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Essays on Time-Consistent Asset Allocation and Participating Life Insurance Contracts

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Overview of Research Papers

Research papers included in this dissertation

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Co-Authorship

Mitja Stadje

Mitja Stadje is a full professor at the Institute of Insurance Science at Ulm University (Helmholtzstraße 20, 89081 Ulm). He received his doctoral degree from Princeton University, USA, in 2009 and is a professor in Ulm since 2015. His contact is: mitja.stadje@uni-ulm.de.

Research Context and Summary of Research Papers

1 Field of Research

This cumulative thesis contributes to the field of portfolio optimization with time-consistent asset allocation and addressing problems which need less regularity compared to the standard literature. Specifically, it introduces a regularization of the Itô-Ventzell formula and explores optimizations for participating life insurance contracts.

The concept of asset allocation dates back to 1952, when Markowitz [57] was the first to study such a problem in a static, one period framework. The key characteristic of such optimization problems is the trade-off between maximizing the expected portfolio return and penalizing the associated risk. A particularly challenging aspect is determining how to appropriately measure risk, as there is no unique accepted method. As a result, there exist a variety of different risk measures, such as variance and Value at Risk, with different features. Hence, each risk measure has its own set of advantages and disadvantages, and regulators enforce different risk measure depending on the industry. For example, insurance companies are required to use Value at Risk as a standard risk measure. Since Markowitz, a continuous stream of publications have generalized his framework in multiple directions. These include both static models, see, for instance, Campbell et al. [15], Rockafeller and Ursayev [61], Ruszczyńsky and Vanderbei [62], Alexander and Baptista [3], Jin et al. [46], Gülpınar and Rustem [38], Adam et al. [1], or He et al. [40], and dynamic models, see, for instance, Basak and Chabakauri [7], Brandt [13], Fu et al. [34], Zeng and Li [66], Hu et al. [44], Czichowsky [22], or Björk et al. [9]. In dynamic settings, time-inconsistency arises, meaning that the optimal strategy changes over time, even when the strategy is executed correctly. There are three main approaches to handle time-inconsistency: The "pre-committed" investor maximizes the objective function once at the outset, see, for instance, Zhou and Li [67]; the "naive" investor continuously optimizes, see, for instance, Basak and Chabakauri [7]; and the "sophisticated" investor optimizes only over strategies she/he would actually follow, see, for instance, Björk and Murgoci [10].

Developed by Ventzell [65] in 1965, the Itô-Ventzell formula generalizes the Itô formula to stochastic flows, which can be described as random functions. In the literature, there is still

an ongoing stream of generalizations of the Itô and the Itô-Ventzell formula, see, for instance, Föllmer et al. [33], Elworthy et al. [30], da Prato et al. [23], Kunita [52], Flandoli and Russo [32], Krylov [51], Karachanskaya [49], Kohlmann et al. [50], or dos Reis and Platonov [26]. However, applying these formulas often leads ro challenges regarding the regularity resp. smoothness of the underlying functions. To address these issues, e.g., Gozzi and Russo [37] regularized the classical Itô formula to $C^{0,1}$ instead of $C^{1,2}$. Several above mentioned generalizations of the Itô formula have also been regularized until today, see, for instance, Bouchard et al. [11], or Bouchard and Vallet [12], though many potential regularizations remain open for further research.

Participating life insurance contracts involve surplus participation of the policyholder, typically structured as a call option (or similar). As a result, the payoffs are mostly neither differentiable nor convex or concave, making some traditional optimization approaches inapplicable. Despite this, many insurance companies offer a wide range of such contracts. In Europe, approximately a quarter of the gross premiums in 2022 are spent by policyholder on these types of contracts, according to the European Insurance Overview 2023 [27] issued by the European Insurance and Occupational Pensions Authority (EIOPA). Consequently, a growing body of literature addresses optimization problems related to surplus participation, see, for instance, Bryis and de Varenne [14], Bacinello and Persson [6], Gatzert and Kling [35], Schmeiser and Wagner [63], Lin et al. [55], Chen et al. [17], Mirza and Wagner [59], Nguyen and Stadje [60], He et al. [39], or Dong et al. [25]. One common approach to solve these optimization problems, particularly from the perspective of the equity holders, involves using Lagrange multipliers, see, for instance, Basak and Shapiro [8], Cuoco et al. [21], Chen et al. [17], Nguyen and Stadje [60], Mi et al. [58], or Avanzi et al. [5]. When considering the insurance company's perspective, there are different approaches to apply like Leland's model, which originally determines the optimal capital structure of a firm. The basic version of this model was developed by Leland [53] in 1994 for an infinite time horizon, and by Leland and Toft [54] in 1996 for a finite time horizon. This model has since been widely generalized to various fields, see, for instance, Ju et al. [48], Liu et al. [56], Hennessy and Tserlukevich [42], Agarwal et al. [2], Elkamhi et al. [29], Glasserman and Nouri [36], Hugonnier et al. [45], He and Milbradt [41], Chen et al. [18], Ambrose et al. [4], Della Seta et al. [24], Carey and Gordy [16], or Hilpert et al. [43].

2 Motivation and Objectives

How can an investor optimally allocate capital across different assets? This is a fundamental question for both individuals and institutions, including banks and insurance companies. However, there is no general answer due to varying personal preferences or regulatory requirements. In particular, dealing with risk aversion in optimization, which can be addressed in multiple

ways, leads to significant variations in the result. For instance, insurance companies are required by Solvency II regulations to use Value at Risk, a quantile-based risk measure. Additionally, institutional investors, such as insurance companies, face the challenge of distributing the total wealth between equity holders and policyholders. Hence, the optimal strategy is affected by the taken perspective. A prominent example of this issue is participating life insurance contracts, where policyholders receive additional payouts if the portfolio performs well. This surplus participation is typically modeled as a call option, which introduces the difficulty of non-smooth payoffs, i.e., some standard approaches require modification or are inapplicable. Therefore, it is essential to address optimal investment problems under quantile-based risk measure, to generalize mathematical theorems such that less smoothness of the underlying functions can be assumed, and to analyze participating life insurance contracts from various perspectives.

Dynamic asset allocation models introduce time-inconsistency, leading to new challenges in determining optimal investment strategies, particularly for "sophisticated" investors. Mathematically, this relates to a non-cooperative interpersonal game, where defining optimality is a central question. The main approach in this context is to consider a Nash-subgame equilibrium as the optimal strategy. However, extending this concept to continuous time is not straightforward, as single points in time form null sets. A natural generalization was introduced by Ekeland and Lazrak [28], who introduced coalitions of players in a short time period and derived the equilibrium by taking the limit of the period length to 0. Björk and Murgoci [10] developed a general theory for such problems, but their framework does not include quantile-based risk measure like Value at Risk. Moreover, the classical Black-Scholes market admits several disadvantages compared to the more general Lévy processes, as discussed by Cont and Tankov [20]. Furthermore, once an optimal equilibrium strategy is determined, it is crucial to understand its characteristics and implications. This leads to the following research questions:

1. How can one calculate an optimal equilibrium strategy for a "sophisticated" investor, considering the reward and risk of her/his investment in a financial market modeled by α-stable Lévy processes for quantile-based risk measures?

2. What properties does such an optimal strategy have?

As mentioned in the previous subsection, the literature lacks several regularizations for multiple generalizations of the Itô formula, including the Itô-Ventzell formula (to $C^{0,1}$). Most of the regularizations are made for weak Dirichlet processes, which are the sum of a local martingale and an adapted process with zero covariation with all continuous local martingales, including semimartingales. This is because the convoluted process, in general, is no longer a semimartingale, see, for instance, Gozzi and Russo [37], who generalized the classical Itô formula to $C^{0,1}$ for a more detailed discussion on this. In total, we get the following research question:

3. Is it possible to regularize the Itô-Ventzell formula to $C^{0,1}$ for weak Dirichlet processes?

When considering participating life insurance contracts, different perspectives – policyholders, insurance companies, and equity holders – can be adopted to optimize the strategy. This thesis focuses on the two most reasonable perspectives: the insurance company and the equity holders, starting with the perspective of the equity holders. For optimization, we take the mean-variance approach which is widely used in the industry due to its comparably simple interpretation. However, this approach leads to time-inconsistency, and the structure of variance means that most standard methods require modification or are not applicable. Therefore, this field remains underexplored, and we focus on "pre-committed" investors. One notable adaptation that allows Lagrange multipliers to work for mean-variance optimization was developed by Zhou and Li [67], although this comes with the trade-off that an additional parameter arises, making the existence of this parameter and the budget constraint is non-trivial. Hence, we consider the following research questions (for pre-committed investors):

4. Can explicit formulas be derived for the optimal terminal wealth and the optimal strategy when considering a mean-variance investor offering participating life insurance contracts?

5. If so, show the existence of all parameters and how does the structure of the optimal terminal wealth and the optimal strategy differ from those of an expected utility optimization?

Next, we examine the perspective of insurance company. One approach is to generalize and adapt Leland's model [53, 54] to explain the capital structure of a life insurance company. Although various generalizations of this model exist, there is a gap in the literature when it comes to applying it to participating life insurance contracts. When considering this model, the first step is to determine the bankruptcy-triggering value before computing the optimal participation rates and regular payment rates. Therefore, the objectives of this thesis regarding the optimization of participating life insurance contracts from the insurance company's perspective are:

6. How can the bankruptcy-triggering value be determined for participating life insurance contracts using Leland's finite time horizon model?

7. Why do insurance companies offer contracts that provide both guarantees and a surplus participation?

3 Summary of Research Papers

Research Paper 1: Time-consistent asset allocation for risk measures in a Lévy market

In the first paper, we analyze an asset allocation problem where we maximize reward against a risk penalization function, which encompasses most standard risk measures. The financial marketis modeled using α-stable Lévy processes, a generalization of the Black-Scholes model that accounts also for jumps. To solve this time-inconsistent problem, we derive a Hamilton-Jacobi Bellman (HJB) equation, where the Nash-subgame equilibrium represents the optimal strategy. This paper was presented at the IME conference 2023 in Edinburgh, Scotland, and at the Ifa-IVW Workshop 2023 in Ulm, Germany. It is a joint work with Mitja Stadje and has been published in the *European Journal of Operational Research*.

In the model, we focus on an asset allocation problem that measures gains and losses relative to a risk-free benchmark, rather than on terminal wealth. The risk is quantified using a law-invariant, cash- or shift-invariant, and positively homogeneous risk measure, which is then incorporated into a non-decreasing and convex C^1-function. As mentioned earlier, this formulation includes most standard risk measures, such as (relative) Value at Risk, all coherent risk measures, standard deviation, variance, and generalized deviation risk measures. For a comprehensive overview of such risk measures, see, for instance, Jorion [47]. A key feature of such optimization problems is their time-consistency, as discussed in works such as Basak and Chabakauri [7], Cheridito and Stadje [19], or Björk and Murgoci [10]. In theses settings, it becomes apparent that the dynamic programming principle no longer holds, which makes decision-making more complex. One potential solution, which we adopt in this paper, is to consider a "sophisticated" investor, who, in the words of Strotz [64], seeks to find the best strategy "among those that he will actually follow".

The paper first provides an analysis in discrete time and subsequently in continuous time. In both cases, we show that the optimal strategy corresponds to a Nash-subgame equilibrium solving an HJB equation. In discrete time, we demonstrate that the optimal (equilibrium) solution is deterministic and uniformly bounded in L^2. Furthermore, under appropriate assumptions, we show that the optimal strategy in continuous time is continuous, deterministic, and the uniform limit of discrete strategies with decreasing time steps. Notably, we also present a so-called verification theorem to confirm that the solution to the HJB is indeed a Nash-subgame equilibrium. Additionally, we provide explicit recursive formulas in certain special cases, including the classical mean-variance optimization. In the latter case, we find that our optimal strategy is identical to the one presented by Björk and Murgoci [10], who focused on terminal wealth.

In the numerical analysis, we demonstrate that Value at Risk leads to less risky investments for shorter maturities compared to variance. Furthermore, we show that a time-consistent "sophisticated" investor adopts a more conservative investment strategy than a "pre-committed" investor, i.e., the "sophisticated" investor allocates less capital to the risky asset. This effect is more pronounced for shorter maturities than for longer ones and is stronger for Value at Risk-based risk measures than for variance.

In summary, this paper addresses research questions 1 and 2. We show that a Hamilton-Jacobi-Bellman equation can be used to determine the optimal equilibrium strategy for a "sophisticated" investor, considering a broad range of risk measures, including quantile-based risk measure such as Value at Risk, under an α-stable Lévy market. Furthermore, we demonstrate that this optimal equilibrium strategy is uniformly bounded in L^2, deterministic under appropriate assumptions, and a Nash-equilibrium in both discrete and continuous time. We also provide results for special cases and present a numerical analysis of the optimal strategy.

This paper is a significant extension of the master's thesis by Felix Fießinger [31], with some identical or similar definitions. The proofs of Theorems 4.17, 5.8, and 5.11, Proposition 6.1, and Lemmas 5.3 and 5.4 build on similar ideas as those in the corresponding results of this master's thesis. However, the master's thesis only considers the special case of a one-dimensional Brownian Motion with Value at Risk as the risk measure and a constant risk aversion parameter, whereas the research paper extends the analysis to multi-dimensional α-stable Lévy processes for more general risk measures, with a non-constant risk aversion parameter.

Research Paper 2: The $C^{0,1}$ Itô-Ventzell formula for weak Dirichlet processes

The second paper proves an extension of the Itô-Ventzell formula for weak Dirichlet processes, showing that the regularity of the underlying stochastic flow can be reduced from $C^{1,2}$ to $C^{0,1}$ under some mild additional assumptions. This extension is particularly useful in situations where higher differentiability is difficult to establish or not applicable. This paper is a joint work with Mitja Stadje and has been published in *Electronic Communications in Probability*.

The paper is build upon this central result. Specifically, we show for F, a stochastic flow with characteristics (β, γ), defined as

$$F(t,x) = F(0,x) + \int_0^t \beta(r,x)\mathrm{d}r + \int_0^t \gamma(r,x)\mathrm{d}W_r,$$

where W denotes a Brownian Motion, and X being a weak Dirichlet process that:

$$F(t, X_t) = F(0, X_0) + \int_0^t \gamma(r, X_r)\mathrm{d}W_r + \int_0^t F_x(r, X_r)\mathrm{d}M_r + \mathcal{B}^X(F)_t,$$

where $\mathcal{B}^X$ is a weak zero energy process under certain mild assumptions on the stochastic flow. Note that a weak Dirichlet process, a slight generalization of semimartingales, is the sum of a local martingale and a weak zero energy process. A stochastic flow, in turn, is a generalization of functions that also incorporates randomness, and weak zero energy processes are "orthogonal" to continuous local martingales, meaning that their quadratic variation is zero.

In the final section, we provide several applications of this result. First, we can use it to derive stochastic calculus formulas for quadratic variations. Second, we apply it to develop a more general solution theory for stochastic elliptic partial differential equations. Third, we analyze the dynamics of a so-called real wealth process in a market where a large investor is present who influences the price with her/his trading actions.

Thus, this paper addresses research question 3. We demonstrate that it is indeed possible to regularize the Itô-Ventzell formula for weak Dirichlet processes to $C^{0,1}$. Moreover, we present examples of how to apply this new version in various context, including the development of a more general solution theory for stochastic elliptic partial differential equations.

Research Paper 3: Mean-Variance Optimization for Participating Life Insurance Contracts

In this paper, we take the perspective of the equity holders of an insurance company to solve a portfolio selection problem using the classical mean-variance optimization framework. Unlike traditional approaches, we do not consider the classical terminal wealth of the portfolio. Instead, we focus on the terminal wealth of the equity holders when offering a participating life insurance contract. Additionally, we derive and show explicit formulas for the optimal terminal wealth and the optimal strategy. This paper has been presented at the Ulm-Fudan Conference 2024 in Shanghai, China and at the Ifa-IVW Workshop 2024 in Ulm, Germany. It is a joint work with Mitja Stadje and has been published in the journal *Insurance: Mathematics and Economics*.

We primarily analyze two types of participating life insurance contracts: one with a protected guarantee and one with a non-protected guarantee, representing unlimited or limited liability for the insurer, respectively. Consequently, the payoff structure of these contracts is generally neither concave nor convex, and is not differentiable, which makes the standard concavification approach infeasible. Using a similar method to Zhou and Li [67], which complicates the existence proof for the parameters, we provide analytic results for the optimal terminal wealth and

the optimal strategy within the Black-Scholes market model. As discussed in the overview of the first paper, time-inconsistency arises when optimizing mean-variance problems. In contrast to that paper, we now consider a "pre-commitment" investor, i.e., an investor who optimizes once at the beginning and then adheres ro it for the remainder of the investment horizon.

In our numerical analysis, we observe that the equity holders tend to invest more conservatively, i.e., they reduce their investment in the risky asset, if the economy performs poorly or when the remaining maturity is shorter. When comparing the contracts with a protected and a non-protected guarantee, we find the intuitive result that a protected guarantee leads the insurer to adopt less risky strategies. We also observe that the mean-variance optimization yields a different shape of the optimal solution compared to expected utility optimization, with an S-shaped and exponential utility function. Specifically, mean-variance optimal strategies typically decrease as maturity shortens, which is not the case for expected utility optimization. Moreover, we compare the payoffs in terms of the price density for the two contract types, protected and non-protected guarantee, with each other, against those derived from expected utility optimization with an S-shaped utility function, as well as for different participation rates.

In summary, this paper addresses research questions 5 and 6. We provide closed form solutions for the optimal terminal wealth and the optimal strategy for general participation contracts, including both protected and non-protected guaranteed participating life insurance contracts. Additionally, we offer a system of non-linear equations to determine all arising parameters, with a detailed proof of their existence. A comparison with expected utility optimizations highlights certain special characteristics of the mean-variance approach, auch as the tendency for the optimal strategy to become more conservative as maturity decreases.

Research Paper 4: Optimal Capital Structure for Life Insurance Companies Offering Surplus Participation

In Paper 4, we again consider an insurance company offering participating life insurance contracts. Unlike the third research paper, where we focused on the perspective of the equity holders, this time we adopt the viewpoint of the insurance company itself. To model this, we adapt Leland's finite time horizon model, as presented in Leland and Toft [54], to incorporate participating life insurance contracts and guaranteed payments. The main result is that, without tax benefits, the insurance company has no incentive to offer participating contracts to policyholders. However, as the tax benefit increases, it becomes optimal for the insurance company to include surplus participation. This paper was presented at the Ifa-IVW Workshop 2025 in Ulm, Germany, and is a joint work with Mitja Stadje.

Leland's model was originally developed to determine optimal leverage of a company. In our case, we adapt it to an insurance context, and introduce option-like payoffs, such as those arising from surplus participation. The model accounts for tax benefits derived from both the guaranteed payoff flow and the surplus participation, and includes bankruptcy costs. These cost arise because, from the insurance company's perspective, equity holders are considered to be optimal investors with limited liability, i.e., they shut down the business if the equity value becomes negative. We first determine the bankruptcy-triggering value using a so-called smooth pasting condition. Subsequently, we derive formulas to optimally determine the participation rate and the guaranteed regular payment rate, both individually and when optimizing both together. In the first case, we also provide existence proofs. Furthermore, we outline the conditions under which it is optimal for the insurance company to offer both surplus participation and guarantees.

In the numerical analysis, we first discuss the assumptions made in the model and demonstrate that they are typically satisfied in practice. We then show that the guaranteed regular payment rate has a greater influence on the bankruptcy-triggering value than the participation rate. Furthermore, we find that the optimal participation rate is primarily determined by contract duration, the tax benefit from the participation component, and the volatility of the asset value process of the insurance company. Finally, we demonstrate that the so-called asset substitution effect diminishes when surplus participation is added.

This paper thus addresses research questions 6 and 7. We generalize Leland's finite time horizon model to incorporate participating life insurance contracts and provide a formula for determining the bankruptcy-triggering value within this extended model. We analyze the assumptions involved in the model and offer economic interpretations. Furthermore, we provide equations to determine the optimal participation rate and the optimal regular payment rate, along with a theoretical and numerical analysis of the parameters and their economic implications. Finally, we conduct a sensitivity analysis of the parameters affecting the optimal participation rate.

Bibliography

[1] Alexandre Adam, Mohamed Houkari, and Jean-Paul Laurent. Spectral risk measures and portfolio selection. *Journal of Banking & Finance*, 32(9):1870–1882, 2008.

[2] Sumit Agarwal, Brent W Ambrose, Hongming Huang, and Yildiray Yildirim. The term structure of lease rates with endogenous default triggers and tenant capital structure: theory and evidence. *Journal of Financial and Quantitative Analysis*, 46(2):553–584, 2011.

[3] Gordon J Alexander and Alexandre M Baptista. A comparison of var and cvar constraints on portfolio selection with the mean-variance model. *Management Science*, 50(9):1261–1273, 2004.

[4] Brent W Ambrose, Thomas Emmerling, Henry H Huang, and Yildiray Yildirim. Capital structure and the substitutability versus complementarity nature of leases and debt. *Review of Finance*, 23(3):659–695, 2019.

[5] Benjamin Avanzi, Hayden Lau, and Mogens Steffensen. Optimal reinsurance design under solvency constraints. *Scandinavian Actuarial Journal*, 2024(4):383–416, 2024.

[6] Anna Rita Bacinello and Svein-Arne Persson. Design and pricing of equity-linked life insurance under stochastic interest rates. *The Journal of Risk Finance*, 3(2):6–21, 2002.

[7] Suleyman Basak and Georgy Chabakauri. Dynamic mean-variance asset allocation. *The Review of Financial Studies*, 23(8):2970–3016, 2010.

[8] Suleyman Basak and Alexander Shapiro. Value-at-risk-based risk management: optimal policies and asset prices. *The Review of Financial Studies*, 14(2):371–405, 2001.

[9] Tomas Björk, Mariana Khapko, and Agatha Murgoci. On time-inconsistent stochastic control in continuous time. *Finance and Stochastics*, 21:331–360, 2017.

[10] Tomas Björk and Agatha Murgoci. A general theory of markovian time inconsistent stochastic control problems. *Available at SSRN 1694759*, 2010.

[11] Bruno Bouchard, Grégoire Loeper, and Xiaolu Tan. A $c^{0,1}$-functional itô's formula and its applications in mathematical finance. *Stochastic Processes and their Applications*, 148:299–323, 2022.

[12] Bruno Bouchard and Maximilien Vallet. Itô-dupire's formula for $c^{0,1}$-functionals of cádlág weak dirichlet processes. *arXiv preprint arXiv:2110.03406*, 2021.

[13] Michael W Brandt. Portfolio choice problems. In *Handbook of Financial Econometrics: Tools and Techniques*, pages 269–336. Elsevier, 2010.

[14] Eric Briys and François De Varenne. On the risk of insurance liabilities: debunking some common pitfalls. *Journal of Risk and Insurance*, pages 673–694, 1997.

[15] Rachel Campbell, Ronald Huisman, and Kees Koedijk. Optimal portfolio selection in a value-at-risk framework. *Journal of Banking & Finance*, 25(9):1789–1804, 2001.

[16] Mark Carey and Michael B Gordy. The bank as grim reaper: Debt composition and bankruptcy thresholds. *Journal of Financial Economics*, 142(3):1092–1108, 2021.

[17] An Chen, Thai Nguyen, and Mitja Stadje. Optimal investment under var-regulation and minimum insurance. *Insurance: Mathematics and Economics*, 79:194–209, 2018.

[18] Nan Chen, Paul Glasserman, Behzad Nouri, and Markus Pelger. Contingent capital, tail risk, and debt-induced collapse. *The Review of Financial Studies*, 30(11):3921–3969, 2017.

[19] Patrick Cheridito and Mitja Stadje. Time-inconsistency of var and time-consistent alternatives. *Finance Research Letters*, 6(1):40–46, 2009.

[20] Rama Cont and Peter Tankov. Financial modelling with jump processes. *Chapman and Hall/CRC*, 2004.

[21] Domenico Cuoco, Hua He, and Sergei Isaenko. Optimal dynamic trading strategies with risk limits. *Operations Research*, 56(2):358–368, 2008.

[22] Christoph Czichowsky. Time-consistent mean-variance portfolio selection in discrete and continuous time. *Finance and Stochastics*, 17:227–271, 2013.

[23] Giuseppe Da Prato, Arnulf Jentzen, and Michael Röckner. A mild itô formula for spdes. *Transactions of the American Mathematical Society*, 372(6):3755–3807, 2019.

[24] Marco Della Seta, Erwan Morellec, and Francesca Zucchi. Short-term debt and incentives for risk-taking. *Journal of Financial Economics*, 137(1):179–203, 2020.

[25] Yinghui Dong, Sang Wu, Wenxin Lv, and Guojing Wang. Optimal asset allocation for participating contracts under the var and pi constraint. *Scandinavian Actuarial Journal*, 2020(2):84–109, 2020.

[26] Gonçalo Dos Reis and Vadim Platonov. Itô-wentzell-lions formula for measure dependent random fields under full and conditional measure flows. *Potential Analysis*, pages 1–32, 2022.

[27] EIOPA. European insurance overview 2023. Technical report, European Insurance and Occupational Pensions Authority (EIOPA), 2023.

[28] Ivar Ekeland and Ali Lazrak. Being serious about non-commitment: subgame perfect equilibrium in continuous time. *arXiv preprint math/0604264*, 2006.

[29] Redouane Elkamhi, Jan Ericsson, and Christopher A Parsons. The cost and timing of financial distress. *Journal of Financial Economics*, 105(1):62–81, 2012.

[30] K David Elworthy, Aubrey Truman, and Huaizhong Zhao. Generalized itô formulae and space-time lebesgue–stieltjes integrals of local times. *Séminaire de Probabilités XL*, pages 117–136, 2007.

[31] Felix Fießinger. Dynamic time-consistent value at risk asset allocation. *Master Thesis at Ulm University*, 2022.

[32] Franco Flandoli and Francesco Russo. Generalized integration and stochastic odes. *The Annals of Probability*, 30(1):270–292, 2002.

[33] Hans Föllmer, Philip Protter, and Albert N Shiryayev. Quadratic covariation and an extension of itô's formula. *Bernoulli*, pages 149–169, 1995.

[34] Chenpeng Fu, Ali Lari-Lavassani, and Xun Li. Dynamic mean–variance portfolio selection with borrowing constraint. *European Journal of Operational Research*, 200(1):312–319, 2010.

[35] Nadine Gatzert and Alexander Kling. Analysis of participating life insurance contracts: A unification approach. *Journal of Risk and Insurance*, 74(3):547–570, 2007.

[36] Paul Glasserman and Behzad Nouri. Contingent capital with a capital-ratio trigger. *Management Science*, 58(10):1816–1833, 2012.

[37] Fausto Gozzi and Francesco Russo. Weak dirichlet processes with a stochastic control perspective. *Stochastic Processes and their Applications*, 116(11):1563–1583, 2006.

[38] Nalan Gülpınar and Berç Rustem. Worst-case robust decisions for multi-period mean–variance portfolio optimization. *European Journal of Operational Research*, 183(3):981–1000, 2007.

[39] Lin He, Zongxia Liang, Yang Liu, and Ming Ma. Weighted utility optimization of the participating endowment contract. *Scandinavian Actuarial Journal*, 2020(7):577–613, 2020.

[40] Xue Dong He, Hanqing Jin, and Xun Yu Zhou. Dynamic portfolio choice when risk is measured by weighted var. *Mathematics of Operations Research*, 40(3):773–796, 2015.

[41] Zhiguo He and Konstantin Milbradt. Dynamic debt maturity. *The Review of Financial Studies*, 29(10):2677–2736, 2016.

[42] Christopher A Hennessy and Yuri Tserlukevich. Taxation, agency conflicts, and the choice between callable and convertible debt. *Journal of Economic Theory*, 143(1):374–404, 2008.

[43] Christian Hilpert, Stefan Hirth, and Alexander Szimayer. The information value of distress. *Management Science*, 70(1):78–97, 2024.

[44] Ying Hu, Hanqing Jin, and Xun Yu Zhou. Time-inconsistent stochastic linear–quadratic control. *SIAM Journal on Control and Optimization*, 50(3):1548–1572, 2012.

[45] Julien Hugonnier, Semyon Malamud, and Erwan Morellec. Credit market frictions and capital structure dynamics. *Journal of Economic Theory*, 157:1130–1158, 2015.

[46] Hanqing Jin, Harry Markowitz, and Xun Yu Zhou. A note on semivariance. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 16(1):53–61, 2006.

[47] Philippe Jorion. *Value at risk: the new benchmark for managing financial risk.* The McGraw-Hill Companies, Inc., 2007.

[48] Nengjiu Ju, Robert Parrino, Allen M Poteshman, and Michael S Weisbach. Optimal capital structure. *Journal of Financial and Quantitative Analysis*, 40(2), 2005.

[49] Elena V Karachanskaya. On a generalization of the itô-wentzell formula for system of generalized itô's sde and the stochastic first integral. *Transfer*, 1(2):3, 2011.

[50] Michael Kohlmann, Siyuan Li, and Xiong Dewen. A generalized itô-ventzell formula to derive forward utility models in a jump market. *Stochastic Analysis and Applications*, 31(4):632–662, 2013.

[51] Nikolai Vladimirovich Krylov. On the itô-wentzell formula for distribution-valued processes and related topics. *arXiv preprint arXiv:0904.2752*, 2009.

[52] Hiroshi Kunita. *Stochastic flows and stochastic differential equations*, volume 24. Cambridge University Press, 1997.

[53] Hayne E Leland. Corporate debt value, bond covenants, and optimal capital structure. *The journal of finance*, 49(4):1213–1252, 1994.

[54] Hayne E Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The journal of finance*, 51(3):987–1019, 1996.

[55] Hongcan Lin, David Saunders, and Chengguo Weng. Optimal investment strategies for participating contracts. *Insurance: Mathematics and Economics*, 73:137–155, 2017.

[56] Sheen X Liu, Howard Qi, and Chunchi Wu. Personal taxes, endogenous default, and corporate bond yield spreads. *Management Science*, 52(6):939–954, 2006.

[57] Harry Markowitz. Portfolio selection. *Journal of Finance*, 7:77–91, 1952.

[58] Hui Mi, Zuo Quan Xu, and Dongfang Yang. Optimal management of dc pension plan with inflation risk and tail var constraint. *arXiv preprint arXiv:2309.01936*, 2023.

[59] Charbel Mirza and Joël Wagner. Policy characteristics and stakeholder returns in participating life insurance: which contracts can lead to a win-win? *European Actuarial Journal*, 8:291–320, 2018.

[60] Thai Nguyen and Mitja Stadje. Nonconcave optimal investment with value-at-risk constraint: An application to life insurance contracts. *SIAM journal on control and optimization*, 58(2):895–936, 2020.

[61] R Tyrrell Rockafellar and Stanislav Uryasev. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26(7):1443–1471, 2002.

[62] Andrzej Ruszczynski and Robert J Vanderbei. Frontiers of stochastically nondominated portfolios. *Econometrica*, 71(4):1287–1297, 2003.

[63] Hato Schmeiser and Joël Wagner. A proposal on how the regulator should set minimum interest rate guarantees in participating life insurance contracts. *Journal of Risk and Insurance*, 82(3):659–686, 2015.

[64] Robert Henry Strotz. Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies*, 23(3):165–180, 1955.

[65] Alexander D Wentzell. On equations of the theory of conditional markov processes. *Teoriya Veroyatnostei i ee Primeneniya*, 10(2):390–393, 1965.

[66] Yan Zeng and Zhongfei Li. Optimal time-consistent investment and reinsurance policies for mean-variance insurers. *Insurance: Mathematics and Economics*, 49(1):145–154, 2011.

[67] Xun Yu Zhou and Duan Li. Continuous-time mean-variance portfolio selection: A stochastic lq framework. *Applied Mathematics and Optimization*, 42(1):19–33, 2000.

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Interfaces with other disciplines

Time-consistent asset allocation for risk measures in a Lévy market ☆

Felix Fießinger *, Mitja Stadje

Ulm University, Institute of Insurance Science and Institute of Mathematical Finance, Faculty of Mathematics and Economics, Helmholtzstraße 20, 89081, Ulm, Germany

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ABSTRACT

Focusing on gains & losses relative to a risk-free benchmark instead of terminal wealth, we consider an asset allocation problem to maximize time-consistently a mean-risk reward function with a general risk measure which is (i) law-invariant, (ii) cash- or shift-invariant, and (iii) positively homogeneous, and possibly plugged into a general function. Examples include (relative) Value at Risk, coherent risk measures, variance, and generalized deviation risk measures. We model the market via a generalized version of the multi-dimensional Black–Scholes model using α-stable Lévy processes and give supplementary results for the classical Black–Scholes model. The optimal solution to this problem is a Nash subgame equilibrium given by the solution of an extended Hamilton–Jacobi–Bellman equation. Moreover, we show that the optimal solution is deterministic under appropriate assumptions.

1. Introduction

In this paper, we study an optimal asset allocation problem of the following form

$$V(t,x) = \sup_{u\in\mathcal{A}_{t,x}} \left\{\mathbb{E}_{t,x}[\mathcal{T}(X_T^u - xe^{r(T-t)})] - Risk_{t,x}(X_T^u - xe^{r(T-t)})\right\}, \qquad (1.1)$$

where x is the wealth at time t, $\mathcal{A}_{t,x}$ is the set of all time-consistent, Markovian strategies at time t, X^u is the wealth process driven by an α-stable Lévy process when the strategy u is followed, r is the discounting rate, $\mathcal{T} \in C^1$ is a continuous, non-decreasing, and concave reward function for gains & losses. The risk function is given by $Risk_{t,x}(Y) = \lambda_t F(\rho_{t,x}(Y))$, where $\lambda : [0,T] \to \mathbb{R}_{\geq 0}$ is the time-dependent and continuous risk aversion function with values in $(0,\infty)$, where ρ is a law-invariant, positively homogeneous, and translation- or shift-invariant dynamic risk measure and $F \in C^1$ is a non-decreasing convex function. For tractability reasons, we do not allow λ to be state-dependent, which means that an investor can change his/her risk aversion only deterministically. Since we only assume that ρ is law-invariant, translation or shift-invariant, and positively homogeneous, our results encompass most standard or alternative risk measures like (relative) Value at Risk, Average Value at Risk, coherent risk measures, standard deviation, or generalized deviation measures. On all these (classes of) risk measures, there is an extensive literature, but for space reasons, we only state Jorion (2007), Föllmer and Schied (2016), or Rockafellar, Uryasev, and Zabarankin (2002). We emphasize that ρ does not need to be additive, sub-additive, convex, or continuous. By allowing ρ to be plugged in the convex function F, our risk part in (1.1) also includes examples such as variance or semi-variance. Time-consistency means that the strategy which is optimal at time t, say $(u_s)_{t\leq s}$ will also be optimal from later points on. On the other hand, a time-inconsistent investor does not satisfy such a dynamic programming principle and may initiate a strategy at time

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* Corresponding author.
E-mail addresses: felix.fiessinger@uni-ulm.de (F. Fießinger), mitja.stadje@uni-ulm.de (M. Stadje).

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t (because at time t its whole path attains the maximum in (1.1)) knowing fully well that (s)he will deviate from this strategy from a later point on. Following psychological theories like prospect theory (Kahneman & Tversky, 1979) and benchmark investment theories (see, for instance, Zhao, 2007 or Pirvu & Schulze, 2012), the investor at time t in (1.1) is assumed to consider future *changes* in wealth $X_T^u - xe^{r(T-t)}$ (in \$-values at time T) instead of the terminal wealth X_T^u. We could generalize the benchmark by multiplying and adding deterministic, time-dependent C^1-functions (i.e., consider $a_t(X_t - xe^{r(T-t)}) + b_t$ instead of $X_t - xe^{r(T-t)}$ in (1.1)), which would structurally not change our results in the case of uniformly bounded strategies. Conceptually, this approach is in line with the so-called "growth-optimal portfolio", also known as the Kelly criterion in the context of portfolio theory, optimal decision under risk, game theory, information theory, and insurance, see Wei and Xu (2021) and the references therein. When $\mathcal{T}$ is the identity, considering the terminal expected wealth instead of gain & losses in the reward function in (1.1) leads to identical solutions. We further remark that defining risk as *changes* in values between two dates is a rather classical approach of assessing the riskiness of a position, see, for instance, Jorion (2007), the way risk is measured in the Basel III (BCBS, 2010) or Solvency II (EIOPA, 2014) regulations for banks and insurance companies, respectively, or the literature on return risk measures; see, for instance, Bellini, Laeven, and Rosazza Gianin (2018).

We solve problem (1.1) recursively in discrete time and show that as the time grid gets finer, the discrete-time maximizers converge weakly along a subsequence to continuous time. For shift-invariant risk measures, the limit is a Nash subgame equilibrium under certain assumptions in continuous time. This kind of optimization is a non-cooperative interpersonal game in which the investor plays against reincarnations of himself/herself at other times. There is no direct generalization of Nash equilibria in continuous time since a single time point is a null set. Instead, we follow here the generalization of Ekeland and Lazrak (2006), who allowed the investor at time t to form a coalition with all players $s \in [t, t+\varepsilon]$ and derive the equilibrium strategy by taking $\varepsilon \to 0$. We also prove that, in general, a Nash subgame equilibrium in continuous time of (1.1) satisfies an extended Hamilton–Jacobi–Bellman (HJB) equation. Moreover, we derive explicit solutions for some special cases. We illustrate the applicability of our approach in a few numerical examples showing that Value at Risk leads to less risky investments for shorter maturities than variance. Furthermore, compared to a pre-commitment investor, we show that a time-consistent investor invests more conservatively, i.e., less in the risky asset. This effect is more pronounced for shorter than for longer maturities and more for Value at Risk based risk measures than for variance.

We model the stock returns by an α-stable Lévy processes under some limitations. In physics, α-stable Lévy processes are also called Lévy flights and are used to model physical phenomena, see Cont and Tankov (2004, p.99). The marginal distribution of an α-stable Lévy process are sometimes also called L-stable distributions. Using L-stable distributions to model prices goes back at least to Mandelbrot (1963, 1967), and Fama (1965). Lévy processes also play an important role in operations research for modeling asset returns; see, for instance, Kallsen (2000), Feng and Linetsky (2008), Kaishev and Dimitrova (2009), Kardaras (2009), Nowak and Romaniuk (2010), Fu and Yang (2012), Laeven and Stadje (2014), Fu, Li, Li, and Wu (2017), Ma, Wong, and Zhang (2021), or Černý, Czichowsky, and Kallsen (2023). The induced model is more general than the classical Black–Scholes model and is favorable for several reasons, see Cont and Tankov (2004, pp. 1–16). For example, contrary to Lévy processes (with jumps), the Black–Scholes model does not admit heavy tails, which are empirically observed in financial markets, does not allow for large, sudden movements of the asset prices, and, furthermore, induces a complete financial market in the sense that all risks can be perfectly hedged, which might not be realistic.

In 1952, Markowitz (1952) was the first to study a (static) asset allocation problem of the form (1.1) in *one period* using portfolio returns as basic objects and introducing a trade-off between the expected portfolio return and its risk. In this case, ρ in (1.1) becomes the standard deviation and $F(x) = \max\{0, x\}^2$ so that risk is identified with the variance of a portfolio. Up to date, variance is the most prevalent risk measure and the cornerstone of most finance literature. Many alternative risk measures have been proposed to overcome potential shortfalls of variance. For instance, for regulatory purposes in the financial industry, Value at Risk (VaR) has been the most prominent risk measure. VaR is the amount of money to hold in reserve such that the financial institution can cover its losses in say, 99.5%, of the possible future developments of the market over a prescribed time horizon. VaR is the standard risk measure for insurance companies and banks, with its use being prescribed by Basel III (BCBS, 2010) or Solvency II (EIOPA, 2014) regulations. Jorion (2007) also defines the shift-invariant version of the VaR, the so-called relative Value at Risk (RVaR). He states that RVaR for money situations is conceptually more appropriate than VaR since it accounts more suitable for the time value of money. Moreover, he states that RVaR is more conservative and more consistent with definitions of unexpected loss. In the context of portfolio choice, static mean-risk models with various alternative risk measures have been analyzed, see for instance, Campbell, Huisman, and Koedijk (2001), Rockafellar and Uryasev (2002), Ruszczynski and Vanderbei (2003), Alexander and Baptista (2004), Jin, Markowitz, and Yu Zhou (2006), Gülpınar and Rustem (2007), Adam, Houkari, and Laurent (2008), or He, Jin, and Zhou (2015). Additionally, Markowitz's original single-period static framework has been extended dynamically to discrete and to continuous time; see, for instance, Basak and Chabakauri (2010), Brandt (2010), Fu, Lari-Lavassani, and Li (2010), Zeng and Li (2011), Hu, Jin, and Zhou (2012), Czichowsky (2013), or Björk, Khapko, and Murgoci (2017).

In a dynamic setting, the issue of time-consistency arises. Most static risk measures like variance or VaR are time-inconsistent when used in a dynamic context; see, for instance, Basak and Chabakauri (2010), Cheridito and Stadje (2009), or Föllmer and Schied (2016). Mathematically, this entails that a dynamic programming principle does not hold, making the problem of optimal portfolio choice difficult and often intractable. Furthermore, conceptually time-consistency seems an essential element of rational decision-making, and lacking it seems unreasonable; see, for instance, Björk and Murgoci (2010), Balter and Pelsser (2021), or Balter, Mahayni, and Schweizer (2021).[1]

If a given preference is time-inconsistent or if the regulator prescribes a time-inconsistent risk measure, there are three different approaches discussed in the literature: the pre-committed investor only maximizes once in the beginning, see, for instance, Li and Ng (2000), Zhou and Li (2000), Lim and Zhou (2002), or Xie, Li, and Wang (2008), the naive investor continuously optimizes, always using his actual preferences, see Basak and Chabakauri (Basak & Chabakauri, 2010) for references, and the sophisticated investor optimizes, taking into account that (s)he acts time-inconsistent. We use the third approach and ensure time consistency by analyzing an investor who, in the words of Strotz (1955), aims to find the best strategy "among those that he will actually follow". This means that we consider an interpersonal, non-cooperative game of the investor with himself/herself. Building on Strotz (1955), Basak and Chabakauri (2010) in a seminal work gave an equivalent time-consistent solution for optimizing a mean–variance problem using the conditional variance formula. This and the more technical work by Björk and Murgoci (2010) which generalized this approach to problems involving general functions with outer and inner expectations, initiated a large stream of publications,

[1] In the case of using mean–variance in (1.1), recent literature has actually shown paradoxes arising from time-consistency, see Van Staden, Dang, and Forsyth (2018) and Bensoussan, Wong, and Yam (2019). See also the discussion in Bosserhoff and Stadje (2021). Of course, since mean–variance does not respect first-order stochastic dominance, it also lends itself to various other paradoxes.

see, for instance, Wang and Forsyth (2011), Cui, Li, Wang, and Zhu (2012), Czichowsky (2013), Björk, Murgoci, and Zhou (2014), Bensoussan, Wong, Yam, and Yung (2014), Chiu and Wong (2015), Wu and Chen (2015), Vigna et al. (2016), Wei and Wang (2017), Lindensjö (2019), Cui, Gao, Shi, and Zhu (2019), Bosserhoff and Stadje (2021), or Hernández and Possamaï (2023). For a discussion and comparison between a pre-committed and a time-consistent investor, see Balter et al. (2021). While for a mean–variance problem, the reward function can naturally be expressed as a functional of expectations, this is generally not the case for a risk measure as ours, which is merely assumed to be law-invariant, positively homogeneous, and translation or shift invariant. Consequently, to the best of our knowledge, results on time-consistent optimization in continuous time have not been obtained in the context of general and possibly quantile-based risk measures. Using stability results for α-stable Lévy processes, we tackle the optimization problem by showing that the optimal solution necessarily is deterministic and can be computed recursively in discrete time. From there, an extension to continuous-time can be shown using the structure of α-stable processes. Using time-consistency (leading to the boundedness of the optimal strategies), Alaoglu's theorem, convex structures, Slutky's theorem, and Fatou's generalized lemma, we show that the discrete-time solutions weakly converge to a solution which dominates the discrete-time solutions in mean-risk (given by (1.1)). Under specific assumptions, we show that this limit is also a Nash subgame equilibrium using Arzelà-Ascoli's theorem, with uniform convergence of the control functions. For mean–variance, convergence results have been shown by Czichowsky (2013). We emphasize that even for VaR, arguably the most important risk measure used in industry, apart from Forsyth (2020) and Cui et al. (2019), who do a numerical analysis in discrete time, the time-consistent asset allocation problem has not been studied much.

The paper is structured as follows. In Section 2, we recall the definition of α-stable Lévy processes and briefly recall their properties. In Section 3, we introduce the financial market. Section 4 treats the discrete-time optimization case, while Section 5 analyzes the continuous-time problem. In Section 6, some specific examples are considered, and finally, in Section 7, we present some numerical results.

2. Basics about α-stable Lévy processes

Let us describe some important properties of α-stable Lévy processes used in the sequel. The following description is based on Çınlar (2011, pp.313,314,329-339).

A process L is a Lévy process if L is an adapted càdlàg process with independent and stationary increments. Let $(\Omega, \mathcal{H}, \mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P})$ be a d-dimensional stochastic basis, where $\mathcal{F}$ is the filtration generated by a Lévy process L and let $\alpha \in (0,2]\backslash\{1\}$. We assume that L is α-stable meaning that $t^{-\frac{1}{\alpha}} L_t \overset{d}{=} L_1$ for all $t \in (0,\infty)$, whereby we exclude the case $L = 0$ a.s. If $\alpha = 2$, then L is a Brownian Motion, i.e., for any $z > 0$ we have $zL_t \sim \mathcal{N}(0, z^2 t)$. To highlight this special case, we then denote the process by W instead of L. If $\alpha \in (0,2)\backslash\{1\}$, the associated Lévy measure ν has infinite measure and is constructed as follows: Fix a $c \in (0,\infty)$ and a probability measure $\tilde{\sigma}$ on the unit *sphere* in $\mathbb{R}^d$, denoted by $\mathbb{S}^d$ (we write $\mathbb{S}$ if $d = 1$). Then, for every Borel function $f \geq 0$, the Lévy measure ν is given by (see also Proposition 3.15 in Cont and Tankov (2004)):

$$\nu f = \int_{\mathbb{R}_{\geq 0}} \int_{\mathbb{S}^d} \frac{c}{w^{\alpha+1}} f(wv)\tilde{\sigma}(\mathrm{d}v)\mathrm{d}w. \tag{2.1}$$

The associated Lévy process L is called symmetric, if $L \overset{d}{=} -L$ which is equivalent to the symmetry of ν, i.e., $\nu(B) = \nu(-B)$ for all $B \subset \mathbb{R}^d$ Borel. Recall that for an α-stable Lévy process with $0 < \alpha < 2$ only θ-moments with $\theta < \alpha$ exist, and marginal distributions are still α-stable. In the case of $\alpha = 2$, we have a Brownian Motion, and, of course, all real and exponential moments exist. Multiplying the Lévy process with a constant preserves α-stability.

Appendix A gives a more detailed introduction to Lévy processes and their stochastic integral.

3. Model setup and basic definitions

We use the following notations for this paper: We denote the time dependence in the discrete case as an index and in the continuous case as a variable. Moreover, we write $\langle\cdot,\cdot\rangle_A$ for the scalar product concerning the symmetric and positive definite matrix $A \in \mathbb{R}^{d\times d}$, i.e., $\langle x, y\rangle_A := x^\intercal A y$ for $x, y \in \mathbb{R}^d$, where $\cdot^\intercal$ denotes the transposed vector. If A is the identity matrix, we write $\langle\cdot,\cdot\rangle$. In addition, we denote by $a^{i\cdot}$ the i'th row of the matrix A and by $a^{\cdot j}$ the j'th column of A. Furthermore, we denote by $\mathcal{L}^\alpha(\Omega,\mathcal{H},\mu)$ the set of all $\mathcal{H}$-measurable and α-times integrable functions, i.e., $\mathcal{L}^\alpha(\Omega,\mathcal{H},\mu) = \{f : \Omega \to \mathbb{R} : f \text{ is measurable}, \int_\Omega |f(x)|^\alpha \mu(\mathrm{d}x) < \infty\}$ for the measure space $(\Omega,\mathcal{H},\tilde{\nu})$. In particular, $\mathcal{L}^0(\Omega,\mathcal{H},\mu)$ denotes the set of all measurable functions. Often, "$\mathcal{H}$" will be omitted in the notation if the context is clear.

3.1. Model setup

We consider a finite investment period, which starts at time 0 and ends at time T. Let L_t be a d-dimensional α-stable Lévy process with $\alpha \in (0,2]\backslash\{1\}$ defined with the constant $c \in (0,\infty)$ and the probability measure $\tilde{\sigma}$ (see Eq. (2.1)). Then, we consider a financial market with d risky assets S^i, $i \in \{1,\dots,d\}$, and one risk free asset B. The price dynamics are given by

$$\mathrm{d}S_t^i = \mu_t^i S_t^i \mathrm{d}t + \sum_{j=1}^d \sigma_t^{ij} S_t^i \mathrm{d}L_t^j \text{ and } \mathrm{d}B_t = rB_t\mathrm{d}t, \quad i = 1,\dots,d, \tag{3.1}$$

where μ_t^i and σ_t^{ij} are deterministic and continuous functions describing an additional drift, the volatility, and the dependency of the risky assets, whereas the constant r denotes the short rate of the risk-free asset. We assume for all $i,j \in \{1,\dots,d\}$ and for all $t \in [0,T]$ that $\sigma_t^{ij} > 0$, $\sigma_t^{ij} = \sigma_t^{ji}$, and σ_t is positive definite and uniformly bounded away from zero. Let u_t^i be the amount of money invested in asset S^i until time t. Then, $u_t = (u_t^1,\dots,u_t^d)^\intercal$ is a d-dimensional vector. We assume that u may only take values in a given closed and convex set $\mathcal{V} \subset \mathbb{R}^d$. Define then the d-dimensional, measurable, closed, and convex control space $\{\mathcal{U},\mathcal{G}_U\}$ such that the control functions in $\mathcal{U}$ are $\mathcal{F}$-adapted, predictable, Markovian, and $\mathcal{L}^\alpha((0,T],\mathrm{d}\mathbb{P}\times\mathrm{d}s)$-integrable. For instance, if short-selling is prohibited, then $\mathcal{V} \subset \mathbb{R}^d_{\geq 0}$. The wealth process X^u is given by

$$\mathrm{d}X_t^u = [rX_t^u + \sum_{i=1}^d u_t^i(\mu_t^i - r)]\mathrm{d}t + \sum_{i,j=1}^d u_t^i \sigma_t^{ij} \mathrm{d}L_t^j. \tag{3.2}$$

For $\alpha = 2$, (3.2) is also called the Black–Scholes model. In this case, we assume that the correlation matrix process $R_t := (R_t^{jl})_{jl}$ with $R_t^{jl} := \mathrm{Corr}(W_t^j, W_t^l)$ is positive definite and uniformly bounded away from zero (see Skintzi & Refenes, 2005).

Moreover, we define the measurable control space $\{\tilde{\mathcal{U}},\mathcal{G}_{\tilde{U}}\} \subset \{\mathcal{U},\mathcal{G}_U\}$ with piecewise constant Markovian control functions, add a "det" in the index of $\mathcal{U}$ resp. $\tilde{\mathcal{U}}$ if we restrict to deterministic control functions and we use the short notation $\mathbb{E}^u_{t,x}[Y] := \mathbb{E}[Y|X_t^u = x]$ (similarly also for variances and risk measures). As Björk and Murgoci (2010), we will typically omit the u in $\mathbb{E}^u_{t,x}$.

3.2. Risk measures

In this subsection, we specify the properties of the risk measure from problem (1.1).

Definition 3.1. Let $\mathcal{R}$ be a closed vector space with $\mathcal{R} \supset \mathcal{L}^\alpha(\mathcal{F}_T)$. We can interpret $\mathcal{R}$ as all possible portfolio gains (and losses). Then, we call the mapping $\rho : \mathcal{R} \to \mathbb{R}\cup\{+\infty\}$ a risk measure if it satisfies the following properties:

(a) law-invariance: $\rho_0(Y_1) = \rho_0(Y_2)$ for $Y_1 \overset{d}{=} Y_2$ under $\mathbb{P}$, i.e., the distribution of the changes of a portfolio determines the value of the risk measure,

(b) positive homogeneity: $\rho_0(\beta Y) = \beta\rho_0(Y)$ for all $\beta \geq 0$, i.e., the risk of a portfolio is proportional to its size,

(c1) cash-invariance: $\rho_0(Y + m) = \rho_0(Y) - m$ for all $m \in \mathbb{R}$, i.e., the addition of a sure capital amount to the portfolio reduces the risk by exactly this amount, or

(c2) shift-invariance: $\rho_0(Y + m) = \rho_0(Y)$ for all $m \in \mathbb{R}$, i.e., changing the portfolio by a fixed capital amount does not influence the risk.

Let ρ_t be the dynamic version of ρ. Specially, we define $\rho'(H_Y) := \rho(Y)$, where H_Y denotes the cdf of Y, and notice that by (a), ρ' is well defined. We then set $\rho_t(Y) = \rho'(H_Y^t)$ with $H_Y^t(s) := \mathbb{P}(Y \leq s|\mathcal{F}_t)$ and $\rho_{t,x}^u(Y) = \rho'(H_Y^{t,x,u})$ with $H_Y^{t,x,u}(s) := \mathbb{P}(Y \leq s|X_t^u = x)$. We will typically omit the u also in the notation of $\rho_{t,x}^u$. Moreover, we denote: $\varrho^{\bar{L}_1} := \rho_0(\bar{L}_1)$ for a Lévy process $\bar{L}$.

The case with (c1) refers to measuring the risk by the capital amount needed so that the financial institution is "safe", whereas (c2) refers to measuring the deviation of the portfolio. For background on risk measures satisfying (c1), we simply refer to the textbook by Föllmer and Schied (2016) and the many references therein. For background of risk measures satisfying (c2), we refer to Rockafellar (1972). Note that ρ is not necessarily continuous; see, e.g., Example 3.2(i) below.

Example 3.2. The following examples satisfy the previous definition's properties (a)–(c). Examples (i) and (ii) satisfy (c1), whereas examples (iii) - (v) satisfy (c2).

(i) Value at Risk (VaR): $\mathrm{VaR}_t^p(Y) := \mathrm{essinf}\{m \in \mathcal{L}^0(\mathcal{F}_t) \mid \mathbb{P}(Y + m < 0|\mathcal{F}_t) \leq p\}$.

(ii) All coherent risk measures. A risk measure ρ is coherent if it is additionally monotone, i.e., $\rho_0(Z) \geq \rho_0(Y)$ if $Z \leq Y$ a.s., and sub-additive, i.e., $\rho_0(Z + Y) \leq \rho_0(Z) + \rho_0(Y)$. One example is given by Average Value at Risk (AVaR): $\mathrm{AVaR}_t^p(Y) := \frac{1}{p}\int_0^p \mathrm{VaR}_t^q(Y)\mathrm{d}q$ which is the basis, for instance, of the Swiss solvency test.

(iii) Relative Value at Risk (RVaR): $\mathrm{RVaR}_t^p(Y) := \mathrm{VaR}_t^p(Y - \mathbb{E}[Y])$ (see Jorion, 2007) which is the shift-invariant version of VaR considering the deviation from the mean.

(iv) Standard deviation: $sd_t(Y) := \sqrt{\mathrm{Var}_t(Y)}$ with Var_t denoting the conditional variance.

(v) All generalized deviation risk measures (see Rockafellar et al., 2002, Pistorius & Stadje, 2017 and Stadje, 2020). A generalized deviation risk measure additionally to (a), (b), and (c2) is assumed to be positive, i.e., $\rho_0(Y) > 0$ for all non-constant Y, and $\rho_0(Y) = 0$ for all constant risks Y, and sub-additive.

To also include risk measures like variance, in the sequel, we allow the risk measure to be plugged into a convex C^1 function F.

4. Analysis of (1.1) in discrete time

4.1. Definition of the model

At every point, we maximize the expected target value of the future net gains penalized through the composition of the function F and the risk measure ρ.

We divide $[0, T]$ in N intervals with $N + 1$ grid points: $0 = t_0 < t_1 < \cdots < t_N = T$. We denote by δ the mesh size, i.e., $\delta = \max_{i\in\{1,\ldots,N\}}(t_i - t_{i-1})$. Without loss of generality, we assume that the grid points are equidistant. We consider constant control functions in each interval and define the control function $u_{t_n}^i$ at all time points setting $u_{t_n}^i = u_{t_n-h}^i$ for all $h \in [0, \delta)$, for all $n \in \{1, \ldots, N\}$ and for all $i \in \{1, \ldots, d\}$.

In the first step, let us set the step size $\delta = 1$ and write n instead of t_n. Thus, u_n^i describes the investment in the risky asset i in the interval $(n - 1, n]$.

To get an equivalent structure as the introduced problem (1.1) (see Proposition 4.3), we define the value functional J_n as follows: Let $(n, x) \in \{0, \ldots, T - 1\} \times \mathbb{R}$ and a control law u be fixed. Then, we define J_n as:

$$J_n(x, u) = \mathbb{E}_{n,x}[\mathcal{T}(X_T^u - xe^{r(T-n)})] - \lambda_n F(\rho_{n,x}(X_T^u - xe^{r(T-n)})), \tag{4.1}$$

where $\mathcal{T}$, F, and ρ are as previously described in the introduction and Section 3. Without loss of generality, see Remark 4.15, we ex-ante restrict ourselves to Markovian strategies.

From this definition, we see that this is a generalization of the mean–variance problem, which we get using $\mathcal{T}(x) = x$, $F(x) = \max\{0, x\}^2$ and the standard deviation for ρ.

To ensure that J_n is well-defined, we have to make the following assumption:

Assumption 1. If $\mathcal{V} \subset \mathbb{R}_{\geq 0}^d$ (resp. $\mathcal{V} \subset \mathbb{R}_{\leq 0}^d$), then $\mathbb{E}[|\mathcal{T}(a + bL_t)|] < \infty$ for all $a \in \mathbb{R}$, $b > 0$ (resp. $b < 0$), and for all $t \in (0, T]$. Otherwise, $\mathbb{E}[|\mathcal{T}(a + bL_t)|] < \infty$ for all $a, b \in \mathbb{R}$ and for all $t \in (0, T]$.

For $\alpha = 2$, Assumption 1 is in particular true if there exists $K_1, K_2, K_3 \in \mathbb{R}_{\geq 0}$ such that $|\mathcal{T}(x)| \leq K_1 + K_2 e^{K_3|x|}$, since in this case L_1 is normally distributed. For $\alpha < 2$, it is sufficient that there exists $K_1, K_2 > 0$ and an $\varepsilon > 0$ such that $|\mathcal{T}(x)| \leq K_1 + K_2|x|^{\alpha-\varepsilon}$ since only the moments up to order $\theta < \alpha$ exist in this case. If $\mathcal{V} \subset \mathbb{R}_{\geq 0}^d$ (resp. $\mathcal{V} \subset \mathbb{R}_{\leq 0}^d$), the latter inequality only has to hold for positive (resp. negative) values of x.

Assumption 2. It holds that $\varrho^{\bar{L}_1} = \rho_0(\bar{L}_1) < \infty$ for all $c > 0$ and the corresponding α-stable Lévy process $\bar{L}$ with Lévy measures given by (2.1).

Next, one of the following three alternatives is assumed to hold:

Assumption 3a. L is symmetric.

Assumption 3b. L has only upward or downward jumps. In addition, $\mathcal{V} \subset \mathbb{R}_{\geq 0}^d$, i.e., we restrict the optimization to non-negative control functions, i.e., short selling is prohibited and $u \geq 0$.

If $\alpha = 0.5$ and L has only upward jumps, the distribution is often called a Lévy distribution; see, for instance, Cont and Tankov (2004).

Assumption 3c. $d = 1$, i.e., we only have one risky asset. In addition, $\mathcal{V} \subset \mathbb{R}_{\geq 0}^d$, i.e., we restrict the optimization to non-negative control functions, i.e., short selling is prohibited and $u \geq 0$.

In the case $\alpha = 2$, we have a Brownian Motion, which is always symmetric. Hence, distinguishing between Assumptions 3a, 3b, and 3c is only needed in the case $\alpha < 2$. In the following, we denote by $\tilde{\lambda}_{min}$ the smallest eigenvalue of $\sigma_s e^{r(T-s)}$ for $s \in [0, T]$ if $\alpha < 2$ (or denote by $\tilde{\lambda}_{min}^2$ the smallest eigenvalue of $\sigma_s R_s \sigma_s e^{2r(T-s)}$ for $s \in [0, T]$ if $\alpha = 2$) and by $\tilde{\mu}_{max} := \max\{|\mu_s^i - r|e^{r(T-s)}|s \in [0, T], i \in \{1, \ldots, d\}\}$. It holds that $\tilde{\lambda}_{min} > 0$.

Next, one of the following four alternatives is assumed to hold:

Assumption 4a. There exists an $M > 0$ such that $\mathcal{V} \subseteq [-M, M]^d$, i.e., the strategies are uniformly bounded.

We remark that under Assumption 4a, the market may admit arbitrage (which, however, cannot be scaled up). For instance, for $d \geq 2$ suppose that $\tilde{\sigma}$ has a positive probability only at the two points on the first axis, i.e., $\tilde{\sigma}((1, 0, \ldots, 0)) = \tilde{\sigma}((-1, 0, \ldots, 0)) = \frac{1}{2}$. Then, the marginal Lévy process in all other directions is a deterministic drift process given by $\mu^i t$. In particular, the market admits arbitrage if $\mu^i \not\equiv r$ for an $i \geq 2$. To exclude arbitrage and such examples while not relying on a boundedness assumption, we need $\tilde{\sigma}$ to be non-degenerated on the sphere. In particular, we need some measure in every axis direction. In the Black–Scholes market, non-degeneracy assumptions are standard, i.e., for instance, it is common there to assume that $\sigma^\intercal\sigma$ has a full rank and σ is bounded away from zero where σ is the volatility matrix.

Assumption 4b. $d = 1$, i.e., we only have one risky asset, $\alpha \in (1,2)$, $\lim_{x\to\infty} \mathcal{T}'(x) = 0$, and $\lambda_n \equiv \lambda > 0$. If ρ is cash-invariant, we additionally assume that $\tilde{\mu}_{max} < \varrho^{\hat{L}_1} \min_{s\in[0,T]}\{\sigma_s e^{r(T-s)}\}$.

Note that the last inequality is usually satisfied for the Black–Scholes model and $\rho = \mathrm{VaR}^\alpha$ with typically $\alpha \geq 90\%$, $\mu - r \leq 0.08$, and $\sigma \geq 0.1$. Recall for the following assumption that a hyperspherical cap with polar angle β is a portion of the sphere cut off by a hyperplane, where β is the angle between the vectors from the center of the hypersphere to the pole of the cap and the edge of the hyperdisk forming the base of the cap.

Assumption 4c. $d \geq 2$, $\alpha \in (1,2)$, $\lim_{x\to\infty} \mathcal{T}'(x) = 0$, $\lambda_n \equiv \lambda > 0$, and $\varrho^{\hat{L}_1} > 0$. In addition, we assume that there exist $\hat{\varepsilon} > 0$, $\tilde{\varepsilon} \in (0, \frac{\pi}{24})$ such that either (a) $\tilde{\sigma}$ has a density bigger than $\hat{\varepsilon}$ on the unit ball around a vector on the sphere with respect to the radial distance[2] with radius $12\tilde{\varepsilon}$, or (b) $\tilde{\sigma}$ has a point mass bigger than $\hat{\varepsilon}$ at $(d+1)$ points such that each point has a distance of at least $\tilde{\varepsilon}$ to every $(d-1)$-dimensional hyper-plane spanned by the other d points. If ρ is cash-invariant, we additionally assume that $\varrho^{\hat{L}_1}\tilde{\lambda}_{min}\min\{1,\mathbb{A}_d\}^{1/\alpha}\cos(\tilde{\varepsilon})\hat{\varepsilon}^{1/\alpha} > \tilde{\mu}_{max}$, where $\mathbb{A}_d$ denotes the surface area of a hyperspherical cap of $\mathbb{S}^d$ with polar angle $\tilde{\varepsilon}$.

If $\tilde{\sigma}$ has full support on the whole sphere and a continuous density, it fulfills the assumption stipulated in Assumption 4c for ρ being shift-invariant.

Assumption 4d. $\alpha = 2$, $\lim_{x\to\infty} \mathcal{T}'(x) = 0$, $\lambda_n \equiv \lambda > 0$, and $\varrho^{W_1} > 0$. If ρ is cash-invariant, we additionally assume that $\varrho^{W_1}\tilde{\lambda}_{min} > \tilde{\mu}_{max}$.

Note that the last inequality is satisfied for $\rho = \mathrm{VaR}^\alpha$ in a multidimensional Black–Scholes setting with typical parameter values.

4.2. Infinitesimal generator and Nash equilibrium

Definition 4.1. Let $f : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ be a function sequence. We define the infinitesimal generator A^u in discrete time as $(\mathrm{A}^u f)_n(x) = \mathbb{E}_{n,x}[f_{n+1}(X^u_{n+1}) - f_n(x)]$.

Definition 4.2. Consider a fixed control law $\hat{u}$. Fix an arbitrary point $(n,x) \in \mathbb{N}\times\mathbb{R}$ with $n < T$ and take an arbitrary control law $u \in \mathcal{U}$. We define then for the time points $n, n+1, \dots, T-1$ the control law $\bar{u}$ as follows for any $y \in \mathbb{R}$: $\bar{u}_k(y) = \begin{cases} \hat{u}_k(y) & \text{for } k = n+1,\dots,T-1, \\ u & \text{for } k = n \end{cases}$.
Then the control law $\hat{u}$ is a perfect subgame Nash equilibrium if for all fixed (n,x) it holds that $\sup_{u\in\mathcal{U}} J_n(x,\bar{u}) = J_n(x,\hat{u})$. If such an equilibrium exists, then we also define the equilibrium value function V by $V_n(x) = J_n(x,\hat{u})$.

The following proposition is well-known and not difficult to prove. It shows that an optimal time-consistent strategy indeed corresponds to a Nash equilibrium.

Proposition 4.3. *Let $\mathcal{A}_{T,x} := \{u_T = u(T,x) | u_T \in \mathcal{V}\}$. Next, we recursively define u_j^* as $u_j^* := \arg\max_{u\in\mathcal{A}_{j,x}} \mathbb{E}_{n,x}[\mathcal{T}(X_T^u - xe^{r(T-n)})] - \lambda_n F(\rho_{n,x}(X_T^u - xe^{r(T-n)}))$ for $j = n+1,\dots,T$ and $\mathcal{A}_{n,x}$ by $\mathcal{A}_{n,x} := \{(u_n, u_{n+1}, \dots, u_T) | u_n = u(n,x), u_n \in \mathcal{V}, u_j = u_j^* \text{ for } j = n+1,\dots,T\}$. Then, the control law $\hat{u}$ is a perfect subgame Nash equilibrium if and only if $\hat{u}$ is the* arg max *of problem (1.1).*

Proof. The arg max may be either restricted to u bounded by M (under Assumption 4a) or by Proposition 4.13 to $\mathcal{U} \cap \{u \mid \sum_{i=n}^T |u_i|^\alpha \leq M_n^\alpha\}$ (for appropriate $M_n > 0$). Hence, the maximum exists, and the proof follows by induction. □

[2] The radial distance gives the distance between two points on $\mathbb{S}^d$ along the radial dimension, i.e., the length of the shortest path on $\mathbb{S}^d$.

4.3. Properties of the optimal control function

This section proves that the optimal control function is deterministic and exists. It is unique if $\mathcal{T}$ or $-F$ is strictly concave and $\alpha > 1$. Here, we use the definition of stochastic integrals for functions from $\mathbb{L}$ (the set of adapted and càglàd processes, see Appendix A.2) since the discrete model restricts to piecewise constant control functions with left limits.

Theorem 4.4. *The supremum in (1.1) exists and may (and will in the sequel) be taken only over deterministic strategies. It is unique if $\mathcal{T}$ or $-F$ is strictly concave and $\alpha > 1$.*

Before proving this theorem, we need some additional definitions and results:

Remark 4.5. Under Assumptions 3a and 3b, $\int_{\mathbb{S}^d} \|\langle a,v\rangle\|^\alpha \mathrm{sign}(\langle a,v\rangle)\tilde{\sigma}(\mathrm{d}v)(\int_{\mathbb{S}^d}\|\langle a,v\rangle\|^\alpha\tilde{\sigma}(\mathrm{d}v))^{-1}$ is independent of $a \in \mathbb{R}^d_{\geq 0}$, and we will in the sequel refer to this number as $p-q$ with $p+q=1$. It holds that $p-q=1$ (resp. $p-q=-1$) if L has only upward (resp. downward) jumps and $p-q=0$ if L is symmetric. This will help us to identify later certain stochastic integrals with one-dimensional Lévy processes using Eq. (A.2).

Definition 4.6. Under Assumptions 3a and 3b, if $\alpha \in (0,2]\backslash\{1\}$, let $\tilde{L}$ be a one-dimensional α-stable Lévy process with characteristic function given by (A.2) with the same constant c in the Lévy measure as L and (as in Remark 4.5) with $p-q = 2p-1 = \int_{\mathbb{S}^d}\|\langle a,v\rangle\|^\alpha \mathrm{sign}(\langle a,v\rangle)\tilde{\sigma}(\mathrm{d}v)$ $(\int_{\mathbb{S}^d}\|\langle a,v\rangle\|^\alpha\tilde{\sigma}(\mathrm{d}v))^{-1}$, where $p-q$ is as in Eq. (A.2).

If $\alpha = 2$, let $\tilde{L}$ be a one-dimensional Brownian Motion, i.e., $\tilde{L} \overset{d}{=} W^j$ for all $j \in \{1,\dots,d\}$.

Definition 4.7. We define the Hilbert space H_t as $(\mathbb{R}^d, \langle\cdot,\cdot\rangle_{R_t})$ with R_t as in Section 3.1. Then, we set $\|x\|_{R_t} := \sqrt{\langle x,x\rangle_{R_t}} = \sqrt{xR_tx^\intercal}$ for a row vector x and with a slight abuse of notation write $\|x\|_{\mathcal{L}^2((a,b],\mathrm{d}t;H_t)} = \sqrt{\int_a^b \|x\|^2_{R_t}\mathrm{d}t}$ with $a < b$.

Definition 4.8. We define the two functions m and w as follows for all $s \in (0,T]$

$$m_s(u) := \begin{cases} u_s^\intercal(\mu_s - r)e^{r(T-s)} & \text{if the risk measure } \rho \text{ is cash-invariant,} \\ 0 & \text{if the risk measure } \rho \text{ is shift-invariant,} \end{cases}$$

$${}^\alpha w_s(u) := \begin{cases} \|u_s^\intercal\sigma_s v e^{r(T-s)}\|_{\mathcal{L}^\alpha(\mathbb{S}^d,\tilde{\sigma}(\mathrm{d}v))} & \text{if } \alpha < 2. \\ \|u_s^\intercal\sigma_s e^{r(T-s)}\|_{R_s} & \text{if } \alpha = 2. \end{cases}$$

Moreover, we define $\hat{L} := \begin{cases} \tilde{L} & \text{if } \alpha < 2 \text{ \& Ass. 3a or 3b (see Definition 4.6).} \\ L^1 & \text{if } \alpha < 2 \text{ \& Ass. 3c.} \\ W^1 & \text{if } \alpha = 2. \end{cases}$

If $d = 1$, then ${}^\alpha w_s(u) = |u_s^1\sigma_s^{11}e^{r(T-s)}|$ in all cases since then it holds that $R_s = (1)$ and

$$\|u_s^\intercal\sigma_s v e^{r(T-s)}\|_{\mathcal{L}^\alpha(\mathbb{S}^1,\tilde{\sigma}(\mathrm{d}v))} = |u_s^1\sigma_s^{11}e^{r(T-s)}|\sqrt[\alpha]{\int_{\mathbb{S}^1}|v|^\alpha\tilde{\sigma}(\mathrm{d}v)} = |u_s^1\sigma_s^{11}e^{r(T-s)}|. \tag{4.2}$$

Note that for $\alpha < 1$, $\|\cdot\|_{\mathcal{L}^\alpha}$ is only a quasi-norm.

The following lemmas show that the stochastic integral of an α-stable Lévy process is still α-stable. We state the proofs in Appendix B.

Lemma 4.9. *Let $a = (a^1,\dots,a^d)^\intercal \in \mathbb{R}^d_{\geq 0}$, $\alpha < 2$ and $0 \leq s < t$. Then it holds in the case of Assumption 3a or 3b that $\langle a, L_t - L_s\rangle \overset{d}{=} \sqrt[\alpha]{\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde{\sigma}(\mathrm{d}v)}(\tilde{L}_t - \tilde{L}_s)$.*

Lemma 4.10. *Let f be a deterministic and continuous function. Moreover, in the case of Assumption 3b or 3c, let f additionally be non-negative. Then it holds for all $t \in [0,T)$ that $\int_{t+}^{T} f(s)\mathrm{d}\hat{L}_s \overset{d}{=} \sqrt[\alpha]{\int_t^T |f(s)|^\alpha \mathrm{d}s}\hat{L}_1$.*

Proposition 4.11. *It holds $\rho_{n,x}(X_T^u - xe^{r(T-n)})$ $= -\int_n^T m_s(u)\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(u)\|_{L^\alpha((n,T],\mathrm{d}s)}$ for all $u \in \mathcal{U}_{det}$.*

Proof. We start with the case of Assumption 3a or 3b. First of all, we can rewrite the controlled Markov process X with Itô's lemma for semimartingales in integral form (see Protter (2005, pp. 81–82)). Hence, we get for all $u \in \mathcal{U}$:

$$\begin{aligned}
X_T^u - X_n^u e^{r(T-n)} &= \int_{n+}^{T} -re^{r(T-s_-)}X_{s-}^u\mathrm{d}s + \int_{n+}^{T} e^{r(T-s_-)}\mathrm{d}X_s^u + 0 \\
&\quad + \sum_{t<s\le T}\{e^{r(T-s)}X_s^u - e^{r(T-s_-)}X_{s_-}^u - 0 - e^{r(T-s_-)}\Delta X_s^u\} \\
&= \int_{n+}^{T} re^{r(T-s)}\Delta X_s^u\mathrm{d}s + \sum_{i=1}^{d}\int_{n+}^{T} u_{s-}^i(\mu_s^i - r)e^{r(T-s)}\mathrm{d}s \\
&\quad + \sum_{i,j=1}^{d}\int_{n+}^{T} u_{s-}^i\sigma_s^{ij}e^{r(T-s)}\mathrm{d}L_s^j \\
&= \sum_{i=1}^{d}\int_{n}^{T} u_s^i(\mu_s^i - r)e^{r(T-s)}\mathrm{d}s + \sum_{i,j=1}^{d}\int_{n+}^{T} u_s^i\sigma_s^{ij}e^{r(T-s)}\mathrm{d}L_s^j,
\end{aligned} \tag{4.3}$$

since X_s^u is a càdlàg process (due to L_s being càdlàg) and has therefore only countably many points of discontinuity. Thus, the points of discontinuity form a Lebesgue null set.

Then, we get from (4.3) combined with the definition of m (see Definition 4.8) and the cash- or shift-invariance of ρ that

$$\rho_{n,x}(X_T^u - xe^{r(T-n)}) = -\int_n^T m_s(u)\mathrm{d}s + \rho_{n,x}\Big(\sum_{i,j=1}^{d}\int_{n+}^{T} u_s^i\sigma_s^{ij}e^{r(T-s)}\mathrm{d}L_s^j\Big).$$

Now, since $u \in L^\alpha$ and deterministic, there exist deterministic and continuous ${}^l u$ such that ${}^l u \xrightarrow{L^\alpha} u$. Then, (A.3) implies that

$$\begin{aligned}
\sum_{i,j=1}^{d}\int_{n+}^{T} {}^l u_s^i\sigma_s^{ij}e^{r(T-s)}\mathrm{d}L_s^j &\overset{\mathbb{P}}{=} \sum_{i,j=1}^{d}\lim_{K\to\infty}\sum_{k=1}^{K} {}^l u_{t_k^K}^i\sigma_{t_k^K}^{ij}e^{r(T-t_k^K)}(L_{t_k^K}^j - L_{t_{k-1}^K}^j) \\
&= \lim_{K\to\infty}\sum_{k=1}^{K}\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} {}^l u_{t_k^K}^i\sigma_{t_k^K}^{ij}e^{r(T-t_k^K)}\Big)(L_{t_k^K}^j - L_{t_{k-1}^K}^j)
\end{aligned} \tag{4.4}$$

with the equidistant grid points t_k, i.e., $n = t_0^K < t_1^K < \cdots < t_K^K = T$.

We must distinguish the cases $\alpha < 2$ and $\alpha = 2$. We start with the case $\alpha < 2$:

We apply Lemma 4.9 on the term inside of the sum over k, i.e.:

$$\begin{aligned}
&\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} {}^l u_{t_k^K}^i\sigma_{t_k^K}^{ij}e^{r(T-t_k^K)}\Big)(L_{t_k^K}^j - L_{t_{k-1}^K}^j) \\
&\qquad \overset{d}{=} \sqrt[\alpha]{\int_{\mathbb{S}^d}\Big|\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} {}^l u_{t_k^K}^i\sigma_{t_k^K}^{ij}e^{r(T-t_k^K)}\Big)v^j\Big|^\alpha\tilde{\sigma}(\mathrm{d}v)}(\tilde{L}_{t_k^K} - \tilde{L}_{t_{k-1}^K}).
\end{aligned} \tag{4.5}$$

We know for random variables $Y_i, Z_i, i \in \{1,\dots,d\}$: If $Y_i \overset{d}{=} Z_i$, $Y_i \perp\!\!\!\perp Y_j$, and $Z_i \perp\!\!\!\perp Z_j$ for all $i \neq j$, then also $\sum_{i=1}^d Y_i \overset{d}{=} \sum_{i=1}^d Z_i$. It is:

$$\begin{aligned}
&\rho_{n,x}\Big(\sum_{i,j=1}^{d}\int_{n+}^{T} u_s^i\sigma_s^{ij}e^{r(T-s)}\mathrm{d}L_s^j\Big) \\
&= \rho_{n,x}\Big(\lim_{l\to\infty}\sum_{i,j=1}^{d}\int_{n+}^{T} {}^l u_s^i\sigma_s^{ij}e^{r(T-s)}\mathrm{d}L_s^j\Big) \\
&= \rho_{n,x}\Big(\lim_{l\to\infty}\lim_{K\to\infty}\sum_{k=1}^{K}\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} {}^l u_{t_k^K}^i\sigma_{t_k^K}^{ij}e^{r(T-t_k^K)}\Big)(L_{t_k^K}^j - L_{t_{k-1}^K}^j)\Big) \\
&= \rho_{n,x}\Big(\lim_{l\to\infty}\lim_{K\to\infty}\sum_{k=1}^{K}\sqrt[\alpha]{\int_{\mathbb{S}^d}\Big|\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} {}^l u_{t_k^K}^i\sigma_{t_k^K}^{ij}e^{r(T-t_k^K)}\Big)v^j\Big|^\alpha\tilde{\sigma}(\mathrm{d}v)}(\tilde{L}_{t_k^K} - \tilde{L}_{t_{k-1}^K})\Big) \\
&= \rho_{n,x}\Big(\lim_{l\to\infty}\int_{n+}^{T}\sqrt[\alpha]{\int_{\mathbb{S}^d}\Big|\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} {}^l u_s^i\sigma_s^{ij}e^{r(T-s)}\Big)v^j\Big|^\alpha\tilde{\sigma}(\mathrm{d}v)}\mathrm{d}\tilde{L}_s\Big) \\
&= \rho_0\Big(\lim_{l\to\infty}\int_{n+}^{T}\sqrt[\alpha]{\int_{\mathbb{S}^d}\Big|\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} {}^l u_s^i\sigma_s^{ij}e^{r(T-s)}\Big)v^j\Big|^\alpha\tilde{\sigma}(\mathrm{d}v)}\mathrm{d}\tilde{L}_s\Big) \\
&= \rho_0\Big(\lim_{l\to\infty}\sqrt[\alpha]{\int_{n+}^{T}\Big|\sqrt[\alpha]{\int_{\mathbb{S}^d}\Big|\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} {}^l u_s^i\sigma_s^{ij}e^{r(T-s)}\Big)v^j\Big|^\alpha\tilde{\sigma}(\mathrm{d}v)}\Big|^\alpha\mathrm{d}s\tilde{L}_1}\Big) \\
&= \rho_0\Big(\sqrt[\alpha]{\int_{n+}^{T}\int_{\mathbb{S}^d}\Big|\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} u_s^i\sigma_s^{ij}e^{r(T-s)}\Big)v^j\Big|^\alpha\tilde{\sigma}(\mathrm{d}v)\mathrm{d}s}\tilde{L}_1\Big) \\
&= \varrho^{\tilde{L}_1}\Big\|\sum_{j=1}^{d}\Big(\sum_{i=1}^{d} u_s^i\sigma_s^{ij}e^{r(T-s)}\Big)v^j\Big\|_{L^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}v))} \\
&= \varrho^{\tilde{L}_1}\|u_s^\intercal\sigma_s v e^{r(T-s)}\|_{L^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}v))}
\end{aligned}$$

with Applebaum (2009, p.237) and ${}^l u \xrightarrow{L^\alpha} u$ in the first equality, (4.4) in the second equality, (4.5) in the third equality, the independent increments of L and the definition of ρ in the fifth equality, Lemma 4.10 in the sixth equality due to ${}^l u$ being deterministic, bounded, and continuous, ${}^l u \xrightarrow{L^\alpha} u$ in the seventh equality, and the positive homogeneity and the law-invariance of ρ in the eighth equality.

The proof works with similar ideas if $\alpha = 2$ or if Assumption 3c holds. □

Proposition 4.12. *Let $\alpha > 1$. If the control function u is deterministic, then the functional J_n is concave in u. If $\mathcal{T}$ or $-F$ is strictly concave, J_n is also strictly concave.*

We state the proof of this and the following two propositions in Appendix B.

Proposition 4.13. *The optimal deterministic strategies are uniformly bounded in $(\mathcal{L}^\alpha, \|\cdot\|_{\mathcal{L}^\alpha})$ by an $M > 0$ independent of $\delta > 0$.*

Proposition 4.14. *If the control function u is deterministic, then the functional $J_n(x,u)$ is continuous in $u \in \tilde{\mathcal{U}}_{det}$ for every x. (Note that since $\tilde{\mathcal{U}}_{det}$ is the set of deterministic discrete-time strategies, u may be seen as a vector in $\mathbb{R}^{d\times(T-n)}$).*

Now, we are able to prove our main result of this section, i.e., Theorem 4.4:

Proof of Theorem 4.4. We prove this theorem by backward induction. Let $M > 0$ be greater than the constant from Proposition 4.13.

$n = T-1$:

We know due to (4.3) and as the control function is piecewise constant:

$$X_T^u - X_{T-1}^u e^{r(T-n)} = u_T^\intercal\int_{T-1}^{T}(\mu_s - r)e^{r(T-s)}\mathrm{d}s + u_T^\intercal\int_{(T-1)+}^{T}\sigma_s e^{r(T-s)}\mathrm{d}L_s. \tag{4.6}$$

For a vector $u = (u^1,\dots,u^d)^\intercal \in \mathbb{R}^d$, denote $\|u\|_\alpha = \sum_{i=1}^d |u^i|^\alpha$. We define $u^* \in \mathcal{V} \subset \mathbb{R}^d$ as:

$$\begin{aligned}
u^* &\in \operatorname*{arg\,max}_{\|u\|_\alpha\le M, u\in\mathcal{V}}\Big\{\mathbb{E}\Big[\mathcal{T}\Big(u^\intercal\int_{T-1}^{T}(\mu_s - r)e^{r(T-s)}\mathrm{d}s + u^\intercal\int_{(T-1)+}^{T}\sigma_s e^{r(T-s)}\mathrm{d}L_s\Big)\Big] \\
&\qquad - \lambda_{T-1}F\Big(\rho\Big(u^\intercal\int_{T-1}^{T}(\mu_s - r)e^{r(T-s)}\mathrm{d}s + u^\intercal\int_{(T-1)+}^{T}\sigma_s e^{r(T-s)}\mathrm{d}L_s\Big)\Big)\Big\} \\
&= \operatorname*{arg\,max}_{\|u\|_\alpha\le M, u\in\mathcal{V}}\Big\{\mathbb{E}\Big[\mathcal{T}\Big(u^\intercal\Big(\int_{T-1}^{T}(\mu_s - r)e^{r(T-s)}\mathrm{d}s + \int_{(T-1)+}^{T}\sigma_s e^{r(T-s)}\mathrm{d}L_s\Big)\Big)\Big] \\
&\qquad - \lambda_{T-1}F\Big(-\int_{T-1}^{T} m_s(u)\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(u)\|_{L^\alpha((T-1,T],\mathrm{d}s)}\Big)\Big\} \\
&=: \operatorname*{arg\,max}_{\|u\|_\alpha\le M, u\in\mathcal{V}}\tilde{V}(u),
\end{aligned} \tag{4.7}$$

where we used Proposition 4.13 (with the strategy $u_{T-1}\mathbb{1}_{(T-1,T]}$ and u being then identified with u_{T-1} in $\mathbb{R}^d$) and Proposition 4.11 since u in Eq. (4.7) is deterministic. Now, it follows from Proposition 4.14 (and $\mathcal{V}$ being closed and convex) that $\tilde{V}$ is continuous in u on a compact set and hence has a global maximum, i.e., u^* exists, and by Proposition 4.13 is independent of M. If $\alpha > 1$ and $\mathcal{T}$ or $-F$ are strictly concave, Proposition 4.12 gives us the uniqueness. Next, let $u = u_{T-1}\mathbb{1}_{(T-1,T]}$ be a Markovian strategy with $u_{T-1} = u(T-1, X^u_{T-1})$. Note that for all $x \in \mathbb{R}$, we have $\tilde{V}(u(T-1,x)) \uparrow_{M\to\infty} \mathbb{1}_{|u(T-1,x)|\le M}\tilde{V}(u(T-1,x)) \le \tilde{V}(u^*)$. Hence, it follows that $V(T-1,x) = \operatorname{esssup}_{u(T-1,x)\in\mathcal{V}}\{\mathbb{E}_{T-1,x}[\mathcal{T}(X^u_T - xe^r)] - Risk_{T-1,x}(X^u_T - xe^r)\} = \operatorname{esssup}_{u(T-1,x)\in\mathcal{V}}\tilde{V}(u(T-1,x)) \le \tilde{V}(u^*) \le V(T-1,x)$ a.s. for V defined as in (1.1). Hence, all the inequalities must be equalities, and $u^* \in \mathcal{V} \subset \mathbb{R}^d$ attains the supremum for $t = T-1$ in (1.1). Thus, the arg max of V from (1.1) can be taken only over deterministic control functions at time $T-1$. Furthermore, the equilibrium $\hat{u}_T = u^*$ attains this maximum. Since the first inequality is strict if $u(T-1,x) \neq u^*$ provided that either $\mathcal{T}$ or $-F$ is strictly concave and $\alpha > 1$, uniqueness follows.

arbitrary n: This can be done analogously to the first case. Using that the equilibrium value attains the optimum for all later time points, $\hat{u}_{n+1}$ is also the (unique) equilibrium. □

Remark 4.15. If one considers the time-consistent, ex-ante *non-Markovian* problem

$$V(t) = \operatorname{esssup}_{u\in\hat{\mathcal{A}}_t}\left\{\mathbb{E}_t[\mathcal{T}(X^u_T - X^u_t e^{r(T-t)})] - Risk_t(X^u_T - X^u_t e^{r(T-t)})\right\},$$

with $\hat{\mathcal{A}}_t$ being the set of time-consistent, (possibly non-Markovian) *adapted* strategies, one can show as in the proof of Theorem 4.4 that the supremum is attained in the deterministic strategy u^* from the proof of Theorem 4.4. This justifies the restriction to Markovian strategies in (1.1).

