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On hypergeometric systems of differential equations

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Introduction

This thesis is about the *hypergeometric system of differential equations*

$$(\Lambda - t\,I)\,y'(t) \;=\; A\,y(t)\,, \tag{1}$$

with an arbitrary $(n \times n)$-matrix A and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ of the same dimension. It is a linear system with singularities at $\lambda_1, \ldots, \lambda_n$ and infinity. Although the system looks very simple, fundamental solutions are only known in dimension $n = 2$ and special cases in higher dimension. In dimension 2 the solutions can be expressed by means of the *hypergeometric function*, and that is the reason for the name of the system. There is a close relation via Laplace Transform between the hypergeometric system and the so-called *confluent hypergeometric system*

$$z x'(z) \;=\; (z\,\Lambda + A)\,x(z)\,,$$

which has a regular singularity at the origin and an irregular one at infinity. The system is called like this, since there is also a relation by means of a confluence process (see [25]).

For the confluent system there always exists a formal fundamental solution of the kind

$$\hat{X}(z) \;=\; \hat{F}(z)\,z^L\,e^{z\Lambda}\,,$$

where $\hat{F}(z)$ denotes a formal power series in z^{-1}. This formal power series is 1-summable (see appendix A), hence there are proper fundamental solutions $X_\nu(z), \nu \in \mathbb{Z}$, so that $\hat{X}(z)$ is the asymptotic expansion of $X_\nu(z)$ in certain sectors S_ν. Since $X_\nu(z)$ is a fundamental solution for all $\nu \in \mathbb{Z}$, there exist constant matrices V_ν satisfying

$$X_{\nu-1}(z) \;=\; X_\nu(z)\,V_\nu\,.$$

These matrices are called the *Stokes' Multipliers* and they describe the *Stokes' Phenomenon*. This means that solutions exist which in sufficiently small sectors show the same asymptotic behavior, but, in general, depend upon the sector. Reversely, one can say that one solution shows different asymptotic behavior in different sectors.

Most of the time, we assume that the eigenvalues of Λ are distinct. Then the singularities of the hypergeometric system are all regular. If we assume in addition

that the diagonal entries of A are not integers, there exist *characteristic constants* c_{jk} so that

$$y_k(t) \;=\; c_{jk}\, y_j(t) + \mathrm{reg}(t-\lambda_j)\,,\; 1 \le j,k \le n\,, \tag{2}$$

for certain solutions $y_1(t),\ldots,y_n(t)$ of (1), where $y_j(t)$ is singular at λ_j, $1 \le j \le n$, and $\mathrm{reg}(t)$ denotes an arbitrary function which is holomorphic near the origin.

Because of the irregular singularity at infinity, the confluent system is rather more interesting than the hypergeometric system. In particular, one wants to obtain the Stokes' Multipliers which describe the change in behavior of its solutions at infinity. However, by means of the intimate relation to the hypergeometric system, it does not matter, which of the both systems we investigate, since results for one of them can be carried over to analogous results for the other one. Particularly, the characteristic constants c_{jk} and the Stokes' Multipliers are related so intimately that the knowledge of the constants c_{jk} is equivalent to the one of the Stokes' Multipliers.

From equation (2) it follows directly that $c_{jj} = 1$, for all $j = 1,\ldots,n$. Since we can interchange rows and columns of the matrices Λ and A by simple transformations of the solutions, we obtain that it suffices to determine just one of the constants c_{jk} for $j \neq k$ as a function of Λ and A. If we know, for example, c_{n1} in terms of Λ and A, any other constant c_{jk} $(j \neq k)$ then can be expressed by the same formula depending on $\tilde{\Lambda}$ and $\tilde{A}$, where $\tilde{\Lambda}$, resp. $\tilde{A}$, arises after interchanging the first and the k-th, as well as the n-th and the j-th row and column of Λ, resp. A. Because of that, we will concentrate on c_{n1}, when we consider a general hypergeometric system with distinct eigenvalues. It is an interesting fact that if $\hat{\Lambda}$ and $\hat{A}$ are obtained after interchanging any two rows and columns of Λ and A, which are neither the first nor the n-th ones, causes that c_{n1} remains to be the same, that means $c_{n1}(\Lambda, A) = c_{n1}(\hat{\Lambda}, \hat{A})$. In this sense, $c_{n1}(\Lambda, A)$ can be regarded as a symmetric function in some of the variables.

As we mentioned above, the solutions of a 2-dimensional hypergeometric system can be expressed by means of the hypergeometric function. Furthermore, the corresponding characteristic constants can be represented in terms of the *Gamma function*. Both functions are well-known and characterized by a differential resp. difference equation and several further properties. It would be desirable to find and characterize such a function which can be used to express the solutions resp. constants in each higher dimension. It has been shown in [7] that such a function exists for the solutions, and it has been given by means of a power series. Furthermore, this function has been represented as solution of an integral equation. However, it is still possible that there might be a more natural choice for such a function. For the characteristic constants the situation is different, since no corresponding choice for such a function has been made so far, although it is clear from above that *one* such choice can be $c_{n1}(\Lambda, A)$.

In this thesis, the solutions and characteristic constants will be given as power series in a newly introduced parameter ε and their coefficients will be characterized. More precisely, these coefficients can be expressed in terms of functions which will be introduced and analyzed at the beginning of the work. Hence we will name these functions *coefficient functions*. The introduction of the new parameter is based on the author's diploma thesis [22]. Thereby the upper triangular part of A (except of the diagonal entries) will be multiplied with ε. Observe that this coincides with the choice of investigating the constant c_{n1}. Then both the solutions and the constants are entire functions in ε, and therefore the radius of convergence of the considered power series is infinity

The method to regard the solutions and constants as functions of several entries of A, has already been used by *W. Balser* in [1] and [2]. There he expressed the solutions as power series in $n(n-1)$ variables. The advantage of the approach in this thesis is that there always appear power series in only one variable, and this leads to simpler formulas.

This work is organized as follows:

In Chapter 1, we define the *coefficient functions of the first kind* f_k for $k \geq 0$, where each function f_k depends on variables $t, \lambda_1, \ldots, \lambda_k$ and $l_1, \ldots, l_k$. Here we make certain restrictions on t and λ_j ($j \in \mathbb{N}$) to assure the existence of the functions. In particular, we assume $\lambda_j \neq 0$ and $t \neq \lambda_j$ for all j. Later we will analyze the behavior of f_k when some of the values λ_j tend to zero resp. when t tends to λ_j. This is needed to show the relation to the coefficient functions of the second kind and to express the solutions of a hypergeometric system in terms of these functions. It follows directly that each f_k solves an inhomogeneous differential equation (of order 1) with respect to t. Later we obtain that it also solves a homogeneous one of order $k+1$. Furthermore, we will show that f_k solves a difference equation with respect to l_j, for all $j = 1, \ldots, k$, and thereof we derive differential equations with respect to λ_j, again for all $j = 1, \ldots, k$.

In the next section of Chapter 1, we introduce - under certain assumptions to λ_j and l_j - the *coefficient functions of the second kind* d_k for $k \geq 1$, where each coefficient function d_k depends on $\lambda_1, \ldots, \lambda_k$ and $l_1, \ldots, l_k$ but not on t. Again we obtain for each function d_k difference equations with respect to l_j, $j = 1, \ldots, k$. Furthermore, we show that the functions f_k and d_k are intimately related so that it suffices to know one of them. By means of this relation, we derive for the function d_k differential equations with respect to λ_j, $j = 1, \ldots, k$, too.

In the last section of the first chapter, we characterize the coefficient function d_k (and thereof f_k) as the single solution of the difference equations with respect to l_j that has certain additional properties.

In the second chapter, the hypergeometric and the confluent hypergeometric system of differential equations will be introduced. Moreover, we will consider two scalar transformations which allows us to choose one of the diagonal entries of both Λ and A arbitrarily.

Chapter 3 is about hypergeometric systems in triangular form. That means the matrix A is always assumed to be a lower triangular matrix. First we assume in addition that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of Λ are distinct. In the first section, we further restrict the matrix A to have a very special form. Then we will give solutions in this case by means of the coefficient functions (of the first kind) f_k and the scalar transformations which have been introduced in Chapter 2. Thereof we obtain formulas for the constants c_{jk}. In these formulas, c_{jk} will be given in terms of the coefficient functions (of the second kind) d_k. In the next section, we consider the system with a general triangular matrix A. The solutions and constants then can be represented as linear combinations of the solutions resp. constants for the special triangular system.

The last section in this chapter is about triangular systems with multiple eigenvalues. Under some restrictions, the solutions can be expressed by the same formulas as in the case of distinct eigenvalues.

In Chapter 4, we will finally be concerned with the hypergeometric system in general form, but we assume again that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct. Then we will introduce the new parameter ε and consider the solutions and the characteristic constant c_{n1} as power series in this parameter. With the results from Section 3.3, we will be able to give formulas for their coefficients in terms of the coefficient functions of the first resp. second kind.

Chapter 1

Coefficient functions

In this chapter, we will introduce and discuss two recursively defined sequences of functions f_k and d_k which are functions in several variables. Since we cannot write all of these functions in explicit form by means of already known functions, the functions seem to be higher transcendental functions. Hence our aim will be to characterize these functions as much as possible, e. g., by means of differential and difference equations.

In the first section, we will define the functions f_k. First these functions are considered in one variable t, but later they will be considered as functions in several variables, namely $t, \lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k$. Then we will discuss the holomorphy of these functions. As function in t, each f_k solves an inhomogeneous differential equation of order 1. We will see that it also solves difference equations with respect to the variable l_j, $j = 1, \ldots, k$. From these difference equations we will obtain differential equations with respect to λ_j, $j = 1, \ldots, k$. Furthermore, we will derive a homogeneous differential equation of order $k + 1$, for which f_k is a solution as function in t. Finally we will discuss the analytic behavior of f_k in some cases which will arise in the remaining chapters.

The functions d_k then will be introduced in the second section. These functions do not depend on t but on $\lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k$. Again we will discuss the holomorphy of the functions and then derive difference equations with respect to l_j, $j = 1, \ldots, k$, which are solved by d_k.

We will see in the third section that there is a close relation between the functions f_k and d_k, and thereof we obtain that d_k solves differential equations with respect to λ_j, $j = 1, \ldots, k - 1$, too.

In the last section of this chapter, we characterize the coefficient function d_k as the single solution of the difference equations with respect to l_j that has certain additional properties.

The motivation of investigating these functions will become clear in the following chapters, since it will be analyzed there how solutions of the so-called *hypergeometric* and *confluent hypergeometric system of differential equations* can be expressed in terms of the functions f_k, and the corresponding characteristic constants resp. *Stokes' Multipliers* in terms of the functions d_k. More precisely there exist power series expansions, and their coefficients are represented by means of these functions. Because of this, we will call the functions f_k and d_k *coefficient functions of the first resp. second kind.*

1.1 Coefficient functions of the first kind

In this section, we will introduce the *coefficient functions of the first kind* f_k. First these functions are studied as functions in one variable, but later they will be considered as functions in several variables $t, \lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k$. Then the holomorphy of the functions will be analyzed. Afterwards, we will derive several difference equations for them with respect to l_j, $j = 1, \ldots, k$ and thereof differential equations with respect to λ_j, $j = 1, \ldots, k$. Furthermore, we will consider the analytic behavior of f_k in some special cases, since this is needed to express solutions of the already mentioned hypergeometric system of differential equations in terms of these functions.

1.1.1 Definition

Let G be a bounded region which is starshaped with respect to the origin. By G^c we denote the complement of G in $\mathbb{C}$ (i.e. $G^c = \mathbb{C} \setminus G$) and by $\mathring{G}^c$ the interior of G^c. Then for an arbitrary, but fixed $u \in \mathring{G}^c$, we make a chioice for the argument of u and determine the branch of the logarithm of $(u - z)$ for all $z \in G$ so that

$$\log(u - z)\big|_{z=0} := \log(u)\,.$$

This is possible, since G is a simply connected region.

For $t \in G, \lambda_j \in \mathring{G}^c, l_j \in \mathbb{C}, j = 1, 2, \ldots$, let the coefficient functions (of the first kind) $f_k(t)$ for $k \geq 0$ be recursively defined by

$$\begin{aligned} f_0(t) &\equiv 1\,, \\ f_k(t) &= (\lambda_k - t)^{-l_k} \int_0^t (\lambda_k - w)^{l_k - 1} f_{k-1}(w)\, dw\,,\ k \geq 1\,, \end{aligned} \tag{1.1}$$

where we take the integrals along a straight line joining 0 to t. Of course, according to Cauchy's integral formula, we can also choose an arbitrary path from 0 to t which lies inside of G. The branch of the logarithm of $(\lambda_j - t)$ shall be chosen

according to the definition above. However, observe that in view of the function f_k, it does not matter which branch we choose. This follows from the fact that choosing another branch causes that $(\lambda_j - t)^{-l_j}$ carries over to $(\lambda_j - t)^{-l_j} \cdot e^{-2\mu\pi i l_j}$ (for $\mu \in \mathbb{Z}$) and on the other side the term $(\lambda_j - w)^{l_j - 1}$ will be multiplied with $e^{2\mu\pi i l_j}$ for arbitrary $t, w \in G$.

Remark 1: Since we are integrating from 0 to t, $f_k(0) = 0$ for all $k \geq 1$, and the order of the zero equals k. □

Remark 2: For $k = 1$, the function f_k can be explicitly computed:

$$f_1(t) \;=\; \begin{cases} -\frac{1}{l_1}\left(1 - (1 - t/\lambda_1)^{-l_1}\right) & , \quad l_1 \neq 0 \\ -\log(1 - t/\lambda_1) & , \quad l_1 = 0 \end{cases},$$

where we choose the principle value of $\log(1 - t/\lambda_1)$. Considering this function as function in l_1, it is clear that f_1 is holomorphic in $\mathbb{C} \setminus \{0\}$. Because of

$$-\frac{1}{l_1}\left(1 - (1 - t/\lambda_1)^{-l_1}\right) \;=\; \frac{(1 - t/\lambda_1)^{-l_1} - 1}{l_1}$$

$$\stackrel{(l_1 \to 0)}{\longrightarrow} \quad \frac{\partial}{\partial l_1} e^{-l_1 \log(1 - t/\lambda_1)} \Big|_{l_1 = 0} \;=\; -\log(1 - t/\lambda_1)\,,$$

it follows that f_1 is continuous in $l_1 = 0$. From the *Riemann removable singularities theorem* (see appendix B) we obtain now that f_1 is holomorphic in $l_1 = 0$, i.e. it is an entire function in l_1. As we will see in the following lemma, this statement even holds for arbitrary $k \in \mathbb{N}$, that means f_k is an entire function in l_k for each k. □

Lemma 1.1
For all $k \geq 1$, the function f_k is holomorphic in $G \times (\mathring{G}^c)^k \times \mathbb{C}^k$ as function in several variables.

Proof: It is obvious that $f_k(t)$ is holomorphic in G (as function in t) for all $k \geq 0$. Furthermore, if we consider the function f_k as function in λ_j, for an arbitrary $j = 1, \ldots, k$, it is holomorphic in $\mathring{G}^c$. This follows from the fact that in this case the branch points of the function are on the line from 0 to t which lies in G, and $f_k(\lambda_j e^{2\pi i}) = f_k(\lambda_j)$ for all $\lambda_j \in \mathring{G}^c$, since all the choices of the branch of the logarithm were equivalent like we explained above. Because of the same reason, f_k is continuos for $\lambda_j \in G^c$. Moreover, for each $k \geq 1$ and $j = 1, \ldots, k$, the function f_k is an entire function in l_j. To obtain this, we observe that f_k is an entire function in l_j for $j = 1, \ldots, k-1$, if the same holds for f_{k-1}. So it suffices to show that f_k is an entire function in l_k for all k. The case $k = 1$ has been shown in the previous remark. For an arbitrary $k \geq 1$, f_{k-1} does not depend on l_k, and

since $(\lambda_k - w)^{l_k-1} = e^{(l_k-1)\log(\lambda_k-w)}$, it follows that $\int_0^t (\lambda_k - w)^{l_k-1} f_{k-1}(w)\,dw$ is an entire function in l_k. Because of $(\lambda_k - t)^{-l_k} = e^{-l_k \log(\lambda_k-t)}$ we finally have that f_k is an entire function in l_k, too. Now the statement of the lemma follows by means of *Hartogs' theorem* (see appendix B). □

Remark 3: Later we will see that each f_k is a solution of a differential equation and therefore each vector function $(f_0(t), \ldots, f_n(t))$ $(n \in \mathbb{N})$ solves a homogeneous system of differential equations. Because of $f_k(0) = 0$ for all k, the statement of the previous lemma then follows by means of [27], Theorem 24.1, too. □

In the following, the function f_k shall be considered as function in several variables $t, \lambda_1, \ldots, \lambda_k$ and $l_1, \ldots, l_k$. According to the previous lemma, if we do not say something else, we always consider f_k in $G \times (\mathring{G}^c)^k \times \mathbb{C}^k$. Furthermore, we also use the notation $f_k(t; \lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k)$, and we constitute that f_k' always denotes the derivative of f_k with respect to t.

1.1.2 Difference equations

It can be checked directly that the functions f_k solve the following differential equations with respect to t:

$$\begin{aligned} f_0' &= 0\,, \\ (\lambda_k - t)\, f_k' &= l_k f_k + f_{k-1}\,,\ k \geq 1\,. \end{aligned} \tag{1.2}$$

By means of these equations we will derive several difference equations with respect to l_j, $j = 1, \ldots, k$, in this subsection. These difference equations will be used in the next subsection to obtain differential equations with respect to λ_j, $j = 1, \ldots, k$.

Remark 4: To simplify the representation of the later formulas we will introduce the following notation:

By ℓ we denote the k-tupel $\ell := (l_1, \ldots, l_k)$, while λ is the abbreviation of $\lambda := (\lambda_1, \ldots, \lambda_k)$. Furthermore, for an arbitrary k-tupel $v = (v_1, \ldots, v_k)$ we define

$$v^{(j)} := (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k)\,,\ 1 \leq j \leq k\,,$$

$$v \pm e_\mu := (v_1, \ldots, v_{\mu-1}, v_\mu \pm 1, v_{\mu+1}, \ldots, v_k)\,,\ 1 \leq \mu \leq k\,,$$

and

$$v \pm e := (v_1 \pm 1, \ldots, v_k \pm 1)\,.$$

According to this definition, for $1 \leq j \neq \mu \leq k$ the term $(v + e_\mu)^{(j)}$ denotes the $(k-1)$-tupel

$$(v_1, \ldots, v_{\mu-1}, v_\mu + 1, v_{\mu+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k) \qquad (j > \mu)$$

resp.

$$(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{\mu-1}, v_\mu + 1, v_{\mu+1}, \ldots, v_k) \qquad (j < \mu)\,.$$

□

Theorem 1.1
For all $k \geq 2$, the function f_k is a solution of the following difference equation with respect to l_k:

$$\begin{aligned}(l_k - l_{k-1} + 1)\,(\lambda_k - t)\,f_k(t;\lambda,\ell+e_k) - l_k\,(\lambda_k - \lambda_{k-1})\,f_k(t;\lambda,\ell) \\ = \;(\lambda_k - t)\,f_{k-1}(t;\lambda^{(k-1)},(\ell+e_k)^{(k-1)}) - (\lambda_{k-1} - t)\,f_{k-1}(t;\lambda^{(k)},\ell^{(k)})\,.\end{aligned} \tag{1.3}$$

Proof: With the identity $(\lambda_k - w) = (\lambda_k - \lambda_{k-1} + \lambda_{k-1} - w)$ we get from (1.1)

$$\begin{aligned}l_k\,(\lambda_k - t)\,f_k(t;\lambda,\ell+e_k) \;&=\; l_k\,(\lambda_k - \lambda_{k-1})\,f_k(t;\lambda,\ell) \\ &\quad + l_k\,(\lambda_k - t)^{-l_k}\int_0^t (\lambda_k - w)^{l_k-1}\,(\lambda_{k-1} - w)\,f_{k-1}(w;\lambda^{(k)},\ell^{(k)})\,dw\,.\end{aligned}$$

Thereof, integration by parts and then using equation (1.2) lead to

$$\begin{aligned}&l_k\,(\lambda_k - t)\,f_k(t;\lambda,\ell+e_k) - l_k\,(\lambda_k - \lambda_{k-1})\,f_k(t;\lambda,\ell) \\ &\quad= -(\lambda_{k-1} - t)\,f_{k-1}(t;\lambda^{(k)},\ell^{(k)}) + (l_{k-1} - 1)\,(\lambda_k - t)\,f_k(t;\lambda,\ell+e_k) \\ &\qquad + (\lambda_k - t)\,f_{k-1}(t;\lambda^{(k-1)},(\ell+e_k)^{(k-1)})\,,\end{aligned}$$

which is equivalent to the statement. □

Remark 5: In the previous theorem, the case $k = 1$ did not appear. However, in the following lemma, the case $k = 1$ leads to a difference equation for f_1 with respect to l_1. □

Lemma 1.2
For all $k \geq 1$, the following equation is valid:

$$\begin{aligned}(l_1 + 1)\,(\lambda_k - t)\,f_k(t;\lambda,\ell+e) - \lambda_1\,l_k\,f_k(t;\lambda,\ell) \;&= \\ \lambda_1\,f_{k-1}(t;\lambda^{(k)},\ell^{(k)}) - \;(\lambda_k - t)\,f_{k-1}(t;\lambda^{(1)},(\ell+e)^{(1)})\,.&\end{aligned} \tag{1.4}$$

Proof: The proof will be made by induction with respect to k. For $k = 1$ and $l_1 \neq 0, -1$ the equation is valid because of $f_1(t;\lambda_1,l_1) = -\frac{1}{l_1} + \frac{\lambda_1^{l_1}(\lambda_1 - t)^{-l_1}}{l_1}$. If $l_1 = 0$, the equation above is given by

$$f_1(t;\lambda_1,1) \;=\; \lambda_1\,(\lambda_1 - t)^{-1}\,f_0(t) - f_0(t)\,,$$

which is true, since both sides are equal to $\lambda_1(\lambda_1 - t)^{-1} - 1$ and if $l_1 = -1$, we have to prove

$$\lambda_1 \, (\lambda_1 - t)^{-1} \big(\, - f_1(t; \lambda_1, -1) + 1 \big) - 1 \; = \; 0 \, ,$$

but this follows directly after inserting the explicit formula for f_1. If the equation holds for $k-1$ for some $k \geq 2$, inserting the hypothesis into (1.1) and then using the differential equation (1.2) lead to

$$\begin{aligned} (l_1 + 1) \, (\lambda_k - t) \, f_k(t; \lambda, \ell + e) \; = \\ (\lambda_k - t)^{-l_k} \Bigg(\lambda_1 \int_0^t (\lambda_k - w)^{l_k} \, f'_{k-1}(w; \lambda^{(k)}, l^{(k)}) \, dw \\ - \int_0^t (\lambda_k - w)^{l_k} \, f_{k-2}(w; \lambda^{(1,k)}, (\ell + e)^{(1,k)}) \, dw \Bigg) \, , \end{aligned}$$

where $\lambda^{(1,k)} := (\lambda_2, \ldots, \lambda_{k-1})$ and $(\ell + e)^{(1,k)} := (l_2 + 1, \ldots, l_{k-1} + 1)$. From this we get the statement via integration by parts. □

Together with the following lemma, equation (1.4) can be used to obtain other difference equations where just one of the values l_j is increasing.

