Universität Ulm

Fakultät für Mathematik und Wirtschaftswissenschaften

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Institut für Versicherungswissenschaften

Erratum zur Dissertation

Essays on Time-Consistent Asset Allocation and Participating Life Insurance Contracts

Dissertation

zur Erlangung des Doktorgrades Dr.rer.nat. der Fakultät für Naturwissenschaften/Mathematik und Wirtschaftswissenschaften der Universität Ulm

vorgelegt von

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Tag der Promotion: 08.10.2025

Table of Content

1 Corrigendum to Paper 1: Time-consistent asset allocation for risk measures in a Lévy market 1
2 Corrigendum to Paper 4: Optimal Capital Structure for Life Insurance Companies Offering Surplus Participation 9

1 Corrigendum to Paper 1: Time-consistent asset allocation for risk measures in a Lévy market

Following the publication of the paper *Time-Consistent Asset Allocation for Risk Measures in a Lévy Market*, we identified a few minor inaccuracies that merit clarification. In comparison to the examiner-reviewed version, this section represents an additional inclusion. These are outlined and corrected in the following:

1. The proof of Proposition 4.16 addresses only the more challenging case in which, during the induction step, all strategies $\hat{u}_{n+2}, \dots, \hat{u}_T$ vanish. We now consider the complementary situation, namely when this assumption does not hold. Specifically, suppose that $\|\hat{u}_s^1 \sigma_s^{11} e^{r(T-s)}\|_{\mathcal{L}^\alpha((n+1,T],\mathrm{d}s} > 0$. Substituting $0+$ into $\frac{\partial}{\partial \hat{u}_{n+1}^1} J_n$, and recalling the first equation stated in the proof of Proposition 4.16, we obtain:

$$\begin{aligned}\tfrac{\partial}{\partial \hat{u}_{n+1}^1} J_n(0+) =& \mathbb{E}_{n,x}[\mathcal{T}'(\textstyle\int_{n+1}^T \hat{u}_s^1(\mu_s^1 - r)e^{r(T-s)}\mathrm{d}s + \int_{(n+1)+}^T \hat{u}_s^1 \sigma_s^{11} e^{r(T-s)}\mathrm{d}L_s^1) \\ &\cdot (\textstyle\int_n^{n+1}(\mu_s^1 - r)e^{r(T-s)}\mathrm{d}s + \int_{n+}^{n+1} \sigma_s^{11} e^{r(T-s)}\mathrm{d}L_s^1)] \\ &- \lambda_n F'(-\textstyle\int_{n+1}^T \hat{u}_s^1(\mu_s^1 - r)e^{r(T-s)}\mathrm{d}s + \varrho^{\hat{L}_1}\|\hat{u}_s^1\sigma_s^{11}e^{r(T-s)}\|_{\mathcal{L}^\alpha((n+1,T],\mathrm{d}s)}) \\ &\cdot (-\textstyle\int_n^{n+1}(\mu_s^1 - r)e^{r(T-s)}\mathrm{d}s).\end{aligned}$$

   Since L has independent increments, the expectation can be decomposed accordingly. Furthermore, by symmetry of L, we observe that $\mathbb{E}_{n,x}[\int_{(n+1)+}^T \hat{u}_s^1 \sigma_s^{11} e^{r(T-s)}\mathrm{d}L_s^1] = 0$. Thus, the expression simplifies to

$$\begin{aligned}\tfrac{\partial}{\partial \hat{u}_{n+1}^1} J_n(0+) =& \textstyle\int_n^{n+1}(\mu_s^1 - r)e^{r(T-s)}\mathrm{d}s \\ &\cdot (\mathbb{E}_{n,x}[\mathcal{T}'(\textstyle\int_{n+1}^T \hat{u}_s^1(\mu_s^1 - r)e^{r(T-s)}\mathrm{d}s + \int_{(n+1)+}^T \hat{u}_s^1 \sigma_s^{11} e^{r(T-s)}\mathrm{d}L_s^1) \\ &+ \lambda_n F'(-\textstyle\int_{n+1}^T \hat{u}_s^1(\mu_s^1 - r)e^{r(T-s)}\mathrm{d}s + \varrho^{\hat{L}_1}\|\hat{u}_s^1\sigma_s^{11}e^{r(T-s)}\|_{\mathcal{L}^\alpha((n+1,T],\mathrm{d}s)})).\end{aligned}$$

   Finally, since by assumption $\mu_s^1 \geq r$ for all $s \in [0,T]$, $\lambda_n \geq 0$ for all n, and both T and F are non-decreasing functions (so that $T', F' \geq 0$), it follows that $\frac{\partial}{\partial \hat{u}_{n+1}^1} J_n \geq 0$.

2. In Definition 5.1, the dependence on the infinitesimal generator was omitted, which may lead to minor misunderstandings in the interpretation of the subsequent proofs. We therefore revise the definition as follows:

   Definition 5.1. *Definition 5.1 Let $f_{\hat{u}} : [0,T] \times \mathcal{R} \to \mathbb{R}$ be a function sequence for a strategy $\hat{u}$. Moreover, we define the strategy $u_h^{v,t}$ for $v \in \mathcal{V}$ by* $u_h^{v,t}(s) = \begin{cases} \hat{u}_s & \text{for } t+h \leq s \leq T, \\ v & \text{for } t+h \leq s \leq T. \end{cases}$

a) The infinitesimal generator $\mathrm{A}^{\hat{u}}$ *is defined as* $(\mathrm{A}^{\hat{u}} f_{\hat{u}})(t,x) = \lim_{h\to 0} \frac{1}{h}(\mathbb{E}_{t,x}[f_{u_h^{v,t}}(t+h, X^u_{t+h}) - f_{h_h^{v,t}}(t,x)])$.

b) Moreover, we define the operator A^u_h, $h \in \mathbb{R}$, *as* $(\mathrm{A}^u_h f)(t,x) = \mathbb{E}_{t,x}[f(t+h, X^u_{t+h}) - f(t,x)]$.

By definition, it holds $(\mathrm{A}^{\hat{u}} f)(t,x) = \lim_{h\to 0} \frac{1}{h}(\mathrm{A}_h^{u_h^{v,t}} f)(t,x)$.

Furthermore, we use this definition of $u_h^{v,t}$ to replace the definition of u_h in Definition 5.2 and Subsection 5.3.

3. In Chapters 5 and 6, under Assumption 4a — that is, in the case of bounded strategies — we require $\mathcal{V}$ to be a d-dimensional rectangle rather than merely a closed and convex subset of $[-M, M]^d$.

4. In the proof of Theorem 5.9, we additionally require continuity of the optimizer u with respect to the parameter C_2 in order to exclude discontinuous "jump" phenomena at the clipping boundary, i.e., at points where the unconstrained maximizer of the function f from the equation following (5.6) would exceed the upper bound M in some coordinates. Since we restrict u to lie in $[0, M]^d$, this raises the question of why replacing components larger than M by M preserves optimality in the constrained case.

   First, in the case $\beta = \alpha$, equation (5.7) shows that the partial derivatives are separable across dimensions. This separability directly implies that componentwise clipping yields an optimal solution (e.g., via the Karush–Kuhn–Tucker (KKT) conditions). If $\beta > \alpha$, the situation is different: the objective function f is strictly concave, because all parameters are positive (or positive definite). Together with the fact that $u \in [0, M]^d$, a compact and C_2-independent feasible set, Berge's Maximum Theorem guarantees that the argmax correspondence is non-empty, compact-valued, and upper hemicontinuous in C_2. Strict concavity then ensures uniqueness of the maximizer, from which continuity of u with respect to C_2 follows. Since u is uniformly Lipschitz continuous on $(0, M)^d$ and continuous on $[0, M]^d$, a standard boundary-extension argument shows that u is in fact uniformly Lipschitz continuous on the entire domain $[0, M]^d$. This indeed excludes any possible discontinuous "jump" phenomena.

