BW04, §2]). Therefore let k from now on be an algebraically closed field with $\operatorname{char} k > 2$.

Definition 3.1.6 A *hypergeometric deformation datum* is a normalized special deformation datum (g, ω) with $\sigma_i(\omega) = \frac{p+1}{p-1}$ for all $i \in \mathbb{B}_{\text{new}}$.

Remark 3.1.7 By Proposition 2.2.1 (d) the ramification invariant $\sigma(\bar{g}) = \sigma(\omega) = \frac{p+1}{p-1}$ determines a genus-0 new tail cover uniquely. It is the new tail cover given by the equation

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^{p+1}}{\bar{z}}.$$

This correspondence is important and we will use it in Section 4.2 in the construction of special G-deformation data. This correspondence guarantees that a hypergeometric deformation datum and the new tail cover satisfy the compatibility conditions in Definition 3.1.4 (ii). For more details on the compatibility conditions we refer to Remark 3.2.7.

Let (g, ω) be a hypergeometric deformation datum. Recall that by assumption $\mathbb{B}_0 = \{1, 2, 3\}$. For $i \in \mathbb{B}_0$ we define integers $0 \leq a_i < p - 1$ such that

$$a_i \equiv (p-1)\sigma_i(\omega) \qquad \bmod p - 1. \tag{3.1.2}$$

The tuple (a_1, a_2, a_3) is called the *signature* of the hypergeometric deformation datum (g, ω). By [BW04, §3, (7)] we have

$$|\mathbb{B}_{\text{new}}| = (p - 1 - a_1 - a_2 - a_3)/2. \tag{3.1.3}$$

In particular, $\sum_{i=1}^{3} a_i$ is an integer between 0 and $p - 1$.

The following result of [BW04] shows the existence of hypergeometric deformation data with signature (a_1, a_2, a_3) under mild restrictions.

Proposition 3.1.8 *(a) Let (g, ω) be a hypergeometric deformation datum. Let (a_1, a_2, a_3) be its signature, and suppose that $\sum_{i=1}^{3} a_i < p - 1$. Then $u(x) := \prod_{i \in \mathbb{B}_{\text{new}}} (x - \tau_i)$ is a solution of the hypergeometric differential equation*

$$x(x-1)u'' + [(A + B + 1)x - C]u' + ABu = 0,$$

where $A = (1 + a_1 + a_2 + a_3)/2, B = (1 + a_1 + a_2 - a_3)/2$ and $C = 1 + a_1$. The degree of u in x is $d := (p - 1 - a_1 - a_2 - a_3)/2$.

(b) Let $0 \leq a_1, a_2, a_3 < p - 1$ be integers with $a_1 + a_2 + a_3 \leq p - 1$. We assume that $a_1 + a_2 + a_3$ is even. Let $H = \mathbb{Z}/m\mathbb{Z}$, where

$$m = \begin{cases} (p-1)/2, & \text{if } a_1, a_2, a_3 \text{ are all even} \\ p - 1, & \text{otherwise.} \end{cases}$$

Choose an injective character $\chi : H \to \mathbb{F}_p^*$. There exists a hypergeometric deformation datum (g, ω) of signature (a_1, a_2, a_3) and type (H, χ). The deformation datum (g, ω) is uniquely determined by (a_1, a_2, a_3), up to multiplying the differential ω by an element of $\mathbb{F}_p^*$.

Proof: See [BW04, Proposition 3.2]. □

Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $2 \leq e_1 \leq e_2 \leq e_3 \leq d < 2p$ and $e_3 < 2p - 1$. Note that $e_3 < 2p - 1$ is not a typing error. The reason for this seemingly peculiar condition will be explained in the proof of Proposition 3.1.9. Define $d_i := \{e_i, p\}$. We impose the additional condition $d - p \geq \sum_{i=1}^{3}(d_i - p)$ on $\mathcal{C}$. In Chapter 4 we prove that the condition $d - p \geq \sum_{i=1}^{3}(d_i - p)$ is equivalent to the unique three-point cover of type $\mathcal{C}$ having bad reduction (Theorem 4.1.1).

Using Proposition 3.1.8 (b) we now link abstract ramification types (with the above conditions) with hypergeometric deformation data. To be more precise, we prove that we can define a tuple of integers

$(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$ with the help of $\mathcal{C}$, such that there exists a hypergeometric deformation datum with signature $(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$. Moreover, we prove that the existence of this hypergeometric deformation datum is equivalent to the condition $d - p \geq \sum_{i=1}^{3}(d_i - p)$. Hence, with Theorem 4.1.1 we obtain a link between the existence of hypergeometric deformation data and bad reduction of three-point covers. This link is a new result for three-point covers in the single-cycle case (Remark 1.2.4) with $d < 2p$.

Note that [BW04] Bouw and Wewers also proved a link between the existence of hypergeometric deformation data and bad reduction, but for a different class of three-point covers ([BW04, Theorem 6.6]). Hence the link is not really unexpected, but a similar result in a different case.

Proposition 3.1.9 *Let $\mathcal{C} = (d; e_1, e_1, e_3)$ be an abstract ramification type with $2 < p \leq d < 2p$ and $2 \leq e_1 \leq e_2 \leq e_3 \leq d$ with $e_3 < 2p - 1$. Let*

$$\sigma_i = \begin{cases} \frac{p-e_i}{p-1} & , \text{ if } e_i < p \\ 0 & , \text{ if } e_i = p \\ \frac{e_i-p}{p-1} & , \text{ if } e_i > p. \end{cases}$$

Let $0 \leq a_i(\mathcal{C}) < p - 1$ with $a_i(\mathcal{C}) \equiv (p-1)\sigma_i \mod p - 1$. Then the following holds.

(a) The integers $a_i(\mathcal{C})$ and $d_i := \max\{e_i, p\}$ satisfy

$$(p - 1 - a_1(\mathcal{C}) - a_2(\mathcal{C}) - a_3(\mathcal{C}))/2 = d - p - \sum_{i=1}^{3}(d_i - p). \tag{3.1.4}$$

(b) $\sum_{i=1}^{3} a_i(\mathcal{C}) \leq p - 1$ if and only if $\sum_{i=1}^{3}(d_i - p) \leq d - p$.

(c) There exists a hypergeometric deformation datum (g, ω) with signature

$$(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$$

if and only if $\sum_{i=1}^{3}(d_i - p) \leq d - p$. It is unique up to multiplying ω with an element in $\mathbb{F}_p^*$.

Proof: We prove (a) by analyzing four different cases. These cases are determined by the number of e_i greater than p. For the sake of simplicity we write a_i for $a_i(\mathcal{C})$. Recall that since $\mathcal{C}$ is an abstract ramification type, the integers e_i satisfy $\sum e_i = 2d + 1$.

Since $e_3 < 2p - 1$ we have $a_i = 0$ if and only if $e_i = p$. In this case we can simply write $e_i - p$ for a_i. Note that if we allowed e_3 to be $2p - 1$, then $a_3(\mathcal{C})$ would be 0 as well, since $0 \leq a_3(\mathcal{C}) < p - 1$ with $a_3(\mathcal{C}) \equiv (p-1)\sigma_3 \equiv (2p - 1 - p) \equiv 0 \mod p - 1$. Hence we get the simple and unique expression

$$a_i = |e_i - p|.$$

1. Case: $e_i \leq p$ **for** $i = 1, 2, 3$

We have $a_i = p - e_i$ and $d_i = p$ for all i and it follows that

$$\begin{aligned}(p - 1 - a_1 - a_2 - a_3)/2 &= (p - 1 + (e_1 + e_2 + e_3) - 3p)/2 \\ &= (-2p - 1 + (2d + 1))/2 \\ &= d - p = d - p - \sum_{i=1}^{3}(d_i - p).\end{aligned}$$

2. Case: $e_1 \leq e_2 \leq p < e_3$

We have $a_i = p - e_i$ and $d_i = p$ for $i = 1, 2$ and $a_3 = e_3 - p$ and $d_3 = e_3$. This yields

$$\begin{aligned}(p - 1 - a_1 - a_2 - a_3)/2 &= (p - 1 - (p - e_1) - (p - e_2) - (e_3 - p))/2 \\ &= (-1 + e_1 + e_2 - e_3)/2 \\ &= (-1 + (e_1 + e_2 + e_3) - 2e_3)/2 \\ &= (2d - 2e_3)/2 \\ &= d - e_3 = d - p - (e_3 - p) = d - p - \sum_{i=1}^{3}(d_i - p).\end{aligned}$$

3. Case: $e_1 \leq p < e_2 \leq e_3$

We have $a_i = e_i - p$ and $d_i = e_i$ for $i = 2, 3$ and $a_1 = p - e_1$ and $d_1 = p$. This yields

$$\begin{aligned}(p-1-a_1-a_2-a_3)/2 &= (p-1-(p-e_1)-(e_2-p)-(e_3-p))/2\\ &= (2p-1+e_1-e_2-e_3)/2\\ &= (2p-1-(e_1+e_2+e_3)+2e_1)/2\\ &= (2p-1-(2d+1)+2e_1)/2\\ &= p-d-1+e_1.\end{aligned}$$

Moreover, we compute that

$$\begin{aligned}d-p-\sum_{i=1}^{3}(d_i-p) &= d-p-(e_2-p)-(e_3-p)\\ &= d+p-(e_1+e_2+e_3)+e_1\\ &= d+p-(2d+1)+e_1\\ &= p-d-1+e_1.\end{aligned}$$

4. $e_i > p$ **for** $i = 1, 2, 3$

We have $a_i = e_i - p$ and $d_i = e_i - p$ for all i. On the one hand:

$$\begin{aligned}(p-1-a_1-a_2-a_3)/2 &= (p-1-(e_1-p)-(e_2-p)-(e_3-p))/2\\ &= (4p-1-(e_1+e_2+e_3))/2\\ &= (4p-1-(2d+1))/2\\ &= 2p-d-1,\end{aligned}$$

and on the other hand

$$\begin{aligned}d-p-\sum_{i=1}^{3}(d_i-p) &= d-p-(e_1-p)-(e_2-p)-(e_3-p)\\ &= d+2p-(e_1+e_2+e_3)\\ &= d+2p-(2d+1)\\ &= 2p-d-1.\end{aligned}$$

We have seen that the formula is true in every case. Thus the proof of (a) is complete.

Assertion (b) is a simple consequence of (a) as follows. Note that equation (3.1.4) is equivalent to

$$\sum_{i=1}^{3} a_i(\mathcal{C}) = (p-1) + 2\sum_{i=1}^{3}(d_i-p) - 2(d-p).$$

Thus $\sum_{i=1}^{3} a_i(\mathcal{C}) \leq p-1$ if and only if $2\sum_{i=1}^{3}(d_i-p) \leq 2(d-p)$ or equivalently $\sum_{i=1}^{3}(d_i-p) \leq d-p$.

We now prove (c). Note that since $a_i = |e_i - p|$,

$$\sum_{i=1}^{3} a_i \equiv \sum_{i=1}^{3} e_i - 3p \equiv 2d+1-3p \equiv 1-p \equiv 0 \mod 2, \tag{3.1.5}$$

since we have assumed $p > 2$. In particular, $\sum_{i=1}^{3} a_i$ is even. Thus the assertion follows from Proposition 3.1.8 (b). □

Corollary 3.1.10 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $e_1 \leq e_2 \leq e_3 \leq d < 2p$ and $e_3 < 2p-1$ and $\sum_{i=1}^{3}(d_i-p) \leq d-p$. Let $a_i(\mathcal{C})$ be defined as in Proposition 3.1.9. Then the hypergeometric deformation datum (g, ω) with signature $(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$ satisfies*

$$|\mathbb{B}_{\text{new}}| = d-p-\sum_{i=1}^{3}(d_i-p).$$

Proof: By equation (3.1.3) we have

$$|\mathbb{B}_{\text{new}}| = (p - 1 - a_1(\omega) - a_2(\omega) - a_3(\omega))/2.$$

By Proposition 3.1.9 (b) this number is equal to $d - p - \sum_{i=1}^{3}(d_i - p)$. □

Remark 3.1.11 Recall that if a primitive tail cover $\bar{g} : \bar{Y} \to \bar{X}$ is given by the Galois closure of

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^e + \bar{z}^p & , \text{ if } e < p \\ \frac{\bar{z}^e}{\bar{z}^{e-p}-1} & , \text{ if } e > p, \end{cases}$$

then by Lemma 2.4.1 (a) we have that

$$\sigma(\bar{g}) = \frac{|e - p|}{p - 1}.$$

Hence the σ_i in Proposition 3.1.9 can be thought of as ramification invariants of the primitive tail covers. Moreover, note that $d = \max\{e, p\}$ is the degree of the primitive tail cover. Proposition 3.1.9 thus essentially states that there exists a hypergeometric deformation datum such that the "ramification invariant" in the critical points is equal to the ramification invariant of any inertia group of the primitive tail covers above. Similar to Remark 3.1.7 this is linked with the compatibility conditions in Definition 3.1.4. See Remark 3.2.7 for more details.

We now want to determine the numbers $m_j(\omega)$ and $h_j(\omega)$ from equation (3.1.1).

Remark 3.1.12 Note that in Proposition 3.1.9 (c) the character of the hypergeometric deformation datum (g, ω) is injective!

Proposition 3.1.13 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $\sum_{i=1}^{3}(d_i - p) \leq d - p$ and $e_j < 2p-1$. Moreover, let (g, ω) be a hypergeometric deformation datum with signature $(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$ (see Proposition 3.1.9 for the definition of $a_i(\mathcal{C})$). Then*

$$h_j(\omega) = \frac{a_j(\mathcal{C})}{\gcd(a_j(\mathcal{C}), p-1)}, \quad m_j(\omega) = \frac{p-1}{\gcd(a_j(\mathcal{C}), p-1)}, \quad \text{if } a_j(\mathcal{C}) \neq 0.$$

Proof: The character of (g, ω) is injective and hence by [BW06, Proposition 2.2. (i)], $\gcd(h(\omega), m(\omega)) = 1$. Let $a_j(\mathcal{C}) \neq 0$. The definition of $a_j(\mathcal{C})$ in Proposition 3.1.9 together with the fact that $e_j < 2p - 1$ implies that

$$a_j(\mathcal{C}) = (p - 1)\sigma_j.$$

Note that by (3.1.2)

$$a_j(\mathcal{C}) = (p-1)\sigma_j \equiv (p-1)\sigma_j(\omega) = (p-1)\frac{h_j(\omega)}{m_j(\omega)} \mod (p-1).$$

Recall that $\mathbb{B}_0 = \mathbb{B}_{\text{prim}} \cup \mathbb{B}_{\text{wild}}$. Since $\sigma_j(\omega) < 1$ for all $j \in \mathbb{B}_0$, it follows that

$$a_j(\mathcal{C}) = (p-1)\frac{h_j(\omega)}{m_j(\omega)}.$$

This is equivalent to

$$\frac{a_j(\mathcal{C})}{(p-1)} = \frac{h_j(\omega)}{m_j(\omega)}.$$

Since $\gcd(h_j(\omega), m_j(\omega)) = 1$, it follows that

$$h_j(\omega) = \frac{a_j(\mathcal{C})}{\gcd(a_j(\mathcal{C}), p-1)}, \quad m_j(\omega) = \frac{p-1}{\gcd(a_j(\mathcal{C}), p-1)}.$$

□

3.2 Structure of stable reduction of three-point covers with mild reduction

Throughout this section we use the notations and assumptions of Convention 1.2.5. Additionally, let R be a complete discrete valuation ring with quotient field K and residue field k of characteristic $p > 0$. We assume that k is algebraically closed.

The goal of this section is to recall the structure theorem of the stable reduction of three-point covers which is due to Wewers ([Wew03, Theorem 2.6]). For this we recall the definition of mild reduction and introduce tails. This will provide the link between the stable reduction of three-point covers and tail covers.

This whole section is based upon [Wew03].

We follow the definition in [Wew03, Definition 1.2] word for word with a slight change of notation.

Definition 3.2.1 Let $f : Y \to X = \mathbb{P}^1$ be a G-Galois three-point cover and $\bar{f} : \bar{Y} \to \bar{X} = \mathbb{P}^1_k$ its stable reduction. For a point $\eta \in \bar{Y}$ (closed or generic) we denote by G_η the decomposition group of η and by I_η the inertia group. We say that f has *mild reduction* if it has bad reduction and if for every $\eta \in \bar{Y}$

(i) The p-part of I_η has at most order p.

(ii) If $I_\eta \neq \{1\}$, then $p \nmid |G_\eta / I_\eta|$.

Remark 3.2.2 Note that if $d < 2p$ then $p^2 \nmid |S_d|$. In particular, if $\mathcal{C} = (d; e_1, e_2, e_3)$ denotes a ramification type with $d < 2p$ and $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ denotes the unique genus-0 Belyi morphism of type $\mathcal{C}$ (Lemma 1.2.6), then the Galois closure $f : Y \to X$ of φ (Definition 1.2.1 (c)) has mild reduction if it has bad reduction.

The following is taken from [Wew03, 1.1.2] with slightly different notation.

Let $f_K : Y_K \to X_K = \mathbb{P}^1_K$ be a Galois three-point cover with bad reduction. By Theorem 1.6.3 there exists a finite extension $\tilde{K}/K$ (with ring of integers $\tilde{R}$) such that there exists a stable model $f_{\tilde{R}} : \mathcal{Y} \to \mathcal{X}$ over $\tilde{R}$. By [Knu83] there exists a canonical contraction morphism $q_{\tilde{R}} : \mathcal{X} \to X \otimes_R \tilde{R}$. Moreover, there exists a unique irreducible component $\bar{X}_0 \subset \bar{X}$ such that the induced map $\bar{q} : \bar{X} \to \bar{X}_0 := X \otimes_R k$ is an isomorphism. We will identify this component of $\bar{X}$ with $\bar{X}_0$.

Definition 3.2.3 The component $\bar{X}_0$ of $\bar{X}$ is called the *original component*.

We now proceed with the definition of tails. The following definition can be found in [Wew03, Definition 1.3].

Definition 3.2.4 Let $f : Y \to X = \mathbb{P}^1$ be a Galois three-point cover with mild reduction and let $\bar{f} : \bar{Y} \to \bar{X}$ be its stable reduction.

(i) An irreducible component $\bar{X}_v$ of $\bar{X}$ which is not the original component and which is connected to the rest of $\bar{X}$ in a single point is called a *tail*. The point where $\bar{X}_v$ meets the rest of $\bar{X}$ is denoted by $\bar{\infty}_v$.

(ii) If a tail $\bar{X}_v$ contains one of the points $\bar{0}, \bar{1}, \bar{\infty} \in \bar{X}$, i.e. the specialization of the branch points of f, then $\bar{X}_v$ is called *primitive*.

(iii) If a tail is not primitive, it is called *new*.

(iv) An irreducible component $\bar{X}_v$ which is not a tail is called an *interior component*.

The following result establishes the connection between tails and tail covers. We cite the result in the form found in [Wew03, Lemma 1.4]. This result is essentially due to Raynaud (see [Ray99]).

Lemma 3.2.5 (Raynaud) *Suppose that $f : Y \to X = \mathbb{P}^1$ is any Galois three-point cover with mild reduction. For each component $\bar{X}_v \subset \bar{X}$ we choose an irreducible component $\bar{Y}_v \subset \bar{Y}$ above $\bar{X}_v$. Let G_v (resp. I_v) denote the decomposition (resp. inertia) group of $\bar{Y}_v$. Then the following holds.*

(a) *If $\bar{X}_v$ is a tail, then the inertia group I_v is trivial. In other words, the map $\bar{Y}_v \to \bar{X}_v$ is a Galois cover, with Galois group G_v. If $\bar{X}_v$ is a primitive tail (resp. new tail), then the cover $\bar{g}_v : \bar{Y}_v \to \bar{X}_v$ is a primitive tail cover (resp. new tail cover).*

(b) *If $\bar{X}_v$ is not a tail, then the inertia group I_v is cyclic of order p. In this case the cover $\bar{Y}_v \to \bar{X}_v$ factors through the relative k-linear Frobenius $F : \bar{Y}_v \to \bar{W}_v := \bar{Y}_v^{(p)}$. The resulting map $\bar{W}_v \to \bar{X}_v$ is a Galois cover, with Galois group $H_v := G_v/I_v$. The map $\bar{Z}_v \to \bar{X}_v$ is étale outside the set of points which are either singular points of $\bar{X}$ or belong to the set $\{\bar{0}, \bar{1}, \bar{\infty}\} \subset \bar{X}$.*

(c) *For $x \in \{0, 1, \infty\}$ let e be the ramification index of $f : Y \to X$ above x. Suppose that e is prime to p. Then the specialization $\bar{x}$ of x lies on a primitive tail $\bar{X}_v$ and the ramification index of the primitive tail cover $\bar{Y}_v \to \bar{X}_v$ at $\bar{x}$ is e.*

(d) *With the same notations as in (c) assume that $e = pe'$. Then $\bar{x}$ lies on a component $\bar{X}_v$ which is not a tail. Moreover, the Galois cover $\bar{W}_v \to \bar{X}_v$ has ramification index e' at $\bar{x}$.*

We now state the structure theorem about $\bar{X}$.

Theorem 3.2.6 (Wewers) *Let $f : Y \to X = \mathbb{P}^1_K$ be any Galois three-point cover with mild reduction. Then $\bar{X}$ consists only of the original component $\bar{X}_0$ and tails $\bar{X}_v$.*

Proof: See [Wew03, Theorem 2.6]. □

Theorem 3.2.6 says that $\bar{X}$ has the structure of a comb (see Figure 3.2.1). Moreover, we see that tail covers are an essential part of the stable reduction of three-point covers.

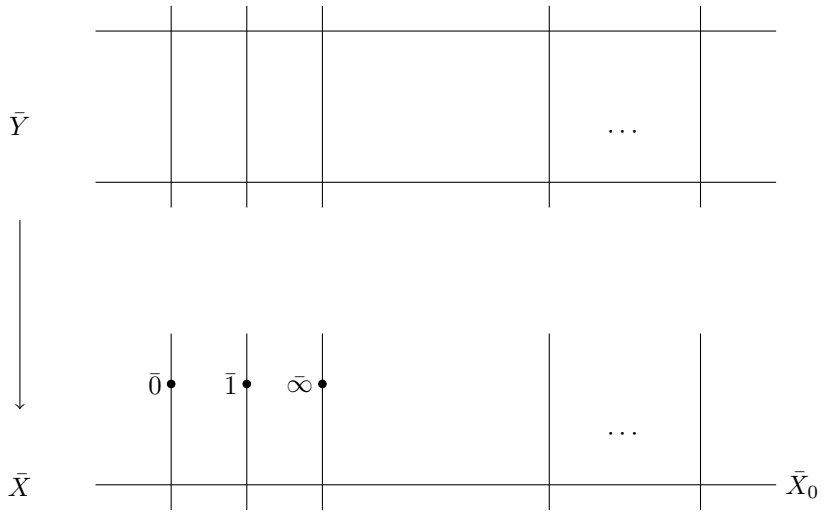

Figure 3.2.1: Stable reduction of a three-point cover with mild reduction.

Note that Theorem 3.2.6 does not say which tail covers occur, how many there are and where the intersection points of the tails with the original component are.

Remark 3.2.7 We now explain the reason for the compatibility conditions in the definition of special G-deformation data (Definition 3.1.4). By Theorem 3.2.6 $\bar{X}$ is a comb. Every tail intersects the original component in one single point. At these intersection points $\bar{f}$ is ramified (Lemma 1.6.2 (c)).

One the one hand one can regard the intersection points as points on $\bar{X}_0$. The ramification information of these points is encoded in the deformation datum.

On the other hand one can regard these intersection points as points on the corresponding tail $\bar{X}_j$. The ramification information at an intersection point is now encoded in the inertia group of a point $\bar{y} \in \bar{Y}$ which lies above the intersection point.

Since $\bar{f}$ is well defined, both descriptions of the ramification information must coincide. This leads to the compatibility conditions.

We now demonstrate in an example that the tail covers we defined in Lemma 2.3.1 really occur in the stable reduction. This is also a demonstration of the structure Theorem 3.2.6.

Example 3.2.8 Let $\mathcal{C} = (7; 4, 5, 6)$ and $p = 5$. Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the unique genus-0 Belyi morphism with ramification type $\mathcal{C}$ (Lemma 1.2.6) and $f : Y \to X$ the Galois closure. Recall that by Example 1.2.7 φ is given by

$$(x, z) \mapsto x = \varphi(z) = \frac{z^4\left(-10z^3 + 42z^2 - 63z + 35\right)}{7z - 3} \tag{3.2.1}$$

and by Example 1.6.6 f has bad reduction to characteristic 5.

What do we expect? What should the stable reduction of f look like?

Combining Lemma 3.2.5 and Theorem 3.2.6 we see that there can only be primitive tail covers for $e_1 = 4$ and $e_3 = 6$, because $e_2 = 5$ is divisible by $p = 5$. Moreover, by Theorem 3.2.6 there are possibly several new tail covers as well. In Chapter 4 in Theorem 4.2.2, we prove that the deformation datum of a three-point cover with $p \leq d < 2p$ is hypergeometric (except if $e_j = 2p-1$, which we discuss separately) and is given by the signature of Proposition 3.1.9. Although we have not proven this fact yet, we use it.

Define $a_i(\mathcal{C})$ as in Proposition 3.1.9 that is: $a_1(\mathcal{C}) = 1, a_2(\mathcal{C}) = 0, a_3(\mathcal{C}) = 1$. By Proposition 3.1.9 there exists a hypergeometric deformation datum with signature $(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$. Moreover, by Corollary 3.1.10 we see that

$$|\mathbb{B}_{\text{new}}| = 7 - 5 - (6 - 5) - (5 - 5) - (5 - 5) = 1.$$

Thus we expect to see 2 primitive tail covers corresponding to $x = 0, x = \infty$ and 1 new tail cover.

We now compute the tail covers. We refer to Section 5.2 for more details on the strategy of the computation. We begin with the primitive tail covers.