Next, we analyze if the optimal solution is non-negative. In the case of Assumptions 3b and 3c, we already assumed this property; hence, we only need to consider the case of Assumption 3a:

Proposition 4.16. *Under Assumption 3a: Let $d = 1$, $\alpha > 1$, $\mu_t \ge r$ for all $t \in [0,T]$, and $\mathcal{T}$ be such that $\mathbb{E}[|\mathcal{T}'(a + bL^1_t)|] < \infty$ for all $a, b \in \mathbb{R}$ and for all $t \in (0,T]$. Further assume that $F'(0) = 0$. Then, every deterministic optimal control function is non-negative.*

We state this rather technical proof in Appendix B. If $d > 1$, the proposition is no longer valid since some assets might serve as a hedge against others.

4.4. The HJB equation in discrete time

Based on the results from above, we can now restrict the analysis to deterministic control functions.

Theorem 4.17. *If an equilibrium control law $\hat{u}$ exists, the equilibrium value function V satisfies:*

$$\begin{aligned}\sup_{u\in\mathcal{V}}\{&(\mathrm{A}^u V)_n(x) - (\mathrm{A}^u\mathbb{E}_\cdot[\mathcal{T}(X^{\bar{u}}_T - X^u_\cdot e^{r(T-\cdot)})])_n(x)\\ &+(\mathrm{A}^u(\lambda_\cdot F(\rho_\cdot(X^{\bar{u}}_T - X^u_\cdot e^{r(T-\cdot)})))_n(x)\} = 0 \qquad (4.8)\\ &V_T(x) = \mathcal{T}(0) - \lambda_T F(0),\end{aligned}$$

where $\bar{u}_s = \begin{cases} u & ,\text{if } s = n \\ \hat{u}_s & ,\text{if } s \ge n+1 \end{cases}$ and the supremum is attained at $u = \hat{u}_n$, see Definition 4.1 for the definition of A.

Let m, w and $\hat{L}$ be as in Definition 4.8. Then we have the following identities for $0 \le n \le T-1$:

$$\begin{aligned}&(\mathrm{A}^{\hat{u}}\mathbb{E}_\cdot[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_\cdot e^{r(T-\cdot)})])_n(x)\\ &= \mathbb{E}_{n,x}[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_{n+1}e^{r(T-n-1)}) - \mathcal{T}(X^{\hat{u}}_T - xe^{r(T-n)})]\\ &(\mathrm{A}^{\hat{u}}(\lambda_\cdot F(\rho_\cdot(X^{\hat{u}}_T - X^{\hat{u}}_\cdot e^{r(T-\cdot)}))))_n(x)\\ &= \lambda_{n+1}[F(-\int_{n+1}^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((n+1,T],\mathrm{d}s)})\\ &\quad - F(-\int_n^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((n,T],\mathrm{d}s)})]\\ &\quad -(\lambda_n - \lambda_{n+1})F(-\int_n^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((n,T],\mathrm{d}s)})\end{aligned}$$

with $\lambda_T F(\rho_{T,x}(X^{\hat{u}}_T - xe^{r(T-T)})) = \lambda_T F(0)$.

The proof of this theorem is rather technical, and we give it in Appendix B.

5. Analysis in continuous time

5.1. Definition of the model

We now extend the discrete-time result into continuous time. Hence, we maximize the value functional J defined as follows with fixed $(t,x) \in [0,T)\times\mathbb{R}$ and a $\mathcal{L}^\alpha(\mathrm{d}\mathbb{P}\times\mathrm{d}s)$-integrable, fixed control law u:

$$J(t,x,u) = \mathbb{E}_{t,x}[\mathcal{T}(X^u_T - xe^{r(T-t)})] - \lambda_t F(\rho_{t,x}(X^u_T - xe^{r(T-t)})). \qquad (5.1)$$

We use a convergence approach to formally obtain the extended HJB equation. However, we first define the infinitesimal generator and Nash equilibria in continuous time and prove some properties of the optimal control function.

5.2. Infinitesimal generator and Nash equilibrium

Following the same references as in Section 4.2, we define:

Definition 5.1. Let $f : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ be a function sequence.

1. We define the operator P^u_h, $h \in \mathbb{R}$, as $(\mathrm{P}^u_h f)(t,x) = \mathbb{E}_{t,x}[f(t+h, X^u_{t+h})]$.
2. The infinitesimal generator A^u is defined as $\mathrm{A}^u = \frac{\partial \mathrm{P}^u_h}{\partial h}\Big|_{h=0}$.
3. Moreover, we define the operator A^u_h, $h \in \mathbb{R}$, as $(\mathrm{A}^u_h f)(t,x) = \mathbb{E}_{t,x}[f(t+h, X^u_{t+h}) - f(t,x)]$.

Definition 5.2. We consider a fixed control law $\hat{u}$, fix an arbitrary point $(t,x) \in \mathbb{N}\times\mathbb{R}$ with $t < T$, an $h > 0$ and an arbitrary control law $u \in \mathcal{U}$. Then, we define for the time interval $[t,T]$ the control law u_h as follows for any $y \in \mathbb{R}$: $u_h(s,y) = \begin{cases} \hat{u}(s,y) & \text{for } t+h \le s \le T \\ u & \text{for } t \le s < t+h \end{cases}$. If it holds that $\liminf_{h\to 0}\frac{J(t,x,\hat{u}) - J(t,x,u_h)}{h} \ge 0$ for all $u \in \mathcal{U}$, then we call the control law $\hat{u}$ a Nash equilibrium control law. If such an equilibrium exists, then we also define the equilibrium value function V by $V(t,x) = J(t,x,\hat{u})$ and call $\hat{u}$ an optimal control function.

Ekeland and Lazrak (2006), and Ekeland and Pirvu (2008) introduced this definition for Nash equilibria in continuous time. It is also used, for instance, in Björk and Murgoci (2010), Hu et al. (2012), Björk et al. (2014), and in other references for time-consistency given in the introduction. As already stated in the introduction, this generalization of Nash equilibria allows the player t to form a coalition with all players $s \in [t, t+\varepsilon]$ and the equilibrium is taken by $\varepsilon \to 0$. This construction is necessary since the decision of player t alone influences only the strategy at the null set $\{t\}$ and hence has no influence.

Assumption 5. $\alpha > 1$ or an optimal solution $\hat{u}$ exists which is continuous and deterministic.

This assumption is needed in the following since for $\alpha < 1$, the L^α norm is only a quasi norm, and the respective space is no longer locally convex.

5.3. Derivation of the HJB in continuous time

Unlike the discrete-time case, we state the HJB equation only as a definition and prove it afterwards in a Verification Theorem. In this subsection, we show the motivation for the HJB equation:

As in Björk and Murgoci (2010), we assume there exists an equilibrium law $\hat{u}$ and proceed as follows: First, we take a point (t,x) and an $h > 0$ small. Second, we define u_h in $[t,T]$ as $u_h(s,y) = \begin{cases} u & \text{for } t \le s < t+h, \quad y \in \mathbb{R} \\ \hat{u}(s,y) & \text{for } t+h \le s \le T, \quad y \in \mathbb{R} \end{cases}$ and third, for h small enough, we expect to have: $J(t,x,u_h) \le J(t,x,\hat{u})$.

Theorem 4.17 for the interval from t to $t+h$ gives us:

$$(\mathrm{A}_h^u V)(t,x) - (\mathrm{A}_h^u \mathbb{E}_\cdot[\mathcal{T}(X_T^{u_h} - X_\cdot^u e^{r(T-\cdot)})])(t,x) + (\mathrm{A}_h^u(\lambda_\cdot F(\rho_\cdot(X_T^{u_h} - X_\cdot^u e^{r(T-\cdot)}))))(t,x) \le 0.$$

Next, we divide by h and take the limit of h to 0:

$$(\mathrm{A}^u V)(t,x) - (\mathrm{A}^u \mathbb{E}_\cdot[\mathcal{T}(X_T^{\hat{u}} - X_\cdot^u e^{r(T-\cdot)})])(t,x) + (\mathrm{A}^u(\lambda_\cdot F(\rho_\cdot(X_T^{\hat{u}} - X_\cdot^u e^{r(T-\cdot)}))))(t,x) \le 0.$$

Before defining the extended HJB, we want to specify the formulas for the infinitesimal generators, which we infer using a limiting approach from the constraints of the HJB in discrete time (Theorem 4.17):

Lemma 5.3. *Let u be continuous and deterministic. Moreover, let Assumption 6 and 7 below hold. Then, we have* $(\mathrm{A}^{\hat{u}}\mathbb{E}_\cdot[\mathcal{T}(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)})])(t,x) = \frac{\partial}{\partial h}\mathbb{E}[\mathcal{T}(X_T^{\hat{u}} - X_{t+h}^{\hat{u}} e^{r(T-t-h)})]\big|_{h=0}$.

Lemma 5.4. *Let u be continuous and deterministic, and m, w, and $\hat{L}$ be as in Definition 4.8. Moreover, let Assumption 6 and 7 below hold. Then, it holds that*

$$\begin{aligned}(\mathrm{A}^{\hat{u}}(\lambda_\cdot F(\rho_\cdot(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)}))))(t,x) &= \lambda_t' F\Big(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}\Big) \\ &\quad + \lambda_t F'\Big(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}\Big) \\ &\quad \cdot [m_t(\hat{u}) - \varrho^{\hat{L}_1\,\alpha} w_t(\hat{u})\Big(\int_t^T |^\alpha w_s(\hat{u})|^\alpha \mathrm{d}s\Big)^{\frac{1}{\alpha}-1}].\end{aligned}$$

The proofs of these lemmas are given in Appendix B.

5.4. The extended HJB equation in continuous time

We define the extended HJB equation by using the formal derivation of the previous Section 5.3 and the formulas from Lemmas 5.3 and 5.4 for the constraints.

Definition 5.5 (*Extended HJB Equation*)**.** The extended HJB equation for the Nash equilibrium problem is for $0 \le t \le T$ defined as

$$\begin{aligned}\sup_{u\in\mathcal{V}}\{(\mathrm{A}^u V)(t,x) - (\mathrm{A}^u \mathbb{E}_\cdot[\mathcal{T}(X_T^{\hat{u}} - X_\cdot^u e^{r(T-\cdot)})])(t,x) \\ +(\mathrm{A}^u(\lambda_\cdot F(\rho_\cdot(X_T^{\hat{u}} - X_\cdot^u e^{r(T-\cdot)}))))(t,x)\} = 0, \qquad (5.2) \\ V(T,x) = \mathcal{T}(0) - \lambda_T F(0),\end{aligned}$$

under the constraints:

$$(\mathrm{A}^{\hat{u}}\mathbb{E}_\cdot[\mathcal{T}(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)})])(t,x) = \frac{\partial}{\partial h}\mathbb{E}[\mathcal{T}(X_T^{\hat{u}} - X_{t+h}^{\hat{u}} e^{r(T-t-h)})]|_{h=0} \text{ and}$$

$$\begin{aligned}&(\mathrm{A}^{\hat{u}}(\lambda_\cdot F(\rho_\cdot(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)}))))(t,x) \\ &= \lambda_t' F\Big(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}\Big) \\ &\quad + \lambda_t F'\Big(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}\Big) \\ &\quad \cdot [m_t(\hat{u}) - \varrho^{\hat{L}_1\,\alpha} w_t(\hat{u})\Big(\int_t^T |^\alpha w_s(\hat{u})|^\alpha \mathrm{d}s\Big)^{\frac{1}{\alpha}-1}],\end{aligned}$$

with m, w, $\hat{L}$ as in Definition 4.8, and $\hat{u}$ taking the supremum in (5.2).

Remark 5.6. The constraints in Definition 5.5 always hold if the strategy under question is deterministic and continuous.

5.5. Properties of the optimal control function

This subsection proves that the continuous-time optimal control function is again deterministic under some assumption. Note that by Assumption 5, we assume $\alpha > 1$.

Proposition 5.7. *If u is deterministic, then the functional J from (5.1) is concave in u. If $\mathcal{T}$ or $-F$ is strictly concave, J is strictly concave.*

Proof. This proof is identical to the proof of Proposition 4.12. □

Theorem 5.8. *There exists a sequence ${}^\delta\hat{u}$ of optimal control functions with step size δ and a $\hat{u} \in \mathcal{U}_{det}$ such that ${}^\delta\hat{u} \xrightarrow{\delta\to 0} \hat{u}$ in $L^\alpha((0,T),\mathrm{d}t)$ and $\limsup_{\delta\to 0} J_t(x, {}^\delta\hat{u}) \le J(t,x,\hat{u})$.*

Furthermore, if $\hat{u}$ is continuous, $\mathcal{T}(x) = x$, $F(x) = \max\{0,x\}^\alpha$ and ρ shift-invariant, then $\hat{u}$ is a deterministic continuous-time equilibrium control function.

Proof. First of all, Theorem 4.4 applied to the interval $(t, t+\delta]$ gives us a deterministic optimal control function, denoted by ${}^\delta\hat{u}$, for each δ. Since all ${}^\delta\hat{u}$ are uniformly bounded in the reflexive Banach space $\mathcal{L}^\alpha((0,T],\mathrm{d}t)$ by M due to Proposition 4.13, a corollary of the theorem of Banach–Alaoglu gives us weak convergence of ${}^\delta\hat{u}$ by possibly switching to a subsequence. Denote the corresponding weak limit by $\tilde{u} \in \mathcal{L}^\alpha((0,T],\mathrm{d}t)$. Since M bounds the norm of all ${}^\delta\hat{u} \in \mathcal{U}_{det}$ for all $\delta > 0$, weak limits of deterministic processes are deterministic, and $\mathcal{U}$ is weakly closed since it is convex and closed, we get $\tilde{u} \in \mathcal{U}_{det}$.

Next, we prove the claim $\limsup_{\delta\to 0} J_t(x, {}^\delta\hat{u}) \le J(t,x,\tilde{u})$: Due to ${}^\delta\hat{u}$ and $\tilde{u}$ being deterministic, the functional J is concave by Proposition 5.7. Hence, the claim reduces to the weak upper semi-continuity for deterministic function sequences. This is equivalent to the strong upper semi-continuity in $\mathcal{L}^\alpha((0,T],\mathrm{d}t)$ since J is concave in $\mathcal{L}^\alpha((0,T],\mathrm{d}t)$. So suppose that ${}^\delta\hat{u}$ converges strongly to $\tilde{u}$ in $\mathcal{L}^\alpha((0,T],\mathrm{d}t)$.

First, we notice that by possibly switching to a subsequence ${}^\delta\hat{u}$ converges $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-a.s. to $\tilde{u}$. Then, note that $|{}^\delta\hat{u}_s^\intercal \sigma_s v e^{r(T-s)}|^\alpha$ $(\le C\|{}^\delta\hat{u}_s\|^\alpha$ for some $C > 0)$ is dominated by a converging sequence and thus uniformly integrable. Hence, it follows using the exact same steps as in the proof of Proposition 4.11 by replacing the risk measure $\rho(\cdot)$ with equality in distribution that under Assumption 3a or 3b with $\alpha < 2$ for all $t \in (0,T]$

$$\int_t^T \sum_{i,j=1}^d {}^\delta\hat{u}_s^i \sigma_s^{ij} e^{r(T-s)}\mathrm{d}L_s^j \overset{d}{=} \sqrt[\alpha]{\int_{t+}^T |\sqrt[\alpha]{\int_{\mathbb{S}^d} |\sum_{j=1}^d (\sum_{i=1}^d {}^\delta\hat{u}_s^i \sigma_s^{ij} e^{r(T-s)}) v^j|^\alpha \tilde{\sigma}(\mathrm{d}v)}|^\alpha \mathrm{d}s\, \tilde{L}_1}.$$

Analog to the proof of Proposition 4.11, a similar result also holds if $\alpha = 2$ and in the case of Assumption 3c. Then, Slutsky's theorem implies that for all $i,j \in \{1,\dots,d\}$ and for all $t \in (0,T]$

$$\int_t^T {}^\delta\hat{u}_s^i \sigma_s^{ij} e^{r(T-s)}\mathrm{d}L_s^j \xrightarrow[\delta\to 0]{d} \int_t^T \tilde{u}_s^i \sigma_s^{ij} e^{r(T-s)}\mathrm{d}L_s^j, \qquad (5.3)$$

where $\xrightarrow{d}$ denotes convergence in distribution. Now, Skohorod's representation theorem implies the existence of a common probability space such that the convergence holds almost surely. Furthermore, note that by possibly switching to a subsequence also ${}^\delta\hat{u}_s^i(\mu_s^i - r)e^{r(T-s)}$ is a.s. convergent to $\tilde{u}_s^i(\mu_s^i - r)e^{r(T-s)}$ and uniformly integrable (by the same arguments as above). Hence, the Vitali convergence theorem implies for all $i \in \{1,\dots,d\}$:

$$\int_t^T {}^\delta\hat{u}_s^i(\mu_s^i - r)e^{r(T-s)}\mathrm{d}s \xrightarrow{\delta\to 0} \int_t^T \tilde{u}_s^i(\mu_s^i - r)e^{r(T-s)}\mathrm{d}s. \qquad (5.4)$$

Hence, using (4.3) we obtain: $X_T^{{}^\delta\hat{u}} - X_t^{{}^\delta\hat{u}} e^{r(T-t)} \xrightarrow[\delta\to 0]{a.s.} X_T^{\tilde{u}} - X_t^{\tilde{u}} e^{r(T-t)}$. This implies that

$$\mathcal{T}(X_T^{{}^\delta\hat{u}} - X_t^{{}^\delta\hat{u}} e^{r(T-t)}) \xrightarrow[\delta\to 0]{a.s.} \mathcal{T}(X_T^{\tilde{u}} - X_t^{\tilde{u}} e^{r(T-t)}) \tag{5.5}$$

due to the continuity of $\mathcal{T}$. Now we know for the positive part $\mathcal{T}_+$ of the function $\mathcal{T}$ that $\mathcal{T}_+(x) \le (\mathcal{T}(0) + \mathcal{T}'(0)\cdot x)_+ \le |\mathcal{T}(0)| + |\mathcal{T}'(0)\cdot x|$ since $\mathcal{T}$ is concave. Let $\varepsilon > 0$ such that $1+\varepsilon < \alpha$. Then, we get:

$$\begin{aligned}
&\sup_{\delta>0} \mathbb{E}[|X_T^{{}^\delta\hat{u}} - X_t^{{}^\delta\hat{u}} e^{r(T-t)}|^{1+\varepsilon}] \\
&= \sup_{\delta>0} \mathbb{E}[|\sum_{i=1}^d \int_t^T {}^\delta\hat{u}_s^i(\mu_s^i - r)e^{r(T-s)}\mathrm{d}s + \sum_{i,j=1}^d \int_t^T {}^\delta\hat{u}_s^i \sigma_s^{ij} e^{r(T-s)}\mathrm{d}L_s^j|^{1+\varepsilon}] \\
&\le \sup_{\delta>0} |\int_t^T {}^\delta\hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s|^{1+\varepsilon} \\
&\quad + \begin{cases} \sup_{\delta>0} \|{}^\delta\hat{u}_s^\intercal \sigma_s \upsilon e^{r(T-s)}\|^{1+\varepsilon}_{\mathcal{L}^\alpha((t,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}\upsilon))} \mathbb{E}|\hat{L}_1|^{1+\varepsilon} & \text{if } \alpha < 2 \\ \sup_{\delta>0} \|{}^\delta\hat{u}_s^\intercal \sigma_s e^{r(T-s)}\|^{1+\varepsilon}_{\mathcal{L}^2((t,T],\mathrm{d}s;H_s)} \mathbb{E}|\hat{L}_1|^{1+\varepsilon} & \text{if } \alpha = 2 \end{cases}.
\end{aligned}$$

For getting the second part of this estimation, we use again the exact same steps as in the proof of Proposition 4.11 by replacing the risk measure $\rho(\cdot)$ with $\mathbb{E}[|\cdot|^{1+\varepsilon}]$.

Since ${}^\delta\hat{u}$ is uniformly bounded in $\mathcal{L}^\alpha$, the first norm in the last term is uniformly bounded by the argument from above. Hence, $X_T^{{}^\delta\hat{u}} - X_t^{{}^\delta\hat{u}} e^{r(T-t)}$ is integrable for all ε with $1+\varepsilon < \alpha$. This property also holds for $|\mathcal{T}'(0)\cdot(X_T^{{}^\delta\hat{u}} - X_t^{{}^\delta\hat{u}} e^{r(T-t)})|$, i.e., $\sup_{\delta>0}\mathbb{E}|\mathcal{T}'(0)\cdot(X_T^{{}^\delta\hat{u}} - X_t^{{}^\delta\hat{u}} e^{r(T-t)})|^{1+\varepsilon} < \infty$. Due to $\mathcal{T}_+(x) \le |\mathcal{T}(0)| + |\mathcal{T}'(0)\cdot x|$, it follows directly that $\sup_{\delta>0}\mathbb{E}|\mathcal{T}(X_T^{{}^\delta\hat{u}} - X_t^{{}^\delta\hat{u}} e^{r(T-t)})_+|^{1+\varepsilon} < \infty$. This implies the uniform integrability of $\{(\mathcal{T}(X_T^{{}^\delta\hat{u}} - X_t^{{}^\delta\hat{u}} e^{r(T-t)}))_+\}_{\delta>0}$, because of the lemma of de la Vallée Poussin with $\phi(x) = x^{1+\varepsilon}$. Thus, a general version of Fatou's lemma (see Shiryaev, 2007, p.4) leads to:

$$\begin{aligned}
&\limsup_{\delta\to 0} \mathbb{E}_{t,x}[\mathcal{T}(X_T^{{}^\delta\hat{u}} - xe^{r(T-t)})] \\
&\le \mathbb{E}_{t,x}[\limsup_{\delta\to 0} \mathcal{T}(X_T^{{}^\delta\hat{u}} - xe^{r(T-t)})] \overset{(5.5)}{=} \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\tilde{u}} - xe^{r(T-t)})].
\end{aligned}$$

We must still prove the upper semi-continuity for the second part of J. Therefore, we start with the term inside of F. As above, we note that $|{}^\delta\hat{u}_s^\intercal \sigma_s \upsilon e^{r(T-s)}|^\alpha$ (resp. $|{}^\delta\hat{u}_s^\intercal \sigma_s e^{r(T-s)}|^2$) is deterministic, uniformly integrable with respect to $\mathrm{d}t\times\tilde{\sigma}(\mathrm{d}\upsilon)$, and ${}^\delta\hat{u}$ converges $\mathrm{d}t$-a.s. to $\tilde{u}$. Hence, we get by the Vitali convergence theorem:

$$\begin{aligned}
\lim_{\delta\to 0} \|{}^\delta\hat{u}_s^\intercal \sigma_s \upsilon e^{r(T-s)}\|_{\mathcal{L}^\alpha((t,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}\upsilon))} &= \|\tilde{u}_s^\intercal \sigma_s \upsilon e^{r(T-s)}\|_{\mathcal{L}^\alpha((t,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}\upsilon))}, \\
\lim_{\delta\to 0} \|{}^\delta\hat{u}_s^\intercal \sigma_s e^{r(T-s)}\|_{\mathcal{L}^2((t,T],\mathrm{d}s;H_s)} &= \|\tilde{u}_s^\intercal \sigma_s e^{r(T-s)}\|_{\mathcal{L}^2((t,T],\mathrm{d}s;H_s)}.
\end{aligned}$$

These equations and (5.4) imply that $\lim_{\delta\to 0}\|^\alpha w_s({}^\delta\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)} = \|^\alpha w_s(\tilde{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}$ and $\lim_{\delta\to 0} m_s({}^\delta\hat{u}) = m_s(\tilde{u})$. Then, it follows from this combined with Proposition 4.11 and ${}^\delta\hat{u}$ and $\tilde{u}$ being deterministic that

$$\begin{aligned}
\rho_{t,x}(X_T^{{}^\delta\hat{u}} - xe^{r(T-t)}) &= -\int_t^T m_s({}^\delta\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s({}^\delta\hat{u})\|_{\mathcal{L}^\alpha(t,T],\mathrm{d}s}, \\
\rho_{t,x}(X_T^{\tilde{u}} - xe^{r(T-t)}) &= -\int_t^T m_s(\tilde{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\tilde{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}
\end{aligned}$$

with m, w and $\hat{L}$ as in Definition 4.8.

Since we already discussed the continuity property of these integrals, and they are inserted into the continuous functions $\sqrt[\alpha]{\cdot}$ and/or $F(\cdot)$, it follows that $\lim_{\delta\to 0} F(\rho_{t,x}(X_T^{{}^\delta\hat{u}} - xe^{r(T-t)})) = F(\rho_{t,x}(X_T^{\tilde{u}} - xe^{r(T-t)}))$. Combining the two parts gives us the strong upper semi-continuity and the first claim.

For the second claim, the additional assumptions imply that $J(t,x,u) = \int_t^T u_s(\mu_s - r)e^{r(T-s)}\mathrm{d}s - \lambda_t(\varrho^{\hat{L}_1})^\alpha\|^\alpha w_s(\tilde{u})\|^\alpha_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}$ with w and $\hat{L}$ as in Definition 4.8 and u being deterministic. In particular, this equation holds if u is ${}^\delta\hat{u}$ or $\tilde{u}$. We observe that for $F(x) = \max\{0,x\}^\alpha$, the problem is time-consistent. Now, we define by ${}_\beta\upsilon$ a β-optimal solution for all $\beta > 0$, i.e., $J(t,x,{}_\beta\upsilon) \ge \sup_{u\in\mathcal{U}} J(t,x,u) - \beta$. Then, we define by ${}^\delta_\beta\upsilon$ a discretization of ${}_\beta\upsilon$ with step size $\delta > 0$, such that ${}^\delta_\beta\upsilon$ converges strongly to ${}_\beta\upsilon$ in $\mathcal{L}^\alpha((0,T),\mathrm{d}\mathbb{P}\times\mathrm{d}t)$ for $\delta\to 0$ and $\lim_{\delta\to 0} J_t^\delta(x,{}^\delta_\beta\upsilon) = J(t,x,{}_\beta\upsilon)$. Such a discretization exists due to monotone class arguments. By construction, ${}^\delta_\beta\upsilon$ is piecewise constant and bounded. Because ${}^\delta\hat{u}$ is the optimal control function for each δ, we get that $J_t^\delta(x,{}^\delta_\beta\upsilon) \le J_t^\delta(x,{}^\delta\hat{u})$ for all $\beta > 0$. Taking the limes superior of $\delta\to 0$ results in $J(t,x,{}_\beta\upsilon) \le J(t,x,\tilde{u})$ for all $\beta > 0$. Since ${}_\beta\upsilon$ is β-optimal and the inequality holds for all $\beta > 0$, $\tilde{u}$ is an optimal control function and hence an equilibrium function due to the time-consistency property of the problem. □

For the following theorem, we need in higher dimensions that L is isotropic. That means that for $\alpha < 2$, the probability measure $\tilde{\sigma}$ is the uniform distribution on $\mathbb{S}^d$, and for $\alpha = 2$, that R_s is the identity.

Theorem 5.9. *Let (a) $d = 1$ and $\sigma_t \in C^1$; or (b) L be isotropic, $\sigma_t = \bar{\sigma}_t \cdot Id$ with $\bar{\sigma}_t$ being a deterministic C^1-function which is bounded away from 0, and $\mathcal{V} \subset \mathbb{R}_{\ge 0}$. Moreover, let $\mu_{min} := \min_{t\in[0,T]}\mu_t > r$, $\lambda_{min} := \min_{t\in[0,T]}\lambda_t > 0$, $\mu_t, \lambda_t \in C^1$, $\mathcal{T}(x) = x$, $F(x) = \max\{0,x\}^\beta$ with $\beta \ge \alpha > 1$, and ρ shift-invariant. If either $\beta = \alpha$ or under Assumption 4a, there exists a sequence ${}^\delta\hat{u}$ of optimal control functions with step size δ and a continuous $\hat{u} \in \mathcal{U}_{det}$ such that ${}^\delta\hat{u}_t \xrightarrow{\delta\to 0} \hat{u}_t$ uniformly in t and $\lim_{\delta\to 0} J_t(x,{}^\delta\hat{u}) = J(t,x,\hat{u})$.*

Proof. We show this statement with the theorem of Arzelà-Ascoli. Specifically, we show that ${}^\delta\hat{u}$ is equi-continuous and bounded on the time grid t_i^δ. Henceforth, we therefore always assume that $t, s, t+\delta, s+\delta, \dots$ are on the time grid. In this proof, we denote by $|\cdot|$ the classical l_α-norm for vectors. First, we note that if L is isotropic and $\sigma_t = \bar{\sigma}_t \cdot Id$, then it holds that $\int_{\mathbb{S}^d} |u^\intercal \sigma_s \upsilon e^{r(T-s)}|^\alpha \tilde{\sigma}(\mathrm{d}\upsilon) = C_{\alpha d}\bar{\sigma}_t|u|^\alpha e^{\alpha r(T-s)}$ with $C_{\alpha d} = \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{\alpha+d}{2})\Gamma(\frac{1}{2})}$ if $\alpha < 2$ by Çınlar (2011, pp.332,337,339). Additionally, we define $C_{2d} := 1 =: C_{\alpha 1}$ for $\alpha = 2$ resp. $d = 1$. Under this theorem's assumptions using (4.3), the optimization problem in discrete time (4.1) with step size δ reduces to

$$\begin{aligned}
&\arg\max_{u\in[-M,M]^d} \{u^\intercal\mu_\delta + \int_{t+\delta}^T {}^\delta\hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s \\
&\quad - \lambda_t(\varrho^{\hat{L}_1}\sqrt[\alpha]{\sigma_\delta|u|^\alpha + C_{\alpha d}\int_{t+\delta}^T |{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s})^\beta\},
\end{aligned} \tag{5.6}$$

where $\mu_\delta := \int_t^{t+\delta}(\mu_s - r)e^{r(T-s)}\mathrm{d}s$ and $\sigma_\delta := \int_t^{t+\delta}\sigma_s^\alpha e^{\alpha r(T-s)}\mathrm{d}s$ if $d = 1$ and $\sigma_\delta := C_{\alpha d}\int_t^{t+\delta}\bar{\sigma}_s^\alpha e^{\alpha r(T-s)}\mathrm{d}s$ if $d \ge 2$. Proposition 4.16 (if $d = 1$ and Assumption 3a holds) or $\mathcal{V} \subset \mathbb{R}_{\ge 0}$ (otherwise) imply that u is non-negative. Hence, we can take the arg max over $[0,M]^d$. Now, we define

$$f_{C_1,C_2}(u) := u^\intercal\mu_\delta + C_1 - \lambda_t(\varrho^{\hat{L}_1}\sqrt[\alpha]{\sigma_\delta|u|^\alpha + C_2})^\beta.$$

We note that the arg max of f is independent of C_1, and we get our main problem in discrete time with step size δ if $C_2^\delta(t) = C_{\alpha d}\int_{t+\delta}^T |{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s$. To show the equicontinuity of ${}^\delta\hat{u}$ (to apply the theorem of Arzelà-Ascoli), we note that for $t < s$ with $|s-t| < \tilde{\delta}$, it holds if $\beta > \alpha$ that $|C_2^\delta(s) - C_2^\delta(t)| = C_{\alpha d}\int_{t+\delta}^{s+\delta} |{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s \le C_{\alpha d}\tilde{\delta}d^\alpha M^\alpha \sigma_{max}^\alpha e^{\alpha r T}$ independent of δ using Assumption 4a with $\sigma_{max} := \max_{t\in[0,T]}\sigma_t$ if $d = 1$ and $\sigma_{max} := \max_{t\in[0,T]}\bar{\sigma}_t$ else. As λ_t, μ_t, and σ_t are Lipschitz continuous (since they are in C^1), it remains to show the equicontinuity of ${}^\delta\hat{u}$ as a function of C_2, $\delta^{-1}\sigma_\delta$, $\delta^{-1}\mu_\delta$, and λ_t. Without loss of generality, we assume in the sequel that $\delta^{-1}\sigma_\delta$, $\delta^{-1}\mu_\delta$, and λ_t do not depend on t so that ${}^\delta\hat{u}$ actually only depends on C_2. For instance, for $\delta^{-1}\sigma_\delta$, we can replace f by $(f - C_1)(\frac{1}{\delta^{-1}\sigma_\delta})^{\frac{\beta}{\alpha}}$ which has the same arg max as f and then use that $\delta^{-1}\sigma_\delta$ is Lipschitz (with Lipschitz constant independent of δ), bounded, and bounded away from 0 so that we can replace C_2 by $\frac{C_2}{\delta^{-1}\sigma_\delta}$ and μ_δ by $\tilde{\mu}_\delta := \mu_\delta(\delta^{-1}\sigma_\delta)^{-\beta/\alpha}$. Note that $\delta^{-1}\tilde{\mu}_\delta$ is also Lipschitz (with Lipschitz constant independent of δ), bounded, and bounded away from 0. For $\delta^{-1}\mu_\delta$, and λ_t, the following steps would work similarly using that $\sigma_t, \mu_t, \lambda_t \in C^1$ and $\delta^{-1}\sigma_\delta$, $\delta^{-1}\mu_\delta$, and λ_t are bounded and bounded away from 0 so that

the boundedness of the derivative (analogously to (5.11)) is preserved. Specifically, B^i_δ, which will be defined later, would depend on t in a Lipschitz continuous way which does not change the nature of the proof, see also the online version of this article. Thus, from now on, we consider u simply as a function of C_2 and set a δ in the index to express the step size directly. In the following, we prove that the optimal control function ${}^\delta\hat{u}$ is uniformly Lipschitz continuous in C_2 with Lipschitz constant independent of δ, which directly implies the equicontinuity in t. Note that u_δ denotes the arg max of f for given C_1 and C_2. Thus, it holds that ${}^\delta\hat{u}_t = u_\delta$ with the correct C_1 and C_2. Now, let $\beta > \alpha$ and $k \in \{1, \dots, d\}$ fixed. If $u^i_\delta \in (0, M)$ for all $i \in \{1, \dots, k\}$ and $u^i_\delta = M$ for $i > k$, then it holds for $i \in \{1, \dots, k\}$ that

$$\frac{\mathrm{d}}{\mathrm{d}u^i_\delta} f_{C_1,C_2}(u_\delta(C_2)) = \mu^i_\delta - \lambda_t(\rho^{\hat{L}_1})^\beta \beta(\sigma_\delta|u_\delta(C_2)|^\alpha + C_2)^{\frac{\beta-\alpha}{\alpha}} \sigma_\delta(u^i_\delta(C_2))^{\alpha-1} = 0. \tag{5.7}$$

Define $B^i_\delta := (\frac{\mu^i_\delta}{\sigma_\delta\lambda_t(\rho^{\hat{L}_1})^\beta\beta})^{\frac{\alpha}{\beta-\alpha}}$, and $q := \frac{\alpha(\alpha-1)}{\beta-\alpha} > 0$. Then rearranging (5.7) implies

$$B^i_\delta = ((u^1_\delta(C_2))^\alpha + \cdots + (u^k_\delta(C_2))^\alpha + (d-k)M^\alpha)\sigma_\delta(u^i_\delta(C_2))^q + C_2(u^i_\delta(C_2))^q. \tag{5.8}$$

Since the implicit function theorem combined with Lemma B.3 in the appendix implies the existence of the derivative of u as a function of C_2, we get by taking this derivative (where we suppress again the C_2): $0 = (\alpha(u^1_\delta)^{\alpha-1}(u^1_\delta)' + \cdots + \alpha(u^k_\delta)^{\alpha-1}(u^k_\delta)')\sigma_\delta(u^i_\delta)^q + ((u^1_\delta)^\alpha + \cdots + (u^k_\delta)^\alpha + (d-k)M^\alpha)\sigma_\delta q(u^i_\delta)^{q-1}(u^i_\delta)' + (u^i_\delta)^q + C_2 q(u^i_\delta)^{q-1}(u^i_\delta)'$. Then, multiplying $u^i_\delta((u^1_\delta)^\alpha + \cdots + (u^k_\delta)^\alpha + (d-k)M^\alpha)$, inserting (5.8) twice and canceling terms leads to:

$$\begin{aligned}
0 &= \alpha(\sum_{j=1}^k (u^j_\delta)^{\alpha-1}(u^j_\delta)')u^i_\delta(B^i_\delta - C_2(u^i_\delta)^q) + D_\delta q(u^i_\delta)' B^i_\delta + D_\delta(u^i_\delta)^{q+1} \\
&= \alpha(u^1_\delta)^{\alpha-1}(u^1_\delta)'u^i_\delta(B^i_\delta - C_2(u^i_\delta)^q) + \alpha(\sum_{j=2}^k (u^j_\delta)^{\alpha-1}(u^j_\delta)')u^i_\delta(B^i_\delta - C_2(u^i_\delta)^q) \\
&\quad + D_\delta q(u^i_\delta)' B^i_\delta + D_\delta(u^i_\delta)^{q+1},
\end{aligned} \tag{5.9}$$

where $D_\delta := \sum_{j=1}^k (u^j_\delta)^\alpha + (d-k)M^\alpha$. Now, we show that $(u^1_\delta)'$ is bounded independent of δ and k. First, we assume that $k \geq 2$ (the proof for $k = 1$ is actually an easier version of the following). Then, we multiply the equations for $i \in \{2, \dots, k\}$ with $(u^i_\delta)^{\alpha-1}u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)(B^i_\delta)^{-1}$ and sum over all those i to receive:

$$\begin{aligned}
0 =& \alpha(u^1_\delta)^\alpha(u^1_\delta)'(B^1_\delta - C_2(u^1_\delta)^q)(\sum_{i=2}^k (u^i_\delta)^\alpha(B^i_\delta - C_2(u^i_\delta)^q)(B^i_\delta)^{-1}) \\
&+ (\alpha\sum_{j=2}^k (u^j_\delta)^{\alpha-1}(u^j_\delta)')u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)(\sum_{i=2}^k (u^i_\delta)^\alpha(B^i_\delta - C_2(u^i_\delta)^q)(B^i_\delta)^{-1}) \\
&+ D_\delta q u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)(\sum_{i=2}^k (u^i_\delta)^{\alpha-1}(u^i_\delta)') \\
&+ D_\delta u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)(\sum_{i=2}^k (u^i_\delta)^{q+\alpha}(B^i_\delta)^{-1}) \\
=& (\sum_{i=2}^k (u^i_\delta)^\alpha(B^i_\delta - C_2(u^i_\delta)^q)(B^i_\delta)^{-1}) \\
&\quad \cdot [\alpha(u^1_\delta)^\alpha(u^1_\delta)'(B^1_\delta - C_2(u^1_\delta)^q) + \underline{\alpha(\sum_{j=2}^k (u^j_\delta)^{\alpha-1}(u^j_\delta)')u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)}] \\
&+ D_\delta q\alpha^{-1}[\underline{\alpha(\sum_{j=2}^k (u^j_\delta)^{\alpha-1}(u^j_\delta)')u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)}] \\
&+ D_\delta u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)(\sum_{i=2}^k (u^i_\delta)^{q+\alpha}(B^i_\delta)^{-1}).
\end{aligned} \tag{5.10}$$

Then, rearranging (5.9) for $i = 1$ gives us $\alpha(\sum_{j=2}^k (u^j_\delta)^{\alpha-1}(u^j_\delta)')u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q) = -\alpha(u^1_\delta)^\alpha(u^1_\delta)'(B^1_\delta - C_2(u^1_\delta)^q) - D_\delta q(u^1_\delta)' B^1_\delta - D_\delta(u^1_\delta)^{q+1}$. Inserting this twice into (5.10) (underlined terms) leads to:

$$\begin{aligned}
0 =& (\sum_{i=2}^k (u^i_\delta)^\alpha(B^i_\delta - C_2(u^i_\delta)^q)(B^i_\delta)^{-1})[-D_\delta q(u^1_\delta)' B^1_\delta - D_\delta(u^1_\delta)^{q+1}] \\
&+ D_\delta q\alpha^{-1}[-\alpha(u^1_\delta)^\alpha(u^1_\delta)'(B^1_\delta - C_2(u^1_\delta)^q) - D_\delta q(u^1_\delta)' B^1_\delta - D_\delta(u^1_\delta)^{q+1}] \\
&+ D_\delta u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)(\sum_{i=2}^k (u^i_\delta)^{q+\alpha}(B^i_\delta)^{-1})
\end{aligned}$$

Now, canceling D_δ in each term and solving for $(u^1_\delta)'$ gives us:

$$\begin{aligned}
(u^1_\delta)' =& \frac{u^1_\delta(B^1_\delta - C_2(u^1_\delta)^q)(\sum_{i=2}^k (u^i_\delta)^{q+\alpha}(B^i_\delta)^{-1}) - (u^1_\delta)^{q+1}(\sum_{i=2}^k (u^i_\delta)^\alpha(B^i_\delta - C_2(u^i_\delta)^q)(B^i_\delta)^{-1})}{qB^1_\delta(\sum_{i=2}^k (u^i_\delta)^\alpha(B^i_\delta - C_2(u^i_\delta)^q)(B^i_\delta)^{-1}) + q(u^1_\delta)^\alpha(B^1_\delta - C_2(u^1_\delta)^q) + D_\delta q^2\alpha^{-1}B^1_\delta} \\
&- \frac{D_\delta\alpha^{-1}q(u^1_\delta)^{q+1}}{qB^1_\delta(\sum_{i=2}^k (u^i_\delta)^\alpha(B^i_\delta - C_2(u^i_\delta)^q)(B^i_\delta)^{-1}) + q(u^1_\delta)^\alpha(B^1_\delta - C_2(u^1_\delta)^q) + D_\delta q^2\alpha^{-1}B^1_\delta}.
\end{aligned} \tag{5.11}$$

We note that $B^i_\delta - C_2(u^i_\delta)^q \geq 0$ for all i by (5.8) since the u^i are non-negative. Moreover, there exist $B_d, B_u \in (0, \infty)$ independent of δ and i such that $B_d \leq B^i_\delta \leq B_u$ using the original and additional assumptions of the model parameters. Combining these properties also implies that $B^i_\delta - C_2(u^i_\delta)^q \leq B_u$. Now, since $q, \alpha > 0$, $(u^1_\delta)'$ is uniformly bounded if the nominator of the first line in (5.11) divided by D_δ is uniformly bounded since the first two terms in the denominator are non-negative and the last term divided by D_δ is bounded away from 0 due to $q^2\alpha^{-1}B^1_\delta \geq q^2\alpha^{-1}B_d > 0$ independent of δ. If $k < d$, $D_\delta \geq M^\alpha$ and the claim follows directly. If $k = d$, we observe that $D_\delta = \sum_{i=1}^d (u^i_\delta)^\alpha$ and $B^i_\delta - C_2(u^i_\delta)^q = (\sum_{j=1}^d (u^j_\delta)^\alpha)\sigma_\delta(u^i_\delta)^q = D_\delta\sigma_\delta(u^i_\delta)^q$ by (5.8). Hence, the D_δ cancels out and the nominator is uniformly bounded due to $(B^i_\delta)^{-1}$, $B^i_\delta - C_2(u^i_\delta)^q$, u^i_δ, and D_δ being all bounded independent of δ. By symmetry, we can show analogously that $(u^i_\delta)'$ is uniformly bounded for every $i \in \{2, \dots, k\}$.

From (5.7), it follows that $\frac{\mathrm{d}}{\mathrm{d}u^i_\delta} f(0) = \mu^i_\delta > 0$. Thus, the arg max cannot be attained at $u^i_\delta = 0$. Since the u^i are uniform Lipschitz continuous in C_2 and constant if $u^i = M$, we get the uniform Lipschitz continuity of u_δ. Hence, u_δ is equicontinuous for $\beta > \alpha$. If $\alpha = \beta$, we conclude from (5.7) that ${}^\delta\hat{u}^i_t = \sqrt[\alpha-1]{\frac{\mu^i_\delta}{\sigma_\delta\alpha\lambda_t(\rho^{\hat{L}_1})^\beta}}$ (possibly being additional truncated at M under Assumption 4a). Now since λ, $\delta^{-1}\sigma_\delta$, and $\delta^{-1}\mu_\delta$ are in δ uniformly bounded away from 0 and uniformly bounded, u is equicontinuous.

It remains to show the uniform boundedness in t. If $\beta \neq \alpha$, this property is given by Assumption 4a. Now, we define $\sigma_{min} := \min_{t\in[0,T]}\sigma_t$ if $d = 1$ resp. $\sigma_{min} := C_{\alpha d}\min_{t\in[0,T]}\bar{\sigma}_t$ if $d \geq 1$. If $\alpha = \beta$, we conclude as before that ${}^\delta\hat{u}_t = \sqrt[\alpha-1]{\frac{\mu^i_\delta}{\sigma_\delta\alpha\lambda_t(\rho^{\hat{L}_1})^\beta}} \leq \sqrt[\alpha-1]{\frac{\max_{t\in[0,T],i\in\{1,\dots,d\}}(u^i_t - r)e^{rT}}{\sigma^\alpha_{min}\alpha\lambda_{min}(\rho^{\hat{L}_1})^\beta}} < \infty$ independent of δ. Hence, u is uniformly bounded.

The theorem of Arzelà-Ascoli implies the existence of a uniform convergent subsequence to a continuous limit. The convergence of the value functional J for this subsequence follows directly since J is continuous in the control function under the theorem's assumptions. □

5.6. Verification theorem

To prove the verification theorem, we need the following assumption:

Assumption 6. If $\mathcal{V} \subset \mathbb{R}^d_{\geq 0}$ (resp. $\mathcal{V} \subset \mathbb{R}^d_{\leq 0}$), it holds that $\mathbb{E}[|L_t\mathcal{T}'(a + bL_t)|] < \infty$ for all $a \in \mathbb{R}$, $b > 0$ (resp. $b < 0$), and for all $t \in (0, T]$. Otherwise, it holds that $\mathbb{E}[|L_t\mathcal{T}'(a + bL_t)|] < \infty$ for all $a, b \in \mathbb{R}$ and for all $t \in (0, T]$.

Assumption 7. $\lambda_t \in C^1$.

Proposition 5.10. *Let u be deterministic and continuous. Then, it holds that $(\lambda_\cdot F(\rho_\cdot(X^u_T - X^u_\cdot e^{r(T-\cdot)}))) \in C^{1,2}$ and $\mathbb{E}_\cdot[\mathcal{T}(X^u_T - X^u_\cdot e^{r(T-\cdot)})] \in C^{1,2}$.*

We give this proof in Appendix B.

Assumption 8. $V \in C^{1,2}$.

Assumption 9. The optimal equilibrium function $\hat{u}$ is continuous and deterministic.

If the assumptions of Theorem 5.9 are fulfilled, this assumption is redundant.

Following the steps of Björk and Murgoci (2010), we prove that our extended HJB equation, which we derived by a limiting approach, is correct.

Theorem 5.11. *Assume $\tilde{V}$ is a solution of the extended HJB, and the control law $\hat{u}$ attains the supremum in Eq. (5.2). Then $\hat{u}$ is an equilibrium law, and $\tilde{V}$ is the corresponding value function.*

Proof. *Step 1: Let us show that $\tilde{V}(t,x) = J(t,x,\hat{u})$:*

It holds by assumption that

$$(\mathrm{A}^{\hat{u}}\tilde{V})(t,x) - (\mathrm{A}^{\hat{u}}\mathbb{E}_{\cdot}[\mathcal{T}(X_T^{\hat{u}} - X_{\cdot}^{\hat{u}}e^{r(T-\cdot)})])(t,x) + (\mathrm{A}^{\hat{u}}(\lambda_{\cdot}F(\rho_{\cdot}(X_T^{\hat{u}} - X_{\cdot}^{\hat{u}}e^{r(T-\cdot)}))))(t,x) = 0. \tag{5.12}$$

Next, we replace (t,x) by $(s, X_s^{\hat{u}})$, integrate over "ds" in (t,T), and take the conditional expectation. Now, from Dynkin's theorem, we conclude that $\mathbb{E}_{t,x}[\tilde{V}(T, X_T^{\hat{u}})] = \tilde{V}(t,x) + \mathbb{E}_{t,x}[\int_t^T (\mathrm{A}^{\hat{u}}\tilde{V})(s, X_s^{\hat{u}})\mathrm{d}s]$. Due to Assumption 9 (which can be replaced by the assumptions in Theorem 5.9) and Proposition 5.10, we use Dynkin again for the other two terms from Eq. (5.12). Solving this equation for $\tilde{V}$ leads to

$$\begin{aligned}\tilde{V}(t,x) =& \mathbb{E}_{t,x}[\tilde{V}(T, X_T^{\hat{u}})] - \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\hat{u}} - X_{\cdot}^{\hat{u}}e^{r(T-\cdot)})(T, X_T^{\hat{u}})] \\ &+ \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\hat{u}} - X_{\cdot}^{\hat{u}}e^{r(T-\cdot)})(t,x)] \\ &+ \mathbb{E}_{t,x}[\lambda_T F(\rho(X_T^{\hat{u}} - X_{\cdot}^{\hat{u}}e^{r(T-\cdot)}))(T, X_T^{\hat{u}})] \\ &- \lambda_t F(\rho(X_T^{\hat{u}} - X_{\cdot}^{\hat{u}}e^{r(T-\cdot)}))(t,x).\end{aligned}$$

Lastly, we insert the final value to infer:

$$\begin{aligned}\tilde{V}(t,x) =& \mathcal{T}(0) - \lambda_T F(0) - \mathcal{T}(0) + \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\hat{u}} - xe^{r(T-t)})] \\ &+ \lambda_T F(0) - \lambda_t F(\rho(X_T^{\hat{u}} - xe^{r(T-t)})) \\ =& \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\hat{u}} - xe^{r(T-t)})] - \lambda_t F(\rho(X_T^{\hat{u}} - xe^{r(T-t)})) = J(t,x,\hat{u}).\end{aligned}$$

Step 2: Let us show that $\hat{u}$ is an equilibrium function:

First, we construct a control law u_h as in Definition 5.2. Now, (B.1) applied to the interval t to $t+h$ gives us (suppressing the h in u_h):

$$\begin{aligned}J(t,x,u) =& \mathbb{E}_{t,x}[\tilde{V}(t+h, X_{t+h}^u)] - \mathbb{E}_{t,x}[\mathcal{T}(X_T^u - X_{t+h}^u e^{r(T-t-h)}) - \mathcal{T}(X_T^u - xe^{r(T-t)})] \\ &+ \mathbb{E}_{t,x}[\lambda_{t+h}F(\rho_{t+h,X_{t+h}^u}(X_T^u - X_{t+h}^u e^{r(T-t-h)})) - \lambda_t F(\rho_{t,x}(X_T^u - xe^{r(T-t)}))],\end{aligned} \tag{5.13}$$

where we used $J(t+h, X_{t+h}^u, u) = \tilde{V}(t+h, X_{t+h}^u)$, which follows from the definition of Nash equilibria (see Definition 5.2). Since $\tilde{V}$ is a solution of the extended HJB as in Definition 5.5, it holds that for all $u \in \mathcal{U}$

$$(\mathrm{A}^u\tilde{V})(t,x) - (\mathrm{A}^u\mathcal{T}(X_T^u - X_{\cdot}^u e^{r(T-\cdot)}))(t,x) + (\mathrm{A}^u(\lambda_{\cdot}F(\rho_{\cdot}(X_T^u - X_{\cdot}^u e^{r(T-\cdot)}))))(t,x) \le 0.$$

A discretization of this equation leads to

$$\begin{aligned}&\mathbb{E}_{t,x}[\tilde{V}(t+h, X_{t+h}^u)] - \tilde{V}(t,x) - \mathbb{E}_{t,x}[\mathcal{T}(X_T^u - X_{t+h}^u e^{r(T-t-h)}) - \mathcal{T}(X_T^u - xe^{r(T-t)})] \\ &+ \mathbb{E}_{t,x}[\lambda_{t+h}F(\rho_{t+h,X_{t+h}^u}(X_T^u - xe^{r(T-t-h)})) - \lambda_t F(\rho_{t,x}(X_T^u - xe^{r(T-t)}))] \le \mathrm{o}(h).\end{aligned} \tag{5.14}$$

Since $\tilde{V}(t,x) = J(t,x,\hat{u})$, it follows that $J(t,x,\hat{u}) \ge J(t,x,u) + \mathrm{o}(h)$ from (5.13) and (5.14), which implies that $\liminf_{h\to 0} \frac{J(t,x,\hat{u}) - J(t,x,u)}{h} \ge 0$. Definition 5.2 yields the proof. □

Now, we come back to the setting of Theorem 5.9 to show that this limit of the deterministic equilibria is also an optimal equilibrium in continuous time.

Theorem 5.12. *Let the assumptions of Theorem 5.9 hold. Then, $\hat{u}$ from Theorem 5.9 is an optimal equilibrium function.*

Proof. First, note that for convenience of the reader, we denote $\hat{u}$ from Theorem 5.9 by $\tilde{u}$. For this proof, we rewrite the HJB equations in discrete and continuous time using the special setting properties given by the assumptions. We remark that for continuous and deterministic optimal control functions, the infinitesimal generator equals the derivative with respect to t using Proposition 5.10. Now, similar to the calculations in the proof of Proposition 5.10, we get for $\bar{u}$ with $\bar{u}_s = \hat{u}_s$ for $s > t$, where $\hat{u}$ denotes the optimal solution, and $\bar{u}_t = u$ for an arbitrary, but fixed $u \in \mathcal{V}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J(t,x,\bar{u}) =& -u^\intercal(\mu_t - r)e^{r(T-t)} \\ &+ \lambda_t(\varrho^{\hat{L}_1}\sqrt[\alpha]{C_{\alpha d}})^\beta \frac{\beta}{\alpha}\Big(\int_t^T |\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s\Big)^{\frac{\beta-\alpha}{\alpha}} |u^\intercal\sigma_t e^{r(T-t)}|^\alpha \\ &- \lambda_t'(\varrho^{\hat{L}_1}\sqrt[\alpha]{C_{\alpha d}})^\beta\Big(\int_t^T |\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s\Big)^{\frac{\beta}{\alpha}},\end{aligned}$$

where $C_{\alpha d}$ is defined as in the proof of Theorem 5.9. Here, we used that L is isotropic or $d = 1$. Hence, the continuous HJB Eq. (5.2) becomes after noting that $V(t,x) = J(t,x,\hat{u})$ for $\hat{u}$ being optimal (since the λ_t'-part cancels):

$$\begin{aligned}&\sup_{u\in\mathcal{V}}\{(u^\intercal - \hat{u}_t^\intercal)(\mu_t - r)e^{r(T-t)} - \lambda_t(\varrho^{\hat{L}_1}\sqrt[\alpha]{C_{\alpha d}})^\beta \frac{\beta}{\alpha} \\ &\cdot\Big(\int_t^T |\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s\Big)^{\frac{\beta-\alpha}{\alpha}}(|u^\intercal\sigma_t e^{r(T-t)}|^\alpha - |\hat{u}_t^\intercal\sigma_t e^{r(T-t)}|^\alpha)\} = 0.\end{aligned} \tag{5.15}$$

Thus, we note that $\tilde{u}$ solves the HJB Eq. (5.2) if and only if the previous equation holds with $\tilde{u}$ instead of $\hat{u}$. Similarly, we can reduce the discrete HJB Eq. (4.8) to:

$$\begin{aligned}&\sup_{u\in\mathcal{V}}\Big\{(u^\intercal - {}^\delta\hat{u}_t^\intercal)\int_t^{t+\delta}(\mu_s - r)e^{r(T-s)}\mathrm{d}s \\ &- \lambda_t(\varrho^{\hat{L}_1}\sqrt[\alpha]{C_{\alpha d}})^\beta\Big((|u|^\alpha\int_t^{t+\delta}|\sigma_s^{11}e^{r(T-s)}|^\alpha\mathrm{d}s + \int_{t+\delta}^T |{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha\mathrm{d}s\Big)^{\frac{\beta}{\alpha}} \\ &- (|{}^\delta\hat{u}_t|^\alpha\int_t^{t+\delta}|\sigma_s^{11}e^{r(T-s)}|^\alpha\mathrm{d}s + \int_{t+\delta}^T |{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha\mathrm{d}s)^{\frac{\beta}{\alpha}}\Big)\Big\} = 0.\end{aligned}$$

Here, we also used that $|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha = |\hat{u}_s|^\alpha|\sigma_s^{11}e^{r(T-s)}|^\alpha$ since either $d = 1$ or σ is the identity times possibly a constant. Now, we divide the equation by δ and show that for every arbitrary but fixed $u \in \mathcal{V}$ by taking the limit of $\delta \to 0$, we get the continuous HJB with $\tilde{u}$ instead of $\hat{u}$ on the left side. Note that for every fixed $u \in \mathcal{V}$ the discrete HJB equation holds with a "≤ 0" instead of a "$= 0$".

First, it holds that $\lim_{\delta\to 0}\delta^{-1}(u^\intercal - {}^\delta\hat{u}_t^\intercal)\int_t^{t+\delta}(\mu_s - r)e^{r(T-s)}\mathrm{d}s = (u^\intercal - \tilde{u}_t^\intercal)(\mu_t - r)e^{r(T-t)}$ since ${}^\delta\hat{u}$ converges to $\tilde{u}$ by Theorem 5.9 and μ is continuous. For the second term, the argumentation is trickier, and for the sake of simplicity we introduce $\sigma_\delta := \int_t^{t+\delta}|\sigma_s^{11}e^{r(T-s)}|^\alpha\mathrm{d}s$ and $C_\delta(x) := \int_{t+\delta}^T |x_s^\intercal\sigma_s e^{r(T-s)}|^\alpha\mathrm{d}s$. Now, we get

$$\begin{aligned}&\frac{(|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))^{\frac{\beta}{\alpha}} - (|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))^{\frac{\beta}{\alpha}}}{\delta} \\ &= \frac{(|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))^{\frac{\beta}{\alpha}} - (|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))^{\frac{\beta}{\alpha}}}{(|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u})) - (|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))} \\ &\cdot\frac{(|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u})) - (|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))}{\delta}.\end{aligned}$$

The second term converges to $(|u|^\alpha - |\tilde{u}|^\alpha)|\sigma_t^{11}e^{r(T-t)}|^\alpha = |u^\intercal\sigma_t e^{r(T-t)}|^\alpha - |\hat{u}_t^\intercal\sigma_t e^{r(T-t)}|^\alpha$ for $\delta \to 0$ due to the convergence of ${}^\delta\hat{u}$ to $\tilde{u}$ and σ being continuous if $d = 1$ or a constant times the identity if $d \ge 2$. On the first term, we apply the mean value theorem of differentiation and thus get a ξ_δ between $|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u})$ and $|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u})$ such that the first term is equal to $\frac{\beta}{\alpha}\xi_\delta^{\frac{\beta-\alpha}{\alpha}}$. Since both boundary values for ξ_δ converge to $C_0(\tilde{u})$ due to the convergence of ${}^\delta\hat{u}$ to $\tilde{u}$, also ξ_δ converges to $C_0(\tilde{u})$. Overall, we get the convergence of the inner part of the left side from the discrete to

the continuous case for a fixed u. Hence, for a fixed u, we have (5.15) with a "$\leq$" for $\tilde{u}$. Now, we take the supremum over all $u \in \mathcal{V}$ which leads to the continuous HJB Eq. (5.15) with a "≤ 0" instead of a "$= 0$". However, choosing $u = \tilde{u}_t$ gives immediately a "0" in the equation by the previously shown convergence results for the discrete optimal strategies. Hence, $\tilde{u}$ solves the continuous HJB Eq. (5.2). Moreover, the constraints hold by Remark 5.6. Thus, $\tilde{u}$ is an optimal equilibrium function by Theorem 5.11, and the theorem follows. □

6. Formulas for the optimal control function in special cases

In the following, we consider an exponential target function $\mathcal{T}(x) = c^{-1}(1 - \exp(-\gamma x))$ with $c, \gamma > 0$ in the first two subsections and the identity for $\mathcal{T}$ in the third one. Since the exponential target function is strictly concave, the optimal control function in discrete time is unique (see Theorem 4.4). Moreover, we consider a multidimensional Black–Scholes model in the first, a multidimensional pure upward jump Lévy model in the second, and a one-dimensional general Lévy model in the third subsection. We provide proofs in Appendix B. Note that under Assumption 4a, the optimal strategies need to be truncated at $\pm M$. However, for the ease of exposition, we will assume in the sequel that under Assumption 4a M is large enough.

6.1. Multidimensional Black–Scholes model with an exponential target function

This section considers the Black–Scholes model, i.e., $\alpha = 2$. Since Brownian Motions are symmetric, we are in the case of Assumption 3a.

6.1.1. An explicit recursion

We give a formula for calculating the optimal control function in discrete and continuous time. Recall that we restricted in discrete time the optimization to piecewise constant control functions.

Proposition 6.1. *The optimal solution in discrete time is given by going backward in time from $n = T-1$ to $n = 0$ solving the following equation for $\hat{u}^k_{n+1}$ for all $k \in \{1, \dots, d\}$:*

$$\begin{aligned}0 =& -c^{-1}\exp(-\gamma\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s + \frac{\gamma^2}{2}\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|^2_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)})\\ &\cdot(-\gamma\int_n^{n+1}(\mu_s^k - r)e^{r(T-s)}\mathrm{d}s + \gamma^2\int_n^{n+1}\langle\hat{u}_s^\intercal\sigma_s e^{r(T-s)}, \sigma_s^{k\cdot}e^{r(T-s)}\rangle_{R_s}\mathrm{d}s)\\ &-\lambda_n F'(-\int_n^T m_s(\hat{u})\mathrm{d}s + \varrho^{W_1^1}\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)})\\ &\cdot(-\mathbb{1}_{\{\rho\text{ cash-invariant}\}}\int_n^{n+1}(\mu_s^k - r)e^{r(T-s)}\mathrm{d}s\\ &+\varrho^{W_1^1}\frac{\int_n^{n+1}\langle\hat{u}_s^\intercal\sigma_s e^{r(T-s)}, \sigma_s^{k\cdot}e^{r(T-s)}\rangle_{R_s}\mathrm{d}s}{\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)}}),\end{aligned} \tag{6.1}$$

with m as in Definition 4.8.

Remark 6.2. Since J is not differentiable in 0, it is not ensured that (6.1) admits a solution. Whenever there is no solution, the optimal control function is then given by 0 for this asset at this time point. Note that this remark also applies to all following results, where the solution is not given analytically.

Remark 6.3. The continuous-time analog of (6.1) is given by going backward in time solving the following equation for $\hat{u}_t^k$ for all $k \in \{1, \dots, d\}$:

$$\begin{aligned}0 =& -c^{-1}\exp(-\gamma\int_t^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s + \frac{\gamma^2}{2}\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|^2_{\mathcal{L}^2((t,T],\mathrm{d}s;H_s)})\\ &\cdot(-\gamma(\mu_t^k - r)e^{r(T-t)} + \gamma^2\langle\hat{u}_t^\intercal\sigma_t e^{r(T-t)}, \sigma_t^{k\cdot}e^{r(T-t)}\rangle_{R_t})\\ &-\lambda_t F'(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{W_1^1}\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|_{\mathcal{L}^2((t,T],\mathrm{d}s;H_s)})\\ &\cdot(-\mathbb{1}_{\{\rho\text{ cash-invariant}\}}(\mu_t^k - r)e^{r(T-t)} + \varrho^{W_1^1}\frac{\langle\hat{u}_t^\intercal\sigma_t e^{r(T-t)}, \sigma_t^{k\cdot}e^{r(T-t)}\rangle_{R_t}}{\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|_{\mathcal{L}^2((t,T],\mathrm{d}s;H_s)}}),\end{aligned}$$

with m as in Definition 4.8. This can be seen by replacing n and $n+1$ by t and $t+\delta$, dividing by δ, and taking the limit $\delta \to 0$.

6.1.2. No risk penalization

The following paragraph considers the special case where we do not penalize risk, i.e., $F \equiv 0$. In this case, we can calculate the optimal control function analytically:

Proposition 6.4. *In discrete time, the optimal control function can be calculated by solving for each point $n \in \{0, \dots, T-1\}$ the following d-dimensional system, where the k'th element is given by:*

$$\hat{u}^k_{n+1} = \frac{\int_n^{n+1}(\mu_s^k - r)e^{r(T-s)}\mathrm{d}s - \gamma\sum_{i=1,i\neq k}^d \hat{u}^i_{n+1}\int_n^{n+1}\sigma_s^{i\cdot}R_s\sigma_s^{\cdot k}e^{2r(T-s)}\mathrm{d}s}{\gamma\int_n^{n+1}\sigma_s^{k\cdot}R_s\sigma_s^{\cdot k}e^{2r(T-s)}\mathrm{d}s}. \tag{6.2}$$

Remark 6.5. The continuous-time analog of (6.2) for $\hat{u}$ can be calculated by solving for each point $t \in (0, T]$ the following d-dimensional system, where the k'th element is given by $\hat{u}_t^k = (\mu_t^k - r - \gamma\sum_{i=1,i\neq k}^d \hat{u}_t^i\sigma_t^{i\cdot}R_t\sigma_t^{\cdot k}e^{2r(T-t)})(\gamma\sigma_t^{k\cdot}R_t\sigma_t^{\cdot k}e^{2r(T-t)})^{-1}$. Again, this can be seen by replacing n and $n+1$ by t and $t+\delta$, dividing by δ and taking the limit $\delta \to 0$.

6.2. An explicit recursion for a pure upward Lévy model with an exponential target function

In this subsection, we take for L a pure upward jump Lévy process, i.e., $\tilde{\sigma}(\upsilon) > 0$ only if $\upsilon^i \geq 0$ for all $i \in \{1, \dots, d\}$. We restrict to such Lévy processes for the exponential target function since the Laplace function $\mathbb{E}[e^{-tL}]$ only exists for such processes if $t \geq 0$.

Proposition 6.6. *The optimal solution in discrete time is given by going backward in time from $n = T-1$ to $n = 0$ solving the following equation for $\hat{u}^k_{n+1}$ for all $k \in \{1, \dots, d\}$:*

$$\begin{aligned}0 =& -c^{-1}\exp\big(-\gamma\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s\\ &+\frac{c_\alpha\gamma^\alpha}{-\cos(\frac{\pi\alpha}{2})}\|\hat{u}_s^\intercal\sigma_s\upsilon e^{r(T-s)}\|^\alpha_{\mathcal{L}^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}\upsilon))}\big)\\ &\cdot(-\gamma\int_n^{n+1}(\mu_s^k - r)e^{r(T-s)}\mathrm{d}s\\ &+\frac{\alpha c_\alpha\gamma^\alpha}{-\cos(\frac{\pi\alpha}{2})}\int_n^{n+1}\int_{\mathbb{S}^d}|\hat{u}_s^\intercal\sigma_s\upsilon e^{r(T-s)}|^{\alpha-1}\sigma_s^{k\cdot}\upsilon e^{r(T-s)}\tilde{\sigma}(\mathrm{d}\upsilon)\mathrm{d}s)\\ &-\lambda_n F'(-\int_n^T m_s(\hat{u})\mathrm{d}s + \varrho^{\tilde{L}_1}\|\hat{u}_s^\intercal\sigma_s\upsilon e^{r(T-s)}\|_{\mathcal{L}^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}\upsilon))})\\ &\cdot\big(-\mathbb{1}_{\{\rho\text{ cash-invariant}\}}\int_n^{n+1}(\mu_s^k - r)e^{r(T-s)}\mathrm{d}s\\ &+\varrho^{\tilde{L}_1}\frac{\int_n^{n+1}\int_{\mathbb{S}^d}|\hat{u}_s^\intercal\sigma_s\upsilon e^{r(T-s)}|^{\alpha-1}\sigma_s^{k\cdot}\upsilon e^{r(T-s)}\tilde{\sigma}(\mathrm{d}\upsilon)\mathrm{d}s}{\|\hat{u}_s^\intercal\sigma_s\upsilon e^{r(T-s)}\|^{1-\frac{1}{\alpha}}_{\mathcal{L}^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}\upsilon))}}\big),\end{aligned} \tag{6.3}$$

with m as in Definition 4.8 and c_α as in (A.1).

Remark 6.7. The continuous-time analog of (6.3) is given by going backward in time solving the following equation for $\hat{u}_t^k$ for all $k \in \{1, \dots, d\}$:

$$\begin{aligned}0 =& -c^{-1}\exp\big(-\gamma\int_t^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s\\ &+\frac{c_\alpha\gamma^\alpha}{-\cos(\frac{\pi\alpha}{2})}\|\hat{u}_s^\intercal\sigma_s\upsilon e^{r(T-s)}\|^\alpha_{\mathcal{L}^\alpha((t,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}\upsilon))}\big)\end{aligned}$$

$$\cdot(-\gamma(\mu_t^k - r)e^{r(T-t)} + \frac{\alpha c_\alpha \gamma^\alpha}{-\cos(\frac{\pi\alpha}{2})}\int_{\mathbb{S}^d} |\hat{u}_t^\intercal \sigma_t v e^{r(T-t)}|^{\alpha-1}\sigma_t^{k\cdot} v e^{r(T-t)}\tilde{\sigma}(\mathrm{d}v))$$

$$- \lambda_t F'(-\int_t^T m_s(\hat{u})\mathrm{d}s + \rho^{\hat{L}_1}\|\hat{u}_s^\intercal\sigma_s v e^{r(T-s)}\|_{L^\alpha((t,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}v))})$$

$$\cdot\big(-\mathbb{1}_{\{\rho \text{ cash-invariant}\}}(\mu_t^k - r)e^{r(T-t)}$$

$$+ \rho^{\hat{L}_1}\frac{\int_{\mathbb{S}^d}|\hat{u}_t^\intercal\sigma_t v e^{r(T-t)}|^{\alpha-1}\sigma_t^{k\cdot} v e^{r(T-t)}\tilde{\sigma}(\mathrm{d}v)}{\|\hat{u}_s^\intercal\sigma_s v e^{r(T-s)}\|^{1-\frac{1}{\alpha}}_{L^\alpha((t,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}v))}}\big),$$

with m as in Definition 4.8 and c_α as in (A.1). Again, this can be seen by replacing n and $n+1$ by t and $t+\delta$, dividing by δ and taking the limit $\delta \to 0$.

6.3. Mean-fractional central moment in a one-dimensional Lévy model

In this subsection, we use the identity as the target function, i.e., $\mathcal{T}(x) = x$, and fractional central moments as the risk measure, i.e., $\rho(L) = \sqrt[\eta]{\mathbb{E}|L - \mathbb{E}[L]|^\eta} = \sqrt[\eta]{\mathbb{E}|L|^\eta}$ (since $\mathbb{E}[L] = 0$) and $F(x) = \max\{0,x\}^\eta$ with $1 < \eta < \alpha$ if $1 < \alpha < 2$ and $1 < \eta \leq \alpha$ if $\alpha = 2$, which are shift-invariant. This includes the variance if $\eta = \alpha = 2$. Moreover, we compare the mean–variance optimal solution from our setting (i.e., considering gains and losses) to the mean–variance optimal solution which optimizes terminal wealth (see, e.g., Basak & Chabakauri, 2010, Björk & Murgoci, 2010, Forsyth, 2020, Lim & Zhou, 2002, Lindensjö, 2019, Pedersen & Peskir, 2016, Wang & Forsyth, 2011). Following Björk and Murgoci (2010, p.57), we set $d = 1$, $\mu_t \equiv \mu$, $\sigma_t \equiv \sigma$, and $\lambda_t \equiv \lambda$. In this subsection, we give the optimal solution in closed formulas. We get $F(\rho^{\hat{L}_1}) = F(\rho^{W_1^1}) = 2^{\eta/2}\frac{\Gamma(\frac{\eta+1}{2})}{\sqrt{\pi}}$ if $\alpha = 2$ (see Winkelbauer, 2012), and $F(\rho^{\hat{L}_1}) = \frac{2}{\pi}\Gamma(\eta+1)\sin(\frac{\pi\eta}{2})\int_0^\infty \frac{1-e^{-c_\alpha|t|^\alpha}}{t^{\eta+1}}\mathrm{d}t$ if $\alpha < 2$ (see Lin & Hu, 2018), where Γ denotes the Gamma function and c_α is as in (A.1). In particular, it holds that $F(\rho^{\hat{L}_1}) = F(\rho^{W_1^1}) = 1$ if $\eta = \alpha = 2$.

Proposition 6.8. *The unique discrete-time optimal solution $\hat{u}$ is given by going backward in time from $n = T-1$ to $n = 0$ solving the following equation for $\hat{u}_{n+1}$*

$$0 = (\mu - r)\int_n^{n+1} e^{r(T-s)}\mathrm{d}s \tag{6.4}$$

$$- sign(\hat{u}_{n+1})\eta\lambda F(\rho^{\hat{L}_1})\sigma^\eta\Big(\int_n^T |\hat{u}_s|^\alpha e^{\alpha r(T-s)}\mathrm{d}s\Big)^{\frac{\eta}{\alpha}-1}|\hat{u}_{n+1}|^{\alpha-1}\int_n^{n+1} e^{\alpha r(T-s)}\mathrm{d}s.$$

Remark 6.9. The continuous-time analog of (6.4) for $\hat{u}$ is given by $\hat{u}_t = \big(\frac{\mu-r}{\eta\lambda F(\rho^{\hat{L}_1})\sigma^\eta}\big)^{\frac{1}{\alpha-1}} e^{-r(T-t)}\big(\int_t^T |\hat{u}_s|^\alpha e^{\alpha r(T-s)}\mathrm{d}s\big)^{\frac{\alpha-\eta}{\alpha(\alpha-1)}}$ for all $t \in (0,T]$. Also this can be seen by replacing n and $n+1$ by t and $t+\delta$, dividing by δ and taking the limit $\delta \to 0$.

From Remark 6.9, we conclude that the continuous-time optimal solution for mean–variance optimization in the Black–Scholes model, i.e., $\eta = \alpha = 2$, is given by $\hat{u}_t = \frac{\mu-r}{2\lambda\sigma^2}e^{-r(T-t)}$ which is the same as in Björk and Murgoci (2010, p.57) (with a different notation). Hence, optimizing gains and losses gives identical solutions as optimizing terminal wealth for mean–variance in the Black–Scholes model.

7. Numerical results

In the following section, we calculate numerical values of the optimal control function, applying the results from the previous sections. For the solutions shown, we use VaR (see Example 3.2(i)) as the main risk measure and compare it to mean–variance optimization. In the case of VaR, it holds that $\rho^{W_1^1} = -z_p = z_{1-p}$ for $\alpha = 2$, where z_β denotes the β-quantile of a standard normal distribution. If $\alpha < 2$, we use a simulation based on the algorithm of Misiorek and Weron (2012) to calculate VaR.

So, we start to consider a two-dimensional geometric Brownian Motion with the risk-free interest rate $r = 0.02$, expected return $\mu := \mu_t \equiv \begin{pmatrix}0.08\\0.06\end{pmatrix}$, volatility matrix $\sigma := \sigma_t \equiv \begin{pmatrix}0.20 & 0.10\\0.10 & 0.15\end{pmatrix}$, and correlation matrix $R := R_t \equiv \begin{pmatrix}1 & 0.5\\0.5 & 1\end{pmatrix}$. We set $T = 10$ with a step size of $\delta = 0.01$, i.e., there are 1000 intermediate points. Moreover, we take $\lambda := \lambda_t \equiv 0.25$ for the risk aversion parameter and $p = 0.01 = 1\%$ for the risk level. In the one-dimensional optimizations, we just take the first asset from this model.

In Fig. 1, we compare the time-consistent optimal investments under Value-at-Risk (with different target and/or F functions) to the optimal investment under mean–variance. In the two-dimensional Black–Scholes model in the left plot, we illustrate this for the cases that $F \equiv 0$, $F(x) = x$ or $F(x) = \max\{x,0\}^2$ when $\mathcal{T}$ is the exponential target function with parameters $c = \gamma = 1$, and $F(x) = x$ or $F(x) = \max\{x,0\}^2$ when $\mathcal{T}$ is the identity target function $\mathcal{T}(x) = x$. In the one-dimensional Lévy model in the middle plot, we replace the Brownian Motion with a pure upward jump α-stable Lévy process with $c = 0.05$. We give the result for the exponential target function and the Value at Risk with $F(x) = \max\{x,0\}^2$. In the one-dimensional Lévy model of the right plot, we use a pure upward jump, a symmetric, and a pure downward jump Lévy model with $c = 0.5$, the identity target function and the Value at Risk with $F(x) = \max\{x,0\}^2$. We display the optimal solutions as a fraction of the mean–variance optimal solution (with $\alpha = 2$). The fraction for the second asset in the 2-dimensional Black–Scholes model is not shown since the curves differ only slightly.

The left plot of Fig. 1 shows that the investment with VaR optimization non-increases over time compared to the mean–variance optimal investment in all cases. In particular, under a shorter time horizon, the mean–variance investor tends to take riskier positions than the optimal VaR investor. Moreover, we notice that the optimal investment into the risky asset is zero if $F(x) = x$. Furthermore, we see that the exponential utility investment without risk penalization has an investment structure similar to mean–variance optimization. This stems from the similar structure of these problems, which can be seen by considering the Taylor series of the exponential utility function. Hence, we conclude that VaR optimization leads to more conservative investments for shorter maturities and comparably larger investments in the risky asset than mean–variance. On top of that, we notice that using the exponential target function compared to the identity leads to a more conservative investment. This is reasonable since when taking $\gamma = c$ and their limit to 0, the exponential target function reduces to the identity. If these values are positive, the investor is risk-averse, whereas the investor is risk-neutral in the identity case.

The other plots of Fig. 1 show that the style of the investment strategy is different when using Lévy processes instead of the Brownian Motion. Contrary to Brownian Motions, Lévy processes show the feature that compared to the Black–Scholes mean–variance optimization, the riskiness increases for very short maturities compared to longer maturities. Furthermore, the middle plot displays no monotonicity in the investment behavior concerning α. This stems from the observation that Value at Risk of $\hat{L}_1$ is actually not monotone in α. In fact, Value at Risk value describes a U-shape for this parametrization. Moreover, we observe that for smaller α's, the investment strategy gets comparably riskier with shorter maturities. This effect decreases and inverses for higher α's as observed for $\alpha = 2$ in the left plot. From the right plot, we can observe that a model with pure upward jumps tends to entail riskier strategies than one with pure downward jumps. This stems from the fact that the Value at Risk of $\hat{L}_1$ is smaller for pure upward jumps, and hence, the penalization of the risk is smaller. In addition, when comparing the middle and the right plots, we observe that a higher c in the Lévy model induces less risky strategies.

In Fig. 2, we compare the Nash equilibrium strategy of a sophisticated investor to a pre-committed investor who optimizes the problem only once in the beginning and sticks to this strategy until maturity. We consider only the first risky asset from the parametrization at the start of the section and set $T = 1$. We analyze mean–variance optimization in the left plot and VaR optimization using the exponential target function with $\gamma = c = 0.25$ (and $F(x) = \max\{0,x\}^2$) in the right plot. Both

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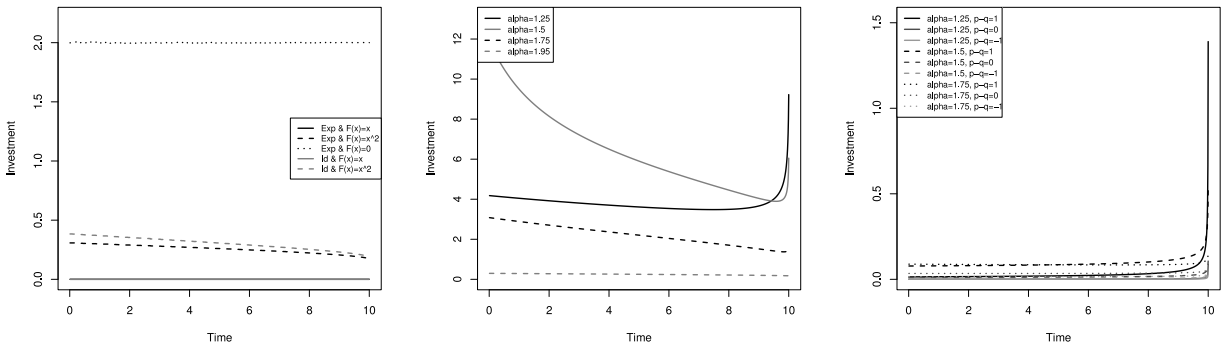

Fig. 1. Comparison of optimal control functions using the exponential or identity target function with different F functions to the mean–variance optimization for the 2-dimensional Black–Scholes model (left) and the one-dimensional Lévy model (middle and right).

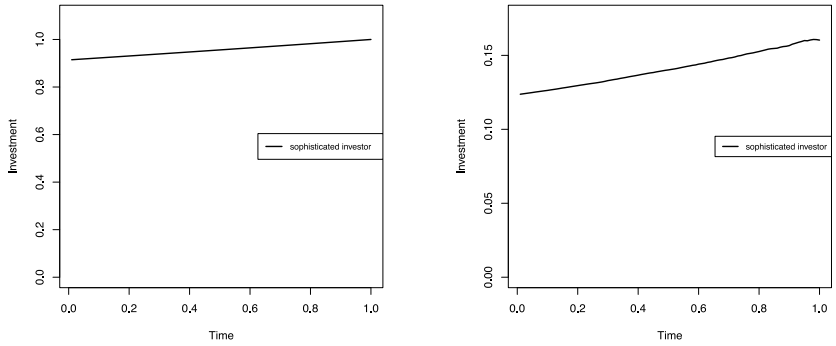

Fig. 2. Comparison of Nash equilibrium to pre-commitment optimal strategies for mean–variance (left) and mean-VaR^2 (right) optimization.

plots give the investments as a fraction of the optimal pre-commitment investment. In the mean–variance case, the pre-commitment strategy is taken from Zhou and Li (2000). In the VaR case, we calculate the pre-commitment solution for the exponential target function in the same way as Basak and Shapiro (2001) did for power utility. Note that the pre-commitment strategies are not deterministic. Hence, we take the expected value of the optimal strategy instead. Since this cannot be done explicitly for VaR, we use, for this case, the average of a Monte-Carlo-Simulation with $N = 100\,000$.

Fig. 2 shows that time-consistent investment is more conservative than pre-commitment investment and that this difference gets smaller for shorter time horizons. The difference between the solutions is more significant for mean-VaR^2 than for mean–variance.

8. Conclusion

In this paper, we derived an HJB equation for a multidimensional asset allocation problem focusing on gains & losses with rather general risk measures in an α-stable Lévy model. We showed that under appropriate conditions an optimal strategy in the sense of a Nash subgame equilibrium exists and is deterministic. We also proved a convergence result from discrete to continuous time of the optimal strategies. In continuous time, we needed additional assumptions on the model. A numerical analysis shows that Value at Risk leads to less risky investments compared to mean–variance for short maturities and that a time-consistent investor invests less risky than a pre-commitment investor.

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CRediT authorship contribution statement

Felix Fießinger: Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Mitja Stadje:** Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Lévy processes and their stochastic integral

In this section, we describe more properties of Lévy processes as a continuation of Section 2 and construct their stochastic integral. The description is based on Çınlar (2011, pp. 314–315,322–323), and the stochastic integral construction on Protter (2005, pp. 51–65).

A.1. Lévy processes

Let L be a Lévy process, i.e., it is adapted to $\mathcal{F}$, has càdlàg paths for almost all ω with $L_0(\omega) = 0$, and has independent increments with $L_{t+u} - L_t \overset{d}{=} L_u$ for all $t, u \in \mathbb{R}_{\geq 0}$. The last property is equivalent to having independent and stationary increments. Let L be α-stable with $\alpha \in (0, 2]\backslash\{1\}$. Then, the characteristic function φ of L is given by

$$\varphi_{L_t}(r) = \mathbb{E}[\exp(i\langle r, L_t\rangle)] = \exp(-tc_\alpha \int_{\mathbb{S}^d} |\langle r, v\rangle|^\alpha [1 - i\tan(\frac{\pi\alpha}{2})\text{sign}(\langle r, v\rangle)]\tilde{\sigma}(\mathrm{d}v)) \quad \text{(A.1)}$$

where $r \in \mathbb{R}^d$, $c_\alpha := c\frac{\Gamma(1-\alpha)}{\alpha}\cos(\frac{\pi\alpha}{2})$, and $\Gamma(\cdot)$ denotes the Gamma function. If $\alpha \in (1,2]$, it holds that $\mathbb{E}[L_t] = 0$ and that L_t is a martingale if L_t is additionally symmetric. In the one-dimensional case, the characteristic function reduces to

$$\varphi_{L_t}(r) = \mathbb{E}[\exp(irL_t)] = \exp(-tc_\alpha|r|^\alpha[1 - i(p-q)\tan(\frac{\pi\alpha}{2})\operatorname{sign}(r)]), \quad \text{(A.2)}$$

where p is the (conditional) probability of an upward jump and q is the (conditional) probability of a downward jump. In particular, $p+q=1$.

A.2. Stochastic integral for Lévy processes

Every Lévy process is a semimartingale; hence, we use the definition of stochastic integrals for semimartingales here. *A sequence of processes $(X^n)_{n\geq 1}$ converges in ucp to the process X, if $\sup_{0\leq s\leq t}|X^n_s - X_s| \xrightarrow{\mathbb{P}} 0$ for all $t>0$* (Protter, 2005, Cf. p.57). We define $\mathbb{L}$ as the set of all adapted processes with càglàd (left-continuous with right limits) paths and $\mathbb{D}$ as the set of all adapted processes with càdlàg (right-continuous with left limits) paths. Moreover, we write $\mathbb{L}_{ucp}$ for $\mathbb{L}$ endowed with the *ucp*-topology and $\mathbb{L}^{det}_{sup}$ for all functions from $\mathbb{L}_{ucp}$ which are deterministic. For $\mathbb{D}$, we use analogous notations. *We call the process H simple predictable if we can write H as $H_t = H_0\mathbb{1}_{\{0\}}(t) + \sum_{j=1}^n H_j\mathbb{1}_{(T_{j-1},T_j]}(t)$ where $0 = T_0 \leq \cdots \mathcal{T}_n < \infty$ is a sequence of stopping times and for all $j \in \{0,\dots,n\}$ holds: $H_j \in \mathcal{F}_{T_j}$* (Protter, 2005, Cf. p.51). Let $\mathbf{S}$ be the set of all simple predictable processes.