5. An additional result, Lemma 5.A, is required directly after Proposition 5.10. This lemma supports the logical flow and completeness of the arguments in the subsequent theorems:

   Lemma 5.A. *Let* u *be continuous and deterministic. If the infinitesimal generator from*

Definition 5.1 is build with $v = u_t$, then it holds:

$$(\mathrm{A}^u\mathbb{E}_\cdot[\mathcal{T}(X_T^u - X_\cdot^u e^{r(T-\cdot)})])(t,x) = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_{t,x}[\mathcal{T}(X_T^u - xe^{r(T-t)})],$$
$$(\mathrm{A}^u(\lambda_\cdot F(\rho_\cdot(X_T^u - X_\cdot^u e^{r(T-\cdot)}))))(t,x) = \tfrac{\mathrm{d}}{\mathrm{d}t}\lambda_t F(\rho_{t,x}(X_T^u - xe^{r(T-t)})).$$

Proof of Lemma 5.A. We provide the proof of the first equation only, as the proof of the second follows analogously. Using the notation $w := u_h^{u_t,t}$, we obtain:

$$\begin{aligned}
(\mathrm{A}^u\mathbb{E}_\cdot[\mathcal{T}(X_T^u - X_\cdot^u e^{r(T-\cdot)})])(t,x) &= \lim_{h\to 0}\tfrac{1}{h}(\mathrm{A}_h^w\mathbb{E}_\cdot[\mathcal{T}(X_T^u - X_\cdot^u e^{r(T-\cdot)})])(t,x)\\
&= \lim_{h\to 0}\tfrac{1}{h}\mathbb{E}_{t,x}[\mathbb{E}_{t+h,X_{t+h}^w}[\mathcal{T}(X_T^w - X_{t+h}^{u_t} e^{r(T-(t+h))})] - \mathbb{E}_{t,x}[\mathcal{T}(X_T^w - xe^{r(T-t)})]]\\
&= \lim_{h\to 0}\tfrac{1}{h}\mathbb{E}[\mathcal{T}(a_{t+h} + b_{t+h}\hat{L}_1)\\
&\quad - \mathcal{T}(\textstyle\int_t^{t+h} u_t^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s + a_{t+h} + \sqrt[\alpha]{\int_t^{t+h} {}^\alpha w_s(u_t)^\alpha \mathrm{d}s + \int_{t+h}^T {}^\alpha w_s(u)^\alpha \mathrm{d}s}\hat{L}_1)],
\end{aligned}$$

where a_t, b_t, $\hat{L}_1$, and w are defined as in the proof of Proposition 5.10. We now expand the expression inside the expectation by $\pm\mathcal{T}(a_t + b_t\hat{L}_1)$, and analyze the resulting terms individually. Using an argument similar to the one employed to interchange limits in the proof of Proposition 5.10, we obtain for the first term:

$$\lim_{h\to 0}\tfrac{1}{h}\mathbb{E}[\mathcal{T}(a_{t+h} + b_{t+h}\hat{L}_1) - \mathcal{T}(a_t + b_t\hat{L}_1)] = \tfrac{\partial\mathbb{E}[\mathcal{T}(a_t + b_t\hat{L}_1)]}{\mathrm{d}t}.$$

For the second term, we again apply a similar limit-interchange argument to get:

$$\mathbb{E}[\lim_{h\to 0}\frac{\mathcal{T}(a_{t+h}+b_{t+h}\hat{L}_1) - \mathcal{T}(\int_t^{t+h} u_t^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s + a_{t+h} + \sqrt[\alpha]{\int_t^{t+h} {}^\alpha w_s(u_t)^\alpha \mathrm{d}s + \int_{t+h}^T {}^\alpha w_s(u)^\alpha \mathrm{d}s}\hat{L}_1)}{a_t + b_t\hat{L}_1 - (\int_t^{t+h} u_t^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s + a_{t+h} + \sqrt[\alpha]{\int_t^{t+h} {}^\alpha w_s(u_t)^\alpha \mathrm{d}s + \int_{t+h}^T {}^\alpha w_s(u)^\alpha \mathrm{d}s}\hat{L}_1)}$$
$$\cdot \frac{a_t + b_t\hat{L}_1 - (\int_t^{t+h} u_t^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s + a_{t+h} + \sqrt[\alpha]{\int_t^{t+h} {}^\alpha w_s(u_t)^\alpha \mathrm{d}s + \int_{t+h}^T {}^\alpha w_s(u)^\alpha \mathrm{d}s}\hat{L}_1)}{h}].$$

By Assumption 6, the first fraction converges to $\mathcal{T}'(a_t + b_t\hat{L}_1) < \infty$. We now decompose the second fraction as follows:

$$\frac{a_t - a_{t+h} - \int_t^{t+h} u_t^\intercal(\mu_s - r)e^{r(T-s)}\mathrm{d}s}{h} + \frac{\sqrt[\alpha]{b_t^\alpha} - \sqrt[\alpha]{\int_t^{t+h} {}^\alpha w_s(u_t)^\alpha \mathrm{d}s + \int_{t+h}^T {}^\alpha w_s(u)^\alpha \mathrm{d}s}}{h}\hat{L}_1.$$

The first summand converges to zero by the definition of a_t and the mean value theorem for integrals, since u is continuous. For the second summand, observe that for $\alpha > 1$ and $\min\{a,b\} > 0$, the inequality $|\sqrt[\alpha]{c_1} - \sqrt[\alpha]{c_2}| \leq \frac{1}{\alpha}\min\{c_1,c_2\}^{1/\alpha-1}|c_2 - c_1|$ follows directly from a simple application of the mean value theorem. If $c_{min} := \min\{b_t^\alpha, \int_t^{t+h} {}^\alpha w_s(u_t)^\alpha \mathrm{d}s + \int_{t+h}^T {}^\alpha w_s(u)^\alpha \mathrm{d}s\} = 0$, then the second summand is clearly equal to zero by the definition

of w (see Definition 4.8). Hence, we may assume that $c_{min} > 0$, which yields:

$$\begin{aligned}|\frac{\sqrt[\alpha]{b_t^\alpha} - \sqrt[\alpha]{\int_t^{t+h} {}^\alpha w_s(u_t)^\alpha \mathrm{d}s + \int_{t+h}^T {}^\alpha w_s(u)^\alpha \mathrm{d}s}}{h}| &\leq \tfrac{1}{\alpha} c_{min}^{1/\alpha - 1} |b_t^\alpha - (\int_t^{t+h} {}^\alpha w_s(u_t)^\alpha \mathrm{d}s + \int_{t+h}^T {}^\alpha w_s(u)^\alpha \mathrm{d}s)| \\ &\leq \tfrac{1}{\alpha} c_{min}^{1/\alpha - 1} \int_t^{t+h} |{}^\alpha w_s(u)^\alpha - {}^\alpha w_s(u_t)^\alpha| \mathrm{d}s.\end{aligned}$$

Since u is continuous, this summand also converges to zero, and the claim follows. □