Tail for $x = 0, e_1 = 4$

Normalizing the coordinates x and z by $z = 5\tilde{z}$ and $x = 5^5\tilde{x}$ we get

$$\tilde{x} = \frac{\tilde{z}^4\left(-10 \cdot 5^2 \cdot \tilde{z}^3 + 42 \cdot 5 \cdot \tilde{z}^2 - 63\tilde{z} + 7\right)}{7 \cdot 5\tilde{z} - 3} \equiv \frac{\tilde{z}^4\left(-3\tilde{z} + 2\right)}{-3} \equiv \tilde{z}^4 + \tilde{z}^5 \mod 5.$$

We see that we obtain a primitive tail cover of Lemma 2.3.1 (i).

Tail for $x = \infty, e_3 = 6$

To compute the tail cover in ∞ we change coordinates on X and Z such that ∞ lies in 0. Replacing x with $1/x$ and z with $1/z$ we get

$$\tilde{\varphi}(z) = \bar{x} = \frac{z^6(-3z + 7)}{35z^3 + 63z^2 + 42z - 10}.$$

Note that $\tilde{\varphi}$ is the genus-0 Belyi morphism of the type $\tilde{\mathcal{C}} = (7; 6, 5, 4)$.

Substituting $x = 5^5\tilde{x}$ and $z = 5\tilde{z}$ we compute

$$\tilde{x} = \frac{\tilde{z}^6\left(-3 \cdot 5\tilde{z} + 7\right)}{35 \cdot 5^2\tilde{z}^3 - 63 \cdot 5\tilde{z}^2 + 42\tilde{z} - 2} \equiv \frac{2\tilde{z}^6}{2\tilde{z} - 2} \equiv \frac{\tilde{z}^6}{\tilde{z} - 1} \mod 5.$$

We obtain the other primitive tail cover of Lemma 2.3.1 (i) with $a = 1$. In Chapter 4 we show that the primitive tail covers of Lemma 2.3.1 (for $a = 1$) are the only primitive tail covers that can occur if $d < 2p$. This however is not trivial, as there are potentially many different primitive tail covers of degree $> p$.

New tail cover

We already computed that there exists only one new tail cover (under the assumption that the underlying deformation datum is hypergeometric). Moreover, the uniqueness of the three-point covers in the single-cycle case forces that the new tail cover has a well-defined position. But where is the new tail cover?

The position of the new tails is determined by the deformation datum, which is described in Proposition 3.1.8 (a). We use

$$u = x - c \quad , A = (1 + 1 + 0 + 1)/2 = 3/2 \quad , B = (1 + 1 + 0 - 1)/2 = 1/2 \quad , C = 1 + 1 = 2.$$

Then the differential equation is

$$x(x-1)u'' + [(A+B+1)x - C]u' + ABu = (3x-2) + \frac{3}{4}(x-c) = 0,$$

Note that this computation is done modulo 5 and hence

$$0 = (3x-2) + \frac{3}{4}(x-c) = 3x - 2 - 3x + 3c$$

or equivalently $c = -1$. In particular, the new tail cover lies above $x = -1$. Hence we change coordinates $x = \tilde{x} - 1$. Further, we substitute $z = \mu\tilde{z} + \gamma$ where μ and γ are yet unknown.

After the above substitution the denominator of $\varphi(\mu\tilde{z} + \gamma) + 1$, with φ as in (3.2.1), is given by

$$7\mu\tilde{z} + (7\gamma - 3).$$

We define $\gamma = -1$. This implies that $7\gamma - 3 \equiv 0 \mod 5$ vanishes.

We then compute that

$$\tilde{x} = \frac{-10\mu^7\tilde{z}^7 + 112\mu^6\tilde{z}^6 - 525\mu^5\tilde{z}^5 + 1330\mu^4\tilde{z}^4 - 1960\mu^3\tilde{z}^3 + 1680\mu^2\tilde{z}^2 - 770\mu\tilde{z} + 140}{7\mu\tilde{z} - 10}.$$

In Table 3.2.1 we have listed the valuation of the coefficients, where we let ε denote the valuation $v(\mu)$ (where v is normalized such that $v(p) = v(5) = 1$).

	z^7	z^6	z^5	z^4	z^3	z^2	z^1	z^0
numerator	$1+7\varepsilon$	6ε	$2+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1
denominator							ε	1

Table 3.2.1: Valuation of the coefficients of $\varphi(\mu z - 1) + 1$.

Recall that by Proposition 2.2.1 (d) a new tail cover with $\sigma = \frac{p+1}{p-1}$ is given by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^{p+1}}{\bar{z}}. \tag{3.2.2}$$

Since we want to find a new tail cover as in (3.2.2), Table 3.2.1 shows that if we choose $6\varepsilon = 1$ or equivalently $\varepsilon = 1/6$, then the coefficients of z^6 and z^0 in the numerator have the same valuation and all the other coefficients have strictly larger valuation. Moreover, the constant term in the denominator also has strictly larger valuation. We set $z = \sqrt[6]{5}\tilde{z} - 1$. Note that since $0 < \varepsilon < 6\varepsilon = 1$ allows us to cancel out μ in numerator and denominator. Again substituting $\tilde{x} = \left(\sqrt[6]{5}\right)^5 \tilde{x}_2$ and reducing modulo $\sqrt[6]{5}$ we get

$$\tilde{x}_2 \equiv \frac{\tilde{z}^6 - 1}{\tilde{z}} \mod \sqrt[6]{5}.$$

Again substituting $\tilde{x}_2 = -\hat{x}_3$ and $\tilde{z} = -\hat{z}$ we finally get

$$\hat{x} \equiv \frac{1 + \hat{z}^6}{\hat{z}} \mod \sqrt[6]{5}.$$

Summarizing the whole substitution, we see that

$$x = -\left(\sqrt[6]{5}\right)^5 \hat{x} - 1, \quad z = -\sqrt[6]{5}\hat{z} - 1.$$

We see that this is the new tail cover with $\tilde{d} = p + 1 = 6$ from Proposition 2.2.1. Note that the computation shows that we have to enlarge R.

We now recall the so-called vanishing cycle formula. This formula holds regardless which deformation datum occurs in the stable reduction. This is particularly useful in the proof of Theorem 4.2.2 (c). The following version of the vanishing cycle formula is a citation of [Wew03, Corollary 1.11] with some minor but obvious changes in the notation. We use the notation of Definition 3.1.1 for the deformation datum corresponding to the stable reduction $\bar{f}$ (as in Remark 3.1.5).

Theorem 3.2.9 *Let $f : Y \to X = \mathbb{P}^1$ be a three-point cover with mild reduction and $\bar{f} : \bar{Y} \to \bar{X}$ its stable reduction. Let $\bar{g}_j : \bar{Y}_j \to \bar{X}_j = \mathbb{P}^1_k$ be the tail covers of $\bar{f}$. Then the following holds.*

$$\sum_{j \in \mathbb{B}_{\text{prim}}} \sigma_j + \sum_{j \in \mathbb{B}_{\text{new}}} (\sigma_j - 1) = 1.$$

Remark 3.2.10 Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $d < 2p$ and $e_j < 2p - 1$. Suppose that

$$\sigma_j = \frac{|e_j - p|}{p - 1}, \quad \text{for } j \in \mathbb{B}_{\text{prim}}.$$

If $e_j = p$, define $\sigma_j = 0 = \frac{|e_j - p|}{p-1}$. Suppose that

$$\sigma_j = \frac{p + 1}{p - 1}, \quad \text{for } j \in \mathbb{B}_{\text{new}}$$

and define $a_j(\mathcal{C}) = (p - 1)\sigma_j$ for $j \in \mathbb{B}_{\text{prim}}$ as in Proposition 3.1.9. Then

$$1 = \sum_{j \in \mathbb{B}_{\text{prim}}} \sigma_j + \sum_{j \in \mathbb{B}_{\text{new}}} (\sigma_j - 1)$$

if and only if

$$(p - 1) = \sum_{j \in \mathbb{B}_{\text{prim}}} a_j(\mathcal{C}) + 2|\mathbb{B}_{\text{new}}|.$$

Solving for $|\mathbb{B}_{\text{new}}|$ we get

$$|\mathbb{B}_{\text{new}}| = (p - 1 - a_1(\mathcal{C}) - a_2(\mathcal{C}) - a_3(\mathcal{C}))/2.$$

This links Proposition 3.1.9 with the vanishing-cycle formula.

3.3 Lifting result

In this section we recall the lifting result from Wewers (Theorem 3.3.4). In some sense, this result is a converse result to Theorem 3.2.6 and is the result on which our characterization of the stable reduction in the case $d < 2p$ in Chapter 4 is based.

We use the notation and assumptions of Convention 1.2.5. Moreover, K is a complete discretely valued field with characteristic 0 and k its residue field with characteristic $p > 0$. We assume k to be algebraically closed.

All the following definitions can be found in [Wew03, Definition 2.14, Definition 2.15].

Definition 3.3.1 (i) A *stable G-map* over a scheme S is a homomorphism $f : Y \to X$ of marked semistable curves over S together with an S-linear action of G on Y such that Y is stably marked (Definition 1.6.4) and f commutes with the G-action.

(ii) Two stable G-maps $\bar{f}_i : \bar{Y}_i \to \bar{X}_i$ are said to be isomorphic if there exist isomorphisms $\varphi : Y_1 \xrightarrow{\sim} Y_2$ and $\psi : X_1 \to X_2$ of marked curves such that $f_2 \circ \varphi = \psi \circ f_1$ and such that φ is G-equivariant.

We now recall the construction of a stable G-map via a special G-deformation datum (Definition 3.1.4). This construction can be found in [Wew03] after Definition 2.14. Because the description of the construction written in [Wew03] is very comprehensive and short, we cite this construction word for word with minor changes in the notation to better suit the notation in this thesis. We refer to Section 1.5 for a description of the induced curve.

Construction 3.3.2 Let $(g_0, \omega_0; \bar{g}_j)$ be a special G-deformation datum. Define

$$\bar{X} := (\bar{X}_0 \amalg \coprod_j \bar{X}_j)/\sim .$$

Here the equivalence relation $\sim$ identifies the critical point $\tau_j \in \bar{X}_0$ with the distinguished point $\bar{\infty}_j \in \bar{X}_j$. The topological space $\bar{X}$ carries a unique structure of a semistable k-curve such that $\bar{X}_0$ and $\bar{X}_j$ are the

irreducible components of $\bar{X}$. We define marked points $\bar{x}_j$ on $\bar{X}$ (indexed by the set $\mathbb{B}_0 = \mathbb{B}_{\text{prim}} \cup \mathbb{B}_{\text{wild}}$), as follows. For $j \in \mathbb{B}_{\text{prim}}$ we let $\bar{x}_j \in \bar{X}_j$ be the unique tame branch point of the tail cover $\bar{g}_j$. For $j \in \mathbb{B}_{\text{wild}}$ we set $\bar{x}_j := \tau_j$.

To define the curve $\bar{Y}$, let $\bar{Y}_0$ be the smooth projective curve with function field $k(\bar{W}_k)^{1/p}$ (recall that $\bar{W}_k$ is the curve defined by $g_0 : \bar{W}_k \to \bar{X}_0$). The action of H_0 on $\bar{W}_k$ extends uniquely to an action on $\bar{Y}_0$. We let G_0 act on $\bar{Y}_0$ via the quotient map $G_0 \to H_0$. Denote the induced map $\bar{Y}_0 \to \bar{X}_0$ by $\bar{f}_0$. Furthermore, we define

$$\bar{Y} := (\text{Ind}_{G_0}^G(\bar{Y}_0) \amalg \coprod_j \text{Ind}_{G_j}^G(\bar{Y}_j))/\sim .$$

The equivalence relation $\sim$ is defined as follows: for $\eta' \in \text{Ind}_{G_0}^G(\bar{Y}_0)$ and $\eta'' \in \text{Ind}_{G_j}^G(\bar{Y}_j)$ we have $\eta' \sim \eta''$ if and only if there exists an element $\alpha \in G$ such that $\alpha(\eta') \in \bar{Y}_0$ lies above $\xi_j \in \bar{W}_k$ and $\alpha(\eta'') = \eta_j \in \bar{Y}_j$. Again, $\bar{Y}$ carries a canonical structure of a semistable curve over k. It follows from condition (i) of Definition 3.1.4 that $\bar{Y}$ is connected. Let $\bar{f} : \bar{Y} \to \bar{X}$ be the unique map which commutes with the natural G-action on $\bar{Y}$ and whose restriction to $\bar{Y}_0$ (resp. to $\bar{Y}_j$) is equal to $\bar{f}_0$ (resp. to $\bar{g}_j$). The marked points of $\bar{Y}$ are, by definition, those which lie over a marked point of $\bar{X}$. Thus, a point on $\bar{Y}$ is marked either if it is the G-translate of a ramification point of a tail cover $\bar{g}_j$ or if it is a G-translate of a point on $\bar{Y}_0$ which lies above a wild critical point. One checks that $\bar{f} : \bar{Y} \to \bar{X}$ is a stable G-map. We call $\bar{X}_0 = \mathbb{P}^1_k$ the original component of $\bar{f}$.

Finally we are able to define special G-maps. The following definition can be found in [Wew03, Definition 2.15].

Definition 3.3.3 A stable G-map $\bar{f} : \bar{Y} \to \bar{X}$ over k is called *special* if there exists a special G-deformation datum $(g_0, \omega_0; \bar{g}_j)$ such that $\bar{f}$ is isomorphic to the stable G-map constructed in Construction 3.3.2.

We can now state the lifting result of Wewers. This result can be found in [Wew03, Corollary 4.6].

Let K_0 denote the fraction field of $W(k)$ and $\bar{K}$ the algebraic closure of K_0.

Theorem 3.3.4 (Wewers) *Let $\bar{f} : \bar{Y} \to \bar{X}$ be a special G-map defined over k. Then there exists a three-point G-cover $f : Y \to X := \mathbb{P}_{\bar{K}}$ whose stable reduction is isomorphic to $\bar{f}$.*

Remark 3.3.5 Theorem 3.3.4 shows that if we find any special G-deformation datum which gives rise to three-point cover with ramification type $\mathcal{C} = (d; e_1, e_2, e_3)$, then we know the stable reduction.

The following example demonstrates an application of Theorem 3.3.4. It basically involves all ideas for the proof of Theorem 4.2.2 where the stable reduction of three-point covers with $d < 2p$ is described. However, it might seem that many ideas in the proof are made up out from thin air. The following example demonstrates that this is not the case and motivates all the ideas of the proof of Theorem 4.2.2.

Example 3.3.6 Let $\mathcal{C} = (7; 4, 5, 6)$ and $p = 5$. Let $f : Y \to X = \mathbb{P}^1$ be the unique Galois three-point cover with ramification type $\mathcal{C}$. We denote the Galois group of f by G. In Example 3.2.8 we assumed that the underlying deformation datum is hypergeometric. We also computed all the tail covers of the stable reduction, which are given as the Galois closure of the maps defined by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^4 + \bar{z}^5 & \text{, for } e = 4 \\ \frac{\bar{z}^6}{\bar{z}-1} & \text{, for } e = 6 \\ \frac{1+\bar{z}^6}{\bar{z}} & \text{, the new tail cover.} \end{cases} \tag{3.3.1}$$

We now show that this is all the information needed to describe the stable reduction of f! To do so we want to use Theorem 3.3.4 and Remark 3.3.5. Therefore, we pretend that we don't know that these tail covers actually occur in the stable reduction. We simply use these tail covers and the hypergeometric deformation datum and prove that we can regard this whole datum as a special G-deformation datum. Note that by Remark 3.3.5 we know that if this special G-deformation datum gives rise to a three-point cover with ramification type $\mathcal{C} = (7; 4, 5, 6)$, then we already know that it is the unique special G-deformation datum of f. In particular, the stable reduction of f is known.

Let (g_0, ω_0) be the hypergeometric deformation datum with signature $(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C})) = (1, 0, 1)$ (see Proposition 3.1.9 for the definition of the $a_i(\mathcal{C})$). Define for $i = 1, 3, 4$ (note the indices!) the tail covers $\bar{g}_j : \bar{Y}_j \to \bar{X}_j = \mathbb{P}^1_k$ as the Galois closure of $\bar{\varphi}_j : \bar{Z}_j = \mathbb{P}^1_k \to \bar{X}_j = \mathbb{P}^1_k$, with

$$\begin{aligned}(\bar{x}_1, \bar{z}_1) &\mapsto \bar{x}_1 = \bar{z}_1^4 + \bar{z}_1^5,\\ (\bar{x}_3, \bar{z}_3) &\mapsto \bar{x}_3 = \frac{\bar{z}_3^6}{\bar{z}_3 - 1},\\ (\bar{x}_4, \bar{z}_4) &\mapsto \bar{x}_4 = \frac{1 + \bar{z}_4^6}{\bar{z}_4}.\end{aligned}$$

For convenience we denote this whole datum by $(g_0, \omega_0; \bar{g}_j)$. Moreover, we denote the degree of $\bar{\varphi}_j$ by d_j, i.e. $d_1 = 5, d_3 = 6, d_4 = 6$.

We now discuss how it is possible to regard $(g_0, \omega_0; \bar{g}_j)$ as special G-deformation datum. To do this we need to realize what actually needs to be done!

Definition 3.1.4 (i) tells us that the group G must be generated by G_0 and G_j where the G_j are the Galois groups of the tail covers and G_0 is the automorphism group of the original component. Here we encounter the first problem. We don't know the groups G_j.

However, what we do know is that $G \subset S_7$ and all $G_j \subset S_{d_j}$ are transitive groups on d (resp. d_j) numbers (Remark 1.1.7). This shows that for G to be generated by the groups G_j (and G_0) we have to embed the G_j (and G_0) in S_7 in an appropriate way! But since we don't know the G_j, we cannot be certain that $G = \langle G_j \mid j \in \mathbb{B}_{\text{new}} \cup \mathbb{B}_{\text{prim}} \cup \{0\}\rangle$, even if we embedded G_j appropriately (whatever that means). This is another problem. We will discuss this problem later during this example.

Definition 3.1.4 (ii) now tells us that the deformation datum and the tail covers must satisfy strong compatibility conditions. In Remark 3.2.7 we explained the reason for the compatibility conditions. Most of these are automatically satisfied if we embed G_j in S_7 appropriately.

We see that for both conditions of Definition 3.1.4 we have to embed the groups G_0 and G_j in G appropriately. The obvious question now is: "what is an appropriate embedding of G_j in S_7?"

Part of the idea for the embedding can be extracted from the definition of special G-deformation data itself. Note that Definition 3.1.4 (ii) requires that for every tail cover $\bar{g}_j$ there exists a point $\bar{y}_j \in \bar{Y}_j$ with inertia group

$$I_{\bar{y}_j} = I_0 \rtimes H_j, \tag{3.3.2}$$

where I_0 is a group of order p and is fixed throughout all the tail covers. This shows that every G_j has to permute at least $\operatorname{supp}(I_0)$ (see Remark 1.1.7). In particular, if $d_j = p$, then $G_j \subset S(\operatorname{supp}(I_0))$.

But what if $d_j > p$? What is with the other $d_j - p$ leaves?

Note that $|\operatorname{supp}(I_0)| = 5$ and $d - p = 7 - 5 = 2 = (d_3 - p) + (d_4 - p)$. In particular, we can number the $d_j - p$ branches of the non-Galois cover disjointly (additionally to $\operatorname{supp}(I_0)$).

Now that the idea is clear, we define G_0 and prove that we can regard the tail covers as pointed G_j-tail which satisfy (3.3.2). Then we embed the G_j just as described above.

We define

$$I_0 := \langle (1, \ldots, 5)\rangle.$$

Note that in the construction of the hypergeometric deformation datum we have $m = p - 1 = 4$ (Proposition 3.1.8 with $(a_1, a_2, a_3) = (1, 0, 1)$). We let $H_0 = \mathbb{Z}/4\mathbb{Z}$ normalize I_0 in S_p, i.e. we define

$$G_0 = I_0 \rtimes H_0 \subset N_{S_p}(I_0) \subset S_p = S(\{1, \ldots, p\}).$$

Note that G_0 is uniquely defined by the order of H_0 and by I_0.

For every tail cover $\bar{g}_j : \bar{Y}_j \to \bar{X}_j = \mathbb{P}^1_k$ let $\bar{y}_j \in \bar{Y}_j$ be a wild ramification point. We number the leaves of $\bar{\varphi}_j$ such that the Sylow p-subgroup of $I_{\bar{y}_j}$ is equal to I_0. The prime-to-p part of $I_{\bar{y}_j}$ (denoted by H_j) normalizes I_0. Moreover, by Theorem 2.3.11 (c) we have $I_{\bar{y}_j} \subset S_p$. Hence we obtain

$$I_{\bar{y}_j} = I_0 \rtimes H_j \subset N_{S_p}(I_0) \subset S_p. \tag{3.3.3}$$

In particular, we can regard $\bar{g}_j$ as pointed G_j-tail cover which satisfies (3.3.2). Note that $I_{\bar{y}_j} \subset S_p$ is uniquely determined by its order and I_0.

We now embed the G_j in $S_7 = S(\{1, \dots, 7\})$ such that they are transitive on d_j numbers with $I_{\bar{y}_j} \subset G_j$ as in equation (3.3.3), and number the remaining $d_j - p$ leaves such that

$$\begin{aligned} G_1 &\subset S(\{1, \dots, 5\}) \\ G_3 &\subset S(\{1, \dots, 5, 6\}) \\ G_4 &\subset S(\{1, \dots, 5, 7\}). \end{aligned} \tag{3.3.4}$$

Define $\tilde{G} := \langle G_0, G_1, G_2, G_4 \rangle \subset S_7$. During this example we will see that $\tilde{G} = G$ which is absolutely nontrivial. Note that since all G_j are transitive on the sets from (3.3.4), $\tilde{G}$ is transitive on $\{1, \dots, 7\}$.

We claim that $(g_0, \omega_0; \bar{g}_j)$ with the embeddings from above forms a special $\tilde{G}$-deformation datum.

Definition 3.1.4 (i) is clear from the definition of $\tilde{G}$. We now prove that part (ii) of Definition 3.1.4 is also satisfied.

Recall that the ramification type of the tail covers is computed in Proposition 2.5.2. The tuples $(m(\omega), g(\omega))$ of the hypergeometric deformation datum are described in Proposition 3.1.13. One easily checks that

$$(m_i(\omega), h_i(\omega)) = (m_i(\bar{g}_i), h_i(\bar{g}_i)) \text{ for all } i.$$

By the choice of the embedding of the groups G_j in S_7 we know that there exists points $\bar{y}_j \in \bar{Y}_j$ with

$$I_{\bar{y}_j} = I_0 \rtimes H_j.$$

We know that H_j is uniquely determined by the order $m_j(\bar{g}_j)$ and the choice of I_0.

We compute that $m_1(\bar{g}_1) = 4, m_3(\bar{g}_3) = 4, m_4(\bar{g}_4) = 2$ and realize that all $m_j(\bar{g})$ divide $m = 4$. Hence

$$H_j = H_0 \cap I_{\bar{y}_j}.$$

This proves that the tail covers and the hypergeometric deformation datum really form a special $\tilde{G}$-deformation datum, which we call $(g, \omega; \bar{g}_j)$.

How can we describe the stable reduction? Why is $\tilde{G} = G$? Since we will use the same idea in Section 4.2 in the general setup, we state it in an extra remark, for better reference.

Remark 3.3.7 Let $\bar{\tilde{f}} : \bar{\tilde{Y}} \to \bar{X}$ be the special $\tilde{G}$-map defined by Construction 3.3.2 corresponding to $(g, \omega; \bar{g}_j)$. By Theorem 3.3.4 there exists a lift of $\bar{\tilde{f}}$ which we denote by $\tilde{f} : \tilde{Y} \to X = \mathbb{P}^1_K$. However, Theorem 3.3.4 does not say that $\tilde{f}$ has anything to do with f or $\mathcal{C}$ at all! It is by no means obvious that f and $\tilde{f}$ are isomorphic! The following however is clear. If $\bar{\tilde{f}}$ is isomorphic to $\bar{f}$, then the lift $\tilde{f}$ is the Galois closure of a genus-0 Belyi morphism. We make this precise as follows.

Let H_{i_0} be the stabilizer of i_0 in $\tilde{G}$, i.e. $H_{i_0} = \tilde{G} \cap S(\{1, \dots, i_0 - 1, i_0 + 1, \dots, d\})$. Since $\tilde{G}$ is transitive on $\{1, \dots, d = 7\}$ by construction, it follows that $[\tilde{G} : H_{i_0}] = d = 7$. If $\bar{\tilde{f}} \simeq \bar{f}$, then the genus of $\bar{\tilde{Y}}/H_{i_0}$ is 0.

Conversely: If $\bar{\tilde{Y}}/H_{i_0}$ has genus 0, then $\tilde{Y}/H_{i_0} \to X$ is a genus-0 Belyi morphism of degree $d = 7$. Moreover, since we know $\bar{\tilde{f}}$ (and in particular the tail covers (we have chosen them!)), we know that $\tilde{Y}/H_{i_0} \to X$ has at least one ramification point with ramification index $e_1 = 4$ and one with ramification point with ramification index $e_3 = 6$. Since there is no tail cover for $e_2 = 5$, we know by Lemma 3.2.5 that there exists at least one point with ramification index $p = 5$ (this can also be seen from ω). By Riemann-Hurwitz it follows that $\tilde{Y}/H_{i_0} \to X$ has exactly these three ramification points and no more. In particular, $\tilde{Y}/H_{i_0} \to X$ is a genus-0 Belyi morphism with ramification type $\mathcal{C} = (7; 4, 5, 6)$. Since genus-0 Belyi morphisms with in the single-cycle case are unique up to normalization (Lemma 1.2.6), we see that the corresponding genus-0 Belyi-morphisms of f and $\tilde{f}$ are the same up to normalization. Since $\tilde{f}$ and f are the Galois closures of the genus-0 Belyi-morphisms it follows that $\tilde{f} \simeq f$ and $G \simeq \tilde{G}$. Moreover, $\bar{\tilde{f}} \simeq \bar{f}$.