Lemma 1.3
For all $2 \leq j \leq k$, we have:

$$\begin{aligned} &(l_j - l_{j-1} + 1) \, (\lambda_k - t) \, f_k(t; \lambda, \ell + e_j + \ldots + e_k) - (\lambda_j - \lambda_{j-1}) \, l_k \, f_k(t; \lambda, \ell) \\ &= \; (\lambda_j - \lambda_{j-1}) \, f_{k-1}(t; \lambda^{(k)}, \ell^{(k)}) - (\lambda_k - t) \, f_{k-1}(t; \lambda^{(j)}, (\ell + e_{j+1} + \ldots + e_k)^{(j)}) \\ &\quad + (\lambda_k - t) \, f_{k-1}(t; \lambda^{(j-1)}, (\ell + e_j + \ldots + e_k)^{(j-1)}) \, . \end{aligned} \tag{1.5}$$

Proof: We will show the statement by a "backward induction". That means, we substitute $j = k - m$ and prove the statement by induction with respect to m. For $m = 0$, the statement equals (1.3) and therefore is true. Now we assume that for some $m = k - j \geq 1$ the equation is valid for $m - 1$. Inserting the hypothesis into (1.1) and then using the differential equation (1.2), we get

$$\begin{aligned} (l_j - l_{j-1} + 1) \, (\lambda_k - t) \, f_k(t; \lambda, \ell + e_j + \ldots + e_k) \; = \\ (\lambda_j - \lambda_{j-1}) \, (\lambda_k - t)^{-l_k} \int_0^t (\lambda_k - w)^{l_k} \, f'_{k-1}(w; \lambda^{(k)}, \ell^{(k)}) \, dw \\ - (\lambda_k - t) \, f_{k-1}(t; \lambda^{(j)}, (\ell + e_{j+1} + \ldots + e_k)^{(j)}) \\ + (\lambda_k - t) \, f_{k-1}(t; \lambda^{(j-1)}, (\ell + e_j + \ldots + e_k)^{(j-1)}) \end{aligned}$$

and via integration by parts we complete the proof. □

From the previous difference equations we obtain the following equations so that we finally can write f_k as solution of a difference equation with respect to l_j for all $1 \leq j \leq k$.

Theorem 1.2
For all $k \geq 2$, f_k solves the difference equation with respect to l_1:

$$\begin{aligned}(\lambda_2 - \lambda_1)\,(l_1+1)\,f_k(t;\lambda,\ell+e_1) \; &- \lambda_1\,(l_2-l_1)\,f_k(t;\lambda,\ell) \\ &= \; \lambda_1\,f_{k-1}(t;\lambda^{(2)},\ell^{(2)}) - \lambda_2\,f_{k-1}(t;\lambda^{(1)},\ell^{(1)})\end{aligned} \tag{1.6}$$

and for $2 \leq j \leq k-1$, the difference equation with respect to l_j:

$$\begin{aligned}&(\lambda_{j+1}-\lambda_j)\,(l_j - l_{j-1}+1)\,f_k(t;\lambda,\ell+e_j) \; - (\lambda_j-\lambda_{j-1})\,(l_{j+1}-l_j)\,f_k(t;\lambda,\ell) \\ &= (\lambda_j-\lambda_{j-1})\,f_{k-1}(t;\lambda^{(j+1)},\ell^{(j+1)}) - (\lambda_{j+1}-\lambda_{j-1})\,f_{k-1}(t;\lambda^{(j)},\ell^{(j)}) \\ &\quad + (\lambda_{j+1}-\lambda_j)\,f_{k-1}(t;\lambda^{(j-1)},(\ell+e_j)^{(j-1)})\,.\end{aligned} \tag{1.7}$$

Proof: From equation (1.4) we get for an arbitrary $k \geq 2$

$$\begin{aligned}(\lambda_2-\lambda_1)\,(l_1+1)\,f_k(t;\lambda,\ell+e_1) \; &= (\lambda_1-\lambda_2)\,f_{k-1}(t;\lambda^{(1)},\ell^{(1)}) \\ &\lambda_1\,(\lambda_k-t)^{-1}\,(\lambda_2-\lambda_1)\,(l_k-1)\,f_k(t;\lambda,\ell-e_2-\ldots-e_k) \\ &+ \lambda_1\,(\lambda_k-t)^{-1}\,(\lambda_2-\lambda_1)\,f_{k-1}(t;\lambda^{(k)},(\ell-e_2-\ldots-e_k)^{(k)})\end{aligned} \tag{1.8}$$

and from equation (1.5) (with $j=2$) we have

$$\begin{aligned}&\lambda_1\,(\lambda_k-t)^{-1}\,(\lambda_2-\lambda_1)\,(l_k-1)\,f_k(t;\lambda,\ell-e_2-\ldots-e_k) \; = \\ &\qquad \lambda_1\,(l_2-l_1)\,f_k(t;\lambda,\ell) + \lambda_1\,f_{k-1}(t;\lambda^{(2)},\ell^{(2)}) - \lambda_1\,f_{k-1}(t;\lambda^{(1)},\ell^{(1)}) \\ &\qquad - \lambda_1\,(\lambda_k-t)^{-1}\,(\lambda_2-\lambda_1)\,f_{k-1}(t;\lambda^{(k)},(\ell-e_2-\ldots-e_k)^{(k)})\,.\end{aligned} \tag{1.9}$$

Now, inserting (1.9) in (1.8) directly leads to equation (1.6). Furthermore, from equation (1.5) it follws that

$$\begin{aligned}&(\lambda_{j+1}-\lambda_j)\,(l_j-l_{j-1}+1)\,f_k(t;\lambda,\ell+e_j) \; = \\ &\quad (\lambda_j-\lambda_{j-1})\,(\lambda_k-t)^{-1}\,(\lambda_{j+1}-\lambda_j)\,(l_k-1)\,f_k(t;\lambda,\ell-e_{j+1}-\ldots-e_k) \\ &\quad - (\lambda_{j+1}-\lambda_j)\,f_{k-1}(t;\lambda^{(j)},\ell^{(j)}) + (\lambda_{j+1}-\lambda_j)\,f_{k-1}(t;\lambda^{(j-1)},(\ell+e_j)^{(j-1)}) \\ &\quad + (\lambda_j-\lambda_{j-1})\,(\lambda_k-t)^{-1}\,(\lambda_{j+1}-\lambda_j)\,f_{k-1}(t;\lambda^{(k)},(\ell-e_{j+1}-\ldots-e_k)^{(k)})\end{aligned}$$

for all $2 \leq j \leq k-1$. On the other hand, we obtain by the same equation

$$\begin{aligned}(\lambda_j-\lambda_{j-1})\,(\lambda_k-t)^{-1}\,(\lambda_{j+1}-\lambda_j)\,(l_k-1)\,f_k(t;\lambda,\ell-e_{j+1}-\ldots-e_k) \;=\\ (\lambda_j-\lambda_{j-1})\,\Big(\,(l_{j+1}-l_j)\,f_k(t;\lambda,\ell)+f_{k-1}(t;\lambda^{(j+1)},\ell^{(j+1)})-f_{k-1}(t;\lambda^{(j)},\ell^{(j)})\,\Big)\\ -\,(\lambda_j-\lambda_{j-1})\,(\lambda_k-t)^{-1}\,(\lambda_{j+1}-\lambda_j)\,f_{k-1}(t;\lambda^{(k)},(\ell-e_{j+1}-\ldots-e_k)^{(k)})\,,\end{aligned}$$

and these two identities imply equation (1.7). □

1.1.3 Differential equations

We already know that each coefficient function f_k is a solution of a differential equation with respect to t. We will now see that it also solves differential equations with respect to λ_j, for all $j = 1,\ldots,k$. To obtain them, we use the difference equations from the previous subsection.

Lemma 1.4
For all $1 \leq j \leq k$, f_k is a solution of the following differential-difference equation:

$$\frac{\partial}{\partial\lambda_j}\,f_k(t;\lambda,\ell) \;=\; f_k'(t;\lambda,\ell-e_j-\ldots-e_k)-f_k'(t;\lambda,\ell-e_{j+1}-\ldots-e_k)\,. \tag{1.10}$$

Proof: Setting $m := k-j$ we will prove the statement by induction with respect to m. Since f_{k-1} does not depend on λ_k, we have

$$\begin{aligned}\frac{\partial}{\partial\lambda_k}\,f_k(t;\lambda,\ell) \;&=\; -\,l_k\,(\lambda_k-t)^{-l_k-1}\int_0^t(\lambda_k-w)^{l_k-1}\,f_{k-1}(w;\lambda^{(k)},\ell^{(k)})\,dw\\ &\quad+\,(l_k-1)\,(\lambda_k-t)^{-l_k}\int_0^t(\lambda_k-w)^{l_k-2}\,f_{k-1}(w;\lambda^{(k)},\ell^{(k)})\,dw\\ &=\;-\,\frac{l_k}{\lambda_k-t}\,f_k(t;\lambda,\ell)+\frac{l_k-1}{\lambda_k-t}\,f_k(t;\lambda,\ell-e_k)\end{aligned}$$

and via (1.2) the statement for $m = 0$. Assuming that the formula is true for $m-1$ ($1 \leq m \leq k-1$), after inserting the hypothesis into (1.1) and integration by parts, we get the statement using again equation (1.2). □

By means of the differential-difference equation (1.10) and several of the difference equations from the previous subsection, we obtain the following theorem:

Theorem 1.3
For $k \geq 2$, f_k is a solution of the differential equation with respect to λ_k:

$$\begin{aligned}\frac{\partial}{\partial\lambda_k} f_k(t;\lambda,\ell) \;&=\; \left(\frac{l_k - l_{k-1}}{\lambda_k - \lambda_{k-1}} - \frac{l_k}{\lambda_k - t}\right) f_k(t;\lambda,\ell) \\ &+ \frac{\lambda_{k-1} - t}{(\lambda_k - t)(\lambda_k - \lambda_{k-1})}\, f_{k-1}(t;\lambda^{(k)},\ell^{(k)}) - \frac{1}{\lambda_k - \lambda_{k-1}}\, f_{k-1}(t;\lambda^{(k-1)},\ell^{(k-1)})\,,\end{aligned} \tag{1.11}$$

and of the one with respect to λ_1:

$$\begin{aligned}\frac{\partial}{\partial\lambda_1} f_k(t;\lambda,\ell) \;&=\; \left(\frac{l_1}{\lambda_1} - \frac{l_2 - l_1}{\lambda_2 - \lambda_1}\right) f_k(t;\lambda,\ell) \\ &+ \frac{\lambda_2}{(\lambda_2 - \lambda_1)\lambda_1}\, f_{k-1}(t;\lambda^{(1)},\ell^{(1)}) - \frac{1}{\lambda_2 - \lambda_1}\, f_{k-1}(t;\lambda^{(2)},\ell^{(2)})\,.\end{aligned} \tag{1.12}$$

Furthermore, for $2 \leq j \leq k-1$, it solves the following differential equation with respect to λ_j:

$$\begin{aligned}\frac{\partial}{\partial\lambda_j} f_k(t;\lambda,\ell) \;&=\; \left(\frac{l_j - l_{j-1}}{\lambda_j - \lambda_{j-1}} - \frac{l_{j+1} - l_j}{\lambda_{j+1} - \lambda_j}\right) f_k(t;\lambda,\ell) \\ &+ \frac{\lambda_{j+1} - \lambda_{j-1}}{(\lambda_{j+1} - \lambda_j)(\lambda_j - \lambda_{j-1})}\; f_{k-1}(t;\lambda^{(j)},\ell^{(j)}) \\ &- \frac{1}{\lambda_j - \lambda_{j-1}}\; f_{k-1}(t;\lambda^{(j-1)},\ell^{(j-1)}) \;- \frac{1}{\lambda_{j+1} - \lambda_j}\, f_{k-1}(t;\lambda^{(j+1)},\ell^{(j+1)})\,.\end{aligned} \tag{1.13}$$

Proof: By means of the differential equation (1.2), we get from equation (1.5) for all $2 \leq j \leq k$:

$$\begin{aligned}(\lambda_j - \lambda_{j-1})\, f_k'(t;\lambda,\ell - e_j - \ldots - e_k) \;&=\; \\ &(l_j - l_{j-1})\, f_k(t;\lambda,\ell) + f_{k-1}(t;\lambda^{(j)},\ell^{(j)}) - f_{k-1}(t;\lambda^{(j-1)},\ell^{(j-1)})\,.\end{aligned}$$

Furthermore, the same differential equation and equation (1.4) lead to

$$\lambda_1\, f_k'(t;\lambda,\ell - e) \;=\; l_1\, f_k(t;\lambda,\ell) + f_{k-1}(t;\lambda^{(1)},\ell^{(1)})\,,$$

for all $k \geq 1$. Then we obtain the statement after inserting these two identities into the differential-difference equation (1.10). □

1.1.4 Homogeneous differential equation of higher order

In this subsection, we will show that each coefficient function f_k is not only a solution of an inhomogeneous differential equation of order 1 as function in t, but also a solution of a homogeneous differential equation of order $k+1$.

Theorem 1.4
Let δ_λ be the differential operator defined by

$$\delta_\lambda := (\lambda - t)\,\frac{d}{dt}\,.$$

Then for $k \geq 1$, f_k solves the differential equation

$$\frac{d}{dt}\,(\delta_{\lambda_1} - l_1)(\delta_{\lambda_2} - l_2)\ldots(\delta_{\lambda_k} - l_k)\,f_k \;=\; 0\,, \tag{1.14}$$

with initial values $f_k^{(k)}(0) = (\lambda_1 \cdot \ldots \cdot \lambda_k)^{-1}$ and $f_k^{(j)}(0) = 0$, $0 \leq j \leq k-1$.

Proof: Since for all $k \geq 1$, the differential equation (1.2) is equivalent to $(\delta_{\lambda_k} - l_k)\,f_k \;=\; f_{k-1}$, it follows iteratively that

$$(\delta_{\lambda_1} - l_1)(\delta_{\lambda_2} - l_2)\ldots(\delta_{\lambda_k} - l_k)\,f_k \;=\; f_0 \;\equiv\; 1\,,$$

and derivation of both sides then leads to equation (1.14). To get the initial values, we observe that $f_k(0) = 0$, for all $k \geq 1$ (see Remark 1). Furthermore, after repeated differentiation of (1.2), we get for arbitrary $j \in \mathbb{N}_0$ and $k \geq 1$

$$(\lambda_k - t)\,f_k^{(j+1)} \;=\; (l_k + j) f_k^{(j)} + f_{k-1}^{(j)}\,,$$

and, in particular,

$$f_k^{(j+1)}(0) \;=\; \frac{l_k + j}{\lambda_k}\,f_k^{(j)}(0) + \frac{1}{\lambda_k}\,f_{k-1}^{(j)}(0)\,.$$

By means of this equation, the initial values specified above follow by a simple induction. □

Remark 6: Because of

$$\frac{d}{dt}\,(\delta_\lambda - l)\,f \;=\; (\lambda - t)f'' - f' - lf' \;=\; (\delta_\lambda - l - 1)\,\frac{d}{dt}\,f\,,$$

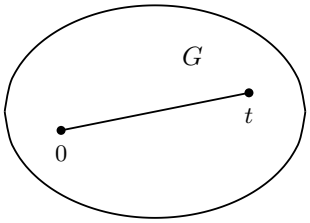
the differential equation (1.14) is equivalent to

$$(\delta_{\lambda_1} - l_1 - 1)(\delta_{\lambda_2} - l_2 - 1)\ldots(\delta_{\lambda_k} - l_k - 1)\,f_k' \;=\; 0\,.$$

□

1.1.5 Global analytic behavior

Considering f_k as function in λ_j, for an arbitrary but fixed $j, 1 \leq j \leq k$, and a fixed $t \in G$, we saw in the first subsection that f_k is holomorphic in $\mathring{G}^c$. Furthermore, for a fixed $t \in \mathbb{C}$ the region G can be arbitrarily chosen as long as it is starshaped with respect to the origin and contains the point t (e.g. like in the figure below).

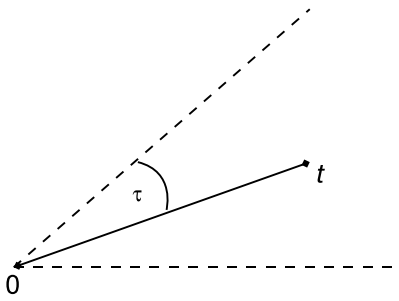


In the following, we do not want to restrict λ_j any longer to be in $\mathring{G}^c$ and in particular we want to know what happens when λ_j tends to the origin. Here we want to avoid the line from 0 to t so that we can still choose the same argument of $(\lambda_j - t)$ like we did at the beginning of this thesis. We just consider the case $\lambda_j \to 0$ if $l_j = 0$, since this is the only case that is needed in the remaining chapters.

Theorem 1.5
Let $l_j = 0$ for some $j \geq 2$. Then for an arbitrary, but fixed $t \in G$, the limit $\lim_{\lambda_j \to 0} f_j(t)$ - where λ_j tends to zero outside of the sector with bisection direction $\varrho := \arg(t)$ and opening 2τ for some small, but fixed $\tau > 0$ - exists and equals

$$f_j^{(\lambda_j=0)}(t) \;:=\; \int_0^t (-w)^{-1}\, f_{j-1}(w)\, dw\,. \tag{1.15}$$

Moreover, the limit $\lim_{\lambda_j \to 0} f_k(t)$ exists for all k.



Proof: We consider the difference

$$f_j(t) - f_j^{(\lambda_j=0)}(t) \;=\; \int_0^t (1 - (1 - \lambda_j/w)^{-1})\, \frac{f_{j-1}(w)}{w}\, dw\,. \tag{1.16}$$

Since λ_j is not in the sector described above, we obtain $-\lambda_j/w = re^{i\varphi}$ with $r = |\lambda_j|/|w|$ and $|\varphi| \leq \pi - \tau$. Hence there exists $\delta > 0$ so that $\cos(\varphi) \geq -1 + \delta$. Then we can estimate the absolute value of $(1 - \lambda_j/w)$ as follows:

$$|1 - \lambda_j/w|^2 \;=\; |1 + re^{i\varphi}|^2 \;=\; 1 + 2r\,\cos(\varphi) + r^2 \;\geq\; (1-r)^2 + 2\delta r\,. \qquad (1.17)$$

Therefore we obtain

$$\left|\, 1 - (1 - \lambda_j/w)^{-1} \,\right|^2 \;=\; \left| \frac{re^{i\varphi}}{1 + re^{i\varphi}} \right|^2 \;\leq\; \frac{r^2}{(1-r)^2 + 2\delta r} \;=:\; K(r)\,. \qquad (1.18)$$

Obviously, $K(r)$ is continuous on $\mathbb{R}_+$, $K(r) \to 1\,(r \to \infty)$ and $K(r) \to 0\,(r \to 0)$, hence it follows that $K(r)$ is bounded on $\mathbb{R}_+$. Now we split the integral from equation (1.16) into

$$\int_0^{\tilde{\eta}} (1-(1-\lambda_j/w)^{-1})\,\frac{f_{j-1}(w)}{w}\,dw + \int_{\tilde{\eta}}^{t} (1-(1-\lambda_j/w)^{-1})\,\frac{f_{j-1}(w)}{w}\,dw \;=:\; I_1 + I_2\,,$$

where $\tilde{\eta} = \eta e^{i\varrho}$, with an arbitrary but fixed $\eta > 0$, and without loss of generality we assume $\eta < 1$. Then for all w on the path of integration of I_2 we have $|w| \geq 1$ and therefore $r \leq |\lambda_j|$. Hence it follows from (1.18) that in this case $(1-(1-\lambda_j/w)^{-1})$ converges uniformly to 0 for $\lambda_j \to 0$ and so we can interchange integral and limit and obtain $I_2 \to 0$ for $\lambda_j \to 0$, and this holds for each $\eta > 0$. Now we consider I_1. Since $f_{j-1}(w)$ is holomorphic on the path of integration and has a zero at the origin, it follows that $f_{j-1}(w)/w$ is holomorphic, too, and therefore the maximum

$$M \;:=\; \max\left\{ \frac{f_{j-1}(w)}{w} :\; w = \alpha e^{i\varrho},\; 0 \leq \alpha \leq |t| \right\}$$

exists. Then because of the boundedness of $K(r)$, there exists a constant $C > 0$ so that

$$|I_1| \;\leq\; \eta\, C\, M\,.$$

Since we can choose an arbtirary small η, we obtain that $I_1 \to 0$, too, and that means the convergence of $f_j(t)$ to $f_j^{(\lambda_j=0)}(t)$. Now we show that

$$\lim_{\lambda_j \to 0} f_{j+1}(t) \;=\; (\lambda_{j+1} - t)^{-l_{j+1}} \int_0^t (\lambda_{j+1} - u)^{l_{j+1}-1}\, f_j^{(\lambda_j=0)}(u)\,du\,.$$

It suffices to show that the convergence of $f_j(u)$ to $f_j^{(\lambda_j=0)}(u)$ is uniform on the path of integration, because then we can again interchange integral and limit. Since I_1 did not depend on t, we just have to consider I_2 with u instead of t. For all u on the path of integration we have $|u| \leq |t|$. Since $(1 - (1 - \lambda_j/w)^{-1})$

converges uniformly to 0, we can estimate the absolute value of this term by an arbitrary small ε. Hence we obtain

$$|I_2| \;\leq\; |u|M\varepsilon \;\leq\; |t|M\varepsilon\,,\ \forall\, u\,,$$

and so the uniform convergence of $f_{j+1}(t)$ for $\lambda_j \to 0$. Analogously, one can show the existence of $\lim_{\lambda_j \to 0} f_k(t)$ for each higher k. □

Remark 7: Observe that $f_j^{(\lambda_j=0)}(0) = 0$ is still valid. However, the order of the zero did not increase, but is the same as the one of f_{j-1}. □

Now we want to analyze how the function f_k behaves (as function in λ_j) when λ_j is no longer restricted to be in $\mathring{G}^C$, but is crossing the line from 0 to t. Here we always assume that $\lambda_j \neq 0, t$. We denote the function $f_k(t)$ by $f_k^{(\lambda_j,+)}(t)$ when λ_j tends to the line from the left-hand side when looking from 0 to t, and when λ_j tends to the line from the other side by $f_k^{(\lambda_j,-)}(t)$. For all $1 \leq j \leq k$, $f_k^{(\lambda_j,+)}(t)$ and $f_k^{(\lambda_j,-)}(t)$ both are solutions of the appropriate inhomogeneous differential equation in Theorem 1.3, that is why the difference $f_k^{(\lambda_j,+)}(t) - f_k^{(\lambda_j,-)}(t)$ solves the corresponding homogeneous differential equation and therefore is of the form

$$f_k^{(\lambda_1,+)}(t) - f_k^{(\lambda_1,-)}(t) = K_1 \cdot \lambda_1^{l_1}\,(\lambda_1 - \lambda_2)^{l_2 - l_1}\,,$$
$$f_k^{(\lambda_k,+)}(t) - f_k^{(\lambda_k,-)}(t) = K_k \cdot (\lambda_k - \lambda_{k-1})^{l_k - l_{k-1}} \cdot (\lambda_k - t)^{-l_k}\,,$$

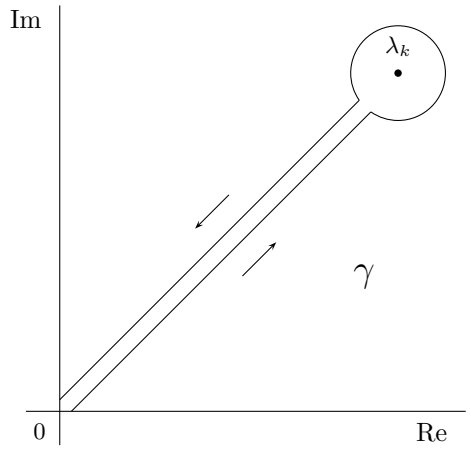
and for $2 \leq j \leq k-1$

$$f_k^{(\lambda_j,+)}(t) - f_k^{(\lambda_j,-)}(t) = K_j \cdot (\lambda_j - \lambda_{j-1})^{l_j - l_{j-1}} \cdot (\lambda_j - \lambda_{j+1})^{l_{j+1} - l_j}$$

with K_j not depending on λ_j, but maybe on t (and the other parameters). For the case $j = k$ we will give one more formula for this difference. According to (1.1), the functions $f_k^{(\lambda_k,+)}(t)$ and $f_k^{(\lambda_k,-)}(t)$ just differ by the choice of the path of integration, and therefore

$$f_k^{(\lambda_k,+)}(t) - f_k^{(\lambda_k,-)}(t) \;=\; (\lambda_k - t)^{-l_k} \int_\gamma (\lambda_k - w)^{l_k - 1}\, f_{k-1}(w)\, dw\,,$$

where γ is the loop contour that begins at the origin, encircles the point λ_k in the positive sense and returns to the origin (see figure below).