6. Consequently, minor adjustments to the proofs of Theorems 5.11 and 5.12 are necessary to incorporate the revised definition and the additional lemma:

Proof of Theorem 5.11. Step 1: Let us show that $\tilde{V}(t,x) = J(t,x,\hat{u})$:

It holds by assumption that

$$(\mathrm{A}^{\hat{u}}\tilde{V})(t,x) - (\mathrm{A}^{\hat{u}}\mathbb{E}_\cdot[\mathcal{T}(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)})])(t,x) + (\mathrm{A}^{\hat{u}}(\lambda_\cdot F(\rho_\cdot(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)}))))(t,x) = 0. \tag{1}$$

Next, we replace (t,x) by $(s, X_s^{\hat{u}})$, integrate over $\mathrm{d}s$ in (t,T), and take the conditional expectation. Now, from Dynkin's theorem, which we are allowed to use due to Lemma 5.A and Assumption 8, we conclude that $\mathbb{E}_{t,x}[\tilde{V}(T, X_T^{\hat{u}})] = \tilde{V}(t,x) + \mathbb{E}_{t,x}[\int_t^T (\mathrm{A}^{\hat{u}}\tilde{V})(s, X_s^{\hat{u}})\mathrm{d}s]$. Due to Assumption 9 (which can be replaced by the assumptions in Theorem 5.9) and Proposition 5.10, we use Dynkin again for the other two terms from ((1)). Solving this equation for $\tilde{V}$ leads to

$$\begin{aligned}\tilde{V}(t,x) =& \mathbb{E}_{t,x}[\tilde{V}(T, X_T^{\hat{u}})] - \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)})(T, X_T^{\hat{u}})] + \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)})(t,x)] \\ &+ \mathbb{E}_{t,x}[\lambda_T F(\rho(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)}))(T, X_T^{\hat{u}})] - \lambda_t F(\rho(X_T^{\hat{u}} - X_\cdot^{\hat{u}} e^{r(T-\cdot)}))(t,x).\end{aligned}$$

Lastly, we insert the final value to infer:

$$\begin{aligned}\tilde{V}(t,x) =& \mathcal{T}(0) - \lambda_T F(0) - \mathcal{T}(0) + \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\hat{u}} - x e^{r(T-t)})] + \lambda_T F(0) - \lambda_t F(\rho(X_T^{\hat{u}} - x e^{r(T-t)})) \\ =& \mathbb{E}_{t,x}[\mathcal{T}(X_T^{\hat{u}} - x e^{r(T-t)})] - \lambda_t F(\rho(X_T^{\hat{u}} - x e^{r(T-t)})) = J(t,x,\hat{u}).\end{aligned}$$

Step 2: Let us show that $\hat{u}$ is an equilibrium function:
First, we construct a control law u_h as in Definition 5.2. Now, (B.1) applied to the interval t to $t+h$ gives us (suppressing the h and the v,t in $u_h^{v,t}$):

$$\begin{aligned}J(t,x,u) =& \mathbb{E}_{t,x}[\tilde{V}(t+h, X_{t+h}^u)] - \mathbb{E}_{t,x}[\mathcal{T}(X_T^u - X_{t+h}^u e^{r(T-t-h)}) - \mathcal{T}(X_T^u - x e^{r(T-t)})] \\ &+ \mathbb{E}_{t,x}[\lambda_{t+h} F(\rho_{t+h, X_{t+h}^u}(X_T^u - X_{t+h}^u e^{r(T-t-h)})) - \lambda_t F(\rho_{t,x}(X_T^u - x e^{r(T-t)}))],\end{aligned} \tag{2}$$

where we used $J(t+h, X^u_{t+h}, u) = \tilde{V}(t+h, X^u_{t+h})$, which follows from the definition of Nash equilibria (see Definition 5.2). Since $\tilde{V}$ is a solution of the extended HJB as in Definition 5.5, it holds that for all $u \in \mathcal{U}$

$$(\mathrm{A}^u\tilde{V})(t,x) - (\mathrm{A}^u\mathcal{T}(X^u_T - X^u_\cdot e^{r(T-\cdot)}))(t,x) + (\mathrm{A}^u(\lambda_\cdot F(\rho_\cdot(X^u_T - X^u_\cdot e^{r(T-\cdot)}))))(t,x) \leq 0.$$

A discretization of this equation leads to

$$\begin{aligned}\mathbb{E}_{t,x}[\tilde{V}(t+h, X^u_{t+h})] - \tilde{V}(t,x) - \mathbb{E}_{t,x}[\mathcal{T}(X^u_T - X^u_{t+h}e^{r(T-t-h)}) - \mathcal{T}(X^u_T - xe^{r(T-t)})] \qquad (3)\\ + \mathbb{E}_{t,x}[\lambda_{t+h}F(\rho_{t+h,X^u_{t+h}}(X^u_T - xe^{r(T-t-h)})) - \lambda_t F(\rho_{t,x}(X^u_T - xe^{r(T-t)}))] \leq \mathrm{o}(h).\end{aligned}$$

Since $\tilde{V}(t,x) = J(t,x,\hat{u})$, it follows that $J(t,x,\hat{u}) \geq J(t,x,u) + \mathrm{o}(h)$ from (2) and (3), which implies that $\liminf_{h\to 0} \frac{J(t,x,\hat{u}) - J(t,x,u)}{h} \geq 0$. Definition 5.2 yields the proof. □

Proof of Theorem 5.12. First, note that for convenience of the reader, we denote $\hat{u}$ from Theorem 5.9 by $\tilde{u}$. For this proof, we rewrite the HJB equations in discrete and continuous time using the special setting properties given by the assumptions. We remark that for continuous and deterministic optimal control functions, the infinitesimal generator equals the derivative with respect to t using Lemma 5.A Now, similar to the calculations in the proof of Proposition 5.10, we get for $\bar{u}$ with $\bar{u}_s = \hat{u}_s$ for $s > t$, where $\hat{u}$ denotes the optimal solution, and $\bar{u}_t = u$ for an arbitrary, but fixed $u \in \mathcal{V}$:

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}J(t,x,\bar{u}) = &- u^\intercal(\mu_t - r)e^{r(T-t)} + \lambda_t(\varrho^{\hat{L}_1}\sqrt[\alpha]{C_{\alpha d}})^\beta \tfrac{\beta}{\alpha}(\textstyle\int_t^T |\hat{u}_s^\intercal \sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s)^{\frac{\beta-\alpha}{\alpha}} |u^\intercal\sigma_t e^{r(T-t)}|^\alpha\\ &- \lambda_t'(\varrho^{\hat{L}_1}\sqrt[\alpha]{C_{\alpha d}})^\beta(\textstyle\int_t^T |\hat{u}_s^\intercal \sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s)^{\frac{\beta}{\alpha}},\end{aligned}$$

where $C_{\alpha d}$ is defined as in the proof of Theorem 5.9. Here, we used that L is isotropic or $d = 1$. Hence, the first part of the continuous HJB equation (5.2) becomes after noting that $V(t,x) = J(t,x,\hat{u})$ for $\hat{u}$ being optimal (since the λ_t'-part cancels):

$$\begin{aligned}\sup_{u\in\mathcal{V}}\{&(u^\intercal - \hat{u}_t^\intercal)(\mu_t - r)e^{r(T-t)} \qquad (4)\\ &- \lambda_t(\varrho^{\hat{L}_1}\sqrt[\alpha]{C_{\alpha d}})^\beta \tfrac{\beta}{\alpha}(\textstyle\int_t^T |\hat{u}_s^\intercal \sigma_s e^{r(T-s)}|^\alpha \mathrm{d}s)^{\frac{\beta-\alpha}{\alpha}}(|u^\intercal\sigma_t e^{r(T-t)}|^\alpha - |\hat{u}_t^\intercal\sigma_t e^{r(T-t)}|^\alpha)\} = 0.\end{aligned}$$