Let L be a Lévy process and $H \in \mathbf{S}$. Then the stochastic integral of H with respect to L is defined as $\int_0^t H_s\mathrm{d}L_s = H_0L_0 + \sum_{j=1}^n H_j(L_{T_j} - L_{T_{j-1}})$ with H constructed as above and $T_j \leq t$ for all j. This is a mapping from $\mathbf{S}$ to $\mathbb{D}$. We define the general stochastic integral as the extension of this from $\mathbb{L}_{ucp}$ to $\mathbb{D}_{ucp}$. If we want to exclude the 0, we write $\int_{0+}^t H_s\mathrm{d}L_s$. For this paper, an important property of the stochastic integral over $\mathbb{L}$ is that it holds for $Y \in \mathbb{L}^{det}_{sup}$ and $t>0$ that

$$\int_{0+}^t Y^{\pi_n}_s\mathrm{d}L_s \xrightarrow{\mathbb{P}} \int_{0+}^t Y_s\mathrm{d}L_s \quad \text{(A.3)}$$

where $\pi_n : 0 = t_0 \leq \cdots \leq t_n = t$ is the equidistant partition of $[0,T]$ in n intervals. Note: $\int_{0+}^t Y^{\pi_n}_s\mathrm{d}L_s = \sum_{j=1}^n Y_{t_j}(L_{t_j} - L_{t_{j-1}})$.

One can generalize this definition to predictable processes such that the stochastic integral is linear in the integrand, see Applebaum (2009, p.237) and the references therein.

Appendix B. Lemmas and proofs

Proof of Lemma 4.9. First, we define $Z_{t-s} = \langle a, L_t - L_s\rangle$. Then we get from (A.1) since $r \in \mathbb{R}$:

$$\begin{aligned}
\varphi_{Z_{t-s}}(r) &= \mathbb{E}[\exp(irZ_{t-s})]\\
&= \mathbb{E}[\exp(ir\langle a, L_t - L_s\rangle)]\\
&= \mathbb{E}[\exp(i\langle ra, L_t - L_s\rangle)]\\
&= \exp\{-(t-s)c_\alpha\int_{\mathbb{S}^d}|\langle ra,v\rangle|^\alpha[1 - i\tan(\tfrac{\pi\alpha}{2})\operatorname{sign}(\langle ra,v\rangle)]\tilde\sigma(\mathrm{d}v)\}\\
&= \exp\{-(t-s)c_\alpha|r|^\alpha[\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde\sigma(\mathrm{d}v)\\
&\quad - i\tan(\tfrac{\pi\alpha}{2})\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\operatorname{sign}(r)\operatorname{sign}(\langle a,v\rangle)\tilde\sigma(\mathrm{d}v)]\}\\
&= \exp\Big\{-(t-s)c_\alpha|r|^\alpha\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde\sigma(\mathrm{d}v)[1\\
&\quad - i\tan(\tfrac{\pi\alpha}{2})\operatorname{sign}(r)\frac{\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\operatorname{sign}(\langle a,v\rangle)\tilde\sigma(\mathrm{d}v)}{\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde\sigma(\mathrm{d}v)}]\Big\}\\
&= \varphi_{\tilde L_{(t-s)\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde\sigma(\mathrm{d}v)}}(r),
\end{aligned}$$

when comparing this formula with (A.2), since $p-q := \int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\operatorname{sign}(\langle a,v\rangle)\tilde\sigma(\mathrm{d}v)(\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde\sigma(\mathrm{d}v))^{-1}$ is independent of a (see Remark 4.5) under our assumptions.

Finally, we obtain $\tilde L_{(t-s)\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde\sigma(\mathrm{d}v)} \overset{d}{=} \sqrt[\alpha]{\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde\sigma(\mathrm{d}v)}\tilde L_{(t-s)} \overset{d}{=} \sqrt[\alpha]{\int_{\mathbb{S}^d}|\langle a,v\rangle|^\alpha\tilde\sigma(\mathrm{d}v)}(\tilde L_t - \tilde L_s)$ due to the α-stability of $\tilde L$. □

Proof of Lemma 4.10. Here, we must distinguish the cases $\alpha = 2$ and $\alpha < 2$. If $\alpha = 2$, then the Lévy process is a Brownian Motion, and hence, this lemma follows from the well-known properties of a Brownian Motion and Definition 4.6 as

$$\int_{t+}^T f(s)\mathrm{d}W^1_s \overset{d}{=} \mathcal{N}(0, \int_{t+}^T f(s)^2\mathrm{d}s) \overset{d}{=} \sqrt{\int_{t+}^T f(s)^2\mathrm{d}s}W^1_1 \overset{d}{=} \sqrt{\int_{t+}^T f(s)^2\mathrm{d}s}\tilde L_1.$$

In the case $\alpha < 2$, we have $\hat L = \tilde L$. Define $Z := \int_{t+}^T f(s)\mathrm{d}\tilde L_s\,(\sqrt[\alpha]{\int_t^T|f(s)|^\alpha\mathrm{d}s})^{-1}$, and let us show that $Z \overset{d}{=} \tilde L_1$. We use the time equidistant discretizations $t = s_0 < \cdots < s_n = T$, $\Delta\tilde L_{s_k} = \tilde L_{s_k} - \tilde L_{s_{k-1}}$ and $\Delta s_k = s_k - s_{k-1}$ for $k \in \{1,\dots,n\}$ to define $Z_n := (\sum_{k=1}^n f(s_k)\Delta\tilde L_{s_k})(\sqrt[\alpha]{\sum_{k=1}^n|f(s_k)|^\alpha\Delta s_k})^{-1}$. We prove the result in three steps:

In Step 1 we show $\varphi_{Z_n} = \varphi_{\tilde L_1}$: Using that the $\Delta\tilde L_{s_k}$'s are independent for different k's, and (A.2), we get :

$$\begin{aligned}
\varphi_{Z_n}(r) &= \mathbb{E}[\exp(irZ_n)] = \mathbb{E}\big[\exp\big(i\frac{r}{\sqrt[\alpha]{\sum_{k=1}^n|f(s_k)|^\alpha\Delta s_k}}\sum_{k=1}^n f(s_k)\Delta\tilde L_{s_k}\big)\big]\\
&= \prod_{k=1}^n\mathbb{E}\big[\exp\big(i\frac{rf(s_k)}{\sqrt[\alpha]{\sum_{l=1}^n|f(s_l)|^\alpha\Delta s_l}}\Delta\tilde L_{s_k}\big)\big]\\
&= \prod_{k=1}^n\exp\big(-\Delta s_kc_\alpha|\frac{rf(s_k)}{\sqrt[\alpha]{\sum_{l=1}^n|f(s_l)|^\alpha\Delta s_l}}|^\alpha\\
&\qquad\cdot[1 - i(p-q)\tan(\tfrac{\pi\alpha}{2})\operatorname{sign}(\frac{rf(s_k)}{\sqrt[\alpha]{\sum_{l=1}^n|f(s_l)|^\alpha\Delta s_l}})]\big)\\
&= \exp(-c_\alpha\frac{1}{\sum_{l=1}^n|f(s_l)|^\alpha\Delta s_l}\sum_{k=1}^n|f(s_k)|^\alpha\Delta s_k|r|^\alpha[1 - i(p-q)\tan(\tfrac{\pi\alpha}{2})\operatorname{sign}(r)])\\
&= \exp(-c_\alpha|r|^\alpha[1 - i(p-q)\tan(\tfrac{\pi\alpha}{2})\operatorname{sign}(r)]) = \mathbb{E}[\exp(ir\tilde L_1)] = \varphi_{\tilde L_1}(r),
\end{aligned}$$

where we used that either $f \geq 0$ in the case of Assumptions 3b and 3c or $p-q=0$ in the case of Assumption 3a.

In Step 2 we prove $Z_n \xrightarrow{\mathbb{P}} Z$: For this purpose, we define $\phi_n = \sum_{k=1}^n|f(s_k)|^\alpha\mathbb{1}_{(s_{k-1},s_k]}$ and $\chi_n = \sum_{k=1}^n f(s_k)\mathbb{1}_{(s_{k-1},s_k]}$. We see that both functions are deterministic, $\phi_n \xrightarrow{n\to\infty} |f|^\alpha$, and $\chi_n \xrightarrow{n\to\infty} f$ pointwise on $(t,T]$ since f is continuous and deterministic. Now, the convergence of Riemann sums and (A.3) give us with $\tilde\lambda$ being the Lebesgue measure:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^n|f(s_k)|^\alpha\Delta s_k &= \lim_{n\to\infty}\int_{(t,T]}\phi_n\mathrm{d}\tilde\lambda = \int_t^T|f(s)|^\alpha\mathrm{d}s,\\
\sum_{k=1}^n f(s_k)\Delta\tilde L_{s_k} &= \int_{(t,T]}\chi_n\mathrm{d}\tilde L_s \xrightarrow{\mathbb{P}} \int_{t+}^T f(s)\mathrm{d}\tilde L_s.
\end{aligned}$$

These two limits give us the claim of Step 2.

In Step 3 we finally show the claim $Z \overset{d}{=} \tilde L_1$: We know $\lim_{n\to\infty} Z_n \overset{\mathbb{P}}{=} Z$ from Step 2, which implies the convergence of the characteristic functions, i.e., $\lim_{n\to\infty}\varphi_{Z_n} = \varphi_Z$. Due to Step 1, we then get $\varphi_{\tilde L_1} = \lim_{n\to\infty}\varphi_{\tilde L_1} = \lim_{n\to\infty}\varphi_{Z_n} = \varphi_Z$ which is equivalent to the claim. □

Proof of Proposition 4.12. We prove these propositions with properties for compositions of concave and convex functions, see Zălinescu (2002, p.43).

First of all, we show that the first term of J_n is concave in u:

We know from (4.3):

$$X^u_T - X^u_ne^{r(T-n)} = \sum_{i=1}^d\int_n^T u^i_s(\mu^i_s - r)e^{r(T-s)}\mathrm{d}s + \sum_{i,j=1}^d\int_{n+}^T u^i_s\sigma^{ij}_se^{r(T-s)}\mathrm{d}L^j_s.$$

Due to the linearity of the integrals, $X^u_T - X^u_ne^{r(T-n)}$ is a linear mapping. Since $\mathcal{T}$ is a concave function and the expected value is linear, it follows that $\mathbb{E}_{n,x}[\mathcal{T}(X^u_T - xe^{r(T-n)})]$ is concave in u.

The next step is to show that $\rho_{n,x}(X^u_T - xe^{r(T-n)})$ is convex in u. We know from Proposition 4.11 that $\rho_{n,x}(X^u_T - xe^{r(T-n)}) = -\int_n^T m_s(u)\mathrm{d}s +$

$$\begin{cases} \varrho^{\hat{L}_1}\|u_s^\intercal\sigma_s v e^{r(T-s)}\|_{\mathcal{L}^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}v))} & \text{if } \alpha<2.\\ \varrho^{W_1^1}\|u_s^\intercal\sigma_s e^{r(T-s)}\|_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)} & \text{if } \alpha=2. \end{cases}$$

The first term is linear since m_s is linear. Furthermore, since $\alpha>1$ by assumption, the second term consists in each case of an $\mathcal{L}^\alpha$-norm, which is convex due to the positive homogeneity and the triangular inequality of norms. Hence, the whole second term is convex in u as well. Since F is non-decreasing and convex, $F(\rho_{n,x}(X_T^{\hat{u}}-xe^{r(T-n)}))$ is convex in u. Then we know that $-\lambda_n F(\rho_{n,x}(X_T^u-xe^{r(T-n)}))$ is concave in u and so the functional J_n is concave in u as the sum of concave functions.

If $\mathcal{T}$ or $-F$ is strictly concave, then the respective term is also strictly concave since σ_t is positive definite. Hence, J_n is strictly concave as well as the sum of a strictly concave and a concave function. □

Proof of Proposition 4.13. In this proof, we denote by ${}^\delta\hat{u}$ the optimal strategy with respect to the step size δ. However, before showing Proposition 4.13, we prove the following two lemmas:

Lemma B.1. *If the control function u is deterministic and $\lambda_n\equiv\lambda$, then the functional J_n is always non-negative in the optimum, i.e., $J_n(\hat{u})\geq 0$ for all $n\in\{0,\dots,T-1\}$.*

Proof. We show this proposition by induction, where the induction start is similar to the induction step. Thus, let $n\in\{0,\dots,T-1\}$. Now, (4.3) and the assumption that λ is constant imply that $J_n(x,u=(\hat{u}_{n+1},\hat{u}_{n+2},\dots,\hat{u}_T)^\intercal)\geq J_n(x,u=(0,\hat{u}_{n+2},\dots,\hat{u}_T)^\intercal)=J_{n+1}(x,u=(\hat{u}_{n+2},\dots,\hat{u}_T)^\intercal)\geq 0$ by the induction step where the equality follows by Proposition 4.11. □

Lemma B.2. $\|{}^\alpha w_s({}^\delta\hat{u})\|^\alpha_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}\geq\tilde{\lambda}^\alpha_{min}\min\{1,\mathbb{A}_d\}\cos^\alpha(\tilde{\varepsilon})\hat{\varepsilon}\|{}^\delta\hat{u}_s\|^\alpha_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}$ *holds under Assumption 4c.*

Proof. First, we can assume without loss of generality that ${}^\alpha\hat{u}_s\neq 0$ for all s. Otherwise, since ${}^\alpha\hat{u}_s$ is piecewise constant, the right hand side of the equation is already "0" in this interval and can be ignored in the following proof.

Now, we show the lemma in the case that $\tilde{\sigma}$ has a density which is bigger than $\hat{\varepsilon}$ on the $(d-1)$-dimensional unit ball around a vector with respect to the radial distance with radius $12\tilde{\varepsilon}$. We denote this vector by $w\in\mathbb{S}^d$, the unit ball on the sphere around w by O_w, and the radial distance between two vectors by d. Moreover, we define the distance $\tilde{d}$ as the maximum norm on the sphere, i.e., using polar coordinates, one can describe a point in $\mathbb{S}^d$ by $d-1$ angles. For the maximal norm $\tilde{d}(x,y)$ on the sphere, the standard (componentwise) maximal norm is then applied to the $d-1$ angles between the two vectors x and y in their polar coordinates (see the set A in Fig. B.3).

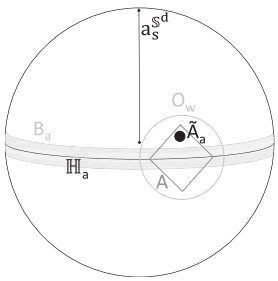

Fig. B.3. Graphical illustration of the proof for $d=3$ if $w\in\mathbb{H}_a$. If $w\notin\mathbb{H}_a$, the situation is simpler.

Then, it holds that $A:=\{\hat{w}\in\mathbb{S}^d|\tilde{d}(\hat{w},w)\leq 6\tilde{\varepsilon}\}\subset O_w$ by construction. We can describe A as some $(d-1)$-dimensional hyper-'cube' with curved sides. Define $a_s^{\mathbb{S}^d}({}^\delta\hat{u}):=\frac{{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}}{\|{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|_2}\in\mathbb{S}^d$. Then, the set of vectors on $\mathbb{S}^d$ which are orthogonal to $a_s^{\mathbb{S}^d}$ form a $(d-2)$-dimensional curved hyperplane, denoted by $\mathbb{H}_a$. By orthogonality, for all $\tilde{w}\in\mathbb{H}_a$ the angle between $a_s^{\mathbb{S}^d}({}^\delta\hat{u})$ and $\tilde{w}$ is $\frac{\pi}{2}$. Now, define $B_a:=\{\hat{w}\in\mathbb{S}^d|dist(\hat{w},\mathbb{H}_a)\leq\tilde{\varepsilon}\}\subset\mathbb{S}^d$, where $dist$ denotes the radial distance between a vector and a set on $\mathbb{S}^d$, i.e., $dist(x,C)=\inf_{y\in C}d(x,y)$ for $x\in\mathbb{S}^d$ and $C\subset\mathbb{S}^d$. Hence, B_a is a hyperspherical segment. By construction, there exists a point $\hat{v}_a\in A\backslash B_a$ such that $\tilde{A}_a:=\{\hat{w}\in\mathbb{S}^d|d(\hat{w},\hat{v}_a)\leq\tilde{\varepsilon}\}\subset A\backslash B_a$. In particular, it holds for all $\hat{w}\in\tilde{A}_a$ that the angle between $\hat{w}$ and $a_s^{\mathbb{S}^d}$ is at least $\tilde{\varepsilon}$ away from $\frac{\pi}{2}$ and the density of $\tilde{\sigma}$ is bigger or equal than $\hat{\varepsilon}$. Moreover, note that the surface area of $\tilde{A}_a$ is equal to $\mathbb{A}_d$ with $\mathbb{A}_d$ as in Assumption 4c. Hence, we get since $|a\cdot v|=|\cos(\phi)||a|$ for $v\in\mathbb{S}^d$ and ϕ denoting the angle between a and v:

$$\begin{aligned}\|{}^\alpha w_s({}^\delta\hat{u})\|^\alpha_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}&=\int_0^T\int_{\mathbb{S}^d}|{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}v|^\alpha\tilde{\sigma}(\mathrm{d}v)\mathrm{d}s\\&\geq|\cos|^\alpha(\tilde{\varepsilon})\int_0^T|{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha\int_{\tilde{A}_a}\tilde{\sigma}(\mathrm{d}v)\mathrm{d}s\\&\geq\mathbb{A}_d\cos^\alpha(\tilde{\varepsilon})\hat{\varepsilon}\int_0^T|{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha\mathrm{d}s\\&\geq\mathbb{A}_d\cos^\alpha(\tilde{\varepsilon})\hat{\varepsilon}\tilde{\lambda}^\alpha_{min}\int_0^T|{}^\delta\hat{u}_s^\intercal|^\alpha\mathrm{d}s,\end{aligned}$$

i.e., the claim follows. For the other case, the argumentation is similar, just without $\mathbb{A}_d$. □

Now, we are ready to prove Proposition 4.13:

In the case of Assumption 4a, the claims follow directly by possibly increasing M. Otherwise, we show this proposition by contradiction, i.e., we assume that $\|{}^\delta\hat{u}\|_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}\xrightarrow{\delta\to 0}\infty$ by possibly switching to a subsequence. Now, since $\lim_{x\to\infty}\mathcal{T}'(x)=0$, for all $\gamma_{\mathcal{T}}>0$ exists a $C_\gamma>0$ such that $\mathcal{T}(x)\leq C_\gamma+\gamma_{\mathcal{T}}x$ and since F is convex, there exists a $C_F\leq 0$ and a $\gamma_F>0$ such that $F(x)\geq C_F+\gamma_F x$. Also, note that $\|\cdot\|_{\mathcal{L}^1((0,T],\mathrm{d}s)}\leq C_\alpha\|\cdot\|_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}$ with $C_\alpha:=T^{1-1/\alpha}$ since $\alpha>1$ by assumption. Then, we get:

$$\begin{aligned}J_0(x,u)=&\mathbb{E}_{0,x}[\mathcal{T}(\int_0^T{}^\delta\hat{u}_s^\intercal(\mu_s-r)e^{r(T-s)}\mathrm{d}s+\int_{0+}^T{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\mathrm{d}L_s)]\\&-\lambda F(-\int_0^T m_s({}^\delta\hat{u})\mathrm{d}s+\varrho^{\hat{L}_1}\|{}^\alpha w_s({}^\delta\hat{u})\|_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)})\\\leq&C_\gamma+\gamma_{\mathcal{T}}\tilde{\mu}_{max}C_\alpha\|{}^\delta\hat{u}_s\|_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}-\lambda C_F\\&+\lambda\gamma_F\int_0^T|m_s({}^\delta\hat{u})|\mathrm{d}s-\lambda\gamma_F\varrho^{\hat{L}_1}\|{}^\alpha w_s({}^\delta\hat{u})\|_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)},\end{aligned}$$

where we recall that $\tilde{\mu}_{max}=\max\{|\mu_s^i-r|e^{r(T-s)}|s\in[0,T],i\in\{1,\dots,d\}\}$. We used Proposition 4.11 in the last inequality. Now, we consider Assumption 4c. If ρ is shift-invariant, we get with Lemma B.2:

$$\begin{aligned}J_0(x,u)\leq&C_\gamma+\gamma_{\mathcal{T}}\tilde{\mu}_{max}C_\alpha\|{}^\delta\hat{u}_s\|_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}-\lambda C_F\\&-\lambda\gamma_F\varrho^{\hat{L}_1}\tilde{\lambda}_{min}\min\{1,\mathbb{A}_d\}^{1/\alpha}\cos(\tilde{\varepsilon})\hat{\varepsilon}^{1/\alpha}\|{}^\delta\hat{u}_s\|_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}.\end{aligned}$$

Choosing $\gamma_{\mathcal{T}}<\frac{\lambda\gamma_F\varrho^{\hat{L}_1}\tilde{\lambda}_{min}\min\{1,\mathbb{A}_d\}^{1/\alpha}\cos(\tilde{\varepsilon})\hat{\varepsilon}^{1/\alpha}}{\tilde{\mu}_{max}C_\alpha}$ leads to $J_0(x,u)\xrightarrow{\delta\to 0}-\infty$ which is a contradiction to Lemma B.1 and thus we get the claim. If ρ is cash-invariant, we get with Lemma B.2 that

$$\begin{aligned}J_0(x,u)\leq&C_\gamma-\lambda C_F+\|{}^\delta\hat{u}_s\|_{\mathcal{L}^\alpha((0,T],\mathrm{d}s)}(\gamma_{\mathcal{T}}\tilde{\mu}_{max}C_\alpha+\lambda\gamma_F\tilde{\mu}_{max}\\&-\lambda\gamma_F\varrho^{\hat{L}_1}\tilde{\lambda}_{min}\min\{1,\mathbb{A}_d\}^{1/\alpha}\cos(\tilde{\varepsilon})\hat{\varepsilon}^{1/\alpha}).\end{aligned}$$

Now, due to the assumption that $\varrho^{\hat{L}_1}\tilde{\lambda}_{min}\min\{1,\mathbb{A}_d\}^{1/\alpha}\cos(\tilde{\varepsilon})\hat{\varepsilon}^{1/\alpha}>\tilde{\mu}_{max}$, it is possible to choose $0<\gamma_{\mathcal{T}}<\frac{\lambda\gamma_F(\varrho^{\hat{L}_1}\tilde{\lambda}_{min}\min\{1,\mathbb{A}_d\}^{1/\alpha}\cos(\tilde{\varepsilon})\hat{\varepsilon}^{1/\alpha}-\tilde{\mu}_{max})}{\tilde{\mu}_{max}C_\alpha}$ and the contradiction can be derived similarly. The case of Assumption 4b is a simpler variant of this argument where (4.2) replaces Lemma B.2. Also, Assumption 4d leads to a simpler variant after noticing that $\|{}^\alpha w_s({}^\delta\hat{u})\|^2_{\mathcal{L}^2((0,T],\mathrm{d}s)}\geq\tilde{\lambda}^2_{min}\|{}^\delta\hat{u}_s\|^2_{\mathcal{L}^2((0,T],\mathrm{d}s)}$ by the definition of $\tilde{\lambda}^2_{min}$. □

Proof of Proposition 4.14. Plugging Eq. (4.3) and Proposition 4.11 into Eq. (4.1) leads to:

$$J_n(x,u)=\mathbb{E}_{n,x}[\mathcal{T}(\int_n^T u_s^\intercal(\mu_s-r)e^{r(T-s)}\mathrm{d}s+\int_{n+}^T u_s^\intercal\sigma_s e^{r(T-s)}\mathrm{d}L_s)]$$

$$- \lambda_n F(-\int_n^T m_s(u)\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(u)\|_{\mathcal{L}^\alpha((n,T],\mathrm{d}s)}).$$

That m and w are continuous in u follows directly from their definitions. Hence, the second part of J_n is continuous in u since $F \in C^1$. In the first part, it follows similar to the proof of Proposition 4.11 that $\int_{n+}^T u_s^\intercal \sigma_s e^{r(T-s)}\mathrm{d}L_s \overset{d}{=} \|^\alpha w_s(u)\|_{\mathcal{L}^\alpha((n,T],\mathrm{d}s)}\hat{L}_1$. Moreover, if $u^k \xrightarrow{k\to\infty} u$, u and u^k are uniformly bounded converging vectors and therefore uniformly bounded. Hence, Assumption 1 and Lebesgue's convergence theorem (since $\mathcal{T}$ is monotone) also give us the continuity of J_n in u for the first part. □

Proof of Proposition 4.16. Denote the optimal deterministic control function by $\hat{u}$. We show the desired property by backward induction. Therefore, we show at each time point $n \in \{0,\dots,T-1\}$ that the derivative of J_n with respect to $\hat{u}_{n+1}$ at the point 0+ is non-negative. Combining this with the concavity of J_n (Proposition 4.12) and the independence of x (Theorem 4.4), i.e., we can identify $J_n(x,\hat{u}) = J_n(\hat{u})$ (see Eq. (4.1)), giving the claim. Indeed, for a maximum of a differentiable one-dimensional concave function, the derivative is then non-negative for all smaller values and non-positive for bigger values.

The induction start is similar to the induction step. Hence, we only show the induction step for an arbitrary $n \in \{0,\dots,T-1\}$:

Using Eq. (4.3), where the existence of the expected value is guaranteed due to the second assumption of this theorem, the monotonicity of $\mathcal{T}$ and $\mathcal{T}'$, and Proposition 4.11, it holds that

$$\begin{aligned}
\frac{\partial}{\partial \hat{u}^1_{n+1}} J_n(\hat{u}) =& \mathbb{E}_{n,x}[\mathcal{T}'(\int_n^T \hat{u}^1_s(\mu^1_s - r)e^{r(T-s)}\mathrm{d}s + \int_{n+}^T \hat{u}^1_s\sigma^{11}_s e^{r(T-s)}\mathrm{d}L^1_s)\\
&\cdot(\int_n^{n+1}(\mu^1_s - r)e^{r(T-s)}\mathrm{d}s + \int_{n+}^{n+1}\sigma^{11}_s e^{r(T-s)}\mathrm{d}L^1_s)]\\
&- \lambda_n F'(-\int_n^T \hat{u}^1_s(\mu^1_s - r)e^{r(T-s)}\mathrm{d}s\\
&+ \varrho^{\hat{L}_1}\|\hat{u}^1_s\sigma^{11}_s e^{r(T-s)}\|_{\mathcal{L}^\alpha((n,T],\mathrm{d}s)})\\
&\cdot(-\int_n^{n+1}(\mu^1_s - r)e^{r(T-s)}\mathrm{d}s\\
&+ \varrho^{\hat{L}_1}\frac{|\hat{u}^1_{n+1}|^{\alpha-1}\mathrm{sign}(\hat{u}^1_{n+1})\int_n^{n+1}|\sigma^{11}_s e^{r(T-s)}|^\alpha \mathrm{d}s}{\|\hat{u}^1_s\sigma^{11}_s e^{r(T-s)}\|^{\alpha-1}_{\mathcal{L}^\alpha(n,T],\mathrm{d}s}}).
\end{aligned}$$

We show that the last fraction is bounded. Hence, the second part of the term converges to 0 for u going to 0+. We get with the Hölder inequality using $p = \frac{\alpha}{\alpha-1}$ and $q = \alpha$:

$$\begin{aligned}
&\int_n^{n+1}|\hat{u}^1_{n+1}|^{\alpha-1}\mathrm{sign}(\hat{u}^1_{n+1})|\sigma^{11}_s e^{r(T-s)}|^\alpha \mathrm{d}s\\
&\le \||\hat{u}^1_{n+1}\sigma^{11}_s e^{r(T-s)}|^{\alpha-1}|\sigma^{11}_s e^{r(T-s)}|\|_{\mathcal{L}^1((n,n+1],\mathrm{d}s)}\\
&\le \||\hat{u}^1_{n+1}\sigma^{11}_s e^{r(T-s)}|^{\alpha-1}\|_{\mathcal{L}^p((n,n+1],\mathrm{d}s)}\|\sigma^{11}_s e^{r(T-s)}\|_{\mathcal{L}^q((n,n+1],\mathrm{d}s)}\\
&\le \|\hat{u}^1_s\sigma^{11}_s e^{r(T-s)}\|^{\alpha-1}_{\mathcal{L}^\alpha((n,T],\mathrm{d}s)}\|\sigma^{11}_s e^{r(T-s)}\|_{\mathcal{L}^\alpha((n,n+1],\mathrm{d}s)}.
\end{aligned}$$

Thus, the fraction is bounded by $\|\sigma^{11}_s e^{r(T-s)}\|_{\mathcal{L}^\alpha((n,n+1],\mathrm{d}s)}$ ($\le \max_{\{s\in[0,T]\}}\sigma^{11}_s e^{rT}T$). Since L^1 is symmetric by assumption, the Lévy measure ν is also symmetric, and L^1 is therefore a martingale. Thus, we get the following when inserting 0+ into the derivative:

$$\begin{aligned}
\frac{\partial}{\partial \hat{u}^1_{n+1}} J_n(0+) &= \mathbb{E}_{n,x}[\mathcal{T}'(0)(\int_n^{n+1}(\mu^1_s - r)e^{r(T-s)}\mathrm{d}s + \int_{n+}^{n+1}\sigma^{11}_s e^{r(T-s)}\mathrm{d}L^1_s)] - 0\\
&= \mathcal{T}'(0)\int_n^{n+1}(\mu^1_s - r)e^{r(T-s)}\mathrm{d}s \ge 0
\end{aligned}$$

due to the assumption $\mu_t \ge r$ for all t, and $\mathcal{T}$ being an increasing function. □

Proof of Theorem 4.17. We start with the main formula of the HJB equation. We notice that the value functional J_n (see Eq. (4.1)) satisfies the following recursion formula:

$$\begin{aligned}
J_n(x,u) =& \mathbb{E}_{n,x}[J_{n+1}(X^u_{n+1},u)] - \mathbb{E}_{n,x}[\mathbb{E}_{n+1,X^u_{n+1}}[\mathcal{T}(X^u_T - X^u_{n+1}e^{r(T-n-1)})]\\
&- \mathcal{T}(X^u_T - xe^{r(T-n)})]\\
&+ \mathbb{E}_{n,x}[\lambda_{n+1}F(\rho_{n+1,X^u_{n+1}}(X^u_T - X^u_{n+1}e^{r(T-n-1)}))\\
&- \lambda_n F(\rho_{n,x})(X^u_T - xe^{r(T-n)})].
\end{aligned} \tag{B.1}$$

This characterization and the definition of V (see Definition 4.2) imply that

$$\begin{aligned}
&\sup_{u\in\mathcal{V}}\{\mathbb{E}_{n,x}[J_{n+1}(X^u_{n+1},\bar{u})] - V_n(x)\\
&\quad- \mathbb{E}_{n,x}[\mathbb{E}_{n+1,X^u_{n+1}}[\mathcal{T}(X^{\bar{u}}_T - X^u_{n+1}e^{r(T-n-1)})] - \mathcal{T}(X^{\bar{u}}_T - xe^{r(T-n)})]\\
&\quad+ \mathbb{E}_{n,x}[\lambda_{n+1}F(\rho_{n+1,X^u_{n+1}}(X^{\bar{u}}_T - X^u_{n+1}e^{r(T-n-1)})) - \lambda_n F(\rho_{n,x}(X^{\bar{u}}_T - xe^{r(T-n)}))]\} = 0.
\end{aligned}$$

By the definition of Nash equilibria (see Definition 4.2), we get $\mathbb{E}_{n,x}[J_{n+1}(X^u_{n+1},\bar{u})] = V_{n+1}(X^u_{n+1})$. With that and the infinitesimal operator A^u, this recursion simplifies to the main part of our HJB equation:

$$\begin{aligned}
&\sup_{u\in\mathcal{V}}\{(\mathrm{A}^u V)_n(x) - (\mathrm{A}^u\mathbb{E}_\cdot[\mathcal{T}(X^{\bar{u}}_T - X^u_\cdot e^{r(T-\cdot)})])_n(x)\\
&\quad+ (\mathrm{A}^u(\lambda_\cdot F(\rho_\cdot(X^{\bar{u}}_T - X^u_\cdot e^{r(T-\cdot)}))))_n(x)\} = 0.
\end{aligned}$$

For the final value, we get:

$$\begin{aligned}
V_T(x) = J_T(X^{\hat{u}}_T,\hat{u}) &= \mathbb{E}_{T,X^u_T}[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_T e^{r(T-T)})] - \lambda_T F(\rho_{T,X^u_T}(X^{\hat{u}}_T - X^{\hat{u}}_T e^{r(T-T)}))\\
&= \mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_T) - \lambda_T F(\rho_{T,X^u_T}(0)) = \mathcal{T}(0) - \lambda_T F(0).
\end{aligned}$$

It remains to prove the formulas for the constraints where the first one is simply the definition of the infinitesimal generator (see Definition 4.1). We show the formula for the second constraint by backward induction:

$n = T$: $\lambda_T F(\rho_{T,x}(X^{\hat{u}}_T - xe^{r(T-T)})) = \lambda_T F(\rho_{T,x}(0)) = \lambda_T F(0)$.

$n = T-1$: This is a more straightforward case than for $T-2$ and will be omitted.

$n = T-2$: Due to the optimal control function being deterministic (Theorem 4.4), we can use Proposition 4.11 to calculate:

$$\begin{aligned}
&(\mathrm{A}^{\hat{u}}(\lambda_\cdot F(\rho_\cdot(X^{\hat{u}}_T - X^{\hat{u}}_\cdot e^{r(T-\cdot)}))))_{T-2}(x)\\
&\quad= \lambda_{T-1}\mathbb{E}_{T-2,x}[F(-\int_{T-1}^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((T-1,T],\mathrm{d}s)})]\\
&\qquad- \lambda_{T-2}F(-\int_{T-2}^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((T-2,T],\mathrm{d}s)})\\
&\quad= \lambda_{T-1}[F(-\int_{T-1}^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((T-1,T],\mathrm{d}s)})\\
&\qquad\quad- F(-\int_{T-2}^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((T-2,T],\mathrm{d}s)})]\\
&\qquad- (\lambda_{T-2} - \lambda_{T-1})F(-\int_{T-2}^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((T-2,T],\mathrm{d}s)}).
\end{aligned}$$

for all other n: Similarly. □

Proof of Lemma 5.3. Using the definition for the period from t to $t+h$, it holds that

$$\begin{aligned}
&(\mathrm{A}^{\hat{u}}\mathbb{E}_\cdot[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_\cdot e^{r(T-\cdot)})])(t,x)\\
&\quad= \lim_{h\to0} h^{-1}(\mathrm{A}^{\hat{u}}_h\mathbb{E}_\cdot[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_\cdot e^{r(T-\cdot)})])(t,x)\\
&\quad= \lim_{h\to0} h^{-1}\mathbb{E}_{t,x}[\mathbb{E}_{t+h,X^{\hat{u}}_{t+h}}[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_{t+h}e^{r(T-t-h)})] - (\mathcal{T}(X^{\hat{u}}_T - xe^{r(T-t)}))]\\
&\quad= \lim_{h\to0} h^{-1}(\mathbb{E}[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_{t+h}e^{r(T-t-h)})] - \mathbb{E}[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_t e^{r(T-t)})])\\
&\quad= \frac{\partial}{\partial h}\mathbb{E}[\mathcal{T}(X^{\hat{u}}_T - X^{\hat{u}}_{t+h}e^{r(T-t-h)})]|_{h=0},
\end{aligned}$$

where we used in the third equation that the term inside the expected value is deterministic. The existence of the derivative follows immediately from Proposition 5.10. □

Proof of Lemma 5.4. It holds with Proposition 4.11 and the second constraint of Theorem 4.17 used for the interval from t to $t+h$ that

$$(\mathrm{A}^{\hat{u}}(\lambda_\cdot F(\rho_\cdot(X^{\hat{u}}_T - X^{\hat{u}}_\cdot e^{r(T-\cdot)}))))(t,x)$$

$$
\begin{aligned}
&= \lim_{h\to 0} h^{-1}(\mathrm{A}^{\hat{u}}_h(\lambda_\cdot F(\rho_\cdot(X^{\hat{u}}_T - X^{\hat{u}}_\cdot e^{r(T-\cdot)})))(t,x)\\
&= \lim_{h\to 0} \lambda_{t+h} h^{-1}[F(-\int_{t+h}^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t+h,T],\mathrm{d}s)})\\
&\qquad - F(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)})]\\
&\quad - \lim_{h\to 0} h^{-1}(\lambda_t - \lambda_{t+h}) F(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)})\\
&= \lambda_t F'(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)})\\
&\qquad \cdot [m_t(\hat{u}) - \varrho^{\hat{L}_1\,\alpha} w_t(\hat{u})(\int_t^T |^\alpha w_s(\hat{u})|^\alpha \mathrm{d}s)^{\frac{1}{\alpha}-1}]\\
&\qquad + \lambda_t' F(-\int_t^T m_s(\hat{u})\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(\hat{u})\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}).
\end{aligned}
$$

The chain rule may be used since F is in C^1 and the optimal control function and the other functions are continuous. □

Lemma B.3. *Let $k \in \mathbb{N}$, $u \in (0,M)^k$, $\alpha, q > 0$, $K_1, K_3 \geq 0$, $K_2 > 0$, and $K_4 \in \mathbb{R}$. Now, consider $f = (f_1, \dots, f_k)$ defined by: $f_i(u) := (u_1^\alpha + \cdots + u_k^\alpha + K_1)K_2 u_i^q + K_3 u_i^q + K_4$. Then the Jacobi matrix J_f is invertible.*

Proof. It holds for $i \neq j$ that $\frac{\partial f_i}{\partial u_j} = \alpha u_j^{\alpha-1} K_2 u_i^q$, and

$$
\frac{\partial f_i}{\partial u_i} = \alpha u_i^{\alpha-1} K_2 u_i^q + q(u_1^\alpha + \cdots + u_k^\alpha + K_1)K_2 u_i^{q-1} + K_3 q u_i^{q-1}.
$$

Define $D = (d_1, \dots, d_k)$ as the invertible diagonal matrix with positive diagonal values $d_i = q(u_1^\alpha + \cdots + u_k^\alpha + K_1)K_2 u_i^{q-1} + K_3 q u_i^{q-1} > 0$ for all $i \in \{1, \dots, k\}$ (where for simplicity we only specify the non-zero values) and the matrix $M = (m_{ij})_{ij}$ with $m_{ij} = \alpha u_j^{\alpha-1} K_2 u_i^q$ for all $i, j \in \{1, \dots, k\}$. Then, $J_f = M + D$. Note that $M = \alpha K_2 v w^\intercal$ where v and w are two column vectors with positive entries $v_i = u_i^{\alpha-1}$ and $w_i = u_i^q$ for $i \in \{1, \dots, k\}$ and $vw^\intercal$ is the outer product. Now, the Matrix determinant lemma resp. the Sherman–Morrison formula implies that J_f is also invertible. □

Proof of Proposition 5.10. For the first claim, we note that $\rho_{t,x}(X_T^u - xe^{r(T-t)})$ does not depend on x since we know from Proposition 4.11 that $\rho_{t,x}(X_T^u - xe^{r(T-t)}) = -\int_t^T m_s(u)\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(u)\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}$. Moreover, $\rho_{t,x}(X_T^u - xe^{r(T-t)})$ is continuously differentiable in t if u is continuous, so it follows that $\frac{\partial}{\partial t}\rho_{t,x}(X_T^u - xe^{r(T-t)}) = m_t(u) - \varrho^{L_1^1\,\alpha} w_t(u)(\alpha(\int_t^T {}^\alpha w_s(u)\mathrm{d}s)^{1-\frac{1}{\alpha}})^{-1}$. Hence, we get:

$$
\begin{aligned}
\frac{\partial}{\partial t} F(\rho_{t,x}(X_T^u - xe^{r(T-t)})) =& (m_t(u) - \varrho^{L_1^1\,\alpha} w_t(u)(\alpha(\int_t^T {}^\alpha w_s(u)\mathrm{d}s)^{1-\frac{1}{\alpha}})^{-1})\\
&\cdot F'(-\int_t^T m_s(u)\mathrm{d}s + \varrho^{\hat{L}_1}\|^\alpha w_s(u)\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}).
\end{aligned}
$$

By definition, we know that F is in C^1, and m and w are continuous. In addition, we know that u is continuous by assumption, and $\lambda_t \in C^1$ by Assumption 7. Hence, we get the first claim.

For proving the second claim, we have from (4.3) and Lemmas 4.9 and 4.10 used as in the proof of Proposition 4.11:

$$
X_T^u - xe^{r(T-t)} = \sum_{i=1}^d \int_t^T u_s^i(\mu_s^i - r)e^{r(T-s)}\mathrm{d}s + \sum_{i,j=1}^d \int_{t+}^T u_s^i \sigma_s^{ij} e^{r(T-s)}\mathrm{d}L_s^j \overset{d}{=} a_t + b_t \hat{L}_1,
$$

where $a_t := \int_t^T u_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s$, $b_t := \|^\alpha w_s(u)\|_{\mathcal{L}^\alpha((t,T],\mathrm{d}s)}$, and m, w, and $\hat{L}$ are given in Definition 4.8.

First, we notice again that $X_T^u - xe^{r(T-t)}$ does not depend on x, and hence, of course, $X_T^u - xe^{r(T-t)} \in C^\infty$ and $\mathbb{E}_{t,x}[\mathcal{T}(X_T^u - xe^{r(T-t)})] \in C^\infty$ as functions of x. Furthermore, we get with u being deterministic by assumption, the cdf $F_{\hat{L}_1}$ of $\hat{L}_1$ and Assumptions 1 and 6:

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathbb{E}_{t,x}[\mathcal{T}(X_T^u - xe^{r(T-t)})] &= \frac{\partial}{\partial t}\mathbb{E}[\mathcal{T}(a_t + b_t\hat{L}_1)]\\
&= \frac{\partial}{\partial t}\int_{-\infty}^\infty \mathcal{T}(a_t + b_t y)\mathrm{d}F_{\hat{L}_1}(y)\\
&= a_t' \cdot \int_{-\infty}^\infty \mathcal{T}'(a_t + b_t y)\mathrm{d}F_{\hat{L}_1}(y) + b_t' \cdot \int_{-\infty}^\infty y\mathcal{T}'(a_t + b_t y)\mathrm{d}F_{\hat{L}_1}(y)\\
&= a_t'\mathbb{E}[\mathcal{T}'(a_t + b_t\hat{L}_1)] + b_t'\mathbb{E}[\hat{L}_1\mathcal{T}'(a_t + b_t\hat{L}_1)].
\end{aligned}
$$

Since u is continuous by assumption and $b_t \geq 0$ with $b_t \equiv 0$ if and only if $u_t \equiv 0$ by definition, we can conclude that $a_t, b_t \in C^1$. Moreover, $\mathcal{T} \in C^1$ and $\mathcal{T}$ and $\mathcal{T}'$ are monotone by the model setup. Hence, Assumptions 1 and 6 allow us to apply Lebesgue's dominated convergence theorem, which then yields that $\mathbb{E}[\mathcal{T}'(a_t + b_t\hat{L}_1)]$ resp. $\mathbb{E}[\hat{L}_1\mathcal{T}'(a_t + b_t\hat{L}_1)]$ is continuous in t which implies the second claim. □

Proof of Proposition 6.1. Similar steps as in the proof of Proposition 4.11, (4.3) lead to

$$
X_T^{\hat{u}} - X_n^{\hat{u}} e^{r(T-n)} \sim \mathcal{N}(\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s, \|\hat{u}_s^\intercal \sigma_s e^{r(T-s)}\|^2_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)}).
$$

Thus: $e^{-\gamma(X_T^{\hat{u}} - X_n^{\hat{u}} e^{r(T-n)})} \sim \mathcal{LN}(-\gamma\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s, \gamma^2\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|^2_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)})$. Hence, with the formula for the expected value of log-normal distributions, we get:

$$
\begin{aligned}
\mathbb{E}_{n,x}[\mathcal{T}(X_T^{\hat{u}} - xe^{r(T-n)})] = c^{-1}(1 - \exp(-\gamma\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s\\
+ \frac{\gamma^2}{2}\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|^2_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)})).
\end{aligned}
$$

With this and Proposition 4.11, the functional J_n from Eq. (4.1) becomes:

$$
\begin{aligned}
J_n(\hat{u}) =& c^{-1}(1 - \exp(-\gamma\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}ds + \frac{\gamma^2}{2}\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|^2_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)}))\\
&- \lambda_n F(-\int_n^T m_s(\hat{u})\mathrm{d}s + \varrho^{W_1^1}\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)}).
\end{aligned}
$$

Taking the partial derivative for $\hat{u}_{n+1}^k$ and setting it equal to 0 gives the claim. □

Proof of Proposition 6.4. We prove this proposition by backward induction. To avoid repetition, we only show the induction step for arbitrary n:

We get the optimum by setting the derivative equal to 0. From Proposition 6.1, we get the partial derivative:

$$
\begin{aligned}
0 \overset{!}{=}& \frac{\partial}{\partial \hat{u}_{n+1}^k} J_n(\hat{u})\\
=& -c^{-1}\exp(-\gamma\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s + \frac{\gamma^2}{2}\|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}\|^2_{\mathcal{L}^2((n,T],\mathrm{d}s;H_s)})\\
&\cdot(-\gamma\int_n^{n+1}(\mu_s^k - r)e^{r(T-s)}\mathrm{d}s + \gamma^2\int_n^{n+1}\langle\hat{u}_s^\intercal\sigma_s e^{r(T-s)}, \sigma_s^{k\cdot}e^{r(T-s)}\rangle_{R_s}\mathrm{d}s)\\
\Leftrightarrow\ \gamma\int_n^{n+1}(\mu_s^k - r)e^{r(T-s)}\mathrm{d}s =& \gamma^2\sum_{i=1}^d \hat{u}_{n+1}^i\int_n^{n+1}\langle\sigma_s^{i\cdot}e^{r(T-s)}, \sigma_s^{k\cdot}e^{r(T-s)}\rangle_{R_s}\mathrm{d}s,
\end{aligned}
\tag{B.2}
$$

because of the assumption that $\hat{u}$ is piecewise constant in each component. Solving (B.2) for $\hat{u}_{n+1}^k$ gives us the claim. □

Proof of Proposition 6.6. As above:

$$
X_T^{\hat{u}} - X_n^{\hat{u}} e^{r(T-n)} \overset{d}{=} \int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s + \|\hat{u}_s^\intercal\sigma_s v e^{r(T-s)}\|^\alpha_{\mathcal{L}^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}v))}\tilde{L}_1.
$$

Hence, it holds when using the Lagrange function of $\tilde{L}_1$ in the second step (see Janson, 2011, p.9):

$$
\begin{aligned}
\mathbb{E}[e^{-\gamma(X_T^{\hat{u}} - X_n^{\hat{u}} e^{r(T-n)})}] &= e^{-\gamma\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s}\mathbb{E}[e^{-\gamma\|\hat{u}_s^\intercal\sigma_s v e^{r(T-s)}\|^\alpha_{\mathcal{L}^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}v))}\tilde{L}_1}]\\
&= e^{-\gamma\int_n^T \hat{u}_s^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s} e^{\frac{c_\alpha\gamma^\alpha}{-\cos(\frac{\pi\alpha}{2})}\|\hat{u}_s^\intercal\sigma_s v e^{r(T-s)}\|^\alpha_{\mathcal{L}^\alpha((n,T]\times\mathbb{S}^d,\mathrm{d}s\times\tilde{\sigma}(\mathrm{d}v))}}.
\end{aligned}
$$

Thus, plugging this result into (4.1) for $\mathcal{T}$ being the exponential target function, using Proposition 4.11, taking the partial derivative for $\hat{u}_{n+1}^k$ and setting it equal to 0 gives the claim. □

Proof of Proposition 6.8. Let s, $\mathcal{T}$, F, and ρ as in the description before the proposition. Then J_n given by (4.1) leads to $J_n(u) = (\mu - r)\int_n^T u_s e^{r(T-s)}\mathrm{d}s - \lambda F(\varrho^{\hat{L}_1})\sigma^\eta(\int_n^T |u_s|^\alpha e^{\alpha r(T-s)}\mathrm{d}s)^{\eta/\alpha}$ where we used (4.3) and Proposition 4.11. Moreover, we dropped the dependency on x in the argumentation due to J_n being independent of x. Since u is piecewise constant, we can take the derivative with respect to u_{n+1}. Setting this derivative equal to 0 to find the maximum gives us the claim. □

References

Adam, A., Houkari, M., & Laurent, J.-P. (2008). Spectral risk measures and portfolio selection. *Journal of Banking & Finance*, *32*(9), 1870–1882.

Alexander, G. J., & Baptista, A. M. (2004). A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model. *Management Science*, *50*(9), 1261–1273.

Applebaum, D. (2009). *Lévy processes and stochastic calculus*. Cambridge University Press.

Balter, A. G., Mahayni, A., & Schweizer, N. (2021). Time-consistency of optimal investment under smooth ambiguity. *European Journal of Operational Research*, *293*(2), 643–657.

Balter, A. G., & Pelsser, A. (2021). Quantifying ambiguity bounds via time-consistent sets of indistinguishable models. *Systems & Control Letters*, *149*, Article 104877.

Basak, S., & Chabakauri, G. (2010). Dynamic mean-variance asset allocation. *The Review of Financial Studies*, *23*(8), 2970–3016.

Basak, S., & Shapiro, A. (2001). Value-at-risk-based risk management: optimal policies and asset prices. *The Review of Financial Studies*, *14*(2), 371–405.

BCBS (2010). *Basel III: A global regulatory framework for more resilient banks and banking systems*: *Technical report*, BCBS (Basel Committee on Banking Supervision).

Bellini, F., Laeven, R. J., & Rosazza Gianin, E. (2018). Robust return risk measures. *Mathematics and Financial Economics*, *12*, 5–32.

Bensoussan, A., Wong, K. C., & Yam, S. C. P. (2019). A paradox in time-consistency in the mean–variance problem? *Finance and Stochastics*, *23*, 173–207.

Bensoussan, A., Wong, K. C., Yam, S. C. P., & Yung, S.-P. (2014). Time-consistent portfolio selection under short-selling prohibition: From discrete to continuous setting. *SIAM Journal on Financial Mathematics*, *5*(1), 153–190.

Björk, T., Khapko, M., & Murgoci, A. (2017). On time-inconsistent stochastic control in continuous time. *Finance and Stochastics*, *21*, 331–360.

Björk, T., & Murgoci, A. (2010). A general theory of Markovian time inconsistent stochastic control problems. Available at SSRN 1694759.

Björk, T., Murgoci, A., & Zhou, X. Y. (2014). Mean–variance portfolio optimization with state-dependent risk aversion. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, *24*(1), 1–24.

Bosserhoff, F., & Stadje, M. (2021). Time-consistent mean-variance investment with unit linked life insurance contracts in a jump-diffusion setting. *Insurance: Mathematics & Economics*, *100*, 130–146.

Brandt, M. W. (2010). Portfolio choice problems. In *Handbook of financial econometrics: tools and techniques* (pp. 269–336). Elsevier.

Campbell, R., Huisman, R., & Koedijk, K. (2001). Optimal portfolio selection in a value-at-risk framework. *Journal of Banking & Finance*, *25*(9), 1789–1804.

Černý, A., Czichowsky, C., & Kallsen, J. (2023). Numeraire-invariant quadratic hedging and mean–variance portfolio allocation. *Mathematics of Operations Research*, *49*(2), 752–781.

Cheridito, P., & Stadje, M. (2009). Time-inconsistency of VaR and time-consistent alternatives. *Finance Research Letters*, *6*(1), 40–46.

Chiu, M. C., & Wong, H. Y. (2015). Dynamic cointegrated pairs trading: Mean–variance time-consistent strategies. *Journal of Computational and Applied Mathematics*, *290*, 516–534.

Çınlar, E. (2011). *Probability and stochastics*: *vol. 261*, Springer.

Cont, R., & Tankov, P. (2004). *Financial modelling with jump processes*. Chapman and Hall/CRC.

Cui, X., Gao, J., Shi, Y., & Zhu, S. (2019). Time-consistent and self-coordination strategies for multi-period mean-conditional value-at-risk portfolio selection. *European Journal of Operational Research*, *276*(2), 781–789.

Cui, X., Li, D., Wang, S., & Zhu, S. (2012). Better than dynamic mean-variance: Time inconsistency and free cash flow stream. *Mathematical Finance: an International Journal of Mathematics, Statistics and Financial Economics*, *22*(2), 346–378.

Czichowsky, C. (2013). Time-consistent mean-variance portfolio selection in discrete and continuous time. *Finance and Stochastics*, *17*, 227–271.

EIOPA (2014). *Technical specifications for the Solvency II preparatory phase - Part I*: *Technical report*, EIOPA.

Ekeland, I., & Lazrak, A. (2006). Being serious about non-commitment: subgame perfect equilibrium in continuous time. arXiv preprint math/0604264.

Ekeland, I., & Pirvu, T. A. (2008). Investment and consumption without commitment. *Mathematics and Financial Economics*, *2*(1), 57–86.

Fama, E. F. (1965). The behavior of stock-market prices. *Journal of Business*, *38*(1), 34–105.

Feng, L., & Linetsky, V. (2008). Pricing options in jump-diffusion models: an extrapolation approach. *Operations Research*, *56*(2), 304–325.

Föllmer, H., & Schied, A. (2016). *Stochastic finance*: *vol. 4*, De Gruyter Graduate.

Forsyth, P. A. (2020). Multiperiod mean conditional value at risk asset allocation: Is it advantageous to be time consistent? *SIAM Journal on Financial Mathematics*, *11*(2), 358–384.

Fu, C., Lari-Lavassani, A., & Li, X. (2010). Dynamic mean–variance portfolio selection with borrowing constraint. *European Journal of Operational Research*, *200*(1), 312–319.

Fu, M. C., Li, B., Li, G., & Wu, R. (2017). Option pricing for a jump-diffusion model with general discrete jump-size distributions. *Management Science*, *63*(11), 3961–3977.

Fu, J., & Yang, H. (2012). Equilibruim approach of asset pricing under Lévy process. *European Journal of Operational Research*, *223*(3), 701–708.

Gülpınar, N., & Rustem, B. (2007). Worst-case robust decisions for multi-period mean–variance portfolio optimization. *European Journal of Operational Research*, *183*(3), 981–1000.

He, X. D., Jin, H., & Zhou, X. Y. (2015). Dynamic portfolio choice when risk is measured by weighted VaR. *Mathematics of Operations Research*, *40*(3), 773–796.

Hernández, C., & Possamaï, D. (2023). Time-inconsistent contract theory. arXiv preprint arXiv:2303.01601.

Hu, Y., Jin, H., & Zhou, X. Y. (2012). Time-inconsistent stochastic linear–quadratic control. *SIAM Journal on Control and Optimization*, *50*(3), 1548–1572.

Janson, S. (2011). Stable distributions. arXiv preprint arXiv:1112.0220.

Jin, H., Markowitz, H., & Yu Zhou, X. (2006). A note on semivariance. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, *16*(1), 53–61.

Jorion, P. (2007). *Value at risk: the new benchmark for managing financial risk*. The McGraw-Hill Companies, Inc.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometria*, *47*(2), 263–292.

Kaishev, V. K., & Dimitrova, D. S. (2009). Dirichlet bridge sampling for the variance gamma process: pricing path-dependent options. *Management Science*, *55*(3), 483–496.

Kallsen, J. (2000). Optimal portfolios for exponential Lévy processes. *Mathematical Methods of Operations Research*, *51*, 357–374.

Kardaras, C. (2009). No-free-lunch equivalences for exponential Lévy models under convex constraints on investment. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, *19*(2), 161–187.

Laeven, R. J., & Stadje, M. (2014). Robust portfolio choice and indifference valuation. *Mathematics of Operations Research*, *39*(4), 1109–1141.

Li, D., & Ng, W.-L. (2000). Optimal dynamic portfolio selection: Multiperiod mean-variance formulation. *Mathematical Finance*, *10*(3), 387–406.

Lim, A. E., & Zhou, X. Y. (2002). Mean-variance portfolio selection with random parameters in a complete market. *Mathematics of Operations Research*, *27*(1), 101–120.

Lin, G. D., & Hu, C.-Y. (2018). Absolute moments in terms of characteristic functions. arXiv preprint arXiv:1809.07460.

Lindensjö, K. (2019). A regular equilibrium solves the extended HJB system. *Operations Research Letters*, *47*(5), 427–432.

Ma, J., Wong, T.-K. L., & Zhang, J. (2021). Time-consistent conditional expectation under probability distortion. *Mathematics of Operations Research*, *46*(3), 1149–1180.

Mandelbrot, B. B. (1963). The variation of certain speculative prices. *Journal of Business*, *36*(4), 394–419.

Mandelbrot, B. B. (1967). The variation of some other speculative prices. *Journal of Business*, *40*(4), 393–413.

Markowitz, H. (1952). Portfolio selection. *Journal of Finance*, *7*, 77–97.

Misiorek, A., & Weron, R. (2012). Heavy-tailed distributions in VaR calculations. In *Handbook of Computational Statistics: Concepts and Methods* (pp. 1025–1059). Springer.

Nowak, P., & Romaniuk, M. (2010). Computing option price for levy process with fuzzy parameters. *European Journal of Operational Research*, *201*(1), 206–210.

Pedersen, J. L., & Peskir, G. (2016). Optimal mean–variance selling strategies. *Mathematics and Financial Economics*, *10*, 203–220.

Pirvu, T. A., & Schulze, K. (2012). Multi-stock portfolio optimization under prospect theory. *Mathematics and Financial Economics*, *6*, 337–362.

Pistorius, M., & Stadje, M. (2017). On dynamic deviation measures and continuous-time portfolio optimization. *27*, (6), (pp. 3342 – 3384).

Protter, P. E. (2005). *Stochastic integration and differential equations*: *vol. 21*, Springer Science & Business Media.

Rockafellar, R. T. (1972). *Convex analysis*. Princeton University Press.

Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, *26*(7), 1443–1471.

Rockafellar, R. T., Uryasev, S. P., & Zabarankin, M. (2002). Deviation measures in risk analysis and optimization. University of Florida, Department of Industrial & Systems Engineering Working Paper, 2002-7.

Ruszczynski, A., & Vanderbei, R. J. (2003). Frontiers of stochastically nondominated portfolios. *Econometrica*, *71*(4), 1287–1297.

Shiryaev, A. N. (2007). *Optimal stopping rules*: *vol. 8*, Springer Science & Business Media.

Skintzi, V. D., & Refenes, A.-P. N. (2005). Implied correlation index: A new measure of diversification. *Journal of Futures Markets: Futures, Options, and Other Derivative Products*, *25*(2), 171–197.

Stadje, M. (2020). Two results on dynamic extensions of deviation measures. *Journal of Applied Probability*, *57*(2), 531–540.

Strotz, R. H. (1955). Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies*, *23*(3), 165–180.

Van Staden, P. M., Dang, D.-M., & Forsyth, P. A. (2018). Time-consistent mean–variance portfolio optimization: A numerical impulse control approach. *Insurance: Mathematics & Economics*, *83*, 9–28.

Vigna, E., et al. (2016). On time consistency for mean-variance portfolio selection. In *Collegio Carlo Alberto Notebook: vol. 476*.

Wang, J., & Forsyth, P. A. (2011). Continuous time mean variance asset allocation: A time-consistent strategy. *European Journal of Operational Research*, *209*(2), 184–201.

Wei, J., & Wang, T. (2017). Time-consistent mean–variance asset–liability management with random coefficients. *Insurance: Mathematics & Economics*, *77*, 84–96.

Wei, P., & Xu, Z. Q. (2021). Dynamic growth-optimum portfolio choice under risk control. arXiv preprint arXiv:2112.14451.

Winkelbauer, A. (2012). Moments and absolute moments of the normal distribution. arXiv preprint arXiv:1209.4340.

Wu, H., & Chen, H. (2015). Nash equilibrium strategy for a multi-period mean–variance portfolio selection problem with regime switching. *Economic Modelling*, *46*, 79–90.

Xie, S., Li, Z., & Wang, S. (2008). Continuous-time portfolio selection with liability: Mean–variance model and stochastic LQ approach. *Insurance: Mathematics & Economics*, *42*(3), 943–953.

Zălinescu, C. (2002). *Convex analysis in general vector spaces*. World scientific.

Zeng, Y., & Li, Z. (2011). Optimal time-consistent investment and reinsurance policies for mean-variance insurers. *Insurance: Mathematics & Economics*, *49*(1), 145–154.

Zhao, Y. (2007). A dynamic model of active portfolio management with benchmark orientation. *Journal of Banking & Finance*, *31*(11), 3336–3356.

Zhou, X. Y., & Li, D. (2000). Continuous-time mean-variance portfolio selection: A stochastic lq framework. *Applied Mathematics and Optimization*, *42*(1), 19–33.

2 The $C^{0,1}$ Itô-Ventzell formula for weak Dirichlet processes

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The $C^{0,1}$ Itô-Ventzell formula for weak Dirichlet processes

Felix Fießinger* Mitja Stadje*

Abstract

This paper proves an extension of the Itô-Ventzell formula that applies to stochastic flows in $C^{0,1}$ for continuous weak Dirichlet processes. We apply this theorem, for example, to give a representation result for strong solutions of time-dependent elliptic SPDEs, to derive formulas for quadratic variations, and to relax assumptions in a financial mathematics context.

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1 Introduction

In this paper, we show an extension of the Itô-Ventzell formula for continuous weak Dirichlet processes, such that the underlying stochastic flow only needs to be in C^1 instead of being in C^2 in the spacial argument under an additional mild assumption. The Itô-Ventzell formula is a generalization of the Itô formula for stochastic flows. Specifically, let $X = M + A$ be a continuous semimartingale, where M is a local martingale and A is a process of finite variation, and let $F(t,x)$ be a stochastic flow with local characteristics $(\beta(t,x), \gamma(t,x))$, i.e., $F(t,x)$ is a random field and admits the Itô dynamics

$$F(t,x) = F(0,x) + \int_0^t \beta(r,x)\mathrm{d}r + \int_0^t \gamma(r,x)\mathrm{d}W_r,$$

where W is a Brownian Motion. Moreover, we assume that F is a continuous progressively measurable process for fixed x and twice differentiable in x for fixed t and ω. Note that the dependence of F, β, and γ on ω is suppressed. Then the Itô-Ventzell formula states that $F(t,X_t)$ is also a continuous semimartingale and has the following decomposition:

$$F(t,X_t) = F(0,X_0) + \int_0^t \gamma(s,X_s)\mathrm{d}W_s + \int_0^t F_x(s,X_s)\mathrm{d}M_s + \mathcal{B}^X(F)_t,$$

where

$$\mathcal{B}^X(F)_t = \int_0^t \beta(s,X_s)\mathrm{d}s + \int_0^t F_x(s,X_s)\mathrm{d}A_s + \frac{1}{2}\int_0^t F_{xx}(s,X_s)\mathrm{d}[X]_s + \int_0^t \gamma_x(s,X_s)\mathrm{d}[X,W]_s.$$

*Institute of Insurance Science & Institute of Mathematical Finance, Faculty of Mathematics and Economics, Ulm University, Ulm, Germany. E-mail: felix.fiessinger@uni-ulm.de,mitja.stadje@uni-ulm.de

$C^{0,1}$ Itô-Ventzell formula

The Itô-Ventzell formula links stochastic processes to stochastic partial differential equations (SPDEs) and has various applications. It is used, for instance, to analyze various optimal control problems, to determine localization errors of SPDEs, for generalized stochastic forward integrals, for forward performance processes, and to develop pricing models. We apply our new version of the Itô-Ventzell formula to derive stochastic calculus formulas and to obtain a representation result for strong solutions of time-dependent stochastic elliptic PDEs. Here, we generalize naturally the solution theory of SPDEs to the setting of the paper. Moreover, we apply our result in a financial mathematics context to obtain the dynamics of the portfolio of a large investor whose trading influences prices.

Multiple works generalize the classical Itô formula in different directions, see, for instance, Föllmer et al. [12], Elworthy et al. [9] or da Prato et al. [7]. We focus on the extension discovered by Ventzell [26] in 1965, the Itô-Ventzell formula, which was generalized later by Kunita [17]. Since then, multiple works extended the Itô-Ventzell formula, see, for instance, Flandoli and Russo [11], Krylov [16], Karachanskaya [14], Kohlmann et al. [15], or dos Reis and Platonov [8].

Our results hold for weak Dirichlet processes, the sum of a local martingale and an adapted process with zero covariation with all continuous local martingales, including semimartingales as a special case. Since the classical theorem of stochastic integration is defined only with respect to semimartingales, we use an extension of stochastic integration developed by Russo and Vallois [20, 21, 22, 23, 24], Coquet et al. [6], and Bandini and Russo [1, 2]. Considering weak Dirichlet processes is not unusual in generalizations of Itô's-formula which require milder smoothness conditions on the function F, see Gozzi and Russo [13] who generalized the classical Itô formula to $C^{0,1}$, Bouchard et al. [4] for the functional Itô formula, or Bouchard and Vallet [5] for the Itô-Dupire formula.

The paper is structured as follows: In Section 2, we introduce weak Dirichlet processes, generalized stochastic integration, and stochastic flows. Section 3 gives our result, namely the extension of the Itô-Ventzell formula to stochastic flows in $C^{0,1}$, while the proof of this extension is shown in Section 4. Section 5 gives some applications of our extended version of the Itô-Ventzell formula.

2 Basic definitions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis where the filtration $\mathcal{F}$ is generated by a Brownian Motion W satisfying the usual conditions and $[0,T]$ is the observation period. We define $f_x(t,x)$ to be the partial derivative of f in the second argument and denote higher derivatives similarly. In this paper, we use the space of progressively measurable and continuous processes equipped with the topology of uniform convergence on compacts in probability. We denote by $C^{n,m+\alpha}$ the set of functions which are n-times continuously differentiable in the first and m-times in the second argument where the mth derivative is $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-a.s. locally α-Hölder continuous with $\alpha>0$. If the mth derivative is just jointly continuous, we simply write $C^{n,m}$, and $C^{0,0}$ denotes the set of all jointly continuous functions. Furthermore, we denote by $\mathcal{L}^1(\mu)$ (resp. $\mathcal{L}^2(\mu)$) the space of integrable (resp. square integrable) random variables/measurable functions with respect to a measure μ.

Definition 2.1. *The sequence of progressively measurable processes (H^n) converges uniformly on compacts in probability (ucp) to a process H if $\sup_{s\in[0,t]}|H^n_s - H_s| \xrightarrow{n\to\infty} 0$ in probability for all $t>0$.*

Our processes may not be semimartingales and may not admit a quadratic (co-) variation. For this purpose, we need generalizations of these concepts, see Gozzi and

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Russo [13, p.1567]:

Definition 2.2. *Let X and Y be two continuous processes of the same dimension. Then, we define for all $t \in [0,T]$:*

$$[X,Y]_t = \lim_{\varepsilon \downarrow 0} \int_0^t \tfrac{(X_{r+\varepsilon}-X_r)(Y_{r+\varepsilon}-Y_r)}{\varepsilon} \mathrm{d}r,$$
$$\int_0^t X_r \mathrm{d}^- Y_r = \lim_{\varepsilon \downarrow 0} \int_0^t X_r \tfrac{Y_{r+\varepsilon}-Y_r}{\varepsilon} \mathrm{d}r,$$

if those limits exist in the sense of ucp-convergence. We call $[X,Y]_t$ the covariation, and $\int_0^t X_r \mathrm{d}^- Y_r$ the forward integral. We use the short notation for the covariation: $[X]_t := [X,X]_t$.

Definition 2.2 ensures that the covariation and the forward integral are again continuous processes and that the covariation is bilinear and symmetric. In particular, the (forward) integral against the covariation is well-defined. For a more detailed explanation of the covariation and the integral against it, see Russo and Vallois [22, pp. 83-97].

Remark 2.3. Due to Definition 2.2, the functions inside of the covariation and the integral need to be defined on $[0, T+\varepsilon]$ instead of $[0,T]$. Consequently, we use the convention that these processes or functions are constantly continued on $(T, T+\varepsilon]$, e.g., we define $X_r = X_T$ for all $r > T$.

Definition 2.2 ensures that the covariation and the forward integral are continuous. If X and Y are semimartingales, then the concepts above coincide with the quadratic covariation and the usual notion of a stochastic integral. While semimartingales are of finite quadratic variation, this is, of course, not true for general continuous processes. Hence, we define the following:

Definition 2.4. *Let X be a continuous and adapted process. If $[X]_t < \infty$ for all $t \in [0,T]$, then we call X a finite quadratic variation (FQV) process.*

The following definition is standard, see, for instance, Russo and Vallois [20, 21, 22, 23, 24], Coquet et al. [6], Bandini and Russo [1, 2] or Gozzi and Russo [13, p.1570]:

Definition 2.5. *A real continuous and adapted process X is called a weak Dirichlet process if X is decomposable into a local martingale M and a weak zero energy process A, i.e., $X = M + A$.*

For the process A, it holds: $A_0 = 0$ and $[A,N]_t = 0$ for all continuous local martingales N and $t \in [0,T]$.

In general, this decomposition is not unique. However, when M is required to be continuous, there exists a unique decomposition. In the following, we always use the latter decomposition. In particular, A is continuous, since by assumption X and M are. If X is additionally a semimartingale, A is a continuous finite variation process.

Example 2.6. The following processes are weak-Dirichlet processes according to Gozzi and Russo [13, pp.1570-1571], and Errami and Russo [10, p.267]:

- all semimartingales,
- all Dirichlet processes X, i.e., X is decomposable into a local martingale and a zero quadratic variation process,
- the process $X_t = \int_0^t G(t,r) \mathrm{d}M_r$, where $G : \{0 \le r \le t \le 1\} \to \mathbb{R}$ is a continuous $(\mathcal{F}_r)$-measurable random field and M is a local continuous $(\mathcal{F}_t)$-martingale.

Contrary to the classical Itô formula, the Itô-Ventzell formula considers stochastic flows, which are defined in the following definition. Similar, but slightly different definitions can be found, e.g., in Kohlmann et al. [15, p.7] or dos Reis and Platonov [8, p.6].

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Definition 2.7. *Let $F(t,x) = F(\omega,t,x) : \Omega \times [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be a stochastic flow with local characteristics $(\beta(t,x),\gamma(t,x))$, i.e., F is a random field which admits the Itô dynamics*

$$F(t,x) = F(0,x) + \int_0^t \beta(r,x)\mathrm{d}r + \int_0^t \gamma(r,x)\mathrm{d}W_r, \tag{2.1}$$

where

(a) W is the Brownian motion which generates the filtration $\mathcal{F}$,

(b) $F(\cdot,x)$ is a progressively measurable process for all fixed $x \in \mathbb{R}^d$,

(c) $\beta : \Omega \times \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$ and $\gamma : \Omega \times \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d$ are previsible random fields,

(d) $\gamma(t,x)$ is $\mathrm{d}\mathbb{P} \times \mathrm{d}t$-a.s. locally α-Hölder continuous in x for an $\alpha > 0$, and

(e) for any $K \subset \mathbb{R}^d$ compact and for any $S \geq 0$ it holds a.s. that $\int_0^S \sup_{x\in K} |\beta(t,x)|\mathrm{d}t < \infty$, and $\int_0^S \sup_{x\in K} |\gamma(t,x)|^2\mathrm{d}t < \infty$.

The dependence on ω will be suppressed throughout this paper unless it is necessary for the steps of a proof.

Remark 2.8. The local α-Hölder continuity of γ is necessary to ensure through Kolmogorov's continuity theorem that the stochastic integral $\int_0^t \gamma(r,x)\mathrm{d}W_r$ is a.s. well-defined for all (t,x). We refer to Kunita [17] for details.

3 Main result

The following result states the Itô-Ventzell formula for stochastic flows in $C^{0,1}$:

Theorem 3.1. *Let X be a continuous weak Dirichlet FQV process and $F \in C^{0,1}$ be a stochastic flow with local characteristics $(\beta(t,x),\gamma(t,x))$. Furthermore, we assume:*

1a $\int_0^T \sup_{s\in[0,T]} |\beta(s,X_t)|\mathrm{d}t < \infty$ a.s., or

1b there exist a $B_1 > 0$ such that $\int_0^T \sup_{s\in[0,T]} \mathbb{E}[|\beta(s,X_t)|]\mathrm{d}t \leq B_1$.

2 for all $K \subset \mathbb{R}^d$ compact and $p \geq 1$, we suppose the existence of a $C_p \in \mathbb{R}$ such that for any $x,y \in K$ holds that $\mathbb{E}[\sup_{t\in[0,T]} |\gamma(t,x) - \gamma(t,y)|^p] \leq C_p|x-y|^p$ and $\mathbb{E}[\sup_{t\in[0,T]} |\gamma(t,0)|^p] < \infty$.

3 $\gamma(t,X_t)$ is a.s. càdlàg (right continuous with left limits).

Then it holds for all $t \in [0,T]$:

$$F(t,X_t) = F(0,X_0) + \int_0^t \gamma(r,X_r)\mathrm{d}W_r + \int_0^t F_x(r,X_r)\mathrm{d}M_r + \mathcal{B}^X(F)_t,$$

where $\mathcal{B}^X$ is linear and continuous (in F) with respect to the topology of ucp convergence and $F(t,X_t)$ is again a continuous weak Dirichlet process. Moreover,

(a) $\mathcal{B}^X$ is a weak zero energy process, and

(b) if $F(t,x) \in C^{0,2}$, $\beta(t,x) \in C^{0,1}$, and $\gamma(t,x) \in C^{0,1+\alpha}$ for an $\alpha \in (0,1)$, then it holds:

$$\mathcal{B}^X(F)_t = \int_0^t \beta(r,X_r)\mathrm{d}r + \int_0^t F_x(r,X_r)\mathrm{d}^- A_r + \int_0^t \gamma_x(r,X_r)\mathrm{d}[X,W]_r + \tfrac{1}{2}\int_0^t F_{xx}(r,X_r)\mathrm{d}[X]_r.$$

In the last two integrals of Theorem 3.1 (b) both the integrand and the integrator are matrices. These integrals are, as usual, defined as the trace of the matrix of the corresponding integrals. The assumptions of Theorem 3.1 are satisfied (through Assumption 1a) if the mappings $(t,x) \mapsto \beta(t,x)$ and $(t,x) \mapsto \gamma(t,x)$ are a.s. continuous, $\sup_{t\in[0,T]} |\gamma(t,0)|$ is p-integrable for all $p \in [1,\infty)$ and $x \mapsto \gamma(\cdot,x)$ is a.s. locally Lipschitz continuous with a Lipschitz constant independent of t which is integrable in ω.

The theorem generalizes the result of Gozzi and Russo [13, pp.1571-1574] to stochastic flows.

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. Moreover, assume that $F(\cdot,X)$ is a local martingale. Then, it holds for all $t \in [0,T]$:*

$$F(t,X_t) = F(0,X_0) + \int_0^t \gamma(r,X_r)\mathrm{d}W_r + \int_0^t F_x(r,X_r)\mathrm{d}M_r.$$

Proof. Using Theorem 3.1, we get:

$$\mathcal{B}^X(F)_t = F(t,X_t) - F(0,X_0) - \int_0^t \gamma(r,X_r)\mathrm{d}W_r - \int_0^t F_x(r,X_r)\mathrm{d}M_r.$$

Since $F(\cdot,X)$, M, and W are local martingales, $\mathcal{B}^X(F)$ is a local martingale as well, and Theorem 3.1 states that $\mathcal{B}^X(F)$ is a continuous weak zero energy process. Thus, $[\mathcal{B}^X(F)] = 0$ which implies that $\mathcal{B}^X(F) \equiv 0$. Hence, Theorem 3.1 finalizes the proof. □

Remark 3.3. Theorem 3.1 in Krylov [16] provides a distribution-valued version of the Itô-Ventzell formula. The main difference between his and our version of the Itô-Ventzell formula is that in Theorem 3.1 in Krylov [16], one needs the existence of a second derivative in the space direction (in the distributional sense) of the function F, while we only need the first derivative. On the other hand, Krylov [16] considers possibly infinitely many space directions and does not need joint continuity in time and space directions and the previsibility of the characteristics. Contrary, we need less regularity for the characteristics, i.e., we do not need to assume that β and γ are continuous. Regarding the integrability, our Theorem 3.1 and Theorem 3.1 in Krylov [16] have somewhat similar assumptions.

4 Proof of Theorem 3.1

Before proving the theorem, we need some preliminary results. First of all, we assume $d = 1$ to simplify the notation. In the whole section, we use the convention from Remark 2.3 for the constant continuation of processes and functions.

By recalling Proposition 1.2 of Russo and Vallois [22, p.89], it holds:

Lemma 4.1. *Let X and Y be two FQV processes and H be a càdlàg process defined on the interval $[0,T]$. Then, it holds that $\int_0^t H(r)\frac{(X_{r+\varepsilon}-X_r)(Y_{r+\varepsilon}-Y_r)}{\varepsilon}\mathrm{d}r \xrightarrow[ucp]{\varepsilon\to 0} \int_0^t H(r)\mathrm{d}[X,Y]_r$.*

Lemma 4.2. *Let X, Y be two FQV processes, $t \in [0,T]$. Moreover, let Z^ε be a measurable process for all $\varepsilon > 0$ such that $\sup_{r\in[0,T]} |Z_r^\varepsilon| \xrightarrow[a.s.]{\varepsilon\to 0} 0$. Then, it holds that $\int_0^t Z_r^\varepsilon \frac{(X_{r+\varepsilon}-X_r)(Y_{r+\varepsilon}-Y_r)}{\varepsilon}\mathrm{d}r \xrightarrow[ucp]{\varepsilon\to 0} 0$.*

Proof. First, we note that Z^ε is a.s. bounded for all $\varepsilon > 0$ (with a bound possibly depending on ω), i.e., the integral is well-defined. Then, we get:

$$\begin{aligned}\sup_{t\in[0,T]} \left|\int_0^t Z_r^\varepsilon \tfrac{(X_{r+\varepsilon}-X_r)(Y_{r+\varepsilon}-Y_r)}{\varepsilon}\mathrm{d}r\right| &\le \int_0^T |Z_r^\varepsilon \tfrac{(X_{r+\varepsilon}-X_r)(Y_{r+\varepsilon}-Y_r)}{\varepsilon}|\mathrm{d}r \\ &\le \sup_{r\in[0,T]} |Z_r^\varepsilon| \cdot \int_0^T |\tfrac{(X_{r+\varepsilon}-X_r)(Y_{r+\varepsilon}-Y_r)}{\varepsilon}|\mathrm{d}r,\end{aligned}$$

and the lemma follows from Lemma 4.1. □

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For the following lemma, we need the concept of boundedness in probability, see, e.g., van der Vaart [25]:

Definition 4.3. *Let I be an index set. A set of random variables $(X_\delta)_{\delta\in I}$ is bounded in probability (also known as uniformly tight) if there exists for all $\varepsilon > 0$ an $M > 0$ such that $\sup_{\delta\in I} \mathbb{P}(|X_\delta| > M) < \varepsilon$.*

Note that $\mathcal{L}^1$ boundedness implies via Markov's inequality boundedness in probability.

Lemma 4.4. *Let X be a continuous FQV process, $t \in [0,T]$, and Z^ε be a measurable and a.s. bounded process for all $\varepsilon > 0$ such that $\int_0^T |Z_r^\varepsilon| \mathrm{d}r$ is bounded in probability in $\varepsilon > 0$. Then, it holds that $\int_0^t Z_r^\varepsilon (X_{r+\varepsilon} - X_r)\mathrm{d}r \xrightarrow[ucp]{\varepsilon\to 0} 0$.*

Proof. Let $\omega \in \Omega$ be arbitrary such that $X(\omega)$ is continuous. Thus, $X(\omega)$ is uniformly continuous on $[0,T]$, i.e., $\sup_{r\in[0,T]} |X_{r+\varepsilon}(\omega) - X_r(\omega)| \xrightarrow[a.s.]{\varepsilon\to 0} 0$. Hence, we get:

$$\begin{aligned}\sup_{t\in[0,T]} \left|\int_0^t Z_r^\varepsilon (X_{r+\varepsilon} - X_r)\mathrm{d}r\right| &\le \int_0^T |Z_r^\varepsilon (X_{r+\varepsilon} - X_r)|\mathrm{d}r \\ &\le \int_0^T |Z_r^\varepsilon|\mathrm{d}r \cdot \sup_{r\in[0,T]} |X_{r+\varepsilon}(\omega) - X_r(\omega)| \xrightarrow[\mathbb{P}]{\varepsilon\to 0} 0. \quad \square\end{aligned}$$

Lemma 4.5. *Let X be a FQV process, $t \in [0,T]$, and Z^ε be a measurable and a.s. bounded process for all $\varepsilon > 0$ such that $\int_0^T |Z_r^\varepsilon|^2 \mathrm{d}r \xrightarrow[\mathbb{P}]{\varepsilon\to 0} 0$. Then, it holds that $\int_0^t Z_r^\varepsilon \frac{X_{r+\varepsilon} - X_r}{\sqrt{\varepsilon}} \mathrm{d}r \xrightarrow[ucp]{\varepsilon\to 0} 0$.*

Proof. We get with Cauchy-Schwarz:

$$\sup_{t\in[0,T]} \left|\int_0^t Z_r^\varepsilon \tfrac{X_{r+\varepsilon} - X_r}{\sqrt{\varepsilon}} \mathrm{d}r\right| \le \int_0^T |Z_r^\varepsilon \tfrac{X_{r+\varepsilon} - X_r}{\sqrt{\varepsilon}}| \mathrm{d}r \le \sqrt{\int_0^T |Z_r^\varepsilon|^2 \mathrm{d}r} \sqrt{\int_0^T \tfrac{(X_{r+\varepsilon} - X_r)^2}{\varepsilon} \mathrm{d}r}.$$

The first factor converges in probability to 0 for $\varepsilon \to 0$ due to the assumption of the lemma. The second factor converges in probability to $\sqrt{[X]_T}$ for $\varepsilon \to 0$ by Definition 2.2. $\square$

Lemma 4.6. *In the setting of Theorem 3.1, it holds for all $t \in [r,T]$ and for all $\mathcal{F}_r$-measurable random variables G such that $\int_0^t |\beta(s,G)| \mathrm{d}s < \infty$ a.s. and $\int_0^t |\gamma(s,G)|^2 \mathrm{d}s < \infty$ a.s.:*

$$\int_r^t \beta(s,x)\mathrm{d}s\big|_{x=G} = \int_r^t \beta(s,G)\mathrm{d}s, \qquad \int_r^t \gamma(s,x)\mathrm{d}W_s\big|_{x=G} = \int_r^t \gamma(s,G)\mathrm{d}W_s.$$

Proof. The first property follows directly from the ω-wise definition of the "$\mathrm{d}s$"-integral. The second property is an immediate consequence of Lemma 3.3 in Russo and Vallois [23]. $\square$

Lemma 4.7. *Consider the setting of Theorem 3.1 and assume that X takes only values in a compact set $K \subset \mathbb{R}$. Then Assumption 2 implies that there exist $B_2 > 0$ such that $\sup_{s\in[0,T]} \mathbb{E}[\gamma(s,X_t)^{2+\varrho}] \le B_2$ for all $t \in [0,T]$ and for all $\varrho > 0$.*

Proof. The lemma follows from Assumption 2 in Theorem 3.1. $\square$

Now, we are ready to prove our main result:

Proof of Theorem 3.1. We can write $\mathcal{B}^X$ as

$$\mathcal{B}^X(F)_t = F(t,X_t) - F(0,X_0) - \int_0^t \gamma(r,X_r)\mathrm{d}W_r - \int_0^t F_x(r,X_r)\mathrm{d}M_r.$$

Hence, $\mathcal{B}^X$ is linear and continuous (in F) with respect to the ucp topology. The property that $F(t,X_t)$ is a weak Dirichlet process follows directly from $\mathcal{B}^X$ being a weak zero energy process (which we show later) since M and W are local martingales.

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If $F(t,x) \in C^{0,2}$ and $\gamma(t,x), \beta(t,x) \in C^{0,1}$ with α-Hölder-continuous derivatives in x, then Flandoli and Russo [11, p.279] give us the Itô-Ventzell formula for weak Dirichlet processes:

$$\begin{aligned} F(t,X_t) = F(0,X_0) + \int_0^t \beta(r,X_r)\mathrm{d}r + \int_0^t \gamma(r,X_r)\mathrm{d}W_r + \int_0^t F_x(r,X_r)\mathrm{d}^-X_r \\ + \int_0^t \gamma_x(r,X_r)\mathrm{d}[X,W]_r + \tfrac{1}{2}\int_0^t F_{xx}(r,X_r)\mathrm{d}[X]_r. \end{aligned}$$

Next note that $\int_0^t F_x(r,X_r)\mathrm{d}^-X_r$ exists. Due to $\int_0^t F_x(r,X_r)\mathrm{d}^-M_r = \int_0^t F_x(r,X_r)\mathrm{d}M_r$ coinciding with the classical stochastic integral, $\int_0^t F_x(r,X_r)\mathrm{d}^-A_r$ also exists. Moreover, when carefully scanning the to our setting adapted proof of the Itô-Ventzell formula in Flandoli and Russo [11, pp.279-280], we can reduce the regularity assumptions of β and γ due to Lemma 4.6. Thus, it is sufficient to assume that $\beta(t,x) \in C^{0,1}$ (without Hölder-continuity of $\beta_x(t,x)$ in x) and $\gamma(t,x) \in C^{0,1+\alpha}$ (with only local α-Hölder-continuity of $\gamma_x(t,x)$ in x) in case (b).

Hence, it remains to prove the case (a) where $F \in C^{0,1}$ and the Assumptions 1-3 hold:

First of all, since X is continuous, we can use a classical localization argument and assume that there exists a compact set $K \subset \mathbb{R}$ such that $X_t(\omega) \in K$ for all $t \in [0,T]$ and almost all $\omega \in \Omega$. Then, by using again classical localization arguments, we can assume that $\sup_{x\in K}|\beta(t,x)| \in \mathcal{L}^1(\Omega\times[0,T],\mathrm{d}\mathbb{P}\times\mathrm{d}t)$, $\sup_{x\in K}|\gamma(t,x)| \in \mathcal{L}^2(\Omega\times[0,T],\mathrm{d}\mathbb{P}\times\mathrm{d}t)$ both from Definition 2.7(e). Now, the prerequisites of Lemma 4.6 are fulfilled, and thus all of the (stochastic) integrals in this proof are well-defined.