Remark 8: Now we specify that if there is a value λ_j on the straight line from 0 to t (but $\lambda_j \neq 0, t$), the path of integration shall be chosen so that λ_j lies on the left side on it (when looking from 0 to t). This means, in this case, we constitute that $f_k(t) = f_k^{(\lambda_j,+)}(t)$. □

Remark 9: According to Theorem 1.5 and the considerations above, we can enlarge the domain of definition for f_k in the following sense: For $j = 1, \ldots, k$, the variable λ_j is permitted to lie in the interior of the straight line segment from 0 to t, and if for some $j \geq 2$, the value l_j equals 0, then λ_j may be equal to 0, too. If nothing else is said, In the following, we will consider this enlarged domain of definition. □

Now we want to analyze, how the functions f_k behave (as function in t) near the values $\lambda_1, \ldots, \lambda_k$. Here we consider the enlarged domain of definition for f_k, and then t shall tend to a value $\lambda_j \in \mathring{G}^c$ along a path which avoids the other values λ_ν. The case $\lambda_j = 0$ is not considered here, since it is already known that f_k is holomorphic at the origin. Furthermore, by $\mathrm{reg}(t)$ we will denote a function which is holomorphic in a neighborhood of the origin.

Theorem 1.6
For an arbitrary, but fixed $\lambda_j \in \mathring{G}^c$ ($j \in \mathbb{N}$), we assume that $l_j \notin -\mathbb{N}_0$ and that there exists no $\nu \in \mathbb{N}$ so that $\lambda_\nu = \lambda_{\nu+1} = \lambda_j$. Furthermore, we suppose that $\lambda_\nu = \lambda_j$ implies $l_\nu = l_j$. Then

$$f_k(t) \;=\; \mathrm{reg}(t - \lambda_j) + (\lambda_j - t)^{-l_j}\, \mathrm{reg}(t - \lambda_j)\;, \quad \forall\, k \geq j\;.$$

Proof: Without loss of generality, we assume that

$$j \;=\; \min\{\, \nu \in \mathbb{N} :\; \lambda_\nu = \lambda_j \,\}\,. \tag{1.19}$$

Now we will prove the statement by induction with respect to k and we will show that for $\lambda_k \neq \lambda_j$ even the following identity holds:

$$f_k(t) \;=\; \operatorname{reg}(t-\lambda_j) + (\lambda_j - t)^{1-l_j} \operatorname{reg}(t-\lambda_j)\,. \tag{1.20}$$

First we consider the case $k = j$. Because of (1.19), it is clear that for all $k < j$, $f_k(t)$ can be analytically continued to λ_j and therefore $f_k(t) = \operatorname{reg}(t-\lambda_j)$ for $k < j$. In particular, $f_{j-1}(w)$ is holomorphic at λ_j, hence there exists a power series expansion like $f_{j-1}(w) = \sum_{n=0}^{\infty} a_n\,(w-\lambda_j)^n$. Inserting this into (1.1) then leads to

$$f_j(t) \;=\; (t-\lambda_j)^{-l_j} \int_0^t \sum_{n=0}^{\infty} a_n\,(w-\lambda_j)^{n+l_j-1}\,dw\,,$$

and thereof we get after termwise integration and observing that $l_j \notin -\mathbb{N}_0$

$$f_j(t) \;=\; \operatorname{reg}(t-\lambda_j) + (\lambda_j - t)^{-l_j} \operatorname{reg}(t-\lambda_j)\,.$$

Now we assume that the statement is true for $k-1$, for some $k \geq j+1$, and we consider the case $\lambda_k \neq \lambda_j$ which implies that $(\lambda_k - w)^{l_k-1} \;=\; \operatorname{reg}(w-\lambda_j)$. From the hypothesis we then obtain

$$\begin{aligned} f_k(t) \;&= \operatorname{reg}(t-\lambda_j) \int_0^t \operatorname{reg}(w-\lambda_j) + (\lambda_j - w)^{-l_j} \operatorname{reg}(w-\lambda_j)\,dw \\ &= \operatorname{reg}(t-\lambda_j) + \operatorname{reg}(t-\lambda_j) \int_0^t \sum_{n=0}^{\infty} a_n\,(w-\lambda_j)^{n-l_j}\,dw\,, \end{aligned}$$

with suitable a_n. Thereof, termwise integration directly leads to (1.20). Now we consider the case $\lambda_k = \lambda_j$ (and $l_k = l_j$). According to the assumptions of the theorem we know that $\lambda_{k-1} \neq \lambda_j$ and so it follows from the previous case that

$$f_{k-1}(t) \;=\; \operatorname{reg}(t-\lambda_j) + (\lambda_j - t)^{1-l_j} \operatorname{reg}(t-\lambda_j)\,.$$

This leads to

$$f_k(t) \;=\; (t-\lambda_j)^{-l_j} \int_0^t \Big(\sum_{n=0}^{\infty} a_n\,(w-\lambda_j)^{n+l_j-1} + \sum_{n=0}^{\infty} b_n\,(w-\lambda_j)^n \Big)\,dw$$

with suitable a_n and b_n, and, once more, we obtain the statement after termwise integration and observing that $l_j \notin -\mathbb{N}_0$. □

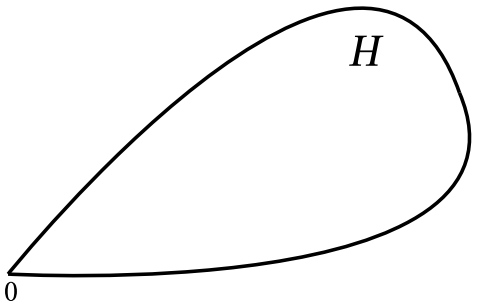
1.2 Coefficient functions of the second kind

In this section, we will introduce the *coefficient functions of the second kind* d_k for $k \geq 1$, where each function d_k depends on the variables $\lambda_1, \ldots, \lambda_k$ and

$l_1, \ldots, l_k$. Again we will discuss their holomorphy and derive difference equations with respect to l_j, while in the following section we will see that there is a close relation to the coefficient functions of the first kind f_k. Thereof we obtain later differential equations with respect to λ_j, too. The meaning of the functions d_k will become clear in the remaining chapters, since it will be shown there that the characteristic constants of a hypergeometric system of differential equations can be expressed by means of these functions.

1.2.1 Definition

Let H be a bounded region so that the origin is a boundary point of H but can be reached from each inner point of H by a straight line which lies in H. Furthermore, we assume that there exists a direction ϱ so that the ray from 0 to infinity along $\arg(z) = \varrho$ is disjoint from H. By $\mathring{H}^c$ we denote the interior of $\mathbb{C} \setminus H$. Observe that the assumptions assure that H is a simply connected region.



From the definition it follows that we can choose the argument of z for an arbitrary $z \in H$ so that $\arg(z) \in (\varrho, \varrho + 2\pi)$. Furthermore, for an arbitrary, but fixed $u \in \mathring{H}^c$, we make a choice for the argument of u and determine the branch of the logarithm of $(u - z)$ for all $z \in H$ so that

$$\lim_{z \to 0} \log(u - z) \;=\; \log(u)\,.$$

Now we want to define the coefficient functions (of the second kind) d_k for $k \geq 1$ recursively by

$$\begin{aligned} d_1(\lambda_1, l_1) \;&=\; \frac{\lambda_1^{l_1}}{l_1}\,, \\ d_k(\lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k) \;&=\; (\lambda_{k-1} - \lambda_k)^{l_k - l_{k-1}} \\ \times \int_0^{\lambda_k} &(\lambda_{k-1} - w)^{l_{k-1} - l_k - 1}\, d_{k-1}(\lambda_1, \ldots, \lambda_{k-2}, w, l_1, \ldots, l_{k-2}, l_k)\, dw\,,\; k \geq 2\,, \end{aligned} \tag{1.21}$$

where for an arbitrary, but fixed $k \geq 1$, we assume that $\lambda_k \in H$, $\lambda_1, \ldots, \lambda_{k-1} \in \mathring{H}^c$, $l_1, \ldots, l_{l_{k-1}} \in \mathbb{C}$ and l_k is in the (open) right half-plane and the path of integration shall be the straight line from 0 to λ_k (or any other path inside of H). The assumption $\Re(l_k) > 0$ assures that $d_1(w, l_k)$ tends to zero for $w \to 0$ and since $\lambda_1, \ldots, \lambda_{k-1}$ are not in H, the existence of the integral follows iteratively.

Lemma 1.5
For all $k \geq 1$, the function d_k is holomorphic in $(\mathring{H}^c)^{k-1} \times H \times \mathbb{C}^{k-1} \times D$ as function in several variables, where D denotes the (open) right half-plane. Moreover, d_k is a continuous function of λ_k on the boundary of H.

Proof: Considering the function d_k as function in λ_k, it is obvious that the possible branch points of this function are $0, \lambda_1, \ldots, \lambda_{k-1}$ which are not elements of H. Since H is simply connected, it follows that d_k is holomorphic in H (as function in λ_k). Furthermore, $\lambda_1, \ldots, \lambda_{k-1}$ are not in $\overline{H}$ and because of $\Re(l_k) > 0$ the limit $\lim_{\lambda_k \to 0} \lambda_k^{l_k}$ exists, and therefore d_k is continuous for λ_k in $\overline{H}$ (where $\overline{H}$ denotes the closure of H). On the other hand, as function in λ_j, $j = 1, \ldots, k-1$, the branch points of d_k are on the straight line from 0 to λ_k which lies in H. Hence d_k is holomorphic for λ_j in $\mathring{H}^c$, $j = 1, \ldots, k-1$. Finally, we want to consider d_k as function in $l_j, j = 1, \ldots, k$. It is obvious that d_1 is holomorphic for l_1 in the right half-plane.. From this it follows iteratively for all $k \geq 1$ that d_k is holomorphic for $l_k \in D$. Since $(\lambda_{k-1} - w)^{l_{k-1} - l_k - 1}$ and $(\lambda_{k-1} - \lambda_k)^{l_k - l_{k-1}}$ are exponential functions as functions in l_{k-1} and $d_{k-1}(\lambda_1, \ldots, \lambda_{k-2}, w, l_1, \ldots, l_{k-2}, l_k)$ does not depend on l_{k-1} it is clear that d_k is an entire function in l_{k-1}. Thereof we obtain iteratively that d_k is an entire function in l_j for all $j = 1, \ldots, k-1$. Now the statement of the lemma follows by Hartogs' theorem. □

Remark 10: According to the previous lemma, in the following we always consider the function d_k for $(\lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k) \in (\mathring{H}^c)^{k-1} \times H \times \mathbb{C}^{k-1} \times D$, if we do not say something else. □

Remark 11: The function d_2 can be expressed by means of the *hypergeometric function* (see appendix B) in the following way:

$$d_2(\lambda_1, \lambda_2, l_1, l_2) \;=\; \frac{\lambda_2^{l_2+1} \lambda_1^{-1}}{l_2(l_2+1)} \, F(l_1 + 1, 1; l_2 + 2; \lambda_2/\lambda_1) \, .$$

This follows, since substituting w by $\lambda_2(1-u)/(1 - u\lambda_2/\lambda_1)$, the definition of d_k leads to

$$d_2(\lambda_1, \lambda_2, l_1, l_2) \;= \frac{1}{l_2} \, \frac{\lambda_2^{l_2+1}}{\lambda_1} \int_0^1 (1-u)^{l_2} \, (1 - \frac{\lambda_2}{\lambda_1} u)^{-l_1 - 1} \, du \, .$$

Because of $\lambda_2 \in H$ and $\lambda_1 \in \mathring{H}^c$, it follows that λ_1 can not be on the straight line from 0 to λ_2, which means that $\lambda_2/\lambda_1 \notin [1, \infty)$. Furthermore we have

$\Re(l_2+2) > 2 > 1$ and that is why we can use the integral representation (B.2) of the hypergeometric function and thereof we obtain the statement. □

Using again the notation from Remark 4, the function d_k obviously solves for each $k \geq 2$ the following differential equation with respect to λ_k:

$$(\lambda_k - \lambda_{k-1}) \frac{\partial}{\partial \lambda_k} d_k(\lambda, \ell) \;=\; (l_k - l_{k-1})\, d_k(\lambda, \ell) - d_{k-1}(\lambda^{(k-1)}, \ell^{(k-1)})\,, \qquad (1.22)$$

with $d_k = 0$ if $\lambda_k = 0$. Later we will see that d_k also solves differential equations with respect to λ_j for all $j = 1, \ldots, k-1$.

1.2.2 Difference equations

We will now obtain several difference equations with respect to l_j in this subsection. Later we will show that the coefficient function d_k can be characterized as the single solution of these difference equations that has certain additional properties. Again we use the notation which we introduced in Remark 4 (p.4).

Lemma 1.6
For $k \geq 2$, the function d_k satisfies

$$(l_1+1)\, d_k(\lambda, \ell + e) - \lambda_1\, l_k\, d_k(\lambda, \ell) \;=\; -d_{k-1}(\lambda^{(1)}, (\ell+e)^{(1)})\,. \qquad (1.23)$$

Proof: From the definition it follows directly that

$$\lambda_1 \frac{\partial}{\partial \lambda_1} d_1(\lambda_1, l_1) \;=\; l_1\, d_1(\lambda_1, l_1)\,, \qquad (1.24)$$

and

$$(l_1+1)\, d_1(\lambda_1, l_1+1) \;=\; l_1\, \lambda_1\, d_1(\lambda_1, l_1)\,. \qquad (1.25)$$

Integration by parts and (1.24) lead to

$$\begin{aligned}(l_2 - l_1)\, d_2(\lambda_1, \lambda_2, l_1+1, l_2+1) \;&=\; d_1(\lambda_2, l_2+1) \\ &- (l_2+1)\,(\lambda_1 - \lambda_2)^{l_2 - l_1} \int_0^{\lambda_2} (\lambda_1 - w)^{l_1 - l_2 - 1}\, \frac{\lambda_1 - w}{w}\, d_1(w, l_2+1)\, dw\,,\end{aligned}$$

and from this we get with (1.25)

$$(l_1+1)\, d_2(\lambda_1, \lambda_2, l_1+1, l_2+1) \;=\; (l_2\, \lambda_1)\, d_2(\lambda_1, \lambda_2, l_1, l_2) - d_1(\lambda_2, l_2+1)\,,$$

that is the statement in the case $k = 2$. For a general $k \geq 3$, the statement follows immediately after inserting the induction hypothesis for $k-1$ into (1.21). □

Theorem 1.7
For $k \geq 2$, d_k is a solution of the following difference equation with respect to l_k:

$$\begin{aligned}(l_k - l_{k-1} + 1)\, d_k(\lambda, \ell + e_k) - l_k\, (\lambda_k - \lambda_{k-1})\, d_k(\lambda, \ell) \\ = \; d_{k-1}(\lambda^{(k-1)}, (\ell + e_k)^{(k-1)})\,.\end{aligned} \tag{1.26}$$

Proof: Using integration by parts and identity (1.24), we get

$$\begin{aligned}(l_2 - l_1 + 1)\, d_2(\lambda_1, \lambda_2, l_1, l_2 + 1) \;&=\; d_1(\lambda_2, l_2 + 1) \\ &- (l_2 + 1)\, (\lambda_1 - \lambda_2)^{l_2 - l_1 + 1} \int_0^{\lambda_2} (\lambda_1 - w)^{l_1 - l_2 - 1}\, \frac{1}{w}\, d_1(w, l_2 + 1)\, dw\,,\end{aligned}$$

and after inserting (1.25) this means the statement in the case $k = 2$. Assuming that for a general $k \geq 3$ the equation is true for $k - 1$, integration by parts and the differential equation (1.22) lead to

$$\begin{aligned}(l_k - l_{k-1} + 1)\, d_k(\lambda, \ell + e_k) \;&=\; d_{k-1}(\lambda^{(k-1)}, (\ell + e_k)^{(k-1)}) \\ &-(\lambda_{k-1} - \lambda_k)^{l_k - l_{k-1} + 1} \int_0^{\lambda_k} (\lambda_{k-1} - w)^{l_{k-1} - l_k - 1} \\ &\times \left(\frac{l_k - l_{k-2} + 1}{w - \lambda_{k-2}}\, d_{k-1}(\lambda_1, \ldots, \lambda_{k-2}, w, l_1, \ldots, l_{k-2}, l_k + 1) \right. \\ &\left. - \frac{1}{w - \lambda_{k-2}}\, d_{k-2}(\lambda_1, \ldots, \lambda_{k-2}, w, l_1, \ldots, l_{k-2}, l_k + 1) \right) dw\,.\end{aligned}$$

From this we get the statement for k after inserting the hypothesis for $k - 1$. □

Remark 12: Since $d_1 = \lambda_1^{l_1}\, l_1^{-1}$ can be analytically continued for $l_1 \in \mathbb{C} \setminus \{0\}$, it follows by means of the difference equation (1.26) that for all $k \geq 2$ the function d_k can be analytically continued for $l_k \in \mathbb{C} \setminus -\mathbb{N}_0$. □

Lemma 1.7
For $2 \leq j \leq k - 1$, the following equation is valid:

$$\begin{aligned}&(l_j - l_{j-1} + 1)\, d_k(\lambda, \ell + e_j + \ldots + e_k) - (\lambda_j - \lambda_{j-1})\, l_k\, d_k(\lambda, \ell) \\ &\quad = \; d_{k-1}(\lambda^{(j-1)}, (\ell + e_j + \ldots + e_k)^{(j-1)}) - d_{k-1}(\lambda^{(j)}, (\ell + e_{j+1} + \ldots + e_k)^{(j)})\,.\end{aligned} \tag{1.27}$$

Proof: We will substitute j by $k - m$ and prove the equation by induction with respect to m ($1 \leq m \leq k - 2$). With integration by parts, differential

equation (1.22) and $(\lambda_{k-1} - w) = (\lambda_{k-1} - \lambda_{k-2}) + (\lambda_{k-2} - w)$ we have

$$\begin{aligned}
(l_{k-1} - l_k)\, d_k(\lambda, \ell + e_{k-1} + e_k) \;&=\; -d_{k-1}(\lambda^{(k-1)}, (\ell + e_k)^{(k-1)}) \\
&+ (\lambda_{k-1} - \lambda_{k-2})\, (\lambda_{k-1} - \lambda_k)^{l_k - l_{k-1}} \\
&\quad \times \int_0^{\lambda_k} (\lambda_{k-1} - w)^{l_{k-1} - l_k - 1}\, \frac{\partial}{\partial w}\, d_{k-1}(\lambda_1, \ldots, \lambda_{k-2}, w, l_1, \ldots, l_{k-2}, l_k + 1)\, dw \\
&- (l_k - l_{k-2} + 1)\, (\lambda_{k-1} - \lambda_k)^{l_k - l_{k-1}} \\
&\quad \times \int_0^{\lambda_k} (\lambda_{k-1} - w)^{l_{k-1} - l_k - 1}\, d_{k-1}(\lambda_1, \ldots, \lambda_{k-2}, w, l_1, \ldots, l_{k-2}, l_k + 1)\, dw \\
&+ (\lambda_{k-1} - \lambda_k)^{l_k - l_{k-1}} \\
&\quad \times \int_0^{\lambda_k} (\lambda_{k-1} - w)^{l_{k-1} - l_k - 1}\, d_{k-2}(\lambda_1, \ldots, \lambda_{k-3}, w, l_1, \ldots, l_{k-3}, l_k + 1)\, dw\,.
\end{aligned}$$

Using again integration by parts on the first integral, we see that the first summand equals

$$\frac{\lambda_{k-1} - \lambda_{k-2}}{\lambda_{k-1} - \lambda_k} \left[(l_{k-1} - l_k - 1)\, d_k(\lambda, \ell + e_k) + d_{k-1}(\lambda^{(k-1)}, (\ell + e_k)^{(k-1)}) \right],$$

while the remaining summands are equal to

$$\begin{aligned}
-(l_k - l_{k-2} + 1)\, d_k(\lambda, \ell + e_{k-1} + e_k) - d_{k-1}(\lambda^{(k-1)}, (\ell + e_k)^{(k-1)}) \\
+ d_{k-1}(\lambda^{(k-2)}, (\ell + e_{k-1} + e_k)^{(k-2)})\,.
\end{aligned}$$

Now we finally get the statement in the case $m = 1$ after inserting the difference equation (1.26). For a general m, $2 \le m \le k-2$, inserting the equation for $m-1$ into (1.21) directly leads to the statement. □

The previous difference equations imply the following ones:

Theorem 1.8
For $k \ge 2$, the function d_k solves the following difference equation with respect to l_1:

$$\begin{aligned}
(\lambda_2 - \lambda_1)\, (l_1 + 1)\, d_k(\lambda, \ell + e_1) \;&-\; \lambda_1\, (l_2 - l_1)\, d_k(\lambda, \ell) \\
&= \lambda_1\, d_{k-1}(\lambda^{(2)}, \ell^{(2)}) - \lambda_2\, d_{k-1}(\lambda^{(1)}, \ell^{(1)})\,,
\end{aligned} \tag{1.28}$$

and for $2 \le j \le k-2$, the difference equation with respect to l_j:

$$\begin{aligned}
&(\lambda_{j+1} - \lambda_j)\, (l_j - l_{j-1} + 1)\, d_k(\lambda, \ell + e_j) - (\lambda_j - \lambda_{j-1})\, (l_{j+1} - l_j)\, d_k(\lambda, \ell) \\
&\quad = (\lambda_j - \lambda_{j-1})\, d_{k-1}(\lambda^{(j+1)}, \ell^{(j+1)}) - (\lambda_{j+1} - \lambda_{j-1})\, d_{k-1}(\lambda^{(j)}, \ell^{(j)}) \\
&\qquad + (\lambda_{j+1} - \lambda_j)\, d_{k-1}(\lambda^{(j-1)}, (\ell + e_j)^{(j-1)})\,.
\end{aligned} \tag{1.29}$$

If $k \geq 3$, then it is a solution of the difference equation with respect to l_{k-1}:

$$(\lambda_k - \lambda_{k-1})\,(l_{k-1} - l_{k-2} + 1)\,d_k(\lambda, \ell + e_{k-1}) \; - (\lambda_{k-1} - \lambda_{k-2})\,(l_k - l_{k-1})\,d_k(\lambda, \ell)$$
$$= \; (\lambda_k - \lambda_{k-1})\,d_{k-1}(\lambda^{(k-2)}, (\ell + e_{k-1})^{(k-2)}) - (\lambda_k - \lambda_{k-2})\,d_{k-1}(\lambda^{(k-1)}, \ell^{(k-1)})\,. \tag{1.30}$$

Proof: From equation (1.23) it follows that

$$(\lambda_2 - \lambda_1)\,(l_1 + 1)\,d_k(\lambda, \ell + e_1) \; =$$
$$\lambda_1\,(\lambda_2 - \lambda_1)\,(l_k - 1)\,d_k(\lambda, \ell - e_2 - \ldots - e_k) - (\lambda_2 - \lambda_1)\,d_{k-1}(\lambda^{(1)}, \ell^{(1)})\,,$$

and from (1.27) we have

$$\lambda_1\,(\lambda_2 - \lambda_1)\,(l_k - 1)\,d_k(\lambda, \ell - e_2 - \ldots - e_k) \; =$$
$$\lambda_1\,(l_2 - l_1)\,d_k(\lambda, \ell) - \lambda_1\,d_{k-1}(\lambda^{(1)}, \ell^{(1)}) + \lambda_1\,d_{k-1}(\lambda^{(2)}, \ell^{(2)})\,.$$

Now these two equations lead to (1.28). Observing that (1.27) implies both

$$(\lambda_{j+1} - \lambda_j)\,(l_j - l_{j-1} + 1)\,d_k(\lambda, \ell + e_j) \; =$$
$$(\lambda_j - \lambda_{j-1})\,(\lambda_{j+1} - \lambda_j)\,(l_k - 1)\,d_k(\lambda, \ell - e_{j+1} - \ldots - e_k)$$
$$+ (\lambda_{j+1} - \lambda_j)\,d_{k-1}(\lambda^{(j-1)}, (\ell + e_j)^{(j-1)}) - (\lambda_{j+1} - \lambda_j)\,(d_{k-1}(\lambda^{(j)}, \ell^{(j)})$$

and

$$(\lambda_j - \lambda_{j-1})\,(\lambda_{j+1} - \lambda_j)\,(l_k - 1)\,d_k(\lambda, \ell - e_{j+1} - \ldots - e_k) \; =$$
$$(\lambda_j - \lambda_{j-1})\,(l_{j+1} - l_j)\,d_k(\lambda, \ell) - (\lambda_j - \lambda_{j-1})\,d_{k-1}(\lambda^{(j)}, \ell^{(j)})$$
$$+ (\lambda_j - \lambda_{j-1})\,d_{k-1}(\lambda^{(j+1)}, \ell^{(j+1)})\,,$$

we obtain equation (1.29). From (1.27) we also have

$$(\lambda_k - \lambda_{k-1})\,(l_{k-1} - l_{k-2} + 1)\,d_k(\lambda, \ell + e_{k-1}) \; =$$
$$(\lambda_{k-1} - \lambda_{k-2})\,(\lambda_k - \lambda_{k-1})\,(l_k - 1)\,d_k(\lambda, \ell - e_k)$$
$$+ (\lambda_k - \lambda_{k-1})\,d_{k-1}(\lambda^{(k-2)}, (\ell + e_{k-1})^{(k-2)}) - (\lambda_k - \lambda_{k-1})\,d_{k-1}(\lambda^{(k-1)}, \ell^{(k-1)})$$

and from (1.26)

$$(\lambda_{k-1} - \lambda_{k-2})\,(\lambda_k - \lambda_{k-1})\,(l_k - 1)\,d_k(\lambda, \ell - e_k) \; =$$
$$(\lambda_{k-1} - \lambda_{k-2})\,(l_k - l_{k-1})\,d_k(\lambda, \ell) - (\lambda_{k-1} - \lambda_{k-2})\,d_{k-1}(\lambda^{(k-1)}, \ell^{(k-1)})\,,$$

and this leads to equation (1.30) and completes the proof. □

1.3 Relation between the coefficient functions

In this section, we will analyze the coherence between the coefficient functions of the first resp. second kind f_k and d_k and we will see that there is a close relation. Therefore we consider again a region H with the same properties as in Subsection 1.2.1. Observe that for an arbitrary but fixed $k \geq 1$ with $\lambda_1, \ldots, \lambda_{k-1} \in \mathring{H}^c$, $\lambda_k \in H$, and l_k in the right half-plane, the conditions of Theorem 1.6 for $j = k$ are satisfied, and therefore the limit

$$\tilde{d}_k(\lambda, \ell) \;:=\; \lim_{t \to \lambda_k} (\lambda_k - t)^{l_k} \, f_k(t; \lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k) \tag{1.31}$$

exists. From the definition of f_k, we obtain that

$$\begin{aligned} \tilde{d}_k(\lambda, \ell) \;&=\; \int_0^{\lambda_k} (\lambda_k - w)^{l_k - 1} \, f_{k-1}(w; \lambda^{(k)}, \ell^{(k)}) \, dw \\ &=\; \int_0^{\lambda_k} u^{l_k - 1} \, f_{k-1}(\lambda_k - u; \lambda^{(k)}, \ell^{(k)}) \, du \,. \end{aligned}$$

For the given region H and fixed $\lambda_1, \ldots, \lambda_{k-1} \in \mathring{H}^c$, it is possible to find a region G of the kind which we considered in the first section so that $H \subset G$ and $\lambda_1, \ldots, \lambda_{k-1} \in \mathring{G}^c$. Hence we know from the first section that f_{k-1} is holomorphic in G and in particular for every fixed u on the path of integration, $f_{k-1}(\lambda_k - u)$ is holomorphic for $\lambda_k \in H$. Since $\lambda_k^{l_k - 1}$ is also holomorphic in H it follows that $\tilde{d}_k(\lambda, \ell)$ is holomorphic in H (as function in λ_k).

Considering the first integral representation of $\tilde{d}_k$ for $k \geq 2$ and $\Re(l_k) > 1$ we obtain after using the chain rule that the following differential-difference equation is valid:

$$\begin{aligned} \frac{\partial}{\partial \lambda_k} \tilde{d}_k(\lambda, \ell) \;&=\; (l_k - 1) \int_0^{\lambda_k} (\lambda_k - w)^{l_k - 2} \, f_{k-1}(w; \lambda^{(k)}, \ell^{(k)}) \, dw \\ &=\; (l_k - 1) \, \tilde{d}_k(\lambda, \ell - e_k) \,. \end{aligned} \tag{1.32}$$

The difference equation (1.3) for f_k leads to another one for $\tilde{d}_k$ for all $k \geq 2$, since multiplying both sides of (1.3) with $(\lambda_k - t)^{l_k}$ and observing the limit $t \to \lambda_k$ we obtain the following difference equation with respect to l_k:

$$\begin{aligned} (l_k - l_{k-1} + 1) \, \tilde{d}_k(\lambda, \ell + e_k) - l_k(\lambda_k - \lambda_{k-1}) \, \tilde{d}_k(\lambda, \ell) \\ =\; \tilde{d}_{k-1}(\lambda^{(k-1)}, (\ell + e_k)^{(k-1)}) \,. \end{aligned}$$

Inserting this difference equation into (1.32), it follows for all $k \geq 2$ that $\tilde{d}_k$ is a solution of the differential equation (1.22) (which has been solved by d_k) and it can be checked directly that this is also true in the case $k = 1$. Obviously,

$\tilde{d}_k \to 0$ for $\lambda_k \to 0$ and so we obtain $d_k = \tilde{d}_k$ for all $k \geq 1$. However, $\tilde{d}_k$ solves the differential equation just for $\Re(l_k) > 1$, but the identity theorem for holomorphic functions implies that d_k and $\tilde{d}_k$ are the same functions for all l_k with $\Re(l_k) > 0$.

This means that the following relationships between the coefficient functions f_k and d_k holds true:

Theorem 1.9
For all $k \geq 1$, $\lambda_1, \ldots, \lambda_{k-1} \in \mathring{H}^c$, $\lambda_k \in H$, $l_1, \ldots, l_{k-1} \in \mathbb{C}$ and l_k in the right half-plane, the functions d_k and f_k are related by

$$d_k(\lambda, \ell) \;=\; \lim_{t \to \lambda_k} (\lambda_k - t)^{l_k} \, f_k(t; \lambda, \ell)\,, \tag{1.33}$$

which directly leads to

$$d_k(\lambda, \ell) \;=\; \int_0^{\lambda_k} (\lambda_k - w)^{l_k - 1} \, f_{k-1}(w; \lambda^{(k)}, \ell^{(k)})\, dw\,, \tag{1.34}$$

and

$$f_{k-1}(\lambda_k; \lambda_1, \ldots, \lambda_{k-1}, l_1, \ldots, l_{k-1}) \;=\; \frac{\partial}{\partial \lambda_k} \, d_k(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k, l_1, \ldots, l_{k-1}, 1)\,.$$

Remark 13: If the assumptions from Theorem 1.6 are satisfied for $j = k$ and additionally $\Re(l_k) > 0$, the limit (1.31) exists, and hence d_k is well-defined in this case. So, according to Remark 9, we can enlarge the domain of definition for the functions d_k, too, so that λ_j is no longer restricted to be in $\mathring{H}^c$ and may even be equal to λ_k. Because of Theorem 1.6 we just have to assume that $\lambda_\nu = \lambda_k$ implies $l_\nu = l_k$ for all $\nu \geq 1$ and that there exists no $\nu \in \mathbb{N}$ so that $\lambda_\nu = \lambda_{\nu+1} = \lambda_k$. Then λ_j $(1 \leq j \leq k-1)$ may lie on the straight line from 0 to λ_k (but $\lambda_j \neq 0$). In this case we specify again that the path of integration shall be chosen so that λ_j lies on the left side (when looking from 0 to λ_k). Furthermore, for $j \geq 2$, λ_j may be equal to zero, if $l_j = 0$. If we do not say something else, in the following, we will always consider the function d_k in this enlarged domain of definition. □

1.3.1 Further differential equations

By means of the relationship between d_k and f_k and the differential equations for f_k which we considered in the first section, we obtain further differential equations for d_k with respect to λ_j, for each $j = 1, \ldots, k-1$.

Lemma 1.8
For $1 \leq j \leq k-1$, d_k solves the following differential-difference equation:

$$\begin{aligned} \frac{\partial}{\partial \lambda_j} \, d_k(\lambda, \ell) \;=\; & (l_k - 1)\, d_k(\lambda, \ell - e_j - \ldots - e_k) \\ & - (l_k - 1)\, d_k(\lambda, \ell - e_{j+1} - \ldots - e_k)\,. \end{aligned} \tag{1.35}$$

Proof: From (1.34) and (1.10) we have for all $1 \leq j \leq k-1$:

$$\begin{aligned}\frac{\partial}{\partial\lambda_j} d_k(\lambda,\ell) \;=\; \int_0^{\lambda_k} (\lambda_k - w)^{l_k-1} \Big(& f'_{k-1}(w;\lambda^{(k)},(\ell - e_j - \ldots - e_{k-1})^{(k)}) \\ & - f'_{k-1}(w;\lambda^{(k)},(\ell - e_{j+1} - \ldots - e_{k-1})^{(k)}) \Big)\, dw\,.\end{aligned} \tag{1.36}$$

Furthermore, via integration by parts it follows that

$$\int_0^{\lambda_k} (\lambda_k - w)^{l_k-1} f'_{k-1}(w;\lambda^{(k)},(\ell)^{(k)})\, dw \;=\; (l_k - 1)\, d_k(\lambda, \ell - e_k)$$

for $\Re(l_k) > 1$. Inserting this identity into (1.36), we obtain the statement of the lemma. □

Using (1.35) and the difference equations from Subsection 1.2.2, we then obtain

Theorem 1.10
For $k \geq 3$, the function d_k solves the differential equation with respect to λ_1:

$$\begin{aligned}\frac{\partial}{\partial\lambda_1} d_k(\lambda,\ell) \;=\; & \left(\frac{l_1}{\lambda_1} - \frac{l_2 - l_1}{\lambda_2 - \lambda_1} \right) d_k(\lambda,\ell) \\ & + \frac{\lambda_2}{(\lambda_2 - \lambda_1)\lambda_1}\, d_{k-1}(\lambda^{(1)},\ell^{(1)}) - \frac{1}{\lambda_2 - \lambda_1}\, d_{k-1}(\lambda^{(2)},\ell^{(2)})\end{aligned}$$

and the one with respect to λ_{k-1}:

$$\begin{aligned}\frac{\partial}{\partial\lambda_{k-1}} d_k(\lambda,\ell) \;=\; & \left(\frac{l_{k-1} - l_{k-2}}{\lambda_{k-1} - \lambda_{k-2}} - \frac{l_k - l_{k-1}}{\lambda_k - \lambda_{k-1}} \right) d_k(\lambda,\ell) \\ & + \frac{\lambda_k - \lambda_{k-2}}{(\lambda_k - \lambda_{k-1})(\lambda_{k-1} - \lambda_{k-2})}\, d_{k-1}(\lambda^{(k-1)},\ell^{(k-1)}) \\ & - \frac{1}{\lambda_{k-1} - \lambda_{k-2}}\, d_{k-1}(\lambda^{(k-2)},\ell^{(k-2)})\,.\end{aligned}$$

Furthermore, for $2 \leq j \leq k-2$, it is a solution of the differential equation with respect to λ_j:

$$\begin{aligned}\frac{\partial}{\partial\lambda_j} d_k(\lambda,\ell) \;=\; & \left(\frac{l_j - l_{j-1}}{\lambda_j - \lambda_{j-1}} - \frac{l_{j+1} - l_j}{\lambda_{j+1} - \lambda_j} \right) d_k(\lambda,\ell) \\ & + \frac{\lambda_{j+1} - \lambda_{j-1}}{(\lambda_{j+1} - \lambda_j)(\lambda_j - \lambda_{j-1})}\, d_{k-1}(\lambda^{(j)},\ell^{(j)}) - \frac{1}{\lambda_j - \lambda_{j-1}}\, d_{k-1}(\lambda^{(j-1)},\ell^{(j-1)}) \\ & - \frac{1}{\lambda_{j+1} - \lambda_j}\, d_{k-1}(\lambda^{(j+1)},\ell^{(j+1)})\,.\end{aligned}$$

Proof: From equation (1.23), we obtain for all $k \geq 2$ (and $\Re(l_k) > 1$)

$$(l_k - 1)\, d_k(\lambda, \ell - e) \;=\; \frac{l_1}{\lambda_1}\, d_k(\lambda, \ell) + \frac{1}{\lambda_1}\, d_{k-1}(\lambda^{(1)}, \ell^{(1)})$$

and equation (1.26) is equivalent to

$$(l_k - 1)\, d_k(\lambda, \ell - e_k) \;=\; \frac{l_k - l_{k-1}}{\lambda_k - \lambda_{k-1}}\, d_k(\lambda, \ell) - \frac{1}{\lambda_k - \lambda_{k-1}}\, d_{k-1}(\lambda(k-1), \ell^{(k-1)})\,.$$

Furthermore, for all $2 \leq j \leq k-1$, we have from equation (1.27)

$$\begin{aligned}(l_k - 1)\, d_k(\lambda,&\ell - e_j - \ldots - e_k) \;=\; \frac{l_j - l_{j-1}}{\lambda_j - \lambda_{j-1}}\, d_k(\lambda, \ell) \\ &- \frac{1}{\lambda_j - \lambda_{j-1}}\, d_{k-1}(\lambda^{(j-1)}, \ell^{(j-1)}) + \frac{1}{\lambda_j - \lambda_{j-1}}\, d_{k-1}(\lambda^{(j)}, \ell^{(j)})\,.\end{aligned}$$

Now, inserting these identities into the differential-difference equation (1.35) leads to the given differential equations. □

1.4 Characterization of the coefficient functions

In this section, we will give additionally conditions so that the coefficient function (of the second kind) d_k is uniquely determined among the solutions of the difference equations with respect to l_j which we derived in Subsection 1.2.2.

At first let us consider the function d_k for $k \geq 2$ as solution of the difference equation (in l_k) (1.26) and assume that $|\lambda_k| < |\lambda_k - \lambda_{k-1}|$. It is well-known that the general solution of an inhomogeneous difference equation is the sum of a special solution of this equation and the general solution of the corresponding homogeneous equation. In our case this homogeneous difference equation is given by

$$(l_k - l_{k-1} + 1)\, d_k(\lambda, \ell + e_k) \;=\; l_k\, (\lambda_k - \lambda_{k-1})\, d_k(\lambda, \ell)\,,$$

and the general solution thereof is

$$\omega(l_k)\, \frac{\Gamma(l_k)}{\Gamma(l_k - l_{k-1} + 1)}\, (\lambda_k - \lambda_{k-1})^{l_k}\,,$$

with an arbitrary 1-periodic function $\omega(l_k)$. According to *Stirling's formula* (B.1), for $\Re(l_k) \to \infty$, this solution grows like $(l_k)^{l_{k-1}-1}\, (\lambda_k - \lambda_{k-1})^{l_k}$, if $\omega(l_k) \not\equiv 0$. In the previous section we showed that

$$d_k(\lambda, \ell) = \int_0^{\lambda_k} (\lambda_k - w)^{l_k - 1}\, f_{k-1}(w; \lambda^{(k)}, \ell^{(k)})\, dw\,,$$

and since $f_{k-1}(w;\lambda^{(k)},\ell^{(k)})$ does not depend upon l_k, the coefficient function d_k cannot grow faster than $K|\lambda_k|^{l_k}$ with a constant K which is independent of l_k. So, because of the assumption $|\lambda_k| < |\lambda_k - \lambda_{k-1}|$, the general solution of the homogeneous difference equation grows too fast, if $\omega(l_k) \not\equiv 0$. Thereof it follows that $\omega(l_k) \equiv 0$ and d_k is uniquely determined as the only solution of the inhomogeneous equation which grows slow enough.

Now we consider d_k as solution of the difference equation (1.29) resp. (1.30) in l_j for $2 \le j \le k-1$ and assume that $|\lambda_j - \lambda_{j+1}| < |\lambda_j - \lambda_{j-1}|$. The corresponding homogeneous difference equation is

$$(\lambda_{j+1} - \lambda_j)\,(l_j - l_{j-1} + 1)\,d_k(\lambda, \ell + e_j) \;=\; (\lambda_j - \lambda_{j-1})\,(l_{j+1} - l_j)\,d_k(\lambda,\ell)$$

and the general solution thereof is given by

$$\omega(l_j)\,\frac{\Gamma(l_j - l_{j+1})}{\Gamma(l_j - l_{j-1} + 1)}\,\left(\frac{\lambda_j - \lambda_{j-1}}{\lambda_j - \lambda_{j+1}}\right)^{l_j},$$

where $\omega(l_j)$ again denotes an arbitrary 1-periodic function. So for $\Re(l_j) \to \infty$, the solution grows like $(l_j - l_{j+1})^{l_{j-1}-l_{j+1}-1}\left(\frac{\lambda_j-\lambda_{j-1}}{\lambda_j-\lambda_{j+1}}\right)^{l_j}$, if $\omega(l_j) \not\equiv 0$. Since d_k can be represented by (1.34), there also exists a representation of the kind

$$d_k(\lambda,\ell) \;=\; \int_0^{\lambda_k} \dots \int_0^{\tilde v}\int_0^{\tilde u}\int_0^{v} \left(\frac{\lambda_j - u}{\lambda_j - v}\right)^{l_j} \dots\, du\, dv\, d\tilde u \,\dots\, dw\,,$$

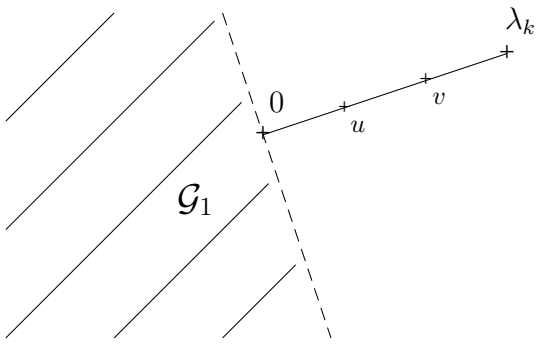
where the terms denoted by "..." do not depend upon l_j and therefore can be estimated by constants if $\Re(l_j) \to \infty$. The remaining term can be estimated by

$$\left(\max_{"0\le u\le v\le\lambda_k"} \left|\frac{\lambda_j - u}{\lambda_j - v}\right|\right)^{l_j},$$

where the notation "$0 \le u \le v \le \lambda_k$" means that v lies on the - when possible linear - path of integration from the origin to λ_k and u of the one from the origin to v. For simplification, we consider the problem for fixed λ_k just for λ_j in the half plane

$$\mathcal{G}_1 \;:=\; \left\{\, \lambda_j \in \mathbb{C} :\, \arg(\lambda_j) \in \left(d - \frac{\pi}{2}, d + \frac{\pi}{2}\right)\right\},$$

where d is defined by $d := -\arg(\lambda_k)$.