The other two parts are similar. Thus, we note that $\tilde{u}$ solves the HJB equation (5.2) if and only if the previous equation holds with $\tilde{u}$ instead of $\hat{u}$. Similarly, we can reduce the

first part of the discrete HJB equation (4.8) to:

$$\begin{aligned}\sup_{u\in\mathcal{V}}\{(u^\intercal - {}^\delta\hat{u}_t^\intercal)\int_t^{t+\delta}(\mu_s - r)e^{r(T-s)}\mathrm{d}s \\ - \lambda_t(\varrho^{\hat{L}_1}\sqrt[\alpha]{C_{\alpha d}})^\beta((|u|^\alpha\int_t^{t+\delta}|\sigma_s^{11}e^{r(T-s)}|^\alpha\mathrm{d}s + \int_{t+\delta}^T|{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha\mathrm{d}s)^{\frac{\beta}{\alpha}} \\ - (|{}^\delta\hat{u}_t|^\alpha\int_t^{t+\delta}|\sigma_s^{11}e^{r(T-s)}|^\alpha\mathrm{d}s + \int_{t+\delta}^T|{}^\delta\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha\mathrm{d}s)^{\frac{\beta}{\alpha}})\} = 0.\end{aligned}$$

Here, we also used that $|\hat{u}_s^\intercal\sigma_s e^{r(T-s)}|^\alpha = |\hat{u}_s|^\alpha|\sigma_s^{11}e^{r(T-s)}|^\alpha$ since either $d = 1$ or σ is the identity times possibly a constant. Again, the two other parts are similar. Now, we divide the equation by δ and show that for every arbitrary but fixed $u \in \mathcal{V}$ by taking the limit of $\delta \to 0$, we get the continuous HJB with $\tilde{u}$ instead of $\hat{u}$ on the left side. Note that for every fixed $u \in \mathcal{V}$ the discrete HJB equation holds with a "≤ 0" instead of a "$= 0$". First, it holds that $\lim_{\delta\to 0}\delta^{-1}(u^\intercal - {}^\delta\hat{u}_t^\intercal)\int_t^{t+\delta}(\mu_s - r)e^{r(T-s)}\mathrm{d}s = (u^\intercal - \tilde{u}_t^\intercal)(\mu_t - r)e^{r(T-t)}$ since ${}^\delta\hat{u}$ converges to $\tilde{u}$ by Theorem 5.9 and μ is continuous. For the second term, the argumentation is trickier, and for the sake of simplicity we introduce $\sigma_\delta := \int_t^{t+\delta}|\sigma_s^{11}e^{r(T-s)}|^\alpha\mathrm{d}s$ and $C_\delta(x) := \int_{t+\delta}^T|x_s^\intercal\sigma_s e^{r(T-s)}|^\alpha\mathrm{d}s$. Now, we get

$$\begin{aligned}&\frac{(|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))^{\frac{\beta}{\alpha}} - (|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))^{\frac{\beta}{\alpha}}}{\delta} \\ &\quad= \frac{(|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))^{\frac{\beta}{\alpha}} - (|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))^{\frac{\beta}{\alpha}}}{(|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u})) - (|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))}\cdot\frac{(|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u})) - (|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u}))}{\delta}.\end{aligned}$$

The second term converges to $(|u|^\alpha - |\tilde{u}|^\alpha)|\sigma_t^{11}e^{r(T-t)}|^\alpha = |u^\intercal\sigma_t e^{r(T-t)}|^\alpha - |\hat{u}_t^\intercal\sigma_t e^{r(T-t)}|^\alpha$ for $\delta \to 0$ due to the convergence of ${}^\delta\hat{u}$ to $\tilde{u}$ and σ being continuous if $d = 1$ or a constant times the identity if $d \geq 2$. On the first term, we apply the mean value theorem of differentiation and thus get a ξ_δ between $|u|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u})$ and $|{}^\delta\hat{u}_t|^\alpha\sigma_\delta + C_\delta({}^\delta\hat{u})$ such that the first term is equal to $\frac{\beta}{\alpha}\xi_\delta^{\frac{\beta-\alpha}{\alpha}}$. Since both boundary values for ξ_δ converge to $C_0(\tilde{u})$ due to the convergence of ${}^\delta\hat{u}$ to $\tilde{u}$, also ξ_δ converges to $C_0(\tilde{u})$. Overall, we get the convergence of the inner part of the left side from the discrete to the continuous case for a fixed u after noticing that the argumentation is similar for the other two terms. Hence, for a fixed u, we have (4) with a "$\leq$" for $\tilde{u}$. Now, we take the supremum over all $u \in \mathcal{V}$ which leads to the continuous HJB equation (4) with a "≤ 0" instead of a "$= 0$". However, choosing $u = \tilde{u}_t$ gives immediately a "0" in the equation by the previously shown convergence results for the discrete optimal strategies. Hence, $\tilde{u}$ solves the continuous HJB equation (5.2). Moreover, the constraints hold by Remark 5.6. Thus, $\tilde{u}$ is an optimal equilibrium function by Theorem 5.11, and the theorem follows. □

7. Finally, we need to adjust the right plot of Figure 2 and slightly revise its interpretation as follows:

   Figure 2 illustrates that time-consistent investment strategies tend to be more conserva-

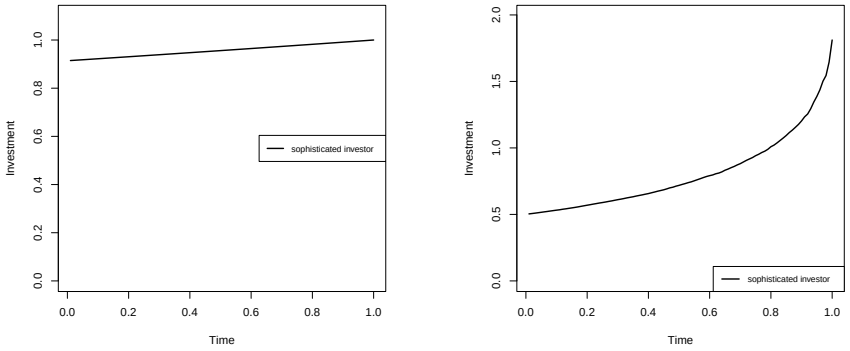

Figure 2: Comparison of Nash equilibrium to pre-commitment optimal strategies for mean-variance (left) and mean-VaR^2 (right) optimization.

tive than pre-commitment strategies over longer time horizons. For shorter maturities, however, this pattern holds only under the mean-variance criterion, where the difference between the two approaches remains relatively small. In contrast, under the mean-VaR^2 framework, the time-consistent sophisticated strategy is noticeably riskier for short-term horizons than the pre-commitment strategy.

2 Corrigendum to Paper 4: Optimal Capital Structure for Life Insurance Companies Offering Surplus Participation

This section presents additional results and corrections of the paper *Optimal Capital Structure for Life Insurance Companies Offering Surplus Participation* relative to the examiner-reviewed version of this thesis. These new results and corrections, as they appear in the updated arXiv version of the paper, are outlined below. For completeness, certain proofs and definitions are reproduced verbatim where necessary.