Let N be a continuous local martingale. Since N, M, and W are semimartingales, we know:

$$\begin{aligned} [\textstyle\int_0^\cdot F_x(r,X_r)\mathrm{d}M_r, N] &= \textstyle\int_0^\cdot F_x(r,X_r)\mathrm{d}[M,N]_r, \\ [\textstyle\int_0^\cdot \gamma(r,X_r)\mathrm{d}W_r, N] &= \textstyle\int_0^\cdot \gamma(r,X_r)\mathrm{d}[W,N]_r. \end{aligned}$$

Hence, it remains to show that $[F(\cdot,X),N] = \int_0^\cdot F_x(r,X_r)\mathrm{d}[M,N]_r + \int_0^\cdot \gamma(r,X_r)\mathrm{d}[W,N]_r$. Using Definition 2.2, we consider the *ucp* limit of

$$\begin{aligned} &\textstyle\int_0^t [F(r+\varepsilon,X_{r+\varepsilon}) - F(r,X_r)]\frac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r \\ &\quad = \textstyle\int_0^t [F(r+\varepsilon,X_{r+\varepsilon}) - F(r+\varepsilon,X_r)]\frac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r + \int_0^t [F(r+\varepsilon,X_r) - F(r,X_r)]\frac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r \\ &\quad =: I_1(t,\varepsilon) + I_2(t,\varepsilon). \end{aligned}$$

We first show: $\lim_{\varepsilon\to 0} I_1(t,\varepsilon) = \int_0^t F_x(r,X_r)\mathrm{d}[M,N]_r$. It is

$$\begin{aligned} I_1(t,\varepsilon) &= \textstyle\int_0^t F_x(r,X_r)(X_{r+\varepsilon}-X_r)\frac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r \\ &\quad + \textstyle\int_0^t (F_x(r+\varepsilon,X_r) - F_x(r,X_r))(X_{r+\varepsilon}-X_r)\frac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r \\ &\quad + \textstyle\int_0^t \int_0^1 \{F_x(r+\varepsilon,X_r+\lambda(X_{r+\varepsilon}-X_r)) - F_x(r+\varepsilon,X_r)\}\mathrm{d}\lambda(X_{r+\varepsilon}-X_r)\frac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r \\ &=: I_{1,1}(t,\varepsilon) + R_{1,1}(t,\varepsilon) + R_{1,2}(t,\varepsilon). \end{aligned}$$

We know that $r \mapsto F_x(r,X_r)$ is càdlàg since $F \in C^{0,1}$ and X is a.s. continuous. Thus, Lemma 4.1 implies that $\lim_{\varepsilon\to 0} I_{1,1}(t,\varepsilon) = \int_0^t F_x(r,X_r)\mathrm{d}[X,N]_r = \int_0^t F_x(r,X_r)\mathrm{d}[M,N]_r$ since $[A,N]_r \equiv 0$. Hence, it remains to show: $\lim_{\varepsilon\to 0} R_{1,1}(t,\varepsilon) = 0$ and $\lim_{\varepsilon\to 0} R_{1,2}(t,\varepsilon) = 0$ in *ucp*. For this purpose, we define for $r \in [0,T]$ and $\varepsilon > 0$:

$$\begin{aligned} Y_r^\varepsilon &:= F_x(r+\varepsilon,X_r) - F_x(r,X_r), \\ Z_r^\varepsilon &:= \textstyle\int_0^1 \{F_x(r+\varepsilon,X_r+\lambda(X_{r+\varepsilon}-X_r)) - F_x(r+\varepsilon,X_r)\}\mathrm{d}\lambda. \end{aligned}$$

Let $\omega \in \Omega$ be arbitrary such that $r \to X_r(\omega)$ is continuous. Hence, there exists a convex compact set K such that $X_r(\omega) \in K$ for all $r \in [0,T]$. Since $F_x(t,x)$ is continuous, we

know that $F_x(t,x)$ is uniformly continuous on $[0,T]\times K$. Thus, for all $L>0$ there exists a $0<\rho_\omega\leq 1$ such that it holds for all $r\in[0,T]$ and for all $0<\varepsilon\leq\rho_\omega$: $|Y_r^\varepsilon(\omega)|\leq L$. (Note that $\|(r+\varepsilon,X_r(\omega))-(r,X_r(\omega))\|_2=\varepsilon\leq\rho_\omega$.) In particular, we get for any arbitrary $L>0$ a $0<\rho_\omega\leq 1$ such that $\sup_{r\in[0,T]}|Y_r^\varepsilon(\omega)|\leq L$ for all $0<\varepsilon\leq\rho_\omega$. Hence, $\sup_{r\in[0,T]}|Y_r^\varepsilon(\omega)|\xrightarrow[a.s.]{\varepsilon\to 0}0$.

Next, note that it holds that $X_r(\omega)+\lambda(X_{r+\varepsilon}(\omega)-X_r(\omega))\in K$ for all $\lambda\in[0,1]$ since $X_r(\omega),X_{r+\varepsilon}(\omega)\in K$. Moreover, $X_r(\omega)$ is uniformly continuous in r for $r\in[0,T]$, i.e., for all $\tilde{L}>0$ there exists a $0<\tilde{\rho}_\omega\leq 1$ such that for all $r\in[0,T]$ and for all $0<\varepsilon\leq\tilde{\rho}_\omega$ it holds: $|X_{r+\varepsilon}(\omega)-X_r(\omega)|\leq\tilde{L}$. Let $L>0$ arbitrary. Since $F_x(t,x)$ is uniformly continuous on $[0,T]\times K$, there exists a $\delta>0$ such that for all $x,y\in K$ with $\|(t,x)-(t,y)\|_2\leq\delta$ it holds that $|F_x(t,x)-F_x(t,y)|\leq L$ for all $t\in[0,T]$ fixed. Now choose $\tilde{L}=\delta$ and define $\rho_\omega:=\min\{\delta,\tilde{\rho}_\omega\}$. Then

$$\|(r+\varepsilon,X_r(\omega)+\lambda(X_{r+\varepsilon}(\omega)-X_r(\omega)))-(r+\varepsilon,X_r(\omega))\|_2=\lambda|X_{r+\varepsilon}(\omega)-X_r(\omega)|\leq\delta$$

for all $0<\varepsilon\leq\rho_\omega$, all $\lambda\in[0,1]$, and all $r\in[0,T]$. Hence, it holds that

$$|F_x(r+\varepsilon,X_r(\omega)+\lambda(X_{r+\varepsilon}(\omega)-X_r(\omega)))-F_x(r+\varepsilon,X_r(\omega))|\leq L$$

for all $0<\varepsilon\leq\rho_\omega$, all $\lambda\in[0,1]$, and all $r\in[0,T]$. This implies that $|Z_r^\varepsilon(\omega)|\leq L$ for all $0<\varepsilon\leq\rho_\omega$ and all $r\in[0,T]$. In particular, we get $\sup_{r\in[0,T]}|Z_r^\varepsilon(\omega)|\leq L$ for all $0<\varepsilon\leq\rho_\omega$. Since we chose $L>0$ and $\omega\in\Omega$ arbitrary, $\sup_{r\in[0,T]}|Z_r^\varepsilon|\xrightarrow[a.s.]{\varepsilon\to 0}0$. Now, Lemma 4.2 yields that $\lim_{\varepsilon\to 0}R_{1,1}(t,\varepsilon)=0$ and $\lim_{\varepsilon\to 0}R_{1,2}(t,\varepsilon)=0$ in *ucp*.

Let us next show that $\lim_{\varepsilon\to 0}I_2(t,\varepsilon)=\int_0^t\gamma(r,X_r)\mathrm{d}[W,N]_r$. We use (2.1) combined with Lemma 4.6 (whose conditions are fulfilled due to the localization arguments at the beginning) in I_2 to write:

$$\begin{aligned}
I_2(t,\varepsilon)&=\int_0^t\int_r^{r+\varepsilon}\beta(s,X_r)\mathrm{d}s\tfrac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r+\int_0^t\int_r^{r+\varepsilon}\gamma(s,X_r)\mathrm{d}W_s\tfrac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r\\
&=\int_0^t\int_r^{r+\varepsilon}\gamma(r,X_r)\mathrm{d}W_s\tfrac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r+\int_0^t\int_r^{r+\varepsilon}\beta(s,X_r)\mathrm{d}s\tfrac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r\\
&\quad+\int_0^t\int_r^{r+\varepsilon}[\gamma(s,X_r)-\gamma(r,X_r)]\mathrm{d}W_s\tfrac{N_{r+\varepsilon}-N_r}{\varepsilon}\mathrm{d}r\\
&=:I_{2,1}(t,\varepsilon)+R_{2,1}(t,\varepsilon)+R_{2,2}(t,\varepsilon).
\end{aligned}$$

For $I_{2,1}$, we get due to Lemma 4.1 and Assumption 3:

$$I_{2,1}(t,\varepsilon)=\int_0^t\gamma(r,X_r)\tfrac{(W_{r+\varepsilon}-W_r)(N_{r+\varepsilon}-N_r)}{\varepsilon}\mathrm{d}r\xrightarrow[ucp]{\varepsilon\to 0}\int_0^t\gamma(r,X_r)\mathrm{d}[W,N]_r.$$

For the two terms remaining, we define for $r\in[0,T]$ and $\varepsilon>0$:

$$\widetilde{Y}_r^\varepsilon:=\tfrac{1}{\varepsilon}\int_r^{r+\varepsilon}\beta(s,X_r)\mathrm{d}s,\qquad\qquad\widetilde{Z}_r^\varepsilon:=\tfrac{1}{\sqrt{\varepsilon}}\int_r^{r+\varepsilon}[\gamma(s,X_r)-\gamma(r,X_r)]\mathrm{d}W_s.$$

First, we note that both $\widetilde{Y}^\varepsilon$ and $\widetilde{Z}^\varepsilon$ are a.s. bounded for all $\varepsilon>0$ since $\sup_{x\in K}|\beta(t,x)|\in\mathcal{L}^1(\Omega\times[0,T],\mathrm{d}\mathbb{P}\times\mathrm{d}t)$, $\sup_{x\in K}|\gamma(t,x)|\in\mathcal{L}^2(\Omega\times[0,T],\mathrm{d}\mathbb{P}\times\mathrm{d}t)$, and X is a.s. continuous.

We continue with $R_{2,1}$: In the case of Assumption 1a, classical localization arguments give us the existence of a $\tilde{B}_1>0$ such that $\int_0^T\sup_{s\in[0,T]}|\beta(s,X_t)|\mathrm{d}t\leq\tilde{B}_1$ a.s. and we get a.s.:

$$\int_0^T|\widetilde{Y}_r^\varepsilon|\mathrm{d}r\leq\int_0^T\tfrac{1}{\varepsilon}\int_r^{r+\varepsilon}|\beta(s,X_r)|\mathrm{d}s\mathrm{d}r\leq\int_0^T\sup_{s\in[0,T]}|\beta(s,X_r)|\mathrm{d}r\leq\tilde{B}_1.$$

In the case of Assumption 1b, we get:

$$\begin{aligned}
\mathbb{E}[|\int_0^T|\widetilde{Y}_r^\varepsilon|\mathrm{d}r|]&\leq\mathbb{E}[\int_0^T\tfrac{1}{\varepsilon}\int_r^{r+\varepsilon}|\beta(s,X_r)|\mathrm{d}s\mathrm{d}r]\\
&=\int_0^T\tfrac{1}{\varepsilon}\int_r^{r+\varepsilon}\mathbb{E}[|\beta(s,X_r)|]\mathrm{d}s\mathrm{d}r\leq\int_0^T\sup_{s\in[0,T]}\mathbb{E}[|\beta(s,X_r)|]\mathrm{d}r\leq B_1,
\end{aligned}$$

i.e., $\int_0^T |\widetilde{Y}_r^\varepsilon| \mathrm{d}r$ is bounded in $\mathcal{L}^1(\mathrm{d}\mathbb{P})$. Hence, in both cases it holds that $\int_0^T |\widetilde{Y}_r^\varepsilon| \mathrm{d}r$ is bounded in probability and Lemma 4.4 implies that $\lim_{\varepsilon\to 0} R_{2,1} = 0$ in *ucp*.

Finally, we analyze $R_{2,2}$: First, we get for $r \in [0,T]$ fixed:

$$\widetilde{Z}_r^\varepsilon = \tfrac{W_{r+\varepsilon}-W_r}{\sqrt{\varepsilon}} \cdot \tfrac{1}{W_{r+\varepsilon}-W_r} \int_r^{r+\varepsilon} [\gamma(s,X_r) - \gamma(r,X_r)] \mathrm{d}W_s.$$

The first fraction is $\mathcal{N}(0,1)$-distributed for all ε, and hence converges in distribution to $Z \sim \mathcal{N}(0,1)$ for $\varepsilon \to 0$. The second part converges in probability to 0 for $\varepsilon \to 0$ due to Protter [19, p.96]. Note that we are allowed to use that result since, for $s \geq r$, $H_s := \gamma(s,X_r) - \gamma(r,X_r)$ is adapted and continuous due to γ being continuous in the first argument. Hence, Slutsky implies that $\widetilde{Z}_r^\varepsilon \xrightarrow[d]{\varepsilon\to 0} 0$. Due to 0 being constant, this convergence also holds in probability.

In the next step, we want to show that this convergence also holds in $L^2(\mathrm{d}\mathbb{P})$. For this purpose, we show that $|\widetilde{Z}_r^\varepsilon|^2$ is uniformly integrable. Let $\varrho > 0$ be arbitrary. Then, we get using Mao [18, p.39] (with $p = 2+\varrho$):

$$\begin{aligned}\mathbb{E}[|\widetilde{Z}_r^\varepsilon|^{2+\varrho}] &= \tfrac{1}{\varepsilon^{1+\frac{\varrho}{2}}} \mathbb{E}[|\int_r^{r+\varepsilon} [\gamma(s,X_r) - \gamma(r,X_r)] \mathrm{d}W_s|^{2+\varrho}] \\ &\leq \tfrac{1}{\varepsilon^{1+\frac{\varrho}{2}}} (\tfrac{(2+\varrho)(1+\varrho)}{2})^{\frac{2+\varrho}{2}} \varepsilon^{\frac{\varrho}{2}} \mathbb{E}[\int_r^{r+\varepsilon} |\gamma(s,X_r) - \gamma(r,X_r)|^{2+\varrho} \mathrm{d}s].\end{aligned}$$

We define $\tilde{C} := (\frac{(2+\varrho)(1+\varrho)}{2})^{\frac{2+\varrho}{2}}$ and $C := \tilde{C} \cdot 2^{2+\varrho}$. In particular, $\tilde{C} \leq C$. Now, we get for all $r \in [0,T]$ using Lemma 4.7:

$$\begin{aligned}\mathbb{E}[|\widetilde{Z}_r^\varepsilon|^{2+\varrho}] &\leq \tilde{C} \cdot \tfrac{1}{\varepsilon} \int_r^{r+\varepsilon} \mathbb{E}|\gamma(s,X_r) - \gamma(r,X_r)|^{2+\varrho} \mathrm{d}s \qquad (4.1)\\ &\leq \tilde{C} \sup_{s\in[0,T]} \mathbb{E}[|\gamma(s,X_r) - \gamma(r,X_r)|^{2+\varrho}] \leq C \sup_{s\in[0,T]} \mathbb{E}[|\gamma(s,X_r)|^{2+\varrho}] \leq C \cdot B_2.\end{aligned}$$

Thus, $\widetilde{Z}_r^\varepsilon \xrightarrow[\mathcal{L}^2(\mathrm{d}\mathbb{P})]{\varepsilon\to 0} 0$ for all $r \in [0,T]$. Analogously to (4.1), we also get $\sup_{r\in[0,T]} \mathbb{E}[|\widetilde{Z}_r^\varepsilon|^2] \leq C \cdot B_2$. Hence, the bounded convergence theorem gives us that $\int_0^T \mathbb{E}|\widetilde{Z}_r^\varepsilon|^2 \mathrm{d}r \xrightarrow{\varepsilon\to 0} 0$. Thus, $\int_0^T |\widetilde{Z}_r^\varepsilon|^2 \mathrm{d}r \xrightarrow[\mathbb{P}]{\varepsilon\to 0} 0$ and Lemma 4.5 implies that $\lim_{\varepsilon\to 0} R_{2,2} = 0$ in *ucp* which gives us the proof of the Theorem. □

5 Applications

In this section, we apply Theorem 3.1 to derive a novel stochastic calculus formula for quadratic variations and a representation result for strong solutions of SPDEs. Moreover, we provide an example from financial mathematics.

5.1 Formulas for quadratic variations

In this subsection, we give in Proposition 5.1 a novel stochastic calculus formula for quadratic variations. Afterwards, we point out a special case. Note that we cannot use the Itô-Ventzell formula in $C^{0,2}$ or $C^{1,2}$ since the function F defined below is not twice continuously differentiable in the space direction. We remark further that Proposition 5.1 is not a special case of the Meyer-Itô formula (see, for instance, Protter [19]) which is only applicable for delta-convex functions, i.e., for the difference of two continuous convex functions. However, the class of C^1-functions and the class of delta-convex functions are not included in each other, see, e.g., Zelený [27].

Proposition 5.1. *Let W be a one-dimensional Brownian Motion, $X = M + A$ be a continuous weak Dirichlet FQV process, and $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot,x)$ is previsible for all $x \in \mathbb{R}^d$, $f(t,W_t)$ is a.s. càdlàg, and f is continuously differentiable in x with the derivative being Hölder-continuous (for an $\alpha > 0$) for all $t \in [0,T]$. Moreover,*

we assume that for all $K \subset \mathbb{R}$ compact exists a $C_K(\omega) \geq 0$ with $\mathbb{E}[C_K^2] < \infty$ such that $\sup_{(t,x)\in[0,T]\times K}(|f(\omega,t,x)| + |f_x(\omega,t,x)|) \leq C_K(\omega)$. Then, it holds that

$$\left[\int_0^t f(r,x)\mathrm{d}W_r\big|_{x=X_t}, W_t\right] = \int_0^t f(r,X_r)\mathrm{d}r + \int_0^t \int_0^r f_x(s,x)\mathrm{d}W_s\big|_{x=X_r}\mathrm{d}[M,W]_r.$$

Proof. First, we define $F(t,x) := \int_0^t f(r,x)\mathrm{d}W_r$ (where we suppress the ω as usual). Then, F is a stochastic flow in the sense of Definition 2.7 with local characteristics $\beta(t,x) = 0$ and $\gamma(t,x) = f(t,x)$. The assumptions of Theorem 3.1 are fulfilled since continuous differentiability implies local Lipschitz continuity. Moreover, it holds that $F_x(t,x) = \int_0^t f_x(r,x)\mathrm{d}W_r$ due to the stochastic Leibniz integral rule. Then, Theorem 3.1 implies with the decomposition $X = M + A$:

$$\int_0^t f(r,x)\mathrm{d}W_r\big|_{x=X_t} = 0 + \int_0^t f(r,X_r)\mathrm{d}W_r + \int_0^t \int_0^r f_x(s,x)\mathrm{d}W_s\big|_{x=X_r}\mathrm{d}M_r + \mathcal{B}^X(F)_t,$$

where $\mathcal{B}^X(F)_t$ is a weak zero energy process. Hence, we get for any continuous local martingale $\tilde{M}$:

$$\left[\int_0^t f(r,x)\mathrm{d}W_r\big|_{x=X_t}, \tilde{M}_t\right] = \int_0^t f(r,X_r)\mathrm{d}[W,\tilde{M}]_r + \int_0^t \int_0^r f_x(s,x)\mathrm{d}W_s\big|_{x=X_r}\mathrm{d}[M,\tilde{M}]_r.$$

Finally, choosing $\tilde{M} := W$ yields the proposition. □

Corollary 5.2. *Let W be a one-dimensional Brownian Motion, and $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then, we get that:*

$$[f(W_t)W_t, W_t] = \int_0^t f(W_r)\mathrm{d}r + \int_0^t f'(W_r)W_r\mathrm{d}r.$$

Proof. This corollary follows immediately from Proposition 5.1 after setting $X = W$ with $\mathrm{d}[W]_r = \mathrm{d}r$. Moreover, note that $\int_0^t \int_0^r f_x(x)\mathrm{d}W_s\big|_{x=X_r}\mathrm{d}[M,W]_r = \int_0^t f'(X_r)W_r\mathrm{d}[M,W]_r$ in this setting. □

For Corollary 5.2, we could have alternatively tried to use an Itô-formula with weaker differentiability conditions.

5.2 A representation result for strong solutions of SPDEs

In this subsection, the overall target is to give a representation result for strong solutions of stochastic elliptic PDE for a special type of weak Dirichlet processes. We start by giving a generalization of the solution theory of SPDEs (naturally generalizing the theory of Gozzi and Russo [13] who considered PDEs).

Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be two continuous functions. Then, we define the time-dependent elliptic operator $L_0(t) : D(L_0) \subset C^0([0,T] \times \mathbb{R}^d) \to C^0([0,T] \times \mathbb{R}^d)$ with $D(L_0) = C^{0,2}([0,T] \times \mathbb{R}^d)$ by $L_0(t)F(t,x) := b(t,x)F_x(t,x) + \frac{1}{2}\mathrm{tr}(\sigma^\intercal(t,x)F_{xx}(t,x)\sigma(t,x)$, where tr denotes the trace of a matrix. We then consider the stochastic inhomogeneous time-dependent elliptic problem for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and almost all $\omega \in \Omega$:

$$L_0(t)F(\omega,t,x) = h(\omega,t,x), \tag{5.1}$$

where $h(\omega,\cdot) \in C^0([0,T] \times \mathbb{R}^d)$ almost surely.

Definition 5.3. *We say that F with local characteristics $(\beta(t,x),\gamma(t,x))$ is a strict solution to the stochastic elliptic problem* (5.1) *if for a.e. $\omega \in \Omega$ it holds that $F(\omega,\cdot) \in D(L_0)$, $\beta(\omega,\cdot) \in C^{0,1}([0,T] \times \mathbb{R})$, $\gamma(\omega,\cdot) \in C^{0,1+\alpha}([0,T] \times \mathbb{R}^d)$ for an $\alpha \in (0,1)$, and* (5.1) *holds.*

$C^{0,1}$ Itô-Ventzell formula

Analogously to Gozzi and Russo [13] who defined a strong solution for PDEs, we define a strong solution for SPDEs as follows:

Definition 5.4. *We say that a stochastic flow F with local characteristics $(\beta(t,x),\gamma(t,x))$ is a strong solution to the stochastic elliptic problem* (5.1) *if there exists a sequence $(F^n)_{n\in\mathbb{N}}$ with local characteristics $(\beta^n(\omega,t,x),\gamma^n(\omega,t,x))$ and a sequence $(h^n)_{n\in\mathbb{N}}$ such that for every $n\in\mathbb{N}$ and a.e. $\omega\in\Omega$ holds:*

1. *$F^n(\omega,\cdot)\in D(L_0)$, and $h^n(\omega,\cdot)\in C^0([0,T]\times\mathbb{R}^d)$,*
2. *$\beta^n(\omega,\cdot)\in C^{0,1}([0,T]\times\mathbb{R})$, and $\gamma^n(\omega,\cdot)\in C^{0,1+\alpha}([0,T]\times\mathbb{R}^d)$ for an $\alpha\in(0,1)$,*
3. *F^n is a strict solution of $L_0(t)F^n(\omega,t,x)=h^n(\omega,t,x)$ for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and a.e. $\omega\in\Omega$,*
4. *$F^n(\omega,\cdot)\xrightarrow{n\to\infty}F(\omega,\cdot)$ and $h^n(\omega,\cdot)\xrightarrow{n\to\infty}h(\omega,\cdot)$ in $C^0([0,T]\times\mathbb{R}^d)$ (equipped with the supremum-norm on compact sets), and*
5. *$\beta^n(\omega,\cdot)\xrightarrow{n\to\infty}\beta(\omega,\cdot)$ in $\mathcal{L}^1(K;\mathrm{d}t)$ for all compact sets $K\subset[0,T]\times\mathbb{R}$.*

Note that this a generalization of the notion of a strict solution since, of course, every strict solution is also a strong solution by taking the constant sequences $F^n:=F$ and $h^n:=h$. Our representation theorem states then:

Theorem 5.5. *Let $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ be two continuous functions, and F (with local characteristics $(\beta(\omega,t,x),\gamma(\omega,t,x))$) be a strong solution of problem* (5.1) *where we denote the approximation sequence by F^n (with local characteristics $(\beta^n(\omega,t,x),\gamma^n(\omega,t,x))$). Define $X_t:=x_0+\int_0^t\sigma(r,X_r)\mathrm{d}W_r+A_t$, where $x_0\in\mathbb{R}^d$ and A is a continuous weak zero energy process such that X is a continuous FQV process. Furthermore, we assume:*

1. *$F(\omega,\cdot)\in C^{0,1}([0,T]\times\mathbb{R}^d)$ for a.e. $\omega\in\Omega$,*
2. *β and γ fulfill the assumptions of Theorem 3.1 with $\gamma(\omega,\cdot)\in C^{0,1}([0,T]\times\mathbb{R}^d)$,*
3. *γ^n satisfy Assumption 2 of Theorem 3.1 for all $n\in\mathbb{N}$, and*
4. *it holds in ucp that*

$$\lim_{n\to\infty}\Big|\int_0^t(F_x(r,X_r)-F_x^n(r,X_r))\mathrm{d}^-A_r\Big|+\Big|\int_0^t\mathrm{tr}((\gamma_x(r,X_r)-\gamma_x^n(r,X_r))\sigma(r,X_r))\mathrm{d}r\Big|+\Big|\int_0^t(F_x(r,X_r)-F_x^n(r,X_r))b(r,X_r)\mathrm{d}r\Big|=0. \tag{5.2}$$

In particular, all of these integrals are assumed to exist.

Then, we have that (suppressing the dependence on ω)

$$\begin{aligned}F(t,X_t)=&F(0,X_0)+\int_0^t\gamma(r,X_r)\mathrm{d}W_r+\int_0^tF_x(r,X_r)\sigma(r,X_r)\mathrm{d}W_r+\int_0^t\beta(r,X_r)\mathrm{d}r\\&+\int_0^tF_x(r,X_r)\mathrm{d}^-A_r+\int_0^t\mathrm{tr}(\gamma_x(r,X_r)\sigma(r,X_r))\mathrm{d}r+\int_0^th(r,X_r)\mathrm{d}r\\&-\int_0^tb(r,X_r)F_x(r,X_r)\mathrm{d}r.\end{aligned}$$

Proof. Since F is a strong solution of (5.1) with approximating sequences F^n and h^n, we get with the second part of Theorem 3.1 (suppressing the dependence on ω):

$$\begin{aligned}
F^n(t,X_t) =& F^n(0,X_0) + \int_0^t \gamma^n(r,X_r)\mathrm{d}W_r + \int_0^t F^n_x(r,X_r)\sigma(r,X_r)\mathrm{d}W_r + \int_0^t \beta^n(r,X_r)\mathrm{d}r \\
&+ \int_0^t F^n_x(r,X_r)\mathrm{d}^- A_r + \int_0^t \mathrm{tr}(\gamma^n_x(r,X_r)\sigma(r,X_r))\mathrm{d}r \\
&+ \frac{1}{2}\int_0^t \mathrm{tr}(\sigma^\intercal(r,X_r)F^n_{xx}(r,X_r)\sigma(r,X_r))\mathrm{d}r \\
=& F^n(0,X_0) + \int_0^t \gamma^n(r,X_r)\mathrm{d}W_r + \int_0^t F^n_x(r,X_r)\sigma(r,X_r)\mathrm{d}W_r + \int_0^t \beta^n(r,X_r)\mathrm{d}r \\
&+ \int_0^t F^n_x(r,X_r)\mathrm{d}^- A_r + \int_0^t \mathrm{tr}(\gamma^n_x(r,X_r)\sigma(r,X_r))\mathrm{d}r + \int_0^t L_0(r)F^n(r,X_r)\mathrm{d}r \\
&- \int_0^t b(r,X_r)F^n_x(r,X_r)\mathrm{d}r. \qquad (5.3)
\end{aligned}$$

Now, we define:

$$\begin{aligned}
M^n_t :=& F^n(t,X_t) - F^n(0,X_0) - \int_0^t \beta^n(r,X_r)\mathrm{d}r - \int_0^t F^n_x(r,X_r)\mathrm{d}^- A_r \\
&- \int_0^t \mathrm{tr}(\gamma^n_x(r,X_r)\sigma(r,X_r))\mathrm{d}r - \int_0^t L_0(r)F^n(r,X_r)\mathrm{d}r + \int_0^t b(r,X_r)F^n_x(r,X_r)\mathrm{d}r, \\
M_t :=& F(t,X_t) - F(0,X_0) - \int_0^t \beta(r,X_r)\mathrm{d}r - \int_0^t F_x(r,X_r)\mathrm{d}^- A_r \\
&- \int_0^t \mathrm{tr}(\gamma_x(r,X_r)\sigma(r,X_r))\mathrm{d}r - \int_0^t h(r,X_r)\mathrm{d}r + \int_0^t b(r,X_r)F_x(r,X_r)\mathrm{d}r. \qquad (5.4)
\end{aligned}$$

Note that for almost all $\omega \in \Omega$, it holds that $L_0(r)F^n(r,X_r(\omega)) = h^n(r,X_r(\omega)) \xrightarrow{n\to\infty} h(r,X_r(\omega))$ uniformly in r due to 4 in Definition 5.4 and $(r,X_r(\omega))$ taking values in a compact set for $r \in [0,t]$. Then, by the assumptions of a strong solution and by (5.2), it holds that $M^n_t \xrightarrow{n\to\infty} M_t$ in ucp. Moreover, it follows from (5.3) that M^n_t is a local martingale. Proposition 4.4 of Gozzi and Russo [13] implies then that M is also a local martingale. Furthermore, note that M_t is continuous in t since X, F, and A are continuous. Then, we define the weak zero energy process

$$\begin{aligned}
\tilde{\mathcal{B}}^X(F)_t :=& \int_0^t \beta(r,X_r)\mathrm{d}r + \int_0^t F_x(r,X_r)\mathrm{d}^- A_r + \int_0^t \mathrm{tr}(\gamma_x(r,X_r)\sigma(r,X_r))\mathrm{d}r \\
&+ \int_0^t h(r,X_r)\mathrm{d}r - \int_0^t b(r,X_r)F_x(r,X_r)\mathrm{d}r.
\end{aligned}$$

Rearranging (5.4) implies that $F(t,X_t) = F(0,X_0) + M_t + \tilde{\mathcal{B}}^X(F)_t$. Next, we apply the first part of Theorem 3.1 on F to get that $F(t,X_t) = F(0,X_0) + \int_0^t \gamma(r,X_r)\mathrm{d}W_r + \int_0^t F_x(r,X_r)\sigma(r,X_r)\mathrm{d}W_r + \mathcal{B}^X(F)_t$, where $\tilde{M}_t := \int_0^t \gamma(r,X_r)\mathrm{d}W_r + \int_0^t F_x(r,X_r)\sigma(r,X_r)\mathrm{d}W_r$ is a continuous local martingale. Finally, since M_t and $\tilde{M}_t$ are continuous, the decomposition of the weak Dirichlet process $F(t,X_t)$ is unique. In particular, we get that $M_t = \tilde{M}_t$ and $\mathcal{B}^X(F)_t = \tilde{\mathcal{B}}^X(F)_t$ which implies the claim. □

5.3 An example from financial mathematics

In this subsection, we consider a simplified version of a model introduced in Bank and Baum [3] which describes a wealth process of a large investor with reduced differentiability assumptions. The large investor directly influences the price with the amount of shares she/he buys/sells.

$C^{0,1}$ Itô-Ventzell formula

For this purpose, let $T > 0$ and define for $0 \leq t \leq T$ a price fluctuation process $P_t^\vartheta = P(t, \vartheta)$, $\vartheta \in \mathbb{R}$, of the risky asset as a stochastic flow with local characteristics $(\beta(t,x), \gamma(t,x))$ as in Definition 2.7 with $\beta(t,x) \in C^{0,0}$ and $\gamma(t,x) \in C^{0,0+\alpha}$ for an $\alpha \in (0,1)$. The parameter ϑ denotes the number of shares the large investor holds. Moreover, we assume that P fulfills the assumptions of Theorem 3.1. (This is a relaxation compared to Bank and Baum [3], where second order differentiability is needed.) Let $X = (X_t)_{t\in[0,T]}$ be a (self-financing[1]) strategy for the large investor describing the number of shares held. Assuming that X is a semimartingale, according to Bank and Baum [3, p.5], the discounted bank account value, say Ξ^X, of the large investor evolves as

$$\Xi_t^X = \Xi_0 - \int_0^t P(r, X_r)\mathrm{d}X_r - [P(\cdot, X), X]_t, \quad \text{for } 0 \leq t \leq T, \tag{5.5}$$

where the integral models the fluctuations of the portfolio value until time t and the quadratic variation term accounts for the sensitivity of the prices regarding the large investor's orders at time t. By splitting an order into smaller packages, Bank and Baum [3] argue that when liquidating the portfolio, the amount obtained by the large investor is given by $L(t,\vartheta) = \int_0^\vartheta P(t,x)\mathrm{d}x$, where ϑ denotes the large investor's number of shares before liquidation. The real (or asymptotically realizable) wealth V^X which can be achieved by the strategy X until time t is consequently given by

$$V_t^X = \Xi_t^X + L(t, X_t). \tag{5.6}$$

Proposition 5.6. *Let X be a continuous self-financing strategy which is a semimartingale. The dynamics of the real wealth process V^X is then given by:*

$$V_t^X = V_0^X + \int_0^t \beta^L(r, X_r)\mathrm{d}r + \int_0^t \gamma^L(r, X_r)\mathrm{d}X_r - \frac{1}{2}\int_0^t P_x(r, X_r)\mathrm{d}[X]_r,$$

where $\beta^L(t,\vartheta) = \int_0^\vartheta \beta(t,x)\mathrm{d}x$ and $\gamma^L(t,\vartheta) = \int_0^\vartheta \gamma(t,x)\mathrm{d}x$.

Proof. First, we can use the standard decomposition $X = M + A$ with M being a martingale and A a finite variation process since X is a semimartingale. Next, we notice by the definition of L, that L is also a stochastic flow with local characteristics $(\beta^L(t,x), \gamma^L(t,x))$ in the sense of Definition 2.7 and fulfills the assumption of Theorem 3.1 (b). Indeed, we are allowed to interchange the integrals when determining the local characteristics of L since $\beta(s,x) \in C^{0,0}$ and thus β is bounded on $[0,t] \times [0,\vartheta]$ for a.e. $\omega \in \Omega$ fixed. Furthermore, $\gamma(s,x) \in C^{0,0+\alpha}$, and we can therefore apply the stochastic Fubini theorem, given in Lemma 2.6 resp. Lemma 2.7 in Krylov [16]), for the stochastic integral. Hence, we get with Theorem 3.1 (b):

$$\begin{aligned}
L(t, X_t) =& L(0, X_0) + \int_0^t \beta^L(r, X_r)\mathrm{d}r + \int_0^t \gamma^L(r, X_r)\mathrm{d}W_r + \int_0^t L_x(r, X_r)\mathrm{d}M_r \\
&+ \int_0^t L_x(r, X_r)\mathrm{d}^- A_r + \int_0^t \gamma_x^L(r, X_r)\mathrm{d}[X, W]_r + \frac{1}{2}\int_0^t L_{xx}(r, X_r)\mathrm{d}[X]_r \\
=& L(0, X_0) + \int_0^t \beta^L(r, X_r)\mathrm{d}r + \int_0^t \gamma^L(r, X_r)\mathrm{d}W_r + \int_0^t P(r, X_r)\mathrm{d}X_r \\
&+ \int_0^t \gamma(r, X_r)\mathrm{d}[X, W]_r + \frac{1}{2}\int_0^t P_x(r, X_r)\mathrm{d}[X]_r,
\end{aligned} \tag{5.7}$$

since $L_x = P$ with $\gamma_x^L = \gamma$. Next, we apply Theorem 3.1 (a) to P which gives us:

$$P(t, X_t) = P(0, X_0) + \int_0^t \gamma(r, X_r)\mathrm{d}W_r + \int_0^t P_x(r, X_r)\mathrm{d}M_r + \mathcal{B}^X(P)_t.$$

[1] That is, assuming that there are no intermediate capital injections.

Hence, it follows that:

$$\begin{aligned}[P(\cdot, X), X]_t &= \int_0^t \gamma(r, X_r)\mathrm{d}[X, W]_r + \int_0^t P_x(r, X_r)\mathrm{d}[X, M]_r \\ &= \int_0^t \gamma(r, X_r)\mathrm{d}[X, W]_r + \int_0^t P_x(r, X_r)\mathrm{d}[X]_r,\end{aligned}$$

since $[A, M]_r = 0 = [A]_r$ due to A being a finite variation process. Thus, we get for the bank account value Ξ (defined in (5.5)):

$$\Xi_t^X = \Xi_0 - \int_0^t P(r, X_r)\mathrm{d}X_r - \int_0^t \gamma(r, X_r)\mathrm{d}[X, W]_r - \int_0^t P_x(r, X_r)\mathrm{d}[X]_r. \tag{5.8}$$

Next, we plug (5.7) and (5.8) into (5.6), then several terms cancel out and the claim follows. □

References

[1] Elena Bandini and Francesco Russo, *Weak dirichlet processes with jumps*, Stochastic Processes and their Applications **127** (2017), no. 12, 4139–4189. MR3718109

[2] Elena Bandini and Francesco Russo, *Weak dirichlet processes and generalized martingale problems*, arXiv preprint arXiv:2205.03099 (2022). MR4682671

[3] Peter Bank and Dietmar Baum, *Hedging and portfolio optimization in financial markets with a large trader*, Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics **14** (2004), no. 1, 1–18. MR2030833

[4] Bruno Bouchard, Grégoire Loeper, and Xiaolu Tan, *A $c^{0,1}$-functional itô's formula and its applications in mathematical finance*, Stochastic Processes and their Applications **148** (2022), 299–323. MR4397143

[5] Bruno Bouchard and Maximilien Vallet, *Itô-dupire's formula for $c^{0,1}$-functionals of cádlág weak dirichlet processes*, arXiv preprint arXiv:2110.03406 (2021).

[6] François Coquet, Adam Jakubowski, Jean Mémin, and Leszek Słominski, *Natural decomposition of processes and weak dirichlet processes*, In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX (2006), 81–116. MR2276891

[7] Giuseppe Da Prato, Arnulf Jentzen, and Michael Röckner, *A mild itô formula for spdes*, Transactions of the American Mathematical Society **372** (2019), no. 6, 3755–3807. MR4009384

[8] Gonçalo Dos Reis and Vadim Platonov, *Itô-wentzell-lions formula for measure dependent random fields under full and conditional measure flows*, Potential Analysis (2022), 1–32. MR4647952

[9] K David Elworthy, Aubrey Truman, and Huaizhong Zhao, *Generalized itô formulae and space-time lebesgue–stieltjes integrals of local times*, Séminaire de Probabilités XL (2007), 117–136. MR2409002

[10] Mohammed Errami and Francesco Russo. n-covariation, generalized dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Processes and their Applications*, 104(2):259–299, 2003. MR1961622

[11] Franco Flandoli and Francesco Russo, *Generalized integration and stochastic odes*, The Annals of Probability **30** (2002), no. 1, 270–292. MR1894108

[12] Hans Föllmer, Philip Protter, and Albert N Shiryayev, *Quadratic covariation and an extension of itô's formula*, Bernoulli (1995), 149–169. MR1354459

[13] Fausto Gozzi and Francesco Russo, *Weak dirichlet processes with a stochastic control perspective*, Stochastic Processes and their Applications **116** (2006), no. 11, 1563–1583. MR2269216

[14] Elena V Karachanskaya, *On a generalization of the itô-wentzell formula for system of generalized itô's sde and the stochastic first integral*, transfer **1** (2011), no. 2, 3.

[15] Michael Kohlmann, Li Siyuan, and Dewen Xiong, *A generalized itô-ventzell formula to derive forward utility models in a jump market*, Stochastic Analysis and Applications **31** (2013), no. 4, 632–662. MR3175789

[16] Nikolai Vladimirovich Krylov, *On the itô-wentzell formula for distribution-valued processes and related topics*, arXiv preprint arXiv:0904.2752 (2009). MR2800911

[17] Hiroshi Kunita, *Stochastic flows and stochastic differential equations*, vol. 24, Cambridge university press, 1997. MR1472487

[18] Xuerong Mao, *Stochastic differential equations and applications*, Elsevier, 2007. MR2380366

[19] Philip E Protter, *Stochastic integration and differential equations*, vol. 21, Springer Science & Business Media, 2005. MR2020294

[20] Francesco Russo and Pierre Vallois, *Intégrales progressive, rétrograde et symétrique de processus non adaptés*, Comptes rendus de l'Académie des sciences. Série 1, Mathématique **312** (1991), no. 8, 615–618. MR1101044

[21] Francesco Russo and Pierre Vallois, *Forward, backward and symmetric stochastic integration*, Probability theory and related fields **97** (1993), 403–421. MR1245252

[22] Francesco Russo and Pierre Vallois, *The generalized covariation process and itô formula*, Stochastic Processes and their applications **59** (1995), no. 1, 81–104. MR1350257

[23] Francesco Russo and Pierre Vallois, *Itô formula for c^1-functions of semimartingales*, Probability theory and related fields **104** (1996), 27–41. MR1367665

[24] Francesco Russo and Pierre Vallois, *Elements of stochastic calculus via regularization*, Séminaire de Probabilités XL, Springer, 2007, pp. 147–185. MR2409004

[25] Aad W Van der Vaart, *Asymptotic statistics*, vol. 3, Cambridge university press, 2000. MR1652247

[26] Alexander D Wentzell, *On equations of the theory of conditional markov processes*, Teoriya Veroyatnostei i ee Primeneniya **10** (1965), no. 2, 390–393.

[27] Miroslav Zelenỳ, *An example of a $\mathcal{C}^{1,1}$ function, which is not a dc function*, Commentationes Mathematicae Universitatis Carolinae **43** (2002), no. 1, 149–154. MR1903313

3 Mean-Variance Optimization for Participating Life Insurance Contracts

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Mean-variance optimization for participating life insurance contracts

Felix Fießinger *, Mitja Stadje

University of Ulm, Institute of Insurance Science and Institute of Mathematical Finance, Faculty of Mathematics and Economics, Ulm, Germany

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ABSTRACT

This paper studies the equity holders' mean-variance optimal portfolio choice problem for (non-)protected participating life insurance contracts. We derive explicit formulas for the optimal terminal wealth and the optimal strategy in the multi-dimensional Black-Scholes model, showing the existence of all necessary parameters. Moreover, we provide a numerical analysis of the Black-Scholes market. The equity holders on average increase their investment into the risky asset in bad economic states and decrease their investment over time.

1. Introduction

This paper investigates a mean-variance optimization from the perspective of the equity holders of an insurance company for the two standard designs of participating life insurance contracts, i.e., with a protected or non-protected guarantee. In a participating life insurance contract, the policyholder gets a (possibly protected) guarantee and participates proportionally at maturity from the portfolio value exceeding a pre-defined threshold higher than the guarantee. The policyholders receive at least the guarantee value at maturity if the guarantee is protected. If the guarantee is not protected, then the policyholders get, at most, the portfolio value where, initially, the insurance company gives additional equity to the premium for the investment, i.e., the equity has only limited liability.

The portfolio theory research goes back to the 1950s with the pioneering work of Markowitz (1952, 1959), who used variance to measure the riskiness of stock returns in a one-period setting. Mean-variance was later also analyzed in dynamic settings, see for instance Hakansson (1971), Samuelson (1975) or Merton (1972, 1975), and has been extended in various directions. For a more detailed overview of mean-variance portfolio optimizations, see the literature review from Zhang et al. (2018). In these works, the management is typically assumed to maximize the mean-variance of the entire portfolio, and no distinction between equity holders and debt holders or policyholders is made.

This paper aims to provide an explicit formula for the optimal terminal wealth (from the perspective of the equity holders of an insurance company), show that the optimal solution exists (which is non-trivial since the solution includes an additional parameter), and prove an analytical formula for the optimal strategy. Moreover, we discuss the characteristics of the terminal wealth and the optimal strategy in a numerical analysis.

While the optimal investment problem for participating life insurance contracts was solved for expected utility (EU), an analysis for mean-variance is lacking, which is the contribution of our paper. This analysis is relevant as mean-variance is widely spread in the industry, and most finance papers and textbooks use it as the benchmark to measure risk; see, for instance, Cochrane (2009). The reason is that mean-variance provides a straightforward interpretation of risk vs. reward, admits tractable statistical properties, and also, due to historical contingencies, has become the leading standard taught in business schools, making it easier for a risk manager to justify its use compared to specifying a utility function. Finally, due to the nonlinearity of variance, the mathematical analysis differs from EU. In particular, one needs to specify an equivalent problem which, contrary to EU-maximization, adds an additional parameter λ to the model. The existence of such a λ is not apparent, and we are only able to derive its existence and compute it in a semi-explicit way in the case of the Black-Scholes model.

As discussed in the beginning, the insurance company offering a participating life insurance contract pays back a surplus to the policyholder

* Corresponding author.
E-mail addresses: felix.fiessinger@uni-ulm.de (F. Fießinger), mitja.stadje@uni-ulm.de (M. Stadje).

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in good economic states. Insurance policies with profit participation play an essential role in the life insurance sector. According to the European Insurance Overview 2023 (EIOPA, 2023), issued by the European Insurance and Occupational Pensions Authority (EIOPA), policyholders spent 2022 around a quarter of their gross premiums on profit participation insurance policies in the life sector (includes life, health, and pension insurance). In Croatia, Italy, and Belgium, these policies have a market share of over 50% in the life sector. Several publications are concerned with the valuation and hedging of such insurance policies, e.g., Briys and De Varenne (1997), Bacinello and Persson (2002), Gatzert and Kling (2007), Schmeiser and Wagner (2015) or Mirza and Wagner (2018). In recent years, the study of optimal investments for participating life insurance contracts in continuous time started. Lin et al. (2017) analyzed 2017 the optimal investment for an EU-problem for a specific S-shaped utility function. Afterwards, Nguyen and Stadje (2020) examined a similar setting for more general S-shaped utility functions under a Value at Risk constraint and mortality risk. He et al. (2020) made another generalization by considering a weighted utility function between the insurer and the policyholders. Moreover, Dong et al. (2020) added to the EU-problem a Value at Risk and a portfolio insurance constraint. Chen et al. (2018) considered Value at Risk, expected shortfall, and Average Value at Risk constraints. To the best of our knowledge, we are the first to consider participating insurance contracts in continuous time under mean-variance optimization.

A significant factor of an optimal policy design is the payoff structure: For a participating life insurance contract, the payoff has at least one point where it is non-differentiable, e.g., at the threshold where the proportional surplus participation starts. (The payoff of the non-protected product also has a second point of non-differentiability at the guarantee value.) At these points, the payoff changes its slope, resulting in a payoff that is generally neither convex nor concave (see Fig. 2 for an illustration of the insurer's payoff). In such situations, one often uses concavification techniques; see, for instance, Larsen (2005) or Reichlin (2013). Liang et al. (2021) used a generalization of this method. For alternative approaches in non-concave portfolio optimization, see, for instance, Kraft and Steffensen (2013), Dai et al. (2019), or Qian and Yang (2023). In this work, we use the specific structure of a payoff in a complete market to carefully compare different solution candidates and show that the optimal terminal wealth actually has a closed and, up to the implicit parameters whose existence proof and computation in our setting is delicate, simple form. We also use a standard Lagrangian approach to get the optimal terminal wealth and use the price density process to give the analytic formula for the optimal strategy. This method was used to solve several other optimization problems in complete markets; see, for instance, Basak and Shapiro (2001), Cuoco et al. (2008), Chen et al. (2018), Chen et al. (2019), Nguyen and Stadje (2020), Mi et al. (2023), Chen et al. (2024), or Avanzi et al. (2024). To use this Lagrangian approach for mean-variance, we start to give an equivalent problem in the spirit of Zhou and Li (2000). The ansatz described above then leads to having two multipliers instead of only one Lagrangian multiplier. We obtain a system of non-linear equations with several variables yielding implicit functions whose properties, through various intermediate results, then entail the existence of a solution. Although mean-variance does not respect first-order stochastic dominance, we do get similar results compared to EU-optimization regarding the general form of the optimal solution, but with the terminal wealth having a simpler structure. In particular, in a complete market, the optimal terminal wealth is piecewise linear in the price density. Compared to the papers discussed above, the main difference of our ansatz is to consider an optimization functional, where we include an option-like payoff of the participation instead of adding further conditions to a non-option-like payoff. We show in Proposition 3.6 that our optimization corresponds to a target problem, where one aims to be as close as possible to certain exogenously given constants. This alternative formulation is close to Avanzi et al. (2024), who minimize a variance-like function under solvency conditions. However, while they face a 1-target problem our optimization corresponds to a 2-target problem, where the exogenously given target depends on the region of the terminal wealth. An additional constraint ensures that mass can only be shifted from one region to another at a rate which depends on the amount of participation. Target-based optimization is common in the literature as an alternative to mean-variance, see for instance Avanzi et al. (2024), Menoncin and Vigna (2017), or Li and Forsyth (2019). Finally, our numerical results show a somewhat different investment behavior compared to EU-maximization, in particular for S-shaped utility functions. Specifically, we observe that the investment gets more conservative with shorter maturities, and the equity holders on average increase their investment into the risky asset in bad economic states. Moreover, the insurer invests more riskily when offering a non-protected participation life insurance product than when offering a protected one due to a reduced downside potential. Since variance is the most widely used risk measure in industry and the cornerstone of most modern portfolio theory and the finance literature, these results might be potentially relevant from a descriptive point of view.

Section 2 describes participating life insurance contracts and briefly introduces the functional to optimize. In Section 3, we explicitly show the optimal terminal wealth, the optimal strategy, and the existence of the necessary parameters in the Black-Scholes model. Section 4 analyzes some numerical results for the Black-Scholes model. Finally, Section 5 concludes the paper.

2. Model setup

2.1. Participating life insurance contracts

Participating insurance contracts or, in general, insurance contracts with some profit participation play a crucial role in the life sector. Fig. 1 shows that, on average, around a quarter of the gross premiums 2022 in the life sector are spent on policies with profit participation. In some countries, like Croatia, Belgium, or Italy, even more than 50% of the gross premiums are invested in such policies.

This paper focuses on two standard designs of participating life insurance contracts. Both products offer a guarantee value G and a proportional surplus participation rate α_2 when the portfolio value exceeds a threshold, which we denote by k_2. Note that k_2 is always higher or equal to G. The difference between the two designs is that the first product offers a non-protected guarantee, whereas the second provides a protected one. Protected means, in this case, that the policyholders get at least their guarantee value, independent of the economic situation at maturity. In contrast, in the non-protected case, the insurance company declares bankruptcy if the portfolio value is below G. In this case, the policyholders only get the portfolio value. For the portfolio, it is essential to note that the initial portfolio value x_0 is the sum of the premiums from the policyholders plus some initial capital from the insurer (i.e., the equity holders). There is no rule on how to set the guarantee value. One possible example of setting G is to take the sum of the premiums in addition to a guaranteed interest rate below the risk-free interest rate. Moreover, product designers often set the threshold k_2 as the sum of the premiums divided by the share of the policyholders in the portfolio.

Hence, we conclude the following payoffs V for the policyholders (pol) and the insurer (ins) for the contracts with non-protected (non) resp. protected (pro) guarantees and terminal portfolio value $X_T \geq 0$[1]:

$$V_{\text{pol}}^{\text{non}}(X_T) = \begin{cases} X_T & \text{if } X_T < G, \\ G & \text{if } G \leq X_T < k_2, \\ G + \alpha_2(X_T - k_2) & \text{if } X_T \geq k_2, \end{cases}$$

[1] For simplicity, we assume that the management or the regulator does not allow the total wealth to become negative. However, if either we extend the following definitions also for $X_T < 0$, or let the insurer cover in both cases all losses stemming from a negative terminal portfolio value, all our results hold (see also Remark 3.3c).

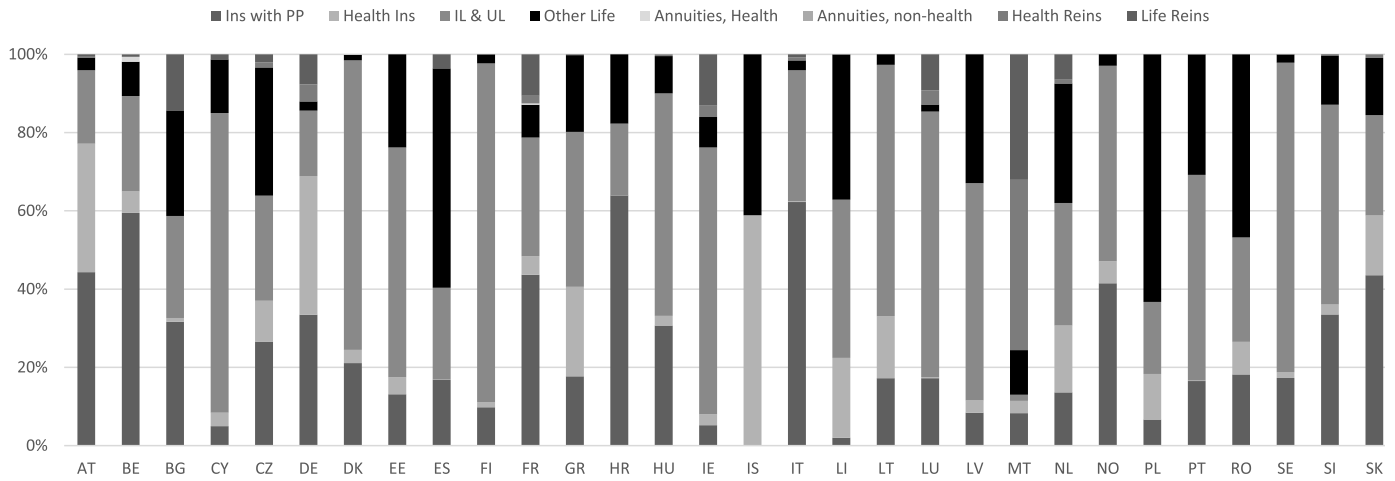

Fig. 1. Market share in 2022 of the gross premium separated by the line of business in the life sector. Data source: European Insurance Overview from the EIOPA (2023).

$$V_{\text{pol}}^{\text{pro}}(X_T) = \begin{cases} G & \text{if } X_T < k_2, \\ G + \alpha_2(X_T - k_2) & \text{if } X_T \geq k_2, \end{cases}$$

$$V_{\text{ins}}^{\text{non}}(X_T) = \begin{cases} 0 & \text{if } X_T < G, \\ X_T - G & \text{if } G \leq X_T < k_2, \\ X_T - G - \alpha_2(X_T - k_2) & \text{if } X_T \geq k_2, \end{cases}$$

$$V_{\text{ins}}^{\text{pro}}(X_T) = \begin{cases} X_T - G & \text{if } X_T < k_2, \\ X_T - G - \alpha_2(X_T - k_2) & \text{if } X_T \geq k_2. \end{cases}$$

Note that solely $V_{\text{ins}}^{\text{pro}}$ can attain negative values. For a more detailed explanation of participating insurance, we refer to Nguyen and Stadje (2020).

2.2. Optimization functional

Before defining the optimization functional, let us introduce the basic financial market, which in Section 3 and 4 will become the Black-Scholes model.

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space with time horizon $T > 0$. The filtration is generated by the d-dimensional Brownian Motion W satisfying the usual conditions. We consider an arbitrage-free market with a risk-free asset B and a deterministic interest rate $r_t \geq 0$ following the price process $\mathrm{d}B_t = B_t r_t \mathrm{d}t$, and d risky assets S^i, $i \in \{1, \dots, d\}$ with adapted price processes. Let the dynamic strategy u represent the fraction of wealth invested in the corresponding risky asset, while the remaining money is invested in the risk-free asset with the corresponding wealth process X_t and initial value $X_0 = x_0$. We denote by $\mathcal{U}$ the set of all admissible strategies given by all u, which are progressively measurable and induce an integrable X_T.

Next, we introduce a general functional to optimize, the previously defined participating life insurance products being special cases. We look for the optimal strategy $\hat{u} \in \mathcal{U}$ such that

$$J(0, T, \hat{u}, x_0) = \sup_{u\in\mathcal{U}} J(0, T, u, x_0), \tag{2.1}$$

where the value functional J is defined as

$$J(0, T, u, x_0) := \mathbb{E}[F(0, T, u, x_0)] - \gamma \mathrm{Var}(F(0, T, u, x_0))$$

with the risk aversion parameter $\gamma > 0$. Hence, we are looking for the mean-variance optimal strategy for the (continuous) function F, which we define for $s < t$ as

$$\begin{aligned} F(s, t, u, x) &:= \alpha\left((X_t - k_1)_+ - k_0\right) - \alpha_2(X_t - k_2)_+ \\ &= \begin{cases} -\alpha k_0 & X_t < k_1, \\ \alpha(X_t - k_1 - k_0) & k_1 \leq X_t < k_2, \\ \tilde{\alpha}(X_t - k_2) + \alpha(k_2 - k_1 - k_0) & X_t \geq k_2, \end{cases} \end{aligned}$$

where $X_s = x$, $0 \leq k_0, k_1 \leq k_2 < \infty$ with $k_0 + k_1 \leq k_2$, $k_2 > 0$, $0 \leq \alpha_2 < \alpha < \infty$ with $\tilde{\alpha} := \alpha - \alpha_2$. Note that $0 < \tilde{\alpha} \leq \alpha$ holds. We also write $F(X_T)$ instead of $F(0, T, u, x)$ when the trading strategy is clear (for instance, for the optimal terminal wealth $\hat{X}_T$), suppressing the initial value.

Now, one can observe that if $\alpha = 1$, $k_0 = 0$, and $k_1 = G$, the function F reduces to the payoff of the insurer for the non-protected participating life insurance contract $V_{\text{ins}}^{\text{non}}$. If $\alpha = 1$, $k_0 = G$, and $k_1 = 0$, F reduces to the payoff of the insurer for the protected product $V_{\text{ins}}^{\text{pro}}$. In the following Fig. 2, we show the payoff of the insurer for these two variants of participating life insurance contracts.

Remark 2.1. Since the problem in this paper is a variant of a mean-variance optimization, it inherits also the time-inconsistency of the classical mean-variance approach, see, for instance, Basak and Chabakauri (2010) or Björk and Murgoci (2010). There are several ways to approach this time-inconsistency. We take in this paper the perspective of a so-called pre-committed investor. Such an investor optimizes once in the beginning and sticks to this strategy, i.e., the chosen strategy of the investor is not dynamically reoptimized. Note that our problem is not covered by the general theory from Björk and Murgoci (2010) which assumes a sophisticated investor, i.e., she/he knows that her/his optimal strategy changes over time and only optimizes over all strategies she/he would actually follow.

3. Optimization in a Black-Scholes market

This section assumes that the underlying financial market follows the Black-Scholes model, i.e., the market is complete. Then, the price dynamics for the d risky assets S^i, $i \in \{1, \dots, d\}$ are given by

$$\mathrm{d}S_t^i = S_t^i \mu_t^i \mathrm{d}t + S_t^i \sigma_t^i \mathrm{d}W_t,$$

where W is the Brownian Motion generating the filtration $\mathcal{F}$, μ_i is the deterministic drift of the i'th asset and σ^{ij} is the deterministic volatility between the i'th and the j'th asset. We assume that σ_t is bounded, bounded away from zero and invertible. Then, there exists a unique price density process ξ with the wealth process X and the price density process ξ admitting the following dynamics

$$\mathrm{d}X_t = X_t\left[r_t + u_t^T(\mu_t - r)\right]\mathrm{d}t + X_t u_t^T \sigma_t \mathrm{d}W_t, \tag{3.1}$$

$$\mathrm{d}\xi_t = -\xi_t r_t \mathrm{d}t - \xi_t \kappa_t^T \mathrm{d}W_t, \tag{3.2}$$

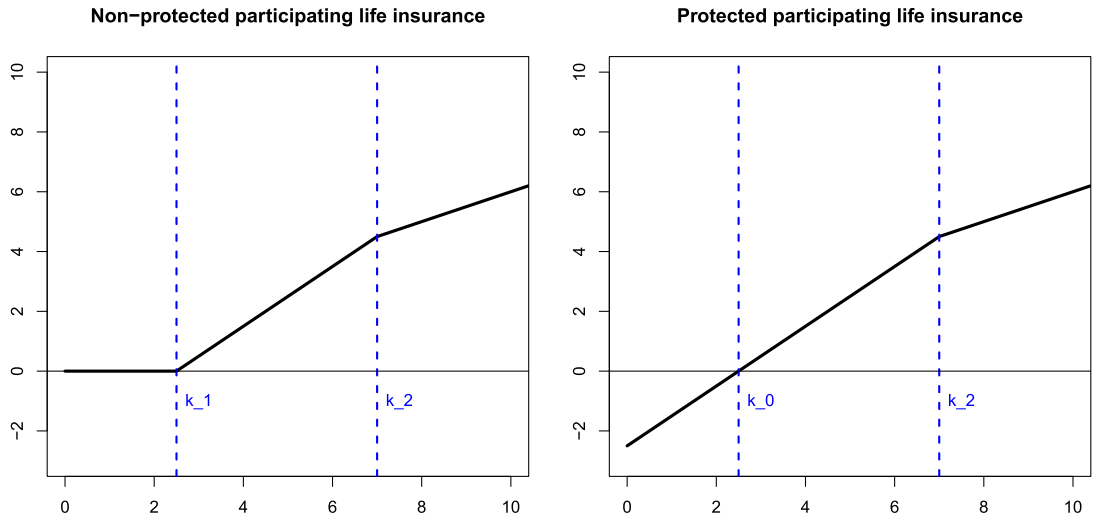

Fig. 2. Payoffs for the insurer with $k_1 = 2.5$ (left) resp. $k_0 = 2.5$ (right), $k_2 = 7$, $\alpha = 1$, $\alpha_2 = 0.5$.

with $X_0 = x_0$ and $\xi_0 = 1$, where $\kappa_t = (\sigma_t)^{-1}(\mu_t - r_t)$ is the Sharpe ratio process and $\cdot^T$ denotes the transpose of a vector. The term $\xi_T(\omega)$, $\omega \in \Omega$, can be interpreted as the Arrow-Debreu value per probability unit in state ω at time T. Note that $\xi_T(\omega)$ can be written as a decreasing function of the stock price, and therefore attains high values in times of a bad economy and low values in times of a good economy. We assume that the processes r_t and μ_t are integrable and the processes σ_t and κ_t are square-integrable over $[0,T]$ to ensure that the previous SDEs and all of the following integrals are well-defined.

3.1. Derivation of the optimal terminal wealth

The mean-variance optimization (2.1) is challenging to solve directly due to the term $(\mathbb{E}[X])^2$ in the decomposition formula $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. Hence, we show in the following Lemma 3.1 that if an optimal strategy exists, we can alternatively look for the optimal strategy considering the value functional $\tilde{J}$ which we define as:

$$\tilde{J}(0,T,u,x_0) := \mathbb{E}[\lambda F(0,T,u,x_0) - \gamma F(0,T,u,x_0)^2] \tag{3.3}$$

with $\lambda = 1 + 2\gamma\mathbb{E}\left[F(0,T,\hat{u},x_0)\right]$ where $\hat{u}$ is the optimal strategy.

Lemma 3.1. *If $\hat{u}$ is an optimal strategy for J, it is also an optimal strategy for $\tilde{J}$.*

Consequently, the optimal terminal wealth also coincides for the two problems maximizing J resp. $\tilde{J}$. This result is a slight generalization of Theorem 3.1 in Zhou and Li (2000), who showed this lemma in the case of F being the identity. We will include the proof in Appendix A for the reader's convenience.

Since we have a complete market, we optimize with the following three steps. First, we use a Lagrangian approach to find the optimal terminal wealth $\hat{X}$ (using the alternative problem). Second, we derive the optimal strategy, and third, we determine the λ and the Lagrangian multiplier y since the optimal terminal wealth and strategy depend on these values initially. In the following, we suppress the dependence on λ and y for the sake of simplicity in the notation unless stated otherwise in some proofs, and we use the convention that $(a,a] = \emptyset$ and $[a,b] = \emptyset$ if $b < a$.

Theorem 3.2. *The optimal terminal wealth $\hat{X}_T$ is given by:*

$$\hat{X}_T := \begin{cases} k_2 + \dfrac{\lambda\tilde{\alpha} - y\xi_T}{2\gamma\tilde{\alpha}^2} - \dfrac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0) & \xi_T \in (0, \xi_1^*], \\ k_2 & \xi_T \in (\tilde{\alpha}\hat{\xi}, \xi_2^*], \\ k_0 + k_1 + \dfrac{\lambda\alpha - y\xi_T}{2\gamma\alpha^2} & \xi_T \in (\alpha\hat{\xi}, \xi_3^*], \\ 0 & \text{else,} \end{cases} \tag{3.4}$$

where y is the Lagrangian multiplier which solves $\mathbb{E}[\xi_T \hat{X}_T(y)] = \xi_0 x_0$, $\lambda = 1 + 2\gamma\mathbb{E}\left[F(0,T,\hat{u},x_0)\right]$, and

$$\hat{\xi} := \max\left\{0, \frac{\lambda - 2\gamma\alpha(k_2 - k_1 - k_0)}{y}\right\},$$

$$\bar{\xi} := \frac{\lambda\alpha}{y} + \frac{2\gamma\alpha^2 k_0}{y},$$

$$\tilde{\xi}_1^* := \tilde{\alpha}\hat{\xi} - \frac{2\gamma\tilde{\alpha}}{y}\left(\sqrt{\max\left\{0, (\alpha(k_0+k_1) - \alpha_2 k_2)^2 - \alpha^2 k_0^2 + \frac{\lambda}{\gamma}(\alpha k_1 - \alpha_2 k_2)\right\}} - \tilde{\alpha}k_2\right),$$

$$\xi_1^* := \max\left\{0, \min\left\{\tilde{\alpha}\hat{\xi}, \tilde{\xi}_1^*\right\}\right\},$$

$$\tilde{\xi}_2^* := \frac{\alpha\lambda}{y} - \frac{\gamma\alpha^2(k_2-k_1)^2 - 2\gamma\alpha^2 k_0(k_2-k_1) + \lambda\alpha k_1}{y k_2},$$

$$\xi_2^* := \max\left\{\tilde{\alpha}\hat{\xi}, \min\left\{\alpha\hat{\xi}, \tilde{\xi}_2^*\right\}\right\},$$

$$\xi_3^* := \max\left\{\alpha\hat{\xi}, \bar{\xi} - \frac{2\gamma\alpha^2}{y}\left(\sqrt{k_1^2 + k_1\left(2k_0 + \frac{\lambda}{\gamma\alpha}\right)} - k_1\right)\right\}.$$

In particular, such $\hat{u}, t, y$ exist. Moreover, let ξ^* be defined by:

$$\xi^* := \begin{cases} \xi_3^* & \text{, if } \xi_3^* > \alpha\hat{\xi}, \\ \xi_2^* & \text{, if } \xi_3^* = \alpha\hat{\xi}, \xi_2^* > \tilde{\alpha}\hat{\xi}, \\ \xi_1^* & \text{, if } \xi_3^* = \alpha\hat{\xi}, \xi_2^* = \tilde{\alpha}\hat{\xi}. \end{cases}$$

Then, it holds that $\xi^* > 0$ and $\hat{X}_T > 0$ for $\xi \in (0, \xi^*)$ and $\hat{X}_T = 0$ for $\xi > \xi^*$.

Remark 3.3.

(a) The terminal wealth $\hat{X}_T$ denotes the wealth before distributing the wealth to the insurer and the policyholders. The terminal wealth of the insurer is given by $F(\hat{X}_T) = \alpha((\hat{X}_T - k_1)_+ - k_0) - \alpha_2(\hat{X}_T - k_2)_+$. In particular, in the case of a protected participating life insurance contract, i.e., $\alpha = 1$, $k_1 = 0$, the insurer makes a loss if $\hat{X}_T < k_0$. In the case of a non-protected participating life insurance contract, the insurer cannot make a loss by construction.

(b) If $\alpha_2 = \alpha$ resp. $\tilde{\alpha} = 0$, the surplus over k_2 is fully distributed to the policyholders. In this case, it holds that $\tilde{\xi}_1^* = 0$ and $\tilde{\alpha}\hat{\xi} = 0$, i.e., the result is similar with the exception that the first case, $\hat{X}_T = k_2 + \frac{\lambda\tilde{\alpha} - y\xi_T}{2\gamma\tilde{\alpha}^2} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)$, does not need to be considered. The proofs are similar, but we exclude this case for the ease of exposition.

(c) We can generalize the result for $X_T \in \mathbb{R}$ when we extend the function F to $\mathbb{R}_{<0}$ as $F(X_T) = \alpha(X_T - k_0)$. Then, the optimal terminal wealth has the same structure as before, but with the additional case that $\hat{X}_T = k_0 + \frac{\lambda\alpha - y\xi_T}{2\gamma\alpha^2}$ if $\xi_T > \bar{\xi}$. Hence, the restriction to $X_T \geq 0$ is not crucial for the structure of the solution. Note that $F(\hat{X}_T(\xi_T = \bar{\xi})) = 0$, and that for $\xi_T > \bar{\xi}$ the function F is linearly decreasing in ξ_T. Again, the proofs are similar to before, but we nevertheless restrict to $X_T \geq 0$ to avoid too technical proofs.

(d) The representation of $\hat{X}_T$ also holds in a general complete market (without necessarily using a Black-Scholes market model) if we assume the existence of the multipliers y and λ. However, that these exist is then not clear. Note that the price density process, ξ, in a different complete market would change, but still admit a geometric form. For example, if we replace the Brownian Motion W by a Poisson process N with parameter λ_2 (in the Black-Scholes model), the price density admits a dynamic of the form (see, e.g., Cont and Tankov (2004, p.303)):

$$\mathrm{d}\xi_{t-} = \xi_{t-}(\lambda_2 - \lambda_1 - r_t)\mathrm{d}t - \xi_{t-}\ln\tfrac{\lambda_2}{\lambda_1}\mathrm{d}N_t,$$

where $\lambda_1 > 0$ is an appropriately chosen constant.

(e) The way how to compute the parameters λ and y is given in the proof of Proposition A.1.

From the definition formula of $\hat{X}_T$, we can derive the following two propositions, which are proven in Appendix A:

Proposition 3.4. *It holds that* $(0, \xi_1^*] \cup (\tilde{\alpha}\hat{\xi}, \xi_2^*] \cup (\alpha\hat{\xi}, \xi_3^*] = (0, \xi^*]$ *with these constants defined in Theorem 3.2, i.e., the three intervals of the terminal wealth are connected.*

Proposition 3.5. *It holds that* $\hat{X}_T$ *as a function of* ξ *is continuous and non-increasing in* $(0, \xi^*) \cup (\xi^*, \infty)$ *with* ξ^* *as in Theorem 3.2. If* $k_1 > 0$*, then* $\hat{X}_T$ *is always discontinuous at* ξ^**.*

Therefore, $\hat{X}_T$ as a function of ξ_T is non-increasing, taking the value 0 for $\xi_T > \xi^*$ and one possible discontinuity point. See the black lines in Fig. 3 for a visualization.

In the following proposition, we give an alternative formulation for our optimization problem. Again, we give the proof in Appendix A.

Proposition 3.6. *Let* $\tilde{L}(x) = \begin{cases} \gamma\alpha^2 k_0^2, & \text{if } x < k_1, \\ \gamma\alpha^2(x - k_1 - k_0)^2, & \text{if } k_1 \leq x < k_2, \\ \gamma\tilde{\alpha}^2(x - k_2 + \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0))^2, & \text{if } x \geq k_2, \end{cases}$

and consider the optimization problem $\min_{u \in \mathcal{U}} \mathbb{E}[\tilde{L}(X_T)]$ *under the constraint* $\mathbb{E}[\alpha(X_T(\lambda) - k_1)\mathbb{1}_{k_1 \leq X_T(\lambda) < k_2} + \tilde{\alpha}(X_T(\lambda) - k_2 + \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1))\mathbb{1}_{X_T(\lambda) \geq k_2}] = \alpha k_0$ *and the budget constraint* $\mathbb{E}[\xi_T X_T(y)] = \xi_0 x_0$*, where* λ *and* y *are the respective Lagrangian multipliers.*

Then, the optimal terminal wealth is identical to the one of our main problem, i.e., given by (3.4).

Note that the loss function $\tilde{L}$ is continuous and non-decreasing in x, and that the definition and therefore the interpretation of the parameter λ is different than for mean-variance. However, Proposition 3.6 still gives an equivalent formulation. As already stated in the introduction, this formulation is a 2-target problem. Depending on the region of the terminal wealth, we want to minimize the distance to different exogenously given target values ($k_1 + k_0$ if the terminal wealth $X \in [k_1, k_2)$ respectively to $k_2 - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)$ if $X \geq k_2$). Moreover, the first constraint entails that one can only reduce mass in one region when one simultaneously increases the mass in another region where the reduction and the enlargement are, in general, not of equal size. Especially, the weight of the region $[k_2, \infty)$ is heavily influenced by the participation rate α_2 through the pre-factor $\tilde{\alpha}(= \alpha - \alpha_2)$.

Proof of Theorem 3.2. We use the Lagrangian multiplier method to prove this theorem. Therefore, choose $\lambda > 0$ and $y > 0$ as in Proposition A.1 in the appendix and we define the Lagrangian function L for $X \geq 0$ as

$$\begin{aligned} L(X, y) &:= \lambda F(0, T, u, x_0) - \gamma F(0, T, u, x_0)^2 - y\xi X \qquad (3.5) \\ &= \lambda\left[\alpha\left((X - k_1)_+ - k_0\right) - \alpha_2(X - k_2)_+\right] \\ &\quad - \gamma\left[\alpha\left((X - k_1)_+ - k_0\right) - \alpha_2(X - k_2)_+\right]^2 - y\xi X \\ &= \lambda\Big[\alpha(X - k_1 - k_0)\mathbb{1}_{k_1 \leq X < k_2} + \tilde{\alpha}(X - k_2)\mathbb{1}_{X \geq k_2} \\ &\quad + \alpha(k_2 - k_1 - k_0)\mathbb{1}_{X \geq k_2} - \alpha k_0 \mathbb{1}_{X < k_1}\Big] \\ &\quad - \gamma\Big[\alpha(X - k_1 - k_0)\mathbb{1}_{k_1 \leq X < k_2} + \tilde{\alpha}(X - k_2)\mathbb{1}_{X \geq k_2} \\ &\quad + \alpha(k_2 - k_1 - k_0)\mathbb{1}_{X \geq k_2} - \alpha k_0 \mathbb{1}_{X < k_1}\Big]^2 - y\xi X, \\ &= \lambda\alpha(X - k_1 - k_0)\mathbb{1}_{k_1 \leq X < k_2} \\ &\quad + \lambda\left[\tilde{\alpha}(X - k_2) + \alpha(k_2 - k_1 - k_0)\right]\mathbb{1}_{X \geq k_2} - \lambda\alpha k_0 \mathbb{1}_{X < k_1} \\ &\quad - \gamma\alpha^2(X - k_1 - k_0)^2\mathbb{1}_{k_1 \leq X < k_2} \\ &\quad - \gamma\left[\tilde{\alpha}(X - k_2) + \alpha(k_2 - k_1 - k_0)\right]^2\mathbb{1}_{X \geq k_2} - \gamma\alpha^2 k_0^2 \mathbb{1}_{X < k_1} \\ &\quad - y\xi X \\ &= \begin{cases} -\lambda\alpha k_0 - \gamma\alpha^2 k_0^2 - y\xi X & X \in [0, k_1), \\ \lambda\alpha(X - k_1 - k_0) - \gamma\alpha^2(X - k_1 - k_0)^2 - y\xi X & X \in [k_1, k_2), \\ \lambda\left[\tilde{\alpha}(X - k_2) + \alpha(k_2 - k_1 - k_0)\right] & \\ \quad -\gamma\left[\tilde{\alpha}(X - k_2) + \alpha(k_2 - k_1 - k_0)\right]^2 - y\xi X & X \in [k_2, \infty), \end{cases} \end{aligned}$$

where we suppress the ω, write X instead of X_T, ξ instead of ξ_T, and use y as the multiplier.

The following paragraph considers L as a function of X. Obviously, L is not smooth in k_1 and k_2, but it is continuous in $[0, \infty)$. Hence, we optimize the piecewise concave function L in the regions $[0, k_1]$, $[k_1, k_2]$, and $[k_2, \infty)$ separately and compare afterwards the maximal points. Now, for the optimization, we get:

$$\frac{\partial L}{\partial X} = \begin{cases} -y\xi, & X \in (0, k_1), \\ \lambda\alpha - 2\gamma\alpha^2(X - k_1 - k_0) - y\xi, & X \in (k_1, k_2), \\ \lambda\tilde{\alpha} - 2\gamma\tilde{\alpha}\left[\tilde{\alpha}(X - k_2) + \alpha(k_2 - k_1 - k_0)\right] - y\xi, & X \in (k_2, \infty). \end{cases}$$

Hence, it follows that we get the following possible maximal points (if they are in the respective interval): $X_1 = 0$, $X_2 = k_0 + k_1 + \frac{\lambda\alpha - y\xi}{2\gamma\alpha^2}$, and $X_3 = k_2 + \frac{\lambda\tilde{\alpha} - y\xi}{2\gamma\tilde{\alpha}^2} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)$. To have X_2 and X_3 in the correct interval, we get the following conditions (since $\xi > 0$):

$$\begin{aligned} &X_2 \in [k_1, k_2] \quad \Leftrightarrow \\ &\xi \in \left[\frac{\lambda\alpha - 2\gamma\alpha^2(k_2 - k_1 - k_0)}{y}, \frac{\lambda\alpha}{y} + \frac{2\gamma\alpha^2 k_0}{y}\right] \\ &\Leftrightarrow \xi \in \left[\alpha\hat{\xi}, \bar{\xi}\right], \qquad (3.6) \end{aligned}$$

$$X_3 \in [k_2,\infty) \quad \Leftrightarrow \quad \xi \in \left(0, \frac{\lambda\tilde{\alpha} - 2\gamma\tilde{\alpha}\alpha(k_2-k_1-k_0)}{y}\right] = \left(0, \tilde{\alpha}\hat{\xi}\right]. \quad (3.7)$$

Moreover, for the maximum of L, we must consider the different boundary values of the corresponding intervals. The first boundary value k_1 is always dominated by X_1 due to L being non-increasing in $[0,k_1]$. The second boundary value k_2 is dominated by X_2 or by X_3 if ξ is in one of the intervals from (3.6) or (3.7). If $\xi > \bar{\xi}$, then L is non-increasing in $[0,\infty)$ since then $\frac{\partial L}{\partial X} \le 0$ for all $X \ge 0$ and hence k_2 is dominated by X_1. If $\xi \in [\tilde{\alpha}\hat{\xi}, \alpha\hat{\xi}]$, then L has a local maximum in k_2 since $\frac{\partial L}{\partial X} \ge 0$ for $X \in (k_1,k_2)$ and $\frac{\partial L}{\partial X} \le 0$ for $X > k_2$. Hence, we add

$$X_4 = k_2 \quad \text{for} \quad \xi \in [\tilde{\alpha}\hat{\xi}, \alpha\hat{\xi}] \quad (3.8)$$

to the list of potential maximum points. We note that the intervals of ξ for X_2, X_3, and X_4 are disjoint except the lower (resp. upper) boundary points of X_2 (resp. X_3) with the upper (resp. lower) boundary point of X_4. However, in this case X_2 (resp. X_3) coincides with X_4. It is remarkable that due to the disjointness of the related intervals for X_2, X_3, and X_4, we do not have to compare these values themselves with each other (for the potential maximum of L). Thus, we replace the closed intervals for ξ by left-open, right-closed intervals and denote by X^* the combined solution of X_2, X_3, and X_4 with $X^* = 0$ for $\xi > \bar{\xi}$, i.e.,

$$X^* := \begin{cases} k_2 + \frac{\lambda\tilde{\alpha} - y\xi}{2\gamma\tilde{\alpha}^2} - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) & \xi \in (0, \tilde{\alpha}\hat{\xi}], \\ k_2 & \xi \in (\tilde{\alpha}\hat{\xi}, \alpha\hat{\xi}], \\ k_0 + k_1 + \frac{\lambda\alpha - y\xi}{2\gamma\alpha^2} & \xi \in (\alpha\hat{\xi}, \bar{\xi}], \\ 0 & \xi \in (\bar{\xi}, \infty). \end{cases}$$

Note that X^* takes the same values as $\hat{X}_T$ in (3.4), but the intervals are not truncated. Moreover, we observe that X^* is continuous in ξ on $(0,\bar{\xi}]$. Therefore, we only have to compare $L(X^*, y)$, y fixed, with $L(X_1, y) = -\gamma\alpha^2 k_0^2 - \lambda\alpha k_0$. Thus, we compare these values in the following separately for the three cases, i.e., depending on ξ, which value out of $\{X_2, X_3, X_4\}$ X^* attains. Note that we consider, from now on, L as a function of ξ. To emphasize this, we write $L(X, y, \xi)$.

Case 1: $\xi \in (\alpha\hat{\xi}, \bar{\xi}]$, i.e., $X^* = X_2 \in [k_1, k_2]$:
Then, we get:

$$\begin{aligned} L(X_2, y, \xi) &= \lambda\alpha\frac{\lambda\alpha - y\xi}{2\gamma\alpha^2} - \gamma\alpha^2\left(\frac{\lambda\alpha - y\xi}{2\gamma\alpha^2}\right)^2 - y\xi(k_0+k_1) - y\xi\frac{\lambda\alpha - y\xi}{2\gamma\alpha^2} \\ &= \frac{y^2}{4\gamma\alpha^2}\xi^2 - y\left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right)\xi + \frac{\lambda^2}{4\gamma}. \end{aligned} \quad (3.9)$$

Then $L(X_2, y, \xi) \ge -\gamma\alpha^2 k_0^2 - \lambda\alpha k_0 = L(X_1, y, \xi)$ if $\xi \le \xi_-$ or $\xi \ge \xi_+$ with

$$\begin{aligned} \xi_\pm &= \frac{2\gamma\alpha^2}{y^2}\left(y\left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right)\right. \\ &\quad \left. \pm\sqrt{y^2\left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right)^2 - \frac{y^2}{\gamma\alpha^2}\left(\frac{\lambda^2}{4\gamma} + \gamma\alpha^2 k_0^2 + \lambda\alpha k_0\right)}\right) \\ &= \frac{\lambda\alpha}{y} + \frac{2\gamma\alpha^2(k_0+k_1)}{y} \pm \frac{2\gamma\alpha^2}{y^2}\sqrt{y^2k_1^2 + \frac{y^2k_1\lambda}{\gamma\alpha} + 2y^2k_0k_1} \\ &= \frac{\lambda\alpha}{y} + \frac{2\gamma\alpha^2(k_0+k_1)}{y} \pm \frac{2\gamma\alpha^2}{y}\sqrt{k_1^2 + k_1\left(2k_0 + \frac{\lambda}{\gamma\alpha}\right)} \\ &= \bar{\xi} + \frac{2\gamma\alpha^2}{y}\left(\pm\sqrt{k_1^2 + k_1\left(2k_0 + \frac{\lambda}{\gamma\alpha}\right)} + k_1\right). \end{aligned}$$

We notice that $\xi_+ > \bar{\xi}$ and $\max\{\alpha\hat{\xi}, \xi_-\} = \xi_3^*$ where ξ_3^* is defined as in Theorem 3.2. If $\xi_- < \alpha\hat{\xi}$, X_2 is not in the interval $[k_1,k_2)$ which entails that $(\alpha\hat{\xi}, \xi_3^*] = \emptyset$. Otherwise, it holds that $\xi_- = \xi_3^*$ by which we get (3.4) for this case.

Case 2: $\xi \in (0, \tilde{\alpha}\hat{\xi}]$, i.e., $X^* = X_3 \in [k_2,\infty)$:
Without loss of generality, we assume that $\hat{\xi} > 0$, since otherwise $(0, \tilde{\alpha}\hat{\xi}] = \emptyset$ and there is nothing to show. Then it holds:

$$\begin{aligned} L(X_3, y, \xi) &= \lambda\left(\frac{\lambda\tilde{\alpha} - y\xi}{2\gamma\tilde{\alpha}} - \tilde{\alpha}\frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) + \alpha(k_2-k_1-k_0)\right) \\ &\quad - \gamma\left(\tilde{\alpha}\frac{\lambda\tilde{\alpha} - y\xi}{2\gamma\tilde{\alpha}^2} - \tilde{\alpha}\frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) + \alpha(k_2-k_1-k_0)\right)^2 \\ &\quad - y\xi k_2 - y\xi\frac{\lambda\tilde{\alpha} - y\xi}{2\gamma\tilde{\alpha}^2} + y\xi\frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) \\ &= \frac{y^2}{4\gamma\tilde{\alpha}^2}\xi^2 - y\left(k_2 - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) + \frac{\lambda}{2\gamma\tilde{\alpha}}\right)\xi + \frac{\lambda^2}{4\gamma}. \end{aligned} \quad (3.10)$$

Note that $k_2 - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) = \frac{\alpha}{\tilde{\alpha}}(k_0+k_1) - \frac{\alpha_2}{\tilde{\alpha}}k_2$. First we assume that the discriminant, i.e., the term under the square root in the following equation, is non-negative. Then, we have $L(X_3, y, \xi) \ge -\gamma\alpha^2 k_0^2 - \lambda\alpha k_0 = L(X_1, y, \xi)$ if $\xi \le \xi_-$ or $\xi \ge \xi_+$ with

$$\begin{aligned} \xi_\pm &= \frac{2\gamma\tilde{\alpha}^2}{y^2}\left(y\left(k_2 - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) + \frac{\lambda}{2\gamma\tilde{\alpha}}\right)\right) \\ &\quad \pm \tfrac{2\gamma\tilde{\alpha}^2}{y^2}\sqrt{y^2\left(k_2 - \tfrac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) + \tfrac{\lambda}{2\gamma\tilde{\alpha}}\right)^2 - \tfrac{y^2}{\gamma\tilde{\alpha}^2}\left(\tfrac{\lambda^2}{4\gamma} + \gamma\alpha^2k_0^2 + \lambda\alpha k_0\right)} \\ &= \frac{2\gamma\tilde{\alpha}^2}{y^2}\left(y\left(k_2 - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0) + \frac{\lambda}{2\gamma\tilde{\alpha}}\right)\right) \\ &\quad \pm \tfrac{2\gamma\tilde{\alpha}^2}{y^2}\sqrt{y^2\left(\tfrac{\alpha^2}{\tilde{\alpha}^2}(2k_0k_1+k_1^2) + \tfrac{\alpha_2^2}{\tilde{\alpha}^2}k_2^2 - 2\tfrac{\alpha\alpha_2}{\tilde{\alpha}^2}(k_0+k_1)k_2 + \tfrac{\lambda}{\gamma\tilde{\alpha}^2}(\alpha k_1 - \alpha_2 k_2)\right)} \\ &= \frac{\lambda\tilde{\alpha}}{y} - \frac{2\gamma\alpha\tilde{\alpha}(k_2-k_1-k_0)}{y} + \frac{2\gamma\tilde{\alpha}^2k_2}{y} \\ &\quad \pm \frac{2\gamma\tilde{\alpha}}{y}\sqrt{\alpha^2(2k_0k_1+k_1^2) + \alpha_2^2k_2^2 - 2\alpha\alpha_2(k_0+k_1)k_2 + \frac{\lambda}{\gamma}(\alpha k_1 - \alpha_2 k_2)} \\ &= \tilde{\alpha}\hat{\xi} + \tfrac{2\gamma\tilde{\alpha}}{y}\left(\pm\sqrt{(\alpha(k_0+k_1) - \alpha_2k_2)^2 - \alpha^2k_0^2 + \tfrac{\lambda}{\gamma}(\alpha k_1 - \alpha_2 k_2)} + \tilde{\alpha}k_2\right). \end{aligned}$$

Again, we notice that $\xi_+ > \tilde{\alpha}\hat{\xi}$ and since the discriminant is non-negative, we get $\xi_- = \tilde{\xi}_1^*$ where we can ignore this case if $\xi_- \le 0$ due to the assumption that ξ has to be in the interval $(0, \tilde{\alpha}\hat{\xi}]$. Now, if the discriminant is negative, then $L(X_3, y, \xi) \ge -\gamma\alpha^2k_0^2 - \lambda\alpha k_0$ for all ξ and $\tilde{\xi}_1^* > \tilde{\alpha}\hat{\xi}$. Hence, X_3 is optimal for all $\xi \in (0, \tilde{\alpha}\hat{\xi}]$ and it holds that $\xi_1^* = \tilde{\alpha}\hat{\xi}$ which implies (3.4) for this case.

