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Retirement Planning - An Analysis From Views of Clients and Insurance Companies

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Contents

Overview of Research Papers **vii**

Co-Authorship **ix**

Research Context and Summary of Research Papers **1**

1 Field of Research 1
2 Motivation and Objectives 3
3 Summary of Research Papers 6

Research Papers

1 Tail Index-Linked Annuity: A Longevity Risk Sharing Retirement Plan **19**

2 Collective Longevity Swap: A Novel Longevity Risk Transfer Solution and Its Economic Pricing **59**

3 Asymmetric Information and Longevity Risk Transfer **85**

4 What to Offer If Consumers Do Not Want What They Need? A Simultaneous Evaluation Approach with an Application to Retirement Savings Products **137**

5 The Benefits of Return Smoothing in Insurer's Cover Funds – Analyses From a Client's Perspective **171**

Curriculum Vitae **205**

Overview of Research Papers

Research papers included in this dissertation

1. Chen, A., Li, H., and Schultze M. (2022). Tail index-linked annuity: A longevity risk sharing retirement plan. *Scandinavian Actuarial Journal*, 2022:2, 139-164.

2. Chen, A., Li, H., and Schultze M. (2022). Collective Longevity Swap: a Novel Longevity Risk Transfer Solution and Its Economic Pricing. *Journal of Economic Behavior & Organization*, 201, 227-249.

3. Chen, A., Li, H., and Schultze M. (2022). Asymmetric Information and Longevity Risk Transfer. Submitted to *Economic Modelling* (revise and resubmit).

4. Ruß, J., Schelling, S., and Schultze M. (2022). What to Offer If Consumers Do Not Want What They Need? A Simultaneous Evaluation Approach with an Application to Retirement Savings Products. Resubmitted to the *European Actuarial Journal* (under review).

5. Ruß, J., Schelling, S., and Schultze M. (2022). The Benefits of Return Smoothing in Insurer's Cover Funds – Analyses From a Client's Perspective. Submitted to *The European Journal of Finance* (under review).

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Research Context and Summary of Research Papers

1 Field of Research

This cumulative thesis contributes to research on retirement planning and longevity risk management by taking the perspective of policyholders as well as of life insurers into account.

As a consequence of technological improvement within the last decades, the life expectancy in many OECD countries has been increasing continuously (OECD, 2021). In most countries, this results in a demographic transition, i.e., the ratio of old to young people is constantly increasing. This development leads to a major challenge for old-age provision systems, in particular pay-as-you-go systems. Most OECD countries follow a three-pillar system. Countries that have been very generous with the first pillar (public pension based on a pay-as-you-go system) cannot maintain a high guaranteed pension payment. As a consequence, occupational pensions (second pillar) and private retirement savings (third pillar) become increasingly important for clients in the future to maintain the desired standard of living in the retirement phase. Simultaneously, this evolution has also an impact on life insurers and pension plans, as higher life expectancy leads to an increased longevity risk exposure, i.e., the risk that policyholders live longer than expected. To overcome these problems, new retirement products, evaluation methods, and reinsurance products need to be designed, developed, and analyzed.

The retirement phase is the focus of the first three research papers. There are various methods for a life insurer to deal with longevity risk. Tontines or pooled annuities, where the longevity risk is shared between the policyholders have been proposed by the literature (cf. e.g., Milevsky and Salisbury, 2015, 2016; Piggott et al., 2005; Qiao and Sherris, 2013) and thus the life insurer does not bear any longevity risk. However, the policyholders receive rather volatile annuity payments, which are typically disliked by risk-averse policyholders compared to constant annuity payments with the same expected benefit. Due to this drawback, combinations of such products with traditional annuities have been proposed (Chen et al., 2019, 2020). Nevertheless, these combinations still bear a substantial longevity risk exposure for the insurer. In addition, indexed annuities have been discussed in the literature. These annuities regularly adjust the annuity payments based on a realization of a mortality index, cf. e.g., (Luthy et al., 2001, Pitacco,

2016, Richter and Weber, 2011, Denuit et al., 2011, Denuit et al., 2015). But still, similar to tontines and pooled annuities, the annuity payments are volatile. Another option for managing longevity risk is to transfer the longevity risk via a longevity derivative to a reinsurer or the capital market. Various longevity derivatives have been discussed and proposed by the literature (Blake et al., 2006; Dawson et al., 2010; Dowd et al., 2006; Fung et al., 2014). However, in reality, the total volume of such longevity risk transactions is relatively low since they are rather expensive (Li and Luo, 2012). Moreover, when pricing such longevity derivatives perfect information is assumed in the aforementioned literature. Yan and Hong, 2015 show that there exists asymmetric information in some reinsurance markets, which indicates that information asymmetry should be taken into account when pricing longevity derivatives. The first three papers of this thesis contribute to the literature about longevity risk management by proposing an indexed annuity and a novel longevity derivative, novel economic pricing approaches, and by analyzing the effect of asymmetric information on the longevity risk transfer market.

The last two research papers focus on the retirement savings phase from the perspective of a client. There is a large amount of literature about the optimal design (based on normative models) of retirement savings products, e.g., Nielsen and Steffensen, 2008, Branger et al., 2010, Chen et al., 2015, Chen et al., 2020. However, in reality, behavioral economics observed that there are deviations between the actual and optimal decisions of clients, cf. Benartzi and Thaler, 2007 or Richter et al., 2019 and such deviations are suboptimal from a rational point of view (Kahneman and Klein, 2009). The perceived attractiveness of choices is modeled by descriptive models and analyzed in the context of retirement savings products, e.g., Dierkes et al., 2010 or Ebert et al., 2012. Nevertheless, the two modeling approaches are usually not considered together.

As in the aforementioned literature we use normative models (e.g., Expected Utility Theory (EUT)) to evaluate the objective utility and descriptive models (e.g., (Multi) Cumulative Prospect Theory) to assess the subjective attractiveness of different choices. We propose a method to combine the two different preference functions to simultaneously assess the perceived attractiveness and objective utility of choices and find preferable compromises. Also, we analyze the effect of return smoothing in different cover funds of a life insurer, where the generic smoothing mechanism is inspired by Korn and Wagner, 2019. We evaluate the objective utility (terminal wealth distribution) and the subjective attractiveness (short-term fluctuations) of various actuarially fair investment products with different features. The results in both papers contribute to the field of retirement planning by providing novel evaluation approaches and investment product components, which help to identify and design preferable retirement savings products and thus can help consumers make objectively better choices.

2 Motivation and Objectives

In the last decades, life expectancy has risen sharply and will likely increase further. On the one hand, this can lead to lower state pensions and thus private retirement savings become more important. On the other hand, this leads to an increased longevity risk exposure for life insurers and pension plans and together with the low-interest rate environment to an increased market value of pension liabilities and high capital requirements. Therefore, appropriate risk management solutions for life insurers and pension plans are crucial.

Due to the aforementioned reasons, life insurers and pension funds are in need of an appropriate risk management of longevity risk. Further, a high longevity risk exposure yields high capital requirements for the life insurer triggering high costs. In general, there are two practical options. Firstly, to limit the longevity risk exposure of the life insurer. This is possible for life insurers by offering retirement products, which include a longevity risk sharing component. There are various alternatives to conventional annuities proposed in the literature, e.g., pooled annuities (Piggott et al., 2005), different types of tontines (Chen et al., 2019, 2020; Milevsky and Salisbury, 2015) or indexed annuities (Denuit et al., 2011, 2015; Richter and Weber, 2011). All of these products can reduce the longevity risk exposure of the life insurer, but leave the policyholder with uncertain and volatile payments, especially in advanced retirement ages. Thus, the following research questions emerge:

1. How to design an annuity product which reduces the longevity risk exposure of the life insurer and thus reduces the loss in presence of a longevity shock?

2. Can an annuity product with a longevity risk sharing component yield a higher lifetime utility for the policyholder than a traditional annuity?

The second option for the risk management of a life insurer is to pass the longevity risk to a third party, usually a reinsurance company or capital market investors. There is a large amount of literature about different longevity risk transfer solutions, e.g., longevity swaps or bonds, cf. Blake et al., 2006 or Dowd et al., 2006. Blake et al., 2019 find that the most popular longevity risk transfer product is the indemnity longevity swap. In an indemnity longevity swap, the hedger pays a fixed periodic payment and receives a floating payment dependent on the actual mortality level in its pension portfolio and thus the longevity risk is completely eliminated from the hedger's portfolio. However, this product is rather expensive and therefore only very large annuity providers can afford such a transaction, cf. Li and Luo, 2012. A natural substitute could be an index-based longevity swap since it yields more stable payments, has better liquidity and thus can be preferable to the capital market, cf. Cairns and El Boukfaoui, 2021. An index-based swap is similar to an indemnity swap but the floating payment depends on the level of an mortality index. As a consequence, the hedger remains with the population basis risk, i.e., the

mortality experience of the index and the hedger's portfolio can differ. This is one reason why such products are not very popular so far. Moreover, pricing such longevity derivatives cause several problems. Due to a limited number of transactions and an illiquid longevity market, it is difficult to find the arbitrage-free price of longevity derivatives under a risk-neutral measure (Milevsky et al., 2006; Wang, 2000), which is in particular not unique. Another possibility is to use an economic approach to price longevity derivatives, which do not rely on the risk-neutral approach, cf. Zhou et al., 2015a or Zhou et al., 2015b. However, in these cases, it is not possible to compute closed-form solutions and hence extensive numerical calculations must be performed to determine a price. Applying an economic pricing approach, it is important to take account of asymmetric information between the insurer and the reinsurer while in the previously mentioned studies it is assumed that all contract partners have perfect information about each other. However, in reality, the hedger likely has more precise information about their liability process, i.e., the pension beneficiaries, than for example the capital market investors. In addition, most pension plans have a small portfolio, cf. United States Department of Labor, 2021, and hence it is rather difficult for a capital market investor to accurately model and estimate future survival probabilities of a pension plan. These aspects can lead to information asymmetry and adverse selection, which can harm the profitability of the longevity risk transfer market or can even result in losses for the reinsurer or the capital market investor. Inspired by the aforementioned observations, the following research questions arise, which are answered in the second and third research paper:

3. How can we design and price a longevity swap, which offers a good hedging opportunity for the hedger, but is also simultaneously profitable for the reinsurer? And if so, is it superior to already existing longevity swaps?

4. How can longevity swaps be priced in the presence of information asymmetry such that the hedger has no incentive to misrepresent its true risk type?

5. Is the effect of information asymmetry on the profitability of the reinsurer negligible?

The last two research papers focus on the retirement savings phase. For the clients, it is important that insurers offer retirement savings products with a high objective utility based on a rational and normative model, e.g., EUT. But it is just as important that consumers actually buy such products. However, Benartzi and Thaler, 2007 and Richter et al., 2019 have shown that there are systematic deviations between the product leading to the highest expected utility and the actually chosen product by the consumer. In particular, this applies also to retirement savings products. In this context, these deviations can result in a reduction of the standard of living in the retirement phase, since consumers receive lower pension payments. These discrepancies can be explained by behavioral biases such as the availability bias or "Framing" (Tversky and Kahneman, 1973, 1981). Based on such behavioral biases Tversky and Kahnemann (1979; 1992) have developed the Cumulative Prospect Theory (CPT), which measures the subjective

attractiveness of different choices. However, Ruß and Schelling, 2018 observe that CPT cannot accurately describe the observed decisions in the retirement savings market due to the long investment horizon of such products. Therefore, they extend CPT to the Multi Cumulative Prospect Theory (MCPT), which evaluates interim gains and losses and thus can accurately describe observed decisions for the retirement savings market. Accordingly, the next research question is formulated as follows:

6. How can we identify and design retirement savings products that create a high (albeit not the maximum possible) objective utility while at the same time being subjectively of high (albeit not maximum possible) attractiveness?

In typical life insurance contracts, policyholders participate in a surplus distribution, which is allocated from the collective investment of the life insurer. This collective investment is referred to as insurer's cover fund (ICF). Special features of the ICF compared to traditional market-based investments are the various smoothing mechanisms, which reduce short-term fluctuations even without a guaranteed rate of return, cf. Ruß and Schelling, 2021. However, such a collective investment is usually accompanied by a low stock ratio due to solvency requirements and high guarantees from the past. Also, the policyholder typically cannot choose between multiple cover funds, since the life insurer offers only a single ICF. In this context, the following research questions concerning the ICF are:

7. What is the effect of smoothing in an ICF compared to a similar market-based investment?

8. Can choosing between multiple ICFs be advantageous for the consumer, i.e., making retirement savings more subjectively attractive and/or objectively beneficial for the consumer?

In the next section, a summary of the five research papers is provided.

3 Summary of Research Papers

Research Paper 1: Tail Index-Linked Annuity: A Longevity Risk Sharing Retirement Plan

In the first paper, we propose an innovative annuity product focusing on longevity risk sharing. We refer to this annuity as tail index-linked annuity (TILA). The proposed TILA pays variable payments, which depend on a reference survival index. In particular, the payments are equal to the nominal amount if the survival index is lower than a predefined threshold and the payments are reduced if the survival index exceeds the threshold. Using a stochastic multi-population mortality model, calibrated on real-world mortality data, we show that the proposed TILA does not only improve the expected utility of the policyholders but also the longevity risk exposure of the life insurer. This paper is a joint work with An Chen and Hong Li and is published in the *Scandinavian Actuarial Journal*. The paper has been presented at the 2020 Actuarial Research Conference, online.

The concept of indexed annuities is to adjust the annuity payments depending on a mortality index, which transfers a part of the longevity risk from the insurer to the policyholder. This concept has already been discussed in the literature, e.g., Richter and Weber, 2011 or Denuit et al., 2011. While these existing indexed annuities can diminish the longevity risk exposure of the insurer, the resulting payments to the policyholder are uncertain in most years and could be volatile for advanced ages. Our proposed TILA, however, pays a fixed nominal benefit most of the time and only in very unfavorable evolutions, when the reference index is exceeding the predetermined threshold, the annual benefit is reduced (but not less than a minimum benefit). Consequently, we describe in the first part of the paper the payoff function of the TILA and derive the corresponding fair value based on a risk-neutral pricing method. Next, we compute and analyze fair combinations of the nominal and minimum payments, where we assume that the thresholds are fixed by the insurer to lower the dimensions of the parameter space.

For the numerical analysis, we calibrate the gravity model proposed by Dowd et al., 2011 on U.S. and Canadian uni-sex mortality data, where the reference index is modeled by the merged population and the survival probabilities of the policyholders are based on the Canadian population. Due to less credible data, we extrapolate the survival probabilities from age 100 up to 120. We find that the remaining life expectancy is notably higher for the policyholder compared to the reference population. In order to price the TILA, we apply the Sharpe ratio approach proposed by Milevsky et al., 2006 to determine the risk-neutral measure. Therefore, we adjust the process of survival probabilities such that the Sharpe ratio coincides with the observed market values from Bauer et al., 2010.

In the first part of the results, we show that the expected utility of the TILA is higher than for a regular annuity and that this observation holds for all levels of risk aversion. This is due to the fact that the nominal and minimum payment can be adjusted based on the level of risk aversion of the policyholder and that the implicit risk loading is substantially lower for the TILA compared to a regular annuity as a consequence of the longevity risk sharing. Further, we observe that in presence of a longevity shock the loss of a life insurer can be significantly reduced if TILAs are part of the pension portfolio. This is true even if the share of TILAs in the insurer's portfolio is not high. Lastly, we verify our findings in various sensitivity analyses, especially, for different Sharpe ratios, thresholds, age of retirement and level of interest rates.

To sum up, this paper answers the first two research questions. With the proposed TILA it is possible to share the longevity risk between the life insurer and the policyholder. In particular, the losses in presence of a longevity shock can be massively reduced for the life insurer compared to a regular annuity portfolio. This is possible since the nominal benefit is paid out in nearly all cases and only reduced in very unfavorable scenarios. Therefore, the TILA yields a higher lifetime utility than a traditional annuity for policyholders with various degrees of risk aversion.

Research Paper 2: Collective Longevity Swap: A Novel Longevity Risk Transfer Solution and Its Economic Pricing

The second paper proposes a collective longevity swap between a reinsurer and a group of hedgers, which helps to reduce the longevity risk exposure of the hedgers. This swap combines advantageous features of popular longevity swaps, i.e., indemnity swap and index-based swap. To price the collective longevity swap and other longevity-linked derivatives, a new economic framework is introduced based on a principal-agent model. Using a stochastic multi-population mortality model and real-world mortality data, we show that in a competitive market the collective longevity swap yields a higher expected profit than comparable longevity swaps and that the majority of hedgers experience a utility improvement while the remainders are not worse off. This paper is a joint work with An Chen and Hong Li and is published in the *Journal of Economic Behavior & Organization*. The paper has been presented at the 24th International Congress on Insurance: Mathematics and Economics, online.

There are various index-based longevity-linked securities discussed in the literature, which can reduce the longevity risk exposure of a life insurer or pension fund, cf. Blake et al., 2006 or Dawson et al., 2010. Despite the variety of derivatives, there have hardly been any such transactions, in reality, cf. Blake et al., 2019 since they are either too expensive or do not cover population basis risk, i.e., the risk that payments received by the hedger do not perfectly match mortality experiences of its portfolio. Inspired by the *combined annuity* from Sun Life Financial,

we propose the collective longevity swap, where the floating payments are aggregated for the reinsurer making them more predictable and stable, and due to a predetermined splitting rule, the floating payment can perfectly hedge the liabilities of the hedgers. As a consequence, we first define the floating (paid by the reinsurer) and the fixed payments (paid by the hedger) and show that with an appropriate splitting rule, the collective longevity swap is similar to an index-based swap for the reinsurer and an indemnity swap for the hedger.

Similar to Zhou et al., 2015a and Zhou et al., 2015b, we use an economic approach to price the collective swap removing the necessity to determine the market price of longevity risk. In particular, we assume a principal-agent model, where the participation amount and the risk loading are chosen such that the reinsurer (principal) maximizes its expected profit and the hedgers (agents) are participating in the swap if they are not worse off after entering the swap in terms of utility. Based on this pricing approach we derive analytical solutions for the optimal participation amount and the optimal risk loading. To compare the results with different longevity swaps, we derive the analytical solutions of an indemnity swap and an index-based swap. Based on the analytical solutions, we find that in a monopolistic market setting the indemnity swap yields the highest expected profit, the collective swap the second highest, and the index-based swap the lowest. This is because the collective swap charges a universal risk loading. However, a monopolistic market setting is rather unrealistic and in a competitive market setting the collective swap yields the highest expected profit for the reinsurer, since it creates a utility gain for some hedgers.

In a numerical analysis, we verify the theoretical findings by using the coherent functional demographic model (Co-FDM) proposed by Hyndman et al., 2013. We calibrate the multi-population mortality model to uni-sex mortality data from the UK and Japan and extrapolate the age-dependent variables up to age 120. Further, we analyze the effect of the risk aversion of the hedger and the impact of the portfolio size in a competitive market. In all analyses, we find that the collective swap yields the highest expected profit and that some hedgers have a utility gain.

All in all, this research paper answers the third research question by introducing the collective longevity swap combining the advantages of an indemnity swap for the hedger, i.e., it can eliminate the longevity risk completely and of an index-based swap, i.e., the payments to the hedger are more stable and thus a lower risk-loading can be charged by the reinsurer. In a realistic competitive market setting, we show that the collective longevity swap can offer a higher expected profit for the reinsurer compared to the indemnity or index-based swap. In addition, the collective swap yields for most hedgers a utility gain compared to the other two considered swaps.

Research Paper 3: Asymmetric Information and Longevity Risk Transfer

The third paper analyzes the effect of asymmetric information in the longevity risk transfer market. In particular, as in the previous paper, we consider longevity swaps and an economic pricing framework. In contrast to the literature about longevity derivatives, we assume information asymmetry between the hedger and the hedge provider. Based on the information asymmetry and the resulting incentive constraints, we compute for indemnity swaps the optimal risk premiums and hedge demands in a separating equilibrium under both a monopolistic and a competitive market setting. Further, in a numerical analysis, we find that the expected profit of the hedge provider and the hedger's expected utility can be significantly increased if the information asymmetry is appropriately addressed. This paper is a joint work with An Chen and Hong Li and submitted to *Economic Modelling* (revise and resubmit).

There is a wide variety of literature about designing and pricing longevity derivatives, which mitigate the longevity risk exposure of life insurers and pension plans, cf. e.g., Blake et al., 2014; Coughlan et al., 2011; Wills and Sherris, 2010. However, they usually assume that all parties have perfect information about the longevity risk of the hedger. But in reality, this assumption is rather unrealistic, since the hedger could have more information about its policyholders and thus the capital market investor selects an incorrect model to forecast future mortality rates. That is why information asymmetry can lead to adverse selection possibly resulting in financial losses for the investor hence information asymmetry must be taken into account when pricing longevity derivatives. Therefore, we describe the setup and the incorporation of the information asymmetry in the model in the first part of the paper. Following the second research paper, we use a principal-agent model to price an indemnity swap. But in contrast to the previous paper, we add incentive constraints so that the hedger reveals its true risk type and buys the swap designed for the corresponding risk type. Moreover, we consider a monopolistic and a competitive market setting. Based on the constraints, we solve the optimization problem for both market settings and compute closed form solutions for the optimal risk premiums and hedge demands for each risk type. To evaluate the impact of information asymmetry, we compare the optimal solutions under two different scenarios. In the first scenario, the hedge provider has perfect information and in the second case, the hedge provider ignores information asymmetry and charges the same risk loading for both risk types.

Next, we perform an illustrative numerical analysis of the optimal solutions. For this, we calibrate an Age-Period-Cohort-Improvement (APCI) model to forecast future survival rates on real-world mortality data from the US and the UK. The UK represents the high-risk type since the simulated expected liabilities and the variance of the liabilities are higher than for the US. In a monopolistic market setting, we find that the high-risk type will opt for a full hedge, while the low-risk type will pick a partial hedge, which is in line with the theoretical findings about adverse selection, cf. Stiglitz, 1977. Further, we observe that information asymmetry

leads to a lower expected profit for the hedge provider, where the difference is increasing for more heterogeneous risk types. However, we also find that addressing information asymmetry can lead to a higher expected profit and a lower downside risk for the hedge provider. In a competitive market setting, we find that taking information asymmetry into account when pricing longevity swaps improves the utility of the low-risk type hedger, which is the only risk type affected by information asymmetry.

Summarizing, this paper answers research questions 4 and 5. In particular, we show how to price an indemnity longevity swap in a principal-agent pricing framework in presence of information asymmetry. The corresponding optimal contract parameters characterize a separating equilibrium, which means that the hedger has no incentive to misrepresent its true risk type. Moreover, taking information asymmetry into account in the pricing process can significantly increase the expected profit of the hedge provider or the expected utility of the hedger, which implies that information asymmetry cannot be neglected.

Research Paper 4: What to Offer If Consumers Do Not Want What They Need? A Simultaneous Evaluation Approach with an Application to Retirement Savings Products

In the fourth paper, we present a general framework to assess simultaneously two different preference formulations (e.g., EUT, CPT or MCPT) to find suitable compromises. In particular, we propose a method to identify favorable compromises between products, which are subjectively attractive and those, which lead to a high objective utility level and hence are "good" for consumers. We apply this approach to the market of retirement savings products and show that a promotion of such favorable compromises can help that consumers make better decisions compared to observed decisions. This paper is a joint work with Jochen Ruß and Stefan Schelling and resubmitted to the *European Actuarial Journal*. The paper has been presented at the 2021 ARIA Annual Meeting (online) and at the annual meeting of the German Association for Insurance Science (DVfVW) 2022 (online).

The design of optimal retirement savings products has already been discussed in depth in the existing literature (e.g., Branger et al., 2010, Chen et al., 2015)). But it has been observed that consumers' actual decisions deviate from the optimal decisions based on EUT, cf. Benartzi and Thaler, 2007 or Richter et al., 2019. This is due to several behavioral patterns influencing the actual choices made by consumers. This shows that the objective benefit and the subjective attractiveness should be taken into account when designing products. This can be done by our proposed combined preference function. In the first part of the paper, we introduce and discuss reasonable properties, which should be fulfilled by our proposed simultaneous evaluation approach. Based on these properties, we suggest an explicit combined preference function.

After proposing a combined preference function, we apply this function to identify favorable retirement savings products in a numerical analysis. We use EUT to measure the objective utility and MCPT introduced by Ruß and Schelling, 2018 to measure the subjective attractiveness. We show that our proposed approach works particularly well for individuals with a low and medium loss aversion, i.e., we identify products with a high objective utility and high subjective attractiveness (although not the maximum possible). These compromises combine features of the product with the highest objective utility and the product with the highest subjective attractiveness. However, for a heavy loss aversion, the favorable compromise coincides with the most attractive product and thus the approach reaches its limits. Finally, we find that it is possible to identify favorable compromises if various parameters of the model setup are changed. Our findings hold for different reasonable parameterizations of the combined preference function as well as for different capital market parameters. Moreover, we demonstrate that the simultaneous evaluation approach is also applicable to different preference formulations, e.g., CPT, and it is possible to identify favorable compromises.

To review, this paper proposes an answer to the sixth research question by creating a combined preference function which simultaneously evaluates two different preference functions such that the resulting favorable product has a rather high utility and high attractiveness although not the maximum one. Further, it is possible to identify components of products, which are responsible for a high utility and a high attractiveness. Moreover, promoting such compromises can result in better decisions for the consumer and ultimately in a higher standard of living in the retirement phase. Lastly, it is noteworthy that this method can be adapted to other preference functions and situations.

Research Paper 5: The Benefits of Return Smoothing in Insurer's Cover Funds – Analyses From a Client's Perspective

The last paper analyzes the benefits of return smoothing in an insurer's cover funds from a consumer's perspective. Particularly, we study the effect of multiple segregated cover funds on the short-term risk-return profile and on the terminal benefit risk-return profile. Further, we analyze the objective utility and subjective attractiveness of static and dynamic investment products based on an insurer's cover fund and compare them with similar market-based products. We derive the numerical results in a capital market with stochastic interest rates and stocks. This paper is a joint work with Jochen Ruß and Stefan Schelling and submitted to *The European Journal of Finance*.

Participating life insurance products usually take part in the return of an insurer cover fund (ICF). Ruß and Schelling, 2021 and Goecke, 2013 show that the return smoothing mechanism without additional guarantees in an ICF can significantly reduce the short-term fluctuation of

the investment. Thus, return smoothing can be similarly attractive to consumers as products with guarantees. In this paper, we study a setting, where an insurer can offer (multiple segregated) cover funds with different stock ratios. We compare the ICFs with other market-based low-risk products, in particular a rolling bond and a terminal bond investment.

In a two-factor Hull-White model (which allows a perfect calibration to the observed bond prices), we find that ICFs yield a significantly better short-term risk-return profile than similar market-based products while at the same time having a similar risk-return profile for the terminal payoff.

In the second part of the paper, we extend the study to static and dynamic investment products, where the low-risk asset is either an ICF or a similar market-based low-risk asset. We observe that dynamic investment products based on an ICF outperform dynamic market-based products, i.e., for a similar risk profile a dynamic ICF has a significantly higher expected return. Moreover, we compare the objective utility and the subjective attractiveness of the investment products. To measure the objective utility we use the Expected Utility Theory framework and to measure the subjective attractiveness we use a variety of frameworks based on Cumulative Prospect Theory. We find that in most cases static ICF products with rather high stock ratios yield a higher objective utility and higher subjective attractiveness than market-based investment products. These findings are confirmed in various sensitivity analyses, particularly in a higher interest rate environment and different return smoothing mechanisms. Hence, offering multiple ICFs with different stock ratios can help the insurer to offer retirement savings products with a higher objective utility level, which are also attractive to a wide variety of consumers.

All in all, this paper answers research questions 7 and 8. The paper shows in a realistic capital market setting that an ICF and the corresponding smoothing mechanism can significantly reduce short-term volatility compared to similar market-based investments while hardly influencing the terminal risk-return characteristics. Further, offering investment strategies based on multiple segregated ICFs can help the insurer to offer more attractive and optimal retirement savings products for different types of consumers.

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Tail Index-Linked Annuity: A Longevity Risk Sharing Retirement Plan

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Abstract

This paper proposes an innovative retirement product with a focus on longevity risk sharing, a contract we refer to as tail index-linked annuity (TILA). Specifically, the proposed TILA pays out variable annual payments, which will be equal to a regular nominal amount when a reference survival index is lower than a predetermined threshold (i.e., normal evolution of longevity risk), and a reduced, index-dependent payment when the threshold is passed (i.e., highly unfavorable evolution of longevity risk). The proposed TILA aims at not only improving the benefits of the policyholders, which has been the focus in recent literature on innovative retirement products, but also reducing the longevity risk exposure of the insurer, particularly for advanced retirement ages. Using real-world mortality data and a stochastic multi-population mortality model, we find that the proposed TILA leads to higher expected lifetime utility than regular annuities for policyholders with different degrees of risk aversions. Meanwhile, numerical analysis shows that the proposed TILA could greatly mitigate the solvency risk of the insurer, leading to a substantially lower loss probability and expected (tail-) loss than regular annuities in the presence of a longevity shock, and therefore could reduce the insurer's required solvency capital under the latest solvency regulations.

Keywords: Indexed annuity, longevity risk, multi-population mortality, retirement product design
JEL: G22, G13, J11

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1 Introduction

With population ageing around the globe, longevity risk has become a high-profile risk for governments and many financial institutions. In particular, unexpected mortality improvements have put severe financial strain on annuity providers. Longevity risk, together with some other market conditions like low interest rates and the tightening solvency regulations,[1] forces the insurers to set aside rather high solvency capital for retirement products like conventional annuities. This has the consequence that the already-expensive annuity products become more expensive and less favored by annuitants, although these conventional products can provide them with efficient protection against the longevity risk (Mitchell, 2002; Pang & Warshawsky, 2010; Horneff *et al.*, 2010). Therefore, innovative retirement products that can achieve better longevity risk sharing between life insurers and policyholders are needed for the development of the retirement financial market.

As potential alternatives to the conventional annuities, various products have been proposed in both academia and the insurance industry, with two prominent examples being pooled annuity funds and tontines. In a nutshell, these products create a longevity sharing pool of policyholders, where a member's share of asset would devolve to the other policyholders in the pool when they die. In other words, each remaining policyholder would receive higher payments when the total number of policyholders alive is lower. As a result, in contrast to regular annuities, most of the longevity risk is shared by the policyholders, and insurers merely act as the administrator of the fund. For existing studies of tontines and pooled annuity funds, see, among others, Piggott *et al.* (2005), Qiao & Sherris (2013), Donnelly (2015), Milevsky & Salisbury (2015), and Milevsky & Salisbury (2016). Despite being able to free most of the longevity risk from the insurers, and thus only require minor administration fees from the policyholders, tontines and pooled annuity funds might leave policyholders with rather volatile annuity payments, especially at the advanced ages.

Recognizing the distinct features of the conventional annuities and tontines, recent studies have attempted to combine the advantages of these two types of products. For example, Chen *et al.* (2019) and Chen *et al.* (2020) propose a combination of the tontines and regular annuities such as *tonuities* and *antines*. Taking a tonuity as an example, policyholders receive tontine-type

[1] Note that no uniform international standards exist for the regulation of insurance companies. For instance, in the United States, the insurers are regulated at the state level with some input from the Federal Insurance Office and the National Association of Insurance Commissioners. In EU, all the member countries are subjected to the Solvency II regulation implemented starting in January 2016. Despite their differences in terms of implementation, the main idea of the solvency regulation is very similar. For more information about Solvency II, see `https://www.eiopa.europa.eu/browse/solvency-2_en`. Concerning longevity risk, readers can consult new longevity risk factors in the U.S. for more details, see `https://content.naic.org/sites/default/files/call_materials/Longevity%20Risk%2011-25-19%20Materials.pdf`, and alternatively see `https://content.naic.org/`.

payoffs up to a predetermined age (say 85), after which the remaining assets in the pool will be transferred to regular annuities for the policyholders. Moreover, Chen & Rach (2019) propose a new class of products called tontines with minimum guarantees. In particular, whenever the underlying tontine payment performs worse than the guaranteed level, the policyholder will end up with the guarantee payment. Such innovative products are potentially appealing to policyholders, as they could be cheaper than conventional annuities, while at the same time ensure more secure payments, either over the advanced ages or throughout the whole term of the contract. Indeed, such products are shown to yield higher lifetime expected utility for policyholders. However, insurers providing these products still have to bear a substantial part of longevity risks, especially of policyholders at the very advanced ages. In fact, existing studies suggest that uncertainties of the mortality rates at the advanced ages (say, 85 onward) have been the major longevity risk concerns for life insurers (Blake *et al.*, 2014). Therefore, while these innovative products could be more appealing than both conventional annuities and tontines, they still entail large capital reserves, especially under the tightening solvency capital provision rules all over the world.

Besides the aforementioned products, the concept of indexed annuity has been discussed in the literature. The essential idea of indexed annuity is to frequently adjust the annuity benefits depending on the realization of a mortality index, which means that (part of) the longevity risk embedded in the annuity products can be transferred to the policyholders (Luthy *et al.*, 2001; Pitacco, 2016). Based on the philosophy of group self-annuityzation (GSA), Richter & Weber (2011) discuss annuity contracts with benefits linked to actual mortality experience of the policyholders. In this case, policyholders will bear their own longevity risk. Moreover, Denuit *et al.* (2011) propose life annuities with payments depending on a longevity index linked to unexpected longevity improvements of a reference population. Their approach is in contrast with a GSA, where annuitants bear only the systematic longevity risk. The idea of Denuit *et al.* (2011) is extended to accounting for deferred periods by Denuit *et al.* (2015). Similar to pooled annuity funds and tontines, although the existing indexed annuities could mitigate the longevity risk exposure of the insurer, annuity payments are uncertain in each year and could be volatile at the advanced ages.[2]

This paper extends the existing literature on the search of attractive retirement products, with the focus on longevity risk. In particular, we propose an innovative tail index-linked annuity (TILA) with the aim to improve the expected utility of the policyholders, while mitigating the insurer's loss in unfavorable scenarios in which policyholders live (much) longer than expected. The TILA provides a variable payment scheme, with payments dependent on the realization of a reference survival index. The proposed TILA pays out a regular nominal payment in

[2]Denuit *et al.* (2011) impose floors and caps to reduce the volatility of annuity benefits. However, annuity payments could still be uncertain in most years, unless the bound is hit.

the years when the realized value of a reference survival index is within a predetermined (and time-varying) threshold, and only pays out reduced, index-dependent payments in the highly unfavorable scenarios when the threshold is passed. The survival index is constructed by the mortality experiences of a reference population whose mortality development is positively correlated with the annuity portfolio, and is a measure of longevity risk. Specifically, in each year, a lower survival index indicates lower survival rates in the reference population and thus on average lower number of policyholders alive in the annuity portfolio. Therefore, in the "normal" years when the survival index is low, the TILA works like a regular annuity and pays out fixed nominal payments, while in the "bad" years when the survival index is higher than the threshold, the insurer may need to cover more policyholders than expected, but with reduced payments to each policyholder. In this way, longevity risk could be effectively shared between the insurer and the policyholders. As will be shown later, the longevity risk sharing feature will lead to lower longevity risk loadings of the proposed TILA, compared to the regular annuity. This could potentially make the proposed TILA more attractive for policyholders and provide more incentives for individuals to buy these annuity products. In this sense, it could help to mitigate the annuity puzzle, which is a term used to describe the discrepancy between the theoretical demand for annuities (see, for instance, Yaari, 1965; Peijnenburg *et al.*, 2016) and the fact that few households voluntarily purchase an annuity (see, among many others, Hu & Scott, 2007; Inkmann *et al.*, 2011; Lockwood, 2012). Let us note that there has been a large and growing literature explaining the annuity puzzle, e.g., Beshears *et al.* (2014), Agnew *et al.* (2008) and Brown *et al.* (2017), and one of the interpretations is that individuals perceive annuities as too expensive. In our paper, we introduce a variety of products which lead to a better longevity sharing between the insurer and policyholders, but at the same time allow lower prices than regular annuities. Furthermore, by properly choosing the thresholds, the policyholders are expected to receive fixed nominal payments in most years, which is in contrast to the existing indexed annuities. Annuity payments will not be further reduced when the survival index reaches a second (also predetermined and time-varying) threshold, in which case policyholders will end up with a minimum protected payment. These features make the payment scheme of the TILA rather flexible, and could meet the risk preferences of different policyholders and ensure their financial safety in the worst-case scenarios. It is worth noting that the payoff structure of the proposed TILA has similar features as the Swiss Re bond issued in 2003 (Lin & Cox, 2008), which is also an indexed-based mortality-linked security with reduced principal payment in unfavorable scenarios. However, the proposed TILA differs from the Swiss Re bond in that it focuses on longevity risk rather than mortality risks, and, more importantly, it is a long-term retirement product (in contrast to the 3-year term of the Swiss Re bond) so that the thresholds are time-varying. Further, the users of Swiss Re bonds are pension funds and life insurers, rather than individual consumers. Finally, the proposed TILA includes the regular annuity with constant payments as a special case when the nominal payment is equal to the minimum payment.

While most of the recent literature on the innovative retirement products focuses on improving only the benefits of either the policyholders or the insurers, the proposed TILA aims at benefiting both parties. In particular, we show that the proposed TILA could reduce the longevity risk exposure of the insurer, while simultaneously preserving its attractiveness to the policyholders. Moreover, in contrast to many of the aforementioned studies, which use either a simplified and predetermined mortality law or rather simple stochastic models, this paper utilizes real-world mortality data and a statistical multi-population mortality model for the numerical analysis. Multi-population mortality models have attracted vast attention in the mortality forecasting literature over the past decade. Unlike traditional, single-population mortality models, such as the widely used Gompertz-law (Gompertz, 1825), multi-population models are able to capture the dependence structure among multiple populations, and thus better predict the joint mortality dynamics of the underlying populations. For studies on multi-population mortality models, see, for example, Li & Lee (2005), Cairns *et al.* (2011), and Hyndman *et al.* (2013).

In this paper, we illustrate the proposed TILA using U.S. and Canadian single-age, uni-sex mortality data, and the gravity two-population mortality model proposed by Dowd *et al.* (2011). We let the merged population of the U.S. and Canada represent the reference population, and the Canadian population represent the population of the annuity portfolio. As a result, we find that the proposed TILA leads to higher expected utility levels than a benchmark regular annuity for policyholders with different risk preferences. Moreover, the TILA has a lower implicit risk loading, which means that the amount of longevity risk contained in the TILA is lower than the regular annuity. In addition to the policyholder's perspective, more importantly, the benefits of the proposed TILA from the insurer's point of view are also assessed. In our stylized model, we compare an insurer which sells exclusively regular annuity products, and an insurer which sells a mix of the regular annuity and our TILA product. It can be seen that the proposed TILA contains less risk than the regular annuity for the insurer, especially in the scenarios where mortality rates are lower than expected. We find that portfolios with a higher fraction of TILA are more robust against longevity risks. In this sense, the TILA provides good potentials for the insurers to eliminate the longevity risk. On the whole, the proposed TILA is attractive from both the policyholders' and the insurers' perspective, with a better risk sharing between the two parties.

The remainder of the paper is organized as follows. Section 2 introduces the tail index-linked annuity and the payment features of the TILA. Section 3 describes the mortality model, as well as the estimation, the risk-neutral pricing technique, and the calibration method. Section 4 conducts the numerical analysis of the TILA from the perspective of the policyholder and the insurer. Section 5 provides sensitivity analysis with respect to the risk-neutral pricing measure and the contract parameters. Finally, Section 6 concludes the paper.

2 The tail index-linked annuity

In this section, we introduce the tail index-linked annuity (TILA) product, which looks to achieve a better risk sharing scheme between the insurer and the policyholders. We first introduce the notations and the definition of the TILA and then discuss its pricing formula, as well as the different payment structures that can be achieved by varying parameter combinations of the contract.

2.1 Notations and preliminaries

For ease of exposition, this paper considers a life insurer offering annuity products to a pool of homogeneous policyholders, i.e., all policyholders have the same age at inception, with independently and identically distributed (i.i.d.) remaining life times.[3] The TILA provides whole life annual payments which will be equal to a nominal amount when mortality experience of a reference population is "normal," but will be reduced in "unfavorable" years when the corresponding mortality experiences are substantially better than expected. The most important parameters of a TILA contract are a survival index, which quantifies the mortality experience of the reference population, and two (time-varying) thresholds which define how annuity payments are determined based on the index. Before proceeding to the formal definition, we first introduce the notations used throughout this paper:

- l_x: the initial number (in year 0) of policyholders in the annuity portfolio;
- l_{x+t}: the number of policyholders alive in year t;
- T_x: the remaining future lifetime of the policyholders;
- Y_t: the survival index in year t;
- B: the nominal annual payment;
- $0 \leq z \leq B$: the minimum annual payment;
- L_t: the deterministic lower threshold for Y_t;
- $U_t > L_t$: the deterministic upper threshold for Y_t.

Note that a higher survival index Y_t means a lower mortality rate (mortality improvement), and is thus more "unfavorable" from the insurer's perspective.

The payment structure of the TILA is defined as follows. Firstly, in the "normal" years when Y_t is below L_t, policyholders receive the nominal annuity payment B. Secondly, in the highly

[3] Policyholders with different mortality distributions can be incorporated in a straightforward manner.

"unfavorable" years when Y_t exceeds U_t, annuity payments will be reduced to the minimum amount z. Finally, when Y_t lies between L_t and U_t, the annuity payment will be a linear function of Y_t, bounded by B and z. In general, the thresholds (L_t, U_t) could be either constant or time-varying. However, since the TILA is a long-term annuity product and Y_t is time-varying, it is more natural to let the thresholds be time-varying as well. In particular, we let (L_t, U_t) be the percentiles of Y_t for each t, so that the (best estimated) probability of the annuity payments being reduced or reaching the minimum level could be easily understood by the policyholders. Detailed specifications of the thresholds will be discussed in Section 4. Mathematically, let $\{\mathcal{B}(t)\}_{t>0}$ be the payment stream of the TILA, and the time-t annuity payment is given by

$$\mathcal{B}(t) = \begin{cases} B & Y_t \leq L_t, \\ z + \frac{B-z}{U_t - L_t}(U_t - Y_t) & L_t < Y_t < U_t, \\ z & Y_t \geq U_t, \end{cases} \tag{2.1}$$

or more compactly as

$$\mathcal{B}(t) = z + \frac{B-z}{U_t - L_t}(\max\{U_t - Y_t, 0\} - \max\{L_t - Y_t, 0\}). \tag{2.2}$$

We see that the TILA permits a rather flexible payment structure depending on the set of parameters (B, z, L_t, U_t), with the potential to meet the risk preferences of different policyholders. For example, given L_t and U_t, a less risk averse policyholder may prefer higher payments in normal years (higher B) at the cost of lower payments in bad years (lower z), while a more risk-averse policyholder may prefer more stable payments throughout their life (close B and z). Also, a policyholder could choose to receive lower B and z in exchange of higher L_t and U_t, i.e., their benefits will only be reduced at very bad times, or the other way around. In the extreme case when $B \downarrow z$, the TILA reduces to a regular annuity, and L_t and U_t have no impact on the annuity payments. The proposed TILA can be considered as a "reversed" variable annuity product, which pays out a minimum guarantee payment z and a surplus when the survival index performs well. The total payment (including the guarantee payment and the surplus) is capped by B. Since this paper focuses on the longevity risk sharing potential of the TILA, our product differs from general variable annuities which typically contain equity risk. We assume a constant risk-free interest rate and exclude other financial market risks.

In general, the reference population could be any population related to the annuity portfolio. For example, Y_t could be the survival probabilities of the annuity portfolio itself. In this case, whether policyholders will receive reduced annuity payments depends entirely on their own mortality experience. Alternatively, Y_t could be based on a larger benchmark population whose mortality is positively correlated with the annuity portfolio, such as the national population or

the population of all annuitants in the country which the insurer operates.[4] The potential of using such a benchmark population in retirement product design and longevity risk hedging for pension plans and life insurers has been widely studied in recent literature (see, among others, Coughlan *et al.*, 2011; Blake *et al.*, 2013, 2014; Cairns & El Boukfaoui, 2019). Although mortality experiences of the benchmark populations may not be perfectly correlated with that of the annuity portfolio, these populations typically have (much) longer and more credible data and more stable historical mortality experiences. Therefore, linking Y_t to such populations could lead to more stable and predictable future annuity payments. In the next subsection, we discuss a general valuation framework in which Y_t depends on a population other than the annuity portfolio. The special case where Y_t depends on the annuity portfolio itself can be achieved by assuming the two populations are identical.

Arguably, the payoff structure of the proposed TILA is more complex than conventional annuity products with fixed payments. However, we remark that there are some real-world annuity products with more complex structures. Firstly, in the Dutch occupational pension system, pension payments are indexed and allowed to be cut (Broeders & Chen, 2010). As pointed out by a study conducted by MERCER CFA Institute about Global pension index, in which the pension systems of 39 countries are compared, the Netherlands is front-runner of the table as the Dutch pension system is an almost fully-funded one and contains a quasi-mandatory occupational plan. In this sense, the index-linked pension schemes are implemented in real-world applications. Further, due to the the social security and occupational arrangements, there is not much space left for very large traditional annuity market (Rusconi, 2008). Secondly, in the newly introduced target pension in Germany, the payments are even more complex, determined in a rather complicated way on the funding ratio of the pension fund (Chen & Rach, 2020). This pension reform was initiated in 2018 and will take some time until this new pension scheme is widespread in occupational pension schemes. Besides the aforementioned pension schemes, many variable annuities with guarantee riders have rather complex structures as well (Bacinello *et al.*, 2011). According to the individual annuity report of LIMRA, variable annuity has taken a market share between 42% to 48% of total individual annuity sales in the U.S. between 2016 to 2020 (LIMRA, 2021). Hence, the individual retirement market seems to be tolerant to complicated products.

2.2 Valuation of the tail index-linked annuity

In this subsection, we introduce the valuation framework of the tail index-linked annuity. Since the market for insurance is incomplete, there is no unique price measure resulting from an arbitrage-free pricing. In our case, if the longevity risk can be fully hedged, e.g., by trading in

[4]Examples of such populations include the CMI population in UK (`https://www.actuaries.org.uk/learn-and-develop/continuous-mortality-investigation/cmi-mortality-and-morbidity-tables`).

the longevity-linked securities, a complete financial market would result and the risk-neutral probability measure Q could be determined uniquely. So far, however, there have been only few longevity-linked securities traded in the market. As a result, it is not practical to rely on market data for pricing purposes and alternative methods are required to select the risk-neutral measure. Various methods have been introduced to price mortality- or longevity-linked securities, including the Wang's transformation (Lin & Cox, 2005, 2008) and the Sharpe ratio approach (Milevsky *et al.*, 2006; Chen *et al.*, 2019; Chen & Rach, 2019). In this paper, we calibrate the risk-neutral measure using the Sharpe ratio approach proposed by Milevsky *et al.* (2006). In particular, we determine the risk-neutral distribution of mortality rates assuming a given Sharpe ratio of the regular annuity portfolio written on the pool of policyholders. A more detailed explanation is given in Section 3.3.

Under a risk-neutral measure Q the TILA is fairly priced if the single up-front premium W_0 is equal to the initial market value of the future annuity payments, i.e.,

$$W_0 = \mathbb{E}_Q\left[\sum_{t=1}^{\omega_x} \mathcal{B}(t) D_t \mathbb{1}_{\{T_x>t\}}\right], \tag{2.3}$$

where ω_x is the largest attainable remaining life time of a policyholder aged x, and $D_t = e^{-rt}$ is the discount factor at time t. Since this paper focuses on the longevity risk sharing potential of the TILA, we assume that the risk-free interest rate r is constant over time. Equation (2.3) can then be written as

$$\begin{aligned}
W_0 =& \mathbb{E}_Q\left[\sum_{t=1}^{\omega_x} \mathcal{B}(t) \cdot D_t \cdot \mathbb{1}_{\{T_x>t\}}\right] \\
=& \sum_{t=1}^{\omega_x} e^{-rt} \mathbb{E}_Q\left[\mathbb{1}_{\{T_x>t\}}\left(z + \frac{B-z}{U_t - L_t}(\max\{U_t - Y_t, 0\} - \max\{L_t - Y_t, 0\})\right)\right] \\
=& \sum_{t=1}^{\omega_x} e^{-rt}\Bigg(\mathbb{E}_Q\left[B \cdot \mathbb{1}_{\{Y_t \le L_t\}}\mathbb{1}_{\{T_x>t\}}\right] + \mathbb{E}_Q\left[z \cdot \mathbb{1}_{\{Y_t \ge U_t\}}\mathbb{1}_{\{T_x>t\}}\right] \\
&+ \mathbb{E}_Q\left[\left(z + \frac{B-z}{U_t - L_t}(U_t - Y_t)\right)\mathbb{1}_{\{U_t > Y_t > L_t\}}\mathbb{1}_{\{T_x>t\}}\right]\Bigg) \\
=& \sum_{t=1}^{\omega_x} e^{-rt}\Bigg(B \cdot \mathbb{E}_Q\left[\mathbb{1}_{\{Y_t \le L_t\}}\mathbb{1}_{\{T_x>t\}}\right] + z \cdot \mathbb{E}_Q\left[\mathbb{1}_{\{Y_t \ge U_t\}}\mathbb{1}_{\{T_x>t\}}\right] \\
&+ z \cdot \mathbb{E}_Q\left[\mathbb{1}_{\{U_t > Y_t > L_t\}}\mathbb{1}_{\{T_x>t\}}\right] + \frac{B-z}{U_t - L_t}\mathbb{E}_Q\left[(U_t - Y_t)\mathbb{1}_{\{U_t > Y_t > L_t\}}\mathbb{1}_{\{T_x>t\}}\right]\Bigg) \\
=& \sum_{t=1}^{\omega_x} e^{-rt}\Bigg(B \cdot \mathbb{E}_Q\left[\mathbb{1}_{\{Y_t \le L_t\}}\mathbb{1}_{\{T_x>t\}}\right] + z \cdot \mathbb{E}_Q\left[\mathbb{1}_{\{Y_t > L_t\}}\mathbb{1}_{\{T_x>t\}}\right]
\end{aligned}$$

$$+\frac{B-z}{U_t-L_t}\mathbb{E}_Q\Big[(U_t-Y_t)\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]\Bigg). \tag{2.4}$$

2.3 Fair combination of parameters

As discussed earlier, the TILA allows for a versatile payment structure with contract parameters $(B, z, (U_t, L_t)_{t=1,\dots,\omega_x})$. In particular, a set of parameters is fair as long as it satisfies the valuation equation (2.3). This flexibility allows us to examine the relation between parameters. However, due to the high dimension of the parameter space, an exhaustive pairwise analysis is impractical. Moreover, in real-world applications, it would be impractical to ask a policyholder to determine every single parameter of the contract. Therefore, we focus on the case where $(U_t, L_t)_{t=1,\dots,\omega_x}$ are fixed by the insurer, and the policyholder only needs to choose the combination of (B, z) that best reflects their risk preferences. Given the above assumptions, we can express B as a function of z using Equation (2.4), i.e.,

$$B(z)=\frac{W_0-\sum_{t=1}^{\omega_x}e^{-rt}z\left(\mathbb{E}_Q\Big[\mathbb{1}_{\{Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]-\mathbb{E}_Q\Big[\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]\right)}{\sum_{t=1}^{\omega_x}e^{-rt}\left(\mathbb{E}_Q\Big[\mathbb{1}_{\{Y_t\le L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]+\mathbb{E}_Q\Big[\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]\right)}. \tag{2.5}$$

From (2.5) we can observe the following patterns:

a. The function $B(z)$ is monotonically decreasing in z, with the first derivative given by

$$\frac{\partial B}{\partial z}=B'(z)=\frac{-\sum_{t=1}^{\omega_x}e^{-rt}\left(\mathbb{E}_Q\Big[\mathbb{1}_{\{Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]-\mathbb{E}_Q\Big[\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]\right)}{\sum_{t=1}^{\omega_x}e^{-rt}\left(\mathbb{E}_Q\Big[\mathbb{1}_{\{Y_t\le L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]+\mathbb{E}_Q\Big[\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]\right)}\le 0. \tag{2.6}$$

The derivative is negative, since $\mathbb{E}_Q\Big[\mathbb{1}_{\{Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]\ge\mathbb{E}_Q\Big[\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]$ (because $\frac{U_t-Y_t}{U_t-L_t}<1$ for all $U_t>Y_t>L_t$).

b. If there is no minimum payment, i.e., $z=0$, then the nominal guarantee has an upper bound at

$$B(0)=\frac{W_0}{\sum_{t=1}^{\omega_x}e^{-rt}\left(\mathbb{E}_Q\Big[\mathbb{1}_{\{Y_t\le L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]+\mathbb{E}_Q\Big[\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x>t\}}\Big]\right)}. \tag{2.7}$$

$B(0)$ is an upper bound because $B(z)$ is decreasing in z and we have assumed that there is no negative minimum payment.

c. If $z \uparrow B$, then we get that

$$W_0 = \sum_{t=1}^{\omega_x} e^{-rt} \cdot z \cdot \mathbb{E}_Q\Big[\mathbb{1}_{\{T_x>t\}}\Big] = \sum_{t=1}^{\omega_x} z e^{-rt}\, {}_t\tilde{p}_x^{(2)}, \tag{2.8}$$

where ${}_t\tilde{p}_x^{(2)}$ denotes the t-year survival probability of a policyholder under the risk-neutral measure Q. This shows that in case the nominal payment and the minimum payment are identical, they will be both equal to the fair payment of a regular annuity with a constant payment. Stated differently, the TILA is reduced to a regular annuity.

3 Mortality model

In the following, we use the gravity mortality model proposed by Dowd *et al.* (2011) to model the survival index Y_t and the survival rates of the policyholders. Based on the estimated model parameters, we then compute the risk-neutral probability measure with the Sharpe ratio approach introduced by Milevsky *et al.* (2006) to value regular annuities and the proposed TILA. Section 3.1 outlines the model specification, Section 3.2 shows some estimation results under the real-world measure, and Section 3.3 proposes a method to determining the mortality dynamics under the risk-neutral probability measure based on the Sharpe ratio approach.

3.1 The gravity mortality model

The gravity model is studied by Dowd *et al.* (2011, 2019), which extends the traditional Age-Period-Cohort (APC) model to describe the joint mortality dynamics of two populations. Under the gravity model, the two populations studied are assumed to be of different sizes. Also, mortality of the large population is assumed to evolve solely based on its own historical trend, while mortality of the small population is dependent on the large population. Without loss of generality, letting populations 1 and 2 be the benchmark population and the annuity portfolio, respectively, the gravity model is given by

$$\log m_{t,x}^{(1)} = \beta_x^{(1)} + n_a^{-1}\kappa_t^{(1)} + n_a^{-1}\gamma_c^{(1)}, \tag{3.1}$$
$$\log m_{t,x}^{(2)} = \beta_x^{(2)} + n_a^{-1}\kappa_t^{(2)} + n_a^{-1}\gamma_c^{(2)}. \tag{3.2}$$

For population $i = 1, 2$, $\log m_{t,x}^{(i)}$ is the logarithm of the central death rate at age x and year t; $\beta_x^{(i)}$ is the age effect, i.e., the average mortality of age x and population i over time; $\kappa_t^{(i)}$ is the period effect, i.e., the aggregate mortality level of population i in year t, and $\gamma_c^{(i)}$ is the cohort effect, i.e., the mortality pattern of the cohort in population i born in year $c = t - x$. Finally, n_a is the number of ages in the sample.

After fitted to historical data, Equations (3.1) - (3.2) can be used to generate the predictive distribution of future mortality rates. To this end, dynamics of $(\kappa_t^{(i)})_t$ and $(\gamma_c^{(i)})_c$ need to be specified. In this paper, we follow the specifications used in Dowd *et al.* (2011). Firstly, the dynamics of the period effects is given by

$$\kappa_t^{(1)} = \kappa_{t-1}^{(1)} + \mu^{(1)} + C^{(11)} Z_t^{(1)}, \tag{3.3}$$

$$\kappa_t^{(2)} = \kappa_{t-1}^{(2)} + \phi^{(k)}(\kappa_{t-1}^{(1)} - \kappa_{t-1}^{(2)}) + \mu^{(2)} + C^{(21)} Z_t^{(1)} + C^{(22)} Z_t^{(2)}, \tag{3.4}$$

where $(Z_t^{(1)}, Z_t^{(2)})$ are 2-dimensional i.i.d. standard normal random variables under the real-world measure. From Equations (3.3) - (3.4), we see that $\kappa_t^{(1)}$ is a random walk with drift depending solely on its own history, whereas the evolution of $\kappa_t^{(2)}$ is determined by two components: a random walk with drift on its own history $(\kappa_{t-1}^{(2)} + \mu^{(2)})$, and the spread relative to population 1 $(\phi^{(k)}(\kappa_{t-1}^{(1)} - \kappa_{t-1}^{(2)}))$. In particular, the spread term introduces dependency between the long-term trends of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$: a positive $\phi^{(k)}$ will pull the spread between $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ towards zero. Moreover, because $\kappa_t^{(1)}$ does not depend on $\kappa_t^{(2)}$, it is a "gravitational pull" of $\kappa_t^{(2)}$ towards $\kappa_t^{(1)}$.

Similarly, the cohort effects are assumed to be dependent and follow a gravitational specification as well. Specifically, the joint dynamics of $\gamma_c^{(1)}$ and $\gamma_c^{(2)}$ is given by

$$\gamma_c^{(1)} = (1 + \alpha^{(\gamma^1)})\gamma_{c-1}^{(1)} - \alpha^{(\gamma^1)}\gamma_{c-2}^{(1)} + \mu^{(\gamma^1)}(1 - \alpha^{(\gamma^1)}) + C^{(\gamma^{11})} Z_c^{(\gamma^1)}, \tag{3.5}$$

$$\begin{aligned}\gamma_c^{(2)} = {} & (1 + \alpha^{(\gamma^2)} - \phi^{(\gamma)})\gamma_{c-1}^{(2)} - \alpha^{(\gamma^2)}\gamma_{c-2}^{(2)} + \phi^{(\gamma)}\gamma_{c-1}^{(1)} + \mu^{(\gamma^2)}(1 - \alpha^{(\gamma^2)}) \\ & + C^{(\gamma^{21})} Z_c^{(\gamma^1)} + C^{(\gamma^{22})} Z_c^{(\gamma^2)},\end{aligned} \tag{3.6}$$

where $(Z_c^{(\gamma^1)}, Z_c^{(\gamma^2)})$ are 2-dimensional i.i.d. standard normal random variables under the real-world measure. In Equations (3.5) - (3.6), $\mu^{(\gamma^1)}$ (resp. $\mu^{(\gamma^2)}$) is the mean reversion level of $\Delta\gamma_c^{(1)}$ (resp. $\Delta\gamma_c^{(2)}$), and $\alpha^{(\gamma^1)}$ (resp. $\alpha^{(\gamma^2)}$) is the AR(1) parameter for $\Delta\gamma_c^{(1)}$ (resp. $\Delta\gamma_c^{(2)}$). Similar to the period effects, $\gamma_c^{(1)}$ evolves solely based on its own history, while $\gamma_c^{(2)}$ is dependent on the spread relative to $\gamma_c^{(1)}$ via the gravity parameter $\phi^{(\gamma)}$. In particular, a positive $\phi^{(\gamma)}$ will pull the spread between the two $\gamma_c^{(1)}$ and $\gamma_c^{(2)}$ towards 0. For more detailed interpretations of Equations (3.3) - (3.6), we refer to Dowd *et al.* (2011).

3.2 Estimation results

In real world applications, an insurer could use its own annuity portfolio as population 2. However, company-level mortality experiences are typically confidential and thus not available to the public. Therefore, in this paper we illustrate the proposed tail index-linked annuity using the U.S. and Canadian national population. Specifically, we let population 1 be the merged population of the U.S. and Canada, and population 2 be the Canadian population. The national

populations of the U.S. and Canada in 2019 were approximately 332 million and 36 million, respectively, which make the size of population 1 about 10 times as population 2.[5] Because of this, we believe that these two populations could reasonably approximate the situation of a smaller annuity portfolio and a larger benchmark population. Mortality data are downloaded from the Human Mortality Database.[6] In particular, uni-sex data from 1950 to 2016 and ages 50 to 100 are used to estimate the gravity mortality model.

As a preliminary examination, the historical accumulated survival probabilities of the cohort born in 1900 in each population are displayed in Figure 1.[7] We see that the historical survival probabilities of the Canadian cohort were higher. This observation is consistent with the empirical findings that mortality of annuity portfolios is in general lower than the corresponding reference population (see, for example, Plat (2009); Dowd *et al.* (2011)). The estimated period effects of the two populations are shown in Figure 2. The steeper slope of $\kappa_t^{(2)}$ indicates a faster aggregate mortality decline in population 2 over the sample. To gain more insights of the aggregate mortality trends of the two populations, we summarize the parameter estimates in Equations (3.3) - (3.4) in Table 1. Indeed, we see that $\mu^{(2)}$ is larger than $\mu^{(1)}$ by approximately 22% (in absolute value), which indicates a faster mortality decline in Canada. Moreover, the gravity parameter ϕ is positive, which supports the empirical dependence of $\kappa_t^{(2)}$ on $\kappa_t^{(1)}$. A summary of more detailed estimation results is shown in Appendix A.

	$\mu^{(i)}$	ϕ	C	
Population 1	-0.5698	0.0290	$C^{(11)} = 0.7060$	
Population 2	-0.6956		$C^{(21)} = 0.3711$	$C^{(22)} = 0.4600$

Table 1: The estimated parameters in Equations (3.3) - (3.4) fitted to the merged (population 1) and Canadian (population 2) uni-sex population over 1950 to 2016.

In the estimation procedure we include mortality data up to 100 because exposures at more advanced ages were rather small and their mortality data were less credible. For example, this can be a result of age misreporting by older persons (Gavrilov & Gavrilova, 2019). However, insurance applications typically require mortality tables that include older ages, for example, up to 120. To fill this gap, we extrapolate our mortality table to age 120. To do this, we first extrapolate the estimated age effects to 120, then project future mortality rates by extrapolating κ and γ using their respective time-series process. In order to extrapolate the age effects $\beta_x^{(1)}$ and $\beta_x^{(2)}$, we assume that $\beta_x^{(i)}$ follows a linear trend, i.e., $\beta_{x+1}^{(i)} = \beta_x^{(i)} + c^{(i)} + \eta_x^{(i)}$ for $i = 1, 2$, where $c^{(i)} \in \mathbb{R}$ is a constant and $\eta_x^{(i)}$ are i.i.d. normal random variables under measure P. We estimate that $c^{(1)} = 0.0856$ and $c^{(2)} = 0.0913$. The estimated and extrapolated $\beta_x^{(i)}$'s are given

[5]Source: United States Census Bureau (`https://www.census.gov/popclock/`).

[6]`https://www.mortality.org/`.

[7]Mortality patterns of other cohorts are qualitatively similar.

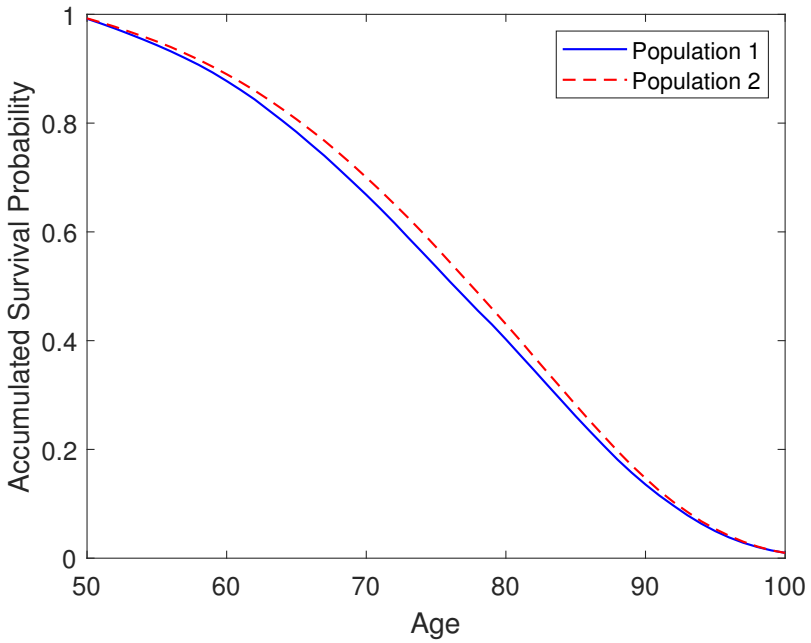


Figure 1: Historical accumulated survival probability of the merged (population 1) and Canadian (population 2) cohort born in 1900.

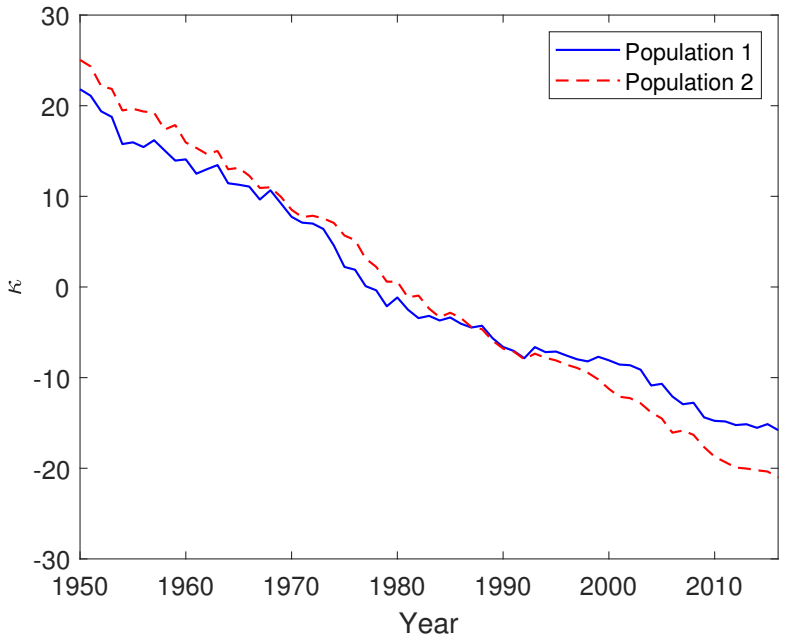


Figure 2: Period effect of the merged (population 1) and Canadian (population 2) using uni-sex population from 1950 to 2016, age 50 to 100.

in Figure A.1 in Appendix A. With the estimated parameters, we compute the remaining life expectancy of a 65-year old individual at time 0 under P. In particular, this remaining life expectancy is 22.13 years in the merged population and 23.91 years in Canada.

3.3 Mortality rates under the risk-neutral measure

In the following, we apply the Sharpe ratio approach to determine the risk-neutral measure Q, which depends on the portfolio size of the insurer. In the longevity risk pricing literature, it is typical to characterize the risk-neutral measure by the longevity risk premiums added to the mortality trends, i.e., the drift terms of the period effects (see, for example, Cairns *et al.*, 2006; Li, 2018). Formally, we denote by $(\tilde{Z}_t^{(1)}, \tilde{Z}_t^{(2)})$ a 2-dimensional i.i.d. normal random variables under Q such that

$$\tilde{Z}_t^{(1)} = \lambda_1 + Z_t^{(1)}, \quad \tilde{Z}_t^{(2)} = \lambda_2 + Z_t^{(2)},$$

where λ_1 and λ_2 are the market price of longevity risk associated with $Z_t^{(1)}$ and $Z_t^{(2)}$, respectively. Under the chosen decomposition for the co-variance matrix of $(Z_t^{(1)}, Z_t^{(2)})$, λ_1 is the longevity risk premium of population 1 and λ_1 and λ_2 together constitute the longevity risk premium of population 2. Given this specification, the dynamics of the period effects under Q is given by

$$\kappa_t^{(1)} = \kappa_{t-1}^{(1)} + \mu^{(1)} + C^{(11)}\tilde{Z}_t^{(1)} = \kappa_{t-1}^{(1)} + \tilde{\mu}^{(1)} + C^{(11)}Z_t^{(1)}, \tag{3.7}$$

and

$$\begin{aligned}\kappa_t^{(2)} &= \kappa_{t-1}^{(2)} + \phi^{(k)}(\kappa_{t-1}^{(1)} - \kappa_{t-1}^{(2)}) + \mu^{(2)} + C^{(21)}\tilde{Z}_t^{(1)} + C^{(22)}\tilde{Z}_t^{(2)} \\ &= \kappa_{t-1}^{(2)} + \phi^{(k)}(\kappa_{t-1}^{(1)} - \kappa_{t-1}^{(2)}) + \tilde{\mu}^{(2)} + C^{(21)}Z_t^{(1)} + C^{(22)}Z_t^{(2)},\end{aligned} \tag{3.8}$$

where $\tilde{\mu}^{(1)} = \mu^{(1)} + C^{(11)}\lambda_1$ and $\tilde{\mu}^{(2)} = \mu^{(2)} + C^{(21)}\lambda_1 + C^{(22)}\lambda_2$ are the drift terms of the period effects under Q.