1. We introduce an additional assumption: $k \geq V_0$. This ensures that the threshold for surplus participation is at least as large as the initial portfolio value. This condition is generally satisfied in surplus participation contracts, as the baseline for determining surplus is typically set to the initial value with an appropriate safety margin.

2. Updated and new definitions of $\bar{\alpha}$ and $\tilde{\alpha}$: The definition of $\bar{\alpha}$ is revised, and a new parameter $\tilde{\alpha}$ is introduced, as detailed in the updated formulas. These definitions are required for the subsequent proofs and the formulation of uniqueness conditions.

$$\bar{\alpha} := \begin{cases} \nu(1-\tau_2-e^{-\nu T})^{-1}, & \text{if } 1-\tau_2-e^{-\nu T}>0, \\ \infty, & \text{else,} \end{cases}$$
$$\tilde{\alpha} := \begin{cases} \frac{1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2}{A_4-\tau_2 A_3}, & \text{if } A_4-\tau_2 A_3>0, \\ \infty, & \text{else.} \end{cases}$$

3. Several lemmas have been added to support the revised proofs. These are necessary to validate the updated arguments in Theorem 3.1 and related propositions.

   Lemma B.A. *Let $a, b, y \in \mathbb{R}$, and $x > 0$. Define $h(x) := \frac{\ln(x)\varphi(a\ln(x)+b)}{x^y}$. Then, we get that $\lim_{x\to 0}|h(x)| = 0$.*

   Proof. We get with the substitution $z = \ln(x)$:

   $$\lim_{x\to 0}|h(x)| = \lim_{z\to-\infty}|\tfrac{z\varphi(az+b)}{e^{yz}}| = \lim_{z\to-\infty}|ze^{-yz}\tfrac{1}{2\pi}e^{-\frac{1}{2}(az+b)^2}| = \tfrac{1}{2\pi}\lim_{z\to-\infty}|z|e^{-\frac{1}{2}(az+b)^2-yz} = 0,$$

   since $\lim_{z\to-\infty} -\frac{1}{2}(az+b)^2 - yz = -\infty$. □

   Lemma B.B. *In our framework, the optimal bankruptcy-triggering is bounded away from 0.*

Proof. Assume, for the sake of contradiction, that $V_B = 0$. From (3.1), (3.2), (3.6), and the formula following (3.6), it follows that both the bankruptcy costs and the value of bankruptcy vanish under this assumption. Consequently, we obtain

$$E(V) = V + TB_1(V) + TB_2(V) - VG(V) - VP(V) - VSP(V),$$

where TB_1 and TB_2 denote the tax benefits associated with the guarantee and the surplus participation (cf. (3.3), (3.4)), VG the value of the guarantee, VP the value of the principal payment, and VSP the value of the surplus participation. By assumption, the guarantee and the surplus participation is worth at least as much as its corresponding tax benefit, i.e., $TB_1 \leq VG$ and $TB_2 \leq VSP$. Moreover, since $V_B = 0 < V_S$ for all s, it follows that $F^V, G^V \equiv 0$ and hence $I_1^V, I_2^V \equiv 0$. Thus, we obtain that $E(V; 0, T) \leq V - VP(V) = V - P\frac{1-e^{-rT}}{rT}$. As $P > 0$ by assumption, this inequality implies that as $V \to V_B = 0$, equity becomes negative. This contradicts the requirement that shareholders would liquidate the firm before equity turns negative. Therefore, the assumption $V_B = 0$ is inconsistent, and the boundary solution cannot represent the optimal bankruptcy-triggering value. This proof also shows that V_B is bounded away from 0. □

Lemma B.C. *Let $R(V)$ be the right-hand side of (3.15) with $V = V_B$. If $\lim_{V\to\infty} R(V) > 0$ (resp. $\downarrow 0$), then R has a unique zero root. If $\lim_{V\to\infty} R(V) < 0$ (resp. $\uparrow 0$), then R has at most two zero roots. If $R(V) = 0$ for all V large enough, then $R(V) = 0$ for all $V \geq k$ and there is up to one zero root smaller than k.*

Notation: $\lim_{V\to\infty} R(V) \downarrow 0$ (resp. $\uparrow 0$), if $\lim_{V\to\infty} R(V) = 0$ and there exists a $\tilde{V}$ such that $R(V) < 0$ (resp. $R(V) > 0$) for all $V \geq \tilde{V}$.

Proof. Taking the derivative yields $R'(V) = \frac{1}{V^2}H(V)$, as follows from (B.10) and (B.11), where

$$\begin{aligned} H(V) := &\left(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)\right) \\ &+ \tau_2\alpha 2k\int_T^\infty e^{-rt}\left(\lambda_2\Phi(d_2(\min\{\tfrac{V}{k},1\},t)) + \frac{\varphi(d_2(\min\{\frac{V}{k},1\},t))}{\sigma\sqrt{t}}\right)\mathrm{d}t \\ &- \alpha(1-\tau_2)2k\int_0^T e^{-rt}\left(\lambda_2\Phi(d_2(\min\{\tfrac{V}{k},1\},t)) + \frac{\varphi(d_2(\min\{\frac{V}{k},1\},t))}{\sigma\sqrt{t}}\right)\mathrm{d}t. \end{aligned}$$

First, observe that $\lim_{V\to 0} H(V) = \left(\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)\right) > 0$ by Lemma B.10, and that H is constant for $V \geq k$. Moreover, we note that if $H(k) = 0$, then $R(V) = 0$ for all $V \geq k$.

In what follows, we establish the stated properties by considering several cases. First, in (i), we treat the case in which H has no zeros in the interval $(0, k]$. Next, in (ii), we analyze the case where H has exactly one zero in $(0, k)$, distinguishing the subcases (ii.a) $H(k) \neq 0$ and (ii.b) $H(k) = 0$. Finally, in (iii), we show that H cannot have more than one zero in $(0, k)$. The argument is completed by noting that H is constant on $[k, \infty)$.

(i) If H has no zero in $(0, k]$, then H has no zero in $(0, \infty)$, i.e. $H > 0$, and consequently R is strictly increasing in V. Hence, if $\lim_{V\to\infty} R(V) \leq 0$, then no zero root exists, and if $\lim_{V\to\infty} R(V) > 0$, a unique zero root exists.

(ii.a) Let $H(k) \neq 0$. If H has exactly one zero in $(0, k)$, then there exists $\tilde{V}$ such that $H(V) > 0$ for all $V < \tilde{V}$ and $H(V) < 0$ for all $V > \tilde{V}$. By the first part of the proof of Theorem 3.1 (using that $R = h_2$ with h_2 as in (C.4)), we have $\lim_{V\to 0} R(V) = -\infty$. If $\lim_{V\to\infty} R(V) > 0$ (resp. $\downarrow 0$), the zero root of (3.15) is unique. Indeed, in intervals where R is strictly increasing or decreasing, there can be at most one zero. Since R decreases to a positive value, no additional zero can occur in this region where R is decreasing. Hence, R has a unique zero (existence follows by the intermediate value theorem) as H is continuous and constant for $V \geq k$. If H would have a zero at $V = k$, then $H(V) = 0$ for all $V \geq k$, i.e. R is constant for $V \geq k$. Since we assumed here that $\lim_{V\to\infty} R(V) > 0$ (resp. $\downarrow 0$), we conclude that $R(V) > 0$ for all $V \geq k$. Thus, if H has at most one zero in $(0, k)$, then R has at most one local extremum in $(0, k)$[1], which must be a maximum since $\lim_{V\to 0} R(V) = -\infty$. Therefore, $R(V) > 0$ for all $V \geq k$ implies that the zero of R in $(0, \infty)$ is unique. Now, if $\lim_{V\to\infty} R(V) < 0$ (resp. $\uparrow 0$), there can be at most two zero roots of R with a similar argumentation as before. (ii.b) Furthermore, if $H(k) = 0$, i.e., $H(V) = 0$ and $R(V) = 0$ for all $V \geq k$, then a similar argument shows that there is at most one zero root in $(0, k)$.