Case 3: $\xi \in (\tilde{\alpha}\hat{\xi}, \alpha\hat{\xi}]$, i.e., $X^* = X_4 = k_2$:
Now, we get:

$$L(X_4, y, \xi) = \lambda\alpha(k_2-k_1-k_0) - \gamma\alpha^2(k_2-k_1-k_0)^2 - y\xi k_2. \quad (3.11)$$

Then $L(X_4, y, \xi) \ge -\gamma\alpha^2k_0^2 - \lambda\alpha k_0 = L(X_1, y, \xi)$ if

$$\xi \le \frac{\lambda\alpha}{y} - \frac{\gamma\alpha^2(k_2-k_1)^2 - 2\gamma\alpha^2k_0(k_2-k_1) + \lambda\alpha k_1}{yk_2} = \tilde{\xi}_2^*.$$

Hence, it follows that X_4 is optimal for $\xi \in (\tilde{\alpha}\hat{\xi}, \tilde{\xi}_2^*]$ which implies (3.4) in this case as well. Hence, (3.4) holds in total.

Finally, we must have that $\xi^* > 0$ since otherwise $(0, \xi_1^*] \cup (\tilde{\alpha}\hat{\xi}, \xi_2^*] \cup (\alpha\hat{\xi}, \xi_3^*] = \emptyset$ which implies that $\hat{X}_T \equiv 0$. Hence, the budget constraint $\mathbb{E}[\xi_T\hat{X}_T(y)] = \xi_0 x_0$ cannot be fulfilled, which is a contradiction to Proposition A.1 whose proof is given below. The property of $\hat{X}_T$ follows directly from Proposition 3.4 and $y < \infty$. □

We deduce from Theorem 3.2:

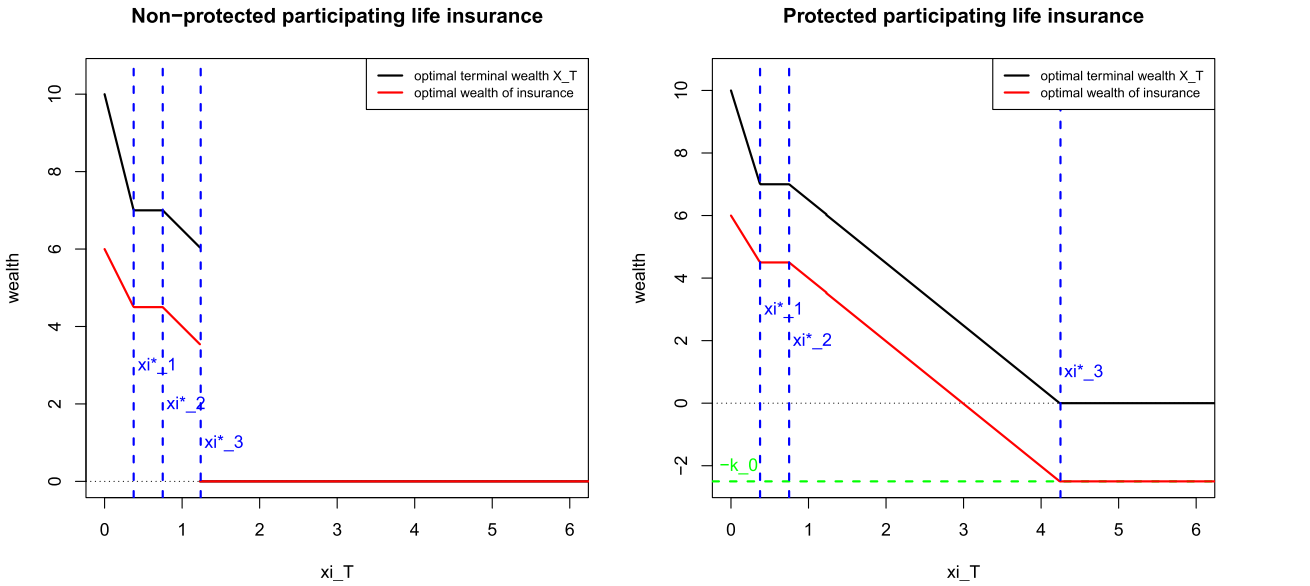

Fig. 3. The optimal value $\hat{X}_T$ and the optimal terminal wealth of the insurance company as a function of ξ_T with $k_1 = 2.5$ (left) resp. $k_0 = 2.5$ (right), $k_2 = 7$, $\alpha = 1$, $\alpha_2 = 0.5$, $\lambda = 3$, $y = 1$, $\gamma = 0.25$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Corollary 3.7. *The optimal payoff of the insurer is given by:*

$$F(\hat{X}_T) = \begin{cases} \dfrac{\lambda\tilde{\alpha} - y\xi_T}{2\gamma\tilde{\alpha}} & \xi_T \in (0, \xi_1^*], \\ \alpha(k_2 - k_1 - k_0) & \xi_T \in (\tilde{\alpha}\hat{\xi}, \xi_2^*], \\ \dfrac{\lambda\alpha - y\xi_T}{2\gamma\alpha} & \xi_T \in (\alpha\hat{\xi}, \xi_3^*], \\ -\alpha k_0 & \text{else,} \end{cases}$$

where $\hat{\xi}$, ξ_1^*, ξ_2^*, and ξ_3^* are defined as in Theorem 3.2.

Proof. This proof follows immediately by simple calculations when we plug (3.4) into the function F. □

In Fig. 3, we derive the shape of the optimal value $\hat{X}_T$ and the corresponding wealth of the insurance company as a function of ξ_T.

3.2. Derivation of the optimal strategy

Next, we derive the optimal strategy for our problem where the proof is somewhat technical and therefore given in Appendix A.

Theorem 3.8. *The optimal solution $\hat{u}$ is non-negative and given by:*

$$\hat{u}_t = (\sigma_t^T)^{-1}\kappa_t \frac{v_t}{\hat{X}_t},$$

where

$$\begin{aligned} v_t = {} & \left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)\right) \frac{e^{-\int_t^T r_s \mathrm{d}s}}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}} \varphi\left(d_1\left(\xi_1^*, t\right)\right) \\ & + \frac{y}{2\gamma\tilde{\alpha}^2}\xi_t e^{\int_t^T -(2r_s - \|\kappa_s\|^2)\mathrm{d}s}\left[\Phi\left(d_2\left(\xi_1^*, t\right)\right) - \frac{1}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}}\varphi\left(d_2\left(\xi_1^*, t\right)\right)\right] \\ & + k_2 \frac{e^{-\int_t^T r_s \mathrm{d}s}}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}}\left(\varphi\left(d_1\left(\xi_2^*, t\right)\right) - \varphi\left(d_1\left(\tilde{\alpha}\hat{\xi}, t\right)\right)\right) \\ & + \left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) \frac{e^{-\int_t^T r_s \mathrm{d}s}}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}}\left(\varphi\left(d_1\left(\xi_3^*, t\right)\right) - \varphi\left(d_1\left(\alpha\hat{\xi}, t\right)\right)\right) \\ & + \frac{y}{2\gamma\alpha^2}\xi_t e^{\int_t^T -(2r_s - \|\kappa_s\|^2)\mathrm{d}s}\left[\left(\Phi\left(d_2\left(\xi_3^*, t\right)\right) - \Phi\left(d_2\left(\alpha\hat{\xi}, t\right)\right)\right)\right. \\ & \left. - \frac{1}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}}\left(\varphi\left(d_2\left(\xi_3^*, t\right)\right) - \varphi\left(d_2\left(\alpha\hat{\xi}, t\right)\right)\right)\right], \end{aligned}$$

$$\begin{aligned} \hat{X}_t = {} & \left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)\right) e^{-\int_t^T r_s \mathrm{d}s}\Phi\left(d_1\left(\xi_1^*, t\right)\right) \\ & - \frac{y}{2\gamma\tilde{\alpha}^2}\xi_t e^{\int_t^T -(2r_s - \|\kappa_s\|^2)\mathrm{d}s}\Phi\left(d_2\left(\xi_1^*, t\right)\right) \\ & + k_2 e^{-\int_t^T r_s \mathrm{d}s}\left(\Phi\left(d_1\left(\xi_2^*, t\right)\right) - \Phi\left(d_1\left(\tilde{\alpha}\hat{\xi}, t\right)\right)\right) \\ & + \left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) e^{-\int_t^T r_s \mathrm{d}s}\left(\Phi\left(d_1\left(\xi_3^*, t\right)\right) - \Phi\left(d_1\left(\alpha\hat{\xi}, t\right)\right)\right) \\ & - \frac{y}{2\gamma\alpha^2}\xi_t e^{\int_t^T -(2r_s - \|\kappa_s\|^2)\mathrm{d}s}\left(\Phi\left(d_2\left(\xi_3^*, t\right)\right) - \Phi\left(d_2\left(\alpha\hat{\xi}, t\right)\right)\right) \end{aligned}$$

with

$$d_1(x,t) = \frac{\ln x - \ln\xi_t + \int_t^T (r_s - \frac{\|\kappa_s\|^2}{2})\mathrm{d}s}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}},$$

$$d_2(x,t) = \frac{\ln x - \ln\xi_t + \int_t^T (r_s - \frac{3\|\kappa_s\|^2}{2})\mathrm{d}s}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}} = d_1(x,t) - \sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s},$$

where Φ denotes the cdf and ϕ the density of a standard normal distributed random variable. The values $\hat{\xi}$, ξ_1^*, ξ_2^*, and ξ_3^* are defined as in Theorem 3.2.

4. Numerical results

In the following section, we discuss the numerical results for the Black-Scholes model with a given calibration of the parameters with a setting as in Section 3. The first two figures show the impact of different values for the risk aversion resp. the participation rate and of different optimization strategies (EU instead of mean-variance) on the optimal terminal wealth as a function of ξ_T. The remaining four figures show simulation outcomes for the optimal terminal wealth and the optimal strategy. We analyze the results for the non-protected resp. the protected participation life insurance product, and compare those (a) to

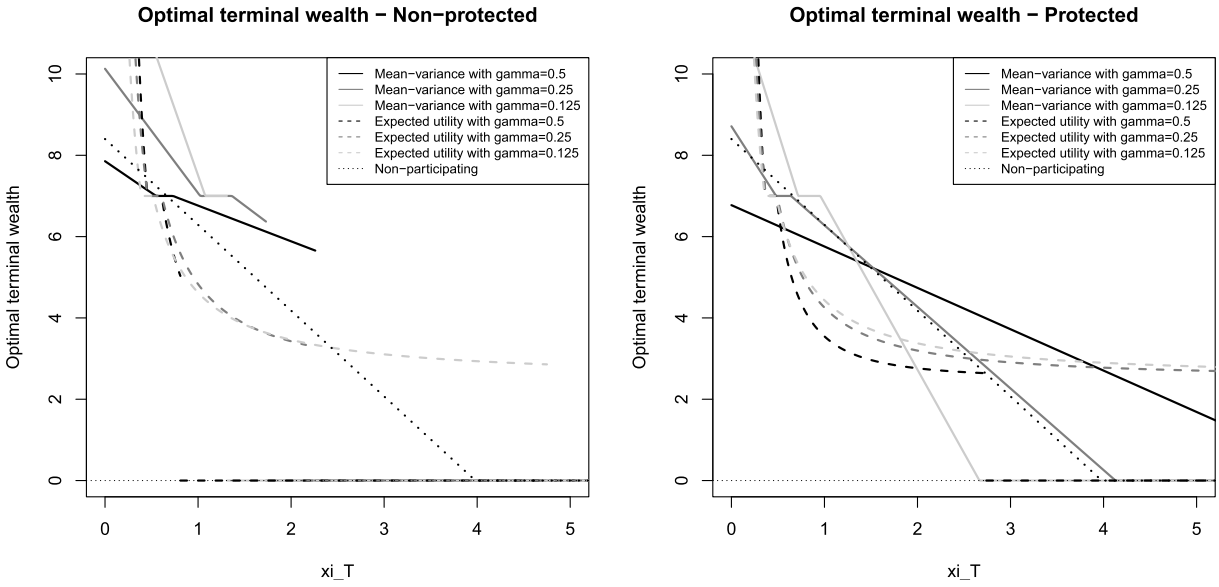

Fig. 4. Comparison of the optimal terminal wealth as a function of ξ_T of mean-variance with EU optimized participating life insurance products and no participation in the case of a not protected guarantee (left) and a protected one (right). Note that the risk aversion parameters are not comparable for mean-variance and EU.

each other, (b) to a product with no surplus participation, and (c) to the results with an EU-optimization.

For the participating life insurance contract, we use the parameters $k_0 = 0$, $k_1 = 2.5$, $k_2 = 7$, $\alpha = 1$, and $\alpha_2 = 0.25$ for the non-protected insurance product. We change the values of k_0 and k_1, i.e., $k_0 = 2.5$ and $k_1 = 0$, to get the protected product. The initial wealth, i.e., the sum of the premiums and the initial capital from the insurance company, is given by $x_0 = 4$, the time horizon by $T = 10$, and the risk aversion by $\gamma = 0.25$. We assume a constant risk-free interest rate of $r := r_t \equiv 0.02$ for the financial market and consider one risky asset with constant mean $\mu := \mu_t \equiv 0.08$ and volatility $\sigma_t \equiv 0.2$. In particular, the Sharpe ratio is also constant and given by $\kappa_t \equiv 0.3$. For the implementation, we used a time step of $\delta = 0.01$, i.e., we have 1000 intermediate points. Moreover, we simulate 10000 realizations of the Brownian Motion.

In Fig. 4, we present the optimal terminal wealth as a function of ξ_T for different values of the risk aversion parameter γ. (Note that the risk aversion parameters are not comparable for the different optimization strategies.) For both the non-protected and the protected case, we compare the optimal terminal wealth stemming from our mean-variance optimization with the EU-optimization for participating life insurance contracts from Lin et al. (2017). To measure the utility, we use, as Lin et al., the following S-shaped utility function: $U(x) = \begin{cases} x^{\tilde{\gamma}} & x \geq 0, \\ -\tilde{\lambda}(-x)^{-\tilde{\gamma}} & x < 0 \end{cases}$ with $\tilde{\lambda} = 2$ and different values for $\tilde{\gamma}$ (Lin et al. (2017) used $\tilde{\gamma} = 0.5$ in their numerical analysis). Moreover, we added the case that there is no additional surplus participation of the policyholders, i.e., $\alpha_2 = 0$, $k_0 = 0$, and $k_1 = 0$. Both participating optimizations share that the optimal wealth function has three points of significant behavioral change, i.e., at ξ_1^*, ξ_2^* and ξ_3^* as in Theorem 3.2. This theorem shows that the optimal wealth decreases before ξ_1^* and between ξ_2^* and ξ_3^*, and that the optimal wealth is constant elsewhere. The main difference in the shape of the functions is that we have a piecewise linearity in the mean-variance case but a total non-linearity in the EU case. This effect is already apparent from the respective formulas. Noticeably, the distance between the second and third point of behavioral change is much higher in the EU-optimization than in the mean-variance one. Moreover, the figure shows a moral hazard of the insurance companies offering a non-protected participating life insurance contract since the insurance companies favor a portfolio value of 0 over a portfolio value at the guarantee value level. We can observe this effect for both the mean-variance and the EU-optimization in the figure due to the drop-down of the optimal terminal wealth to zero. Protected products have no such effect when optimizing mean-variance since the insurance company always profits from higher portfolio values. While this effect also occurs when optimizing EU, the probability is smaller since the drop-down is at a far higher value (compared to the non-protected product).

In Fig. 5, we show the influence of the participation rate α_2 on the optimal terminal wealth as a function of ξ_T. We can see that a higher rate α_2 leads to a more extended plateau at $k_2 = 7$. This effect is not surprising when checking the formula (3.4) of the optimal terminal wealth $\hat{X}_T$ since a higher α_2 leads to a smaller $\tilde{\alpha}$. Thus, the intermediate interval $(\tilde{\alpha}\hat{\xi}, \xi_2^*)$ gets bigger since for our parametrization, it always holds that $\xi_2^* = \alpha\hat{\xi}$. Moreover, we can observe that ξ^* (defined in Theorem 3.2 as the point where in the non-protected case the optimal terminal wealth drops down to 0) is increasing for an increasing participation rate α_2. This effect stems from the different values of the variables λ and y when changing the participation rate. Additionally, the figure shows that for minimal values of ξ_T, the optimal terminal wealth is higher for large participation rates, which we can derive directly from the formula of $\hat{X}_T$ in (3.4). For the economic interpretation, we infer that a higher participation rate of the policyholders leads to a higher portfolio value in bad economic states, because for a higher α_2, the upside potential of the insurer is reduced. Hence, the optimal strategy should be less risky, which implies a higher portfolio value. For good economic states, the opposite effect happens. A somewhat surprising result is that in very good economic states (happening with an extremely low probability), the optimal terminal wealth is higher for higher participation rates, which occurs due to changes in the Lagrange multiplier of the budget constraint.

In both, Fig. 4 and Fig. 5, we can observe a jump of the optimal terminal wealth in the non-protected case, which is typical for problems with option-like constraints, see for instance Basak and Shapiro (2001), Chen et al. (2018), Nguyen and Stadje (2020), Chen et al. (2024), or Kraft and Steffensen (2013). The reason is that for certain outcomes, the insurance company's incentive change and therefore it can gain additional utility by changing to a different regime.[2] Moreover, the piecewise linearity observed also occurs in, e.g., Avanzi et al. (2024) who considered a minimization of a quadratic function under solvency constraints in-

[2] For instance, if the entire wealth for certain outcomes is transferred to the policyholders, the insurance company will target a terminal wealth of zero in these states, in order to gain additional wealth in states where it benefits more. (Note that the overall expectation is fixed due to the budget constraint.)

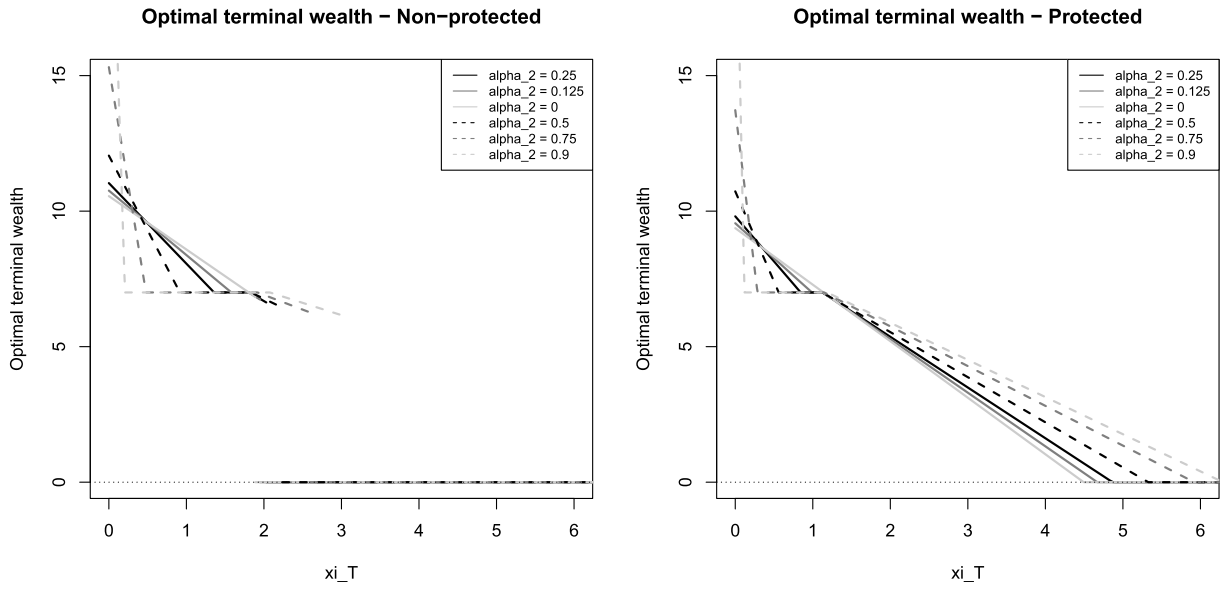

Fig. 5. Influence of the participation rate α_2 on the optimal terminal wealth as a function of ξ_T of a not protected (left) and a protected (right) participation life insurance.

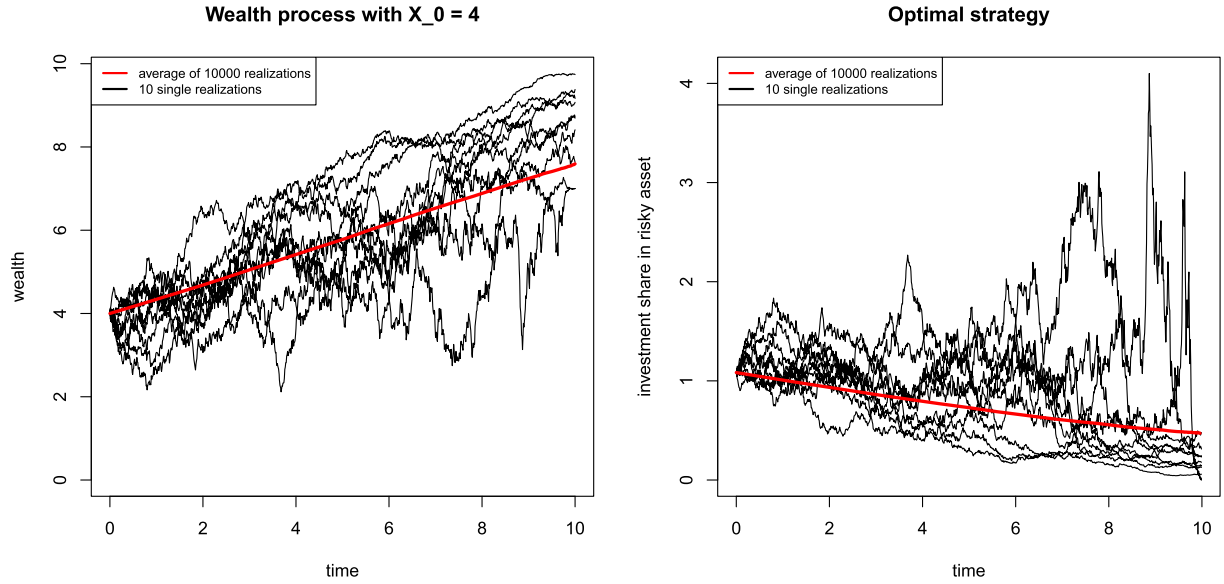

Fig. 6. Optimal wealth and optimal strategy for the non-protected participating life insurance product.

cluding option-like constraints as the expected shortfall. This piecewise linearity is typical for problems with some kind of quadratic penalties, whereas the option-like constraints entail the existence of the plateaus and the jumps.

In Fig. 6, we show the optimal wealth process and the optimal share into the risky asset over time for the non-protected participating life insurance product. In both figures, the red line shows the average of the 10000 realizations, and each of the ten black lines shows a single realization. Note that for the optimal strategy, we used a weighted average with the weight given by the absolute investment amount (which also holds for all other averages in this section). Furthermore, we get $\lambda \approx 3.423$ and $y \approx 0.860$. The average optimal wealth process develops from 4 to approximately 7.6, where the increase is slightly larger in the beginning than in the end. A terminal wealth of about 7.6 corresponds to a terminal wealth for the insurer of 4.95 and a wealth of 2.65 for the policyholder. Note that this is higher than the guarantee value, which is 2.5. When exercising the optimal strategy, we start with high investment into the risky asset (approximately 110% of our wealth), which decreases relatively constant over time. The final optimal investment share into the risky asset is around 45%. When analyzing the strategies in more detail, one observes that the most risky investments are taken when the economy has poorly evolved until then, i.e., when ξ_t is high, while the least risky investments are taken in cases where the economy has developed well. These results are reasonable since when the wealth is below the guarantee, the insurer has nothing to lose anymore, while the wealth is already over the second threshold k_2 (i.e., the policyholder also participates in the surplus over k_2), the upside potential of the insurer is reduced, whereas the downside potential is not (or only slightly).

In Fig. 7, we show the optimal wealth process and the optimal share into the risky asset over time for the protected participating life insurance product. Again, the red line shows the average of the 10000 realizations, and each of the ten black lines shows a single realization. Moreover, the variables λ and y are given by $\lambda \approx 2.893$ and $y \approx 1.003$. In this case, the optimal value evolves only to around 6.7, corresponding to a terminal wealth of the insurance company of 4.2 since we are below k_2, i.e., there is no surplus participation of the policyholder. The opti-

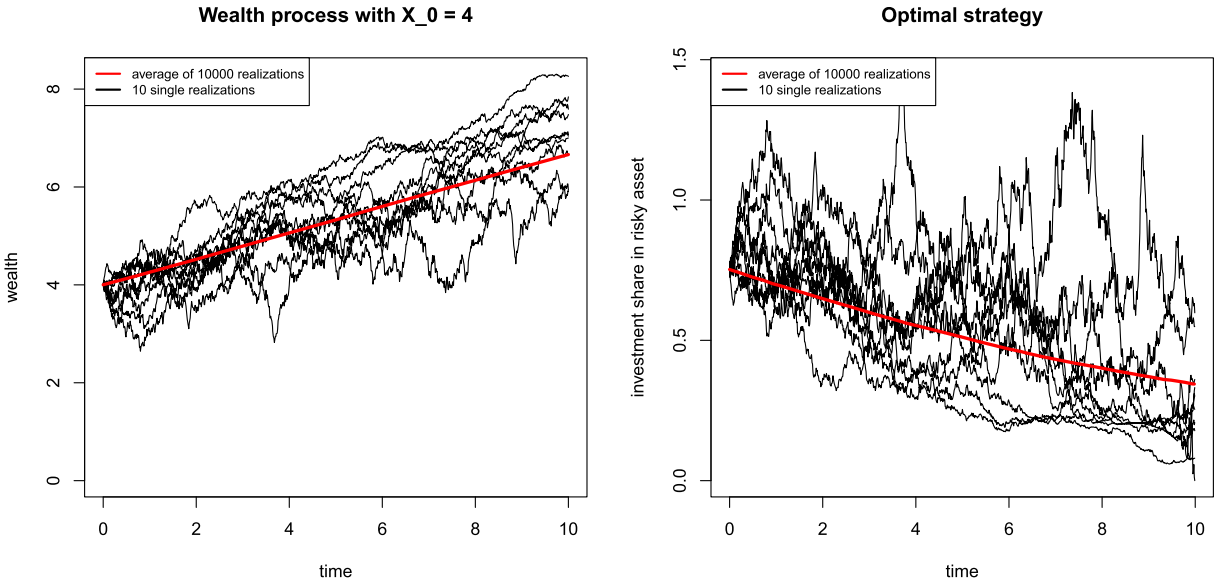

Fig. 7. Optimal wealth and optimal strategy for the protected participating life insurance product.

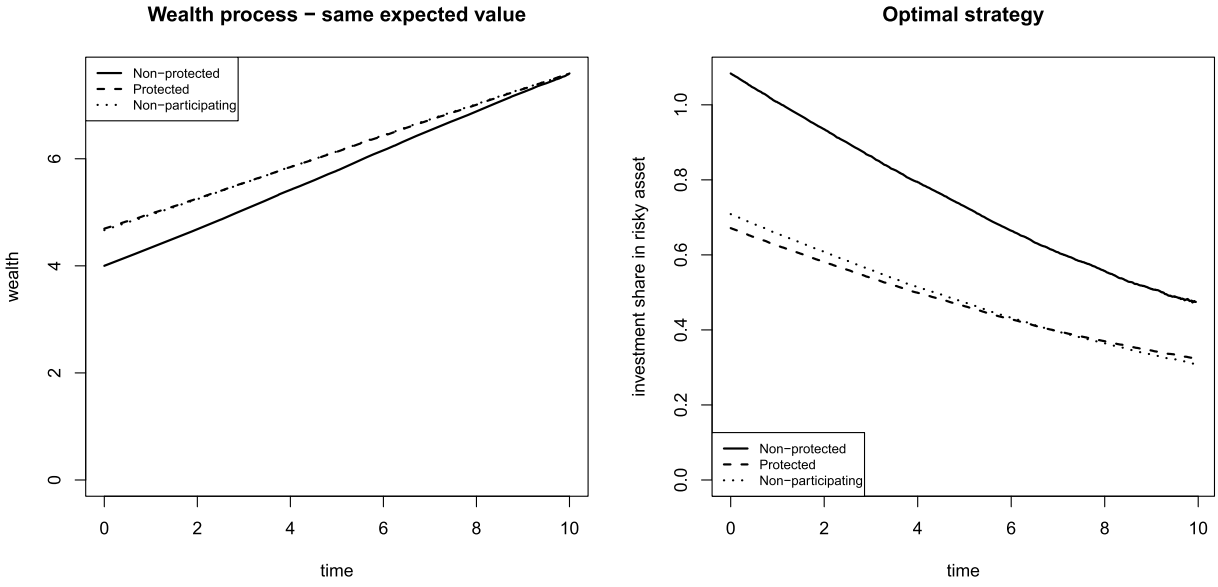

Fig. 8. Comparison of the optimal wealth (left) and strategy (right) between a (non-)protected participating life insurance product and a non-participating investment.

mal strategy starts risky but not as risky as in the non-protected case, with around 75% of the wealth invested into the risky asset. It also decreases over time, but the reduction is smaller, and the final optimal investment percentage is at around 35% of the wealth. As in the non-protected case, the realizations in the best economic states correspond with the least risky investments and vice versa. One of the shown strategies is rather inconspicuous over most of the time but has a major change close to the final time point $T = 10$, i.e., a drop-down from $u_{9.93} = 0.2$ to $u_{9.99} = 0.00001$. Such an effect (also in the other way, i.e., an upward move) happens for several realizations when their wealth for time points t, which are close to the maturity T, is close to the second threshold k_2 since the optimal terminal wealth as a function of ξ has a small plateau at k_2, i.e., if $\xi_t \in (\tilde{\alpha}\hat{\xi}, \xi_2^*]$. Hence, for $\xi_t \approx \tilde{\alpha}\hat{\xi}$ (resp. $\xi_t \approx \tilde{\alpha}\xi_2^*$) for t close to T, the upside (resp. downside) potential is reduced and the optimal strategy is to invest safely (resp. risky).

In Fig. 8, we compare the optimal wealth before splitting it between the policyholders and the insurance company and the optimal strategy of our two participating life insurance products with mean-variance to a non-participating investment, i.e., we set $\alpha_2 = 0$, $k_0 = 0$, and $k_1 = 0$. Moreover, we set the initial wealth such that the mean of $\hat{X}_T$ is approximately equal to 7.59. Therefore, we get the following initial values: $x_0 = \begin{cases} 4 & \text{non-protected product,} \\ 4.692 & \text{protected product,} \\ 4.665 & \text{no participation.} \end{cases}$ The lines show again an average of 10000 realizations each. The figure shows that the investor makes the riskiest investment when offering a non-protected insurance product and approximately identical investments when offering one of the other two products. When offering the protected product compared to the non-participation product, it is remarkable that the investor invests slightly safer in the beginning and somewhat riskier close to maturity. This investment behavior leads to the highest wealth gain for the non-protected product since, on average, the risky asset performs better than the risk-less asset (due to $\mu > r$). For the other two products, the wealth gain is approximately equal. The protected life insurance investment strategy is relatively close to the non-participating strategy since a fixed-guarantee payment only minimally influences the optimal strategy. Then, the only remaining difference in the payoff shape is when the wealth is above k_2, which happens in our chosen parametrization either relatively late

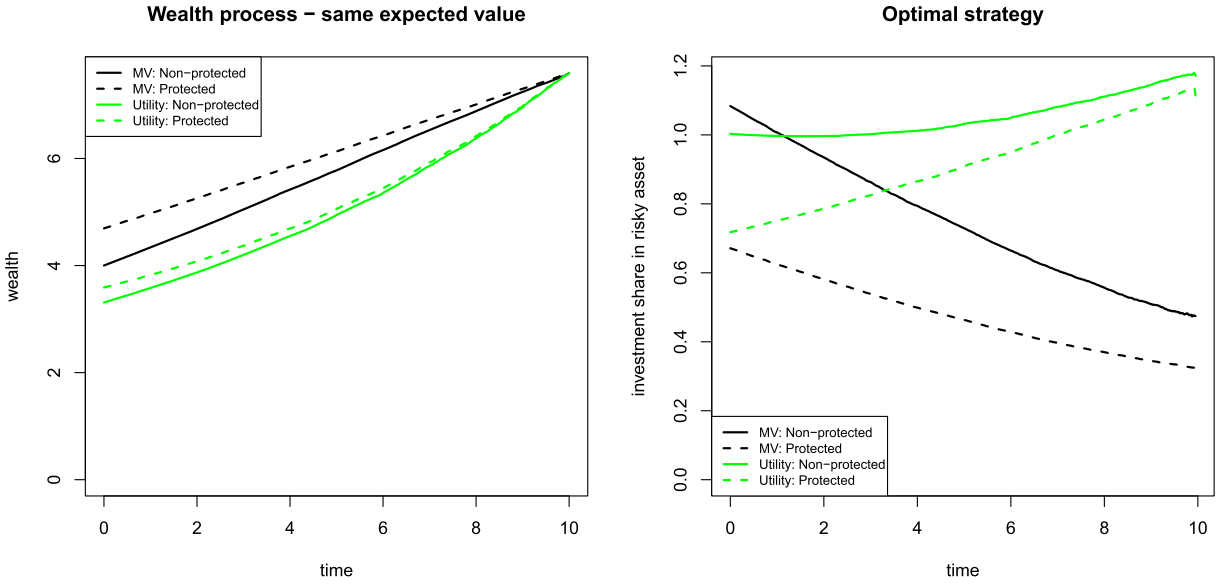

Fig. 9. Comparison of the optimal wealth (left) and the optimal strategy (right) between a (non-)protected participating life insurance contract optimizing mean-variance and EU with S-shaped utility from Lin et al. (2017).

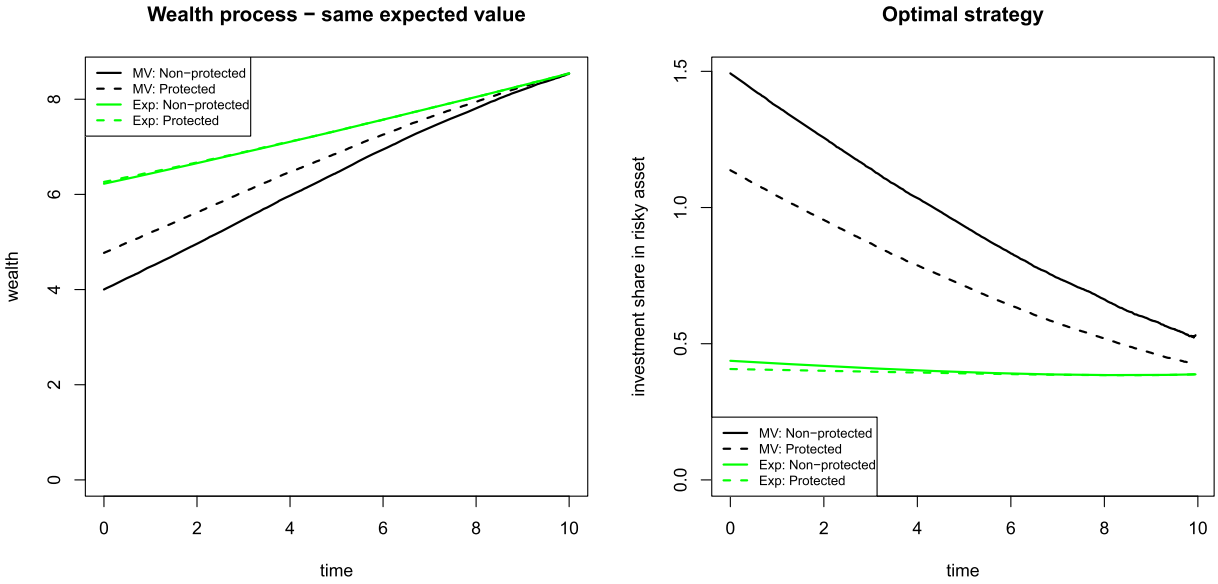

Fig. 10. Comparison of the optimal wealth (left) and the optimal strategy (right) between a (non-)protected participating life insurance contract optimizing mean-variance, or EU with an exponential utility function.

or not at all (see Fig. 7). The payoff structure for the non-protected insurance product differs highly in the values due to the reduced downside potential below k_1.

In Fig. 9, we compare the optimal wealth and strategy for the non-protected and the protected life insurance contract when considering mean-variance resp. EU-optimization. We take the result for optimizing EU from Lin et al. (2017) as in Fig. 4 with parameter values $\tilde{\gamma} = 0.125$ and $\tilde{\lambda} = 2$. As in the other figures, the lines show the average of 10000 realizations, and we state the total wealth (before splitting it between the policyholders and the insurance company). As in the previous figure, we again choose the initial value such that the terminal wealth is approximately equal for all products, i.e., we take $x_0 = \begin{cases} 4 & \text{non-protected product,} \\ 4.692 & \text{protected product} \end{cases}$ for the mean-variance optimization and $x_0 = \begin{cases} 3.31 & \text{non-protected product,} \\ 3.59 & \text{protected product} \end{cases}$ for the EU-optimization. Both optimizations have in common that the insurance company offering a non-protected product invests riskier than the insurance company offering the protected product due to the reduced downside potential. When comparing the strategies, we observe that the strategies optimizing EU become riskier over time (except for the final time points). In contrast, the strategies optimizing mean-variance get less risky over time, as already discussed. Since we chose the two (not comparable) risk aversion factors such that the initial investments in the risky asset are similar, the wealth gain of the EU-optimization is higher. The decreasing portion invested into the risky asset is typical for a pre-commitment mean-variance optimization strategy (this can be seen by implementing the optimal mean-variance strategy by Zhou and Li (2000) when using typical ranges for the parameters), which is structurally different from EU-optimization strategies.

In Fig. 10, we show another comparison for the optimal wealth and strategy for the non-protected and the protected life insurance contract when considering mean-variance resp. EU-optimization. Here, we take for the EU-optimization an exponential utility function $U(x) = 1 - e^{-\hat{\gamma}x}$,

where $\hat{\gamma} = 0.5$ denotes the risk aversion of the investor. Note that of course this risk aversion parameter is not directly comparable to the risk aversion parameter from our mean-variance model. (For small risks though U is actually related to mean-variance by a Taylor-expansion.) The optimal wealth and strategy are calculated based on Korn and Trautmann (1999). Due to their assumptions, we change the risk-free interest rate to $r = 0$. Again, we show the average of 10000 realizations, state the total wealth (before splitting it between the policyholders and the insurance company), and choose the initial value such that the terminal wealth is approximately equal for all products, i.e., we take $x_0 = \begin{cases} 4 & \text{non-protected product,} \\ 4.77 & \text{protected product} \end{cases}$ for the mean-variance optimization and $x_0 = \begin{cases} 6.224 & \text{non-protected product,} \\ 6.262 & \text{protected product} \end{cases}$ for the EU-optimization. We obtain again that the insurance company offering a non-protected product invests riskier than the insurance company offering the protected product due to the reduced downside potential, where the difference between the strategies is larger for the mean-variance optimization. In particular, we observe for exponential EU-optimization that the optimal strategy is relatively similar for the non-protected and the protected life insurance products. Furthermore, these strategies are decreasing for longer maturities and slightly increasing for short maturities, and are closer to the mean-variance optimization than the ones discussed in Fig. 9 for S-shaped utility. However, the change in the strategies for exponential EU-maximization over time is less than for mean-variance optimization.

5. Conclusion

In this paper, we derived explicit analytic formulas for general contracts, including participating life insurance contracts, when optimizing mean-variance in the multi-dimensional Black-Scholes market. Moreover, by showing the existence of all arising parameters, we showed the existence of the optimal solution. A numerical analysis shows that the mean-variance optimal strategy compared to the EU optimal strategy becomes more conservative the shorter the maturity is and is increased in particular in bad economic states. Future research directions include possibly generalizing this approach to other stock market models.

CRediT authorship contribution statement

Felix Fießinger: Writing – original draft, Methodology, Investigation, Implementation, Formal analysis, Conceptualization. **Mitja Stadje:** Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

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Declaration of competing interest

None

Appendix A. Proofs

Proof of Lemma 3.1. We show this lemma by contradiction. In particular, we assume that a $\hat{u}$ exists, which is an optimal solution for J, but not for $\tilde{J}$. Consequently, there exists a strategy u which is better than $\hat{u}$ for $\tilde{J}$, i.e.,

$$\begin{aligned}&\tilde{J}(0,T,u,x_0) - \tilde{J}(0,T,\hat{u},x_0) > 0\\ &\Leftrightarrow \lambda\left(\mathbb{E}\left[F(0,T,u,x_0) - F(0,T,\hat{u},x_0)\right]\right)\\ &\quad - \gamma\left(\mathbb{E}\left[F(0,T,u,x_0)^2 - F(0,T,\hat{u},x_0)^2\right]\right) > 0. \end{aligned} \tag{A.1}$$

The next step is to define the function $G(x,y) := y - \gamma x + \gamma y^2$. We observe that G is a convex function, and it holds that

$$\begin{aligned}&G\left(\mathbb{E}\left[F(0,T,u,x_0)^2\right], \mathbb{E}\left[F(0,T,u,x_0)\right]\right)\\ &= \mathbb{E}\left[F(0,T,u,x_0)\right] - \gamma \mathrm{Var}\left(F(0,T,u,x_0)\right) = J(0,T,u,x_0).\end{aligned}$$

We note that $\frac{\partial}{\partial x}G(x,y) = -\gamma$ and $\frac{\partial}{\partial y}G(x,y) = 1 + 2\gamma y$. Then, the convexity of G implies

$$\begin{aligned}&G\left(\mathbb{E}\left[F(0,T,u,x_0)^2\right], \mathbb{E}\left[F(0,T,u,x_0)\right]\right)\\ &\geq G\left(\mathbb{E}\left[F(0,T,\hat{u},x_0)^2\right], \mathbb{E}\left[F(0,T,\hat{u},x_0)\right]\right)\\ &\quad + \begin{pmatrix} -\gamma \\ 1+2\gamma\mathbb{E}\left[F(0,T,\hat{u},x_0)\right] \end{pmatrix} \cdot \begin{pmatrix} \mathbb{E}\left[F(0,T,u,x_0)^2\right] - \mathbb{E}\left[F(0,T,\hat{u},x_0)^2\right] \\ \mathbb{E}\left[F(0,T,u,x_0)\right] - \mathbb{E}\left[F(0,T,\hat{u},x_0)\right] \end{pmatrix}\\ &= G\left(\mathbb{E}\left[F(0,T,\hat{u},x_0)^2\right], \mathbb{E}\left[F(0,T,\hat{u},x_0)\right]\right)\\ &\quad - \gamma\left(\mathbb{E}\left[F(0,T,u,x_0)^2\right] - \mathbb{E}\left[F(0,T,\hat{u},x_0)^2\right]\right)\\ &\quad + (1 + 2\gamma\mathbb{E}\left[F(0,T,\hat{u},x_0)\right])\left(\mathbb{E}\left[F(0,T,u,x_0)\right] - \mathbb{E}\left[F(0,T,\hat{u},x_0)\right]\right)\\ &> G\left(\mathbb{E}\left[F(0,T,\hat{u},x_0)^2\right], \mathbb{E}\left[F(0,T,\hat{u},x_0)\right]\right), \end{aligned} \tag{A.2}$$

where we used (A.1) and $\lambda = 1 + 2\gamma\mathbb{E}\left[F(0,T,\hat{u},x_0)\right]$. Hence, (A.2) implies that $J(0,T,u,x_0) > J(0,T,\hat{u},x_0)$, i.e., $\hat{u}$ is not optimal, which is a contradiction and implies the lemma. □

Proof of Proposition 3.4. We prove this proposition by showing the following two statements from which the claim follows directly: (i) If $\xi_3^* > \alpha\hat{\xi}$, then it holds that $\xi_2^* = \alpha\hat{\xi}$, and (ii) if $\xi_2^* > \tilde{\alpha}\hat{\xi}$, then it holds that $\xi_1^* = \tilde{\alpha}\hat{\xi}$.

We start with the proof of (i). First, if $\hat{\xi} = 0$, then it holds that $\xi_2^* = 0$ by definition, i.e., we get the claim. Hence, we assume that $\hat{\xi} > 0$. Next, we reformulate the condition $\xi_3^* > \alpha\hat{\xi}$ to get a condition for λ. When plugging in the formulas as defined in Theorem 3.2, it holds that $\xi_3^* > \alpha\hat{\xi}$ if and only if

$$\frac{2\gamma\alpha^2(k_0+k_1) - 2\gamma\alpha^2\sqrt{k_1^2 + k_1\left(2k_0 + \frac{\lambda}{\gamma\alpha}\right)}}{y} > -\frac{2\gamma\alpha^2(k_2 - k_1 - k_0)}{y}. \tag{A.3}$$

Hence, it follows that $y < \infty$ and $k_2 > \sqrt{k_1^2 + 2k_0k_1 + \frac{\lambda k_1}{\gamma\alpha}}$ which is again equivalent to $\frac{\lambda k_1}{\gamma\alpha} < k_2^2 - k_1^2 - 2k_0k_1$. Thus, we get that $\xi_3^* > \alpha\hat{\xi}$ holds if and only if $k_1 = 0$ or $\lambda < \gamma\alpha\left(\frac{k_2^2}{k_1} - k_1 - 2k_0\right)$ since $k_2 > 0$. Before showing the implication (i), let us also reformulate the equality $\xi_2^* = \alpha\hat{\xi}$, which we want to show. First, $\xi_2^* = \alpha\hat{\xi}$ if and only if $\tilde{\xi}_2^* \geq \alpha\hat{\xi}$ which is equivalent to:

$$-\frac{\gamma\alpha^2(k_2-k_1)^2 - 2\gamma\alpha^2k_0(k_2-k_1) + \lambda\alpha k_1}{yk_2} \geq -\frac{2\gamma\alpha^2(k_2-k_1-k_0)}{y}.$$

Since $y < \infty$ (otherwise (A.3) would not hold), this is equivalent to:

$$2\gamma\alpha^2\left(k_0(k_2-k_1) + k_2(k_2-k_1-k_0)\right) - \gamma\alpha^2(k_2^2 - 2k_1k_2 + k_1^2) \geq \lambda\alpha k_1.$$

This again is equivalent to:

$$\begin{aligned}\lambda k_1 &\leq \gamma\alpha\left(2k_0k_2 - 2k_0k_1 + 2k_2^2 - 2k_1k_2 - 2k_0k_2 - k_2^2 + 2k_1k_2 - k_1^2\right)\\ &= \gamma\alpha\left(k_2^2 - k_1^2 - 2k_0k_1\right).\end{aligned}$$

Now, if $k_1 = 0$ or if $\lambda < \gamma\alpha\left(\frac{k_2^2}{k_1} - k_1 - 2k_0\right)$ the inequality is fulfilled since $k_2 > 0$. Thus, statement (i) follows.

Second, we prove (ii). By defintion, it holds that $\hat{\xi} > 0$ in order to get $\xi_2^* > \tilde{\alpha}\hat{\xi}$. As in the proof of (i), we start by reformulating the first

condition. We note that $\xi_2^* > \tilde{\alpha}\hat{\xi}$ if and only if $\tilde{\xi}_2^* > \tilde{\alpha}\hat{\xi}$ which is equivalent to:

$$\frac{\alpha\lambda}{y} - \frac{\gamma\alpha^2(k_2-k_1)^2 - 2\gamma\alpha^2 k_0(k_2-k_1) + \lambda\alpha k_1}{yk_2} > \frac{\tilde{\alpha}\lambda}{y} - \frac{2\gamma\alpha\tilde{\alpha}(k_2-k_1-k_0)}{y}.$$

From this inequality, it follows that $y < \infty$ and

$$\lambda(\alpha-\tilde{\alpha})k_2 - \lambda\alpha k_1 + \gamma\alpha^2\left(-k_2^2 + 2k_1k_2 - k_1^2 + 2k_0k_2 - 2k_0k_1\right) > 2\gamma\alpha\tilde{\alpha}\left(-k_2^2 + k_1k_2 + k_0k_2\right).$$

Since $\alpha - \tilde{\alpha} = \alpha_2$, this is equivalent to:

$$\lambda(\alpha_2k_2 - \alpha k_1) > -\gamma\alpha\left(k_2^2(\tilde{\alpha}-\alpha_2) + 2k_1k_2\alpha_2 + 2k_0k_2\alpha_2 - \alpha k_1^2 - 2\alpha k_0k_1\right). \tag{A.4}$$

As before, we also reformulate the equality $\xi_1^* = \tilde{\alpha}\hat{\xi}$. This equality is indeed true if $\tilde{\xi}_1^* \geq \tilde{\alpha}\hat{\xi}$. If the term under the square root in the definition of $\tilde{\xi}_1^*$ (see Theorem 3.2) is 0, then $\tilde{\xi}_1^* \geq \tilde{\alpha}\hat{\xi}$ is fulfilled since $\frac{2\gamma\tilde{\alpha}^2k_2}{y} \geq 0$. Now, if this term is positive, $\tilde{\xi}_1^* \geq \tilde{\alpha}\hat{\xi}$ is equivalent to:

$$\frac{2\gamma\tilde{\alpha}^2k_2}{y} \geq \frac{2\gamma\tilde{\alpha}}{y}\sqrt{(\alpha(k_0+k_1)-\alpha_2k_2)^2 - \alpha^2k_0^2 + \frac{\lambda}{\gamma}(\alpha k_1 - \alpha_2k_2)}.$$

Then, since $y < \infty$, this is equivalent to:

$$\tilde{\alpha}^2k_2^2 \geq \alpha^2k_0^2 + 2\alpha^2k_0k_1 + \alpha^2k_1^2 - 2\alpha\alpha_2k_0k_2 - 2\alpha\alpha_2k_1k_2 + \alpha_2^2k_2^2 - \alpha^2k_0^2 + \frac{\lambda}{\gamma}(\alpha k_1 - \alpha_2k_2),$$

which is again equivalent to:

$$\lambda(\alpha_2k_2 - \alpha k_1) \geq -\gamma\alpha\left(k_2^2\left(\frac{\tilde{\alpha}^2-\alpha_2^2}{\alpha}\right) + 2k_1k_2\alpha_2 + 2k_0k_2\alpha_2 - \alpha k_1^2 - 2\alpha k_0k_1\right).$$

Now, we get that $\frac{\tilde{\alpha}^2-\alpha_2^2}{\alpha} = \frac{(\tilde{\alpha}-\alpha_2)(\tilde{\alpha}+\alpha_2)}{\alpha} = \tilde{\alpha} - \alpha_2$ since $\tilde{\alpha} + \alpha_2 = \alpha$. Hence, the inequality is fulfilled if $\lambda(\alpha_2k_2 - \alpha k_1) > -\gamma\alpha(k_2^2(\tilde{\alpha}-\alpha_2) + 2k_1k_2\alpha_2 + 2k_0k_2\alpha_2 - \alpha k_1^2 - 2\alpha k_0k_1)$ which by equation (A.4) implies statement (ii) and finishes the proof. □

Proof of Proposition 3.5. Using the two properties proven in the proof of Proposition 3.4, we only have to check the boundary values $\xi_1^* = \tilde{\alpha}\hat{\xi}$ and $\xi_2^* = \alpha\hat{\xi}$ for continuity. However, this follows immediately by plugging in the values. Now, if $k_1 > 0$, we see that $\hat{X}_T > 0$ if $\xi^* \neq \xi_3^*$. If $\xi^* = \xi_3^*$, by the definition of ξ^* in Theorem 3.2, it actually holds that $\xi_3^* > \alpha\hat{\xi}$. We note that

$$\begin{aligned}\hat{X}_T(\xi_3^*) &= k_0 + k_1 + \frac{\lambda\alpha - \lambda\alpha - 2\gamma\alpha^2k_0 + 2\gamma\alpha^2\left(\sqrt{k_1^2 + k_1\left(2k_0 + \frac{\lambda}{\gamma\alpha}\right)} - k_1\right)}{2\gamma\alpha^2} \\ &= k_0 + k_1 - k_0 - k_1 + \sqrt{k_1^2 + k_1\left(2k_0 + \frac{\lambda}{\gamma\alpha}\right)} \\ &= \sqrt{k_1\left(k_1 + 2k_0 + \frac{\lambda}{\gamma\alpha}\right)}.\end{aligned}$$

Hence, $\hat{X}_T(\xi_3^*) > 0$ if $k_1 > 0$ and $\lambda > -2\gamma\alpha k_0 - \gamma\alpha k_1$. From the proof of Proposition A.3, we see that $\lambda > C$ with $C \geq -2\gamma\alpha k_0$. Thus, $\hat{X}_T(\xi_3^*) > 0$ if $k_1 > 0$. □

Proof of Proposition 3.6. For this proof, we show that the alternative formulation of this proposition admits the same Lagrange function as our main problem given in (3.5). Therefore, we first reformulate the problem as a maximization problem by replacing $\min_{u\in\mathcal{U}} \mathbb{E}[\tilde{L}(X_T)]$ with $\max_{u\in\mathcal{U}} \mathbb{E}[-\tilde{L}(X_T)]$. Then, we get for the Lagrange function with $X \geq 0$:

$$\begin{aligned}&L(X,y,\lambda) \\ &= -\tilde{L}(X) - \lambda[\alpha k_0 - \alpha(X-k_1)\mathbb{1}_{k_1\leq X<k_2} \\ &\quad - \tilde{\alpha}(X - k_2 + \tfrac{\alpha}{\tilde{\alpha}}(k_2-k_1))\mathbb{1}_{X\geq k_2}] - y\xi X \\ &= \begin{cases} -\gamma\alpha^2k_0^2 - \lambda\alpha k_0 - y\xi X, & \text{if } X\in[0,k_1), \\ -\gamma\alpha^2(X-k_1-k_0)^2 - \lambda\alpha k_0 + \lambda\alpha(X-k_1) - y\xi X, & \text{if } X\in[k_1,k_2), \\ -\gamma\tilde{\alpha}^2(X-k_2+\frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0))^2 - \lambda\alpha k_0 & \\ \quad +\lambda\tilde{\alpha}(X-k_2+\frac{\alpha}{\tilde{\alpha}}(k_2-k_1)) - y\xi X, & \text{if } X\in[k_2,\infty), \end{cases} \\ &= \begin{cases} -\gamma\alpha^2k_0^2 - \lambda\alpha k_0 - y\xi X, & \text{if } X\in[0,k_1), \\ -\gamma\alpha^2(X-k_1-k_0)^2 + \lambda\alpha(X-k_1-k_0) - y\xi X, & \text{if } X\in[k_1,k_2), \\ -\gamma(\tilde{\alpha}(X-k_2)+\alpha(k_2-k_1-k_0))^2 & \\ \quad +\lambda(\tilde{\alpha}(X-k_2)+\alpha(k_2-k_1-k_0)) - y\xi X, & \text{if } X\in[k_2,\infty). \end{cases}\end{aligned}$$

By comparing the terms with (3.5), we observe that the Lagrange functions are identical which gives us the proof. □

Proof of Theorem 3.8. From (3.1) and (3.2), we conclude using Itô's formula that $\xi_t\hat{X}_t$ is a martingale. Hence, we get using Theorem 3.2:

$$\begin{aligned}\hat{X}_t &= \mathbb{E}\left[\frac{\xi_T}{\xi_t}\hat{X}_T\middle|\mathcal{F}_t\right] \\ &= \left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0)\right)\mathbb{E}\left[\frac{\xi_T}{\xi_t}\mathbb{1}_{\xi_T\leq\xi_1^*}\middle|\mathcal{F}_t\right] \\ &\quad - \frac{y}{2\gamma\tilde{\alpha}^2}\mathbb{E}\left[\frac{\xi_T^2}{\xi_t}\mathbb{1}_{\xi_T\leq\xi_1^*}\middle|\mathcal{F}_t\right] + k_2\mathbb{E}\left[\frac{\xi_T}{\xi_t}\mathbb{1}_{\tilde{\alpha}\hat{\xi}<\xi_T\leq\xi_2^*}\middle|\mathcal{F}_t\right] \\ &\quad + \left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right)\mathbb{E}\left[\frac{\xi_T}{\xi_t}\mathbb{1}_{\alpha\hat{\xi}<\xi_T\leq\xi_3^*}\middle|\mathcal{F}_t\right] \\ &\quad - \frac{y}{2\gamma\alpha^2}\mathbb{E}\left[\frac{\xi_T^2}{\xi_t}\mathbb{1}_{\alpha\hat{\xi}<\xi_T\leq\xi_3^*}\middle|\mathcal{F}_t\right].\end{aligned} \tag{A.5}$$

Now, the claim for $\hat{X}_t$ follows from the formula for the conditional expectation of log-normal distributions (see (B.1)) using that it holds conditionally on $\mathcal{F}_t$:

$$\frac{\xi_T}{\xi_t} \sim \mathcal{LN}\left(-\int_t^T r_s + \frac{\|\kappa_s\|^2}{2}\mathrm{d}s, \int_t^T \|\kappa_s\|^2\,\mathrm{d}s\right),$$

$$\frac{\xi_T^2}{\xi_t} \sim \mathcal{LN}\left(\ln\xi_t - 2\int_t^T r_s + \frac{\|\kappa_s\|^2}{2}\mathrm{d}s, 4\int_t^T \|\kappa_s\|^2\,\mathrm{d}s\right),$$

where $\mathcal{LN}$ denotes a log-normal distribution.

The next step is to calculate the volatility process of $\hat{X}_t$, i.e., the term before the $\mathrm{d}W_t$ in its SDE, denoted by $\sigma_{\hat{X}_t}$ using (3.2). There, we get $\sigma_{\hat{X}_t} = (-\kappa_t^T\xi_t)\frac{\partial\hat{X}_t}{\partial\xi_t} = \kappa_t^Tv_t$, where we used that $\frac{\partial d_1(x)}{\partial\xi_t} = \frac{\partial d_2(x)}{\partial\xi_t} = \frac{1}{-\xi_t\sqrt{\int_t^T\|\kappa_s\|^2\,\mathrm{d}s}}$. Now, we get the optimal strategy $\hat{u}_t$ by comparing the volatility processes from $\hat{X}_t$ which is $\sigma_{\hat{X}_t} = \hat{X}_t\hat{u}^T\sigma_t$. Hence, the claim follows. □

A.1. Proposition A.1

Since the equation $\mathbb{E}[\xi_T \hat{X}_T(y)] = \xi_0 x_0$ depends on λ and $\lambda = 1 + 2\gamma\mathbb{E}\left[F(0,T,\hat{u},x_0)\right]$ depends on y, we have to solve these two equations together. We show that these variables always exist in the following Proposition A.1.

Proposition A.1. *There always exists a solution for λ and y and an equation system that can be numerically solved to determine them.*

We split this proof into two lemmas. In the first Lemma A.2, we give the equation system to solve for the two parameters y and λ, where we added y and λ as a superscript to help the reader follow the interdependence. In the second Lemma A.3, we show that these parameters exist:

Lemma A.2. *The variables y and λ are the solution of the following equation system:*

$$
\begin{aligned}
x_0 &= \left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)\right) e^{-\int_0^T r_s \mathrm{d}s} \Phi\left(d_1\left(\xi_1^{*,y,\lambda}, 0\right)\right) \\
&\quad - \frac{y}{2\gamma\tilde{\alpha}^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s} \Phi\left(d_2\left(\xi_1^{*,y,\lambda}, 0\right)\right) \\
&\quad + k_2 e^{-\int_0^T r_s \mathrm{d}s} \left(\Phi\left(d_1\left(\xi_2^{*,y,\lambda}, 0\right)\right) - \Phi\left(d_1\left(\tilde{\alpha}\hat{\xi}^{y,\lambda}, 0\right)\right)\right) \\
&\quad + \left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) e^{-\int_0^T r_s \mathrm{d}s} \left(\Phi\left(d_1\left(\xi_3^{*,y,\lambda}, 0\right)\right) - \Phi\left(d_1\left(\alpha\hat{\xi}^{y,\lambda}, 0\right)\right)\right) \\
&\quad - \frac{y}{2\gamma\alpha^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s} \left(\Phi\left(d_2\left(\xi_3^{*,y,\lambda}, 0\right)\right) - \Phi\left(d_2\left(\alpha\hat{\xi}^{y,\lambda}, 0\right)\right)\right), \\
\lambda &= 1 - 2\gamma\alpha k_0 + 2\gamma\alpha k_0 \left(\Phi\left(d_0\left(\xi_3^{*,y,\lambda}, 0\right)\right) - \Phi\left(d_0\left(\alpha\hat{\xi}^{y,\lambda}, 0\right)\right)\right. \\
&\quad \left. + \Phi\left(d_0\left(\xi_2^{*,y,\lambda}, 0\right)\right) - \Phi\left(d_0\left(\tilde{\alpha}\hat{\xi}^{y,\lambda}, 0\right)\right) + \Phi\left(d_0\left(\xi_1^{*,y,\lambda}, 0\right)\right)\right) \\
&\quad + \lambda\left(\Phi\left(d_0\left(\xi_3^{*,y,\lambda}, 0\right)\right) - \Phi\left(d_0\left(\alpha\hat{\xi}^{y,\lambda}, 0\right)\right) + \Phi\left(d_0\left(\xi_1^{*,y,\lambda}, 0\right)\right)\right) \\
&\quad - y\left(\frac{1}{\alpha}\Phi\left(d_1\left(\xi_3^{*,y,\lambda}, 0\right)\right) - \frac{1}{\alpha}\Phi\left(d_1\left(\alpha\hat{\xi}^{y,\lambda}, 0\right)\right) + \frac{1}{\tilde{\alpha}}\Phi\left(d_1\left(\xi_1^{*,y,\lambda}, 0\right)\right)\right) \\
&\quad + 2\gamma\alpha(k_2 - k_1 - k_0)\left(\Phi\left(d_0\left(\xi_2^{*,y,\lambda}, 0\right)\right) - \Phi\left(d_0\left(\tilde{\alpha}\hat{\xi}^{y,\lambda}, 0\right)\right)\right)
\end{aligned}
$$

with

$$
d_0(x,t) = \frac{\ln x - \ln \xi_t + \int_t^T r_s + \frac{\|\kappa_s\|^2}{2} \mathrm{d}s}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}}.
$$

The values $\hat{\xi}^{y,\lambda}$, $\xi_1^{*,y,\lambda}$, $\xi_2^{*,y,\lambda}$, and $\xi_3^{*,y,\lambda}$ are defined as in Theorem 3.2 and the functions d_1 and d_2 are defined as in Theorem 3.8.

Proof. The first formula follows from the definition $\mathbb{E}[\xi_T \hat{X}_T(y)] = \xi_0 x_0 = x_0$ and (A.5) for $t = 0$. The second formula follows from the definition $\lambda = 1 + 2\gamma\mathbb{E}\left[F(0,T,\hat{u},x_0)\right]$ and the following properties for arbitrary $0 \le a < b < \infty$:

$$
\begin{aligned}
\hat{X}_T &= \left(k_2 + \frac{\lambda\tilde{\alpha} - y\xi_T}{2\gamma\tilde{\alpha}^2} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)\right) \mathbb{1}_{\xi_T \in (0, \xi_1^*]} + k_2 \mathbb{1}_{\xi_T \in [\tilde{\alpha}\hat{\xi}, \xi_2^*]} \\
&\quad + \left(k_0 + k_1 + \frac{\lambda\alpha - y\xi_T}{2\gamma\alpha^2}\right) \mathbb{1}_{\xi_T \in [\alpha\hat{\xi}, \xi_3^*]}, \\
\ln \xi_T &\sim \mathcal{N}\left(-\int_0^T r_s + \frac{\|\kappa_s\|^2}{2} \mathrm{d}s, \int_0^T \|\kappa_s\|^2 \mathrm{d}s\right),
\end{aligned}
$$

$$
\mathbb{E}[\xi_T \mathbb{1}_{\xi_T \in [a,b]}] = \Phi\left(d_1(b,0)\right) - \Phi\left(d_1(a,0)\right), \tag{A.6}
$$

$$
\mathbb{E}[\mathbb{1}_{\xi_T \in [a,b]}] = \mathbb{P}(\xi_T \in [a,b]) = \Phi\left(d_0(b,0)\right) - \Phi\left(d_0(a,0)\right). \tag{A.7}
$$

Moreover, we conclude from (3.4) (combined with Proposition 3.5) that $\mathbb{1}_{\hat{X}_T \ge k_2} = \mathbb{1}_{\xi_T \in (0, \xi_1^*]} + \mathbb{1}_{\xi_T \in (\tilde{\alpha}\hat{\xi}, \xi_2^*]}$, $\mathbb{1}_{\hat{X}_T \in [k_1, k_2)} = \mathbb{1}_{\xi_T \in (\alpha\hat{\xi}, \xi_3^*]}$, and $\mathbb{1}_{\hat{X}_T < k_1} = \mathbb{1}_{\hat{X}_T = 0} = \mathbb{1}_{\xi_T \notin (0, \xi_1^*] \cup (\tilde{\alpha}\hat{\xi}, \xi_2^*] \cup (\alpha\hat{\xi}, \xi_3^*]}$.

Indeed, we get:

$$
\begin{aligned}
&\mathbb{E}\left[F(0,T,\hat{u},x_0)\right] \\
&= \mathbb{E}\left[-\alpha k_0 \mathbb{1}_{\hat{X}_T < k_1} + \alpha(\hat{X}_T - k_1 - k_0)\mathbb{1}_{\hat{X}_T \in [k_1, k_2)} + \tilde{\alpha}(\hat{X}_T - k_2)\mathbb{1}_{\hat{X}_T \ge k_2}\right] \\
&\quad + \mathbb{E}\left[\alpha(k_2 - k_1 - k_0)\mathbb{1}_{\hat{X}_T \ge k_2}\right] \\
&= -\alpha k_0 \mathbb{P}(\hat{X}_T < k_1) + \alpha\mathbb{E}[(\hat{X}_T - k_1 - k_0)\mathbb{1}_{\xi_T \in (\alpha\hat{\xi}, \xi_3^*]}] \\
&\quad + \mathbb{E}\left[\left(\tilde{\alpha}(\hat{X}_T - k_2) + \alpha(k_2 - k_1 - k_0)\right)\left(\mathbb{1}_{\xi_T \in (0, \xi_1^*]} + \mathbb{1}_{\xi_T \in (\tilde{\alpha}\hat{\xi}, \xi_2^*]}\right)\right] \\
&= -\alpha k_0 + \alpha k_0 \left(\mathbb{P}(\xi_T \in (0, \xi_1^*]) + \mathbb{P}(\xi_T \in (\tilde{\alpha}\hat{\xi}, \xi_2^*]) + \mathbb{P}(\xi_T \in (\alpha\hat{\xi}, \xi_3^*])\right) \\
&\quad + \frac{\lambda}{2\gamma}\mathbb{P}(\xi_T \in (\alpha\hat{\xi}, \xi_3^*]) - \frac{y}{2\gamma\alpha}\mathbb{E}\left[\xi_T \mathbb{1}_{\xi_T \in (\alpha\hat{\xi}, \xi_3^*]}\right] + \frac{\lambda}{2\gamma}\mathbb{P}(\xi_T \in (0, \xi_1^*]) \\
&\quad - \frac{y}{2\gamma\tilde{\alpha}}\mathbb{E}\left[\xi_T \mathbb{1}_{\xi_T \in (0, \xi_1^*]}\right] + \alpha(k_2 - k_1 - k_0)\mathbb{P}(\xi_T \in (\tilde{\alpha}\hat{\xi}, \xi_2^*]),
\end{aligned}
$$

where we used the above-discussed results in the second and (3.4) in the third equation. The claim follows with (A.6) and (A.7). □

Lemma A.3. *The equation system from Lemma A.2 admits a solution.*

Proof. For this existence proof, we make the following two definitions by writing the two equations as functions depending on y and λ:

$$
\begin{aligned}
&f_1(y,\lambda) \\
&:= -x_0 + \left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)\right) e^{-\int_0^T r_s \mathrm{d}s} \Phi\left(d_1\left(\xi_1^*, 0\right)\right) \\
&\quad - \frac{y}{2\gamma\tilde{\alpha}^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s} \Phi\left(d_2\left(\xi_1^*, 0\right)\right) \\
&\quad + k_2 e^{-\int_0^T r_s \mathrm{d}s} \left(\Phi\left(d_1\left(\xi_2^*, 0\right)\right) - \Phi\left(d_1\left(\tilde{\alpha}\hat{\xi}, 0\right)\right)\right) \\
&\quad + \left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) e^{-\int_0^T r_s \mathrm{d}s} \left(\Phi\left(d_1\left(\xi_3^*, 0\right)\right) - \Phi\left(d_1\left(\alpha\hat{\xi}, 0\right)\right)\right) \\
&\quad - \frac{y}{2\gamma\alpha^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s} \left(\Phi\left(d_2\left(\xi_3^*, 0\right)\right) - \Phi\left(d_2\left(\alpha\hat{\xi}, 0\right)\right)\right),
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
&f_2(y,\lambda) \\
&:= 1 - 2\gamma\alpha k_0 + 2\gamma\alpha k_0 \left(\Phi\left(d_0\left(\xi_3^*, 0\right)\right) - \Phi\left(d_0\left(\alpha\hat{\xi}, 0\right)\right) + \Phi\left(d_0\left(\xi_2^*, 0\right)\right)\right. \\
&\quad \left. - \Phi\left(d_0\left(\tilde{\alpha}\hat{\xi}, 0\right)\right) + \Phi\left(d_0\left(\xi_1^*, 0\right)\right)\right) \\
&\quad + \lambda\left(-1 + \Phi\left(d_0\left(\xi_3^*, 0\right)\right) - \Phi\left(d_0\left(\alpha\hat{\xi}, 0\right)\right) + \Phi\left(d_0\left(\xi_1^*, 0\right)\right)\right) \\
&\quad - y\left(\frac{1}{\alpha}\Phi\left(d_1\left(\xi_3^*, 0\right)\right) - \frac{1}{\alpha}\Phi\left(d_1\left(\alpha\hat{\xi}, 0\right)\right) + \frac{1}{\tilde{\alpha}}\Phi\left(d_1\left(\xi_1^*, 0\right)\right)\right) \\
&\quad + 2\gamma\alpha(k_2 - k_1 - k_0)\left(\Phi\left(d_0\left(\xi_2^*, 0\right)\right) - \Phi\left(d_0\left(\tilde{\alpha}\hat{\xi}, 0\right)\right)\right).
\end{aligned}
$$

Note that we suppress again the dependence of $\hat{\xi}$ and ξ_i^*, $i \in \{1,2,3\}$, on y and λ. By definition, the equation system from Lemma A.2 is solved if there exist λ^*, y^* such that $f_1(y^*, \lambda^*) = 0 = f_2(y^*, \lambda^*)$.

To show this, we define

$$
C := \begin{cases} -2\gamma\alpha k_0 & x_0 e^{\int_0^T r_s \mathrm{d}s} \le k_1, \\ 2\gamma\alpha(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1 - k_0) & x_0 e^{\int_0^T r_s \mathrm{d}s} \in (k_1, k_2], \\ 2\gamma\left(\tilde{\alpha} x_0 e^{\int_0^T r_s \mathrm{d}s} + \alpha_2 k_2 - \alpha(k_0 + k_1)\right) & x_0 e^{\int_0^T r_s \mathrm{d}s} > k_2, \end{cases}
$$

and $h : (C, \infty) \to \mathbb{R}_{\ge 0}$ an arbitrary continuous function with $\lim_{x \to C} h(x) = 0$ if $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_1$ resp. $\liminf_{x \to C} h(x) \ge 0$ if $x_0 e^{\int_0^T r_s \mathrm{d}s} \le k_1$, and $\lim_{x \to \infty} h(x) = \infty$. Then, we make the following four statements, which are shown at the end of this proof:

$$
\lim_{y \to 0} f_1(y,\lambda) > 0 \text{ for all } \lambda > C, \tag{A.9}
$$

$$
\lim_{y \to \infty} f_1(y,\lambda) \le -x_0 \text{ for all } \lambda > C, \tag{A.10}
$$

$$
\liminf_{\lambda \searrow C} f_2(h(\lambda),\lambda) \ge 1, \tag{A.11}
$$

$$
\lim_{\lambda \to \infty} f_2(h(\lambda),\lambda) = -\infty. \tag{A.12}
$$

Due to the continuity of $y \mapsto f_1(y,\cdot)$ and $\lambda \mapsto f_2(h(\lambda),\lambda)$, there exists for each $\lambda > C$ a $y^*_\lambda \in [0,\infty)$ and for all functions h a $\lambda^*_h \in (C,\infty)$ such that $f_1(y^*_\lambda,\lambda)=0$ and $f_2(h(\lambda^*_h),\lambda^*_h)=0$. If there exists more than one solution, then we take y^*_λ as the smallest one of those (well-defined due to y being bounded from below and the continuity of f_1 and f_2). Moreover, we have the following four statements, which are also shown at the end of this proof:

$$\lambda \mapsto y^*_\lambda \text{ is continuous in } \lambda \text{ on } (C,\infty), \tag{A.13}$$

$$\lim_{\lambda \searrow C} y^*_\lambda = 0 \text{ if } x_0 e^{\int_0^T r_s \mathrm{d}s} > k_1, \tag{A.14}$$

$$\liminf_{\lambda \searrow C} y^*_\lambda \geq 0 \text{ if } x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_1, \tag{A.15}$$

$$\lim_{\lambda \to \infty} y^*_\lambda = \infty. \tag{A.16}$$

Now, if we consider y^*_λ as a function of λ, it fulfills the assumptions of h. Hence, there exists a λ^* such that $f_2(y^*_{\lambda^*},\lambda^*)=0$. On the other hand, by the definition of $\lambda \mapsto y_\lambda$, we have that $f_1(y^*_{\lambda^*},\lambda^*)=0$, which implies the claim.

To finalize the proof, we show the remaining eight statements afterwards. But first, we note that it holds by definition (see Theorem 3.2):

$$0 \leq \xi^*_1 \leq \tilde{\alpha}\hat{\xi} \leq \xi^*_2 \leq \alpha\hat{\xi} \leq \xi^*_3 \leq \bar{\xi}. \tag{A.17}$$

Proof of (A.9). First of all, we claim that

$$\xi^*_3 > 0 \text{ if } \lambda > C, \tag{A.18}$$

$$\xi^*_1 > 0 \text{ if and only if } \lambda > 2\gamma\alpha(k_2 - k_1 - k_0), \tag{A.19}$$

which we will show directly after the proof of (A.9). So for now assume that (A.18) and (A.19) hold, and let $\lambda > C$.

By (A.18), $\xi^*_3 \xrightarrow{y\to 0} +\infty$ if $\lambda > C$. Moreover, it follows directly from the definition that

$$\hat{\xi} \xrightarrow{y\to 0} \begin{cases} 0, & \text{if } \lambda \leq 2\gamma\alpha(k_2-k_1-k_0), \\ +\infty, & \text{if } \lambda > 2\gamma\alpha(k_2-k_1-k_0). \end{cases} \tag{A.20}$$

Thus, if $\lambda \leq 2\gamma\alpha(k_2-k_1-k_0)$ then it holds that (i) $\Phi(d_1(\xi^*_3,0)) - \Phi(d_1(\alpha\hat{\xi},0)) \xrightarrow{y\to 0} 1$ and $\Phi(d_1(\xi^*_1,0)) \xrightarrow{y\to 0} 0$. On the other hand, if $\lambda > 2\gamma\alpha(k_2-k_1-k_0)$, then it holds that (ii) $\Phi\left(d_1\left(\xi^*_1,0\right)\right) \xrightarrow{y\to 0} 1$ as (A.19) entails that $\xi^*_1 \xrightarrow{y\to 0} \infty$. Now, if $x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_1$, it holds that (iii) $\left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) e^{-\int_0^T r_s \mathrm{d}s} > x_0$ for $\lambda > C = -2\gamma\alpha k_0$. Furthermore, if $x_0 e^{\int_0^T r_s \mathrm{d}s} \in (k_1,k_2]$, we deduce that (iv) $\left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) e^{-\int_0^T r_s \mathrm{d}s} > x_0$ for $\lambda > C = 2\gamma\alpha(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1 - k_0)$ and, if $x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_2$, it follows directly that (v) $\left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0)\right) e^{-\int_0^T r_s \mathrm{d}s} > x_0$ for $\lambda > 2\gamma\alpha(k_2-k_1-k_0) (\geq C)$. Finally, if $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_2$, then it holds that $C > 2\gamma\alpha(k_2-k_1-k_0)$ and (vi) $\left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0)\right) e^{-\int_0^T r_s \mathrm{d}s} > x_0$ for $\lambda > C = 2\gamma\left(\tilde{\alpha} x_0 e^{\int_0^T r_s \mathrm{d}s} + \alpha_2 k_2 - \alpha(k_0+k_1)\right)$. Note that $\tilde{\alpha}\hat{\xi} \leq \xi^*_2 \leq \alpha\hat{\xi}$ (see (A.17)) and hence (vii) ξ^*_2 shows the same limiting behavior as $\hat{\xi}$ for $y \to 0$. Summarizing these results, we conclude for $\underline{\lambda > C}$:

(a) Properties (i), (ii), (v), and (vi) imply:

$$\lim_{y\to 0}\left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2-k_1-k_0)\right) e^{-\int_0^T r_s \mathrm{d}s} \Phi\left(d_1\left(\xi^*_1,0\right)\right)$$
$$\begin{cases} > x_0 & \text{if } \lambda > 2\gamma\alpha(k_2-k_1-k_0) \text{ and } x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_2, \\ > x_0 & \text{if } x_0 e^{\int_0^T r_s \mathrm{d}s} > k_2, \\ \geq 0 & \text{else.} \end{cases}$$

(b) One can see directly that $\lim_{y\to 0} \frac{y}{2\gamma\tilde{\alpha}^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s} \Phi\left(d_2\left(\xi^*_1,0\right)\right) = 0$.

(c) It holds that $\lim_{y\to 0} k_2 e^{-\int_0^T r_s \mathrm{d}s}\left(\Phi\left(d_1\left(\xi^*_2,0\right)\right) - \Phi\left(d_1\left(\tilde{\alpha}\hat{\xi},0\right)\right)\right) = 0$ due to (vii).

(d) Properties (i), (iii), (iv), $\lambda > -2\gamma\alpha k_0$, and (A.17) imply:

$$\lim_{y\to 0}\left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) e^{-\int_0^T r_s \mathrm{d}s}\left(\Phi\left(d_1\left(\xi^*_3,0\right)\right) - \Phi\left(d_1\left(\alpha\hat{\xi},0\right)\right)\right)$$
$$\begin{cases} > x_0 & \text{if } \lambda \in (C, 2\gamma\alpha(k_2-k_1-k_0)] \text{ and } x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_2, \\ \geq 0 & \text{else.} \end{cases}$$

(e) One can see directly that $\lim_{y\to 0} \frac{y}{2\gamma\alpha^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s}\left(\Phi\left(d_2\left(\xi^*_3,0\right)\right) - \Phi\left(d_2\left(\alpha\hat{\xi},0\right)\right)\right) = 0$.

Note that since either $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_2$, or $x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_2$ and $\lambda > 2\gamma\alpha(k_2-k_1-k_0)$ or $\lambda \in (C, 2\gamma\alpha(k_2-k_1-k_0)]$, we always have in (a) or in (d) "$> x_0$". Thus, the claim follows. □

Proof of (A.18). We show this equation by considering the three different cases of C:

Case 1: $x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_1$, i.e., $C = -2\gamma\alpha k_0$:

Let $\lambda > C$, i.e., there exists an $\varepsilon > 0$ such that $\lambda = C + 2\gamma\alpha\varepsilon$. First, we notice that $\xi^*_3 > 0$ if $\tilde{\xi}^*_3 := \frac{\lambda\alpha}{y} + \frac{2\gamma\alpha^2 k_0}{y} - \frac{2\gamma\alpha^2}{y}\left(\sqrt{k_1^2 + k_1\left(2k_0 + \frac{\lambda}{\gamma\alpha}\right)} - k_1\right) > 0$. Now, it holds if we plug in λ:

$$\begin{aligned}\tilde{\xi}^*_3 &= \frac{2\gamma\alpha^2}{y}\left(-k_0 + \varepsilon + k_0 + k_1 - \sqrt{k_1^2 + k_1\left(2k_0 - 2k_0 + 2\varepsilon\right)}\right) \\ &= \frac{2\gamma\alpha^2}{y}\left(k_1 + \varepsilon - \sqrt{(k_1+\varepsilon)^2 - \varepsilon^2}\right).\end{aligned}$$

Since $y,\gamma,\alpha,\varepsilon > 0$ and $k_1 \geq 0$, we get that $\tilde{\xi}^*_3 > 0$ if and only if $(k_1+\varepsilon)^2 > \left(\sqrt{(k_1+\varepsilon)^2 - \varepsilon^2}\right)^2$, which is equivalent to $\varepsilon^2 > 0$. Thus, the claim, i.e., $\xi^*_3 > 0$ for $\lambda > C$, follows.

Case 2: $x_0 e^{\int_0^T r_s \mathrm{d}s} \in (k_1,k_2]$, i.e., $C = 2\gamma\alpha(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1 - k_0)$:

Let $\lambda > C$, i.e., there exists an $\varepsilon > 0$ such that $\lambda = C + 2\gamma\alpha\varepsilon$. As in the first case, we get that $\xi^*_3 > 0$ if $\tilde{\xi}^*_3 > 0$ (defined as in Case 1). Then, we get after plugging in λ:

$$\begin{aligned}\tilde{\xi}^*_3 &= \frac{2\gamma\alpha^2}{y}\left(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1 - k_0 + \varepsilon + k_0 + k_1\right. \\ &\quad \left. - \sqrt{k_1^2 + k_1\left(2k_0 + 2x_0 e^{\int_0^T r_s \mathrm{d}s} - 2k_1 - 2k_0 + 2\varepsilon\right)}\right) \\ &= \frac{2\gamma\alpha^2}{y}\left(x_0 e^{\int_0^T r_s \mathrm{d}s} + \varepsilon\right. \\ &\quad \left. - \sqrt{-k_1^2 + 2k_1 x_0 e^{\int_0^T r_s \mathrm{d}s} - \left(x_0 e^{\int_0^T r_s \mathrm{d}s}\right)^2 + \left(x_0 e^{\int_0^T r_s \mathrm{d}s}\right)^2 + 2\varepsilon k_1}\right) \\ &= \frac{2\gamma\alpha^2}{y}\left(x_0 e^{\int_0^T r_s \mathrm{d}s} + \varepsilon\right. \\ &\quad \left. - \sqrt{-\left(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1\right)^2 + \left(x_0 e^{\int_0^T r_s \mathrm{d}s}\right)^2 + 2\varepsilon k_1}\right).\end{aligned}$$

Since $y,\gamma,\alpha,\varepsilon > 0$ and $x_0 e^{\int_0^T r_s \mathrm{d}s} \geq 0$, we get that $\tilde{\xi}^*_3 > 0$ if and only if

$$\left(x_0 e^{\int_0^T r_s \mathrm{d}s} + \varepsilon\right)^2 > -\left(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1\right)^2 + \left(x_0 e^{\int_0^T r_s \mathrm{d}s}\right)^2 + 2\varepsilon k_1.$$

Since $\left(x_0 e^{\int_0^T r_s \mathrm{d}s} + \varepsilon\right)^2 = \left(x_0 e^{\int_0^T r_s \mathrm{d}s}\right)^2 + 2\varepsilon x_0 e^{\int_0^T r_s \mathrm{d}s} + \varepsilon^2$, this is equivalent to:

$$0 < \left(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1\right)^2 - 2\varepsilon k_1 + 2\varepsilon x_0 e^{\int_0^T r_s \mathrm{d}s} + \varepsilon^2$$
$$= \left(\left(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1\right) + \varepsilon\right)^2.$$

Now, the claim, i.e., $\xi_3^* > 0$ for $\lambda > C$, follows.