As discussed earlier, we rely on the Sharpe ratio approach proposed by Milevsky *et al.* (2006) to determine the longevity risk premium (λ_1, λ_2). In particular, for each population $i = 1, 2$, we assume that there is a hypothetical regular annuity portfolio written on $l_x^{(i)}$ policyholders of the cohort x at time 0. Note that we use a regular annuity to determine the longevity risk loading, as there is no market data for our proposed TILA, although admittedly the market data for regular annuities are rather scarce. Following the assumptions made throughout this paper, these $l_x^{(i)}$ policyholders have i.i.d remaining life times, conditional on the survival probabilities ${}_tp_x^{(i)}$, $t = 1, 2, ..., \omega_x$. Denoted by $X(l_x^{(i)})$, the (random) present value of the total annuity payment resulting from the portfolio of population i, its Sharpe ratio is then given by $\frac{\mathbb{E}_Q[X(l_x^{(i)})] - \mathbb{E}_P[X(l_x^{(i)})]}{\sqrt{\mathrm{Var}_P[X(l_x^{(i)})]}}$. With the Sharpe ratio approach, the risk-neutral measure Q is determined

by letting the Sharpe ratio of the two annuity portfolios equal to a preset value η. Stated differently, (λ_1, λ_2) is derived such that:

$$\eta = \frac{\mathbb{E}_Q[X(l_x^{(i)})] - \mathbb{E}_P[X(l_x^{(i)})]}{\sqrt{\mathrm{Var}_P[X(l_x^{(i)})]}}, \quad i = 1, 2. \tag{3.9}$$

Next, we describe the procedure of deriving the risk-neutral measure Q. First, denote the t-year survival probability under Q for population 1 and 2 by ${}_t\tilde{p}_x^{(1)}(\lambda_1)$ and ${}_t\tilde{p}_x^{(2)}(\lambda_1, \lambda_2)$. Note that the risk-neutral survival probabilities of population 1 are independent of λ_2, cf. Equations (3.7) to (3.8). Denote the σ-algebra containing ${}_tp_x^{(i)}$ until time t by $\mathcal{F}_t^{(i)}$ and $\mathcal{F}^{(i)} = \bigcup_{t=1}^{\omega_x} \mathcal{F}_t^{(i)}$ for $i = 1, 2$. Then the expectation and the variance of the annuity portfolios can be written as:

$$\begin{aligned}
\mathbb{E}_P[X(l_x^{(i)})] &= l_x^{(i)} \sum_{t=1}^{\omega_x} e^{-rt}\, {}_tp_x^{(i)}, \quad i = 1, 2, \\
\mathbb{E}_Q[X(l_x^{(1)})] &= l_x^{(1)} \sum_{t=1}^{\omega_x} e^{-rt}\, {}_t\tilde{p}_x^{(1)}(\lambda_1), \\
\mathbb{E}_Q[X(l_x^{(2)})] &= l_x^{(2)} \sum_{t=1}^{\omega_x} e^{-rt}\, {}_t\tilde{p}_x^{(2)}(\lambda_1, \lambda_2), \\
\mathrm{Var}_P[X(l_x^{(i)})] &= (l_x^{(i)})^2 \mathrm{Var}_P\left[\sum_{t=1}^{\omega_x} e^{-rt}\, {}_tp_x^{(i)}\right] + l_x^{(i)} \mathbb{E}_P\left[\mathrm{Var}_P\left(\sum_{t=1}^{\omega_x} e^{-rt} \mathbb{1}_{\{T_x^{(i)} > t\}} \middle| \mathcal{F}^{(i)}\right)\right], \quad i = 1, 2,
\end{aligned}$$

where $T_x^{(i)}$ is the random lifetime variable of a policyholder with initial age x in population i. In particular, the first and second component of $\mathrm{Var}_P[X(l_x^{(i)})]$ represent the systematic and the diversifiable longevity risk in population i, respectively. Furthermore, when the portfolio size increases to infinity, i.e., when the diversifiable longevity risk disappears, $\mathrm{SD}_P[X(l_x^{(i)})]$ reduces to

$$\lim_{l_x^{(i)} \to \infty} \frac{1}{l_x^{(i)}} \mathrm{SD}_P[X(l_x^{(i)})] = \mathrm{SD}_P\left[\sum_{t=1}^{\omega_x} e^{-rt}\, {}_tp_x^{(i)}\right] \tag{3.10}$$

where SD_P stands for the standard deviation under measure P. Based on the formulas above, we are able to calibrate (λ_1, λ_2) with the chosen values of $(\eta, l_x^{(1)}, l_x^{(2)})$. First, we select λ_1 such that the Sharpe ratio equation (3.9) holds for $i = 1$. Then, given the selected λ_1, we determine λ_2 such that the Sharpe ratio equation holds for $i = 2$. The choice of $(\eta, l_x^{(1)}, l_x^{(2)})$ is discussed in the numerical analysis in Section 4, and the simulation procedure for solving the Sharpe ratio equations is delegated to Appendix B.

4 Numerical Analysis

This section discusses the numerical evaluation of the proposed TILA. In particular, the payoff structure of the TILA is illustrated in Section 4.1, and the TILA is compared to a regular annuity from the perspective of the policyholders in Section 4.2 and from the perspective of the insurer in Section 4.3. The base case parameter setup used in this section is given in Table 2. Also, the survival index is set to be the realized t-year survival probability of the cohort age 65 in year 0 of population 1, i.e., $Y_t = {}_tp_x^{(1)}$. In deriving the risk-neutral pricing measure Q, we let $x = 65$ and $l_x^{(2)} = 1,000$. Further, we let $l_x^{(1)} = \infty$ because population 1 is the survival index and thus only systematic longevity risk will be priced. Finally, the choice of $\eta = 10\%$ is motivated by Bauer *et al.* (2010), who found that UK pension annuities have a Sharpe ratio between 4% and 12%. Based on 10,000 simulations, the risk-neutral drift terms are estimated as $\tilde{\mu}^{(1)} = -0.5890$ and $\tilde{\mu}^{(2)} = -0.7151$. They are smaller than their counterparts under P, and lead to higher survival probabilities under Q for all x and t, i.e.,

$$ {}_t\tilde{p}_x^{(i)} \geq {}_tp_x^{(i)} \tag{4.1}$$

for $i = 1, 2$.

Initial age	Pool Size	Annuity Price	Risk-free rate	Sharpe ratio
$x_0 = 65$	$l_{x_0}^{(2)} = 1,000$	$W_0 = 100$	$r = 2\%$	$\eta = 10\%$

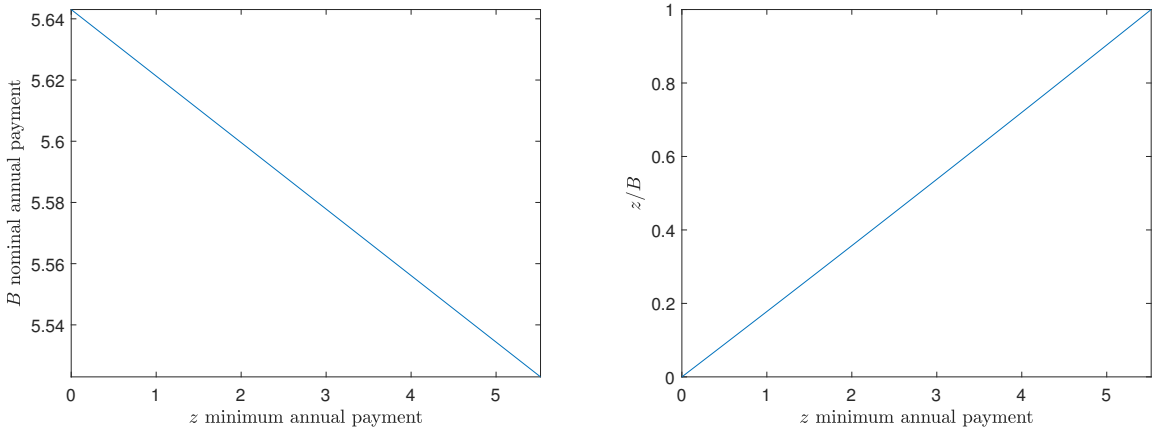
Table 2: Base case parameter setup.

4.1 Fair combination of parameters

In this subsection, we present numerical results on the fair combination of parameters of the TILA contract. As discussed earlier, we focus on the relationship between B and z with fixed $(U_t, L_t)_{t=1,\ldots,\omega_x}$. In particular, we let the thresholds be $L_t = Q_{Y_t}(0.95)$ and $U_t = Q_{Y_t}(0.995)$ for all t, where Q_{Y_t} is the quantile function of Y_t under measure Q. In other words, the thresholds are chosen as the 95%-th and 99.5%-th quantile of the survival index under Q and represent unfavorable situations where the realized survival probability of the reference population is significantly higher than expected. Moreover, the chosen thresholds are easy to understand and comprehensible for the policyholders. Sensitivity of the results with respect to L_t and U_t is discussed in Section 5.

Given the chosen $(U_t, L_t)_{1,\ldots,\omega_x}$, Figure 3 displays the fair relation between B and z (left panel), as well as the ratio of z/B and z (right panel) calculated by Equation (2.5). A detailed simulation procedure is described in Appendix B. Three observations can be made from the

figure. Firstly, as shown in Section 2.3, B is decreasing in z, which means that the policyholders have to accept a lower nominal payment in normal years in exchange for a higher minimum payment in the unfavorable years. Secondly, z has to be substantially reduced for a slight increase of B because the probability of receiving B is rather high in most years. This has led to an increasing z/B ratio over z, observed in Figure 3(b). Therefore, only the tail longevity risk is shared using the chosen thresholds. Finally, in the extreme case, we have $B = z = 5.5230$, which is equal to the annual payment of the regular annuity.



(a) B in dependence of z

(b) z/B in dependence of z

Figure 3: The dependence structure between the fair nominal annual payment B (left panel) and the ratio z/B (right panel) on the minimum annual payment z. Based on $L_t = Q_{Y_t}(95\%)$, $U_t = Q_{Y_t}(99.5\%)$ and the base case parameters.

4.2 Policyholder's perspective

In this subsection, we analyze how different fair combinations of B and z could meet the need of policyholders with different risk preference. We compare the TILA to the regular annuity with constant annual payments z^A issued to the same pool of policyholders and has the same price at time 0. The annual payment z^A is defined through the following pricing equation:

$$W_0 = \sum_{i=1}^{\omega_x} e^{-rt}\ z^A\ \mathbb{E}_Q[1_{T_x>t}] = \sum_{i=1}^{\omega_x} e^{-rt}\ z^A\ {}_t\tilde{p}_x^{(2)},$$

and is equal to 5.5230 given the assumed parameter values.

Let $u(x)$ be the utility function and ρ be the subjective discount factor of the policyholder, the policyholder's expected discounted lifetime utility is then given by:

$$\begin{aligned} &U(\{\mathcal{B}(t)\}_{t>0}) \\ =&\mathbb{E}_P\left[\sum_{t=1}^{\omega_x} e^{-\rho t} u(\mathcal{B}(t)) \mathbb{1}_{\{T_x>t\}}\right] \end{aligned}$$

$$= \sum_{t=1}^{\omega_x} e^{-\rho t} \mathbb{E}_P \left[\left(u(B) \mathbb{1}_{\{Y_t \leq L_t\}} + u(z) \mathbb{1}_{\{Y_t \geq U_t\}} + u\left(z + \frac{B-z}{U_t - L_t}(U_t - Y_t) \right) \mathbb{1}_{\{U_t > Y_t > L_t\}} \right) \mathbb{1}_{\{T_x > t\}} \right]. \tag{4.2}$$

Equation (4.2) can be obtained in a similar way to Equation (2.4). The objective of the policyholder is to maximize the expected discounted lifetime utility over all possible fair combinations of (B, z). In other words, the policyholder has the following optimization problem

$$\max_{(B,z)} U(\{\mathcal{B}(t)\}_{t>0}) \quad \text{subject to } W_0 = \mathbb{E}_Q \left[\sum_{t=1}^{\omega_x} e^{-rt} \mathcal{B}(t) \mathbb{1}_{\{T_x > t\}} \right]. \tag{4.3}$$

Since the maximum operator is not differentiable, we cannot find an explicit solution for this optimization problem. Therefore, we only consider certain parameterized choices of (B, z) Specifically, we consider a grid of 101 equidistantly possible values of z between 0 and z^A. For each z, the corresponding fair value of B is determined by Equation (2.5). We then calculate the expected discounted lifetime utility for each combination, and the optimal combination is the one yielding the maximal utility level. Moreover, to make the utility comparable, we compute the certainty equivalents for each optimal choice of parameters, which is given by

$$CE(\{\mathcal{B}(t)\}_{t>0}) = u^{-1}\left(\frac{U(\{\mathcal{B}(t)\}_{t\geq 0})}{\sum_{t=1}^{\omega_x} e^{-\rho t} \mathbb{E}_P[\mathbb{1}_{\{T_x > t\}}]} \right), \tag{4.4}$$

where u^{-1} is the inverse of the utility function. The certainty equivalent describes the constant annual payment which leads to the same utility as the uncertain payment stream determined by the optimal (B, z). In other words, it translates the optimal expected discounted lifetime utility to a dollar amount. Note that the certainty equivalent of a regular annuity is the annual payment z^A.

In the numerical analysis, we assume that the utility of a policyholder is described by the power utility function, i.e., $u(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$. Also, the subjective discount factor is assumed to be equal to the risk-free interest rate, i.e., $\rho = r$. In Table 3, we demonstrate the optimal (B, z)-combination and the resulting certainty equivalent of the TILA for policyholders with different risk aversions. For comparison, we state also the certainty equivalent of the regular annuity. Note first that the tail index-linked annuity delivers a higher certainty equivalent for all the policyholders considered. In this sense, policyholders do like some volatility in their annuity payoffs. However, this superiority of the TILA over the regular annuities diminishes in risk aversion. It is rather natural that a more risk-averse policyholder prefers a more stable payment stream. This effect can also be confirmed by the optimal choices of (B, z). As the payments of the TILA lie within $[z, B]$, a narrower interval implies a more stable payment stream.

γ	0.5	2	4	6	8	10
B	5.5722	5.5374	5.5302	5.5278	5.5266	5.5254
z	3.2586	4.8602	5.1916	5.3021	5.3573	5.4125
CE^{TILA}	5.5270	5.5241	5.5236	5.5234	5.5233	5.5232
CE^{A}	5.5230	5.5230	5.5230	5.5230	5.5230	5.5230

Table 3: The optimal choice of B and z in dependence of the risk aversion and the corresponding certainty equivalent of the index-based and the regular annuity.

In order to determine what has driven the attractiveness of the TILA over the regular annuity, we exhibit the implied risk loading through the choice of the risk-neutral pricing measure Q. We denote this loading by θ and define it as

$$\theta(X) = \frac{\mathbb{E}_Q[X]}{\mathbb{E}_P[X]} - 1, \tag{4.5}$$

where X is a risk. In our case, we are interested in the risk loading for the TILA and the regular annuity. The magnitude of this loading reflects the longevity risk contained in the retirement products.

Table 4 exhibits the implicit risk loading for the optimal choices of (B, z) of the TILA resulting from Table 3. Overall, we observe that the risk loading is lower for the tail index-linked annuities than the regular annuity. It implies that less longevity risk is contained in the tail index-linked annuity products. As a less risk-averse policyholder allows more volatility in his annuity payments, the lower bound z in the TILA can be substantially lower. Therefore, by (partially) giving up the higher guaranteed payments, the policyholder bears some longevity risk, unlike in the regular annuity case where all the longevity risk is born by the insurer. This has the consequence that the risk loading is lower for a less risk-averse policyholder.

γ	0.5	2	4	6	8	10
θ^{TILA}	0.08%	0.17%	0.19%	0.20%	0.20%	0.21%
θ^{A}	0.21%	0.21%	0.21%	0.21%	0.21%	0.21%

Table 4: The implicit risk loading for the optimal tail index-linked annuity in dependence of the risk aversion (θ^{TILA}) and the regular annuity (θ^{A}).

4.3 Insurer's perspective

In this subsection, we analyze whether the tail index-linked annuity could help to reduce the insurer's loss, especially in the presence of a longevity shock. Also, as shown in the last subsection, policyholders with different risk aversion levels will prefer the TILA with different (B, z)'s.

This will, in turn, affect the profitability of the insurer. Therefore, in this subsection we consider two life insurers which sell annuities to a pool of $l_{65} = 1,000$ homogeneous policyholders with different risk aversions. For both insurers, we assume that the risk aversion of the policyholder is exponentially distributed with mean 2.15. Further, we assume that the risk aversion of a policyholder is not lower than 0.35, so we add 0.35 to the simulated risk aversions. This assumption is based on Conine *et al.* (2017). The mean of the risk aversion of the policyholders is 2.5, which is in line with Chiappori & Paiella (2011). Moreover, we assume that policyholders who are less risk averse have a greater wealth level to invest in annuities, which is in line with Riley Jr & Chow (1992). In particular, the least risk averse policyholder has an initial wealth of 100, the most risk averse policyholder has an initial wealth of 10, and the i-th risk averse policyholder has an initial wealth of $w_i = 100 - \frac{(100-10)\cdot(i-1)}{(n-1)}$. The portfolios of the two insurers are given below:

- Insurer one sells exclusively regular annuities with fixed payments.
- Insurer two sells the same regular annuity as insurer one to the al_{65}, with $a \in [0,1]$, most risk averse policyholders, and $(1-a)l_{65}$ policies of the TILA to the rest.

Depending on their wealth level, each policyholder will receive an unique annuity contract. For example, a policyholder with initial wealth of 100 could get an annual payment of z^A if they opt for the regular annuity, while a policyholder with initial wealth of 50 could only get an annuity with payments of $\frac{1}{2}z^A$.

Denote the present random liability of insurer one (resp. insurer two) by L^A (resp. L^{MIX}). Based on the above assumptions, the liabilities are given by

$$L^A = \sum_{t=1}^{\omega_x} e^{-rt} \sum_{i=1}^{l_x} \mathbb{1}_{\{T_{x_i} > t\}} \, z_i^A, \tag{4.6}$$

$$\begin{aligned} L^{MIX} = &\sum_{t=1}^{\omega_x} e^{-rt} \sum_{i=1}^{l_x^A} \mathbb{1}_{\{T_{x_i} > t\}} \, z_i^A \\ &+ \sum_{t=1}^{\omega_x} e^{-rt} \left[\sum_{i=l_x^A+1}^{l_x^A + l_x^{TILA}} \mathbb{1}_{\{T_{x_i} > t\}} \left(z_i + \frac{B_i - z_i}{U_t - L_t} (\max\{U_t - Y_t, 0\} - \max\{L_t - Y_t, 0\}) \right) \right]. \end{aligned} \tag{4.7}$$

In Equation (4.7), l_x^A (resp. l_x^{TILA}) denotes the number of policyholders who invest at time $t = 0$ in a regular annuity (resp. TILA). Also, z_i^A, z_i and B_i denote the annuity payments to the i-th policyholder, dependent on the chosen annuity product and the wealth they possess. Based on the above assumptions, it holds that $l_x = l_x^A + l_x^{TILA}$, i.e., the initial number of policyholder is the same for both insurers. As it is less natural to assume a utility function for the insurance company, we consider other important quantities of interest from the insurer's

perspective. Inspired by, for example, Cairns (2013) and Kling *et al.* (2014), we consider the (random) present value of future losses, which is defined as

$$\tilde{L}^X = L^X - \sum_{i=1}^{l_x} w_i = L^X - \mathbb{E}_Q[L^X] \tag{4.8}$$

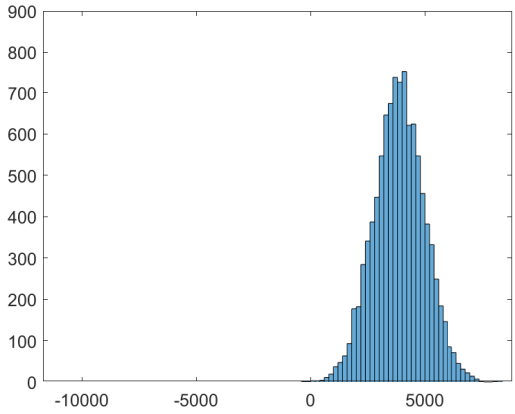
for $X \in \{A, MIX\}$.

Latest insurance solvency regulations require insurers to test their balance sheets against various stress-test scenarios. For instance, in the Canadian solvency regulation, a 10 -20% decrease of mortality rates (depending on the type of annuity business) is assumed for a longevity shock.[8] The U.S. regulation assumes a stress on mortality improvement between 16% and 40% (depending on the age). This results in lower mortality rates between 0.7% and 6%. In Solvency II, a longevity shock is defined by an annual decrease of death probabilities by 20%.

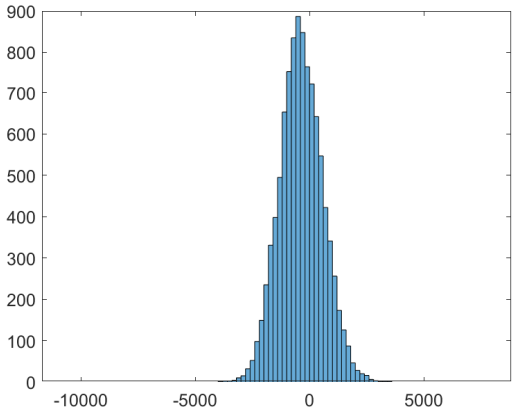
Figure 4 displays the distribution of future loss of insurer one and insurer two with $a = 0.9$, 0.5, and 0.1 under the Solvency II longevity shock and generated with 10,000 simulations under the real-world measure P. The loss distribution of the regular annuity portfolio has a positive mean, which indicates that insurer one will on average suffer a loss under the longevity shock. This result is intuitive, as more policyholders will be alive under the longevity shock, which will in turn increase the total payout of the regular annuity portfolio. On the other hand, all loss distributions of the mixed portfolios have negative means, meaning that insurer two is still expected to make profits under the longevity shock. The reason is that payments of the TILA will be reduced due to the higher values of Y_t, although more policyholders will be alive under the longevity shock. It seems that, for the insurers, the "positive" effect of lower payments dominates the "negative" effect caused by more outliving policyholders. Also, the expected loss is more negative when the fraction of the regular annuity is lower, which means that a higher percentage of the TILA will make the portfolio more robust to longevity shocks. Finally, the loss distribution is more spread out for the regular annuity portfolio and becomes more concentrated when the fraction of the TILA is large. The reason is that payments of the regular annuities are constant and independent of the survival probabilities of policyholders, while payments of the TILA are dependent on survival probabilities of both the annuity portfolio and the reference population. Thus the policyholders would get reduced payments in most years, resulting in a lower deviation of the aggregate annuity payouts.

The above observation is confirmed in Table 5, which reports the expectation of losses (first row) and the probability of incurring loss for each portfolio (second row). Indeed, it can be

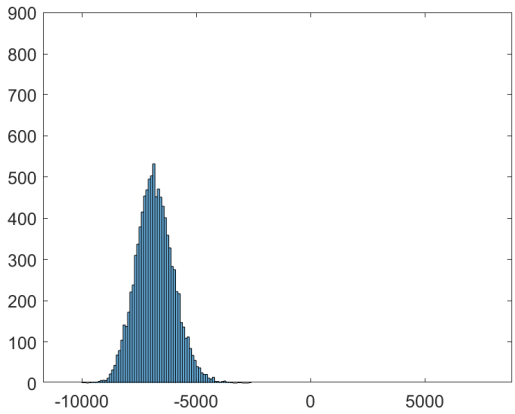
[8] For more information about the Canadian regulation, see `https://www.osfi-bsif.gc.ca/Eng/fi-if/rg-ro/gdn-ort/gl-ld/Pages/LICAT18_chpt6.aspx`



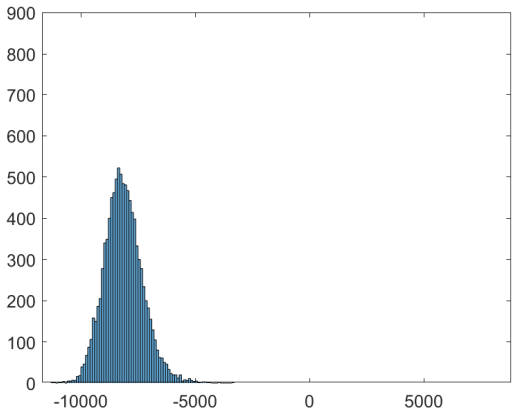
(a) Regular annuity portfolio



(b) Mixed annuity portfolio with $a = 0.9$



(c) Mixed annuity portfolio with $a = 0.5$



(d) Mixed annuity portfolio with $a = 0.1$

Figure 4: Histogram of the random present value of future losses for a portfolio of regular annuities and mixed portfolios with different level of a under the Solvency II longevity shock. Generated with 10,000 simulations.

seen that the expected loss of the regular annuity portfolio is positive, while all the three mixed portfolios have a negative expected loss with the absolute value increasing in a. Further, the probability of an incurring loss is very close to 1 for the regular annuity portfolio, which means that it is likely that the portfolio becomes insolvent. While this is also possible for the mixed portfolios, the probability is close to 0.

Besides the mean characteristics, it is also interesting to look at the percentiles of the loss distributions. Firstly, the fourth row in Table 5 shows the conditional expected loss per contract, i.e., the expected amount of loss when it is incurred, for the four portfolios. We see that the three mixed portfolios have very low or no conditional expected losses, which are substantially lower than that of the regular annuity portfolio. Finally, the 99.5% VaR of the loss distributions are shown in the fifth row in Table 5. Similarly, the 99.5% VaR is quite large for the regular annuity portfolio, and decreases substantially with a.

X	Regular annuities	Mixed portfolio $a = 0.9$	Mixed portfolio $a = 0.5$	Mixed portfolio $a = 0.1$
$\mathbb{E}_P\left[\tilde{L}_X\right]$	3884.6	-361.1	-6798.6	-8134.1
$P\left(\tilde{L}_X > 0\right)$	99.98%	34.37%	0%	0%
$\frac{1}{l_x}\mathbb{E}_P\left[\tilde{L}_X \middle\vert \tilde{L}_X > 0\right]$	3.89	0.65	0	0
$VaR_{P,99.5\%}\left(\tilde{L}_X\right)$	6781.6	2162.9	-4395.0	-5444.6

Table 5: The expected future loss, probability of incurring loss, conditional expected loss per contract, and the 99.5% Value at Risk of the regular annuity portfolio and the mixed portfolios with a= 0.1, 0.5, and 0.9.

5 Sensitivity analysis

In this section, we analyze the effect of the target Sharpe ratio (η), the thresholds, the initial age of the policyholder and the risk-free interest rate on the optimal choice of B and z, the policyholder's utility and the distribution of future losses under the Solvency II longevity shock.

5.1 The Sharpe ratio

In Section 4, we determine the risk-neutral measure Q by assuming a Sharpe ratio of $\eta = 10\%$. In this subsection, we examine the performance of the TILA by imposing a different Sharpe ratio. In particular, we let $\eta = 5\%$. This choice of Sharpe ratio may be reasonable if we assume

that the Sharpe ratio of the annuity market is at the lower end of the 4% to 12% range estimated by Bauer *et al.* (2010). Under this assumption and using 10,000 simulations, we obtain the estimated values: $\tilde{\mu}^{(1)} = -0.5805$ and $\tilde{\mu}^{(2)} = -0.7064$.

Table 6 displays the optimal combinations of (B, z), the certainty equivalent of the tail index-linked annuities and the regular annuity. Similar to Section 4, we see that the TILAs lead to higher certainty equivalent amounts than the regular annuity for all degrees of risk aversion, with the difference being smaller for more risk averse policyholders. The last two rows of Table 6 show the implicit risk loading of the TILA versus the regular annuity. Due to the lower Sharpe ratio, the implicit risk loading of the regular annuity is around 0.09% lower than in the base case in Section 4. Also, the implicit risk loading of the tail index-linked annuities are still lower than the regular annuity, indicating lower longevity risks contained in the TILA products. Finally, the summary characteristics of the loss distribution of the TILA and the regular annuity under the Solvency II longevity shock are reported in Table 7. Again, similar patterns are observed as in Section 4, i.e., the TILAs have lower mean and 99.5% VaR of loss, as well as lower probability of incurring loss than the regular annuity.

γ	0.5	2	4	6	8	10
B	5.5582	5.5366	5.5318	5.5306	5.5306	5.5294
z	4.1462	5.1412	5.3624	5.4176	5.4176	5.4729
CE^{TILA}	5.5296	5.5286	5.5284	5.5283	5.5283	5.5283
CE^{A}	5.5282	5.5282	5.5282	5.5282	5.5282	5.5282
θ^{TILA}	0.07%	0.1%	0.11%	0.11%	0.11%	0.12%
θ^{A}	0.12%	0.12%	0.12%	0.12%	0.12%	0.12%

Table 6: The optimal choice of B and z in dependence of the risk aversion, the corresponding certainty equivalent, and the implicit risk loading of the tail index-based and the regular annuity. For a Sharpe ratio of 5%, $L_t = Q_{Y_t}(95\%)$ and $U_t = Q_{Y_t}(99.5\%)$.

X	Regular annuity	Mixed portfolio $a = 0.9$	Mixed portfolio $a = 0.5$	Mixed portfolio $a = 0.1$
$\mathbb{E}_P\left[\tilde{L}_X\right]$	3940.1	1316.3	-2478.5	-3246.4
$P\left(\tilde{L}_X > 0\right)$	99.98%	90.69%	0.36%	0.03%
$\frac{1}{n}\mathbb{E}_P\left[\tilde{L}_X \middle\vert \tilde{L}_X > 0\right]$	3.94	1.50	0.27	0.28
$VaR_{P,99.5\%}\left(\tilde{L}_X\right)$	6839.8	4003.9	-99.9	-917.4

Table 7: The expected future loss, probability of incurring loss, conditional expected loss per contract, and the 99.5% Value at Risk of the regular annuity portfolio and the mixed portfolios with $a = 0.1$, 0.5, and 0.9. For a Sharpe ratio of 5%, $L_t = Q_{Y_t}(95\%)$ and $U_t = Q_{Y_t}(99.5\%)$.

5.2 Thresholds

In Section 4, we let the thresholds be $L_t = Q_{Y_t}(95\%)$ and $U_t = Q_{Y_t}(99.5\%)$. In this section, we investigate the sensitivity of the choice of thresholds. In particular, we let $L_t = Q_{Y_t}(97.5\%)$ and $U_t = Q_{Y_t}(99\%)$. These new thresholds represent a risk sharing scheme where both the nominal payment and the minimum payment are paid out more frequently. The resulting numbers are qualitatively similar to the base case parameter setup discussed in Section 4 and are shown in Tables 8 to 9. In particular, the TILA is still preferred by policyholders with different risk aversions, and is more robust to the Solvency II longevity shock from the insurer's perspective.

γ	0.5	2	4	6	8	10
B	5.5546	5.5316	5.5278	5.5259	5.5249	5.5249
z	3.7004	5.0259	5.2469	5.3573	5.4125	5.4125
CE^{TILA}	5.5257	5.5237	5.5234	5.5233	5.5232	5.5232
CE^{A}	5.5230	5.5230	5.5230	5.5230	5.5230	5.5230
θ^{TILA}	0.12%	0.19%	0.20%	0.20%	0.21%	0.21%
θ^{A}	0.21%	0.21%	0.21%	0.21%	0.21%	0.21%

Table 8: The optimal choice of B and z in dependence of the risk aversion, the corresponding certainty equivalent, and the implicit risk loading of the tail index-based and the regular annuity. For a Sharpe ratio of 10%, $L_t = Q_{Y_t}(97.5\%)$ and $U_t = Q_{Y_t}(99\%)$.

X	Regular annuities	Mixed portfolio $a = 0.9$	Mixed portfolio $a = 0.5$	Mixed portfolio $a = 0.1$
$\mathbb{E}_P\left[\tilde{L}_X\right]$	3884.6	398.8	-4839.7	-5926.1
$P\left(\tilde{L}_X > 0\right)$	99.98%	65.64%	0%	0%
$\frac{1}{n}\mathbb{E}_P\left[\tilde{L}_X \middle\vert \tilde{L}_X > 0\right]$	3.89	0.93	0	0
$VaR_{P,99.5\%}\left(\tilde{L}_X\right)$	6781.6	-3004.4	-2568.9	-3602.1

Table 9: The expected future loss, probability of incurring loss, conditional expected loss per contract, and the 99.5% Value at Risk of the regular annuity portfolio and the mixed portfolios with $a = 0.1$, 0.5, and 0.9. For a Sharpe ratio of 10%, $L_t = Q_{Y_t}(97.5\%)$ and $U_t = Q_{Y_t}(99\%)$.

5.3 The risk-free interest rate

So far, we have assumed that the risk-free interest rate is 2%, which seems rather optimistic for the current prevailing low interest rate environment. In this section, we investigate the performance of the TILA with an ultra-low interest rate, i.e., $r = 0\%$. With the lower interest rate, the expected value of an annuity portfolio is different, as well as the Sharpe ratio.

Consequentially, a new risk-neutral measure Q is derived with estimated $\tilde{\mu}^{(1)} = -0.5885$ and $\tilde{\mu}^{(2)} = -0.7134$ (based on 10,000 simulations). Note that the influence of the interest rate on the drift terms is almost negligible and they differ only by 0.0005 and 0.0017 compared to base case setup, cf. Section 3.2. Although the drift terms differ only slightly, the certain equivalents of the regular annuity and the TILA are noticeably lower than in the base case. This is due to the fact that later payments are not discounted, cf. Table 10. Further, the risk loadings are larger, since the estimated mortality improvements are not discounted. However, the TILA still yields a higher utility for policyholders with all degrees of risk aversion.

γ	0.5	2	4	6	8	10
B	4.2078	4.1804	4.1749	4.1731	4.1722	4.1722
z	2.4183	3.6691	3.9193	4.0027	4.0444	4.0444
CE^{TILA}	4.1728	4.1704	4.1699	4.1698	4.1697	4.1697
CE^{A}	4.1695	4.1695	4.1695	4.1695	4.1695	4.1695
θ^{TILA}	0.13%	0.23%	0.25%	0.26%	0.26%	0.26%
θ^{A}	0.27%	0.27%	0.27%	0.27%	0.27%	0.27%

Table 10: The optimal choice of B and z in dependence of the risk aversion, the corresponding certainty equivalent, and the implicit risk loading of the tail index-based and the regular annuity. For a Sharpe ratio of 10%, $r = 0\%$, $L_t = Q_{Y_t}(95\%)$ and $U_t = Q_{Y_t}(99.5\%)$.

In Table 11, we observe that under a lower interest rate the expected loss and the Value at Risk is higher for all portfolios. This is because all future payments, which are higher than expected due to higher survival rates, are not discounted. Moreover, the portfolio with 10% TILA is expected to make a loss and the probability of a loss is doubled compared to the base case. Therefore, larger fractions of TILA in an annuity portfolio become more important to avoid a loss under a longevity shock when the interest rate is lower.

X	Regular annuities	Mixed portfolio $a = 0.9$	Mixed portfolio $a = 0.5$	Mixed portfolio $a = 0.1$
$\mathbb{E}_P\left[\tilde{L}_X\right]$	5052.0	613.9	-6166.0	-7579.4
$P\left(\tilde{L}_X > 0\right)$	99.99%	68.76%	0%	0%
$\frac{1}{n}\mathbb{E}_P\left[\tilde{L}_X \middle\vert \tilde{L}_X > 0\right]$	5.05	1.23	0	0
$VaR_{P,99.5\%}\left(\tilde{L}_X\right)$	8958.3	3942.0	-3082.0	-4155.0

Table 11: The expected future loss, probability of incurring loss, conditional expected loss per contract, and the 99.5% Value at Risk of the regular annuity portfolio and the mixed portfolios with $a = 0.1$, 0.5, and 0.9. For a Sharpe ratio of 10%, $r = 0\%$, $L_t = Q_{Y_t}(95\%)$ and $U_t = Q_{Y_t}(99.5\%)$.

5.4 The age of the policyholder

Due to the increasing life expectancy around the world, people are expected to work longer and thus to retire at an older age. For example, the legal retirement has been set to 67 for the German working population born after 1964.[9] Therefore, in this section we examine the performance of the TILA with a higher initial age. In particular, we let $x_0 = 70$ and estimate that $\tilde{\mu}^{(1)} = -0.5907$ and $\tilde{\mu}^{(2)} = -0.7178$ based on 10,000 simulations. These values are slightly higher than for the base case setup since the maximum remaining life time of the cohort $x_0 = 70$ is shorter so the longevity risk premiums need to be higher to reach the target Sharpe ratio.

The optimal choice of B and z, the certainty equivalent values and the implicit risk loading are shown in Table 12. We observe that the certainty equivalents are higher than those in the base case for both annuities. This is due to the fact that policyholders with an initial age of 70 have a lower expected remaining lifetime and thus receive higher payments with the same initial wealth. Also, the implicit risk loading is higher for all levels of risk aversion and both annuities since the drift terms under Q are higher (in absolute values) than those in the base case setup. Consequently, the mortality improvements are higher under Q and thus the implicit risk loading.

The distribution of future losses under a Solvency II longevity shock is shown in Table 13. We observe a similar pattern as in the base case, i.e., only the regular annuity portfolio is expected to make a loss, whereas the mixed portfolios are expected to make profit under a Solvency II longevity shock. Also, by introducing the TILA in the portfolio, the probability of making loss could be significantly reduced. Finally, due to the higher annual payments the expected loss and the Value at Risk differ more substantially among the annuity portfolios.

γ	0.5	2	4	6	8	10
B	6.5075	6.4658	6.4560	6.4532	6.4518	6.4518
z	3.6752	5.6095	6.0608	6.1897	6.2542	6.2542
CE^{TILA}	6.4529	6.4491	6.4484	6.4481	6.4480	6.4479
CE^{A}	6.4477	6.4477	6.4477	6.4477	6.4477	6.4477
θ^{TILA}	0.09%	0.19%	0.21%	0.22%	0.22%	0.22%
θ^{A}	0.23%	0.23%	0.23%	0.23%	0.23%	0.23%

Table 12: The optimal choice of B and z in dependence of the risk aversion, the corresponding certainty equivalent, and the implicit risk loading of the tail index-based and the regular annuity. For a Sharpe ratio of 10%, $x_0 = 70$, $L_t = Q_{Y_t}(95\%)$ and $U_t = Q_{Y_t}(99.5\%)$.

[9] People can retire early, but their pension paid by the government will be reduced. For more information, see `https://www.deutsche-rentenversicherung.de/DRV/EN/Home/home_node.html`.

X	Regular annuities	Mixed portfolio $a = 0.9$	Mixed portfolio $a = 0.5$	Mixed portfolio $a = 0.1$
$\mathbb{E}_P\left[\tilde{L}_X\right]$	4542.9	-79.7	-7159.8	-8641.1
$P\left(\tilde{L}_X > 0\right)$	99.99%	46.13%	0%	0%
$\frac{1}{n}\mathbb{E}_P\left[\tilde{L}_X \middle\vert \tilde{L}_X > 0\right]$	4.54	0.84	0	0
$VaR_{P,99.5\%}\left(\tilde{L}_X\right)$	7636.9	2694.3	-4712.2	-6153.0

Table 13: The expected future loss, probability of incurring loss, conditional expected loss per contract, and the 99.5% Value at Risk of the regular annuity portfolio and the mixed portfolios with $a = 0.1$, 0.5, and 0.9. For a Sharpe ratio of 10%, $x_0 = 70$, $L_t = Q_{Y_t}(95\%)$ and $U_t = Q_{Y_t}(99.5\%)$.

6 Conclusion and Outlook

With population ageing around the globe, longevity risk is becoming an increasingly important issue for both the retirees and the life insurers. On the one hand, longevity risk makes the annuity payouts of traditional annuity providers more uncertain, and thus increases the financial costs for them to issue annuities. On the other hand, these additional costs will then be (partially) transferred to potential policyholders, which makes annuity products less affordable. In this paper, we present an innovative index-linked annuity (TILA), which could potentially lead to a better longevity risk sharing scheme between policyholders and life insurers. In particular, the payouts of the TILA are dependent on the realization of a survival index linked to the mortality experiences of a reference population: it will pay out a nominal amount when the survival index is "normal", and a reduced, index-dependent amount in the highly unfavorable situations where the survival index is higher than a predetermined threshold. The payment reduction feature could be interpreted as a longevity risk hedge from the insurer's perspective, and could lower the annuity payouts when the number of policyholders alive is higher than expected. On the other hand, as a compensation, policyholders will receive a higher nominal amount than an otherwise identical regular annuity in the normal years, which could potentially benefit the policyholders as well.

The attractiveness of the proposed TILA is illustrated using the two-population gravity mortality model proposed by Dowd *et al.* (2011). Firstly, using an expected utility framework, we show that policyholders could optimize the payment structure of the TILA based on their risk aversion. This achieves higher utility levels by investing in the TILA rather than in the regular annuity. These optimal payment structures also lead to lower implicit risk loadings, meaning that policyholders could receive higher expected annuity payments for each dollar they invest in the TILA rather than the regular annuity. Secondly, the proposed TILA is shown to be attractive from the insurer's point of view. In particular, the TILA leads to drastically lower

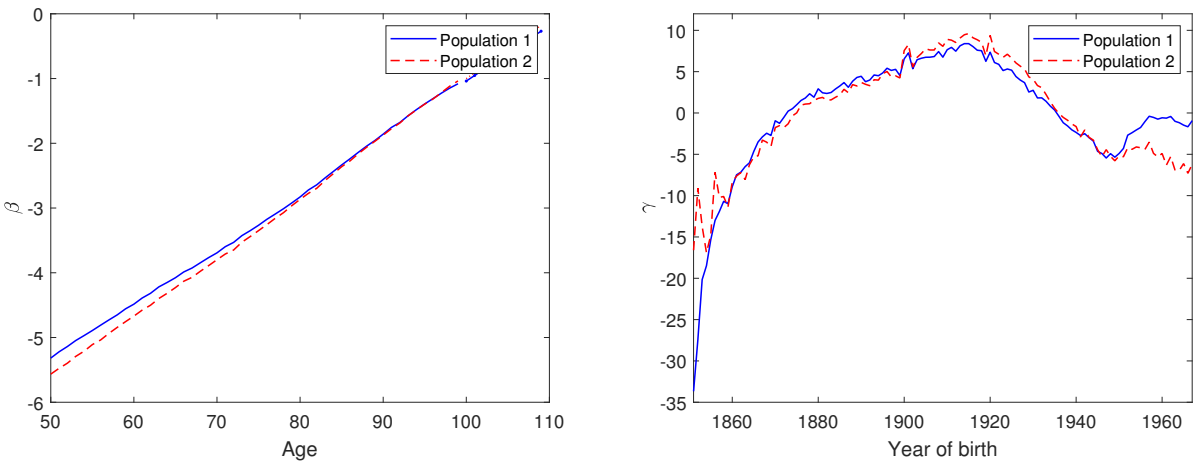
(conditional) expected loss and 99.5% Value at Risk of loss than the regular annuity under the Solvency II longevity shock. Consequently, the proposed TILA provides a new type of retirement product which contains less longevity risk than the regular annuity for the insurer, while being beneficial to policyholders at the same time.

This paper has been focused on the longevity risk exclusively. It will be interesting, in future research, to extend our analysis to unit-linked annuity products, where financial market risks play a substantial role as well. Furthermore, it would be interesting to explore the possibility of combining the proposed TILA with the index-linked longevity derivative products, such as the survival swaps and swaptions considered in Dawson *et al.* (2010), to reach a more efficient and robust longevity risk-return trade off for the insurer.

Appendices

A Estimation results of the gravity model

Figure A.1 displays the estimated age effects (left panel) and the cohort effects (right panel) of the gravity model fitted to the merged population (population 1) and the Canadian (population 2) uni-sex data over age 50 to 100 and year 1950 to 2016. In particular, the age effects are extrapolated linearly for age 101 to 120 (the dotted-solid and the dotted linear). We note that the extrapolated age effects become positive after age 112 for the merged population and 111 for Canada, respectively. A positive age effect indicates an (in-sample) average death rate higher than 100% of the corresponding age. As noted by Dowd & Blake (2019), an empirical death rate can exceed 100% because of the possibility of measurement errors in the exposure data, which is more likely among the very advanced ages. Dowd & Blake (2019) obtain positive extrapolated age effects for Australian male population at the very advanced ages as well. Therefore, we will proceed with the positive extrapolated age effects in our numerical analysis.



(a) Age effects (b) Cohort effects

Figure A.1: Age effects and cohort effects of the merged and Canadian uni-sex data, 1950 to 2016.

B Simulation procedure

This section summarizes the formulas used in the simulation procedure. During the simulation procedure, we simulate N trajectories of survival probabilities under both measure P and Q for both populations. Specifically, denote by ${}_tp_x^{(i)}(n)$ the t-year survival probability of the cohort with initial age x in population i under P in the n-th simulated trajectory for $i = 1, 2$, $t = 1, ..., \omega_x$, and $n = 1, ..., N$. Furthermore, given the longevity risk premium (λ_1, λ_2), denote by ${}_t\tilde{p}_x^{(1)}(\lambda_1, n)$ and ${}_t\tilde{p}_x^{(2)}(\lambda_1, \lambda_2, n)$ the corresponding survival probabilities in population

1 and 2, respectively, under measure Q in the n-th simulated trajectory for $t = 1, ..., \omega_x$ and $n = 1, ..., N$. In the following derivations, we denote by $\mathcal{F}_t^{(i)}$ the σ-algebra containing ${}_tp_x^{(i)}$ until t, and $\mathcal{F}^{(i)} = \bigcup_{t=1}^{\omega_x} \mathcal{F}_t^{(i)}$ for $i = 1, 2$.

B.1 Calibration of the risk-neutral pricing measure

Given the portfolio sizes $l_x^{(1)}$ and $l_x^{(2)}$, the expectation formulas used in Section 3.3 are given by:

$$\mathbb{E}_P[X(l_x^{(i)})] = \mathbb{E}_P\left[\sum_{j=1}^{l_x^{(i)}}\sum_{t=1}^{\omega_x} e^{-rt}\mathbb{1}_{\{T_{x,j}^{(i)}>t\}}\right] = \frac{l_x^{(i)}}{N}\sum_{t=1}^{\omega_x} e^{-rt}\sum_{n=1}^{N} {}_tp_x^{(i)}(n), \quad \text{for } i = 1, 2, \tag{B.1}$$

$$\mathbb{E}_Q[X(l_x^{(1)})] = \mathbb{E}_Q\left[\sum_{j=1}^{l_x^{(1)}}\sum_{t=1}^{\omega_x} e^{-rt}\mathbb{1}_{\{T_{x,j}^{(1)}>t\}}\right] = \frac{l_x^{(1)}}{N}\sum_{t=1}^{\omega_x}\sum_{n=1}^{N} e^{-rt}\,{}_t\tilde{p}_x^{(1)}(\lambda_1, n), \tag{B.2}$$

$$\mathbb{E}_Q[X(l_x^{(2)})] = \mathbb{E}_Q\left[\sum_{j=1}^{l_x^{(2)}}\sum_{t=1}^{\omega_x} e^{-rt}\mathbb{1}_{\{T_{x,j}^{(2)}>t\}}\right] = \frac{l_x^{(2)}}{N}\sum_{t=1}^{\omega_x} e^{-rt}\sum_{n=1}^{N} {}_t\tilde{p}_x^{(2)}(\lambda_1, \lambda_2, n). \tag{B.3}$$

The variance formula is given by:

$$\begin{aligned}
&\mathrm{Var}_P[X(l_x^{(i)})] = \mathrm{Var}_P\left[\sum_{j=1}^{l_x^{(i)}}\sum_{t=1}^{\omega_x} e^{-rt}\mathbb{1}_{\{T_{x,j}^{(i)}>t\}}\right] \qquad (B.4)\\
&= \mathrm{Var}_P\left[\mathbb{E}_P\left(\sum_{j=1}^{l_x^{(i)}}\sum_{t=1}^{\omega_x} e^{-rt}\mathbb{1}_{\{T_{x,j}^{(i)}>t\}}\middle|\mathcal{F}^{(i)}\right)\right] + \mathbb{E}_P\left[\mathrm{Var}_P\left(\sum_{j=1}^{l_x^{(i)}}\sum_{t=1}^{\omega_x} e^{-rt}\mathbb{1}_{\{T_{x,j}^{(i)}>t\}}\middle|\mathcal{F}^{(i)}\right)\right]\\
&= (l_x^{(i)})^2\left[\sum_{t=1}^{\omega_x} e^{-2rt}\mathrm{Var}_P({}_tp_x^{(i)}) + 2\sum_{s=2}^{\omega_x}\sum_{u<s} e^{-r(s+u)}\mathrm{Cov}_P({}_sp_x^{(i)}, {}_up_x^{(i)})\right]\\
&+ l_x^{(i)}\mathbb{E}_P\left[\sum_{t=1}^{\omega_x} e^{-2rt}\,{}_tp_x^{(i)}(1 - {}_tp_x^{(i)}) + 2\sum_{s=2}^{\omega_x}\sum_{u<s} e^{-r(s+u)}({}_sp_x^{(i)} - {}_sp_x^{(i)}\,{}_up_x^{(i)})\right]\\
&= (l_x^{(i)})^2\left[\sum_{t=1}^{\omega_x}\frac{e^{-2rt}}{N-1}\sum_{n=1}^{N}\left({}_tp_x^{(i)}(n) - \frac{1}{N}\sum_{l=1}^{N} {}_tp_x^{(i)}(l)\right)^2\right.\\
&\left.+ 2\sum_{s=2}^{\omega_x}\sum_{u<s}\frac{e^{-r(s+u)}}{N-1}\sum_{n=1}^{N}\left({}_sp_x^{(i)}(n) - \frac{1}{N}\sum_{l=1}^{N} {}_sp_x^{(i)}(l)\right)\left({}_up_x^{(i)}(n) - \frac{1}{N}\sum_{l=1}^{N} {}_up_x^{(i)}(l)\right)\right]\\
&+ \frac{l_x^{(i)}}{N}\sum_{n=1}^{N}\left[\sum_{t=1}^{\omega_x} e^{-2rt}\,{}_tp_x^{(i)}(n)\big(1 - {}_tp_x^{(i)}(n)\big) + 2\sum_{s=2}^{\omega_x}\sum_{u<s} e^{-r(s+u)}\big({}_sp_x^{(i)}(n) - {}_sp_x^{(i)}(n)\,{}_up_x^{(i)}(n)\big)\right],
\end{aligned}$$

where $T_{x,j}^{(i)}$ is the lifetime variable of policyholder j of population i. The third equality in Equation (B.4) holds because:

$$\begin{aligned}
&\mathrm{Var}_P\left(\sum_{j=1}^{l_x^{(i)}}\sum_{t=1}^{\omega_x} e^{-rt}\mathbb{1}_{\{T_{x,j}^{(i)}>t\}}\middle|\mathcal{F}^{(i)}\right)\\
=&l_x^{(i)}\mathrm{Var}_P\left(\sum_{t=1}^{\omega_x} e^{-rt}\mathbb{1}_{\{T_x^{(i)}>t\}}\middle|\mathcal{F}^{(i)}\right)\\
=&l_x^{(i)}\left(\sum_{t=1}^{\omega_x}\mathrm{Var}_P(e^{-rt}\mathbb{1}_{\{T_x^{(i)}>t\}}|\mathcal{F}^{(i)}) + 2\sum_{s=2}^{\omega_x}\sum_{u<s}\mathrm{Cov}_P(e^{-rs}\mathbb{1}_{\{T_x^{(i)}>s\}}, e^{-ru}\mathbb{1}_{\{T_x^{(i)}>u\}}|\mathcal{F}^{(i)}))\right),
\end{aligned}$$

with

$$\begin{aligned}
\mathrm{Var}_P(\mathbb{1}_{\{T_x^{(i)}>t\}}|\mathcal{F}^{(i)}) =&\mathbb{E}_P[(\mathbb{1}_{\{T_x^{(i)}>t\}}|\mathcal{F}^{(i)})^2] - (\mathbb{E}_P(\mathbb{1}_{\{T_x^{(i)}>t\}}|\mathcal{F}^{(i)}))^2\\
=&{}_tp_x^{(i)} - ({}_tp_x^{(i)})^2, \qquad \text{and:}\\
\mathrm{Cov}_P(\mathbb{1}_{\{T_x^{(i)}>s\}}, \mathbb{1}_{\{T_x^{(i)}>u\}}|\mathcal{F}^{(i)}) =&\mathbb{E}_P[\mathbb{1}_{\{T_x^{(i)}>s\}}\mathbb{1}_{\{T_x^{(i)}>u\}}|\mathcal{F}^{(i)}] - \mathbb{E}_P[\mathbb{1}_{\{T_x^{(i)}>s\}}|\mathcal{F}^{(i)}]\mathbb{E}_P[\mathbb{1}_{\{T_x^{(i)}>u\}}|\mathcal{F}^{(i)}]\\
=&{}_sp_x^{(i)} - {}_up_x^{(i)}{}_sp_x^{(i)}, \qquad \text{for } u<s.
\end{aligned}$$

Using Equations (B.1) to (B.4), the longevity risk premium (λ_1, λ_2) can be calibrated using the method introduced in Section 3.3.

B.2 Simulation of the TILA

This section describes the formulas needed to simulate the present value of the TILA, cf. Equations (2.4) and (4.2). Specifically, the key steps involve the simulation of the expectations of (multiplication of) lifetime indicator functions:

$$\begin{aligned}
\mathbb{E}_P\left[\mathbb{1}_{\{Y_t\le L_t\}}\mathbb{1}_{\{T_x^{(2)}>t\}}\right] &= \mathbb{E}_P\left[\mathbb{E}_P\left(\mathbb{1}_{\{Y_t\le L_t\}}\mathbb{1}_{\{T_x^{(2)}>t\}}\middle|\mathcal{F}_t^{(1)}\right)\right] = \mathbb{E}_P\left[\mathbb{1}_{\{Y_t\le L_t\}}{}_tp_x^{(2)}\right]\\
&= \frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{\{Y_t(n)\le L_t\}}{}_tp_x^{(2)}(n),\\
\mathbb{E}_P\left[\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x^{(2)}>t\}}\right] &= \mathbb{E}_P\left[\mathbb{E}_P\left(\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}\mathbb{1}_{\{T_x^{(2)}>t\}}\middle|\mathcal{F}_t^{(1)}\right)\right]\\
&= \mathbb{E}_P\left[\frac{U_t-Y_t}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t>L_t\}}{}_tp_x^{(2)}\right]\\
&= \frac{1}{N}\sum_{n=1}^{N}\frac{U_t-Y_t(n)}{U_t-L_t}\mathbb{1}_{\{U_t>Y_t(n)>L_t\}}{}_tp_x^{(2)}(n).
\end{aligned}$$

$Y_t(n)$ and ${}_tp_x^{(2)}(n)$ are the n-th simulated value of ${}_tp_x^{(1)}$ and ${}_tp_x^{(2)}$, respectively. All other expected values are calculated similarly. The expectations under the risk-neutral pricing measure Q can be calculated using the same formulae but with variables $\tilde{Y}_t(n)$'s and ${}_t\tilde{p}_x^{(2)}(n)$'s simulated using the distributions under Q.

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2 Collective Longevity Swap: A Novel Longevity Risk Transfer Solution and Its Economic Pricing

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Collective longevity swap: A novel longevity risk transfer solution and its economic pricing

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ABSTRACT

To mitigate the hedger's longevity risk exposure, this paper proposes a collective longevity swap between a reinsurer (hedge provider) and a group of hedgers (pension plans and annuity providers), and an economic framework to price longevity risk and longevity swaps. Combining the appealing features of two widely discussed longevity swaps, i.e., the indemnity swap and index-based swap, the collective longevity swap requires that reinsurer's payments are based on the longevity risk of the aggregate portfolio of all hedgers, whereas each hedger could receive indemnity payments on their own portfolio. In a general principal-agent pricing model under the incomplete market setting, we determine the optimal risk premiums and hedge rates that maximize the reinsurer's expected profit under a set of the hedgers' participation constraints. We find that, in a competitive market, the proposed collective swap could lead to a higher expected profit for the reinsurer than the indemnity and the index-based swap, and simultaneously improve the benefits of the majority of hedgers without worsening the benefits of the rest. Finally, using a stochastic multi-population mortality model and real-world mortality data, a series of numerical analyses are performed to verify the theoretical findings.

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1. Introduction

We are living in an aging world. With the rising human life expectancy, the elderly population is playing an increasingly important role in global economy. Taking the U.S. as an example, the 50-plus cohort is composed of 117.4 million people (35% of the total population). This figure will rise to 132.4 million by 2030 and, by 2050, 157.3 million people (cf. The longevity Economy Outlook, (Accius and Suh, 2019)). Meanwhile, 56 cents of every dollar spent in the U.S. in 2018 was attributable to the 50-plus population, and this share is set to increase to 61 cents by 2050. Therefore, providing retirement security is vitally important not only for the elderly, but also for the stability of the global economy. However, this is not an easy task, as the ability for the major players in the retirement system, such as defined benefit pension funds and annuity providers (hereafter referred to as hedgers), to deliver retirement payments has been impaired by longevity risk, the risk that people live longer than expected. On the other hand, as the major channel of insuring longevity risk for the hedgers, the longevity risk transfer market is still in its infancy. In particular, as discussed later, the market is incomplete and not many longevity risk transfer solutions are currently available, especially for the small- and medium-size hedgers. In this paper,

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we propose an innovative longevity risk solution, which we call the collective longevity swap, which has the potential to reduce the cost for smaller hedgers to mitigate or eliminate their longevity risk exposure. Moreover, an economic pricing method based on the principal-agent framework is developed to price the proposed collective longevity swap, as well as other longevity-linked securities, in an incomplete market where a unique risk-neutral pricing measure does not exist.

While longer life expectancy itself is favorable for us all, it leads to potentially more years spent in retirement and extra financial obligations for defined benefit pension schemes and life annuity providers. Michaelson and Mulholland (2014) estimated that the accrued retirement obligations of the world's developed economies as of 2013 was between $60 and $80 trillion. Moreover, each additional year of unanticipated life expectancy at age 65, roughly equivalent to a 1% increase in mortality improvements, can increase pension liabilities by 4% to 5%. On the other hand, as the major channel for insuring longevity risk, the longevity risk transfer market is underdeveloped. Between 2007 and 2020, there has been only around $620 billion of notional value in pension risk transfer transactions in the United States, the United Kingdom, and Canada, the three major markets of longevity risk transactions.[1] This transaction volume is rather small compared to other mature risk transfer markets. Taking the interest risk market for example, the notional value of outstanding over-the-counter derivatives was $610 trillion at end-June 2021 according to the Bank for International Settlements.[2] Due to the lack of longevity risk transfer solutions, most of the retirement obligations are left uninsured, especially for the smaller hedgers. This makes the retirement security of numerous elderly vulnerable to unexpected increases in their life expectancy.

By far the most successful longevity risk transfer solutions are indemnity insurance products, including pension buy-out and buy-in, and indemnity longevity swap (Blake et al., 2019). With these insurance contracts, the hedger receives indemnity payments from the hedge provider which perfectly match either the actual pension/annuity payments (pension buy-out and buy-in), or the realized mortality in their portfolio (the indemnity longevity swap). In this way, the hedger fully removes the longevity risk exposure from their liability. However, although such insurance contracts could provide a perfect hedge, they are typically rather expensive and are only affordable to very large pension plans and annuity providers (Li and Luo, 2012).

Another class of longevity risk transfer solutions, which are often referred to as index-based longevity risk securities, have been introduced in mid 2000s. In contrast to the indemnity insurance, the payoffs of the index-based longevity-linked securities are linked to the realization of a public survival index, such as the survival rates of the national population. Theoretically speaking, the index-based products could provide more predictable payments as the public index is likely to be more predictable and stable than mortality experience of a (small) pension portfolio. As a result, they could be cheaper than indemnity insurance. Various types of index-based longevity-linked securities have been discussed in the recent literature, including the longevity bond (Blake et al., 2006), q-forward (Li and Luo, 2012), survivor forward (Fung et al., 2014), and longevity swap (Dowd et al., 2006; Dawson et al., 2010). We refer to Blake (2018) and Kessler (2019) for recent overviews of these products. Despite the theoretical advantages of index-based longevity-linked securities, however, the transaction volume of such products has been disappointingly low. Up to now, there have been only a few index-based securities successfully issued (Blake et al., 2019), and most of these index-based securities remain in academic discussions. One reason behind the lack of popularity of index-based securities is population basis risk, i.e., the risk that payments received by the hedger do not perfectly match the realized mortality of its portfolio. As a result, it is difficult for the hedgers to credibly quantify the hedge effectiveness, i.e., how much longevity risk exposure could be reduced, by trading the index-based longevity-linked securities. This risk is particularly pronounced for small hedgers, as it is rather difficult to accurately predict the correlation between the mortality of a small portfolio with the survival index, which is based on the mortality experience of a much larger population (Cairns et al., 2014). In this sense, although the above instruments can play a substantial role in transferring longevity risk to the better suited agents, the transactions related to these instruments cannot be fully arranged yet in the current longevity market.

The high prices of indemnity insurance solutions and the uncertain hedge effectiveness result from the index-based longevity securities have motivated the insurance industry to seek alternative longevity risk transfer solutions with better trade off between cost and hedge effectiveness. In 2015, Sun Life Financial in Canada issued a *combined annuity* with the total value of $530 million CAD to two anonymous, previously unrelated defined-benefit pension plans. The issued *combined annuity* was similar to a pension buy-in, but the payments will cover the liabilities of two pension plans jointly. The transaction was the first, and so far the only of its kind in North America. It is estimated that the combined transaction had reduced the total price by at least $20 million compared with two separate pension buy-ins to each of the pension plans.[3]

Based on the recent academic advancement and market development of longevity risk transfer, this paper proposes an innovative longevity-linked security, which we refer to as the *collective longevity swap*, as well as a general economic pricing framework for longevity-linked securities. The collective longevity swap formalizes the *combined annuity* introduced above in a more rigid and general context. The collective swap combines the appealing features of both the indemnity insurance and index-based securities. Specifically, it is a contract between a hedge provider and a group of hedgers (pension plans and annuity providers), and could be either an over-the-counter insurance contract, or a publicly traded security. For ease of exposition, we refer to the hedge provider as a reinsurer in the sequel. However, we remark that the hedge provider could

[1] Source: LIMRA, Hymans Robertson, LCP and Prudential analysis as of December 31, 2020. For a history of the development of this market, we refer to Blake et al. (2019).

[2] Source: https://www.bis.org/publ/otc_hy2111.pdf. Retrieved on May 29, 2022.

[3] Source: https://financialpost.com/investing/investing-pro/a-canadian-first-a-combined-annuity-buy-in-leads-to-big-savings-for-two-canadian-pension-funds. Retrieved on December 5, 2020.

also be a capital market investor, such as an investment bank. With the collective longevity swap, the reinsurer exchanges periodic cash flows with the group of hedgers, with both the fixed-leg and the floating-leg linked to mortality experience of the *aggregate* portfolio of all hedgers. On the one hand, the floating payments made by the reinsurer are defined on the aggregate portfolio, and are thus more stable and predictable than those from the indemnity insurance written on a single portfolio. In this sense, the collective swap is equivalent to an index-based swap from the reinsurer's perspective. On the other hand, with a predetermined splitting rule, the floating payments will be internally split among the individual hedgers. As will be shown later, with appropriate splitting rule and hedge rates, payments received by each hedger could perfectly match their pension/annuity payouts. In this sense, the collective swap is equivalent to an indemnity swap from each hedger's perspective.

While proposing the collective longevity swap can be considered as the first key contribution of the paper, the second key contribution lies in the application of an economic pricing model to determine the value of such products. There have been so far only few longevity-linked securities openly traded in the market. Therefore, it is not practical to rely on market data for pricing purpose. In other words, the market for insurance is incomplete, and alternative methods are required to price the longevity-linked products. In this paper, we set ourselves in a principal-agent model, where the reinsurer is the principal and the hedgers are the agents. Based on the mean-variance preferences and some functional form for the risk premia, the optimal risk premiums and hedge rate of the collective swap are determined such that the expected profit of the reinsurer (principal) is maximized under a set of participation constraints for the hedgers (agents), which ensure that the hedgers are not worse off in terms of utility if they participate in the transaction. Closed-form expressions of the risk premiums and the optimal hedge rates are derived for the collective swap.

To set up a reasonable comparison, the proposed principal-agent model is also applied to indemnity- and index-based longevity swaps. Under a competitive market setting, we show that the proposed collective swap could lead to a higher expected profit of the reinsurer, compared to the indemnity and the index-based swap. Furthermore, perfect hedge could be achieved for all hedgers using the collective swap, while the risk premiums charged from the majority of hedgers are lower than those of the individual indemnity swaps. As a result, the hedgers who would have not purchased an indemnity insurance due to the high price might find the collective swap attractive. Note that, while the principal agent model has been a popular tool for insurance welfare analysis and contract design (Landsberger and Meilijson, 1994; Olivella and Schroyen, 2014; Hu et al., 2018), to our best knowledge, it has not been used to price realistic longevity-linked securities before. In this sense, our paper enriches the literature on economic pricing of mortality- or longevity-risked products by applying the principal-agent model. Compared to the abundant arbitrage-free pricing literature (see Section 3 for more discussions), there have been only a handful of papers on the economic pricing of longevity-linked securities. For example, Zhou et al. (2015a) propose a Tâtonnement approach and Zhou et al. (2015b) a Nash bargaining approach. These two approaches only consider one (representative) buyer in the market and rely on simulation procedures to determine the price and trading amount for the products. It is not clear whether optimal solutions still exist when there are multiple heterogeneous buyers in these settings. In contrast, the proposed principal-agent model allows for multiple hedgers with different portfolios and risk types, and is therefore able to price the proposed collective swap or other combined annuities alike. Furthermore, closed-form solutions are derived, which allows an analytical comparison among various longevity-linked securities.

The theoretical results in this paper are illustrated in a context of two hedgers using real-world mortality data and realistic mortality models. Specifically, we approximate the portfolios of the two hedgers using the single-age, uni-sex mortality data from Japan and the United Kingdom, and project future mortality rates of the two portfolios using the Coherent Mortality Forecasting Model proposed by Hyndman et al. (2013). In line with the theoretical results, we find that the expected profit of the reinsurer and the utility level of the hedgers could be substantially improved under various choice of portfolio sizes and degrees of risk aversion of the hedgers. All in all, the proposed collective swap seems to be an attractive longevity risk transfer solution, both from the reinsurer's and the hedgers' perspective, and the principal-agent pricing framework could serve as a sensible tool to price longevity risk without the need of abundant market price data. Finally, we remark that although we focus on the pricing of longevity swaps in this paper, the proposed principal-agent pricing framework is applicable to a variety of longevity risk transfer solutions, including pension buy-out and buy-in, traditional proportional or stop-loss reinsurance products, and the index-based longevity-linked derivatives.