(iii) Finally, we show that H has no more than one zero in $(0, k)$. Therefore, let $\varepsilon \in (0, k)$ be small and consider H on $(0, k - \varepsilon]$. Then, it holds with $\frac{d_2(\frac{V}{k},t)}{\sigma\sqrt{t}} = \frac{\ln(\frac{V}{k})}{\sigma^2 t} + \lambda_2$ and $\varphi'(x) = -x\varphi(x)$ that

$$\begin{aligned}\frac{\partial}{\partial V}\Big(\lambda_2\Phi(d_2(\tfrac{V}{k},t)) + \frac{\varphi(d_2(\frac{V}{k},t))}{\sigma\sqrt{t}}\Big) &= \lambda_2\varphi(d_2(\tfrac{V}{k},t))\frac{1}{\sigma\sqrt{t}V} - \varphi(d_2(\frac{V}{k},t))\frac{d_2(\frac{V}{k},t)}{\sigma\sqrt{t}}\frac{1}{\sigma\sqrt{t}V} \\ &= \tfrac{(-\ln(\frac{V}{k}))}{\sigma^3 t^{3/2}V}\varphi(d_2(\frac{V}{k},t)) =: \hat{f}(t,V).\end{aligned}$$

By Lemma B.A, $\lim_{V\to 0} \hat{f}(t, V) = 0$. With $\tilde{\varepsilon} := -\ln\big(\frac{k-\varepsilon}{k}\big) > 0$, we obtain

$$\lim_{V\to k-\varepsilon} \hat{f}(t,V) = \tfrac{\tilde{\varepsilon}}{\sigma^3 t^{3/2}(k-\varepsilon)}\tfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{\tilde{\varepsilon}^2}{\sigma^2 t} - 2\lambda_2\tilde{\varepsilon} + \lambda_2^2\sigma^2 t)}.$$

[1]We adopt the convention that local extrema cannot occur at the boundary points of open intervals.

Hence, there exists $\hat{F}(t)$ with $|\hat{f}(t,V)| \le \hat{F}(t)$ for all $V \in (0, k-\varepsilon]$ and $\int_0^\infty \hat{F}(t)\,\mathrm{d}t < \infty$, so that the derivative and the integral can be interchanged. Therefore, for $V \in (0, k-\varepsilon]$ we have

$$H'(V) = 2k\alpha\tau_2 \int_0^\infty \tfrac{e^{-rt}(-\ln(\frac{V}{k}))}{\sigma^3 t^{3/2} V}\varphi(d_2(\frac{V}{k},t))\mathrm{d}t - 2k\alpha \int_0^T \tfrac{e^{-rt}(-\ln(\frac{V}{k}))}{\sigma^3 t^{3/2} V}\varphi(d_2(\frac{V}{k},t))\mathrm{d}t.$$

Now define:

$$\eta(V) := \frac{2k\alpha \int_0^T \frac{e^{-rt}(-\ln(\frac{V}{k}))}{\sigma^3 t^{3/2} V}\varphi(d_2(\frac{V}{k},t))\mathrm{d}t}{2k\alpha\tau_2 \int_0^\infty \frac{e^{-rt}(-\ln(\frac{V}{k}))}{\sigma^3 t^{3/2} V}\varphi(d_2(\frac{V}{k},t))\mathrm{d}t} = \frac{\int_0^T \frac{e^{-rt}}{t^{3/2}}\varphi(d_2(\frac{V}{k},t))\mathrm{d}t}{\tau_2 \int_0^\infty \frac{e^{-rt}}{t^{3/2}}\varphi(d_2(\frac{V}{k},t))\mathrm{d}t} =: \frac{B_1(V)}{B_2(V)}.$$

Next, we show that η is increasing, i.e. $\eta' > 0$. Interchanging differentiation and integration is again justified by the same reasoning as above. We obtain $\eta' = \frac{B_1' B_2 - B_1 B_2'}{B_2^2}$, so that $\eta' > 0$ if and only if $B_1' B_2 - B_1 B_2' > 0$. Indeed,

$$B_1'(V)B_2(V) = \tfrac{\tau_2}{\sigma V}\int_0^T \int_0^\infty \tfrac{e^{-rt}}{t^2}\varphi(d_2(\tfrac{V}{k},t))(-d_2(\tfrac{V}{k},t))\tfrac{e^{-rs}}{s^{3/2}}\varphi(d_2(\tfrac{V}{k},s))\mathrm{d}s\,\mathrm{d}t,$$
$$B_1(V)B_2'(V) = \tfrac{\tau_2}{\sigma V}\int_0^T \int_0^\infty \tfrac{e^{-rt}}{t^{3/2}}\varphi(d_2(\tfrac{V}{k},t))\tfrac{e^{-rs}}{s^2}\varphi(d_2(\tfrac{V}{k},s))(-d_2(\tfrac{V}{k},s))\mathrm{d}s\,\mathrm{d}t.$$

Splitting the integral from 0 to ∞ at T, the contributions over $[0,T]^2$ cancel, leaving

$$B_1' B_2 - B_1 B_2' = \tfrac{\tau_2}{\sigma V}\int_0^T \int_T^\infty \frac{e^{-rt}e^{-rs}}{s^{3/2}t^{3/2}}\varphi(d_2(\tfrac{V}{k},t))\varphi(d_2(\tfrac{V}{k},s))\Big[\frac{d_2(\frac{V}{k},s)}{\sqrt{s}} - \frac{d_2(\frac{V}{k},t)}{\sqrt{t}}\Big]\mathrm{d}s\,\mathrm{d}t > 0,$$

since $0 < t < s$ and $\psi(t) := \frac{d_2(\frac{V}{k},t)}{\sqrt{t}} = \frac{\ln(\frac{V}{k})}{t} + \lambda_2\sigma$ is increasing in t (as $\ln(\frac{V}{k}) < 0$).