We see that all we have to do to prove that we have found the unique special G-deformation datum of f, is to verify that $\bar{Z} := \bar{\tilde{Y}}/H_{i_0}$ has genus 0. Here we use Proposition 1.5.4. By [Zha92, Proposition 2.1], the

genus of $\bar{Z}$ is given by the formula

$$g(\bar{Z}) = 1 + \sum_{\bar{W} \subset \bar{Z} \text{ irred.}} g(\bar{W}) - \lambda + \gamma,$$

where λ is the number of irreducible components and γ the number of double points.

From Construction 3.3.2 we know that

$$\tilde{\bar{Y}} := (\mathrm{Ind}_{G_0}^{\tilde{G}}(\bar{Y}_0) \amalg \coprod_j \mathrm{Ind}_{G_j}^{\tilde{G}}(\bar{Y}_j))/\sim$$

and

$$\bar{X} := (\bar{X}_0 \amalg \coprod_j \bar{X}_j)/\sim .$$

Define $\mathbb{B} := \{1, 3, 4\}$ and

$$\mathfrak{Z}_j := \mathrm{Ind}_{G_0}^{\tilde{G}}(\bar{Y}_j).$$

Then obviously

$$\tilde{\bar{f}}^{-1}(\bar{X}_j) = \mathfrak{Z}_j \quad \text{for all } j \in \mathbb{B}.$$

We now analyze the maps

$$\mathfrak{Z}_j/H_{i_0} \to \bar{X}_j \quad \text{for } j \in \mathbb{B} \cup \{0\}.$$

This analysis is done with Proposition 1.5.4.

For $j \in \mathbb{B}$ we use Proposition 1.5.4 with $\ell = d_j$ and $d = 7$. Thus there are $7 - d_j$ components of degree 1 and one component of degree d_j above $\bar{X}_j$ in $\bar{Z}$. Thus there are

$$\lambda_j := 1 + (7 - d_j) = \begin{cases} 3, & \text{for } j = 1 \\ 2, & \text{for } j = 3 \\ 2, & \text{for } j = 4 \end{cases}$$

irreducible components above $\bar{X}_j$ in $\bar{Z}$. Moreover, all irreducible components of degree 1 are isomorphic to $\mathbb{P}^1_k$ and hence have genus 0. The unique irreducible component of degree d_j has also genus 0 because $\bar{Y}_j \to \bar{X}_j$ is a genus-0 tail cover. This shows that all the irreducible components above any tail have genus 0.

Recall that $\bar{\varphi}_j$ is ramified in exactly one point above $\bar{\infty}$. This combined with the fact that singular points are mapped to singular points (see Lemma 1.6.2 (c)) implies that there are

$$\gamma_j = 3 (= 1 + d - p)$$

double points above $\bar{\infty}_j$ in $\bar{Z}$.

For $j = 0$ we use Proposition 1.5.4 with $\ell = p$ and $d = 7$. Thus there is one irreducible component with degree $p = 5$ and $7 - 5 = 2$ irreducible components of degree 1 above $\bar{X}_0$ in $\bar{Z}$. Thus we have that

$$\lambda_0 = 1 + (7 - 5) = 3.$$

If $\bar{W} \subset \mathfrak{Z}_0/H_{i_0}$ is an irreducible component of degree 1 above $\bar{X}_0$, then $\bar{W} \simeq \mathbb{P}^1_k$. In particular, $\bar{W}$ has genus 0 and the map $\bar{W} \to \bar{X}_0$ is separable. That the irreducible component with degree p above $\bar{X}_0$ has genus 0 can be seen as follows. Since the idea is the same in the more general setup, we state the reasoning in an extra remark, for better reference.

Remark 3.3.8 Recall that the map $\bar{Y}_0 \to \bar{X}_0$ (Construction 3.3.2) is inseparable and has degree $p \cdot m = 5 \cdot 4 = 20$. The transitivity of $\tilde{G}$ on $\{1, \ldots, d = 7\}$ implies that not all subgroups of $\tilde{G}$ which are conjugate to I_0 (in $\tilde{G}$), are subgroups of H_{i_0}. Therefore, at least one irreducible component $\bar{W}_0$ lies inseparably above $\bar{X}_0$ and the degree of $\bar{W}_0 \to \bar{X}_0$ is $p = 5$. Since the degree of the inseparable map $\bar{W}_0 \to \bar{X}_0$ is $p = 5$, this map is purely inseparable and thus is given by the Frobenius. Since the Frobenius is a bijection on points, it follows that $\bar{W}_0 \simeq \mathbb{P}^1_k$ and has genus 0.

We summarize that all irreducible components above $\bar{X}_0$ have genus 0, i.e. $\sum_{\bar{w}\subset\bar{Z}} g(\bar{Z}) = 0$ and

$$\lambda = \sum \lambda_i = 10$$
$$\gamma = \sum \gamma_i = 9.$$

It follows

$$g(\bar{Z}) = 1 + 0 - 10 + 9 = 0.$$

In Figure 3.3.1 the structure of $\bar{Z}$ is illustrated.

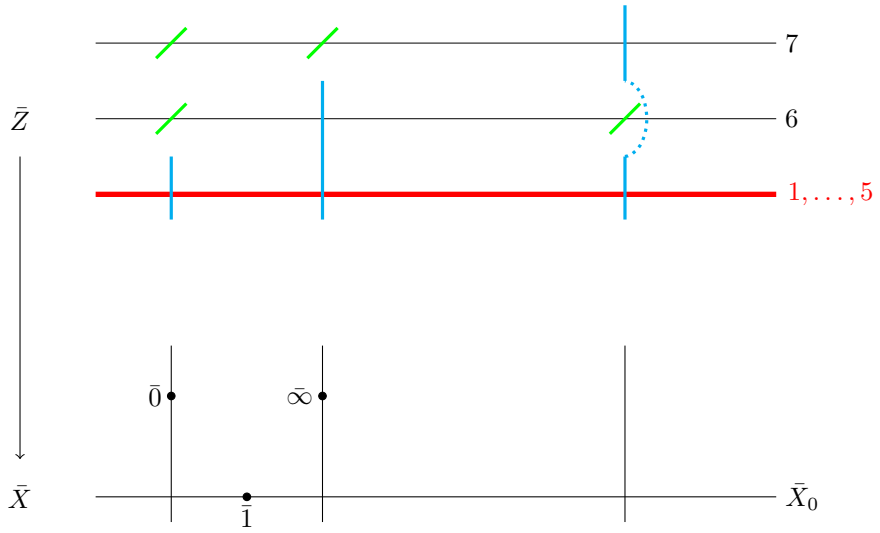

Figure 3.3.1: The map $\bar{Z} \to \bar{X}$ has degree 7. Every horizontal line in $\bar{Z}$ represents a leaf. The thick red line represents the "reserved" $p = 5$ leaves for the wild ramification, whereas the thin black lines represent common leaves. The vertical blue line above each tail is the unique irreducible component in $\bar{Z}$ above the tail with degree d_j. The blue dotted arc is just a visualization of the fact that the irreducible component above the new tail with degree $d_j = 6$ is connected but does not intersect the leaf with the number 6. Hence the groups G_j act transitively on the leaves which are intersected with the blue line. The green lines represent the irreducible components above the tails which have degree 1 above the tail. These are the leaves which are fixed by G_j. Observe that there are 9 double points and 10 irreducible components in $\bar{Z}$. One immediately sees that $g(\bar{Z}) = 0$.

4 The guaranteed mild case $d < 2p$

Throughout this whole chapter we fix the following setup.

Setup 4.0.1 (i) Let R be a complete discrete valuation ring with a discrete valuation v, K the fraction field with $\operatorname{char} K = 0$ and k the residue field of R which we assume to have characteristic $p > 0$. Further, we assume that k is algebraically closed.

(ii) All curves are assumed to be smooth, projective and absolutely irreducible.

(iii) Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type. We generally assume that $e_1 \leq e_2 \leq e_3 \leq d < 2p$ if we do not explicitly state otherwise.

(iv) Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism over K with three single ramification points and $\mathcal{C} = (d; e_1, e_2, e_3)$ its ramification type. We always assume that φ is normalized as follows:

- $\varphi(0) = 0$ with ramification index e_1,
- $\varphi(1) = 1$ with ramification index e_2 and
- $\varphi(\infty) = \infty$ with ramification index e_3.

(v) Every mention of three-point covers always means the single-cycle case (Remark 1.2.4).

In this chapter we characterize the stable reduction of three-point covers with $d < 2p$. The reason for the restriction to $d < 2p$ is simple. In this case we are guaranteed to have mild reduction as opposed to the case $d \geq 2p$ (Remark 3.2.2).

The structure of this chapter is the following. In Section 4.1 we prove a purely combinatorial result about bad reduction (Theorem 4.1.1), based on the characterization of good reduction by Osserman ([Oss08, Theorem 2.4] or Theorem 4.1.7). The obtained condition however is very easy to check. With this characterization of bad reduction in the case $d < 2p$, we are able to prove that the existence of hypergeometric deformation data is equivalent to bad reduction (except when $e_3 = 2p - 1$) (Corollary 4.1.3). In Section 5.3 we further discuss a connection between Ossermans characterization of good reduction and tail covers.

In Section 4.2 we state the main result of this thesis, Theorem 4.2.2 and prove part (a) and (b) of this theorem.

In Section 4.3 we prove Theorem 4.2.2 (c) and thus finish the characterization of the stable reduction for $d < 2p$.

4.1 Combinatorial result for bad reduction

Recall that we use the notations and assumptions of Setup 4.0.1.

In this section we prove a combinatorial result for bad reduction of a Galois three-point cover f with $p \leq d < 2p$ based on Osserman's characterization of three-point covers with good reduction ([Oss08, Theorem 2.4] or Theorem 4.1.7). This result is a great simplification of Osserman's quite technical result in the case $d < 2p$. Note however that Osserman's result is much more general and applies to arbitrary d. In Chapter 5 we use his result in a more general setup for $d \geq 2p$, where we cannot prove a similar simplification result.

We now state the combinatorial result and postpone the proof to the end of this section. The reason is that the result and the involved notation is very simple (and thus can be understood immediately), but the notation required to prove the result is rather complicated.

Theorem 4.1.1 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract genus-0 ramification type. Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the genus-0 Belyi morphism of type $\mathcal{C}$ and $f : Y \to X$ the Galois closure of φ. Define $d_i := \max\{e_i, p\}$. Then the following is equivalent:*

(i) f has bad reduction to characteristic p,

(ii) the integers d_i satisfy

$$\sum_{i=1}^{3}(d_i - p) \leq d - p.$$

Example 4.1.2 Let $\mathcal{C} = (7; 4, 5, 6)$ and $p = 5$. Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the unique genus-0 Belyi morphism of type $\mathcal{C}$ and $f : Y \to X$ the Galois closure. In Example 1.6.6 we verified with explicit computation that f has bad reduction to characteristic 5. Observe that the same result can be obtained with Theorem 4.1.1. We compute

$$\sum_{i=1}^{3}(d_i - p) = \sum_{i=1}^{3}(d_i - 5) = 1 < d - p = 7 - 5 = 2.$$

Theorem 4.1.1 has a very important consequence.

Corollary 4.1.3 *Assume that* $\operatorname{char} k = p > 2$. *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $e_3 < 2p - 1$, $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ the genus-0 Belyi morphism of type $\mathcal{C}$ and $f : Y \to X$ its Galois closure. There exists a hypergeometric deformation datum (g, ω) with signature $(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$ (see the Proposition 3.1.9 for the definition of $a_i(\mathcal{C})$) if and only if f has bad reduction to characteristic p. The hypergeometric deformation datum (g, ω) is unique up to multiplying ω with a unit from $\mathbb{F}_p^*$.*

Proof: By Theorem 4.1.1 f has bad reduction to characteristic p if and only if $\sum_{i=1}^{3}(d_i - p) \leq d - p$.

By Proposition 3.1.9 (c) the existence of a hypergeometric deformation datum (g, ω) with the signature $(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$ is equivalent to $\sum_{i=1}^{3}(d_i - p) \leq d - p$.

The assertion follows. □

Remark 4.1.4 (i) In Example 3.3.6 we computed the stable reduction of $f : Y \to X = \mathbb{P}^1$ in the case $\mathcal{C} = (7; 4, 5, 6)$ and $p = 5$. To be able to compute the stable reduction we had to choose the right deformation datum (note that it is unique by Remark 3.3.5) and we chose the hypergeometric deformation datum. Corollary 4.1.3 motivates very effectively why this choice was sensible in the first place. Corollary 4.1.3 states that the existence of hypergeometric deformation data with the given signature is directly linked with bad reduction.

(ii) The restriction $e_3 < 2p - 1$ in Corollary 4.1.3 leads to the following small inconvenience. To describe the stable reduction of three-point covers, Theorem 3.3.4 implies that we need a special deformation datum for every abstract ramification type. With the help of Proposition 3.1.9 we can construct a hypergeometric deformation datum for all abstract ramification types except for the ones with $e_3 = 2p-1$. In this case we need to make an explicit computation and construct another deformation datum. In Section 4.3 we discuss this problem in more detail.

The following notations and definitions can be found in [Oss08, Chapter2].

Notation 4.1.5 (see Notation 2.1 in [Oss08]) Let e be a positive integer prime to p. Then $\bar{e}^{[m,u]} := \lceil e/p^m \rceil$ and $\bar{e}^{[m,d]} := \lfloor e/p^m \rfloor$. Furthermore, define $e^{[m,u]} := p^m \bar{e}^{[m,u]} - e$ and $e^{[m,d]} := e - p^m \bar{e}^{[m,d]}$.

The following definition of numerical p-admissibility is a slight variant of [Oss08, Definition 2.2] but is equivalent to it. The only difference between our definition of numerical p-admissibility and [Oss08, Definition 2.2] is that we avoid the so-called triangle inequality, i.e. $e_i \leq e_j + e_k$ for all distinct $i, j, k \in \{1, 2, 3\}$. This is possible because we assume that $e_i \leq d$ (see Setup 4.0.1). This is equivalent to the triangle inequality.

Definition 4.1.6 (variant of Definition 2.2 in [Oss08]) Let (e_1, e_2, e_3) be a triple of positive integers prime to p. Let d be an integer satisfying $2d + 1 = e_1 + e_2 + e_3$. Assume further that $e_i \leq d$ for all i. Then the triple (e_1, e_2, e_3) is called *numerically p-admissible* if for any $m > 0$, and any $S \subset \{1, 2, 3\}$ such that:

(i) $p^m \leq d$;

(ii) $e_i > p^m$ for all $i \in S$;

(iii) $\sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]}$ is odd;

the following inequality is always satisfied:

$$\sum_{i\in S} e^{[m,d]} + \sum_{i\notin S} e^{[m,u]} \geq p^m. \tag{4.1.1}$$

We can now state Osserman's characterization of good reduction.

Theorem 4.1.7 *Suppose we are given positive integers d, e_1, e_2, e_3 with $2d+1 = e_1+e_2+e_3$, each e_i prime to p and the $e_i \leq d$ for all i, together with distinct points Q_1, Q_2, Q_3 on $\mathbb{P}^1$. Then the triple (e_1, e_2, e_3) is numerically p-admissible if and only if there exists a (necessarily unique up to automorphism) separable cover $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d branched over each Q_i with a single ramification point of index e_i.*

Proof: See the proof of [Oss08, Theorem 2.4]. □

Now we prove the hardest part of Theorem 4.1.1.

Proposition 4.1.8 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract genus-0 ramification type such that $p \nmid e_1e_2e_3$ and $p \leq d < 2p$. Then the following is equivalent:*

(i) (e_1, e_2, e_3) is p-admissible

(ii) $\sum_{i=1}^{3}(d_i - p) > d - p$.

Proof: Note that in the assertion of Proposition 4.1.8 we assume that p does not divide any e_i. In particular, to prove Theorem 4.1.1 (where we allow that several of the e_j are equal to p) we have to analyze these cases separately. This is done in Lemma 4.1.10.

First we show that for (e_1, e_2, e_3) to be p-admissible, exactly two of the e_i have to be greater than p. The idea of the proof is simple. We analyze the following four different cases: no e_i is greater than p, exactly one e_i is greater than p, exactly two e_i are greater than p and finally all e_i are greater than p (this corresponds to the cases in the proof of Proposition 3.1.9). In each case we check whether (e_1, e_2, e_3) is p-admissible. We will see that only for the case that exactly two e_i are greater than p, (e_1, e_2, e_3) can be p-admissible at all.

Note that $m = 1$ is the only relevant choice for m in Definition 4.1.6, since $d < 2p$.

During this proof we simply say "condition (x)" for the appropriate condition (x) of Definition 4.1.6.

1. Case: $e_i < p$ **for** $i = 1, 2, 3$

Table 4.1.1 contains all information needed to verify p-admissibility.

	$\bar{e}_i^{[m,u]}$	$\bar{e}_i^{[m,d]}$	$e^{[m,u]}$	$e^{[m,d]}$
$i = 1$	1	0	$p - e_1$	e_1
$i = 2$	1	0	$p - e_2$	e_2
$i = 3$	1	0	$p - e_3$	e_3

Table 4.1.1: All e_i are less than p

Since all e_i are less than p, condition (ii) implies that we only have to check $S = \emptyset$. The table shows that condition (iii) 4.1.6 is satisfied, but

$$\begin{aligned}\sum_{i\in S} e_i^{[m,d]} + \sum_{i\notin S} e_i^{[m,u]} &= \sum_{i\notin S} e_i^{[m,u]} \\ &= 3p - (e_1 + e_2 + e_3) \\ &= 3p - (2d + 1) \\ &< 3p - 2p - 1 = p - 1.\end{aligned}$$

We see that if all e_i are less than p, then (e_1, e_2, e_3) is not p-admissible.

2. Case: $e_1 \leq e_2 < p < e_3$

Table 4.1.2 contains all information needed to verify p-admissibility.

	$\bar{e}_i^{[m,u]}$	$\bar{e}_i^{[m,d]}$	$e^{[m,u]}$	$e^{[m,d]}$
$i=1$	1	0	$p-e_1$	e_1
$i=2$	1	0	$p-e_2$	e_2
$i=3$	2	1	$2p-e_3$	e_3-p

Table 4.1.2: Exactly one e_i is greater than p

Condition (ii) is satisfied for all subsets of $\{3\}$.

For $S=\emptyset$ condition (iii) is not satisfied. Thus we only have to check $S=\{3\}$. We get

$$\begin{aligned}\sum_{i\in S} e_i^{[m,d]} + \sum_{i\notin S} e_i^{[m,u]} &= e_3^{[m,d]} + e_1^{[m,u]} + e_2^{[m,u]} \\ &= e_3 - p + p - e_1 + p - e_2 \\ &= p - (e_1+e_2+e_3) + 2e_3 \\ &= p - 2d - 1 + 2e_3.\end{aligned}$$

Now $e_3 \leq d$ (Definition 1.2.1 (d)) and therefore $p-2d-1+2e_3 \leq p-1$.

We see that if exactly one e_i is greater than p, then (e_1, e_2, e_3) is not p-admissible.

3. Case: $e_1 < p < e_2 \leq e_3$

Table 4.1.3 contains all information needed to calculate p-admissibility.

	$\bar{e}_i^{[m,u]}$	$\bar{e}_i^{[m,d]}$	$e^{[m,u]}$	$e^{[m,d]}$
$i=1$	1	0	$p-e_1$	e_1
$i=2$	2	1	$2p-e_2$	e_2-p
$i=3$	2	1	$2p-e_3$	e_3-p

Table 4.1.3: Exactly two e_i are greater than p

Condition (ii) is obviously satisfied for $S \subset \{2,3\}$.

For $S=\{2\}$ or $\{3\}$ condition (iii) is not satisfied and we do not have to check these sets further.

For $S=\emptyset$

$$\sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = 5$$

is odd and

$$\begin{aligned}\sum_{i\in S} e_i^{[m,d]} + \sum_{i\notin S} e_i^{[m,u]} &= p - e_1 + 2p - e_2 + 2p - e_3 \\ &= 5p - 2d - 1 \\ &\geq 5p - 2(2p-1) - 1 \\ &= p + 1 > p.\end{aligned}$$

Now we have to check $S=\{2,3\}$. In this case

$$\sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = 3,$$

is odd and

$$\sum_{i\in S} e_i^{[m,d]} + \sum_{i\notin S} e_i^{[m,u]} = (e_2-p) + (e_3-p) + (p-e_1) = e_2 + e_3 - e_1 - p.$$

We see that (e_1, e_2, e_3) is p-admissible if and only if $e_2 + e_3 - e_1 - p \geq p$. This is equivalent to

$$e_2 + e_3 - e_1 \geq 2p. \tag{4.1.2}$$

By Definition 1.2.1 (e) $e_2 + e_3 + e_1 = 2d + 1$ and thus (4.1.2) is equivalent to

$$(e_2 + e_3 - e_1) + 2e_1 = 2d + 1 \geq 2p + 2e_1.$$

This on the other hand is equivalent to

$$e_1 \leq d - p + 1/2,$$

but since e_1 is an integer this is equivalent to $e_1 < d - p + 1$.

We see that if exactly 2 of the e_i are greater than p, then (e_1, e_2, e_3) is p-admissible if and only if $e_1 \leq d - p + 1/2 < d - p + 1$.

4. Case: $e_i > p$ **for** $i = 1, 2, 3$

Table 4.1.4 contains all information needed to verify p-admissibility.

	$\bar{e}_i^{[m,u]}$	$\bar{e}_i^{[m,d]}$	$e^{[m,u]}$	$e^{[m,d]}$
$i = 1$	2	1	$2p - e_1$	$e_1 - p$
$i = 2$	2	1	$2p - e_2$	$e_2 - p$
$i = 3$	2	1	$2p - e_3$	$e_3 - p$

Table 4.1.4: All e_i are greater than p

To prove that (e_1, e_2, e_3) is not p-admissible, it suffices to show that there exists a subset $S \subset \{1, 2, 3\}$, such that the conditions (ii) and (iii) are satisfied, but the last condition (4.1.1) is violated.

We choose $S = \{1, 2, 3\}$. Then of course $e_i > p^m$ for all $i \in S$, and

$$\sum_{i \in S} \bar{e}_i^{[m,d]} + \sum_{i \notin S} \bar{e}_i^{[m,u]} = 3.$$

Thus S satisfies condition (ii) and (iii). By Definition 1.2.1 (d) $e_1 + e_2 + e_3 = 2d + 1$. Therefore,

$$\begin{aligned} \sum_{i \in S} e_i^{[m,d]} + \sum_{i \notin S} e_i^{[m,u]} &= \sum_{i=1}^{3} (e_i - p) \\ &= e_1 + e_2 + e_3 - 3p \\ &= 2d + 1 - 3p \\ &\leq 2(2p - 1) + 1 - 3p = p - 1. \end{aligned}$$

We see that (e_1, e_2, e_3) is not p-admissible if all e_i are greater than p.

We see that for (e_1, e_2, e_3) to be p-admissible, exactly two e_i have to be greater than p and in this case (e_1, e_2, e_3) is p-admissible if and only if $e_1 < d - p + 1$.

Now let (e_1, e_2, e_3) be p-admissible. We have seen that this is the case if and only if $e_1 < p < e_2 \leq e_3$ and $e_1 < d - p + 1$. Note that the latter inequality is equivalent to $d - p + 1 - e_1 > 0$. For the sake of completeness we note that $d_2 = e_2$ and $d_3 = e_3$. This yields

$$\begin{aligned} \sum_{i=1}^{3} (d_i - p) &= e_3 - p + e_2 - p \\ &= (e_3 + e_2 + e_1) - 2p - e_1 \\ &= 2d + 1 - 2p - e_1 \\ &= (d - p + 1 - e_1) + d - p > d - p, \end{aligned}$$

which proves that (i) implies (ii).

Now we prove the that (ii) implies (i).

Let (e_1, e_2, e_3) be not p-admissible. We show that $\sum_{i=1}^{3}(d_i - p) \leq d - p$.

1. Case: $e_i < p$ **for** $i = 1, 2, 3$

We have $d_i = p$ for all i. Moreover $d - p \geq 0$ and $\sum_{i=1}^{3}(d_i - p) = 0$. We see $\sum_{i=1}^{3}(d_i - p) \leq d - p$.

2. Case: $e_1 \leq e_2 < p < e_3$

We have $d_3 = e_3$ and $d_1 = d_2 = p$. This yields $\sum_{i=1}^{3}(d_i - p) = e_3 - p \leq d - p$, since $e_3 \leq d$.

3. Case: $e_1 < p < e_2 \leq e_3$

We have $d_1 = p, d_2 = e_2$ and $d_3 = e_3$ and

$$\sum_{i=1}^{3}(d_i - p) = e_2 - p + e_3 - p = e_2 + e_3 + e_1 - 2p - e_1 = 2d + 1 - e_1 - 2p.$$

We have seen that (e_1, e_2, e_3) is not p-admissible if and only if $e_1 \geq d - p + 1$ (cf. the proof of "(i) implies (ii)"). Therefore

$$2d + 1 - e_1 - 2p = (d - p + 1 - e_1) + d - p \leq d - p.$$

4. Case: $e_i > p$ **for** $i = 1, 2, 3$

Then $d_i = e_i$ for all i, and

$$\begin{aligned}\sum_{i=1}^{3}(d_i - p) = e_1 + e_2 + e_3 - 3p &= 2d + 1 - 3p \\ &= 2(d - p) + 1 - p.\end{aligned}$$

Now $2(d - p) + 1 - p \leq d - p$ if and only if $d \leq 2p - 1$, which is true. □

Remark 4.1.9 (a) In Section 5.3 we show that there exists a correspondence between the sets $S \subset \{1, 2, 3\}$ which satisfy condition (ii) and (iii) of Definition 4.1.6 but do not satisfy (4.1.1), to the degree of tail covers. The computations are very similar to the computations in the proof of Proposition 4.1.8.

(b) Note that Proposition 4.1.8 implies that Theorem 4.1.1 is true if none of the ramification indices is equal to p. Hence to complete the proof of Theorem 4.1.1 we need to analyze the case $p \mid e_1e_2e_3$ in more detail. To be more precise, we need to prove that if $p \mid e_1e_2e_3$, then $\sum_{i=1}^{3}(d_i - p) \leq d - p$.