The remainder of this paper is organized as follows. Section 2 introduces the concept of the collective longevity swap. Section 3 describes and characterizes the principal-agent pricing framework, as well as presents the optimal solution for the collective swap. Section 4 discusses the optimal risk premiums and hedge rates for the indemnity and index-based swap in a monopolistic and competitive market example. Section 5 presents the mortality model and the simulation methods of future mortality rates. Section 6 presents the numerical analyses. Section 7 concludes the paper. Some detailed proofs and graphic illustrations are provided in the appendix.

2. The collective longevity swap

This paper considers a reinsurer offering a collective longevity swap to a group of N hedgers. For ease of exposition, we assume that each hedger holds an annuity portfolio consisting of a pool of homogeneous policyholders, i.e., all policyholders within a portfolio have the same age x at inception, with independently and identically distributed (i.i.d.) remaining life times. The hedger pays an annuity benefit of 1 to each policyholder alive at the end of each year. Each portfolio could have

a different portfolio size and mortality law of its policyholders. In this paper, we consider the case where the mortality laws of different portfolios are positively correlated with each other. The assumption of positively correlated mortality laws of different hedgers can be justified by many real world scenarios. For example, the hedgers could be occupational pension plans of the same industry, or annuity providers operating in the same region.[4] Before proceeding to the definition of the collective longevity swap, we first introduce the notations used throughout this paper:

- $l_x(n)$: the initial number (in year 0) of policyholders in the n-th hedger's portfolio;
- $l_{x+t}(n)$: the number of remaining policyholders in the n-th hedger's portfolio in year t, $t \in \{1, \dots, \omega_x\}$, where ω_x is the maximal remaining life time for the policyholders aged x;
- $l_{x+t} = \sum_{n=1}^{N} l_{x+t}(n)$: the total number of policyholders alive in year t;
- ${}_tp_x = \frac{l_{x+t}}{l_x}$: the (random) accumulated aggregate survival probability in year t;
- ${}_t\hat{p}_x = \mathbb{E}_{\mathbb{P}}[{}_tp_x] = \frac{\hat{l}_{x+t}}{l_x}$: the expected t-year aggregate survival probability under the real-world measure $\mathbb{P}$;
- ${}_tp_x(n) = \frac{l_{x+t}(n)}{l_x(n)}$ and ${}_t\hat{p}_x(n) = \frac{\hat{l}_{x+t}(n)}{l_x(n)}$: the random and expected t-year survival probability of hedger n's portfolio under the real-world measure $\mathbb{P}$.

Based on the above notations, the time-0 random present value of hedger n's annuity portfolio is given by:

$$\left(A_n - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right),$$

where A_n is the initial premium income collected from the policyholders, and is exogenously given. Moreover, we assume a constant interest rate r. In real world applications, A_n would be determined using the hedger's pricing assumptions, which could be different from the mortality assumptions used in the hedging procedure discussed in the sequel. Furthermore, because the numbers of future remaining policyholders, $l_{x+t}(n)$'s, are uncertain, and it deals with annuity products, the hedgers are exposed to longevity risk.

In this paper, we propose a collective longevity swap between the N hedgers and a reinsurer with the aim to mitigate the hedgers' longevity risk exposure. The key idea of the collective swap is that the reinsurer, which acts as the hedge provider, provides protection against the uncertainty of the aggregate portfolio l_{x+t}, $t = 1, \dots, \omega_x$, rather than each individual annuity portfolio. The total payoff from the reinsurer will then be distributed to the N hedgers, based on the predetermined splitting rule discussed below. With an appropriate splitting rule, payments received by each hedger could perfectly match their pension payments. Through these specific arrangements, the collective swap takes advantages of the index-based and indemnity swaps. The collective swap is equivalent to an index-based swap from the reinsurer's perspective, and is equivalent to an indemnity swap from each hedger's perspective.

Formally, in each year t, the reinsurer pays a floating payment of $M \cdot S_t$ in exchange of a fixed payment of $(1+\pi_t) \cdot M \cdot H_t$, with M being the notional amount, S_t and H_t the floating and fixed rate of the swap respectively. $(\pi_t)_{t=1,\cdot,\omega_x}$ are the longevity risk premium loadings charged by the reinsurer for providing the longevity protection, and should be agreed by both the reinsurer and the hedgers at time 0. In other words, the reinsurer will make a net total payout of $M\big(S_t - (1+\pi_t)H_t\big)$ in year t if the floating payment is higher than the fixed payment, and will otherwise receive a net total payment of $M\big((1+\pi_t)H_t - S_t\big)$ from the N hedgers. In general, the payments of the reinsurer, both the fixed and floating leg, could be any payoff functions defined on the aggregate mortality experience of all the N hedgers' portfolios. A popular choice in the existing literature is to let S_t be the realized survival probability ${}_tp_x$ of the aggregate portfolio in year t, and H_t the corresponding best estimate ${}_t\hat{p}_x$ (Dowd et al., 2006; Dawson et al., 2010). The payment the reinsurer shall provide with these specifications of S_t and H_t is given by $M({}_tp_x - (1+\pi_t){}_t\hat{p}_x)$. On the other side, each individual hedger's payment will depend on the realized and the expected survival probability of its own portfolio. Specifically, in each year t, hedger n will pay the reinsurer a fixed payment of $(1+\pi_t)M_n\,{}_t\hat{p}_x(n)$, with M_n being the hedger-specific notional amount, in exchange of a floating payment $M_n\,{}_tp_x(n)$.

Based on the assumptions above, the net payment to hedger n in year t is thus $M_n\big[{}_tp_x(n) - (1+\pi_t){}_t\hat{p}_x(n)\big]$. As a result, summing up the net payments received by the N hedgers gives:

$$\sum_{n=1}^{N} M_n\big[{}_tp_x(n) - (1+\pi_t){}_t\hat{p}_x(n)\big]. \tag{2.1}$$

Comparing (2.1) with the payment the reinsurer shall provide, i.e., $M({}_tp_x - (1+\pi_t){}_t\hat{p}_x)$, these two payments do not necessarily coincide. However, an appropriate choice of the hedger's nominal hedge amounts $(M_n)_{n=1,\dots,N}$ can make these two payments equal. In the following, we assume that M_n is a function of the initial portfolio size $l_x(n)$, $M_n := z^C l_x(n)$, where z^C denotes the hedge rate. In particular, $z^C = 0$ refers to the case of no hedge, whereas $z^C = 1$ is the case of full hedge.[5]

[4] The assumption that policyholders have different mortality distributions within the same portfolio can be incorporated in a straightforward manner.

[5] When $z^C < 0$, the hedgers will act as the hedger provider, i.e., their longevity risk exposure will be increased by trading the swap. $z^C > 1$ refers to the case of "over-hedge", in which the floating payments are more than sufficient to offset the uncertainties in the annuity payments, and will thus introduce excessive volatility to the hedged portfolios.

Equation (2.1) can then be rewritten as:

$$\begin{aligned}\sum_{n=1}^{N} M_n\left[{}_t p_x(n) - (1+\pi_t)\,{}_t\hat{p}_x(n)\right] &= \sum_{n=1}^{N} z^C\left[l_{x+t}(n) - (1+\pi_t)\hat{l}_{x+t}(n)\right] \\ &= z^C\left[l_{x+t} - (1+\pi_t)\hat{l}_{x+t}\right] \\ &= M\left[{}_t p_x - (1+\pi_t)\,{}_t\hat{p}_x\right], \end{aligned} \tag{2.2}$$

where $M = z^C\, l_x = \sum_{n=1}^{N} z^C\, l_x(n) = \sum_{n=1}^{N} M_n$. Note that this amount in (2.2) is exactly the net amount paid by the reinsurer in year t. Equation (2.2) addresses a given time point t. Theoretically, the hedge rate z^C can vary when different time points are considered. However, we will show in the next section that a constant z^C would still be optimal even if it is allowed to be time-varying.

3. The principal-agent pricing framework

After specifying the swap contract payments, a natural question arises: How shall the longevity risk premium loadings, $(\pi_t)_{t=1,\ldots,\omega_x}$, and the notional amounts, $(M_n)_{n=1,\ldots,N}$, be chosen such that both the reinsurer and the N hedgers are interested in trading the collective longevity swap and benefit from this innovative product? After clarifying this question, we can further compare the collective swap with alternative longevity swap contracts.

Many attempts to price longevity risk have been made in the existing literature, among which the most popular ones are based on arbitrage-free pricing. For example, Wang et al. (2015) choose the risk premiums to make the longevity swap initially fair, i.e., the time-0 value of all future fixed payments is equal to the time-0 expected value of all future floating payments. For example, Lin and Cox (2005) and Lin et al. (2013) apply the so-called Wang transformation to distort the best estimate cumulative distribution function of the future lifetime, such that the distorted distribution could represent the risk-neutral pricing measure. In similar spirits, various studies incorporate longevity risk premiums by adjusting the mortality trends of the best estimate mortality distribution (see, among many others, Cairns et al., 2006; Dowd et al., 2006; Cox et al., 2006; Dawson et al., 2010). Arguably, these arbitrage-free methods would work well if the market is liquid and sufficiently large number of longevity-linked securities and insurance policies are used for pricing. However, so far longevity-linked transactions have been rare, and hence the market is far from being complete. In fact, the pricing results could be rather sensitive to the securities or the distortion approach used for pricing (Bauer et al., 2010). As an alternative to arbitrage-free pricing, there have been recent attempts to price longevity risk using economic methods. As already pointed out e.g. by Zhou et al. (2015a) and Zhou et al. (2015b), the economic pricing approaches do not require the market prices of longevity-linked securities or insurance, and thus do not suffer from the lack of market price data.

In this paper, we apply an economic approach to price the collective longevity swap. We set ourselves in a principal agent model, where the reinsurer is the principal and the N hedgers are the agents. The proposed principal-agent model belongs to the category of economic pricing, where the optimal risk premiums and hedge rates are determined by maximizing the reinsurer's expected profit under a set of participation constraints of the hedgers. We rely on the principal-agent model for pricing mainly because we deal with an incomplete market setting. In practice, the assumption of a complete market for insurance contracts is usually violated (see e.g. Chen and Hieber (2016)). Due to the non-full tradability of the insurance risk, it is often not possible to fully hedge this risk. In our setting, the longevity risk market is still in its infancy developing stage compared to other major financial markets, such as the interest rate or exchange rate derivative market. Hence, it is typically impossible to fully hedge the inherent longevity risk, and regular pricing rules are not applicable. Furthermore, in order to make the cash flows comparable, we discount all the payments to time zero. In other words, we assume that the hedgers (resp. reinsurer) are concerned about the time-0 present value of their hedged liabilities (resp. profit). This setup belongs to the value-hedging category, as discussed in Coughlan et al. (2011), Cairns (2013), and Cairns et al. (2014).

3.1. Characterization of the optimization problem

In this subsection, we introduce the general optimization problem for the collective longevity swap. We assume that the reinsurer is risk neutral, and is interested in maximizing its expected profit. Moreover, the reinsurer has no annuity-linked liability, and its profit solely depends on the aggregate net payments from trading the longevity swap:

$$PR_R^C = z^C\, l_x \sum_{t=1}^{\omega_x} e^{-rt}\left[(1+\pi_t)\,{}_t\hat{p}_x - {}_t p_x\right]. \tag{3.1}$$

The optimal risk premiums will be determined under a set of participation constraints, which ensures that the hedgers are willing to participate in the swap transaction. In order to come to the participation constraint for hedger n, let us first write down its random net profit function:

$$PR_n^C = \left(A_n - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right) + z^C\, l_x(n) \sum_{t=1}^{\omega_x} e^{-rt}\left[{}_t p_x(n) - (1+\pi_t)\,{}_t\hat{p}_x(n)\right],$$

i.e., the time-0 net profit of hedger n is the summation of its annuity portfolio value and the net payment from participating in the collective longevity swap. In particular, hedger n's expected profit becomes a constant when $z^C = 1$:

$$PR_n^C = A_n - \sum_{t=1}^{\omega_x} e^{-rt}(1+\pi_t)\hat{l}_{x+t}(n).$$

Therefore, we call $z^C = 1$ the "full hedge", because the profit contains no longevity risk in this case. Furthermore, we assume that hedger n has the following mean-variance preferences:

$$U_n(PR_n^C) = \mathbb{E}_{\mathbb{P}}[PR_n^C] - \frac{1}{2}\gamma_n Var_{\mathbb{P}}[PR_n^C], \tag{3.2}$$

for $n = 1, \ldots, N$, where γ_n describes the degree of risk aversion of hedger n. Mean-variance preference is a widely used preference in decision theory and portfolio choice literature (see e.g. Ormiston and Schlee, 2001). In particular, it is shown to be fully compatible with the expected utility theory under certain conditions (Sinn, 1990). Further, it can approximate decisions of a wide variety of (concave) utility functions (Kroll et al., 1984; Markowitz, 2014).

In this paper, we assume that the reinsurer has perfect information about γ_n, and the distribution of $l_{x+t}(n)$ for each n and t. The principal-agent optimization problem is then given by:

$$\begin{aligned} &\max_{(\pi_t)_{t=1,\ldots,\omega_x}, z^C} \mathbb{E}_P[PR_R^C] \text{ subject to} \\ &U_n(PR_n^C) \geq \bar{U}_n \text{ for all \$}n\text{\$}. \end{aligned} \tag{3.3}$$

In Problem (3.3), the reinsurer chooses the loadings $(\pi_t)_{t=1,\ldots,\omega_x}$ and the total notional amount $M = z^C l_x$ to maximize its expected profit, provided that all hedgers are weakly better off by participating in the collective longevity swap. Specifically, the utility hedger n achieves from participating in the collective longevity swap should be at least as high as the reservation utility $\bar{U}_n$. In general, there are many choices for the reservation utility. In this paper, we let $\bar{U}_n$ be the utility of hedger n in the absence of the collective swap, i.e.,

$$\bar{U}_n = \mathbb{E}_P\left[A_n - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right] - \frac{1}{2}\gamma_n Var_P\left[A_n - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right].$$

In other words, the hedgers shall be at least indifferent in terms of utility between trading in the collective longevity swap or not. The solution of the optimization problem (3.3) is summarized in the following proposition.

Proposition 1 (Solutions of the optimal contracting in a collective swap)**.** *The solutions to the optimization problem (3.3) are given by:*

a) $z^{(C,*)} = 0$, $\pi_k^{(*)} = \frac{\sum_{n=1}^N \lambda_n^C \gamma_n \mathcal{V}_n}{(\sum_{n=1}^N \lambda_n^C e^{-rk} l_{x+k}(n)) - e^{-rk} l_{x+k}}$ *for all* $k = 1, \ldots, \omega_x$ *and* $\lambda_n^C \geq 0$ *for all hedgers,*

b) $z^{(C,*)} = 1$, $\sum_{t=1}^{\omega_x} e^{-rt}\pi_t^{(*)}\hat{l}_{x+t}(n') = \frac{1}{2}\gamma_{n'}\mathcal{V}_{n'}$ *for* $\forall n' \in \mathcal{N}'$*, where* $\mathcal{N}'$ *is the set of hedgers whose participation constraint is binding,*

where $\mathcal{V}_n = Var\left(\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right)$.

Proof. The proof can be found in Appendix A.1.1. □

The optimal contract between the reinsurer and the hedgers is either "no hedge" ($z^{(C,*)} = 0$) or "full hedge" ($z^{(C,*)} = 1$). In particular, Solution a) has $z^{(C,*)} = 0$, meaning that the reinsurer's expected profit will be 0 with certainty independently of the risk loadings. On the other hand, Solution b) has $z^{(C,*)} = 1$ and $\pi_t^{(*)} > 0$ for all t, and thus the reinsurer's expected profit will be positive. Therefore, in the following discussions, we focus on the non-trivial Solution b).

While $z^{(C,*)}$ is uniquely determined, there is no unique solution for the time-dependent risk loadings $\pi_t^{(*)}$ in Solution b). Therefore, additional assumptions need to be imposed to determine the values of $(\pi_t^{(*)})_{t=1,\ldots,\omega_x}$. For example, in the case of constant risk loadings, it holds that $\pi_1^{(*)} = \pi_2^{(*)} = \ldots = \pi_{\omega_x}^{(*)} = \frac{1}{2}\gamma_n\mathcal{V}_n / \left(\sum_{t=1}^{\omega_x} e^{-rt}\hat{l}_{x+t}(n')\right)$. In this paper, we illustrate the pricing framework based on the following assumption:

$$\pi_t = \alpha^C \tilde{\pi}_t^C, \tag{3.4}$$

where $\alpha^C \geq 0$ is the constant risk premium, and $\tilde{\pi}_t^C$, $t = 1, 2, \ldots, \omega_x$ is a series of time-varying variables known at time 0. With this specification, only the optimal value of the risk premium α^C needs to be determined.

Through this simplification, the number of control variables are reduced from $\omega_x + 1$ to 2, and the optimization problem becomes:

$$\begin{aligned} &\max_{\alpha^C, z^C} \mathbb{E}_P[PR_R^C], \quad \text{subject to} \\ &U_n(PR_n^C) \geq \bar{U}_n \text{ for all\$}n\text{\$}. \end{aligned} \tag{3.5}$$

The profit function of the reinsurer and each hedger n respectively can then be rewritten as:

$$PR_R^C = z^C l_x \sum_{t=1}^{\omega_x} e^{-rt}\Big[(1+\alpha^C\tilde{\pi}_t^C)_t\hat{p}_x - {}_tp_x\Big], \tag{3.6}$$

and

$$\begin{aligned} PR_n^C &= \left(A_n - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right) + z^C l_x(n) \sum_{t=1}^{\omega_x} e^{-rt}\Big[{}_tp_x(n) - (1+\alpha^C\tilde{\pi}_t^C)_t\hat{p}_x(n)\Big] \\ &= A_n - \sum_{t=1}^{\omega_x} e^{-rt}\Big[(1+\alpha^C\tilde{\pi}_t^C)z^C\,\hat{l}_{x+t}(n) + (1-z^C)l_{x+t}(n)\Big]. \end{aligned} \tag{3.7}$$

Hereby we have used $\hat{l}_{x+t}(n) = l_x(n)\cdot{}_t\hat{p}_x(n)$ and $l_{x+t}(n) = l_x(n)\cdot{}_tp_x(n)$. For ease of exposition, we assume that the hedge rate z^C is a constant over time in the current setting. We show in Appendix A.1.2 that this assumption does not reduce the generality of the optimization problem considered. In particular, when the hedge rates can be time-dependent, the optimal hedge rate will still be constant. Finally, we remark that the imposed functional form of $(\pi_t)_{t=1,\dots,\omega_x}$ is only an illustration. The proposed principal-agent pricing framework can be implemented with any specification of $(\pi_t)_{t=1,\dots,\omega_x}$ such that the equality $\sum_{t=1}^{\omega_x} e^{-rt}\pi_t^{(*)}\,\hat{l}_{x+t}(n') = \frac{1}{2}\gamma_{n'}\mathcal{V}_{n'}$ is satisfied.

Next, we proceed to solve the optimization problem (3.5). Firstly, the associated Lagrangian function is given by:

$$\mathcal{L} = z^C \sum_{t=1}^{\omega_x} e^{-rt}\alpha^C\tilde{\pi}_t^C\,\hat{l}_{x+t} + \sum_{n=1}^{N}\lambda_n^C(U_n(PR_n^C) - \bar{U}_n), \tag{3.8}$$

with $\lambda_n^C \geq 0$, $n = 1,\dots,N$ being the Lagrangian multipliers. Then, based on the Lagrangian function, the solutions to the optimization problem (3.5) are collected in the following corollary.

Corollary 1. *The solutions to the optimization problem (3.5) are given by:*

a) $z^{(C,*)} = 0$, $\alpha^{(C,*)} = \frac{\sum_{n=1}^{N}\lambda_n^C\gamma_n\mathcal{V}_n}{(\sum_{n=1}^{N}\lambda_n^C B_n^C) - B^C}$ *and* $\lambda_n^C \geq 0$ *for all hedgers,*

b) $z^{(C,*)} = 1$, $\alpha^{(C,*)} = \frac{\gamma_{n'}\mathcal{V}_{n'}}{2B_{n'}^C}$ *for* $\forall n' \in \mathcal{N}'$, *where* $\mathcal{N}'$ *is the set of hedgers whose participation constraint is binding.*

$B^C = \sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}_t^C\,\hat{l}_{x+t}$ *and* $B_n^C = \sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}_t^C\,\hat{l}_{x+t}(n)$.

Proof. The proof can be found in Appendix A.1.3. □

To provide more intuitions about the optimal solutions, we consider the simplified scenario where there is only one hedger in the market. In this case, the reinsurer will choose the optimal α^C and z^C such that the participation constraint of the hedger is binding. The reason is that, if the participation constraint is not binding, i.e., the hedger's utility level with the swap is strictly higher than the reservation utility, then the reinsurer could always increase α^C and/or z^C to extract more surpluses, until the hedger becomes indifferent between hedging or not. When the participation constraint is binding, we have:

$$\begin{aligned} & U_1(PR_1^C) = \bar{U}_1 \\ \Leftrightarrow\ & 0 = -\alpha^C z^C B_1^C - \frac{1}{2}\gamma_1(1-z^C)^2\mathcal{V}_1 + \frac{1}{2}\gamma_1\mathcal{V}_1, \\ \Leftrightarrow\ & 0 = \gamma_1\mathcal{V}_1(z^C)^2 + 2(\alpha^C B_1^C - \gamma_1\mathcal{V}_1)z^C, \\ \Leftrightarrow\ & 0 = z^C\Big(\gamma_1\mathcal{V}_1 z^C + 2(\alpha^C B_1^C - \gamma_1\mathcal{V}_1)\Big). \end{aligned} \tag{3.9}$$

Therefore, for the participation constraint to be binding, the hedger should either choose no hedge ($z^C = 0$) or let the hedge rate be $z^C = 2 - \frac{2\alpha^C B_1^C}{\gamma_1\mathcal{V}_1}$, i.e., z^C is a decreasing function of α^C with slope $-\frac{B_1^C}{\gamma_1\mathcal{V}_1}$, and starting at 2 when $\alpha^C = 0$. Consequently, the reinsurer's optimization problem can be reformulated as:

$$\begin{aligned} &\max_{\alpha^C, z^C}\ \mathbb{E}_P[PR_R^C] \text{ subject to} \\ &z^C = 0 \text{ or } z^C = 2 - \frac{2\alpha^C B_1^C}{\gamma_1\mathcal{V}_1}. \end{aligned} \tag{3.10}$$

When $z^C = 0$, $\mathbb{E}_P[PR_R^C] = 0$. When $z^C = 2 - \frac{2\alpha^C B_1^C}{\gamma_1\mathcal{V}_1}$, the optimization problem becomes:

$$\max_{\alpha^C}\ 2B_1^C\frac{\gamma_1\mathcal{V}_1\alpha^C - B_1^C(\alpha^C)^2}{\gamma_1\mathcal{V}_1}.$$

Letting the first order derivative of the objective function with respect to α^C be zero leads to $\alpha^{(C,*)} = \frac{\gamma_1 \mathcal{V}_1}{2B_1^C}$, and consequentially $z^{(C,*)} = 1$. Therefore, in this simplified scenario, the reinsurer will increase α^C until the participation constraint of the hedger is binding. Moreover, due to the mean-variance preference, the hedger will choose a full hedger at the binding point, irrespective of its degree of risk aversion. The degree of risk aversion is fully reflected in the optimal risk premium instead.

The intuition above applies to the general situation when there are more than one hedgers. However, in this case, the reinsurer can only increase the risk premium α^C until the first set of binding participation constraints emerge, and the resulting $\alpha^{(C,*)}$ and $z^{(C,*)}$ are given in Solution b) of Corollary 1. This observation is reasonable, as when all hedgers bargain with the reinsurer as a group, the optimal risk premium $\alpha^{(C,*)}$ should be acceptable to every hedger. Therefore, $\alpha^{(C,*)}$ in our optimization problem is chosen such that the hedger who is willing to accept the lowest risk premium is indifferent between hedging or not. Finally, the set of hedgers willing to accept the lowest risk premium (with binding participation constraints) are given by $\mathcal{N}'$.

From Solution b), we see that the optimal $\alpha^{(C,*)}$ increases with the risk aversion parameter ($\frac{\gamma_{n'}}{2}$), as well as the ratio of the variance over the expectation of the present value of total annuity payments ($\frac{\mathcal{V}_{n'}}{B_{n'}^C}$) of the binding hedgers. Furthermore, as shown in Appendix A.1, the set of hedgers with binding participation constraint is never empty, i.e., the optimal solution is guaranteed to exist. In particular, when $\mathcal{N}' = \{n'\}$ has only one element, then the optimal risk premium is given by $\alpha^{(C,*)} = \frac{\gamma_{n'}\mathcal{V}_{n'}}{2B_{n'}^C} = \min_{n\in\{1,2,\dots,N\}} \frac{\gamma_n \mathcal{V}_n}{2B_n^C}$; and when $\mathcal{N}'$ has more than one element, we would have $\alpha^{(C,*)} = \frac{\gamma_{n'}\mathcal{V}_{n'}}{2B_{n'}^C} = \frac{\gamma_{m'}\mathcal{V}_{m'}}{2B_{m'}^C}$ for all $n', m' \in \mathcal{N}'$. In other words, there is a unique $\alpha^{(C,*)}$ such that the participation constraints of all hedgers $\mathcal{N}'$ are simultaneously binding. Moreover, we observe that the "non-binding" hedgers will benefit from the collective swap, since their utility levels will be higher with trading in the collective swap. In particular, we see that the "binding" hedgers will be those with the lowest risk aversion parameter γ_n or the least volatile normalized portfolio $\frac{\mathcal{V}_n}{B_n^C}$, or both. Hence, the hedgers with a higher degree of risk aversion and/or a more volatile annuity portfolio will benefit from the collective swap.

4. Indemnity swap and index-based swap

As the collective longevity swap provides a nice combination of the indemnity- and index-based longevity swap, in this section, we apply the principal agent optimization problem to these two widely discussed longevity swaps in the literature. In contrast to the proposed collective swap, both the indemnity and the index-based swap are negotiated hedger by hedger. In other words, each hedger will negotiate with the reinsurer individually, and choose its specific optimal hedge rate, z_n^*.

4.1. The indemnity longevity swap

An indemnity swap (IN-swap) is a contract between the reinsurer and a hedger, which exchanges payments depending on the mortality experience of the hedger's own annuity portfolio. Specifically, via the indemnity swap, hedger n will pay the reinsurer a fixed payment of $\big(1+\pi_t^{IN}(n)\big)M_{n\,t}\hat{p}_x(n)$ in exchange of the floating payment $M_{n\,t}p_x(n)$, with the hedger-specific notional amount given by $M_n = z_n^{IN} l_x(n)$. Similar to Sections 2 and 3, we consider a market with one reinsurer and N hedgers. Further, the premium loadings are assumed to be $\pi_t^{IN}(n) = \alpha_n^{IN}\tilde{\pi}_t^{IN}(n)$, with $\tilde{\pi}_t^{IN}(n)$, $t = 1,\dots,\omega_x$ determined at time 0. Note that we allow the loadings to be hedger-specific as they will be determined for each hedger individually when bargaining with the reinsurer.

In the following, we take hedger n as an example. The optimal contract between another hedger and the reinsurer can be worked out similarly. Given the assumptions above, the profit functions of hedger n and the reinsurer are given by:

$$PR_n^{IN} = \left(A_n - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right) + z_n^{IN} l_x(n) \sum_{t=1}^{\omega_x} e^{-rt}\Big[{}_tp_x(n) - \big(1+\alpha_n^{IN}\tilde{\pi}_t^{IN}(n)\big){}_t\hat{p}_x(n)\Big], \tag{4.1}$$

$$PR_R^{IN} = z_n^{IN} l_x(n) \sum_{t=1}^{\omega_x} e^{-rt}\Big[\big(1+\alpha_n^{IN}\tilde{\pi}_t^{IN}(n)\big){}_t\hat{p}_x(n) - {}_tp_x(n)\Big]. \tag{4.2}$$

Note that the profit represented in (4.2) is that achieved by the reinsurer in the specific contract with hedger n. The principal-agent optimization problem for hedger n using the indemnity swap is then defined as:

$$\max_{\alpha_n^{IN}, z_n^{IN}} \ \mathbb{E}_P[PR_R^{IN}], \quad \text{subject to}$$
$$U_n(PR_n^{IN}) \geq \bar{U}_n. \tag{4.3}$$

Naturally, instead of N participation constraints as in the collective swap, we have only one participation constraint for hedger n. The corresponding Lagrangian function is:

$$\mathcal{L} = z_n^{IN} \sum_{t=1}^{\omega_x} e^{-rt}\alpha_n^{IN}\tilde{\pi}_t^{IN}(n)\cdot\hat{l}_{x+t}(n) + \lambda^{IN}\big(U_n(PR_n^{IN}) - \bar{U}_n\big). \tag{4.4}$$

The solutions to the optimization problem (4.3) are summarized in the following proposition.

Proposition 2 (Solutions of the optimal contracting with an indemnity swap)**.** *The optimization problem (4.3) has the following solutions:*

a) $z_n^{(IN,*)} = 0$, $\alpha_n^{(IN,*)} = \frac{\lambda^{IN}\gamma_n\nu_n}{B^{IN}(n)(\lambda^{IN}-1)}$, $\lambda^{IN} \geq 0$ *and* $\lambda^{IN} \neq 1$,

b) $z_n^{(IN,*)} = 1$, $\alpha_n^{(IN,*)} = \frac{\gamma_n\nu_n}{2B^{IN}(n)}$ *and* $\lambda^{IN} = 1$,

where $B^{IN}(n) = \sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}_t^{IN}(n)\hat{l}_{x+t}(n)$.

Proof. The proof can be found in Appendix A.2. □

Similar to the collective swap, the solutions of Problem (4.3) are either no hedger ($z_n^{(IN,*)} = 0$) or full hedge ($z_n^{(IN,*)} = 1$). Compared to Corollary 1, we see that the optimal risk premium α_n^{IN} in Solution b) has the same functional form. However, because the reinsurer negotiates with each hedger separately in the case of the indemnity swap, in each negotiation, the reinsurer makes the hedger's participation constraint binding. As a result, all hedgers will be indifferent between hedging or not, and the reinsurer can exploit all the hedgers' surpluses. Finally, if we assume $\tilde{\pi}_t^C = \tilde{\pi}_t{}^{IN}(n)$ for all n and t, then the indemnity swap becomes identical to the simplified scenario of collective swap with one hedger discussed in Section 3.1, and the optimal risk premium of the collective swap will be equal the minimum of $\alpha_n^{(IN,*)}$ over all n's, i.e., $\alpha^{(C,*)} = \min_{n\in\{1,\dots,N\}} \alpha_n^{(IN,*)}$.

4.2. The index-based longevity swap

Another longevity-linked derivative widely discussed in the literature is the index-based longevity swap (IB swap). With this swap, the reinsurer and the hedger exchange periodic payments based on the evolution of a survival index, usually the realized survival rates of a larger reference population, e.g. population of the region or the country which the hedger operates in. The IB swap is argued to have several theoretical advantages compared to the IN swap. Firstly, as mortality of the reference population is more stable than individual annuity portfolios, payments of the IB swaps are more predictable than those of the IN swaps. Secondly, realized survival rates of the reference population are publicly available, and thus are more transparent than individual annuity portfolios. Thirdly, as the survival index is likely to be correlated the portfolios of most life insurer and pension plans in the same region, the IB swap could be treated as a standardized product, and is potentially attractive to more hedgers. Consequently, the IB swaps would theoretically have a higher demand and lower price than the IN swaps.

In general, the reference population could be any population, as long as its survival rates are correlated with those of the annuity portfolios. In order to facilitate comparison, however, we let the reference population be the aggregate population of all the N annuity portfolios in our analysis, i.e., the index is defined on the aggregate survival probabilities ${}_tp_x = \frac{l_{x+t}}{l_x}$, $t = 1, \dots, T$. Formally, via the IB swap, hedger n will pay the reinsurer a fixed payment of $\left(1 + \pi_t^{IB}(n)\right)M_n\,{}_t\hat{p}_x$, in exchange of a floating payment $M_n\,{}_tp_x$ at each time t. We assume the notional amount to be $M_n = z_n^{IB}l_x$, with z_n^{IB} the hedger-specific hedge rate. Similar to the indemnity swap, the reinsurer will negotiate separately with each hedger on the optimal hedge rate and the risk premium. Finally, the same functional form is imposed on $\pi_t^{IB}(n)$: $\pi_t^{IB}(n) = \alpha_n^{IB}\tilde{\pi}_t^{IB}(n)$, where $\tilde{\pi}_t^{IB}(n)$, $t = 1, 2, \dots, \omega_x$ are the hedger-specific loadings.

Let us again take hedger n as an example to derive the optimal contract. Given the assumptions above, the profit functions of hedger n and the reinsurer are given by:

$$PR_n^{IB} = \left(A_n - \sum_{t=1}^{\omega_x} e^{-rt}l_{x+t}(n)\right) + z_n^{IB}l_x\sum_{t=1}^{\omega_x} e^{-rt}\left[{}_tp_x - (1 + \alpha_n^{IB}\tilde{\pi}_t^{IB}(n))\,{}_t\hat{p}_x\right], \tag{4.5}$$

$$PR_R^{IB} = z_n^{IB}l_x\sum_{t=1}^{\omega_x} e^{-rt}\left[(1 + \alpha_n^{IB}\tilde{\pi}_t^{IB}(n))_t\hat{p}_x - {}_tp_x\right]. \tag{4.6}$$

The principal agent optimization for the IB swap is then defined as:

$$\max_{\alpha_n^{IB}, z_n^{IB}} \mathbb{E}_P[PR_R^{IB}], \quad \text{subject to}$$
$$U_n(PR_n^{IB}) \geq \bar{U}_n, \tag{4.7}$$

and the corresponding Lagrangian function is given by:

$$\mathcal{L} = z_n^{IB}\alpha_n^{IB}\sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}_t^{IB}(n)\,\hat{l}_{x+t} + \lambda^{IB}(U_n(PR_n^{IB}) - \bar{U}_n). \tag{4.8}$$

The solutions to the optimization problem (4.7) are summarized in the following proposition. The principal agent problem for other hedgers can be solved analogously.

Proposition 3 (Solutions of the optimal contracting with an index-based swap)**.** *The optimization problem described in (4.7) has the following solutions:*

a) $z_n^{(IB,*)} = 0$, $\alpha_n^{(IB,*)} = \frac{\lambda^{IB}\gamma_n Cov(\mathcal{D}_n,\mathcal{D})}{B^{IB}(n)(\lambda^{IB}-1)}$, $\lambda^{IB} \geq 0$ *and* $\lambda^{IB} \neq 1$,

b) $z_n^{(IB,*)} = \frac{Cov\left(\mathcal{D}_n,\mathcal{D}\right)}{\mathcal{V}}$ *and* $\alpha_n^{(IB,*)} = \frac{\gamma_n Cov\left(\mathcal{D}_n,\mathcal{D}\right)}{2B^{IB}(n)}$ *and* $\lambda^{IB} = 1$,

where $B^{IB}(n) = \sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}_t^{IB}(n)\,\hat{l}_{x+t}$, $\mathcal{V} = Var\left(\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}\right)$, $\mathcal{D} = \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}$ *and* $\mathcal{D}_n = \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)$.

Proof. The proof can be found in the Appendix A.3. □

From Proposition 3, we see that the optimal hedge rate $z_n^{(IB,*)}$ depends positively on the covariance between the hedger's annuity portfolio and the reference population. Similar to the collective swap and the indemnity swap, the degree of risk aversion γ_n is fully captured by the optimal risk premium, and not reflected in $z_n^{(IB,*)}$. Moreover, similar to the indemnity swap, all consumer surpluses are extracted by the reinsurer, and all hedgers are indifferent between hedging or not with the optimal $z_n^{(IB,*)}$ and $\alpha_n^{(IB,*)}$. However, in contrast to the indemnity swap, $z_n^{(IB,*)}$ is in general smaller than 1. The reason is that the hedger's portfolio is a subset of the reference population when $N > 1$, and thus $Cov\left(\mathcal{D}_n, \mathcal{D}\right)$ should be smaller than $\mathcal{V}$. In fact, $z_n^{(IB,*)}$ can be rewritten as:

$$z_n^{(IB,*)} = \rho_n \sqrt{\frac{\mathcal{V}_n}{\mathcal{V}}},$$

where ρ_n is the correlation coefficient between $\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}$ and $\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)$. Given the construction of the survival index, and the assumption that all annuity portfolios are positively correlated, we have $\rho_n \in (0,1)$ when $N > 1$.[6] The hedged variance of hedger n's profit is given by:

$$Var\left(z_n^{(IB,*)} \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t} - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right) = (1-\rho_n^2)\mathcal{V}_n. \tag{4.9}$$

Therefore, the hedged variance will be positive, meaning that there are still remaining longevity risks in the hedger's portfolio after hedging, i.e., the population basis risk.

In the extreme case when $N = 1$, we have $l_{x+t}(1) = l_{x+t}$ for all t. Consequently, we have $\rho_1 = 1$, $\mathcal{V}_1 = \mathcal{V}$, and $z_1^{(IB,*)} = 1$. In fact, the index-based swap is reduced to the indemnity swap in this extreme case, as the survival index is identical to the hedger's annuity portfolio. Indeed, the optimal risk premium in this case becomes:

$$\alpha_1^{(IB,*)} = \frac{\gamma_1 Cov\left(\mathcal{D}_1, \mathcal{D}\right)}{2B^{IB}(n)} = \frac{\gamma_1 \mathcal{V}_1}{2B^{IN}(n)},$$

which is the optimal risk premium of the indemnity swap written on the hedger's portfolio with $\tilde{\pi}_t^{IN}(n) = \tilde{\pi}_t^{IB}(n)$, $t = 1, 2, \ldots, \omega_x$. Finally, since the reference population is assumed to be the aggregation of all annuity portfolios in our analysis, summing up all the optimal $z_n^{(IB,*)}$'s leads to:

$$\sum_{n=1}^{N} z_n^{(IB,*)} = \sum_{n=1}^{N} \frac{Cov\left(\mathcal{D}_n, \mathcal{D}\right)}{\mathcal{V}} = \frac{Cov\left(\sum_{n=1}^{N} \mathcal{D}_n, \mathcal{D}\right)}{\mathcal{V}} = \frac{Cov\left(\mathcal{D}, \mathcal{D}\right)}{\mathcal{V}} = \frac{\mathcal{V}}{\mathcal{V}} = 1.$$

4.3. Theoretical comparison of the three swaps

In this subsection, we compare the benefits of the three swaps discussed so far to the reinsurer. Following the discussions above, we assume that there are N hedgers in the market, and the reinsurer has perfect information of the mortality laws of the hedgers' portfolios, as well as their risk types.

4.3.1. Monopolistic market

First, we consider a monopolistic market where there is only one reinsurer. In this case, the reinsurer can freely decide which swap to offer. As a result, it will offer the swap which could maximize its expected profit. Assume that all hedgers will participate in the swap when they are indifferent, the expected profits of the reinsurer by offering each of the three swaps are, respectively, given by:

$$\mathbb{E}[PR_R^C] = z^{(C,*)}\alpha^{(C,*)}B^C = \frac{\gamma_{n'}\mathcal{V}_{n'}}{2B_{n'}^C}\sum_{n=1}^{N} B_n^C, \tag{4.10}$$

[6] Note that if a hedger is negatively correlated with the index, i.e., not exposed to longevity risk, the corresponding hedge rate would be negative. As a consequence it is possible that there exists a hedger with a $z_n^{(IB,*)} > 1$.

$$\mathbb{E}[PR_R^{IN}] = \sum_{n=1}^{N} z_n^{(IN,*)} \alpha_n^{(IN,*)} B^{IN}(n) = \sum_{n=1}^{N} \frac{1}{2} \gamma_n \mathcal{V}_n, \tag{4.11}$$

$$\mathbb{E}[PR_R^{IB}] = \sum_{n=1}^{N} z_n^{(IB,*)} \alpha_n^{(IB,*)} B^{IB}(n) = \sum_{n=1}^{N} \frac{Cov(\mathcal{D}_n, \mathcal{D})^2}{2\mathcal{V}} = \sum_{n=1}^{N} \frac{1}{2} \rho_n^2 \gamma_n \mathcal{V}_n. \tag{4.12}$$

From Section 3.1, we know that $\frac{\gamma_{n'}\mathcal{V}_{n'}}{2B_{n'}^C} \leq \frac{\gamma_n \mathcal{V}_n}{2B_n^C}$ for any n, and thus $\frac{\gamma_{n'}\mathcal{V}_{n'}}{2B_{n'}^C} \sum_{n=1}^{N} B_n^C \leq \sum_{n=1}^{N} \frac{\gamma_n \mathcal{V}_n}{2B_n^C} B_n^C = \sum_{n=1}^{N} \frac{1}{2} \gamma_n \mathcal{V}_n$. Therefore, we have:

$$\mathbb{E}[PR_R^C] = \frac{\gamma_{n'}\mathcal{V}_{n'}}{2B_{n'}^C} \sum_{n=1}^{N} B_n^C \leq \sum_{n=1}^{N} \frac{\gamma_n \mathcal{V}_n}{2B_n^C} B_n^C = \mathbb{E}[PR_R^{IN}].$$

Furthermore, for the index-based swap, we have:

$$\mathbb{E}[PR_R^{IB}] = \sum_{n=1}^{N} \frac{1}{2} \rho_n^2 \gamma_n \mathcal{V}_n \leq \sum_{n=1}^{N} \frac{1}{2} \gamma_n \mathcal{V}_n = \mathbb{E}[PR_R^{IN}],$$

since $\rho_n \in (0, 1)$. Therefore, the monopolistic reinsurer will choose to offer the indemnity swap. Furthermore, with the indemnity swap, the reinsurer extracts all the consumer surpluses from the hedgers, and the participation constraints of all hedgers are binding with the hedger-specific optimal risk premiums and hedge rates. This observation is consistent with the classic monopolistic insurance market studies, which argue that, under perfect information, the optimal contract between the reinsurer and each hedger is the one such that the reinsurer's expected profit is maximized, and each insurer is indifferent between hedging or not (Landsberger and Meilijson, 1994). Both the collective swap and the index-based swap will not be offered by the reinsurer, as they lead to lower expected profits. As discussed earlier, a universal risk premium will be charged to all hedgers in case of the collective swap, so that not all consumer surpluses could be extracted. For the index-based swap, although the risk premium is hedger specific and all participation constraints are binding, the hedgers are only willing to offer a lower risk premium because the index-based swap cannot fully eliminate the risk of future annuity payments due to the population basis risk.

4.3.2. Competitive market

Next, we consider a competitive market which includes more than one reinsurers. We assume that all the three swaps are offered in the market, and the hedgers will invest in the swap which gives them the highest utility. By examining the hedgers' utility functions, we see that while the binding hedgers (the one belonging to $\mathcal{N}'$) are indifferent among the three swaps, all other hedgers would prefer the collective swap. In general, N' depends on the set of hedgers and their portfolio properties $\mathcal{D}$ and $\mathcal{V}$. However, once a collective swap is fixed/concluded it holds that N' is fixed for the time of the contract. Here, we consider the market consisting of three reinsurers, each selling the collective swap, the indemnity swap, and the index-based swap, respectively, we compare the expected profit of these three reinsurers in the following hypothetical scenarios:

1. All hedgers participate in the collective swap.
2. Only the binding hedgers participate in the indemnity swap, and other hedgers choose not to hedge with indemnity swap, as moving to the collective swap increases their utility.
3. Only the binding hedgers participate in the index-based swap, and other hedgers choose not to hedge with index-based swap, as moving to the collective swap increases their utility.

Arguably, the above three scenarios are not exhaustive. For example, it is possible that a part of the binding hedgers choose the indemnity swap, while the remaining hedgers choose the index-based swap. Note that in this case the reinsurer would offer a collective swap with new contract parameters for the remaining hedgers. However, in order to limit the number of scenarios and facilitate the comparison of the three swaps, we focus on the three representative scenarios above.

For every binding hedger, it holds that $\frac{\gamma_{n'}\mathcal{V}_{n'}}{2B_{n'}^C} = K$, where $K > 0$ is a constant, cf. Corollary 1 b). We then get the following expected profits for the three scenarios above:

- Scenario 1:

$$\mathbb{E}[PR_R^{(1)}] = \alpha^{(C,*)} \sum_{t=1}^{\omega_x} e^{-rt} \tilde{\pi}_t^C \hat{l}_{x+t} = KB^C = K \sum_{t=1}^{\omega_x} e^{-rt} \tilde{\pi}_t{}^C \hat{l}_{x+t}. \tag{4.13}$$

- Scenario 2:

$$\mathbb{E}[PR_R^{(2)}] = \sum_{n' \in \mathcal{N}'} \alpha_{n'}^{(IN,*)} B^{IN}(n') = \frac{1}{2} \sum_{n' \in \mathcal{N}'} \gamma_{n'} \mathcal{V}_{n'} = K \sum_{n' \in \mathcal{N}'} B_{n'}^C$$

$$= K \sum_{t=1}^{\omega_x} e^{-rt} \tilde{\pi}_t^C \sum_{n' \in \mathcal{N}'} \hat{l}_{x+t}(n'). \tag{4.14}$$

- Scenario 3:

$$\begin{aligned}\mathbb{E}[PR_R^{(3)}] &= \sum_{n' \in \mathcal{N}'} z_{n'}^{(IB,*)} \alpha_{n'}^{(IB,*)} B^{IB}(n') = \frac{1}{2} \sum_{n' \in \mathcal{N}'} \gamma_{n'} \frac{Cov(\mathcal{D}_{n'}, \mathcal{D})}{\mathcal{V}} Cov(\mathcal{D}_{n'}, \mathcal{D}) \\ &= \frac{1}{2} \sum_{n' \in \mathcal{N}'} \gamma_{n'} \rho_{n'}^2 \mathcal{V}_{n'}. \end{aligned} \tag{4.15}$$

From the equations above, we observe that $\mathbb{E}[PR_R^{(3)}] < \frac{1}{2} \sum_{n' \in \mathcal{N}'} \gamma_{n'} \mathcal{V}_{n'} = \mathbb{E}[PR_R^{(2)}]$. Moreover, it holds that $\mathbb{E}[PR_R^{(2)}] \leq \mathbb{E}[PR_R^{(1)}]$, since $\sum_{n' \in \mathcal{N}'} \hat{l}_{x+t}(n') \leq \hat{l}_{x+t} = \sum_{n=1}^{N} \hat{l}_{x+t}(n)$, with equality holds only if $\mathcal{N}'$ includes all the N hedgers. As a result, we see that the collective swap is the optimal choice for the reinsurer in the competitive market, as it maximizes the reinsurer's expected profit. Finally, we see that the index-based swap is never optimal in our analysis, because the expected profit of selling it is always lower than that of the indemnity swap. Finally, we remark that the competitive market setting is more realistic in the current insurance industry, as there are a large number of international reinsurers operating in different jurisdictions all over the world (Best, 2020). Moreover, up to now, there have been over twenty reinsurers participating in the longevity swap and longevity risk transfer transactions worldwide.[7]

5. Mortality model

In the past twenty years, numerous mortality models have been proposed to model and project human mortality rates. Most of these models impose a parametric structure on mortality rates of multiple ages and populations, and rely on a factor representation for forecasting. For a more detailed overview of the existing mortality models, we refer to Blake and Cairns (2021) and the references therein. In this paper, we illustrate the proposed collective swap and the principal agent problem using the coherent functional demographic model (Co-FDM) proposed by Hyndman et al. (2013). Consider a data set of N populations, each representing a hedger and containing I ages and T years of observations. Further, denote by $\log u_{n,i,t}$ the logarithm of central death rate for age i and year t of population $n = 1, \ldots, N$. The Co-FDM model assumes that there is a smooth function, $f_{n,i,t}$, underlying the observed $\log u_{n,i,t}$ with the dynamics given by:

$$\log u_{n,i,t} = \log f_{n,i,t} + \varepsilon_{n,i,t}, \tag{5.1}$$

where $\varepsilon_{n,i,t}$ is an independently and identically distributed normal random variable with mean 0 and variance $\sigma_{n,i,t}$. The smooth function $f_{n,i,t}$ and the variance $\sigma_{n,i,t}$ are then estimated using the constrained weighted penalized regression splines (Hyndman and Ullah, 2007). Then, define the products and ratios of the smoothed rates for each sex:

$$\begin{aligned}\log \theta_{i,t} &= \frac{1}{N} \sum_{n=1}^{N} \log f_{n,i,t}, \\ \log \tau_{n,i,t} &= \log f_{n,i,t} - \log \theta_{i,t}, \quad n = 1, \ldots, N. \end{aligned}$$

Specifically, $\log \theta_{i,t}$ is the mean of the logarithm of the smoothed rates and thus represents the joint behavior of all populations, and the ratio function $\log \tau_{n,i,t}$ represents the deviation of population n's mortality from the joint behavior $\log \theta_{x,t}$. Therefore, the smooth functions can be represented by $\log f_{n,i,t} = \log \theta_{i,t} + \log \tau_{n,i,t}$ for each n, i, and t. After constructing the joint behavior and the ratio functions, the following functional time series models are imposed:

$$\begin{aligned}\log \theta_{i,t} &= u_i^{\theta} + \sum_{h=1}^{H} \phi_{h,i} K_{h,t} + e_{i,t}, \\ \log \tau_{n,i,t} &= u_{n,i}^{\tau} + \sum_{l=1}^{L} \Phi_{n,l,i} k_{n,l,t} + w_{n,i,t}, \quad n = 1, \ldots, N. \end{aligned} \tag{5.2}$$

The dynamics of the smoothed mortality function $f_{n,i,t}$ for each n is then given by:

$$\log f_{n,i,t} = \log \theta_{i,t} + \log \tau_{n,i,t} = u_{n,i} + \sum_{h=1}^{H} \phi_{h,i} K_{h,t} + \sum_{l=1}^{L} \Phi_{n,l,i} k_{n,l,t} + z_{n,i,t}, \tag{5.3}$$

where $u_{n,i} = u_i^{\theta} + u_{n,i}^{\tau}$ is the average mortality over time for age i and population n, and $z_{n,i,t} = e_{i,t} + w_{n,i,t}$ is the normal error term with mean 0 and variance $\sigma_{n,i}^2$.

[7] Source: https://www.artemis.bm/longevity-swaps-and-longevity-risk-transfers/.

Equation (5.2) can be estimated using principal component analysis. Mortality forecasts can then be obtained based on Equation (5.3) by extrapolating the time-varying common factors $K_{h,t}$, $h = 1, \ldots, H$, and the population-specific factors $k_{n,l,t}$, $n = 1, \ldots, N$, $l = 1, \ldots, L$. In general, the joint behaviors $\log \theta_{i,t}$ are non-stationary for each i, reflecting the ongoing mortality improvements. Consequentially, the common factors will be non-stationary. Meanwhile, to ensure coherent forecast, i.e., forecasts of the age-specific death rates for any two populations do not diverge in the long-run, the population-specific factors should be stationary. Following Hyndman et al. (2013), we forecast the common factors using the autoregressive fractionally integrated moving-average (ARFIMA)(p, d, q) process, and the population specific factors using the stationary autoregressive moving-average (ARMA)(p, q) processes. We estimate the Co-FDM model using the R package "demography", where the optimal order of the time-series specifications are determined by goodness-of-fit to the data.

The Co-FDM model is fitted to the Japanese and British uni-sex mortality data from 1956 to 2016 and ages between 0 and 100, since high quality historical mortality data with a sufficiently long observation window are available for both countries on the Human Mortality Database.[8] In our analysis, we let $H = L = 1$, and leave out the indices l and h of the parameters for ease of exposition. In this paper, we assume that the maximally attainable age is 120. However, for ages above 100 the mortality data have considerably smaller exposure and larger measurement errors, and are thus of lower credibility (Gatzert, 2019). Therefore, after estimating the Co-FDM model, we extrapolate the age-dependent variables for both hedgers. Specifically, the following extrapolation methods are used:[9]

- $u_{n,i}$ is extrapolated with a quadratic regression for each hedger using the estimated parameters of older people, i.e., $i \geq 60$.
- $\Phi_{n,i}$ is extrapolated using a random walk with drift.
- The variance of $z_{n,i,t}$ is extrapolated using the average of the estimated parameters of very old people, i.e., $i \geq 90$.
- ϕ_i is extrapolated using a random walk with drift.

The estimated and extrapolated values of the age-dependent variables are shown in Appendix B. After the extrapolation, we simulate the future smooth mortality functions using the time-series specification in Equation (5.3). After simulating the smooth mortality rates, accumulated survival probabilities of the policyholders in both annuity portfolios can be generated. Specifically, the t-year survival probability for population n can be approximated by (see, for example, Pitacco et al., 2009):

$$ {}_tp_x(n) \approx e^{-\sum_{i=x}^{x+t-1} f_{n,i,i-x+1}}. $$

In the following numerical analysis, we first simulate S sets of survival probabilities. Then, based on each simulated set of survival probabilities, V trajectories of the number of policyholders alive in each portfolio in each year will be simulated. First, for each simulation $s = 1, \ldots, S$, we first simulate a trajectory of ${}_tp_x^{(s)}(n)$ for $t = 1, \ldots, \omega_x$ and $n = 1, \ldots N$. Denote this trajectory by $\boldsymbol{p}^{(s)}$. Conditional on $\boldsymbol{p}^{(s)}$, we then simulate V trajectories of $l_{x+t}(n)$'s: let $l_{x+t}^{(s,v)}(n)$ be the simulated v-th trajectory based on $\boldsymbol{p}^{(s)}$. Specifically, $l_{x+t}^{(s,v)}(n)$ is assumed to follow the Binomial distribution given by:

$$ l_{x+t}^{(s,v)}(n) \sim Bin\Big(l_{x+t-1}^{(s,v)}(n), {}_tp_x^{(s)}(n) / {}_{t-1}p_x^{(s)}(n)\Big), $$

for $v = 1, ., V$, $t = 1, \ldots, \omega_x$ and $n = 1, \ldots N$. The aggregate portfolio is then given by $l_{x+t}^{(s,v)} = \sum_{n=1}^{N} l_{x+t}^{(s,v)}(n)$. Based on the simulated $l_{x+t-1}^{(s,v)}(n)$'s, distributions of the reinsurer's profit, as well as the hedgers' utility levels with each swap can be calculated. The simulation procedure is described in Appendix C.

6. Numerical analysis

In this section, we numerically evaluate the theoretical results discussed in this paper. In particular, using real-world mortality data, we compare the expected profit of the insurers and the utility level of the hedgers, under both the monopolistic and the competitive market. To simplify the discussion, we consider a market with $N = 2$ hedgers, and use uni-sex mortality data of Japan (hedger 1) and United Kingdom (hedger 2) as illustration. We include Japan and the U.K. in our numerical analysis because Japan has the highest life expectancy at 65 among the OECD countries as of 2019,[10] and has one of the largest private corporate sector pension liabilities around the world, while the U.K. is the earliest, and by far the largest longevity risk transfer market (Blake, 2018). Therefore, these two countries are both relevant for the study of longevity risk transfer.

We simulate the future mortality experiences in both annuity portfolios using $S = 1,000$ and $V = 1,000$ following the previously described method. Based on our simulation, the life expectancy at 65 is 23.1 years in Japan and 21.1 years in the United Kingdom, respectively. Further, the initial risk loadings of the three swaps are given by $\tilde{\pi}_t^{C} = \tilde{\pi}_t^{JB}(n) = \sigma(l_{x+t})$ and $\tilde{\pi}_t^{IN}(n) = \sigma(l_{x+t}(n))$, respectively, for $t = 1, \ldots, \omega_x$, $n = 1, 2$. The base case parameter setup of the numerical study is summarized in Table 1. Note that the value for γ_n is in line with typical values of this parameter, cf. (Zhang et al., 2009; Li et al., 2017).

[8] Source: https://www.mortality.org/.

[9] For more detailed discussions on old age mortality extrapolations, see, for example, Dowd and Blake (2019) and Dowd et al. (2019).

[10] Source: https://data.oecd.org/healthstat/life-expectancy-at-65.htm.

Table 1
Base case parameter setup.

Initial age	Pool Size	
$x_0 = 65$	$l_{x_0}(1) = 1000$	$l_{x_0}(2) = 1000$
Risk-free rate	Risk aversion	
$r = 2\%$	$\gamma_1 = 0.05$	$\gamma_2 = 0.05$

Table 2
Optimal contract parameters of the three swaps (top panel) and the expected profits of the reinsurer in the monopolistic market and the competitive market (bottom panel).

Optimal Contracts					
Collective Swap		Indemnity Swap		Index-based Swap	
$z^{(C,*)} =$ 1	$\alpha^{(C,*)} =$ 0.0075	$z_1^{(IN,*)} = 1$ $z_2^{(IN,*)} = 1$	$\alpha_1^{(IN,*)} = 0.0093$ $\alpha_2^{(IN,*)} = 0.0105$	$z_1^{(IB,*)} = 0.44$ $z_2^{(IB,*)} = 0.56$	$\alpha_1^{(IB,*)} = 0.0018$ $\alpha_2^{(IB,*)} = 0.0023$
Expected Profit					
Monopolistic Market			Competitive Market		
$\mathbb{E}[PR_R^C]$	$\mathbb{E}[PR_R^{IN}]$	$\mathbb{E}[PR_R^{IB}]$	$\mathbb{E}[PR_R^{(1)}]$	$\mathbb{E}[PR_R^{(2)}]$	$\mathbb{E}[PR_R^{(3)}]$
3811.2	4271.0	1075.7	3811.2	2000.0	402.3

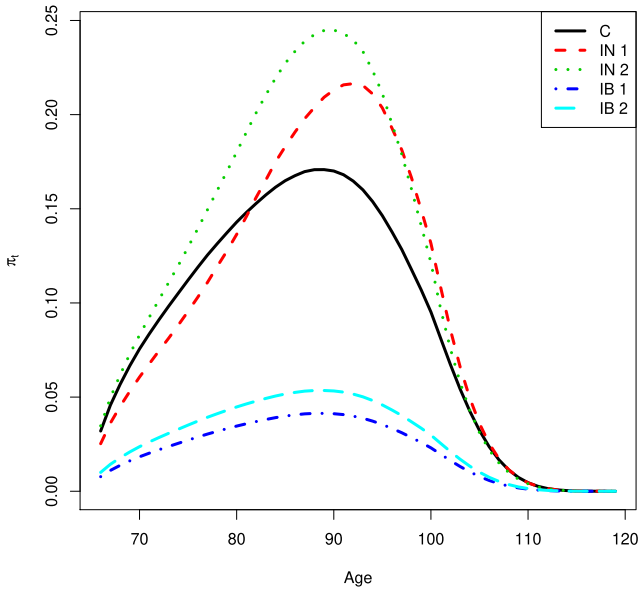


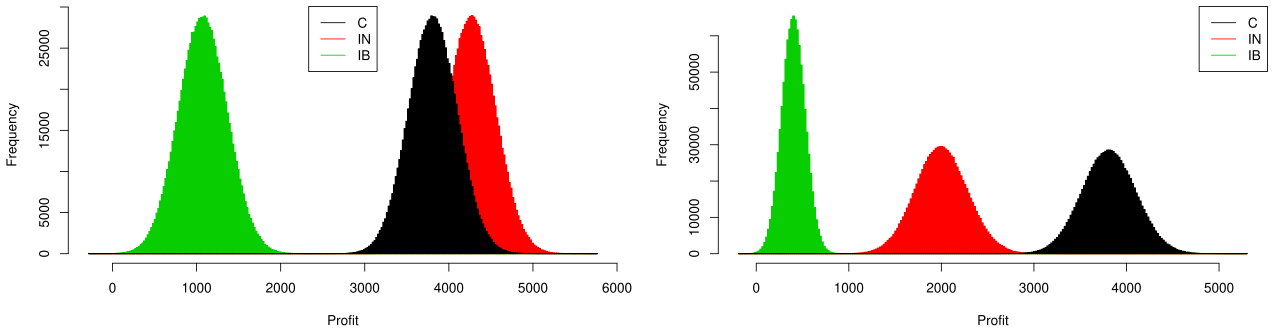
Fig. 1. The optimal risk premiums for each swap (C, IN, IB) and both hedgers.

The choice of the value for risk-free rate accounts for the current low interest rate environments in many countries and the ongoing increasing trend. For example, the average risk-free rate on investments in the United Kingdom in the year 2020 was 1.1% (see Statista (2020)). We assume a slightly higher one to incorporate the rising trend. Based on the initial parameters, we compute:

$$\frac{\gamma_1 \mathcal{V}_1}{2B_1^C} = 0.0075 \text{ and } \frac{\gamma_2 \mathcal{V}_2}{2B_2^C} = 0.0094. \tag{6.1}$$

Hence, the present value of hedger 1's portfolio (based on the Japanese population) has a lower variance to mean ratio ($\frac{\mathcal{V}_1}{B_1^C}$), and its participation constraint is binding when $\gamma_1 = \gamma_2$. The expected profits of the reinsurer for each swap, based on the optimal solutions of the principal-agent problem and the assumptions in Section 4.3, are given in Table 2.

We observe that the reinsurers' expected profits are in line with the theoretical results in Section 4.3. In particular, the indemnity swap is most attractive in the monopolistic market, whereas the collective swap leads to significantly higher expected profit than the other two swaps in the competitive market. The optimal risk loadings π_t^*'s of the three swaps are shown in Fig. 1 for both hedgers. The following observations can be made. First, the optimal risk loadings are higher for hedger 2, the non-binding hedger, for the indemnity and the index-based swap at most ages. This holds for the indemnity swap with ages below 100 and the index-based swap with ages below 110. This observation is in line with the theoretical



(a) Monopolistic Market (b) Competitive Market

Fig. 2. The histogram of realized profits of the reinsurer by issuing the collective swap (black), the indemnity swap (red), and the index-based swap (green) in the monopolistic market (left panel) and the competitive market (right panel). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

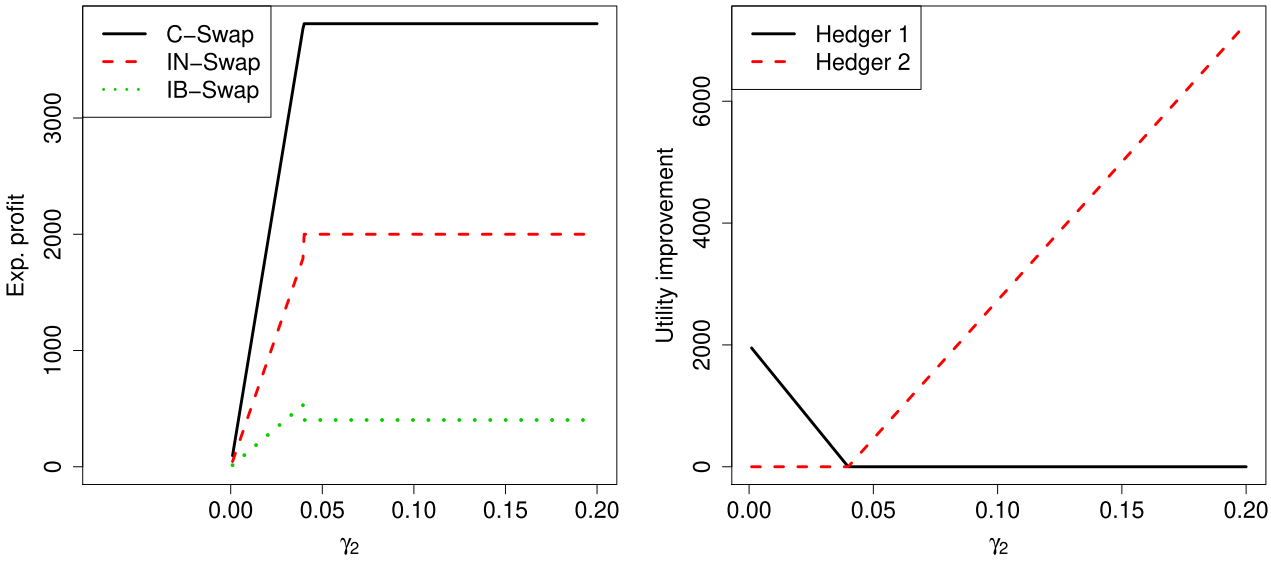
analysis in Sections 3 and 4 that the non-binding hedger is willing to accept a higher risk premium because its portfolio is more volatile for in most years. For the extremely old ages, there will be more policyholders alive in hedger 1's portfolio, which makes $\tilde{\pi}_t^{IN}(1)$ and $\tilde{\pi}_t^{IB}(1)$ larger than $\tilde{\pi}_t^{IN}(2)$ and $\tilde{\pi}_t^{IB}(2)$. However, the number of policyholders alive is rather small in these years for both hedgers, so the differences in risk premiums are quite minor. Second, the optimal risk loadings of the indemnity swap are higher than those of the collective swap over all ages for the non-binding hedger, and over the majority of ages for the binding hedger. This observation is not surprising. For the non-binding hedger, since it has a more volatile annuity portfolio than the aggregate portfolio, we expect $\tilde{\pi}_t^{IN}(2)$ to be larger than $\tilde{\pi}_t^{C}$, and thus $\pi_t^{(IN,*)}(2) > \pi_t^{(C,*)}$ for all t's. For the binding hedger, although its annuity portfolio is less volatile, and thus $\tilde{\pi}_t^{IN}(1)$ is likely to be smaller than $\tilde{\pi}_t^{C}$ for the majority of years, their difference becomes smaller as the policyholders get older, and thus in later years $\pi_t^{(IN,*)}(1)$ becomes larger than $\pi_t^{(C,*)}$ because $\alpha_1^{(IN,*)} > \alpha^{(C,*)}$. Third, the optimal risk loadings are the smallest for the index-based swap, because the risk premiums $\alpha_n^{(IB,*)}$ are the smallest among the three swaps for $n = 1, 2$. Finally, we see that all risk loadings are increasing in the early years with the peaks reached around age 90, and then decrease afterwards. The reason is that the number of policyholders alive in both portfolios is the most uncertain when the policyholders are close to 90. In particular, the survival probabilities are more certain in the early years, while the number if policyholders alive are lower in the later years. Both effects will lead to lower $\sigma(l_{x+t}(n))$'s, and thus lower optimal risk loadings.

Fig. 2 displays the distributions of reinsurer's profits with different swaps in the monopolistic market (left panel) and the competitive market (right panel). We observe that in the monopolistic market the profit volatilities of all swaps are the same, which is in line with the theoretical results. However, the volatility of the profit of the reinsurer is highest for the collective swap (peak on the right side) and lowest for the index-based swap (peak on the left side) in the competitive market, cf. Fig. 2 (b). This observation is reasonable, since the profit of the collective swap depends on remaining policyholders of the aggregate portfolio, while the profits of the other two swaps depend only on the remaining policyholders of the binding portfolio. In case of the index-based swap, the optimal hedge rate is comparably low (c.f. the top panel of Table 2), due to the population basis risk.

6.1. Impact of risk aversion

In the previous numerical analysis, we assume $\gamma_1 = \gamma_2$, and thus only one hedger's participation constraint is binding. If we want to make the participation constraint binding for both hedgers,we can for example adjust the risk aversion of the second hedger by setting $\frac{\gamma_1 \nu_1}{2B_1^C} = \frac{\gamma_2 \nu_2}{2B_2^C}$. In this case, we get that $\gamma_2 = 0.0399$, $\mathbb{E}[PR_R^C] = 3811.2$, $\mathbb{E}[PR_R^{IN}] = 3811.2$ and $\mathbb{E}[PR_R^{IB}] = 939.4$. This is in line with our theoretical result that the expected profit of the index-based swap is less than that of the collective swap, and the expected profits of the indemnity swap and the collective swap are equal if the participation constraints are binding for all hedgers.