It follows that $H'(V) = 2k\alpha\tau_2 \int_0^\infty \frac{e^{-rt}(-\ln(\frac{V}{k}))}{\sigma^3 t^{3/2} V}\varphi(d_2(\frac{V}{k},t))\mathrm{d}t(1-\eta(V))$. Since k, α, τ_2, σ, $(-\ln(\frac{V}{k}))$, and V are all non-negative, the sign of H' is determined solely by the factor $(1-\eta)$, i.e., $\mathrm{sign}(H') = \mathrm{sign}(1-\eta)$. As $(1-\eta)$ is strictly decreasing, it follows that $H'(V)$ can undergo at most one sign change on $(0, k-\varepsilon]$, and this change must be from positive to negative. Passing to the limit as $\varepsilon \to 0$ yields that H' admits at most one sign change on the full interval $(0,k)$, again necessarily from positive to negative. Consequently, H may have at most one local maximum and no local minimum in $(0,k)$. Furthermore, by Lemma B.3, Lemma B.10, and using the facts that $\lim_{x\to-\infty}\Phi(x) = 0$ and $|\Phi| \le 1$, we obtain that $\lim_{V\to 0} H(V) = \left(\frac{2(P-\frac{G}{r})A_1}{rT} + 2\frac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)\right) > 0$. Since H starts positive at the boundary $V = 0$, possesses at most one local maximum, and no local minimum in $(0,k)$, it can have at most one zero in this interval. □

Lemma B.D. *The partial derivatives* $\frac{\partial}{\partial V_B}c_{do}(V,k,V_B,T)$, $\frac{\partial}{\partial\alpha}c_{do}(V,k,V_B,T)$, *and* $\frac{\partial}{\partial g}c_{do}(V,k,V_B,T)$ *are continuous with respect to the variable of differentiation.*

Proof. According to the proofs of Lemmas B.16 and B.18, the functions $V_B'(\alpha)$ and $V_B'(g)$ are continuous. Using equations (B.14), (B.15), (B.18), and (B.19), we observe that the continuity of the respective partial derivatives follows provided continuity holds at the point $V_B = k$. To verify this, we compare the corresponding expressions at $V_B = k$ and note that continuity at $V_B = k$ is ensured if the following two conditions are satisfied: (i) $(\frac{V_B}{V})^{2\lambda_1}\varphi(d_1(\frac{V_B}{V}, T)) = \varphi(d_1(\frac{V}{V_B}, T))$ and (ii) $(\frac{V_B}{V})^{2\lambda_1-2}\varphi(d_2(\frac{V_B^2}{Vk}, T)) = \varphi(d_2(\frac{V}{V_B}, T))$.

We begin with the proof of (i). From (A.3), it follows that $d_1(x, T) = \frac{\ln x}{\sigma\sqrt{t}} + \sigma\lambda_1\sqrt{t}$. Then, using this expression, we compute:

$$\begin{aligned}
(\tfrac{V_B}{V})^{2\lambda_1}\varphi(d_1(\tfrac{V_B}{V}, T)) &= e^{2\lambda_1(\ln V_B - \ln V)} \cdot \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{(\ln V_B - \ln V)^2}{\sigma^2 t} + 2\lambda_1(\ln V_B - \ln V) + \sigma^2\lambda_1^2 t)} \\
&= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{(\ln V_B - \ln V)^2}{\sigma^2 t} - 2\lambda_1(\ln V_B - \ln V) + \sigma^2\lambda_1^2 t)} \\
&= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{(\ln V - \ln V_B)^2}{\sigma^2 t} + 2\lambda_1\sigma\sqrt{t}\frac{(\ln V - \ln V_B)}{\sigma\sqrt{t}} + \sigma^2\lambda_1^2 t)} = \varphi(d_1(\tfrac{V}{V_B}, T)).
\end{aligned}$$

The proof of (ii) follows analogously, using the facts that $\lambda_1 - 1 = \lambda_2$ (see (A.9)) and $d_2(x, T) = \frac{\ln x}{\sigma\sqrt{t}} + \sigma\lambda_2\sqrt{t}$. □

4. Lemma B.D replaces and corrects Remark B.17 and Remark B.19.

5. Update of Theorem 3.1: There, V_B is determined as the minimum of V_0 and the smallest solution of equation (3.15), rather than the largest. The updated proof is identical until the definition of (C.5). Afterwards, it looks as follows:

   Proof. By Lemma B.B, the optimal bankruptcy-triggering value is bounded away from 0.

   Now, formula (3.15) follows directly from (C.4), and we obtain formula (3.16) from (B.3) and (B.4), where we used that $\lambda_1 - 1 = \lambda_2$. If the solution of (3.15) exceeds V_0, the insurance company declares bankruptcy immediately. Therefore, we can equivalently set $V_B = V_0$ in this case, without affecting the timing of the bankruptcy declaration, while ensuring that V_B represents the asset value (before subtracting the bankruptcy costs) at the time of bankruptcy.

   It still remains to discuss which solution of (3.15) is the best one and the existence. First, we note that the optimal bankruptcy-triggering value V_B has to be a minimum of $V \to E(V, V_B)$ as a maximum violates the condition $E(V) \geq 0$ for all $V \geq V_B$ since $E(V_B, V_B) = 0$ (see also the discussion at the beginning of Section 3.3). Moreover, remember that for the optimal-bankruptcy triggering value V_B, the function $V \to E(V, V_B)$ cannot be negative.

   Now, if $\lim_{V\to\infty} h_2(V) > 0$ (resp. $\downarrow 0$), then Lemma B.C implies the uniqueness of the zero root of (3.15), which trivially corresponds to a minimum of $V \to E(V, V_B)$ (as

$\lim_{V\to 0} h_2(V) = -\infty$). If $\lim_{V\to\infty} h_2(V) < 0$ (resp. $\uparrow 0$), then Lemma B.C implies that there exist up to two zero roots of (3.15). If h_2 has no zero root, then $E'_V(V,V) < 0$ for all $V > 0$ as $\lim_{V\to 0} h_2(V) = -\infty$. Hence, every choice of a bankruptcy-triggering value would lead to a negative equity at a value larger than this point which contradicts our assumption of non-negative equity and immediate bankruptcy is optimal. Thus, we can set $V_B = V_0$, i.e., immediate bankruptcy[2]. If h_2 has one zero root V_B^*, then we have again the unique solution. If h_2 has two zero roots, by (C.5) only the first zero root corresponds to a minimum of the function $V \to E(V, V_B)$. In the remaining case that $\lim_{V\to\infty} h_2(V) = 0$ and that h_2 is constant from a certain V on, Lemma B.C implies that $h_2(V) = 0$ for all $V \geq k$. Lemma B.C entails that there is at most one zero root smaller than k which corresponds analogously to before to a minimum of the function $V \to E(V, V_B)$. The zero roots larger than k can be ignored, as we may take the minimum of the optimal solution and V_0 (since $k \geq V_0$ by assumption). Summarizing, we have shown the statement that the minimum of the smallest solution of the smooth pasting condition and V_0 is the optimal bankruptcy-triggering value. In particular, note that at most one solution of (3.15) is a possible bankruptcy-triggering value. □

6. Update of the proof of Proposition 3.2: Due to the modifications in Theorem 3.1, the first paragraph of the proof of Proposition 3.2 is updated. The rest of the proof remains unchanged.

   Proof. We first note from the previous proof and Lemma B.C that even when parameters change, no additional relevant zero roots can occur or vanish as long as a solution exists. If τ_2 increases, we observe that the right-hand side of (3.15) also increases. From the proof of Theorem 3.1, we know that the smallest solution of (3.15) corresponds to a sign transition from "$-$" to "$+$" (or to a local minimum, with the function being positive to the right-hand side of the zero root). Therefore, as the graph shifts upwards, the zero root decreases showing that V_B is increasing in τ_2 if a zero root exists. If no zero root exists, it holds that $V_B = V_0$. Now, if τ_2 decreases, no zero root can occur. If τ_2 increases and equation (3.15) gets a zero root, this leads to the case that $V_B \leq V_0$ (as V_B is the minimum of this zero root and V_0). □

7. Left- vs. right-continuity update: The statement and proof of Proposition 3.3 are adjusted to establish left-continuity and right-hand limits instead of right-continuity and left-hand limits. Accordingly, Assumption 1 is revised to require right-continuity. Only the first two paragraphs of the proof are updated; the remaining argument is unchanged.