Lemma 4.1.10 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $p \leq d < 2p$ and $p \mid e_1e_2e_3$. Define $d_i := \max\{e_i, p\}$. Then we have*

$$\sum_{i=1}^{3}(d_i - p) \leq d - p.$$

Proof: If $e_3 = p$, then $d_i = p$ for all i because (e_1, e_2, e_3) is ordered (cf. Setup 4.0.1). Then $\sum_{i=1}^{3}(d_i - p) \leq d - p$ is trivially true.

Now let $e_2 = p$. Then $d_1 = d_2 = p$ and $d_3 = e_3$ and hence $\sum_{i=1}^{3}(d_i - p) = e_3 - p \leq d - p$ because $e_3 \leq d$.

Now suppose that $e_1 = p$. Then $d_i = e_i$ for all i and in particular,

$$\begin{aligned}\sum_{i=1}^{3}(d_i - p) &= (e_2 - p) + (e_3 - p) \\ &= (e_1 + e_2 + e_3) - 2p - e_1 \\ &= (2d + 1) - 2p - e_1.\end{aligned}$$

Using that $e_1 = p$ we see that $2d - 2p - e_1 + 1 = 2d - 3p + 1$. Now $2d - 3p + 1 \leq d - p$ if and only if $d \leq 2p - 1$ which is true by assumption on d. □

Proof of Theorem 4.1.1: The proof follows immediately from Proposition 4.1.8 and Lemma 4.1.10. □

4.2 Special G-deformation data and lifting

Recall that we use the notations and assumptions of Setup 4.0.1.

Additionally, we fix the following notation.

Notation 4.2.1
- Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $\sum_{i=1}^{3}(d_i - p) \leq d - p$. We denote by $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ the genus-0 Belyi morphism with ramification type $\mathcal{C}$ and by $f : Y \to X$ the Galois closure of φ. The Galois group of f is denoted by G.
- If $p \nmid e_j$ for $j \in \{1, 2, 3\}$, we denote the primitive tail cover with tame ramification index e_j by $\bar{g}_j : \bar{Y}_j \to \bar{X}_j$ and its Galois group by G_j. We use the the same notation on all (potentially) occurring new tail covers, however with indices $j \geq 4$. For all tail cover $\bar{g}_j$ the corresponding map between genus-0 curves is denoted by $\bar{\varphi}_j : \bar{Z}_j = \mathbb{P}^1_k \to \bar{X}_j = \mathbb{P}^1_k$. Moreover, we denote the degree of $\bar{\varphi}_j$ by d_j.

In this section we characterize the stable reduction of three-point covers with $d < 2p$ and $e_3 < 2p - 1$. We begin by stating the general result (with no restriction on e_3), which can be understood very easily. In this section we prove (a) and (b) of Theorem 4.2.2 whereas the proof of part (c) is in Section 4.3.

Theorem 4.2.2 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $p \leq d < 2p, 1 < e_i \leq d$ and $\sum_{i=1}^{3}(d_i - p) \leq d - p$. Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the genus-0 Belyi morphism with ramification type $\mathcal{C}$ and $f : Y \to X = \mathbb{P}^1$ its Galois closure. Let G denote the Galois group of f and denote the stable reduction of f by $\bar{f}$. Then the following holds.*

(a) If $p \nmid e_j$, then the primitive tail cover for e_j is given by the Galois closure of

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^p + \bar{z}^{e_j} & , \text{ if } e_j < p \\ \frac{\bar{z}^{e_j}}{\bar{z}^{e_j - p} - 1} & , \text{ if } e_j > p. \end{cases}$$

(b) If $\operatorname{char} k = p > 2$ and $e_3 < 2p - 1$, then the deformation datum of f is hypergeometric and there are $d - p - \sum_{i=1}^{3}(d_i - p)$ new tail covers which are given by the Galois closure of

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^{p+1}}{\bar{z}}.$$

(c) For $e_3 = 2p - 1$ (for any $p > 0$) there exists only one three-point cover with bad reduction. It is the three-point cover with ramification type $\mathcal{C} = (2p-1; p, p, 2p-1)$. The primitive tail cover for $e_3 = 2p-1$ is as in (a) and the deformation datum is given by

$$\omega = -\frac{dx}{x(x-1)}.$$

Theorem 4.2.2 (b) suggests that there are at least two cases to analyze, one being $p = 2$ and the other $p > 2$. However, (c) solves $p = 2$ as well. We make this precise as follows.

Let $p = 2$. We determine the number of three-point covers with bad reduction where $d < 2p$. Note that since $d < 2p$ it follows $d \leq 3$. Moreover, we are interested in real three-point covers, which means that we do not allow $e_i = 1$ and hence $d = 2$ is not possible. Now let $d = 3$. Then the formula of Remark 1.2.4 is

$$2 \cdot 3 + 1 = 7 = e_1 + e_2 + e_3$$

with $1 < e_i \leq 3$. Hence there exists exactly one ramification type for $p = 2$, namely $\mathcal{C} = (3; 2, 2, 3)$. It is obvious that the three-point cover with ramification type $\mathcal{C}$ has bad reduction. Using $p = 2$ we see $\mathcal{C} = (3; 2, 2, 3) = (2p - 1; p, p, 2p - 1)$. This is Theorem 4.2.2 (c) with $p = 2$.

In Section 4.3 we prove (c). For the rest of this section we assume that $p > 2$ and $e_3 < 2p - 1$.

We now construct a special $\tilde{G}$-deformation datum for any $\mathcal{C} = (d; e_1, e_2, e_3)$ with $e_3 < 2p - 1$ and $p > 2$ analogous to Example 3.3.6.

Construction 4.2.3 Let $p > 2$ and $\mathcal{C} = (d; e_1, e_2, e_3)$ a ramification type with $p \leq d < 2p$ and $1 < e_1 \leq e_2 \leq e_3 \leq d$ such that $\sum_{i=1}^{3}(d_i - p) \leq d - p$ and $e_3 < 2p - 1$.

For $1 \leq j \leq 3$ such that $e_j \neq p$ let $\bar{g}_j : \bar{Y}_j \to \bar{X}_j = \mathbb{P}^1_k$ be the primitive tail cover defined by the Galois closure of the $\bar{\varphi}_j : \bar{Z}_j = \mathbb{P}^1_k \to \bar{X}_j$ which are defined by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^{e_j} + \bar{z}^p & , \text{ if } e_j < p \\ \frac{\bar{z}^{e_j}}{\bar{z}^{e_j - p} - 1} & , \text{ if } e_j > p. \end{cases}$$

In Remark 3.1.11 we summarized that the ramification invariant $\sigma_j(\bar{g}_j)$ is

$$\sigma_j(\bar{g}_j) = \frac{|e_j - p|}{p - 1}.$$

Let $0 \leq a_j(\mathcal{C}) < p - 1$ be such that

$$a_j(\mathcal{C}) \equiv (p-1)\sigma_j(\bar{g}_j) \mod p - 1.$$

Since $e_3 < 2p - 1$ we know that $a_j(\mathcal{C}) = (p-1)\sigma_j(\bar{g}_j)$. Define m by

$$m = \begin{cases} (p-1)/2 & , \text{ if all } a_j \text{ are even (all } e_j \text{ are odd)} \\ p - 1 & , \text{ otherwise} \end{cases}$$

and $H_0 = \mathbb{Z}/m\mathbb{Z}$. Let (g_0, ω_0) be the hypergeometric deformation datum with signature $(a_1(\mathcal{C}), a_2(\mathcal{C}), a_3(\mathcal{C}))$ (see Proposition 3.1.9).

Define $I_0 = \langle (1 \ \dots \ p) \rangle$ and

$$G_0 := I_0 \rtimes H_0 \subset N_{S_p}(I_0) \subset S_p = S(\{1, \dots, p\}).$$

Note that $G_0 \subset S_p = S(\{1, \dots, p\})$ is uniquely defined by the order of H_0 and I_0.

For $4 \leq j \leq d - p - \sum_{i=1}^{3}(d_i - p) + 3$ let $\bar{g}_j : \bar{Y}_j \to \bar{X}_j = \mathbb{P}^1_k$ be the new tail cover defined by the Galois closure of the cover $\bar{\varphi}_j : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ given by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{1 + \bar{z}^{p+1}}{\bar{z}}. \tag{4.2.1}$$

For any tail cover $\bar{g}_j$ let d_j denote the degree of $\bar{\varphi}_j$, i.e.

$$d_j = \begin{cases} \max\{p, e_j\} & , \text{if } 1 \leq j \leq 3 \\ p + 1 & , \text{if } j \geq 4. \end{cases}$$

We now describe the embedding of $G_j = \mathrm{Gal}(\bar{g}_j)$ in $S_d = S(\{1, \dots, d\})$.

Because $\bar{g}_j$ is the Galois closure of $\bar{\varphi}_j : \bar{Z}_j = \mathbb{P}^1_k \to \bar{X}_j = \mathbb{P}^1_k$ (Notation 4.2.1), we have that $G_j \subset S_{d_j}$ is transitive on d_j numbers (Remark 1.1.7).

For each tail cover $\bar{g}_j : \bar{Y}_j \to \bar{X}_j = \mathbb{P}^1_k$ let $\bar{y}_j \in \bar{Y}_j$ be a wild ramification point. We number the leaves of $\bar{\varphi}_j$ such that the Sylow p-subgroup of $I_{\bar{y}_j}$ is equal to I_0. Let $H_j \subset I_{\bar{y}_j}$ be the prime-to-p part. Note that H_0 normalizes I_0 and by Theorem 2.3.11 (c) we have $I_{\bar{y}_j} \subset S_p$. Since the order H_j is known (Proposition 2.5.2), this defines

$$I_{\bar{y}_j} = I_0 \rtimes H_j \subset N_{S_p}(I_0) \subset S_p = S(\{1, \dots, p\}) \tag{4.2.2}$$

uniquely.

To define the embedding of G_j in S_d, we use the idea of Example 3.3.6. Recall that for every tail cover $\bar{g}_j$ we have a point $\bar{y}_j \in \bar{Y}_j$ with $I_{\bar{y}_j} = I_0 \rtimes H_j$. In particular, the leaves with the numbers $1, \dots, p$ are already "reserved" for wild ramification. Thus we only have to choose how to number the $d_j - p$ other leaves. Recall that by assumption we have $\sum_{i=1}^{3}(d_i - p) \leq d - p$ and by Corollary 3.1.10 there are

$$|\mathbb{B}_{\mathrm{new}}| = d - p - \sum_{i=1}^{3}(d_i - p)$$

new tail covers which have degree $p+1$. This shows that for every tail cover $\bar{g}_j$ we can choose $d_j - p$ unique ascending numbers (compare this to Example 3.3.6). We make this precise as follows. We number the leaves of $\bar{\varphi}_j$ such that $I_{\bar{y}_j} = I_0 \rtimes H_j \subset G_j$ and[3]

$$\begin{aligned} G_1 &\subset S(\{1,\dots,p,\dots,d_1\}) \\ G_2 &\subset S(\{1,\dots,p,d_1+1,\dots,d_1+d_2-p\}) \\ G_3 &\subset S(\{1,\dots,p,d_1+d_2-p+1,\dots,d_1+d_2+d_3-2p\}) \\ &\vdots \\ G_{3+j} &\subset S(\{1,\dots,p,d_1+d_2+d_3-2p+j\}) \text{ for } 1 \le j \le d-p-\sum_{i=1}^{3}(d_i-p). \end{aligned} \tag{4.2.3}$$

Note that $d_1+d_2+d_3-2p+(d-p-\sum_{i=1}^{3}(d_i-p)) = d$. We define

$$\tilde{G} := \langle G_j \mid j \in \mathbb{B} \cup \{0\} \rangle.$$

Note that $\tilde{G} \subset S_d$ is transitive on $\{1,\dots,d\}$ because the G_j are transitive on the sets of (4.2.3) and the groups G_j have a nontrivial intersection.

For convenience we denote the whole datum by $(g_0, \omega_0; \bar{g}_j)$.

Proposition 4.2.4 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be a ramification type as in Construction 4.2.3.*

The datum $(g_0, \omega_0; \bar{g}_j)$ from Construction 4.2.3 is a special $\tilde{G}$-deformation datum.

Proof: We freely use the notation introduced in Construction 4.2.3

To verify that $(g_0, \omega_0; \bar{g}_j)$ is a special $\tilde{G}$-deformation datum, we have to check Definition 3.1.4.

By definition of $\tilde{G}$, Definition 3.1.4 (i) is satisfied.

Moreover, combining Proposition 2.5.2 and 3.1.13 yields that $\bar{g}_j$ for $1 \le j \le 3$ has ramification type $(m_j(\bar{g}_j), h_j(\bar{g}_j)) = (m_j(\omega_0), h_j(\omega_0))$. Proposition 2.2.1 (d) states that there exists a unique new tail cover corresponding to $\sigma = \frac{p+1}{p-1}$, and this new tail cover is given by (4.2.1). Hence we have $(m_j(\bar{g}_j), h_j(\bar{g}_j)) = (m_j(\omega_0), h_j(\omega_0))$ for the new tail covers, as well.

By (4.2.2) we have $I_{\bar{y}_j} = I_0 \rtimes H_j$, since we chose the numbering of the leaves of $\bar{\varphi}_j$ such that the Sylow p-subgroup of $I_{\bar{y}_j}$ is equal to I_0, and chose the embedding G_j in S_d such that $I_{\bar{y}_j} \subset G_j$.

That we have $H_j = H_0 \cap I_{\bar{y}_j}$ can be seen as follows. Note that $I_{\bar{y}_j}$ and G_0 are subgroups of $N_{S_p}(I_0)$ and hence are uniquely determined by the order of the group itself and I_0. Thus we only have to show that $|H_j| \mid |H_0|$.

Recall from Proposition 2.5.2 that for $1 \le j \le 3$ we have $|H_j| = m_j(\bar{g}_j) = (p-1)/\gcd(p-e_j, p-1)$, and $|H_j| = (p-1)/2$ for $j \ge 4$.

If $|H_0| = p-1$, it follows immediately that $|H_j| \mid |H_0|$ for all $j \in \mathbb{B}$. If $|H_0| = (p-1)/2$, then all e_j are odd. Since $p > 2$ this implies that $p - e_j$ is even. In particular, $2 \mid \gcd(p-e_j, p-1)$ and hence $|H_j| \mid |H_0|$ for all $j \in \mathbb{B}$ as well.

We see that the datum $(g_0, \omega_0; \bar{g}_j)$ defines a special $\tilde{G}$-deformation datum. □

The second part of the proof of Theorem 4.2.2 (for any $p > 0$) consists of the analysis of the special $\tilde{G}$-map $\bar{\tilde{f}} : \bar{\tilde{Y}} \to \bar{X}$ from Construction 3.3.2. Repeating the idea in Remark 3.3.7 in Example 3.3.6 word for word (with the obvious changes), we see that we only have to show that $\bar{Z} := \bar{\tilde{Y}}/H_{i_0}$ has genus 0, where $H_{i_0} = \tilde{G} \cap S(\{1,\dots,i_0-1,i_0+1,\dots,d\})$ is the one-point-stabilizer of i_0 in $\tilde{G}$.

[3] If $e_j = p$, then there exists no primitive tail cover corresponding to e_j by Lemma 3.2.5. In particular, there exists no G_j and no d_j. To make the embedding (4.2.3) meaningful we simply omit this group and choose the numbers ascending without any shift of $d_j - p$ (since there is no d_j). In Example 3.3.6 in (3.3.4) we exactly did this, when there was no group G_2.

Proposition 4.2.5 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $d < 2p$ such that $\sum_{i=1}^{3}(d_i - p) \leq d - p$ and $e_3 < 2p - 1$. Let $(g, \omega; \bar{g}_j)$ be the special $\tilde{G}$-deformation datum of Construction 4.2.3 and $\bar{\tilde{f}} : \bar{\tilde{Y}} \to \bar{X}$ the special $\tilde{G}$-map from Construction 3.3.2. Let $H_{i_0} = \tilde{G} \cap S(\{1, \ldots, i_0 - 1, i_0 + 1, \ldots, d\}) \subset \tilde{G}$ be the one-point stabilizer of i_0 in $\tilde{G}$ and define $\bar{Z} := \bar{\tilde{Y}}/H_{i_0}$. If λ denotes the number of irreducible components of $\bar{Z}$ and γ the number of double points, then*

$$\lambda = |\mathbb{B}_{\text{prim}}| + |\mathbb{B}_{\text{new}}|(d-p) + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) + d - p + 1,$$
$$\gamma = |\mathbb{B}_{\text{prim}}| + |\mathbb{B}_{\text{new}}|(d-p) + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) + d - p.$$

In particular, $\gamma = \lambda - 1$.

Moreover, all component of $\bar{Z}$ have genus 0 and hence

$$g(\bar{Z}) = 0.$$

Proof: Recall that the special $\tilde{G}$-map in Construction 3.3.2 is defined as

$$\bar{\tilde{f}} : \bar{Y} \to \bar{X},$$

where

$$\bar{\tilde{Y}} = (\text{Ind}_{G_0}^{\tilde{G}}(\bar{Y}_0) \amalg \coprod_j \text{Ind}_{G_j}^{\tilde{G}}(\bar{Y}_j))/\sim$$

and

$$\bar{X} = (\bar{X}_0 \amalg \coprod_j \bar{X}_j)/\sim .$$

For $j \in \mathbb{B} \cup \{0\}$ obviously

$$\bar{\tilde{f}}^{-1}(\bar{X}_j) = \text{Ind}_{G_j}^{\tilde{G}}(\bar{Y}_j).$$

We define $\mathfrak{Z}_j = \text{Ind}_{G_j}^{\tilde{G}}(\bar{Y}_j)$ and now analyze the maps

$$\mathfrak{Z}_j/H_{i_0} \to \bar{X}_j = \mathbb{P}^1_k.$$

For $j \in \mathbb{B}$ we use Proposition 1.5.4 with $\tilde{G} = \tilde{G}, S = G_j, \ell = d_j$ and $d = d$. Thus there are $d - d_j$ components of degree 1 and one component of degree d_j above $\bar{X}_j$ in $\bar{Z}$.

Thus the number of irreducible components above each tail $\bar{X}_j \subset \bar{X}$ in $\bar{Z}$ is

$$\lambda_j := 1 + (d - d_j).$$

Moreover, if $\bar{W} \subset \mathfrak{Z}_j/H_{i_0}$ is an irreducible component of degree 1 above $\bar{X}_j$, then trivially $\bar{W} \simeq \bar{X}_j \simeq \mathbb{P}^1_k$ and $\bar{W}$ has genus 0. If $\bar{W} \to \bar{X}_j$ has degree d_j, then $\bar{W} \simeq \mathbb{P}^1_k$ because the tail cover $\bar{Y}_j \to \bar{X}_j$ is a genus-0 tail cover.

We see that all irreducible components in $\bar{Z}$ above any given tail have genus 0.

We now compute the number of double points above $\bar{\infty}_j \in \bar{X}_j$.

Recall that $\bar{\varphi}_j$ is ramified in exactly one point above $\bar{\infty}$ (see Theorem 2.3.11 (a) for the primitive tail covers, and (b) for the new tail cover with $d = p + 1$). This combined with the fact that singular points are mapped to singular points (Lemma 1.6.2 (c)) implies that there are $1 + d_j - p$ double points above $\bar{\infty}_j$ in the preimage of $\bar{\varphi}_j$.

Moreover, since $\bar{Z}$ is connected, it follows that every component above any given tail with degree 1 intersects the rest of $\bar{Z}$ in at least one point. That it is exactly one point follows from the fact that singular points are mapped to singular points (Lemma 1.6.2 (c)) and that $\bar{X}$ is a comb (Theorem 3.2.6).

We conclude that the number of double points above $\bar{\infty}$ in $\bar{Z}$ is

$$\gamma_j := 1 + (d_j - p) + (d - d_j).$$

Note that one could easily simplify the term for γ_j (it is obviously $1+d-p$). However, one loses the information which summands occurred in γ_j. In the following, when we compute the number of all double points of $\bar{Z}$, this information will be needed!

For $j=0$ we use Proposition 1.5.4 with $\ell = p$ and $d = d$. Thus there are $d-p$ components of degree 1 and one component of degree p above $\bar{X}_0$ in $\bar{Z}$.

Hence, the number of irreducible components above the original component is

$$\lambda_0 = 1 + (d-p).$$

Moreover, if $\bar{W} \subset \mathfrak{Z}_0/H_{i_0}$ is an irreducible component of degree 1 above $\bar{X}_0$, then $\bar{W} \simeq \mathbb{P}^1_k$. In particular, $\bar{W}$ has genus 0 and the map $\bar{W} \to \bar{X}_0$ is separable.

Repeating the argument from Remark 3.3.8 word for word with the obvious changes, we see that the irreducible component of degree p above $\bar{X}_0$ in $\bar{Z}$ has genus 0.

Note that since $\bar{X}$ is a comb, all double points of $\bar{Z}$ have already been counted previously, when we analyzed the covers $\bar{\mathfrak{Z}}_j/H_{i_0}$ with $j \in \mathbb{B}$.

We summarize: the number of components above $\bar{X}$ is

$$\begin{aligned}
\lambda &= \sum_{j \in \mathbb{B} \cup \{0\}} \lambda_j = \sum_{j \in \mathbb{B}_{\text{prim}}} \lambda_j + \sum_{j \in \mathbb{B}_{\text{new}}} \lambda_j + \lambda_0 \\
&= \left(|\mathbb{B}_{\text{prim}}| + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) \right) + |\mathbb{B}_{\text{new}}| \left(1 + (d - (p+1))\right) + (1 + d - p) \\
&= |\mathbb{B}_{\text{prim}}| + |\mathbb{B}_{\text{new}}|(d-p) + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) + d - p + 1
\end{aligned}$$

and the number of double points is

$$\begin{aligned}
\gamma &= \sum_{j \in \mathbb{B}} \gamma_j = \sum_{j \in \mathbb{B}_{\text{prim}}} \gamma_j + \sum_{j \in \mathbb{B}_{\text{new}}} \gamma_j \\
&= \left(|\mathbb{B}_{\text{prim}}| + \sum_{j \in \mathbb{B}_{\text{prim}}} (d_j - p) + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) \right) + |\mathbb{B}_{\text{new}}|(1 + d - p).
\end{aligned}$$

Recall that $|\mathbb{B}_{\text{new}}| = d - p - \sum_{i=1}^{3}(d_i - p) = d - p - \sum_{j \in \mathbb{B}_{\text{prim}}}(d_j - p)$. We get

$$\begin{aligned}
\gamma &= \left(|\mathbb{B}_{\text{prim}}| + \sum_{j \in \mathbb{B}_{\text{prim}}} (d_j - p) + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) \right) + |\mathbb{B}_{\text{new}}|(1 + d - p) \\
&= |\mathbb{B}_{\text{prim}}| + \sum_{j \in \mathbb{B}_{\text{prim}}} (d_j - p) + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) + |\mathbb{B}_{\text{new}}| + |\mathbb{B}_{\text{new}}|(d-p) \\
&= |\mathbb{B}_{\text{prim}}| + \sum_{j \in \mathbb{B}_{\text{prim}}} (d_j - p) + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) + \left(d - p - \sum_{j \in \mathbb{B}_{\text{prim}}} (d_j - p) \right) + |\mathbb{B}_{\text{new}}|(d-p) \\
&= |\mathbb{B}_{\text{prim}}| + |\mathbb{B}_{\text{new}}|(d-p) + \sum_{j \in \mathbb{B}_{\text{prim}}} (d - d_j) + d - p \\
&= \lambda - 1.
\end{aligned}$$

Moreover, by [Zha92, Proposition 2.1] the genus of $\bar{Z}$ is given by

$$g(\bar{Z}) = 1 + \sum_{\bar{W} \subset \bar{Z} \text{ irred.}} g(\bar{W}) - \lambda + \gamma,$$

and hence

$$g(\bar{Z}) = 1 + \sum_{\bar{W} \subset \bar{Z} \text{ irred.}} 0 - \lambda + (\lambda - 1) = 0.$$

□

In Figure 4.2.1 the structure of $\bar{Z}$ is illustrated.

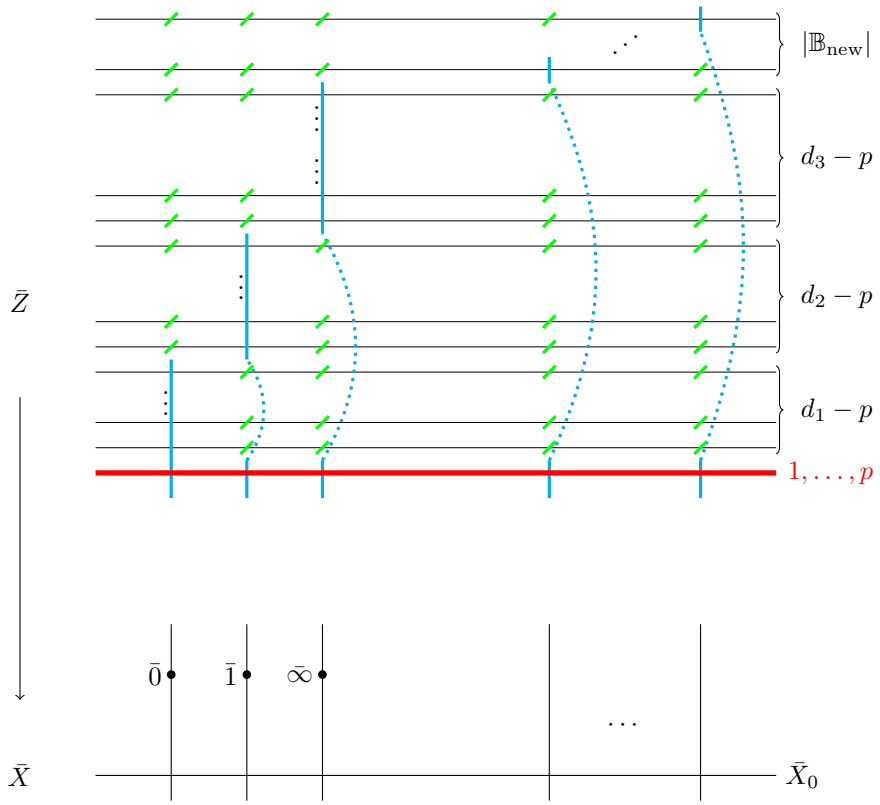

Figure 4.2.1: The map $\bar{Z} \to \bar{X}$. Every horizontal line in $\bar{Z}$ represents a leaf. The thick red line represents the "reserved" p leaves for wild ramification, whereas the thin black lines represent common leaves. The vertical blue line above each tail is the unique irreducible component in $\bar{Z}$ above the tail with degree d_j. Every blue dotted arc is a visualization of the fact that the component is connected but does not intersect the other leaves. Hence the groups G_j act transitively on the leaves which are intersected with the blue line. The green lines represent the degree-1-components above each tail. Note that the picture shows that the genus-0 map of every tail cover is only ramified in one point above ∞.