We then vary the risk aversion of the second hedger (γ_2) from 0.001 to 0.2 and calculate the corresponding expected profit of the reinsurer. An illustration is given in Fig. 3 (a). Firstly, we see that the reinsurer achieves lowest expected profit from the index-based swap, and the highest from the collective swap. This holds for all the chosen levels of hedger 2's risk aversion. Moreover, the expected profit is increasing for all swaps over γ_2, until it reaches around 0.04, and remains constant after that. This is due to the fact that after this value it holds that $\frac{\gamma_1 \nu_1}{2B_1} < \frac{\gamma_2 \nu_2}{2B_2}$, which implies that the only binding hedger is the first hedger. Furthermore, the expected profit of the collective swap is growing most rapidly, as both hedgers participate in this swap, unlike in the other two swaps. In particular, the increasing magnitude of the expected profit of the reinsurer is rather small in an index-based swap, since the risk premium is lower than that for the indemnity swap, cf. Section 4.3. Finally, we see that the expected profit has a jump at $\gamma_2 \approx 0.04$ for the indemnity swap and the index-based swap. This is because the binding hedger switches from hedger 2 to hedger 1 when γ_2 passes the critical point. The jump



(a) Expected profit (b) Utility improvement

Fig. 3. The expected profit of the reinsurer for different swaps (left panel), and the utility improvement of each hedger with the collective swap (right panel) over the degrees of risk aversion for hedger 2.

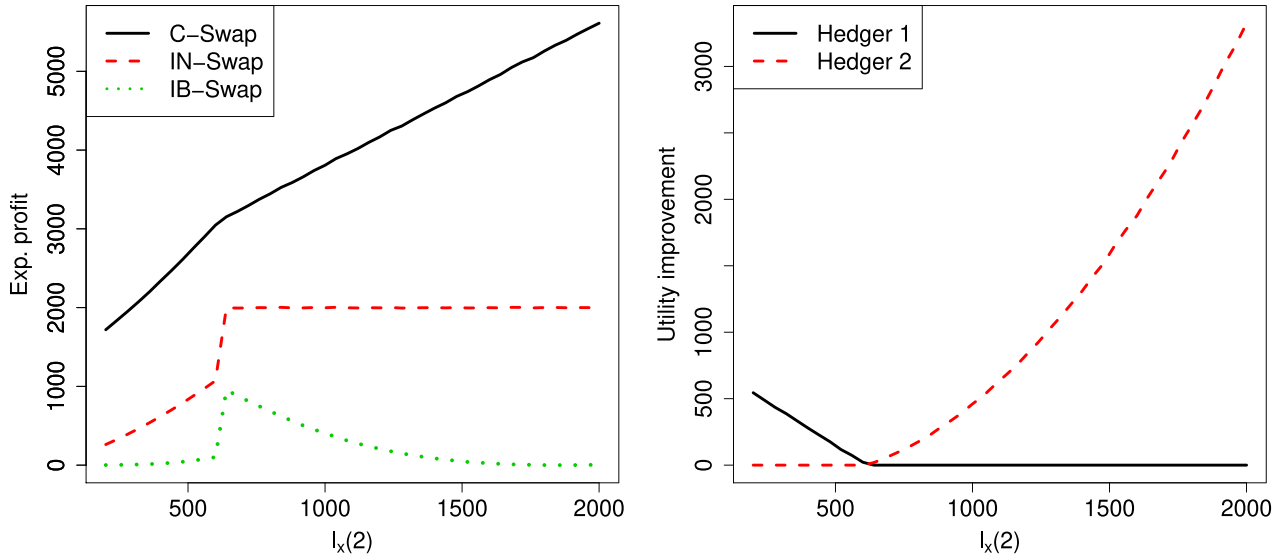
is upward for the indemnity swap because B_1^C is greater than B_2^C, cf. Equation (4.14), and is downward for the index-based swap because $\rho_1^2\mathcal{V}_1$ is smaller than $\rho_2^2\mathcal{V}_2$, cf. Equation (4.15).

Next, we show the utility improvements, i.e., the difference between the optimal utility level and the reservation utility of both hedgers with the collective swap as a function of γ_2 in Fig. 3 (b). Firstly, we see that the utility improvement is positive for the non-binding hedger (hedger 1 for $\gamma_2 < 0.04$ and hedger 2 for $\gamma_2 > 0.04$), and is 0 for the binding hedger. This is intuitive, as the binding hedger is indifferent in utility term between participating in the swap or not, and thus cannot improve its utility. On the other hand, the non-binding hedger can benefit from the collective swap because participating in the swap would make it the hedger strictly better off. Further, when hedger 1 is the binding hedger, the utility improvement of hedger 2 increases with its degree of risk aversion. The reason is that while the optimal risk loading is fixed at $\frac{\gamma_1\mathcal{V}_1}{2B_1^C}$ in this case, hedger 2 would be willing to pay more for the swap when it is more risk averse, and thus will enjoy a higher surplus with the optimal risk loading.

6.2. Impact of portfolio size

Now, we evaluate the impact of the relative portfolio sizes of the hedgers by varying the size of the second portfolio from 200 to 2000, while keeping the first portfolio size fixed at 1000. We then calculate the corresponding expected profit of the reinsurer. An illustration is given in Fig. 4. Similar to the previous sensitivity analysis with respect to γ_2, we observe from Fig. 4 (a) that the expected profit is lowest with the index-based swap, followed by the indemnity swap, and highest with the collective swap. Moreover, the binding hedger switches from hedger 2 to hedger 1 at round $l_x(2) = 600$. While the expected profits of all the three swaps are increasing in $l_x(2)$ when hedger 2 is the binding hedger, their patterns become rather different after the switching point. Firstly, the expected profit of the collective swap is monotonically increasing in $l_x(2)$, but with different slopes before and after the switch of the binding hedger. From Equation (4.13), we see that the expected profit of the collective swap is a linear function of the aggregate portfolio size, with the slope being the optimal risk premium $\alpha^{(C,*)}$. Before the switch of the binding hedger, the optimal risk premium is given by $\frac{\gamma_2\mathcal{V}_2}{2B_2^C}$, which is larger than the optimal risk premium after the switch, $\frac{\gamma_1\mathcal{V}_1}{2B_1^C}$. As a result, the increase of expected profit over $l_x(2)$ (and therefore the aggregate portfolio) is slower when hedger 1 becomes the binding hedger. Secondly, the expected profit of the indemnity swap is constant after the switching point, as it becomes independent of $l_x(2)$ when hedger 2 is no longer the binding hedger, and therefore will not purchase the indemnity swap. Finally, the expected profit of the index-based swap is decreasing with $l_x(2)$ after the switching point. The reason is that as $l_x(2)$ becomes bigger, the correlation between hedger 1's portfolio and the survival index becomes smaller, and thus the hedge rate. Consequentially, the expected profit is smaller with a larger $l_x(2)$ (c.f. Eq. 4.15).

Fig. 4 (b) displays the utility improvement from participating the collective swap of both hedgers with $l_x(2)$. The observation here is rather similar to the sensitivity analysis of risk aversion above. Specifically, the utility improvement of hedger 1 (as the non-binding hedger) is decreasing with $l_x(2)$ up to the switching point, while the utility improvement of hedger 2 is increasing with $l_x(2)$ after the switching point.



(a) Expected profit (b) Utility improvement

Fig. 4. The expected profit of the reinsurer for different swaps (left panel), and the utility improvement of each hedger with the collective swap (right panel) over the portfolio size of hedger 2.

7. Conclusion

In the past few decades, longevity risk transfer has become an important topic for defined benefit pension plans and annuity providers. So far, pension buy-in, buy-out, and indemnity longevity insurance are the most popular solutions. However, such services often come with rather high premiums, and thus are only affordable to very large pension plans and life insurers. As an alternative, innovative capital market solutions of longevity risk transfer such as longevity-linked securities have been developed, with longevity swaps being the leading product. This paper proposes a collective longevity swap, which provides the advantages of both the indemnity swap and the index-based swap, the two most widely discussed longevity swaps in the literature.

With the proposed collective swap, the reinsurer (hedge provider) exchanges periodic cash flows with a group of hedgers (pension plans or annuity providers), with both the fixed and the floating payments defined on the aggregate mortality experiences of all hedgers. In this way, the reinsurer assumes the aggregate longevity risks of the whole group of hedgers, rather than the portfolio-specific longevity risk of a single hedger, similar to index-based longevity swaps. On the other hand, the payments received by each hedger are defined on their own annuity portfolios, similar to an indemnity-based longevity swap. The hedgers could exactly offset the uncertainties in their annuity portfolios, and thus eliminate their longevity risk exposures. Therefore, the proposed collective swap could improve the longevity risk hedging effectiveness compared to the index-based swap, with which full elimination of longevity risks is in general impossible due to population basis risks.

Using a principal agent framework, we derive the optimal theoretical risk premiums and hedging rates of the collective longevity swap. Furthermore, we find that a part of the hedgers could benefit from the collective swap by strictly increasing their utility. This is in contrast to the indemnity swap and the index-based swap, in which cases all hedgers will be indifferent between hedging or not in terms of utility at the optimal risk premiums. Moreover, we show that in a (realistic) competitive market setting, the reinsurer can achieve a higher expected profit by offering the collective swap compared to the indemnity and the index-based swaps. The proposed collective swap is illustrated with real-world mortality data and the Coherent Mortality Forecasting Model proposed by Hyndman et al. (2013). The numerical results confirm our theoretical results. Specifically, the collective swap yields a higher expected profit for the reinsurer. Furthermore, utility improvement after entering the collective swap is significant for some hedgers, with the magnitude depending on their risk aversions and relative portfolio sizes. This could already be observed after considering only two hedgers. All in all, our analysis suggests that the collective longevity swap has the potential of achieving a better longevity risk sharing between the reinsurer and the hedgers.

Potential, but more complex aspects of future research can include, for instance, the study of the impact of asymmetric information between the reinsurer and hedgers. A dynamic competitive market is a nice subject for further research, too. Further, we can allow for model risks, i.e., the mortality laws of the hedgers are estimated with errors, and study its impact on the considered optimal contracting framework.

Declaration of Competing Interest

The authors whose names are listed immediately below certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakersbureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affi liations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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Appendix A. Proofs

This section collects the proofs of the propositions and corollary in this paper.

A1. Collective longevity swap

A1.1. Proof of proposition 1

The optimal solution shall satisfy the following first order conditions and complementary slackness conditions

$$0 = \frac{\partial \mathcal{L}}{\partial z^C} = \sum_{t=1}^{\omega_x} e^{-rt}\pi_t l_{x+t} - \sum_{n=1}^{N} \lambda_n^C \left(\sum_{t=1}^{\omega_x} e^{-rt}\pi_t l_{x+t}(n) + \gamma_n (z^C - 1)\mathcal{V}_n \right), \tag{A.1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \pi_k^C} = z^C e^{-rk} l_{x+k} - z^C \sum_{n=1}^{N} \lambda_n^C e^{-rk} l_{x+k}(n) \quad \text{for all } k = 1, \ldots, \omega_x, \tag{A.2}$$

$$0 = \lambda_n^C (U_n(PR_n^C) - \bar{U}_n) \quad \text{for all } n. \tag{A.3}$$

Given the first order conditions, (A.1) - (A.3), two sets of solution exists. The first set of solution is $z^{(C,*)} = 0$, $\pi_k^{(C,*)} = \frac{\sum_{n=1}^{N} \lambda_n^C \gamma_n \mathcal{V}_n}{(\sum_{n=1}^{N} \lambda_n^C e^{-rk} l_{x+k}(n)) - e^{-rk} l_{x+k}}$ for all $k = 1, \ldots, \omega_x$, and $\lambda_n^C > 0$ for all n, in which case all participation constraints are binding. Note that in the first solution the reinsurer's expected profit would be $\mathbb{E}_P[PR_R^C] = 0$. The second set of solutions is given by $z^C \neq 0$, in which case we get with (A.2)

$$0 = l_{x+k} - \sum_{n=1}^{N} \lambda_n^C l_{x+k}(n) \Leftrightarrow l_{x+k} = \sum_{n=1}^{N} \lambda_n^C l_{x+k}(n) \quad \text{for all } k = 1, \ldots, \omega_x. \tag{A.4}$$

We plug this into (A.1) and we find

$$\begin{aligned} 0 &= \sum_{t=1}^{\omega_x} e^{-rt}\pi_t \sum_{n=1}^{N} \lambda_n^C l_{x+t}(n) - \sum_{n=1}^{N} \lambda_n^C \sum_{t=1}^{\omega_x} e^{-rt}\pi_t l_{x+t}(n) + \sum_{n=1}^{N} \lambda_n^C \gamma_n (1 - z^C)\mathcal{V}_n \\ \Leftrightarrow\ 0 &= \sum_{t=1}^{\omega_x} \sum_{n=1}^{N} e^{-rt}\pi_t \lambda_n^C l_{x+t}(n) - \sum_{n=1}^{N} \sum_{t=1}^{\omega_x} \lambda_n^C e^{-rt}\pi_t l_{x+t}(n) + \sum_{n=1}^{N} \lambda_n^C \gamma_n (1 - z^C)\mathcal{V}_n \\ \Leftrightarrow\ z^{(C,*)} &= 1. \end{aligned} \tag{A.5}$$

Suppose the participation constraint for hedger n is binding. Then, using $z^{(C,*)} = 1$, we have:

$$\begin{aligned} & U_n(PR_n^C) = \bar{U}_n \\ \Longleftrightarrow\ & A_n - \sum_{t=1}^{\omega_x} e^{-rt} \left[(1 + \pi_t^C)\, \hat{l}_{x+t}(n) \right] = A_n - \sum_{t=1}^{\omega_x} e^{-rt} \hat{l}_{x+t}(n) - \frac{1}{2}\gamma_n \mathcal{V}_n \\ & \Longleftrightarrow\ \sum_{t=1}^{\omega_x} e^{-rt}\pi_t\, \hat{l}_{x+t}(n) = \frac{1}{2}\gamma_n \mathcal{V}_n. \end{aligned} \tag{A.6}$$

Therefore, in order for the participation constraints to hold, the optimal π_t for $t = 1, \ldots, \omega_x$ has to be the minimal $\sum_{t=1}^{\omega_x} e^{-rt}\pi_t\, \hat{l}_{x+t}(n) = \frac{1}{2}\gamma_n \mathcal{V}_n$ for a fixed π_t, $t = 1, \ldots, \omega_x$ among all n's.

A1.2. Time-varying hedge rates

This subsection shows that the optimal hedge rate will still be constant even if they are allowed to be time-varying. Assuming that z_t^C is not constant over time, the random profit functions of the reinsurer and hedger n are then given by:

$$PR_R^C = l_x \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \Big[(1+\alpha^C \tilde{\pi}_t^C)_t \hat{p}_x - {}_t p_x \Big].$$

$$PR_n^C = \left(A_n - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n) \right) + l_x(n) \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \Big[{}_t p_x(n) - (1+\alpha^C \tilde{\pi}_t^C)_t \hat{p}_x(n) \Big].$$

The utility function of hedger n is given by:

$$\begin{aligned} U_n(PR_n^C) &= E_{\mathbb{P}}[PR_n^C] - \frac{1}{2}\gamma_n Var_{\mathbb{P}}[PR_n^C] \\ &= A_n - \left(\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n) + \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \alpha^C \tilde{\pi}_t^C l_{x+t}(n) \right) - \frac{1}{2}\gamma_n Var\left(\sum_{t=1}^{\omega_x} e^{-rt}(1-z_t^C) l_{x+t}(n) \right) \end{aligned}$$

The resulting Lagrangian function is:

$$\begin{aligned} \mathcal{L} &= \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \alpha^C \tilde{\pi}_t^C \hat{l}_{x+t} + \sum_{n=1}^{N} \lambda_n^C (U_n(PR_n^C) - \bar{U}_n), \text{ where} \\ & U_n(PR_n^C) - \bar{U}_n \\ &= -\sum_{t=1}^{\omega_x} e^{-rt} z_t^C \alpha^C \tilde{\pi}_t^C l_{x+t}(n) - \frac{1}{2}\gamma_n \left(\sum_{t=1}^{\omega_x} \sum_{s=1}^{\omega_x} e^{-rt} e^{-rs} (1-z_t^C)(1-z_s^C) Cov_{t,s}(n) - \mathcal{V}_n \right), \end{aligned}$$

where $Cov_{t,s}(n) = Cov(l_{x+t}(n), l_{x+s}(n))$. The first-order conditions are given by:

$$\begin{aligned} 0 =& \frac{\partial \mathcal{L}}{\partial z_t^C} = \alpha^C e^{-rt} \tilde{\pi}_t^C \hat{l}_{x+t} \\ & - \sum_{n=1}^{N} \lambda_n^C \left(\alpha^C e^{-rt} \tilde{\pi}_t^C \hat{l}_{x+t}(n) + \gamma_n \Big(\sum_{s=1}^{\omega_x} e^{-rt} e^{-rs} (z_s^C - 1) Cov_{t,s}(n) \Big) \right) \\ & \text{for all} t. \end{aligned} \qquad \text{FOC}(z_t^C)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha^C} = \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \tilde{\pi}_t^C \hat{l}_{x+t} - \sum_{n=1}^{N} \lambda_n^C \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \tilde{\pi}_t^C l_{x+t}(n) \qquad \text{FOC}(\alpha^C)$$

$$0 = \lambda_n^C (U_n(PR_n^C) - \bar{U}_n) \quad \text{for all} n. \qquad \text{CS}$$

From FOC(α^C) we get $\sum_{t=1}^{\omega_x} e^{-rt} z_t^C \tilde{\pi}_t^C \hat{l}_{x+t} = \sum_{n=1}^{N} \lambda_n^C \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \tilde{\pi}_t^C l_{x+t}(n)$. Multiplying FOC($z_t^C$) by z_t^C for each t, and summing all the resulting equations results in:

$$0 = \alpha^C \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \tilde{\pi}_t^C \hat{l}_{x+t} - \alpha^C \sum_{n=1}^{N} \lambda_n^C \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \tilde{\pi}_t^C \hat{l}_{x+t}(n) - \sum_{n=1}^{N} \lambda_n^C \gamma_n \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \Big(\sum_{s=1}^{\omega_x} e^{-rs} (z_s^C - 1) Cov_{t,s}(n) \Big). \tag{A.7}$$

As a result,

$$0 = \sum_{n=1}^{N} \lambda_n^C \gamma_n \sum_{t=1}^{\omega_x} e^{-rt} z_t^C \left(\sum_{s=1}^{\omega_x} e^{-rs} (z_s^C - 1) Cov_{t,s}(n) \right). \tag{A.8}$$

Now, given the assumption that $z_t^C \in [0,1]$ for all t, we see that a) $z_t^C = 0$ for all t is a solution; and b) $z_t^C = 1$ for all t is also a solution. We observe that $\sum_{s=1}^{\omega_x} e^{-rs} (z_s^C - 1) Cov_{t,s}(n) \le 0$ for all t if $z_t^C \in [0,1]$ for all t. Since $\lambda_n^C \gamma_n \ge 0$ and $z_t^C \ge 0$, the sum contains no (strictly) positive values. It follows that if the sum is 0, all the summands must also be zero. This is only possible if all z_t^C are 0 or 1 for all t. If this is not the case, then there exists $z_k^C \in [0,1)$ and all other $z_t^C = 1$, and consequently $e^{-rk}(z_k^C - 1) Cov_{k,k}(n) < 0$. If there exists $z_k^C \in (0,1]$ and all other $z_t^C = 0$, then

$$\sum_{t=1}^{\omega_x} e^{-rt} z_t^C \left(\sum_{s=1}^{\omega_x} e^{-rs} (z_s^C - 1) Cov_{t,s}(n) \right) = e^{-rk} z_k^C \left(\sum_{s=1}^{\omega_x} e^{-rs} (z_s^C - 1) Cov_{k,s}(n) \right) < 0.$$

This shows that z_t^C is constant over time.

A1.3. Proof of corollary 1

The optimal solution shall satisfy the following first order conditions and complementary slackness conditions of the Lagrangian function described in (3.8)

$$0 = \frac{\partial \mathcal{L}}{\partial z^C} = \alpha^C B^C - \sum_{n=1}^{N} \lambda_n^C \left(\alpha^C B_n^C + \gamma_n (z^C - 1)\mathcal{V}_n\right), \tag{A.9}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha^C} = z^C B^C - z^C \sum_{n=1}^{N} \lambda_n^C B_n^C, \tag{A.10}$$

$$0 = \lambda_n^C (U_n(PR_n^C) - \bar{U}_n) \quad \text{for all } n. \tag{A.11}$$

Given the first order conditions, (A.9) - (A.11), two sets of solution exists. The first set of solution is $z^{(C,*)} = 0$, $\alpha^{(C,*)} = \frac{\sum_{n=1}^{N} \lambda_n^C \gamma_n \mathcal{V}_n}{(\sum_{n=1}^{N} \lambda_n^C B_n^C) - B^C}$, and $\lambda_n^C > 0$ for all n, in which case all participation constraints are binding. Note that in the first solution the reinsurer's expected profit would be $\mathbb{E}_P[PR_R^C] = 0$. The second set of solutions is given by $z^C \neq 0$, in which case we plug (A.10) into (A.9) and we get

$$\begin{aligned} & 0 = \alpha^C (B^C - \sum_{n=1}^{N} \lambda_n^C B_n^C) + (1 - z^C) \sum_{n=1}^{N} \lambda_n^C \gamma_n \mathcal{V}_n \\ \iff & 0 = (1 - z^C) \sum_{n=1}^{N} \lambda_n^C \gamma_n \mathcal{V}_n \\ \iff & z^{(C,*)} = 1. \end{aligned} \tag{A.12}$$

Suppose the participation constraint for hedger n is binding. Then, using $z^{(C,*)} = 1$, we have:

$$\begin{aligned} & U_n(PR_n^C) = \bar{U}_n \\ \iff & A_n - \sum_{t=1}^{\omega_x} e^{-rt} \left[(1 + \alpha^C \tilde{\pi}_t^C) \hat{l}_{x+t}(n) \right] = A_n - \sum_{t=1}^{\omega_x} e^{-rt} \hat{l}_{x+t}(n) - \frac{1}{2} \gamma_n \mathcal{V}_n \\ \iff & \frac{1}{2} \gamma_n \mathcal{V}_n - \alpha^C B_n^C = 0 \\ \iff & \alpha_n^{(C,*)} = \frac{\gamma_n \mathcal{V}_n}{2 B_n^C}. \end{aligned} \tag{A.13}$$

Therefore, in order for the participation constraints to hold, the optimal α^C has to be the minimal $\frac{\gamma_n \mathcal{V}_n}{2B_n^C}$ among all n's, i.e.,

$$\alpha^{(C,*)} = \min_{n \in \{1,2,\ldots,N\}} \frac{\gamma_n \mathcal{V}_n}{2 B_n^C}, \tag{A.14}$$

and the set of hedgers whose participation constraints are binding is given by

$$\mathcal{N}' = \arg\min_{n \in \{1,2,\ldots,N\}} \frac{\gamma_n \mathcal{V}_n}{2 B_n^C}. \tag{A.15}$$

Note that, for a given z^C, a larger $\alpha^{C'}$ would lead to a lower $U_n(PR_n^C)$ for any n, as it would decrease the expected profit of the hedger. Therefore, for any $i \in \mathcal{N}'$ we have

$$U_i(PR_i^C) - \bar{U}_i = \frac{1}{2} \gamma_n \mathcal{V}_n - \alpha^{C'} B_n^C < 0, \text{ since } \alpha^{C'} > \alpha^{(C,*)} = \min_{n \in \{1,2,\ldots,N\}} \frac{\gamma_n \mathcal{V}_n}{2 B_n^C} = \frac{\gamma_i \mathcal{V}_i}{2 B_i^C}.$$

A2. Proof of proposition 2 (indemnity swap)

The first order conditions of the Lagrangian function represented in (4.4) are given by

$$0 = \frac{\partial \mathcal{L}}{\partial z_n^{IN}} = \alpha_n^{IN} B^{IN}(n) - \lambda^{IN} \alpha_n^{IN} B^{IN}(n) - \lambda^{IN} \gamma_n (z_n^{IN} - 1)\mathcal{V}_n, \tag{A.16}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha_n^{IN}} = z_n^{IN} B^{IN}(n) - \lambda^{IN} z_n^{IN} B^{IN}(n), \tag{A.17}$$

$$0 = \lambda^{IN}(U_n(PR_n^{IN}) - \bar{U}_n). \tag{A.18}$$

There are multiple sets of solutions to the first order conditions (A.16) - (A.18). The first set of solutions is $z_n^{(IN,*)} = 0$, $\alpha_n^{(IN,*)} = 0$ and $\lambda^{IN} = 0$. Consequently, the expected profit of the reinsurer for the indemnity swap with hedger n is 0. The next set of solutions is $z_n^{(IN,*)} = 0$, $\alpha_n^{(IN,*)} = \frac{\lambda^{IN}\gamma_n\mathcal{V}_n}{B^{IN}(n)(\lambda^{IN}-1)}$ and $\lambda^{IN} > 0$, $\lambda^{IN} \neq 1$. As for the first solution the expected profit of the reinsurer is 0. For the last set of solutions, we assume $z^{IN} \neq 0$ and $\lambda^{IN} > 0$. In this case, by Equation (A.17) we get

$$B^{IN}(n) = \lambda^{IN}B^{IN}(n) \quad \rightarrow \quad \lambda^{IN} = 1.$$

Combining the above equation and Condition (A.16) gives

$$0 = -\gamma_n(z_n^{IN} - 1)\mathcal{V}_n \rightarrow z_n^{(IN,*)} = 1. \tag{A.19}$$

With $z_n^{(IN,*)} = 1$, Condition (A.18) is equal to

$$\begin{aligned} 0 &= \frac{1}{2}\gamma_n\mathcal{V}_n - \alpha^{IN}B^{IN}(n) \\ \iff \alpha_n^{(IN,*)} &= \frac{\gamma_n\mathcal{V}_n}{2B^{IN}(n)}. \end{aligned} \tag{A.20}$$

A3. Proof of proposition 3 (index-based swap)

The corresponding first order conditions of the Lagrangian function described in (4.8) are

$$0 = \frac{\partial\mathcal{L}}{\partial z_n^{IB}} = \alpha_n^{IB}B^{IB}(n) + \lambda^{IB}\left(-\alpha_n^{IB}B^{IB}(n)\right) - \lambda^{IB}\gamma_n\left[z_n^{IB}\mathcal{V} - Cov\left(\mathcal{D}_n, \mathcal{D}\right)\right] \quad \text{for all } n, \tag{A.21}$$

$$0 = \frac{\partial\mathcal{L}}{\partial \alpha_n^{IB}} = B^{IB}(n)z_n^{IB} - \lambda^{IB}z_n^{IB}B^{IB}(n), \tag{A.22}$$

$$0 = \lambda^{IB}(U_n(PR_n^{IB}) - \bar{U}_n) \quad \text{for all } n. \tag{A.23}$$

Solutions to the first order conditions (A.21) - (A.23) can be categorized into multiple sets. The first set includes solutions with $\lambda^{IB} = 0$. We get that $z_n^{(IB,*)} = 0$ and $\alpha_n^{(IB,*)} = 0$.[11] For the second set we assume that $\lambda^{IB} > 0$, $\lambda^{IB} \neq 1$ and $z_n^{IB} = 0$ then we get $\alpha_n^{(IB,*)} = \frac{\lambda^{IB}\gamma_n Cov(\mathcal{D}_n,\mathcal{D})}{B^{IB}(n)(\lambda^{IB}-1)}$ from (A.21). The last set of solutions we assume that $\lambda^{IB} > 0$ and $z_n^{IB} \neq 0$ then we get from (A.22) that

$$B^{IB}(n) = \lambda^{IB}B^{IB}(n) \rightarrow \lambda^{IB} = 1. \tag{A.24}$$

Combining the above equation and Condition (A.21) gives

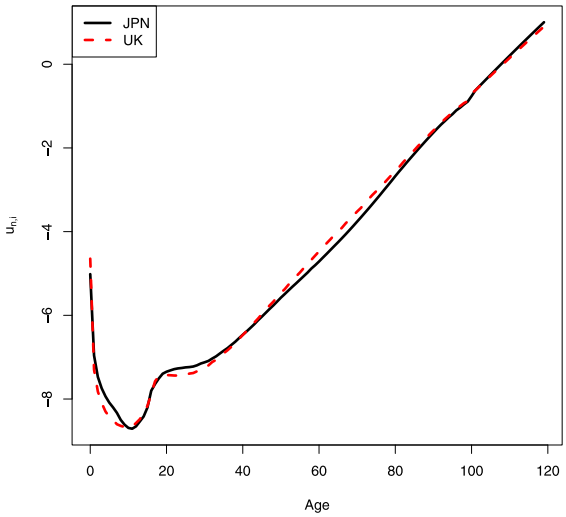
$$0 = -\gamma_n\left(z_n^{IB}\mathcal{V} - Cov(\mathcal{D}_n, \mathcal{D})\right) \rightarrow z_n^{(IB,*)} = \frac{Cov(\mathcal{D}_n, \mathcal{D})}{\mathcal{V}}. \tag{A.25}$$

With $z_n^{(IB,*)} = Cov\left(\mathcal{D}_n, \mathcal{D}\right)/\mathcal{V}$, Condition (A.23) is equal to

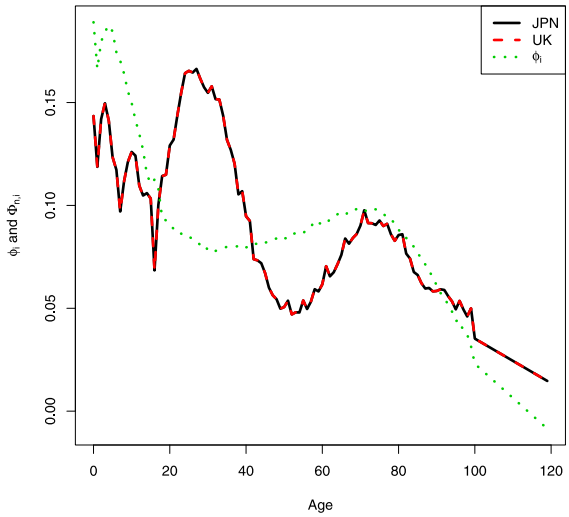
$$\begin{aligned} 0 &= -\alpha_n^{IB}B^{IB}(n) - \frac{1}{2}\gamma_n\left[z_n^{IB}\mathcal{V} - 2Cov\left(\mathcal{D}_n, \mathcal{D}\right)\right] \\ &= -\alpha_n^{IB}B^{IB}(n) - \frac{1}{2}\gamma_n\left[Cov\left(\mathcal{D}_n, \mathcal{D}\right) - 2Cov\left(\mathcal{D}_n, \mathcal{D}\right)\right] \\ \iff \alpha_n^{(IB,*)} &= \frac{\gamma_n Cov\left(\mathcal{D}_n, \mathcal{D}\right)}{2B^{IB}(n)}. \end{aligned} \tag{A.26}$$

[11] $z_n^{(IB,*)}$ can be derived from (A.22) and $\alpha_n^{(IB,*)}$ from (A.21).

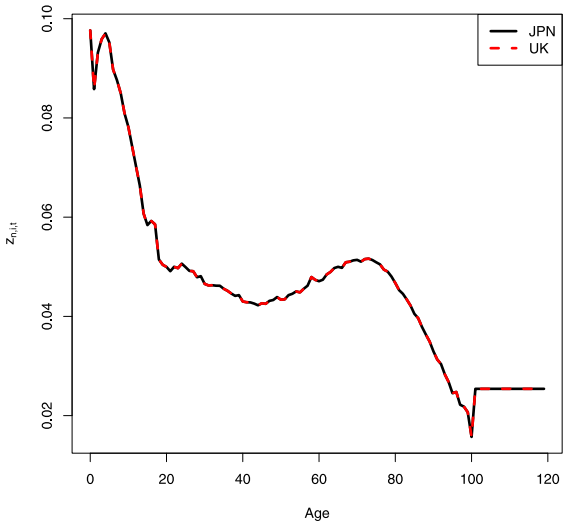
Appendix B. Estimation and extrapolation results



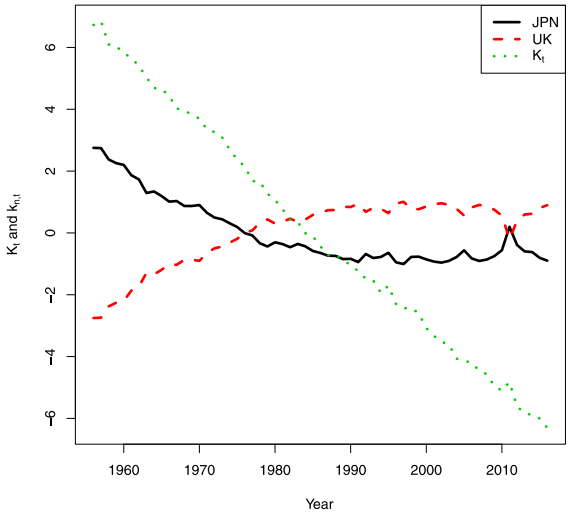
(a) The estimated and extrapolated values of $u_{n,i}$ for both countries.



(b) The estimated and extrapolated values of $\Phi_{n,i}$ for both countries and ϕ_i.



(c) The estimated and extrapolated values of $z_{n,i,t}$ for both countries.



(d) The estimated values of $k_{n,t}$ for both countries and K_t.

Note that the extrapolated values of $u_{n,i}$ are very similar, since both values are estimated with a quadratic linear regression. Moreover, the extrapolated and estimated values of $\Phi_{n,i}$ and the variance of $z_{n,i,t}$ are the same for both countries. This is because of the estimation process of the mortality model for two countries.

Appendix C. Simulation procedure

In this section we introduce the simulation procedure of the distribution of the reinsurer's and the hedgers' profits.

Given the simulated number of policyholders alive, we proceed to simulate the expectations and variance involved in the optimization problems discussed in Section 3 and 4. Take $B^{IN}(n) = \sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}_t^{IN}(n)\hat{l}_{x+t}(n)$ for example. We first calculate the conditional mean:

$$B^{(IN,s)}(n) = \mathbb{E}\Big[\sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}_t^{(IN)}(n)\, l_{x+t}(n)|\boldsymbol{p}^{(\mathbf{s})}\Big] = \frac{1}{V}\sum_{v=1}^{V}\sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}_t^{(IN)}(n) l_{x+t}^{(s,v)}(n),$$

using the V trajectories of $l_{x+t}^{(s,v)}(n)$, $t = 1, \ldots, \omega_x$ conditional on $\boldsymbol{p}^{(s)}$. The unconditional mean is then given by:

$$B^{(IN)}(n) = \mathbb{E}\left[B^{(IN,s)}(n)\right] = \frac{1}{S}\sum_{s=1}^{S} B^{(IN,s)}(n).$$

For the simulation of variances, we utilize the law of total variance. Take $\mathcal{V}_n = Var\left(\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n)\right)$ for example. We first simulate the conditional variance:

$$\mathcal{V}_n^{(s)} = Var\left(\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}(n) \middle| \boldsymbol{p}^{(s)}\right) = \frac{1}{V}\sum_{v=1}^{V}\left(\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}^{(s,v)}(n)\right)^2 - \left(\frac{1}{V}\sum_{v=1}^{V}\sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}^{(s,v)}(n)\right)^2.$$

The unconditional variance is then given by:

$$\begin{aligned}\mathcal{V}_n &= Var\left(B^{IN}(n)\right) + E\left[\mathcal{V}_n\right] \\ &= \frac{1}{S}\sum_{s=1}^{S}\left(B^{(IN,s)}(n)\right)^2 - \left(\frac{1}{S}\sum_{s=1}^{S} B^{(IN,s)}(n)\right)^2 + \frac{1}{S}\sum_{s=1}^{S}\mathcal{V}_n^{(s)}.\end{aligned}$$

Finally, the distributions of reinsurer's profit can be calculated using the simulated $l_{x+t}^{(s,v)}(n)$, $s = 1, \ldots, S$, $v = 1, \ldots, V$, $t = 1, \ldots, \omega_x$, $n = 1, \ldots, N$.

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3 Asymmetric Information and Longevity Risk Transfer

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Asymmetric Information and Longevity Risk Transfer

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Abstract

This paper investigates the optimal transfer of longevity risk between capital market investors and longevity risk exposure holders (such as defined benefit pension plans and life insurers) under information asymmetry. Since longevity risk is a new risk to the capital market, the investors may have less knowledge of forecasting longevity-linked cash flows than the risk exposure holders. A principal-agent model is proposed to incorporate such information asymmetry in longevity-linked transactions. Using indemnity longevity swaps as an example of the risk transfer solutions, we derive the analytical solution to the optimal risk premiums and incentive-compatible hedge demands under both a monopolistic and a competitive market setting. The theoretical results are evaluated using real-world mortality data in extensive numerical analyses. We find that the hedge provider's expected profit and the hedger's expected utility could be substantially improved when information asymmetry is properly addressed in longevity risk transactions.

Keywords: Longevity Risk; Adverse selection; Longevity Swap; Economic Pricing; Principal-Agent-Model
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1 Introduction

In the past few decades, unprecedented rise in human life expectancy has been witnessed. While it is per se a very pleasant development, the increasing life expectancy has led to a sequence of societal problems related to longevity risk, i.e. the risk that people live longer than expected. In particular, the increasing uncertainty surrounding future longevity forecasts has made the liability risk management more challenging for the global retirement system (see, for example, Bielecki *et al.*, 2016; Pestieau & Racionero, 2016; Wang *et al.*, 2016). As a result, there is a growing need for transferring the longevity risk exposure for institutions bearing longevity risk, including defined benefit pension plans, annuity providers, and insurance intermediaries (Blake *et al.*, 2019). However, this is not an easy task, as the global longevity risk exposure has far exceeded the capacity of the insurance sector, which is the dominating market of longevity risk transfer thus far. Take the pension liabilities for example. The accrued pension obligations in the United States, the United Kingdom, and Canada, the three largest pension and longevity transfer markets in the world, are estimated to be $42.2 trillion at the end of 2020 (OECD, 2021). On the other hand, there has been only around $620 billion of notional value in pension risk transfer transactions in these three countries between 2007 (the launch year of the longevity transfer market) and 2020.[1] Compared to other mature risk transfer markets, this transaction volume is rather small.[2]

Acknowledging the limited capacity of the reinsurance sector, existing studies suggest that capital market investors could potentially act as a complement to the reinsurance sector by providing hedge supplies to the market (see, among many others, Blake, 2018; Kessler, 2021; Blake & Cairns, 2021). Longevity risk is thought to have little correlation with traditional financial risks and provides diversification benefits. Therefore, it could be an attractive asset class to general capital market investors, including hedge funds, endowments, sovereign wealth, and other institutional investors, if the return is attractive. However, despite its advantages, longevity risk is a new and lesser-known risk to general market investors. In particular, longevity risk exposure holders, such as pension funds, life insurers, and insurance intermediaries (reinsurers, pension buy-out firms, etc) may have better knowledge of their risk than the potential investors due to continuously monitoring of their portfolio and better expertise of forecasting longevity-linked cash flows, etc (Biffis & Blake, 2010). The resulting information asymmetries could make longevity risk less attractive to capital market investors, and possibly prevent them from entering the market. In this paper, we focus on the information asymmetry in longevity risk transfer by developing a principal-agent framework to investigate the optimal transactions

[1] Source: LIMRA, Hymans Robertson, LCP and Prudential analysis as of December 31, 2020. For a history of the development of this market, we refer to Blake *et al.* (2019).

[2] For example, in the interest risk market, the notional value of outstanding over-the-counter derivatives was $610 trillion at end-June 2021 according to the Bank for International Settlements. Source: `https://www.bis.org/publ/otc_hy2111.pdf`. Retrieved May 29, 2022.

in different information and market settings.

As one of the most studied classical economic problems, information asymmetry and the resultant adverse selection of the agents have been widely discussed in economies and financial markets (Capasso & Mavrotas, 2010; Tran *et al.*, 2019; Hu & Prigent, 2019; Aney & Banerji, 2022), as well as in the insurance market. For example, in their seminal work, Rothschild & Stiglitz (1978) lay the theoretical foundation of the market equilibrium between an insurer and individuals under information asymmetry. Specifically, they characterize a self-selecting, separating equilibrium between the insurer and two types (high- and low-risk type) of individuals, in which both the prices and the quantities of insurance are specified for both types. The conditions under which the separating equilibrium exists are also discussed. The adverse selection issue has later been investigated in different fields of insurance, including general insurance (Cohen & Siegelman, 2010), automobile insurance (Puelz & Snow, 1994; Dionne *et al.*, 2001), long-term care (Sloan & Norton, 1997), health insurance (Cutler & Reber, 1998), and agriculture insurance (Makki & Somwaru, 2001), just to name a few.

When it comes to the annuity market, abundant literature has discussed the issue of adverse selection at the policyholder level, i.e., healthier people are more likely to purchase annuities (see, among many others, Davies & Kuhn, 1992; Finkelstein & Poterba, 2004; Hosseini, 2015; Chen *et al.*, 2022b). However, although there have been plenty of studies on institutional longevity risk transactions (see, for example, Zhou *et al.*, 2015; Blake *et al.*, 2019; Kessler, 2021, and the references therein), only a few of them focus on information asymmetry. For example, Biffis & Blake (2010) study the securitization and tranching of longevity exposures between the exposure holder and a number of uninformed capital market investors. Further, Biffis & Blake (2013) investigate pension buyout transactions between pension funds and capital market investors, through insurers operating as intermediaries. Biffis & Blake (2014) consider a similar framework consisting of pension funds, insurance intermediaries, and end investors, and propose a principal-at-risk bond as a promising format for transferring longevity risk.

In this paper, we utilize the principal-agent framework proposed by Rothschild & Stiglitz (1978) to investigate the longevity risk transaction between a number of hedge providers (principal) and a hedger (agent) under information asymmetry. To resemble capital market transactions, we assume that the hedge providers are uninformed capital market investors interested in taking longevity risk exposures in exchange for a return, and the hedger is an informed longevity risk exposure holder, which could be a defined-benefit pension fund, a life insurer, or an insurance intermediary (such as reinsurance companies or pension buy-out firms). We assume that the hedger's exposure contains private information not observed by the hedge provider, from whose perspective the hedger can be either a low- or a high-risk type. The hedge providers then offer two types of longevity risk securities tailor-made to each risk type, and determines the optimal

prices and quantities of both securities using the principal-agent model. The current longevity risk transfer market is incomplete and not many longevity risk transfer solutions are available, especially for the small- and medium-size risk exposure holders. This makes risk-neutral pricing, which is applied in more liquid markets, not applicable to longevity risk transfer products. The principal-agent model offers a convenient valuation framework in this situation. Inactive trading also makes it hard for the hedger to find counterparties. This motivates us to consider both a monopolistic and a competitive market setting. The monopolistic market setting represents the case in which only one hedge provider is interested in taking the longevity risk, whereas the competitive market setting is appropriate when a (large) number of hedge providers are interested in taking the risk. In both market settings, a set of participation and incentive constraints is imposed to ensure that both risk types will participate in the market and reveal their real types.

The *indemnity longevity swaps* is used as an example of the longevity-linked securities in our analysis. An indemnity longevity swap involves periodic exchanges of cash flows between the hedger and the hedge provider. The cash flows are linked to the mortality experience of the hedger's portfolio, and could mitigate, or even eliminate the hedger's longevity risk. In contrast to traditional longevity risk transfer solutions, such as pension buy-in (also known as the bulk annuity) and buy-out,[3] which cover all risks of the hedger's portfolio (including investment risks and operation risks, etc), indemnity longevity swaps focus on the transfer of longevity risk, and thus could be much cheaper for the hedger. Due to its cost efficiency, indemnity longevity swap has gained substantial popularity in the global longevity risk transfer market in the past decade (Blake & Cairns, 2021).

In order to evaluate the impact of information asymmetry, we investigate the principal-agent model in three cases. The first case corresponds to the first-best situation with perfect information, in which the hedge provider negotiates with each risk type and settles the deal separately. The second case is our benchmark analysis where a separating equilibrium is derived in the presence of information asymmetry. The third case is a Stackelberg game, where the hedge providers do not address information asymmetry, but instead sell a single swap to both risk types. With a mean-variance preference for the hedgers, analytical solutions are derived in all the three cases under both market settings. Some key observations are as follows. First, in the monopolistic market, the high-risk type hedger entails a full hedge in the first two cases, while the low-risk type opts for a partial hedge. This is consistent with the findings in Stiglitz (1977). Moreover, the equilibrium price of the swap for the high-risk type under asymmetric information is lower than the price under perfect information, while the opposite is true for the low-risk type. This indicates that the high-risk type is subsidized by the low-risk type in the

[3]For more discussions of buy-in and buy-out, we refer to Cox *et al.* (2018).

presence of information asymmetry. In the competitive market, both swaps have the actuarially fair prices and the hedge provider breaks even. In the benchmark analysis, the low-risk type hedger has a lower utility improvement than the first-best level due to partial hedge, while the high-risk type hedger has the same utility improvement. Finally, the Stackelberg game leads to the worst results in both market settings — the lowest expected profit for the hedge provider under monopoly, and the lowest utility improvement for the low-risk type hedger under competition. In particular, we find that the low-risk type hedger will not participate in the swap transaction in the Stackelberg game in the competitive market, and thus has zero utility improvement. The optimal contracts are also derived with an exponential utility for the hedger. Although analytical solutions are not available in some cases, the results above also hold with the exponential utility.

The theoretical results are supplemented with extensive numerical analyses using real-world mortality data and a sophisticated mortality model. In particular, mortality data of the United Kingdom and the United States are fitted to the Age-Period-Cohort-Improvement mortality model, which is used by the Continuous Mortality Investigation[4] mortality projections committee to generate life tables for UK life insurers and pension funds. Using the simulated future survival probabilities from the fitted model, the optimal prices and quantities of the longevity swap are evaluated under both market settings. In the monopolistic market, while the benchmark contracts lead to only a slightly lower expected profit for the hedge provider than the first-best value, they result in a distribution of profit substantially higher in mean and all percentiles (e.g., the 0.5% percentile)[5] than the Stackelberg game. Similarly, in the competitive market, the separating equilibrium contracts lead to a significantly higher utility improvement for the low-risk type (and slightly lower than in the perfect information case), compared to the Stackelberg game. All in all, our analyses suggest that taking the potential information asymmetry into account could substantially improve the profitability and meanwhile reduce the uncertainty of the longevity risk business for the hedge providers. The numerical analyses are repeated using an exponential utility for the hedger, and qualitatively very similar results are obtained.

The contribution of this paper is three-fold. First, the analytical equilibrium solutions from the proposed principal-agent framework provide a convenient way to evaluate the impact of information asymmetry on the optimal prices and quantities of the longevity swaps. The equi-

[4]Continuous Mortality Investigation is an organization supported by the Institute and Faculty of Actuaries in UK. Source: `https://www.actuaries.org.uk/learn-and-develop/continuous-mortality-investigation/about-cmi`.

[5]Despite some undesirable properties (Embrechts *et al.*, 2018), Value-at-Risk is still one of the most frequently used risk measures in industry. The 0.5% percentile is a key statistics in (re)insurance solvency capital management under the new international solvency and accounting regulations, such as Solvency II and IFRS 17 (England *et al.*, 2019).

librium pricing formulas also shed light on the longevity risk pricing problem, which is itself a hot topic in the literature (see, for example, Denuit *et al.*, 2007; Wills & Sherris, 2010; Leung *et al.*, 2018, and the references therein). The equilibrium pricing formulas also allow for straightforward analysis of the impact of different characteristics of the hedger's portfolio on the price of the swaps. Second, our analysis provides conditions under which the longevity swap market exists in the presence of information asymmetry. In particular, we find that the market will break down when the outstanding liabilities of the two risk types are too different. This finding could be valuable for real-world hedge providers operating in or planning to enter the longevity risk transfer market. It also indicates the importance of mitigating information asymmetry to the further development of the longevity risk transfer market. Third, this paper contributes to the existing studies on longevity risk transactions under information asymmetry by considering both the monopolistic and the competitive market setting in an integrated framework. The monopolistic market could be more appropriate when only one hedge provider is interested in taking the longevity risk of the hedger considered. This could be the case in the current longevity risk transfer market where transactions are rare. Finally, we remark that although the indemnity longevity swap is chosen as an illustration, the proposed framework can be applied in a straightforward manner to a variety of longevity hedging instruments, including the widely-used pension buy-out and bulk annuity.

The remainder of this paper is organized as follows. Section 2 introduces the preliminaries. Section 3 describes the principal-agent framework and discusses the equilibrium contracts. Section 4 introduces the mortality data and model, and discusses the numerical analyses. Section 5 provides a sensitivity analysis assuming an exponential utility for the hedger. Finally, Section 6 concludes. Proofs and graphical illustrations of the mortality data and model are summarized in the appendix.

2 Preliminaries

In this section, we introduce the preliminaries and consider how a contract on an indemnity longevity swap can be arranged between a hedge provider (swap seller) and a hedger (swap buyer). For ease of exposition, we assume that the hedger holds a life annuity portfolio with a stream of in-arrears fixed unitary payments to l_x policyholders who belong to the same cohort with age x in year 0 and have homogeneous mortality, i.e., their future survival probabilities are governed by the same mortality law.[6] Due to longevity risks prevailing in the annuities products, the hedger is interested in purchasing longevity swaps to transfer a part of the risk to the hedge provider. Let us first introduce the notations used throughout this paper:

[6]We have chosen the simplest setting to keep the paper stay focused on the main topic. The annuity business of a real life insurer could be more complex, for example, it may include multiple cohorts from different risk classes. Such generalizations can be incorporated by our analysis in a straightforward manner.

- l_x: the initial number (in year 0) of policyholders in the hedger's portfolio;
- l_{x+t}: the random number of remaining policyholders in the hedger's portfolio in year t, $t \in \{1, ..., \omega_x\}$, where ω_x is the maximal remaining life time for the policyholders aged x;
- $\hat{l}_{x+t}$: the time-0 expected value of l_{x+t};
- ${}_tp_x = \frac{l_{x+t}}{l_x}$: the (random) t-year survival probability of the hedger's portfolio;
- ${}_t\hat{p}_x = \frac{\hat{l}_{x+t}}{l_x}$: the time-0 expected value of ${}_tp_x$.

A more detailed table summarizing the notations used in this paper is provided in Appendix A. The time-0 random present value of the hedger's annuity portfolio is given by:

$$\left(A - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}\right),$$

where A is the exogenous, initial premium collected from the policyholders. Further, we assume for simplicity a constant annual interest rate r for all t. In real world applications, A would be determined using the hedger's pricing assumptions, which could be different from the (reserving) mortality assumptions used in the hedging procedure discussed in the sequel. Furthermore, because the numbers of future remaining policyholders, l_{x+t}'s, are uncertain, the hedgers are exposed to longevity risk.

We assume there are two types of hedgers, a low (L)- and a high (H)-risk type. The future mortality rates of the hedger's portfolio depend on its risk type. Specifically, we use ${}_tp_x^L$ and ${}_tp_x^H$, $t = 1, \cdots, \omega_x$ to denote the conditional survival probability of the low- and high-risk hedger, which are random variables underlying different probability distributions. The underlying mortality models will be discussed in Section 4.1. Further, we assume that the hedge provider knows the probability distribution of both sets of survival probabilities, but does not know which type the hedger is. More specifically, the hedge provider knows that the hedger is a low-risk type with a probability of ϵ, which is strictly between 0 and 1, and a high-risk type with probability $1 - \epsilon$. This assumption is equivalent to the case where there are multiple hedgers with the same initial portfolio size, with ϵ being the proportion of hedgers of the low-risk type.

We consider indemnity longevity swaps between the hedger and the hedge provider, with the aim to mitigate the hedger's longevity risk exposure. Specifically, the hedge provider offers protection against the uncertainty of the portfolio of the hedger over its lifespan. Formally, the hedge provider provides two swaps at time 0, one designed for the low-risk type and the other for the high-risk type. By acquiring the swap, the hedger of risk type i pays a fixed payment of $(1 + \pi_t^i) \cdot M^i \cdot FIX_t^i$, with M^i being the notional amount in exchange of a floating payment $M^i \cdot FLT_t^i$, with M^i being the notional amount, and FIX_t^i and FLT_t^i the fixed and

the floating rate of the swap respectively for $i = L, H$. In general, the risk premium loading at each time t can be negotiated separately between the two counterparties. However, this will lead to substantial complexity to our economic model, as well as computational burden to the numerical analysis. Therefore, in this paper we follow Chen *et al.* (2022a) and assume the following functional form for the risk loading:

$$\pi_t^L = \alpha_L^A \, \tilde{\pi}_t^L,$$
$$\pi_t^H = \alpha_H^A \, \tilde{\pi}_t^H.$$

That is, the risk loading π_t^i can be presented as the product of a constant and a time-varying component, where $\alpha_L^A \geq 0$ and $\alpha_H^A \geq 0$ are the constant components which will become two choice variables in our optimal contracting, and $\tilde{\pi}_t^L$ and $\tilde{\pi}_t^H$ the time-varying parts determined at time 0 for the low- and the high-risk type, respectively. In general, the payments of the hedge provider, both the fixed and floating leg, could be any cash flows linked to the mortality experience of the hedger's portfolio. In this paper, we follow a popular choice in the existing literature and let FLT_t^i be the t-year realized survival probability, ${}_tp_x^i$, $i = L, H$, and FIX_t^i be the corresponding best estimate, i.e., ${}_t\hat{p}_x^L$ for the low-risk type and ${}_t\hat{p}_x^H$ for the high-risk type (Dawson *et al.*, 2010). In this paper, we consider both a monopolistic and a competitive market setting. In the monopolistic market, a hedger can only conclude a swap with one hedge provider, while in the competitive market, a hedger can choose between multiple hedge providers offering the same types of swaps.

In order to focus on the annuity portfolios, we assume that the hedge providers have no pre-existing liability, and their profit solely depends on the aggregate net payments from trading the longevity swaps. Therefore, in the competitive market, the hedge providers can be considered homogeneous since they have identical profit functions. Further, when both types of swaps are available to the hedger, we consider for the moment that the hedger will purchase the contract design for its risk type, i.e., it will not have the incentive to misrepresent and take the contract designed for the other type.[7] Formally, let $M^i = z_i^A \, l_x$, where z_i^A is the hedge rate of the type-i hedger, the expected profit of the hedge provider is given by:

$$\begin{aligned}
&\mathbb{E}_P[PR_{HP}^A] \\
&= \mathbb{E}_P\left[z_L^A \, \mathbb{1}_{\{L\}} \, l_x \sum_{t=1}^{\omega_x} e^{-rt} \left[(1+\pi_t^L)_t\hat{p}_x^L - {}_tp_x^L \right] + z_H^A \, \mathbb{1}_{\{H\}} l_x \sum_{t=1}^{\omega_x} e^{-rt} \left[(1+\pi_t^H)_t\hat{p}_x^H - {}_tp_x^H \right] \right] \\
&= \epsilon \, z_L^A \, \alpha_L^A \, l_x \sum_{t=1}^{\omega_x} e^{-rt} \tilde{\pi}_t^L \, {}_t\hat{p}_x^L + (1-\epsilon) \, z_H^A \, \alpha_H^A \, l_x \sum_{t=1}^{\omega_x} e^{-rt} \tilde{\pi}_t^H \, {}_t\hat{p}_x^H \qquad (2.1) \\
&= \epsilon \, z_L^A \, \alpha_L^A \, \mathcal{B}^L + (1-\epsilon) \, z_H^A \, \alpha_H^A \, \mathcal{B}^H,
\end{aligned}$$

[7] In principle, the hedger of risk type i can deviate to the contract designed for the other type. We ignore this possible choice for now, and will later work out the conditions which justify the negligence.

with $\mathcal{B}^i = \sum_{t=1}^{\omega_x} e^{-rt} \tilde{\pi}_t^i \hat{l}_{x+t}^i$ for $i = L, H$, and $\mathbb{1}_{\{L\}}$ (resp. $\mathbb{1}_{\{H\}}$) the indicator function which equals to 1 if the hedger is the low- (resp. high-) risk type and 0 otherwise. The optimization problem in both market settings will be discussed in the next section.

3 The Principal-Agent Problem

This section introduces the principal-agent model, which is widely used in economic analysis (see, for example, Mu *et al.*, 2016; Aguirre & Beitia, 2017), between the hedge providers and the hedger under information asymmetry. Compared to the mature risk transfer markets, such as the interest risk or the credit risk market, the longevity risk market is underdeveloped and the analysis of this is somewhat underappreciated. The market is incomplete and not many longevity risk transfer solutions are currently available, especially for the small- and medium-size hedgers. The fact that this market is far from efficient and transparent makes risk-neutral pricing, which is applied to the other more liquid derivatives, not applicable. Hence, the principal-agent model, which is utility based, is a good candidate for valuing longevity-linked products. Sections 3.1 and 3.2 discuss the monopolistic and the competitive market settings, respectively, and Section 3.3 investigates the case in which the hedge providers do not address information asymmetry through a Stackelberg game. Discussions of the theoretical results are provided in Section 3.4.

3.1 Monopolistic Market Setting

In the monopolistic setting, there is only one hedge provider offering the swaps. We follow typical principal-agent optimization problems (see, for example, Stiglitz, 1977; McAfee & McMillan, 1986), and assume that the swap transactions will be executed in the following order.

1. The hedge provider announces the two types of swaps, with the specified risk premiums and hedge rates (α_i^A, z_i^A), $i = L, H$, and commits to them. The superscript A stands for **A**symmetric information.

2. The hedger announces its type.

3. The hedger will enter the swap transaction if doing so does not lower its utility level. The hedger will buy the swap which gives it the higher utility level.

The optimal contract parameters will be determined by the hedge provider under a set of participation and incentive constraints, which ensure that the hedger is willing to participate in the swap transaction and will not misrepresent its type. In order to come to these constraints, let us first write down the random net profit function of both types as a function of the hedge rate and the risk premium. Assuming that the hedger can only purchase swaps of one type,

the random profit function of the type-i hedger purchasing the type-j swap with risk premium α_j^A and hedge rate z_j^A is given by:

$$PR^i(\alpha_j^A, z_j^A) = \left(A - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}^i\right) + z_j^A \, l_x \sum_{t=1}^{\omega_x} e^{-rt} \left[{}_tp_x^i - (1+\alpha_j^A \, \tilde{\pi}_t^j)\,{}_t\hat{p}_x^j\right]. \tag{3.1}$$

The superscript i in PR^i means the hedger is of type i, and (α_j^A, z_j^A) means the hedger purchases the type-j swap (j could be the same as i). In the following sections, we consider the mean-variance preference for the hedger:

$$U\big(PR^i(\alpha_j^A, z_j^A)\big) = \mathbb{E}_{\mathbb{P}}\big[PR^i(\alpha_j^A, z_j^A)\big] - \frac{1}{2}\gamma Var_{\mathbb{P}}\big[PR^i(\alpha_j^A, z_j^A)\big], \; i, j = L, H,$$

where γ describes the degree of risk aversion.[8] Mean-variance preference is a widely used preference in decision theory and portfolio choice literature (see e.g. Ormiston & Schlee, 2001). In particular, it is shown to be fully compatible with the expected utility theory under certain conditions (Sinn, 1990). Furthermore, it can approximate decisions of a wide variety of (concave) utility functions (Kroll *et al.*, 1984; Markowitz, 2014). In this paper, we consider a natural characterization of the risk types: the high-risk type hedger's outstanding liability is higher in both the expectation and variance than that of the low-risk type. The robustness of the results obtained with the mean-variance preference will be examined with the exponential utility function in Section 5.

Before getting to the optimal contracting under information asymmetry, we derive the optimal solution under *perfect information*, i.e., when the hedge provider knows exactly the risk type of the hedger. With perfect information, Chen *et al.* (2022a) show that it is optimal for the hedge provider to negotiate with each type of hedger separately, and sell them the swap designed for their own type. The resulting optimal hedge rates and the risk premiums, which we refer to as the first-best solution, are summarized in the following Proposition.

Proposition 1 (Optimal contracting with perfect information)**.** *The optimization problem under perfect information for each hedger $i \in \{L, H\}$ is given by:*

$$\max_{\alpha_i^P, z_i^P} \; \mathbb{E}_P[PR_{HP}^P] \quad \textit{subject to} \tag{3.2}$$

$$U\big(PR_i^P(\alpha_i^P, z_i^P)\big) \geq \bar{U}^i, \tag{3.3}$$

[8] The hedger can be risk-averse due to multiple reasons, including the damages to its reputational risk and the stakeholder values in the case of failing to meet the annuity commitment, the cost of bankruptcy/reorganization, or solvency constraints.

where

$$PR^P_{HP} = z_i^P\, l_x^i \sum_{t=1}^{\omega_x} e^{-rt}\Big[\big(1+\alpha_i^P\,\tilde{\pi}_t^i\big)_t\hat{p}_x^i - {}_tp_x^i\Big],$$

$$PR_i^P(\alpha_i^P, z_i^P) = \left(A - \sum_{t=1}^{\omega_x} e^{-rt} l_{x+t}^i\right) + z_i^P\, l_x^i \sum_{t=1}^{\omega_x} e^{-rt}\Big[{}_tp_x^i - \big(1+\alpha_i^P\,\tilde{\pi}_t^i\big)_t\hat{p}_x^i\Big],\ \textit{and}$$

$\tilde{\pi}_t^i$ *is the hedger specific risk loading. The optimal solution is given by*

$$z_i^{(P,*)} = 1,\ \alpha_i^{(P,*)} = \frac{\gamma\mathcal{V}^i}{2\mathcal{B}^i},\ \textit{where}\ \mathcal{V}^i = l_x^2\ Var\left(\sum_{t=1}^{\omega_x} e^{-rt}{}_tp_x^i\right).$$

Proof. The proof can be found in Proposition 2 in Chen *et al.* (2022a). □

The superscript P in the proposition above refers to **P**erfect information. With perfect information, we see that both risk types opt for a full indemnity hedge ($z_i^{(P,*)} = 1,\ i = L, H$). Moreover, the risk premiums depend only on the hedger's own portfolio, since each contract is negotiated individually.

Next, we proceed to deriving the optimal contracts *in presence* of information asymmetry. To this end, we introduce the constraints for the hedge provider's optimization problem. First, given a risk type (high/low), the hedger owns different investment possibilities. Hence, in general, it is not necessarily the case that a hedger will honestly announce its type, and purchase the "right" swap. Instead, the hedger will not misrepresent its type only if the following incentive constraints are satisfied:

$$U\big(PR^i(\alpha_i^A, z_i^A)\big) \geq U\big(PR^i(\alpha_j^A, z_j^A)\big),\ \text{ with } i,j = L,H \text{ and } i \neq j,$$

that is, the hedger could receive a higher, or at least equal, utility level by choosing the swap designed for its type. Second, both risk types are assumed to have a reservation utility level, denoted by $\bar{U}^L$ and $\bar{U}^H$, respectively, representing the corresponding type's utility level without trading any swap. The hedger will participate in the swap transaction only if doing so does not lead to an utility level lower than the reservation level, namely, if the following participation constraints are satisfied:

$$U\big(PR^i(\alpha_i^A, z_i^A)\big) \geq \bar{U}^i,\ \text{ with } i,j = L,H.$$

Under the constraints above, the optimization problem of the hedge provider is given by:

$$\max_{\alpha_L^A, z_L^A, \alpha_H^A, z_H^A} \mathbb{E}_P[PR^A_{HP}] \quad \text{subject to} \tag{3.4}$$

$$U\big(PR^L(\alpha_L^A, z_L^A)\big) \geq \bar{U}^L, \tag{PC1}$$

$$U\big(PR^H(\alpha_H^A, z_H^A)\big) \geq \bar{U}^H, \tag{PC2}$$
$$U\big(PR^L(\alpha_L^A, z_L^A)\big) \geq U\big(PR^L(\alpha_H^A, z_H^A)\big), \tag{IC1}$$
$$U\big(PR^H(\alpha_H^A, z_H^A)\big) \geq U\big(PR^H(\alpha_L^A, z_L^A)\big). \tag{IC2}$$

Typically, (PC1) and (IC2) will be binding at the optimal solution, if it exists. This means that the low-risk type will be indifferent between participating or not, and the high-risk type will have no incentive to deviate to the contract offered to the low-risk type at the optimal solution. Classic literature on adverse selection (Rothschild & Stiglitz, 1978) shows that, if the above conditions are not satisfied, the hedge provider could always increase its expected profit by changing the prices of the two products. As for the conditions (PC2) and (IC1), they are satisfied with inequality and do not need to be taken into consideration. After the optimal contracts are derived, we will show that these two conditions are indeed satisfied.

We proceed with the optimal contracting problem as follows: we first assume that the two conditions (PC1) and (IC2) are binding, based on which the first order and complementary slackness conditions are derived. In other words, we assume the existence of the separating equilibrium, where the hedgers will choose the swaps tailor-made to their type. We then examine the existence of the solution which satisfies conditions (PC2) and (IC1). With (PC1) and (IC2) binding, we get the following Lagrangian function:

$$\begin{aligned}\mathcal{L} = {} & \epsilon\, z_L^A\, \alpha_L^A\, \mathcal{B}^L + (1-\epsilon)\, z_H^A\, \alpha_H^A\, \mathcal{B}^H + \lambda_1^A \Big(U\big(PR^L(\alpha_L^A, z_L^A)\big) - \bar{U}^L \Big) \\ & + \lambda_2^A \Big(U\big(PR^H(\alpha_H^A, z_H^A)\big) - U\big(PR^H(\alpha_L^A, z_L^A)\big) \Big)\end{aligned} \tag{3.5}$$

with non-negative λ_1^A and λ_2^A. Based on the Lagrangian function, the solution to the optimization problem (3.4) is summarized in the following Proposition.

Proposition 2 (Optimal contracting with adverse selection)**.** *The solution to the optimization problem* (3.4) *is given by:*

- $z_L^{(A,*)} = 1 + \frac{(1-\epsilon)(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)}{\gamma\left((1-\epsilon)\mathcal{V}^H - \mathcal{V}^L\right)}$, $\alpha_L^{(A,*)} = \frac{\gamma \mathcal{V}^L}{2\mathcal{B}^L}\left(1 - \frac{(1-\epsilon)(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)}{\gamma\left((1-\epsilon)\mathcal{V}^H - \mathcal{V}^L\right)}\right)$,
- $z_H^{(A,*)} = 1$, $\alpha_H^{(A,*)} = \frac{\gamma \mathcal{V}^H}{2\mathcal{B}^H} + \frac{z_L^{(A,*)}}{\mathcal{B}^H}(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) + \frac{\gamma}{2\mathcal{B}^H}\big((z_L^{(A,*)})^2 - 2z_L^{(A,*)}\big)(\mathcal{V}^H - \mathcal{V}^L)$,
- $\lambda_1^{(A,*)} = 1$ *and* $\lambda_2^{(A,*)} = 1 - \epsilon$,

with $\hat{\mathcal{D}}^i = \sum_{t=1}^{\omega_x} e^{-rt} \hat{l}_{x+t}^i$, $\mathcal{B}^i = \sum_{t=1}^{\omega_x} e^{-rt} \tilde{\pi}_t^i\, \hat{l}_{x+t}^i$, *and* $\mathcal{V}^i = l_x^2\, Var(\sum_{t=1}^{\omega_x} e^{-rt}{}_t p_x^i)$ *for* $i = L, H$.

Proof. The proof can be found in Appendix C.1. □

Based on the characterization of low and high-risk type, it holds that $\hat{\mathcal{D}}^H > \hat{\mathcal{D}}^L$ and $\mathcal{V}^H > \mathcal{V}^L$. Furthermore, we only consider optimal hedge rates that are between 0 and 1, i.e., $z_L^{(A,*)} \in [0, 1]$

and $z_H^{(A,*)} \in [0,1]$. This assumption resembles the real-world situations in which the hedge provider offers partial or full longevity hedge to the life insurers or pension funds, and does not offer short-sell or over-hedge opportunities of the indemnity swap (or other reinsurance products). This assumption is also typical in the reinsurance literature, such as Højgaard & Taksar (1998) and Gu *et al.* (2010).