[2] As (3.15) admits no solution in this case, the smooth-pasting condition cannot be applied. The absence of a solution initially corresponds to $V_B = \infty$, which implies immediate bankruptcy regardless of the initial value V_0. Since choosing $V_B = V_0$ also results in immediate bankruptcy, we may set $V_B = V_0$ without affecting the outcome.

Proof. To show that V_B is monotonically increasing in α, let $0 \leq \alpha_1 < \alpha_2$, and we aim to prove that $V_B(\alpha_1) < V_B(\alpha_2)$. By Theorem 3.1, we know that:

$$\begin{aligned} 0 =& 1 + \rho(\lambda_2 + \lambda_3) + 2(1-\rho)A_2 - \tfrac{1}{V_B(\alpha_1)}\left(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)\right) \\ &+ \tau_2\alpha_1\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B(\alpha_1),t)}{\partial V}\Big|_{V=V_B(\alpha_1)}\mathrm{d}t - \alpha_1\int_0^T \tfrac{\partial c_{do}(V,k,V_B(\alpha_1),t)}{\partial V}\Big|_{V=V_B(\alpha_1)}\mathrm{d}t. \end{aligned}$$

Using Lemma B.10, we get, in particular, that:

$$\begin{aligned} 0 >& 1 + \rho(\lambda_2 + \lambda_3) + 2(1-\rho)A_2 - \tfrac{1}{V_B(\alpha_1)}\left(\tfrac{2(P-\frac{G}{r})A_1}{rT} + 2\tfrac{G}{r}A_2 - \tau_1\frac{G}{r}(\lambda_2+\lambda_3)\right) \\ &+ \tau_2\alpha_2\int_0^\infty \tfrac{\partial c_{do}(V,k,V_B(\alpha_1),t)}{\partial V}\Big|_{V=V_B(\alpha_1)}\mathrm{d}t - \alpha_2\int_0^T \tfrac{\partial c_{do}(V,k,V_B(\alpha_1),t)}{\partial V}\Big|_{V=V_B(\alpha_1)}\mathrm{d}t. \end{aligned}$$

Loosely speaking, Lemma B.10 implies that, if V_B remains constant, every point of the right-hand side's graph decreases as α increases. As in the previous proof, we know that there exists at most one relevant zero root. Hence, to maintain equality, it must hold that $V_B(\alpha_1) < V_B(\alpha_2)$. This establishes the claim, since V_B is defined as the minimum of this zero root (if it exists) and V_0. □

8. Proposition 3.4 update:

Proposition 3.4. *A sufficient condition for our assumption that $V \to E(V)$ being non-negative is that $\alpha < \bar{\alpha}$, where $\bar{\alpha}$ is as defined in 2. of this Corrigendum.*

Proof. As in the proof of Theorem 3.1, we begin by noting that the non-negativity of the mapping $V \mapsto E(V)$ is implied by the condition $\lim_{V\to\infty} E(V;V_B) \geq 0$ for any fixed V_B. Fix V_B. As $V \to \infty$, the probability of bankruptcy converges to zero, and therefore the limit of the equity value becomes independent of V_B. Using the formula after (3.6), (3.6), and (3.2), we obtain:

$$\begin{aligned} &\lim_{V\to\infty} E(V;V_B) \\ &\quad = \lim_{V\to\infty}[V + \tau_1\tfrac{G}{r} + \tau_2\alpha\int_0^\infty c(V,k,t)\mathrm{d}t - \tfrac{G}{r} - (P-\tfrac{G}{r})\tfrac{1-e^{-rT}}{rT} - \alpha\int_0^T c(V,k,t)\mathrm{d}t]. \end{aligned}$$

Thus, if $\lim_{V\to\infty}[V + \tau_2\alpha\int_0^\infty c(V,k,t)\mathrm{d}t - \alpha\int_0^T c(V,k,t)\mathrm{d}t] = \infty$, then indeed $\lim_{V\to\infty} E(V;V_B) \geq 0$. To verify this condition, we observe that for all $s \in (0,\infty]$, the dominated convergence theorem applies since $r, \nu > 0$ and it holds that $\lim_{V\to\infty}\Phi(d_{1/2}(\frac{V}{k},t)) = 1$. Hence,

$$\begin{aligned} \lim_{V\to\infty}\int_0^s c(V,k,t)\mathrm{d}t &= \lim_{V\to\infty}\int_0^s Ve^{-\nu t}\Phi(d_1(\tfrac{V}{k},t)) - ke^{-rt}\Phi(d_2(\tfrac{V}{k},t))\mathrm{d}t \\ &= \lim_{V\to\infty}[V\int_0^s e^{-\nu t}\mathrm{d}t - k\int_0^s e^{-rt}\mathrm{d}t] = \lim_{V\to\infty}[V\tfrac{1}{\nu}(1-e^{-\nu s}) - k\tfrac{1}{r}(1-e^{-rs})]. \end{aligned}$$

Consequently, $\lim_{V\to\infty}[V+\tau_2\alpha\int_0^\infty c(V,k,t)\mathrm{d}t-\alpha\int_0^T c(V,k,t)\mathrm{d}t]=\infty$ holds provided that $1+\tau_2\alpha\frac{1}{\nu}-\alpha\frac{1}{\nu}(1-e^{-\nu T})>0$, which is equivalent to $\alpha<\bar{\alpha}$, with $\bar{\alpha}$ given in 2. of this Corrigendum. □

9. Corollary 3.5 update: The assumption $\alpha<\tilde{\alpha}$ is now included both in the statement and in the proof of Corollary 3.5, where $\tilde{\alpha}$ is defined as in item 2 of this Corrigendum. All other aspects of Corollary 3.5 remain unchanged.

10. New Corollary in Section 3: A new corollary establishes a sufficient condition for uniqueness of the solution to equation (3.15):

    Corollary 3.A. *A sufficient condition to have a unique solution of (3.15) is that $\alpha<\tilde{\alpha}$, where $\tilde{\alpha}$ is as defined in item 2 of this Corrigendum.*

    Proof. We obtain from the proof of Theorem 3.1 by using Lemma B.C that (3.15) has a unique solution if $\lim_{V_B\to\infty}h_2(V_B)>0$, where h_2 is defined in (C.4). Applying (C.5)yields that if $1+\rho(\lambda_2+\lambda_3)+2(1-\rho)A_2+\tau_2\alpha A_3-\alpha A_4>0$, (3.15) has a unique solution. This inequality is equivalent to $\alpha<\tilde{\alpha}$, with $\tilde{\alpha}$ given in item 2 of this Corrigendum implying the claim. □

11. Proof of Proposition 4.1 update:

    Proof. If $\bar{\alpha}>1$, the proposition follows directly from the continuity of v in α. Therefore, let us assume that $\bar{\alpha}\leq 1$. First, we note that for $\alpha>\bar{\alpha}$, the assumption that $V\to E(V)$ is non-negative does not hold, as then $\lim_{V\to\infty}E(V;V_B)=-\infty$ for all V_B fixed. This, however, is excluded by assumption. Thus, we can restrict our analysis to the case where $\alpha\in[0,\bar{\alpha}]$. Now the proof of Theorem 3.1 and of Proposition 3.3 imply that the only relevant solution of (3.15) is continuous in its parameters as long as it exists. For $\alpha<\bar{\alpha}$, a solution always exists since then $\lim_{V\to\infty}E(V;V_B)=\infty$ for all V_B fixed. Thus, the left-continuity of V_B in α and the continuity of v in α completes the proof. □