Case 3: $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_2$, i.e., $C = 2\gamma\left(\tilde{\alpha} x_0 e^{\int_0^T r_s \mathrm{d}s} + \alpha_2 k_2 - \alpha(k_0 + k_1)\right)$:

Then, we get immediately that $\hat{\xi} > 0$ for $\lambda > C (> 2\gamma\alpha(k_2 - k_1 - k_0)$ since $\alpha_2 + \tilde{\alpha} = \alpha)$. Hence, it holds that $\xi_3^* > 0$ due to $\xi_3^* \geq \alpha\hat{\xi}$ (see (A.17)). □

Proof of (A.19). First, let $\lambda > 2\gamma\alpha(k_2 - k_1 - k_0)$, i.e., there exists an $\varepsilon > 0$ such that $\lambda = 2\gamma\alpha(k_2 - k_1 - k_0) + 2\gamma\tilde{\alpha}\varepsilon$. We note that $\hat{\xi} > 0$ for this λ which implies that $\xi_1^* > 0$ if and only if $\tilde{\xi}_1^* > 0$. Now, if the term under the square root in the maximum in the formula of $\tilde{\xi}_1^*$ (for the definition, see Theorem 3.2) is negative, the claim follows immediately. Hence, we assume that the term is non-negative. If we plug in λ into $\tilde{\xi}_1^*$, we get:

$$\begin{aligned}\tilde{\xi}_1^* &= \frac{2\gamma\tilde{\alpha}}{y}\left(\frac{\lambda}{2\gamma} - \alpha(k_2 - k_1 - k_0) + \tilde{\alpha}k_2\right.\\ &\quad\left. - \sqrt{(\alpha(k_0 + k_1) - \alpha_2 k_2)^2 - \alpha^2 k_0^2 + \frac{\lambda}{\gamma}(\alpha k_1 - \alpha_2 k_2)}\right)\\ &= \frac{2\gamma\tilde{\alpha}}{y}\left(\tilde{\alpha}\varepsilon + \tilde{\alpha}k_2\right.\\ &\quad\left. - \sqrt{\alpha_2^2 k_2^2 + 2\alpha^2 k_1 k_2 - 2\alpha\alpha_2 k_2^2 - \alpha^2 k_1^2 + 2\alpha\tilde{\alpha}k_1\varepsilon - 2\tilde{\alpha}\alpha_2 k_2\varepsilon}\right)\\ &= \frac{2\gamma\tilde{\alpha}^2}{y}\left(k_2 + \varepsilon\right.\\ &\quad\left. - \frac{1}{\tilde{\alpha}}\sqrt{k_2^2\alpha_2(\alpha_2 - 2\alpha) + \alpha^2 k_1(2k_2 - k_1) + 2\tilde{\alpha}\varepsilon(\alpha k_1 - \alpha_2 k_2)}\right).\end{aligned}$$

Since $y, \gamma, \tilde{\alpha}, \varepsilon > 0$ and $k_2 \geq 0$, it holds that $\tilde{\xi}_1^* > 0$ if and only if

$$k_2^2 + 2\varepsilon k_2 + \varepsilon^2 > \frac{1}{\tilde{\alpha}^2}\left(k_2^2\alpha_2(-\tilde{\alpha} - \alpha) + \alpha^2 k_1(2k_2 - k_1)\right.$$
$$\left. + 2\tilde{\alpha}\varepsilon(\alpha k_1 - \alpha_2 k_2)\right).$$

This is equivalent to:

$$L := k_2^2\left(1 + \frac{\alpha_2}{\tilde{\alpha}} + \frac{\alpha\alpha_2}{\tilde{\alpha}^2}\right) + 2\varepsilon\left(k_2\left(1 + \frac{\alpha_2}{\tilde{\alpha}}\right) - k_1\frac{\alpha}{\tilde{\alpha}}\right) + \varepsilon^2$$
$$- \frac{\alpha^2}{\tilde{\alpha}^2}k_1(2k_2 - k_1) > 0.$$

Now, we notice with $\alpha = \tilde{\alpha} + \alpha_2$ that

$$1 + \frac{\alpha_2}{\tilde{\alpha}} + \frac{\alpha\alpha_2}{\tilde{\alpha}^2} = \frac{\tilde{\alpha}^2 + \alpha_2\tilde{\alpha} + \tilde{\alpha}\alpha_2 + \alpha_2^2}{\tilde{\alpha}^2} = \frac{(\tilde{\alpha} + \alpha_2)^2}{\tilde{\alpha}^2} = \frac{\alpha^2}{\tilde{\alpha}^2}.$$

Then, it follows with L as above since $1 + \frac{\alpha_2}{\tilde{\alpha}} = \frac{\alpha}{\tilde{\alpha}}$:

$$\frac{\tilde{\alpha}^2}{\alpha^2}L = k_2^2 + 2\frac{\tilde{\alpha}}{\alpha}\varepsilon(k_2 - k_1) + \frac{\tilde{\alpha}^2}{\alpha^2}\varepsilon^2 - 2k_1k_2 + k_1^2$$
$$= (k_2 - k_1)^2 + 2\frac{\tilde{\alpha}}{\alpha}\varepsilon(k_2 - k_1) + \left(\frac{\tilde{\alpha}}{\alpha}\varepsilon\right)^2 = \left((k_2 - k_1) + \frac{\tilde{\alpha}}{\alpha}\varepsilon\right)^2.$$

Hence, $L > 0$. Second, we note that $\hat{\xi} = 0$ (and thus also ξ_1^*) for $\lambda \leq 2\gamma\alpha(k_2 - k_1 - k_0)$ by definition. Hence, the claim $\xi_1^* > 0$ if and only if $\lambda > 2\gamma\alpha(k_2 - k_1 - k_0)$ follows. □

Proof of (A.10). By definition, it holds that $\bar{\xi} \xrightarrow{y\to\infty} 0$. Then, we get the claim due to (A.17) and $\lim_{x\to 0}\Phi(d_i(x,0)) = 0$ for $i \in \{1,2\}$. □

Proof of (A.11). Recall that h is a continuous function from $(C,\infty) \to \mathbb{R}_{\geq 0}$ with some limiting properties at C and ∞, and we plug this function into f_2 with $y = h(\lambda)$. To start, we give a small overview of the following proof: To show (A.11), we must find the limiting behavior of ξ_1^*, $\tilde{\alpha}\hat{\xi}$, ξ_2^*, $\alpha\hat{\xi}$, and ξ_3^* when $\lambda \searrow C$. However, this behavior heavily depends on $x_0 e^{\int_0^T r_s \mathrm{d}s}$, i.e., the different values of C and the limiting properties of h. Hence, we have to distinguish several cases. First, we differentiate between the different limiting properties for $\lambda \searrow C$ depending if $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_1$ is true. If this is true, we first analyze the limit of ξ_3^*. After that, we have to separate the cases depending if C is bigger, smaller, or equal to $2\gamma\alpha(k_2 - k_1 - k_0)$. For the equality case, we even have to separate along the limiting behavior of $\hat{\xi}$ to derive the limiting behavior for ξ_1^* and ξ_2^*. If $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_1$ is not true, we only have to separate along the possible limiting values of h.

Case 1: $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_1$, i.e., $\lim_{x\to C} h(x) = 0$:

Since $\lim_{\lambda\searrow C} h(\lambda) = 0$ and $C > -2\gamma\alpha k_0$, it holds that (i) $\xi_3^* \to +\infty$ for $\lambda \searrow C$, i.e., $y = h(\lambda) \xrightarrow{\lambda\searrow C} 0$. Indeed, it holds: If $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_2$, we get that $C > 2\gamma\alpha(k_2 - k_1 - k_0)$ and hence $\hat{\xi} \xrightarrow{\lambda\searrow C} +\infty$ since $y = h(\lambda) \xrightarrow{\lambda\searrow C} 0$. Now, (A.17) implies that $\xi_3^* \xrightarrow{\lambda\searrow C} +\infty$. If $x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_2$, it holds that $C = 2\gamma\alpha\left(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1 - k_0\right)$ since $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_1$ by assumption and we get that:

$$\xi_3^* \geq \frac{\alpha\lambda + 2\gamma\alpha^2\left(k_0 + k_1 - \sqrt{k_1^2 + k_1(2k_0 + \frac{\lambda}{\alpha\gamma})}\right)}{h(\lambda)} =: \frac{D}{h(\lambda)}.$$

When calculating the limit for $\lambda \searrow C$ for D, it holds that

$$\lim_{\lambda\to C} D = 2\gamma\alpha^2\left(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1 - k_0 + k_0 + k_1\right.$$
$$\left. - \sqrt{k_1^2 + k_1(2k_0 + 2x_0 e^{\int_0^T r_s \mathrm{d}s} - 2k_1 - 2k_0)}\right)$$
$$= 2\gamma\alpha^2\left(x_0 e^{\int_0^T r_s \mathrm{d}s} - \sqrt{-k_1^2 + 2k_1 x_0 e^{\int_0^T r_s \mathrm{d}s}}\right).$$

Now, we see that $\lim_{\lambda\searrow C} D > 0$ if and only if $\left(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1\right)^2 > 0$ which is true by assumption. Thus, $\xi_3^* \xrightarrow{\lambda\searrow C} +\infty$ since $y = h(\lambda) \xrightarrow{\lambda\searrow C} 0$, i.e., (i) is proven.

Next, if $C > 2\gamma\alpha(k_2 - k_1 - k_0)$, we get that ((ii).1) $\xi_1^* \xrightarrow{\lambda\searrow C} +\infty$ due to (A.19) and $y = h(\lambda) \xrightarrow{\lambda\searrow C} 0$ and thus also ((ii).2) $\alpha\hat{\xi}, \xi_2^*, \tilde{\alpha}\hat{\xi} \xrightarrow{\lambda\searrow C} +\infty$ due to (A.17). If $C < 2\gamma\alpha(k_2 - k_1 - k_0)$, we get that ((iii).1) $\alpha\hat{\xi} \xrightarrow{\lambda\searrow C} 0$ due to (A.20) and $y = h(\lambda) \xrightarrow{\lambda\searrow C} 0$ which can be factored out. Hence, also ((iii).2) $\xi_1^*, \tilde{\alpha}\hat{\xi}, \xi_2^* \xrightarrow{\lambda\searrow C} 0$ due to (A.17). The remaining case that $C = 2\gamma\alpha(k_2 - k_1 - k_0)$ needs a closer look. First, we note that then $k_2 = x_0 e^{\int_0^T r_s \mathrm{d}s}$ and hence $k_2 > k_1$ by the assumption of Case 1. There, we have to distinguish three more cases for the limiting behavior of ξ_1^*, $\tilde{\alpha}\hat{\xi}$, ξ_2^*, and $\alpha\hat{\xi}$ depending on the behavior of $\hat{\xi}$. Note that $\hat{\xi}$ can have multiple accumulation points for $\lambda \searrow C$ (all in $[0,\infty]$) despite $\hat{\xi}$ being continuous in λ. Now, let λ_n be a sequence converging to the lim inf, i.e., $\lim_{n\to\infty} f_2(h(\lambda_n), \lambda_n) = \liminf_{\lambda\searrow C} f_2(h(\lambda), \lambda)$. Note that $\lambda_n \xrightarrow{n\to\infty} C$ from above. We show (A.11) then for each accumulation point separately by possibly switching to subsequences, i.e., we can assume without loss of generality that $\lim_{n\to\infty} \hat{\xi}^{\lambda_n}$ exists (in $[0,\infty]$) and have to consider the following cases:

Case 1.1: $\hat{\xi}^{\lambda_n} \xrightarrow{n\to\infty} 0$:

Then, it follows immediately that (iv) $\xi_1^{*,\lambda_n}, \xi_2^{*,\lambda_n} \xrightarrow{n\to\infty} 0$ due to (A.17).

Case 1.2: $\hat{\xi}^{\lambda_n} \xrightarrow{n\to\infty} +\infty$:

Here, we first note that when applying the same argument as in the proof of (A.19) (with $\varepsilon = 0$) to the second term in the definition of $\tilde{\xi}_1^{*,\lambda_n}$,

this second term is non-negative for $\lambda = C$, i.e., $\tilde{\xi}_1^{*,\lambda_n} \geq \tilde{\alpha}\hat{\xi}^{\lambda_n}$ when taking the limit $n \to \infty$. Then, we obtain ((v).1) $\xi_1^{*,\lambda_n} \xrightarrow{n\to\infty} +\infty$. In particular, we get ((v).2) $\xi_1^{*,\lambda_n}, \tilde{\alpha}\hat{\xi}^{\lambda_n}, \xi_2^{*,\lambda_n}, \alpha\hat{\xi}^{\lambda_n} \xrightarrow{n\to\infty} +\infty$ due to (A.17).

Case 1.3: $\hat{\xi}^{\lambda_n} \xrightarrow{n\to\infty} c \in (0,\infty)$:

First, since $k_2 > k_1$, we get by plugging in $\lambda = 2\gamma\alpha(k_2 - k_1 - k_0)$ into $\tilde{\xi}_2^*$ that $\tilde{\xi}_2^{*,\lambda_n} \xrightarrow{n\to\infty} +\infty$. The reason is that $y_n = h(\lambda_n) \xrightarrow{n\to\infty} 0$ and the nominator in $\tilde{\xi}_2^{*,\lambda_n}$ converges to a positive number, i.e.:

$$\begin{aligned}
&\alpha\lambda k_2 - \gamma\alpha^2(k_2-k_1)^2 + 2\gamma\alpha^2 k_0(k_2-k_1) - \lambda_n\alpha k_1 \\
&\xrightarrow{n\to\infty} 2\gamma\alpha^2 k_2(k_2-k_1-k_0) - \gamma\alpha^2(k_2-k_1)^2 + 2\gamma\alpha^2 k_0(k_2-k_1) \\
&\quad - 2\gamma\alpha^2 k_1(k_2-k_1-k_0) \\
&= 2\gamma\alpha^2(k_2-k_1-k_0)(k_2-k_1) - \gamma\alpha^2(k_2-k_1)(k_2-k_1-2k_0) \\
&= \gamma\alpha^2(k_2-k_1)(2k_2-2k_1-2k_0-k_2+k_1+2k_0) \\
&= \gamma\alpha^2(k_2-k_1)^2 > 0.
\end{aligned}$$

Hence, ((vi).1) $\xi_2^{*,\lambda_n} \xrightarrow{n\to\infty} \alpha c$. Second, we note that $\tilde{\xi}_1^{*,\lambda_n} \geq \tilde{\alpha}\hat{\xi}^{\lambda_n}$ for $n \to \infty$ by the same argument as in Case 1.2. Thus, ((vi).2) $\xi_1^{*,\lambda_n} \xrightarrow{n\to\infty} \tilde{\alpha}c$.

Summarizing, we get in all subcases of Case 1 with $C = 2\gamma\alpha(k_2 - k_1 - k_0)$, that $\Phi(d_0(\xi_3^*, 0)) \xrightarrow{\lambda\searrow C} 1$ since $\lim_{x\to\infty}\Phi\left(d_0(x,0)\right) = 1$ due to (i) and $-\Phi(d_0(\alpha\hat{\xi}^{\lambda_n},0)) + \Phi(d_0(\xi_2^{*,\lambda_n},0)) - \Phi(d_0(\tilde{\alpha}\hat{\xi}^{\lambda_n},0)) + \Phi(d_0(\xi_1^{*,\lambda_n},0)) \xrightarrow{n\to\infty} 0$ due to either (ii), (iii), (iv), (v), or (vi) depending on C. Hence, it holds that $\Phi(d_0(\xi_3^{*,\lambda_n},0)) - \Phi(d_0(\alpha\hat{\xi}^{\lambda_n},0)) + \Phi(d_0(\xi_2^{*,\lambda_n},0)) - \Phi(d_0(\tilde{\alpha}\hat{\xi}^{\lambda_n},0)) + \Phi(d_0(\xi_1^{*,\lambda_n},0)) \xrightarrow{n\to\infty} 1$ since $\lim_{x\to\infty}\Phi\left(d_0(x,0)\right) = 1$ and $\lambda_n \xrightarrow{n\to\infty} C$ from above. In particular, we get that $\lambda(-1 + \Phi(d_0(\xi_3^{*,\lambda_n},0)) - \Phi(d_0(\alpha\hat{\xi}^{\lambda_n},0)) + \Phi(d_0(\xi_1^{*,\lambda_n},0))) + 2\gamma\alpha(k_2 - k_1 - k_0)(\Phi(d_0(\xi_2^{*,\lambda_n},0)) - \Phi(d_0(\tilde{\alpha}\hat{\xi}^{\lambda_n},0))) \xrightarrow{n\to\infty} 0$ since $C = 2\gamma\alpha(k_2 - k_1 - k_0)$ and $\lambda_n \xrightarrow{n\to\infty} C$ from above. Thus, the claim follows.

Case 2: $x_0 e^{\int_0^T r_s \mathrm{d}s} \leq k_1$, i.e., $\liminf_{x\searrow C} h(x) \geq 0$ and $C = -2\gamma\alpha k_0$:

For this case, we have to distinguish two more subcases depending on the possible limit of h. However, h can have multiple accumulation points. As before, we consider each accumulation point separately, denote by λ_n a sequence converging to the liminf of h, and assume, without loss of generality by possibly switching to a subsequence, that $\lambda_n \xrightarrow{n\to\infty} C$ from above and $\lim_{n\to\infty} h(\lambda_n)$ exists (in $[0,\infty]$). This gives us the following cases:

Case 2.1: $\lim_{n\to\infty} h(\lambda_n) > 0$:

Then, we get that $\bar{\xi}^{\lambda_n} \xrightarrow{n\to\infty} 0$ since $\bar{\xi}^{\lambda_n} = \frac{\alpha(\lambda_n - C)}{h(\lambda_n)}$ for $C = -2\gamma\alpha k_0$ which implies that $\hat{\xi}^{\lambda_n}, \xi_1^{*,\lambda_n}, \xi_2^{*,\lambda_n}, \xi_3^{*,\lambda_n} \xrightarrow{n\to\infty} 0$ using (A.17). Thus, it holds that $\lim_{n\to\infty} f_2(h(\lambda_n),\lambda_n) = 1 - 2\gamma\alpha k_0 - C = 1$ using that $\lim_{x\to 0}\Phi(d_i(x,0)) = 0$ for $i \in \{0,1\}$.

Case 2.2: $\lim_{n\to\infty} h(\lambda_n) = 0$:

Here, we first rewrite f_2 into:

$$\begin{aligned}
f_2(h(\lambda_n),\lambda_n) &:= 1 + 2\gamma\alpha k_0\Big(-1 + \Phi\left(d_0\left(\xi_3^{*,\lambda_n},0\right)\right) \\
&\quad - \Phi\left(d_0\left(\alpha\hat{\xi}^{\lambda_n},0\right)\right) + \Phi\left(d_0\left(\xi_1^{*,\lambda_n},0\right)\right)\Big) \\
&\quad + \lambda_n\Big(-1 + \Phi\left(d_0\left(\xi_3^{*,\lambda_n},0\right)\right) \\
&\quad - \Phi\left(d_0\left(\alpha\hat{\xi}^{\lambda_n},0\right)\right) + \Phi\left(d_0\left(\xi_1^{*,\lambda_n},0\right)\right)\Big) \\
&\quad - h(\lambda_n)\Big(\frac{1}{\alpha}\Phi\left(d_1\left(\xi_3^{*,\lambda_n},0\right)\right) - \frac{1}{\alpha}\Phi\left(d_1\left(\alpha\hat{\xi}^{\lambda_n},0\right)\right) \\
&\quad + \frac{1}{\tilde{\alpha}}\Phi\left(d_1\left(\xi_1^{*,\lambda_n},0\right)\right)\Big) \\
&\quad + 2\gamma\alpha(k_2-k_1)\Big(\Phi\left(d_0\left(\xi_2^{*,\lambda_n},0\right)\right) - \Phi\left(d_0\left(\tilde{\alpha}\hat{\xi}^{\lambda_n},0\right)\right)\Big) \\
&= 1 + (\lambda_n - C)\Big(-1 + \Phi\left(d_0\left(\xi_3^{*,\lambda_n},0\right)\right) \\
&\quad - \Phi\left(d_0\left(\alpha\hat{\xi}^{\lambda_n},0\right)\right) + \Phi\left(d_0\left(\xi_1^{*,\lambda_n},0\right)\right)\Big) \\
&\quad - h(\lambda_n)\Big(\frac{1}{\alpha}\Phi\left(d_1\left(\xi_3^{*,\lambda_n},0\right)\right) - \frac{1}{\alpha}\Phi\left(d_1\left(\alpha\hat{\xi}^{\lambda_n},0\right)\right) \\
&\quad + \frac{1}{\tilde{\alpha}}\Phi\left(d_1\left(\xi_1^{*,\lambda_n},0\right)\right)\Big) \\
&\quad + 2\gamma\alpha(k_2-k_1)\Big(\Phi\left(d_0\left(\xi_2^{*,\lambda_n},0\right)\right) - \Phi\left(d_0\left(\tilde{\alpha}\hat{\xi}^{\lambda_n},0\right)\right)\Big),
\end{aligned}$$

where we used that $C = -2\gamma\alpha k_0$. Now, the claim follows, i.e., $\liminf_{\lambda\searrow C} f_2(h(\lambda),\lambda) \geq 1$ since $\lim_{n\to\infty} h(\lambda_n) = 0$, $\lambda_n \xrightarrow{n\to\infty} C$ from above, $\xi_2^{*,\lambda_n} \geq \tilde{\alpha}\hat{\xi}^{\lambda_n}$, and $\Phi(d_0(\cdot,0))$ being non-decreasing. □

Proof of (A.12). For this proof, we have to consider three cases depending on the limiting behavior of $\frac{\lambda}{h(\lambda)}$ for $\lambda \to \infty$. As in the proof of (A.11), we have here possibly multiple accumulation points that we treat separately and denote by λ_n the sequence converging to the lim sup. Note that $\lambda_n \xrightarrow{n\to\infty} \infty$ in this case. Hence, we can assume, without loss of generality by possibly switching to a subsequence, that $\lim_{n\to\infty}\frac{\lambda_n}{h(\lambda_n)}$ exists (in $[0,\infty]$) and consider the three cases that $\lim_{n\to\infty}\frac{\lambda_n}{h(\lambda_n)} = 0$, $\lim_{n\to\infty}\frac{\lambda_n}{h(\lambda_n)} = c \in (0,\infty)$, and $\lim_{n\to\infty}\frac{\lambda_n}{h(\lambda_n)} = \infty$. Note that the factor to the right of "λ" in f_2 is negative, while the factor to the right of "y" in f_2 is positive by (A.17).

Case 1: $\lim_{n\to\infty}\frac{\lambda_n}{h(\lambda_n)} = 0$:

It holds that $\bar{\xi}^{\lambda_n} \xrightarrow{n\to\infty} 0$ using $y_n = h(\lambda_n)$ and the assumption. Therefore, we also get that $\hat{\xi}^{\lambda_n}, \xi_1^{*,\lambda_n}, \xi_2^{*,\lambda_n}, \xi_3^{*,\lambda_n} \xrightarrow{n\to\infty} 0$ due to (A.17). Thus, the claim follows using $\lim_{x\to 0}\Phi(d_i(x,0)) = 0$ for $i \in \{0,1\}$ and $h(\lambda_n) \xrightarrow{n\to\infty} +\infty$.

Case 2: $\lim_{n\to\infty}\frac{\lambda_n}{h(\lambda_n)} = c \in (0,\infty)$:

In this case, it holds by assumption that $\hat{\xi}^{\lambda_n} \xrightarrow{n\to\infty} c$, and $\bar{\xi}^{\lambda_n}, \xi_3^{*,\lambda_n} \xrightarrow{n\to\infty} \alpha c$. Moreover, it holds that $\lim_{n\to\infty}\xi_2^{*,\lambda_n} \in [\tilde{\alpha}c, \alpha c]$ by (A.17) and $\xi_1^{*,\lambda_n} \xrightarrow{n\to\infty} \tilde{\alpha}c$ since $\frac{\lambda_n - \sqrt{l(\lambda_n)}}{h(\lambda_n)} \xrightarrow{n\to\infty} c$ for any affine function l. Next, we note that $\Phi(d_i(ac,0)) \in (0,1)$ for all $a > 0$ and $i \in \{0,1\}$. In total, we get that $\lim_{n\to\infty} -1 + \Phi(d_0(\xi_3^{*,\lambda_n},0)) - \Phi(d_0(\alpha\hat{\xi}^{\lambda_n},0)) + \Phi(d_0(\xi_1^{*,\lambda_n},0)) = -1 + \Phi(d_0(\tilde{\alpha}c,0)) < 0$ and $\lim_{n\to\infty}\frac{1}{\alpha}\Phi(d_1(\xi_3^{*,\lambda_n},0)) - \frac{1}{\alpha}\Phi(d_1(\alpha\hat{\xi}^{\lambda_n},0)) + \frac{1}{\tilde{\alpha}}\Phi(d_1(\xi_1^{*,\lambda_n},0)) = \frac{1}{\tilde{\alpha}}\Phi(d_1(\tilde{\alpha}c,0)) > 0$. Thus, the claim follows since $\lambda_n, h(\lambda_n) \xrightarrow{n\to\infty} +\infty$.

Case 3: $\lim_{n\to\infty}\frac{\lambda_n}{h(\lambda_n)} = \infty$:

In this case, we get that $\xi_1^{*,\lambda_n} \xrightarrow{n\to\infty} \infty$ as $\lim_{n\to\infty}\frac{\lambda_n - \sqrt{l(\lambda_n)}}{h(\lambda_n)} = \infty$ for any affine function l. Hence, it holds that $\hat{\xi}^{\lambda_n}, \bar{\xi}^{\lambda_n}, \xi_2^{*,\lambda_n}, \xi_3^{*,\lambda_n} \xrightarrow{n\to\infty} \infty$ due to (A.17). Thus, we get the claim using $\lim_{n\to\infty} h(\lambda_n) = \infty$, $\lim_{n\to\infty}\lambda_n = \infty$, and $\lim_{x\to\infty}\Phi(d_i(x,0)) = 1$ for $i \in \{0,1\}$. □

Proof of (A.13). To show this, it is sufficient to show that (i) $\lambda \mapsto f_1(\cdot,\lambda)$ is strictly increasing, (ii) $y \mapsto f_1(y,\cdot)$ is strictly decreasing and (iii) f_1 is jointly continuous on $\mathbb{R}_{>0}\times(C,\infty)$. Indeed, let $\lambda, \lambda_n > C$ with $\lambda_n \xrightarrow{n\to\infty} \lambda$. Due to the existence of the zero root (see the main part of the proof) and the uniqueness of the zero root (by (ii)), there exist then unique $y_\lambda^*, y_{\lambda_n}^* \in (0,\infty)$ such that $f_1(y_\lambda^*,\lambda) = 0 = f_1(y_{\lambda_n}^*,\lambda_n)$. Then, we have to show that $y_{\lambda_n}^* \xrightarrow{n\to\infty} y_\lambda^*$. Due to (i), (ii), and $\lambda_n \xrightarrow{n\to\infty} \lambda$, it holds that $y_{\lambda_n}^* \in [y^*_{\max_{k\in\mathbb{N}}\{\lambda,\lambda_k\}}, y^*_{\min_{k\in\mathbb{N}}\{\lambda,\lambda_k\}}]$ (maximum and minimum exist due to the convergence of λ_n to λ). Then, there exist a subsequence $y^*_{\lambda_{n_l}}$ and a $\tilde{y}$ such that $y^*_{\lambda_{n_l}} \xrightarrow{l\to\infty} \tilde{y}$. Moreover, we know that $f_1(y_\lambda^*,\lambda) = 0 = f_1(y^*_{\lambda_{n_l}},\lambda_{n_l}) \xrightarrow{l\to\infty} f_1(\tilde{y},\lambda)$ due to the joint continuity of f_1. Now, the uniqueness of the zero root implies that $\tilde{y} = y_\lambda^*$. Hence, all subsequences converge to y_λ^* and thus also the sequence itself. There-

fore, (A.13) would follow provided we can show (i), (ii), and (iii) from the beginning:

First, the joint continuity (i.e., property (iii)) follows directly from the definition of f_1.

Second, we derive property (i), i.e., the strict monotonicity in λ, from the formula of $\hat{X}_T$ (see Theorem 3.2), which gives us the claim. Note that we add in the following paragraph a superscript to $\hat{X}_T$, ξ_1^*, ξ_2^*, and ξ_3^* when we take these values for a certain fixed λ.

Indeed, if, for all $\omega \in \Omega$, we can show that $\hat{X}_T^{\lambda_1}(\xi_T(\omega)) \geq \hat{X}_T^{\lambda_2}(\xi_T(\omega))$ with a strict inequality for a set with positive probability for all $\lambda_1 > \lambda_2 (> C)$, then also $f_1(\cdot, \lambda_1) > f_1(\cdot, \lambda_2)$ since $f_1 = \mathbb{E}[\xi_T \hat{X}_T] - x_0$ and $\xi_T > 0$. Therefore, let $\lambda_1 > \lambda_2 (> C)$: Due to (A.18) and Proposition 3.4, we conclude that for all $\lambda > C$ (and hence for λ_1 and λ_2) $\hat{X}_T \not\equiv 0$, i.e., $\hat{X}_T(\xi_T) > 0$ for ξ_T small enough. The formula of $\hat{X}_T$ (see Theorem 3.2) implies that for fixed y the slope remains unchanged in each interval $(0, \xi_1^*]$, $(\tilde{\alpha}\hat{\xi}, \xi_2^*]$, resp. $(\alpha\hat{\xi}, \xi_3^*]$ when changing λ, but the interval boundaries change. Note that the slope (as a function of ξ_T) is strictly negative in $(0, \xi_1^*]$ and $(\alpha\hat{\xi}, \xi_3^*]$, and constant otherwise. Therefore, due to $\hat{X}_T$ being non-increasing and a non-increasing function getting larger when being shifted to the right, it is sufficient to show that all interval boundaries do not decrease when λ increases and at least one boundary value strictly increases when λ increases:

It follows directly from the definition of $\hat{X}_T$ that $\lim_{\xi_T \to 0} \hat{X}_T^{\lambda_1}(\xi_T) > \lim_{\xi_T \to 0} \hat{X}_T^{\lambda_2}(\xi_T) (> 0)$. Hence, due to having the same slope and the continuity of $\hat{X}_T^{\lambda_1}$ (resp. $\hat{X}_T^{\lambda_2}$) in ξ_T on $(0, \xi_1^{*,\lambda_1}]$ (resp. $(0, \xi_1^{*,\lambda_2}]$), we conclude that $\xi_1^{*,\lambda_1} > \xi_1^{*,\lambda_2}$ since $\hat{X}_T^{\lambda_1}(\xi_T = \xi_1^{*,\lambda_1}) = k_2 = \hat{X}_T^{\lambda_2}(\xi_T = \xi_1^{*,\lambda_2})$, i.e., the interval is strictly increasing in λ. Next, we observe immediately from its definition that $\hat{\xi}^{\lambda_1} \geq \hat{\xi}^{\lambda_2}$. For ξ_2^* and ξ_3^*, we show this property by proving that their derivatives with respect to λ are non-negative. Let $\tilde{\xi}_2^*$ be defined as in Theorem 3.2 and $\tilde{\xi}_3^* := \bar{\xi} - \frac{2\gamma\alpha^2}{y}\left(\sqrt{k_1^2 + k_1\left(2k_0 + \frac{\lambda}{\gamma\alpha}\right)} - k_1\right)$ with $\bar{\xi}$ as in Theorem 3.2. Then, we get:

$$\frac{\partial}{\partial\lambda}\tilde{\xi}_2^* = \frac{\alpha}{y} - \frac{\alpha k_1}{y k_2} = \frac{\alpha}{y}\left(1 - \frac{k_1}{k_2}\right) \geq 0$$

$$\Rightarrow \frac{\partial^-}{\partial\lambda}\xi_2^* \geq 0,$$

$$\frac{\partial}{\partial\lambda}\tilde{\xi}_3^* = \begin{cases} \frac{\alpha}{y} & \text{if } k_1 = 0, \\ \frac{\alpha}{y} - \frac{2\gamma\alpha^2}{y}\frac{\frac{1}{2}\cdot\frac{k_1}{\gamma\alpha}}{\sqrt{k_1^2 + k_1(2k_0 + \frac{\lambda}{\gamma\alpha})}} = \frac{\alpha}{y}\left(1 - \frac{k_1}{\sqrt{k_1^2 + k_1(2k_0 + \frac{\lambda}{\gamma\alpha})}}\right) & \text{if } k_1 > 0, \end{cases}$$

$$\geq 0,$$

$$\Rightarrow \frac{\partial^-}{\partial\lambda}\xi_3^* \geq 0,$$

where $\frac{\partial^-}{\partial\lambda}$ denotes the left side derivative with respect to λ. Summarizing, ξ_1^* strictly increases and ξ_2^* and ξ_3^* non-decrease in λ. Thus, property (i) follows.

Third, we prove property (ii), i.e., that f_1 is strictly decreasing in y. Let $\hat{X}_T(y)$ be a function of y as in Theorem 3.2. It follows directly from the definitions of $\hat{\xi}$, ξ_1^*, ξ_2^*, and ξ_3^* that they are non-increasing in y for fixed λ since they are non-negative by definition and $\frac{1}{y}$ can be factored out. Now, let $y_1 < y_2$ and $\xi_T > 0$ arbitrary. Then, the claim follows if $\hat{X}_T(y_1, \xi_T) \geq \hat{X}_T(y_2, \xi_T)$ and $\hat{X}_T(y_1, \xi_T) > \hat{X}_T(y_2, \xi_T)$ on a set with positive probability. If ξ_T is in the same interval for both values y_1 and y_2, "$\geq$" is obvious due to (3.4). Moreover, we get "$>$" if $\xi_T \in (0, \xi_1^*] \cup (\alpha\hat{\xi}, \xi_3^*]$ (attained with positive probability). Since $\hat{\xi}$, ξ_1^*, ξ_2^*, and ξ_3^* are non-increasing in y and $\hat{X}_T$ is non-increasing in ξ_T (check Proposition 3.5), and for all $y \geq 0$ holds that $\hat{X}_T(y, a) > \hat{X}_T(y, b) > \hat{X}_T(y, c) > \hat{X}_T(y, d)$ for $a \in (0, \xi_1^*)$, $b \in (\tilde{\alpha}\hat{\xi}, \xi_2^*]$, $c \in (\alpha\hat{\xi}, \xi_3^*)$, and $d > \xi_3^*$, the claim also follows if ξ_T is in different intervals for y_1 and y_2. □

Proof of (A.14). For the proof of this equation, we have to consider the two cases $x_0 e^{\int_0^T r_s \mathrm{d}s} < k_2$ and $x_0 e^{\int_0^T r_s \mathrm{d}s} \geq k_2$. Note that if $x_0 e^{\int_0^T r_s \mathrm{d}s} = k_2$, we get that $C = 2\gamma\alpha(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1 - k_0) = 2\gamma\left(\tilde{\alpha} x_0 e^{\int_0^T r_s \mathrm{d}s} + \alpha_2 k_2 - \alpha(k_0 + k_1)\right)$. Moreover, notice that $x_0 e^{\int_0^T r_s \mathrm{d}s} > k_1$ by assumption.

Case 1: $x_0 e^{\int_0^T r_s \mathrm{d}s} < k_2$, i.e., $C = 2\gamma\alpha(x_0 e^{\int_0^T r_s \mathrm{d}s} - k_1 - k_0)$:

For $\lambda_\varepsilon = C + 2\gamma\alpha e^{\int_0^T r_s \mathrm{d}s}\varepsilon$, it holds that $\hat{\xi} = 0$ for ε small enough and hence also $\xi_1^* = \xi_2^* = 0$ due to (A.17). However, it holds with (A.18) that $\xi_3^* > 0$. Note that $\lim_{x \to 0} \Phi(d_j(x, 0)) = 0$ for $j \in \{0, 1, 2\}$. Thus, f_1 reduces for all $\varepsilon > 0$ small enough to

$$\begin{aligned} &f_1(y, C + 2\gamma\alpha e^{\int_0^T r_s \mathrm{d}s}\varepsilon) \\ &= -x_0 + \left(k_0 + k_1 + \frac{C + 2\gamma\alpha e^{\int_0^T r_s \mathrm{d}s}\varepsilon}{2\gamma\alpha}\right) e^{-\int_0^T r_s \mathrm{d}s} \Phi\left(d_1\left(\xi_3^*, 0\right)\right) \\ &\quad - \frac{y}{2\gamma\alpha^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s} \Phi\left(d_2\left(\xi_3^*, 0\right)\right) \qquad \text{(A.21)} \\ &= -x_0 + (x_0 + \varepsilon)\Phi\left(d_1\left(\xi_3^*, 0\right)\right) - \frac{y}{2\gamma\alpha^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s} \Phi\left(d_2\left(\xi_3^*, 0\right)\right). \end{aligned}$$

Now, if $\limsup_{\varepsilon \to 0} y_{\lambda_\varepsilon}^* > 0$, it follows that $\liminf_{\varepsilon \to 0} f_1(y_{\lambda_\varepsilon}^*, \lambda_\varepsilon) < 0$ (since $\Phi\left(d_1\left(\xi_3^*, 0\right)\right) < 1$ for all $y > 0$) which is a contradiction to $0 = \liminf_{\varepsilon \to 0} 0 = \liminf_{\varepsilon \to 0} f_1(y_{\lambda_\varepsilon}^*, \lambda_\varepsilon)$ by definition of $y_{\lambda_\varepsilon}^*$. Thus, the claim follows, i.e., $\lim_{\lambda \to C} y_\lambda^* = 0$, when taking the limit of $\varepsilon \to 0$ (and hence $\lambda \to C$) on both sides, since $y_\lambda^* \geq 0$. Note that for every y, the existence of a solution λ_y of (A.21) was already ensured in the main part of the proof.

Case 2: $x_0 e^{\int_0^T r_s \mathrm{d}s} \geq k_2$, i.e., $C = 2\gamma\left(\tilde{\alpha} x_0 e^{\int_0^T r_s \mathrm{d}s} + \alpha_2 k_2 - \alpha(k_0 + k_1)\right)$:

For $\lambda_\varepsilon = C + 2\gamma\tilde{\alpha} e^{\int_0^T r_s \mathrm{d}s}\varepsilon$, it holds that:

$$\begin{aligned} &f_1(y, \lambda_\varepsilon) = -x_0 + (x_0 + \varepsilon)\Phi\left(d_1\left(\xi_1^*, 0\right)\right) \\ &\quad - \frac{y}{2\gamma\tilde{\alpha}^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s} \Phi\left(d_2\left(\xi_1^*, 0\right)\right) \\ &\quad + k_2 e^{-\int_0^T r_s \mathrm{d}s}\left(\Phi\left(d_1\left(\xi_2^*, 0\right)\right) - \Phi(d_1(\tilde{\alpha}\hat{\xi}, 0))\right) \\ &\quad + \left(\frac{\tilde{\alpha}}{\alpha}(x_0 + \varepsilon) + \frac{\alpha_2}{\alpha} k_2 e^{-\int_0^T r_s \mathrm{d}s}\right)\left(\Phi\left(d_1\left(\xi_3^*, 0\right)\right) - \Phi(d_1(\alpha\hat{\xi}, 0))\right) \\ &\quad - \frac{y}{2\gamma\alpha^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s}\left(\Phi\left(d_2\left(\xi_3^*, 0\right)\right) - \Phi(d_2(\alpha\hat{\xi}, 0))\right) \\ &\leq -x_0 + x_0\left(\Phi\left(d_1\left(\xi_1^*, 0\right)\right) + \Phi\left(d_1\left(\xi_2^*, 0\right)\right) - \Phi(d_1(\tilde{\alpha}\hat{\xi}, 0))\right. \\ &\quad \left. + \Phi\left(d_1\left(\xi_3^*, 0\right)\right) - \Phi(d_1(\alpha\hat{\xi}, 0))\right) \\ &\quad + \varepsilon\left(\Phi\left(d_1\left(\xi_1^*, 0\right)\right) + \Phi\left(d_1\left(\xi_3^*, 0\right)\right) - \Phi(d_1(\alpha\hat{\xi}, 0))\right) \\ &\quad - \frac{y}{2\gamma} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s}\left(\frac{1}{\alpha^2}\Phi\left(d_2\left(\xi_3^*, 0\right)\right) - \frac{1}{\alpha^2}\Phi(d_2(\alpha\hat{\xi}, 0))\right. \\ &\quad \left. + \frac{1}{\tilde{\alpha}^2}\Phi\left(d_2\left(\xi_1^*, 0\right)\right)\right) \\ &\leq -x_0 + x_0 + \varepsilon - \frac{y}{2\gamma\tilde{\alpha}^2} e^{\int_0^T -2r_s + \|\kappa_s\|^2 \mathrm{d}s}\left(\Phi\left(d_2\left(\xi_3^*, 0\right)\right) - \Phi(d_2(\alpha\hat{\xi}, 0))\right. \\ &\quad \left. + \Phi\left(d_2\left(\xi_1^*, 0\right)\right)\right), \end{aligned}$$

where we used that $k_2 e^{-\int_0^T r_s \mathrm{d}s} \leq x_0$, $\tilde{\alpha} + \alpha_2 = \alpha$, and $\frac{\tilde{\alpha}}{\alpha} \leq 1$ in the first inequality. Moreover, we used (A.17) and $\Phi\left(d_1\left(\xi_3^*, 0\right)\right) \leq 1$ for all $y > 0$ in the second inequality. Due to $\xi_3^* > 0$ and (A.17), it holds that $\Phi\left(d_2\left(\xi_3^*, 0\right)\right) - \Phi(d_2(\alpha\hat{\xi}, 0)) + \Phi\left(d_2\left(\xi_1^*, 0\right)\right) > 0$. Hence, the claim follows, i.e., $\lim_{\varepsilon \to 0} y_{\lambda_\varepsilon}^* = 0$ by the same argument as in Case 1. Note that the existence of a solution was already ensured in the main part of the proof. □

Proof of (A.15). There is nothing to show since we already have that $y_\lambda^* \in [0, \infty)$. □

Proof of (A.16). We prove this by contradiction. Therefore, we assume that there exists an $L > 0$ such that $\limsup_{\lambda\to\infty} y_\lambda^* \le L$. Under this assumption, it holds that $\xi_1^* \xrightarrow{\lambda\to\infty} \infty$ since y_λ^* is bounded and $\lambda - \sqrt{l(\lambda)} \xrightarrow{\lambda\to\infty} \infty$ for all affine functions l. Then (A.17) implies that $\hat{\xi}, \bar{\xi}, \xi_2^*, \xi_3^* \xrightarrow{\lambda\to\infty} \infty$ and we know that $\Phi(d_1(\xi_3^*, 0)) \ge \Phi(d_1(\alpha\hat{\xi}, 0))$. However, then $\lim_{\lambda\to\infty} \inf_{0\le y\le L} f_1(y, \lambda) = \infty$ which is a contradiction to y_λ^* being the zero root of f_1. □

Since all statements are proved, we have shown the lemma. □

Appendix B. Additional lemma

The following lemma restates a well-known result for log-normal distributions, which is used, e.g., in deriving the pricing formula of a put or call in a Black-Scholes model. The proof is just a straightforward calculation. However, we will give it for the sake of completeness.

Lemma B.1. *Let $0 \le a \le b \le +\infty$ and $X \sim \mathcal{LN}(\mu, \sigma^2)$, where $\mathcal{LN}$ denotes a log-normal distribution. Then, it holds that:*

$$\mathbb{E}\left[X \mathbb{1}_{X\in[a,b]}\right] = e^{\mu+\frac{\sigma^2}{2}} \left(\Phi\left(\frac{\ln b - \mu - \sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln a - \mu - \sigma^2}{\sigma}\right)\right), \tag{B.1}$$

where Φ denotes the cdf of a standard normal distribution with $\Phi(+\infty) = 1$ and $\Phi(-\infty) = 0$. The formula remains unchanged when we replace the interval $[a, b]$ by $(a, b]$, $[a, b)$, or (a, b).

Proof. It holds with $y := \ln x$ and $z := \frac{y-\mu-\sigma^2}{\sigma}$:

$$\begin{aligned}
&\mathbb{E}\left[X \mathbb{1}_{X\in[a,b]}\right] \\
&= \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(y - \frac{(y-\mu)^2}{2\sigma^2}\right) \mathrm{d}y \\
&= \exp\left(-\frac{1}{2\sigma^2}\left(\mu^2 - (\mu+\sigma^2)^2\right)\right) \int_{\frac{\ln a-\mu-\sigma^2}{\sigma}}^{\frac{\ln b-\mu-\sigma^2}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2}z^2\right) \mathrm{d}z \\
&= \exp\left(\mu + \frac{\sigma^2}{2}\right) \left(\Phi\left(\frac{\ln b - \mu - \sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln a - \mu - \sigma^2}{\sigma}\right)\right). \quad \square
\end{aligned}$$

Data availability

No data was used for the research described in the article.

References

Avanzi, Benjamin, Lau, Hayden, Steffensen, Mogens, 2024. Optimal reinsurance design under solvency constraints. Scand. Actuar. J. 2024 (4), 383–416.
Bacinello, Anna Rita, Persson, Svein-Arne, 2002. Design and pricing of equity-linked life insurance under stochastic interest rates. J. Risk Finance 3 (2), 6–21.
Basak, Suleyman, Chabakauri, Georgy, 2010. Dynamic mean-variance asset allocation. Rev. Financ. Stud. 23 (8), 2970–3016.
Basak, Suleyman, Shapiro, Alexander, 2001. Value-at-risk-based risk management: optimal policies and asset prices. Rev. Financ. Stud. 14 (2), 371–405.
Björk, Tomas, Murgoci, Agatha, 2010. A general theory of Markovian time inconsistent stochastic control problems. Available at SSRN 1694759.
Briys, Eric, De Varenne, François, 1997. On the risk of insurance liabilities: debunking some common pitfalls. J. Risk Insur., 673–694.
Chen, An, Nguyen, Thai, Stadje, Mitja, 2018. Optimal investment under var-regulation and minimum insurance. Insur. Math. Econ. 79, 194–209.
Chen, An, Hieber, Peter, Nguyen, Thai, 2019. Constrained non-concave utility maximization: an application to life insurance contracts with guarantees. Eur. J. Oper. Res. 273 (3), 1119–1135.
Chen, An, Stadje, Mitja, Zhang, Fangyuan, 2024. On the equivalence between value-at-risk-and expected shortfall-based risk measures in non-concave optimization. Insur. Math. Econ. 117, 114–129.
Cochrane, John, 2009. Asset Pricing, revised edition. Princeton University Press.
Cont, Rama, Tankov, Peter, 2004. Financial Modelling with Jump Processes. Chapman and Hall/CRC.
Cuoco, Domenico, He, Hua, Isaenko, Sergei, 2008. Optimal dynamic trading strategies with risk limits. Oper. Res. 56 (2), 358–368.
Dai, Min, Kou, Steven, Qian, Shuaijie, Wan, Xiangwei, 2019. Non-concave utility maximization without the concavification principle. Available at SSRN 3422276.
Dong, Yinghui, Wu, Sang, Lv, Wenxin, Wang, Guojing, 2020. Optimal asset allocation for participating contracts under the var and pi constraint. Scand. Actuar. J. 2020 (2), 84–109.
EIOPA, 2023. European insurance overview 2023. Technical report. European Insurance and Occupational Pensions Authority (EIOPA).
Gatzert, Nadine, Kling, Alexander, 2007. Analysis of participating life insurance contracts: a unification approach. J. Risk Insur. 74 (3), 547–570.
Hakansson, Nils H., 1971. Capital growth and the mean-variance approach to portfolio selection. J. Financ. Quant. Anal. 6 (1), 517–557.
He, Lin, Liang, Zongxia, Liu, Yang, Ma, Ming, 2020. Weighted utility optimization of the participating endowment contract. Scand. Actuar. J. 2020 (7), 577–613.
Korn, Ralf, Trautmann, Siegfried, 1999. Optimal control of option portfolios and applications: optimale steuerung von optionsportefeuilles und anwendungen. OR Spektrum 21, 123–146.
Kraft, Holger, Steffensen, Mogens, 2013. A dynamic programming approach to constrained portfolios. Eur. J. Oper. Res. 229 (2), 453–461.
Larsen, Kasper, 2005. Optimal portfolio delegation when parties have different coefficients of risk aversion. Quant. Finance 5 (5), 503–512.
Li, Yuying, Forsyth, Peter A., 2019. A data-driven neural network approach to optimal asset allocation for target based defined contribution pension plans. Insur. Math. Econ. 86, 189–204.
Liang, Zongxia, Liu, Yang, Ma, Ming, et al., 2021. A unified formula of the optimal portfolio for piecewise hara utilities. arXiv preprint. arXiv:2107.06460.
Lin, Hongcan, Saunders, David, Weng, Chengguo, 2017. Optimal investment strategies for participating contracts. Insur. Math. Econ. 73, 137–155.
Markowitz, Harry, 1952. Portfolio selection. J. Finance 7 (1), 77–91.
Markowitz, Harry M., 1959. Portfolio Selection: Efficient Diversification of Investments. Yale University Press.
Menoncin, Francesco, Vigna, Elena, 2017. Mean–variance target-based optimisation for defined contribution pension schemes in a stochastic framework. Insur. Math. Econ. 76, 172–184.
Merton, Robert C., 1972. An analytic derivation of the efficient portfolio frontier. J. Financ. Quant. Anal. 7 (4), 1851–1872.
Merton, Robert C., 1975. Optimum consumption and portfolio rules in a continuous-time model. In: Stochastic Optimization Models in Finance. Elsevier, pp. 621–661.
Mi, Hui, Xu, Zuo Quan, Yang, Dongfang, 2023. Optimal management of dc pension plan with inflation risk and tail var constraint. arXiv preprint. arXiv:2309.01936.
Mirza, Charbel, Wagner, Joël, 2018. Policy characteristics and stakeholder returns in participating life insurance: which contracts can lead to a win-win? Eur. Actuar. J. 8, 291–320.
Nguyen, Thai, Stadje, Mitja, 2020. Nonconcave optimal investment with value-at-risk constraint: an application to life insurance contracts. SIAM J. Control Optim. 58 (2), 895–936.
Qian, Shuaijie, Yang, Chen, 2023. Non-concave utility maximization with transaction costs. arXiv preprint. arXiv:2307.02178.
Reichlin, Christian, 2013. Utility maximization with a given pricing measure when the utility is not necessarily concave. Math. Financ. Econ. 7 (4), 531–556.
Samuelson, Paul A., 1975. Lifetime portfolio selection by dynamic stochastic programming. In: Stochastic Optimization Models in Finance, pp. 517–524.
Schmeiser, Hato, Wagner, Joël, 2015. A proposal on how the regulator should set minimum interest rate guarantees in participating life insurance contracts. J. Risk Insur. 82 (3), 659–686.
Zhang, Yuanyuan, Li, Xiang, Guo, Sini, 2018. Portfolio selection problems with Markowitz's mean–variance framework: a review of literature. Fuzzy Optim. Decis. Mak. 17, 125–158.
Zhou, Xun Yu, Li, Duan, 2000. Continuous-time mean-variance portfolio selection: a stochastic lq framework. Appl. Math. Optim. 42, 19–33.

4 Optimal Capital Structure for Life Insurance Companies Offering Surplus Participation

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Optimal Capital Structure for Life Insurance Companies Offering Surplus Participation*

Felix Fießinger† and Mitja Stadje‡,§

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Abstract

We adapt Leland's dynamic capital structure model to the context of an insurance company selling participating life insurance contracts explaining the existence of life insurance contracts which provide both a guaranteed payment and surplus participation to the policyholders. Our derivation of the optimal participation rate reveals its pronounced sensitivity to the contract duration and the associated tax rate. Moreover, the asset substitution effect, which describes the tendency of equity holders to increase the riskiness of a company's investment decisions, decreases when adding surplus participation.

Keywords: portfolio insurance, Leland's model, participating life insurance contracts, surplus participation

JEL: C61, C68, G11, G22, G33

MSC: 90B50, 91B06, 91B50, 91G05, 91G10, 91G50

1 Introduction

This paper analyzes the surplus participation and the guaranteed payment in life insurance contracts from the insurance company's capital structure perspective. These contracts are classified as participating life insurance contracts and encompass all contract types in which the policyholder receives some form of surplus participation together with a guaranteed payment stream. We consider contracts with proportional surplus participation above a pre-determined level. Additionally, we allow the contract to provide both a guarantee payment and a final lump sum payment. We model the insurance company using an extension of Leland's model, originally developed to determine the optimal debt structure of a company. To the best of our knowledge, this is the first dynamic capital structure model which includes surplus participation for policyholders.

The aim of this paper is to analyze the reasons for the extensive use of participating insurance contracts in the life insurance sector from a capital structure perspective, and to show how the optimal participation and guaranteed payment can be computed. Our findings indicate that tax benefits are crucial for incentivizing the insurance company to offer participating contracts to the policyholders. We apply these results to a basic setting, discuss underlying assumptions, and conduct sensitivity analysis on various parameters. Moreover, we show that the asset substitution effect, which describes the tendency of equity holders, in the presence of debtors, to increase the riskiness of a company's investment decisions beyond the level they normally would, is less pronounced when adding surplus participation.

*Declarations of interest: none †Ulm University, Institute of Insurance Science and Institute of Mathematical Finance, Faculty of Mathematics and Economics, Ulm, Germany. Email: felix.fiessinger@uni-ulm.de ‡Ulm University, Institute of Insurance Science and Institute of Mathematical Finance, Faculty of Mathematics and Economics, Ulm, Germany. Email: mitja.stadje@uni-ulm.de §Corresponding author

Our basic framework adapts Leland's model from the insurance company's perspective, incorporating surplus participation into the original model. Leland's model, introduced in 1994 [42], was designed to determine the optimal leverage of companies by deriving the optimal capital structure between equity and debt. Initially simple, this model has closed-form solutions. A key feature of this setting is that equity holders are not assumed to be tied to their investments; rather they can liquidate the business if the company's asset value falls too low and obligations to debt holders become unsustainable. In this strand of literature, equity holders shut down the company if the value of the equity becomes negative. Leland then derives a bankruptcy-triggering value, such that for all asset values higher than this threshold, equity remains non-negative. By construction, the bankruptcy-triggering value has to be determined endogenously in the analysis. Various generalizations and adaptions of this model exist. Initially, Leland and Toft [44] adapted the original framework to finite-time debt. In this paper, we build upon this adaptation, as we also consider contracts with finite durations. Further generalizations include Leland [43], who incorporated capital restructuring, Goldstein et al. [25], who allowed the company to increase their debt level, and Manso et al. [48], who introduced performance-dependent coupon levels. He and Milbradt [29] examined the debt structure within a dynamic framework, where the firm has the flexibility to adjust its debt maturity structure in response to evolving market conditions. Hilpert et al. [32] extended the model by incorporating asymmetric information between the firm and debt holders. They also introduced learning dynamics in the market over time and considered performance-sensitive debt. Other significant contributions continue to build on Leland's model in various directions, including those by Ju et al. [35], Liu et al. [46], Hennessy and Tserlukevich [30], Agarwal et al. [1], Elkamhi et al. [21], Glasserman and Nouri [24], Hugonnier et al. [33], Chen et al. [14], Ambrose et al. [4], Della Seta et al. [16], Carey and Gordy [11], and many others.

In the life sector of insurance, policies offering some profit participation are widespread. Profit participation is typically paid during favorable economic conditions, with a proportional participation, as in this paper, being the common example. In the life sector of the insurance market, which also includes pension and health insurance, according to the European Insurance Overview 2023 [20], published by the European Insurance and Occupational Pensions Authority (EIOPA), approximately a quarter of the total gross premium in Europe are spent on contracts with some form of profit participation. In countries like Croatia, Italy, or Belgium, this proportion exceeds 50 %. Research in this area is also ongoing, with many contributions, see, for instance, Bryis and de Varenne [10], Bacinello and Persson [5], Gatzert and Kling [23], Schmeiser and Wagner [57], Lin et al. [45], Chen et al. [13], Mirza and Wagner [52], Nguyen and Stadje [53], He et al. [28], Dong et al. [18], or Fießinger and Stadje [22]. To the best of our knowledge, however, we are the first to combine Leland's model and surplus participation providing a possible capital-based explanation for the peculiar structure of the life insurance market, where guaranteed interest is often combined with surplus participation. According to, e.g., Kling et al. [39], the combination of these two obligations to the policyholders are typical in the design of insurance products with surplus participation, whereby this combination is mostly studied focusing on managing the risk of the insurer, see, for instance, Kling et al. [39, 40], Hieber et al. [31] or Schmeiser and Wagner [57]. Starting in the early 2000s, several publications have analyzed surplus participation products in the context of potential insolvency of life insurance companies, beginning with Grosen and Jørgensen [26]. Subsequent works, such as those by Bernard et al. [8], Ballotta et al. [6], and Cheng and Li [15], further explored this topic. However, these studies did not address the determination of the optimal bankruptcy-triggering value or the optimal capital structure, which are the pillars of dynamic capital models.

In the insurance market, there are several products offered with a surplus participation on the financial market result. These product constructions are sometimes called "Zero+Call" which also includes, e.g., equity-indexed annuities. In a "Zero+Call" typed product, the insurer combines guarantees ("Zero"), such as a guaranteed interest rate or a premium refund guarantee,

with surplus participation ("Call") in assets like an index or a special portfolio. In the US insurance market, the "Principal Protected Notes" offered by JPMorgan Chase exemplifies such a structure. Moreover, there are several products in the market available where the insurance company provides access for investments in special markets, typically unavailable to small investors, such as infrastructure, sustainability, or private equity. These products are also incorporated into this paper's model, especially when additional guarantees are provided. Examples of such products are, for instance, "Allianz Index Advantage" in the US, "AXA TwinStar" in France, and "Allianz InvestFlex Green" and "Swiss Life Champion" in Germany. The "Principal Protected Notes" offered by JPMorgan Chase (mainly in the US) can also be combined with funds focused on investment in such specialized markets. Further discussion on such products in the insurance market can be found in Chen et al. [12]. Similar products with comparable features exist in private pension schemes, such as the "Prudential Premier Retirement" in the US, "Manulife UL" in Canada, or "Prudential With-Profits Pension Annuity" in the UK.

Another application of such a model is given in the context of occupational pension schemes, based on defined contribution (DC) plans. In these schemes, the employer commits to contributing a specified amount to a funds or a similar investment vehicle, transferring the investment risk entirely to the employee (in contrast to defined benefit (DB) plans). Now, in some countries, a hybrid model exists that adds a guarantee to a DC plan. Due to these guarantees, employees, on the other hand, do not fully benefit from the returns of the fund, resulting in a product that offers both a guaranteed interest rate and a share in the fund's performance. Such a combination of guaranteed interest and a surplus participation can be chosen, e.g., in the "Allianz Advantage Pensioen" offered by Allianz Nederland Levensverzekering.

We show that the bankruptcy-triggering value is uniquely determined through a non-linear equation and is monotonically decreasing in the tax rates, but monotonically increasing in the surplus participation and the guarantee rate. We give sufficient conditions for the participation and the guarantee rate to be strictly positive. In particular, if the tax rate is sufficiently high, it is always beneficial for the insurance company, from a capital structure perspective, to offer a surplus participation. This provides a possible explanation for the peculiar structure of the life insurance market, where guarantees which match the policyholders' preference for safety, go typically hand in hand with surplus participation. In the numerical analysis, we demonstrate that these conditions are typically satisfied. However, if other payout obligations, like the lump sum payment, are too large, or the tax rates (and therefore the tax benefits) are too low, then and only then it is advantageous for the insurance company not to offer a positive surplus participation. A sensitivity analysis indicates that the surplus participation is mainly exposed to changes in the dividend payout of the insurance company, the contract duration, and the tax rate. Finally, we explore the so-called asset substitution effect, which is a type of agency costs. This effect describes the tendency of equity holders to increase the riskiness of a company's investment decisions, leading to a transfer of value from liabilities to equity. This phenomenon was first identified by Black and Scholes [9] and Jensen and Meckling [49]. Subsequently, Merton [51] and Barnea et al. [7] expanded upon this issue, identifying the core issue as the treatment of equity as a call option. However, when additional features such as guaranteed payments, taxes, and bankruptcy costs are incorporated, equity is no longer a classical call option, and the asset substitution effect weakens, particularly for shorter contract durations. Barnea et al. [7] already proposed that shorter durations diminish shareholders' incentives to increase investment risk. In our framework, we demonstrate that surplus participation further mitigates the asset substitution effect, rendering it a negligible factor. Specifically, we observe that for reasonable contract maturities, up to 50 years, the asset-substitution effect disappears entirely when surplus participation is incorporated. Therefore, agency costs associated with asset substitution are effectively eliminated when such contracts are offered, as in our framework.

The structure of the paper is as follows: Section 2 introduces participating life insurance contracts, outlines the basic model, and discusses the modeling of participation rates using option

theory. In Section 3, we present the liability structure and the insurance company's value, followed by the derivation of the bankruptcy-triggering value. Section 4 derives formulas for the optimal participation rate and the guarantee rate. In Section 5, we perform numerical analysis based on the result from the preceding section. Section 6 concludes the paper. All proofs are included in Appendix C.

2 Basic Concepts and Model Setup

2.1 Participating life insurance contracts

In the life insurance sector, contracts with surplus participation, or more broadly, contracts with profit participation, are commonplace. As illustrated in Figure 1, which presents the market share of various lines of business in the life sector in 2022, approximately one-quarter of the total gross premiums are allocated to insurance contracts with a participation component. Furthermore, in countries such as Croatia, Belgium, and Italy, more than half of all gross premiums are invested in contracts featuring some form of participation.

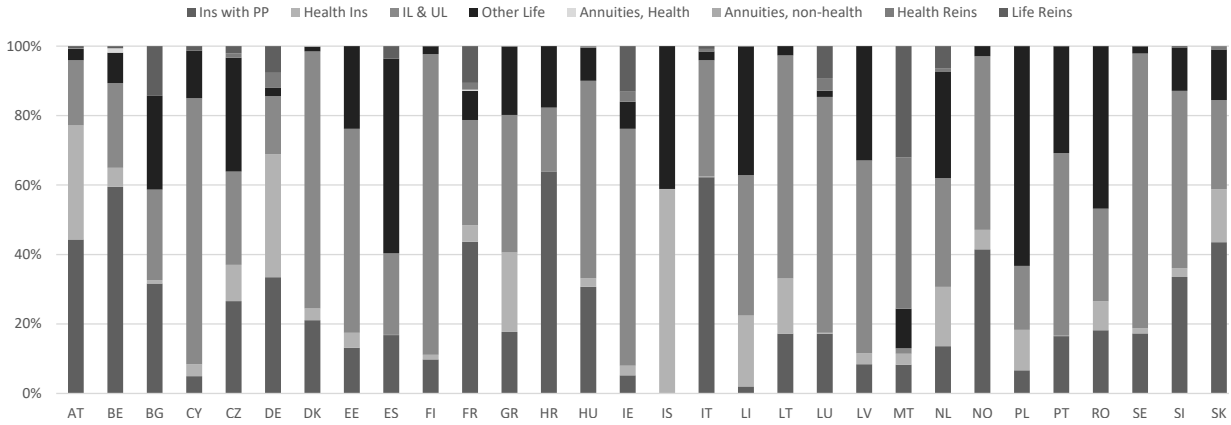

Figure 1: Market share in 2022 of the gross premium separated by the line of business in the life sector. Data source: European Insurance Overview from the EIOPA [20]

In countries like Germany, for instance, profit participation is legally mandated, as stipulated in the so-called "Mindestzuführungsverordnung" (minimum allocation decree), requiring insurance companies to distribute cost-, risk-, and investment-surpluses to policyholder (if any) for all life insurance policies. While such legal requirements are less stringent in many other countries, many contracts still include profit participation, as can be seen from Figure 1.

In this paper, we focus on an insurance company that offers a single type of insurance contract. This contract includes a constant, guarantee rate $g \geq 0$, a deterministic lump sum payment at maturity, and an additional participation on the surplus exceeding a pre-determined threshold $k \geq 0$, with a participation rate $\alpha \in [0, 1]$. The modeling of this participation presents some mathematical complexities, which we address in Subsection 2.3 following the setup of the basic model.

2.2 Model Setup

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space with time horizon $T > 0$, where the Brownian Motion W generates the filtration, satisfying the usual conditions. Additionally, let $\mathbb{Q}$ denote the pricing measure. We consider a market with a default-free, risk-free asset B offering an interest rate $r > 0$. Following the approach of Leland and Toft [44] (and its various generalizations, such as Ju et al. [35], Liu et al. [46], or Hennessy and Tserlukevich [30]), we model the asset

value of the insurance company by the stochastic differential equation:

$$\mathrm{d}V_t = (\mu_t(V_t) - \nu)V_t\mathrm{d}t + \sigma V_t \mathrm{d}W_t,$$

where μ_t represents the insurance company's total expected rate of return, $\nu > 0$ is the constant fraction paid out (to equity holders and policyholders together), and $\sigma > 0$ is the (constant) volatility. Moreover, we consider a constant default-triggering value $V_B \in [0, V_0]$, which determines the threshold at which the insurance company decides to default, if the asset value V falls below V_B. We denote by f^{V_0} the density and by F^{V_0} the cumulative distribution function of the first passage time of the asset value V to the bankruptcy-triggering value V_B under risk-neutral valuation, i.e., under $\mathbb{Q}$, where the drift rate of V is given by $r - \nu$. For clarity, we explicitly state the dependence of f and F on the initial value V_0 as an upper index.

Furthermore, we let τ_1 (resp. τ_2) represent the tax rate of the insurance company on the guaranteed payment (resp. the surplus participation), and ρ denote the fraction of the asset value that is lost in the event of bankruptcy. Throughout the paper, we ignore that, apart from policyholders, there might be additional debt holders. Note that if only the company's profits are taxed, then it holds that $\tau_1 = \tau_2$ and both are equal to the corporate tax rate. The total value of the insurance company is denoted by v, which is partitioned into the equity value E and the liability value L. The liability value L encompasses the value of the payments to policyholders, including guaranteed rates, lump sum payments and surplus participation. It is important to note that the total value does not equate to the asset value of the insurance company due to tax benefits and bankruptcy costs. In event of bankruptcy, the remaining asset value, $(1-\rho)V_B$, will be distributed solely among the policyholders.

2.3 Modeling participation rates with barrier options

By the construction of the participation part in this contract, as described in Subsection 2.1, the surplus participation at maturity is given by $\alpha(V_T - k)_+$, where $(x)_+ := \max\{x, 0\}$. This surplus participation is solely paid if the insurance company remains solvent, i.e., if $V_t \geq V_B$ for all $t \in [0, T]$, or equivalently $\min_{t\in[0,T]} V_t \geq V_B$. First, we observe that the payout of the surplus participation, without considering the bankruptcy condition, is equivalent to the payout of a call option. For now assume that the bankruptcy-triggering value V_B is constant (which will be shown in Subsection 3.3). Then the value of the surplus participation can be modeled as a so-called Down-and-Out-Call option. Specifically, the barrier is set as V_B, the strike price as k, and the asset value as V. Consequently, the value of the Down-and-Out Call option, corresponding to the surplus participation, with maturity T and dividend rate ν, is given by:

$$c_{do}(V_0, k, V_B, T) = \mathbb{E}^{\mathbb{Q}}[(V_T - k)_+ \mathbb{1}_{\{\min_{s\in[0,T]} V_s \geq V_B\}}].$$

For further details on barrier options, we refer to Appendix A.1 or to Hull [34].

3 Insurance company setup and determination of the bankruptcy-triggering value

For the following analysis, we assume that the insurance company continuously sells contracts with identical features over time, such that the portfolio remains stationary. Specifically, as long as the insurance company remains solvent, at each point in time, the value of maturing contracts is equal to the value of the newly issued contracts. Additionally, we assume that the maturities of the contracts are uniformly distributed within each interval $[s, s+T]$. Without loss of generality, we set $s = 0$ in the formulas. These assumptions align with those made in Leland's model for finite maturity debt.

Let G denote the total amount of guaranteed payments per year, and $P > 0$ the total amount of lump sum payments at maturity. Based on the assumptions above, both G and P are time-independent, in contrast to the surplus participation component. The constant guarantee rate is given by $g = \frac{G}{T}$, the constant yearly lump sum payment rate is $p = \frac{P}{T}$, and the insurance company pays out a total of $G + \frac{P}{T}$ along with the random surplus participation per year. In practice, the model remains valid even if the insurer pays out the rates at later times (e.g., monthly or yearly) and invests the money in the risk-free asset during the intermediate time period.

In the analysis, we restrict the parameters to those that reflect reasonable market conditions for equity. By "reasonable market conditions", we refer to a market in which (a) limited liability holds, i.e., the equity value, $E(V)$, cannot be negative, and (b) an increase in the asset value of the insurance company results in a non-decreasing equity value, i.e., $V \to E(V)$ is non-decreasing. If this condition were violated, a maximum value for the equity holders would exist, and the equity holders would, when the price is at its maximum, be actually unable to sell their shares at that price, as the buyer would have no upside potential indicating that such a price cannot arise in a competitive market. Although without restrictions on the parameters, excessively high surplus participation promises could lead to such a scenario, an insurance company offering such products would face challenges in attracting investors. Thus, for the remainder of this paper, we assume that equity is non-decreasing in the asset value and non-negative.

Additionally, we assume that the liability value of the guaranteed payment exceeds the tax benefit associated with this payment[1]. This assumption is also necessary in the absence of surplus participation, such as in the basic model of Leland and Toft [44], where it is implicitly assumed, even though not explicitly stated.[2] If this assumption is not satisfied, we find that V_B is decreasing in G, leading to the paradoxical situation that it would be advantageous for the policyholders, the insurance company, and the equity holders to increase the guaranteed payments to infinity due to the high tax benefit. Therefore, we exclude this possibility for the remainder of the paper. Numerical analysis indicates that this assumption may not hold if the tax rate is close to 100% and the lump sum payment is small. However, as the contract duration, T, approaches infinity, the present value of the liability associated with the guaranteed payment will always exceed the tax benefit derived from it, regardless of the parameter choices.

We also make a similar assumption regarding surplus participation: namely, that the liability value of the surplus participation exceeds the tax benefit associated with it[1]. If this assumption does not hold, we again encounter paradoxical situations where increasing surplus participation to infinity would be advantageous for the policyholders, the insurance company, and the equity holders due to the high tax benefit. Our analysis indicates that this assumption may not hold true if, for instance, the tax rate is excessively high. However, as long as the contract duration is finite, the assumption is always valid. As such, we exclude this possibility from further analysis in this paper.

For simplicity, from this point forward, we will replace the initial value V_0 with V in the notation.

3.1 Liability Structure

In this subsection, we derive formulas for the liability associated with a fixed bankruptcy-triggering level V_B. Additionally, we restate the results from Leland and Toft [44] concerning the non-participation component of the liabilities. That V_B is in the optimum chosen constantly follows from the stationarity assumption and will be demonstrated in Subsection 3.3. Before defining the liabilities for the entire portfolio, we begin by considering the liability stemming from a portfolio with maturity t, denoted by l. We let τ represent the stopping time at which the asset value

[1] From a mathematical perspective, this excess is not required to hold globally. It suffices for this condition to be valid when the insurance company's asset value is close to the bankruptcy-triggering value. [2] In the case of no surplus participation, we later demonstrate (see 3.17) that the optimal bankruptcy-triggering value, V_B, is affine-linear in the guaranteed payment G. Thus, this assumption ensures that V_B is non-decreasing and, consequently, non-negative.

of the insurance company V_t hits the bankruptcy triggering value V_B, i.e., $\tau := \inf_{s\geq 0}\{V_s \leq V_B\}$:

$$\begin{aligned} l(V;V_B,t) =& \mathbb{E}^{\mathbb{Q}}\left[\int_0^t e^{-rs} g \mathbb{1}_{\{\min_{r\in[0,s]} V_r \geq V_B\}} \mathrm{d}s\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-rt} p \mathbb{1}_{\{\min_{s\in[0,t]} V_s \geq V_B\}}\right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}(1-\rho)V_B \mathbb{1}_{\tau\leq t}\right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}\alpha(V_t-k)_+ \mathbb{1}_{\{\min_{s\in[0,t]} V_s \geq V_B\}}\right] \\ =& \int_0^t e^{-rs} g(1-F^V(s))\mathrm{d}s + e^{-rt}p(1-F^V(t)) + \int_0^t e^{-rs}(1-\rho)V_B f^V(s)\mathrm{d}s \\ &+ \alpha c_{do}(V,k,V_B,t) \\ =& \frac{g}{r} + e^{-rt}\left(p-\frac{g}{r}\right)(1-F^V(t)) + \left((1-\rho)V_B - \frac{g}{r}\right)G^V(t) + \alpha c_{do}(V,k,V_B,t), \end{aligned} \tag{3.1}$$

where $G^V(t) := \int_0^t e^{-rs} f^V(s)\mathrm{d}s$. The last step follows from integration by parts of the first term. In this equation, the first term corresponds to the guaranteed payment, the second term represents the final lump sum payment, the third term reflects the remaining asset value in event of bankruptcy, and the fourth term accounts for the surplus participation.

Now, we proceed with the liability of the entire portfolio with maturity T, i.e., the total liability value L, which is given by:

$$\begin{aligned} L(V;V_B,T) &= \int_0^T l(V;V_B,t)\mathrm{d}t \\ &= \tfrac{G}{r} + (P-\tfrac{G}{r})(\tfrac{1-e^{-rT}}{rT} - I_1^V(T)) + \left((1-\rho)V_B - \tfrac{G}{r}\right) I_2^V(T) + \alpha \textstyle\int_0^T c_{do}(V,k,V_B,t)\mathrm{d}t, \end{aligned} \tag{3.2}$$

where $I_1^V(T) := \frac{1}{T}\int_0^T e^{-rt}F^V(t)\mathrm{d}t$ and $I_2^V(T) := \frac{1}{T}\int_0^T G^V(t)\mathrm{d}t$. The functions F^V, G^V, I_1^V, and I_2^V admit explicit formulas, which are provided in Appendix A.2.

3.2 Total value and equity value

As outlined in the model setup in Subsection 2.2, the total value of the insurance company, v, is the sum of the actual value and the tax benefit TB, minus the lost value in the event of bankruptcy, denoted by BC. Specifically, we have:

$$v = V + TB - BC.$$

By definition of the model, the tax benefit TB can be decomposed into two components: $TB = TB_1 + TB_2$, where TB_1 represents the tax benefit arising from the guaranteed payment, and TB_2 corresponds to the tax benefit from the participation component. Given the assumption of the stationarity of the portfolio over time, we will consider the case where $T = \infty$ for the valuation of the tax benefit TB and the bankruptcy costs BC. In this context, TB_1 is given by $\tau_1 G$, where G is the total guaranteed payment, as long as the insurance company remains solvent, i.e., as long as $\min_{s\in[0,t]} V_s \geq V_B$. A similar interpretation applies to TB_2, with τ_2 replacing τ_1 and the value of the participation component substituting the value of the guarantee. Finally, BC represents the value of the bankruptcy costs. Thus, we get with $S := \inf\{t > 0 : V_t \leq V_B\}$:

$$TB_1 = \tau_1 \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty e^{-rt} G \mathbb{1}_{V_s \geq V_B\ \forall s\in[0,t]}\mathrm{d}t\right], \tag{3.3}$$

$$TB_2 = \tau_2 \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty e^{-rt}\alpha(V_t-k)_+ \mathbb{1}_{V_s \geq V_B\ \forall s\in[0,t]}\mathrm{d}t\right], \tag{3.4}$$

$$BC = \rho \mathbb{E}^{\mathbb{Q}}\left[e^{-rS} V_B \mathbb{1}_{S<\infty}\right]. \tag{3.5}$$

For the non-participation terms TB_1 and BC, and applying Tonelli's theorem for TB_2, one obtains:

$v(V;V_B) = V + TB_1 + TB_2 - BC$

$$= V + \tau_1 \tfrac{G}{r}(1 - (\tfrac{V_B}{V})^{\lambda_2+\lambda_3}) + \tau_2 \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty e^{-rt}\alpha(V_t - k)_+ \mathbb{1}_{V_s \geq V_B \, \forall s\in[0,t]} \mathrm{d}t\right] - \rho V_B (\tfrac{V_B}{V})^{\lambda_2+\lambda_3}$$
$$= V + \tau_1 \tfrac{G}{r}(1 - (\tfrac{V_B}{V})^{\lambda_2+\lambda_3}) + \tau_2 \alpha \int_0^\infty c_{do}(V, k, V_B, t)\mathrm{d}t - \rho V_B (\tfrac{V_B}{V})^{\lambda_2+\lambda_3}. \tag{3.6}$$

The value of the equity E is simply given by:

$$E(V; V_B, T) = v(V; V_B) - L(V; V_B, T).$$

3.3 Derivation of the bankruptcy-triggering value V_B

We employ the smooth-pasting condition, also known as low-contact rule, to derive the equilibrium bankruptcy-triggering value V_B, which is given by the largest solution of

$$\frac{\partial E(V; V_B, T)}{\partial V}\Big|_{V=V_B} = 0. \tag{3.7}$$

This condition maximizes both the equity and the insurance company's value with respect to V_B, under the condition of the limited liability for equity holders (i.e., equity holders can always walk away), ensuring that $E(V) \geq 0$ for all $V \geq V_B$. Furthermore, it holds that $E(V) = 0$ for all $V \leq V_B$. Using the smooth-pasting condition, V_B is chosen endogenously ex post via a maximization. $E(V) \geq 0$ for all $V \geq V_B$ guarantees that $\frac{\partial^2}{\partial V^2} E(V; V_B, T)\big|_{V=V_B} \geq 0$ and that $\frac{\partial E(V;V_B,T)}{\partial V_B} = 0$ for any level of V. Note that the solution to equation (3.7) is independent of time, t, i.e., V_B is constant in the analysis. If multiple solutions exist, we select the largest solution, as this is the only one consistent with the limited liability of equity. This implies that the equity value E is increasing in the insurance company's asset value V for all $V \geq V_B$.

For a more detailed derivation of the smooth pasting condition and the equivalency of this condition with $\frac{\partial E(V;V_B,T)}{\partial V_B} = 0$, we refer to Merton [50], Dixit [17], Dumas [19], Leland [42], and He and Milbradt [29]. However, we provide a brief discussion of the intuition behind this condition: The insurance company would set V_B as low as possible in order to maximize its value and prefers to avoid bankruptcy (because of the bankruptcy costs). Conversely, equity holders want to ensure that the equity value is always non-negative. Due to their limited liability, they will liquidate the insurance company (i.e., stop payments) if the equity becomes negative. The equity holders determine the level of V_B after the insurance company has finalized its liability structure. It is important to note that if V_B is (in theory) set too low, the equity value would become negative if the insurance company's assets are low, as the guaranteed payment would become too costly. This also explains the underlying minimization problem in the smooth-pasting condition, as the equity holders minimize V_B to the lowest possible value such that the equity capital remains non-negative. Additionally, due to the absolute priority rule, the value of equity is (theoretically) 0 for every $V \leq V_B$. The term "low-contact rule" refers to the boundary condition that for equity E, seen as a function of V and V_B, the set where $V = V_B$ determines a boundary where the function E is defined. We then use the fact that $h(V) := E(V, V) = 0$ for all $V \geq 0$ and thus $\frac{\partial}{\partial V} h(V) = 0$.

From this point forward, we will assume, as previously discussed, that the liability value of the guaranteed payment and of the surplus participation exceeds the associated tax benefit. Consequently, based on the framework outlined in the preceding subsections, the following inequalities are assumed to hold throughout the rest of this paper:

$$\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\int_0^t e^{-rs} g \mathbb{1}_{\{\min_{r\in[0,s]} V_r \geq V_B\}} \mathrm{d}s\right] \mathrm{d}t \geq T B_1, \tag{3.8}$$

$$\int_0^T c_{do}(V, k, V_B, t)\mathrm{d}t \geq \tau_2 \int_0^\infty c_{do}(V, k, V_B, t)\mathrm{d}t. \tag{3.9}$$

In the following theorem, we apply this smooth-pasting condition and provide a formula where the solution yields the bankruptcy-triggering value V_B. Before proceeding, we introduce

the following shorthand notations with λ_1 as defined in (A.3) and λ_2, λ_3 as defined in (A.9):

$$A_1 := \tfrac{\lambda_2-\lambda_3}{2} + \lambda_3\Phi(\lambda_3\sigma\sqrt{T}) - \lambda_2 e^{-rT}\Phi(\lambda_2\sigma\sqrt{T}) > 0, \tag{3.10}$$

$$A_2 := \tfrac{\lambda_2-\lambda_3}{2} - \tfrac{1}{2\lambda_3\sigma^2 T} + (\lambda_3 + \tfrac{1}{\lambda_3\sigma^2 T})\Phi(\lambda_3\sigma\sqrt{T}) + \tfrac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}} > 0, \tag{3.11}$$

$$A_3 := \tfrac{\lambda_1}{\nu} + \tfrac{1}{\sigma\nu}\sqrt{\lambda_1^2\sigma^2 + 2\nu} > 0, \tag{3.12}$$

$$A_4 := \tfrac{\lambda_1}{\nu}(1 - 2e^{-\nu T}\Phi(\lambda_1\sigma\sqrt{T})) + \tfrac{1}{\sigma\nu}\sqrt{\lambda_1^2\sigma^2 + 2\nu}(2\Phi(\sqrt{\lambda_1^2\sigma^2 + 2\nu}\sqrt{T}) - 1) > 0, \tag{3.13}$$

$$\bar{\alpha} := \begin{cases} \frac{1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2}{A_4-\tau_2 A_3}, & \text{if } A_4 - \tau_2 A_3 > 0. \\ \infty, & \text{else.} \end{cases} \tag{3.14}$$

The constants A_1, A_2, A_3, and A_4 are indeed non-negative, as shown in Lemma B.12. Additionally, we adopt the convention that $[0, \bar{\alpha}] = [0, \infty)$ when $\bar{\alpha} = \infty$.

Theorem 3.1. *The bankruptcy-triggering value V_B is determined as the minimum of V_0 and the largest solution of the following formula:*

$$\begin{aligned} 0 = 1 + \rho(\lambda_2 + \lambda_3) + 2(1-\rho)A_2 - \tfrac{1}{V_B}\Big(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\tfrac{G}{r}(\lambda_2+\lambda_3)\Big) \\ + \tau_2\alpha\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t - \alpha\int_0^T \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t, \end{aligned} \tag{3.15}$$

where λ_2, λ_3 are as in (A.9),

$$\begin{aligned} \tfrac{\partial}{\partial V}c_{do}(V,k,V_B,t)\big|_{V=V_B} = 2e^{-\nu t}\Big(\lambda_1\Phi(d_1(\min\{\tfrac{V_B}{k},1\},t)) + \tfrac{\varphi(d_1(\min\{\frac{V_B}{k},1\},t))}{\sigma\sqrt{t}}\Big) \\ - \tfrac{2ke^{-rt}}{V_B}\Big(\lambda_2\Phi(d_2(\min\{\tfrac{V_B}{k},1\},t)) + \tfrac{\varphi(d_2(\min\{\frac{V_B}{k},1\},t))}{\sigma\sqrt{t}}\Big), \end{aligned} \tag{3.16}$$

and d_1 as in (A.3). In particular, this formula is well-defined and a solution exists with $V_B > 0$.

Note that the two (lengthy) terms in the first line of equation (3.15) are positive, as demonstrated in Lemma B.13. This theorem ensures that a solution for the bankruptcy-triggering value always exists. However, if the solution of (3.15) is larger than V_0, we can set $V_B = V_0$, because if $V_B \geq V_0$, the company declares bankruptcy immediately. Furthermore, the theorem reveals that the bankruptcy-triggering value depends on the chosen contract maturity, which aligns with the basic model of Leland and Toft [44]. However, in models with flow-based bankruptcy or a positive net worth covenant, the bankruptcy-triggering value is independent of the maturity, as shown in works by, e.g., Kim et al. [36], Longstaff and Schwartz [47], or Ross [55].

In the following figure, Figure 2, we provide a graphical illustration of the solution of formula (3.15). In the left plot, we depict the right-hand side of this formula, where the intersection of the graph with the horizontal axis at 0 corresponds to V_B. In the right plot, we show the first line and the negative of the second line from (3.15). The intersection of these two lines represents V_B. This plot illustrates the most typical scenario, where a unique solution to (3.15) exists.

In the following propositions, we explore the influence of several parameters on the optimal bankruptcy-triggering value and provide a sufficient mathematical condition for the assumption that $V \to E(V)$ is non-decreasing.

Proposition 3.2. *The optimal bankruptcy-triggering value V_B is monotonically decreasing in the tax rates τ_1 and τ_2. Moreover, if $P - \frac{G}{r} \leq 0$, we find that V_B is monotonically increasing in the contract maturity T.*

This result seems reasonable, as larger tax rates increase the equity value through a larger tax benefit, which leads to a lower bankruptcy-triggering value. On the other hand, a longer contract duration results in earlier payments of the surplus participation, which increases the liability value and consequently raises the bankruptcy-triggering value.

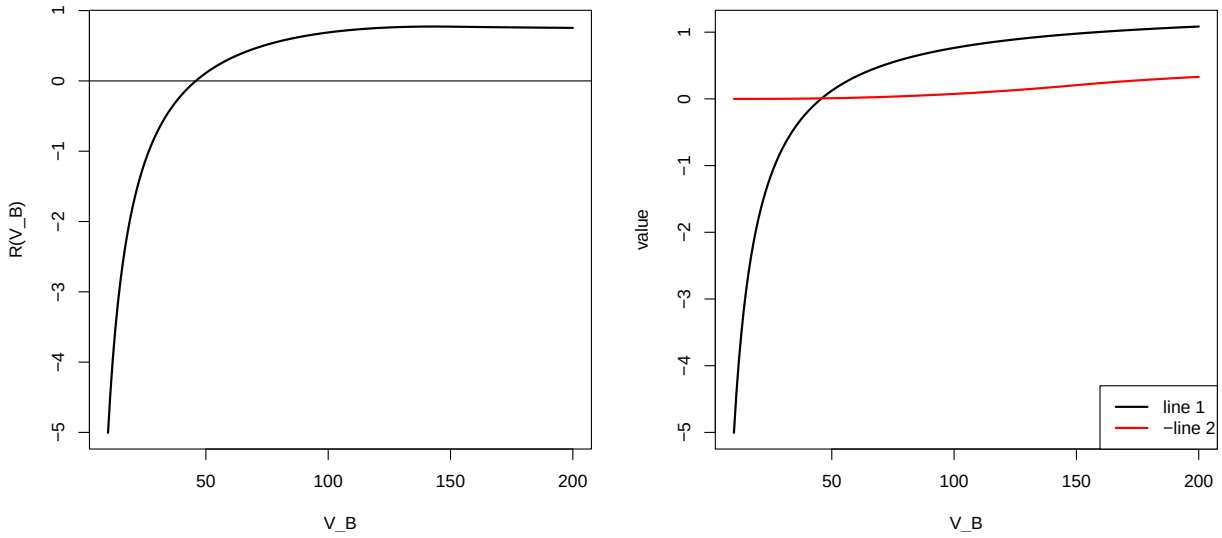

Figure 2: Plot of the right hand side of (3.15) (left) and of its first line and the negative of its second line (right). Note that the intersection with 0 (left) resp. of the two lines (right) is V_B.

Proposition 3.3. *The optimal bankruptcy-triggering value V_B is monotonically increasing and right-continuous in both the surplus participation rate α and the guaranteed payment G. Furthermore, the left-limits exist.*

As both a larger surplus participation and a larger guaranteed payment increase the payment obligations to the policyholders, equity holders will opt for a larger bankruptcy-triggering value to offset the increased liabilities.

The following proposition demonstrates that $V \to E(V)$ is actually non-decreasing (as assumed) as long as the participation rate is not unreasonably high.