From Proposition 2, we see that the optimal risk premium $\alpha_H^{(A,*)}$ of the high-risk type is lower than the first-best case, which means the high-risk type hedger enjoys a cheaper swap due to information asymmetry. Consequently, the high-risk type hedger will opt for a full hedge, as in the first-best case. On the other hand, the optimal hedge rate (resp. risk premium) of the low-risk type hedger is reduced (resp. increased) by the same percentage adjustment factor, $\frac{(1-\epsilon)(\hat{\mathcal{D}}^H-\hat{\mathcal{D}}^L)}{\gamma\left(\mathcal{V}^L-(1-\epsilon)\mathcal{V}^H\right)}$, respectively. When this adjustment factor is bounded by 0 and 1, it holds that $z_L^{(A,*)} \in [0,1]$ and $\alpha_H^{(A,*)}$ lower than the first-best value. In other words, in order to prevent the high-risk type from deviating, the hedge provider needs to charge a higher risk premium for the low-risk type swap, which causes the low-risk type hedger to choose a lower hedge rate. To keep the adjustment factor within the desired range, the following conditions need to hold:

$$\mathcal{V}^H < \frac{\mathcal{V}^L}{(1-\epsilon)}, \text{ and} \tag{3.6}$$

$$\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L \leq \frac{\gamma\left(\mathcal{V}^L-(1-\epsilon)\mathcal{V}^H\right)}{1-\epsilon}. \tag{3.7}$$

Several observations can be made. First, from Constraint (3.6), we see that $\mathcal{V}^H$ cannot be substantially larger than $\mathcal{V}^L$ for a fixed ϵ. Second, the difference in $\mathcal{V}^H$ and $\mathcal{V}^L$ can be larger if ϵ is large, i.e., the probability of the hedger being the high-risk type is small. Third, Constraint (3.7) indicates that, the difference between $\hat{\mathcal{D}}^H$ and $\hat{\mathcal{D}}^L$ cannot exceed a threshold dependent on $\mathcal{V}^H$, $\mathcal{V}^L$, ϵ, and γ. That is, when the expected liability of the two risk types are too different, $\alpha_L^{(A,*)}$ has to be so high that the low-risk type finds the longevity swap not as attractive as a hedging instrument. In fact, when Constraint (3.7) is violated, $\alpha_H^{(A,*)}$ would be larger than its first-best value, and the participation constraint of the high-risk type hedger would be violated:

$$\begin{aligned}
&U(PR^H(\alpha_H^{(A,*)}, z_H^{(A,*)})) - \bar{U}^H = -\alpha_H^{(A,*)}\, z_H^{(A,*)}\, B^H - \frac{1}{2}\gamma((z_H^{(A,*)})^2 - 2z_H^{(A,*)})\mathcal{V}^H \\
&= -\alpha_H^{(A,*)}\, B^H + \frac{\gamma\mathcal{V}^H}{2} \\
&= -\left(\frac{z^L}{\mathcal{B}^H}(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) + \frac{\gamma}{2\mathcal{B}^H}\left((z_L^{(A,*)})^2 - 2z_L^{(A,*)}\right)(\mathcal{V}^H - \mathcal{V}^L)\right) < 0,
\end{aligned}$$

and thus no solution exists to the principal-agent problem. In other words, when the expected liability of the high-risk type hedger is too large, the hedge provider would have to charge higher risk premiums for both types to ensure that they do not deviate. However, due to the

overly high-risk premium, even the high-risk type hedger would not be willing to purchase any swap at all, and thus the market collapses.

All in all, the optimal hedge rate of the low-risk type will fall outside the $[0, 1]$ interval when the two risk types are "too" different, i.e., when $\mathcal{D}^H$ and/or $\mathcal{V}^H$ are substantially larger than $\mathcal{D}^L$ and/or $\mathcal{V}^L$. This finding is consistent with the widely discussed result that information asymmetry may cause market failure (see, among many others, Stiglitz, 2000; Spence, 2002, and the references therein). Similar findings are documented for the longevity risk transfer market, see e.g. Biffis & Blake (2013). Since the two risk types are determined only by information not observed by the hedge provider, they are not likely to be substantially different from each other after all observable information is taken into account. Indeed, in the subsequent numerical analyses, we will see that moderate differences of the mortality law between the two risk types will lead to reasonable values of $z_L^{(A,*)}$.

3.2 Competitive Market Setting

In this subsection, we assume a competitive market setting, i.e., there are multiple hedge providers which offer indemnity longevity swaps to the hedger. Further, the hedge providers offer swaps as long as doing so does not lower their expected profit. This means that the expected profit of a swap is non-negative. This assumption is in line with Rothschild & Stiglitz (1978). Now that there are multiple contract specifications available, the hedger can choose the combination which results in the highest utility. Similar to the monopolistic market setting, we derive the optimal contract parameters in the *absence* of information asymmetry before analyzing the impact of asymmetric information.

As in the monopolistic setting, the expected profit of the hedge provider by offering the indemnity swap designed for risk type i is given by $\mathbb{E}_P[PR_{HP}^{CP}] = z_i^{CP}\alpha_i^{CP}\mathcal{B}^i$. Further, the expected utility and reservation utility of hedger $i \in \{L, H\}$ are given by:

$$\bar{U} = E[u(PR^i(0,0))] = A - \hat{\mathcal{D}}^i - \frac{1}{2}\gamma\mathcal{V}^i \text{ and} \tag{3.8}$$

$$\mathbb{E}_P[u(PR^i(\alpha_i^{CP}, z_i^{CP}))] = A - \hat{\mathcal{D}}^i - z_i^{CP}\alpha_i^{CP}\mathcal{B} - \frac{1}{2}\gamma(1 - z_i^{CP})^2\mathcal{V}^i. \tag{3.9}$$

Note that CP denotes the parameters in the **C**ompetitive market setting under **P**erfect information.

In contrast to the monopolistic market setting, all hedge providers earn zero expected profit in the market equilibrium due to the obvious reason that, if a hedge provider earns a positive expected profit, then a new entrant offering the identical swaps with a slightly lower price will get all the market shares. With the zero profit constraint, the optimization problem of the

hedger is given by:

$$\max_{\alpha_i^{CP}, z_i^{CP}} \ \mathbb{E}_P[u(PR^i(\alpha_i^{CP}, z_i^{CP}))] \quad \text{subject to} \tag{3.10}$$
$$\mathbb{E}_P[PR_{HP}^{CP}(z_i^{CP}, \alpha_i^{CP})] = 0.$$

The optimal solution of the optimization problem above is summarized in the following proposition.

Proposition 3 (Optimal contracting with perfect information in a competitive market)**.** *The optimal solution of the maximization problem* (3.10) *is given by:*

$$z_i^{(CP,*)} = 1,\ \alpha_i^{(CP,*)} = 0.$$

Proof. The derivations are collected in Appendix D.1. □

As in the monopolistic market, we observe that, under perfect information, both risk types obtain a perfect hedge. However, the risk loading is zero in the competitive market setting due to the zero expected profit constraint.

Next, we investigate the impact of information asymmetry between the hedger and the hedge provider. In a competitive insurance market under information asymmetry, Rothschild & Stiglitz (1978) and Dionne *et al.* (2000) show that the high-risk type will always opt for a full hedge, and the optimal risk loading is zero due to the zero profit constraint of the hedge providers. In this case, the low-risk type hedger maximizes its expected utility subject to two constraints. First, the high-risk type has no incentive to deviate to the contract provided to the low-risk type. Second, the expected profit of the hedge provider shall be zero. The optimization problem is thus given by:

$$\max_{z_L^{CA}, \alpha_L^{CA}} \ \mathbb{E}_P\big[u\big(PR_L^{CA}(z_L^{CA}, \alpha_L^{CA})\big)\big], \text{ where} \tag{3.11}$$
$$\mathbb{E}_P\big[u\big(PR^H(\alpha_H^{CA}, z_H^{CA})\big)\big] \geq \mathbb{E}_P\big[u\big(PR^H(\alpha_L^{CA}, z_L^{CA})\big)\big], \tag{IC2}$$
$$\mathbb{E}_P[PR_{HP}^{P}(z_L^{CA}, \alpha_L^{CA})] = 0, \tag{Zero Profit}$$

where $\alpha_H^{CA} = 0$ and $z_H^{CA} = 1$. CA denotes the parameters in the **C**ompetitive market setting under **A**symmetric information. The optimal solution of Problem (3.11) is summarized in the following proposition.

Proposition 4 (Optimal contracting with information asymmetry in a competitive market)**.** *The optimal solution of the maximization problem* (3.11) *is given by:*

$$z_H^{(CA,*)} = 1,\ \alpha_H^{(CA,*)} = 0, \alpha_L^{(CA,*)} = 0,$$

$$z_L^{(CA,*)} = 1 + \frac{\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L - \sqrt{(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)^2 + 2\gamma\mathcal{V}^H(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)}}{\gamma\mathcal{V}^H},$$

$$\lambda_1^{(CA,*)} = \frac{\gamma(1 - z_L^{(CA,*)})\mathcal{V}^L}{\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L + \gamma(1 - z_L^{(CA,*)})\mathcal{V}^H} \in (0,1) \quad \textit{and} \quad \lambda_2^{(CA,*)} = 1 - \lambda_1^{(CA,*)}.$$

Proof. The derivations are collected in Appendix D.2. □

Similar to the monopolistic market, we observe that partial hedge is optimal for the low-risk type. Moreover, $z_L^{(CA,*)} \in [0,1]$ when $\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L - \sqrt{(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)^2 + 2\gamma\mathcal{V}^H(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)} \in [-1,0]$. Further, both risk loadings are zero due to the zero profit constraint. This is consistent with classical economic theory under asymmetric information: The high-risk type obtains full coverage, paying the high price that is fair to him and makes the incentive constraint binding. On the other hand, the low-risk type type receives a partial hedge as a higher hedge rate would attract the high-risk type and thus breaks the incentive constraint.

3.3 Disregarding Information Asymmetry

In this section, we investigate the optimal contracts in both market settings when information asymmetry is neglected by the hedge providers, i.e., we consider the case in which the hedge providers do not know the type of the hedger, neither do they wish to address the adverse selection issue. Instead, they only offer a single longevity swap for both risk-types. To this end, we propose a Stackelberg game to determine the optimal risk loading and hedge rates. The superscript S, which denotes the **S**tackelberg game, is added to the contract parameters in this case.

Stackelberg game is a strategic game widely applied in economics (see, among many others, Maharjan *et al.*, 2013; Sinha *et al.*, 2013; Li & Sethi, 2017). We assume the hedge provider is the Stackelberg leader, and the hedger is the Stackelberg follower. The resulting contract will be a subgame perfect Nash equilibrium, which serves both the hedge provider and hedger best. The Stackelberg game proceeds as follows.

1. The hedge provider stipulates the premium for the longevity swap, α^S.
2. Based on the given α^S, the hedger chooses the hedge rate z_i^S based on its type. The chosen z_i^S will maximize its utility $U\big(PR^i(z_i^S, \alpha^S)\big)$.
3. The third step depends on the underlying market. In case of a monopolistic market, the hedge provider maximizes its expected profit by choosing the optimal risk premium, taking into account the optimal responses of the hedger, i.e. the hedge rate as a function of α^S. In a competitive market, the hedge provider chooses the optimal risk premium such that the expected profit is zero.

Finally, since the hedge provider only offers a common longevity swap in the Stackelberg game, we assume that this common swap is written on a synthetic portfolio, $l_{x+t} = \epsilon\, l^L_{x+t} + (1-\epsilon) l^H_{x+t}$, $t = 1, 2, ..., \omega_x$. In other words, at each time t, the number of policyholders alive in the synthetic portfolio is the average of the low-risk and the high-risk type portfolio, weighted by their respective probabilities. The time-0 best-estimate survival probabilities of this synthetic portfolio is defined as ${}_t\hat{p}_x = \frac{\hat{l}_{x+t}}{l_x}$, $t = 1, 2, ..., \omega_x$.

Monopolistic Market

We now derive the optimal contracts in the monopolistic market setting. In the first step, the hedger chooses a hedge rate z_i^S to maximize its expected utility given an exogenous α^S. The optimization problem has the following form:

$$\begin{aligned} &\max_{z_i^S} \; U\big(PR_i^S(z_i^S, \alpha^S)\big), \text{ where} \\ &PR_i^S = \left(A - \sum_{t=1}^{\omega_x} e^{-rt} l^i_{x+t}\right) + z_i^S l_x \sum_{t=1}^{\omega_x} e^{-rt}\left[{}_tp^i_x - \big(1+\alpha^S\tilde{\pi}^S_t\big){}_t\hat{p}_x\right]. \end{aligned} \tag{3.12}$$

Based on the optimal hedge rate of the hedger, the hedge provider then maximizes its expected profit by choosing the optimal α^S.

At this stage, the hedge provider has two options: it could offer a lower risk premium so that the hedger would purchase the swap regardless of its risk type, or it could set a higher risk premium at which only the high-risk type hedger will be interested in buying the swap. With the former option, the hedge provider enjoys a larger expected transaction volume in exchange of a lower price, whereas the opposite holds for the latter option. A priori, it is unclear which option is better. Hence, we will discuss both of them. The optimization problem of the first option is given by:

$$\begin{aligned} &\max_{\alpha^S} \; \mathbb{E}_P[PR^S_{HP}], \text{ where} \\ &PR^S_{HP} = z^S_L \mathbb{1}_{\{L\}} l_x \sum_{t=1}^{\omega_x} e^{-rt}\left[\big(1+\alpha^S\tilde{\pi}^S_t\big){}_t\hat{p}_x - {}_tp^L_x\right] \\ &\qquad + z^S_H \mathbb{1}_{\{H\}} l_x \sum_{t=1}^{\omega_x} e^{-rt}\left[\big(1+\alpha^S\tilde{\pi}^S_t\big){}_t\hat{p}_x - {}_tp^H_x\right]. \end{aligned} \tag{3.13}$$

With the second option, the hedge provider sets an α^S so high that the low-risk type hedger will find it too expensive to participate. Formally, denote by $\tilde{\alpha}$ the threshold risk premium that makes the low-risk type hedger indifferent between participating or not (i.e., $z_L^{(S,*)} = 0$). If $\alpha^{(S,*)} > \tilde{\alpha}$, the low-risk type hedger will not enter the market. The optimization problem of

the hedge provider in this case is given by:

$$\begin{aligned} &\max_{\alpha^S} \ \mathbb{E}_P[PR^S_{HP}], \text{ subject to} \\ &\quad \alpha^S \geq \tilde{\alpha}, \text{where} \\ PR^S_{HP} &= z^S_H \mathbb{1}_{\{H\}} l_x \sum_{t=1}^{\omega_x} e^{-rt} \Big[\big(1 + \alpha^S \tilde{\pi}^S_t\big)_t \hat{p}_x - {}_t p^H_x \Big]. \end{aligned} \tag{3.14}$$

After solving the two optimization problems above, the hedge provider will then compare the resultant expected profits, and track down the final optimal risk loading α^S. The optimal solution of the optimization problems above is summarized in the following proposition.

Proposition 5 (Optimal contracting of the Stackelberg game)**.** *The optimal solution of the maximization problem* (3.12)-(3.13) *is given by:*

$$\alpha^{(S,*)} = \frac{\frac{1}{2}\gamma \mathcal{V}^H \mathcal{V}^L + \epsilon(\hat{\mathcal{D}}^L - \hat{\mathcal{D}})\mathcal{V}^H + (1-\epsilon)(\hat{\mathcal{D}}^H - \hat{\mathcal{D}})\mathcal{V}^L}{\mathcal{B}^S\big(\epsilon \mathcal{V}^H + (1-\epsilon)\mathcal{V}^L\big)} \text{ and} \tag{3.15}$$

$$z_i^{(S,*)} = 1 + \frac{\hat{\mathcal{D}}^i - \hat{\mathcal{D}} - \alpha^{(S,*)}\mathcal{B}^S}{\gamma \mathcal{V}^i}, \text{ for } i = L, H, \tag{3.16}$$

where $\hat{\mathcal{D}} = \sum_{t=1}^{\omega_x} e^{-rt} \hat{l}_{x+t}$, *and* $\mathcal{B}^S = l_x \sum_{t=1}^{\omega_x} e^{-rt} \tilde{\pi}^S_t {}_t\hat{p}_x$. *The threshold risk premium is* $\tilde{\alpha} = \frac{\hat{\mathcal{D}}^L - \hat{\mathcal{D}} + \gamma \mathcal{V}^L}{\mathcal{B}^S}$.

The optimal solution of the maximization problem (3.12) *and* (3.14) *exists only if* $\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L \geq \gamma \mathcal{V}^L - \frac{1}{2}\gamma \mathcal{V}^H$, *in which case is given by:*

$$\begin{aligned} \alpha^{(S,*)} &= \frac{\frac{1}{2}\gamma \mathcal{V}^H + (\hat{\mathcal{D}}^H - \hat{\mathcal{D}})}{\mathcal{B}^S} \\ z_L^{(S,*)} &= 0 \text{ and } z_H^{(S,*)} = 1 + \frac{\hat{\mathcal{D}}^H - \hat{\mathcal{D}} - \alpha^S \mathcal{B}^S}{\gamma \mathcal{V}^H} = \frac{1}{2}. \end{aligned}$$

Proof. The derivations are collected in Appendix C.2. □

Hence, if $\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L < \gamma \mathcal{V}^L - \frac{1}{2}\gamma \mathcal{V}^H$, i.e., the difference between the expected liability of the two risk types is not too large compared to the difference in the volatility, it is optimal for the hedge provider to set a lower premium and attract both risk types. In fact, the constraint $\alpha^S \geq \tilde{\alpha}$ will be violated when $\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L < \gamma \mathcal{V}^L - \frac{1}{2}\gamma \mathcal{V}^H$ and hence the optimization problem (3.14) has no solution. On the other hand, when $\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L \geq \gamma \mathcal{V}^L - \frac{1}{2}\gamma \mathcal{V}^H$, the hedge provider will compare the two options and select the $\alpha^{(S,*)}$ that leads to the higher expected profit.

Competitive Market

Next, we derive the optimal contract parameters in the competitive market, in which multiple hedge provider offers identical longevity swaps. Note that CS denotes the parameters in the **C**ompetitive market setting under the **S**tackelberg game. Similar, to the monopolistic market, the Stackelberg game follows a two-step optimization procedure. First, the hedger chooses a hedge rate z_i^{CS} to maximize its utility level given an exogenous α^{CS}, cf. maximization problem (3.12). Based on the optimal hedge rate of the hedger, the hedge provider then selects the optimal α^{CS} such that its expected profit is zero, i.e.,

$$\begin{aligned}\mathbb{E}_P[PR_{HP}^{CS}] = \mathbb{E}_P\Bigg[& z_L^{CS}\mathbb{1}_{\{L\}} l_x \sum_{t=1}^{\omega_x} e^{-rt}\left[\left(1+\alpha^{CS}\tilde{\pi}_t^{CS}\right){}_t\hat{p}_x - {}_tp_x^L\right] \\ & + z_H^{CS}\mathbb{1}_{\{H\}} l_x \sum_{t=1}^{\omega_x} e^{-rt}\left[\left(1+\alpha^{CS}\tilde{\pi}_t^{S}\right){}_t\hat{p}_x - {}_tp_x^H\right]\Bigg] = 0. \end{aligned} \tag{3.17}$$

The optimal contracts are summarized in the following proposition.

Proposition 6 (Optimal contracting of the Stackelberg game in a competitive market)**.** *Assuming that $z_i^{CS} \in [0,1]$ for $i = L, H$. The optimal solution of the maximization problem* (3.12) *and the zero profit constraint* (3.17) *is given by:*

$$z_H^{(CS,*)} = 1,\ \alpha^{(CS,*)} = \frac{\hat{\mathcal{D}}^H - \hat{\mathcal{D}}}{\mathcal{B}},\ \textit{and}\ z_L^{(CS,*)} = 0.$$

Proof. The derivations are collected in Appendix D.3. □

In contrast to the monopolistic market setting, the low-risk type will not participate in the competitive market if only a single swap is offered. In particular, when both risk-types participate, there exists no risk loading such that both $z_L^{(CS,*)}$ and $z_H^{(CS,*)}$ are between 0 and 1. In other words, it is either the case in which the low-risk type short sells the swap ($z_L^{(CS,*)} < 0$) or the high-risk type over hedges ($z_H^{(CS,*)} > 1$).

3.4 Discussions

Before numerically investigating the impact of information asymmetry, this subsection summarizes some theoretical observations and comparisons between the hedge provider's expected profit and the hedger's utility improvement based on the optimal contracts derived above.

In the monopolistic market, the expected profit of the hedge provider under perfect information (based on the optimal contracts in Proposition 1) is $\frac{\gamma\mathcal{V}^i}{2}$ for $i = H, L$. From Proposition 2, the

optimal expected profit of the hedge provider in the benchmark analysis is

$$\begin{aligned}&(1-\frac{(1-\epsilon)(\hat{\mathcal{D}}^H-\hat{\mathcal{D}}^L)}{\gamma\big(\mathcal{V}^L-(1-\epsilon)\mathcal{V}^H\big)})\bigg(1+\frac{(1-\epsilon)(\hat{\mathcal{D}}^H-\hat{\mathcal{D}}^L)}{\gamma\big(\mathcal{V}^L-(1-\epsilon)\mathcal{V}^H\big)}\bigg)\frac{\gamma\mathcal{V}^L}{2}\\ =&\frac{\gamma\mathcal{V}^L}{2}-\bigg(\frac{(1-\epsilon)(\hat{\mathcal{D}}^H-\hat{\mathcal{D}}^L)}{\gamma\big(\mathcal{V}^L-(1-\epsilon)\mathcal{V}^H\big)}\bigg)^2\frac{\gamma\mathcal{V}^L}{2}\end{aligned}$$

from the low-risk type hedger, which is smaller than the first-best value $\frac{\gamma\mathcal{V}^L}{2}$. The expected profit from the high-risk type hedger is:

$$\frac{\gamma\mathcal{V}^H}{2}+z_L^{(A,*)}(\hat{\mathcal{D}}^L-\hat{\mathcal{D}}^H)+\frac{\gamma}{2}\big((z_L^{(A,*)})^2-2z_L^{(A,*)}\big)(\mathcal{V}^H-\mathcal{V}^L),$$

which is also smaller than the first-best value $\frac{\gamma\mathcal{V}^H}{2}$ when $z_L^{(A,*)}\in[0,1]$. Therefore, the hedge provider's overall expected profit is lower in the presence of information asymmetry compared to the first-best case of perfect information. Finally, regarding the Stackelberg game, the hedge provider has to allow the hedger to freely decide its hedge rate in order for the market to be functional. This additional freedom of the hedger is expected to lower the hedge provider's expected profit compared to the benchmark analysis, in which the hedge provider is able to decide both the hedge rate and the price of the swaps. We will show that this is indeed the case with the numerical analysis in Section 4.

In the competitive market, it is obvious that the expected profits of all hedge providers are zero in all the three cases. Furthermore, due to the zero profit constraint, all the longevity swaps have the actuarially fair price. In the perfect information and the benchmark analysis, it holds that $\alpha_i^{(CP,*)}=\alpha_i^{(CA,*)}=0$, $i=L,H$ (see Propositions 3 and 4). For the Stackelberg game, the fix leg of the swap depends on the expected survival probabilities of the synthetic portfolio l_{x+t} (see Section 3.3), which are lower than those of the high-risk type hedger's portfolio l_{x+t}^H. Therefore, a positive risk loading is required to make the hedge provider break even, in which case the price of the swap is still actuarially fair to the high-risk type hedger. Combining the actuarially fair price with the fact that the high-risk type hedge always opts for the full hedge, the high-risk type hedger will have the same utility improvement in all the three cases.

For the low-risk type hedger, however, the utility improvement depends on whether information asymmetry exists, and if it does, whether it is appropriately addressed. First, Proposition 4 suggests that the optimal hedge rate of the low-risk type hedger is less than 1 unless in the special case where $\hat{\mathcal{D}}^H-\hat{\mathcal{D}}^L-\sqrt{(\hat{\mathcal{D}}^H-\hat{\mathcal{D}}^L)^2+2\gamma\mathcal{V}^H(\hat{\mathcal{D}}^H-\hat{\mathcal{D}}^L)}=0$. Therefore, the utility improvement of the low-risk type hedger in the benchmark case is bounded from above by the utility improvement in the perfect information case. Second, the low-risk type hedge receives zero utility improvement in the Stackelberg game as it does not participate in the

swap transaction. All in all, we can conclude that, although worse than the first-best case of perfect information, the low-risk type hedger could benefit if information asymmetry is properly addressed.

4 Numerical Analysis

In this section, illustrative numerical analyses are performed to evaluate the optimal longevity swap contracts, as well as their impact on the hedge provider and the hedger under information asymmetry. Section 4.1 introduces the mortality model used to simulate the outstanding liabilities of the hedger, and Sections 4.2 and 4.3 discuss the numerical results, including the optimal contracts and the comparison of the hedger's utility level and the hedge provider's profit in different scenarios under the monopolistic and the competitive market settings, respectively.

4.1 The mortality model

Over the past decades, numerous mortality models have been developed to understand and forecast human mortality (see, among many others, Halicioglu, 2011; O'Hare & Li, 2014). In this paper, we use the Age-Period-Cohort-Improvement (APCI) model to simulate future survival probabilities. The APCI model is used by the CMI Mortality Projections Committee[9] to generate life tables for UK life insurers and pension funds. Specifically, denote by $m_{x,t}$ the annual death rates at age x and in year t, the APCI model describes the logarithm of the death rate as:

$$\ln(m_{x,t}) = \beta_x^{(1)} + \beta_x^{(2)}(t - \bar{t}) + \kappa_t + \gamma_c, \tag{4.1}$$

where $\beta_x^{(1)}$ is the age effect representing the average mortality level at age x, $\beta_x^{(2)}$ is the period effect representing the age-specific exposure to the linear time trend, $\bar{t}$ is a normalization parameter set to the average year in the sample, κ_t is a time-varying parameter representing the residual mortality improvements not captured by the linear trend, and finally, γ_c is the cohort-effect, i.e., the unique mortality pattern shared by the group of people born in the same year.

The parameters of Model (4.1) are estimated by the least squares approach. Specifically, let X be the number of ages and T the number of years in the data, there are $3X + 2T - 1$ parameters to be fitted. The model is fitted separately to mortality data of the United States and England from 1956 to 2016 and ages from 20 to 100. The resulting parameters are then used to simulate future mortality of the low- and the high-risk type hedger, respectively. We do not consider ages above 100 in the estimation, as the number of death is scarce and volatile

[9]Source: CMI Mortality Projections Committee *et al.* (2016).

for these ages, and thus the mortality rates could introduce large estimation errors when included. In the numerical analysis, we assume that the maximally attainable age is 120, which is typically the maximum age of life tables used by insurance companies in practice. In order to generate the mortality rates from 101 to 120, we need to extrapolate the age and the period effects. Specifically, we follow Dowd & Blake (2019) and Dowd *et al.* (2019), and extrapolate $\beta_x^{(1)}$ using a linear regression fitted to the estimated parameters for ages $x \geq 70$, and $\beta_x^{(2)}$ using the estimated parameters for age 100, i.e. $\beta_x^{(2)} = \beta_{100}^{(2)}$ for $x > 100$. In other words, we assume that the log-mortality level increases linearly for ages above 100, while their period effects are identical to that of age 100. We refer to Dowd & Blake (2019) and Dowd *et al.* (2019) for more detailed discussions on old age mortality extrapolation. The estimated parameters and the extrapolated age and period effects for both countries are shown in Appendix B.

In order to forecast future survival probabilities, the mortality improvement factor κ_t needs to be projected. To this end, we model κ_t based on an ARIMA(0,1,0) for both data sets, i.e., as a random walk

$$\kappa_t^* = \kappa_{t-1}^* + \sigma_\kappa^* \varepsilon_t,$$

where σ_κ^* is the standard deviation of the error terms, ε_t are i.i.d. standard normal distributed error-terms, and $* \in \{UK, US\}$. We obtain the estimation results: $\sigma_\kappa^{UK} = 0.020$ and $\sigma_\kappa^{US} = 0.017$. Finally, if only the existing cohorts are considered, as is the case in our analysis, the cohort effect γ_c does not need to be projected.

4.2 The Monopolistic Market Setting

In this subsection, we evaluate the hedge provider's expected profit and the hedger's utility improvement under the different scenarios (perfect information, asymmetric information and the Stackelberg game) in the monopolistic market setting.

Parameter Setup

The parameter setup of the benchmark numerical analysis is summarized in Table 1. In particular, we consider a single cohort with age 65 at time 0, and the hedger's initial portfolio size is 1,000. The probability of the hedger being the low-risk type is set to be $\epsilon = 0.5$, that is, both risk types are equally probable given the observable information of the hedger. The value of γ is in line with the values used in the existing literature (see, for example, Zhang *et al.*, 2009; Li *et al.*, 2017).

$S = 1,000$ future paths of survival probabilities are simulated using the ACPI model fitted to the UK and the US mortality data, respectively. Based on this setup, the simulated life

Risk-free rate	Initial age	Pool size	Risk aversion	Prob. of low-risk type
$r = 2\%$	$x_0 = 65$	$l_{x_0} = 1,000$	$\gamma = 0.05$	$\epsilon = 0.5$

Table 1: Base case parameter setup. [Updated]

expectancy is 20.23 years for the US data (low-risk type), and 21.76 for the UK data (high-risk type). Furthermore, the expectation and the variance of the liabilities, $\hat{\mathcal{D}}$ and $\mathcal{V}$, calculated based on the two countries, are reported in the last two columns of Table 2. We see that the expected liabilities $\hat{D}^H$ based on the UK data is roughly 6% higher and the variance of the liability's present value $\mathcal{V}^H$ is 34% higher than those based on the US data. Based on these results, we let the ACPI model fitted to the US data represent the data generating process of the low-risk type, and the model fitted to the UK data represent the high-risk type. Finally, when calculating the payoff of the swaps, the time-varying risk loading $\tilde{\pi}_t^i$ is set as the standard-deviation premium principle, $\tilde{\pi}_t^i = \sigma\left(\frac{l_{x+t}^i}{\tilde{l}_{x+t}^i}\right)$, $i = L, H$.

Risk type	$\hat{\mathcal{D}}^i$	$\mathcal{V}^i$
L (US data)	15,884	113,981
H (UK data)	16,866	153,296

Table 2: Mean and variance of the outstanding liabilities under the base case setup. [Updated]

Utility Improvement of the Hedger

Besides the expected profit of the hedge provider, we will also compare the utility improvement (compared to no hedge) of the hedger in the three cases: the benchmark **A**symmetric information case, the **P**erfect information case, and the **S**tackelberg game. Given the parameter values above, it holds that $\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L < \gamma\mathcal{V}^L - \frac{1}{2}\gamma\mathcal{V}^H$ and thus the solution of the Stackelberg game is given by (3.15) and (3.16), i.e., both risk types will purchase the longevity swap.

Let us define the utility improvement of the hedger by

$$\hat{U}_i^{\Diamond} = U_i(PR_i^{\Diamond}(\alpha_i^{(\Diamond,*)}, z_i^{(\Diamond,*)})) - \bar{U}_i$$

for $i = L, H$ and $\Diamond \in \{A, P, S\}$, i.e. the utility gain achieved from trading the swap. For the three cases, they are given by:

- Benchmark analysis: $\hat{U}_i^A = -\alpha_i^{(A,*)} z_i^{(A,*)} B^i - \frac{1}{2}\gamma\big((z_i^{(A,*)})^2 - 2z_i^{(A,*)}\big)\mathcal{V}^i$.
- Perfect information: $\hat{U}_i^P = -\alpha_i^{(P,*)} z_i^{(P,*)} B^i - \frac{1}{2}\gamma\big((z_i^{(P,*)})^2 - 2z_i^{(P,*)}\big)\mathcal{V}^i$.
- Stakelberg game: $\hat{U}_i^F = z_i^{(S,*)}\hat{\mathcal{D}}^i - z_i^{(S,*)}(\hat{\mathcal{D}} + \alpha^{(S,*)}\mathcal{B}^S) - \frac{1}{2}\gamma\big((z_i^{(S,*)})^2 - 2z_i^{(S,*)}\big)\mathcal{V}^i$.

Finally, we remark that in the case of perfect information, ϵ would be either 0 or 1 because the hedge provider knows exactly the hedger's type. However, in order to facilitate the comparison with the other two cases, we still consider the following expected profit formula in the perfect information case:

$$\mathbb{E}_P[PR^P_{HP}] = \mathbb{E}_P\Bigg[z^P_L\,\mathbb{1}_{\{L\}}\,l_x\sum_{t=1}^{\omega_x} e^{-rt}\big[(1+\alpha^P_L\tilde{\pi}^L_t)\,{}_t\hat{p}^L_x - {}_tp^L_x\big] + z^P_H\,\mathbb{1}_{\{H\}}l_x\sum_{t=1}^{\omega_x} e^{-rt}\big[(1+\alpha^P_H\tilde{\pi}^H_t)\,{}_t\hat{p}^H_x - {}_tp^H_x\big]\Bigg],$$

with $\mathbb{E}_P[\mathbb{1}_{\{L\}}] = \epsilon$ and $\mathbb{E}_P[\mathbb{1}_{\{H\}}] = 1-\epsilon$.

Numerical Results

The optimal contract parameters, the expected profit of the hedge provider, as well as the utility improvement of both risk types are shown in Table 3. First, we see $\alpha^{(A,*)}_L > \alpha^{(P,*)}_L$ and $\alpha^{(A,*)}_H < \alpha^{(P,*)}_H$. This is consistent with the theoretical results derived in Section 3, i.e., the hedge provider has to increase $\alpha^{(A,*)}_L$ and reduce $\alpha^{(A,*)}_H$ in order to prevent the high-risk type from deviating in the presence of asymmetric information. As a result, the high-risk type hedger benefits from the cheaper swap and enjoys an improvement in expected utility, whereas the low-risk type hedger would opt for a lower hedge rate due to the higher price.

The total present value of the fixed leg, $\Pi^{(\Diamond)}_i = \alpha^{(\Diamond,*)}_i \sum_{t=1}^{\omega_x} e^{-rt}\tilde{\pi}^{(\Diamond)}_t\,\hat{l}^{(\Diamond)}_{x+t}$, is also shown for each risk type for each case. First, we see that the hedge provider obtains the highest expected profit in the case of perfect information. This is not surprising, as the hedge provider can design contracts that exploit all utility improvement from both risk types. Second, and more importantly, taking the adverse selection into account and offering type-dependent swaps leads to much higher expected profit of the hedge provider under information asymmetry. This is because the benchmark principal-agent problem utilizes type-dependent swaps, and can thus extract more profit from both risk types. This is not possible in the Stackelberg game, where only one swap is offered. In particular, we see that Π^A_L is much higher than Π^A_H in the benchmark case, which indicates the value of type-dependent pricing. Another reason that the hedge provider's expected profit is lower in the Stackelberg game is that, in order for the market to exist, the hedge provider has to allow the hedger to select a hedge rate that maximizes its utility level. This is not the case in both the benchmark analysis and the perfect information case. Hence, both risk types end up with larger utility improvement in the Stackelberg game. Finally, the optimal hedge rate of the low-risk type hedger is between 0 and 1 in all cases, meaning that both conditions (3.6) and (3.7) are satisfied with the chosen parameter values.

Optimal Contracts					
Benchmark Analysis		Perfect Information		Stackelberg Game	
$z_L^{(A,*)} = 0.74$	$\alpha_L^{(A,*)} = 8.42$	$z_L^{(P,*)} = 1$	$\alpha_L^{(P,*)} = 6.67$	$z_L^{(S,*)} = 0.35$	$\alpha^{(S,*)} = 6.97$
$z_H^{(A,*)} = 1$	$\alpha_H^{(A,*)} = 4.54$	$z_H^{(P,*)} = 1$	$\alpha_H^{(P,*)} = 7.93$	$z_H^{(S,*)} = 0.65$	
Present Value of the Fixed Leg					
$\Pi_L^A = 3598.8$	$\Pi_H^A = 2194.0$	$\Pi_L^P = 2849.5$	$\Pi_H^P = 3832.4$	$\Pi_L^S = \Pi_H^S = 3196.5$	
Expected Profit					
$\mathbb{E}[PR_{HP}^A] = 2423.2$		$\mathbb{E}[PR_{HP}^P] = 3340.3$		$\mathbb{E}[PR_{HP}^S] = 1525.7$	
Utility improvement of the hedger					
$\hat{U}_L^A = 0$	$\hat{U}_H^A = 1638$	$\hat{U}_L^P = 0$	$\hat{U}_H^P = 0$	$\hat{U}_L^S = 355$	$\hat{U}_H^S = 1604$

Table 3: Optimal contract parameters of the three swaps (top panel), the expected profits of the hedge provider (mid panel) and the utility improvement of the hedger (bottom panel). [Updated]

Besides the expectation, we also investigate the distribution of the hedge provider's profit in the three cases above. The distribution of the hedge provider's profits are displayed in Figure 1 and the 0.5%, 5%, 95%, and 99.5% quantiles are shown in Table 4. We see that the profit distributions is bi-modal for the Stackelberg game and the perfect information case. The reason is that the difference between the risk premium (of the indemnity swaps) and the hedge rate (of the single swap) of the two risk types are substantially different. Consequently, the two risk types result in rather varied profits for the hedge provider. Also, for both distributions, the left peak is higher and more concentrated due to the lower and more certain profit resulting from the low-risk type. Interestingly, the bi-modal phenomenon is not observed in the benchmark case. This is because, while $z_L^{(A,*)}$ is lower than $z_H^{(A,*)}$, $\alpha_L^{(A,*)}$ is much higher than $\alpha_H^{(A,*)}$. These two aspects make the profit resulting from the low-risk type quite close to that from the high-risk type. Finally, the benchmark analysis leads to substantially higher quantiles than the Stackelberg game, cf. Table 4. Hence, we see that properly addressing adverse selection in longevity swap transaction can lead to not only a higher expected profit, but also much lower downside risk for the hedge provider.

Quantile	0.5%	5%	95%	99.5%
A	1237.5	1691.2	2995.9	3252.0
P	2037.4	2416.8	4327.3	4724.7
S	1002.4	1140.8	2070.8	2327.9

Table 4: Selected quantiles of the profit distributions of the hedge provider resulting from the benchmark analysis (A), the perfect information case (P), and the Stackelberg game (S). [Updated]

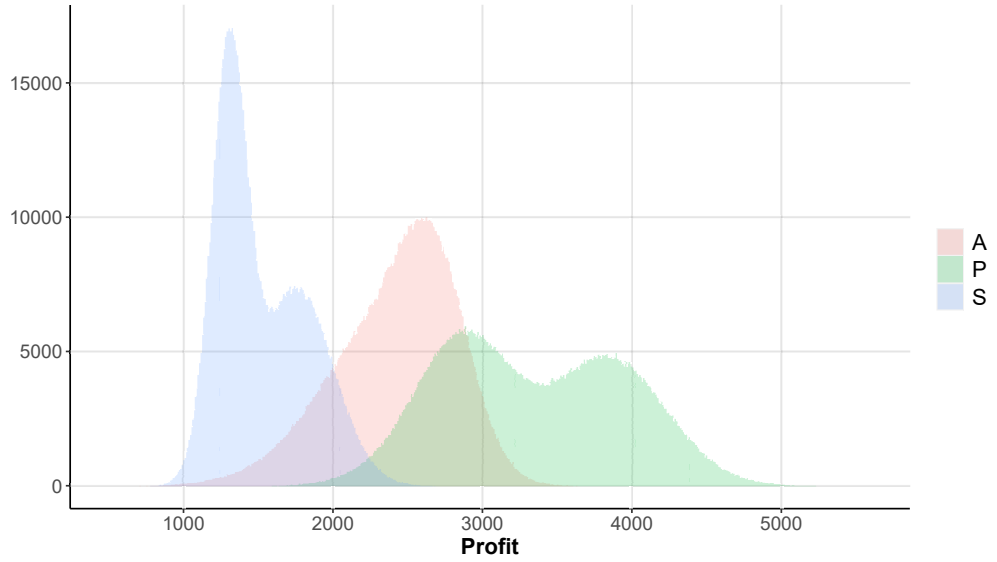


Figure 1: The histogram of the distribution of the profit of the hedge provider for each case, where A denotes the benchmark case, P the perfect information case and S the Stackelberg game. [Updated]

4.3 Competitive Market Setting

In this subsection, we repeat the analysis above under a competitive market setting, where the optimal results are derived in Section 3.2. The numerical results are shown in Table 5. We observe that the risk loading is 0 for both risk types in the benchmark case and the perfect information case. Further, due to information asymmetry, the low-risk type has a lower hedge rate in the benchmark case, resulting in a lower utility improvement compared to the perfect information case. Moreover, the high-risk type hedger has the same utility improvement in all cases since all longevity swap for its type is actuarially fair. Finally, the low-risk type hedger has no utility improvement, as it does not participate in the swap transaction. Consequently, in a competitive market setting, taking information asymmetry into account when pricing longevity swaps can significantly improve the utility of the low-risk type hedger, which is the only risk type affected by information asymmetry.

Optimal Contracts					
Benchmark Analysis		Perfect Information		Stackelberg Game	
$z_L^{(A,*)} = 0.61$	$\alpha_L^{(A,*)} = 0$	$z_L^{(P,*)} = 1$	$\alpha_L^{(P,*)} = 0$	$z_L^{(S,*)} = 0$	$\alpha^{(S,*)} = 1.07$
$z_H^{(A,*)} = 1$	$\alpha_H^{(A,*)} = 0$	$z_H^{(P,*)} = 1$	$\alpha_H^{(P,*)} = 0$	$z_H^{(S,*)} = 1$	
Utility improvement of the hedger					
$\hat{U}_L^A = 2407$	$\hat{U}_H^A = 3832$	$\hat{U}_L^P = 2850$	$\hat{U}_H^P = 3832$	$\hat{U}_L^S = 0$	$\hat{U}_H^S = 3832$

Table 5: Optimal contract parameters of the three swaps (top panel) and the utility improvement of the hedger (bottom panel) in the competitive market setting.

To sum up, we observe that information asymmetry has negative effects in both market settings:

In the monopolistic market, it decreases the expected profit of the hedge provider, while in the competitive market, it decreases the utility improvement of the low-risk type hedger. However, and more importantly, in both market settings, it is beneficial for the hedge provider or the hedger to take information asymmetry into account when conducting longevity risk transfer transactions.

5 Exponential Utility of the Hedger

Sections 3 and 4 have studied the principal agent problems withe a mean-variance preference for the hedger. This section discusses the optimal contracts when the hedger's preference is represented by an exponential utility function. Section 5.1 summarizes the theoretical results, and Section 5.2 provides the numerical analyses.

5.1 Optimal Contracts with the Exponential Utility

There is a large literature arguing that the mean-variance preference can approximate the expected utility maximization for many concave utility functions rather well (Markowitz, 2014; Kroll *et al.*, 1984). Exponential utility, which is also a commonly-used utility function in insurance pricing (Loss & Piaser, 2019), seems to make an exception here: the mean-variance preference does not generate as accurate approximation of the exponential utility as other functions (see, e.g., Markowitz (2014) for further discussions). Hence, we examine the optimal longevity risk transfer solution under the exponential utility to sufficiently generally consider the utility of the hedger. In this section, we aim to find whether the results derived in the previous sections with mean-variance preferences still hold qualitatively for the exponential utility. We consider an exponential utility given by $u(X) = -\frac{1}{\gamma}e^{-\gamma X}$ for the hedger, where $\gamma > 0$ describes the level of the absolute risk aversion and X is the present value of the hedger's annuity portfolio. Consequently, the expected utility and reservation utility of the hedger are given by:

$$\bar{U} = \mathbb{E}_P[u(PR^i(0,0))] = -\frac{1}{\gamma}e^{-\gamma A}E[e^{\gamma \mathcal{D}^i}] \text{ and} \tag{5.1}$$

$$\mathbb{E}_P[u(PR^i(\alpha_i, z_i))] = -\frac{1}{\gamma}e^{-\gamma A}e^{\gamma z_i(\hat{\mathcal{D}}^i + \alpha_i \mathcal{B}^i)}E[e^{\gamma(1-z_i)\mathcal{D}^i}] \tag{5.2}$$

for $i = L, H$, and $\mathcal{D}^i = \sum_{t=1}^{\omega_x} e^{-rt} l^i_{x+t}$ the random present value of liabilities.

Propositions 7 and 8 summarize the optimal contracts in the three cases for the monopolistic and the competitive market setting, respectively. Due to the length of the paper, a sketch of the proofs are collected in Appendix E. Complete proofs are provided in the online supplementary material of this paper. The optimal contracts are qualitatively similar to the ones obtained

with the mean-variance preference. For example, both risk-types will opt for a full hedge under perfect information, and the high-risk type will enjoy a lower risk loading in the presence of information asymmetry. However, due to the complexity of the optimization problems, no analytical solution can be obtained except for the perfect information case.

Proposition 7. *[Optimal contracting under monopoly with the exponential utility]*

1. Perfect Information

The optimal solution of the maximization problem (3.2) *based on the exponential utility is given by*

$$z_i^{(P,*)} = 1,\ \alpha_i^{(P,*)} = \frac{\frac{1}{\gamma}\ln\left(E[e^{\gamma(\mathcal{D}^i-\hat{\mathcal{D}}^i}]\right)}{\mathcal{B}^i} \quad \textit{and} \quad \lambda^{(P,*)} = e^{\gamma A}e^{-\gamma(\hat{\mathcal{D}}^i+\alpha_i^{(P,*)}\mathcal{B}^i)}.$$

2. Benchmark Analysis

The optimal solution of the maximization problem (3.4) *based on the exponential utility is given by:*

$$\begin{aligned}
\lambda_1^{(A,*)} &= \frac{e^{\gamma A}}{E[e^{\gamma\mathcal{D}^L}]}, \quad \lambda_4^{(A,*)} = \frac{(1-\epsilon)e^{\gamma A}e^{-\gamma z_H^{(A,*)}(\hat{\mathcal{D}}^H+\alpha_H^{(A,*)}\mathcal{B}^H)}}{E[e^{\gamma(1-z_H^{(A,*)})\mathcal{D}^H}]},\\
\alpha_L^{(A,*)} &= \frac{\left(\frac{1}{\gamma z_L^{(A,*)}}\ln\left(\frac{E[e^{\gamma\mathcal{D}^L}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]}\right)-\hat{\mathcal{D}}^L\right)}{\mathcal{B}^L}, z_L^{(A,*)} = \tilde{z}^A,\\
\alpha_H^{(A,*)} &= \frac{\left(\frac{1}{\gamma}\left(\ln\big(E[e^{\gamma\mathcal{D}^L}]\big)+\ln\big(\frac{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]}\big)\right)-\hat{\mathcal{D}}^H\right)}{\mathcal{B}^H} \quad \textit{and} \quad z_H^{(A,*)} = 1
\end{aligned}$$

where $\tilde{z}_L^A$ *is the solution of:*

$$\frac{E[\mathcal{D}^L e^{\gamma(1-z_L^A)\mathcal{D}^L}]}{E[e^{\gamma(1-z_L^A)\mathcal{D}^L}]} - \epsilon\hat{\mathcal{D}}^L = (1-\epsilon)\frac{E[\mathcal{D}^H e^{\gamma(1-z_L^A)\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^A)\mathcal{D}^H}]}.$$

$\alpha_H^{(A,*)} < \alpha_H^{(P,*)}$ *when* $z_L^{(A,*)} \in [0,1]$.

3. Stackelberg Game

The optimal solution of the maximization problem (3.12)-(3.13) *based on the exponential utility is given by:*

$$\alpha^{(S,*)}\mathcal{B} = \frac{E[\mathcal{D}^i e^{\gamma(1-z_i^{(S,*)})\mathcal{D}^i}]}{E[e^{\gamma(1-z_i^{(S,*)})\mathcal{D}^i}]} - \hat{\mathcal{D}} \quad \textit{for } i = L,H \quad \textit{and} \tag{5.3}$$

$$\alpha^{(S,*)}\mathcal{B} = \frac{\epsilon\left(\frac{\partial z_L^{(S,*)}}{\partial\alpha^{(S,*)}}\hat{\mathcal{D}}^L - z_L^{(S,*)}\mathcal{B}\right)+(1-\epsilon)\left(\frac{\partial z_H^{(S,*)}}{\partial\alpha^{(S,*)}}\hat{\mathcal{D}}^H - z_H^{(S,*)}\mathcal{B}\right)}{\epsilon\frac{\partial z_L^{(S,*)}}{\partial\alpha^{(S,*)}}+(1-\epsilon)\frac{\partial z_H^{(S,*)}}{\partial\alpha^{(S,*)}}} - \hat{\mathcal{D}}, \tag{5.4}$$

where

$$\frac{\partial z_i^{(S,*)}}{\partial \alpha^{(S,*)}} = \frac{\mathcal{B}}{\gamma} \frac{E[e^{\gamma(1-z_i^{(S,*)})\mathcal{D}^i}]^2}{(E[\mathcal{D}^i e^{\gamma(1-z_i^{(S,*)})\mathcal{D}^i}])^2 - E[(\mathcal{D}^i)^2 e^{\gamma(1-z_i^{(S,*)})\mathcal{D}^i}] E[e^{\gamma(1-z_i^{(S,*)})\mathcal{D}^i}]}.$$

The optimal solution of the maximization problem (3.12) *and* (3.14) *based on the exponential utility is given by:*

$$\alpha^{(S,*)}\mathcal{B} = \frac{E[\mathcal{D}^H e^{\gamma(1-z_H^{(S,*)})\mathcal{D}^H}]}{E[e^{\gamma(1-z_H^{(S,*)})\mathcal{D}^H}]} - \hat{\mathcal{D}}\, z_L^{(S,*)} = 0$$

and $z_H^{(S,*)}$ *is the solution of:*

$$\frac{E[\mathcal{D}^H e^{\gamma(1-z_H^S)\mathcal{D}^H}]}{E[e^{\gamma(1-z_H^S)\mathcal{D}^H}]} = \hat{\mathcal{D}}^H - \frac{z_H^S \mathcal{B}}{\frac{\partial z_H^S}{\partial \alpha^S}}.$$

Proposition 8. *[Optimal contracting under competition with the exponential utility]*

1. Perfect Information

The optimal solution of the maximization problem (3.10) *based on the exponential utility is given by*

$$z_i^{(CP,*)} = 1,\ \alpha_i^{(CP,*)} = 0 \quad \textit{and} \quad \lambda^{CP} = e^{-\gamma A} e^{\gamma \hat{\mathcal{D}}}.$$

2. Benchmark Analysis

The optimal solution of the maximization problem (3.11) *based on the exponential utility is given by*

$$\begin{aligned}
&\alpha_H^{(CA,*)} = 0, z_H^{(CA,*)} = 1, \alpha_L^{(CA,*)} = 0, z_L^{(CA,*)} = \tilde{z}_L^{CA}, \\
&\lambda_1^{(CA,*)} = \frac{\hat{\mathcal{D}}^L E[e^{\gamma(1-z_L^{(CA,*)})\mathcal{D}^L}] - E[\mathcal{D}^L e^{\gamma(1-z_L^{(CA,*)})\mathcal{D}^L}]}{\hat{\mathcal{D}}^L E[e^{\gamma(1-z_L^{(CA,*)})\mathcal{D}^H}] - E[\mathcal{D}^H e^{\gamma(1-z_L^{(CA,*)})\mathcal{D}^H}]} \quad \textit{and} \\
&\lambda_2^{(CA,*)} = (e^{-\gamma A} e^{\gamma z_L^{(CA,*)} \hat{\mathcal{D}}^L} E[e^{\gamma(1-z_L^{(CA,*)})\mathcal{D}^L}] - \lambda_1^{(CA,*)} e^{-\gamma A} e^{\gamma z_L^{(CA,*)} \hat{\mathcal{D}}^L} E[e^{\gamma(1-z_L^{(CA,*)})\mathcal{D}^H}])/\epsilon
\end{aligned}$$

where $\tilde{z}_L^{CA}$ *is the solution of* $e^{\gamma \hat{\mathcal{D}}^H} = e^{\gamma z_L^{(CA,*)} \hat{\mathcal{D}}^L} E[e^{\gamma(1-z_L^{CA,*})\mathcal{D}^H}]$.

3. Stackelberg Game *The optimal solution of the maximization problem* (3.12) *and the zero profit constraint* (3.17) *based on the exponential utility is given by:*

$$z_L^{(CS,*)} = 0, z_H^{(CS,*)} = 1 \quad \textit{and} \quad \alpha\mathcal{B} = \hat{\mathcal{D}}^H - \hat{\mathcal{D}}.$$

5.2 Numerical Analysis

In this subsection, we analyze the effect of asymmetric information when the hedger has an exponential utility. The key quantities of the outstanding liabilities are the same as for the mean-variance preference, cf. Table 2.

Monopolistic market

First, we compute the optimal contract parameters, the present value of the fixed leg, the expected profit and the utility improvement of the hedger in the monopolistic market. Since the magnitude of the exponential utility could be rather big, we have chosen to examine the difference in the log of the optimal and the reservation utility:

$$\mathcal{U}_i^{\Diamond} = \ln\left(U_i(PR_i^{\Diamond}(\alpha_i^{(\Diamond,*)}, z_i^{(\Diamond,*)}))\right) - \ln\left(\bar{U}_i\right)$$

for $i = L, H$ and $\Diamond \in \{A, P, S\}$. The results are shown in Table 6. We observe that the results are qualitatively very similar to the solutions under mean-variance preference. In particular, information asymmetry lowers the expected profit of the hedge provider. More importantly, sufficiently accounting for information asymmetry can significantly improve the expected profit of the hedge providers. Furthermore, under information asymmetry, the low-risk type subsides the high-risk type, since the high-risk type benefits from a lower risk loading compared to the perfect information case and enjoys an utility gain.

Optimal Contracts					
Benchmark Analysis		Perfect Information		Stackelberg Game	
$z_L^{(A,*)} = 0.60$	$\alpha_L^{(A,*)} = 3.37$	$z_L^{(P,*)} = 1$	$\alpha_L^{(P,*)} = 2.65$	$z_L^{(S,*)} = 0.48$	$\alpha^{(S,*)} = 1.89$
$z_H^{(A,*)} = 1$	$\alpha_H^{(A,*)} = 1.26$	$z_H^{(P,*)} = 1$	$\alpha_H^{(P,*)} = 2.64$	$z_H^{(S,*)} = 0.92$	
Present Value of the Fixed Leg					
$\Pi_L^A = 1439.9$	$\Pi_H^A = 610.8$	$\Pi_L^P = 1132.8$	$\Pi_H^P = 1278.2$	$\Pi_L^S = \Pi_H^S = 866.3$	
Expected Profit					
$\mathbb{E}[PR_{HP}^A] = 741.0$		$\mathbb{E}[PR_{HP}^P] = 1205.5$		$\mathbb{E}[PR_{HP}^S] = 500.9$	
Utility improvement of the hedger					
$\mathcal{U}_L^A = 0$	$\mathcal{U}_H^A = 20.0$	$\mathcal{U}_L^P = 0$	$\mathcal{U}_H^P = 0$	$\mathcal{U}_L^S = 9.8$	$\mathcal{U}_H^S = 12.7$

Table 6: Optimal contract parameters of the three swaps (top panel), the expected profits of the hedge provider (mid panel) and the utility improvement of the hedger (bottom panel) with the exponential utility in the monopolistic market setting.

Competitive market setting

Next, we compute the optimal contract parameters, the present value of the fixed leg, and the utility improvement in the competitive market. The results are summarized in Table 7. As in the monopolistic market setting, we observe that the results are qualitatively very similar to the mean-variance results.

Optimal Contracts					
Benchmark Analysis		Perfect Information		Stackelberg Game	
$z_L^{(A,*)} = 0.50$	$\alpha_L^{(A,*)} = 0$	$z_L^{(P,*)} = 1$	$\alpha_L^{(P,*)} = 0$	$z_L^{(S,*)} = 0$	$\alpha^{(S,*)} = 1.07$
$z_H^{(A,*)} = 1$	$\alpha_H^{(A,*)} = 0$	$z_H^{(P,*)} = 1$	$\alpha_H^{(P,*)} = 0$	$z_H^{(S,*)} = 1$	
Present Value of the Fixed Leg					
$\Pi_L^A = 0$	$\Pi_H^A = 0$	$\Pi_L^P = 0$	$\Pi_H^P = 0$	$\Pi_H^S = 490.8$	
Utility improvement of the hedger					
$\mathcal{U}_L^A = 22.1$	$\mathcal{U}_H^A = 38.3$	$\mathcal{U}_L^P = 34.0$	$\mathcal{U}_H^P = 38.3$	$\mathcal{U}_L^S = 0$	$\mathcal{U}_H^S = 38.3$

Table 7: Optimal contract parameters of the three swaps (top panel), the expected profits of the hedge provider (mid panel) and the utility improvement of the hedger (bottom panel) with the exponential utility in the competitive market setting.

6 Conclusion

This paper studies longevity risk transfers between capital market investors (hedge providers) and risk exposure holders, such as defined benefit pension plan, life insurers, or insurance intermediaries (reinsurance companies and pension buy-out firms) under information asymmetry. To cope with the fact that the current longevity risk transfer market is incomplete and transactions have been still rare, a utility-based principal-agent model is proposed for determining the optimal transactions. Furthermore, optimal transactions are investigated under both the competitive market and the monopolistic market setting. The monopolistic market is relevant when only one hedge provider is interested in trading the longevity risk exposure of the hedger, while the competitive market is suitable when multiple hedge providers are interested. With the indemnity longevity swap as an illustration and the mean-variance preference for the hedger, analytical solutions of the optimal price and quantity of the type-dependent swaps are derived in three cases: a first-best perfect information case, an asymmetric information case, and a Stackelberg game in which the hedge providers do not address information asymmetry but instead offer a single swap to both risk types. The theoretical results show that neglecting the information asymmetry could result in a loss of the hedge provider's expected profit (under monopoly) and the low-risk type hedger's utility improvement (under competition).

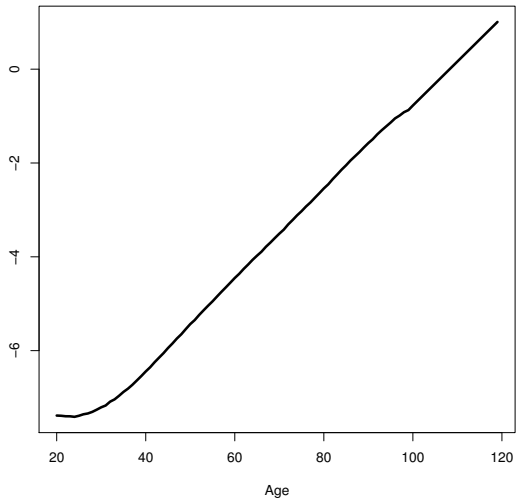
The theoretical results are supplemented by extensive numerical analyses using real world mortality data and the Age-Period-Cohort-Improvement mortality model. We find that while the hedge provider's expected profit or the hedger's utility improvement are only sightly lower than the first-best case if information asymmetry is properly addressed (by offering type-dependent swaps in a separating equilibrium), they are much lower in the Stackelberg game where information asymmetry is not addressed. The numerical analyses are repeated when assuming exponential utility for the hedger, and qualitatively very similar results are obtained.

This work leads to several future research directions. First, although the proposed principal-agent framework is illustrated with a longevity swap, it could be applied in other contexts, including traditional longevity reinsurance and the innovative products discussed in the literature, such as tail longevity risk reinsurance, longevity swaptions, and index-based longevity-linked derivatives. In particular, combinations of traditional reinsurance products, such as proportional reinsurance cover, and capital market solutions can also be considered. Typically, both assets and underwriting risks are transferred with a reinsurance cover. It would be interesting to consider such combinations in a more general framework in which both the assets and the liabilities of the hedger are modeled. Furthermore, our analysis assumes the same degree of risk aversion of both risk types. It might be the case that the high-risk type is more risk averse than the low-risk type. Finally, it will be interesting to analyze the effect of asymmetric information in a multi-period dynamic setting and/or in a setting with a continuum of risk types of hedgers.

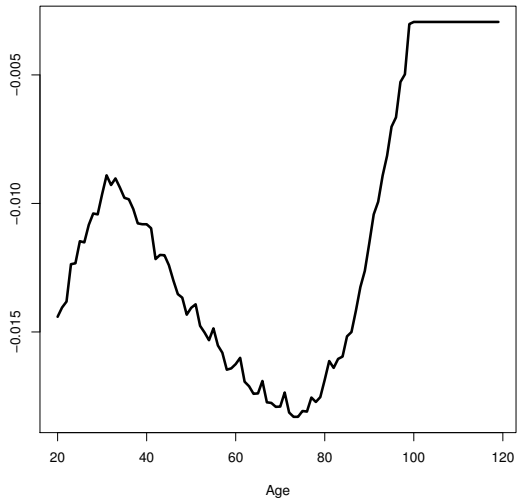
A Key Variable definitions

Variable	Definition
l_{x+t} ($\hat{l}_{x+t}$)	the random (expected) number of policyholders alive in year t
ω_x	maximal remaining life time for the policyholders with x at time 0
${}_tp_x$ (${}_t\hat{p}_x$)	the random (expected) t-year survival probability of the policyholders
r	constant annual interest rate
$\tilde{\pi}_t^i$	the time-varying risk loading at time t of the swap designed for type $i = L, H$
α_i	the constant risk premium of the swap designed for type $i = L, H$
z_i	the hedge rate of the hedger of risk type $i = L, H$
ϵ	probability that the hedger is the low-risk type
$\mathcal{B}^i$	the expected present value of the initial risk loading for risk type i
$U(x)$	the mean-variance utility function
$\bar{U}^i$	reservation utility level of risk type i
$\hat{\mathcal{D}}^i$	the expected present value of the type-i hedger's liability
$\hat{\mathcal{D}}$	the expected present value of the synthetic portfolio's liability in the Stackelberg game
$\mathcal{V}^i$	the variance of the present value of the type-i hedger's liability
Π_i^J	total present value of the fixed leg of risk type i with swap J

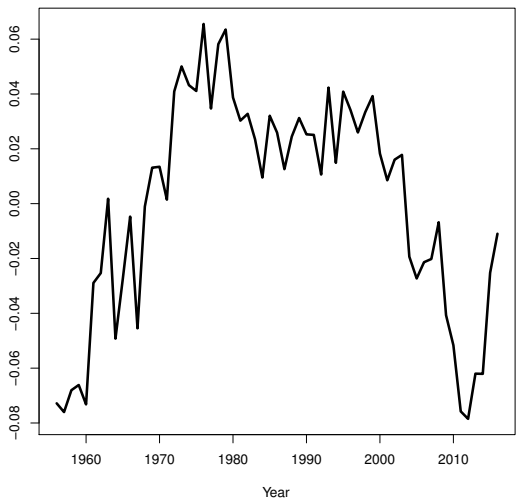
B Estimation and extrapolation results



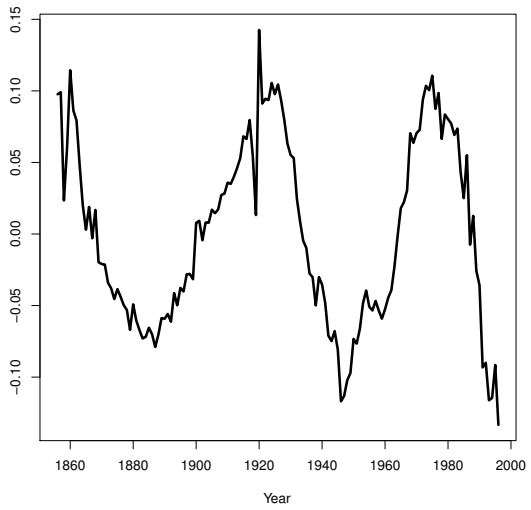
(a) The estimated and extrapolated values of $\beta_x^{(1,UK)}$.



(b) The estimated and extrapolated values of $\beta_x^{(2,UK)}$.



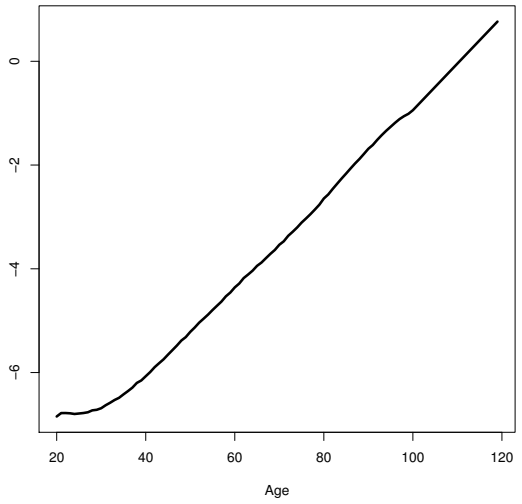
(c) The estimated values of κ_t^{UK}.



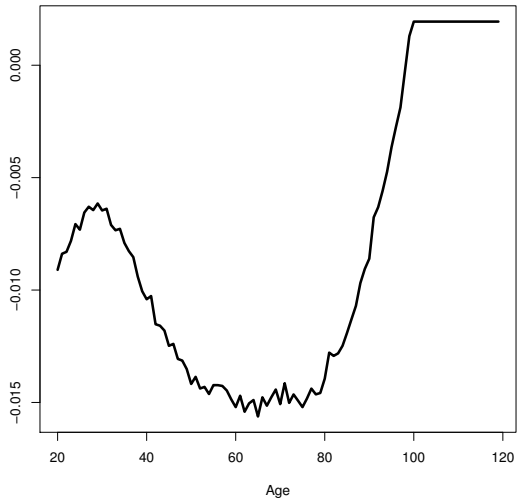
(d) The estimated values of γ_c^{UK}.

Note that a linear extrapolation of $\beta_x^{(2)}$ would result in positive values for very old ages and consequently leads to increasing mortality rates which is unrealistic. Therefore, we set $\beta_x^{(2)}$

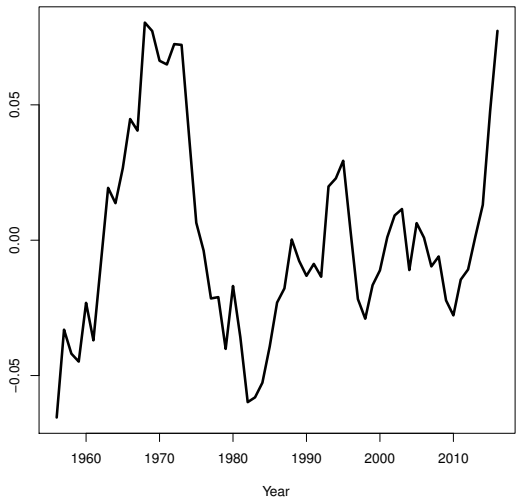
equal to $\beta_{100}^{(2)}$ for very old ages.



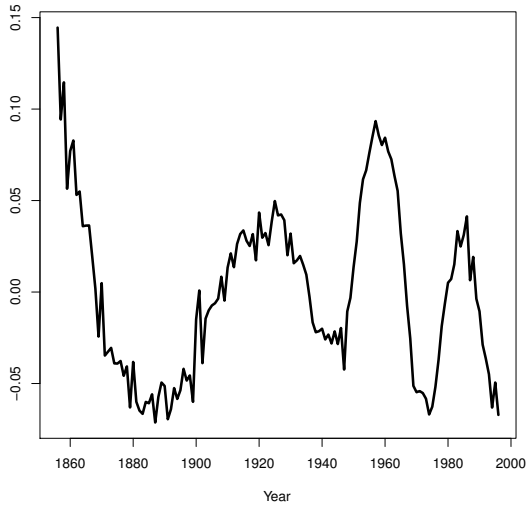
(a) The estimated and extrapolated values of $\beta_x^{(1,US)}$.



(b) The estimated and extrapolated values of $\beta_x^{(2,US)}$.



(c) The estimated values of κ_t^{US}.