For $\lambda_j \in \mathcal{G}_1$ we always have $|\lambda_j - u| \leq |\lambda_j - v|$ and so

$$\max_{"\lambda_1 \leq u \leq v \leq \lambda_k"} \left| \frac{\lambda_j - u}{\lambda_j - v} \right| \; = \; 1 \, .$$

Hence in this case $d_k(\lambda, \ell)$ does not grow at all for $\Re(l_j) \to \infty$. Because of $\left|\frac{\lambda_j - \lambda_{j-1}}{\lambda_j - \lambda_{j+1}}\right| > 1$, the solution of the homogeneous difference equation again grows too fast if $\omega(l_j) \not\equiv 0$. That is why the function d_k again is uniquely determined as the only solution of the difference equation which grows slow enough.

Remark 14: The uniqueness of the coefficient functions (of the second kind) d_k has just been shown under certain assumptions on the position of some λ_j's. However, the identity theorem for holomorphic functions leads to the uniqueness in the general case. Because of the relation which has been shown in Theorem 1.9, the coefficient functions (of the first kind) f_k are unique in this sense, too. □

Chapter 2

The hypergeometric and the confluent hypergeometric system

In this chapter, we will introduce the so-called *hypergeometric system of differential equations* and the *confluent hypergeometric system of differential equations*, since they will play an important role in the remaining chapters. In the first section, we will give the definition of the two systems and in the second section, we will discuss some properties of them.

2.1 Introduction

The *hypergeometric system of differential equations* is a system of the kind

$$(\Lambda - t\,I)\,y'(t) \;=\; A\,y(t)\,, \tag{2.1}$$

where Λ and A are arbitrary $(n \times n)$-matrices. This system has singularities only at the eigenvalues of Λ and at infinity. Mostly, one assumes that Λ is diagonalizable and after using a transformation one can consider Λ as a diagonal matrix.

Remark 15: In the following, we will always consider the system with the assumption that Λ is a diagonal matrix and if we do not say something else, we assume that

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,.$$

□

In the case $n = 2$, it is possible to compute a fundamental solution in terms of the hypergeometric function (see appendix B) and the appropriate formulas can be found, for example, in [4]. The appearance of the hypergeometric function in this case is the reason for the name of the system. For systems in higher dimension there are just special cases with strong restrictions to the entries of A

and Λ where solutions can be expressed in terms of generalized hypergeometric functions (see e.g. [3]). In general, the solutions of this system are believed to be "new" higher transcendental functions.

The hypergeometric system is intimately related via Laplace Transform to the standard differential equation of Poincaré rank one

$$z x'(z) \;=\; (\, z\,\Lambda + A)\, x(z)\,. \tag{2.2}$$

This system has a regular singularity at the origin and an irregular singularity at infinity. It has been shown by *R. Schäfke* [25] that the two systems are also related by means of a confluence process, and therefore (2.2) is called the *confluent hypergeometric system of differential equations*. Besides Schäfke [23, 24, 25], a number of authors investigated the hypergeometric and/or the confluent hypergeometric system, among them *Balser* [1, 2, 3], *Balser, Jurkat, and Lutz* [5, 6], *Balser and Röscheisen* [7], *M. Kohno* [12, 13, 14, 15, 16], *Kohno and Yokoyama* [17], *Okubo and Takano* [18], and *Okubo, Takano, and Yoshida* [19].

2.2 Some properties

In this section, we will give an overview of the basic properties of the hypergeometric and the confluent hypergeometric system. For more informations about terms occuring here, like formal power series, asymptotic expansion or summability, the user is referred to appendix A.

2.2.1 Singular and Stokes' directions, sectors

The elements of the set

$$\{\, d \in \mathbb{R} :\; d \;\equiv\; \arg\,(\lambda_j - \lambda_k) \pmod{2\pi}\,,\; \lambda_j \neq \lambda_k,\; 1 \leq j,k \leq n \,\}$$

are called the *singular directions*. Obviously, the set is discrete and countable, hence it is possible to index its elements so that

$$d_\nu > d_{\nu+1}\;,\; \forall\, \nu \in \mathbb{Z}\,.$$

Now the *Stokes' directions* τ_ν are defined by

$$\tau_\nu \;=\; -d_\nu + \frac{\pi}{2}\,,$$

which then satisfy

$$\tau_\nu < \tau_{\nu+1}\;,\; \forall\, \nu \in \mathbb{Z}\,.$$

Furthermore, we define sectors S_ν by

$$S_\nu \;=\; \{\, z \in \mathbb{C} :\; |z| > 0,\, \tau_\nu - \pi < \arg z < \tau_{\nu+1} \}\;,\; \nu \in \mathbb{Z}\,.$$

The opening of each sector is greater than π, which implies that the sectors are larger than half-planes.

2.2.2 Solutions

For the confluent system (2.2) with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ there always exists a formal fundamental solution of the form

$$\hat{X}(z) \;=\; \hat{F}(z)\, z^L\, e^{z\Lambda}\,, \tag{2.3}$$

where $L = \operatorname{diag}(l_1, \ldots, l_n)$ is the diagonal matrix consisting of the diagonal elements of A, and $\hat{F}(z) = \sum z^{-\mu} F_\mu$ is a formal power series in z^{-1}. If we choose $F_0 := I$, the (matrix) coefficients $F_\mu, \mu \geq 1$ are uniquely determined and can be computed recursively by

$$(\,\Lambda - \lambda_k\, I\,)\, f_k(\mu+1) \;=\; (\,(l_k - \mu)\, I - A\,)\, f_k(\mu)\,, \tag{2.4}$$

where $f_k(\mu) = (f_{1k}(\mu), \ldots, f_{nk}(\mu))$ denotes the k-th column of F_μ. Of course, in the k-th component the left side of this equation is equal to 0, hence it is not possible to compute $f_{kk}(\mu+1)$ directly from $f_k(\mu)$. However, (2.4) leads to

$$\mu \cdot f_{kk}(\mu) \;=\; \sum_{j \neq k} a_{kj}\, f_{jk}(\mu)\,, \tag{2.5}$$

with $A = (a_{jk})_{j,k=1}^n$. Then after we have obtained all the other components $f_{jk}(\mu+1), j \neq k$, in terms of $f_k(\mu)$, we can compute $f_{kk}(\mu+1)$ with equation (2.5). The k-th column of $\hat{X}(z)$ then has the form $\hat{x}_k(z) = \sum_{\mu=0}^{\infty} f_k(\mu)\, z^{-\mu+l_k}\, e^{z\lambda_k}$.

In the case of multiple eigenvalues there is also a formal fundamental solution of the kind (2.3). However, in general, the matrix L is not diagonal.

For the case of distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, we define

$$h_k(\mu) := \frac{f_k(\mu)}{\Gamma(\mu - l_k)}\,, \quad \mu \geq 0\,,$$

and vectors $y_k(t)$ by

$$y_k(t) \;:=\; (\lambda_k - t)^{-(l_k+1)} \sum_{\mu=0}^{\infty} h_k(\mu)\, (\lambda_k - t)^\mu\,.$$

For $l_k \notin \mathbb{N}_0$, the identities $F_0 = I$ and (2.4) lead to

$$\begin{aligned} h_k(0) \;&=\; \frac{e_k}{\Gamma(-l_k)}\,, \\ (\mu - l_k)\,(\,\Lambda - \lambda_k\, I\,)\, h_k(\mu+1) \;&=\; (\,(l_k - \mu)\, I - A\,)\, h_k(\mu)\,, \quad \mu \geq 0\,, \end{aligned}$$

where e_k denotes the k-th unit vector. If $l_k = m \in \mathbb{N}_0$, we obtain

$$y_k(t) \;=\; \sum_{\nu=0}^{\infty} \frac{f_k(\nu+m+1)}{\nu!}\, (\lambda_k - t)^\nu$$

and in this case equation (2.4) implies

$$(\,\Lambda - \lambda_k\, I\,)\, f_k(\nu + m + 2) \;=\; -(\,(\nu+1)\, I + A\,)\, f_k(\mu)\,, \quad \nu \geq 0\,.$$

In both cases it follows that $y_k(t)$ is a solution of

$$(\,\Lambda - t\, I\,)\, y'(t) \;=\; (A + I)\, y(t)\,, \tag{2.6}$$

which is the hypergeometric system (2.1) with $A+I$ instead of A. Since the eigenvalues $\lambda_1, \ldots, \lambda_n$ were assumed to be distinct, the singularities of the hypergeometric system are all regular. Hence the radius of convergence of the power series occuring in $y_k(t)$ is positive, although the previous power series $\sum_{\mu=0}^{\infty} f_k(\mu)\, z^{-\mu}$ has been a formal series.

Now, from $y_k(t)$ we can derive proper solutions of the confluent system (2.2) via Laplace Transform. We assume that $\Re(l_k) < 0$ and for arbitrary $z \in S_\nu$ we choose $d \in (d_{\nu+1}, d_\nu)$ so that $-\pi/2 < \arg(z) + d < \pi/2$. These assumptions assure the existence of the integral

$$x_k^{(\nu)}(z) \;:=\; e^{z\lambda_k} \int_0^{\infty(d)} e^{-zu}\, y_k(\lambda_k - u)\, du\,,$$

with integration along $\arg u = d$. For each $\nu \in \mathbb{Z}$, the function $x_k^{(\nu)}(z)$ is asymptotically equal to $\hat{x}_k(z)$ in the sector S_ν, hence it is a proper solution of (2.2). Furthermore, the matrix $X_\nu(z) = (x_1^{(\nu)}, \ldots, x_n^{(\nu)})$ is the 1-sum of $\hat{X}(z)$ in S_ν and therefore a fundamental solution of (2.2).

2.2.3 Stokes' Multipliers and characteristic constants

In the previous subsection, we saw that there are infinitely many fundamental solutions $X_\nu(z)$ of (2.2), and for all $\nu \in \mathbb{Z}$, $X_\nu(z)$ is asymptotically equal to $\hat{X}(z)$ in the sector S_ν. Hence, there exist constant matrices V_ν, which satisfy

$$X_{\nu-1}(z) \;=\; X_\nu(z)\, V_\nu\,,$$

the so-called *Stokes' Multipliers*. They describe the *Stokes' Phenomenon*, which means that solutions exist which in sufficiently small sectors show the same asymptotic behavior, but, in general, depend upon the sector. Reversely, one can say that one solution shows different asymptotic behavior in different sectors. The problem of calculating Stokes' Multipliers is sometimes referred to as a *lateral connection problem* for the differential equation.

It has been shown by *Balser, Jurkat and Lutz* in [5] that, if the eigenvalues of Λ are distinct and the diagonal entries of A are not integers, the hypergeometric

system (2.6) has at each point $t = \lambda_j$ one singular solution vector (which equals the vector $y_j(t)$ considered in Subsection 2.2.2) and $n-1$ linearly independent regular solution vectors. Furthermore, $y_k(t)$ can be analytically continued along every path which avoids the singularities $\lambda_1, \ldots, \lambda_n$. Hence there exist characteristic constants c_{jk} for all $1 \leq j, k \leq n$ (ignoring details of how to choose a path for the analytic continuation) so that

$$y_k(t) \;=\; c_{jk}\, y_j(t) + \mathrm{reg}(t - \lambda_j)\,, \tag{2.7}$$

where $\mathrm{reg}(t)$ again denotes a function which is holomorphic in a neighborhood of the origin. The constant c_{jk} describes how the solution vector $y_k(t)$ which is known in the neigborhood of λ_k behaves near λ_j.

If one is not interested in all of these constants c_{jk} but just in one (or a few) of them, the assumptions which assure the existence of this constant can be eased. For arbitrary but fixed j, k $(1 \leq j, k \leq n)$ let us assume that the value λ_j is a single eigenvalue of Λ and the j-th diagonal entry of $A + I$ - which shall be denoted by $l_j + 1$ - is not an integer. Analogously to *Balser, Jurkat and Lutz* we write the system (2.6) in the form

$$y'(t) \;=\; (\,\Lambda - t\,I\,)^{-1}\,(A + I)\,y(t)\,,$$

and use the abbreviation

$$B(t) := (\,\Lambda - t\,I\,)^{-1}\,(A + I)\,.$$

The matrix $B(t)$ has a pole of order 1 at λ_j and hence can be expanded as

$$B(t) \;=\; \sum_{\mu=-1}^{\infty} B_\mu\,(t - \lambda_j)^\mu\,.$$

Since we assumed that λ_j is a single eigenvalue, the only nonzero entries of the coefficient matrix B_{-1} are in the j-th row of it and this row equals the j-th row of $-(A + I)$. Hence, B_{-1} has the eigenvalues 0 (with multiplicity $n-1$) and $-(l_j + 1)$ (single eigenvalue) and because of our assumption they do not differ by an integer. From [27], Theorem 5.1 it follows then that there exists a fundamental solution

$$Y(t) \;=\; P(t)\,(t - \lambda_j)^{B_{-1}}\,,$$

where $P(t)$ is a matrix-valued function which is holomorphic at λ_j. Let now T be the regular matrix so that $T\,B_{-1}T^{-1}$ is diagonal. Then $\tilde{Y}(t) = T\,Y(t)\,T^{-1}$ is also a fundamental solution of the considered system. Furthermore the j-th column of $\tilde{Y}(t)$ has a singularity at λ_j and all other columns are holomorphic there. So in this case there also exist one singular solution vector and $n-1$

linearly independent regular solution vectors at $t = \lambda_j$. That is why there exists again the unique constant c_{jk} satisfying (2.7).
In [5] it has also been shown that there is a close relation between the characteristic constants c_{jk} and the Stokes' Multipliers of (2.2) so that the knowledge of the constants c_{jk} is equivalent to the knowledge of the Stokes' Multipliers. In the following chapters, we will show how the constants c_{jk} for certain systems can be expressed by means of the coefficient functions (of the second kind) d_k introduced in the first chapter.

2.2.4 Scalar transformations

We can easily check that, if $x(z)$ is a solution of (2.2), then $\tilde{x}(z) := e^{\lambda z}\, x(z)$ is one of

$$z\tilde{x}'(z) \;=\; (\, z\,(\Lambda + \lambda\, I) + A)\,\tilde{x}(z)\,,$$

and $\bar{x}(z) := z^{\alpha}\, x(z)$ solves

$$z\bar{x}'(z) \;=\; (\, z\,\Lambda + (A + \alpha\, I))\,\bar{x}(z)\,.$$

These are simple scalar transformations, and so it is possible to fix without loss of generality both one of the values λ_j and one of the values l_j, for example we can set $\lambda_1 := 0$ and $l_1 := 0$. Analogously, for the hypergeometric system, the substitution $\tilde{y}(t) := y(t - \lambda)$ causes that $\tilde{y}$ is a solution of

$$(\,(\Lambda + \lambda\, I) - t\, I\,)\,\tilde{y}'(t) \;=\; A\,\tilde{y}(t)\,,$$

if y is one of (2.1). To find a transformation which carries (2.1) over to

$$(\,\Lambda - t\, I\,)\,\bar{y}'(t) \;=\; (A + \alpha\, I)\,\bar{y}(t)\,, \tag{2.8}$$

we will make some considerations. Regarding the relation between the coefficients of $\hat{x}_k(z)$ and $y_k(t)$ it can be seen that multiplying $\hat{x}_k(z)$ with z^{α} causes that $(\lambda_k - t)^{\mu - l_k - 1}/\Gamma(\mu - l_k)$ carries over to $(\lambda_k - t)^{\mu - l_k - \alpha - 1}/\Gamma(\mu - l_k - \alpha)$. For $\alpha \in \mathbb{N}$ it is obviously the same as computing the α-th derivation and multiplying with $(-1)^{\alpha}$. That is why we want to define the generalized derivation ∂^{α} of order α for arbitrary $\alpha \in \mathbb{C}, \beta \notin -\mathbb{N}$ by the relation

$$\partial^{\alpha} :\; \frac{(\lambda_k - t)^{\beta}}{\Gamma(1 + \beta)} \;\mapsto\; e^{-i\pi\alpha}\, \frac{(\lambda_k - t)^{\beta - \alpha}}{\Gamma(1 + \beta - \alpha)}\,,$$

or equivalently

$$\partial^{\alpha} :\; t^{\beta} \;\mapsto\; t^{\beta - \alpha}\, \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta - \alpha)}\,.$$

On power series, this operator shall be applied termwise. With the previous considerations it follows that $\bar{y} := e^{i\pi\alpha}\,(\partial^{\alpha}\, y)$ is a solution of (2.8), if y is one

of (2.1). Of course, $e^{i\pi\alpha}$ is just a constant, but in this case it is necessary to obtain that $(\lambda_k - t)^{\mu-l_k-1}/\Gamma(\mu - l_k)$ really carries over to $(\lambda_k - t)^{\mu-l_k-\alpha-1}/\Gamma(\mu - l_k - \alpha)$ and we have the analogous transformation to the one for the confluent system. In the following chapters, these transformations will help to express solutions of the hypergeometric system in terms of the coefficient functions (of the first kind) f_k.

Remark 16: Observe that for $\alpha \in -\mathbb{N}$, the operator ∂^α is, in fact, equal to termwise integration of power series. □

Chapter 3

Triangular systems

In this chapter, we will consider the hypergeometric system (2.1) in triangular form, that is

$$A = \begin{pmatrix} l_1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_{21} & l_2 & \ddots & & & \vdots \\ a_{31} & a_{32} & l_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & \cdots & a_{n,n-2} & a_{n,n-1} & l_n \end{pmatrix}$$

is a lower triangular matrix, and Λ remains to be the diagonal matrix with the entries $\lambda_1, \ldots, \lambda_n$. In the following two sections, we always assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct, and in the first section we consider the matrix A only in a very special form. Then we will give in both sections formulas for solutions of the respective system. These solutions build a fundamental solution, if we assume in addition that the values $l_1, \ldots, l_n$ are not natural numbers. Furthermore, if we assume that l_j is not an integer for an arbitrary but fixed j, while the other values l_k are not restricted in any way, there exist the characteristic constants c_{jk} for the hypergeometric system for all $k = 1, \ldots, n$ and we will obtain formulas for them, too. In the last section of this chapter, we discuss a certain triangular system with multiple eigenvalues and specify a fundamental solution for this system, too. These systems will be important in chapter 4 and help us to find formulas for a fundamental solution and the characteristic constants of a general hypergeometric system with distinct eigenvalues. Throughout this chapter, we will use the coefficient functions f_k and d_k which have been introduced and analyzed in the first chapter.

3.1 Special triangular systems

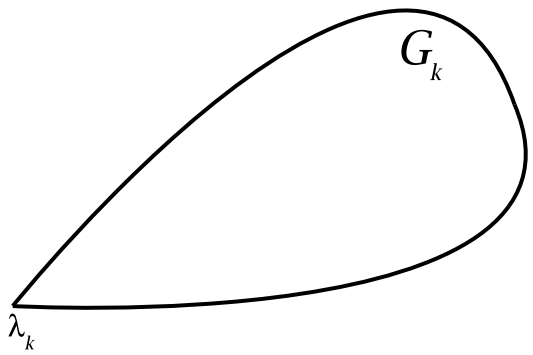
In this section, we will discuss the hypergeometric system (2.1) in a very special triangular form, namely the system with

$$A = \begin{pmatrix} l_1 & 0 & \cdots & \cdots & 0 \\ 1 & l_2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & l_n \end{pmatrix}. \tag{3.1}$$

Furthermore, we assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of Λ are distinct, and we give formulas for solutions of the system by means of the coefficient functions (of the first kind) f_k. These solutions even build a fundamental solution if $l_1, \ldots, l_n$ are not natural numbers. Later we assume that l_j is not an integer for an arbitrary but fixed j, while we make no restrictions on the other values l_k. Then the characteristic constants c_{jk} exist for all k, and we obtain formulas for them by means of the coefficient functions (of the second kind) d_k.

3.1.1 Solutions

First we define regions G_k in the following way: For each $k = 1, \ldots, n$ let G_k be a bounded region so that λ_k is a boundary point of G_k and can be reached from each inner point by a straight line. Furthermore, none of the other values $\lambda_j \neq \lambda_k$ is in G_k. Moreover, we assume that there exists a direction ϱ_k so that the ray from λ_k to ∞ along $\arg(z - \lambda_k)$ is disjoint from G_k. Then we can choose the argument of $(\lambda_k - t)$ to be in the interval $(\varrho_k - \pi, \varrho_k + \pi)$ for all $t \in G_k$.



Remark 17: Observe that for the definiton of G_k, it does not matter if some of the values $\lambda_1, \ldots, \lambda_n$ are equal. □

For each $k = 1, \ldots, n$, we define functions $y_{jk}(t)$ $(k \le j \le n)$ for $t \in G_k$ by

$$y_{kk}(t) \;:=\; \frac{(\lambda_k - t)^{-l_k}}{\Gamma(1 - l_k)}\,,$$

and for $k < j \le n$:

$$y_{jk}(t) \;:=\; e^{i\pi l_k}\,\partial^{l_k}\, f_{j-k}(t - \lambda_k; \lambda_{k+1} - \lambda_k, \ldots, \lambda_j - \lambda_k, l_{k+1} - l_k, \ldots, l_j - l_k)\,,$$

where ∂^{l_k} is the generalized derivation of order l_k which has been defined in Subsection 2.2.4.

Theorem 3.1
For every $k = 1, \ldots, n$, the n-dimensional vector

$$y_k(t) \;:=\; \big(\, \underbrace{0, \ldots, 0}_{k-1 \text{ times}}, y_{kk}(t), \ldots, y_{nk}(t) \,\big)^T$$

is a solution of our special triangular system.

Proof: By means of equation (1.2), the vector

$$h_k(t) \;:=\; \big(\, \underbrace{0, \ldots, 0}_{k-1 \text{ times}}, h_{kk}(t), \ldots, h_{nk}(t) \,\big)^T$$

where $h_{kk}(t) = 1$ and

$$h_{jk}(t) \;:=\; f_{j-k}(t; \lambda_{k+1} - \lambda_k, \ldots, \lambda_j - \lambda_k, l_{k+1} - l_k, \ldots, l_j - l_k)\,,\; k < j \le n\,,$$

satisfies

$$\begin{aligned} h'_{kk}(t) \;&=\; 0\,, \\ (\lambda_j - \lambda_k - t)\, h'_{jk}(t) \;&=\; (l_j - l_k)\, h_{jk}(t) + h_{j-1,k}(t)\,,\; k+1 \le j \le n\,. \end{aligned}$$

With $h_{jk}(t) := 0$ for $1 \le j < k$ this can be extended to

$$\begin{aligned} (-\lambda_k - t)\, h'_{1k}(t) \;&=\; (l_1 - l_k)\, h_{1k}(t)\,, \\ (\lambda_j - \lambda_k - t)\, h'_{jk}(t) \;&=\; (l_j - l_k)\, h_{jk}(t) + h_{j-1,k}(t)\,,\; 2 \le j \le n\,, \end{aligned}$$

which is equivalent to

$$(\,\hat{\Lambda} - t\, I\,)\, h'_k(t) \;=\; \hat{A}\, h_k(t)\,,$$

where $\hat{\Lambda} = \operatorname{diag}(\lambda_1 - \lambda_k, \ldots, \lambda_n - \lambda_k) = \Lambda - \lambda_k\, I$ and

$$\hat{A} \;=\; \begin{pmatrix} l_1 - l_k & 0 & \cdots & \cdots & 0 \\ 1 & l_2 - l_k & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & l_n - l_k \end{pmatrix} \;=\; A - l_k\, I\,.$$

Using the considerations made in Subsection 2.2.4, we see that after applying the two scalar transformations on $h_k(t)$ (that means substituting t by $t-\lambda_k$ and considering the l_k-th derivation times $e^{i\pi l_k}$) this is equivalent to the statement of the theorem. □

Observe that if we restrict the values $l_1,\dots,l_n$ not to be natural numbers, the term $1/\Gamma(1-l_k)$ does not vanish and hence $y_{kk}(t)\not\equiv 0$. This means that $Y(t):=\big(\,y_1(t),\dots,y_n(t)\,\big)$ is a triangular matrix with $\det Y(t)\neq 0$ and therefore leads to

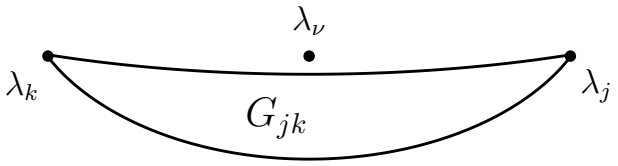
Lemma 3.1
If the values $l_1,\dots,l_n$ are not natural numbers,

$$Y(t)=\big(\,y_1(t),\dots,y_n(t)\,\big)$$

is a fundamental solution of the special triangular hypergeometric system considered here.