Proposition 3.4. *A sufficient condition for our assumption that $V \to E(V)$ being non-decreasing is that $\alpha < \bar{\alpha}$, where $\bar{\alpha}$ is as defined in (3.14).*

Proposition 3.4 also implies (using Lemmas B.12 and B.13) that the tax benefit associated with the tax rate τ_2 significantly influences $\bar{\alpha}$. In particular, we observe that the limit $\bar{\alpha}$ increases with τ_2 as long as $\tau_2 A_3 < A_4$, and is infinity beyond this point. This is intuitive, as a higher tax benefit makes the participation structure more advantageous, and it ensures that the equity remains increasing in the asset value (since the tax benefit is itself increasing in asset value). On the other hand, if $\tau_2 A_3 \geq A_4$, the increase in the value of the tax benefit surpasses the decrease in the value of the liabilities for the surplus participation component, even for arbitrarily high bankruptcy-triggering values. This seems unnatural and a numerical analysis reveals that this scenario does not occur for reasonable parameter values. However, our results cover this situation as well.

In Figure 3, we illustrate how the equity value evolves as a function of the asset value under two conditions: when $V \to E(V)$ is non-decreasing (left) and when it is not (right), i.e., when the participating rate is set too high. If there is no surplus participation, i.e., $\alpha = 0$, $V \to E(V)$ is, of course, increasing.

In most cases, an analytical solution for the bankruptcy-triggering value V_B as defined in Theorem 3.1 is not available. However, if there is no surplus participation, the proof of Theorem 3.1 (or Leland and Toft [44]) yields the following formula:

$$V_B^* = \frac{\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2}, \tag{3.17}$$

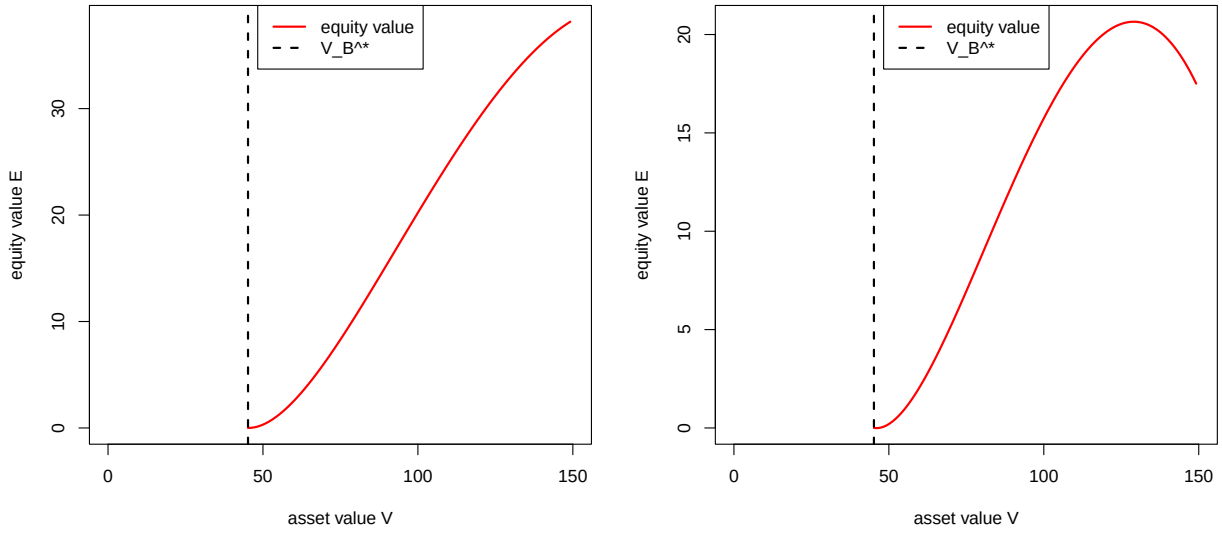

Figure 3: Equity value as a function of the asset value with an α such that $V \to E(V)$ is non-decreasing (left) and if it does not hold (right). In both plots is V_B chosen for the left case.

where λ_2, λ_3 are as defined in (A.9). Moreover, if the portfolio parameters or the market situation are such that the bankruptcy-triggering value lies above the threshold for surplus participation, an analytical solution is provided by the following corollary:

Corollary 3.5. *Define*

$$\hat{V}_B = \frac{\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3) + \tau_2\alpha k A_5 - \alpha k A_6}{1+\rho(\lambda_2+\lambda_3) + 2(1-\rho)A_2 + \tau_2\alpha A_3 - \alpha A_4}, \tag{3.18}$$

where A_1, A_2, A_3, and A_4 are defined as in (3.10), (3.11), (3.12), and (3.13), λ_2, λ_3 are as in (A.9), and

$$A_5 := \tfrac{\lambda_2}{r} + \tfrac{1}{\sigma r}\sqrt{\lambda_2^2\sigma^2 + 2r},$$
$$A_6 := \tfrac{\lambda_2}{r}(1 - 2e^{-rT}\Phi(\lambda_2\sigma\sqrt{T})) + \tfrac{1}{\sigma r}\sqrt{\lambda_2^2\sigma^2+2r}(2\Phi(\sqrt{\lambda_2^2\sigma^2+2r}\sqrt{T}) - 1).$$

If $\hat{V}_B \geq k$, then $\hat{V}_B$ is the largest solution of (3.7) and therefore $V_B = \hat{V}_B$.

4 Optimal rates

In this section, we derive formulas for the optimal participation rate and the optimal guarantee rate. Providing existence results, we begin by fixing one of the two parameters and then conclude with the derivation of the joint optimal values. Based on the definition of $\bar{\alpha}$ in (3.14) and Lemma B.13, it follows that $\bar{\alpha} \geq 0$. For the existence results, we need the following assumption stipulating that small changes in the participation rate resp. the guarantee rate are expected to result in small changes in the bankruptcy-triggering value:

Assumption 1. *The bankruptcy-triggering value V_B, derived from (3.15), is left-continuous in $g \in [0,\infty)$, and left-continuous in $\alpha \in [0,\bar{\alpha})$ if equation (3.15) does not admit a solution for $\alpha = \bar{\alpha}$, or left-continuous in $\alpha \in [0,\bar{\alpha}]$ if equation (3.15) admits a solution for $\alpha = \bar{\alpha}$ (no joint continuity is required).*

Note that Proposition 3.3 already guarantees the right-continuity of V_B with respect to both α and g. Assumption 1 is in particular satisfied if (3.15) admits a unique solution, as the right hand side of this equation is smooth in α and g. A unique solution is guaranteed, for instance,

if $\alpha = 0$ (indicating no surplus participation) or if $V_B \geq k$, based on the analytical solution for V_B (see (3.17) resp. (3.18)). Furthermore, Assumption 1 holds if $\tau_2 \in [0, 1)$ is large enough (for a detailed condition, see Proposition D.1), which corresponds to a sufficiently large tax benefit, leading to an increasing benefit from additional capital. We note that Assumption 1 holds true in all numerical examples we conducted.

4.1 Derivation of the optimal participation rate with a pre-determined guarantee rate

In this subsection, we derive the optimal participation rate α^* when the guarantee rate g is fixed in advance, such that the total insurance company value v in (3.6) is maximized. Therefore, we consider V_B as the largest solution of (3.15) and as a function of α. Note that, in general, $V_B(\alpha)$ does not have an explicit form. Only when $\alpha = 0$, we obtain an explicit form $V_B(0) = V_B^*$ as in (3.17). Hence, (with a slight abuse of notation) our optimization problem is formulated as follows: We seek the optimal participation rate α^* given by:

$$\alpha^* = \underset{\alpha \in [0,1]}{\arg\max}\, v(V; V_B(\alpha)), \tag{4.1}$$

where v is defined as in (3.6).

Proposition 4.1. *There exists an optimal participation rate $\alpha^* \in [0, 1]$.*

The previous proposition asserts that there is an optimal participation rate α^*. The next natural question is under which conditions $\alpha^* > 0$, i.e., when is it advantageous for the insurance company to offer contracts with surplus participation? The following theorem provides an answer to this question.

Theorem 4.2. *There exist $\bar{\tau}, \bar{\bar{\tau}} \in (0, 1)$ with $\bar{\bar{\tau}} \leq \bar{\tau}$ such that it is optimal to choose $\alpha^* > 0$ if $\tau_2 \in (\bar{\tau}, 1]$, and it is optimal to choose $\alpha^* = 0$ if $\tau_2 \in [0, \bar{\bar{\tau}})$.*

Finally, if the following equation (4.2) admits a solution in α, then this solution is equal to the optimal participation rate α^*:

$$\begin{aligned} 0 = &- \tfrac{\tau_1 \frac{G}{r}(\lambda_2+\lambda_3)}{V}\left(\tfrac{V_B(\alpha)}{V}\right)^{\lambda_2+\lambda_3-1} V_B'(\alpha) - \rho(\lambda_2+\lambda_3+1)\left(\tfrac{V_B(\alpha)}{V}\right)^{\lambda_2+\lambda_3} V_B'(\alpha) \\ &+ \tau_2 \int_0^\infty c_{do}(V, k, V_B(\alpha), t)\mathrm{d}t + \alpha\tau_2 \int_0^\infty \tfrac{\partial c_{do}(V,k,V_B(\alpha),T)}{\partial \alpha}\mathrm{d}t, \end{aligned} \tag{4.2}$$

where $\frac{\partial c_{do}(V,k,V_B(\alpha),T)}{\partial \alpha}$ is given in (B.14) resp. (B.15). For an explicit formula of $V_B'(\alpha)$ see (A.10).

The existence of a threshold value $\bar{\tau}$, as stated in the theorem, is plausible, as offering surplus participation becomes more attractive when the tax benefit associated with it is higher. The proof of the theorem further provides an equation for determining $\bar{\tau}$ (by finding the zero root of (C.10)). Interestingly for $\tau_2 < \bar{\tau}$, it is more advantageous for the insurance company to refrain from offering surplus participation rather than offering a small rate which might explain why many contracts do not have any participation element.

From our numerical analysis, we observe that, in general, $\bar{\bar{\tau}} = \bar{\tau}$, $\tau_2 > \bar{\tau}$ (so that $\alpha^* > 0$) and that α^* increases with τ_2. However, if the final lump sum payment P or the guaranteed payment G are too high, $\bar{\tau}$ may actually exceed τ_2 and no participation is offered to policyholders (i.e., $\alpha^* = 0$). The reason is that higher lump sum payments or a larger guarantee rate result in more costly liabilities, while a lower tax rate reduces the tax benefit and thus decreases equity. Consequently, the contract's liabilities must not be too expensive compared to the equity to ensure that a positive participation rate remains optimal. However, typically when optimizing G, the resulting liabilities do not impose an excessive cost on equity, which reinforces that $\tau_2 > \bar{\tau}$ usually holds.

4.2 Derivation of the optimal guarantee rate with a pre-determined participation rate

In this subsection, we derive the optimal guarantee rate g^* when the participation rate α is fixed in advance, such that the total insurance company value v (as defined in (3.6))is maximized. Therefore, we consider V_B as the largest solution of (3.15) and as a function of g. Note that, in general, $V_B(g)$ does not have an explicit form. Also, recall that $g = \frac{G}{T}$. Hence, our optimization problem (with a slight abuse of notation) is as follows: We seek the optimal guarantee rate g^* given by:

$$g^* = \arg\max_{g\in[0,\infty)} v(V; V_B(g)), \tag{4.3}$$

where v is defined in (3.6).

Proposition 4.3. *There exists an optimal guarantee rate $g^* \in [0, \infty)$.*

Next, we address the question of which conditions ensure that $g^* > 0$, i.e., when contracts with a guarantee rate are better for the insurance company than those without such guarantees? This question is answered in the upcoming theorem, but first, we state an assumption made (solely) for this subsection:

Assumption 2. *Let*

$$\begin{aligned}
&\tfrac{2PA_1}{(V_B(0))^2 rT} + \tau_2\alpha \int_0^\infty \tfrac{\partial}{\partial V_B}\big[\tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\big]\big|_{V_B=V_B(0)}\mathrm{d}t \\
&\qquad\qquad - \alpha\int_0^T \tfrac{\partial}{\partial V_B}\big[\tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\big]\big|_{V_B=V_B(0)}\mathrm{d}t \neq 0, \\
&\tfrac{\tau_1 T}{r}\big(1-(\tfrac{V_B(0)}{V})^{\lambda_2+\lambda_3}\big) - \rho(\lambda_2+\lambda_3+1)(\tfrac{V_B(0)}{V})^{\lambda_2+\lambda_3}V_B'(0) + \alpha\tau_2\textstyle\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial g}\big|_{g=0}\mathrm{d}t > 0,
\end{aligned} \tag{4.4}$$

where more explicit formulas are given in (A.9), (A.13), (A.16), (B.10) *resp.* (B.11), *and* (B.18) *resp.* (B.19) *with $g = G = 0$.*

A numerical analysis shows that this assumption already holds for small values of τ_1 (greater than 0.1% in our basic setting).

Theorem 4.4. *It is optimal to choose $g^* > 0$, i.e., it is optimal to provide a contract with a positive guarantee rate.*

Moreover, if the following equation (4.5) *admits a solution for g, then that solution is the optimal guarantee rate g^*:*

$$\begin{aligned}
0 = {}& \tfrac{\tau_1 T}{r}\big(1-(\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\big) - \tfrac{\tau_1 gT(\lambda_2+\lambda_3)}{Vr}(\tfrac{V_B}{V})^{\lambda_2+\lambda_3-1}V_B'(g) - \rho(\lambda_2+\lambda_3+1)(\tfrac{V_B}{V})^{\lambda_2+\lambda_3}V_B'(g) \\
&+ \alpha\tau_2\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B,T)}{\partial g}\mathrm{d}t.
\end{aligned} \tag{4.5}$$

where $\frac{\partial c_{do}(V,k,V_B(g),T)}{\partial g}$ is as in (B.18) *resp.* (B.19). *For an explicit formula of $V_B'(g)$ see* (A.14).

Under some technical conditions, equation (4.5) always admits a solution (see Proposition D.3). To conclude this subsection, we discuss the numerical observations regarding Assumption 2 in more detail:

As in the previous Subsection 4.1, our numerical analysis also shows that Assumption (4.4) is typically fulfilled when optimizing G. However, if the lump sum payment P is too high, or if the tax rate τ_1 is too low, the assumption might not hold. This suggests that when the liabilities become too costly in comparison to equity, it becomes difficult to provide additional promises to policyholders. Additionally, while an arbitrary high participation rate α could theoretically cause (4.4) to fail, the restriction $\alpha \in [0, \bar{\alpha}]$ ensures that in most situations, this range is not sufficient for the assumption to be violated.

4.3 Derivation of the optimal guarantee rate and participation rate

In this subsection, we derive the optimal participation rate α^* and the optimal guarantee rate g^* simultaneously, such that the total insurance company value v (as defined in (3.6)) is maximized. Therefore, we consider V_B as the largest solution of (3.15), and make its dependence on (α, g) explicit. Note that, in general, $V_B(\alpha, g)$ does not have an explicit form. Hence, our optimization problem is then (with a slight abuse of notation) formulated as follows: We seek the optimal rate vector (α^*, g^*) given by:

$$(\alpha^*, g^*) = \underset{(\alpha,g)\in[0,1]\times[0,\infty)}{\arg\max} v(V; V_B(\alpha, g)), \tag{4.6}$$

where v is defined in (3.6).

Proposition 4.5. *Under Assumption 1, there exists an* $(\alpha^*, g^*) \in [0, 1] \times [0, \infty)$ *which is the optimal pair of rates for* $(\alpha, g) \in [0, 1] \times [0, \infty)$.

Proposition 4.6. *If the non-linear equation system consisting of equations* (4.2) *and* (4.5) *admits a solution, then the solution is given by* (α^*, g^*).

This proposition directly follows from Theorems 4.2 and 4.4. From these theorems, we also obtain sufficient conditions for ensuring that $\alpha^* > 0$ and $g^* > 0$.

5 Numerical Results

In this section, we provide a sensitivity analysis on the assumptions for Theorems 4.2 and 4.4, as well as an examination of the optimal participation rate α^* and the optimal guarantee rate g^*. Additionally, we present a plot showing the equity E and liability L values as functions of the asset value V, and discuss how the asset substitution effect changes when adding participation.

For this analysis, we use a basic setting for both the financial market and the contract conditions. Unless stated otherwise, the parameters take the following values: For the financial market, we use a risk-free interest rate $r = 1\%$, a dividend rate $\nu = 5\%$ and a volatility of $\sigma = 20\%$. The contract length is $T = 30$ with lump sum payment $P = 95$, guarantee rate $\frac{G}{P} = 2\%$, initial asset value $V_0 = 100$, and a surplus participation starting at $k = 150$ with participation rate $\alpha = 5\%$, i.e., we have the liability rate $\frac{P}{V_0} = 95\%$ and the surplus initiation rate $\frac{k}{V_0} = 150\%$. The tax rates are $\tau_1 = 35\% = \tau_2$, and we use a loss fraction at bankruptcy of $\rho = 50\%$. These values for the lump sum payment and the dividend are typical for large insurance companies. The high liability capital is also not uncommon for life insurance companies, see, for instance the balance sheets of [2, p.150] and [3, p.30]. The tax rates and the loss fraction are taken from Leland and Toft [44], whereas Chen et al. [12] utilized slightly lower values (20% or 25%). It is worth noting that, as discussed, e.g., by Kling et al. [37, 38], insurance companies commonly employ return smoothing mechanisms in their payments to policyholders. This can be modeled by reducing the volatility σ to $50 - 75\%$ of its original value. As shown in the sensitivity analysis in Figure 8, this would significantly increase the surplus participation rate. However, we maintain $\sigma = 20\%$, as this effect is not typically considered in the majority of the related literature.

We have checked the pre-conditions from Theorems 3.1, 4.2, and 4.4 for all cases presented, and they are satisfied with $\bar{\alpha} \approx 0.10$ and $\bar{\tau} = \bar{\bar{\tau}}$ in the basic setting. However, $\bar{\tau}$ exceeds τ_2, i.e., $\alpha^* = 0$, if, ceteris paribus, the lump sum payment value or the guarantee rate becomes too high ($\frac{P}{V_0} \geq 150\%$ resp. $\frac{G}{P} \geq 7\%$), or if the tax rate on the participation is too low ($\tau_2 \leq 8\%$). Similarly, equation (4.4) from Assumption 2 does not hold, i.e., $g^* = 0$, if, ceteris paribus, the lump sum payment value gets too high ($\frac{P}{V_0} \geq 150\%$) or if the tax rate on the guaranteed payment is too low ($\tau_1 \leq 0.1\%$). These results align with the discussions at the end of Subsections 4.1 and 4.2. Changes in other parameters, however, generally maintain the two conditions, ensuring that $\alpha^* > 0$ resp. $g^* > 0$. For these parameters, we provide an overview in Figure 4, where we

plot the results when varying the parameters simultaneously. The left plot shows the region such that both $\alpha^* > 0$ and $g^* > 0$ hold, depending on the liability ratio $\frac{P}{V_0}$ and the the tax benefit $\tau = \tau_1 = \tau_2$. The right plot shows the region such that $\alpha^* > 0$, depending on the liability ratio $\frac{P}{V_0}$ and the guarantee rate $\frac{G}{P}$. Since we vary G in the right plot, it is not meaningful to impose the condition $g^* > 0$ in this context. In both plots, a white square indicates that the respective optimal rates are positive, whereas a black square indicates that at least one parameter is zero in the optimal case. All other parameters are fixed as in the basic parametrization. From the plots, we confirm the statements made in the discussions at the end of Subsections 4.1 and 4.2, which suggest that the optimal rates remain positive as long as the liabilities are not excessively costly relative to the equity value. This indicates that it becomes challenging to offer additional promises to policyholders when the original promises, like the lump sum payment, are already too costly compared to the receiving tax benefit.

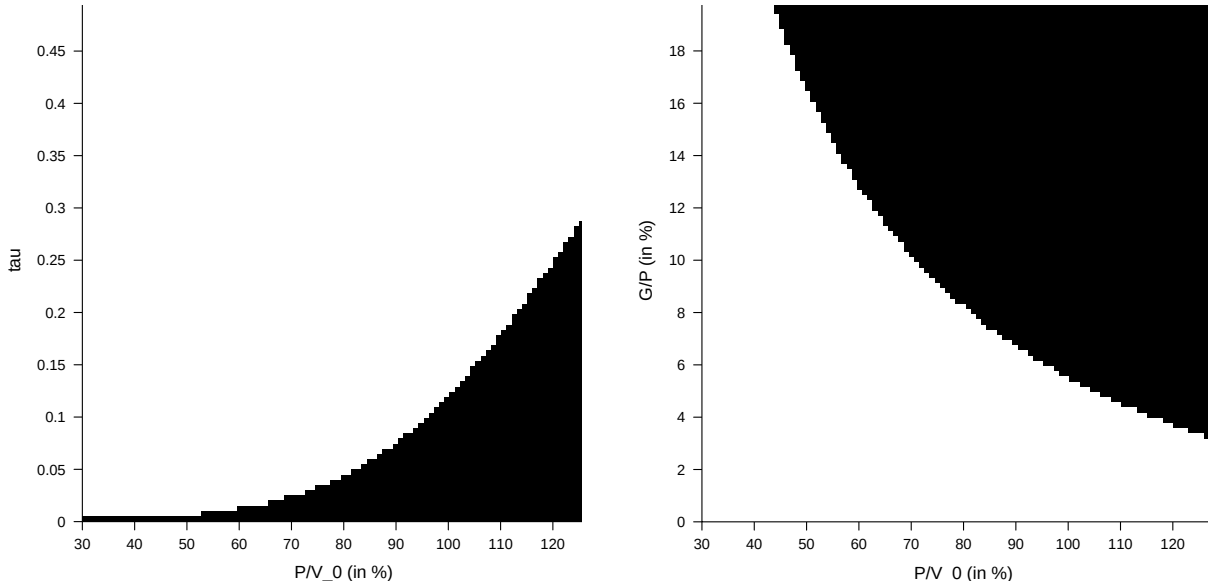

Figure 4: Scatterplot if the optimal rates are positive as functions of the liability ratio $\frac{P}{V_0}$ (in %) and the tax benefit $\tau = \tau_1 = \tau_2$ (left) resp. the guarantee rate $\frac{G}{P}$ (in %) (right). A white square indicates that $\alpha^* > 0$ and (solely in the left plot) that additionally $g^* > 0$ holds.

In Figure 5, we illustrate how the bankruptcy-triggering value V_B varies as a function of the participation rate α (left plot) and the guarantee rate $\frac{G}{P}$ (right plot). For the left plot, we restrict the participation rate $\alpha < \bar{\alpha}$, ensuring that a solution to equation (3.15) exists, i.e., the bankruptcy-triggering value is not set to V_0 which leads to immediate bankruptcy. In both plots, we observe that an increase in either the participation rate or the guarantee rate results in a higher bankruptcy-triggering value V_B. The reason is that both higher participation rates and higher guarantee rates to policyholders increase the total liabilities, which, in turn, brings the equity holders to default earlier in order not to have negative equity. Additionally, we notice that the bankruptcy-triggering value is more sensitive to changes in the guarantee rate than to changes in the participation rate. This can be explained by the fact that the guarantee rate is a fixed obligation, meaning it must be paid regardless of asset performance. In contrast, the participation rate only affects payments when the asset value exceeds a certain threshold (e.g., in the basic parametrization starting at $\frac{k}{V_0} = 150\%$). Given that the probability of surpassing this threshold is rather low when asset values are close to V_B, the participation rate, in this case, has less impact on the bankruptcy-triggering value. From the right plot, we can also observe that when the accumulated guarantee rate $\frac{G}{P}$ reaches approximately 11.1%, the bankruptcy-triggering value V_B exceeds the initial value $V_0 = 100$. In this case, equity holders would be forced to declare bankruptcy immediately, indicating that the guarantee rate has been set unfeasibly high.

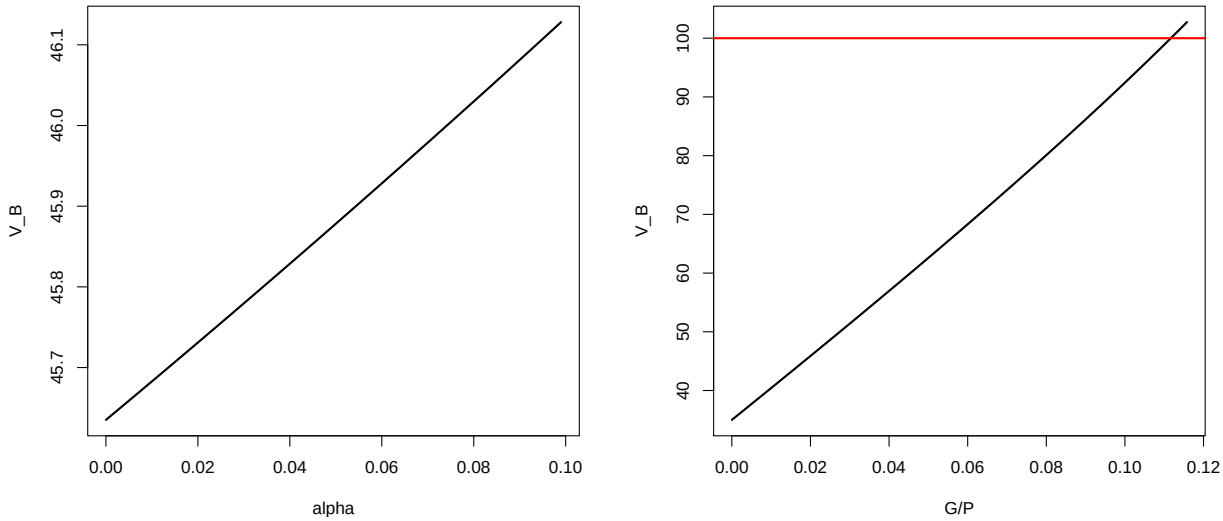

Figure 5: Variation of the bankruptcy-triggering value V_B for different values of the participation rate α (with $\alpha < \bar{\alpha}$) and the guaranteed payment rate $\frac{G}{P}$.

In Figure 6, we show the value of the insurance company v, the equity value E, and the liability value L as functions of the guaranteed payment G. From the plot, we see that in the basic parametrization, the optimal guarantee rate is approximately $\frac{G^*}{P} \approx 1.91\%$ when optimizing for the insurance company value v. However, this optimal value does not correspond to the optimal guarantee rate from either the policyholder's or the equity holder's perspectives. This finding aligns with the result in the case of no surplus participation, as discussed by Lando [41].

In Figure 7, we present the insurance company value v, the equity value E, and the liability value L as functions of the asset value V, for the optimal values α^*, G^*, and V_B^* in the basic parametrization. We find an optimal participation rate of $\alpha^* \approx 0.099$, an optimal guarantee rate $\frac{G^*}{P} \approx 1.91\%$, and the corresponding bankruptcy-triggering value $V_B^* \approx 45.36$. From the plots in Figure 7, we observe that all three values, the insurance company value v, the equity value E and the liability value L, increase with the asset value V, which is expected. However, it is important to note that the increase in the insurance company value is not linear. As the asset value V increases, the growth in the company value starts to decrease slightly. Additionally, we see that the equity value E becomes zero at V_B^*, which is consistent with the smooth-pasting condition. When comparing these results to Lando [41], who analyzed a model without participation, we observe that the equity value E is no longer convex with respect to the asset value. This change is due to the participation costs becoming more significant at higher values of V. As a result, the rate of increase of E in V decreases, and E adopts a concave form as V grows. Moreover, in the absence of convexity, the "option-like" nature of equity diminishes. This is crucial for mitigating the asset substitution effect, which will be discussed in greater detail later.

In Figure 8, we present some results of a sensitivity analysis on the optimal participation rate α^*. The plots display the effects of variations in the dividend rate ν on the left, the contract duration T in the middle, and the tax rate τ_2 on the right. We find that the optimal participation rate is strongly influenced by all three parameters. As the dividend rate increases, the optimal participation rate also increases. This is economically reasonable, as a higher dividend rate offered by the insurance company diminishes the long-term performance of the company's asset process, thereby making it more cost-effective for the insurer to offer a higher participation rate. On the other hand, a longer contract duration decreases the optimal participation rate. This occurs because the longer duration increases the likelihood of experiencing a high surplus participation, assuming positive expected returns over time. Lastly, an increase in the tax rate τ_2 results in

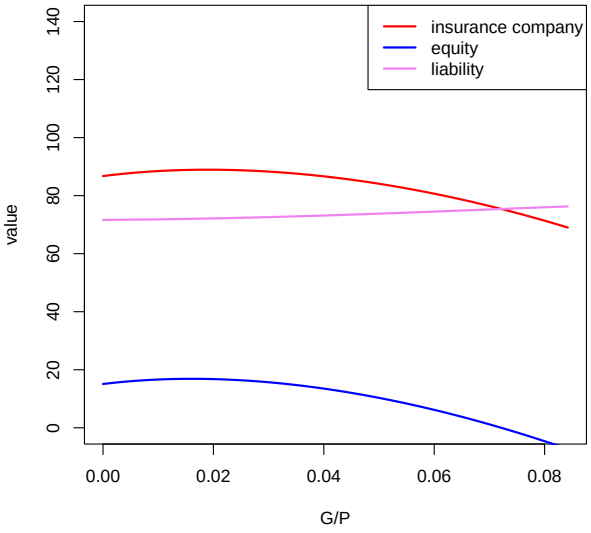

Figure 6: Insurance company value v, liability value L, and equity value E as a function of the guarantee rate $\frac{G}{P}$ in the basic parametrization.

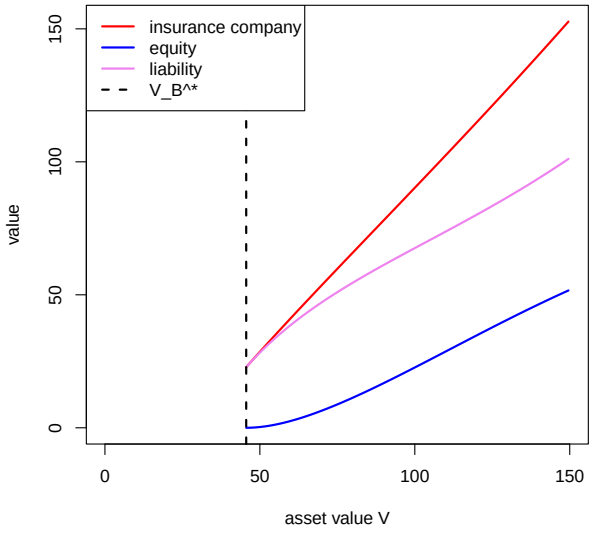

Figure 7: Value of the insurance company, equity and liability as a function of the asset value V. Here, we used the optimal participation rate $\alpha^* \approx 9.9\%$ and the optimal guarantee rate $\frac{G^*}{P} \approx 1.91\%$ to determine $\frac{V_B^*}{V_0} \approx 45.36\%$.

higher optimal participation rates. This is intuitive, as higher tax rates enhance the value of equity, thus making participation more attractive. Conversely, the effect of the tax rate τ_1 on the optimal participation rate is minimal.

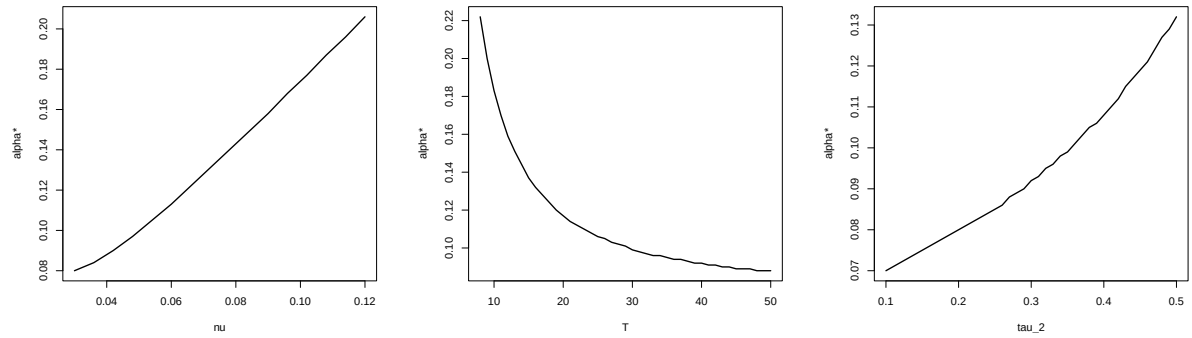

Figure 8: Sensitivity analysis of the optimal participation rate α^* for the dividend rate ν (left), the contract duration T (middle), and the tax rate τ_2 (right).

In the concluding paragraph of this numerical analysis, we examine the asset substitution effect, which describes the tendency of equity holders to increase the riskiness of a company's investment decisions, leading to a transfer of value from liabilities to equity. Figure 9 illustrates the partial derivatives of equity and liability with respect to asset volatility, across different surplus participation rates (top row) and contract durations (bottom row). The asset substitution effect appears in regions where $\frac{\partial}{\partial\sigma}L < 0$ and $\frac{\partial}{\partial\sigma}E > 0$, meaning equity holders seek to increase risk, while policyholders seek to reduce it. In the absence of participation, i.e., $\alpha = 0\%$, we confirm previous findings (see the references in the introduction) that there is an asset substitution effect in a large region (starting at 75 % of the initial asset value for our parametrization). When surplus participation is introduced, transferring some of the incentives for risk-taking to policyholders, the asset substitution effect vanishes for a reasonable contract duration. However, as the contract duration increases (particularly beyond the lifespan of multiple generations), the influence of

surplus participation diminishes, and the asset substitution effect reverts to the case without surplus participation, where the asset substitution effect is present. This last result is plausible, as a longer contract duration delays surplus payments to policyholders, and in the limit ($T = \infty$), no surplus payment occurs at finite time points. Furthermore, Leland and Toft [44] observe that even in the absence of surplus participation, longer maturities exacerbate the asset substitution effect. They contend that, although the option analogy (presented in the introduction) may not be entirely accurate, the adverse incentives linked to longer maturities are indeed magnified. The impacts of parameter changes align with the case of no participation.

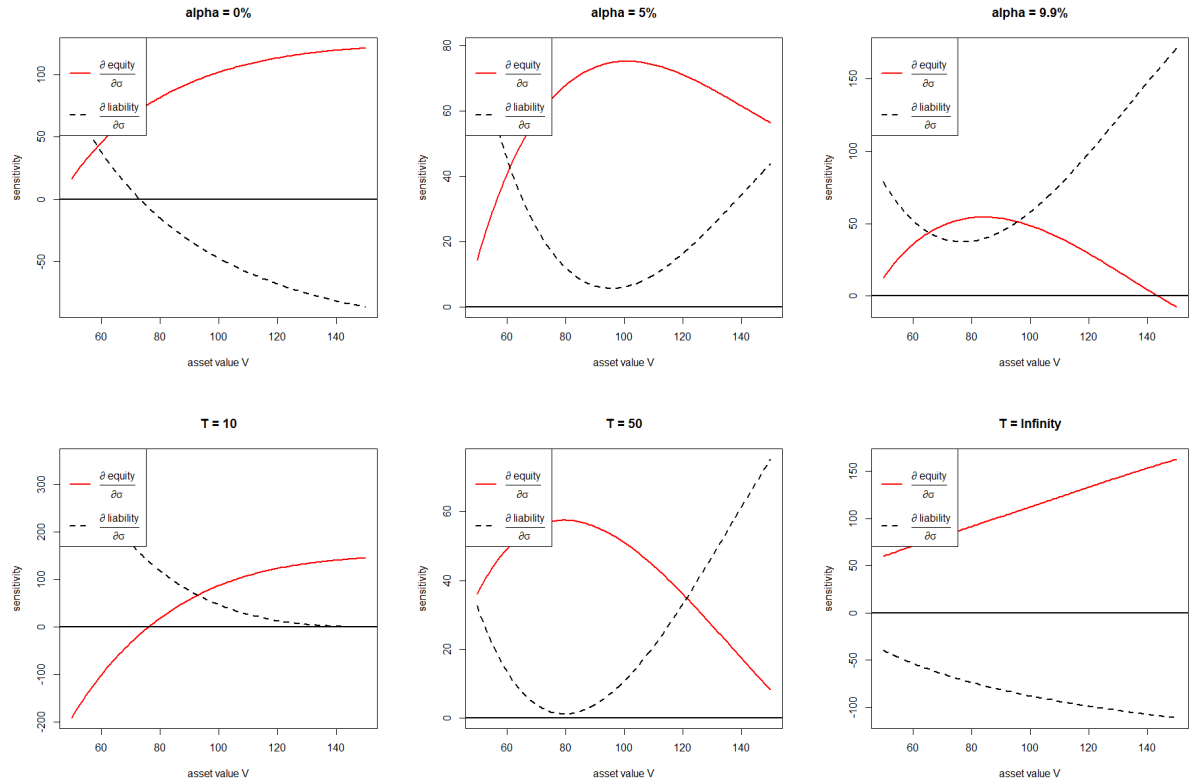

Figure 9: Effect of an increase in the volatility on equity and liabilities when varying the contract duration (top row) and when varying the surplus participation rate (bottom row). The lines show the partial derivative with respect to the volatility.

6 Conclusion

In this paper, we explained the capital structure of life insurance companies and the existence of hybrid contracts that combine participation and guarantee elements generalizing Leland's model to incorporate surplus participation. To this end, we derived formulas for the optimal bankruptcy-triggering value, the optimal participation rate, and the optimal guarantee rate. The numerical analysis demonstrated that the required assumptions are generally satisfied in most cases and that the optimal participation rate is particularly sensitive to changes in the the contract duration and the associated tax rate. Moreover, we showed that the asset substitution effect decreases when adding surplus participation.

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Appendix

A Barrier options and mathematical details on the liability structure and the optimal rates

In this section, we offer deeper insights into barrier options, and we provide more details on the construction and mathematical foundations used to determine the liability structure and the optimal rates.

A.1 Barrier options

The surplus participation component constructed in Subsection 2.1 is modeled as a so-called Down-and-Out-Call Option. Unlike traditional European options, barrier options are path-dependent, meaning their value depends on whether the underlying asset's path reaches a predefined barrier, which in this case is the bankruptcy-triggering value, V_B. Barrier options are classified into two types: "knock-in" and "knock-out" options, see Hull [34]. A "knock-in" option only pays out if the barrier is breached, while a "knock-out" option only pays if the barrier is not hit. Additionally, barrier options are further categorized as "up" or "down" depending on whether the barrier is above or below the initial asset value. In our framework, the value of the surplus participation is equivalent to a Down-and-Out Call option with a barrier at V_B and a strike price of k, where the bankruptcy triggering value V_B is lower than the initial insurance company value V_0. If the asset value hits the barrier V_B, bankruptcy is triggered, and all contracts terminate, meaning no further surplus participation will be paid. Notably, a Down-and-Out Call option is always cheaper than a standard Call option. The pricing formula for barrier options depends on whether the strike price is larger or smaller than the barrier. However, when the strike price equals the barrier, the pricing formulas for both cases coincide.

Now, returning to our setting: Let the barrier be represented by V_B and the strike price by k. By Hull [34], for the asset value V, the values of a classical call option c, of a Down-and-Out Call option $c_{do}^{V_B\leq k}$, when the barrier is below the strike, and of a Down-and-Out Call option $c_{do}^{V_B\geq k}$, when the barrier is above the strike, all with maturity T and dividend rate ν, are given by the following formulas:

$$\begin{aligned}
c(V_0,k,T) &= \mathbb{E}^{\mathbb{Q}}[(V_T-k)_+] \\
&= V_0 e^{-\nu T}\Phi(d_1(\tfrac{V_0}{k},T)) - k e^{-rT}\Phi(d_2(\tfrac{V_0}{k},T)), \\
c_{do}^{V_B\leq k}(V_0,k,V_B,T) &:= \mathbb{E}^{\mathbb{Q}}[(V_T-k)_+\mathbb{1}_{\{\min_{s\in[0,T]}V_s\geq V_B\}}] \qquad \text{(A.1)}\\
&= c(V_0,k,T) - V_0 e^{-\nu T}(\tfrac{V_B}{V_0})^{2\lambda_1}\Phi(d_1(\tfrac{V_B^2}{V_0 k},T)) + k e^{-rT}(\tfrac{V_B}{V_0})^{2\lambda_1-2}\Phi(d_2(\tfrac{V_B^2}{V_0 k},T)), \\
c_{do}^{V_B\geq k}(V_0,k,V_B,T) &:= \mathbb{E}^{\mathbb{Q}}[(V_T-k)_+\mathbb{1}_{\{\min_{s\in[0,T]}V_s\geq V_B\}}] \qquad \text{(A.2)}\\
&= V_0\Phi(d_1(\tfrac{V_0}{V_B},T))e^{-\nu T} - k e^{-rT}\Phi(d_2(\tfrac{V_0}{V_B},T)) - V_0 e^{-\nu T}(\tfrac{V_B}{V_0})^{2\lambda_1}\Phi(d_1(\tfrac{V_B}{V_0},T)) \\
&\quad + k e^{-rT}(\tfrac{V_B}{V_0})^{2\lambda_1-2}\Phi(d_2(\tfrac{V_B}{V_0},T)),
\end{aligned}$$

where Φ denotes the cumulative distribution function of a standard normal distribution and

$$d_{1/2}(x,t) = \frac{\ln x + (r-\nu\pm\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \qquad \lambda_1 = \frac{r-\nu+\frac{\sigma^2}{2}}{\sigma^2}. \qquad \text{(A.3)}$$

By substituting $V_B = k$, we find that $c_{do}^{V_B\leq k}$ and $c_{do}^{V_B\geq k}$ yield the same value when $V_B = k$. Thus, we can express this as:

$$c_{do}(V_0,k,V_B,T) := \begin{cases} c_{do}^{V_B\leq k}(V_0,k,V_B,T) & \text{if } V_B\leq k, \\ c_{do}^{V_B\geq k}(V_0,k,V_B,T) & \text{if } V_B > k, \end{cases} \qquad \text{(A.4)}$$

which is a continuous function in V_B. In particular, note that

$$c_{do}(V_0,k,V_B,T)=\mathbb{E}^{\mathbb{Q}}[(V_T-k)_+\mathbb{1}_{\{\min_{s\in[0,T]}V_s\geq V_B\}}].$$

A.2 Mathematical details on the liability structure

In this brief paragraph, we provide explicit formulas for the functions F^V, G^V, I_1^V, and I_2^V defined in Subsection 3.1 (for the proofs, we refer to Harrison [27], Rubenstein and Reiner [56], and Leland and Toft [44]):

$$F^V(t)=\Phi(-d_3(\tfrac{V}{V_B},t))+(\tfrac{V_B}{V})^{2\lambda_2}\Phi(-d_4(\tfrac{V}{V_B},t)), \tag{A.5}$$

$$G^V(t)=(\tfrac{V_B}{V})^{\lambda_2-\lambda_3}\Phi(-d_5(\tfrac{V}{V_B},t))+(\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\Phi(-d_6(\tfrac{V}{V_B},t)), \tag{A.6}$$

$$I_1^V(T)=\tfrac{1}{rT}(G^V(T)-e^{-rT}F^V(T)), \tag{A.7}$$

$$I_2^V(T)=\tfrac{1}{\lambda_3\sigma\sqrt{T}}\left((\tfrac{V_B}{V})^{\lambda_2-\lambda_3}\Phi(-d_5(\tfrac{V}{V_B},T))d_5(\tfrac{V}{V_B},T)-(\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\Phi(-d_6(\tfrac{V}{V_B},T))d_6(\tfrac{V}{V_B},T)\right), \tag{A.8}$$

where

$$d_{3/4}(x,t)=\frac{\ln x\pm\lambda_2\sigma^2 t}{\sigma\sqrt{t}},\qquad d_{5/6}(x,t)=\frac{\ln x\pm\lambda_3\sigma^2 t}{\sigma\sqrt{t}},$$
$$\lambda_2=\frac{r-\nu-\frac{\sigma^2}{2}}{\sigma^2}(=\lambda_1-1),\qquad \lambda_3=\frac{\sqrt{(\lambda_2\sigma^2)^2+2r\sigma^2}}{\sigma^2}. \tag{A.9}$$

A.3 Formulas for determining the optimal rates

In this subsection, we provide detailed formulas for terms presented in the results of Chapter 4. The correctness of these formulas is demonstrated in the proofs of Theorem 4.2 and 4.4.

For the terms stated in Theorem 4.2, where V_B is expressed as a function of α, we obtain:

$$V_B'(\alpha)=\frac{\int_0^T\frac{\partial c_{do}(V,k,V_B(\alpha),t)}{\partial V}\big|_{V=V_B(\alpha)}\mathrm{d}t-\tau_2\int_0^\infty\frac{\partial c_{do}(V,k,V_B(\alpha),t)}{\partial V}\big|_{V=V_B(\alpha)}\mathrm{d}t}{\frac{1}{V_B^2(\alpha)}\left(\frac{2(P-\frac{G}{r})A_1}{rT}+2\frac{G}{r}A_2-\tau_1\frac{G}{r}(\lambda_2+\lambda_3)\right)+\tau_2\alpha\int_0^\infty\partial c_{do}(t)\mathrm{d}t-\alpha\int_0^T\partial c_{do}(t)\mathrm{d}t}, \tag{A.10}$$

$$\begin{aligned}\partial c_{do}(t)&:=\tfrac{\partial}{\partial V_B(\alpha)}[\tfrac{\partial c_{do}(V,k,V_B(\alpha),t)}{\partial V}\big|_{V=V_B(\alpha)}]\\&=\tfrac{2ke^{-rt}}{V_B^2(\alpha)}\left(\lambda_2\Phi(d_2(\min\{\tfrac{V_B(\alpha)}{k},1\},t))+\tfrac{\varphi(d_2(\min\{\frac{V_B(\alpha)}{k},1\},t))}{\sigma\sqrt{t}}\right),\end{aligned} \tag{A.11}$$

with the special case

$$V_B'(0)=(V_B(0))^2\frac{\int_0^T\frac{\partial c_{do}(V,k,V_B(0),t)}{\partial V}\big|_{V=V_B(0)}\mathrm{d}t-\tau_2\int_0^\infty\frac{\partial c_{do}(V,k,V_B(0),t)}{\partial V}\big|_{V=V_B(0)}\mathrm{d}t}{\left(\frac{2(P-\frac{G}{r})A_1}{rT}+2\frac{G}{r}A_2-\tau_1\frac{G}{r}(\lambda_2+\lambda_3)\right)}. \tag{A.12}$$

For the terms stated in Theorem 4.4, with V_B as a function of g, we find that $V_B(0)$ is the largest solution of

$$\begin{aligned}0=&1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2-\tfrac{2PA_1}{V_B(0)rT}+\tau_2\alpha\int_0^\infty\tfrac{\partial c_{do}(V,k,V_B(0),t)}{\partial V}\big|_{V=V_B(0)}\mathrm{d}t\\&-\alpha\int_0^T\tfrac{\partial c_{do}(V,k,V_B(0),t)}{\partial V}\big|_{V=V_B(0)}\mathrm{d}t.\end{aligned} \tag{A.13}$$

Furthermore, we obtain:

$$V_B'(g)=\frac{\frac{T}{V_B(g)r}(-\frac{2A_1}{rT}+2A_2-\tau_1(\lambda_2+\lambda_3))}{\frac{1}{V_B^2(g)}(\frac{2(P-\frac{G}{r})A_1}{rT}+2\frac{G}{r}A_2-\tau_1\frac{G}{r}(\lambda_2+\lambda_3))+\tau_2\alpha\int_0^\infty\partial c_{do}(t)\mathrm{d}t-\alpha\int_0^T\partial c_{do}(t)\mathrm{d}t}, \tag{A.14}$$

$$\partial c_{do}(t) := \frac{\partial}{\partial V_B(g)}\left[\frac{\partial c_{do}(V,k,V_B(g),t)}{\partial V}\Big|_{V=V_B(g)}\right]$$
$$= \frac{2ke^{-rt}}{(V_B(g))^2}\left(\lambda_2\Phi(d_2(\min\{\frac{V_B(g)}{k},1\},t)) + \frac{\varphi(d_2(\min\{\frac{V_B(g)}{k},1\},t))}{\sigma\sqrt{t}}\right), \tag{A.15}$$

with the special case

$$V_B'(0) = \frac{\frac{T}{V_B(0)r}(-\frac{2A_1}{rT} + 2A_2 - \tau_1(\lambda_2+\lambda_3))}{\frac{2PA_1}{(V_B(0))^2rT} + \alpha(\tau_2\int_0^\infty \partial_0 c_{do}(t)\mathrm{d}t - \int_0^T \partial_0 c_{do}(t)\mathrm{d}t)}, \tag{A.16}$$

$$\partial_0 c_{do}(t) := \frac{\partial}{\partial V_B}\left[\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\Big|_{V=V_B}\right]\Big|_{g=0}$$
$$= \frac{2ke^{-rt}}{(V_B(0))^2}\left(\lambda_2\Phi(d_2(\min\{\frac{V_B(0)}{k},1\},t)) + \frac{\varphi(d_2(\min\{\frac{V_B(0)}{k},1\},t))}{\sigma\sqrt{t}}\right). \tag{A.17}$$

B Technical Lemmas

In this section, we present and prove several technical lemmas that are used in the proofs of the theorems and propositions discussed in the main text. The proofs of these theorems and propositions can be found in Appendix C.

Lemma B.1. *It holds that* $|\int_0^\infty c_{do}(V,k,V_B,t)\mathrm{d}t| < \infty$ *for all* $V > 0$, $k \geq 0$, *and* $V_B \geq 0$.

Proof. It holds:

$$\left|\int_0^\infty c_{do}(V_0,k,V_B,t)\mathrm{d}t\right| = \int_0^\infty \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}(V_t-k)_+ \mathbb{1}_{V_s\geq V_B\,\forall s\in[0,t]}\right]\mathrm{d}t$$
$$\leq \int_0^\infty e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[V_t\right]\mathrm{d}t = \int_0^\infty e^{-rt}V_0e^{(r-\nu)t}\mathrm{d}t = V_0\int_0^\infty e^{-\nu t}\mathrm{d}t = \frac{V_0}{\nu} < \infty,$$

since $\nu > 0$ and where we could drop the absolute value, as everything is non-negative. We denoted $V = V_0$ in accordance with the notation in the main part. □

Lemma B.2. *It holds that* $c_{do}(V,k,V_B,T)$ *is continuously differentiable as a function of* V.

Proof. It is evident from equations (A.1) and (A.2) that both $c_{do}^{V_B\leq k}(V,k,V_B,T)$ and $c_{do}^{V_B\geq k}(V,k,V_B,T)$ are continuously differentiable. Therefore, it suffices to verify whether the derivatives coincide at $V_B = k$. Differentiating equations (A.1) and (A.2) yields:

$$\begin{aligned}\frac{\partial}{\partial V}c_{do}^{V_B\leq k}(V,k,V_B,T) =& e^{-\nu T}\Phi(d_1(\tfrac{V}{k},T)) + Ve^{-\nu T}\varphi(d_1(\tfrac{V}{k},T))\tfrac{1}{\sigma\sqrt{T}V} - ke^{-rT}\varphi(d_2(\tfrac{V}{k},T))\tfrac{1}{\sigma\sqrt{T}V}\\
&- e^{-\nu T}V_B^{2\lambda_1}(1-2\lambda_1)V^{-2\lambda_1}\Phi(d_1(\tfrac{V_B^2}{Vk},T)) - Ve^{-\nu T}(\tfrac{V_B}{V})^{2\lambda_1}\varphi(d_1(\tfrac{V_B^2}{Vk},T))\tfrac{-1}{\sigma\sqrt{T}V}\\
&+ ke^{-rT}V_B^{2\lambda_1-2}(2-2\lambda_1)V^{1-2\lambda_1}\Phi(d_2(\tfrac{V_B^2}{Vk},T))\\
&+ ke^{-rT}(\tfrac{V_B}{V})^{2\lambda_1-2}\varphi(d_2(\tfrac{V_B^2}{Vk},T))\tfrac{-1}{\sigma\sqrt{T}V}\\
=& e^{-\nu T}\Phi(d_1(\tfrac{V}{k},T)) + \tfrac{e^{-\nu T}\varphi(d_1(\frac{V}{k},T))}{\sigma\sqrt{T}} - \tfrac{ke^{-rT}\varphi(d_2(\frac{V}{k},T))}{\sigma\sqrt{T}V}\\
&- (1-2\lambda_1)e^{-\nu T}(\tfrac{V_B}{V})^{2\lambda_1}\Phi(d_1(\tfrac{V_B^2}{Vk},T)) + \tfrac{e^{-\nu T}(\frac{V_B}{V})^{2\lambda_1}\varphi(d_1(\frac{V_B^2}{Vk},T))}{\sigma\sqrt{T}}\\
&+ (2-2\lambda_1)\tfrac{ke^{-rT}}{V}(\tfrac{V_B}{V})^{2\lambda_1-2}\Phi(d_2(\tfrac{V_B^2}{Vk},T)) - \tfrac{ke^{-rT}(\frac{V_B}{V})^{2\lambda_1-2}\varphi(d_2(\frac{V_B^2}{Vk},T))}{\sigma\sqrt{T}V},\end{aligned} \tag{B.1}$$

$$\begin{aligned}\frac{\partial}{\partial V}c_{do}^{V_B\geq k}(V,k,V_B,T) =& e^{-\nu T}\Phi(d_1(\tfrac{V}{V_B},T)) + Ve^{-\nu T}\varphi(d_1(\tfrac{V}{V_B},T))\tfrac{1}{\sigma\sqrt{T}V} - ke^{-rT}\varphi(d_2(\tfrac{V}{V_B},T))\tfrac{1}{\sigma\sqrt{T}V}\\
&- e^{-\nu T}V_B^{2\lambda_1}(1-2\lambda_1)V^{-2\lambda_1}\Phi(d_1(\tfrac{V_B}{V},T)) - Ve^{-\nu T}(\tfrac{V_B}{V})^{2\lambda_1}\varphi(d_1(\tfrac{V_B}{V},T))\tfrac{-1}{\sigma\sqrt{T}V}\\
&+ ke^{-rT}V_B^{2\lambda_1-2}(2-2\lambda_1)V^{1-2\lambda_1}\Phi(d_2(\tfrac{V_B}{V},T))\end{aligned}$$

$$
\begin{aligned}
&+ ke^{-rT}(\tfrac{V_B}{V})^{2\lambda_1-2}\varphi(d_2(\tfrac{V_B}{V},T))\tfrac{-1}{\sigma\sqrt{T}V} \\
&= e^{-\nu T}\Phi(d_1(\tfrac{V}{V_B},T)) + \frac{e^{-\nu T}\varphi(d_1(\frac{V}{V_B},T))}{\sigma\sqrt{T}} - \frac{ke^{-rT}\varphi(d_2(\frac{V}{V_B},T))}{\sigma\sqrt{T}V} \\
&\quad - (1-2\lambda_1)e^{-\nu T}(\tfrac{V_B}{V})^{2\lambda_1}\Phi(d_1(\tfrac{V_B}{V},T)) + \frac{e^{-\nu T}(\frac{V_B}{V})^{2\lambda_1}\varphi(d_1(\frac{V_B}{V},T))}{\sigma\sqrt{T}} \\
&\quad + (2-2\lambda_1)\tfrac{ke^{-rT}}{V}(\tfrac{V_B}{V})^{2\lambda_1-2}\Phi(d_2(\tfrac{V_B}{V},T)) - \frac{ke^{-rT}(\frac{V_B}{V})^{2\lambda_1-2}\varphi(d_2(\frac{V_B}{V},T))}{\sigma\sqrt{T}V}.
\end{aligned} \tag{B.2}
$$

Hence, we obtain $\frac{\partial}{\partial V}c_{do}^{V_B\leq k}(V,k,V_B,T)\big|_{V_B=k} = \frac{\partial}{\partial V}c_{do}^{V_B\geq k}(V,k,V_B,T)\big|_{V_B=k}$ and the claim follows. □

Lemma B.3. *Let $a,b,y\in\mathbb{R}$, and $x>0$. Then, the function $h(x) := \frac{\varphi(a\ln(x)+b)}{x^y}$ is continuous on $(0,\infty)$, and there exists a constant $C>0$ such that $|h(x)|\leq C$ for all $x\in(0,\infty)$. Moreover, we have the following limits: $\lim_{x\to 0}|h(x)|=0$ and $\limsup_{x\to\infty}|h(x)|\leq C$.*

Proof. The continuity of h follows directly from the continuity of φ. Next, we show the existence of a constant $C>0$ such that $|h(x)|\leq C$. First, we note that $h(x)>0$ for all $x\in(0,\infty)$. If we can show that h has a unique extreme point $x^*\in(0,\infty)$ which is a local maximum, then x^* will also be the global maximum of h in $(0,\infty)$, and the main claim follows with $C := h(x^*)$.

To prove the existence of a unique extreme point, we compute the first two derivatives of h:

$$
\begin{aligned}
h'(x) &= \frac{-(a\ln(x)+b)\varphi(a\ln(x)+b)\frac{a}{x}x^y - \varphi(a\ln(x)+b)yx^{y-1}}{x^{2y}} \\
&= -(a^2\ln(x)+ab+y)\frac{\varphi(a\ln(x)+b)}{x^{y+1}}, \\
h''(x) &= -\frac{a^2}{x}\cdot\frac{\varphi(a\ln(x)+b)}{x^{y+1}} \\
&\quad -(a^2\ln(x)+ab+y)\frac{-(a\ln(x)+b)\varphi(a\ln(x)+b)\frac{a}{x}x^{y+1} - \varphi(a\ln(x)+b)(y+1)x^y}{x^{2y+2}} \\
&= \frac{\varphi(a\ln(x)+b)}{x^{y+2}}\left(-a^2+(a^2\ln(x)+ab+y)(a^2\ln(x)+ab+y+1)\right),
\end{aligned}
$$

where we used that $\varphi'(x) = -x\varphi(x)$. Setting $h'(x)\overset{!}{=}0$ is equivalent to the equation $(a^2\ln(x)+ab+y)=0$ since $\varphi(\cdot)>0$ and $x>0$. Solving this equation for x yields the unique solution $x^* = e^{-\frac{y}{a^2}-\frac{b}{a}}>0$. Thus, we have a unique extreme point. Plugging x^* into h'', we obtain:

$$
\begin{aligned}
h''(x^*) &= \frac{\varphi(a\ln(x^*)+b)}{(x^*)^{y+2}}\left(-a^2+(a^2(-\tfrac{y}{a^2}-\tfrac{b}{a})+ab+y)(a^2(-\tfrac{y}{a^2}-\tfrac{b}{a})+ab+y+1)\right) \\
&= \frac{\varphi(a\ln(x^*)+b)}{(x^*)^{y+2}}\left(-a^2+(-y-ab+ab+y)(-y-ab+ab+y+1)\right) \\
&= -a^2\frac{\varphi(a\ln(x^*)+b)}{(x^*)^{y+2}} < 0,
\end{aligned}
$$

since $\varphi(\cdot)>0$ and $x^*>0$. Therefore, x^* is the unique extreme point and a maximum, implying that it is the global maximum of h. Consequently, the main claim follows.

It remains to show that $\lim_{x\to 0}|h(x)|=0$ (since $\limsup_{x\to\infty}|h(x)|\leq C$ follows directly from the first part). We already know from the first part that $\limsup_{x\to 0}|h(x)|\leq C$. Now, we show that $\lim_{x\to 0}|h(x)|=0$ by contradiction. Assume that there exists a sequence $x_n\xrightarrow{n\to\infty}0$ such that $\lim_{n\to\infty}h(x_n)=c\in[-C,C]\backslash\{0\}$. Applying l'Hôpital's rule and using the fact that $\varphi'(x)=-x\varphi(x)$ in the second equation, we get:

$$
c = \lim_{n\to\infty}\frac{\varphi(a\ln(x_n)+b)}{x_n^y}
$$

$$= \lim_{n\to\infty} \frac{-(a\ln(x_n)+b)\varphi(a\ln(x_n)+b)\frac{a}{x_n}}{yx_n^{y-1}}$$
$$= \lim_{n\to\infty} \tfrac{-a}{y}(a\ln(x_n)+b)h(x_n) = -\tfrac{a}{y}c\cdot\lim_{n\to\infty}(a\ln(x_n)+b) = \infty,$$

since $x_n \xrightarrow{n\to\infty} 0$. This leads to a contradiction, and the claim is proven. □

Lemma B.4. *Let $a, y, x > 0$, and $b \in \mathbb{R}$. Then, the function $h(x) := \Phi(-a\ln(x)+b)x^y$ is continuous on $(0,\infty)$ and there exists a constant $C > 0$ such that $|h(x)| \le C$ for all $x \in (0,\infty)$. Furthermore, we have the following properties: $\lim_{x\to 0}|h(x)| = 0$ and $\limsup_{x\to\infty}|h(x)| \le C$. Additionally, we consider the functions $\tilde{h}(x) := \frac{\Phi(-a\ln(x)+b)}{x^y}$ and $\hat{h}(x) := \frac{\Phi(a\ln(x)+b)}{x^y}$. For these functions, it holds that $\lim_{x\to\infty}|\tilde{h}(x)| = 0$ and $\lim_{x\to 0}|\hat{h}(x)| = 0$.*

Proof. First, note that h is obviously continuous in $(0,\infty)$, and therefore bounded (by a possibly larger $C > 0$) if there exists a $C > 0$ such that $\lim_{x\to 0}|h(x)| \le C$ and $\lim_{x\to\infty}|h(x)| \le C$. Moreover, we observe that $h(\cdot) > 0$, $\tilde{h}(\cdot) > 0$, and $\hat{h}(\cdot) > 0$ on $(0,\infty)$. Now, we obtain for a suitable $C > 0$:

$$\lim_{x\to 0}|h(x)| = \lim_{x\to 0}\Phi(-a\ln(x)+b)x^y = 0,$$
$$\limsup_{x\to\infty}|h(x)| = \limsup_{x\to\infty}\frac{\Phi(-a\ln(x)+b)}{x^{-y}}$$
$$\le \limsup_{x\to\infty}\frac{\varphi(-a\ln(x)+b)\frac{-a}{x}}{-yx^{-y-1}} = \tfrac{a}{y}\limsup_{x\to\infty}\frac{\varphi(-a\ln(x)+b)}{x^{-y}} \le C,$$

since $a, y > 0$ and $\lim_{z\to-\infty}\Phi(z) = 0$. Note that we used the generalized rule of de l'Hôpital (see, e.g., Picone [54]) in the second step of the second limit and Lemma B.3 in the last step. Thus, the first claim follows. For the second claim, we have:

$$\lim_{x\to\infty}|\tilde{h}(x)| = \lim_{x\to\infty}\frac{\Phi(-a\ln(x)+b)}{x^y} = 0,$$
$$\lim_{x\to 0}|\hat{h}(x)| = \lim_{x\to 0}\frac{\Phi(a\ln(x)+b)}{x^y} = \lim_{x\to 0}\frac{\varphi(a\ln(x)+b)\frac{a}{x}}{yx^{y-1}} = \tfrac{a}{y}\lim_{x\to 0}\frac{\varphi(a\ln(x)+b)}{x^y} = 0,$$

since $a, y > 0$ and $|\Phi(\cdot)| \le 1$. Note that we used again the rule of de l'Hôpital in the second step of the second limit, and Lemma B.3 in the last step. Thus, the second claim follows. □

Lemma B.5. *For all $k \ge 0$, $V_B \ge 0$, and $T > 0$, there exists a constant $C > 0$ such that $|\frac{\partial}{\partial V}c_{do}(V,k,V_B,T)| \le C(e^{-\nu T}+e^{-rT})$ for all $V \ge V_B$.*

Proof. For this proof, we need to consider two cases: when $V_B = 0$ and when $V_B > 0$. Let us start with $V_B = 0$. Then, the Down-and-Out Call option becomes a classical Call option as $V_t > 0$ for all $t \ge 0$ by the non-negativity of Geometric Brownian Motions, i.e., $c_{do}^{V_B\le k}(V,k,V_B,T) = c(V,k,T)$. Then, the result follows analogously to the proof of Lemma B.2:

$$\tfrac{\partial}{\partial V}c(V,k,T) = e^{-\nu T}\Phi(d_1(\tfrac{V}{k},T)) + \frac{e^{-\nu T}\varphi(d_1(\frac{V}{k},T))}{\sigma\sqrt{T}} - \frac{ke^{-rT}\varphi(d_2(\frac{V}{k},T))}{\sigma\sqrt{T}V}.$$

Thus, Lemma B.3 and $|\Phi(\cdot)| \le 1$ immediately provide the desired result.

Now, consider the case when $V_B > 0$. From equations (B.1) and (B.2), we can conclude that the claim will hold if we show that each individual term in equations (B.1) and (B.2) is bounded by $C(e^{-\nu T}+e^{-rT})$ for some constant $C > 0$, both as $V \to V_B$ and as $V \to +\infty$. Once we establish this, it follows that $|\frac{\partial}{\partial V}c_{do}(V,k,V_B,T)| \le C(e^{-\nu T}+e^{-rT})$, where $C > 0$ may be larger, but still finite. For the case $V \to V_B > 0$, the result holds immediately. For $V \to +\infty$, each individual term is bounded by $C(e^{-\nu T}+e^{-rT})$, with $C > 0$, due to the fact that $|\Phi(\cdot)| \le 1$, Lemma B.3, or Lemma B.4 since $\ln(\frac{V_B^2}{Vk}) = 2\ln(V_B) - \ln(V) - \ln(k)$ (resp. $\ln(\frac{V_B}{V}) = \ln(V_B) - \ln(V)$). □

Lemma B.6. *For all $V_B \geq 0$, and $T > 0$, it holds:*

(a) $\lim_{V_B\to\infty}(\frac{\partial}{\partial V}c_{do}(V,k,V_B,T)\big|_{V=V_B}) = e^{-\nu T}(2\lambda_1\Phi(d_1(1,T)) + \frac{2}{\sigma\sqrt{T}}\varphi(d_1(1,T)))$ *for all* $k \geq 0$,

(b) $\lim_{V_B\to 0}(\frac{\partial}{\partial V}c_{do}(V,k,V_B,T)\big|_{V=V_B}) = 0$ *for all* $k > 0$,

(c) $\frac{\partial}{\partial V}c_{do}(V,0,V_B,T)\big|_{V=V_B} = e^{-\nu T}(2\lambda_1\Phi(d_1(1,T)) + \frac{2}{\sigma\sqrt{T}}\varphi(d_1(1,T)))$.

Proof. First, we obtain from (B.1) and (B.2):

$$\begin{aligned}
\tfrac{\partial}{\partial V}c_{do}^{V_B\leq k}(V,k,V_B,T)\big|_{V=V_B} =& e^{-\nu T}\Phi(d_1(\tfrac{V_B}{k},T)) + \tfrac{e^{-\nu T}\varphi(d_1(\frac{V_B}{k},T))}{\sigma\sqrt{T}} - \tfrac{ke^{-rT}\varphi(d_2(\frac{V_B}{k},T))}{\sigma\sqrt{T}V_B}\\
&- (1-2\lambda_1)e^{-\nu T}\Phi(d_1(\tfrac{V_B}{k},T)) + \tfrac{e^{-\nu T}\varphi(d_1(\frac{V_B}{k},T))}{\sigma\sqrt{T}}\\
&+ (2-2\lambda_1)\tfrac{ke^{-rT}}{V_B}\Phi(d_2(\tfrac{V_B}{k},T)) - \tfrac{ke^{-rT}\varphi(d_2(\frac{V_B}{k},T))}{\sigma\sqrt{T}V_B},\\
=& 2\lambda_1 e^{-\nu T}\Phi(d_1(\tfrac{V_B}{k},T)) + \tfrac{2e^{-\nu T}\varphi(d_1(\frac{V_B}{k},T))}{\sigma\sqrt{T}} - \tfrac{2ke^{-rT}\varphi(d_2(\frac{V_B}{k},T))}{\sigma\sqrt{T}V_B}\\
&+ (2-2\lambda_1)\tfrac{ke^{-rT}}{V_B}\Phi(d_2(\tfrac{V_B}{k},T)), \qquad \text{(B.3)}\\
\tfrac{\partial}{\partial V}c_{do}^{V_B\geq k}(V,k,V_B,T)\big|_{V=V_B} =& e^{-\nu T}\Phi(d_1(1,T)) + \tfrac{e^{-\nu T}\varphi(d_1(1,T))}{\sigma\sqrt{T}} - \tfrac{ke^{-rT}\varphi(d_2(1,T))}{\sigma\sqrt{T}V_B}\\
&- (1-2\lambda_1)e^{-\nu T}\Phi(d_1(1,T)) + \tfrac{e^{-\nu T}\varphi(d_1(1,T))}{\sigma\sqrt{T}}\\
&+ (2-2\lambda_1)\tfrac{ke^{-rT}}{V_B}\Phi(d_2(1,T)) - \tfrac{ke^{-rT}\varphi(d_2(1,T))}{\sigma\sqrt{T}V_B}\\
=& 2\lambda_1 e^{-\nu T}\Phi(d_1(1,T)) + \tfrac{2e^{-\nu T}\varphi(d_1(1,T))}{\sigma\sqrt{T}} - \tfrac{2ke^{-rT}\varphi(d_2(1,T))}{\sigma\sqrt{T}V_B}\\
&+ (2-2\lambda_1)\tfrac{ke^{-rT}}{V_B}\Phi(d_2(1,T)). \qquad \text{(B.4)}
\end{aligned}$$

Now, we conclude for the proof of parts (a) and (b):

$$\begin{aligned}
\lim_{V_B\to\infty}\tfrac{\partial}{\partial V}c_{do}(V,k,V_B,T)\big|_{V=V_B} =& \lim_{V_B\to\infty}\tfrac{\partial}{\partial V}c_{do}^{V_B\geq k}(V,k,V_B,T)\big|_{V=V_B}\\
=& 2\lambda_1 e^{-\nu T}\Phi(d_1(1,T)) + \tfrac{2e^{-\nu T}\varphi(d_1(1,T))}{\sigma\sqrt{T}} - 0 + 0,\\
=& e^{-\nu T}\left(2\lambda_1\Phi(d_1(1,T)) + \tfrac{2}{\sigma\sqrt{T}}\varphi(d_1(1,T))\right),\\
\lim_{V_B\to 0}\tfrac{\partial}{\partial V}c_{do}(V,k,V_B,T)\big|_{V=V_B} =& \lim_{V_B\to 0}\tfrac{\partial}{\partial V}c_{do}^{V_B\leq k}(V,k,V_B,T)\big|_{V=V_B}\\
=& 2\lambda_1 e^{-\nu T}\Phi(d_1(0,T)) + \tfrac{2e^{-\nu T}\varphi(d_1(0,T))}{\sigma\sqrt{T}} - \tfrac{2ke^{-rT}}{\sigma\sqrt{T}}\cdot\lim_{V_B\to 0}\tfrac{\varphi(d_2(\frac{V_B}{k},T))}{V_B}\\
&+ (2-2\lambda_1)ke^{-rT}\cdot\lim_{V_B\to 0}\tfrac{\Phi(d_2(\frac{V_B}{k},T))}{V_B}\\
=& 0,
\end{aligned}$$

due to $d_1(0,T) = -\infty$, $\lim_{z\to-\infty}\varphi(z) = 0$, $\lim_{z\to-\infty}\Phi(z) = 0$, Lemma B.3, and Lemma B.4. Note that the first step (i.e., using the formula of the Down-and-Out Call option for $V_B \geq k$ (resp. $V_B \leq k$) when taking the limit $V_B \to \infty$ (resp. $V_B \to 0$)) is valid because $k > 0$.

For part (c), when $k = 0$, it holds since $V_B \geq 0$:

$$\begin{aligned}
\tfrac{\partial}{\partial V}c_{do}(V,0,V_B,T)\big|_{V=V_B} =& \tfrac{\partial}{\partial V}c_{do}^{V_B\geq k}(V,0,V_B,T)\big|_{V=V_B}\\
=& e^{-\nu T}\Phi(d_1(1,T)) + \tfrac{e^{-\nu T}\varphi(d_1(1,T))}{\sigma\sqrt{T}} - 0\\
&- (1-2\lambda_1)e^{-\nu T}\Phi(d_1(1,T)) + \tfrac{e^{-\nu T}\varphi(d_1(1,T))}{\sigma\sqrt{T}} + 0 - 0\\
=& e^{-\nu T}(2\lambda_1\Phi(d_1(1,T)) + \tfrac{2}{\sigma\sqrt{T}}\varphi(d_1(1,T))).
\end{aligned}$$

□

Lemma B.7. *For all $k \geq 0$, and $T > 0$, there exists a constant $C > 0$ such that $|\frac{\partial}{\partial V} c_{do}(V, k, V_B, T)|\big|_{V=V_B} \leq C(e^{-\nu T} + e^{-rT})$ for all $V = V_B \in (0, \infty)$.*

Proof. By equations (B.3) and (B.4), the claim holds if we can show that each individual term is bounded by $C(e^{-\nu T} + e^{-rT})$ for some constant $C > 0$ in the limits $V_B \to 0$ and $V_B \to +\infty$. Then, in total, we have $|\frac{\partial}{\partial V} c_{do}(V, k, V_B, T)|\big|_{V=V_B} \leq C(e^{-\nu T} + e^{-rT})$ with a possibly larger constant $C > 0$. We now distinguish the cases $k = 0$ and $k > 0$. If $k = 0$, we find that $|\frac{\partial}{\partial V} c_{do}(V, k, V_B, T)|\big|_{V=V_B}$ is constant, and hence bounded. Now, let $k > 0$. For the limit as $V_B \to \infty$, the claim follows directly from equation (B.4) (since $V_B \geq k$ for V_B sufficiently large). For the limit as $V_B \to 0$, we can assume that $|\frac{\partial}{\partial V} c_{do}(V, k, V_B, T)|\big|_{V=V_B} = |\frac{\partial}{\partial V} c_{do}^{V_B \leq k}(V, k, V_B, T)|\big|_{V=V_B}$, since $k > 0$ and $V_B < k$ for V_B sufficiently small. In this case, the claim follows from equation (B.3) using that $|\Phi(\cdot)| \leq 1$, $|\varphi(\cdot)| \leq 1$, and Lemmas B.3 and B.4. □

Lemma B.8. *It holds:*

$$\frac{\partial I_1^V(T)}{\partial V}\big|_{V=V_B} = -\frac{2}{rTV_B}\Big(\frac{\lambda_2-\lambda_3}{2} + \lambda_3\Phi(\lambda_3\sigma\sqrt{T}) - \lambda_2 e^{-rT}\Phi(\lambda_2\sigma\sqrt{T})\Big),$$
$$\frac{\partial I_2^V(T)}{\partial V}\big|_{V=V_B} = -\frac{2}{V_B}\Big(\frac{\lambda_2-\lambda_3}{2} - \frac{1}{2\lambda_3\sigma^2 T} + (\lambda_3 + \frac{1}{\lambda_3\sigma^2 T})\Phi(\lambda_3\sigma\sqrt{T}) + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}}\Big).$$

Proof. By the definition of I_1 (see (A.7)), we begin by differentiating F and G (which are defined in equations (A.5) and (A.6), respectively):

$$\begin{aligned}
\frac{\partial F^V(T)}{\partial V} &= \varphi(-d_3(\tfrac{V}{V_B}, T))\tfrac{-1}{\sigma\sqrt{T}V} - 2\lambda_2 V_B^{2\lambda_2} V^{-2\lambda_2-1}\Phi(-d_4(\tfrac{V}{V_B}, T)) + (\tfrac{V_B}{V})^{2\lambda_2}\varphi(-d_4(\tfrac{V}{V_B}, T))\tfrac{-1}{\sigma\sqrt{T}V} \\
&= -\tfrac{1}{V}\Big(\frac{\varphi(-d_3(\frac{V}{V_B}, T))}{\sigma\sqrt{T}} + 2\lambda_2(\tfrac{V_B}{V})^{2\lambda_2}\Phi(-d_4(\tfrac{V}{V_B}, T)) + (\tfrac{V_B}{V})^{2\lambda_2}\frac{\varphi(-d_4(\frac{V}{V_B}, T))}{\sigma\sqrt{T}}\Big), \\
\frac{\partial G^V(T)}{\partial V} &= (-\lambda_2 + \lambda_3)V_B^{\lambda_2-\lambda_3} V^{-\lambda_2+\lambda_3-1}\Phi(-d_5(\tfrac{V}{V_B}, T)) + (\tfrac{V_B}{V})^{\lambda_2-\lambda_3}\varphi(-d_5(\tfrac{V}{V_B}, T))\tfrac{-1}{\sigma\sqrt{T}V} \\
&\quad + (-\lambda_2 - \lambda_3)V_B^{\lambda_2+\lambda_3} V^{-\lambda_2-\lambda_3-1}\Phi(-d_6(\tfrac{V}{V_B}, T)) + (\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\varphi(-d_6(\tfrac{V}{V_B}, T))\tfrac{-1}{\sigma\sqrt{T}V} \\
&= -\tfrac{1}{V}\Big((\lambda_2 - \lambda_3)(\tfrac{V_B}{V})^{\lambda_2-\lambda_3}\Phi(-d_5(\tfrac{V}{V_B}, T)) + (\tfrac{V_B}{V})^{\lambda_2-\lambda_3}\frac{\varphi(-d_5(\frac{V}{V_B}, T))}{\sigma\sqrt{T}} \\
&\qquad + (\lambda_2 + \lambda_3)(\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\Phi(-d_6(\tfrac{V}{V_B}, T)) + (\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\frac{\varphi(-d_6(\frac{V}{V_B}, T))}{\sigma\sqrt{T}}\Big).
\end{aligned}$$

Hence, using equation (A.7) and the relationships $\varphi(-d_3(1,T)) = \varphi(-d_4(1,T)) = \varphi(\lambda_2\sigma\sqrt{T})$, $\Phi(-d_4(1,T)) = \Phi(\lambda_2\sigma\sqrt{T})$, $\varphi(-d_5(1,T)) = \varphi(-d_6(1,T)) = \varphi(\lambda_3\sigma\sqrt{T})$, and $1 - \Phi(-d_5(1,T)) = \Phi(-d_6(1,T)) = \Phi(\lambda_3\sigma\sqrt{T})$, we obtain:

$$\begin{aligned}
\frac{\partial I_1^V(T)}{\partial V}\big|_{V=V_B} &= \tfrac{1}{rT}\big(\frac{\partial G^V(T)}{\partial V}\big|_{V=V_B} - e^{-rT}\frac{\partial F^V(T)}{\partial V}\big|_{V=V_B}\big) \\
&= -\tfrac{1}{rTV_B}\Big((\lambda_2 - \lambda_3)(1 - \Phi(\lambda_3\sigma\sqrt{T})) + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}} + (\lambda_2 + \lambda_3)\Phi(\lambda_3\sigma\sqrt{T}) + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}} \\
&\qquad - \frac{e^{-rT}\varphi(\lambda_2\sigma\sqrt{T})}{\sigma\sqrt{T}} - 2\lambda_2 e^{-rT}\Phi(\lambda_2\sigma\sqrt{T}) - \frac{e^{-rT}\varphi(\lambda_2\sigma\sqrt{T})}{\sigma\sqrt{T}}\Big) \\
&= -\tfrac{2}{rTV_B}\Big(\tfrac{\lambda_2-\lambda_3}{2} + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}} + \lambda_3\Phi(\lambda_3\sigma\sqrt{T}) - \frac{e^{-rT}\varphi(\lambda_2\sigma\sqrt{T})}{\sigma\sqrt{T}} - \lambda_2 e^{-rT}\Phi(\lambda_2\sigma\sqrt{T})\Big),
\end{aligned}$$

which is the first claim since $\frac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}} = \frac{e^{-rT}\varphi(\lambda_2\sigma\sqrt{T})}{\sigma\sqrt{T}}$. Indeed, we have $\varphi(\lambda_3\sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda_3^2\sigma^2 T}$ and $e^{-rT}\varphi(\lambda_2\sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda_2^2\sigma^2 T - rT}$. Furthermore, by the definition of λ_3, we find that $\frac{1}{2}\lambda_3^2\sigma^2 T = \frac{1}{2}\frac{\lambda_2^2\sigma^4 + 2r\sigma^2}{\sigma^4}\sigma^2 T = \frac{1}{2}\lambda_2^2\sigma^2 T + rT$, which completes the proof of this claim.

For I_2 (as defined in (A.8)), we get:

$$\lambda_3\sigma\sqrt{T}\frac{\partial I_2^V(T)}{\partial V} = (-\lambda_2 + \lambda_3)V_B^{\lambda_2-\lambda_3} V^{-\lambda_2+\lambda_3-1}\Phi(-d_5(\tfrac{V}{V_B}, T))d_5(\tfrac{V}{V_B}, T)$$

$$+ (\tfrac{V_B}{V})^{\lambda_2-\lambda_3}\varphi(-d_5(\tfrac{V}{V_B},T))\tfrac{-d_5(\frac{V}{V_B},T)}{\sigma\sqrt{T}V} + (\tfrac{V_B}{V})^{\lambda_2-\lambda_3}\Phi(-d_5(\tfrac{V}{V_B},T))\tfrac{1}{\sigma\sqrt{T}V}$$
$$- (-\lambda_2-\lambda_3)V_B^{\lambda_2+\lambda_3}V^{-\lambda_2-\lambda_3-1}\Phi(-d_6(\tfrac{V}{V_B},T))d_6(\tfrac{V}{V_B},T)$$
$$- (\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\varphi(-d_6(\tfrac{V}{V_B},T))\tfrac{-d_6(\frac{V}{V_B},T)}{\sigma\sqrt{T}V} - (\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\Phi(-d_6(\tfrac{V}{V_B},T))\tfrac{1}{\sigma\sqrt{T}V}.$$

Thus, using the identities for φ and Φ from above, along with $d_{5/6}(1,T) = \pm\lambda_3\sigma\sqrt{T}$, we obtain:

$$\begin{aligned}\tfrac{\partial I_2^V(T)}{\partial V}\big|_{V=V_B} &= \tfrac{1}{\lambda_3\sigma\sqrt{T}V_B}\Big((-\lambda_2+\lambda_3)(1-\Phi(\lambda_3\sigma\sqrt{T}))\lambda_3\sigma\sqrt{T} - \varphi(\lambda_3\sigma\sqrt{T})\tfrac{\lambda_3\sigma\sqrt{T}}{\sigma\sqrt{T}} + \tfrac{1-\Phi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}}\\ &\qquad - (\lambda_2+\lambda_3)\Phi(\lambda_3\sigma\sqrt{T})\lambda_3\sigma\sqrt{T} - \varphi(\lambda_3\sigma\sqrt{T})\tfrac{\lambda_3\sigma\sqrt{T}}{\sigma\sqrt{T}} - \tfrac{\Phi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}}\Big)\\ &= \tfrac{1}{V_B}\Big(-\lambda_2+\lambda_3-2\lambda_3\Phi(\lambda_3\sigma\sqrt{T}) + \tfrac{1}{\lambda_3\sigma^2T} - \tfrac{2\Phi(\lambda_3\sigma\sqrt{T})}{\lambda_3\sigma^2T} - \tfrac{2\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}}\Big)\\ &= -\tfrac{2}{V_B}\Big(\tfrac{\lambda_2-\lambda_3}{2} - \tfrac{1}{2\lambda_3\sigma^2T} + (\lambda_3+\tfrac{1}{\lambda_3\sigma^2T})\Phi(\lambda_3\sigma\sqrt{T}) + \tfrac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}}\Big),\end{aligned}$$

which is the second claim. □

Lemma B.9. *It holds:*

(a) $\int_0^T \frac{e^{-\nu t}}{\sqrt{t}}\varphi(\lambda_1\sigma\sqrt{t})\mathrm{d}t = \sqrt{\frac{1}{\lambda_1^2\sigma^2+2\nu}}(2\Phi(\sqrt{\lambda_1^2\sigma^2+2\nu}\sqrt{T})-1)$,

(b) $\int_0^\infty \frac{e^{-\nu t}}{\sqrt{t}}\varphi(\lambda_1\sigma\sqrt{t})\mathrm{d}t = \sqrt{\frac{1}{\lambda_1^2\sigma^2+2\nu}}$,

(c) $\int_0^T e^{-\nu t}\Phi(\lambda_1\sigma\sqrt{t})\mathrm{d}t = \frac{1}{2\nu} - \frac{e^{-\nu T}\Phi(\lambda_1\sigma\sqrt{T})}{\nu} + \frac{\lambda_1\sigma}{2\nu}\sqrt{\frac{1}{\lambda_1^2\sigma^2+2\nu}}(2\Phi(\sqrt{\lambda_1^2\sigma^2+2\nu}\sqrt{T})-1)$,

(d) $\int_0^\infty e^{-\nu t}\Phi(\lambda_1\sigma\sqrt{t})\mathrm{d}t = \frac{1}{2\nu} + \frac{\lambda_1\sigma}{2\nu}\sqrt{\frac{1}{\lambda_1^2\sigma^2+2\nu}}$.

Proof. We begin by proving property (a). To do so, we define the function erf as $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\mathrm{d}u$, which is also known as the Gaussian error function. This function has the identity $\mathrm{erf}(x) = 2\Phi(\sqrt{2}x)-1$. Now, considering $\lambda_1 \geq 0$, we proceed with:

$$\begin{aligned}\int_0^T \tfrac{e^{-\nu t}}{\sqrt{t}}\varphi(\lambda_1\sigma\sqrt{t})\mathrm{d}t &= \tfrac{2}{\lambda_1\sigma}\int_0^{\lambda_1\sigma\sqrt{T}}\exp\{-\tfrac{\nu}{\lambda_1^2\sigma^2}s^2\}\varphi(s)\mathrm{d}s\\ &= \tfrac{2}{\lambda_1\sigma}\cdot\tfrac{1}{\sqrt{2\pi}}\int_0^{\lambda_1\sigma\sqrt{T}}\exp\{-s^2(\tfrac{1}{2}+\tfrac{\nu}{\lambda_1^2\sigma^2})\}\mathrm{d}s\\ &= \tfrac{2}{\lambda_1\sigma}\cdot\sqrt{\tfrac{2\lambda_1^2\sigma^2}{\lambda_1^2\sigma^2+2\nu}}\cdot\tfrac{1}{\sqrt{2\pi}}\int_0^{\sqrt{\frac{\lambda_1^2\sigma^2+2\nu}{2\lambda_1^2\sigma^2}}\lambda_1\sigma\sqrt{T}}\exp\{-u^2\}\mathrm{d}u\\ &= \sqrt{\tfrac{1}{\lambda_1^2\sigma^2+2\nu}}\cdot\mathrm{erf}(\sqrt{\tfrac{\lambda_1^2\sigma^2+2\nu}{2}}\sqrt{T}),\end{aligned}$$

where we made the substitutions $s = \lambda_1\sigma\sqrt{t}$ and $u = \sqrt{\frac{\lambda_1^2\sigma^2+2\nu}{2\lambda_1^2\sigma^2}}\cdot s$ in the first, resp. third step. In particular, this establishes claim (a). If $\lambda_1 < 0$, the proof follows similarly, as the two arising negative signs cancel out (from $\mathrm{erf}(-x) = -\mathrm{erf}(x)$ and the calculation of $\frac{\sqrt{\lambda_1^2}}{\lambda_1}$). Property (b) follows by taking the limit $T\to\infty$.