(d) The estimated values of γ_c^{US}.

C Proofs under the Monopolistic Market Setting

C.1 Indemnity Swap with Adverse Selection

Based on (3.4), we have the following participation and incentive constraints:

$$
\begin{aligned}
(PC1): \quad & U(PR^L(\alpha_L^A, z_L^A)) - \bar{U}^L = -\alpha_L^A \, z_L^A \, \mathcal{B}^L - \frac{1}{2}\gamma((z_L^A)^2 - 2z_L^A)\mathcal{V}^L \\
(IC2): \quad & U(PR^H(\alpha_H^A, z_H^A)) - U(PR^H(\alpha_L^A, z_L^A)) \\
& = \alpha_L^A \, z_L^A \, \mathcal{B}^L - \alpha_H^A \, z_H^A \, \mathcal{B}^H + z_L^A(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) - \frac{1}{2}\gamma\bigg((z_H^A)^2 - 2z_H^A - (z_L^A)^2 + 2z_L^A\bigg)\mathcal{V}^H.
\end{aligned}
$$

The optimal solution shall satisfy the following first order conditions and complementary slackness conditions of the Lagrangian function described in (3.5):

$$0 = \frac{\partial \mathcal{L}}{\partial z_L^A} = \epsilon \alpha_L^A \mathcal{B}^L - \lambda_1^A(\alpha_L^A \mathcal{B}^L + \gamma \mathcal{V}^L(z_L^A - 1)) + \lambda_2^A\Big(\alpha_L^A \mathcal{B}^L + \hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H + \gamma\big(z_L^A - 1\big)\mathcal{V}^H\Big) \tag{C.1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial z_H^A} = (1-\epsilon)\alpha_H^A\, \mathcal{B}^H - \lambda_2^A(\alpha_H^A \mathcal{B}^H + \gamma \mathcal{V}^H(z_H^A - 1)) \tag{C.2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha_L^A} = \epsilon z_L^A \mathcal{B}^L - \lambda_1^A\, z_L^A\, \mathcal{B}^L + \lambda_2^A\, z_L^A\, \mathcal{B}^L = z_L^A\, (\epsilon \mathcal{B}^L - \lambda_1^A \mathcal{B}^L + \lambda_2^A \mathcal{B}^L) \tag{C.3}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha_H^A} = (1-\epsilon) z_H^A\, \mathcal{B}^H - \lambda_2^A\, z_H^A\, \mathcal{B}^H \Rightarrow \lambda_2^A = 1 - \epsilon \tag{C.4}$$

$$0 = -\alpha_L^A\, z_L^A\, \mathcal{B}^L - \frac{1}{2}\gamma \mathcal{V}^L z_L^A (z_L^A - 2) \tag{C.5}$$

$$0 = \alpha_L^A\, z_L^A\, \mathcal{B}^L - \alpha_H^A\, z_H^A\, \mathcal{B}^H + z_L^A(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) - \frac{1}{2}\gamma\Big((z_H^A)^2 - 2z_H^A - (z_L^A)^2 + 2z_L^A\Big)\mathcal{V}^H \tag{C.6}$$

With $\lambda_2^A = 1 - \epsilon$ and (C.2) we get

$$\begin{aligned} 0 &= (1-\epsilon)\alpha_H^A\, \mathcal{B}^H - (1-\epsilon)(\alpha_H^A\, \mathcal{B}^H + \gamma \mathcal{V}^H(z_H^A - 1)) \\ &= (1-\epsilon)\gamma \mathcal{V}^H(1 - z_H^A) \Rightarrow z_H^A = 1 \end{aligned} \tag{C.7}$$

If we assume that $z_L^A > 0$ we get from (C.3)

$$\lambda_1^A = \epsilon + 1 - \epsilon = 1.$$

Then we plug this into (C.1)

$$\begin{aligned} 0 &= (\epsilon - 1)\alpha_L^A \mathcal{B}_L^A - \gamma \mathcal{V}^L(z_L^A - 1) + (1-\epsilon)\Big(\alpha_L^A\, \mathcal{B}^L + \hat{D}^L - \hat{D}^H + \gamma\big(z_L^A - 1\big)\mathcal{V}^H\Big)\Big) \\ &= -\gamma \mathcal{V}^L(z_L^A - 1) + (1-\epsilon)(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) + (1-\epsilon)\big(\gamma\big(z_L^A - 1\big)\mathcal{V}^H\big) \\ \Rightarrow z_L^A &= 1 + \frac{(1-\epsilon)(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)}{\gamma\big((1-\epsilon)\mathcal{V}^H - \mathcal{V}^L\big)}. \end{aligned} \tag{C.8}$$

Then from Equation (C.5) we have

$$\begin{aligned} 0 &= -\alpha_L^A\, \mathcal{B}^L - \frac{1}{2}\gamma \mathcal{V}^L(z_L^A - 2) = -\alpha_L^A\, \mathcal{B}^L + \frac{1}{2}\gamma \mathcal{V}^L(2 - z_L^A) \\ \Rightarrow \alpha_L^A &= \frac{\gamma \mathcal{V}^L}{2\mathcal{B}^L}\left(1 - \frac{(1-\epsilon)(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)}{\gamma\big((1-\epsilon)\mathcal{V}^H - \mathcal{V}^L\big)}\right) \end{aligned} \tag{C.9}$$

Finally, from Equations (C.5) to (C.6), we have

$$\begin{aligned} 0 &= -\alpha_H^A \, z_H^A \, \mathcal{B}^H + z_L^A(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) + \frac{1}{2}\gamma\mathcal{V}^H + \frac{1}{2}\gamma\big((z_L^A)^2 - 2z_L^A\big)(\mathcal{V}^H - \mathcal{V}^L) \\ \Rightarrow \alpha^H &= \frac{\gamma\mathcal{V}^H}{2\mathcal{B}^H} + \frac{z_L^A}{\mathcal{B}^H}(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) + \frac{\gamma}{2\mathcal{B}^H}\big((z_L^A)^2 - 2z_L^A\big)(\mathcal{V}^H - \mathcal{V}^L). \end{aligned} \tag{C.10}$$

Next, we check if the inequalities are satisfied. Therefore, we define $\tilde{\Omega} \equiv \frac{(1-\epsilon)(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)}{\gamma\left((1-\epsilon)\mathcal{V}^H - \mathcal{V}^L\right)}$. For the incentive constraint of the first hedger (IC1), we have

$$\begin{aligned} (IC1): \quad & U(PR^L(\alpha_L^A, z_L^A)) - U(PR^L(\alpha_H^A, z_H^A)) \\ =& \alpha_H^A \, z_H^A \, \mathcal{B}^H - \alpha_L^A \, z_L^A \, \mathcal{B}^L + z_H^A(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L) - \frac{1}{2}\gamma\left((z_L^A)^2 - 2z_L^A - (z_H^A)^2 + 2z_H^A\right)\mathcal{V}^L \\ =& \frac{\gamma\mathcal{V}^H}{2} + (1+\tilde{\Omega})(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) + \frac{\gamma}{2}(\tilde{\Omega}^2 - 1)(\mathcal{V}^H - \mathcal{V}^L) - (\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) \\ & - \frac{\gamma\mathcal{V}^L}{2}(1 - \tilde{\Omega}^2) - \frac{\gamma\mathcal{V}^L}{2}(1 - 2\tilde{\Omega} + \tilde{\Omega}^2 - 2 + 2\tilde{\Omega} + 1) \\ =& \tilde{\Omega}(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) + \tilde{\Omega}^2\frac{\gamma}{2}(\mathcal{V}^H - \mathcal{V}^L) \\ >& 0, \end{aligned}$$

if $\tilde{\Omega} < 0$. That is, the low-risk type hedger would prefer the swap tailor-made for its type. The participation constraint for the high-risk type hedger is:

$$\begin{aligned} (PC2): \quad & U(PR^H(\alpha_H^A, z_H^A)) - \bar{U}^H = -\alpha_H^A \, z_H^A \, \mathcal{B}^H - \frac{1}{2}\gamma\big((z_H^A)^2 - 2z_H^A\big)\mathcal{V}^H \\ & = -\alpha_H^A \, \mathcal{B}^H + \frac{\gamma\mathcal{V}^H}{2} \\ & = (1+\tilde{\Omega})(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L) + \frac{\gamma}{2}(1 - \tilde{\Omega}^2)(\mathcal{V}^H - \mathcal{V}^L) \\ & > 0, \end{aligned}$$

if $\tilde{\Omega} > -1$. Note that $\tilde{\Omega} \in [-1, 0]$ if $z_L^A \in [0, 1]$.

C.2 Stackelberg Game

Optimization Problem (3.12)**.** The expected utility of hedger of risk type i is:

$$U(PR_i^S(z_i^S, \alpha^S)) = A - (1 - z_i^S)\hat{\mathcal{D}}^i - z_i^S(\hat{\mathcal{D}} + \alpha^S\mathcal{B}^S) - \frac{1}{2}\gamma\mathcal{V}^i(1 - z_i^S)^2,$$

and the first order condition is given by:

$$0 = \frac{\partial\mathcal{L}}{\partial z_i^S} = \hat{\mathcal{D}}^i - \hat{\mathcal{D}} - \alpha^S\mathcal{B}^S + \gamma\mathcal{V}^i(1 - z_i^S).$$

Optimization Problem (3.13)**.** The first order conditions are given by:

$$
\begin{aligned}
0 = \frac{\partial \mathcal{L}}{\partial \alpha^S} &= \frac{\epsilon \mathcal{B}^S(\hat{\mathcal{D}}^L - \hat{\mathcal{D}})}{\gamma \mathcal{V}^L} + \frac{(1-\epsilon)\mathcal{B}^S(\hat{\mathcal{D}}^H - \hat{\mathcal{D}})}{\gamma \mathcal{V}^H} \\
&+ \mathcal{B}^S + \epsilon \frac{(\hat{\mathcal{D}}^L - \hat{\mathcal{D}})\mathcal{B}^S - 2\alpha^S(\mathcal{B}^S)^2}{\gamma \mathcal{V}^L} + (1-\epsilon)\frac{(\hat{\mathcal{D}}^H - \hat{\mathcal{D}})\mathcal{B}^S - 2\alpha^S(\mathcal{B}^S)^2}{\gamma \mathcal{V}^H} \\
&= \frac{\epsilon(\hat{\mathcal{D}}^L - \hat{\mathcal{D}})}{\gamma \mathcal{V}^L} + \frac{(1-\epsilon)(\hat{\mathcal{D}}^H - \hat{\mathcal{D}})}{\gamma \mathcal{V}^H} + 1 + \epsilon \frac{\hat{\mathcal{D}}^L - \hat{\mathcal{D}} - 2\alpha^S \mathcal{B}^S}{\gamma \mathcal{V}^L} \\
&+ (1-\epsilon)\frac{\hat{\mathcal{D}}^H - \hat{\mathcal{D}} - 2\alpha^S \mathcal{B}^S}{\gamma \mathcal{V}^H} \\
\Leftrightarrow \frac{2\alpha^S \mathcal{B}^S}{\gamma}\left(\frac{\epsilon}{\mathcal{V}^L} + \frac{1-\epsilon}{\mathcal{V}^H}\right) &= 1 + \epsilon \frac{2(\hat{\mathcal{D}}^L - \hat{\mathcal{D}})}{\gamma \mathcal{V}^L} + (1-\epsilon)\frac{2(\hat{\mathcal{D}}^H - \hat{\mathcal{D}})}{\gamma \mathcal{V}^H} \\
\Leftrightarrow \alpha^S \mathcal{B}^S\big(\epsilon \mathcal{V}^H + (1-\epsilon)\mathcal{V}^L\big) &= \frac{1}{2}\gamma \mathcal{V}^H \mathcal{V}^L + \epsilon(\hat{\mathcal{D}}^L - \hat{\mathcal{D}})\mathcal{V}^H + (1-\epsilon)(\hat{\mathcal{D}}^H - \hat{\mathcal{D}})\mathcal{V}^L.
\end{aligned}
$$

Optimization Problem (3.14)**.** The first order conditions are given by:

$$
\begin{aligned}
0 = \frac{\partial \mathcal{L}}{\partial \alpha^S} &= (1-\epsilon)(\mathcal{B}^S + 2(\frac{\mathcal{B}^S(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}) - \alpha^S(\mathcal{B}^S)^2}{\gamma \mathcal{V}^H}) + \lambda \\
0 &= \lambda(\alpha^S - \tilde{\alpha}).
\end{aligned}
$$

Suppose $\alpha^S > \tilde{\alpha}$, then $\lambda = 0$, and the optimal solution is $\alpha^{(S,*)} = \frac{\frac{1}{2}\gamma \mathcal{V}^H + (\hat{\mathcal{D}}^H - \hat{\mathcal{D}})}{\mathcal{B}^S}$. Otherwise, if $\alpha^S = \tilde{\alpha}$, then the solution is simply $\tilde{\alpha}$. Denote by $\mathbb{E}_{\mathbb{P}}\big[PR^S_{HP}(\alpha^{(S,*)})\big]$ and $\mathbb{E}_{\mathbb{P}}\big[PR^S_{HP}(\tilde{\alpha})\big]$ the expected profit of the hedge provider given $\alpha^{(S,*)}$ and $\tilde{\alpha}$, respectively, we have:

$$
\mathbb{E}_{\mathbb{P}}\big[PR^S_{HP}(\alpha^{(S,*)})\big] - \mathbb{E}_{\mathbb{P}}\big[PR^S_{HP}(\tilde{\alpha})\big] = \big(\frac{1}{2}\gamma \mathcal{V}^H - (\gamma \mathcal{V}^L + \hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H)\big)^2.
$$

Hence, $\alpha^{(S,*)}$ is preferred to $\tilde{\alpha}$ whenever $\frac{1}{2}\gamma \mathcal{V}^H \neq \gamma \mathcal{V}^L + \hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H$. However, when $\frac{1}{2}\gamma \mathcal{V}^H < \gamma \mathcal{V}^L + \hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H$, it holds that $\alpha^{(S,*)} < \tilde{\alpha}$, and thus the low-risk type hedger will enter the market as well. In this case, the optimization problem (3.14) no longer reflects the hedge provider's expected profit. Therefore, optimal solution exists only when $\frac{1}{2}\gamma \mathcal{V}^H \geq \gamma \mathcal{V}^L + \hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H$.

D Proofs under the competitive market setting

D.1 Perfect Information

The Lagrangian function of the optimization problem (3.10) is given by

$$
\mathcal{L} = A - \hat{\mathcal{D}} - z^{CP}\alpha^{CP}\mathcal{B} - \frac{1}{2}\gamma(1 - z^{CP})^2 \mathcal{V} + \lambda^{CP}(z^{CP}\alpha^{CP}\mathcal{B} - 0). \tag{D.1}
$$

The optimal solution shall satisfy the following first order conditions and complementary slack-

ness conditions of the Lagrangian function described in (D.1):

$$0 = \frac{\partial \mathcal{L}}{\partial z^{CP}} = -\alpha^{CP}\mathcal{B} + \gamma(1 - z^{CP})\mathcal{V} + \lambda\alpha^{CP}\mathcal{B} \tag{D.2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha^{CP}} = -z^{CP}\mathcal{B} + \lambda^{CP} z^{CP}\mathcal{B} \tag{D.3}$$

$$0 = \lambda^{CP}(\mathbb{E}_P[PR^{CP}_{HP}(z^{CP}, \alpha^{CP})] - 0) = \lambda^{CP}\left(z^{CP}\alpha^{CP}\mathcal{B} - 0\right) \tag{D.4}$$

One solution is $z^{CP} = 0$ which corresponds to no hedge. Now, we assume that $z^{CP} \neq 0$ and $\lambda^{CP} > 0$ and thus we get from (D.4) that $\alpha^{CP} = 0$. From (D.2) we find

$$\gamma(1 - z^{CP})\mathcal{V} = 0 \Leftrightarrow z^{CP} = 1 \tag{D.5}$$

and with (D.3)

$$\lambda^{CP} = 1. \tag{D.6}$$

D.2 Asymmetric Information

The Lagrangian function of the optimization problem (3.11) is given by

$$\begin{aligned}\mathcal{L} =& A - \hat{\mathcal{D}}^L - z_L^{CA}\alpha_L^{CA}\mathcal{B}^L - \frac{1}{2}\gamma(1 - z_L^{CA})^2\mathcal{V}^L + \lambda_1^{CA}\big(E[u(PR^H(\alpha_H^{CA}, z_H^{CA}))] \\ & - E[u(PR^H(\alpha_L^{CA}, z_L^{CA}))]\big) + \lambda_2^{CA}\big(\mathbb{E}_P[PR^{CA}_{HP}(z_L^{CA}, \alpha_L^{CA})] - 0\big).\end{aligned} \tag{D.7}$$

The optimal solution shall satisfy the following first order conditions and complementary slackness conditions of the Lagrangian function described in (D.7) (with $z_H^{CA} = 1$ and $\alpha_H^{CA} = 0$):

$$\begin{aligned}0 = \frac{\partial \mathcal{L}}{\partial z_L^{CA}} =& -\alpha_L^{CA}\mathcal{B}^L + \gamma(1 - z_L^{CA})\mathcal{V}^L + \lambda_1^{CA}\big(\alpha_L^{CA}\mathcal{B}^L + \hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H - \gamma(1 - z_L^{CA})\mathcal{V}^H\big) \\ & + \lambda_2^{CA}\epsilon\alpha_L^{CA}\mathcal{B}^L\end{aligned} \tag{D.8}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha_L^{CA}} = -z_L^{CA}\mathcal{B}^L + \lambda_1^{CA} z_L^{CA}\mathcal{B}^L + \lambda_2^{CA}\epsilon z_L^{CA}\mathcal{B}^L \tag{D.9}$$

$$\begin{aligned}0 &= \lambda_1^{CA}(U(PR^H(0,1) - U(PR^H(\alpha_L^{CA}, z_L^{CA})) \\ &= \lambda_1^{CA}\left(\alpha_L^{CA} z_L^{CA}\mathcal{B}^L + z_L^{CA}(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) - \frac{1}{2}\gamma(2z_L^{CA} - (z_L^{CA})^2 - 1)\mathcal{V}^H\right)\end{aligned} \tag{D.10}$$

$$0 = \lambda_2^{CA}(\mathbb{E}_P[PR^P_{HP}(z_L^{CA}, \alpha_L^{CA})] - 0) = \lambda_2^{CA}\left(\epsilon z_L^{CA}\alpha_L^{CA}\mathcal{B}^L - 0\right) \tag{D.11}$$

We assume that $\lambda_1^{CA}, \lambda_1^{CA} > 0$. Note that the other cases doesn't yield a separating equilibrium. In this case, the hedge provider makes zero expected profit. To ensure (D.11) to hold, $z_L^{CA} > 0$ and $\alpha_L^{CA} = 0$ can be a potential solution to the problem. Supposing these are the solutions, let us further examine the additional conditions. With Condition (D.10) we get (assuming

$\lambda_1^{CA} > 0$):

$$0 = z_L^{CA}(\hat{\mathcal{D}}^L - \hat{\mathcal{D}}^H) - \frac{1}{2}\gamma(2z_L^{CA} - (z_L^{CA})^2 - 1)\mathcal{V}^H, \tag{D.12}$$

which leads to

$$z_L^{(CA,*)} = 1 + \frac{\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L - \sqrt{(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)^2 + 2\gamma\mathcal{V}^H(\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L)}}{\gamma\mathcal{V}^H}. \tag{D.13}$$

We can see that $z_L^{(CA,*)} \in (0,1)$. Further, from (D.8) - (D.9) we find that

$$\lambda_1^{CA} = \frac{\gamma(1 - z_L^{CA})\mathcal{V}^L}{\hat{\mathcal{D}}^H - \hat{\mathcal{D}}^L + \gamma(1 - z_L^{CA})\mathcal{V}^H} \in (0,1) \quad \text{and} \tag{D.14}$$

$$\lambda_2^{CA} = 1 - \lambda_1^{CA}. \tag{D.15}$$

D.3 Stackelberg Game

The optimal solution of (3.12) is given by $z_i^{CS} = 1 + \frac{\hat{\mathcal{D}}^i - \hat{\mathcal{D}} - \alpha^{CS}\mathcal{B}}{\gamma\mathcal{V}^i}$ for $i = L, H$, which is the same as the optimal solution under monopoly. To ensure $z_i^{CS} = [0,1]$, it has to hold that

$$\hat{\mathcal{D}}^H - \hat{\mathcal{D}} \leq \alpha^{CS}\mathcal{B} \leq \hat{\mathcal{D}}^L - \hat{\mathcal{D}} + \gamma\mathcal{V}^L.$$

However, if $\hat{\mathcal{D}}^H - \hat{\mathcal{D}} < \alpha^{CS}\mathcal{B}$, the hedge provider makes a positive profit from both risk types, and therefore the zero profit constraint is violated. Therefore, in order to fulfill the zero profit constraint, only the high-risk type hedger will participate in the market. This yields $z_H^{CS} = 1$ and $\alpha^{CS} = \frac{\hat{\mathcal{D}}^H - \hat{\mathcal{D}}}{\mathcal{B}}$.

E Proofs with the Exponential Utility

E.1 Proposition 7

E.1.1 Perfect Information

The Lagrangian function of the optimization problem (3.2) is given by

$$\mathcal{L} = z^P\alpha^P\mathcal{B} + \lambda^P\left(E[u(PR(\alpha^P, z^P))] - \bar{U}\right). \tag{E.1}$$

The first order conditions of the Lagrangian function represented in (E.1) are given by

$$0 = \frac{\partial\mathcal{L}}{\partial z^P} = \alpha^P\mathcal{B} - \lambda^P\left(e^{-\gamma A}e^{\gamma z^P(\hat{\mathcal{D}}+\alpha\mathcal{B})}\left[(\hat{\mathcal{D}} + \alpha^P\mathcal{B})E[e^{\gamma(1-z^P)\mathcal{D}}] - E[\mathcal{D}e^{\gamma(1-z^P)\mathcal{D}}]\right]\right), \tag{E.2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha^P} = z^P \mathcal{B} + \lambda^P \left(-\frac{1}{\gamma} e^{-\gamma A} \gamma z^P \mathcal{B} e^{\gamma z^P (\hat{\mathcal{D}} + \alpha^P \mathcal{B})} E[e^{\gamma(1-z^P)\mathcal{D}}] \right), \tag{E.3}$$

$$0 = \lambda^P (U(PR(\alpha^P, z^P) - \bar{U}) = \lambda^P \left(-\frac{1}{\gamma} e^{-\gamma A} e^{\gamma z^P (\hat{\mathcal{D}} + \alpha^P \mathcal{B})} E[e^{\gamma(1-z^P)\mathcal{D}}] + \frac{1}{\gamma} e^{-\gamma A} E[e^{\gamma \mathcal{D}}] \right) \tag{E.4}$$

There are multiple sets of solutions to the first order conditions (E.2) - (E.4). The first set of solutions is $z^P = 0$, $\alpha^P = 0$ and $\lambda^P = 0$. Consequently, the expected profit of the hedge provider for the indemnity swap is 0. The next set of solution is given by $z^P = 0$, $\alpha^P = \frac{\lambda^P e^{-\gamma A}\left[(\hat{\mathcal{D}} E[e^{\gamma \mathcal{D}}] - E[\mathcal{D} e^{\gamma \mathcal{D}}]\right]}{\mathcal{B}(1 - \lambda^P e^{-\gamma A} E[e^{\gamma \mathcal{D}}])}$ and $\lambda^P > 0 \neq e^{\gamma A} e^{-\gamma(\hat{\mathcal{D}} + \alpha \mathcal{B})}$. Similar to the first solution, the expected profit of the hedge provider is 0. For the last set of solutions, we assume $z^P \neq 0$ and $\lambda^P > 0$. In this case, (E.3) can be simplified to

$$1 = \lambda^P e^{-\gamma A} e^{\gamma z^P (\hat{\mathcal{D}} + \alpha^P \mathcal{B})} E[e^{\gamma(1-z^P)\mathcal{D}}] \Leftrightarrow \lambda^P = \frac{e^{\gamma A} e^{-\gamma z^P (\hat{\mathcal{D}} + \alpha^P \mathcal{B})}}{E[e^{\gamma(1-z^P)\mathcal{D}}]} > 0. \tag{E.5}$$

Combining the above equation with (E.2), we find

$$0 = \alpha^P \mathcal{B} - \frac{(\hat{\mathcal{D}} + \alpha^P \mathcal{B}) E[e^{\gamma(1-z^P)\mathcal{D}}] - E[\mathcal{D} e^{\gamma(1-z^P)\mathcal{D}}]}{E[e^{\gamma(1-z^P)\mathcal{D}}]} \Leftrightarrow z^P = 1 \tag{E.6}$$

Plug $z^P = 1$ into (E.4) we get

$$e^{\gamma(\hat{\mathcal{D}} + \alpha^P \mathcal{B})} = E[e^{\gamma \mathcal{D}}] \Leftrightarrow \alpha^P = \frac{\frac{1}{\gamma} \ln\left(E[e^{\gamma(\mathcal{D} - \hat{\mathcal{D}}}]\right)}{\mathcal{B}}. \tag{E.7}$$

E.1.2 Asymmetric Information

The Lagrangian function is given by

$$\begin{aligned} \mathcal{L} = {} & \epsilon z_L^A \alpha_L^A \mathcal{B}^L + (1-\epsilon) z_H^A \alpha_H^A \mathcal{B}^H + \lambda_1^A \left(E[u(PR^L(\alpha_L^A, z_L^A))] - \bar{U}^L \right) \\ & + \lambda_4^A \left(E[u(PR^H(\alpha_H^A, z_H^A))] - E[u(PR^H(\alpha_L^A, z_L^A))] \right). \end{aligned} \tag{E.8}$$

The optimal solution shall satisfy the following first order conditions and complementary slackness conditions of the Lagrangian function described in (E.8) (assuming $\lambda_1^A, \lambda_4^A > 0$):

$$\begin{aligned} 0 = \frac{\partial \mathcal{L}}{\partial z_L^A} = {} & \epsilon \alpha_L^A \mathcal{B}^L - \lambda_1^A \left(e^{-\gamma A} e^{\gamma z_L^A (\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L)} \left[(\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L) E[e^{\gamma(1-z_L^A)\mathcal{D}^L}] - E[\mathcal{D}^L e^{\gamma(1-z_L^A)\mathcal{D}^L}] \right] \right) \\ & + \lambda_4^A \left(e^{-\gamma A} e^{\gamma z_L^A (\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L)} \left[(\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L) E[e^{\gamma(1-z_L^A)\mathcal{D}^H}] - E[\mathcal{D}^H e^{\gamma(1-z_L^A)\mathcal{D}^H}] \right] \right) \end{aligned} \tag{E.9}$$

$$\begin{aligned} 0 = \frac{\partial \mathcal{L}}{\partial z_H^A} = {} & (1-\epsilon) \alpha_H^A \mathcal{B}^H - \lambda_4^A \left(e^{-\gamma A} e^{\gamma z_H^A (\hat{\mathcal{D}}^H + \alpha_H^A \mathcal{B}^H)} \left[(\hat{\mathcal{D}}^H + \alpha_H^A \mathcal{B}^H) E[e^{\gamma(1-z_H^A)\mathcal{D}^H}] \right. \right. \\ & \left. \left. - E[\mathcal{D}^H e^{\gamma(1-z_H^A)\mathcal{D}^H}] \right] \right) \end{aligned} \tag{E.10}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha_L^A} = \epsilon z_L^A \mathcal{B}^L - \lambda_1^A e^{-\gamma A} z_L^A \mathcal{B}^L e^{\gamma z_L^A (\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L)} E[e^{\gamma(1-z_L^A)\mathcal{D}^L}]$$
$$+ \lambda_4^A e^{-\gamma A} z_L^A \mathcal{B}^L e^{\gamma z_L^A (\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L)} E[e^{\gamma(1-z_L^A)\mathcal{D}^H}] \tag{E.11}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha_H^A} = (1-\epsilon) z_H^A \mathcal{B}^H - \lambda_4^A e^{-\gamma A} z_H^A \mathcal{B}^H e^{\gamma z_H^A (\hat{\mathcal{D}}^H + \alpha_H^A \mathcal{B}^H)} E[e^{\gamma(1-z_H^A)\mathcal{D}^H}] \tag{E.12}$$

$$0 = \lambda_1^A (U(PR^L(\alpha_L^A, z_L^A) - \bar{U}) = \lambda_1^A \left(-\frac{1}{\gamma} e^{-\gamma A} e^{\gamma z_L^A (\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L)} E[e^{\gamma(1-z_L^A)\mathcal{D}^L}] + \frac{1}{\gamma} e^{-\gamma A} E[e^{\gamma \mathcal{D}^L}] \right) \tag{E.13}$$

$$0 = \lambda_4^A (U(PR^H(\alpha_H^A, z_H^A) - U(PR^H(\alpha_L^A, z_L^A))$$
$$= \lambda_4^A \left(-\frac{1}{\gamma} e^{-\gamma A} e^{\gamma z_H^A (\hat{\mathcal{D}}^H + \alpha_H^A \mathcal{B}^H)} E[e^{\gamma(1-z_H^A)\mathcal{D}^H}] + \frac{1}{\gamma} e^{-\gamma A} e^{\gamma z_L^A (\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L)} E[e^{\gamma(1-z_L^A)\mathcal{D}^H}] \right) \tag{E.14}$$

From Condition (E.12) and $z_H^A \neq 0$ we get:

$$\lambda_4^A = \frac{(1-\epsilon) e^{\gamma A} e^{-\gamma z_H^A (\hat{\mathcal{D}}^H + \alpha_H^A \mathcal{B}^H)}}{E[e^{\gamma(1-z_H^A)\mathcal{D}^H}]}. \tag{E.15}$$

Using the representation of λ_4^A and Condition (E.10) we get:

$$E[\mathcal{D}^H e^{\gamma(1-z_H^A)\mathcal{D}^H}] - \hat{\mathcal{D}}^H E[e^{\gamma(1-z_H^A)\mathcal{D}^H}] = 0 \Leftrightarrow z_H^A = 1. \tag{E.16}$$

With the representation of z_L^A, $\lambda_1^A > 0$ and Condition (E.13) we can compute α_L^A:

$$\alpha_L^A = \frac{\left(\frac{1}{\gamma z_L^A} \ln\left(\frac{E[e^{\gamma \mathcal{D}^L}]}{E[e^{\gamma(1-z_L^A)\mathcal{D}^L}]} \right) - \hat{\mathcal{D}}^L \right)}{\mathcal{B}^L}. \tag{E.17}$$

Finally, we can compute α_H^A with Condition (E.14) and $\lambda_4^A > 0$. From Equations (E.14) and (E.17) we get:

$$\alpha_H^A = \frac{\left(\frac{1}{\gamma} \left(\ln\big(E[e^{\gamma \mathcal{D}^L}]\big) + \ln\big(\frac{E[e^{\gamma(1-z_L^A)\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^A)\mathcal{D}^L}]}\big) \right) - \hat{\mathcal{D}}^H \right)}{\mathcal{B}^H}. \tag{E.18}$$

From (E.17) and (E.18) we get

$$e^{\gamma z_L^A (\hat{\mathcal{D}}^L + \alpha_L^A \mathcal{B}^L)} = \frac{E[e^{\gamma \mathcal{D}^L}]}{E[e^{\gamma(1-z_L^A)\mathcal{D}^L}]} \text{ and } e^{\gamma z_H^A (\hat{\mathcal{D}}^H + \alpha_H^A \mathcal{B}^H)} = \frac{E[e^{\gamma(1-z_L^A)\mathcal{D}^H}] E[e^{\gamma \mathcal{D}^L}]}{E[e^{\gamma(1-z_L^A)\mathcal{D}^L}]}.$$

Plugging the above equations into (E.11) gives $\lambda_1^A = \frac{e^{\gamma A}}{E[e^{\gamma \mathcal{D}^L}]}$. Then, plugging the expressions of λ_1^A and λ_4^A into (E.9) gives:

$$\frac{E[\mathcal{D}^L e^{\gamma(1-z_L^A)\mathcal{D}^L}]}{E[e^{\gamma(1-z_L^A)\mathcal{D}^L}]} - \epsilon\hat{\mathcal{D}}^L = (1-\epsilon)\frac{E[\mathcal{D}^H e^{\gamma(1-z_L^A)\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^A)\mathcal{D}^H}]}. \tag{E.19}$$

Finally, the risk loading of the high-risk type is lower than the risk loading under perfect information, i.e., $\alpha_H^{(A,*)} < \alpha_H^{(P,*)}$ if:

$$\begin{aligned} &\ln\big(E[e^{\gamma\mathcal{D}^L}]\big) + \ln\Big(\frac{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]}\Big) \leq \ln\big(E[e^{\gamma\mathcal{D}^H}]\big), \\ &\quad\Leftrightarrow\quad \ln\Big(\frac{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]}\Big) \leq \ln\Big(\frac{E[e^{\gamma\mathcal{D}^H}]}{E[e^{\gamma\mathcal{D}^L}]}\Big) \end{aligned} \tag{E.20}$$

The inequality (E.20) holds for $z_L^{(A,*)} \in [0,1]$ since:

$$\begin{aligned} &\frac{\partial\Big(\ln\big(\frac{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]}\big)\Big)}{\partial z_L^{(A,*)}} = \frac{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}\times \\ &\gamma\frac{-E[\mathcal{D}^H e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}] + E[\mathcal{D}^L e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}{(E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}])^2} \\ &= \gamma\frac{-E[\mathcal{D}^H e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}] + E[\mathcal{D}^L e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]} \\ &= \gamma\left(\frac{E[\mathcal{D}^L e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^L}]} - \frac{E[\mathcal{D}^H e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}{E[e^{\gamma(1-z_L^{(A,*)})\mathcal{D}^H}]}\right) < 0. \end{aligned} \tag{E.21}$$

E.1.3 Stackelberg Game

The derivative of the expected utility of hedger of risk type i is:

$$\begin{aligned} 0 &= \frac{\partial E\big[u\big(PR_i^S(z_i^S,\alpha^S)\big)\big]}{\partial z_i^S} \\ &= -e^{-\gamma A}e^{\gamma z_i^S(\hat{\mathcal{D}}+\alpha^S\mathcal{B})}\Big[(\hat{\mathcal{D}}+\alpha^S\mathcal{B})E[e^{\gamma(1-z_i^S)\mathcal{D}^i}] - E[\mathcal{D}^i e^{\gamma(1-z_i^S)\mathcal{D}^i}]\Big] \\ &\Leftrightarrow\ (\hat{\mathcal{D}}+\alpha^S\mathcal{B})E[e^{\gamma(1-z_i^S)\mathcal{D}^i}] = E[\mathcal{D}^i e^{\gamma(1-z_i^S)\mathcal{D}^i}] \\ &\Leftrightarrow \alpha^S\mathcal{B} = \frac{E[\mathcal{D}^i e^{\gamma(1-z_i^S)\mathcal{D}^i}]}{E[e^{\gamma(1-z_i^S)\mathcal{D}^i}]} - \hat{\mathcal{D}}. \end{aligned} \tag{E.22}$$

The derivative of (3.13) with respect to α^S is given by:

$$0 = \frac{\partial\mathbb{E}_P[PR_{HP}^S]}{\partial\alpha^S}$$

$$= \epsilon\left(\frac{\partial z_L^S}{\partial \alpha^S}(\hat{\mathcal{D}} + \alpha^S\mathcal{B} - \hat{\mathcal{D}}^L) + z_L^S\mathcal{B}\right) + (1-\epsilon)\left(\frac{\partial z_H^S}{\partial \alpha^S}(\hat{\mathcal{D}} + \alpha^S\mathcal{B} - \hat{\mathcal{D}}^H) + z_H^S\mathcal{B}\right)$$

$$\Leftrightarrow \alpha^S\mathcal{B} = \frac{\epsilon\left(\frac{\partial z_L^S}{\partial \alpha^S}\hat{\mathcal{D}}^L - z_L^S\mathcal{B}\right) + (1-\epsilon)\left(\frac{\partial z_H^S}{\partial \alpha^S}\hat{\mathcal{D}}^H - z_H^S\mathcal{B}\right)}{\epsilon\frac{\partial z_L^S}{\partial \alpha^S} + (1-\epsilon)\frac{\partial z_H^S}{\partial \alpha^S}} - \hat{\mathcal{D}}. \tag{E.23}$$

In the case that only the high-risk type hedger participates in the market, the first order condition is given by:

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha^S} = (1-\epsilon)\left(\frac{\partial z_H^S}{\partial \alpha^S}(\hat{\mathcal{D}} + \alpha^S\mathcal{B} - \hat{\mathcal{D}}^H) + z_H^S\mathcal{B}\right) + \lambda. \tag{E.24}$$

Assuming that $\lambda = 0$, which implies that α^S is greater than $\frac{E[\mathcal{D}^L e^{\gamma\mathcal{D}^L}]}{E[e^{\gamma\mathcal{D}^L}]} - \hat{\mathcal{D}}$, we find that $\alpha^S\mathcal{B} = \hat{\mathcal{D}}^H - \hat{\mathcal{D}} - \frac{z_H^S\mathcal{B}}{\frac{\partial z_H^S}{\partial \alpha^S}}$.

E.2 Proposition 8

E.2.1 Perfect Information

The Lagrangian function of the optimization problem (3.10) is given by

$$\mathcal{L} = E\big[u\big(PR^{CP}(z^{CP}, \alpha^{CP})\big)\big] + \lambda^{CP}\big(\mathbb{E}_P[PR_{HP}^{CP}(z^{CP}, \alpha^{CP})] - 0\big). \tag{E.25}$$

The optimal solution shall satisfy the following first order conditions and complementary slackness conditions of the Lagrangian function described in (E.25):

$$0 = \frac{\partial \mathcal{L}}{\partial z^{CP}}$$
$$= -e^{-\gamma A}e^{\gamma z^{CP}(\hat{\mathcal{D}} + \alpha^{CP}\mathcal{B})}\left[(\hat{\mathcal{D}} + \alpha^{CP}\mathcal{B})E[e^{\gamma(1-z^{CP})\mathcal{D}}] - E[\mathcal{D}e^{\gamma(1-z^{CP})\mathcal{D}}]\right] + \lambda_2^{CP}\alpha^{CP}\mathcal{B} \tag{E.26}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha^{CP}} = -e^{-\gamma A}z^{CP}\mathcal{B}e^{\gamma z^{CP}(\hat{\mathcal{D}} + \alpha^{CP}\mathcal{B})}E[e^{\gamma(1-z^{CP})\mathcal{D}}] + \lambda_2^{CP}z^{CP}\mathcal{B} \tag{E.27}$$

$$0 = \lambda^{CP}(\mathbb{E}_P[PR_{HP}^{CP}(z^{CP}, \alpha^{CP})] - 0) = \lambda^{CP}\left(z^{CP}\alpha^{CP}\mathcal{B} - 0\right). \tag{E.28}$$

One (trivial) solution is $z^{CP} = 0$ which corresponds to no hedge. For a non-trivial solution, we assume that $z^{CP} \neq 0$ and $\lambda^{CP} > 0$, based on which we get from (E.28) that $\alpha^{CP} = 0$. From (E.26) we then find:

$$\hat{\mathcal{D}}E[e^{\gamma(1-z^{CP})\mathcal{D}}] = E[\mathcal{D}e^{\gamma(1-z^{CP})\mathcal{D}}] \Leftrightarrow z^{CP} = 1. \tag{E.29}$$

and with (E.27):

$$\lambda^{CP} = e^{-\gamma A} e^{\gamma \hat{\mathcal{D}}} > 0. \tag{E.30}$$

E.2.2 Asymmetric Information

The Lagrangian function of the optimization problem (3.11) is given by

$$\begin{aligned}\mathcal{L} =& E\big[u\big(PR_L^{CA}(z_L^{CA}, \alpha_L^{CA})\big)\big] + \lambda_1^{CA}\big(E[u(PR^H(\alpha_H^{CA}, z_H^{CA}))] - E[u(PR^H(\alpha_L^{CA}, z_L^{CA}))]\big) \\ &+ \lambda_2^{CA}\big(\mathbb{E}_P[PR_{HP}^P(z_L^{CA}, \alpha_L^{CA})] - 0\big)\end{aligned} \tag{E.31}$$

with $\alpha_H^{CA} = 0$ and $z_H^{CA} = 1$. The optimal solution shall satisfy the following first order conditions and complementary slackness conditions of the Lagrangian function described in (E.31):

$$\begin{aligned}0 =& \frac{\partial \mathcal{L}}{\partial z_L^{CA}} = -e^{-\gamma A} e^{\gamma z_L^{CA}(\hat{\mathcal{D}}^L + \alpha_L^{CA}\mathcal{B}^L)}\Big[(\hat{\mathcal{D}}^L + \alpha_L^{CA}\mathcal{B}^L)E[e^{\gamma(1-z_L^{CA})\mathcal{D}^L}] - E[\mathcal{D}^L e^{\gamma(1-z_L^{CA})\mathcal{D}^L}]\Big] \\ &+ \lambda_1^{CA}\Bigg(e^{-\gamma A} e^{\gamma z_L^{CA}(\hat{\mathcal{D}}^L + \alpha_L^{CA}\mathcal{B}^L)}\Big[(\hat{\mathcal{D}}^L + \alpha_L^{CA}\mathcal{B}^L)E[e^{\gamma(1-z_L^{CA})\mathcal{D}^H}] - E[\mathcal{D}^H e^{\gamma(1-z_L^{CA})\mathcal{D}^H}]\Big]\Bigg) \\ &+ \lambda_2^{CA}\epsilon\alpha_L^{CA}\mathcal{B}^L\end{aligned} \tag{E.32}$$

$$\begin{aligned}0 =& \frac{\partial \mathcal{L}}{\partial \alpha_L^{CA}} = -e^{-\gamma A} z_L^{CA}\mathcal{B}^L e^{\gamma z_L^{CA}(\hat{\mathcal{D}}^L + \alpha_L^{CA}\mathcal{B}^L)}E[e^{\gamma(1-z_L^{CA})\mathcal{D}^L}] \\ &+ \lambda_1^{CA} e^{-\gamma A} z_L^{CA}\mathcal{B}^L e^{\gamma z_L^{CA}(\hat{\mathcal{D}}^L + \alpha_L^{CA}\mathcal{B}^L)}E[e^{\gamma(1-z_L^{CA})\mathcal{D}^H}] + \lambda_2^{CA}\epsilon z_L^{CA}\mathcal{B}^L\end{aligned} \tag{E.33}$$

$$\begin{aligned}0 =& \lambda_1^{CA}(U(PR^H(0,1) - U(PR^H(\alpha_L^{CA}, z_L^{CA})) \\ =& \lambda_1^{CA}\left(-\frac{1}{\gamma}e^{-\gamma A}e^{\gamma\hat{\mathcal{D}}^H} + \frac{1}{\gamma}e^{-\gamma A}e^{\gamma z_L^{CA}(\hat{\mathcal{D}}^L + \alpha_L^{CA}\mathcal{B}^L)}E[e^{\gamma(1-z_L^{CA})\mathcal{D}^H}]\right)\end{aligned} \tag{E.34}$$

$$0 = \lambda_2^{CA}(\mathbb{E}_P[PR_{HP}^P(z_L^{CA}, \alpha_L^{CA})] - 0) = \lambda_2^{CA}\left(\epsilon z_L^{CA}\alpha_L^{CA}\mathcal{B}^L - 0\right) \tag{E.35}$$

Assuming $\lambda_1^{CA} > 0$ and $\lambda_2^{CA} > 0$, then the hedge provider makes zero expected profit. In this case, $z_L^{CA} > 0$ and $\alpha_L^{CA} = 0$ is a potential set of solution such that (E.35) holds. Suppose this is the case, then Condition (E.34) leads to:

$$0 = \lambda_1^{CA}\left(-e^{\gamma\hat{\mathcal{D}}^H} + e^{\gamma z_L^{CA}\hat{\mathcal{D}}^L}E[e^{\gamma(1-z_L^{CA})\mathcal{D}^H}]\right) \overset{\lambda_1^{CA}>0}{\Leftrightarrow} e^{\gamma\hat{\mathcal{D}}^H} = e^{\gamma z_L^{CA}\hat{\mathcal{D}}^L}E[e^{\gamma(1-z_L^{CA})\mathcal{D}^H}] \tag{E.36}$$

and based on this equation we can (numerically) compute z_L^{CA}. Further, With $z_L^{CA} > 0$ and $\alpha_L^{CA} = 0$, we find with Condition (E.32) that:

$$\lambda_1^{CA} = \frac{\hat{\mathcal{D}}^L E[e^{\gamma(1-z_L^{CA})\mathcal{D}^L}] - E[\mathcal{D}^L e^{\gamma(1-z_L^{CA})\mathcal{D}^L}]}{\hat{\mathcal{D}}^L E[e^{\gamma(1-z_L^{CA})\mathcal{D}^H}] - E[\mathcal{D}^H e^{\gamma(1-z_L^{CA})\mathcal{D}^H}]}. \tag{E.37}$$

And consequently through (E.33), we obtain λ_2^{CA}:

$$\begin{aligned} 0 &= -e^{-\gamma A} e^{\gamma z_L^{CA} \hat{\mathcal{D}}^L} E[e^{\gamma(1-z_L^{CA})\mathcal{D}^L}] + \lambda_1^{CA} e^{-\gamma A} e^{\gamma z_L^{CA} \hat{\mathcal{D}}^L} E[e^{\gamma(1-z_L^{CA})\mathcal{D}^H}] + \lambda_2^{CA} \epsilon \\ \Leftrightarrow \lambda_2^{CA} &= (e^{-\gamma A} e^{\gamma z_L^{CA} \hat{\mathcal{D}}^L} E[e^{\gamma(1-z_L^{CA})\mathcal{D}^L}] - \lambda_1^{CA} e^{-\gamma A} e^{\gamma z_L^{CA} \hat{\mathcal{D}}^L} E[e^{\gamma(1-z_L^{CA})\mathcal{D}^H}])/\epsilon. \end{aligned} \tag{E.38}$$

There are other parameter values such that (E.35) holds, for example, $\lambda_2^{CA} > 0$ and/or $z_L^{CA} = 0$. However, one can show that $\lambda_1^{CA} > 0$, $\lambda_2^{CA} > 0$, $z_L^{CA} > 0$, and $\alpha_L^{CA} = 0$ lead to the unique non-trivial solution. Proofs are available upon request.

E.3 Competitive Market - Stackelberg Game

The expected utility of hedger of risk type i is the same as in the monopolistic market setting (E.22), i.e., $\alpha^{CS}\mathcal{B} = \frac{E[\mathcal{D}^i e^{\gamma(1-z_i^{CS})\mathcal{D}^i}]}{E[e^{\gamma(1-z_i^{CS})\mathcal{D}^i}]} - \hat{\mathcal{D}}$. Similar to the case with the mean-variance preference (Section D.3), the zero profit constraint of the hedge provider will be violated when both risk-types participate in the market. Therefore, only the high-risk type participates, with $z_H^{CS} = 1$ and α^{CS} is reduced to $\frac{\mathcal{D}^H - \hat{\mathcal{D}}}{\mathcal{B}}$.

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4 What to Offer If Consumers Do Not Want What They Need? A Simultaneous Evaluation Approach with an Application to Retirement Savings Products

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What to Offer If Consumers Do Not Want What They Need? A Simultaneous Evaluation Approach with an Application to Retirement Savings Products

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Abstract

Standard economic models of rational decision making provide information on how people should decide. In practice, human decisions are influenced by numerous behavioral patterns that lead to systematic deviations from rationally optimal behavior. In the context of retirement savings, this can result in substantial pension gaps, and hence in a reduction of the standard of living in the retirement phase. The aim of this work is to introduce a general framework to (simultaneously) assess and evaluate the objectively rational utility and the subjectively perceived attractiveness. We illustrate the approach by means of an application to retirement savings products. Such a combined approach can help to identify or design retirement savings products that create a high (albeit not the maximum possible) objective utility while at the same time being subjectively of high (albeit not maximum possible) attractiveness. We argue that a focus on such products might lead to improved consumer decisions compared to observed decisions that are often driven by subjective attractiveness (resulting in rather low objective utility).

Keywords: Retirement Savings, Old-Age Provision, Utility Maximization, Behavioral Insurance, Prospect Theory, Simultaneous Evaluation
JEL: D14, G11, G22, G41, J26, J32

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1 Introduction

Decision-making processes depend on many factors, e.g., available choices, personal characteristics and motives, the importance and complexity of the decision, the decision situation, etc., and often involve a trade-off between being accurate and spending as little time and effort as possible, cf., Tversky & Kahneman (1985), Payne *et al.* (1993), and Kahneman & Tversky (2013). Heuristics (consciously and subconsciously applied) often help people to make acceptable decisions with reasonable effort, cf., Kahneman *et al.* (1982) and Gigerenzer (2008). However, especially in the case of complex decisions under uncertainty, like in the context of retirement planning, heuristics may not work very well. As a consequence, intuitively attractive choices may not be very suitable to achieve the objective goal of the decision. In the literature, this is known as systematic error or bias, cf., Tversky & Kahneman (1974). Accordingly, for such decisions there is a considerable risk that the subjectively most attractive choice at the time of decision making may be objectively suboptimal, cf., Kahneman & Klein (2009).

In reality, we observe that people tend to spend more time on important decisions (which can also include seeking for expert advice). For example, many people consult advisors to support their decision-making on retirement planning. This can help to incorporate information in a more accurate way, and, ideally, lead to objectively better decisions, cf., Harvey & Fischer (1997). However, since many heuristics affect the decision process subconsciously, certain objectively disadvantageous attributes of a choice can still have a (strong) subjective attractiveness, cf., Chaiken & Maheswaran (1994). This is particularly relevant for decisions that have to be made only once in a lifetime (or very rarely) for which it is not possible to learn from own experience. For example, myopic loss aversion makes retirement savings products which reduce (or eliminate) short-term losses, e.g., by annual guarantees, which are very popular in many countries, although they can significantly reduce long term returns and are (at least for most people) objectively not optimal, cf., Benartzi & Thaler (1995) and Ruß & Schelling (2018). Further, the results of Ruß & Schelling (2018) indicate that in the case of retirement savings, the objectively optimal choice is perceived as significantly less attractive than the subjectively most attractive choice. In such situations, the subjective attractiveness of specific attributes may dominate the decision to such an extent that even opposing information and expert advice can have only limited impact on the decision. This is demonstrated by the high market share of retirement savings products[1] with annual guarantees in many markets, e.g., Germany, although most of this products are bought after a financial advisor had been consulted.[2]

[1]Note that by retirement savings products, we mean both, products that pay a lump sum at maturity, as well as the deferment period of deferred annuities.

[2]Most retirement savings products in Germany are sold with the involvement of some type of financial advisor – the market share of direct sales in 2020 was only 3.4% (in terms of premiums of new contracts), cf., GDV (2021). When it comes to product types, unit-linked or index-linked products with guarantees have a significant market share. If, for example, we exclude products that are not for retirement savings (like term life insurance, etc.) from the tables on pages 10/11 respectively 8/9 in GDV (2022), their market share is 46% for

In these cases, minimizing the possible negative emotions associated with the choice seems to play an important role (such emotions cannot only arise when making a choice against the subjectively attractive choice but also in the opposite scenario), cf., Larrick (1993), Luce *et al.* (1997), and Luce *et al.* (1999), Loewenstein *et al.* (2001). A "suitable compromise between" the subjectively attractive and the objective choice might be preferable in such a case, since negative emotions will probably be smaller than with the theoretically optimal decision and the decision is objectively better than the subjectively preferred one.

Vast literature shows that people often consider various attributes of choices and tend to make compromises. Most of the literature, however, considers the willingness for compromise in terms of attributes like price vs. quality (rather than in terms of subjective attractiveness vs. objective utility), cf., Simonson (1989) and Simonson & Tversky (1992). Overall, the results of this stream of literature indicate that many people prefer choices which are compromising or middle option in the set of considered options. Verbeke (2006) analyze in the willingness for compromise on taste for health in the case of functional foods, which is closer related to the question analyzed in our paper. Bell *et al.* (1988) describe that people "may be viewed as having divided minds with different aspirations," and "that decision making, even for the individual, is an act of compromise among the different selves". In this spirit, a targeted offer of "compromise products" and tailored advice of these products could help people to make objectively better decision (compared to their subjectively most attractive choice) also in retirement planning. However, it is essential that a suitable compromise exists and that it is included in the set of considered options, otherwise the decision could also be to postpone or to opt for the status quo, cf., Simonson (1992) and Luce (1998).

In this paper, we introduce and discuss an approach which simultaneously considers two different preference formulations; one to asses the preference from an objective, rational point of view, and one to asses the subjective attractiveness at the time of decision making. Our proposed approach is generally applicable (for example with respect to the underlying preference formulations) and is aimed to provide practical aid to identify and design suitable compromises. To the best of our knowledge, there are no similar (quantitative) approaches (e.g., optimization approaches, specific measures or classification systems) in the existing literature which evaluate both aspects simultaneously.

We illustrate our approach by analyzing common unit-linked retirement savings products (with

products with single premium respectively of 56% for products with regular premium payment. Note that the products sold in Germany in this segment mostly fall in the category of static and dynamic hybrid products (cf., Kochanski & Karnarski (2011)) or certain equity-index-linked products (cf., Alexandrova *et al.* (2017)), whereas in this paper we use stylized guaranteed unit-linked products to make our results applicable in a more general framework.

and without terminal guarantees as well as with annual lock-in guarantee features) using the same setting as Ruß & Schelling (2018).[3] We apply Multi Cumulative Prospect Theory (MCPT) which is based on the popular Cumulative Prospect Theory (CPT) introduced by Kahneman & Tversky (1979) and Tversky & Kahneman (1992), and which can explain observed decisions of long investment horizon-investors more accurately, cf., Ruß & Schelling (2018).[4] To assess the objective preference from a rational point of view we consider Expected Utility Theory (EUT).

The application illustrates how products that constitute a suitable compromise between objective utility and subjective attractiveness can be identified and designed. We perform a detailed analysis for various parametrizations of the underlying preference formulations, e.g., with respect to the individual's risk aversion and loss aversion. The results of this application show that loss aversion has a significant influence on the suggested ("favorable") compromise. For moderate values of loss aversion the favorable product combines attributes of the optimal EUT and the preferable MCPT product, which are high stock ratios and low annual guarantees. Hence, in these cases the approach identifies suitable compromises which create a high (albeit not the maximum possible) objective utility while being subjectively appealing. For a high loss aversion, the favorable compromise is very similar to the preferable MCPT product and has an objective utility which is far below the maximum. In this case, our approach indicates limits for the willingness to compromise, i.e., settings in which the subjective attractiveness of certain attributes dominate to such an extent that it is very unlikely that the individual will choose an objectively better product. Especially in these cases, it seems essential to apply further measures to objectively improve the decision, e.g., reducing the impact of loss aversion by suitable information, framing, etc. Overall, the application illustrates how the approach can contribute to the design as well as consulting of retirement savings products to improve decisions.

The remainder of this paper is organized as follows. In Section 2, we motivate the approach, discuss limitations, and introduce the conceptual framework of the simultaneous evaluation. In Section 3, we consider the application to retirement savings. To this end, we specify the

[3]Note that in this paper, we do not consider the application to the annuity payout phase. The specification used for the savings phase cannot be immediately transferred to products paying lifelong or other regular benefits, since, for example, it would require the use of a different certainty equivalent value as the concept of a certainty equivalent return used for the savings phase cannot directly be applied in these circumstances. Moreover, the measurement of subjective (and possibly also objective) attractiveness would need to be adapted. For an alternative possibility to measure objective attractiveness, cf., Alonso-García & Sherris (2019); subjective attractiveness of regular payouts may heavily depend on the way the respective products are framed, cf., Schelling (2018) for potential impact of framing on subjective attractiveness. Nevertheless, the topic becomes increasingly important since innovative (particularly unit-linked guaranteed) products in the retirement phase become more and more popular in many countries, cf., ifa (2021) for an overview over respective products in Germany.

[4]In contrast to CPT, MCPT can explain the demand for products with cliquet or ratch-up guarantees (Ruß & Schelling (2018)), the demand for so-called life-cycle funds which decrease their risk exposure over time (Graf *et al.* (2019)), and the demand for traditional participating life insurance products which make use of collective return smoothing elements (Ruß & Schelling (2021)).

simultaneous evaluation approach, the underlying preference formulations, as well as the capital market and the considered products. Further, we present and discuss the main findings as well as various sensitivity analyses. Section 4 summarizes and gives an outlook for future research.

2 Motivation and Discussion of the Method

In this section, we introduce and discuss the terms of subjective attractiveness and objective utility. Also, we discuss various settings, where our approach is applicable as well as limitations. Further, we identify desirable properties of a "combined preference function" that simultaneously evaluates objective utility and subjective attractiveness. Based on these properties, we develop in the next chapter an exemplary preference function that will be used in the numerical analyses in this paper.

2.1 On the Preference Formulations

Previous research has focused, on the one hand, on how individuals actually make decisions, and on the other hand on how individuals should ideally make decisions. This has given rise to two essential model approaches where the distinction in practice is not always sharp, cf. Slovic *et al.* (1977) and Slovic *et al.* (1988).

Normative models describe how individuals should make decisions. They measure the goodness (often stated in terms of *objective utility*) of a choice using objective and normative criteria under the most complete and correct information set possible. For example, typically normative models fulfill basic principles of rational decision making such as transitivity (if A is better than B, and B is better than C, then A is also better than C). Further, a useful normative model, which is able to describe how an individual should make decisions, can be tuned to the specific characteristics and needs of the individual under consideration (e.g., by the individuals risk aversion or specific constraints). In practice (e.g., for financial decisions), the objective utility of a choice under uncertainty is often measured based on the Expected Utility Theory (EUT).

Descriptive models aim to describe and predict actual decision making based on the *subjective attractiveness*. Subjective attractiveness refers to the individuals' own perception and evaluation of a choice which depends on cognitive capacities and limitations to process information. Mostly subconsciously applied heuristics help individuals to get an (initial) intuitive feeling about choices and to speed up the decision-making process. Common examples are availability, anchoring and the default heuristic, cf., Kahneman *et al.* (1982) and Benartzi & Thaler (2007). While heuristics are often helpful to quickly derive reasonable decisions, they can in some cases lead to severe and systematic biases (i.e., systematic deviations compared to the

objective choice of a normative model), cf., Tversky & Kahneman (1974). In particular, for complex choices under uncertainty, heuristics can lead to cognitive biases like (myopic) loss aversion or probability distortion, which can adversely affect the decision. For example, low probability events can have a disproportionate impact on the decision due to probability distortion. Hence, descriptive models like the popular Cumulative Prospect Theory (CPT) usually incorporate such insights from the field of psychology to better describe actual decision making, cf. Kahneman & Tversky (1979) and Tversky & Kahneman (1992).

It is worth noting that both approaches use models which can only approximate the target variables (objective utility respectively subjective attractiveness). Further, also primal normative models can be modified and applied in such a way that they recognize cognitive limitations and have a better descriptive power. For example, Subjective Expected Utility Theory considers subjective beliefs instead of true probabilities, cf., Savage (1954). Consequently, all applications discussed below depend on a careful and accurate model selection (tailored to the specific application).

2.2 Why and When Do We Need a Compromise?

Normative and descriptive models describe how people should make decisions and how they actually face decisions, respectively. A third stream of literature deals with the prescriptive side of decision making i.e., the question, how to help people to make objectively better decisions recognizing their cognitive biases, cf., Bell *et al.* (1988). One possibility is to present and describe the objective of the decision, the set of choices and related information in such a way that objectively better choices become more appealing. Ideally, such efforts result in objectively optimal decisions. Unfortunately, this is not always the case. Actual decisions can be influenced by subconsciously applied heuristics and negative emotions associated with the choice. Hence, objectively disadvantageous attributes of a choice can have a high subjective attractiveness adversely impacting on the decision. In such cases, compromises which address cognitive concerns but which are objectively better than the subjectively most attractive choice can improve the decision.

Compromises (in the above mentioned sense) are important whenever the objectively best choice is perceived as subjectively very unattractive and at the same time the subjectively attractive choice comes with a significantly lower objective utility. This includes complex decisions under uncertainty where it is difficult for the decision-maker to process relevant information accurately, where cognitive biases have a strong and crucial impact on the decision, and where it is difficult to help people to overcome their cognitive biases. In addition, it is important that the decision-making process allows for information gathering and an act of compromise, i.e., the decision should not be made quickly and solely intuitively. Otherwise, the intuitive choice with

the highest subjective attractiveness will be chosen. This is usually not the case when it comes to important decisions, e.g., when a lot of money is at stake, and where people have a desire to make objectively good decisions which they do not regret later. Also, product providers have an interest to avoid avoid unsatisfied customers and poor reputation in the long run. Hence, they should have an interest to steer their customers away from the subjectively most attractive (but objectively not optimal) product.

In what follows, we will present a model that can help identify suitable compromises in the above sense. This can be helpful for decisions that fulfill the following properties: Firstly, it should be a decision problem in which the objective utility and the subjective attractiveness of the choices can be determined quantitatively. Secondly, our approach is particularly useful when the design of a suitable compromise is not obvious. This is often the case when the choices have a variety of different attributes and specifications. For example, a compromise is rather obvious for an investor who intends to invest his money in bonds and stocks only. In this case, an equal split between bonds and stocks seems to be a reasonable compromise between a pure bond investment (which addresses the need for security) and a pure stock investment (which addresses the desire for gains). This compromise can also be observed to be appealing to many people.[5] However, the identification of a favorable compromise is significantly more difficult, if the set of the considered options also includes, for example, investments with guarantees of different types (e.g., annually or at maturity only) and different levels. Without quantitative methods, it is difficult to assess the combined impacts of the various different attributes on the objective utility and the subjective attractiveness. We therefore believe that particularly in such cases it is important to develop approaches that can identify a suitable compromise, which can then be marketed by product providers and recommended by advisors whenever biases prevent the optimal choice.

2.3 The Simultaneous Evaluation Approach

For the sake of simplicity, we assume a set $\mathcal{C}$ of $n \in \mathbb{N}$ different choices (e.g., products) c_i for $i = 1, \ldots, n$.[6] Let P_1 and P_2 denote two preference formulations. In the remainder of this paper we consider the case that P_1 specifies a normative model and P_2 a descriptive model.[7] Further, we assume that under both preference formulations we can derive unique finite real valued certainty equivalent values (e.g., a fixed amount, a fixed annual return, etc.) for all $c_i \in \mathcal{C}$ and that higher certainty equivalent values are preferred. The use of certainty equivalent values makes it possible to derive quantities under different models that have comparable ranges. We

[5] Balanced funds with a with an approximately equal allocation (50/50 or 60/40) are among the most popular fund investments, cf., Morningstar (2020).

[6] Note that under some additional assumptions the approach can also be extended to a setting with infinitely many choices.

[7] Note that our approach is not restricted to this consideration. For example, one could specify both P_1 and P_2, under EUT, but with different specifications or subjective beliefs.

denote the certainty equivalent values of c_i by $r_i^{P_1}$ and $r_i^{P_2}$. We say that choice c_i *dominates* c_j $(j = 1, \ldots, n)$ if $r_i^{P_1} \geq r_j^{P_1}$ and $r_i^{P_2} \geq r_j^{P_2}$ with at least one of the relations being strict.

Further, we denote the *combined preference function* for the simultaneous evaluation by $K : D_{P_1} \times D_{P_2} \to I$, where D_{P_1}, D_{P_2} are intervals in $\mathbb{R}$ and I is an interval in $\mathbb{R} \cup \{-\infty, \infty\}$. We then use the notation $K(c_i) := K(r_i^{P_1}, r_i^{P_2})$ for the *K-value of choice c_i*. We say that c_i is *strictly preferred* over c_j (denoted as $c_i \succ c_j$) if $K(c_i) > K(c_j)$ and call the relation as *indifferent* if $K(c_i) = K(c_j)$ (denoted as $c_i \sim c_j$). If there is a c_i such that $K(c_i) > K(c_j)$ for all $c_j \in \mathcal{C}$ then we call c_i the *favorable compromise choice*. Next, we introduce some desirable properties of the function K:

1. *Completeness*, i.e., for all $c_i, c_j \in \mathcal{C}$ we have either $c_i \succ c_j$, $c_i \sim c_j$, or $c_j \succ c_i$.

2. *Transitivity*, i.e., for all $c_i, c_j, c_k \in \mathcal{C}$ it holds that if $K(c_i) \geq K(c_j)$ and $K(c_j) \geq K(c_k)$ then $K(c_i) \geq K(c_k)$.

The properties 1) and 2) ensure that the relation $\succ$ defines a *preference order* and are fulfilled since K maps to $\mathbb{R} \cup \{-\infty, \infty\}$.

3. The preference function K suggests only choices which lay on the "efficient frontier", i.e., a choice that is dominated by some other choice can never be the favorable compromise choice, since there exists a better choice. Consequently, K should prevail *preference dominance*, i.e., for all $c_i, c_j \in \mathcal{C}$ it holds that if c_i dominates c_j then $K(c_i) > K(c_j)$, and if $r_i^{P_1} = r_j^{P_1}$ and $r_i^{P_2} = r_j^{P_2}$ then $K(c_i) = K(c_j)$.

4. The combined preference function K is aimed to identify suitable compromises. Hence, it should prefer choices with medium certainty equivalent values under both, P_1 and P_2 over choices with a very high value in one component and a very low value in the other. This can be ensured if the resulting indifference curves of K are (strictly) convex. The property is in line with findings that show that many people prefer choices which are compromising or middle option in the set of considered options, cf. Simonson & Tversky (1992).

5. Lastly, the relation between two choices (c_i and c_j) should not change if other choices are added to or removed from the set of choices $\mathcal{C}$.[8] Note that, while the order of two choices under standard models of rational choice cannot be influenced by the set of choices, this may be different for specific descriptive models, e.g., Prospect Theory (if the reference point or the probability distortion depends on the set of choices), cf., e.g., Guevara &

[8] Formally, this means for all sets of choices $\mathcal{A}$ and $\mathcal{B}$ with $c_i, c_j \in \mathcal{A} \cap \mathcal{B}$ it holds that if $K^{\mathcal{A}}(c_i) \square K^{\mathcal{A}}(c_j)$ then $K^{\mathcal{B}}(c_i) \square K^{\mathcal{B}}(c_j)$ with $\square \in \{>, =\}$. We say that the preference function K specifies preferences which are independent of other choices if this is true for all possible pairs c_i and c_j. The effect that the inclusion of a dominated choice affects the preference order of the remaining choices is documented in the literature as "attraction effect", "decoy effect" or "menu effect", cf., e.g., Huber *et al.* (1982).

Fukushi (2016) for an overview and other examples. Under such a descriptive model, property 5 will be violated (independent of the choice of K).

3 Application for Retirement Savings Products

In this section, we apply our approach to retirement savings products. We closely follow the framework of Ruß & Schelling (2018) and consider several types of common guaranteed products as well as a product without guarantee (Section 3.2). Further, we use the underlying preference formulations discussed therein to describe the objective and the subjective preferences (Section 3.3). Additionally, we specify a concrete choice of the function K (Section 3.4). Finally, we perform extensive numerical analyses (3.5), identify suitable compromises, and discuss the impact of various parameters..

3.1 Motivation for the Application

There is a large amount of literature on the optimal (rational utility-maximizing) design of retirement savings products (e.g., Branger *et al.* (2010), Chen *et al.* (2015)). However, findings from behavioral economics show that numerous human behavioral patterns can cause a deviation between actual and optimal decisions (e.g., Benartzi & Thaler (2007), Beshears *et al.* (2009), Brown *et al.* (2017) or Richter *et al.* (2019) and references therein). Due to its complexity and its long-term nature, decisions in the context of retirement savings seem particularly prone to such deviations. Consequently, the literature shows that there are significant deviations between actual and objectively optimal decisions in retirement savings (cf., e.g., Chen *et al.* (2019), Fuino *et al.* (2020), or Luca *et al.* (2022)). In addition, non-optimal decisions in this context can result in considerable negative consequences on the standard of living in the retirement phase. Therefore, in recent years, an increasing number of studies have examined the question of how to support individuals to make objectively better decisions. For example, framing, i.e., the way products are presented and explained can possibly be used to make products with high objective utility more appealing (cf., Brown *et al.* (2013), Beshears *et al.* (2014) or Brown *et al.* (2021)).