3.1.2 Characteristic constants

The function $y_k(t)$ has been defined in the region G_k. To see that $y_k(t)$ can be analytically continued near another eigenvalue λ_j, for arbitrary but fixed j,k $(1\le j,k\le n)$, we consider a region G_{jk} of the following kind:
Let G_{jk} be a bounded and simply connected region so that λ_k and λ_j are boundary points, and G_{jk} does not contain one of the other eigenvalues. Moreover, if possible, G_{jk} shall contain the straight line from λ_k to λ_j, and if there is a value λ_ν $(\neq\lambda_j,\lambda_k)$ on this line, we choose G_{jk} so that λ_ν lies on the left-hand side of G_{jk}, when looking from λ_k to λ_j. For example, this region can look like this:



Since there are no singularities of $y_k(t)$ in the region G_{jk}, the function $y_k(t)$ has an analytic continuation there. Furthermore, observe that, according to Remark 8 resp. Remark 13 the functions

$$f_{j-k}(t-\lambda_k;\lambda_{k+1}-\lambda_k,\dots,\lambda_j-\lambda_k,l_{k+1}-l_k,\dots,l_j-l_k)$$

and

$$d_{j-k}(\lambda_{k+1}-\lambda_k,\dots,\lambda_j-\lambda_k,l_{k+1}-l_k,\dots,l_j-l_k)$$

are well-defined in the region G_{jk}.

Additionally to the condition that the eigenvalues of Λ are distinct, we assume now that l_j is not an integer. Then there exists a unique constant c_{jk} (see Subsection 2.2.3) so that

$$y_k(t) \;=\; c_{jk}\, y_j(t) + \mathrm{reg}(t-\lambda_j)\,. \tag{3.2}$$

With the results of the previous subsection, we now derive a formula for this constant.

Theorem 3.2
If $j>k$, $\Re(l_j)>0$, $\Re(l_j)>\Re(l_k)$, and $l_j-l_k\notin\mathbb{Z}$, then

$$c_{jk} \;=\; \Gamma(1-l_j+l_k)\, d_{j-k}(\lambda_{k+1}-\lambda_k,\ldots,\lambda_j-\lambda_k,l_{k+1}-l_k,\ldots,l_j-l_k)\,,$$

while $c_{jk}=0$ if $j<k$, and $c_{jk}=1$, if $j=k$.

Proof: From (3.2) we get in particular

$$y_{jk}(t) \;=\; c_{jk}\, y_{jj}(t) + \mathrm{reg}(t-\lambda_j)\,.$$

It is obvious that $c_{jk}=1$, if $j=k$ and it is also easy to see that $c_{jk}=0$ for $j<k$, since $y_{jk}(t)=0$, but $y_{jj}(t)\neq 0$ in this case. In the case $j>k$, we get from the equation above and the definition of the functions $y_{jk}(t)$

$$\begin{aligned} e^{i\pi l_k}\,\partial^{l_k}\, f_{j-k}(t-\lambda_k;\lambda_{k+1}-\lambda_k,\ldots,\lambda_j-\lambda_k,l_{k+1}-l_k,\ldots,l_j-l_k) \\ \;=\; c_{jk}\,\frac{(\lambda_j-t)^{-l_j}}{\Gamma(1-l_j)} + \mathrm{reg}(t-\lambda_j)\,. \end{aligned}$$

Writing $\mathrm{reg}(t-\lambda_j)$ as power series $\sum_\nu \tilde{a}_\nu\,(\lambda_j-t)^\nu$, applying ∂^{-l_k} on both sides and multiplying with $e^{-i\pi l_k}$ leads to

$$\begin{aligned} f_{j-k}(t-\lambda_k;\lambda_{k+1}-\lambda_k,\ldots,\lambda_j-\lambda_k,l_{k+1}-l_k,\ldots,l_j-l_k) \\ \;=\; c_{jk}\,\frac{(\lambda_j-t)^{-l_j+l_k}}{\Gamma(1-l_j+l_k)} + \sum_\nu a_\nu\,(\lambda_j-t)^{\nu+l_k}\,, \end{aligned}$$

with suitable a_ν. Now we multiply both sides with $(\lambda_j-t)^{l_j-l_k}$ and consider the limit $t\to\lambda_j$, which leads to the formula for c_{jk} $(j>k)$ because of $\Re(l_j)>0$, $\Re(l_j)>\Re(l_k)$ and Theorem 1.9. □

Remark 18: The constant c_{jk} can be regarded as function in $\lambda_k,\ldots,\lambda_j,l_k,\ldots,l_j$ and that is why we also use the notation $c_{jk}(\lambda_k,\ldots,\lambda_j,l_k,\ldots,l_j)$. In particular, we consider c_{jk} as function in l_j now. In Theorem 3.2 we showed the formula for c_{jk} under the restriction that $\Re(l_j)>0$ and $\Re(l_j)>\Re(l_k)$. However, by means of the difference equation (1.26) for d_k resp. Remark 12 it follows that c_{jk} can be

analytically continued as long as $l_j - l_k \notin \mathbb{Z}$ and $l_j \notin \mathbb{Z}$. Hence these restrictions can be disregarded. □

Remark 19: By means of the difference and differential equations for the coefficient functions d_k (Subsections 1.2.2 and 1.3.1), one directly obtains similar equations for the "constants" $c_{jk}(\lambda_k, \ldots, \lambda_j, l_k, \ldots, l_j)$. □

3.2 Triangular systems with distinct eigenvalues

In this section, we consider the general triangular system with distinct eigenvalues, that is

$$A = \begin{pmatrix} l_1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_{21} & l_2 & \ddots & & & \vdots \\ a_{31} & a_{32} & l_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & \cdots & a_{n,n-2} & a_{n,n-1} & l_n \end{pmatrix},$$

and we again assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of Λ are distinct. Once more, we give solutions of this system by means of the coefficient functions (of the first kind) f_k. Under the extra assumption that $l_1, \ldots, l_n$ are not natural numbers, these solutions again build a fundamental solution. Furthermore, if l_j is not an integer for an arbitrary but fixed j, while the other values l_k are not restricted in any way, the characteristic constants c_{jk} exist for all $k = 1, \ldots, n$, and we obtain formulas for them by means of the coefficient functions (of the second kind) d_k.

3.2.1 Solutions

First we just assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct, and we use the same regions G_k as in the previous section. To simplify the notation of the later formulas we will make another definition:
For $m, k \geq 1$ and $j \geq k + m$, let $\mathcal{B}^j_{m,k}$ be the set of multi-indices given by

$$\mathcal{B}^m_{k,j} := \{\, p = (p_0, \ldots, p_m) : p_i \in \mathbb{N}_0,\, k = p_0 < p_1 < \ldots < p_m = j \,\}.$$

Furthermore, for $p = (p_0, \ldots, p_m) \in \mathcal{B}^m_{k,j}$ we use the abbreviations

$$f_{m,p}(t) := f_m(t; \lambda_{p_1} - \lambda_k, \ldots, \lambda_{p_m} - \lambda_k, l_{p_1} - l_k, \ldots, l_{p_m} - l_k)$$

and

$$a(p) := \prod_{\mu=1}^{m} a_{p_\mu, p_{\mu-1}},$$

and we define functions $g_{jk}(t)$ for $1 \le k < j \le n$ and t near the origin by

$$g_{jk}(t) \;:=\; \sum_{m=1}^{j-k} \sum_{p \in \mathcal{B}_{k,j}^m} a(p)\, f_{m,p}(t)\,.$$

Theorem 3.3
For all $1 \le k \le n$,

$$g_k(t) \;:=\; (\,1, g_{k+1,k}(t), \ldots, g_{nk}(t)\,)^T$$

is a solution of the $(n-k+1)$*-dimensional system*

$$(\,\hat{\Lambda}_k - t\,I\,)\,g_k'(t) \;=\; \hat{A}_k\, g_k(t)\,, \tag{3.3}$$

where $\hat{\Lambda}_k = diag(0, \lambda_{k+1} - \lambda_k, \ldots, \lambda_n - \lambda_k)$ *and*

$$\hat{A}_k \;=\; \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ a_{k+1,k} & l_{k+1} - l_k & \ddots & & & \vdots \\ a_{k+2,k} & a_{k+2,k+1} & l_{k+2} - l_k & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ a_{nk} & \cdots & \cdots & \cdots & a_{n,n-1} & l_n - l_k \end{pmatrix}.$$

Proof: Equation (3.3) holds if and only if the components $g_{jk}(t), k \le j \le n$ of $g_k(t)$ (where $g_{kk}(t) \equiv 1$) solve the differential equations

$$g_{kk}'(t) \;=\; 0\,,$$

$$(\,\lambda_j - \lambda_k - t\,)\,g_{jk}'(t) \;=\; (\,l_j - l_k\,)\,g_{jk}(t) + \sum_{s=k}^{j-1} a_{js}\, g_{sk}(t)\,,\; j \ge k+1\,.$$

For $j = k$, the equation is obviously true and for $j \ge k+1$, we will prove it by induction with respect to j. With variation of constants it follows that a solution of the differential equation for $j \ge k+1$ is given by

$$g_{jk}(t) \;=\; (\lambda_j - \lambda_k - t)^{-(l_j - l_k)} \int_0^t (\lambda_j - \lambda_k - w)^{l_j - l_k - 1} \sum_{s=k}^{j-1} a_{js}\, g_{sk}(w)\, dw\,. \tag{3.4}$$

If $j = k+1$, we get

$$g_{k+1,k}(t) \;=\; (\lambda_{k+1} - \lambda_k - t)^{-(l_{k+1} - l_k)} \int_0^t (\lambda_{k+1} - \lambda_k - w)^{l_{k+1} - l_k - 1}\, a_{k+1,k}\, dw\,,$$

and so

$$g_{k+1,k}(t) \;=\; a_{k+1,k}\, f_1(t; \lambda_{k+1} - \lambda_k, l_{k+1} - l_k)\,,$$

which equals the statement in this case. Assuming that the formula is true for $k+1,\dots,j-1$ and inserting this hypothesis into equation (3.4) we get

$$\begin{aligned} g_{jk}(t) \;&= a_{jk}\, f_1(t;\lambda_j-\lambda_k, l_j-l_k) \\ &\quad + \sum_{s=k+1}^{j-1} a_{js} \sum_{m=1}^{s-k} \sum_{p\in\mathcal{B}_{k,s}^m} a(p)\, f_{m+1}(t;\lambda_p-\lambda_k\, e, \lambda_j-\lambda_k, l_p-l_k\, e, l_j-l_k)\,, \end{aligned}$$

where $\lambda_p-\lambda_k\, e := (\lambda_{p_1}-\lambda_k,\dots,\lambda_{p_m}-\lambda_k)$ and $l_p-l_k\, e := (l_{p_1}-l_k,\dots,l_{p_m}-l_k)$. Now index shift and interchanging summation lead to

$$\begin{aligned} g_{jk}(t) \;&= a_{jk}\, f_1(t;\lambda_j-\lambda_k, l_j-l_k) \\ &\quad + \sum_{m=2}^{j-k} \sum_{s=m+k-1}^{j-1} \sum_{p\in\mathcal{B}_{k,s}^{m-1}} a_{js}\, a(p)\, f_m(t;\lambda_p-\lambda_k\, e, \lambda_j-\lambda_k, l_p-l_k\, e, l_j-l_k)\,, \end{aligned}$$

with $\lambda_p-\lambda_k\, e = (\lambda_{p_1}-\lambda_k,\dots,\lambda_{p_{m-1}}-\lambda_k)$ and $l_p-l_k\, e = (l_{p_1}-l_k,\dots,l_{p_{m-1}}-l_k)$. Thereof we obtain the statement, observing that

$$\Big(\bigcup_{s=m+k-1}^{j-1} \mathcal{B}_{k,s}^{m-1}\Big) \times \{\, j \,\} \;=\; \mathcal{B}_{k,j}^m\,.$$

□

So let us define for all $k=1,\dots,n$ and $t\in G_k$ the vector function

$$\hat{y}_k(t) := (\,\hat{y}_{1k}(t),\dots,\hat{y}_{nk}(t)\,)^T\,,$$

where

$$\begin{aligned} \hat{y}_{kk}(t) \;&=\; \frac{(\lambda_k-t)^{-l_k}}{\Gamma(1-l_k)}\,, \\ \hat{y}_{jk}(t) \;&=\; e^{i\pi l_k}\,\partial^{l_k}\, g_{jk}(t-\lambda_k)\,, \quad j>k \end{aligned}$$

and $\hat{y}_{jk}(t)=0$ for $j<k$. From the previous theorem and the considerations from Subsection 2.2.4 we obtain then

Theorem 3.4
For each $k=1,\dots,n$, the vector function $\hat{y}_k(t)$ is a solution of the general triangular hypergeometric system.

Assuming again that l_k is not a natural number for all $1\le k\le n$ we know that $1/\Gamma(1-l_k)\neq 0$ and hence $\hat{y}_{kk}(t)$ does not vanish. This leads to

Lemma 3.2
If $l_1,\dots,l_n$ are not natural numbers,

$$\hat{Y}(t) \;:=\; \big(\,\hat{y}_1(t),\dots,\hat{y}_n(t)\,\big)$$

is a fundamental solution of the general triangular hypergeometric system (with distinct eigenvalues).

3.2.2 Characteristic constants

Once more, we want l_j not to be an integer for an arbitrary but fixed j ($1 \leq j \leq n$), while the other values l_k shall not be restricted. For a fixed k we again consider the region G_{jk} which has been defined in the previous section. Then there exist the characteristic constant $\hat{c}_{jk}$ which satisfies

$$\hat{y}_k(t) \;=\; \hat{c}_{jk}\,\hat{y}_j(t) + \mathrm{reg}(t-\lambda_j)\,.$$

For each k, this constant also can be expressed in terms of the coefficient functions (of the second kind) d_k what will be proven in the following theorem:

Theorem 3.5
If $j > k$, $\Re(l_j) > 0$, $\Re(l_j) > \Re(l_k)$, and $l_j - l_k \notin \mathbb{Z}$, then

$$\hat{c}_{jk} \;=\; \Gamma(1-l_j+l_k) \sum_{m=1}^{j-k} \sum_{p\in\mathcal{B}^m_{k,j}} a(p)\,d_{m,p}\,,$$

with the abbreviations

$$d_{m,p} = d_m(\lambda_{p_1}-\lambda_k,\ldots,\lambda_{p_m}-\lambda_k, l_{p_1}-l_k,\ldots,l_{p_m}-l_k)$$

and - as before - $a(p) = \prod_{\mu=1}^{m} a_{p_\mu,p_{\mu-1}}$. Furthermore, $\hat{c}_{jk} = 0$ if $j < k$ and $\hat{c}_{jk} = 1$ in the case $j = k$.

Proof: Again the validity of $\hat{c}_{jj} = 1$ and $\hat{c}_{jk} = 0$ for $j < k$ is obvious. In the case $j > k$, we get analogously to the proof of Theorem 3.2

$$e^{i\pi l_k}\,\partial^{l_k}\,g_{jk}(t-\lambda_k) \;=\; \hat{c}_{jk}\,\frac{(\lambda_j-t)^{-l_j}}{\Gamma(1-l_j)} + \mathrm{reg}(t-\lambda_j)$$

and after applying $e^{-i\pi l_k}\,\partial^{-l_k}$ and multiplying with $(\lambda_j-t)^{l_j-l_k}$ on both sides and the same arguments as in the proof of Theorem 3.2

$$(\lambda_j-t)^{l_j-l_k}\,g_{jk}(t-\lambda_k) \;=\; \frac{\hat{c}_{jk}}{\Gamma(1-l_j+l_k)} + (\lambda_j-t)^{l_j}\,\mathrm{reg}(t-\lambda_j)\,.$$

Since $p_m = j$ for all $p = (p_0,\ldots,p_m) \in \mathcal{B}^m_{k,j}$, $\Re(l_j) > 0$, and $\Re(l_j) > \Re(l_k)$, the limit $t \to \lambda_j$ leads to the statement in this case, again using Theorem 1.9. □

Remark 20: Again the formula has been shown under the restrictions that $\Re(l_j) > 0$ and $\Re(l_j) > \Re(l_k)$. Once more, the difference equations for the coefficient functions d_k allow the analytic continuation for $\hat{c}_{jk}$, and hence these restrictions can be ignored. □

Remark 21: The characteristic constant $\hat{c}_{jk}$ for the general triangular system can also be written as linear combination of the constants for the special triangular system which has been considered in Subsection 3.1.2. Namely, Theorem 3.2 leads to

$$\hat{c}_{jk} \;=\; \sum_{m=1}^{j-k} \sum_{p \in \mathcal{B}_{k,j}^m} a(p)\, c_{m+k,k}(\lambda_{p_0}, \ldots, \lambda_{p_m}, l_{p_0}, \ldots, l_{p_m})\,.$$

□

3.3 Triangular systems with multiple eigenvalues

Now we analyze a certain triangular hypergeometric system

$$(\,\Lambda - t\,I\,)\,y'(t) \;=\; A\,y(t)\,,$$

which is not restricted to have distinct eigenvalues, but satisfies some other conditions. In the face of the next chapter, we consider a system of dimension $s \cdot n$ with an arbitrary but fixed $s \in \mathbb{N}$. Here the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{sn})$ shall satisfy $\lambda_{j+\mu n} = \lambda_j$ for all $\mu = 1, \ldots, s-1$, $j = 1, \ldots, n$, and $\lambda_1, \ldots, \lambda_n$ shall be distinct. Furthermore, for the lower triangular matrix

$$A \;=\; \begin{pmatrix} l_1 & 0 & \cdots & \cdots & 0 \\ a_{21} & l_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ a_{sn,1} & \cdots & \cdots & a_{sn,sn-1} & l_{sn} \end{pmatrix}$$

we assume that $a_{jk} = 0$ if $j - k \geq n$ and that the diagonal elements satisfy $l_{j+\mu n} = l_j$ for all $\mu = 1, \ldots, s-1$, $j = 1, \ldots, n$.

Remark 22: Because of the assumption $a_{jk} = 0$ for $j - k \geq n$, the product $a(p) = \prod_{\mu=1}^m a_{p_\mu, p_{\mu-1}}$ vanishes for many $p \in \mathcal{B}_{k,j}^m$. That is why we consider in the following the subset $\mathcal{C}_{k,j}^m$ of $\mathcal{B}_{k,j}^m$ which is defined by

$$\mathcal{C}_{k,j}^m \;:=\; \{\, p = (p_0, \ldots, p_m) \in \mathcal{B}_{k,j}^m : p_\nu - p_{\nu-1} < n\,,\; \nu = 1, \ldots, m \,\}\,.$$

Observe that for μ, ν with $\lambda_\mu = \lambda_\nu, \mu \neq \nu$ it follows that $|\mu - \nu| \geq n$. Hence we obtain for arbitrary $p \in \mathcal{C}_{k,j}^m$ that $\lambda_{p_\nu} \neq \lambda_{p_{\nu-1}}$, for all $\nu = 1, \ldots, m$ and, in particular, $\lambda_k \neq \lambda_{p_1}$. Furthermore, $\lambda_\mu = \lambda_\nu$ implies $l_\mu = l_\nu$. From these properties and Theorem 1.5, it follows that the function

$$f_m(t; \lambda_{p_1} - \lambda_k, \ldots, \lambda_{p_m} - \lambda_k, l_{p_1} - l_k, \ldots, l_{p_m} - l_k)$$

is well-defined for t near the origin and arbitrary $1 \leq k < j \leq sn$, $1 \leq m \leq j-k$, and $(p_0, \ldots, p_m) \in \mathcal{C}^m_{k,j}$. □

According to the previous remark, we can define functions $g_{jk}(t)$ for all $1 \leq k < j \leq sn$ analogously to the case of distinct eigenvalues by

$$g_{jk}(t) \;:=\; \sum_{m=1}^{j-k} \sum_{p \in \mathcal{C}^m_{k,j}} a(p)\, f_{m,p}(t)\,,$$

where $f_{m,p}(t)$ and $a(p)$ again denote the abbreviations of

$$f_m(t; \lambda_{p_1} - \lambda_k, \ldots, \lambda_{p_m} - \lambda_k, l_{p_1} - l_k, \ldots, l_{p_m} - l_k)$$

resp. $\prod_{\mu=1}^m a_{p_\mu, p_{\mu-1}}$. Furthermore, for $k = 1, \ldots, sn$ and $t \in G_k$ we also define vector functions

$$\tilde{y}_k(t) \;:=\; (\, \tilde{y}_{1k}(t), \ldots, \tilde{y}_{sn,k}(t)\,)^T\,,$$

where

$$\begin{aligned} \tilde{y}_{jk}(t) \;&=\; 0\,, \quad 1 \leq j < k\,, \\ \tilde{y}_{kk}(t) \;&=\; \frac{(\lambda_k - t)^{-l_k}}{\Gamma(1-l_k)} \end{aligned}$$

and

$$\tilde{y}_{jk}(t) \;=\; e^{i\pi l_k}\, \partial^{l_k}\, g_{jk}(t - \lambda_k)\,, \quad k < j \leq sn\,.$$

With the same considerations as in the previous section, we then obtain that for each $k = 1, \ldots, sn$, the vector function $\tilde{y}_k(t)$ is a solution of the system considered in this section. Once more, if we assume in addition that $l_1, \ldots, l_n$ are not natural numbers,

$$\tilde{Y}(t) \;:=\; \big(\, \tilde{y}_1(t), \ldots, \tilde{y}_n(t)\,\big)$$

is a fundamental solution of this system.