4.3 The lifting result in the exceptional case

Recall that we use the notations and assumptions of Setup 4.0.1.

In Section 4.2 we omitted the case $e_3 = 2p-1$ and postponed the proof of Theorem 4.2.2 (c) to the present section. The reason is that for $e_3 < 2p-1$ we knew an appropriate deformation datum, namely the hypergeometric deformation datum and thus could construct a special G-deformation datum.

In the present section we complete the proof of Theorem 4.2.2.

We first show that there exists exactly one three-point cover with $e_3 = 2p-1$ with bad reduction.

Lemma 4.3.1 *Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ the genus-0 Belyi morphism with ramification type $\mathcal{C} = (d; e_1, e_2, 2p-1)$ and $f : Y \to X$ its Galois closure. f has bad reduction if and only if $e_1 = e_2 = p$.*

In particular, there exists exactly one three-point cover with $e_3 = 2p-1$ with bad reduction and its ramification type is $\mathcal{C} = (2p-1; p, p, 2p-1)$.

Proof: Let $e_3 = 2p-1$. Since $e_3 \leq d < 2p$ it follows that $d = 2p-1$. Then the formula of Remark 1.2.4 is

$$2(2p-1) + 1 = e_1 + e_2 + 2p - 1$$

or equivalently

$$2p = e_1 + e_2. \tag{4.3.1}$$

Recall that f has bad reduction if and only if $\sum_{i=1}^{3}(d_i - p) \leq d - p$ (Theorem 4.1.1). This formula now simplifies to

$$(d_1 - p) + (d_2 - p) + (p-1) \leq p - 1.$$

In particular, $d_1 = d_2 = p$. By equation (4.3.1) it follows that $d_1 = d_2 = p$ implies $e_1 = e_2 = p$. In particular, a three-point cover with $e_3 = 2p-1$ has bad reduction if and only if its ramification type is $\mathcal{C} = (2p-1; p, p, 2p-1)$. □

Let $f : Y \to X = \mathbb{P}^1$ be the Galois three-point cover with ramification type $C = (2p-1; p, p, p-1)$. By Lemma 3.2.5 (c) there exists exactly one primitive tail cover in the stable reduction of f. By Theorem 3.2.6 $\bar{X}$ consists of one primitive tail and possibly several new tail covers.

We now construct a special deformation datum which satisfies the compatibility conditions of Definition 3.1.4 with the primitive tail $\bar{g}_3 : \bar{Y}_3 \to \bar{X}_3 = \mathbb{P}^1_k$, which is defined as the Galois closure of

$$(\bar{x}, \bar{z}) \mapsto \frac{\bar{z}^{2p-1}}{\bar{z}^{p-1} - 1}.$$

By Proposition 2.5.2, the primitive tail cover $\bar{g}_3$ has ramification type $(h(\bar{g}_3), m(\bar{g}_3))$ with

$$h(\bar{g}_3) = \frac{e_3 - p}{\gcd(e_3 - p, p-1)} = 1 = \frac{p-1}{\gcd(e_3 - p, p-1)} = m(\bar{g}_3).$$

Hence, a deformation datum satisfying the compatibility conditions of Definition 3.1.4 with $\bar{g}_3$ must satisfy $\sigma_3(\omega) = 1$. Note that the definition of special Deformation data in Definition 3.1.1 does not allow $\sigma_3(\omega) = 1$. Since we used the identical definition as Wewers, this shows that Wewers proved the lifting result Theorem 3.3.4 only for special G-deformation data where the underlying special deformation datum always satisfies $0 < \sigma_i(\omega) < 1$ for all $i \in \mathbb{B}_{\text{prim}}$ (and $1 < \sigma_i(\omega) < 2$ for all $i \in \mathbb{B}_{\text{new}}$. This however is meaningless for this thesis). Nonetheless, by [Ray99, Proposition 3.2.6], Wewers' lifting result 3.3.4 is true in the case $\sigma_3(\omega) = 1$, as well.

We now use the slightly more general definition of special deformation data from [BW04, Definition 2.1]. This definition is identical to ours in Definition 3.1.1 except for the fact that it allows $\sigma(\omega) = 1$ and 2. We adapt the definition of special G-deformation data accordingly.

Using $\sigma(\bar{g}_3) = 1$, the vanishing cycle formula (Theorem 3.2.9) implies that

$$\sum_{j \in \mathbb{B}_{\text{new}}} (\sigma_j - 1) = 0.$$

Since the ramification invariant of new tail covers is always > 1 (see [Ray99, Proposition 1.1.6]), this implies that there exist no new tail covers.

Hence we require a special deformation datum with simple poles in 0 and 1 and a simple zero in ∞.

The differential

$$\omega = -\frac{dx}{x(x-1)}$$

obviously satisfies all requirements and is logarithmic (use $u = x/(x-1)$ and note that $du/u = \omega$). We regard this deformation datum as special deformation datum with the trivial character $\chi : \mathbb{Z}/1\mathbb{Z} \to \mathbb{F}_p^*, 0 \mapsto 1$ and the trivial map $\bar{g} = \mathrm{id}$.

We summarize the easy facts of (g, ω) in the following lemma.

Lemma 4.3.2 *The datum* (id, ω) *with*

$$\omega = -\frac{dx}{x(x-1)}$$

is a normalized, special deformation datum with trivial character. The differential ω *has simple poles in* 0 *and* 1 *and a simple zero in* ∞.

Proof: Trivial. □

Recall that we want to construct a special $\tilde{G}$-deformation datum. Let $\bar{g}_3 : \bar{Y}_3 \to \bar{X}_3 = \mathbb{P}^1_k$ be the primitive tail cover defined as the Galois closure of $\bar{\varphi}_3 : \bar{Z}_3 = \mathbb{P}^1_k \to \bar{X}_3 = \mathbb{P}^1_k$ which is given by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \frac{\bar{z}^{2p-1}}{\bar{z}^{p-1} - 1}.$$

Let (g, ω) be the special deformation datum of Lemma 4.3.2.

Analogous to Construction 4.2.3, we define

$$I_0 = \langle (1 \ \ldots \ p) \rangle.$$

Inspired by $m(\bar{g}) = 1$, we choose $m = 1$ and define

$$G_0 = \langle (1 \ \ldots \ p) \rangle \rtimes \mathbb{Z}/1\mathbb{Z} = \langle (1 \ \ldots \ p) \rangle. \tag{4.3.2}$$

Moreover, let $\bar{y} \in \bar{Y}$ be a wild ramification type and $I_{\bar{y}} \simeq \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/1\mathbb{Z}$ its inertia group. We number the leaves of $\bar{\varphi}_3$ such that the Sylow p-subgroup of $I_{\bar{y}}$ is equal to $\langle (1 \ \ldots \ p) \rangle$. In particular, we get

$$I_{\bar{y}} = \langle (1 \ \ldots \ p) \rangle \rtimes \mathbb{Z}/1\mathbb{Z} = \langle (1 \ \ldots \ p) \rangle. \tag{4.3.3}$$

Proposition 4.3.3 *Let* (id, ω) *be as in Lemma 4.3.2. Moreover, let* $\bar{g}_3 : \bar{Y}_3 \to \bar{X}_3 = \mathbb{P}^1_k$ *be the primitive tail cover defined by*

$$\bar{x} = \frac{\bar{z}^{2p-1}}{\bar{z}^{p-1} - 1}.$$

Define G_0 *and the embedding of* $G_3 = \mathrm{Gal}(\bar{g}_3)$ *as in equation (4.3.2) and (4.3.3) and* $\tilde{G} := \langle G_0, G_3 \rangle$. *The datum* $(\mathrm{id}, \omega, \bar{g}_3)$ *defines a special* $\tilde{G}$*-deformation datum.*

Proof: By Lemma 4.3.2 and 2.5.2 we know that

$$m(\omega) = h(\omega) = m(\bar{g}_3) = h(\bar{g}_3) = 1.$$

Thus Definition 3.1.4 is satisfied and $(\mathrm{id}, \omega, \bar{g}_3)$ is a special $\tilde{G}$-deformation datum. □

We now prove Theorem 4.2.2 (c).

Proof of Theorem 4.2.2 (c): The first part of the assertion follows from Lemma 4.3.1.

Note that in the proof of Proposition 4.2.5 we did not use $\sigma \neq 1$. Hence, repeating the proof of Proposition 4.2.5 word for word proves Theorem 4.2.2 (c). □

5 Computations and examples

In Chapter 4 we discussed three-point covers which have mild reduction due to the fact that $d < 2p$. In this chapter we discuss three-point covers with $d \geq 2p$ and analyze some conditions for mild reduction. The case $d \geq 2p$ however is surprisingly hard and we are far from characterizing the stable reduction in this case. Until now we have not been able to even construct one single three-point cover with mild reduction where $d \geq 2p$. It isn't even clear whether there exists a three-point cover with mild reduction with $d \geq 2p$ at all. But it would be interesting to construct such a three-point cover, since Wewers' lifting result (Theorem 3.3.4) would still be applicable.

Since the case $d \geq 2p$ is utterly unknown, there surely is need for several examples of components of the stable reduction of three-point covers with $d \geq 2p$. Before we can explicitly compute some irreducible components, we need several combinatorial results. We begin this chapter with these combinatorial results and proceed in Section 5.2 with the explicit computation of the mentioned irreducible components for several three-point covers. These irreducible components are the primitive tail covers we defined in Section 2.3 and hence motivate why we analyzed these explicit tail covers. Our hope is that these explicit computation will be helpful for future research on this topic. In Section 5.3 we analyze a connection between Osserman's characterization of good reduction (Theorem 4.1.7) and the degree of tail covers. We explicitly prove that for $d < 2p$ there exists a bijective correspondence (Proposition 5.3.2). We further motivate why this correspondence could by generalized. In Section 5.4 we that many of the three-point covers defined by the abstract ramification types in Section 5.2 have nonmild reduction.

Throughout this whole chapter we fix the following setup and make the following assumptions.

Setup 5.0.1 (i) Let R be a complete discrete valuation ring with a discrete valuation v, K the fraction field of R with $\operatorname{char} K = 0$ and k the residue field of R which we assume to have characteristic $p > 0$. Further we assume that k is algebraically closed. We normalize v such that $v(p) = 1$.

(ii) All curves are assumed to be smooth, projective and absolutely irreducible.

(iii) Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type. We generally assume that $e_1 \leq e_2 \leq e_3 \leq d$ if we do not explicitly state otherwise.

(iv) Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism defined over K, with three single ramification points and $\mathcal{C} = (d; e_1, e_2, e_3)$ its ramification type. We always assume that φ is normalized as follows:

- $\varphi(0) = 0$ with ramification index e_1,
- $\varphi(1) = 1$ with ramification index e_2 and
- $\varphi(\infty) = \infty$ with ramification index e_3.

(v) Every mention of three-point covers always means the single-cycle case.

5.1 Combinatorial preliminaries

In this section we prove several results which are needed to compute several irreducible components (tail covers) for certain three-point covers in Section 5.2. First of all we provide an explicit formula for genus-0 Belyi morphisms (Proposition 5.1.1). This formula will be used in Section 5.2 to compute explicit tail covers. We then recall some basic results about binomial coefficients in positive characteristic and compute the valuation of certain binomial coefficients which arise in the computation of the tail covers.

Proposition 5.1.1 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be a ramification type with $e_3 = d$ (we allow here that $e_1 \geq e_2$). Then the map $\varphi_K : Z = \mathbb{P}^1_K \to X = \mathbb{P}^1_K$ defined by*

$$(x, z) \mapsto x = \varphi_K(z) = cz^{e_1} \sum_{i=0}^{d-e_1} (-1)^{d-e_1-i} \frac{1}{e_1 + i} \binom{d - e_1}{i} z^i$$

is ramified in $z = 0, 1, \infty$ with ramification indices e_1, e_2, e_3 respectively and unramified elsewhere. Moreover, we can choose c such that $\varphi_K(1) = 1$ and thus make φ_K the unique normalized genus-0 Belyi morphism with ramification type $\mathcal{C}$ defined over K.

Proof: Define $\tilde{\varphi}$ by the formula

$$x = z^{e_1} \sum_{i=0}^{d-e_1} (-1)^{d-e_1-i} \frac{1}{e_1+i} \binom{d-e_1}{i} z^i.$$

Then obviously $\tilde{\varphi}$ is fully ramified in $z = \infty$ with ramification index d because $\tilde{\varphi}$ is defined by a polynomial of degree d. Moreover, we get

$$\begin{aligned} \tilde{\varphi}'(z) &= \left(\sum_{i=0}^{d-e_1} (-1)^{d-e_1-i} \frac{1}{e_1+i} \binom{d-e_1}{i} z^{e_1+i} \right)' \\ &= \sum_{i=0}^{d-e_1} (-1)^{d-e_1-i} \frac{e_1+i}{e_1+i} \binom{d-e_1}{i} z^{e_1+i-1} \\ &= z^{e_1-1} \sum_{i=0}^{d-e_1} \binom{d-e_1}{i} z^i (-1)^{d-e_1-i} \\ &= z^{e_1-1}(z-1)^{d-e_1}. \end{aligned} \tag{5.1.1}$$

This shows that $\tilde{\varphi}$ is only ramified in three points, namely $z = 0, 1$ and $z = \infty$. Moreover, one can see that $\tilde{\varphi}(0) = 0$ and that the ramification index in $z = 0$ is e_1 and that the ramification index in $z = 1$ is e_2. The only thing left to show is that $\tilde{\varphi}(1) \neq 0$.

If $\tilde{\varphi}(1)$ were 0, then $\tilde{\varphi}'(z) = z^{e_1-1}(z-1)^{d-e_1}$ would imply that

$$(z-1)^{d-e_1+1} \mid \tilde{\varphi}(z).$$

It follows

$$d = \deg \tilde{\varphi} \geq e_1 + (d - e_1 + 1) = d + 1.$$

Therefore, the assumption $\tilde{\varphi}(1) = 0$ yields a contradiction and is wrong. Thus $\tilde{\varphi}(1) \neq 0$. Taking $c = 1/\tilde{\varphi}(1)$ yields the assertion. □

A similar formula can be found in the unpublished [Esk12, Satz 1.7.2]. For the convenience of the reader we recall this formula but with slight change of notation for better readability.

Proposition 5.1.2 *Let $\mathcal{C} = (d; e_1, e_2, d)$ be an abstract ramification type. Then the normalized genus-0 Belyi morphism is given by*

$$(x, z) \mapsto x = z^{e_1} \sum_{i=0}^{d-e_1} (-1)^i \binom{e_1+i-1}{i} (z-1)^i. \tag{5.1.2}$$

Remark 5.1.3 Note that the formula (5.1.2) has one major disadvantage compared to the formula of Proposition 5.1.1 when changing coordinates. We make this precise as follows.

Formula (5.1.2) involves the terms z^{e_1} and $(z-1)^i$. Substituting the coordinate z by $1 - z$ introduces new binomial coefficients and quickly becomes very confusing.

One advantage of formula (5.1.2) is that one immediately sees that the coefficients of the genus-0 Belyi morphism are integers, which is not trivial. In particular, the valuation of any coefficient of any (normalized) genus-0 Belyi morphism which is fully ramified in one point can never be less than 0.

As mentioned in the introduction of this section, we now recall several results about binomial coefficients in positive characteristic and compute the valuation of certain binomial coefficients.

Lemma 5.1.4 (Lucas' Theorem) *Let $n, k \in \mathbb{N}$ be natural numbers and p a prime. Write $n = \sum_{i=0}^{\ell} n_i p^i$ and $k = \sum_{i=0}^{\ell} k_i p^i$ with $0 \leq n_i, k_i < p$. Then*

$$\binom{n}{k} \equiv \prod_{i=0}^{\ell} \binom{n_i}{k_i} \mod p,$$

where $\binom{n_i}{k_i}$ *is defined as usual, i.e.* $\binom{n_i}{k_i}$ *is zero if and only if* $k_i > n_i$ *and*

$$\binom{n_i}{k_i} = \frac{n_i(n_i-1)\cdots(n_i-(k_i-1))}{k_i!}.$$

Remark 5.1.5 Lemma 5.1.4 was originally proved by Lucas in [Luc78, Section XXI]. For more general results see [Gra97].

Corollary 5.1.6 *Let* $n, k, p \in \mathbb{N}$ *be as in Lemma 5.1.4 and let* $0 \leq k < p \leq n$.

$$\binom{n}{k} \equiv \binom{n_0}{k} \mod p.$$

Proof: By Lemma 5.1.4

$$\binom{n}{k} \equiv \prod_{i=0}^{\ell} \binom{n_i}{k_i} \equiv \binom{n_0}{k} \prod_{i=0}^{\ell} \binom{n_i}{0} \equiv \binom{n_0}{k} \mod p.$$

□

We now compute the valuation of certain binomial coefficients which occur in the computation of the tail covers in Section 5.2.

Let $n \in \mathbb{Z}$ be any non-negative integer and $\tilde{n} = \lfloor n/p \rfloor$. Then trivially

$$v(n!) = v((\tilde{n}p)!) = \sum_{m=0}^{\tilde{n}-1} v((\tilde{n}-m)p). \tag{5.1.3}$$

Moreover, because

$$\binom{ap-1}{i} = \frac{(ap-1)!}{i! \cdot (ap-1-i)!},$$

we have

$$v\binom{ap-1}{i} = v((ap-1)!) - v(i!) - v((ap-1-i)!).$$

By equation (5.1.3) we see that

$$v\left((ap-1-i)!\right) > v\left((ap-1-(i+1))!\right), \text{ and } v((i+1)!) > v(i!)$$

if and only if $i \equiv -1 \mod p$.

The fact that the integers i where $v((ap-1-i)!)$ and $v(i!)$ change always satisfy $i \equiv -1 \mod p$ implies the following result.

Lemma 5.1.7 *Let* $i, j \leq ap-1$ *be integers with* $i, j \not\equiv -1 \mod p$. *Assume that* $rp \leq i, j < (r+1)p$ *for some* r. *Then*

$$v\binom{ap-1}{i} = v\binom{ap-1}{j}.$$

Proof: Since $rp \leq i, j < (r+1)p$ we have that $v(j!) = v(i!) = v((rp)!)$. Moreover, since $i, j \not\equiv -1 \mod p$ we have that $v((ap-1-i)!) = v((ap-1-j)!)$. This shows that

$$\begin{aligned} v\binom{ap-1}{i} &= v((ap-1)!) - v(i!) - v((ap-1-i)!) \\ &= v((ap-1)!) - v(j!) - v((ap-1-j)!) \\ &= v\binom{ap-1}{j}. \end{aligned}$$

□

Corollary 5.1.8 *Let $i, a \in \mathbb{Z}$ be integers with $a > 0$ and $0 \leq i \leq ap-1$. Then the following holds.*

(a) If i is such that there exists an $r \in \mathbb{Z}$ with $rp - 1 < i < (r+1)p - 1$, then

$$v\binom{ap-1}{i} = v\binom{a-1}{r}.$$

(b) If i is such that there exists an $r \in \mathbb{Z}$ with $i = rp - 1$, then

$$v\binom{ap-1}{i} = v\binom{a-1}{r-1}$$

Proof: By Lemma 5.1.7 we have that $v\binom{ap-1}{i} = v\binom{ap-1}{j}$ for all i, j with $rp-1 < i, j < (r+1)p-1$ for some $r \leq a$. Thus it suffices to distinguish the cases $i = rp$ and $i = rp - 1$.

1. Case: $i = rp$

Equation (5.1.3) yields

$$\begin{aligned} v((ap-1)!) &= v\left(\big((a-1)p\big)!\right), \\ v\left((ap-1-rp)!\right) &= v\left(\big((a-r-1)p\big)!\right). \end{aligned}$$

Moreover, we can write

$$\begin{aligned} v\left(\big((a-1)p\big)!\right) &= \sum_{m=1}^{a-1} v\left((a-m)p\right), \\ v\left(\big((a-r-1)p\big)!\right) &= \sum_{m=r+1}^{a-1} v\left((a-m)p\right) \text{ and} \\ v\left((rp)!\right) &= \sum_{m=0}^{r-1} v\left((r-m)p\right). \end{aligned}$$

Thus we get

$$\begin{aligned} v\binom{ap-1}{i} &= v\left((ap-1)!\right) - v\left((ap-1-i)!\right) - v(i!) \\ &= v\left(\big((a-1)p\big)!\right) - v\left(\big((a-r-1)p\big)!\right) - v\left((rp)!\right) \\ &= \sum_{m=1}^{a-1} v\left((a-m)p\right) - \sum_{m=r+1}^{a-1} v\left((a-m)p\right) - \sum_{m=0}^{r-1} v\left((r-m)p\right) \\ &= \sum_{m=1}^{r} v\left((a-m)p\right) - \sum_{m=0}^{r-1} v\left((r-m)p\right) \\ &= \sum_{m=0}^{r-1} v\left((a-(m+1))p\right) - \sum_{m=0}^{r-1} v\left((r-m)p\right) \\ &= \sum_{m=0}^{r-1} v\left(\frac{(a-(m+1))p}{(r-m)p}\right) \\ &= v\left(\frac{(a-1)(a-2)\cdots(a-r)}{r(r-1)\cdots 1}\right) \\ &= v\binom{a-1}{r}. \end{aligned}$$

2. Case: $i = rp - 1$

By equation (5.1.3)

$$v((ap-1)!) = v\left(((a-1)p)!\right)$$
$$v(i!) = v\left(((r-1)p)!\right).$$

Thus one computes

$$\begin{aligned}
v\binom{ap-1}{rp-1} &= v\left((ap-1)!\right) - v\left((rp-1)!\right) - v\left(((a-r)p)!\right) \\
&= v\left(((a-1)p)!\right) - v\left(((r-1)p)!\right) - v\left(((a-r)p)!\right) \\
&= \sum_{m=1}^{a-1} v\left((a-m)p\right) - \sum_{m=1}^{r-1} v\left((r-m)p\right) - \sum_{m=r}^{a-1} v_p\left(((a-m)p)\right) \\
&= \sum_{m=1}^{r-1} v\left((a-m)p\right) - \sum_{m=1}^{r-1} v\left((r-m)p\right) \\
&= \sum_{m=1}^{r-1} v\left(\frac{(a-m)p}{(r-m)p}\right) \\
&= v\left(\frac{(a-1)\cdots(a-(r-1))}{(r-1)\cdots(r-(r-1))}\right) \\
&= v\binom{a-1}{r-1}.
\end{aligned}$$

□

5.2 Explicit computation of tail covers

In this section we compute several irreducible components of the stable reduction of the genus-0 Belyi morphisms with the corresponding ramification types

$$\begin{aligned}
\mathcal{C}_1 &= (ap+1; 3, ap-1, ap+1), \quad a \in \mathbb{N}_{>0} \\
\mathcal{C}_2 &= (ap; e, ap-e+1, ap), \quad (a-1)p < e < ap \\
\mathcal{C}_3 &= (e; e-ap+1, ap, e), \quad ap < e < (a+1)p.
\end{aligned}$$

The inequalities $(a-1)p < e < ap$ and $ap < e < (a+1)p$ may seem strange at first, since we could summarize these inequalities with $\tilde{a}p < e < (\tilde{a}+1)p$. The reason for the seemingly needless additional (and perhaps slightly confusing) inequalities is the form of the irreducible components of the stable reduction we compute. We will see that these irreducible components are the primitive tail covers we defined in Lemma 2.3.1. In particular, by Lemma 2.4.1, the integer a has an interpretation in terms of the ramification invariant (it appears in the denominator in the form $a(p-1)$). If we used the inequality $\tilde{a}p < e < (\tilde{a}+1)p$, this interpretation would be lost.

We now explain the strategy for the computation.

Let $\varphi : Z = \mathbb{P}^1_K \to X = \mathbb{P}^1_K$ be a genus-0 Belyi morphism with ramification type $\mathcal{C} = (d; e_1, e_2, e_3)$ and $f : Y \to X$ its Galois closure. Let $\mathcal{X}_x$ be the model of X over R corresponding to the coordinate x and let $\mathcal{Z}_z$ be the normalization of $\mathcal{X}_x$ in the function field of Z. Since $\mathcal{Z}_z$ is a projective line, it corresponds to a coordinate z on Z (as in [BW14, §4.2]).

Suppose that $\bar{\mathcal{Z}}_z \to \bar{\mathcal{X}}_x$ is one of the cover from Lemma 2.3.1 and let $\bar{Y}_1 \to \bar{\mathcal{X}}_x$ be its Galois closure, in particular, $\bar{Y}_1 \to \bar{\mathcal{X}}_x$ is a primitive tail cover. Then the genus of $\bar{Y}_1$ is > 0 and the uniqueness of the stable model implies that $\bar{Y}_1$ is a component of the stable model.

Hence we want to find coordinates x_1 on X and z_1 on Z such that the reduction of the coefficients of $x_1 = \varphi(z_1)$ is defined and yields a primitive tail cover (see [BW14, §4, §5] for a similar computation and more details).

Recall that all the coordinates of $\mathbb{P}^1$ have the form

$$z = Az_1, \quad x = Bx_1,$$

where A and B are linear fractional maps. Further recall that linear fractional maps are uniquely determined by the images of three distinct points.