Next, we proceed with the proof of property (c). Using (a), we obtain:

$$\begin{aligned}\int_0^T e^{-\nu t}\Phi(\lambda_1\sigma\sqrt{t})\mathrm{d}t &= \big[\tfrac{-1}{\nu}e^{-\nu t}\Phi(\lambda_1\sigma\sqrt{t})\big]_0^T + \int_0^T\tfrac{1}{\nu}e^{-\nu t}\varphi(\lambda_1\sigma\sqrt{t})\tfrac{\lambda_1\sigma}{2\sqrt{t}}\mathrm{d}t\\ &= \tfrac{1}{2\nu} - \tfrac{e^{-\nu T}\Phi(\lambda_1\sigma\sqrt{T})}{\nu} + \tfrac{\lambda_1\sigma}{2\nu}\cdot\sqrt{\tfrac{1}{\lambda_1^2\sigma^2+2\nu}}(2\Phi(\sqrt{\lambda_1^2\sigma^2+2\nu}\sqrt{T})-1)\end{aligned}$$

where we derived the first equation by integration by parts. Property (d) follows directly by taking the limit as $T\to\infty$. □

Lemma B.10. *It holds that* $\frac{-2\frac{G}{r}A_1}{rT} + 2\frac{G}{r}A_2 \geq \tau_1 \frac{G}{r}(\lambda_2 + \lambda_3)$ *and* $\int_0^T \frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B} \mathrm{d}t \geq \tau_2 \int_0^\infty \frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B} \mathrm{d}t$.

Proof. First, we observe that, by definition, the liability value of the guaranteed payment and of the surplus participation (see term 1 and 4 in (3.1)), and the associated tax benefits (see (3.3) and (3.4)) are 0 when $V = V_B$. Therefore, the inequalities in equations (3.8) and (3.9) hold even when we differentiate with respect to V and evaluate at $V = V_B$ obtaining:

$$\frac{\partial}{\partial V}\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\int_0^t e^{-rs} g \mathbb{1}_{\{\min_{r\in[0,s]} V_r \geq V_B\}} \mathrm{d}s\right] \mathrm{d}t \Big|_{V=V_B} \geq \frac{\partial}{\partial V} TB_1 \Big|_{V=V_B}, \tag{B.5}$$

$$\frac{\partial}{\partial V}\int_0^T c_{do}(V,k,V_B,t)\mathrm{d}t \Big|_{V=V_B} \geq \tau_2 \frac{\partial}{\partial V}\int_0^\infty c_{do}(V,k,V_B,t)\mathrm{d}t \Big|_{V=V_B}. \tag{B.6}$$

From (3.2) (with $\alpha = p = 0$ and $\rho = 1$), we have:

$$\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\int_0^t e^{-rs} g \mathbb{1}_{\{\min_{r\in[0,s]} V_r \geq V_B\}} \mathrm{d}s\right] \mathrm{d}t = \frac{G}{r} - \frac{G}{r}\left(\frac{1-e^{-rT}}{rT} - I_1^V(T)\right) - \frac{G}{r} I_2^V(T).$$

Next, using the definitions of A_1 and A_2 in (3.10) and (3.11), along with Lemma B.8, we find that $\frac{\partial I_1^V(T)}{\partial V}\big|_{V=V_B} = -\frac{2A_1}{rTV_B}$ and $\frac{\partial I_2^V(T)}{\partial V}\big|_{V=V_B} = -\frac{2A_2}{V_B}$. Additionally, note that $\frac{\partial (\frac{V_B}{V})^{\lambda_2+\lambda_3}}{\partial V}\big|_{V=V_B} = (-\lambda_2-\lambda_3)V_B^{\lambda_2+\lambda_3} V^{-\lambda_2-\lambda_3-1}\big|_{V=V_B} = -\frac{(\lambda_2+\lambda_3)}{V_B}$. Using the definition of TB_1 from equation (3.6), we obtain the first result after canceling the factor of $\frac{1}{V_B} > 0$ from both sides in (B.5).

For the second claim, we are permitted in (B.6) to interchange the derivative and the integral sign by the Dominated Convergence Theorem, as established by Lemma B.7. This directly leads to the conclusion. □

Lemma B.11. *It holds:*

$$1 + \rho(\lambda_2+\lambda_3) + 2(1-\rho)\left[\frac{\lambda_2-\lambda_3}{2} - \frac{1}{2\lambda_3\sigma^2 T} + \left(\lambda_3 + \frac{1}{\lambda_3\sigma^2 T}\right)\Phi(\lambda_3\sigma\sqrt{T}) + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}}\right] > 0.$$

Proof. It holds:

$$\begin{aligned} B :=& 2\left[\frac{\lambda_2-\lambda_3}{2} - \frac{1}{2\lambda_3\sigma^2 T} + \left(\lambda_3 + \frac{1}{\lambda_3\sigma^2 T}\right)\Phi(\lambda_3\sigma\sqrt{T}) + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}}\right] \\ =& \lambda_2 + \lambda_3 - 2\lambda_3 + 2\lambda_3\Phi(\lambda_3\sigma\sqrt{T}) - \frac{1}{\lambda_3\sigma^2 T} + \frac{2}{\lambda_3\sigma^2 T}\Phi(\lambda_3\sigma\sqrt{T}) + 2\lambda_3\frac{\varphi(\lambda_3\sigma\sqrt{T})}{\lambda_3\sigma\sqrt{T}} \\ =& (\lambda_2+\lambda_3) + 2\lambda_3\left(\Phi(\lambda_3\sigma\sqrt{T}) + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\lambda_3\sigma\sqrt{T}} - 1\right) + \frac{1}{\lambda_3\sigma^2 T}(2\Phi(\lambda_3\sigma\sqrt{T}) - 1). \end{aligned}$$

By definition, $\lambda_3 \geq 0$. Moreover, since $\lambda_3 = \frac{\sqrt{(\lambda_2\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} = \sqrt{\lambda_2^2 + \frac{2r}{\sigma^2}}$, we see that $\lambda_3 > |\lambda_2|$ given that $r > 0$ and $\sigma > 0$. In particular, this implies that $\lambda_2 + \lambda_3 > 0$. Therefore, since $\lambda_3\sigma\sqrt{T} > 0$, we have $2\Phi(\lambda_3\sigma\sqrt{T}) - 1 > 0$. For the middle term, the situation is more intricate. Let $x > 0$ and define $h(x) := \Phi(x) + \frac{\varphi(x)}{x} - 1$. Now, $h'(x) = \varphi(x) + \frac{-\varphi(x)x^2 - \varphi(x)}{x^2} = -\frac{\varphi(x)}{x^2} < 0$, meaning that h is decreasing in x. Since $\lim_{x\to\infty} h(x) = 0$, we conclude that $h(x) > 0$ for all $x \in \mathbb{R}$. Therefore, we obtain the expression that $2\lambda_3\left(\Phi(\lambda_3\sigma\sqrt{T}) + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\lambda_3\sigma\sqrt{T}} - 1\right) > 0$, since $\lambda_3\sigma\sqrt{T} > 0$. Thus, $B > 0$.

Hence, it follows:

$$\begin{aligned} &1 + \rho(\lambda_2+\lambda_3) + 2(1-\rho)\left[\frac{\lambda_2-\lambda_3}{2} - \frac{1}{2\lambda_3\sigma^2 T} + \left(\lambda_3 + \frac{1}{\lambda_3\sigma^2 T}\right)\Phi(\lambda_3\sigma\sqrt{T}) + \frac{\varphi(\lambda_3\sigma\sqrt{T})}{\sigma\sqrt{T}}\right] \\ =& 1 + \rho(\lambda_2+\lambda_3) + (1-\rho)B > 0, \end{aligned}$$

since $\lambda_2 + \lambda_3 > 0$, $B > 0$ as discussed above, and $\rho \in [0,1]$. □

Lemma B.12. *It holds that $A_1, A_2 > 0$ and $A_3 > A_4 > 0$ with A_1, A_2, A_3, and A_4 defined as in equations (3.10), (3.11), (3.12) and (3.13).*

Proof. We begin by proving that $A_1 > 0$: First, we rewrite A_1 as $A_1 = \lambda_3(\Phi(\lambda_3\sigma\sqrt{T}) - \frac{1}{2}) - \lambda_2(e^{-rT}\Phi(\lambda_2\sigma\sqrt{T}) - \frac{1}{2})$. Now, if $\lambda_2 \geq 0$, we obtain the inequality $A_1 \geq \lambda_3(\Phi(\lambda_3\sigma\sqrt{T}) - \frac{1}{2}) - \lambda_2(\Phi(\lambda_2\sigma\sqrt{T}) - \frac{1}{2})$, since $e^{-rT} < 1$. Since $r, \sigma > 0$, it follows from the definition that $\lambda_3 > \lambda_2$. This implies that $\Phi(\lambda_3\sigma\sqrt{T}) - \frac{1}{2} > \Phi(\lambda_2\sigma\sqrt{T}) - \frac{1}{2}$, because $\Phi(\cdot)$ is an increasing function. Therefore, we conclude that $A_1 > 0$ when $\lambda_2 \geq 0$. Next, we consider the case where $\lambda_2 < 0$. In this case, $A_1 > 0$ is equivalent to $\lambda_3(\Phi(\lambda_3\sigma\sqrt{T}) - \frac{1}{2}) > |\lambda_2|(\frac{1}{2} - e^{-rT}\Phi(\lambda_2\sigma\sqrt{T}))$. If $\frac{1}{2} - e^{-rT}\Phi(\lambda_2\sigma\sqrt{T}) \leq 0$, this inequality clearly holds, so assume that $\frac{1}{2} - e^{-rT}\Phi(\lambda_2\sigma\sqrt{T}) > 0$. Since $r, \sigma > 0$, we have $\lambda_3 > |\lambda_2|$, which implies that $A_1 > 0$ if $\Phi(\lambda_3\sigma\sqrt{T}) + e^{-rT}\Phi(\lambda_2\sigma\sqrt{T})) - 1 \geq 0$. This inequality is equivalent to

$$e^{-rT}\Phi(-|\lambda_2|\sigma\sqrt{T})) \geq \Phi(-\lambda_3\sigma\sqrt{T}), \tag{B.7}$$

since $\Phi(-x) = 1 - \Phi(x)$ for all $x \in \mathbb{R}$. Using the substitution $s = \sqrt{t^2 + 2rT}$, we can derive the following identity for any $x > 0$, due to e^{-t^2} being symmetric:

$$e^{-rT}\Phi(-x) = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{-\frac{t^2}{2}}\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-\frac{1}{2}(t^2+2rT)}\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{\sqrt{x^2+2rT}}^{\infty} e^{-\frac{s^2}{2}}\frac{s}{\sqrt{s^2-2rT}}\mathrm{d}s.$$

Thus, we have $e^{-rT}\Phi(-|\lambda_2|\sigma\sqrt{T})) = \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\lambda_2^2\sigma^2T+2rT}}^{\infty} e^{-\frac{s^2}{2}}\frac{s}{\sqrt{s^2-2rT}}\mathrm{d}s = \frac{1}{\sqrt{2\pi}}\int_{\lambda_3\sigma\sqrt{T}}^{\infty} e^{-\frac{s^2}{2}}\frac{s}{\sqrt{s^2-2rT}}\mathrm{d}s$, since $\lambda_2^2\sigma^2T + 2rT = \lambda_3^2\sigma^2T$. Indeed, it holds by definition that $\lambda_3^2\sigma^2 = \frac{\lambda_2^2\sigma^4+2r\sigma^2}{\sigma^4}\sigma^2 = \lambda_2^2\sigma^2 + 2r$. Multiplying by T yields this intermediate statement. Now, a simple rewriting leads to $\Phi(-\lambda_3\sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}}\int_{\lambda_3\sigma\sqrt{T}}^{\infty} e^{-\frac{s^2}{2}}\mathrm{d}s$. Hence, (B.7) is equivalent to

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int_{\lambda_3\sigma\sqrt{T}}^{\infty} e^{-\frac{s^2}{2}}\frac{s}{\sqrt{s^2-2rT}}\mathrm{d}s \geq \frac{1}{\sqrt{2\pi}}\int_{\lambda_3\sigma\sqrt{T}}^{\infty} e^{-\frac{s^2}{2}}\mathrm{d}s, \\ \Leftrightarrow \quad &\int_{\lambda_3\sigma\sqrt{T}}^{\infty} e^{-\frac{s^2}{2}}\left(\frac{s}{\sqrt{s^2-2rT}} - 1\right)\mathrm{d}s \geq 0. \end{aligned}$$

Since $\frac{s}{\sqrt{s^2-2rT}} \geq 1$ (due to $s \geq \lambda_3\sigma\sqrt{T} > 0$ and $(\lambda_3\sigma\sqrt{T})^2 - 2rT = \lambda_2^2\sigma^2T > 0$), the last inequality is indeed correct. Therefore, $A_1 > 0$ follows.

The property that $A_2 > 0$ is directly implied by the proof of Lemma B.11, where $B = 2A_2$ with B defined as in the proof of Lemma B.11.

Next, we show that $A_3, A_4 > 0$: Using Lemma B.9 and the fact that $\left(\frac{\lambda_1^2\sigma}{\nu} + \frac{2}{\sigma}\right)\sqrt{\frac{1}{\lambda_1^2\sigma^2+2\nu}} = \frac{1}{\sigma\nu}\sqrt{\lambda_1^2\sigma^2 + 2\nu}$, we have:

$$A_3 = \int_0^{\infty}\left(e^{-\nu t}(2\lambda_1\Phi(\lambda_1\sigma\sqrt{t}) + \tfrac{2\varphi(\lambda_1\sigma\sqrt{t})}{\sigma\sqrt{t}})\right)\mathrm{d}t, \tag{B.8}$$

$$A_4 = \int_0^{T}\left(e^{-\nu t}(2\lambda_1\Phi(\lambda_1\sigma\sqrt{t}) + \tfrac{2\varphi(\lambda_1\sigma\sqrt{t})}{\sigma\sqrt{t}})\right)\mathrm{d}t. \tag{B.9}$$

We now show that $\lambda_1\Phi(\lambda_1\sigma\sqrt{t}) + \frac{\varphi(\lambda_1\sigma\sqrt{t})}{\sigma\sqrt{t}} > 0$ for all $\lambda_1 \in \mathbb{R}$, $\sigma > 0$, and $t \geq 0$, which directly implies the claim. First, if $\lambda_1 \geq 0$, the result is trivial. If $\lambda_1 < 0$, we obtain: $\lambda_1\Phi(\lambda_1\sigma\sqrt{t}) + \frac{\varphi(\lambda_1\sigma\sqrt{t})}{\sigma\sqrt{t}} = \lambda_1[\Phi(\lambda_1\sigma\sqrt{t}) + \frac{\varphi(\lambda_1\sigma\sqrt{t})}{\lambda_1\sigma\sqrt{t}}]$. Let $y := \lambda_1\sigma\sqrt{t} < 0$, and define $h(y) := \Phi(y) + \frac{\varphi(y)}{y}$. Since $\lambda_1 < 0$, we only need to show that $h(y) < 0$ for all $y \in (-\infty, 0]$. First, we obtain that $\lim_{y\to-\infty} h(y) = 0$ and $\lim_{y\to 0_-} h(y) = -\infty$. Additionally, $h'(y) = \varphi(y) + \frac{-y\varphi(y)y-\varphi(y)}{y^2} = -\frac{\varphi(y)}{y^2} < 0$ which yields the claim.

Finally, we conclude that $A_3 > A_4$, which follows immediately from the fact that $\lambda_1\Phi(\lambda_1\sigma\sqrt{t}) + \frac{\varphi(\lambda_1\sigma\sqrt{t})}{\sigma\sqrt{t}} > 0$ and $T < \infty$. □

Lemma B.13. *The inequalities* $1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2 > 0$ *and* $\frac{2(P-\frac{G}{r})A_1}{rT}+2\frac{G}{r}A_2-\tau_1\frac{G}{r}(\lambda_2+\lambda_3) > 0$ *hold, where* λ_2, λ_3 *are defined in equation* (A.9).

Proof. The first part of the claim has already been established in Lemma B.11 by inserting the definition of A_2 from equation (3.11). The second part of the lemma follows directly from Lemma B.10, with the additional observation that $A_1 > 0$ as established in Lemma B.12, and that $P > 0$ by definition. □

Lemma B.14. *If a solution* V_B *to the equation* (3.15) *exists, then it must satisfy* $V_B \neq 0$.

Proof. To prove this lemma, we demonstrate that the absolute value of the right-hand side of equation (3.15) diverges to ∞ as $V_B \to 0$. According to Lemma B.13, this holds if the integrals in equation (3.15) remain bounded as V_B approaches 0, which is ensured by Lemma B.7 since $\nu > 0$. Therefore, the lemma is proved. □

Lemma B.15. *It holds that* $h(V_B) := \frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}$ *is continuously differentiable. Furthermore, for all* $k \geq 0$ *and* $T > 0$, *there exists a constant* $C > 0$ *such that* $0 \leq \frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,T)}{\partial V}\big|_{V=V_B}] \leq C(e^{-rT}+e^{-\nu T})$ *for all* $V_B > 0$.

Proof. To establish the continuous differentiability of h, it suffices to verify that the derivative is continuous at $V = k$, as these properties follow directly for all other points. From equations (B.3) and (B.4) resp. (3.16), we observe using $2-2\lambda_1 = -2\lambda_2$:

$$\begin{aligned}
\tfrac{\partial}{\partial V_B}[\tfrac{\partial}{\partial V}c_{do}^{V_B\leq k}(V,k,V_B,t)\big|_{V=V_B}] &= 2e^{-\nu t}\Big(\tfrac{\lambda_1\varphi(d_1(\frac{V_B}{k},t))}{\sigma\sqrt{t}V_B}+\tfrac{-\varphi(d_1(\frac{V_B}{k},t))d_1(\frac{V_B}{k},t)}{\sigma^2 tV_B}\Big)\\
&\quad+\tfrac{2ke^{-rt}}{V_B^2}\Big(\lambda_2\Phi(d_2(\tfrac{V_B}{k},t))+\tfrac{\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}}\Big)\\
&\quad-\tfrac{2ke^{-rt}}{V_B}\Big(\tfrac{\lambda_2\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}V_B}+\tfrac{-\varphi(d_2(\frac{V_B}{k},t))d_2(\frac{V_B}{k},t)}{\sigma^2 tV_B}\Big)\\
&= \tfrac{2e^{-\nu t}}{\sigma\sqrt{t}V_B}\Big(\lambda_1\varphi(d_1(\tfrac{V_B}{k},t))-\tfrac{\varphi(d_1(\frac{V_B}{k},t))(\ln(\frac{V_B}{k})+\lambda_1\sigma^2 t)}{\sigma^2 t}\Big)\\
&\quad+\tfrac{2ke^{-rt}}{V_B^2}\Big(\lambda_2\Phi(d_2(\tfrac{V_B}{k},t))+\tfrac{\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}}\Big)\\
&\quad-\tfrac{2ke^{-rt}}{\sigma\sqrt{t}V_B^2}\Big(\lambda_2\varphi(d_2(\tfrac{V_B}{k},t))-\tfrac{\varphi(d_2(\frac{V_B}{k},t))(\ln(\frac{V_B}{k})+\lambda_2\sigma^2 t)}{\sigma^2 t}\Big)\\
&= -\tfrac{2e^{-\nu t}\ln(\frac{V_B}{k})}{\sigma^3 t\sqrt{t}V_B}\varphi(d_1(\tfrac{V_B}{k},t))\\
&\quad+\tfrac{2ke^{-rt}}{V_B^2}\Big(\lambda_2\Phi(d_2(\tfrac{V_B}{k},t))+\tfrac{\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}}(1+\tfrac{\ln(\frac{V_B}{k})}{\sigma^2 t})\Big)\\
&= \tfrac{2ke^{-rt}}{V_B^2}\Big(\lambda_2\Phi(d_2(\tfrac{V_B}{k},t))+\tfrac{\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}}\Big), \qquad \text{(B.10)}\\
\tfrac{\partial}{\partial V_B}[\tfrac{\partial}{\partial V}c_{do}^{V_B\geq k}(V,k,V_B,t)\big|_{V=V_B}] &= \tfrac{2ke^{-rt}}{V_B^2}\Big(\lambda_2\Phi(d_2(1,t))+\tfrac{\varphi(d_2(1,t))}{\sigma\sqrt{t}}\Big). \qquad \text{(B.11)}
\end{aligned}$$

The last step in (B.10) holds since:

$$\begin{aligned}
\tfrac{2e^{-\nu t}\ln(\frac{V_B}{k})}{\sigma^3 t\sqrt{t}V_B}\varphi(d_1(\tfrac{V_B}{k},t)) &= \tfrac{2e^{-\nu t}\ln(\frac{V_B}{k})}{\sigma^3 t\sqrt{t}V_B}\cdot\tfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2 t}[\ln(\frac{V_B}{k})^2+2\ln(\frac{V_B}{k})(r-\nu+\frac{\sigma^2}{2})t+((r-\nu)^2+(r-\nu)\sigma^2+\frac{\sigma^4}{4})t^2]}\\
&= \tfrac{2ke^{-rt}\ln(\frac{V_B}{k})}{\sigma^3 t\sqrt{t}V_B^2}\cdot e^{\ln(\frac{V_B}{k})}\cdot e^{(r-\nu)t}\\
&\quad\cdot\tfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2 t}[\ln(\frac{V_B}{k})^2+2\ln(\frac{V_B}{k})(r-\nu-\frac{\sigma^2}{2})t+((r-\nu)^2-(r-\nu)\sigma^2+\frac{\sigma^4}{4})t^2]}\\
&\quad\cdot e^{-\frac{1}{2\sigma^2 t}[2\ln(\frac{V_B}{k})\sigma^2 t+2(r-\nu)\sigma^2 t^2]}\\
&= \tfrac{2ke^{-rt}\ln(\frac{V_B}{k})}{\sigma^3 t\sqrt{t}V_B^2}\cdot e^{\ln(\frac{V_B}{k})+(r-\nu)t}\cdot e^{-\ln(\frac{V_B}{k})-(r-\nu)t}
\end{aligned}$$

$$\cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\ln(\frac{V_B}{k})+(r-\nu-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}})^2}$$
$$= \frac{2ke^{-rt}\ln(\frac{V_B}{k})}{\sigma^3 t\sqrt{t}V_B^2}\varphi(d_2(\frac{V_B}{k},t)).$$

Now, evaluating (B.10) and (B.11) at $V_B = k \neq 0$ yields:

$$\frac{\partial}{\partial V_B}[\frac{\partial}{\partial V}c_{do}^{V_B\leq k}(V,k,V_B,t)\big|_{V=V_B}]\big|_{V_B=k} = \frac{2e^{-rt}}{k}\Big(\lambda_2\Phi(d_2(1,t)) + \frac{\varphi(d_2(1,t))}{\sigma\sqrt{t}}\Big),$$
$$\frac{\partial}{\partial V_B}[\frac{\partial}{\partial V}c_{do}^{V_B\geq k}(V,k,V_B,t)\big|_{V=V_B}]\big|_{V_B=k} = \frac{2e^{-rt}}{k}\Big(\lambda_2\Phi(d_2(1,t)) + \frac{\varphi(d_2(1,t))}{\sigma\sqrt{t}}\Big).$$

If $k = 0$, Lemma B.6(c) implies that $\frac{\partial}{\partial V_B}[\frac{\partial}{\partial V}c_{do}(V,k,V_B,t)\big|_{V=V_B}] = 0$. This shows that h is continuously differentiable.

For the second claim, we need to show that the limits of $\frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,T)}{\partial V}\big|_{V=V_B}]$ as $V_B \to 0$ and $V_B \to \infty$ are both bounded by $C(e^{-rT} + e^{-\nu T})$ for some constant $C > 0$. The continuity then gives us the result. If $k = 0$, we already know that $\frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,T)}{\partial V}\big|_{V=V_B}] = 0$, which trivially implies the claim. Thus, we assume $k > 0$. For the limit when $V_B \to 0$, we only need to consider the case $V_B \leq k$. Each term can be individually analyzed using Lemma B.3, and Lemma B.4 (with $\ln(\frac{V_B}{k}) = \ln(V_B) - \ln(k)$) to ensure the boundedness of $\frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,T)}{\partial V}\big|_{V=V_B}]$ as $V_B \to 0$. For the limit when $V_B \to \infty$, we only need to consider the case $V_B \geq k$. In this case, we have $\lim_{V_B\to\infty}\frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,T)}{\partial V}\big|_{V=V_B}] = 0$. Combining these results with the continuity of $\frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,T)}{\partial V}\big|_{V=V_B}]$ in V_B, we obtain the claim. □

Lemma B.16. *There exists an $\hat{\alpha} > 0$ and a constant $C_{\hat{\alpha}} > 0$ such that $|V_B'(\alpha)| \leq C_{\hat{\alpha}}$ and $|\frac{\partial c_{do}(V,k,V_B(\alpha),T)}{\partial \alpha}| \leq C_{\hat{\alpha}}(e^{-\nu T} + e^{-rT})$ for all $\alpha \in [0,\hat{\alpha}]$.*

Proof. We begin by taking the derivative of $V_B(\alpha)$. For this, we use the expression from equation (3.15), where we define the right-hand side as $R(\alpha, V_B)$. According to the implicit function theorem (if applicable), we have $V_B'(\alpha) = -\frac{R_\alpha(\alpha,V_B(\alpha))}{R_{V_B}(\alpha,V_B(\alpha))}$, where R_x denotes the partial derivative with respect to x. Next, we evaluate the partial derivatives involved. With $\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}$ as in (3.16) and $\frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]$ as in (B.10) resp. (B.11), we obtain (where we suppress the dependency of V_B on α):

$$R_\alpha(\alpha,V_B) = \tau_2\int_0^\infty \frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t - \int_0^T \frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t, \tag{B.12}$$
$$R_{V_B}(\alpha,V_B) = \frac{1}{V_B^2}\Big(\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)\Big)$$
$$+ \tau_2\alpha\int_0^\infty \frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]\mathrm{d}t - \alpha\int_0^T \frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]\mathrm{d}t, \tag{B.13}$$

where we are allowed to interchange the integral and the derivative due to Lemma B.15. Next, we note that $V_B(0) > 0$ by (3.17) (since $\alpha = 0$ corresponds to no participation). Hence, using Lemma B.13, we have that $R_{V_B}(0, V_B(0)) > 0$. (This property also ensures that we can apply the implicit function theorem.) Consequently, the implicit function theorem guarantees the existence of an $\hat{\alpha} > 0$ such that $V_B(\alpha)$ is continuously differentiable in α for $\alpha \in [0,\hat{\alpha}]$. Thus, we have the continuity of R_{V_B} in α. As a result, possibly after reducing $\hat{\alpha} > 0$, it follows that R_{V_B} is bounded from below by an $\varepsilon > 0$ for all $\alpha \in [0,\hat{\alpha}]$. In particular, there exists a $C_{\hat{\alpha}}$ such that $|V_B'(\alpha)| \leq C_{\hat{\alpha}}$ for all $\alpha \in [0,\hat{\alpha}]$. Moreover, we conclude from equations (A.1) and (A.2) analogously to equations (B.1) and (B.2) (with the dependence of V_B on α suppressed) that:

$$\frac{\partial}{\partial\alpha}c_{do}^{V_B\leq k}(V,k,V_B,T) = -e^{-\nu T}2\lambda_1(\frac{V_B}{V})^{2\lambda_1-1}V_B'(\alpha)\Phi(d_1(\frac{V_B^2}{Vk},T))$$

$$
\begin{aligned}
&- Ve^{-\nu T}(\tfrac{V_B}{V})^{2\lambda_1}\varphi(d_1(\tfrac{V_B^2}{Vk},T))\tfrac{2}{\sigma\sqrt{T}V_B}V_B'(\alpha)\\
&+ \tfrac{k}{V}e^{-rT}(2\lambda_1-2)(\tfrac{V_B}{V})^{2\lambda_1-3}V_B'(\alpha)\Phi(d_2(\tfrac{V_B^2}{Vk},T))\\
&+ ke^{-rT}(\tfrac{V_B}{V})^{2\lambda_1-2}\varphi(d_2(\tfrac{V_B^2}{Vk},T))\tfrac{2}{\sigma\sqrt{T}V_B}V_B'(\alpha), \qquad \text{(B.14)}\\
\tfrac{\partial}{\partial\alpha}c_{do}^{V_B\geq k}(V,k,V_B,T) =\ & Ve^{-\nu T}\varphi(d_1(\tfrac{V}{V_B},T))\tfrac{-1}{\sigma\sqrt{T}V_B}V_B'(\alpha) - ke^{-rT}\varphi(d_2(\tfrac{V}{V_B},T))\tfrac{-1}{\sigma\sqrt{T}V_B}V_B'(\alpha)\\
&- e^{-\nu T}2\lambda_1(\tfrac{V_B}{V})^{2\lambda_1-1}V_B'(\alpha)\Phi(d_1(\tfrac{V_B}{V},T))\\
&- Ve^{-\nu T}(\tfrac{V_B}{V})^{2\lambda_1}\varphi(d_1(\tfrac{V_B}{V},T))\tfrac{1}{\sigma\sqrt{T}V_B}V_B'(\alpha)\\
&+ \tfrac{k}{V}e^{-rT}(2\lambda_1-2)(\tfrac{V_B}{V})^{2\lambda_1-3}V_B'(\alpha)\Phi(d_2(\tfrac{V_B}{V},T))\\
&+ ke^{-rT}(\tfrac{V_B}{V})^{2\lambda_1-2}\varphi(d_2(\tfrac{V_B}{V},T))\tfrac{1}{\sigma\sqrt{T}V_B}V_B'(\alpha). \qquad \text{(B.15)}
\end{aligned}
$$

Next, we consider each term in the expression individually. All of these terms are bounded by $C_{\hat{\alpha}}(e^{-\nu T}+e^{-rT})$, since $V_B(\alpha)$ is bounded (by V by definition) and bounded away from zero for all $\alpha\in[0,\hat{\alpha}]$ (using Lemma B.14 and Proposition 3.3), with an appropriately chosen constant $C_{\hat{\alpha}}$. Therefore, the claim follows. □

Remark B.17. *We observe from equations (B.14) and (B.15) that* $\frac{\partial c_{do}(V,k,V_B(\alpha),T)}{\partial\alpha}$ *is not continuous in* α *(or alternatively in* V_B*) at* $V_B(\alpha)=k$*, meaning there could be a discontinuity at any* α *where* $V_B(\alpha)=k$*.*

Lemma B.18. *Assume*

$$
\begin{aligned}
&\tfrac{2PA_1}{V_B(0)^2rT}+\tau_2\alpha\int_0^\infty \tfrac{\partial}{\partial V_B}[\tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]\big|_{V_B=V_B(0)}\mathrm{d}t\\
&\quad-\alpha\int_0^T \tfrac{\partial}{\partial V_B}[\tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]\big|_{V_B=V_B(0)}\mathrm{d}t\neq 0,
\end{aligned}
$$

where $\frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]$ *is given in (B.10) resp. (B.11). Then, there exists a* $\hat{g}>0$ *and a constant* $C_{\hat{g}}>0$ *such that* $|V_B'(g)|\leq C_{\hat{g}}$ *and* $|\frac{\partial c_{do}(V,k,V_B(g),T)}{\partial g}|\leq C_{\hat{g}}(e^{-\nu T}+e^{-rT})$ *for all* $g\in[0,\hat{g}]$*.*

Proof. This proof follows from an approach similar to the proof of Lemma B.16. We begin by differentiating $V_B(g)$. Again, we use (3.15) and define the right-hand side as $R(g,V_B)$. By applying the implicit function theorem (if applicable), we obtain $V_B'(g)=-\frac{R_g(g,V_B(g))}{R_{V_B}(g,V_B(g))}$, where R_x denotes the partial derivative with respect to x. With $\frac{\partial}{\partial V_B}[\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]$ as in (B.10) resp. (B.11), we obtain (where we suppress the dependency of V_B on g):

$$
\begin{aligned}
R_g(g,V_B) &= -\tfrac{T}{V_Br}\Big(-\tfrac{2A_1}{rT}+2A_2-\tau_1(\lambda_2+\lambda_3)\Big), \qquad \text{(B.16)}\\
R_{V_B}(g,V_B) &= \tfrac{1}{V_B^2}\Big(\tfrac{2(P-\frac{G}{r})A_1}{rT}+2\tfrac{G}{r}A_2-\tau_1\tfrac{G}{r}(\lambda_2+\lambda_3)\Big)\\
&\quad+\tau_2\alpha\int_0^\infty \tfrac{\partial}{\partial V_B}[\tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]\mathrm{d}t-\alpha\int_0^T \tfrac{\partial}{\partial V_B}[\tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}]\mathrm{d}t, \qquad \text{(B.17)}
\end{aligned}
$$

where A_1 and A_2 are defined as in (3.10) and (3.11). By Lemma B.14, we know that $V_B(0)\neq 0$. Therefore, by assumption, we have that $R_{V_B}(0,V_B(0))\neq 0$, which can be verified by substituting $g=0$ into $R_{V_B}(g,V_B(g))$ and comparing it with the assumption of the lemma. (This property also ensures us that we can apply the implicit function theorem.) By the implicit function theorem, we conclude that there exists a $\hat{g}>0$ such that $V_B(g)$ is continuously differentiable in g on $[0,\hat{g}]$. This guarantees the continuity of $R_{V_B}(g,V_B(g))$ in g. Furthermore, possibly after reducing $\hat{g}>0$, it follows that $|R_{V_B}(g,V_B(g))|$ is bounded from below for all $g\in[0,\hat{g}]$. In particular, there exists

a constant $C_{\hat{g}}$ such that $|V_B'(g)| \leq C_{\hat{g}}$ for all $g \in [0, \hat{g}]$. Additionally, similar to Lemma B.15 (where the dependence of V_B on g is omitted), we obtain:

$$\begin{aligned}
\tfrac{\partial}{\partial g} c_{do}^{V_B \leq k}(V,k,V_B,T) = &- e^{-\nu T} 2\lambda_1 (\tfrac{V_B}{V})^{2\lambda_1 - 1} V_B'(g) \Phi(d_1(\tfrac{V_B^2}{Vk}, T)) \\
&- V e^{-\nu T} (\tfrac{V_B}{V})^{2\lambda_1} \varphi(d_1(\tfrac{V_B^2}{Vk}, T)) \tfrac{2}{\sigma\sqrt{T} V_B} V_B'(g) \\
&+ \tfrac{k}{V} e^{-rT} (2\lambda_1 - 2)(\tfrac{V_B}{V})^{2\lambda_1 - 3} V_B'(g) \Phi(d_2(\tfrac{V_B^2}{Vk}, T)) \\
&+ k e^{-rT} (\tfrac{V_B}{V})^{2\lambda_1 - 2} \varphi(d_2(\tfrac{V_B^2}{Vk}, T)) \tfrac{2}{\sigma\sqrt{T} V_B} V_B'(g), \qquad \text{(B.18)} \\
\tfrac{\partial}{\partial g} c_{do}^{V_B \geq k}(V,k,V_B,T) = &\, V e^{-\nu T} \varphi(d_1(\tfrac{V}{V_B}, T)) \tfrac{-1}{\sigma\sqrt{T} V_B} V_B'(g) - k e^{-rT} \varphi(d_2(\tfrac{V}{V_B}, T)) \tfrac{-1}{\sigma\sqrt{T} V_B} V_B'(g) \\
&- e^{-\nu T} 2\lambda_1 (\tfrac{V_B}{V})^{2\lambda_1 - 1} V_B'(g) \Phi(d_1(\tfrac{V_B}{V}, T)) \\
&- V e^{-\nu T} (\tfrac{V_B}{V})^{2\lambda_1} \varphi(d_1(\tfrac{V_B}{V}, T)) \tfrac{1}{\sigma\sqrt{T} V_B} V_B'(g) \\
&+ \tfrac{k}{V} e^{-rT} (2\lambda_1 - 2)(\tfrac{V_B}{V})^{2\lambda_1 - 3} V_B'(g) \Phi(d_2(\tfrac{V_B}{V}, T)) \\
&+ k e^{-rT} (\tfrac{V_B}{V})^{2\lambda_1 - 2} \varphi(d_2(\tfrac{V_B}{V}, T)) \tfrac{1}{\sigma\sqrt{T} V_B} V_B'(g). \qquad \text{(B.19)}
\end{aligned}$$

We now consider each term in the expression individually. All of these terms are bounded by $C_{\hat{g}}(e^{-\nu T} + e^{-rT})$, since V_B is bounded (by V) and bounded away from zero for all $g \in [0, \hat{g}]$ (using Lemma B.14 and Proposition 3.3), with an appropriately chosen constant $C_{\hat{g}}$. Therefore, the claim follows. □

Remark B.19. *We observe from (B.18) and (B.19) that* $\frac{\partial c_{do}(V,k,V_B(g),T)}{\partial g}$ *is also not continuous in g (or alternatively in V_B) at $V_B(g) = k$, meaning that there may be a point of discontinuity for each g such that $V_B(g) = k$.*

C Proof of the main results

In this section, we present the proofs for all theorems and propositions discussed in the main body of the paper.

Proof of Theorem 3.1. To prove this, we must explicitly calculate (3.7) using equations (3.2) and (3.6):

$$\begin{aligned}
\tfrac{\partial E(V;V_B,T)}{\partial V}\big|_{V=V_B} &= \tfrac{\partial v(V;V_B)}{\partial V}\big|_{V=V_B} - \tfrac{\partial L(V;V_B,T)}{\partial V}\big|_{V=V_B} \\
&= \tfrac{\partial V}{\partial V}\big|_{V=V_B} + \tfrac{\partial \tau_1 \frac{G}{r}(1-(\frac{V_B}{V})^{\lambda_2+\lambda_3})}{\partial V}\big|_{V=V_B} + \tfrac{\partial \tau_2 \alpha \int_0^\infty c_{do}(V,k,V_B,t)\mathrm{d}t}{\partial V}\big|_{V=V_B} \\
&\quad - \tfrac{\partial \rho V_B (\frac{V_B}{V})^{\lambda_2+\lambda_3}}{\partial V}\big|_{V=V_B} - \tfrac{\partial \frac{G}{r}}{\partial V}\big|_{V=V_B} - \tfrac{\partial (P - \frac{G}{r})(\frac{1-e^{-rT}}{rT} - I_1^V(T))}{\partial V}\big|_{V=V_B} \\
&\quad - \tfrac{\partial ((1-\rho)V_B - \frac{G}{r}) I_2^V(T)}{\partial V}\big|_{V=V_B} - \tfrac{\partial \alpha \int_0^T c_{do}(V,k,V_B,t)\mathrm{d}t}{\partial V}\big|_{V=V_B} \\
&= 1 - \tau_1 \tfrac{G}{r} \tfrac{\partial (\frac{V_B}{V})^{\lambda_2+\lambda_3}}{\partial V}\big|_{V=V_B} + \tau_2 \alpha \int_0^\infty \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B} \mathrm{d}t \\
&\quad - \rho V_B \tfrac{\partial (\frac{V_B}{V})^{\lambda_2+\lambda_3}}{\partial V}\big|_{V=V_B} - 0 + (P - \tfrac{G}{r}) \tfrac{\partial I_1^V(T)}{\partial V}\big|_{V=V_B} \\
&\quad - ((1-\rho)V_B - \tfrac{G}{r}) \tfrac{\partial I_2^V(T)}{\partial V}\big|_{V=V_B} - \alpha \int_0^T \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B} \mathrm{d}t,
\end{aligned}$$

where we used Lemma B.5 to interchange the integral with the derivative by the Leibniz integral rule (in the measure theoretic version).

Now, in a first step, we demonstrate the existence of a solution V_B by setting the previous equation equal to zero. The definitions of A_1 and A_2 in (3.10) and (3.11), along with Lemma B.8, imply that $\frac{\partial I_1^V(T)}{\partial V}\big|_{V=V_B} = -\frac{2A_1}{rTV_B}$ and $\frac{\partial I_2^V(T)}{\partial V}\big|_{V=V_B} = -\frac{2A_2}{V_B}$. In the initial step, we will

disregard the participating part by setting $\alpha = 0$. Then, using Lemma B.8 and noting that $\frac{\partial(\frac{V_B}{V})^{\lambda_2+\lambda_3}}{\partial V}\big|_{V=V_B} = (-\lambda_2-\lambda_3)V_B^{\lambda_2+\lambda_3}V^{-\lambda_2-\lambda_3-1}\big|_{V=V_B} = -\frac{(\lambda_2+\lambda_3)}{V_B}$, we have:

$$\begin{aligned}0 &= 1 + \tfrac{\tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{V_B} + \tfrac{\rho V_B(\lambda_2+\lambda_3)}{V_B} - \tfrac{2(P-\frac{G}{r})A_1}{rTV_B} + \tfrac{2((1-\rho)V_B-\frac{G}{r})A_2}{V_B}\\ &= 1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2 - \tfrac{1}{V_B}\Big(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\tfrac{G}{r}(\lambda_2+\lambda_3)\Big) =: h_1(V_B). \qquad \text{(C.1)}\end{aligned}$$

Solving this equation for V_B yields:

$$V_B^* = \frac{\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2}, \qquad \text{(C.2)}$$

where Lemma B.13 guarantees that the nominator and the denominator are positive, ensuring that V_B^* is positive and well-defined.

Specifically, we find that $V_B^* > 0$ solves $h_1(V_B) = 0$. Since h is increasing in V_B, we have $h_1(V) < 0$ for all $V < V_B^*$ and $h(V) > 0$ for all $V > V_B^*$. In particular, by Lemma B.11,we obtain:

$$\begin{aligned}\lim_{V\to 0} h_1(V) &= -\infty, \qquad \text{(C.3)}\\ \lim_{V\to\infty} h_1(V) &= 1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2 > 0.\end{aligned}$$

Next, we incorporate the term with the participation component. Consequently, $\frac{\partial E(V;V_B,T)}{\partial V}\big|_{V=V_B} = 0$ is equivalent to

$$0 = h_1(V_B) + \tau_2\alpha\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t - \alpha\int_0^T \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t =: h_2(V_B). \qquad \text{(C.4)}$$

Note that we plug in $V = V_B$ into both functions h_1 and h_2. Since the Dominated Convergence Theorem allows us to interchange the integral and the limit (with its prerequisite demonstrated in Lemma B.5), we can apply Lemma B.6. Furthermore, by utilizing the fact that $d_1(1,t)) = (\frac{r-\nu}{\sigma}+\frac{\sigma}{2})\sqrt{t} = \lambda_1\sigma\sqrt{t}$, and incorporating the equalities from (B.8) and (B.9), we can conclude that:

$$\begin{aligned}\lim_{V_B\to 0} h_2(V_B) &= -\infty,\\ \lim_{V_B\to\infty} h_2(V_B) &= 1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2 + \tau_2\alpha\int_0^\infty \Big(e^{-\nu t}(2\lambda_1\Phi(\lambda_1\sigma\sqrt{t}) + \tfrac{2\varphi(\lambda_1\sigma\sqrt{t})}{\sigma\sqrt{t}})\Big)\mathrm{d}t\\ &\quad - \alpha\int_0^T \Big(e^{-\nu t}(2\lambda_1\Phi(\lambda_1\sigma\sqrt{t}) + \tfrac{2\varphi(\lambda_1\sigma\sqrt{t})}{\sigma\sqrt{t}})\Big)\mathrm{d}t\\ &= 1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2+\tau_2\alpha A_3 - \alpha A_4. \qquad \text{(C.5)}\end{aligned}$$

Additionally, we note that $h_2(V_B) = \frac{\partial E(V;V_B,T)}{\partial V}\big|_{V=V_B}$, and for $V_B \geq k$, there exist constants $C_1, C_2 \in \mathbb{R}$ such that $h_2(V_B) = C_1 + \frac{C_2}{V_B}$ using (B.4) (with Lemma B.7 ensuring well-definedness). If $C_1 = C_2 = 0$, then $h_2(V_B) = 0$ for all $V_B \geq k$, resulting in infinitely many zero-roots. The unboundedness of the zero-roots poses no issue, as by definition then $V_B = V_0$, corresponding to immediate bankruptcy, consistent with the case in which the largest zero-root exceeds V_0. Therefore, assuming $\lim_{V_B\to\infty} h_2(V_B) = 0$, the limit must be approached monotonically from above or below for sufficiently large values of V_B. Given the assumption that $V \to E(V)$ is non-decreasing, we conclude that $\lim_{V_B\to\infty} h_2(V_B) > 0$ or $\lim_{V_B\to\infty} h_2(V_B) \searrow 0$. Indeed, if $\lim_{V_B\to\infty} h_2(V_B) < 0$ or $\lim_{V_B\to\infty} h_2(V_B) \nearrow 0$, then, due to the continuity, there exists a $\hat{V}_B$ large enough such that $h_2(V_B) < 0$ for all $V_B \geq \hat{V}_B$. Moreover, this implies that $\frac{\partial E(V;V_B,T)}{\partial V}\big|_{V=V_B} < 0$ for all $V_B \geq \hat{V}_B$. By the continuity of $\frac{\partial E(V;V_B,T)}{\partial V}$ in V, there exists an $\varepsilon > 0$ such that

$\frac{\partial E(V;V_B,T)}{\partial V} < 0$ for all $V \in (V_B, V_B+\varepsilon)$. However, this contradicts the property that the equity is non-decreasing in the insurance company's value. Therefore, the existence of a solution V_B^* follows from the intermediate value theorem (which can be applied due to Lemma B.2).

Now, formula (3.15) follows directly from (C.4), and we obtain formula (3.16) from (B.3) and (B.4), where we used that $\lambda_1 - 1 = \lambda_2$. If the solution of (3.15) exceeds V_0, the insurance company declares bankruptcy immediately. Therefore, we can equivalently set $V_B = V_0$ in this case, without affecting the timing of the bankruptcy declaration, while ensuring that V_B represents the asset value (before subtracting the bankruptcy costs) at the time of bankruptcy. Finally, we conclude that $V_B \neq 0$ by Lemma B.14, and thus $V_B > 0$ since $V_B \in [0, V_0]$ by definition. □

Proof of Proposition 3.2. If τ_2 increases, we observe that the right-hand side of (3.15) also increases. From the proof of Theorem 3.1, we know that the largest solution of (3.15) corresponds to a sign transition from "−" to "+" (or to a local minimum, with the function being positive to the right-hand side of the zero root). Therefore, as the graph shifts upwards, the zero root decreases showing that V_B is increasing in τ_2.

We know that the positive value $\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)$ decreases if τ_1 increases. (Note that positivity is ensured by Lemma B.13.) Therefore, the zero root V_B must also decrease in order to maintain the equality, ensuring that the right-hand side of equation (3.15) remains zero.

The argument for the contract maturity T follows analogously, but in the opposite direction in both cases. Indeed, an increasing T leads to lower values on the right-hand side of equation (3.15) in the participation component, while simultaneously increasing $\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)$ under the assumption that $P - \frac{G}{r} \leq 0$. Therefore, as T increases, the bankruptcy-triggering value increases as well. □

Proof of Proposition 3.3. To show that V_B is monotonically increasing in α, let $0 \leq \alpha_1 < \alpha_2$, and we aim to prove that $V_B(\alpha_1) < V_B(\alpha_2)$. By Theorem 3.1, we know that:

$$\begin{aligned} 0 = {} & 1 + \rho(\lambda_2+\lambda_3) + 2(1-\rho)A_2 - \tfrac{1}{V_B(\alpha_1)}\Big(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\tfrac{G}{r}(\lambda_2+\lambda_3)\Big) \\ & + \tau_2\alpha_1\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B(\alpha_1),t)}{\partial V}\big|_{V=V_B(\alpha_1)}\mathrm{d}t - \alpha_1\int_0^T \tfrac{\partial c_{do}(V,k,V_B(\alpha_1),t)}{\partial V}\big|_{V=V_B(\alpha_1)}\mathrm{d}t. \end{aligned} \tag{C.6}$$

Using Lemma B.10, we get, in particular, that:

$$\begin{aligned} 0 > {} & 1 + \rho(\lambda_2+\lambda_3) + 2(1-\rho)A_2 - \tfrac{1}{V_B(\alpha_1)}\Big(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\tfrac{G}{r}(\lambda_2+\lambda_3)\Big) \\ & + \tau_2\alpha_2\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B(\alpha_1),t)}{\partial V}\big|_{V=V_B(\alpha_1)}\mathrm{d}t - \alpha_2\int_0^T \tfrac{\partial c_{do}(V,k,V_B(\alpha_1),t)}{\partial V}\big|_{V=V_B(\alpha_1)}\mathrm{d}t. \end{aligned}$$

Since the right-hand side converges to a positive number (or to 0 from above) as $V_B \to \infty$ (see C.5) and as $V_B(\alpha_1)$ is the largest solution of (C.6), we conclude that $V_B(\alpha_2) > V_B(\alpha_1)$.

Second, we demonstrate the right-continuity in α. Let $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a jointly continuous function satisfying $\lim_{x\to 0} f(x,v) = -\infty$, $\lim_{x\to\infty} f(x,v) = c(v) \geq 0$, and assume that for all $v \in [0,\infty)$ there exists a $\tilde{x}(v) \in (0,\infty)$ such that $f(\tilde{x}(v), v) = 0$. Since the right-hand side of (3.15) satisfies the properties of f, it remains to show that $w(v) := \sup\{y \in [0,\infty) : f(y,v) = 0\}$ is right-continuous under the assumption that w is monotonically increasing (which was established in the first part of this proof). Note that the assumption that for all $v \in [0,\infty)$ there exists a $\tilde{x}(v) \in (0,\infty)$ such that $f(\tilde{x}(v), v) = 0$ implies that $\{y \in [0,\infty) : f(y,v) = 0\} \neq \emptyset$ for all $v \in [0,\infty)$. Let $v_1 \geq 0$. Since w is increasing, we know that $w(v_1+\delta) \geq w(v_1)$ for all $\delta > 0$. Now, suppose that there exists an $\varepsilon > 0$ such that $w(v_1+\delta) \geq w(v_1) + \varepsilon$ for all $\delta > 0$. Due to the monotonicity of w, there exists a bound $\tilde{x}$ with $w(v_1) + \varepsilon \leq \tilde{x} = \lim_{\delta\downarrow 0} w(v_1+\delta)$. By definition of w and the joint continuity of f, we have: $0 = f(w(v_1+\delta), v_1+\delta) \xrightarrow{\delta\to 0} f(\tilde{x}, v_1)$. In

particular, $\tilde{x} > w(v_1)$ is a zero root of $y \to f(y, v_1)$. However, this contradicts the definition of $w(v_1)$ as the supremum of the zero root of $y \to f(y, v_1)$. Therefore, no such $\varepsilon > 0$ can exist, and the right-continuity in α follows.

Third, the existence of the left-limits follows directly from the monotonicity.

The proof for the guaranteed payment proceeds similarly. We observe that, under the assumption on the value of the guaranteed payments, the term $\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)$ is increasing in G (as analyzed in the proof of Theorem 3.1), while the other terms depend on G only through V_B. Therefore, an analogous reasoning leads to the same conclusion about the behavior of V_B with respect to G. □

Remark C.1. *Note that left-continuity is not shown in Proposition 3.3 for the following two reasons: (i) If an additional intersection of the right-hand side of (3.15) with 0 occurs when increasing α (resp. g), and this intersection becomes the largest zero root, then the zero root will have a left limit in α (resp. g), but it will be not left-continuous. Specifically, when a new intersection appears and becomes the largest zero root, the left-continuity of the zero root fails. (ii) If there exists an interval of zero roots that includes the largest zero root, left-continuity can fail as well. However, a left limit still exists, and this limit belongs to this interval of zero roots. By definition, however, we select the largest zero root, which lies on the right-hand side of this interval and thus is not the left limit.*

Proof of Proposition 3.4. As in the previous proof of Theorem 3.1, we observe that the assumption of $V \to E(V)$ being non-decreasing is implied by $\lim_{V_B\to\infty} h_2(V_B) = \lim_{V_B\to\infty} \frac{\partial E(V;V_B,T)}{\partial V}\big|_{V=V_B} > 0$ with h_2 defined as in (C.4). Let $\tau_2 A_3 < A_4$. Then, if $\alpha < \bar{\alpha} = \dfrac{1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2}{A_4-\tau_2 A_3}$, we have that $\lim_{V_B\to\infty} h_2(V_B) > 0$ by (C.5), which implies the claim. On the other hand, if $\tau_2 A_3 \geq A_4$, then $\lim_{V_B\to\infty} h_2(V_B) \geq 0$ for all $\alpha \geq 0$ by (C.5) and Lemma B.13, implying the claim. □

Remark C.2. *In the context of the proof of Proposition 3.4 above, we observe that the assumption of $V \to E(V)$ being non-decreasing fails if $\lim_{V_B\to\infty} h_2(V_B) = \lim_{V_B\to\infty} \frac{\partial E(V;V_B,T)}{\partial V}\big|_{V=V_B} < 0$. Analogously to the earlier discussion, this corresponds to the case $\alpha > \bar{\alpha}$. However, if $\lim_{V_B\to\infty} h_2(V_B) = \lim_{V_B\to\infty} \frac{\partial E(V;V_B,T)}{\partial V}\big|_{V=V_B} = 0$, the analysis becomes more delicate. In this case, it is essential to distinguish whether the limit is approached from above or below (cf. the discussion in the proof of Theorem 3.1).*

Proof of Corollary 3.5. Let us assume that $V_B \geq k$, i.e., $\min\{\frac{V_B}{k}, 1\} = 1$. Note that $d_1(1,T) = \lambda_1\sigma\sqrt{T}$ and $d_2(1,T) = \lambda_2\sigma\sqrt{T}$. Then, we get, using (3.16) and Lemma B.9 (once applied in the original version, and once applied with λ_1 replaced by λ_2, and ν replaced by r):

$$\begin{aligned}
\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t &= \tfrac{\lambda_1}{\nu} + \tfrac{\lambda_1^2\sigma}{\nu}\sqrt{\tfrac{1}{\lambda_1^2\sigma^2+2\nu}} + \tfrac{2}{\sigma}\sqrt{\tfrac{1}{\lambda_1^2\sigma^2+2\nu}} \\
&\quad - \tfrac{k}{V_B}\left(\tfrac{\lambda_2}{r} + \tfrac{\lambda_2^2\sigma}{r}\sqrt{\tfrac{1}{\lambda_2^2\sigma^2+2r}} + \tfrac{2}{\sigma}\sqrt{\tfrac{1}{\lambda_2^2\sigma^2+2r}}\right) \\
&= A_3 - \tfrac{k}{V_B}A_5, \qquad \text{(C.7)} \\
\int_0^T \tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t &= \tfrac{\lambda_1}{\nu} - \tfrac{2\lambda_1 e^{-\nu T}\Phi(\lambda_1\sigma\sqrt{T})}{\nu} + \tfrac{\lambda_1^2\sigma}{\nu}\sqrt{\tfrac{1}{\lambda_1^2\sigma^2+2\nu}}(2\Phi(\sqrt{\lambda_1^2\sigma^2+2\nu}\sqrt{T})-1) \\
&\quad + \tfrac{2}{\sigma}\sqrt{\tfrac{1}{\lambda_1^2\sigma^2+2\nu}}(2\Phi(\sqrt{\lambda_1^2\sigma^2+2\nu}\sqrt{T})-1) \\
&\quad - \tfrac{k}{V_B}\Big(\tfrac{\lambda_2}{r} - \tfrac{2\lambda_2 e^{-rT}\Phi(\lambda_2\sigma\sqrt{T})}{r} + \tfrac{\lambda_2^2\sigma}{r}\sqrt{\tfrac{1}{\lambda_2^2\sigma^2+2r}}(2\Phi(\sqrt{\lambda_2^2\sigma^2+2r}\sqrt{T})-1) \\
&\qquad + \tfrac{2}{\sigma}\sqrt{\tfrac{1}{\lambda_2^2\sigma^2+2r}}(2\Phi(\sqrt{\lambda_2^2\sigma^2+2r}\sqrt{T})-1)\Big) \\
&= A_4 - \tfrac{k}{V_B}A_6, \qquad \text{(C.8)}
\end{aligned}$$

where we used that $\left(\frac{\lambda_1^2\sigma}{\nu}+\frac{2}{\sigma}\right)\sqrt{\frac{1}{\lambda_1^2\sigma^2+2\nu}} = \frac{1}{\sigma\nu}\sqrt{\lambda_1^2\sigma^2+2\nu}$ and $\left(\frac{\lambda_2^2\sigma}{r}+\frac{2}{\sigma}\right)\sqrt{\frac{1}{\lambda_2^2\sigma^2+2r}} = \frac{1}{\sigma r}\sqrt{\lambda_2^2\sigma^2+2r}$. Next, we substitute (C.7) and (C.8) into (3.15) and solve for V_B, yielding the formula for $\hat{V}_B$. In particular, if $\hat{V}_B \geq k$, it is the unique solution to (3.15) that satisfies this condition. Therefore, the claim follows. □

Proof of Proposition 4.1. If $\bar{\alpha} > 1$, the proposition follows directly from the continuity of v in α. Therefore, let us assume that $\bar{\alpha} \leq 1$. First, we note that for $\alpha > \bar{\alpha}$, the assumption that $V \to E(V)$ is non-decreasing does not hold, as discussed in Remark C.2. This, however, is excluded by assumption. Thus, we can restrict our analysis to the case where $\alpha \in [0, \bar{\alpha}]$. Now, let h_2 be defined as in (C.4). The definition of $\bar{\alpha}$ in (3.14) implies that $\alpha = \bar{\alpha}$ corresponds to the situation where $\lim_{V_B\to\infty} h_2(V_B) = 0$, see (C.5). From the proof of Theorem 3.1, we know that $\lim_{V_B\to\infty} h_2(V_B) \searrow 0$, ensuring that the assumption of $V \to E(V)$ being non-decreasing is satisfied. Hence, (3.15) always admits a solution for $\alpha \in [0, \bar{\alpha}]$. Finally, the continuity of v in α completes the proof, implying the claim. □

Proof of Theorem 4.2. Before we begin the actual proof, we first show the identities used in the theorem: From the proof of Lemma B.16, we find that $V_B'(\alpha) = -\frac{R_\alpha(\alpha, V_B(\alpha))}{R_{V_B}(\alpha, V_B(\alpha))}$ for $\alpha \in [0, \hat{\alpha}]$, where R_α is defined as in (B.12), R_{V_B} is defined as in (B.13), and $\hat{\alpha} > 0$ is defined as in Lemma B.16. Substituting $\alpha = 0$ gives us (A.12) and (A.10). Finally, (A.11) follows immediately from (B.10) and (B.11).

For the main claim, we consider the total value v of the insurance company as given in (3.6) and take the derivative with respect to α. We restrict ourselves to $\alpha \in [0, \hat{\alpha}]$, with $\hat{\alpha}$ defined as in Lemma B.16. Under these conditions, we can interchange the integral and the derivative (where we suppress the dependency of V_B on α):

$$\begin{aligned}\frac{\partial}{\partial\alpha}v(V;V_B) =& -\frac{\tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{V}\left(\frac{V_B}{V}\right)^{\lambda_2+\lambda_3-1}V_B'(\alpha) - \rho(\lambda_2+\lambda_3+1)\left(\frac{V_B}{V}\right)^{\lambda_2+\lambda_3}V_B'(\alpha)\\ &+\tau_2\int_0^\infty c_{do}(V,k,V_B,t)\mathrm{d}t + \alpha\tau_2\int_0^\infty \frac{\partial c_{do}(V,k,V_B,T)}{\partial\alpha}\mathrm{d}t\\ =& -V_B'(\alpha)\left(\frac{V_B}{V}\right)^{\lambda_2+\lambda_3}\left(\frac{\tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{V_B}+\rho(\lambda_2+\lambda_3+1)\right)\\ &+\tau_2\int_0^\infty c_{do}(V,k,V_B,t)\mathrm{d}t + \alpha\tau_2\int_0^\infty \frac{\partial c_{do}(V,k,V_B,T)}{\partial\alpha}\mathrm{d}t. \end{aligned} \tag{C.9}$$

Setting this equation equal to zero yields (4.2).

Next, we show the existence of a $\bar{\tau}$ as described in the theorem. To do this, we evaluate the above formula at $\alpha = 0$, which gives us:

$$\begin{aligned}\frac{\partial}{\partial\alpha}v(V;V_B)\Big|_{\alpha=0} =& -V_B'(0)\left(\frac{V_B(0)}{V}\right)^{\lambda_2+\lambda_3}\left(\frac{\tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{V_B(0)}+\rho(\lambda_2+\lambda_3+1)\right)\\ &+\tau_2\int_0^\infty c_{do}(V,k,V_B(0),t)\mathrm{d}t. \end{aligned} \tag{C.10}$$

Now, it is optimal to offer a surplus participation for the insurance company if $\frac{\partial}{\partial\alpha}v(V;V_B)\big|_{\alpha=0} > 0$.

Therefore, let us analyze equation (C.10): Lemma B.13 implies that $V_B(0) > 0$ (see (3.17)), and that the denominator of $V_B'(0)$ (see (A.12)) is strictly positive. Moreover, $\left(\frac{\tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{V_B(0)} + \rho(\lambda_2+\lambda_3+1)\right) > 0$, since $\lambda_3 > |\lambda_2|$ by definition, the price of a Down-and-Out Call Option, c_{do}, is non-negative, $V_B(0)$ (see (3.17)) is independent of τ_2, and for $\tau_2 = 1$ the term $(-V_B'(0))$ (see (A.12)) is positive because $T < \infty$. Consequently, we obtain that $\frac{\partial}{\partial\alpha}v(V;V_B)\big|_{\alpha=0} > 0$ for $\tau_2 = 1$,

and that $\frac{\partial}{\partial\alpha}v(V;V_B)\big|_{\alpha=0}$ is continuous in τ_2. This leads to the existence of a $\bar{\tau}\in(0,1)$ such that $\frac{\partial}{\partial\alpha}v(V;V_B)\big|_{\alpha=0}>0$ for $\tau_2\in(\bar{\tau},1)$, implying the claim.

Finally, it remains to demonstrate the existence of $\bar{\bar{\tau}}$. To establish this, we have by (C.9):

$$\frac{\partial}{\partial\alpha}v(V;V_B)\big|_{\tau_2=0}=-V_B'(\alpha)(\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\left(\frac{\tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{V_B}+\rho(\lambda_2+\lambda_3+1)\right)<0,$$

since $\lambda_3>|\lambda_2|$ by definition and $V_B'(\alpha)>0$ (since V_B is increasing in α by Proposition 3.3). Hence, it follows that if $\tau_2=0$, the optimal choice is $\alpha^*=0$. Moreover, the inequality

$$-V_B'(\alpha)(\tfrac{V_B}{V})^{\lambda_2+\lambda_3}\left(\frac{\tau_1\frac{G}{r}(\lambda_2+\lambda_3)}{V_B}+\rho(\lambda_2+\lambda_3+1)\right)<0$$

persists independently of the specific choice of τ_2. Finally, since $\int_0^\infty c_{do}(V,k,V_B,t)\mathrm{d}t+\alpha\int_0^\infty\frac{\partial c_{do}(V,k,V_B,T)}{\partial\alpha}\mathrm{d}t$ in (C.9) is bounded (by Lemmas B.1 and B.16), it follows that there exists a value $\bar{\bar{\tau}}>0$ such that $\frac{\partial}{\partial\alpha}v(V;V_B)<0$ for all $0\le\tau_2<\bar{\bar{\tau}}$, implying that $\alpha^*=0$ for all $0\le\tau_2<\bar{\bar{\tau}}$. Thus, the claim is established. □

Proof of Proposition 4.3. Since v is continuous in g (because V_B is continuous in g), it suffices to show that the supremum is attained in a compact interval of g. First, assume that $-\frac{2A_1}{rT}+2A_2-\tau_1(\lambda_2+\lambda_3)\neq0$. By Lemma B.7, it follows that the term $\tau_2\alpha\int_0^\infty\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t-\alpha\int_0^T\frac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\mathrm{d}t$ is uniformly bounded in V_B, and thus in g. From equation (3.15), we know that $V_B\xrightarrow{g\to\infty}\infty$ (using that $g=\frac{G}{T}$). Therefore, by the continuity of V_B in g, there exists a $\bar{g}\ge0$ such that $V_B(g)\ge V_0$ for all $g\ge\bar{g}$. If $V_B\ge V_0$, however, the insurance company declares bankruptcy immediately. In this case, the insurance company's value is given by $v(V;V_B)=V-\rho V$ for all $V_B\ge V_0$, according to the first equation in (3.6), where $TB_1=TB_2=0$ and $BC=\rho V$ (see (3.3), (3.4), and (3.5)). Therefore, v is constant for $V_B\ge V_0$, and we can consequently set $V_B=V_0$. Hence, we can restrict g to the interval $[0,\bar{g}]$, completing the proof. □

Proof of Theorem 4.4. Note that this proof is similar to the proof of Theorem 4.2.

We begin by showing the identities stated in the theorem: Equation (A.13) follows directly from (3.15). Next, from the proof of Lemma B.18, we obtain that $V_B'(g)=-\frac{R_g}{R_{V_B}}$, where R_g is defined in (B.16) and R_{V_B} is defined in (B.17). Evaluating this expression at $g=0$ and inserting $V_B(0)$ gives us the desired results in (A.14) and (A.16). Finally, the equations (A.15) and (A.17) are immediate consequences of (B.10) and (B.11).

For the main claim (4.5), we again utilize the total value v of the insurance company, as defined in (3.6), and differentiate it with respect to g. We consider $g\in[0,\hat{g}]$, where $\hat{g}$ is defined as in Lemma B.18, such that we can interchange the integral and the derivative (where we suppress the dependency of V_B on g):

$$\begin{aligned}\frac{\partial}{\partial g}v(V;V_B)=&\tfrac{\tau_1T}{r}(1-(\tfrac{V_B}{V})^{\lambda_2+\lambda_3})-\tfrac{\tau_1gT(\lambda_2+\lambda_3)}{Vr}(\tfrac{V_B}{V})^{\lambda_2+\lambda_3-1}V_B'(g)\\&-\rho(\lambda_2+\lambda_3+1)(\tfrac{V_B}{V})^{\lambda_2+\lambda_3}V_B'(g)+\alpha\tau_2\textstyle\int_0^\infty\frac{\partial c_{do}(V,k,V_B,T)}{\partial g}\mathrm{d}t.\end{aligned}\tag{C.11}$$

Setting this equation equal to zero results in the equation (4.5).

It remains to demonstrate that offering a guarantee rate is indeed optimal. To do so, we evaluate this formula at $g=0$, which leads to:

$$\begin{aligned}\frac{\partial}{\partial g}v(V;V_B)\big|_{g=0}=&\tfrac{\tau_1T}{r}(1-(\tfrac{V_B(0)}{V})^{\lambda_2+\lambda_3})-\rho(\lambda_2+\lambda_3+1)(\tfrac{V_B(0)}{V})^{\lambda_2+\lambda_3}V_B'(0)\\&+\alpha\tau_2\int_0^\infty\frac{\partial c_{do}(V,k,V_B,T)}{\partial g}\Big|_{g=0}\mathrm{d}t>0,\end{aligned}\tag{C.12}$$

by assumption. Note that we applied Lemma B.18 to move the point estimation inside the integral. Therefore, it follows that $g^*>0$. □

Proof of Proposition 4.5. This proposition follows directly from Propositions 4.1 and 4.3. Note that the proof of Proposition 4.1 is independent of g, and the proof of Proposition 4.3 holds for all $\alpha \leq 1$. □

D Additional results

In this section, we present additional results that are referenced in the main text.

Proposition D.1. *If (i) $V_B \geq k$ or (ii) $V_B < k$ and the following condition (with d_2 as in (A.3) and λ_2, λ_3 as in (A.9)) holds for all $\alpha \in [0, \min\{\bar{\alpha}, 1\}]$, then the bankruptcy-triggering value V_B is continuous in α:*

$$\begin{aligned}\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\tfrac{G}{r}(\lambda_2+\lambda_3) + \tau_2\alpha\int_0^\infty 2ke^{-rt}\big(\lambda_2\Phi(d_2(\tfrac{V_B}{k},t)) + \tfrac{\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}}\big)\mathrm{d}t \\ - \alpha\int_0^T 2ke^{-rt}\big(\lambda_2\Phi(d_2(\tfrac{V_B}{k},t)) + \tfrac{\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}}\big)\mathrm{d}t > 0, \qquad \text{(D.1)}\end{aligned}$$

Moreover, if (i) $V_B \geq k$ or (ii) $V_B < k$ and (D.1) holds for all $G \in [0, \infty)$, then the bankruptcy-triggering value V_B is continuous in g.

Proof. We begin with the case (i), meaning that $V_B \geq k$. In this case, Corollary 3.5 provides an analytical solution that already ensures the continuity of V_B in α resp. g.

Next, we consider case (ii). In particular, we assume $V_B < k$. To prove this part of the lemma, we aim to establish that the right hand side of (3.15), denoted as $R(V_B)$, is strictly increasing in V_B (treated as an independent variable). This strict monotonicity implies that the zero root is unique. Given that the right hand side of (3.15) is smooth in V_B (treated as an independent variable), as well as in α (resp. g), the uniqueness of the zero root implies that the conditions of the inverse function theorem are locally satisfied around this root. Consequently, by applying this theorem, we conclude that the bankruptcy-triggering value V_B (the unique zero root) is continuous in α (resp. g).
To demonstrate that $R(V_B)$ is increasing in V_B, we will show that R' is positive (using (B.10)). Specifically, it holds that:

$$\begin{aligned}R'(V_B) &= \tfrac{1}{V_B^2}\Big(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\tfrac{G}{r}(\lambda_2+\lambda_3)\Big) \\ &\quad + \tau_2\alpha\int_0^\infty \tfrac{\partial}{\partial V_B}\big[\tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\big]\mathrm{d}t - \alpha\int_0^T \tfrac{\partial}{\partial V_B}\big[\tfrac{\partial c_{do}(V,k,V_B,t)}{\partial V}\big|_{V=V_B}\big]\mathrm{d}t \\ &= \tfrac{1}{V_B^2}\Big(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\tfrac{G}{r}(\lambda_2+\lambda_3)\Big) \\ &\quad + \tau_2\alpha\int_0^\infty \tfrac{2ke^{-rt}}{V_B^2}\Big(\lambda_2\Phi(d_2(\tfrac{V_B}{k},t)) + \tfrac{\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}}\Big)\mathrm{d}t \\ &\quad - \alpha\int_0^T \tfrac{2ke^{-rt}}{V_B^2}\Big(\lambda_2\Phi(d_2(\tfrac{V_B}{k},t)) + \tfrac{\varphi(d_2(\frac{V_B}{k},t))}{\sigma\sqrt{t}}\Big)\mathrm{d}t > 0,\end{aligned}$$

where the last inequality follows from the assumption of the lemma after factoring $\frac{1}{V_B^2} > 0$ out. □

Remark D.2. *Since $\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3) > 0$ (see Lemma B.13), we can conclude that for sufficiently large τ_2, the assumption in equation (D.1) is always satisfied. Economically, this reflects a situation where the tax benefit on surplus participation is sufficiently large that an increase in asset value leads to an increase in equity. In particular, the assumption holds when $\alpha = 0$, i.e., in the absence of a surplus participation.*

Proposition D.3. *Let τ_2 be sufficiently large such that there exists an $\varepsilon > 0$ with $R_{V_B} \geq \varepsilon$ for all $(g, V_B) \in [0, \infty) \times [0, V_0]$ (defined as in (B.17)). Then, under Assumption 2, equation (4.5) always admits a solution.*

Proof. First, we observe that if $\tau_2 = 1$, then $R_{V_B} > 0$ for all $(g, V_B) \in [0, \infty) \times [0, V_0]$, as follows from (B.17), together with Lemmas B.13 and B.15.

Second, note that the right-hand side of (4.5) is nothing else than $\frac{\partial}{\partial g} v(V; V_B)$ (see (C.11)). Thus, we show the existence of an $\tilde{g} \in (0, \infty)$ such that $\frac{\partial}{\partial g} v(V; V_B) = 0$ for $g = \tilde{g}$ to prove the lemma.

Under the assumption made in this lemma, we can choose $\hat{g} = \infty$ in the proof of Lemma B.18, and consequently, also in the proof of Theorem 4.4. By Proposition 3.3, V_B is increasing in g, implying that V_B' is non-negative. Moreover, we know that $V_B \leq V$ by assumption (in Subsection 2.2). Then, from the definition of $V_B'(g)$ (see (A.14)), we conclude that $\lim_{g\to\infty} gV_B'(g) = \lim_{g\to\infty} V_B(g)$ and $V_B'(g) \xrightarrow{g\to\infty} 0$, because $G = gT$ and V_B is increasing and bounded. Moreover, it holds that $V_B(g) \xrightarrow{g\to\infty} V_0$. Indeed, from (3.15), it follows that the largest solution of this equation diverges to infinity as $G \to \infty$, since $\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2 + \lambda_3) \xrightarrow{G\to\infty} \infty$. The reason is that it follows from (3.15) that, if V_B remained bounded, all other terms remain bounded as well, implying that the right hand side cannot be 0 for sufficiently large G. Since, by construction, we set $V_B = V_0$, corresponding to immediate bankruptcy, whenever the largest solution of (3.15) exceeds V_0, it follows that $V_B(g) \xrightarrow{g\to\infty} V_0$. Furthermore, Lemma B.18 implies that we can interchange the integral sign of the last term of (4.5) with the limit of $g \to \infty$. Then, (B.18) and (B.19) imply that the last term of (4.5) converges to 0 for $g \to \infty$ as $V_B'(g) \xrightarrow{g\to\infty} 0$. Therefore, we have $\frac{\partial}{\partial g} v(V; V_B) \xrightarrow{g\to\infty} -\frac{\tau_1(\lambda_2+\lambda_3)}{rT} < 0$ (see (4.5)). Finally, using the continuity of $v(V; V_B)$ in g and (C.12), the intermediate value theorem ensures the desired result. □

References

[1] Sumit Agarwal, Brent W Ambrose, Hongming Huang, and Yildiray Yildirim. The term structure of lease rates with endogenous default triggers and tenant capital structure: theory and evidence. *Journal of Financial and Quantitative Analysis*, 46(2):553–584, 2011.

[2] Allianz Group. Annual report 2023. Technical report, Allianz Group, 2023.

[3] Allianz Lebensversicherungs-AG. Annual report 2023. Technical report, Allianz Lebensversicherungs-AG, 2023.

[4] Brent W Ambrose, Thomas Emmerling, Henry H Huang, and Yildiray Yildirim. Capital structure and the substitutability versus complementarity nature of leases and debt. *Review of Finance*, 23(3):659–695, 2019.

[5] Anna Rita Bacinello and Svein-Arne Persson. Design and pricing of equity-linked life insurance under stochastic interest rates. *The Journal of Risk Finance*, 3(2):6–21, 2002.

[6] Laura Ballotta, Steven Haberman, and Nan Wang. Guarantees in with-profit and unitized with-profit life insurance contracts: Fair valuation problem in presence of the default option. *Journal of Risk and Insurance*, 73(1):97–121, 2006.

[7] Amir Barnea, Robert A Haugen, and Lemma W Senbet. A rationale for debt maturity structure and call provisions in the agency theoretic framework. *The Journal of Finance*, 35(5):1223–1234, 1980.

[8] Carole Bernard, Olivier Le Courtois, and François Quittard-Pinon. Market value of life insurance contracts under stochastic interest rates and default risk. *Insurance: Mathematics and Economics*, 36(3):499–516, 2005.

[9] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[10] Eric Briys and François De Varenne. On the risk of insurance liabilities: debunking some common pitfalls. *Journal of Risk and Insurance*, pages 673–694, 1997.

[11] Mark Carey and Michael B Gordy. The bank as grim reaper: Debt composition and bankruptcy thresholds. *Journal of Financial Economics*, 142(3):1092–1108, 2021.

[12] An Chen, Peter Hieber, and Thai Nguyen. Constrained non-concave utility maximization: An application to life insurance contracts with guarantees. *European Journal of Operational Research*, 273(3):1119–1135, 2019.

[13] An Chen, Thai Nguyen, and Mitja Stadje. Optimal investment under var-regulation and minimum insurance. *Insurance: Mathematics and Economics*, 79:194–209, 2018.

[14] Nan Chen, Paul Glasserman, Behzad Nouri, and Markus Pelger. Contingent capital, tail risk, and debt-induced collapse. *The Review of Financial Studies*, 30(11):3921–3969, 2017.

[15] Chunli Cheng and Jing Li. Early default risk and surrender risk: Impacts on participating life insurance policies. *Insurance: Mathematics and Economics*, 78:30–43, 2018.

[16] Marco Della Seta, Erwan Morellec, and Francesca Zucchi. Short-term debt and incentives for risk-taking. *Journal of Financial Economics*, 137(1):179–203, 2020.

[17] Avinash Dixit. A simplified treatment of the theory of optimal regulation of brownian motion. *Journal of Economic Dynamics and Control*, 15(4):657–673, 1991.

[18] Yinghui Dong, Sang Wu, Wenxin Lv, and Guojing Wang. Optimal asset allocation for participating contracts under the var and pi constraint. *Scandinavian Actuarial Journal*, 2020(2):84–109, 2020.

[19] Bernard Dumas. Super contact and related optimality conditions. *Journal of Economic Dynamics and Control*, 15(4):675–685, 1991.

[20] EIOPA. European insurance overview 2023. Technical report, European Insurance and Occupational Pensions Authority (EIOPA), 2023.

[21] Redouane Elkamhi, Jan Ericsson, and Christopher A Parsons. The cost and timing of financial distress. *Journal of Financial Economics*, 105(1):62–81, 2012.

[22] Felix Fießinger and Mitja Stadje. Mean-variance optimization for participating life insurance contracts. *Insurance: Mathematics and Economics*, 2025.

[23] Nadine Gatzert and Alexander Kling. Analysis of participating life insurance contracts: A unification approach. *Journal of Risk and Insurance*, 74(3):547–570, 2007.

[24] Paul Glasserman and Behzad Nouri. Contingent capital with a capital-ratio trigger. *Management Science*, 58(10):1816–1833, 2012.

[25] Robert Goldstein, Nengjiu Ju, and Hayne Leland. An ebit-based model of dynamic capital structure. *The Journal of Business*, 74(4):483–512, 2001.

[26] Anders Grosen and Peter Løchte Jørgensen. Life insurance liabilities at market value: an analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework. *Journal of Risk and Insurance*, 69(1):63–91, 2002.

[27] J.M. Harrison. *Brownian Motion and Stochastic Flow Systems*. Robert E. Krieger Publishing Company, 1990.

[28] Lin He, Zongxia Liang, Yang Liu, and Ming Ma. Weighted utility optimization of the participating endowment contract. *Scandinavian Actuarial Journal*, 2020(7):577–613, 2020.

[29] Zhiguo He and Konstantin Milbradt. Dynamic debt maturity. *The Review of Financial Studies*, 29(10):2677–2736, 2016.

[30] Christopher A Hennessy and Yuri Tserlukevich. Taxation, agency conflicts, and the choice between callable and convertible debt. *Journal of Economic Theory*, 143(1):374–404, 2008.

[31] Peter Hieber, Ralf Korn, and Matthias Scherer. Analyzing the effect of low interest rates on the surplus participation of life insurance policies with different annual interest rate guarantees. *European Actuarial Journal*, 5(1):11–28, 2015.

[32] Christian Hilpert, Stefan Hirth, and Alexander Szimayer. The information value of distress. *Management Science*, 70(1):78–97, 2024.

[33] Julien Hugonnier, Semyon Malamud, and Erwan Morellec. Credit market frictions and capital structure dynamics. *Journal of Economic Theory*, 157:1130–1158, 2015.

[34] J.C. Hull. *Options, Futures, and Other Derivatives*, volume 10. Pearson Education, 2017.

[35] Nengjiu Ju, Robert Parrino, Allen M Poteshman, and Michael S Weisbach. Optimal capital structure. *Journal of Financial and Quantitative Analysis*, 40(2), 2005.

[36] In Joon Kim, Krishna Ramaswamy, and Suresh Sundaresan. Does default risk in coupons affect the valuation of corporate bonds?: A contingent claims model. *Financial Management*, pages 117–131, 1993.

[37] Alexander Kling, Timon Kramer, and Jochen Ruß. From intertemporal smoothing to intergenerational risk sharing: The effects of different return smoothing mechanisms in life insurance. *Available at SSRN 4744873*.

[38] Alexander Kling, Timon Kramer, Jochen Ruß, and Stefan Schelling. Intertemporal smoothing and intergenerational risk sharing: The impact on objective utility and subjective attractiveness of retirement savings products. *Available at SSRN 5005853*, 2024.

[39] Alexander Kling, Andreas Richter, and Jochen Ruß. The impact of surplus distribution on the risk exposure of with profit life insurance policies including interest rate guarantees. *Journal of Risk and Insurance*, 74(3):571–589, 2007.

[40] Alexander Kling, Andreas Richter, and Jochen Ruß. The interaction of guarantees, surplus distribution, and asset allocation in with-profit life insurance policies. *Insurance: Mathematics and Economics*, 40(1):164–178, 2007.

[41] David Lando. On cox processes and credit risky securities. *Review of Derivatives Research*, 2:99–120, 1998.

[42] Hayne E Leland. Corporate debt value, bond covenants, and optimal capital structure. *The Journal of Finance*, 49(4):1213–1252, 1994.

[43] Hayne E Leland. Agency costs, risk management, and capital structure. *The Journal of Finance*, 53(4):1213–1243, 1998.

[44] Hayne E Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987–1019, 1996.

[45] Hongcan Lin, David Saunders, and Chengguo Weng. Optimal investment strategies for participating contracts. *Insurance: Mathematics and Economics*, 73:137–155, 2017.

[46] Sheen X Liu, Howard Qi, and Chunchi Wu. Personal taxes, endogenous default, and corporate bond yield spreads. *Management Science*, 52(6):939–954, 2006.

[47] Francis A Longstaff and Eduardo S Schwartz. A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3):789–819, 1995.

[48] Gustavo Manso, Bruno Strulovici, and Alexei Tchistyi. Performance-sensitive debt. *The Review of Financial Studies*, 23(5):1819–1854, 2010.

[49] William H Meckling and Michael C Jensen. Theory of the firm. *Managerial Behavior, Agency Costs and Ownership Structure*, 1976.

[50] Robert C Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, pages 141–183, 1973.

[51] Robert C Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.

[52] Charbel Mirza and Joël Wagner. Policy characteristics and stakeholder returns in participating life insurance: which contracts can lead to a win-win? *European Actuarial Journal*, 8:291–320, 2018.

[53] Thai Nguyen and Mitja Stadje. Nonconcave optimal investment with value-at-risk constraint: An application to life insurance contracts. *SIAM Journal on Control and Optimization*, 58(2):895–936, 2020.

[54] M. Picone. Sul teorema di l'Hospital. *Period. Mat., IV. Ser.*, 9:251–253, 1929.

[55] Stephen A Ross. Capital structure and the cost of capital. *Journal of Applied Finance (Formerly Financial Practice and Education)*, 15(1), 2005.

[56] Mark Rubinstein and Eric Reiner. Breaking down the barriers. *Risk Magazine*, 4:28–35, 1991.

[57] Hato Schmeiser and Joël Wagner. A proposal on how the regulator should set minimum interest rate guarantees in participating life insurance contracts. *Journal of Risk and Insurance*, 82(3):659–686, 2015.

Curriculum Vitae

Personal Details

Name	**Felix Fießinger**
Born in	Wangen im Allgäu, 1994

Higher Education

Since 04/2022	**Ulm University**, Ulm Doctoral candidate
04/2020 - 03/2022	**Ulm University**, Ulm Master of Science in Mathematics (1.0)
04/2017 - 03/2020	**Ulm University**, Ulm Bachelor of Science in Mathematics and Management (1.1)

Positions and Practical Experience

Since 05/2025	**Munich Re**, München Consultant Reserving
04/2022 - 03/2025	**Ulm University**, Ulm Research and teaching assistant at the Institute of Insurance Science
10/2021 - 02/2022	**Ulm University**, Ulm Tutor for first semester students
04/2021 - 07/2021	**Allianz Lebensversicherungs AG**, Stuttgart Working student in Life Insurance
10/2021 - 02/2022	**Ulm University**, Ulm Corrector at the Institute of Insurance Science
04/2020 - 07/2020	**Airbus Defense & Space**, Ulm Working student in Radar Technology
10/2018 - 02/2020	**Ulm University**, Ulm Tutor at the Institute of Analysis
08/2019 - 10/2019	**Hannover Re**, Hannover Internship in Life Insurance

Scholarships

10/2018 - 09/2021	**HDI-Scholarship**

Scientific Publications and Working Papers

2025	Fießinger, F., & Stadje, M. (2025). "Time-consistent asset allocation for risk measures in a Lévy market." *European Journal of Operational Research*, 321(2), 676-695. https://doi.org/10.1016/j.ejor.2024.09.049.
2025	Fießinger, F., and Stadje, M. (2025). "The $C^{0,1}$ Itô-Ventzell formula for weak Dirichlet processes." *Electronic Communications in Probability*, 30, 1-15. https://doi.org/10.1214/25-ECP686.
2025	Fießinger, F., and Stadje, M. (2025). "Mean-Variance Optimization for Participating Life Insurance Contracts." *Insurance: Mathematics and Economics*, 122, 230-248. https://doi.org/10.1016/j.insmatheco.2025.03.005.
2025	Fießinger, F., and Stadje, M. (2025). "Optimal Capital Structure for Life Insurance Companies Offering Surplus Participation." *arXiv preprint*. https://doi.org/10.48550/arXiv.2504.12851

Research Conferences (with Contributed Talks)

02/2025	IFA-IVW Workshop at Ulm University, Ulm
09/2024	6th Fudan-Ulm Symposium on Finance and Insurance, Shanghai-Fudan, China
02/2024	IFA-IVW Workshop at Ulm University, Ulm
07/2023	26th International Congress on Insurance: Mathematics and Economics, Edinburgh, United Kingdom
02/2023	IFA-IVW Workshop at Ulm University, Ulm

Ulm, April 2025

Felix Fießinger

Ehrenwörtliche Erklärung

Beim Verfassen dieser Thesis habe ich in der Einleitung, in den Corrigenda zu Paper 1 und 4 sowie für Forschungsprojekt 4 künstliche Intelligenz in Form von ChatGPT verwendet, ausschließlich um die sprachliche Formulierung zu verbessern. Weiterhin habe ich dieselbe künstliche Intelligenz auch für eine Voranalyse der Gleichung (3.15) in Paper 4 genutzt. Die inhaltliche Verantwortung liegt bei mir selbst.
Ich versichere hiermit, dass ich die Arbeit selbstständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die zur Zeit gültige Satzung der Universität Ulm zur Sicherung guter wissenschaftlicher Praxis beachtet habe.
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