Remark 23: Observe that $(\lambda_k - t)^{l_k}\, \tilde{y}_{jk}(t) = \mathrm{reg}(t - \lambda_k)$ and has a zero at λ_k for all $1 \leq k < j \leq sn$ and therefore

$$\lim_{t \to \lambda_k} (\lambda_k - t)^{l_k}\, \tilde{y}_k(t) \;=\; \frac{e_k}{\Gamma(1-l_k)}\,,$$

where e_k denotes the k-th unit-vector. □

Proof: From the first chapter we know that for all $m \geq 1$, the coefficient function $f_m(t)$ is holomorphic at the origin and has a zero there. Hence we obtain for $j > k$ that $g_{jk}(t - \lambda_k)$ is holomorphic at λ_k and vanishes there. With the definition of the generalized derivation (of order l_k) it follows that $\tilde{y}_{jk}(t)$ can be written in the form $(\lambda_k - t)^{-l_k} \sum_{\nu=1}^\infty a_\nu (t - \lambda_k)^\nu$ with suitable a_ν and this directly leads to the statement. □

Chapter 4

General systems

In this chapter, we will finally be concerned with the hypergeometric system (2.1) in general form, that means

$$A \;=\; \begin{pmatrix} l_1 & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1,n} \\ a_{n1} & \cdots & \cdots & a_{n,n-1} & l_n \end{pmatrix}$$

is an arbitrary $(n \times n)$-matrix and Λ is - as before - the diagonal matrix with the entries $\lambda_1, \ldots, \lambda_n$. Again we assume that all eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct. We are primarily interested in the characteristic constants of this system. As we already mentioned in the introduction, we concentrate on one of them, namely c_{n1} and assume that l_n is not an integer to assure the existence of this constant.

Throughout this chapter, we will use the main result of [22] and introduce a new parameter ε. We will consider a generalized system

$$(\,\Lambda - t\,I\,)\,y'(t) \;=\; A(\varepsilon)\,y(t)\,,$$

where $A(\varepsilon)$ depends in a special form on ε. Then the solutions and constants depend also on ε, and it has been shown in [22] that they are entire functions in ε. Hence we can write solutions of the generalized system as power series expansions in this parameter, and by means of Section 3.3, we obtain formulas for their coefficients in terms of the coefficient functions (of the first kind) f_k. Thereof we obtain later formulas for the coefficients of the power series expansion of $c_{n1}(\varepsilon)$ in terms of the coefficient functions (of the second kind) d_k.

4.1 Introducing a parameter

As in the author's diploma thesis [22], we will introduce a parameter ε, and instead of the differential equations (2.1) and (2.2) we will investigate the equations

$$(\Lambda - t\,I)\,y'(t) \;=\; A(\varepsilon)\,y(t)\,, \tag{4.1}$$

and

$$z x'(z) \;=\; (z\,\Lambda + A(\varepsilon))\,x(z)\,. \tag{4.2}$$

Thereby $A(\varepsilon) := U + \varepsilon R$, where

$$U \;:=\; \begin{pmatrix} l_1 & 0 & \cdots & \cdots & 0 \\ a_{21} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ a_{n1} & \cdots & \cdots & a_{n,n-1} & l_n \end{pmatrix}$$

is the lower triangular part of the original matrix A and

$$R \;:=\; \begin{pmatrix} 0 & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

is the upper triangular part of A without the diagonal elements. That means $A(1) = A$, and $A(0)$ is the triangular matrix U. In the considered generic case that all eigenvalues of Λ (that is $\lambda_1, \ldots, \lambda_n$) are distinct, we can apply the main result of [22] and obtain that the fundamental solutions $X(z;\varepsilon;\nu)$ and the corresponding Stokes' Multipliers $V(\varepsilon;\nu)$ of the confluent system (4.2) are entire functions in ε. So we get an analytic transition from the triangular system to the general system. Because of the close relation via Laplace Transform, the matrix solution $Y(t;\varepsilon)$ of (4.1) is an entire function in ε, too. Hence there exists a power series expansion of the kind

$$Y(t;\varepsilon) \;=\; \sum_{s=0}^{\infty} \varepsilon^s\, Y_s(t)\,.$$

Since inserting this power series in (4.1) and comparing coefficients lead to

$$\begin{aligned} (\Lambda - t\,I)\,Y_0'(t) \;&=\; U\,Y_0(t) \\ (\Lambda - t\,I)\,Y_s'(t) \;&=\; U\,Y_s(t) + R\,Y_{s-1}(t)\,, \quad s \geq 1\,, \end{aligned}$$

it follows that for all $s \geq 0$ the matrix $\mathcal{Y}_s(t)$ defined by

$$\mathcal{Y}_s(t) := \begin{pmatrix} Y_0(t) & 0 & \cdots & \cdots & 0 \\ Y_1(t) & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ Y_s(t) & \cdots & \cdots & Y_1(t) & Y_0(t) \end{pmatrix}$$

solves the $(s+1)\cdot n$ -dimensional hypergeometric system

$$(\,\tilde{\Lambda}_s - t\,I\,)\,\mathcal{Y}_s'(t) \;=\; \tilde{A}_s\,\mathcal{Y}_s(t)\,, \tag{4.3}$$

where

$$\tilde{\Lambda}_s \;=\; \begin{pmatrix} \Lambda & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \Lambda \end{pmatrix}$$

and

$$\tilde{A}_s \;=\; \begin{pmatrix} U & 0 & \cdots & \cdots & 0 \\ R & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R & U \end{pmatrix},$$

each in the accordant size.

Remark 24: In the following we denote the diagonal elements of $\tilde{\Lambda}_s$ by $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{(s+1)n}$, the one of $\tilde{A}_s$ by $\tilde{l}_1, \ldots, \tilde{l}_{(s+1)n}$ and the off-diagonal elements of $\tilde{A}_s$ by $\tilde{a}_{\mu\nu}$. Then for all $\mu = 1, \ldots, (s+1)n$, the structure of the matrices leads to $\tilde{\lambda}_\mu = \lambda_{(\mu \mod n)}$ and $\tilde{l}_\mu = l_{(\mu \mod n)}$ with $(\mu \mod n) \in \{1, \ldots, n\}$. Furthermore, $\tilde{a}_{\mu\nu} = 0$ if $\nu > \mu$ or $\mu - \nu \geq n$, and for $0 < \mu - \nu < n$, we obtain $\tilde{a}_{\mu\nu} = a_{(\mu \mod n),(\nu \mod n)}$. That is why we will use the notation

$$\tilde{\mu} := (\mu \mod n)$$

for all indices μ. Moreover, for a given multi-index $p = (p_0, \ldots, p_m) \in \mathcal{C}^m_{k,j}$, the multi-index $(\tilde{p}_0, \ldots, \tilde{p}_m)$ is denoted by $\tilde{p}$. □

For all $s \in \mathbb{N}_0$, the system (4.3) is a triangular system, and for $s \geq 1$ it has multiple eigenvalues. Observing that the system satisfies all conditions from

Section 3.3, we can derive formulas for $\mathcal{Y}_s(t)$ for all $s \geq 0$ from the results given there. So we can finally express all the coefficients

$$Y_s(t) = (y_s^{(1)}(t), \ldots, y_s^{(n)}(t))$$

of the power series expansion of $Y(t;\varepsilon)$ in terms of the coefficient functions (of the first kind) f_k. Therefore we define for each $k = 1, \ldots, n$ and $t \in G_k$ the vector function

$$y_s^{(k)}(t) := (y_s^{(1k)}(t), \ldots, y_s^{(nk)}(t))^T ,$$

where $y_s^{(jk)}(t)$ is given by

$$\begin{aligned}
y_0^{(jk)}(t) &= 0 , \qquad 1 \leq j \leq k-1 , \\
y_0^{(kk)}(t) &= \frac{(\lambda_k - t)^{-l_k}}{\Gamma(1-l_k)} , \\
y_0^{(jk)}(t) &= e^{i\pi l_k}\, \partial^{l_k} \sum_{m=1}^{j-1} \sum_{p \in \mathcal{C}_{k,j}^m} a(\tilde{p})\, f_{m,\tilde{p}}(t-\lambda_k) , \quad k+1 \leq j \leq n ,
\end{aligned}$$

and for $s \geq 1$ with the notation $j_s := sn + j$

$$y_s^{(jk)}(t) = e^{i\pi l_k}\, \partial^{l_k} \sum_{m=1}^{j_s-1} \sum_{p \in \mathcal{C}_{k,j_s}^m} a(\tilde{p})\, f_{m,\tilde{p}}(t-\lambda_k) , \quad 1 \leq j \leq n .$$

Thereby, $f_{m,\tilde{p}}(t-\lambda_k)$ denotes the abbreviation of

$$f_m(t-\lambda_k; \lambda_{\tilde{p}_1} - \lambda_k, \ldots, \lambda_{\tilde{p}_m} - \lambda_k, l_{\tilde{p}_1} - l_k, \ldots, l_{\tilde{p}_m} - l_k)$$

and $a(\tilde{p})$ the one of

$$\prod_{\mu=1}^{m} a_{\tilde{p}_\mu, \tilde{p}_{\mu-1}} .$$

Now the results from Section 3.3 imply

Lemma 4.1
For all $k = 1, \ldots, n$, $\nu = 0, \ldots, s$, the $(s+1) \cdot n$ -dimensional vector function

$$\mathsf{y}_s^{(\nu n + k)}(t) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \\ y_0^{(k)}(t) \\ \vdots \\ y_{s-\nu}^{(k)}(t) \end{pmatrix}$$

solves the hypergeometric system (4.3). Moreover, if the values $l_1, \ldots, l_n$ are not natural numbers, then

$$\mathcal{Y}_s(t) \;:=\; \big(\, \mathsf{y}_s^{(1)}(t), \ldots, \mathsf{y}_s^{((s+1)\cdot n)}(t) \,\big)$$

is a fundamental solution of (4.3).

From this lemma we directly obtain - with the considerations above - the following theorem:

Theorem 4.1
For all $k = 1, \ldots, n$, the vector function

$$y^{(k)}(t;\varepsilon) \;=\; \sum_{s=0}^{\infty} \varepsilon^s \, y_s^{(k)}(t)$$

is a solution of the general hypergeometric system (4.1). Furthermore, if we assume that none of the values $l_1, \ldots, l_n$ is a natural number, then

$$Y(t;\varepsilon) = \big(\, y^{(1)}(t;\varepsilon), \ldots, y^{(n)}(t;\varepsilon) \,\big)$$

is a fundamental solution of (4.1).

4.2 Characteristic constants for general systems

In this section, we consider the characteristic constants (in the case of existence) of the general hypergeometric system (4.1). According to the following remark, we will concentrate on the constant c_{n1}.

Remark 25: It is desirable to find formulas for the constants c_{jk} for all $j, k = 1, \ldots, n, j \neq k$, for which they exist (The case $k = j$ is trivial, since c_{jj} always equals 1). Since we can interchange rows and columns of the matrices Λ and A by simple transformations of the solutions, it suffices if we derive the one for c_{n1} (or anyone else) depending on the entries of Λ and A. Because then c_{jk} can be expressed by the same formula depending on the entries of $\tilde{\Lambda} = P^{-1}\Lambda P$ and $\tilde{A} = P^{-1}AP$, where P is a transformation matrix which causes that certain rows and columns of Λ and A will be interchanged, more precisely $P = (p_{\mu\nu})_{\mu,\nu=1}^{n}$ with

$$p_{\mu\nu} \;=\; \begin{cases} 1 & , \quad \mu = \nu \neq 1, k, j, n \\ 1 & , \quad (\mu,\nu) \in \{\, (1,k), (k,1), (j,n), (n,j) \,\} \\ 0 & , \quad \text{else} \end{cases} \quad .$$

Another interesting fact is that interchanging each two rows and columns of Λ and A which are not the first or the n-th ones causes that c_{n1} remains to be the same value, although the entries have changed their order (that means

$c_{n1}(\Lambda, A) = c_{n1}(\hat{\Lambda}, \hat{A})$). So, in this sense, $c_{n1}(\Lambda, A)$ can be regarded to be a symmetric function. □

Without loss of generality, we set $\lambda_1 := 0$ and $l_1 := 0$, since we can always reach this situation using the scalar transformations from Subsection 2.2.4. The diagonal elements of $A(\varepsilon)$ are the same as the one of A, namely $l_1, \ldots, l_n$. Since the eigenvalues $\lambda_1, \ldots, \lambda_n$ were assumed to be distinct, λ_n particularly is a single eigenvalue. Now we assume in addition that l_n is not an integer, while we do not restrict the values $l_2, \ldots, l_{n-1}$ in any way. Then, according to Subsection 2.2.3, the characteristic constant $c_{n1}(\varepsilon)$ satisfying

$$y^{(1)}(t;\varepsilon) \;=\; c_{n1}(\varepsilon)\, y^{(n)}(t;\varepsilon) + \mathrm{reg}(t-\lambda_n) \tag{4.4}$$

exists. The assumptions even assure the existence of $c_{nk}(\varepsilon)$, for all $k = 1, \ldots, n$, but in the following, we concentrate on $c_{n1}(\varepsilon)$. With the results of the previous section, we will be able to express this constant by means of the coefficient functions (of the second kind) d_k.

Because of the intimate relation between the characteristic constants and the Stokes' Multipliers $V(\varepsilon;\nu)$, it follows that $c_{n1}(\varepsilon)$ is an entire function in ε, too. So we can expand $c_{n1}(\varepsilon)$ in a power series like

$$c_{n1}(\varepsilon) \;=\; \sum_{s=0}^{\infty} \varepsilon^s\, c_s^{(n1)}\,.$$

Inserting the power series expansions for $y^{(1)}(t;\varepsilon)$, $y^{(n)}(t;\varepsilon)$ and $c_{n1}(\varepsilon)$ into equation (4.4), we obtain

$$\sum_{s=0}^{\infty} \varepsilon^s\, y_s^{(1)}(t) \;=\; \sum_{s=0}^{\infty} \varepsilon^s \sum_{q=0}^{s} c_{s-q}^{(n1)}\, y_q^{(n)}(t) + \mathrm{reg}(t-\lambda_n)\,,$$

and for all $s \geq 0$, comparing coefficients then leads to

$$y_s^{(1)}(t) \;=\; \sum_{q=0}^{s} c_{s-q}^{(n1)}\, y_q^{(n)}(t) + \mathrm{reg}(t-\lambda_n)\,. \tag{4.5}$$

Remark 26: Observe that for arbitrary $m \geq 1$, $p \in \mathcal{C}_{1,n_s}^m$, and $\Re(l_n) > 0$, the tupel $(\lambda_{\tilde{p}_1}, \ldots, \lambda_{\tilde{p}_m}, l_{\tilde{p}_1}, \ldots, l_{\tilde{p}_m})$ is in the enlarged domain of definition for d_m (see Remark 13). □

Theorem 4.2
If $\Re(l_n) > 0$ (and $\lambda_1 = l_1 = 0$, $l_n \notin \mathbb{Z}$), the characteristic constant $c_{n1}(\varepsilon)$ corresponding to the general hypergeometric system (4.1) is given by

$$c_{n1}(\varepsilon) \;=\; \sum_{s=0}^{\infty} \varepsilon^s \, c_s^{(n1)} \,,$$

with

$$c_s^{(n1)} \;=\; \Gamma(1-l_n) \sum_{m=1}^{n_s-1} \sum_{p \in \mathcal{C}_{1,n_s}^m} a(\tilde{p}) \, d_{m,\tilde{p}} \,,$$

where again $n_s := (s+1)n$ and $a(\tilde{p}) = \prod_{\mu=1}^{m} a_{\tilde{p}_\mu, \tilde{p}_{\mu-1}}$, while $d_{m,\tilde{p}}$ is the abbreviation of

$$d_m(\lambda_{\tilde{p}_1}, \ldots, \lambda_{\tilde{p}_m}, l_{\tilde{p}_1}, \ldots, l_{\tilde{p}_m}) \,.$$

Proof: If we just consider the n-th row of equation (4.5), we obtain

$$y_s^{(n1)}(t) \;=\; \sum_{q=0}^{s} c_{s-q}^{(n1)} \, y_q^{(nn)}(t) + \mathrm{reg}(t-\lambda_n) \,,$$

which is equivalent to

$$c_s^{(n1)} \, y_0^{(nn)}(t) \;=\; y_s^{(n1)}(t) - \sum_{q=1}^{s} c_{s-q}^{(n1)} \, y_q^{(nn)}(t) + \mathrm{reg}(t-\lambda_n) \,. \tag{4.6}$$

From the previous section we have

$$y_s^{(n1)}(t) \;=\; \sum_{m=1}^{n_s-1} \sum_{p \in \mathcal{C}_{1,n_s}^m} a(\tilde{p}) \, f_{m,\tilde{p}}(t) \,, \; s \geq 0 \,,$$

$$y_0^{(nn)}(t) \;=\; \frac{(\lambda_n - t)^{-l_n}}{\Gamma(1-l_n)} \,,$$

and because of Remark 23 (p. 45) we know that

$$y_q^{(nn)}(t) \;=\; (\lambda_n - t)^{-l_n} \sum_{\nu=1}^{\infty} b_\nu (t-\lambda_n)^\nu \,, \; q \geq 1 \tag{4.7}$$

with suitable b_ν. Inserting the formula for $y_0^{(nn)}(t)$ in (4.6) leads to

$$c_s^{(n1)} \;=\; \Gamma(1-l_n) \, (\lambda_n - t)^{l_n} \left(y_s^{(n1)}(t) - \sum_{q=1}^{s} c_{s-q}^{(n1)} \, y_q^{(nn)}(t) + \mathrm{reg}(t-\lambda_n) \right) .$$

Now we consider the limiting value $t \to \lambda_n$. From (4.7) it follows that

$$(\lambda_n - t)^{l_n}\, y_q^{(nn)}(t) \;\to\; 0 \quad (t \to \lambda_n) \quad \forall\, q \geq 1\,,$$

and because of $\Re(l_n) > 0$

$$(\lambda_n - t)^{l_n}\, \mathrm{reg}(t - \lambda_n) \;\to\; 0 \quad (t \to \lambda_n)\,.$$

Hence we obtain

$$\begin{aligned} c_s^{(n1)} &= \Gamma(1 - l_n) \lim_{t \to \lambda_n} (\lambda_n - t)^{l_n}\, y_s^{(n1)}(t) \\ &= \Gamma(1 - l_n) \sum_{m=1}^{n_s - 1} \sum_{p \in \mathcal{C}_{1,n_s}^m} a(\tilde{p}) \lim_{t \to \lambda_n} (\lambda_n - t)^{l_n}\, f_{m,\tilde{p}}(t)\,, \end{aligned}$$

and by means of Theorem 1.9 this leads to

$$c_s^{(n1)} \;=\; \Gamma(1 - l_n) \sum_{m=1}^{n_s - 1} \sum_{p \in \mathcal{C}_{1,n_s}^m} a(\tilde{p})\, d_{m,\tilde{p}}\,,$$

which completes the proof. □

Remark 27: Once more, the formula for the constant $c_{n1}(\varepsilon)$ has only been shown under the assumption that $\Re(l_n) > 0$. However, with the difference equations for the functions d_k which we derived in the first chapter, this restriction can be ignored again (but not the restriction that l_n is not an integer). □

Appendix A

k-Summability

Given a differential equation, it is in most cases possible to compute a formal solution, that means a power series ansatz leads to some formulas for the coefficients, but in general, the radius of convergence of this power series is equal to zero. We will explain here why such series are still helpful to get proper solutions (of the considered differential equation). For more informations the reader is referred to [4].

A.1 Sector and sectorial region

A *sector* (at infinity) on the Riemann surface of the logarithm is a set of the kind

$$S \;=\; S(d,\alpha,\theta) \;=\; \left\{\, z\in\mathbb{C}:\, |z|>\theta,\, |d-\arg z|<\frac{\alpha}{2}\,\right\},$$

with $d\in\mathbb{R}, \alpha>0$ and $\theta\geq 0$, where d denotes the *bisecting direction*, α the *opening* and θ the *radius* of S. If $\theta=0$ we write for short $S(d,\alpha)$ instead of $S(d,\alpha,0)$ Analogously, a *closed sector* is of the form

$$\bar{S} \;=\; \bar{S}(d,\alpha,\theta) \;=\; \left\{\, z\in\mathbb{C}:\, |z|\geq\theta,\, |d-\arg z|\leq\frac{\alpha}{2}\,\right\},$$

whereas θ is restricted to $\theta>0$. Furthermore a region $\mathcal{G}$ will be called *sectorial region*, if there exist $d,\alpha>0$ so that $\mathcal{G}\subset S(d,\alpha)$ and for every $\beta\in(0,\alpha)$ there exists a $\theta>0$ with $\bar{S}(d,\beta,\theta)\subset\mathcal{G}$. Again we will name d the bisecting direction and α the opening of $\mathcal{G}$ and write $\mathcal{G}(d,\alpha)$. Moreover, by $\mathcal{H}(\mathcal{G})$ we will denote the set of all functions which are holomorphic in $\mathcal{G}$.

A.2 Formal power series, asymptotic expansions, and k-summability

A *formal power series* (in z^{-1}) is a series $\hat{f}(z)=\sum_{n=0}^{\infty} f_n z^{-n}$, $f_n\in\mathbb{C}$, where the coefficients f_n are not restricted in any way and so the radius of convergence of

this series may be equal to zero. If $f \in \mathcal{H}(\mathcal{G})$ and $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^{-n}$ is a formal power series, then one says that $\hat{f}(z)$ is the *asymptotic expansion* of $f(z)$ in $\mathcal{G}$, or $f(z)$ *asymptotically equals* $\hat{f}(z)$ as $z \to \infty$ in $\mathcal{G}$, if to every $N \in \mathbb{N}$ and every closed subsector $\bar{S} \subset \mathcal{G}$ there exists a constant $c = c(N, \bar{S})$ so that

$$\left\| z^N \left(f(z) - \sum_{n=0}^{N-1} f_n \, z^{-n} \right) \right\| \ \leq \ c \, , \, \forall \, z \in \bar{S} \, .$$

In this case, we write for short $f(z) \cong \hat{f}(z)$ in $\mathcal{G}$.

For an arbitrary, but fixed $s > 0$, the formal power series $\hat{f}(z)$ is the *asymptotic expansion of Gevrey order* s of $f(z)$ in $\mathcal{G}$, or $f(z)$ *asymptotically equals* $\hat{f}(z)$ *of Gevrey order* s, if for every closed subsector $\bar{S} \subset \mathcal{G}$ there exist constants $c, K > 0$ so that for every $N \in \mathbb{N}_0$

$$\left\| z^N \left(f(z) - \sum_{n=0}^{N-1} f_n \, z^{-n} \right) \right\| \ \leq \ c \, K^N \, \Gamma(1 + sN) \, , \, \forall \, z \in \bar{S} \, ,$$

and we write for short $f(z) \cong_s \hat{f}(z)$ in $\mathcal{G}$. Given a formal power series $\hat{f}(z)$, $d \in \mathbb{R}$ and $k > 0$, one says that $\hat{f}$ is *k-summable in direction d*, if there exist a sectorial region $\mathcal{G} = \mathcal{G}(d, \alpha)$ with $\alpha > \pi/k$ and a function $f \in \mathcal{H}(\mathcal{G})$ which asymptotically equals $\hat{f}$ of Gevrey order $1/k$ in $\mathcal{G}$. The function f then is called the *k-sum of $\hat{f}$ in direction d*. It is a very important property of the k-sum that if $\hat{f}$ is a formal solution of a differential equation, then its k-sum is a proper solution of the same equation.