To compute the primitive tail cover of f at $x = 0$ it suffices to consider coordinates $x = \pi x_1$ with $\pi \in m_{K'}$, where K'/K is some finite extension, and $m_{K'}$ denotes the maximal ideal of the discrete valuation ring of K'. The reason is that we may normalize the coordinate x such that the branch point with ramification index e_1 has coordinate $x_1 = 0$ and another branch point with ramification index e_j for $j \neq 1$ has coordinate $x_1 = \infty$. The same reasoning applied to z shows that it suffices to consider $z = \mu z_1$ with $\mu \in m_{K'}$. Note that if $\bar{\mathcal{Z}}_{z_1} \to \bar{\mathcal{X}}_{x_1}$ is defined by an equation for a primitive tail cover, then the special fibre is reduced. Moreover, the generic fibre of $\mathcal{Z}_{z_1}$ is smooth and hence $\mathcal{Z}_{z_1}$ is normal. In particular, $\mathcal{Z}_{z_1}$ is a model of Z by [Liu02, Chapter 4.1, Lemma 1.18].

To compute irreducible components above points different from $x = 0$, we change the coordinates on X and Z and let this point be 0 again. We then proceed as described previously.

Hence the whole computation to compute an irreducible component at $x = 0$ consists of finding appropriate

$$x = \pi x_1, \quad z = \mu z_1, \quad \text{with } \pi, \mu \in m_{K'},$$

such that $x_1 = \varphi_1(z_1) \mod m_{K'}$ defines a primitive tail cover.

We now begin with the computation of the irreducible components.

Proposition 5.2.1 *Let $p \neq 2, 3$ and let $a \in \mathbb{N}_{>0}$ be a positive integer, $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism with ramification type $\mathcal{C} = (d = ap + 1; ap - 1, 3, ap + 1)$, $f : Y \to X$ its Galois closure and $\bar{f} : \bar{Y} \to \bar{X}$ the stable reduction of f.*

(a) f has bad reduction modulo p.

(b) Corresponding to $e_1 = ap - 1$ there occurs a primitive tail cover $\bar{Y}_1 \to \bar{X}_1 = \mathbb{P}^1_k$ in the stable reduction of f defined as the Galois closure of

$$\bar{Z}_1 \to \bar{X}_1, \quad (\bar{x}_1, \bar{z}_1) \mapsto x_1 = \bar{z}_1^{ap-1} + \bar{z}_1^{ap}$$

(c) Corresponding to $e_3 = ap + 1$ there occurs a primitive tail cover $\bar{Y}_3 \to \bar{X}_3 = \mathbb{P}^1_k$ in the stable reduction of f defined as the Galois closure of

$$\bar{Z}_3 \to \bar{X}_3, \quad (\bar{x}_3, \bar{z}_3) \mapsto x_3 = \frac{\bar{z}_3^{ap+1}}{\bar{z}_3 - 1}.$$

Proof: By Proposition 5.1.1 the genus-0 Belyi morphism with ramification type $\mathcal{C}$ is given by

$$\begin{aligned} x &= cz^{ap-1} \left(\sum_{i=0}^{2} (-1)^{2-i} \frac{1}{ap - 1 + i} \binom{2}{i} z^i \right) \\ &= c \left(\frac{1}{ap-1} z^{ap-1} - \frac{2}{ap} z^{ap} + \frac{1}{ap+1} z^{ap+1} \right), \end{aligned}$$

where

$$c = \frac{1}{\frac{1}{ap-1} - \frac{2}{ap} + \frac{1}{ap+1}} = \frac{(ap-1)ap(ap+1)}{2}.$$

We conclude that the genus-0 Belyi morphism φ is given by

$$x = \frac{ap(ap+1)}{2} z^{ap-1} - (ap-1)(ap+1) z^{ap} + \frac{ap(ap-1)}{2} z^{ap+1}. \tag{5.2.1}$$

We see that the reduction of the genus-0 Belyi morphism modulo p is $\bar{x} = \bar{z}^{ap}$. From this it follows that f has bad reduction modulo p. Alternatively this may be deduced from Osserman's characterization of three-point covers with good reduction ([Oss08, Theorem 2.4] or Theorem 4.1.7).

Now we prove (b). We use the normalization of φ as in the proof of (a), i.e. φ is given by (5.2.1).

Obviously,

$$v\left(\frac{ap(ap+1)}{2}\right) = v\left(\frac{ap(ap-1)}{2}\right) = v(ap), \quad v\big((ap-1)(ap+1)\big) = 0.$$

Let $z = \mu z_1$ with $v(\mu) = \varepsilon$. Then

$$\begin{aligned} v\left(\frac{ap(ap+1)}{2}\mu^{ap-1}\right) &= v(ap) + (ap-1)\varepsilon, \\ v\big((ap-1)(ap+1)\mu^{ap}\big) &= ap\varepsilon, \\ v\left(\frac{ap(ap-1)}{2}\mu^{ap+1}\right) &= v(ap) + (ap+1)\varepsilon. \end{aligned}$$

We equate $v(ap) + (ap-1)\varepsilon$ with $ap\varepsilon$. Solving for ε we get $\varepsilon = v(ap)$. Since by definition $v(\mu) = \varepsilon$ we may set $\mu = ap$ and we do not need to replace K by an extension.

Substituting $z = apz_1$ into the equation (5.2.1) for φ we get

$$x = (ap)^{ap}\left(\frac{(ap+1)}{2}z_1^{ap-1} - (ap-1)(ap+1)z_1^{ap} + (ap)^2\frac{(ap-1)}{2}z^{ap+1}\right).$$

Substituting $x = (ap)^{ap}x_1$ yields

$$\begin{aligned} x_1 &= \frac{(ap+1)}{2}z_1^{ap-1} - (ap-1)(ap+1)z_1^{ap} + (ap)^2(ap-1)z_1^{ap+1} \\ &\equiv \frac{1}{2}z_1^{ap-1} + z_1^{ap} \mod p. \end{aligned}$$

Since we can change x and z by multiples we see that we can renormalize and obtain

$$\bar{x} \equiv \bar{z}^{ap-1} + \bar{z}^{ap} \mod p.$$

This yields the asserted primitive tail cover for $e = ap - 1$.

We now prove (c). For this we renormalize the genus-0 Belyi morphism of equation (5.2.1) by changing the coordinates such that 0 and ∞ are interchanged, i.e. we compute

$$x = \frac{1}{\varphi(1/z)}.$$

Hence the genus-0 Belyi morphism with ramification type $\hat{C} = (ap+1; ap+1; 3, ap-1)$ is given by

$$x = \frac{z^{ap+1}}{\frac{ap(ap+1)}{2}z^2 - (ap-1)(ap+1)z + \frac{ap(ap-1)}{2}}.$$

We substitute $z = apz_1$ and $x = (ap)^{ap}x_1$ and obtain

$$\begin{aligned} x_1 &= \frac{z_1^{ap+1}}{\frac{(ap)^2(ap+1)}{2}z_1^2 - (ap-1)(ap+1)z_1 + \frac{ap-1}{2}} \\ &\equiv \frac{z_1^{ap+1}}{z_1 - \frac{1}{2}} \mod p. \end{aligned}$$

Changing x and y by a suitable multiple we obtain

$$\bar{x} = \frac{\bar{z}^{ap+1}}{\bar{z} - 1}.$$

□

We now prepare the computation of a component at $x = 1$ corresponding to $e = 3$.

Lemma 5.2.2 *Let $p \neq 2,3$ and let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the normalized genus-0 Belyi morphism with ramification type $\tilde{C} = (ap+1; 3, ap-1, ap+1)$. Then $\tilde{\varphi}$ is given by*

$$(x,z) \mapsto x = cz^3 \sum_{i=0}^{ap-2} (-1)^{ap-2-i} \frac{1}{3+i} \binom{ap-1}{i} z^i = c \sum_{j=0}^{ap+1} c_j z^j, \tag{5.2.2}$$

where c is chosen in such a way that $\tilde{\varphi}(1) = 1$ and $c_0, \ldots, c_2 = 0$. Then the following holds [4].

$$v(c_j) = \begin{cases} v\binom{a-1}{r-1} - v(rp) & , \text{ if } \exists r \in \mathbb{Z} \text{ with } j = rp \\ v\binom{a-1}{r-1} & , \text{ if } \exists r \in \mathbb{Z} \text{ with } j = rp+2 \\ v\binom{a-1}{r} & , \text{ if } \exists r \in \mathbb{Z} \text{ with } rp+2 < j < (r+1)p+2 \text{ and } p \nmid j. \end{cases}$$

Proof: We note that $v\left((-1)^{ap-2-i}\right) = 0$ for every i.

Recall that $j = i+3$. If i is such that there exists an $r \in \mathbb{Z}$ such that $i+3 = rp$, then

$$v(1/(3+i)) = -v(rp).$$

Since $i+3 = rp$ is equivalent to $i = rp-3$ we get

$$(r-1)p - 1 < i < rp - 1.$$

This shows that $i \not\equiv -1 \mod p$. Thus by Corollary 5.1.8 (a)

$$v(c_{i+3}) = v\left((-1)^{ap-1-i} \frac{1}{3+i} \binom{ap-1}{i}\right) = v\binom{ap-1}{r-1} - v(rp).$$

If i is such that there exists an $r \in \mathbb{Z}$ with $i = rp-1$, then by Corollary 5.1.8

$$v(c_{i+3}) = v\left((-1)^{ap-1-i} \frac{1}{3+i} \binom{ap-1}{i}\right) = v\binom{ap-1}{i} = v\binom{a-1}{r-1}.$$

If i is such that $i \not\equiv -3, -1 \mod p$ and there exists an $r \in \mathbb{Z}$ with $rp-1 < i < (r+1)p-1$, then Corollary 5.1.8 yields

$$v(c_{i+3}) = v\left((-1)^{ap-1-i} \frac{1}{3+i} \binom{ap-1}{i}\right) = v\binom{ap-1}{i} = v\binom{a-1}{r}.$$

The assertion now follows with $j = i+3$. □

Remark 5.2.3 In the notation of Lemma 5.2.2 note that if $p \nmid j$, then $v(c_j) \geq 0$ since there are only summands with valuation ≥ 0.

We now prove that if we substitute z with μz and the valuation of μ is large enough, then the coefficients with the least valuation are c_3 and c_p. This result is essential for the computation of the primitive tail cover corresponding to $e = 3$.

Lemma 5.2.4 *Let $p \neq 2,3$. Let $\tilde{\varphi}$ be as in Lemma 5.2.2, and μ be an element in a field extension K' of K with valuation $v(\mu) = \varepsilon$ and $p\varepsilon \geq 1$. Then the following holds.*

(a) $v(c_3\mu^3) < v(c_j\mu^j)$ for all j with $p \nmid j$.

(b) $v(c_p\mu^p) < v(c_{rp}\mu^{rp})$ for all $r \geq 2$.

[4] The reason for these unpleasant conditions is the following. In the case where $j = rp$ and $j = rp+2$ we have $r = \lceil i/p \rceil$ (recall that $j = i+3$). Hence we could take $r = \lceil i/p \rceil$ as definition and hence simplify the notation greatly, since we would have to distinguish only between the cases $j \equiv 0 \mod p$ and $j \equiv 2 \mod p$. However, in the last case, where $j \not\equiv 0, 2 \mod p$, this choice of r is wrong. To see this note that in this last case $rp+2 < j$ and hence $rp < j-2 = i+1$. Now $\lceil i/p \rceil p \geq i$. In particular, in this case the choice $r = \lceil i/p \rceil$ is wrong. In this case we have $r = \lfloor i/p \rfloor$. We thought that distinguishing between these $r = \lceil i/p \rceil$ and $r = \lfloor i/p \rfloor$ in the assertion is more difficult than the actual result, since it would require to introduce more notation.

Proof: Obviously for every j we have

$$v(c_j\mu^j) = jv(\mu) + v(c_j) = j\varepsilon + v(c_j).$$

Note that since $p \neq 3$

$$v(c_3) = v\left((-1)^{ap-2}\frac{1}{3}\binom{ap-1}{0}\right) = 0.$$

Now let $j \geq 0$ be any integer prime to p. By Remark 5.2.3 $p \nmid j$ implies $v(c_j) \geq 0$. Then, because $c_0 = c_1 = c_2 = 0$ and $v(0) = \infty$ it follows that

$$3\varepsilon + v(c_3) = 3\varepsilon \leq j\varepsilon + v(c_j).$$

In particular, $v(c_3\mu^3) < v(c_j\mu^j)$. This proves (a).

Now we prove (b).

By Lemma 5.2.2 (a) we have that

$$v(c_{\ell p}\mu^{\ell p}) = v(c_{\ell p}) + \ell p\varepsilon = v\binom{a-1}{\ell-1} - v(\ell p) + \ell p\varepsilon$$

holds for all $\ell \geq 1$.

Note that

$$v(c_p\mu^p) = -1 + p\varepsilon.$$

Now let $r \geq 2$. Then

$$v\binom{a-1}{r-1} - v(rp) + rp\varepsilon \geq -1 + p\varepsilon \quad \text{if and only if} \quad p\varepsilon(r-1) \geq +v(r) - v\binom{a-1}{r-1}.$$

This is equivalent to

$$p\varepsilon + \frac{1}{r-1}v\binom{a-1}{r-1} \geq \frac{1}{r-1}v(r). \tag{5.2.3}$$

Note that $v(r)/(r-1) \leq 1$ for all $r \geq 2$. In particular, because $p\varepsilon \geq 1$ (by assumption) and $v\binom{a-1}{r-1} \geq 0$, the inequality (5.2.3) is satisfied. This shows the assertion. □

We can finally compute the primitive tail cover for $e = 3$.

Proposition 5.2.5 *Let $p \neq 3$, $a \in \mathbb{N}_{>0}$ and let $\tilde{\varphi}$ be the genus-0 Belyi morphism of type $\tilde{C} = (d = ap+1; 3, ap-1, ap+1)$ as in Lemma 5.2.2. Let $f : Y \to X$ be its Galois closure and $\bar{f} : \bar{Y} \to \bar{X}$ its stable reduction. Corresponding to $e = 3$ there occurs a primitive tail cover $\bar{Y} \to \bar{X} = \mathbb{P}^1_k$ in the stable reduction of f defined as the Galois closure of*

$$\bar{Z} \to \bar{X}, \quad (\bar{x}, \bar{z}) \mapsto \bar{x} = \bar{z}^3 + \bar{z}^p.$$

Proof: In a first step we substitute x with cx and thus eliminate c from the equation $x = \tilde{\varphi}(z)$ in (5.2.2).

We set $z = \mu z_1$ with $v(\mu) = \varepsilon > 0$ and equate

$$v(c_3\mu^3) = v(c_p\mu^p), \tag{5.2.4}$$

since we want the terms z^3 and z^p to survive in reduction. Using Lemma 5.2.2, we compute

$$v(c_3\mu^3) = 3\varepsilon + v(c_3) = 3\varepsilon + v\binom{a-1}{0} = 3\varepsilon \quad \text{and}$$
$$v(c_p\mu^p) = p\varepsilon + v(c_p) = p\varepsilon + v\binom{a-1}{1-1} - v(p) = p\varepsilon - 1.$$

Solving for ε we obtain

$$\varepsilon = \frac{1}{p-3}.$$

Note that this implies that we have to enlarge K, since v is normalized by $v(p) = 1$ (Setup 5.0.1).

Moreover $p\varepsilon = p/(p-3) > 1$. We see that Lemma 5.2.4 yields that all other coefficients have higher valuation.

If we now substitute $x = \mu^3 x_1$ we see that

$$x_1 \equiv (-1)^{ap-1}\frac{1}{3}z^3 + (-1)^{(a-1)p+2}\frac{\mu^{p-3}}{p}\binom{ap-1}{p}z^p \mod \mu.$$

We see that we only have two coefficients left to determine. Since we can change x and y by multiples, we have 2 degrees of freedom. Hence we can renormalize and obtain the tail cover from the assertion. □

Remark 5.2.6 Note that the reason for the long and complicated computation was that $a \in \mathbb{N}_{>0}$ is an arbitrarily large integer.

We now compute several irreducible components of the stable reduction of the three-point cover which is the Galois closure of a genus-0 Belyi morphism with ramification type $\mathcal{C} = (ap; ap-e+1, e, ap)$. Note that in Proposition 5.2.7 (a) the statement is true for every $a \in \mathbb{N}_{>0}$. However, in the proof of Proposition 5.2.7 (b) we need $a < p$.

Proposition 5.2.7 *Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the unique genus-0 Belyi morphism with ramification type $\mathcal{C} = (ap; ap-e+1, e, ap)$ where $(a-1)p < e < ap$ and $a \in \mathbb{N}_{>0}$. Let $f : Y \to X$ be the Galois closure of φ and $\bar{f}$ the stable reduction. Then the following holds.*

(a) Corresponding to $e_2 = e$ there occurs a primitive tail cover $\bar{Y}_2 \to \bar{X}_2 = \mathbb{P}^1_k$ in the stable reduction of f defined as the Galois closure of

$$\bar{Z}_2 \to \bar{X}_2, \quad (\bar{x}_2, \bar{z}_2) \mapsto \bar{x}_2 = \bar{z}_2^e + \bar{z}_2^{ap}$$

(b) Now suppose that $a < p$ and $ap-e+1 \neq p$ (in particular, $ap-e+1 < p$ or equivalently $e > (a-1)p+1$). Corresponding to $e_1 = ap-e+1$ there occurs a primitive tail cover $\bar{Y}_1 \to \bar{X}_1 = \mathbb{P}^1_k$ in the stable reduction of f defined as the Galois closure of

$$\bar{Z}_1 \to \bar{X}_1, \quad (\bar{x}_1, \bar{z}_1) \mapsto \bar{x}_1 = \bar{z}_1^{ap-e+1} + \bar{z}_1^p$$

Proof: We first prove (a). By Proposition 5.1.1 the genus-0 Belyi morphism $\tilde{\varphi}$ with ramification type $\tilde{C} = (ap; e, ap-e+1, ap)$ is given by

$$x = cz^e\left(\sum_{i=0}^{ap-e}(-1)^{ap-e-i}\frac{1}{e+i}\binom{ap-e}{i}z^i\right) =: c\sum_{j=0}^{ap}a_j z^j.$$

Note that since $(a-1)p < e < ap$ we have $0 < ap-e < p$. In particular, by Corollary 5.1.6

$$\binom{ap-e}{i} \not\equiv 0 \mod p \quad \text{for all } 0 \le i \le ap-e.$$

Moreover, since $0 \le i \le ap-e$ we have $e+i \equiv 0 \mod p$ only for $i = ap-e$. Therefore,

$$v(a_{e+i}) = \begin{cases} 0 & , \text{ if } 0 \le i < ap-e \\ -1 & , \text{ if } i = ap-e. \end{cases}$$

We substitute $z = \mu z_1$ where $v(\mu) = \varepsilon > 0$. Then

$$v(a_{e+i}\mu^{e+i}) = \begin{cases} (e+i)\varepsilon & , \text{ if } 0 \le i < ap-e \\ ap\varepsilon - 1 & , \text{ if } i = ap-e. \end{cases}$$

We equate

$$v(a_e\mu^e) = v(a_{ap}\mu^{ap}),$$

and find $\varepsilon = \frac{1}{ap-e}$. Note that this choice of ε (and consequently that of μ) requires that we enlarge the field K, since we have normalized $v(p) = 1$ (Setup 5.0.1). With this choice of μ it immediately follows that all the other coefficients have strictly larger valuation, since $\varepsilon > 0$. In particular, we can change the coordinates x on X and z on Z by multiples such that

$$\bar{x} = \bar{z}^e + \bar{z}^{ap}.$$

We now prove (b). Note that we restrict to $a < p$. By Proposition 5.1.1 the genus-0 Belyi morphism with ramification type $\mathcal{C} = (ap; ap - e + 1, e, ap)$ is given by

$$x = cz^{ap-e+1} \sum_{i=0}^{e-1} (-1)^{e-1-i} \frac{1}{ap-e+1+i} \binom{e-1}{i} z^i = c \sum_{j=0}^{ap} a_j z^j.$$

Since $(a-1)p < e < ap < p^2$ the unique p-adic expansion of $e-1$ is

$$e - 1 = (e - 1 - (a-1)p) + (a-1)p.$$

Moreover $ap - e + 1 + i \equiv 0 \mod p$ if and only if $i \equiv e - 1 \mod p$ or equivalently (for $0 \leq i < e-1$) if $i \in \{e - (a-\ell)p - 1 \mid 1 \leq \ell \leq a\}$. For $i = e - (a-\ell)p - 1$, the p-adic expansion is

$$\begin{aligned} e - (a-\ell)p - 1 &= (e - 1 - (a-1)p) + (a-1)p - (a-\ell)p \\ &= (e - 1 - (a-1)p) + (\ell - 1)p. \end{aligned}$$

From Lucas' Theorem (see Lemma 5.1.4) and the fact that $\ell \leq a < p$ it follows that

$$\binom{e-1}{e-(a-\ell)p-1} \equiv \binom{e-1-(a-1)p}{e-1-(a-1)p} \binom{a-1}{\ell-1} \not\equiv 0.$$

It follows that $v(a_{\ell p}) = v(c) - 1$ for all $1 \leq \ell \leq a$. In particular, substituting z by μz_1, where μ is an element with valuation $\varepsilon > 0$ yields that the coefficients of z^j have the valuation

$$v(a_j) = \begin{cases} -1 + j\varepsilon, & \text{if } p \mid j \\ j\varepsilon + v\binom{e-1}{j}, & \text{if } p \nmid j. \end{cases} \tag{5.2.5}$$

Equating $v(a_{ap-e+1})$ and $v(a_p)$ and solving for ε, we obtain

$$\varepsilon = \frac{1}{e-(a-1)p-1} > 0.$$

Using (5.2.5) (and the fact that $v\binom{e-1}{j} \geq 0$, since $\binom{e-1}{j}$ is an integer), one immediately verifies that with this choice for ε (and consequently for μ), all the other coefficients have strictly larger valuation. Normalizing z and x suitably we obtain

$$\bar{x} = \bar{z}^{ap-e+1} + \bar{z}^{ap}.$$

□

We now compute several components of the stable reduction of the three-point cover defined by the genus-0 Belyi morphism with ramification type $\mathcal{C} = (e; e - ap + 1, ap, e)$. Note that in Proposition 5.2.8 (a) the result is true for any $a \in \mathbb{N}_{>0}$, but the proof of Proposition 5.2.8 (b) requires $a < p$. Note the similarity to Proposition 5.2.7.

Proposition 5.2.8 *Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the unique genus-0 Belyi morphism with ramification type $\mathcal{C} = (e; e - ap + 1, ap, e)$ where $ap < e < (a+1)p$. Let $f : Y \to X$ be the Galois closure of φ and $\bar{f}$ the stable reduction. Then the following holds.*

(a) Corresponding to $e_3 = e$ there occurs a primitive tail cover $\bar{Y}_3 \to \bar{X}_3 = \mathbb{P}^1_k$ in the stable reduction of f defined as the Galois closure of

$$\bar{Z}_3 \to \bar{X}_3, \quad (\bar{x}_3, \bar{z}_3) \mapsto \bar{x}_3 = \frac{\bar{z}_3^e}{\bar{z}_3^{e-ap} - 1}.$$

(b) Now suppose that $a < p$ and $e_1 = e - ap + 1 \neq p$ (in particular, $e - ap + 1 < p$ or equivalently $e < (a+1)p - 1$). Corresponding to $e_1 = e - ap + 1$ there occurs a primitive tail cover $\bar{Y}_1 \to \bar{X}_1 = \mathbb{P}^1_k$ in the stable reduction of f defined as the Galois closure of

$$\bar{Z}_1 \to \bar{X}_1, \quad (\bar{x}_1, \bar{z}_1) \mapsto \bar{x}_1 = \bar{z}_1^{e-ap+1} + \bar{z}_1^p.$$

Proof: To prove (a) let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be a genus-0 Belyi morphism with ramification type $\tilde{C} = (e; ap, e - ap + 1, e)$ and renormalize it. By Proposition 5.1.1 the genus-0 Belyi morphism $\tilde{\varphi}$ with ramification type $\tilde{C}$ is given by

$$x = cz^{ap} \sum_{i=0}^{e-ap} (-1)^{e-ap-i} \frac{1}{ap+i} \binom{e-ap}{i} z^i.$$

We renormalize $\tilde{\varphi}$ to a genus-0 Belyi morphism with ramification type $\hat{\mathcal{C}} = (e; e, e-ap+1, ap)$ by computing $x = 1/\tilde{\varphi}(1/z)$. We compute

$$x = \frac{z^e}{c \sum_{i=0}^{e-ap} (-1)^{e-ap-i} \frac{1}{ap+i} \binom{e-ap}{i} z^{e-ap-i}} =: \frac{z^e}{c \sum_{i=0}^{e-ap} a_i z^i}.$$

Note that $ap < e < (a+1)p$ implies $0 < e - ap < p$. In particular, Corollary 5.1.6 implies that

$$\binom{e-ap}{i} \not\equiv 0 \mod p \quad \text{for all } 0 \leq i \leq e - ap.$$

This proves that the valuation of the coefficients in the denominator are given by

$$v(a_i) = \begin{cases} 0 & , \text{ for } 0 \leq i < e - ap \\ -1 & , \text{ for } i = e - ap. \end{cases}$$

Substituting z by μz_1, where μ is an element with positive valuation we see that that

$$v(a_i \mu^i) = \begin{cases} i\varepsilon & , \text{ for } 0 \leq i < e - ap \\ -1 + (e-ap)\varepsilon & , \text{ for } i = e - ap. \end{cases}$$

Equating $v(a_{e-ap}\mu^{e-ap})$ and $v(a_0)$ yields

$$\varepsilon = \frac{1}{e - ap} > 0.$$

It is easy to see that with this choice of ε (and consequently of μ) all the other coefficients have strictly larger valuation. This proves that we can change the coordinates x on X and z on Z such that

$$\bar{x} = \frac{\bar{z}^e}{\bar{z}^{e-ap} - 1}.$$

We now prove (b). Recall that $a < p$. We use Proposition 5.1.1 with $\mathcal{C} = (e; e - ap + 1, ap - e, e)$, in particular, the genus-0 Belyi morphism is given by

$$x = cz^{e-ap+1} \sum_{i=0}^{ap-1} (-1)^{ap-1-i} \frac{1}{e - ap + 1 + i} \binom{ap-1}{i} z^i = c \sum_{i=0}^{e} a_j z^j.$$

Note that $e - ap + 1 + i \equiv 0 \mod p$ if $i \equiv -e - 1 \mod p$. Since $0 \leq i \leq ap - 1$ it follows that $e - ap + 1 + i \equiv 0 \mod p$ if and only if $i \in \{(a+\ell)p - e - 1 \mid 1 \leq \ell \leq a\}$. We rewrite this and substitute the indices $e - ap + 1 + i$ by j. Then $i \in \{(a+\ell)p - e - 1 \mid 1 \leq \ell \leq a\}$ is equivalent to $p \mid j$.