Nevertheless, existing literature in this field typically either analyzes optimal product design from a rational perspective (e.g., Nielsen & Steffensen (2008)) or the question of the perceived subjective attractiveness of retirement savings products at the time of decision making (e.g., Dierkes *et al.* (2010) or Ebert *et al.* (2012)).[9] We argue that ideally, products should be designed in a way that constitutes a suitable compromise between creating a high (albeit

[9]This also applies to other fields of research, for example, in decision making in financial markets (Kliger & Levy (2009)), energy markets (Häckel *et al.* (2017)) or tax evasion (Dhami & Al-Nowaihi (2007)). Similar, for the broadly related stream of literature which focuses on comparing different risk measures, cf., Benoit *et al.* (2013) or Emmer *et al.* (2015).

not the maximum possible) objective utility and being subjectively appealing. This could help to reduce the deviation between the objective utility of the optimal decision and the decision actually made. Beyond product design, the approach can also help financial advisors by providing information about suitable compromise products. This can help to ensure that a compromise product is in the set of considered options at all.

3.2 Financial Market and Considered Products

We choose the financial market model and corresponding products identical to Ruß & Schelling (2018). Hence, we assume a Black & Scholes (1973) financial market model with a risky asset S following a geometric Brownian motion with drift $\mu > r \geq 0$ and volatility $\sigma > 0$, where r denotes the constant interest rate. The portfolio value process V invests the fraction $\theta \in [0, 1]$ in the risky asset S, and the fraction $1 - \theta$ in the risk free asset B with fixed annual return r. We assume continuous rebalancing.[10] The portfolio value process is the basis for all considered products. For all products, we assume a fixed maturity date T and a single premium of 1 paid at the beginning of the contract. The product that invests in the underlying V without guarantee is referred to as *constant mix (cm)* product. For the guaranteed contracts, the investment premium α describes the fraction of the premium that is allocated to the investment V, while the remaining part $1 - \alpha$ is used to finance the guarantee, where the guarantee rate is denoted by g. We only consider fair contracts with an identical initial arbitrage free price of 1.[11]

We consider three types of guarantees: a terminal guarantee and two different annual guarantee features. The payoff at maturity T of a product with a (terminal) roll-up guarantee feature is given by

$$A_T^{rol} := \max\left(e^{gT}, \alpha V_T\right) = \alpha V_T + \left[e^{gT} - \alpha V_T\right]^+.$$

Further, we consider products with annual guarantee features which aim to protect interim gains. The so-called *ratch-up* guarantee feature is defined by the following payoff:

$$A_T^{rat} := \max\left(e^{gT}, \alpha V_1, \ldots, \alpha V_T\right)$$

This product essentially pays the highest portfolio value at any annual lock-in date or a roll-up with rate g, whichever is higher.

[10] For details on the financial market model we refer to Ruß & Schelling (2018).

[11] To be in line with Ebert *et al.* (2012) and Ruß & Schelling (2018) we only consider products with $\alpha \geq 0.6$ and use closed form solutions to calculate the fair annual guaranteed rate g for a given α. For more details and closed form solutions for the arbitrage free prices of the different products we refer to Ruß & Schelling (2018).

Moreover, we define the *cliquet* guarantee feature by the payoff:

$$A_T^{cli} := \alpha \prod_{i=1}^{T} \max\left(e^g, \frac{V_i}{V_{i-1}}\right).$$

In each period, this product earns the greater of the guaranteed rate and the performance of the underlying portfolio.

3.3 Choice of the Preference Formulations

In this section, we briefly introduce and discuss the considered normative and descriptive models which have been chosen to be in line with Ruß & Schelling (2018).

We use EUT with power utility function $u(x) = x^\gamma$ as a normative model which implies constant relative risk aversion (CRRA) for a risk aversion parameter $\gamma \in (0, 1)$. As descriptive model we use MCPT which can better explain observed behavior of long-term investors than CPT.[12] In particular, there are many popular long-term investment products which neither an EUT-investor nor a CPT-investor would buy (cf., Ebert *et al.* (2012), Ruß & Schelling (2018) or Graf *et al.* (2019)).[13] Ruß & Schelling (2018) show that the demand for these products can be explained by taking into account potential interim value changes which is possible under MCPT. MCPT is based on three main components: Firstly, an S-shaped value function, which allows to distinguish between gains and losses with respect to a certain reference point (which in our case is set to the initial premium paid). The risk attitude is controlled by the parameter a and the loss aversion by the parameter λ. Secondly, a probability distortion function with distortion parameter ζ which particularly takes into account that small probabilities are overweighted in decision making. Thirdly, multiple reference points and evaluation periods which assumes that potential future value fluctuations affect a product's subjective attractiveness already at the time of decision making.

In both models we can derive certainty equivalent returns (CE returns) which describe the fixed annual return that an investor would regard equally desirable as the considered contract, cf., Appendix A. More precisely, under EUT, the CE return can be computed by $r^{EUT} = \frac{1}{T}\ln\left((E[X^\gamma])^{\frac{1}{\gamma}}\right)$, where X represents the terminal payoff of the considered choice. Under MCPT it holds that the CE return (r^{MCPT}) solves the following equation

$$0 = \sum_{t=1}^{T} \rho^t \left(e^{r^{MCPT} t} - e^{r^{MCPT}(t-1)}\right)^a - MCPT(c) \quad \text{if } MCPT(c) \geq 0$$

[12] A more detailed description is given in Appendix A.

[13] Note that while CPT often cannot satisfactorily explain the demand for retirement savings products for the savings period, there is vast literature which shows that it provides an explanation why annuitization rates are very low, cf., Hu & Scott (2007), Brown *et al.* (2008), and Chen *et al.* (2019).

$$0 = -\lambda \sum_{t=1}^{T} \rho^t \left| e^{r^{MCPT} t} - e^{r^{MCPT}(t-1)} \right|^a - MCPT(c) \quad \text{if } MCPT(c) < 0,$$

where $MCPT(c)$ denotes the MCPT value of choice c. Note that for preference formulations which only consider the terminal payoff, like EUT, the CE return corresponds to the internal rate of return of the certainty equivalent payoff. However, MCPT-investors derive value from interim value changes (in our case annually), i.e., from the path which results in the terminal payoff. Consequently, the concept of a certainty equivalent payoff does not make sense here since a representative certainty equivalent value must consider interim value changes.[14] As we need to condense the information into one single number, this can either be done in terms of a constant absolute or relative interim value change. In line with Ruß & Schelling (2018) we use the latter which is easy to apply and interpret in the context of investment decisions.[15] However, we also considered instead a constant absolute annual value change and found that the results are qualitatively very similar.

3.4 The Combined Preference Function

In this subsection, we propose an explicit combined preference function which will be used in our analysis. A natural starting point for the choice of the combined preference function K could be the weighted average or a simple transformation of it. However, it can easily be shown, that this - as well as other "simple" functions - would violate some of the previously stated desired properties. In what follows we specify a combined preference function which fulfills all five properties defined above on the set of considered contracts. While we consider the suggested formulation of K appropriate for our applications, we do not claim that it is the only suitable or the best (in whatever sense) formulation.

Firstly, we restrict the set of considered choices $\mathcal{C}$ to products that yield at least a fixed minimal acceptable certainty equivalent return $b \in [-1, \infty)$ under both preference formulations. If not stated otherwise, we set $b = 0$, i.e., we only consider products which have a non-negative CE return under both preference formulation.[16] This constraint is motivated by the fact that we do not consider products as suitable compromises that are particularly unattractive or objectively disadvantageous. Further, it simplifies the notation of the K-function.[17] Secondly, we define

[14] This is very similar to deriving constant consumption streams for preference models which consider actual consumption over multiple periods, cf., e.g., Cocco *et al.* (2005).

[15] Note that the calculation of the CE return is also possible in a setting with regular premiums, cf., Graf *et al.* (2019).

[16] Note that if b depends on the set of choices (e.g., maximum or minimum) property 5 of the K-function could be violated. This can be seen with similar arguments as used to show that, e.g., descriptive models with a choice-depended reference point (to which gains and losses are evaluated) can explain the attraction effect, cf., e.g., Simonson & Tversky (1992).

[17] This constraint could alternatively also be taken into account by a suitable choice of the K-function for negative values. This constraint could alternatively be taken into account by a suitable choice of the K-function

$K : [b, \infty) \times [b, \infty) \to [0, \infty)$ as

$$K(c_i) := (1 - \omega)\Big(r_i^{P_1} - b\Big)^{\beta} + \omega\Big(r_i^{P_2} - b\Big)^{\beta} \tag{1}$$

with preference weight $\omega \in [0, 1]$. In the application, we focus on the case $\omega = 0.5$, which represents the situation that both preference formulations are equally weighted. Further, $\beta \in (0, 1)$ determines the degree of convexity of the indifference curves[18] and we focus on the case $\beta = 0.5$. It is straightforward to show that the properties 1 - 5 hold for all $c_i \in \mathcal{C}$, cf., Appendix B.

3.5 Numerical Analysis

3.5.1 Base case

For the sake of comparability with Ebert *et al.* (2012) and Ruß & Schelling (2018), we consider a five-year investment horizon ($T = 5$) and the following financial parameters unless stated otherwise: $\mu = 0.06$, $\sigma = 0.3$, $r = 0.03$.[19] In all subsequent figures and tables, a product without guarantee is denoted by $\circ$, a roll-up product is denoted by $+$, a ratch-up product is denoted by $\Diamond$, and a cliquet product is denoted by $*$. We investigate 41 stock ratios θ between 0% and 100% in steps of 2.5%. For products with a guarantee, we consider for each $\theta > 0$ eight levels of α between 0.6 and 0.95 in steps of 0.05. Thus, we investigate 41 different products without guarantee and $40 \cdot 8 = 320$ different fairly priced products for each guarantee type.[20] If not stated otherwise, we assume a probability distortion of $\zeta = 0.65$, which is in line with Tversky & Kahneman (1992), and consider different combinations of the other preference parameters (λ, a, γ). All results are based on Monte-Carlo simulations with sample size of 20,000. The favorable product is then determined by computing the CE return under both preference formulations for all products. Subsequently, for each product in the considered set (i.e., both CE returns are at least 0), the K-value is computed with Formula (1) and the favorable product is the one with the highest K-value.

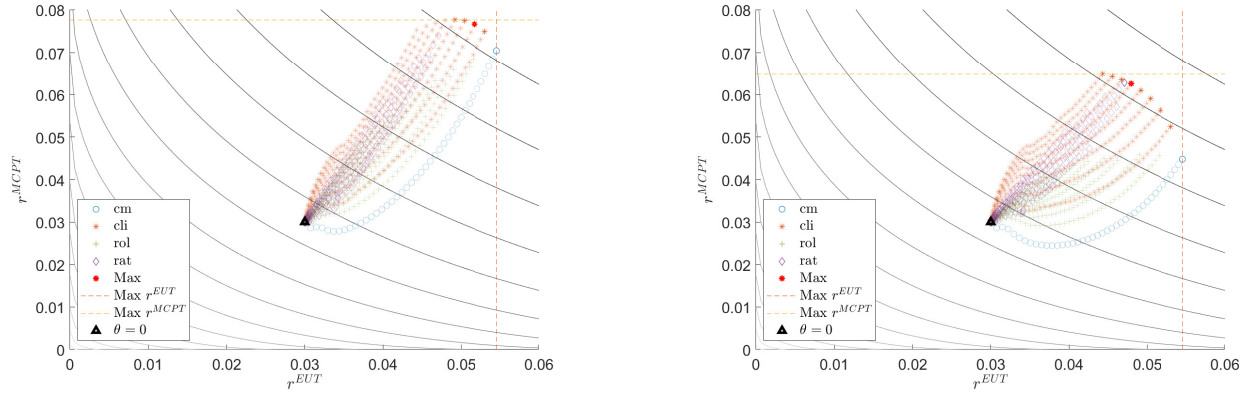
When applying MCPT standalone, more complex guaranteed products, in particular certain cliquet products, are preferred in all considered cases, which is in line with the findings in Ruß & Schelling (2018).

for negative values.

[18] The lower β, the stronger the emphasis on a well balanced compromise.

[19] Detailed descriptions of the product characteristics (guaranteed rates, distribution of terminal values, distribution of annual value changes etc.) can be found in Ebert *et al.* (2012) and Ruß & Schelling (2018). Note that the applications are presented in a rather simple model framework and therefore their aim is not to give actual advise for specific products but rather to illustrate the approach.

[20] Note that fairly priced ratch-up features are not possible for all combinations of θ and α, cf., Ruß & Schelling (2018) for more details.



(a) $\lambda = 1$ (b) $\lambda = 1.25$

(c) $\lambda = 1.75$ (d) $\lambda = 2.25$

Figure 1: The certainty equivalent returns of MCPT and EUT for different levels of loss aversion. The diamond ($\Diamond$) corresponds to ratch-up, the star ($*$) to cliquet, the plus ($+$) to roll-up and the circle ($\circ$) to constant mix products, where the symbols are bold if the corresponding product lies on the efficient frontier. The black triangle denotes a pure investment in the risk-free asset. The red symbol denotes the favorable compromise based on the function K. The dashed vertical line represents the highest EUT CE, while the dashed horizontal line represents the highest MCPT CE.

Based on these results we now investigate favorable compromises under the simultaneous evaluation approach. First, we assume that the risk aversion under EUT coincides with the risk attitude under MCPT, i.e., $\gamma = a$, and we fix it at 0.88, as suggested by Tversky & Kahneman (1992). Note that this is a simplifying assumption as the risk aversion (under EUT) and the risk attitude (under MCPT) are influenced by different aspects. However, we will independently vary γ, a, as well as λ to analyze the impact of the different parameters. Further, we will discuss for which combinations it is possible to find a favorable compromises.

Figure 1 shows the MCPT and EUT CE returns of the products for different levels of loss aversion λ. In this figure, we observe that CE returns of the products follow certain lines, where one line of symbols represents one level of α and all lines start at the same point (black triangle), where $\theta = 0$ and $r^{EUT} = r^{MCPT} = 3\%$. Symbols at the outer end of each line have a higher stock ratio. Moreover, the vertical (horizontal) dotted line shows the highest possible CE under EUT (MCPT). The favorable compromise based on the combined preference function

K is indicated by a red symbol. Further, the solid black lines are indifference curves under the K-measure. We observe that for increasing loss aversion the subjective attractiveness of products which allow for interim losses decreases while the subjective attractiveness of products with a positive annual guarantee remains unchanged. Hence, the favorable compromise also changes with increasing loss aversion.

Without loss aversion ($\lambda = 1$), a cliquet product is favorable (stock ratio of 100%, annual guaranteed rate of -29.15%, i.e., only protecting against very severe annual losses). The stock ratio is the same as for the EUT optimal and MCPT preferable product. However, the guarantee of the favorable compromise is lower than for the MCPT preferable product (-19.58%, i.e., also only protecting against severe annual losses). Compared to the EUT optimal product the EUT CE is slightly lower (5.17% instead of 5.46%) whereas the MCPT CE is higher by about twice the difference (7.66% instead of 7.05%). On the other hand, the EUT CE is 0.25% higher than for the preferable MCPT product while the MCPT CE is only 0.1% lower.

If loss aversion is higher (1.25 or 1.75), the favorable compromise remains a cliquet product with $\theta = 100$, but the guaranteed rate increases to -16.4% or -9.4%, respectively. The MCPT CE returns of the favorable products exceed the MCPT CE returns of the EUT optimal product by a large amount (1.78% respectively 4.66%) while the EUT CE is reduced by a significantly smaller amount (0.66% respectively 1.03%). On the other hand, when the favorable compromise products are compared to the MCPT preferred products, MCPT CE's are only reduced by 0.23% respectively 0.47% whereas EUT CE increases by 0.37% respectively 0.97%. In a nutshell, the favorable compromise has a significantly higher EUT CE than the subjectively most attractive product, but remains attractive for MCPT investors.

For a high loss aversion of 2.25, the favorable compromise coincides with the MCPT preferable product and hence is not a compromise in a strict sense. It is a cliquet product with a positive guarantee (0.14%) and a stock ratio of 50% and comes with an EUT CE of 3.46% and an MCPT CE of 4.79%. This result suggests that a promotion of objectively better products alone might not be sufficient if consumer's subjective evaluations are heavily dominated by a rather high degree of loss aversion. However, in view of the results for a slightly lower loss aversion, it seems promising to take measures to help consumers reduce their loss aversion at least to some degree and combine these measures with product offerings that constitute a suitable compromise for consumers with a slightly lower degree of loss aversion.[21]

So to sum up this first example, the advantages of our approach can be seen particularly in the case $\lambda = 1.75$, where the favorable compromise combining a stock ratio of 100% (as for the

[21] For example, by framing in an appropriate manner, cf., Tversky & Kahneman (1981).

optimal EUT product) and a guarantee (as for the preferable MCPT product) is almost as attractive as the most attractive product, but provides a significantly higher EUT CE. However, a heavy loss aversion can cause the favorable compromise to coincide with the most attractive product under MCPT. This shows the limitations of the approach and leads to the suggestion, that it should be used in combination with approaches that help consumers to overcome their loss aversion at least to some extent.

Next, we analyze the favorable compromise for different values of risk aversion (and attitude) as well as loss aversion. Table 1 displays the optimal product under EUT, the preferred product under MCPT and the favorable compromise under the K-measure for different levels of risk aversion and attitude as well as loss aversion.

We observe that the favorable compromise is always a cliquet or ratch-up product, since these products protect annual gains, which is important under MCPT, cf., also Figure 1. When risk aversion or loss aversion (or both) are rather low (red cells in Table 1) the favorable compromise has a stock ratio of 100% (like the EUT optimal product) and a low annual guarantee (similar to the preferable MCPT product).

For a higher loss aversion and a medium risk aversion (blue cells in Table 1) the favorable compromise is a cliquet product with an annual guaranteed rate slightly above 0 and a stock ratio of 50%. If risk aversion becomes lower, the guaranteed rate becomes negative and the stock ratio increases. This is mainly driven by the fact that for the MCPT preferable product the guaranteed rate decreases and the stock ratio increases as the upside potential is valued higher. On the other hand, if risk aversion becomes higher, the stock ratio decreases and guarantees increase. For very high risk aversion (green cells in Table 1) the favorable compromise is a cliquet product with a guaranteed rate above 1% (1.43% and 1.14%) and a stock ratio of roughly 40%, independent of loss aversion. The compromise has a slightly higher EUT CE than the preferable MCPT product, since the guarantee is lower.

All in all, we observe that the stock ratio (guarantee) of the favorable compromise is decreasing (increasing) for an increasing loss aversion or increasing risk aversion. Further, for a high loss aversion the compromise is very similar or equal to the preferable product of the subjective preference function, because objectively more attractive products come with no or only weak guarantees, which is heavily penalized by a high loss aversion.

3.5.2 Sensitivity analysis

In this section, we present results of various sensitivity analyses.

	EUT optimal	**favorable compromise / preferable MCPT**					
$a = \gamma$		λ	1	1.25	1.75	2.25	2.75
0.6	$\circ(100, 82.5, -, 4.25, [1.47, -1.71])$	K	$*(60, 40, 1.43, 3.26, 3.65)$	$*(60, 40, 1.43, 3.26, 3.65)$	$*(60, 40, 1.43, 3.26, 3.65)$	$*(60, 40, 1.43, 3.26, 3.65)$	$*(60, 40, 1.43, 3.26, 3.65)$
		MCPT	$*(60, 37.5, 1.7, 3.22, 3.67)$	$*(60, 37.5, 1.7, 3.22, 3.67)$	$*(60, 37.5, 1.7, 3.22, 3.67)$	$*(60, 37.5, 1.7, 3.22, 3.67)$	$*(60, 37.5, 1.7, 3.22, 3.67)$
0.65	$\circ(100, 95, -, 4.43, [2.29, -2.61])$	K	$\diamond(75, 100, -5.54, 4.08, 3.19)$	$*(60, 42.5, 1.14, 3.3, 3.75)$	$*(60, 42.5, 1.14, 3.3, 3.75)$	$*(60, 42.5, 1.14, 3.3, 3.75)$	$*(60, 42.5, 1.14, 3.3, 3.75)$
		MCPT	$*(60, 40, 1.43, 3.26, 3.77)$	$*(60, 40, 1.43, 3.26, 3.77)$	$*(60, 40, 1.43, 3.26, 3.77)$	$*(60, 40, 1.43, 3.26, 3.77)$	$*(60, 40, 1.43, 3.26, 3.77)$
0.7	$\circ(100, 100, -, 4.65, [3.26, -3.33])$	K	$\diamond(75, 100, -5.54, 4.21, 4.04)$	$*(60, 45, 0.82, 3.34, 3.88)$	$*(60, 45, 0.82, 3.34, 3.88)$	$*(60, 45, 0.82, 3.34, 3.88)$	$*(60, 45, 0.82, 3.34, 3.88)$
		MCPT	$\diamond(75, 100, -5.54, 4.21, 4.04)$	$*(60, 42.5, 1.14, 3.3, 3.91)$	$*(60, 42.5, 1.14, 3.3, 3.91)$	$*(60, 42.5, 1.14, 3.3, 3.91)$	$*(60, 42.5, 1.14, 3.3, 3.91)$
0.75	$\circ(100, 100, -, 4.88, [4.23, -3.82])$	K	$*(90, 100, -29.15, 4.71, 4.65)$	$\diamond(75, 100, -5.54, 4.35, 3.78)$	$*(60, 47.5, 0.49, 3.39, 4.05)$	$*(60, 47.5, 0.49, 3.39, 4.05)$	$*(60, 47.5, 0.49, 3.39, 4.05)$
		MCPT	$\diamond(75, 100, -5.54, 4.35, 4.94)$	$*(60, 45, 0.82, 3.35, 4.08)$	$*(60, 45, 0.82, 3.35, 4.08)$	$*(60, 45, 0.82, 3.35, 4.08)$	$*(60, 45, 0.82, 3.35, 4.08)$
0.8	$\circ(100, 100, -, 5.1, [5.28, -4.31])$	K	$*(90, 100, -29.15, 4.88, 5.78)$	$\diamond(75, 100, -5.54, 4.48, 4.71)$	$*(60, 50, 0.14, 3.44, 4.28)$	$*(60, 50, 0.14, 3.44, 4.28)$	$*(60, 50, 0.14, 3.44, 4.28)$
		MCPT	$*(75, 100, -16.36, 4.59, 5.89)$	$\diamond(75, 100, -5.54, 4.48, 4.71)$	$*(60, 47.5, 0.49, 3.4, 4.31)$	$*(60, 47.5, 0.49, 3.4, 4.31)$	$*(60, 47.5, 0.49, 3.4, 4.31)$
0.85	$\circ(100, 100, -, 5.33, [6.37, -4.79])$	K	$*(90, 100, -29.15, 5.06, 6.95)$	$\diamond(75, 100, -5.54, 4.62, 5.68)$	$*(60, 50, 0.14, 3.45, 4.6)$	$*(60, 50, 0.14, 3.45, 4.6)$	$*(60, 50, 0.14, 3.45, 4.6)$
		MCPT	$*(80, 100, -19.58, 4.83, 7.06)$	$*(60, 100, -9.38, 4.37, 5.79)$	$*(60, 50, 0.14, 3.45, 4.6)$	$*(60, 50, 0.14, 3.45, 4.6)$	$*(60, 50, 0.14, 3.45, 4.6)$
0.88	$\circ(100, 100, -, 5.46, [7.05, -5.06])$	K	$*(90, 100, -29.15, 5.17, 7.66)$	$*(75, 100, -16.36, 4.8, 6.26)$	$*(60, 100, -9.38, 4.43, 4.32)$	$*(60, 50, 0.14, 3.46, 4.79)$	$*(60, 50, 0.14, 3.46, 4.79)$
		MCPT	$*(80, 100, -19.58, 4.92, 7.76)$	$*(60, 100, -9.38, 4.43, 6.49)$	$*(60, 50, 0.14, 3.46, 4.79)$	$*(60, 50, 0.14, 3.46, 4.79)$	$*(60, 50, 0.14, 3.46, 4.79)$
0.9	$\circ(100, 100, -, 5.55, [7.5, -5.24])$	K	$*(95, 100, -38.05, 5.39, 7.96)$	$*(80, 100, -19.58, 4.98, 6.6)$	$*(60, 100, -9.38, 4.47, 4.84)$	$*(60, 50, 0.14, 3.46, 4.91)$	$*(60, 50, 0.14, 3.46, 4.91)$
		MCPT	$*(80, 100, -19.58, 4.98, 8.23)$	$*(60, 100, -9.38, 4.47, 6.95)$	$*(60, 50, 0.14, 3.46, 4.91)$	$*(60, 50, 0.14, 3.46, 4.91)$	$*(60, 50, 0.14, 3.46, 4.91)$
0.95	$\circ(100, 100, -, 5.78, [8.64, -5.66])$	K	$*(95, 100, -38.05, 5.59, 9.13)$	$*(80, 100, -19.58, 5.12, 7.84)$	$*(60, 100, -9.38, 4.56, 6.12)$	$*(60, 72.5, -3.64, 3.95, 4.91)$	$*(60, 52.5, -0.22, 3.52, 5.23)$
		MCPT	$*(80, 100, -19.58, 5.12, 9.38)$	$*(60, 100, -9.38, 4.56, 8.08)$	$*(60, 100, -9.38, 4.56, 6.12)$	$*(60, 52.5, -0.22, 3.52, 5.27)$	$*(60, 52.5, -0.22, 3.52, 5.23)$
1	$\circ(100, 100, -, 6, [9.76, -6.04])$	K	$*(95, 100, -38.05, 5.79, 10.28)$	$*(85, 100, -23.64, 5.44, 8.89)$	$*(60, 100, -9.38, 4.66, 7.39)$	$*(60, 100, -9.38, 4.66, 5.43)$	$*(60, 57.5, -1, 3.64, 5.54)$
		MCPT	$*(85, 100, -23.64, 5.44, 10.51)$	$*(65, 100, -11.39, 4.81, 9.21)$	$*(60, 100, -9.38, 4.66, 7.39)$	$*(60, 70, -3.17, 3.93, 5.8)$	$*(60, 52.5, -0.22, 3.54, 5.64)$

Table 1: The favorable compromise based on the combined preference function K for different combination of risk aversion, risk attitude and loss aversion. The investment premium, the stock ratio, the guarantee, the EUT CE and the MCPT CE are given in parenthesis, (α in %, θ in %, g in %, EUT CE in %, MCPT CE in %). For the optimal EUT product the MCPT CE is given as a range due to different loss aversions. The diamond ($\diamond$) corresponds to ratch-up, the star ($*$) to cliquet and the circle ($\circ$) to constant mix products. In the green area the favorable compromise has a annual guarantee above 1%. In the red area the favorable compromise has a stock ratio of 100% and a low annual guarantee. In the blue area the favorable compromise has a stock ratio of 50% and a annual guarantee slightly above 0.

Risk aversion and attitude

We will now have a closer look at different combinations of risk aversion (γ) and risk attitude (a) on the favorable compromise by varying a and γ independently. In the base case, for simplicity we had assumed $\gamma = a$. However, while the risk aversion parameter γ in EUT is related to the total payoff, the risk attitude a in MCPT is related to interim value changes. Further, the risk attitude captures risk aversion as well as risk affinity and depends on the reference point, the loss aversion, as well as on the probability weighting. Hence, applying the same values for both does not make them automatically comparable, cf., Harrison & Rutström (2008) for a discussion. Under EUT a risk aversion parameter $\gamma = 0.88$ relates to a constant relative risk aversion of 0.12 which is at the lower end of typical estimates for the relative risk aversion. To account for this we also considered significantly higher relative risk aversions (i.e., smaller values of γ) in the sensitivity analysis. Again, we have performed the analysis for different levels of loss aversion, however, in the remainder we focus on the case of $\lambda = 1.75$ where the results are most interesting. The results are shown in Table 2.

	favorable compromise / preferable MCPT / EUT optimal				
a	γ	0.1	0.3	0.5	0.7
0.65	K	$*(60, 42.5, 1.14, 3.23, 3.75)$	$*(60, 42.5, 1.14, 3.26, 3.75)$	$*(60, 42.5, 1.14, 3.28, 3.75)$	$*(60, 42.5, 1.14, 3.3, 3.75)$
	MCPT	$*(60, 40, 1.43, 3.21, 3.77)$	$*(60, 40, 1.43, 3.23, 3.77)$	$*(60, 40, 1.43, 3.25, 3.77)$	$*(60, 40, 1.43, 3.27, 3.77)$
	EUT	$\circ(100, 37.5, -, 3.56, 0.04)$	$\circ(100, 47.5, -, 3.71, -0.03)$	$\circ(100, 67.5, -, 4, -0.2)$	$\circ(100, 100, -, 4.65, -0.55)$
0.75	K	$*(60, 45, 0.82, 3.25, 4.08)$	$*(60, 47.5, 0.49, 3.31, 4.05)$	$*(60, 47.5, 0.49, 3.34, 4.05)$	$*(60, 47.5, 0.49, 3.38, 4.05)$
	MCPT	$*(60, 45, 0.82, 3.25, 4.08)$	$*(60, 45, 0.82, 3.28, 4.08)$	$*(60, 45, 0.82, 3.31, 4.08)$	$*(60, 45, 0.82, 3.34, 4.08)$
	EUT	$\circ(100, 37.5, -, 3.56, 0.22)$	$\circ(100, 47.5, -, 3.71, 0)$	$\circ(100, 67.5, -, 4, -0.21)$	$\circ(100, 100, -, 4.65, -0.55)$
0.88	K	$*(60, 50, 0.14, 3.29, 4.79)$	$*(60, 50, 0.14, 3.33, 4.79)$	$*(60, 72.5, -3.64, 3.62, 4.54)$	$*(60, 100, -9.38, 4.1, 4.32)$
	MCPT	$*(60, 50, 0.14, 3.29, 4.79)$	$*(60, 50, 0.14, 3.33, 4.79)$	$*(60, 50, 0.14, 3.38, 4.79)$	$*(60, 50, 0.14, 3.42, 4.79)$
	EUT	$\circ(100, 37.5, -, 3.56, 0.75)$	$\circ(100, 47.5, -, 3.71, 0.36)$	$\circ(100, 67.5, -, 4, -0.07)$	$\circ(100, 100, -, 4.65, -0.55)$
0.95	K	$*(60, 92.5, -7.72, 3.19, 5.98)$	$*(60, 100, -9.38, 3.41, 6.12)$	$*(60, 100, -9.38, 3.75, 6.12)$	$*(60, 100, -9.38, 4.1, 6.12)$
	MCPT	$*(60, 100, -9.38, 3.08, 6.12)$	$*(60, 100, -9.38, 3.41, 6.12)$	$*(60, 100, -9.38, 3.75, 6.12)$	$*(60, 100, -9.38, 4.1, 6.12)$
	EUT	$\circ(100, 37.5, -, 3.56, 1.16)$	$\circ(100, 47.5, -, 3.71, 0.78)$	$\circ(100, 67.5, -, 4, 0.28)$	$\circ(100, 100, -, 4.65, -0.55)$

Table 2: The favorable compromise based on the combined preference function K for different combination of risk aversion and risk attitude and for $\lambda = 1.75$. The investment premium, the stock ratio, the guarantee, the EUT CE and the MCPT CE of the products are given in parenthesis, i.e., (α in %, θ in %, g in %, EUT CE in %, MCPT CE in %). The star ($*$) corresponds to cliquet and the circle ($\circ$) to constant mix products. [Updated]

Firstly, we observe that the range of the optimal stock ratio under EUT varies between 37.5% ($\gamma = 0.1$) and 100% ($\gamma = 0.7$) depending on the risk aversion. Also, we observe that the favorable compromise is always a cliquet product with $\alpha = 0.6$ with different stock ratios. While stock ratios are increasing in a, we find that the impact of the risk aversion on the favorable compromise is rather low (at least for $\gamma \geq 0.1$). We observe for nearly all considered combinations of risk aversion and risk attitude, that the favorable compromise is the MCPT preferable product or a very similar product (only with a slightly higher stock ratio) since products with a higher EUT CE return have a significantly lower MCPT CE return. Only for $a = 0.88$ and $\gamma = 0.7$ respectively 0.5, the favorable product is, as in the base case ($a = \gamma =$

0.88), a product with a stock ratio of 100% respectively 72.5% (similar to the EUT optimal product) and the maximum possible guaranteed rate (similar to the MCPT preferred product).

Preference weighting

Next, we have analyzed the effect of preference weighting (ω). We consider the base case with $a = \gamma = 0.88$ and vary λ between 1.25 and 2.75 as well as ω from 0 (pure EUT) to 1 (pure MCPT) for each level of loss aversion, cf., Figure 2. Varying the preference weighting changes the indifference curves which results in different favorable compromises. By increasing ω, the favorable compromise changes along the efficient frontier starting on the vertical line for $\omega = 0$.

We observe that with increasing loss aversion, the value of ω above which the MCPT preferable product constitutes the favorable compromise decreases. Moreover, the favorable compromise comes with lock-in features in most cases (except for $\lambda = 1.25$ and $\omega \leq 0.15$), since loss aversion heavily reduces the attractiveness of products with terminal guarantees only (due to evaluation of annual changes in MCPT). In case of low loss aversion, the favorable compromise always has a stock ratio of 100%. If loss aversion is higher, the stock ratio decreases.

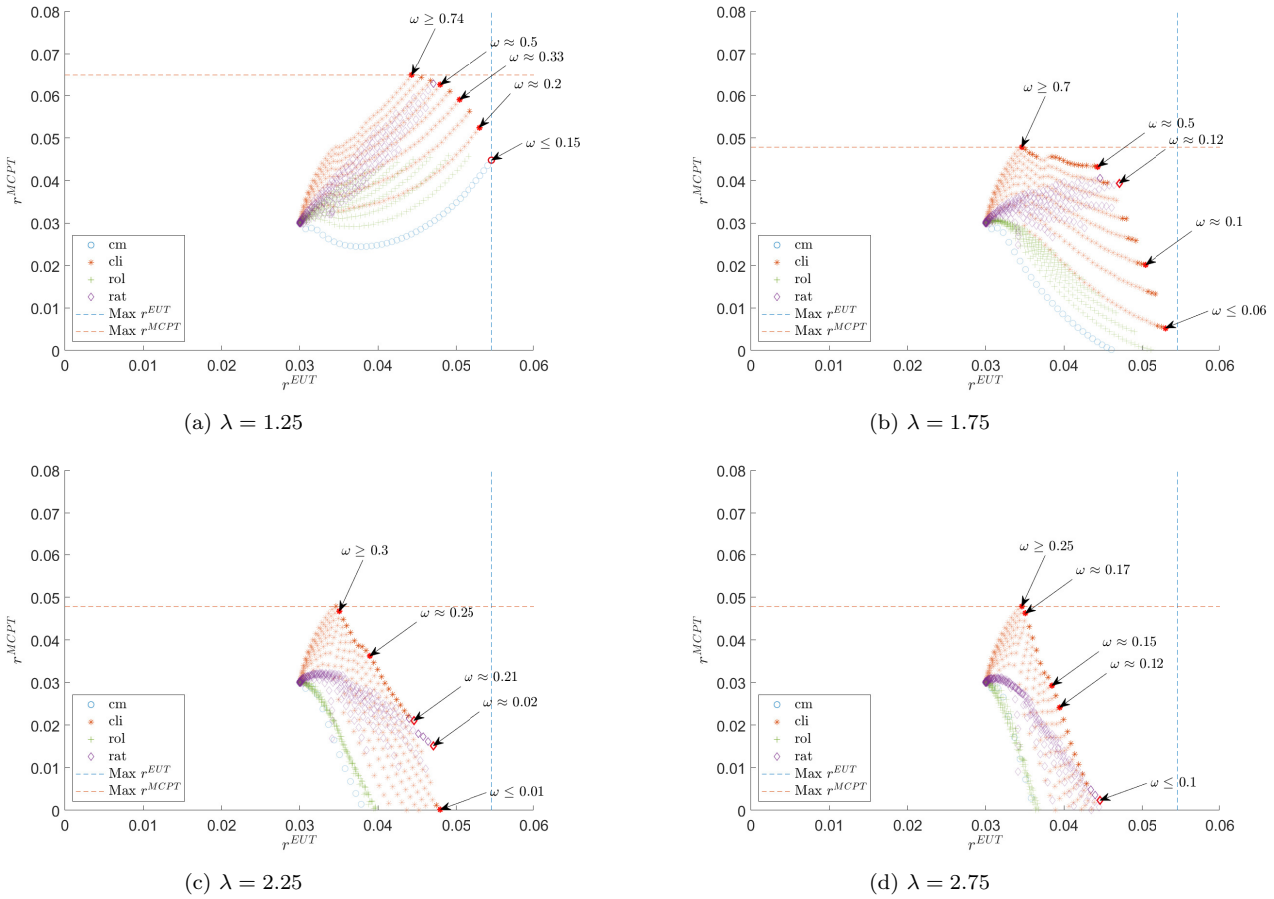
Parameters of the combined preference function

We have also analyzed the effect of different exponents β in the combined preference function K, which also influences the shape of the indifference curves. We find that reasonable choices of β do not significantly influence the favorable compromise.

Moreover, we find that for reasonable choices of the minimum constraint b the favorable compromises do not or only slightly change, i.e., only the stock ratio or the guarantee level of the favorable compromise might be slightly higher or lower.

Probability distortion

Further, we have considered different values for the probability distortion parameter ζ. If $\zeta = 0.5$, the favorable compromise is very similar to the base case, i.e., for medium and low risk aversion and attitude, there is a tendency for higher stock ratios (due to the strong overweighting of high gains). As a consequence, the favorable compromises have a higher EUT CE than in the base case. If $\zeta = 0.8$, the favorable compromise is a pure stock investment in case of no loss aversion and a rather low risk aversion and attitude (which is also the optimal EUT product). For other loss aversion as well as combinations of risk aversion and attitude, the favorable compromise is similar as in the base case (only the stock ratio is slightly lower and the guarantee is slightly higher). The reason is that the probability of large gains is less overweighted and the probability of medium losses is less underweighted, which makes higher



(a) $\lambda = 1.25$

(b) $\lambda = 1.75$

(c) $\lambda = 2.25$

(d) $\lambda = 2.75$

Figure 2: The certainty equivalent returns of MCPT and EUT for different level of loss aversion. The diamond ($\Diamond$) corresponds to ratch-up, the star ($*$) to cliquet, the plus ($+$) to roll-up and the circle ($\circ$) to constant mix products, where the symbols are bold if the corresponding product lies on the efficient frontier. The red symbol denotes the favorable compromise for different values of ω. The dashed vertical line represents the highest EUT CE, while the dashed horizontal line represents the highest MCPT CE.

guarantees and lower stock ratios more attractive.

If there is no probability distortion, the favorable compromise for very risk averse consumers is a product without guarantee and a very low stock ratio (5% or 7.5%) for all levels of loss aversion, because the probability of a loss is close to zero. In this case, the favorable compromise comes with no guarantee (as the optimal EUT product) and a low stock ratio (as the preferable MCPT product). The favorable compromise is a pure stock investment in case of no or low loss aversion and rather low risk aversion and attitude (coinciding with the optimal EUT product). Otherwise, the favorable compromise is a cliquet product with similar guarantee rates (close to zero) as in the base case, but with a lower stock ratio, since the probabilities of large gains are not overweigthed. Again, the guarantees of the favorable compromises are lower than the guarantees of the preferable MCPT products.

Financial market

Next, we have performed sensitivity analyses with respect to the financial market parameters μ, σ and r. If we change σ from 0.3 to 0.1, the favorable compromise is a cliquet product or a product without guarantee. For consumers with loss aversion, the K-measure suggests cliquet products with a stock ratio of 100% and a guaranteed rate between -6.6% and 0.5%, where the guaranteed rate is increasing for an increasing loss aversion. The stock ratios of the favorable compromise are higher than in the base case, because guarantees are less expensive as the volatility of the stock market is lower. Moreover, for high loss aversion, the favorable compromises are similar to the preferable MCPT product. For no loss aversion a pure stock investment (optimal EUT product) is favorable, since the probability of large losses is reduced by the low stock market volatility.

If the risk-free interest rate and the drift of the stock market are lower ($r = 1\%$ and $\mu = 4\%$), as in the base case only cliquet and ratch-up products are favorable. For a low loss aversion and a high risk aversion the favorable compromise is a ratch-up with a stock ratio close to 100% and a low guarantee, i.e., if guarantees are more expensive, only protection against high annual losses is desired. Similarly, if loss aversion remains low and risk aversion is only medium or low, the stock ratio increases up to 100% and guarantees are further reduced to levels between -40.05% and -21.58%. For a high loss aversion, the K-measure suggests a cliquet product with a stock ratio of roughly 30% and a guarantee close to 0. Here, the loss aversion results in guarantees close to zero which can now only be afforded for low stock ratios.

Time horizon

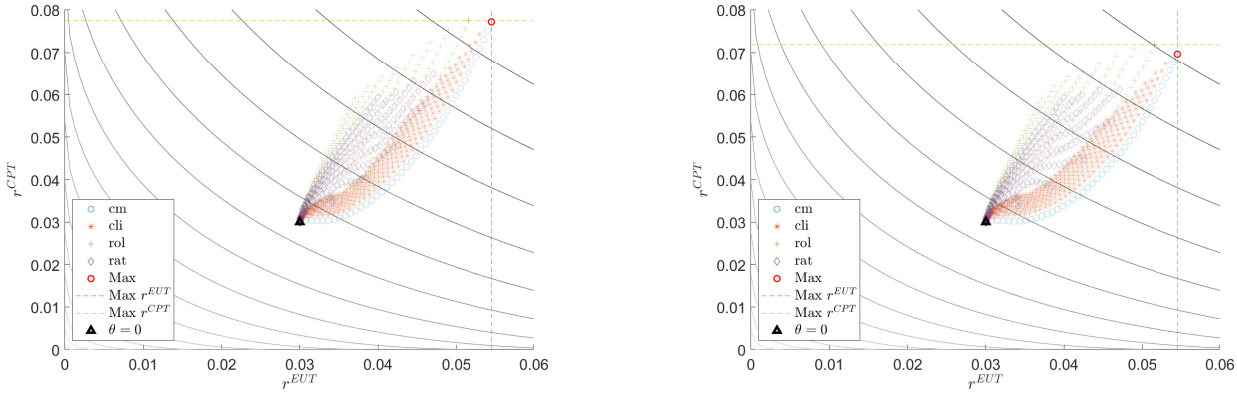
Next, we extended the time horizon of the investment period (T) to 15 years and computed the favorable compromises. We found a similar structure as for the base case: for low and medium loss aversion, the approach works well and identifies suitable compromises with similar characteristics as in the base case whereas for a high loss aversion, the favorable compromise is the product with the highest subjective attractiveness.

Combined preference function

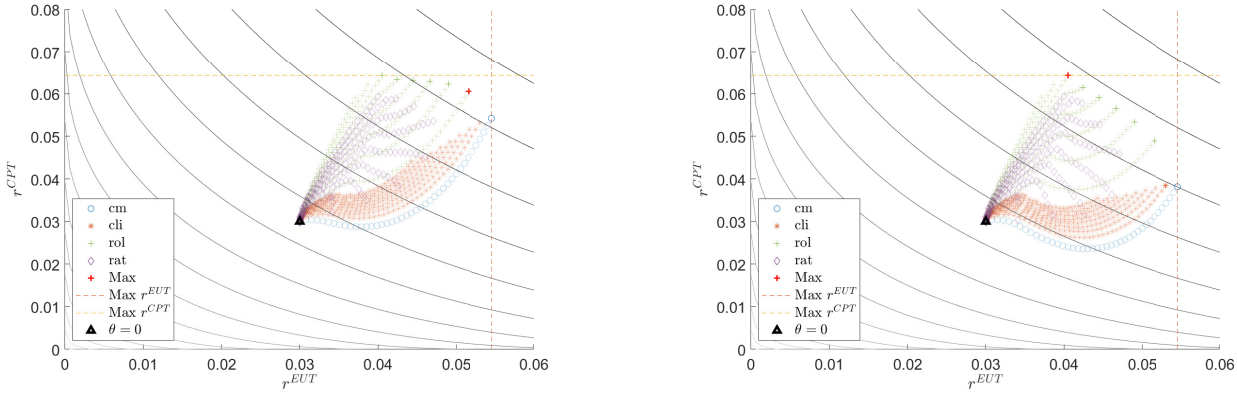
We also analysed the effect of a different combined preference function K which considers the time value of money. More precisely, we consider the following K-function:

$$K(c_i) := \frac{1}{2}\sqrt{\left(1 + r_i^{P_1}\right)^T - 1} + \frac{1}{2}\sqrt{\left(1 + r_i^{P_2}\right)^T - 1}. \tag{2}$$

Note that also this function fulfills the desired properties (1) - (5) and the proofs are similar as for the K-function in the base case. Based on this K-function, we have performed the same analysis as in Section 3.5.1 and we found that the favorable compromises are very similar as



(a) $\lambda = 1$ (b) $\lambda = 1.25$



(c) $\lambda = 1.75$ (d) $\lambda = 2.25$

Figure 3: The certainty equivalent returns of CPT and EUT for different levels of loss aversion. The diamond ($\Diamond$) corresponds to ratch-up, the star ($*$) to cliquet, the plus ($+$) to roll-up and the circle ($\circ$) to constant mix products, where the symbols are bold if the corresponding product lies on the efficient frontier. The black triangle denotes a pure investment in the risk-free asset. The red symbol denotes the favorable compromise based on the function K. The dashed vertical line represents the highest EUT CE, while the dashed horizontal line represents the highest CPT CE.

in the base case setting. This holds for all combinations of risk aversion, risk attitude and loss aversion.

Subjective preference function

Finally, we changed the subjective preference function from MCPT to CPT, where only the terminal value is considered (cf., Appendix A). As in the base case, we investigate which product constitutes the most favorable compromise in the base case with $a = \gamma = 0.88$ and λ between 1 and 2.25. Figure 3 shows the CPT and EUT CE returns of the products under consideration for different levels of loss aversion similar to Figure 1.[22]

In case of no or low loss aversion ($\lambda = 1$ or 1.25), the optimal product under EUT (pure stock

[22]Additionally, we have also analyzed the favorable compromise for different values of risk aversion (and attitude) as well as loss aversion in the CPT case. The results are displayed in Appendix C in Table 3 (in the same fashion as Table 1 in the MCPT case).

investment) and the preferred product under CPT (roll-up with a stock ratio of 100% and only a very weak guaranteed benefit of 0.62) are very similar (and hence differ only marginally in their CE returns). As a consequence, also the favorable compromise is a pure stock investment.

For a medium loss aversion ($\lambda = 1.75$), the favorable compromise still comes with a stock ratio of 100%, but now includes a roll-up feature with a guaranteed benefit of 0.62. Hence it combines features of the optimal EUT product (100% stock ratio) and the preferable CPT product (albeit with a lower guaranteed roll-up benefit than for the preferable CPT product). This again illustrates the benefit of the approach. We find that the objectively optimal product (pure stock investment) is significantly less attractive than the preferred product (CPT CE of 5.42% instead of 6.43%). Consequently, a consumer would very likely avoid this product (even if it is recommended by an advisor). On the other hand, the preferred product under CPT reduces the expected utility heavily compared to the optimal product (EUT CE of 4.05% instead of 5.46%). The favorable compromise product, however, is subjectively more attractive than the EUT-optimal product (CPT CE of 6.06% as opposed to 5.42%) while providing a significantly higher expected utility than the preferred product under CPT (EUT CE of 5.16% as opposed to 4.05%). Hence, promoting this product could help consumers to make an objectively better (although not optimal) choice.

If loss aversion is even higher ($\lambda = 2.25$), the subjective attractiveness under CPT of most products with low or without guarantees, including the EUT optimal product, is reduced heavily. As a consequence the preferred product under CPT coincides with the favorable compromise in this setting. This is a roll-up product with a stock ratio of 100% and a guaranteed benefit above the reference point. The EUT CE of this product is 4.05% and the CPT CE is 6.43%.[23]

To sum up, the proposed combined preference function can in most cases find a favorable compromise which is different to the most attractive product in the setting of retirement saving products. Also, it is possible to adjust the preference theories which describe the normative or descriptive model and still find compromises. Moreover, the results are robust to reasonable parameter changes of the proposed combined preference function (b, β, and ω). Hence, the exact specification of K (which might be rather difficult to estimate) has less influence on the compromise than the (objective and subjective) preference formulations on which there is a large amount of literature.

[23]Since the guaranteed benefit of this product is greater than 1, no losses can occur and hence loss aversion has no impact on this product's CE return.

4 Conclusion and Outlook

In this paper, we have proposed an approach designed to identify choices that constitute a suitable compromise between a theoretically optimal choice (which might be rejected by consumers due to behavioral biases) and the subjectively most attractive choice (which might come with a rather low objective utility). Ideally, such a compromise should be subjectively more attractive than the objectively optimal choice while providing a higher objective utility than the subjectively preferred choice. We have stated desirable properties of such compromises and - based on these properties - proposed a methodology that can be applied using two arbitrary preference functions.

We have illustrated our approach in an applications to retirement savings products where EUT was used as an objective and (M)CPT as a subjective preference function and identified "favorable compromise products". The results of the applications show that the approach can in many cases identify suitable compromises which fit consumers needs while at the same time being subjectively attractive.

Our results indicate that the degree of loss aversion has a particularly high impact on the identified compromise product. For individuals with very low loss aversion, the suggested compromise is identical or very similar to the EUT optimal product. In such cases, no compromise might be required in the first place and thus no actions are necessary to improve the consumer's decision. For moderate values of loss aversion, the approach seems to work particularly well, identifying choices that combine characteristics of the objectively optimal product and the subjectively preferred product. As a consequence, the suggested compromise products are subjectively more attractive than the objectively optimal choice while providing a higher objective utility than the subjectively preferred choice. This is an important result as it shows that we can find suitable compromises for many individuals. Hence, for individuals with reasonable degree of loss aversion, marketing the favorable compromise might already be sufficient to significantly improve consumers' decisions. For high loss aversion, however, the "favorable compromise" is given by the subjectively preferred product. Hence, for such individuals, the willingness to compromise might not be sufficient. We therefore propose to combine our approach with other measures which are suitable to reduce the degree of loss aversion at least to some extent. These measures may consist of appropriate financial advise/education or suitable framing which can reduce the fear of (interim) losses, or even of incentives (government subsidies, etc.) followed by marketing of products which are suitable compromises for consumers with more moderate loss aversion.

Due to the demographic change in many countries, private old-age provision will be increasingly important to maintain a desired standard of living. Consequently, many countries promote private old-age provision (including occupational old-age provision) e.g., by government subsidized

schemes or tax advantages. In this context, the development of retirement savings products that are accepted by consumers and at the same time objectively fit their needs is of high relevance. Hence our findings should be of importance to legislators, product providers, as well as financial advisors.

The paper provides numerous suggestions for further research: Firstly, it is important to investigate under which assumptions (e.g., with respect to the underlying preference formulations and their weights) and to what extent people will actually accept suitable compromises. In this context, it would also be interesting to investigate which additional actions can improve individuals' decision making. Also, it seems worthwhile to analyze whether the strong impact of loss aversion can be confirmed empirically. Secondly, future research should identify suitable compromises based on more realistic product designs (e.g., including products with collective savings elements that play an important role in many countries) and particularly including the decumulation period (i.e., the annuitization decision and the design of annuity products in the payout phase).

A (Multi) Cumulative Prospect Theory

In Cumulative Prospect Theory (CPT) an investment A with (random) final outcomes E is valued with an S-shaped value function v and with respect to a given reference point χ. The gains and losses are described by the random variable $X := E - \chi$. Then the CPT utility is defined as

$$CPT(X) = \int_{-\infty}^{0} v(x) d\left(w\left(F(x)\right)\right) + \int_{0}^{\infty} v(x) d\left(-w\left(1 - F(x)\right)\right), \tag{3}$$

where $F(x) = \mathbb{P}(X \leq x)$ and v is the investor's value-function which is defined as $v(x) := x^a \mathbb{1}\{x \geq 0\} - \lambda |x|^a \mathbb{1}\{x < 0\}$ where $\lambda > 0$ is the loss aversion parameter and $a \in \mathbb{R}_+$ controls the risk appetite. The probability distortion function is given by $w(p) := \frac{p^\zeta}{\left(p^\zeta + (1-p)^\zeta\right)^{\frac{1}{\zeta}}}$ with $\zeta \in (0.28, 1]$, where the lower boundary for ζ is chosen such that w is strictly monotonically increasing for $p \in [0, 1]$.

In Multi Cumulative Prospect Theory (MCPT) the annual gains and losses (X_t) of an investment A are taken into account, i.e., $X_t := A_t - \chi_t$, where $t \in \{1, \ldots, T\}$, T is the maturity of the investment, A_t is the account value at time t and χ_t is the reference point at time t. The MCPT value of investment A is then defined by

$$MCPT(A) := \sum_{t=1}^{T} \eta^t CPT(X_t) \tag{4}$$

with a discounting parameter $\eta \in \mathbb{R}_+$ and with $CPT(X)$ as defined in (3).

B The Properties of the Combined Preference Function

In what follows, we show that the proposed combined preference function fulfills the five properties from Section 2.3 for all choices which are in the consideration set $\mathcal{C}$. The first two properties hold, since the real numbers are ordered (and $K(c_i) \geq 0$ for all $c_i \in \mathcal{C}$). The validation of the second statement of Property 3 is staightforward (i.e., if $r_i^{P_1} = r_j^{P_1}$ and $r_i^{P_2} = r_j^{P_2}$ then $K(c_i) = K(c_j)$). Further, it holds that if c_i dominates c_j then $r_i^{P_1} > r_j^{P_1} \geq b$ and $r_i^{P_2} > r_j^{P_2} \geq b$, since K is increasing in both dimensions as

$$\frac{\partial K}{\partial r_i^{P_1}} = (1 - \omega)\beta (r_i^{P_1} - b)^{\beta - 1} \tag{5}$$

$$\frac{\partial K}{\partial r_i^{P_2}} = \omega\beta (r_i^{P_2} - b)^{\beta - 1}. \tag{6}$$

It follows that $K(c_i) > K(c_j)$. For property 4, we know that choices on an indifference line have the same K-value, this means that if K is fixed then $r_i^{P_2}$ is a function of $r_i^{P_1}$, i.e.,

$$
\begin{aligned}
r_i^{P_2} &= f(r_i^{P_1}) = \left(\frac{K-(1-\omega)(r_i^{P_1}-b)^{\beta}}{\omega}\right)^{1/\beta} + b \\
f'(r_i^{P_1}) &= -\omega^{-1/\beta}\left(K-(1-\omega)(r_i^{P_1}-b)^{\beta}\right)^{1/\beta-1}(1-\omega)(r_i^{P_1}-b)^{\beta-1} \\
f''(r_i^{P_1}) &= -\frac{1-\omega}{\omega^{1/\beta}}\left[-\frac{1-\beta}{\beta}\left(K-(1-\omega)(r_i^{P_1}-b)^{\beta}\right)^{1/\beta-2}\beta(1-\omega)\left((r_i^{P_1}-b)^{\beta-1}\right)^2\right. \\
&\quad \left. + \left(K-(1-\omega)(r_i^{P_1}-b)^{\beta}\right)^{1/\beta-1}(\beta-1)(r_i^{P_1}-b)^{\beta-2}\right] \\
&= \frac{1-\omega}{\omega^{1/\beta}}\left[(1-\beta)\left(K-(1-\omega)(r_i^{P_1}-b)^{\beta}\right)^{1/\beta-2}(1-\omega)\left((r_i^{P_1}-b)^{\beta-1}\right)^2\right. \\
&\quad \left. + \left(K-(1-\omega)(r_i^{P_1}-b)^{\beta}\right)^{1/\beta-1}(1-\beta)(r_i^{P_1}-b)^{\beta-2}\right] > 0,
\end{aligned}
$$

since $r_i^{P_1} - b > 0$, $K-(1-\omega)(r_i^{P_1}-b)^{\beta} > 0$ and $1-\beta > 0$. If we plug in our values $b = 0$, $\omega = 0.5$ and $\beta = 0.5$, we get:

$$
\begin{aligned}
r_i^{P_2} &= f(r_i^{P_1}) = \left(2K-(r_i^{P_1})^{0.5}\right)^2 = 4K^2 - 4K(r_i^{P_1})^{0.5} + r_i^{P_1} \\
f'(r_i^{P_1}) &= -2K(r_i^{P_1})^{-0.5} + 1 \\
f''(r_i^{P_1}) &= K(r_i^{P_1})^{-1.5} > 0
\end{aligned}
$$

Lastly, the definition of the K function as well as the underlying preference functions are independent of the set $\mathcal{C}$ and thus Property 5 holds.

C Additional numerical results

	EUT optimal	**favorable compromise / preferable CPT**					
$a = \gamma$		λ	1	1.25	1.75	2.25	2.75
0.6	∘(100, 82.5, −, 4.25, [2.45, −0.06])	K	+(65, 100, 1.04, 3.61, 4.34)	+(65, 100, 1.04, 3.61, 4.34)	+(65, 100, 1.04, 3.61, 4.34)	+(65, 100, 1.04, 3.61, 4.34)	+(65, 100, 1.04, 3.61, 4.34)
		CPT	+(60, 100, 1.07, 3.51, 4.44)	+(60, 100, 1.07, 3.51, 4.44)	+(60, 100, 1.07, 3.51, 4.44)	+(60, 100, 1.07, 3.51, 4.44)	+(60, 100, 1.07, 3.51, 4.44)
0.65	∘(100, 95, −, 4.43, [3.47, −0.07])	K	+(65, 100, 1.04, 3.65, 4.64)	+(65, 100, 1.04, 3.65, 4.64)	+(65, 100, 1.04, 3.65, 4.64)	+(65, 100, 1.04, 3.65, 4.64)	+(65, 100, 1.04, 3.65, 4.64)
		CPT	+(60, 100, 1.07, 3.54, 4.67)	+(60, 100, 1.07, 3.54, 4.67)	+(60, 100, 1.07, 3.54, 4.67)	+(60, 100, 1.07, 3.54, 4.67)	+(60, 100, 1.07, 3.54, 4.67)
0.7	∘(100, 100, −, 4.65, [4.51, −0.03])	K	∘(100, 100, −, 4.65, 4.51)	+(65, 100, 1.04, 3.7, 4.96)	+(65, 100, 1.04, 3.7, 4.96)	+(65, 100, 1.04, 3.7, 4.96)	+(65, 100, 1.04, 3.7, 4.96)
		CPT	+(65, 100, 1.04, 3.7, 4.96)	+(65, 100, 1.04, 3.7, 4.96)	+(65, 100, 1.04, 3.7, 4.96)	+(65, 100, 1.04, 3.7, 4.96)	+(65, 100, 1.04, 3.7, 4.96)
0.75	∘(100, 100, −, 4.88, [5.41, 0.09])	K	∘(100, 100, −, 4.88, 5.41)	+(95, 100, 0.62, 4.69, 4.81)	+(70, 100, 1.01, 3.88, 5.22)	+(70, 100, 1.01, 3.88, 5.22)	+(70, 100, 1.01, 3.88, 5.22)
		CPT	+(95, 100, 0.62, 4.69, 5.49)	+(65, 100, 1.04, 3.74, 5.3)	+(65, 100, 1.04, 3.74, 5.3)	+(65, 100, 1.04, 3.74, 5.3)	+(65, 100, 1.04, 3.74, 5.3)
0.8	∘(100, 100, −, 5.1, [6.31, 0.66])	K	∘(100, 100, −, 5.1, 6.31)	+(95, 100, 0.62, 4.87, 5.74)	+(70, 100, 1.01, 3.95, 5.68)	+(70, 100, 1.01, 3.95, 5.68)	+(70, 100, 1.01, 3.95, 5.68)
		CPT	+(95, 100, 0.62, 4.87, 6.37)	+(95, 100, 0.62, 4.87, 5.74)	+(70, 100, 1.01, 3.95, 5.68)	+(70, 100, 1.01, 3.95, 5.68)	+(70, 100, 1.01, 3.95, 5.68)
0.85	∘(100, 100, −, 5.33, [7.19, 1.54])	K	∘(100, 100, −, 5.33, 7.19)	∘(100, 100, −, 5.33, 6.41)	+(95, 100, 0.62, 5.05, 5.46)	+(70, 100, 1.01, 4.01, 6.15)	+(70, 100, 1.01, 4.01, 6.15)
		CPT	+(95, 100, 0.62, 5.05, 7.24)	+(95, 100, 0.62, 5.05, 6.65)	+(70, 100, 1.01, 4.01, 6.15)	+(70, 100, 1.01, 4.01, 6.15)	+(70, 100, 1.01, 4.01, 6.15)
0.88	∘(100, 100, −, 5.46, [7.72, 2.17])	K	∘(100, 100, −, 5.46, 7.72)	∘(100, 100, −, 5.46, 6.97)	+(95, 100, 0.62, 5.16, 6.06)	+(70, 100, 1.01, 4.05, 6.43)	+(70, 100, 1.01, 4.05, 6.43)
		CPT	+(95, 100, 0.62, 5.16, 7.74)	+(95, 100, 0.62, 5.16, 7.19)	+(70, 100, 1.01, 4.05, 6.43)	+(70, 100, 1.01, 4.05, 6.43)	+(70, 100, 1.01, 4.05, 6.43)
0.9	∘(100, 100, −, 5.55, [8.06, 2.62])	K	∘(100, 100, −, 5.55, 8.06)	∘(100, 100, −, 5.55, 7.34)	+(95, 100, 0.62, 5.23, 6.45)	+(85, 100, 0.84, 4.72, 6.03)	+(70, 100, 1.01, 4.08, 6.62)
		CPT	+(95, 100, 0.62, 5.23, 8.07)	+(95, 100, 0.62, 5.23, 7.54)	+(85, 100, 0.84, 4.72, 6.64)	+(70, 100, 1.01, 4.08, 6.62)	+(70, 100, 1.01, 4.08, 6.62)
0.95	∘(100, 100, −, 5.78, [8.9, 3.79])	K	∘(100, 100, −, 5.78, 8.9)	∘(100, 100, −, 5.78, 8.23)	+(95, 100, 0.62, 5.42, 7.41)	+(90, 100, 0.75, 5.12, 6.74)	+(80, 100, 0.91, 4.59, 6.69)
		CPT	∘(100, 100, −, 5.78, 8.9)	+(95, 100, 0.62, 5.42, 8.4)	+(90, 100, 0.75, 5.12, 7.49)	+(75, 100, 0.96, 4.36, 7.09)	+(70, 100, 1.01, 4.14, 7.08)
1	∘(100, 100, −, 6, [9.71, 5])	K	∘(100, 100, −, 6, 9.71)	∘(100, 100, −, 6, 9.1)	+(95, 100, 0.62, 5.6, 8.33)	+(95, 100, 0.62, 5.6, 7.41)	+(90, 100, 0.75, 5.27, 6.99)
		CPT	∘(100, 100, −, 6, 9.71)	+(95, 100, 0.62, 5.6, 9.21)	+(90, 100, 0.75, 5.27, 8.35)	+(85, 100, 0.84, 4.97, 7.78)	+(75, 100, 0.96, 4.44, 7.6)

Table 3: The favorable compromise based on the combined preference function K for different combination of risk aversion, risk attitude and loss aversion. The investment premium, the stock ratio, the guaranteed terminal value, the EUT CE and the CPT CE are given in parenthesis, i.e., (α in %, θ in %, $\exp(gT)$, EUT CE in %, CPT CE in %). For the optimal EUT product the CPT CE is given as a range due to different loss aversions. The plus (+) corresponds to roll-up and the circle (∘) to constant mix products. In the blue area the favorable compromise is a pure stock investment. In the red area the favorable compromise is a roll-up with 100% stock ratio and terminal guarantee of 1.04.

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5 The Benefits of Return Smoothing in Insurer's Cover Funds – Analyses From a Client's Perspective

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The Benefits of Return Smoothing in Insurer's Cover Funds – Analyses From a Client's Perspective

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Abstract

Traditional life insurance typically uses some mechanism that is aimed at smoothing the returns of the (collective) assets in the insurer's so-called cover fund. We consider a generic smoothing mechanism and numerically analyze how it impacts the risk-return characteristics of a traditional life insurance contract distinguishing between pathwise volatility (of the annual returns) and the volatility of terminal wealth. We find that pathwise volatility is significantly reduced while the distribution of terminal wealth is hardly affected. We conclude that using multiple segregated cover funds (that come with different asset allocations) as building blocks for more complex products enables insurers to offer a variety of risk-return profiles of terminal wealth in combination with a rather low pathwise volatility (compared to investments without smoothing mechanism). This increases subjective attractiveness for a typical consumer. We consider a variety of such products (static and dynamic investment products) and compare them to similar purely market-based products that do not use an insurer's cover fund. Analyzing risk-return characteristics, (objective) utility, and (subjective) attractiveness under Cumulative Prospect Theory and extensions of it, we conclude that products that become possible by implementing multiple segregated cover funds can increase both, objective utility and subjective attractiveness.

Keywords: Retirement Savings, Collective Assets, Return Smoothing, Insurer's Cover Fund, Cumulative Prospect Theory
JEL: D14, G11, G22, G41, J26, J32

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1 Introduction

Due to the demographic change and the resulting challenges for pay-as-you-go pension systems, private, capital-funded retirement savings become increasingly important. In this segment, many consumers prefer products with relatively high guarantees and low interim fluctuations. In the current low-interest environment, such products can either no longer be offered at all or come with a very limited return potential, which results in a rather low objective utility for consumers. Since the attractiveness of products is crucial for the actual choice of the consumer, insurance companies and other providers of retirement savings products try to offer products that are (subjectively) attractive for consumers without limiting the upside potential too much.

The problems described above particularly apply to participating life insurance products (also referred to as with-profit life insurance products) which are often equipped with year-to-year guarantees.[1] The resulting protection against interim losses is important to many consumers and appears to be one main reason why these products have been very popular in many countries, cf., Ruß & Schelling (2018). Another key characteristic of participating life insurance products is the participation in the return of a "collective investment" of the insurer (which we refer to as the insurer's "cover fund"). The return of this fund is stabilized by various return smoothing mechanisms. The results in Goecke (2013) and Ruß & Schelling (2021) show that collective return smoothing mechanisms alone (that is, without additional guarantees) could already heavily reduce short-term fluctuations while preserving the long-term risk-return characteristics of the underlying fund. Hence, Ruß & Schelling (2021) conclude that products with collective smoothing mechanisms (but without guarantee) might be (subjectively) similarly attractive to long-term investors as products with formal guarantees. Since they can be offered in any interest rate environment without the burdens imposed to the insurer by formal guarantees (e.g., high capital requirement), they might play an important role in overcoming the above mentioned problems. Further, there is a considerable variety of literature dealing with smoothing mechanisms: Hansen & Miltersen (2002) discuss the effect of a smoothed surplus participation on the minimum guarantee rate in Danish pension plans. They analyze fair contract settings between consumers and the insurer and consider the consequences of pooling for a group of inhomogeneous consumers. Guillén *et al.* (2006) define a smoothing mechanism where contingent claims pricing techniques can be used to compute fair market values of contracts. Løchte Jørgensen (2007) considers a specific return smoothing mechanism and shows that the resulting smoothed returns follow approximately a lognormal distribution. Moreover, Maurer *et al.* (2016) analyze the actuarial techniques to smooth reporting of firm assets and liabilities and find that smoothing adds value to both the policyholder and the insurer. Also, Lichtenstern & Zagst (2021) consider smoothing in the post-retirement phase by means of a

[1] Participating life insurance contracts have been discussed extensively in the literature, cf., Bauer *et al.* (2006), Gatzert & Kling (2007), and Bohnert *et al.* (2014).

specific pension adjustment mechanism as well as by a buffer portfolio. They determine the optimal investment strategies during the decumulation phase to maximize the client's expected accumulated utility from the stochastic future pension cash flows and conclude that smoothing can provide remarkable benefits to clients.