A.3 Laplace Operator, formal Borel Operator, and computation of the k-sum

Let $h(u)$ be a function which is continuous at the origin, holomorphic in a sector $S = S(d, \alpha)$ and of exponential growth at most $k > 0$ in S (that is for all $\varphi \in (0, \alpha)$ exist $\zeta, c_1, c_2 > 0$, so that for each u with $|u| \geq \zeta$ and $|d - \arg u| \leq \varphi/2$ the following estimate holds: $||h(u)|| \leq c_1 \exp(c_2 |u|^k))$. Then the *Laplace Operator of order k* (for short $\mathcal{L}_k$) for $h(u)$ is defined by

$$(\mathcal{L}_k \, h)(z) \ = \ z^k \int_0^{\infty(\sigma)} \exp(-(zu)^k) \, h(u) \, du^k \, ,$$

with integration along $\arg u = \sigma$, where σ has to satisfy $|d - \sigma| < \alpha/2$. Given a formal power series in z^{-1}, $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^{-n}$, the operator

$$(\hat{\mathcal{B}}_k \, \hat{f})(z) \ = \ \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + n/k)} \, f_n$$

is called the *formal Borel Operator of order k*.

To compute the k-sum of a summable formal power series, we use the Laplace Operator and the formal Borel Operator. The formal power series $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^{-n}$ is k-summable in direction d if and only if $\hat{g}(u) = (\hat{\mathcal{B}}_k \hat{f})(u)$ converges and leads to a function $h(u)$ which is holomorphic and of exponential growth at most k in a sector $\tilde{S} = S(d, \varepsilon)$ (for a $\varepsilon > 0$). The k-sum of $\hat{f}$ (in direction d) then is given by $f(z) = (\mathcal{L}_k h)(z)$ (with integrating along rays close to d).

Appendix B

Special functions and useful theorems

B.1 The Gamma function

The *Gamma function* $\Gamma(z)$ can be seen to be a extension of the factorial function. For $\Re(z) > 0$, we define

$$\Gamma(z) \;=\; \int_0^\infty e^{-t}\, t^{z-1}\, dt\,,$$

which is also called *Euler's integral of the second kind*. For arbitrary $a, b > 0$, the integral converges uniformly in the strip $\{\, z \in \mathbb{C} : a \leq \Re(z) \leq b \,\}$. And since the integrand is a continuous function with respect to both variables t and z and holomorphic with respect to z (for each fixed t), it follows that $\Gamma(z)$ is holomorphic for $\Re(z) > 0$. Furthermore, the Gamma function satisfies the functional equation $\Gamma(z+1) = z\,\Gamma(z)$ (which has often been used in this work) and because of $\Gamma(1) = 1$, it follows that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. That is why we can consider it as extension of the factorial function. By means of the functional equation the Gamma function can be analytically continued into the left half-plane with exception of the points $0, -1, -2, \ldots$, which are simple poles of the Gamma function. Another interesting property is given in *Stirling's formula*:

$$\Gamma(z) \;\sim\; e^{-z}\, z^z \sqrt{2\pi/z} \qquad (\Re(z) \to \infty) \tag{B.1}$$

We could give a lot of more properties of the Gamma function here, but they are not of importance for this work, and therefore we refer an interested reader to [20], pp. 31-38.

B.2 The hypergeometric function

Another special function is the so-called *hypergeometric function* $F(\alpha, \beta; \gamma; z)$. For $\alpha, \beta \in \mathbb{C}$, $\gamma \in \mathbb{C} \setminus -\mathbb{N}_0$ and $z \in \mathbb{C}$ with $|z| < 1$, it is defined by the series

$$F(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n \, (\beta)_n}{(\gamma)_n \, n!} \, z^n \,,$$

where for arbitrary $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$, the symbol $(x)_n$ denotes the Pochhammer symbol $(x)_0 := 1$ and $(x)_n := x(x+1) \cdot \ldots \cdot (x+n-1)$, $n \geq 1$. $F(\alpha, \beta; \gamma; z)$ solves a scalar differential equation of order 2, namely the *hypergeometric differential equation*

$$y'' + \frac{\gamma - (\alpha + \beta + 1)z}{z(1-z)} \, y' - \frac{\alpha\beta}{z(1-z)} \, y \;=\; 0 \,.$$

Furthermore, the hypergeometric function can be analytically continued for $z \in \mathbb{C} \setminus [1, \infty)$, and if $Re(\gamma) > \Re(\gamma - \beta) > 0$, it has the integral representation

$$F(\alpha, \beta; \gamma; z) \;=\; \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma - \beta)} \int_0^1 u^{\beta-1} \, (1-u)^{\gamma-\beta-1} \, (1 - zu)^{-\alpha} \, du \,. \qquad \text{(B.2)}$$

For more informations about the hypergeometric function see e. g. [11] or [21].

B.3 Riemann removable singularities theorem

A well-known theorem in the complex analysis is the following *removable singularities theorem of Riemann*. For the proof see e.g. [26], p. 90f.

Theorem B.1
Let Ω be an open subset of $\mathbb{C}$ and $z \in \Omega$. Suppose that f is continuous in Ω and holomorphic in $\Omega \setminus \{z\}$. Then f is holomorphic in Ω.

B.4 Hartogs' theorem

Before specifying *Hartogs' theorem*, we have to clear what is meant with holomorphy in several variables. Therefore we want to use the following definition which is in common use (see e.g. [9], p. 2):

Definition 1
Let Ω be an open subset of $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$. Then f is holomorphic in Ω, if for every point $(b_1, \ldots, b_n) \in \Omega$ there exist an open set $U \subset \Omega$ with $(b_1, \ldots, b_n) \in U$ and a power series of the kind

$$\sum_{j_1, \ldots, j_n \geq 0} a_{j_1 \cdots j_n} \, (z_1 - b_1)^{j_1} \cdot \ldots \cdot (z_n - b_n)^{j_n} \,,$$

so that for each $z = (z_1, \ldots, z_n) \in U$, the power series converges to $f(z)$.

Hartogs' theorem, which follows now, declares that partial holomorphy leads to holomorphy in several variables. This is a non-trivial result, remembering that the analogon for real functions is not true, since there exist functions which are partial differentiable but not total differentiable. The proof of the theorem can be found for example in [10], p. 29.

Theorem B.2
If f is a complex-valued function, defined in the open set $\Omega \subset \mathbb{C}^n$, and f is holomorphic in each variable z_j, when the other variables are given arbitrary fixed values, then f is holomorphic in Ω.

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German Summary

Wir betrachten das *hypergeometrische Differentialgleichungssystem*

$$(\Lambda - t\,I)\,y'(t) \;=\; A\,y(t)\,,$$

wobei A eine beliebige $(n \times n)$-Matrix und $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ eine Diagonalmatrix derselben Dimension ist. Es handelt sich hierbei um ein lineares Differentialgleichungssystem mit den Singularitäten $\lambda_1, \ldots, \lambda_n$ und ∞. Obwohl es auf den ersten Blick recht einfach aussieht, sind vollständige Lösungen bisher nur für $n = 2$ und sehr spezielle Fälle in höherer Dimension bekannt. In Dimension 2 kann man die Lösungen mittels der *hypergeometrischen Funktion* darstellen, daher auch der Name des Systems. Es ist durch Laplace-Transformation eng mit dem sogenannten *konfluenten hypergeometrischen Differentialgleichungssystem*

$$zx'(z) \;=\; (z\,\Lambda + A)\,x(z)$$

verknüpft, das eine reguläre Singularität im Ursprung und eine irreguläre bei ∞ besitzt. Der Name des Systems rührt daher, dass die beiden Systeme auch durch einen Konfluenzprozess zusammenhängen (siehe [25]).

Für das konfluente System existiert ein formales Fundamentalsystem in der Form

$$\hat{X}(z) \;=\; \hat{F}(z)\,z^L\,e^{z\Lambda}\,,$$

wobei L eine $(n \times n)$-Matrix und $\hat{F}(z)$ eine formale Potenzreihe in z^{-1} ist. Es ist bekannt, dass diese formale Potenzreihe *1-summierbar* ist, weshalb es eigentliche Fundamentalsysteme $X_\nu(z), \nu \in \mathbb{Z}$, gibt, sodass $\hat{X}(z)$ die asymptotische Entwicklung von $X_\nu(z)$ in gewissen Sektoren S_ν ist. Daher wird der allgemeinere Begriff der *k-Summierbarkeit* in Anhang A erläutert, während der noch stärkere Begriff der *Multisummierbarkeit* (siehe z. B. [8]) hier nicht benötigt wird. Da die $X_\nu(z)$ jeweils Fundamentalsysteme des konfluenten Systems bilden, existieren konstante Matrizen V_ν, sodass

$$X_{\nu-1}(z) \;=\; X_\nu(z)\,V_\nu$$

gilt. Diese Matrizen sind die sogenannten *Stokesschen Multiplikatoren*. Sie beschreiben das *Stokessche Phänomen*, welches besagt, dass unterschiedliche Lösungen des Differentialgleichungssystems das gleiche asymptotische Verhalten

zeigen, wenn man sie im jeweiligen Sektor betrachtet. Umgekehrt bedeutet dies aber auch, dass *eine* Lösung in *verschiedenen* Sektoren unterschiedliches asymptotisches Verhalten aufweist.

In dieser Arbeit wird meistens vorausgesetzt, dass die Eigenwerte $\lambda_1, \ldots, \lambda_n$ von Λ verschieden sind. Unter dieser Voraussetzung ist jede Singularität des hypergeometrischen Systems regulär. Sind außerdem die Diagonalelemente von A nicht ganzzahlig, so existieren *charakteristische Konstanten* c_{jk}, sodass für ein gewisses System von Lösungen $Y(t) = (y_1(t), \ldots, y_n(t))$ des hypergeometrischen Systems gilt:

$$y_k(t) \;=\; c_{jk}\, y_j(t) + \mathrm{reg}(t - \lambda_j)\,,\; 1 \leq j, k \leq n\,,$$

wobei $y_j(t)$ singulär an der Stelle λ_j ist, $1 \leq j \leq n$, und mit $\mathrm{reg}(t)$ eine beliebige im Ursprung holomorphe Funktion bezeichnet sei.

Aufgrund der irregulären Singularität bei ∞ ist das konfluente System eigentlich das interessantere der beiden Systeme. Hierbei möchte man insbesondere die Stokesschen Multiplikatoren kennen, die den Wechsel im Verhalten der Lösungen bei ∞ beschreiben. Durch den engen Zusammenhang mit dem hypergeometrischen System spielt es allerdings keine Rolle, welches der beiden man untersucht, da jedes Resultat, das man für das eine System erhält, in analoger Weise auf das andere System übertragen werden kann. So sind auch die charakteristischen Konstanten des hypergeometrischen Systems so eng mit den Stokesschen Multiplikatoren für das konfluente System verwandt, dass die Kenntnis der c_{jk} äquivalent zur Kenntnis der Stokesschen Multiplikatoren ist.

Aus obiger Gleichung folgt sofort, dass $c_{jj} = 1$ für alle $j = 1, \ldots, n$ gilt. Da man außerdem durch einfache, auf die Lösungen angewendete Transformationen Zeilen und Spalten der Matrizen Λ und A vertauschen kann, wird deutlich, dass es ausreicht, nur eine der Konstanten c_{jk} für feste j, k mit $j \neq k$ zu bestimmen. Kennt man beispielsweise c_{n1} als Funktion von Λ und A, so kann man ein beliebiges c_{jk} ($j \neq k$) durch die gleiche Funktion an den Stellen $\tilde{\Lambda}$ und $\tilde{A}$ darstellen, wobei $\tilde{\Lambda}$ aus Λ (bzw. $\tilde{A}$ aus A) durch Vertauschen der ersten und k-ten, sowie der n-ten und j-ten Zeile und Spalte entsteht. Aufgrund dieser Eigenschaft beschränken wir uns in dieser Arbeit bei der Betrachtung eines allgemeinen hypergeometrischen Systems mit einfachen Eigenwerten auf die Konstante c_{n1}. Entstehen $\hat{\Lambda}$ bzw. $\hat{A}$ aus Λ bzw. A durch Vertauschen zweier Zeilen und Spalten, die weder die ersten noch die n-ten sind, so sieht man, dass der Wert von c_{n1} gleich bleibt, das heißt $c_{n1}(\Lambda, A) = c_{n1}(\hat{\Lambda}, \hat{A})$. Dadurch kann $c_{n1}(\Lambda, A)$ in gewissem Sinne als symmetrische Funktion in einigen der Variablen aufgefasst werden.

Wie bereits erwähnt, kann man in Dimension 2 die Lösungen mit Hilfe der hypergeometrischen Funktion darstellen. Des Weiteren kann man die charakteristischen Konstanten mittels der *Gammafunktion*angeben. Diese beiden Funktionen sind recht gut bekannt und durch eine Differentialgleichung bzw. eine

Differenzengleichung und weitere Eigenschaften charakterisiert. Wünschenswert wäre es, wenn man auch in jeder höheren Dimension jeweils eine Funktion finden und charakterisieren könnte, mit deren Hilfe man die Lösungen bzw. Konstanten ausdrücken kann. In [7] wurde gezeigt, dass eine solche Funktion für die Lösungen existiert und auch eine mögliche in Form einer Potenzreihe angegeben. Außerdem konnte diese Funktion als Lösung einer Integralgleichung dargestellt werden. Es ist jedoch durchaus möglich, dass es eine natürlichere Wahl für solch eine Funktion gibt. Bei den charakteristischen Konstanten ist die Situation anders, da bislang keine entsprechende Wahl für eine solche Funktion getroffen wurde, obwohl aus dem vorigen Absatz folgt, dass $c_{n1}(\Lambda, A)$ *eine* solche Wahl ist.

In dieser Arbeit werden die Lösungen und Konstanten als Potenzreihe in einem neu eingeführten Parameter ε dargestellt und deren Koeffizienten charakterisiert. Diese Koeffizienten lassen sich nämlich in elementarer Weise durch Funktionen ausdrücken, die zu Beginn der Arbeit definiert und untersucht werden. Daher bezeichnen wir diese Funktionen hier auch als *Koeffizientenfunktionen*. Die Einführung des zusätzlichen Parameters basiert auf dem Hauptresultat meiner Diplomarbeit [22]. Hierbei werden in der Matrix A alle Einträge oberhalb der Diagonalen mit ε multipliziert - was übrigens auch dazu passt, dass wir später nur die Konstante c_{n1} betrachten - und sowohl die Lösungen als auch die Konstanten sind ganze Funktionen in ε. Dadurch ist klar, dass die betrachteten Potenzreihen jeweils einen unendlichen Konvergenzradius besitzen.

Die Methode, die Matrix A nicht als konstante Größe anzusehen, wurde auch schon von *W. Balser* in [1] und [2] verwendet. Hierbei wurde jeder Eintrag außerhalb der Diagonalen als Variable betrachtet und die Lösungen als Potenzreihe in mehreren Variablen dargestellt. Im Unterschied dazu, wirkt der Parameter ε in der vorliegenden Dissertation gleichzeitig auf alle Einträge oberhalb der Diagonalen, während er die restlichen Einträge nicht beeinflusst. Daher treten in dieser Arbeit auch nur Potenzreihen in einer Variablen auf, was zu einfacheren Formeln führt.

Die Arbeit ist folgendermaßen gegliedert:

In Kapitel 1 werden zunächst die *Koeffizientenfunktionen erster Art* f_k für $k \geq 0$ rekursiv definiert durch

$$\begin{aligned} f_0(t) &\equiv 1\,, \\ f_k(t) &= (\lambda_k - t)^{-l_k} \int_0^t (\lambda_k - w)^{l_k - 1} f_{k-1}(w)\, dw\,,\ k \geq 1\,. \end{aligned}$$

Hierbei werden gewisse Voraussetzungen an t und λ_j ($j \in \mathbb{N}$) gestellt, um die Existenz der Integrale zu gewährleisten, insbesondere gilt $\lambda_j \neq 0$ und $t \neq \lambda_j$ für

alle j. Später wird noch untersucht, unter welchen Voraussetzungen manche λ_j auch gleich 0 sein dürfen und wie sich die Funktion f_k verhält, wenn t gegen einen der Werte λ_j geht. Diese Eigenschaften benötigt man, um den Zusammenhang mit den Koeffizientenfunktionen zweiter Art herzuleiten und um die Lösungen des hypergeometrischen Systems darzustellen.

Anfangs sind die Funktionen f_k nur Funktionen in einer Variablen (nämlich t) und offensichtlich Lösungen des Differentialgleichungssystems

$$\begin{aligned} f_0' &= 0\,, \\ (\lambda_k - t)\, f_k' &= l_k f_k + f_{k-1}\,,\ k \geq 1\,. \end{aligned}$$

Im Folgenden werden die Werte λ_j und l_j allerdings nicht mehr als fest angenommen, so dass f_k für jedes $k \geq 1$ eine Funktion in mehreren Variablen darstellt. Es werden dann für alle $j = 1, \ldots, k$ Differenzengleichungen bezüglich l_j und daraufhin Differentialgleichungen bezüglich λ_j hergeleitet, die von f_k erfüllt werden. Außerdem wird f_k auch noch als Lösung eines Anfangswertproblems (bezüglich t) der Ordnung $k+1$ dargestellt.

In Abschnitt 1.2 führen wir dann die *Koeffizientenfunktionen zweiter Art* d_k für $k \geq 1$ ebenfalls rekursiv ein, und zwar durch

$$\begin{aligned} d_1(\lambda_1, l_1) &= \frac{\lambda_1^{l_1}}{l_1}\,, \\ d_k(\lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k) &= (\lambda_{k-1} - \lambda_k)^{l_k - l_{k-1}} \\ &\quad \times \int_0^{\lambda_k} (\lambda_{k-1} - w)^{l_{k-1} - l_k - 1}\, d_{k-1}(\lambda_1, \ldots, \lambda_{k-2}, w, l_1, \ldots, l_{k-2}, l_k)\, dw\,,\ k \geq 2\,. \end{aligned}$$

Jede Funktion d_k hängt also nicht von t, sondern nur von den Variablen λ_j und l_j ($j = 1, \ldots, k$) ab. Auch hier werden natürlich wieder gewisse Voraussetzungen an die Variablen gestellt um die Existenz der Funktionen zu gewährleisten. Erneut werden dann für alle $j = 1, \ldots, k$ Differenzengleichungen bezüglich l_j hergeleitet.

Der darauffolgende Abschnitt beschäftigt sich mit dem Zusammenhang zwischen den Funktionen f_k und d_k. Wir zeigen, dass die Kenntnis einer der Funktionenfolgen äquivalent zu der Kenntnis der anderen Folge ist, da die Beziehung

$$d_k(\lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k) = \lim_{t \to \lambda_k} (\lambda_k - t)^{l_k}\, f_k(t; \lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k)$$

gilt, die sofort auch

$$d_k(\lambda_1, \ldots, \lambda_k, l_1, \ldots, l_k) = \int_0^{\lambda_k} (\lambda_k - w)^{l_k - 1}\, f_{k-1}(w; \lambda_1, \ldots, \lambda_{k-1}, l_1, \ldots, l_{k-1})\, dw$$

und

$$f_{k-1}(t;\lambda_1,\ldots,\lambda_{k-1},l_1,\ldots,l_{k-1}) \;=\; \frac{\partial}{\partial t}\, d_k(\lambda_1,\ldots,\lambda_{k-1},t,l_1,\ldots,l_{k-1},1)$$

impliziert. Mittels dieser Formeln kann man auch für die Funktionen d_k Differentialgleichungen bezüglich λ_j für alle $j = 1,\ldots,k$ herleiten. Außerdem können nun die an λ_j und l_j gestellten Voraussetzungen gelockert werden.

Im letzten Abschnitt des ersten Kapitels wird schließlich noch untersucht, unter welchen zusätzlichen Bedingungen die Funktionen d_k (und somit auch f_k) als Lösungen der Differenzengleichungen in l_j eindeutig festgelegt sind.

In Kapitel 2 werden dann die beiden Differentialgleichungssysteme - das hypergeometrische und das konfluente hypergeometrische System - eingeführt. Für diese werden außerdem skalare Transformationen vorgestellt, die es erlauben, jeweils ein Diagonalelement der Matrizen A und Λ frei zu wählen, ohne dass eine Beschränkung der Allgemeinheit vorliegt.

Das 3. Kapitel behandelt ausschließlich dreieckige hypergeometrische Differentialgleichungssysteme, das heißt die Matrix A ist hier immer eine untere Dreiecksmatrix. Zunächst wird zusätzlich noch vorausgesetzt, dass die Eigenwerte $\lambda_1,\ldots,\lambda_n$ von Λ alle verschieden sind. In Abschnitt 3.1 wird die Matrix A sehr speziell gewählt, sodass die meisten ihrer Einträge gleich 0 sind. Für dieses System bilden wir dann Lösungen aus den Koeffizientenfunktionen (erster Art) f_k. Anschließend werden die charakteristischen Konstanten für dieses System im Falle der Existenz mittels der Koeffizientenfunktionen (zweiter Art) d_k ausgedrückt. Im darauffolgenden Abschnitt betrachten wir nun eine allgemeine untere Dreiecksmatrix A. Für diesen Fall können dann die Lösungen und Konstanten als Linearkombinationen der Lösungen bzw. Konstanten des zuvor untersuchten speziellen Systems angegeben werden.

Das Kapitel 3 endet mit der Untersuchung dreieckiger Systeme mit mehrfachen Eigenwerten. Unter gewissen Vorraussetzungen können Lösungen hiervon erneut mit Hilfe der Funktionen f_k dargestellt werden.

Im letzten Kapitel der Arbeit wird dann - wie bereits erwähnt - ein allgemeines hypergeometrisches System mit verschiedenen Eigenwerten untersucht. Dabei wird der zusätzliche Parameter ε eingeführt und die Lösungen sowie die charakteristischen Konstanten (sofern existent) als Potenzreihe in diesem Parameter dargestellt. Deren Koeffizienten werden anschließend mittels der Ergebnisse aus Abschnitt 3.3 bestimmt.

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Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig angefertigt habe. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und noch nicht veröffentlicht. Die Satzung der Universität Ulm zur Sicherung guter wissenschaftlicher Praxis habe ich beachtet.

Ich bin mir bewusst, dass eine unwahre Erklärung rechtliche Folgen haben wird.

Ulm, den 29. Januar 2009 _____