By Corollary 5.1.8 we see that

$$v\binom{ap-1}{i} = \begin{cases} v\binom{a-1}{r} & , \text{ if } \exists r \in \mathbb{Z} \text{ with } rp - 1 < i < (r+1)p - 1 \\ v\binom{a-1}{r-1} & , \text{ if } \exists r \in \mathbb{Z} \text{ with } i = rp - 1. \end{cases}$$

Note that $a < p$ implies $0 \leq r \leq a - 1$ in the first case. Moreover, $i = rp - 1$ can only be true if $r \geq 1$. In particular,

$$v\binom{a-1}{r} = 0 = v\binom{a-1}{r-1}.$$

Hence we can ignore the binomial coefficients. Let μ be an element with valuation ε. Then

$$v(a_j\mu^j) = \begin{cases} j\varepsilon & , \text{ if } p \nmid j \\ -1 + j\varepsilon & , \text{ if } p \mid j. \end{cases}$$

Equating $v(a_{e-ap+1}\mu^{e-ap+1})$ and $v(a_p\mu^p)$ and solving for ε yields

$$\varepsilon = \frac{1}{(a+1)p - e - 1} > 0.$$

Substituting z with an element μ with valuation $\varepsilon = 1/((a+1)p - e - 1)$, it is easy to see that all the other coefficients have strictly larger valuation. This proves that we can change the coordinates x and z appropriately and obtain

$$\bar{x} = \bar{z}^{e-ap+1} + \bar{z}^p.$$

□

Corollary 5.2.9 *For every $e > 1$ and $a > 0$ with $0 < |e - ap| < p$ there exists a Galois three-point cover $f : Y \to X = \mathbb{P}^1$ such that the primitive tail covers defined by the Galois closure of*

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^e + \bar{z}^{ap}, & \text{if } e < ap \\ \frac{\bar{z}^e}{\bar{z}^{e-ap}-1}, & \text{if } e > ap, \end{cases}$$

occur in the stable reduction of f.

Proof: Combine Proposition 5.2.7 (a) and Proposition 5.2.8 (a). □

Remark 5.2.10 (i) The reason that the computation of the tail covers in Proposition 5.2.7 and 5.2.8 is easy compared to the computation of the tail cover in Proposition 5.2.5 is the following. The assumption that $a < p$ makes a result such as in Lemma 5.2.4 (b) obsolete. However, we think that the results of Proposition 5.2.7 and 5.2.8 hold for general a. The only restriction is the more complicated calculation. In Example 5.3.7 we compute for an explicit example that our guess is true. Moreover, in the next section we give a further hint why this guess could be true (Remark 5.3.8).

(ii) We guess that whenever there occurs a tail cover of the form

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \bar{z}^e + \bar{z}^{np},$$

then $0 < |np - e| < p$, i.e. $n = \lceil e/p \rceil$. We do not know a proof, but all the computations implicate this. In the following section we elaborate this and hint at a connection between Osserman's characterization of good reduction (Theorem 4.1.7) and the degree of primitive tail covers.

5.3 p-admissibility and a possible correspondence with the degree of tail covers

In this section we hint at a possible connection between Osserman's characterization of good reduction (Theorem 4.1.7) and the degree of tail covers. We prove a correspondence in the case $d < 2p$ and assert that the same correspondence holds for the case $d \geq 2p$ as well. The correspondence in the case $d < 2p$ is a new result and we obtain a beautiful interpretation for Osserman's definition of p-admissibility (and why the involved formulas are so complicated) in terms of tail covers and their degrees.

We use the notation

$$\begin{aligned} \mathcal{C}_1 &= (ap+1; 3, ap-1, ap+1), \quad a \in \mathbb{N}_{>0} \\ \mathcal{C}_2 &= (ap; e, ap-e+1, ap), \quad (a-1)p < e < ap \\ \mathcal{C}_3 &= (e; e-ap+1, ap, e), \quad ap < e < (a+1)p. \end{aligned}$$

of Section 5.2. We introduce the following useful term.

Definition 5.3.1 Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type and $m > 0$ such that $p^m \leq d$. A set $S \subset \{1, 2, 3\}$ which satisfies Definition 4.1.6 (ii) and (iii) but not the final inequality $\sum_{i \in S} e^{[m,d]} + \sum_{i \notin S} e^{[m,u]} \geq p^m$ is called an *m-failing set*.

We show that there exists a correspondence between failing sets and the degree of tail covers. We proceed as follows.

For any given abstract ramification type $\mathcal{C} = (d; e_1, e_2, e_3)$ we compute the numbers $\bar{e}^{[m,u]}, \bar{e}^{[m,d]}, e^{[m,u]}$ and $e^{[m,d]}$ (Definition 4.1.6), even if $p \mid e_1 e_2 e_3$, and check for p-admissibility. Strictly speaking, this is not allowed by Definition 4.1.6. Note however that Osserman's definition of p-admissibility is meant to find triples (e_1, e_2, e_3), such that there exists a three-point cover in characteristic p. In particular, the genus-0 Belyi morphism in characteristic 0 has good reduction! Therefore it is obvious that Osserman excluded all the integers which are divisible by p, since the appropriate genus-0 Belyi morphism in characteristic 0 certainly would have bad reduction to characteristic p. We however want to exploit Osserman's result for stable reduction and understand the complicated formulas.

We now prove a correspondence between failing sets and the degree of primitive tail covers for $d < 2p$. Recall from Theorem 4.2.2 that every primitive tail cover (for $d < 2p$) is the Galois closure of $\bar{\varphi}$ which is either given by $\bar{x} = \bar{z}^e + \bar{z}^p$ or $\bar{x} = \frac{\bar{z}^e}{\bar{z}^{e-p}-1}$. The degree of φ is $d = \max\{e, p\}$.

Proposition 5.3.2 *Let $\mathcal{C} = (d; e_1, e_2, e_3)$ be an abstract ramification type with $d < 2p$ with $\sum_{i=1}^{3}(d_i - p) \leq d - p$, where $d_i = \max\{e_i, p\}$. Then the following holds.*

(a) There exists a unique 1-failing set.

(b) Let S be the unique 1-failing set of (a). Then we have $d_j > e_j$ if and only if $j \notin S$.

Proof: To prove this result we analyze the number of e_i which are greater than p. Note that we used the idea in the proof of Theorem 4.1.1.

Contrary to the proof of Proposition 4.1.8 we do not assume that the e_j are prime to p. Hence we cannot be sure what the coefficients $\bar{e}_j^{[m,d]}$ are (if $e_j = p$, then $e_j^{[m,d]} = 1$. If however $e_j < p$, then consequently $e_j^{[m,d]} = 0$). Note however that $\bar{e}_i^{[m,u]}$ and $e^{[m,u]}$ remain unchanged! Moreover, if $e_j > p$, then $\bar{e}_j^{[m,d]} = 1$ and $e_j^{[m,d]} = e - p$.

Recall from Theorem 4.1.1 that the inequality $d - p \geq \sum_{i=1}^{3}(d_i - p)$ is equivalent to the fact that (e_1, e_2, e_3) is not p-admissible.

1. Case: $e_i \leq p$ **for** $i = 1, 2, 3$.

If there is no $e_i > p$, then the only set satisfying Definition 4.1.6 (ii) is $S = \emptyset$.

We compute $\bar{e}_i^{[m,u]} = 1$ and $e_i^{[m,u]} = p - e_i$ for $i = 1, 2, 3$.

Analogous to the case 1 in the proof of Proposition 4.1.8 we see that $S = \emptyset$ is a 1-failing set. It is unique for obvious reasons.

2. Case: $e_1 \leq e_2 \leq p < e_3$

It follows that the only sets satisfying Definition 4.1.6 (ii) are $S = \emptyset$ and $S = \{3\}$.

We compute $\bar{e}_i^{[m,u]} = \lceil e_i/p \rceil = 1$ and $e_i^{[m,u]} = p - e_i$ for $i = 1, 2$ and since $p < e_3 < 2p$, we have $\bar{e}_3^{[m,u]} = \lceil e_3/p \rceil = 2$, $e^{[m,u]} = 2p - e_3$ and $\bar{e}_3^{[m,d]} = 1, e_3^{[m,d]} = e_3 - p$.

Analogous to case 2 in the proof of Proposition 4.1.8 we see that $S = \emptyset$ is not a 1-failing set, but $S = \{3\}$ is. Hence $S = \{3\}$ is the unique 1-failing set.

3. Case: $e_1 \leq p < e_2 \leq e_3$

The sets satisfying Definition 4.1.6 (ii) are subsets of $\{2, 3\}$.

We compute $\bar{e}_i^{[m,d]} = 1, \bar{e}_i^{[m,u]} = 2, e_i^{[m,u]} = 2p - e_i, e_i^{[m,d]} = e_i - p$ for $i = 2, 3$ and $\bar{e}_1^{[m,u]} = 1, e_1^{[m,u]} = p - e_1$.

Analogous to case 3 in the proof of Proposition 4.1.8 we conclude that $\emptyset, \{2\}$ and $\{3\}$ are no 1-failing sets.

$S = \{2,3\}$**:** One computes that $\sum_{i\in S} \bar{e}^{[m,d]} + \sum_{i\notin S} \bar{e}^{[m,u]} = 1+1+1 = 3$ is odd.

Since (e_1, e_2, e_3) is not p-admissible (Proposition 4.1.8), we see that $S = \{2,3\}$ is the unique set for which the inequality

$$\sum_{i\in S} e^{[m,d]} + \sum_{i\notin S} e^{[m,u]} \geq p^m$$

can fail at all. Note however that since e_1 is potentially equal to p, this has not to be true. We analyze the cases $e_1 = p$ and $e_1 < p$ separately. But first we compute for both cases

$$\sum_{i\in S} e^{[m,d]} + \sum_{i\notin S} e^{[m,u]} = (e_2 - p) + (e_3 - p) + (p - e_1) = e_2 + e_3 - e_1 - p = e_1 + e_2 + e_3 - p - 2e_1 = 2d + 1 - p - 2e_1.$$

We now have to distinguish between the two cases, namely $e_1 = p$ and $e_1 < p$. If $e_1 = p$, then

$$2d + 1 - p - 2e_1 = 2d + 1 - 3p < p \quad \text{if and only if} \quad 2d < 4p - 1.$$

Since $2d \leq 2p - 1$ we have $2d \leq 2(2p-1) = 4p - 2$. This proves that $S = \{2,3\}$ is a 1-failing set if $e_1 = p$. Note that it is unique!

If $e_1 < p$, then $p \nmid e_1 e_2 e_3$ and by Proposition 4.1.8 (e_1, e_2, e_3) is not p-admissible, since $\sum_{i=1}^{3}(d_i - p) \leq d - p$. Since all other subsets of $\{2,3\}$ did not yield a contradiction to p-admissibility, $S = \{2,3\}$ has to fail. In particular, $S = \{2,3\}$ is the unique 1-failing set.

4. Case: all e_i are greater than p

We compute $\bar{e}_i^{[m,u]} = 2$, $\bar{e}_i^{[m,d]} = 1$, $e_i^{[m,u]} = 2p - e_i$, $e_i^{[m,d]} = e_i - p$ for all $i = 1, 2, 3$.

Note all subsets of $\{1,2,3\}$ satisfy Definition 4.1.6 (ii). We abbreviate the computation, since all cases only differ by the cardinality of S.

$|S| = 0$**:** Then $\sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = 2+2+2 = 6$ is even.

$|S| = 1$**:** For simplicity, we take $S = \{3\}$.

Then $\sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = 1+2+2 = 5$ is odd. One computes

$$\sum_{i\in S} e^{[m,d]} + \sum_{i\notin S} e^{[m,u]} = e_3 - p + 2p - e_1 + 2p - e_2 = 3p + e_3 - e_1 - e_2.$$

Since $e_3 > p$, we have $3p + e_3 - e_1 - e_2 > 4p - e_1 - e_2$. Now $4p - e_1 - e_2 \geq p$ if and only if $e_1 + e_2 \leq 3p$. Recall that since $\mathcal{C}$ is an abstract ramification type, we have that $e_1 + e_2 + e_3 = 2d + 1$ (Remark 1.2.4). This implies that

$$e_1 + e_2 + e_3 = 2d + 1 \leq 2(2p-1) + 1 = 4p - 1 = 3p + p - 1.$$

If $e_1 + e_2$ were $> 3p$, then $e_3 < p$, which contradicts the assumption that all e_i are greater than p. Hence $e_1 + e_2 \leq 3p$. This shows that $S = \{3\}$ satisfies Definition 4.1.6, and hence $S = \{3\}$ is no 1-failing set.

$|S| = 2$**:** We compute $\sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = 1+1+2 = 4$ is even.

$S = \{1,2,3\}$**:**

Analogous to case 4 in the proof of Proposition 4.1.8 we conclude that $S = \{1,2,3\}$ is a 1-failing set. This completes the proof of (a). The proof of (b) is just a summary of (a) and Theorem 4.2.2. □

We now show that it is reasonable to guess that the correspondence between m-failing sets and the degree of tail covers extends for $d \geq 2p$. Recall that the three-point covers corresponding to $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ have bad reduction (see Proposition 5.2.1 (a) for $\mathcal{C}_1$, and note that there exists a ramification index which is divisible by p for C_2 and C_3). This implies for $\mathcal{C}_1$ that there exists at least one m-failing set for some $m > 0$, since no e_i is divisible by p. For the other ramification types this is not clear, but we will see that there exists at least one 1-failing set nonetheless.

We now compute all the 1-failing sets for $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ with $a \geq 2$.

The following computations are lengthy, repetitive and neither hard nor complicated. They are only included for completeness and for verification. It is possible (and probably advisable) to skip these computations and go directly to Remark 5.3.4, where we summarize the results and draw the important conclusions.

1. Case: $\mathcal{C}_1 = (ap+1; 3, ap-1, ap+1), p \neq 2,3$ and $a \geq 2$

	$\bar{e}_i^{[m,u]}$	$\bar{e}_i^{[m,d]}$	$e^{[m,u]}$	$e^{[m,d]}$
$e_1 = 3$	1	0	$p-3$	3
$e_2 = ap-1$	a	$a-1$	1	$p-1$
$e_3 = ap+1$	$a+1$	a	$p-1$	1

Table 5.3.1: $m = 1$, $\mathcal{C}_1 = (ap+1; 3, ap-1, ap+1)$ and $p \neq 2,3$, i.e. $p \geq 5$.

Note that $e_3 = ap+1 \geq p$. Moreover, $e_2 > p$ if and only if $a \geq 2$. Since $p \neq 2,3$ we have $p \geq 5$ and hence $e_1 < p$. Hence all sets $S \subset \{1,2,3\}$ satisfying Definition 4.1.6 (ii) are

$$\emptyset, \{2\}, \{3\}, \{2,3\}, \quad \text{with } 2 \in S \text{ if and only if } a \geq 2.$$

We now proceed to check Definition 4.1.6 (iii) on these sets. If Definition 4.1.6 (iii) is satisfied, we check for the last inequality in Definition 4.1.6. In Table 5.3.1 we have summarized all the numbers needed to check for p-admissibility.

$\boldsymbol{S = \emptyset:}$ $\quad \sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = 1 + a + a + 1 = 2a + 2$ is even

$\boldsymbol{S = \{2\}:}$ $\quad \sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = (a-1) + (1 + a + 1) = 2a + 1$ is odd and

$\quad \sum_{i\in S} e_i^{[m,d]} + \sum_{i\notin S} e_i^{[m,u]} = (p-1) + (p-3+p-1) = 3p-5 > p,$ since $p \geq 5$

$\boldsymbol{S = \{3\}:}$ $\quad \sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = a + (a+1) = 2a + 1$ is odd and

$\quad \sum_{i\in S} e_i^{[m,d]} + \sum_{i\notin S} e_i^{[m,u]} = 1 + (1 + p - 3) = p - 1 < p$

$\boldsymbol{S = \{2,3\}:}$ $\quad \sum_{i\in S} \bar{e}_i^{[m,d]} + \sum_{i\notin S} \bar{e}_i^{[m,u]} = (a - 1 + a) + 1 = 2a$ is even.

We see that $S = \{3\}$ is the unique 1-failing set.

2. Case: $\mathcal{C}_2 = (ap; e, ap-e+1, ap)$ with $(a-1)p < e < ap$ and $a \geq 2$

	$\bar{e}_i^{[m,u]}$	$\bar{e}_i^{[m,d]}$	$e^{[m,u]}$	$e^{[m,d]}$
$e_1 = e$	a	$a-1$	$ap-e$ / $p-1$	$e-(a-1)p$ / 1
$e_2 = ap-e+1$	1	0 / 1	$p-(ap-e+1)$ / 0	$ap-e+1$ / 0
$e_3 = ap$	a	a	0	0

Table 5.3.2: $m = 1$, $\mathcal{C}_2 = (ap; e, ap-e+1, ap)$. The left side of the slashes is in the case $e > (a-1)p+1$ (and consequently $e_2 < p$), the right side of the slashes is in the case $e = (a-1)p+1$ (and consequently $e_2 = p$).

Note that $e_2 \leq p$. Moreover, $e_1, e_3 > p$ if and only if $a \geq 2$. Hence all sets $S \subset \{1,2,3\}$ satisfying Definition 4.1.6 (ii) are

$$\emptyset, \{1\}, \{3\}, \{1,3\}, \quad \text{with } 1 \in S \text{ or } 3 \in S \text{ if and only if } a \geq 2.$$

We now proceed to check Definition 4.1.6 (iii) on these sets. If Definition 4.1.6 (iii) is satisfied, we check for the last inequality in Definition 4.1.6. In Table 5.3.2 we have summarized all the numbers needed to check for p-admissibility.

$$\boldsymbol{S=\emptyset:} \quad \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=a+1+a=2a+1 \quad \text{is odd and}$$
$$\sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=p-1<p$$
$$\boldsymbol{S=\{1\}:} \quad \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=(a-1)+(1+a)=2a \quad \text{is even}$$
$$\boldsymbol{S=\{3\}:} \quad \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=a+(a+1)=2a+1 \quad \text{is odd and}$$
$$\sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=p-1<p$$
$$\boldsymbol{S=\{1,3\}:} \quad \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=(a-1+a)+1=2a \quad \text{is even.}$$

We see that $S=\emptyset$ and $S=\{3\}$ are 1-failing sets.

3. Case: $\mathcal{C}_3=(e;e-ap+1,ap,e)$ with $ap<e<(a+1)p$ and $a\geq 2$

	$\bar{e}_i^{[m,u]}$	$\bar{e}_i^{[m,d]}$	$e^{[m,u]}$	$e^{[m,d]}$
$e_1=e-ap+1$	1	$0 \quad / \quad 1$	$p-(e-ap+1) \quad / \quad 0$	$e-ap+1 \quad / \quad 0$
$e_2=ap$	a	a	0	0
$e_3=e$	$a+1$	a	$(a+1)p-e \quad / \quad 1$	$e-ap \quad / \quad p-1$

Table 5.3.3: $m=1$, $\mathcal{C}_3=(e;e-ap+1,ap,e)$. The left side of the slashes is in the case $e<(a+1)p-1$ (and consequently $e_1<p$), the right side of the slashes is in the case $e=(a+1)p-1$ (and consequently $e_1=p$).

Note that $e_1\leq p$ and $e_3>p$. Moreover, $e_2>p$ if and only if $a\geq 2$. Hence all sets $S\subset\{1,2,3\}$ satisfying Definition 4.1.6 (ii) are

$$\emptyset,\{2\},\{3\},\{2,3\}, \qquad \text{with } e_2\in S \text{ if and only if } a\geq 2.$$

We now proceed to check Definition 4.1.6 (iii) on these sets. If Definition 4.1.6 (iii) is satisfied, we check for the last inequality in Definition 4.1.6. In Table 5.3.3 we have summarized all the numbers needed to check for p-admissibility.

$$\boldsymbol{S=\emptyset:} \quad \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=1+a+a+1=2a+2 \quad \text{is even}$$
$$\boldsymbol{S=\{2\}:} \quad \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=a+(1+a+1)=2a+2 \quad \text{is even}$$
$$\boldsymbol{S=\{3\}:} \quad \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=a+(a+1)=2a+1 \quad \text{is odd and}$$
$$\sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=p-1<p$$
$$\boldsymbol{S=\{2,3\}:} \quad \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=(a+a)+1=2a+1 \quad \text{is odd and}$$
$$\sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=p-1<p.$$

We see that the sets $S=\{3\}$ and $S=\{2,3\}$ are 1-failing sets.

We use the following notation throughout the rest of this section.

Notation 5.3.3 Let $e_j > 1$ and $a_j \geq 1$ be integers with $0 < |e_j - a_j p| < p$. We denote by $\bar{\varphi}_j : \bar{Z}_j = \mathbb{P}^1_k \to \bar{X}_j = \mathbb{P}^1_k$ the cover defined by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^{e_j} + \bar{z}^{a_j p} & , \text{ if } e_j < a_j p \\ \frac{\bar{z}^{e_j}}{\bar{z}^{e_j - a_j p} - 1} & , \text{ if } e_j > a_j p. \end{cases}$$

Let $\bar{g}_j : \bar{Y} \to \bar{X}$ denote the Galois closure of $\bar{\varphi}_j$. As usual we let $d_j = \max\{e_j, a_j p\}$ denote the degree of $\bar{\varphi}_j$.

Remark 5.3.4 In the above computations we computed all the 1-failing sets for the ramification types $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. We summarize the results from the above computations in Table 5.3.4.

$\mathcal{C}$			S
$\mathcal{C}_1 = (ap+1; 3, ap-1, ap+1)$			$\{3\}$
$\mathcal{C}_2 = (ap; e, ap-e+1, ap)$	with	$(a-1)p < e < ap$	$\emptyset, \{3\}$
$\mathcal{C}_3 = (e; e-ap+1, ap, e)$	with	$ap < e < (a+1)p$	$\{3\}, \{2,3\}$

Table 5.3.4: Summary of the computations. The sets S are 1-failing sets (Definition 5.3.1).

In Table 5.3.5 we use the knowledge about the tail covers for $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ (Proposition 5.2.1 - Proposition 5.2.8) and show that it is reasonable to assume that there exists a correspondence between m-failing sets and the degree of tail covers for $d \geq 2p$. Note that we have to restrict to $a < p$ for the primitive tail covers corresponding to $ap - e + 1$ (in type $\mathcal{C}_2$) and $e - ap + 1$ (in type $\mathcal{C}_3$) (see Propositions 5.2.7 (b) and 5.2.8 (b)).

$\mathcal{C}$	d_1	up/down	d_2	up/down	d_3	up/down	S
$\mathcal{C}_1 = (ap+1; 3, ap-1, ap+1)$	p	up	ap	up	e	down	$\{3\}$
$\mathcal{C}_2 = (ap; e, ap-e+1, ap)$	ap	up	$p/-$	up/$-$	$-$	$-$	$\emptyset, \{3\}$
$\mathcal{C}_3 = (e; e-ap+1, ap, e)$	$p/-$	up/$-$	$-$	$-$	e	down	$\{3\}, \{2,3\}$

Table 5.3.5: The integer d_i is the degree of $\bar{\varphi}_i : \bar{Z}_i = \mathbb{P}^1_k \to \bar{X}_i = \mathbb{P}^1_k$ (see Notation 5.3.3). "up" indicates that $d_i = a_i p > e_i$, whereas "down" indicates that $a_i p < e_i = d_i$, where a_j are as in Notation 5.3.3. If the corresponding tail cover does not exist, we denote this with "-". Note that we have to restrict to $a < p$ for the tail covers corresponding to $ap - e + 1$ (in type $\mathcal{C}_2$) and $e - ap + 1$ (in type $\mathcal{C}_3$). The slashes indicate that there is a border case, for which $e_i = p$. For $\mathcal{C}_2$ this is $e = (a-1)p + 1$ and for $\mathcal{C}_3$ this is $e = (a+1)p - 1$. The sets S are the 1-failing sets.

Analyzing Table 5.3.5 we see that "up" corresponds to the complement of S.

Supposition 5.3.5 If $p \nmid e_j$, then $d_j = a_j p > e_j$ ("up") if and only if for every $m > 0$, such that $p^m \leq d$, and for every m-failing set S we have $j \notin S$.

Remark 5.3.6 Note that Supposition 5.3.5 is true for $d < 2p$ by Proposition 5.3.2.

Immediately there come several questions into mind. Is Supposition 5.3.5 only true for the three-point covers with the ramification types $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ with $a < p$ and for the three-point covers with $d < 2p$? Is this correspondence only linked to $m = 1$? The general answers (for $a \geq 2$) are unknown, but there is more evidence that this correspondence might still hold (see Example 5.3.7).

Example 5.3.7 (a) Let $\mathcal{C} = (18; 14, 5, 18)$ and $p = 3$. Note that $\mathcal{C} \neq \mathcal{C}_i$ for $i = 1, 2, 3$. The unique normalized genus-0 Belyi morphism with ramification type $\mathcal{C}$ is given by

$$x = 2380z^{18} - 10080z^{17} + 16065z^{16} - 11424z^{15} + 3060z^{14}. \tag{5.3.1}$$

We now compute the primitive tail covers in $x = 0$ and $x = 1$.

The valuation of the coefficients is summarized in Table 5.3.6.

We multiply the coordinate on Z with μ, where $v(\mu) = 1$. Then it follows that the coefficients of z^{14} and z^{15} have the least valuation, i.e. all other coefficients have strictly larger valuation. Hence we can

	z^{18}	z^{17}	z^{16}	z^{15}	z^{14}
valuation	0	2	3	1	2

Table 5.3.6: Valuation of the coefficients of (5.3.1).

change the coordinates on X and Z such that corresponding to $e = 15$ there occurs a primitive tail cover which is defined by the Galois closure of the cover given by

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \bar{z}^{14} + \bar{z}^{15}.$$

We now compute the tail cover for $e = 5$. The unique normalized genus-0 Belyi morphism with ramification type $\tilde{\mathcal{C}} = (18; 5, 14, 18)$ is given by

$$\begin{aligned} x = &- 2380z^{18} + 32760z^{17} - 208845z^{16} + 816816z^{15} - 2187900z^{14} + 4241160z^{13} - 6126120z^{12} \\ &+ 6683040z^{11} - 5513508z^{10} + 3403400z^{9} - 1531530z^{8} + 477360z^{7} - 92820z^{6} + 8568z^{5}. \end{aligned} \tag{5.3.2}$$

The valuation of the coefficients is in Table 5.3.7.

	z^{18}	z^{17}	z^{16}	z^{15}	z^{14}	z^{13}	z^{12}	z^{11}	z^{10}	z^{9}	z^{8}	z^{7}	z^{6}	z^{5}
valuation	0	2	3	1	2	4	2	3	4	0	2	3	1	2

Table 5.3.7: Valuation of the coefficients of (5.3.2).

Substituting $z = \mu z_1$, where $v(\mu) = 1$, we see that all coefficients have strictly larger valuation than the coefficients of z_1^6 and z_1^5. Normalizing we see that for $e = 5$ there occurs the primitive tail cover defined by the Galois closure of

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \bar{z}^{5} + \bar{z}^{6}.$$

Now we determine the m-failing sets. Since we have $18 > 3^2 = 9$ we have to compute the integers $\bar{e}^{[m,u]}, \bar{e}^{[m,d]}, e^{[m,u]}$ and $e^{[m,d]}$ for $m = 1$ and $m = 2$. We obtain Table 5.3.8.

	$\bar{e}^{[1,u]}$	$\bar{e}^{[1,d]}$	$e^{[1,u]}$	$e^{[1,d]}$	$\bar{e}^{[2,u]}$	$\bar{e}^{[2,d]}$	$e^{[2,u]}$	$e^{[2,d]}$
$e_1 = 5$	2	1	1	2	1	0	4	5
$e_2 = 14$	5	4	1	2	2	1	4	5
$e_3 = 18$	6	6	0	0	2	2	0	0

Table 5.3.8: Numbers for p-admissibility for $\mathcal{C} = (18; 5, 14, 18)$.

First we determine the 1-failing sets.

Since all $e_i > p = 3$, Definition 4.1.6 (ii) is satisfied for every $S \subset \{1, 2, 3\}$. Thus we have to compute Definition 4.1.6 (iii) and (if necessary) the term $\sum_{i \in S} e^{[m,d]} + \sum_{i \notin S} e^{[m,u]}$ and determine whether the set S is a 1-failing set or not, for 8 cases. For the convenience of the reader the whole computation is included. One may easily skip this computation and see the summary below.

$$
\begin{aligned}
\boldsymbol{S=\emptyset:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=2+5+6=13 \quad \text{is odd and}\\
& \sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=1+1=2<3\\
\boldsymbol{S=\{1\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=1+5+6=12 \quad \text{is even}\\
\boldsymbol{S=\{2\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=4+2+6=12 \quad \text{is even}\\
\boldsymbol{S=\{3\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=6+2+5=13 \quad \text{is odd and}\\
& \sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=1+1=2<3\\
\boldsymbol{S=\{1,2\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=(1+4)+6=11 \quad \text{is odd and}\\
& \sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=2+2=4>3\\
\boldsymbol{S=\{1,3\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=(1+6)+5=12 \quad \text{is even}\\
\boldsymbol{S=\{2,3\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=(4+6)+2=12 \quad \text{is even}\\
\boldsymbol{S=\{1,2\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=(1+4)+6=11 \quad \text{is odd and}\\
& \sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=2+2=4>3.
\end{aligned}
$$

We conclude that the 1-failing sets are $S=\emptyset$ and $S=\{3\}$.

We proceed with the computation of the 2-failing sets. Note that $5<3^2=9$ and $14,18>9$. Hence the sets satisfying Definition 4.1.6 (ii) are subsets of $\{2,3\}$.

$$
\begin{aligned}
\boldsymbol{S=\emptyset:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=1+2+2=5 \quad \text{is odd and}\\
& \sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=4+4=8<3^2=9\\
\boldsymbol{S=\{2\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=1+1+2=4 \quad \text{is even}\\
\boldsymbol{S=\{3\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=2+1+2=5 \quad \text{is odd and}\\
& \sum_{i\in S}e_i^{[m,d]}+\sum_{i\notin S}e_i^{[m,u]}=4+4=8<3^2=9\\
\boldsymbol{S=\{2,3\}:}\quad & \sum_{i\in S}\bar{e}_i^{[m,d]}+\sum_{i\notin S}\bar{e}_i^{[m,u]}=(1+2)+2=4 \quad \text{is even.}
\end{aligned}
$$

We conclude that the 2-failing sets are $S=\emptyset$ and $S=\{3\}$.

Summary: We see that the 1-failing sets and 2-failing sets coincide and are $S=\emptyset$ and $S=\{3\}$. We see that Supposition 5.3.5 is satisfied.

(b) Let $\mathcal{C}=\mathcal{C}_2=(12;2,11,12)$ with $p=3, a=4=p+1$ and $e=11$. For $e=11$ we know the tail cover by Proposition 5.2.7 (a). It is the Galois closure defined by

$$(\bar{x},\bar{z})\mapsto \bar{x}=\bar{z}^{11}+\bar{z}^{12}.$$

We now compute the tail cover for $e = 2$.

Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the unique normalized genus-0 Belyi morphism with ramification type $\mathcal{C} = (12; 2, 11, 12)$ (one can use the formula of Proposition 5.1.1 or alternatively the formula of Proposition 5.1.2 to compute φ). Then

$$x = 11z^{12} - 120z^{11} + 594z^{10} - 1760z^9 + 3465z^8 - 4752z^7 + 4620z^6 - 3168z^5 + 1485z^4 - 440z^3 + 66z^2. \quad (5.3.3)$$

The valuation of the coefficients is summarized in Table 5.3.9.

	z^{12}	z^{11}	z^{10}	z^9	z^8	z^7	z^6	z^5	z^4	z^3	z^2
valuation	0	1	3	0	2	3	1	2	3	0	1

Table 5.3.9: Valuation of the coefficients of (5.3.3).

Multiplying z with an element with valuation 1 one easily sees that the coefficients of z^2 and z^3 have the lowest valuation and all other coefficients have strictly larger valuation. Normalizing we see that for $e = 2$ there occurs the primitive tail cover defined as the Galois closure of

$$\bar{x}_1 = \bar{z}_1^2 + \bar{z}_1^3.$$

This yields the expected primitive tail cover.

Note that Table 5.3.4 is still correct, since we have not used $a < p$ in the computation for this table. In particular, we already know the 1-failing sets.

Since $a = 4 > p = 3$ we have to check the 2-failing sets as well. In Table 5.3.10 we have summarized the numbers which we need to compute the 2-failing sets.

	$\bar{e}_i^{[2,u]}$	$\bar{e}_i^{[2,d]}$	$e^{[2,u]}$	$e^{[2,d]}$
$e_1 = 2$	1	0	7	2
$e_2 = 11$	2	1	7	2
$e_3 = 12$	2	1	6	3

Table 5.3.10: The numbers for the computation of the 2-failing sets.

Note that $2 < 3^2 = 9$ and $11, 12 > 9$. Hence Definition 4.1.6 (ii) is satisfied for $S \subset \{2, 3\}$.

$$\begin{aligned}
\boldsymbol{S = \emptyset:} \quad & \sum_{i\in S} \bar{e}^{[m,d]} + \sum_{i\notin S} \bar{e}^{[m,u]} = 1 + 2 + 2 = 5 \quad \text{is odd} \\
& \sum_{i\in S} e^{[m,d]} + \sum_{i\notin S} e^{[m,u]} = 6 + 7 + 7 = 20 > 9 \\
\boldsymbol{S = \{2\}:} \quad & \sum_{i\in S} \bar{e}^{[m,d]} + \sum_{i\notin S} \bar{e}^{[m,u]} = 1 + 1 + 2 \quad \text{is even} \\
\boldsymbol{S = \{3\}:} \quad & \sum_{i\in S} \bar{e}^{[m,d]} + \sum_{i\notin S} \bar{e}^{[m,u]} = 1 + 1 + 2 \quad \text{is even} \\
\boldsymbol{S = \{2,3\}:} \quad & \sum_{i\in S} \bar{e}^{[m,d]} + \sum_{i\notin S} \bar{e}^{[m,u]} = 1 + 1 + 1 = 3 \quad \text{is odd} \\
& \sum_{i\in S} e^{[m,d]} + \sum_{i\notin S} e^{[m,u]} = 2 + 3 + 7 = 12 > 9
\end{aligned}$$

We conclude that there exist no 2-failing sets. In particular, we see that Supposition 5.3.5 is true for at least one three-point cover of type $\mathcal{C}_2$ with $a > p$.

Remark 5.3.8 In Remark 5.2.10 (a) we said that we would give a further hint, why we believe that there is a primitive tail cover defined by $\bar{x} = \bar{z}^{ap-e+1} + \bar{z}^p$ for $\mathcal{C}_2 = (ap; e, ap - e + 1, ap)$ and $\bar{x} = \bar{z}^{e-ap+1} + \bar{z}^p$ for $\mathcal{C}_2 = (e; e - ap + 1, ap, e)$ for any $a \in \mathbb{N}_{>0}$. Recall that the 1-failing sets correspond to these tail covers for $a < p$ (see Remark 5.3.4). For $a > p$ the 1-failing sets remain the same. In Example 5.3.7 (b) for $\mathcal{C} = (12; 2, 11, 12)$ we verified that the correspondence between 1-failing sets and the degree of the tail covers holds. Therefore we guess that the correspondence between 1-failing sets and the degree of the tail cover holds for arbitrary a.

5.4 Nonmild reduction

In this section we show that there exist three-point covers with nonmild reduction.

We refer to Section 2.3 for the structure of the wild inertia group of the primitive tail covers we discuss in Lemma 5.4.1 (see Theorem 2.3.11 for a summary of the results on the structure).

Lemma 5.4.1 *Let a, e, d be positive integers with $e > 1$, $a < p$ and $0 < |e - ap| < p$. Let $\bar{\varphi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be defined by*

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^e + \bar{z}^{ap}, & \text{if } e < ap \text{ or} \\ \frac{\bar{z}^e}{\bar{z}^{e-ap}-1}, & \text{if } e > ap \end{cases}$$

Let $\bar{g} : \bar{Y} \to \bar{X}$ be the Galois closure of $\bar{\varphi}$ and $I_{\bar{y}} = I^{(p)} \rtimes \langle \tau \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^b \rtimes_\chi \mathbb{Z}/m\mathbb{Z}$ the wild inertia group of a wild ramification point. Let $m_1 = |\ker \chi|$ and $m_2 = m/m_1$. Assume that the action of $\langle \tau \rangle$ on $I^{(p)}$ is irreducible. Then the following holds.

$$m_2 = \frac{a(p-1)}{\gcd(e - ap, a(p-1))}.$$

Proof: Since the action of $\langle \tau \rangle$ on $I^{(p)}$ is irreducible, there exists only one lower jump $\neq 0$ (Lemma 1.3.8 (a)). Hence by Lemma 2.4.1 (a) we know that $\sigma = \frac{|e-ap|}{a(p-1)} = \frac{h}{m}$, where h denotes the lower jump $\neq 0$. Moreover, by Lemma 1.3.8 (b) we know that $\gcd(h, m) = m_1$. Define $\tilde{h} := h/m_1$. Then

$$\sigma = \frac{h}{m} = \frac{\tilde{h} m_1}{m_2 m_1} = \frac{\tilde{h}}{m_2},$$

where $\tilde{h}$ and m_2 are relatively prime. Since $|e - ap|/\gcd(e - ap, a(p-1))$ and $a(p-1)/\gcd(e - ap, a(p-1))$ are relatively prime and $m_2 \mid a(p-1)$, it follows that

$$m_2 = \frac{a(p-1)}{\gcd(e - ap, a(p-1))}.$$

□

We now prove a useful criterion which can be used to show that a certain three-point cover has nonmild reduction.

Lemma 5.4.2 *Let a, e, d be positive integers with $e > 1$, $a < p$ and $0 < |e - ap| < p$. Let $\bar{\varphi} : \bar{Z} = \mathbb{P}^1_k \to \bar{X} = \mathbb{P}^1_k$ be defined by*

$$(\bar{x}, \bar{z}) \mapsto \bar{x} = \begin{cases} \bar{z}^e + \bar{z}^{ap}, & \text{if } e < ap \text{ or} \\ \frac{\bar{z}^e}{\bar{z}^{e-ap}-1}, & \text{if } e > ap \end{cases}$$

Let $\bar{g} : \bar{Y} \to \bar{X}$ be the Galois closure of $\bar{\varphi}$ and $I_{\bar{y}} = I^{(p)} \rtimes \langle \tau \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^b \rtimes_\chi \mathbb{Z}/m\mathbb{Z}$ the wild inertia group of a wild ramification point. Assume that $b = 1$. In particular, $I^{(p)}$ can be generated by one single element of order p, which we denote by γ and there exists an $\alpha \in \mathbb{F}_p^*$ such that $\tau\gamma\tau^{-1} = \gamma^\alpha$. Let $m_1 = |\ker \chi|$ and $m_2 = m/m_1$. Then the following holds.

(a) There exists $\beta \in \mathbb{F}_p^*$ with $\beta^a = \alpha$.

(b) $m_2 = \frac{a(p-1)}{\gcd(e-ap, a(p-1))} \mid p - 1$.

Proof: We prove (a).

Let $M(\tau)$ be the matrix representation of the conjugation action of τ on the $\mathbb{F}_p$ vector space $P_a = \langle \rho_1, \ldots, \rho_a \rangle$ (recall the definition in (2.3.1)) as in the proof of Lemma 2.3.10. The assumption that $b = 1$ implies that $M(\tau)$ has at least one eigenvalue in $\mathbb{F}_p^*$. In particular, the characteristic polynomial $t^a - \alpha$ of $M(\tau)$ has at least one root in $\mathbb{F}_p^*$. It follows that there exists a $\beta \in \mathbb{F}_p^*$ with $\beta^a = \alpha$.

We now prove statement (b). By definition, $m_1 = |\ker \chi|$ and hence m_2 is the cardinality of the image of χ. Since $b = 1$, it follows that χ maps to $\mathbb{F}_p^*$. Since the image of χ is a subgroup of $\mathbb{F}_p^*$, it follows that $m_2 \mid (p-1)$.

The assumption that $b = 1$ implies that the action of τ on $I^{(p)}$ is irreducible and hence we can use Lemma 5.4.1. The assertion follows. □

We now use Lemma 5.4.2 (b) to show that the three-point covers with ramification type $\mathcal{C} = (ap+1; 3, ap-1, ap+1)$ have nonmild reduction. Recall from Definition 3.2.1 that if a three-point cover has mild reduction, then p at most exactly divides the order of any inertia group, i.e. if p divides the order, then p^2 does not.

Corollary 5.4.3 *Let $\varphi : Z = \mathbb{P}^1 \to X = \mathbb{P}^1$ be the unique genus-0 Belyi morphism of type $\mathcal{C} = (ap+1; 3, ap-1, ap+1)$ and $f : Y \to X$ its Galois closure. Then f does not have mild reduction for any $a \geq 2$.*

Proof: Recall from Proposition 5.2.1 (c) that the tail cover for $e = ap+1$ is $\bar{g}_3 : \bar{Y}_3 \to \bar{X}_3 = \mathbb{P}^1_k$, where $\bar{g}_3$ is the Galois closure of

$$(\bar{x}_3, \bar{z}_3) \mapsto \bar{x}_3 = \frac{\bar{z}_3^{ap+1}}{\bar{z}_3 - 1}.$$

Let $\bar{y} \in \bar{Y}_3$ be a wild ramification point and $I_{\bar{y}} \simeq (\mathbb{Z}/p\mathbb{Z})^b \rtimes \mathbb{Z}/m\mathbb{Z}$ the inertia group of $\bar{y}$. If the stable reduction of f were mild, then we would have that $b = 1$. Hence, Lemma 5.4.2 would be applicable. We compute that

$$\frac{a(p-1)}{\gcd(e-ap, a(p-1))} = \frac{a(p-1)}{\gcd(1, a(p-1))} = a(p-1) \nmid p-1.$$

Hence $b \neq 1$ and the stable reduction of f is nonmild. This proves that f does not have mild reduction for any $a \geq 2$. □

Remark 5.4.4 If $\gcd(e, a(p-1)) < a < p$, then the proof of Corollary 5.4.3 yields that the three-point covers with ramification type

$$\mathcal{C} = (ap; e, ap-e+1, ap) \quad \text{and} \quad \mathcal{C} = (e; e-ap+1, ap, e)$$

have nonmild reduction as well. We assume that in all the other cases, the stable reduction is nonmild, too. But we don't know a proof yet.

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Zusammenfassung

In der vorliegenden Arbeit betrachten wir die stabile Reduktion einer gewissen Klasse von 3-Punkt-Überlagerungen. Hierbei haben wir stets folgende Situation:

R ist ein vollständiger, diskreter Bewertungsring, K der Quotientkörper von R und k der Restklassenkörper. Wir nehmen an, dass K Charakteristik 0 und k Charakteristik $p > 0$ hat. Wir nehmen ferner an, dass alle Kurven stets glatt, projektiv und absolut irreduzibel sind.

Die 3-Punkt-Überlagerungen in dieser Arbeit sind stets von folgender Machart: $f_K : Y_K \to X_K = \mathbb{P}^1_K$ ist der Galois-Abschluss einer 3-Punkt-Überlagerung $\varphi_K : Z_K = \mathbb{P}^1_K \to X_K = \mathbb{P}^1_K$ vom Grad $d > 1$, d.h. φ_K ist eine endliche Überlagerung vom Grad $d > 1$ die über genau 3 Punkten $0, 1, \infty \in X_K$ verzweigt ist. Zusätzlich setzen wir stets voraus, dass φ_K in genau 3 Punkten $z_1, z_2, z_3 \in Z$ verzweigt ist. Wir bezeichnen die Verzweigungsindizes mit $e_j = e(z_j)$ und nennen $\mathcal{C} = (d; e_1, e_2, e_3)$ den *Verzweigungstyp* von φ_K

Sei G die Galoisgruppe von f_K. Nach eventueller Vergrößerung von K durch eine endliche Körpererweiterung gibt es semistabile Modelle $\tilde{\mathcal{Y}}$ von Y über R, sodass die Verzweigungspunkte in der speziellen Faser zu glatten, paarweise verschiedenen Punkten spezialisieren. Sei nun $\mathcal{Y}$ unter all diesen Modellen das minimale (bezüglich Dominanz) und definiere $\mathcal{X} := \mathcal{Y}/G$. Die Abbildung $f_R : \mathcal{Y} \to \mathcal{X}$ heißt das *stabile Modell* von f_K und die Einschränkung auf die spezielle Faser $\bar{f} : \bar{Y} \to \bar{X}$ heißt die *stabile Reduktion* von f_K. Wir sagen, dass f_K *gute Reduktion* hat, falls $\bar{f}$ separabel ist. Andernfalls sagen wir, dass f_K *schlechte Reduktion* hat.

Die 3-Punkt-Überlagerungen in unserer Situation haben eine sehr starke und reichhaltige Struktur. Dies sieht man bereits daran, dass der Grad d von φ_K zusammen mit den Verzweigungsindizes e_1, e_2, e_3 die 3-Punkt-Überlagerung φ_K, bis auf Normalisierung, eindeutig bestimmen (Lemma 1.2.6). Des Weiteren ist φ_K sogar über $\mathbb{Q}$ definierbar. Natürlich ist mit φ_K auch f_K eindeutig bestimmt.

Die 3-Punkt-Überlagerungen mit guter Reduktion sind vollständig charakterisiert. Osserman hat in [Oss08] ein notwendiges und hinreichendes Kriterium bewiesen, welches nur mit Hilfe des Datums $\mathcal{C} = (d; e_1, e_2, e_3)$ bestimmt, ob es in positiver Charakteristik eine 3-Punkt-Überlagerung vom Grad d gibt, die in genau 3 Punkten mit den Verzweigungsindizes e_1, e_2, e_3 verzweigt ist ([Oss08, Theorem 2.4] oder Theorem 4.1.7). Ossermans Kriterium kann als Resultat zur guten Reduktion interpretiert werden. Die Existenz einer 3-Punkt-Überlagerung mit Verzweigungstyp $\mathcal{C}$ in Charakteristik $p > 0$ ist äquivalent dazu, dass die eindeutige 3-Punkt-Überlagerung in Charakteristik 0 mit Verzweigungstyp $\mathcal{C}$ gute Reduktion hat. Somit haben wir ein Kriterium um zu prüfen wann eine 3-Punkt-Überlagerng gute Reduktion besitzt und wissen, dass die stabile Reduktion einer 3-Punkt-Überlagerung mit guter Reduktion wieder eine 3-Punkt-Überlagerung ist. Doch wie sieht die stabile Reduktion einer 3-Punkt-Überlagerung mit schlechter Reduktion aus?

Für allgemeines d ist nichts bekannt. Für $d < 2p$ gibt es jedoch wichtige Strukturaussagen! Wir erinnern ganz kurz an diese und nehmen deshalb von nun an $d < 2p$ an.

- Wewers hat in [Wew03] gezeigt, dass falls f_K schlechte Reduktion hat, dann ist $\bar{X}$ ein Kamm. Dabei gibt es genau eine zentrale Komponente $\bar{X}_0 \subset \bar{X}$, sodass die Einschränkung von $\bar{f}$ auf eine irreduzible Komponente über $\bar{X}_0$, inseparabel ist. Die Einschränkung von $\bar{f}$ auf auf eine irreduzible Komponente über einer Komponente $\neq \bar{X}_0$ ist separabel und wir nennen eine solche Abbildung eine Überlagerung eines Endes. Von diesen Überlagerungen von Enden gibt es wiederum 2 Arten, abhängig davon wie viele Verzweigungspunkte diese besitzen (siehe Kapitel 3.2 für mehr Details).
- Wewers beweist in [Wew03] des Weiteren ein Hochhebungsergebnis ([Wew03, Corollary 4.6] oder Theorem 3.3.4), welches besagt, dass zu jeder 3-Punkt-Überlagerung mit schlechter Reduktion genau eine eindeutige Menge an Überlagerungen von Enden zugeordnet ist, die mit einem eindeutig bestimmten sogenannten Deformationsdatum gewisse Verträglichkeitsbedingungen erfüllen. Jede Menge von Überlagerungen von Enden, die mit einem Deformationsdatum diese Verträglichkeitsbedingungen erfüllen, können nach Charakteristik 0 "hochgehoben" werden, d.h. dass sie von der stabilen Reduktion einer 3-Punkt-Überlagerung abstammen.

Somit hat Wewers für $d < 2p$ wichtige Strukturaussagen zu der stabilen Reduktion von 3-Punkt-Überlagerungen getroffen. Jedoch sagen die Resultate nicht, wie viele Überlagerungen von Enden es gibt. Sie beschreiben weder die Überlagerungen der Enden noch das Deformationsdatum. Unser Hauptresultat besteht darin, dass wir genau diese Wissenslücke schließen und sämtliche fehlende Informationen bestimmen (Theorem 4.2.2).

Des Weiteren betrachten wir Überlagerungen von Enden als eigenständige Objekte und analysieren für mehrere Klassen dieser Überlagerungen Eigenschaften der wilden Verzweigung (Kapitel 2, Theorem 2.3.11, Lemma 2.4.1). Hierbei verallgemeinern wir die Überlagerungen von Enden zu beliebig hohem Grad und zeigen, dass diese Überlagerungen in der stabilen Reduktion von 3-Punkt-Überlagerungen (mit ebenfalls beliebig hohem Grad) auftreten (Corollary 5.2.9)

Ein weiterer Schwerpunkt dieser Arbeit ist die Vertiefung des Verständnisses von Ossermans Charakterisierung der guten Reduktion (Theorem 4.1.7). Wir vereinfachen sein kombinatorisches Resultat im Fall von $d < 2p$ erheblich (Theorem 4.1.1). Darüberhinaus beweisen wir für $d < 2p$ eine geometrische Interpretation seines Resultats im Falle von schlechter Reduktion (Proposition 5.3.2).

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig angefertigt habe. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht.

Ulm, den July 7, 2015 _____

(Unterschrift)

Danksagung

An dieser Stelle möchte ich mich ganz herzlich bei meiner Betreuerin Frau Prof. Dr. Irene Bouw für das interessante und herausfordernde Thema bedanken. Auch wenn wir bei einigen Kleinigkeiten wie z.B. der Kommasetzung, oft unterschiedlicher Ansicht waren (und sie hatte meistens Recht), konnte ich mich doch stets auf ihre Unterstützung verlassen.

Des Weiteren möchte ich mich bei der Sekretärin der Abteilung, Frau Bernadette Maiwald, bedanken. Sie hat durch ihre freundliche und lustige Art und Weise stets ein gutes und angenehmes Klima verbreitet.

Ich bedanke mich auch ganz herzlich bei allen anderen Mitgliedern der Abteilung für die netten Gespräche, die Bergtouren und die vielen Momente die einem einfach in Erinnerung geblieben sind.

Des Weiteren danke ich ganz herzlich den Ärzten und dem Pflegepersonal der Abteilung für Visceral- und Thoraxchirurgie des Bundeswehrkrankenhauses Ulm. Ohne sie wäre diese Arbeit definitiv nie zustande gekommen.

Ich danke auch ganz herzlich meiner Frau Michaela Eskin dafür, dass sie die Arbeit auf Fehler geprüft hat, mir stets eine große Hilfe war und mir oft den Rücken freigehalten hat.

Ich danke auch meiner Schwester Xenia Boutchard dafür, dass sie sich die Zeit genommen hat meine Einleitung und Zusammenfassung auf sprachliche Richtigkeit zu überprüfen.

Ich danke auch meinen Freunden und meiner Familie die mir geholfen haben in schweren Zeiten auch mal abzuschalten.

Curriculum Vitae

Michael Eskin

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