In reality, insurers' cover funds come with rather low equity exposures (mainly due to high guarantees offered in the past in combination with solvency requirements). Moreover, insurers in many countries typically do not offer different cover funds with different risk-return characteristics. Hence, there is no possibility for the policyholder to choose a cover fund that best meets individual preferences. In this paper, we add to the literature by analyzing a setting where an insurer offers several segregated cover funds with different asset allocation which make use of collective smoothing mechanisms but come without guarantees. These cover funds can be offered standalone or as a "building block" within more complex products.

The main goal of this paper is an analysis from the client's perspective of the resulting products. We consider both, the risk of short-term fluctuations, as well as the distribution of the terminal benefit. We show that the existence of multiple, segregated cover funds enables insurers to offer different risk-return profiles of the terminal benefit. Each of these come with reduced (when compared to investments without smoothing mechanisms) pathwise volatility, which appears highly relevant for the subjective attractiveness.

We also analyze various static and dynamic investment strategies (without formal guarantees but with protection levels) using different collective cover funds as low-risk asset and compare them with products which use market-based safe assets. We find that dynamic products based on cover funds have a significantly higher return potential compared to market-based products with similar short-term risk characteristics. This particularly holds for conservative products (which are relevant for risk averse investors). In addition to risk-return characteristics, we investigate how collective cover funds affect both, the objective utility (measured in an Expected Utility Theory framework) and the subjective attractiveness (measured in several frameworks based on Cumulative Prospect Theory) of these products. We find that in most cases, static products based on cover funds with rather high equity ratios outperform all other considered products in terms of objective utility as well as subjective attractiveness. Sensitivity analyses show that this also holds in an environment of higher interest rates and other choices of the smoothing mechanism. Therefore, our results indicate that the simultaneous offering of multiple, segregated cover funds with different asset allocations would enable insurers to develop a range of retirement savings products including solutions that are at the same time subjectively attractive and objectively preferable for different types of consumers.

The remainder of this paper is organized as follows. In Section 2, we introduce the capital

market model, the insurer's cover fund, as well as the market-based investment strategies. In Section 3, we analyze the impact of smoothing focusing on static investment products. Next, in Section 4, we analyze static as well as dynamic investment products which make use of an insurer's cover fund and compare them with similar market-based products. In particular, we analyze the expected utility and the subjective attractiveness. We also discuss a wide range of sensitivity analyses. Finally, Section 5 concludes and provides an outlook.

2 Model Framework

In this section, we describe our capital market model with stochastic interest rates and stocks. Moreover, we define the insurer's cover fund and the considered market-based investment strategies.

2.1 The Capital Market Model

We assume that the short rate for $t \in [0, T]$ and $T \in \mathbb{N}$ follows a two-factor Hull-White model as described in Brigo & Mercurio (2007). This model is frequently used to model the current low-interest environment, cf., Korn *et al.* (2018). It contains two correlated mean reverting processes and allows a perfect fit to observed bond prices. The dynamics of the stock market is modeled by a generalized Black-Scholes model with a risk premium λ_S and volatility σ_S. A more detailed description can be found in Appendix A.

2.2 Insurer's Cover Fund

Smoothing mechanisms as described in Chapter 1 are often complex, can include elements of management discretion and can differ from country to country. Therefore, inspired by Korn *et al.* (2018), we generically model the impact of smoothing as follows: We denote the insurer's cover fund with a stock ratio $\phi \in [0, 1]$ by ICF^{ϕ} and the smoothing period by $sp \geq 2$ years. The yearly smoothed return $(\tilde{r}_{ICF^{\phi}}(t))$ based on the applied smoothing procedure between time $t-1$ and t (for $t = 1, 2, \dots$) under the real world measure P is given by

$$\tilde{r}_{ICF^{\phi}}(t) = \left(\prod_{k=1}^{sp} (\phi(1 + r_s(t-k)) + (1-\phi)(1 + r_{RB}(t-k)) \right)^{1/sp} - 1, \tag{1}$$

where $r_s(t)$ is the yearly return of the stock market between time $t-1$ and t, cf., Appendix A. Further, $r_{RB}(t)$ denotes the yearly return of a rolling bond investment (RB) between time $t-1$ and t (for $t = 1, 2, \dots$), where the bond is bought and sold $\frac{1}{\Delta t}$ times in a year for $\Delta t \in (0, 1]$,

e.g., monthly ($\frac{1}{\Delta t} = 12$) or daily ($\frac{1}{\Delta t} = 252$). Thus, the return is given by

$$r_{RB}(t) = \prod_{i=1}^{1/\Delta t} (1 + \hat{r}_{RB}(t+i)) - 1 \text{ with } \hat{r}_{RB}(t) = \frac{P(t, t-\Delta t+d)}{P(t-\Delta t, t-\Delta t+d)} - 1, \tag{2}$$

where $d \geq 0$ denotes the term to maturity of the bond investment and $P(t_0, t_1)$ denotes the price of a zero coupon bond at time t_0 with maturity t_1.[2] Note that throughout the paper the annual return of an asset (Ξ) is denoted by r_Ξ and the return in a time interval of Δt is denoted by $\hat{r}_\Xi$. Finally, for the yearly return of the insurer's cover fund ($r_{ICF^\phi}(t, \alpha)$), the smoothed return is adjusted by the surplus participation rate $\alpha \in [0, \infty)$, i.e.,

$$r_{ICF^\phi}(t, \alpha) = \tilde{r}_{ICF^\phi}(t) - (1-\alpha)\max(\tilde{r}_{ICF^\phi}(t), 0) \text{ for } t = 1, 2, \ldots. \tag{3}$$

Hence, α determines the proportion of the positive return which will be credited to the insurer's cover fund. The corresponding return between time $t - \Delta t$ and t can be computed by

$$\hat{r}_{ICF^\phi}(t, \alpha) = \big(1 + r_{ICF^\phi}(t, \alpha)\big)^{\Delta t} - 1. \tag{4}$$

Note that $\hat{r}_{ICF}(t, \alpha)$ is constant during the year.

The participation rate (α) is chosen such that the insurer's cover fund is initially fair if the cover fund is held until maturity (T),[3] i.e.,

$$\mathbb{E}_Q\left[\exp\left(-\int_0^T r^*(s)ds\right)\prod_{t=1}^{T}(1 + r^*_{ICF}(t, \alpha))\right] = 1, \tag{5}$$

where Q denotes the risk-neutral measure, $r^*(s)$ the short rate under Q and r^*_{ICF} the yearly return of the ICF under Q, cf., Appendix A for more details. The fair α usually differs from 1, e.g., since the returns of the past influence $r_{ICF}(t)$ (at least for $t = 1, \ldots, sp$), and can also be greater than one.[4]

[2] We refer to Brigo & Mercurio (2007) for details on pricing zero coupon bonds in this model.

[3] For dynamic products, which may invest in and deinvest from the ICF several times during the investment horizon, this can lead to small deviations compared to fairly priced products. In the following analyses, we neglect this effect. Also note that in a sensitivity analysis below, we consider different values of (initially fair) alphas and the results are qualitatively similar.

[4] In reality, products based on the insurer's cover fund usually come with an annual guarantee. For these products a participation rate of less than 1 primarily compensates for the guarantee. Note that in this paper, we explicitly consider products based on an insurer's cover fund without guarantee as we are interested in the effects of smoothing in collective investments.

2.3 Assets and Products

We analyze different smoothed products based on insurer's cover funds and compare them with products using only market-based assets. For all products we assume a single upfront premium $A_0 > 0$, a term to maturity of T, and no charges. We use different low-risk assets and use stocks as risky assets. In addition to the insurer's cover funds, we consider as low-risk assets a rolling bond investment with a return of $r_{RB}(t)$ as described in the previous subsection as well as an investment in a T-year zero coupon bond (TB). The return from time 0 to time T of a (pure) TB product is given by $1/P(0,T)$ and the return between time $t-\Delta t$ and t by

$$\hat{r}_{TB}(t) = \frac{P(t,T)}{P(t-\Delta t,T)} - 1. \tag{6}$$

We consider two different product groups: Firstly, we examine *static investment*[5] products where the low-risk investments $\Xi \in \{ICF, RB, TB\}$ is mixed with a fixed fraction of stocks ($\theta \in [0,1]$). We denote them as static-ICF, static-RB, and static-TB, respectively and refer to θ as direct stock ratio. The return of a static investment between two rebalancing points in time t and $t-\Delta t$ is given by

$$\hat{r}^{\theta}_{\Xi}(t) = \Big(\theta(1+\hat{r}_s(t)) + (1-\theta)(1+\hat{r}_{\Xi}(t))\Big), \tag{7}$$

where $\hat{r}_s(t)$ denotes the stock return between time $t-\Delta t$ and t. Note that for static-ICF products with $\phi > 0$ the product's total fraction of stocks is given by $\theta^{tot} = \theta + (1-\theta)\phi \geq \theta$.

Secondly, we analyze *dynamic investment* products in the form of CPPI-strategies with a protection level of $PL \leq 1$ in terms of the initial investment $A_0 > 0$. The part of the investment which is not invested in the stock is invested in a low risk investment $\Xi \in \{ICF, RB, TB\}$. The CPPI strategy is implemented as follows:

1. At time t, compute the floor $F_t = PL \cdot A_0 \cdot P(t,T)$, the cushion $C_t = A_t - F_t$, the exposure to the stock investment $E_t = \min(mC_t, A_t)$, where $m \in \{1,2,3,4,5\}$ is the multiplier, and the low risk investment $R_t = A_t - E_t$.

2. Then, $A_{t+\Delta t} = E_t(1+\hat{r}_s(t+\Delta t)) + R_t(1+\hat{r}_{\Xi}(t+\Delta t))$. We repeat that for $t = 0, \ldots, T-\Delta t$.

The average direct stock ratio of a CPPI strategy is given by $\bar{\theta} = \frac{\Delta t}{T}\mathbb{E}\left[\sum_{k=0}^{\frac{T}{\Delta t}-1} E_{k\cdot\Delta t}\right]$ and the total fraction of stocks by $\bar{\theta}^{tot} = \bar{\theta} + (1-\bar{\theta})\phi$, where $\phi = 0$ for market-based CPPI strategies. Note that the probability of achieving a terminal value below the protection level is greater than 0 and typically increasing in the multiplier and the protection level.[6]

[5] Note that we assume a rebalancing at all Δt time points for the static investment products.
[6] More details can be found in Table 12 in Appendix D.

	λ_S	σ_S	d	ϕ	T	Simulations (n)	
	4%	20%	10Y	10%, 30%, 50%	30	50,000	
History	2017/18	2018/19	2019/20	History	2017/18	2018/19	2019/20
Stock	1.28%	5.12%	-5.05%	Rolling Bond	2.33%	7.40%	1.81%

Table 1: The base case parameters of the stock market (λ_S, σ_S), the parameters of the ICF (d, ϕ, T), the number of simulations (n) and the historical stock and rolling bond returns.

3 Impact of Return Smoothing on Risk-Return Characteristics

In this section, we compare static-ICF products using $\phi \in \{10\%, 30\%, 50\%\}$[7] and a smoothing period of 3 ($=sp$) years,[8] with static-RB products as introduced in Section 2. Note that in this section, we only consider these two product types, which are both based on a rolling bond investment, in order to separate the impact of smoothing from other product features. For the market-based products, we assume daily rebalancing and 252 trading days per year and consequently set $\Delta t = 1/252$. Further, we investigate 41 (direct) stock ratios θ between 0% and 100% in steps of 2.5%. To compare the different products, we generate 50,000 Monte Carlo simulations paths and analyze the expected return, the standard deviation of annual returns within a path as well as the standard deviation of the annualized return from 0 to T. For a given asset evolution $(A_t)_{\{t=0,\dots,T\}}$ (which depends on the low-risk asset and stock ratio), we define the standard deviation of annual returns (pathwise volatility $\sigma[r_t]$) as $SD\left[\left(\frac{A_{t+1}-A_t}{A_t}\right)_{\{t=0,\dots,T-1\}}\right]$, where the standard deviation (SD) is taken over all $t = \{1,\dots,T\}$. The standard deviation of the average return ($\sigma[\bar{r}]$) is given by $SD\left[\left(\frac{A_T}{A_0}\right)^{1/T} - 1\right]$. The following results are based on the base case parameter setup presented in Table 1. The parametrization of the short rate model and the stock market are taken from Graf *et al.* (2021) and can be found in Table 9 in Appendix A. To be in line with Graf *et al.* (2021), we model the historical rolling bond returns with the historical Nelson-Siegel-Svensson (NSS) parameters which are given by the Deutsche Bundesbank[9] and for the historical returns of the stock market, we take the returns of the EURO STOXX 50 performance index of the same time intervals.[10] The corresponding returns can also be found in Table 1.

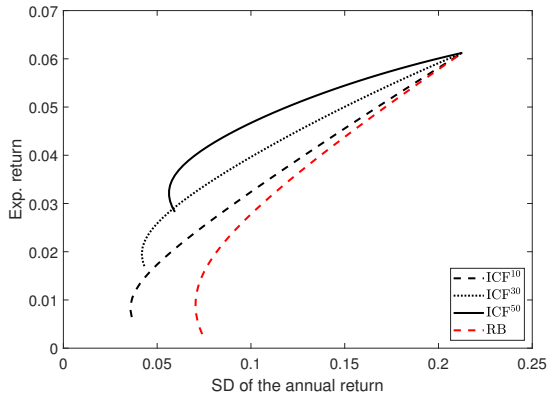
The fair participation rate α is 0.865 for a stock ratio of 10%, 0.924 for $\phi = 30\%$, and in-

[7] We refrain from analyzing higher stock ratios within the ICF since in practice, rather high stock ratios would lead to several risks and issues from the provider's perspective, cf., e.g., our comments on market value adjustments in Section 5. Hence, in practice providers would either refrain from offering such products, or they would use smoothing mechanisms that are specifically tailored to deal with such risks and issues which would make the use of a generic smoothing algorithm as a proxy increasingly unrealistic.

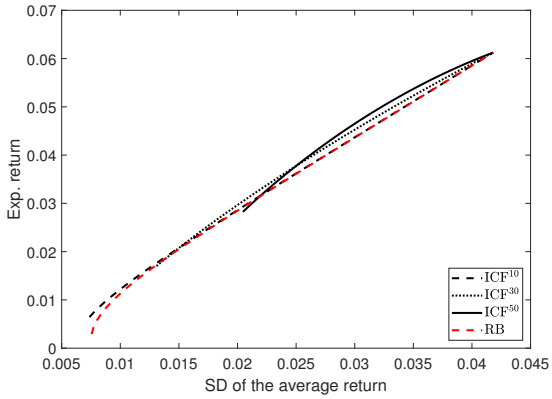
[8] See, Korn *et al.* (2018). Also, in the sensitivity analysis, we show that the results also hold for longer smoothing periods.

[9] The NSS-parameters can be found in Deutsche Bundesbank (2021).

[10] The historical performance of the EURO STOXX 50 is taken from Deutsche Börse AG (2021).



(a) Average SD of annual returns



(b) The SD of the average return

Figure 1: The risk-return characteristics for different static-ICF products and a static-RB product based on the standard deviation of annual returns (left panel) and on the standard deviation of the average return (right panel).

creases to 0.972 for $\phi = 50\%$. It is below 1 because the historical return is above the average return implied by the model under the risk-neutral measure Q. Due to the asymmetric participation in the surpluses and losses, cf., Equation (3), an ICF with a higher stock ratio has a higher participation rate due to the lower volatility of the rolling bond return. This effect is strengthened by the lower historical stock return compared to the historical rolling bond return.

The resulting risk-return characteristics of the different products are shown in Figure 1. In both panels the expected return is shown on the y-axis. Further, on the x-axis the standard deviation of annual returns ("pathwise volatility") is shown in panel (a) and the standard deviation of the annualized return in panel (b).

In both panels, the outer right point of a line corresponds to products with a direct stock ratio of 100% and consequently all lines meet at the same point. The other end of the line corresponds to a direct stock ratio of 0%. We typically observe that static-ICF products with a higher embedded stock ratio (ϕ) have a higher expected return but typically also a higher volatility. As direct stock ratios approach zero, the decreasing expected return comes with an increase in risk (at least pathwise volatility) since diversification effects are no longer used. Most importantly, we find in panel (b) that for a given expected return the standard deviation of the average return is very similar for all four investment types while in panel (a) for a given expected return, the standard deviation of the annual return of static-ICF products are significantly lower compared to static-RB products. In particular, panel (a) shows that the static-RB products are clearly dominated by static-ICF products when considering the annual fluctuations of the product. This already illustrates the potential of ICF based products to reduce pathwise volatility with little effect on the distribution of terminal wealth.

4 Analysis of Static and Dynamic Products

In this section, we analyze static and dynamic investment products introduced in Section 2. For both, we consider five different options for the low-risk asset: ICF^{10}, ICF^{30}, ICF^{50}, a T-year zero coupon bond (TB), and a rolling-bond investment (RB).

4.1 Base Case

Again, we assume daily rebalancing and use 50,000 simulation paths. For the static products we investigate 41 (direct) stock ratios θ between 0% and 100% in steps of 2.5%. For the dynamic products we consider different protection levels $PL \in \{50\%, 70\%, 80\%, 90\%\}$ as well as different multipliers $m = \{1, 2, 3, 4, 5\}$.[11]

4.1.1 Characteristics of the Dynamic Investment Products

Firstly, we analyze the characteristics of the dynamic investment products which have not been discussed in the previous analysis. In Table 2 we display the expected return ($\bar{r}$), the standard deviation of the average return ($\sigma[\bar{r}]$), the standard deviation of annual returns ($\sigma[r_t]$), the corresponding average (direct as well as total) stock ratio over the entire duration ($\bar{\theta}$ and $\bar{\theta}^{tot}$) and the shortfall probability with respect to the protection level ($P_{PL} := P(A_T < PL)$) for the different dynamic products.

As expected, for fixed m, a decreasing protection level results in a higher average stock ratio and consequently in a higher average return (as well as higher standard deviations). The same holds true for a fixed protection level and an increasing multiplier. Further, the shortfall probability is lower for lower protection levels or lower multipliers, except for the dynamic-TB (which always has a shortfall probability very close to 0).

It is noteworthy that for given m and PL, all considered dynamic-ICF products have a higher average direct stock ratio than other dynamic products (and the difference is even larger when considering the ICF product's total stock ratio $\bar{\theta}^{tot}$) which is mainly due to the smoothing effect which reduces fluctuations of the direct stock ratio. This results in higher expected returns for the dynamic-ICF products compared to other dynamic products. The return difference is particularly pronounced for higher protection levels and lower multipliers (i.e., products probably chosen by conservative, risk averse consumers). But this comes with a higher volatility of the terminal benefit, however, still significantly lower than for comparable (same expected return) market-based products, cf. Figure 2 (b). Beyond that, the standard deviations of the annual

[11] Note that in this setting, the maximum possible protection level is given by $1/P(0,T) \approx 0.95$. We examine only protection levels which are significantly below 95% so that significant stock ratios are possible.

	$m=1$ and $PL=90\%$					$m=3$ and $PL=90\%$					$m=5$ and $PL=90\%$				
	ICF with $\phi=$			RB	TB	ICF with $\phi=$			RB	TB	ICF with $\phi=$			RB	TB
	10%	30%	50%			10%	30%	50%			10%	30%	50%		
$\bar{r}$	3.35	3.99	4.47	2.15	0.67	5.42	5.37	5.29	4.23	3	5.52	5.47	5.39	4.43	3.04
$\sigma[\bar{r}]$	2.5	2.81	3.13	2.07	0.95	3.8	3.81	3.89	3.45	2.76	3.96	3.94	4.01	3.64	2.86
$\sigma[r_t]$	8.37	9.14	10.02	8.92	9.36	14.9	14.89	15.04	13.4	12.33	15.44	15.5	15.65	13.74	11.92
$\bar{\theta}$	33.27	35.28	36.27	23.90	12.75	60.39	59.17	57.58	45.27	31.95	61.91	61.04	59.66	45.33	25.22
$\bar{\theta}^{tot}$	39.94	54.70	68.14	23.90	12.75	64.35	71.42	78.79	45.27	31.95	65.72	72.73	79.83	45.33	25.22
P_{PL}	11.0	12.1	14.5	18.7	0	26.0	25.5	26.5	34.7	0	33.4	30.7	30.6	43.3	0
	$m=1$ and $PL=80\%$					$m=3$ and $PL=80\%$					$m=5$ and $PL=80\%$				
$\bar{r}$	3.78	4.29	4.68	2.84	1.88	5.65	5.58	5.49	4.99	4.54	5.71	5.65	5.58	5.13	4.56
$\sigma[\bar{r}]$	2.70	2.97	3.26	2.38	1.76	3.95	3.98	4.06	3.84	3.6	4.09	4.09	4.16	4.03	3.76
$\sigma[r_t]$	9.71	10.2	10.83	9.96	10.24	16.23	16.09	16.1	15.26	14.74	16.66	16.6	16.67	15.54	14.29
$\bar{\theta}$	41.3	42.2	42.3	34.5	29.0	68.6	66.8	64.8	58.8	52.7	69.5	68.2	66.6	58.1	45.0
$\bar{\theta}^{tot}$	47.17	59.54	71.15	34.5	29.0	71.74	76.76	82.40	58.8	52.7	72.55	77.74	83.3	58.1	45.0
P_{PL}	6.1	8.7	11.5	9.1	0	21.6	22.1	23.6	26.9	0	28.8	27.1	27.3	35.4	0
	$m=1$ and $PL=50\%$					$m=3$ and $PL=50\%$					$m=5$ and $PL=50\%$				
$\bar{r}$	4.85	5.08	5.27	4.43	4.08	6.02	5.99	5.95	5.95	5.91	6.04	6.01	5.99	5.96	5.91
$\sigma[\bar{r}]$	3.25	3.42	3.63	3.12	2.93	4.22	4.29	4.38	4.3	4.27	4.29	4.34	4.42	4.39	4.40
$\sigma[r_t]$	13.8	13.84	13.94	13.85	13.96	19.4	19.25	19.17	19.21	19.09	19.59	19.49	19.46	19.31	18.98
$\bar{\theta}$	63.6	63.2	62.3	61.6	60.5	87.9	86.5	85.1	85.6	84.5	88.4	87.3	86.2	85.3	82.3
$\bar{\theta}^{tot}$	67.24	74.24	81.15	61.6	60.5	89.11	90.55	92.55	85.6	84.5	89.56	91.11	93.1	85.3	82.3
P_{PL}	0.9	2.4	4.5	0.8	0	10.7	11.9	13.3	11.3	0	14.4	14.4	15.0	15.7	0

Table 2: Different risk measures for dynamic products with different multipliers (m) and different protection levels (PL) based on different low-risk investments (ICF, RB,TB). All results are given in percent.

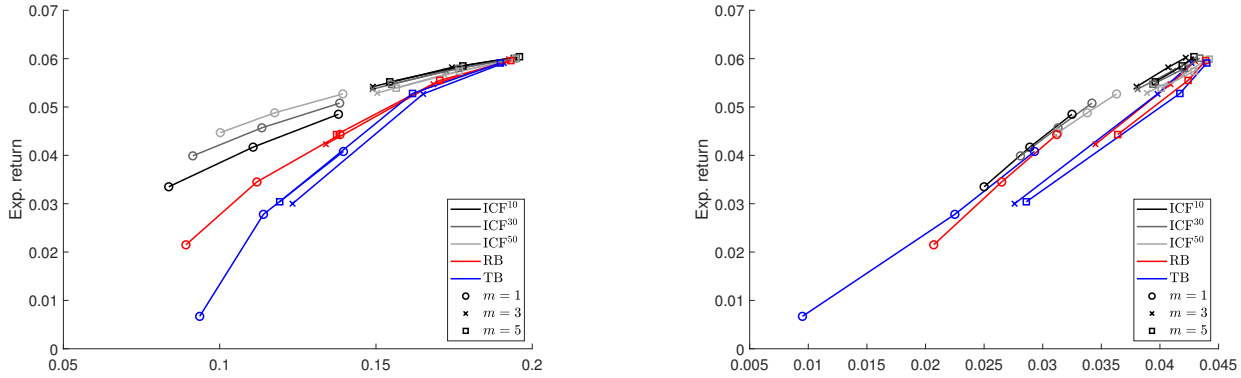
returns are similar for all considered dynamic products.[12] Although the average total stock ratios of the dynamic-ICF products are significantly higher than for the dynamic-RB products, the shortfall probabilities are mostly very similar for both (and in many cases the shortfall probabilities of the dynamic-ICF products are even lower than for the dynamic-RB products).[13]

Figure 2 displays the risk-return characteristics of the dynamic investment strategies from Table 2, where risk is measured by pathwise fluctuations in the left panel and of the terminal wealth in the right panel (as in Figure 1 in Section 3). We observe in the left panel that for a given and rather low pathwise volatility the dynamic-ICFs have a significantly higher expected return. Further, we observe that for a high protection level (lower volatility), the dynamic-ICF^{50} products dominate while for lower protection levels (higher volatility) dynamic-ICF^{10} products dominate.[14] However, with increasing volatility, the ICF and market-based dynamic investment products (and hence also their Risk-Return characteristics) become more similar due to the increasing direct stock ratios. Another interesting effect can be seen in the results for the dynamic-TB products. There, products with a multiplier of $m = 3$ are dominated by

[12]Note that the dynamic-TB is the only considered product with no shortfall risk. However, the results show that this comes with a significantly lower average stock ratio and consequently average return compared to other dynamic products (for m and PL).

[13]Note that this may also be partly driven by the ICF pricing approach and the choice of α, cf., page .

[14]This can also be seen in Table 2, where for dynamic-ICF products with fixed multiplier $m \in \{3, 5\}$ and protection level, the returns decrease (slightly) for a higher ICF stock ratio (ϕ), while the pathwise volatility increases (slightly) for a high protection level $PL = 90\%$ and decreases for a low protection level $PL = 50\%$.



(a) Average SD of annual returns

(b) The SD of the average return

Figure 2: The risk-return characteristics for different dynamic-ICF, dynamic-RB and dynamic-TB products based on the standard deviation of annual returns (left panels) and based on the standard deviation of the average return (right panel). Note that one line corresponds to a group of dynamic products of a fixed type (specified by the color), fixed multiplier (specified by the symbol) and different protection levels.

products with a multiplier of 1 and 5. This can be explained by the impact of the market value changes of the terminal bond on the stock exposure for different multipliers which results in average total stock ratios which first increase in m and then decrease again, cf., Table 2. When risk is measured by variability of terminal wealth (right panel of Figure 2), all products are much closer to the "efficient frontier". However, in particular, for higher standard deviations, market-based products are clearly dominated by ICF-based products. Overall, these results show that dynamic-ICF products have a significant potential to earn a higher return than a comparable dynamic product with market-based low risk assets and similar risk characteristics.

4.1.2 Preferences under EUT, CPT, MCPT and PMCPT

In this section, we analyze the objective expected utility and the subjective attractiveness of the different products. For the former, we apply Expected Utility Theory (EUT) with power utility $u(x) = x^{1-\gamma}/(1-\gamma)$, where $\gamma \geq 0$ $(\neq 1)$ is the risk aversion parameter. To measure the subjective attractiveness, we use Cumulative Prospect Theory (CPT) as described in Kahneman & Tversky (1979) and Multi Cumulative Prospect Theory (MCPT) proposed by Ruß & Schelling (2018) which can better explain observed long term investment decisions than CPT by taking annual fluctuations into account.[15] Also, we include Partial Multi Cumulative Prospect Theory (PMCPT) in our analysis, which is a weighted average of CPT and MCPT, i.e., $PMCPT(X) = \omega CPT(X) + (1-\omega)MCPT(X)$, where X refers to the investment product and $\omega \in [0,1]$ to the weight. Note that for these subjective theories the policyholder is typically assumed to be risk averse for gains and risk seeking for losses with risk attitude $a \in \mathbb{R}_+$. Further, policyholders are assumed to be loss averse, described by a loss aversion parameter $\lambda \geq 1$. Also, we assume that probabilities are distorted based on the distortion function proposed by Prelec

[15] See also Graf *et al.* (2019) and Ruß & Schelling (2021). A more detailed description and definitions can be found in Appendix B.

(1998), which distorts the probabilities for losses (α^-, β^-) and gains (α^+, β^+) differently. To compare the results of the different products, we compute the corresponding certainty equivalent (CE) returns, cf., Ruß & Schelling (2018) cf., Appendix C. The preference parameters used in our analysis can be found in Table 3. The choice of the subjective parameters (except for ω) is based on l'Haridon & Vieider (2019) and is the same for CPT, MCPT and PMCPT. The risk aversion parameter for EUT is in line with Chiappori & Paiella (2011).

γ	a	λ	β^+	α^+	β^-	α^-	ω
2.5	0.88	1.616	1.052	0.767	0.934	0.863	0.5

Table 3: Preference parameters used in the base case.

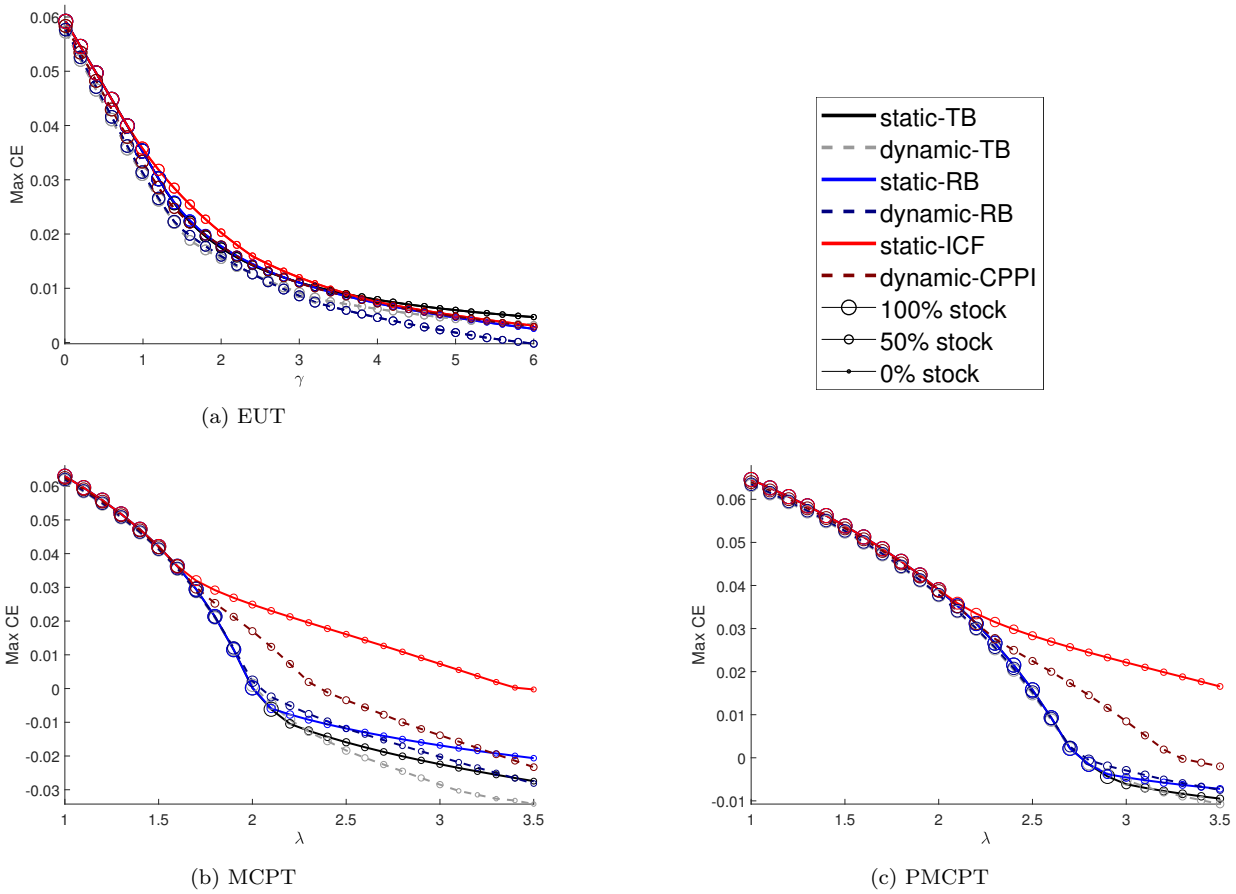
We now analyze all previously mentioned static and dynamic (CPPI) investment products. In Figure 3, we display certainty equivalent values of different product types under different preference theories depending on the risk aversion (panel (a)) and loss aversion (panel (b) and (c)). Due to the very large number of products considered, we only display the version of each product that comes with the highest CE-value. E.g., the value for the static-ICF product displayed at $\gamma = 2.5$ in panel (a) of Figure 3 is from the product with $\phi = 30\%$ and $\theta = 35\%$, since all other combinations of ϕ and θ would result in a lower CE value.[16] Note that for different parameter values of γ, respectively λ, the product with the highest CE of a product type can vary. The size of the circles represents the (average) direct stock ratio of the product with the highest CE. Further, Table 4 shows the product characteristics of the products with the highest CE (of all considered products) under different preference theories and for selected levels of risk and loss aversion, respectively.

	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 6$
EUT	$\theta = 100\%$	static-ICF^{50} $\theta = 57.5\%$	static-ICF^{30} $\theta = 35\%$	static-ICF^{30} $\theta = 25\%$	static-ICF^{30} $\theta = 17.5\%$	static-TB $\theta = 20\%$
	$\lambda = 1.25$	$\lambda = 1.616$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3$	
CPT	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	
MCPT	$\theta = 100\%$	static-ICF^{50} $\theta = 87.5\%$	static-ICF^{50} $\theta = 35\%$	static-ICF^{50} $\theta = 25\%$	static-ICF^{50} $\theta = 20\%$	
PMCPT	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	static-ICF^{50} $\theta = 45\%$	static-ICF^{50} $\theta = 32.5\%$	

Table 4: The product with the highest CE value for different preference theories and for selected levels of risk and loss aversion, respectively.

For a very low risk aversion under EUT ($\gamma < 1$), the CE-maximizing product variants within each product type are very similar and they have a stock ratio of 100% or a high multiplier

[16] Analogously for the dynamic products only the optimal products with respect to ϕ, PL and m are displayed.



(a) EUT

(b) MCPT

(c) PMCPT

Figure 3: The maximum CE of different product types under different preference functions and for different levels of risk or loss aversion. The size of the circle corresponds to the (average) direct stock ratio of the product with the highest CE.

and a low protection level. If risk aversion increases, the CE for all products decreases and the optimal product has a lower direct and total stock ratio, since a risk averse consumer prefers a more stable terminal value. This holds for all product types. E.g., for a medium risk aversion ($\gamma = 2.5$), the static-ICF[30] and a direct stock ratio of 35% ($\theta^{tot} = 54.5\%$) has a higher CE than other product types, while the optimal product for $\gamma = 3.5$ has a direct stock ratio of 17.5% ($\theta^{tot} = 42.25\%$), cf., Table 4. Further, for medium levels of risk aversion the respective CE-maximizing static-ICF has a higher CE than any product of any type. For a very high risk aversion ($\gamma > 4$) the static-TB with a low stock ratio has the highest CE, since a TB comes with a deterministic maturity value.

Under CPT (not displayed in Figure 3), a pure stock investment always yields the highest CE (cf., Table 4) as the probability for large gains is overweighted while the probability of a loss after 30 years is rather low. Note that all considered dynamic products have an average direct and total stock ratio of below 100% and therefore have a lower CPT CE value.[17]

[17] A lower protection level would lead to a higher stock ratio and thus to a higher CPT CE. In particular, a dynamic with protection level of 0% would lead to a stock ratio of 100% and thus the highest CPT CE.

Although annual value changes are punished under MCPT (Figure 3 (b)), for a low loss aversion $\lambda < 1.6$, the product with the highest CE is still a pure stock investment. For higher loss aversion a static-ICF product provides the highest CE (due to smoothed annual returns and hence reduced fluctuations). For $\lambda > 2.3$, static-ICF products with very low direct stock ratios are even the only products with a positive CE. Also, the stock ratio of the product with the highest CE decreases for an increasing loss aversion. Further, for $\lambda > 1.6$, dynamic-ICF products have a higher CE than purely market-based dynamic products. Note also that the most conservative dynamic-ICF still has a total average stock ratio of roughly 40% (for $m = 1$ and $PL = 90\%$, cf., Table 2), which is close to the total stock ratio of the most attractive static-ICF product for rather high loss aversions, cf., Table 4. Nevertheless, dynamic-ICF products typically have a higher fluctuation of the annual value changes than comparable static-ICF products which results in a lower MCPT-CE.[18]

Lastly, we consider PMCPT (cf., Figure 3 (c)), where we take into account annual values changes (MCPT) as well as the total value change (CPT). Overall, the effects are similar as for MCPT. But a pure stock investment now yields the highest CE for λ below 2.1, cf., Table 4. For higher loss aversion a static-ICF product yields (significantly) higher CE values than the other product types. Note that under pure CPT the difference between ICF based products with smoothed returns and comparable other products is only small as return smoothing only slightly impacts the characteristics of the terminal benefit. Therefore, the results under PMCPT are mainly driven by the results under MCPT.

Overall, our results show that under all considered preference theories the static-ICF is in most cases preferred over the market-based products. Also, dynamic-ICF products dominate market-based dynamic investment strategies in most cases (expect for EUT in case of a very high level of risk aversion $\gamma > 5.5$). Further, we observe that in none of the considered settings a dynamic product outperforms all other products (although, under specific settings, e.g., for very high risk aversion or very low risk or loss aversion, we can find dynamic products with a CE that is close to highest CE). Moreover, we note that while for less conservative consumers (low γ respectively λ) a pure stock investment is the objective utility maximizing as well the subjectively preferred product, for more conservative consumers the products with the highest objective utility and the highest subjective attractiveness differ.

[18]Note that in the ICF we do not consider stock ratios above 50%, cf., Footnote 7 in Section 3. However, additional analyses show that for certain combinations of risk attitude (a) and loss aversion (λ) a higher stock ratio in the ICF would result in a subjectively even more attractive product. In practice that means that consumers would demand a product based on an ICF with the highest stock ratio that can feasibly be implemented.

4.2 Sensitivity Analysis

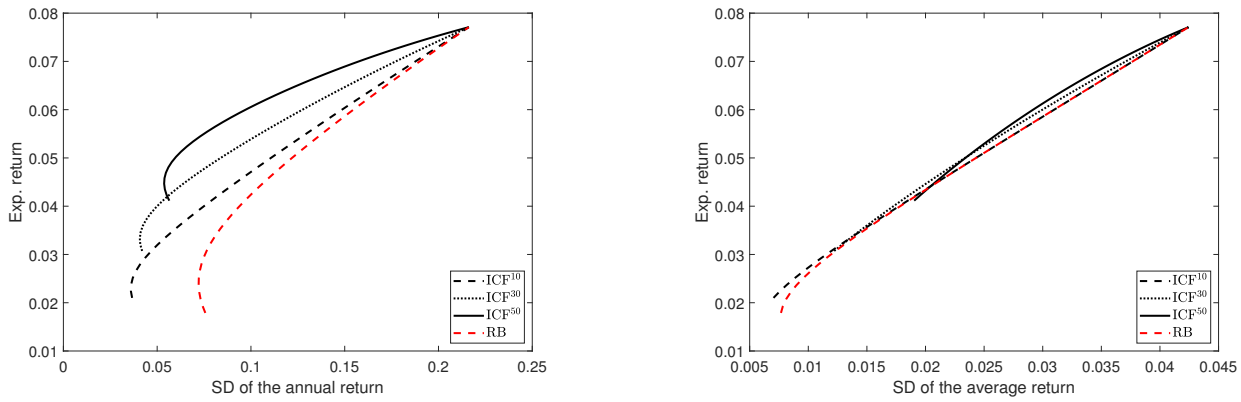
Now, we perform various sensitivity analyses with respect to the interest rates, the probability weighting, the risk attitude, as well as the length of the smoothing period.

4.2.1 Interest Rate Environment and Historical Returns

Firstly, we consider a different initial interest rate environment. In particular, we assume a higher initial market forward rate curve which is given by the historical NSS-parameters of the end of 2014 and the three previous years. The parameters as well as the historical stock return and the return of a rolling bond investment are given in Table 10 in Appendix A. The other parameters of the capital market model remain unchanged. Note that with this change, we increase the 20 year spot rate from -0.165% p.a. to 1.318% p.a. Also, the fair participation rate (α) is 0.8161 for an ICF with 10% stock, 0.8312 for an ICF with 30% stock and 0.8595 for an ICF with 50% stock. The fair participation rates are lower than in the base case because the historical stock and bond returns are significantly higher.

Effects of Smoothing

As in the base case we observe that the cover fund can significantly reduce the volatility of the annual returns while hardly affecting the volatility of the overall return, cf., Figure 4. This indicates that the effects of the return smoothing are qualitatively independent of the initial interest rate environment and the historical returns.



(a) Average SD of annual returns

(b) The SD of the average return

Figure 4: The efficient frontier for different static-ICF and a static-RB investment based on the standard deviation of annual returns and based on the standard deviation of the average return in the interest rate sensitivity scenario.

The Dynamic Investment Products

Again, Table 5 shows results for dynamic products based on the different low-risk assets and for different protection levels and multipliers. Due to the higher interest rates and stock returns

the average direct and total stock ratio is higher for all considered products and the shortfall probability is significantly lower than in the base case. As a consequence of the higher average direct stock ratios, we observe higher expected returns and higher standard deviations. However, as in the base case, the dynamic products based on an ICF with a stock ratio of 10% yield similar or higher returns compared to the dynamic products based on a RB. Also, the shortfall probabilities are very similar in this case.

	$m = 1$ and $PL = 80\%$					$m = 3$ and $PL = 80\%$					$m = 5$ and $PL = 80\%$				
	ICF with $\phi =$			RB	TB	ICF with $\phi =$			RB	TB	ICF with $\phi =$			RB	TB
	10%	30%	50%			10%	30%	50%			10%	30%	50%		
$\bar{r}$	6.47	6.73	6.95	5.94	5.57	7.59	7.56	7.52	7.51	7.47	7.6	7.58	7.55	7.53	7.47
$\sigma[\bar{r}]$	3.33	3.51	3.72	3.14	2.94	4.32	4.39	4.49	4.36	4.33	4.4	4.45	4.53	4.45	4.46
$\sigma[r_t]$	13.98	14.06	14.21	13.93	14.04	19.52	19.36	19.28	19.4	19.26	19.71	19.6	19.57	19.5	19.14
$\bar{\theta}$	63.39	63.28	62.85	60.61	59.36	86.82	85.37	83.98	84.77	83.56	87.26	86.15	85.09	84.38	81.14
θ^{tot}	67.05	74.30	81.43	60.61	59.36	88.14	89.76	91.99	84.77	83.56	88.53	90.31	92.55	84.38	81.14
P_{PL}	1.2	2.7	4.8	0.9	0	12.7	13.9	15.1	11.8	0	16.5	16.6	16.9	16.4	0
	$m = 1$ and $PL = 50\%$					$m = 3$ and $PL = 50\%$					$m = 5$ and $PL = 50\%$				
$\bar{r}$	6.98	7.13	7.25	6.7	6.52	7.69	7.68	7.67	7.68	7.67	7.69	7.68	7.68	7.68	7.67
$\sigma[\bar{r}]$	3.65	3.77	3.91	3.54	3.45	4.32	4.38	4.45	4.34	4.33	4.34	4.39	4.44	4.38	4.38
$\sigma[r_t]$	16.61	16.6	16.6	16.62	16.67	20.91	20.84	20.8	20.89	20.86	21	20.95	20.93	20.95	20.85
$\bar{\theta}$	76.47	76.23	75.78	75.5	75.11	95.53	94.93	94.35	95.09	94.86	95.81	95.34	94.92	95.04	94.33
θ^{tot}	78.82	83.36	87.89	75.5	75.11	95.98	96.45	97.18	95.09	94.86	96.23	96.74	97.46	95.04	94.33
P_{PL}	0.2	0.8	1.8	0.1	0	5.2	6.0	6.8	4.6	0	6.8	7.1	7.5	6.5	0

Table 5: Different risk measures for dynamic products with different multipliers (m) and different protection levels (PL) based on different low-risk investments (ICF, RB,TB) in the interest rate sensitivity scenario. All results are given in percent.

Due to the higher interest rate level, we can also analyze protection levels of 100% and 120%.[19] The results are shown in Table 6. As in the base case the differences between the dynamic-ICF and dynamic-RB are larger for conservative combinations of protection level and multiplier, e.g., $PL = 120\%$ and $m = 1$. In this case the expected return of the dynamic products based on the ICF[10] is higher than for the dynamic-RB while the shortfall probability is higher for the dynamic-RB. Again, the standard deviations of the annual return are very similar. Only the standard deviation of the overall return is slightly higher for the dynamic-ICF[10].

[19] Note that the maximum possible protection level in this case is $PL \approx 148\%$.

	$m = 1$ and $PL = 100\%$					$m = 3$ and $PL = 100\%$					$m = 5$ and $PL = 100\%$				
	ICF with $\phi =$			RB	TB	ICF with $\phi =$			RB	TB	ICF with $\phi =$			RB	TB
	10%	30%	50%			10%	30%	50%			10%	30%	50%		
$\bar{r}$	6.08	6.44	6.74	5.32	4.76	7.49	7.44	7.39	7.24	7.11	7.5	7.46	7.41	7.29	7.12
$\sigma[\bar{r}]$	3.11	3.34	3.58	2.84	2.52	4.25	4.31	4.4	4.25	4.18	4.36	4.39	4.48	4.38	4.36
$\sigma[r_t]$	12.27	12.5	12.82	12.19	12.36	18.3	18.11	18.05	17.93	17.67	18.56	18.43	18.42	18.09	17.39
$\bar{\theta}$	54.58	54.86	54.75	49.75	47.39	79.42	77.72	76.08	74.91	72.23	79.87	78.48	77.16	74.19	67.49
θ^{tot}	59.12	68.4	77.38	49.75	47.39	81.48	84.40	88.04	74.91	72.23	81.88	84.94	88.58	74.19	67.49
P_{PL}	2.6	5.0	7.6	2.6	0	17.9	18.9	20.1	17.9	0	23.2	22.8	23.0	24.4	0
	$m = 1$ and $PL = 120\%$					$m = 3$ and $PL = 120\%$					$m = 5$ and $PL = 120\%$				
$\bar{r}$	5.64	6.13	6.52	4.58	3.71	7.36	7.32	7.27	6.73	6.38	7.37	7.33	7.28	6.86	6.4
$\sigma[\bar{r}]$	2.89	3.16	3.44	2.51	1.96	4.13	4.17	4.27	3.99	3.8	4.26	4.27	4.35	4.17	3.97
$\sigma[r_t]$	10.6	11.05	11.6	10.53	10.78	16.91	16.79	16.82	16.05	15.59	17.25	17.17	17.24	16.3	15.15
$\bar{\theta}$	45.5	46.56	47.08	37.6	33.03	71.36	69.88	68.43	62.34	57.2	71.8	70.59	69.38	61.52	49.85
θ^{tot}	50.95	62.59	73.54	37.6	33.03	74.22	78.92	84.22	62.34	57.2	74.62	79.41	84.69	61.52	49.85
P_{PL}	5.5	8.1	10.8	7.1	0	22.9	23.3	24.2	25.0	0	29.7	28.2	28.1	33.1	0

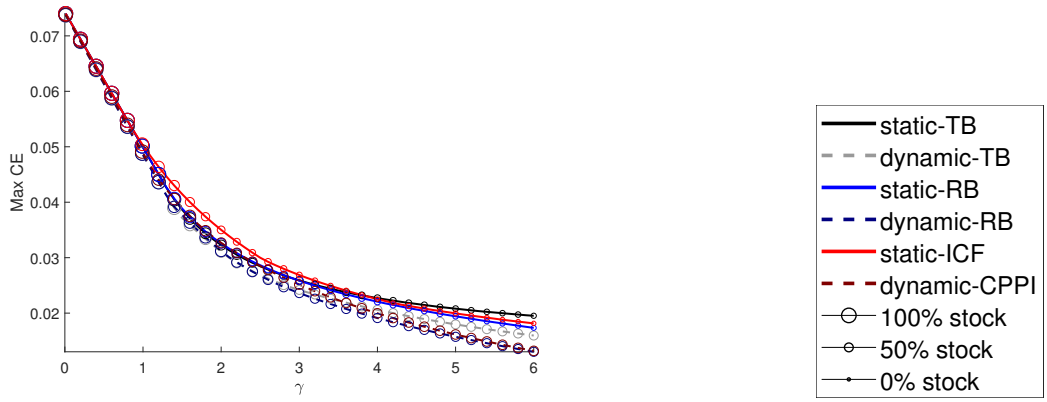
Table 6: Different risk measures for dynamic products with different multipliers (m) and different protection levels (PL) based on different low-risk investments (ICF, RB,TB) in the interest rate sensitivity scenario. All results are given in percent.

Preferences under EUT, CPT, MCPT and PMCPT

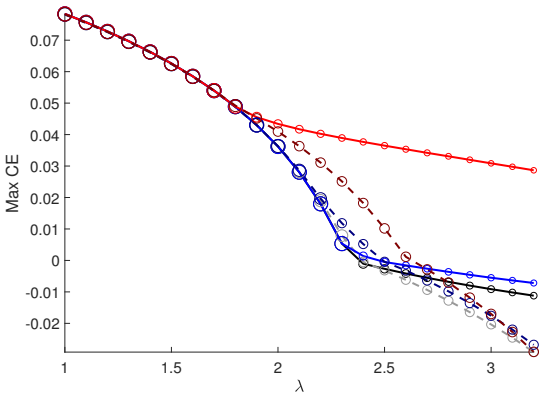
The results under the different preference theories are displayed in Figure 5 and Table 7. Firstly, we observe in Figure 5 that the structure of the results is very similar to the base case. In general, the CE values are higher than in the base case for all products. The products with the highest EUT values are very similar to the base case as the higher interest rates level shifts expected returns upwards but has only little influence on the volatility of the return. For MCPT and PMCT we find that the CE maximizing products have higher stock ratios than in the base case due to the lower probabilities for annual losses.

	$\gamma = 1.5$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 6$
EUT	static-ICF^{50} $\theta = 60\%$	static-ICF^{50} $\theta = 27.5\%$	static-ICF^{30} $\theta = 25\%$	static-ICF^{30} $\theta = 17.5\%$	static-TB $\theta = 20\%$
	$\lambda = 1.25$	$\lambda = 1.616$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3$
CPT	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$
MCPT	$\theta = 100\%$	$\theta = 100\%$	static-ICF^{50} $\theta = 47.5\%$	static-ICF^{50} $\theta = 30\%$	static-ICF^{50} $\theta = 22.5\%$
PMCPT	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	static-ICF^{50} 72.5% Stock	static-ICF^{50} 40% Stock

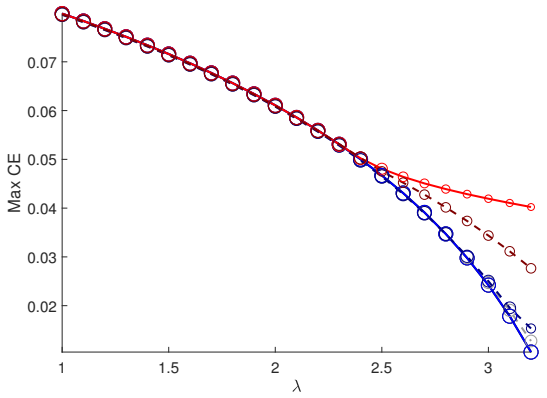
Table 7: The product with the highest CE value for different preference theories and for selected levels of risk and loss aversion in the interest rate sensitivity scenario, respectively.



(a) EUT



(b) MCPT



(c) PMCPT

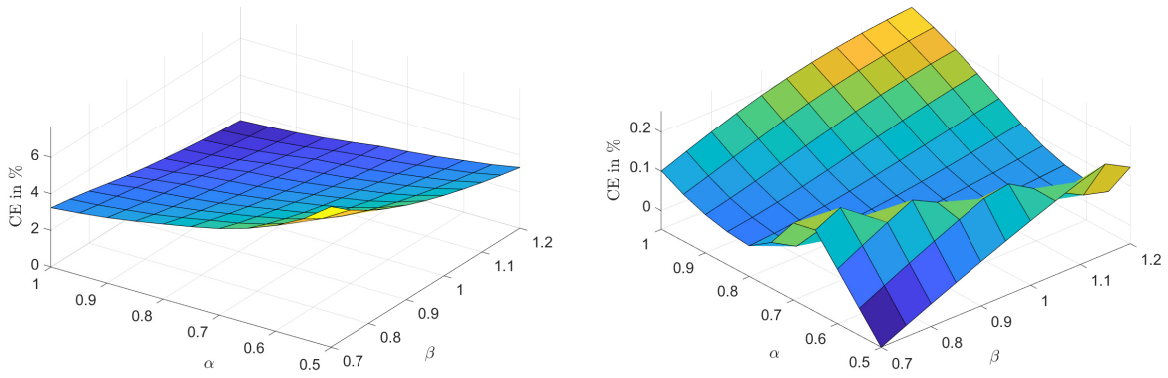
Figure 5: The maximum CE of different product types under different preference functions and for different levels of risk respectively loss aversion in the interest rate sensitivity scenario. The size of the circle corresponds to the (average) direct stock ratio of the product with the highest CE.

4.2.2 Probability Weighting Parameters

Next, we perform a sensitivity analysis with respect to the probability weighting. We fix the loss aversion at $\lambda = 2$ and only vary the probability weighting parameters $\alpha^- = \alpha^+ \in [0.5, 1]$ and $\beta^- = \beta^+ \in [0.7, 1.2]$. While α governs the curvature of the function and hence the tendency to overweight low likelihood events and to underweight high likelihood events (likelihood-insensitivity), the choice of β primarily influences the point where the function (typically inverse S-shaped) crosses the 45-degree line (elevation).[20]

Note that under CPT with $\lambda = 2$ and for all considered specifications of the probability weighting function the ICF-based investment strategy and market-based investment strategy with the highest CE are equal (always a pure stock investment). For MCPT and PMCPT, we display in Figure 6 the difference between the maximum CE of an ICF based investment product (static and dynamic) and the maximum CE of a market-based investment product (static and dynamic with RB and TB). The results are shown in Figure 6 (MCPT in panel (a) and PMCPT

[20]Reducing α increases likelihood-insensitivity and reducing β shifts the crossing point right and upwards.



(a) MCPT (b) PMCPT

Figure 6: The difference between the maximum CE of an ICF-based investment strategy and the maximum of a market-based investment strategy for different levels of probability weighting.

in panel (b)). In panel (a) of Figure 6, we observe that under MCPT the preferred ICF-based product always yields a higher CE than the preferred market-based product. The difference is particularly high for very low values of α and β (i.e., in case of strong overweighting of low-likelihood events). The optimal ICF-product is always a static-ICF^{50} with a direct stock ratio between 27.5% ($\theta^{tot} = 63.75\%$) and 37.5% ($\theta^{tot} = 68.75\%$), where the stock ratio is increasing for increasing α and β. Note that the CE of the preferred ICF-product is decreasing (from 4.9% to 2.3%) for increasing α and β.

Under PMCPT, we observe that for most combinations of α and β, the preferred ICF-product yields a higher CE than the preferred market-based product. For medium and high α and β a static-ICF^{50} with a direct stock ratio between 65% and 95% yields the highest CE (where the stock ratio is decreasing for increasing α and β). Only for very low values of α and β, the market-based product has a higher CE by roughly 0.05%.[21] The shape of the surface is striking and caused by a number of superimposing effects. For example, the type and stock ratio of the optimal ICF-based vs. market-based product changes several times on the surface.

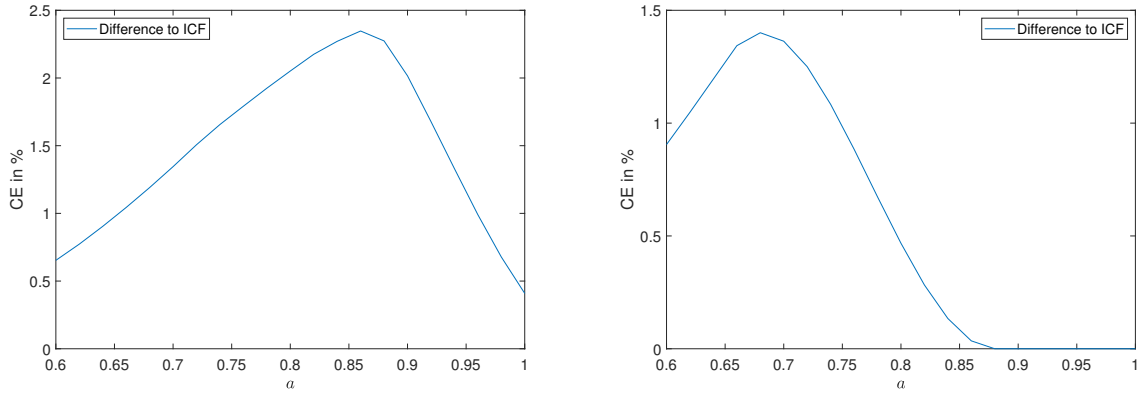
4.2.3 Risk Attitude

We also varied the risk attitude a between 0.6 and 1. We fix $\lambda = 2$ and the probability weighting parameters as stated for the base case. We again compute the difference between the maximum CE of an ICF-based investment product and the maximum of a market-based investment product. The results can be found in Figure 7.[22]

In Figure 7, we observe that for all levels of the risk attitude the maximum CE of the ICF-based

[21] Note that for low values of α and β, the preferred ICF product is a dynamic-ICF based on ICF^{10} with $PL = 80\%$ and $m = 5$, while the preferred market-based product is a dynamic-RB with $PL = 80\%$ and $m = 5$.

[22] Note that under CPT the optimal strategy is again a pure stock investment for all values of a.



(a) MCPT (b) PMCPT

Figure 7: The difference between the maximum CE of an ICF-based investment strategy and the maximum CE of a market-based investment strategy for different level of risk attitude (a) .

investment products is higher or equal than the maximum CE of the market-based investment products. Under MCPT, we observe that an ICF-based investment product yields the highest CE for all values of the risk attitude as ICF products come with significantly smaller annual changes and thus lower probabilities for annual losses. Further, the direct stock ratio of the preferred ICF-products is between 20% and 60% (hence, θ^{tot} is between 60% and 80%) and is increasing for a decreasing risk attitude.

Under PMCPT and for a low risk attitude ($a \geq 0.9$) the market- and the ICF-based products yield the same maximum CE which in both cases is attained by a pure stock investment. Further, we observe that in these cases the CPT CE has a higher influence on the PMCPT CE than for higher risk attitudes (lower values of a). For a higher risk attitude a more stable annual return is more important and CPT has a lower impact on the PMCPT CE than in the case of low risk attitudes. Therefore, in these cases an ICF-based investment product is preferred. Also, the direct stock ratio of the strategy with the highest CE is increasing for a decreasing risk attitude (from 25% ($\theta^{tot} = 62.5\%$) to 100%).

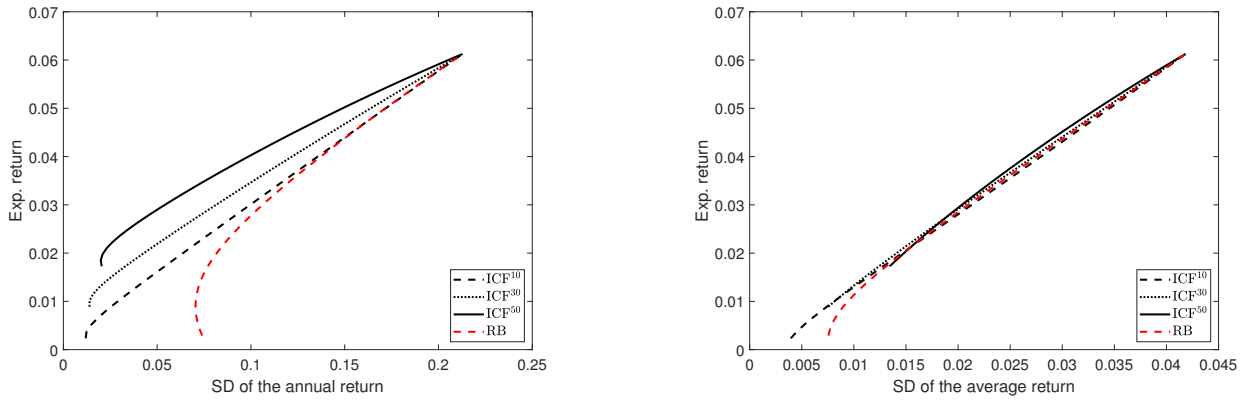
4.2.4 The smoothing period

We have increased the smoothing period from 3 years to 10 years and we repeated the numerical analysis.[23] The fair participation rate is 0.3766 for a stock ratio of 10%, 0.4912 for a stock ratio of 30% and 0.6261 for $\phi = 50\%$. The values of α are significantly lower than for a smoothing period of 3 years because the historical returns are significantly higher.

[23]For this, Equation (1) has been adjusted accordingly. In particular, we use $sp = 10$ years (instead of 3) and the necessary historical stock and bond return are given in Table 11 in Appendix A.

Effects of Smoothing

As for the base case, we observe that smoothing in the insurer's cover fund reduces the volatility of the annual returns while not affecting the overall return, cf., Figure 8. In particular, for a low direct stock ratio the volatility of the annual return is reduced more strongly by the longer smoothing period. Note that the history has only a small influence on the volatility of the annual returns. We also studied other histories and smoothing periods and the results were qualitatively very similar. This indicates that the observed effects do qualitatively not depend on the smoothing period (for reasonable choices).



(a) Average SD of annual returns

(b) The SD of the average return

Figure 8: The efficient frontier for different static-ICF based on a smoothing period of 10 years and a static-RB investment based on the standard deviation of annual returns and based on the standard deviation of the average return.

The Dynamic Investment Products

Again, we have analyzed different figures for dynamic products based on the different low-risk assets and for different protection level and multiplier (similar to Table 2).[24] We observe similar results as in the base case, i.e., ICF products yield higher expected returns and higher average direct stock ratios, in particular for conservative dynamic products (high PL and low m). Also, the standard deviations of annual returns, the average annual returns, and the shortfall probabilities are very similar for the dynamic-ICFs compared to the base case. Consequently, this indicates that, independent of the smoothing period, dynamic-ICFs have a significant potential to earn a higher return than a comparable dynamic-RB.

Preferences under EUT, CPT, MCPT and PMCPT

The results under the different preference theories are displayed in Figure 9 and Table 8. For EUT, we observe similar results as in the base case: static products have a higher CE than

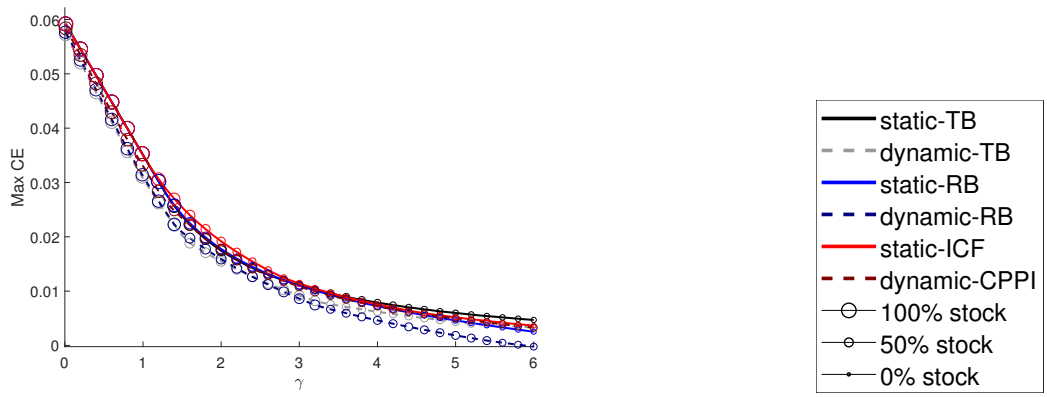
[24]Note that the values for the dynamic-RB and dynamic-TB are identical to the base case setup as they are not influenced by a longer smoothing period.

dynamic products. Further, up to a high risk aversion $\gamma = 4$ a static-ICF product and for $\gamma > 4$ a static-TB yield the highest CE, cf., Figure 9.

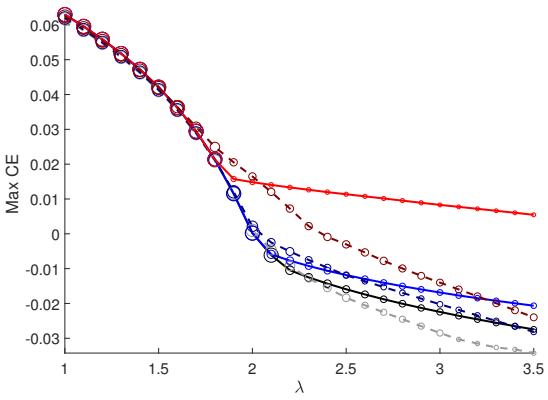
Under MCPT and PMCPT the products based on the insurer's cover fund yield the same or higher CE than the market-based products (as in the base case). However, for a medium level of loss aversion (MCPT $\lambda \approx 2$ and PMCPT $\lambda \approx 2.5$) a dynamic-ICF yields a higher CE than the static-ICFs, cf., Table 8. Note that we also analyzed other smoothing periods (longer than 3 years) and the results are qualitatively very similar. To sum up, also under longer smoothing periods, we observe that the ICF based products outperform the market-based products in many cases (or are at least comparable in terms of CE values).

	$\gamma = 1.5$	$\gamma = 2.5$	$\gamma = 3$	$\gamma = 3.5$	$\gamma = 6$
EUT	Static-ICF^{50} $\theta = 67.5\%$	Static-ICF^{50} $\theta = 35\%$	Static-ICF^{30} $\theta = 30\%$	Static-ICF^{30} $\theta = 22.5\%$	Static-TB $\theta = 20\%$
	$\lambda = 1.25$	$\lambda = 1.616$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3$
CPT	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$
MCPT	$\theta = 100\%$	dynamic-ICF^{10} $PL = 70\%$, $m = 4$	dynamic-ICF^{50} $PL = 90\%$, $m = 1$	Static-ICF^{50} $\theta = 10\%$	Static-ICF^{50} $\theta = 7.5\%$
PMCPT	$\theta = 100\%$	$\theta = 100\%$	$\theta = 100\%$	dynamic-ICF^{50} PL 90%, $m = 1$	Static-ICF^{50} $\theta = 15\%$

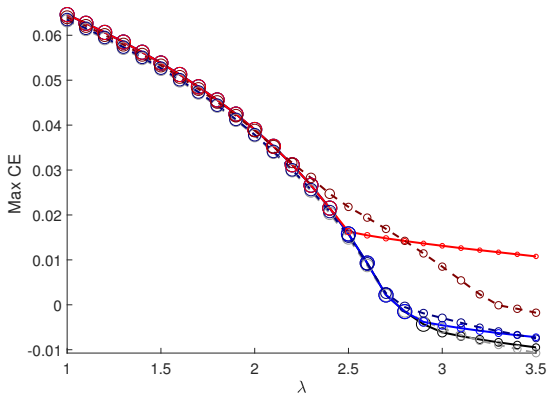
Table 8: The product with the highest CE value for different preference theories and for selected levels of risk and loss aversion in the smoothing period sensitivity scenario, respectively.



(a) EUT



(b) MCPT



(c) PMCPT

Figure 9: The maximum CE of different product types under different preference functions and for different levels of risk and loss aversion in the smoothing period sensitivity scenario, respectively. The size of the circle corresponds to the (average) direct stock ratio of the product with the highest CE.

5 Conclusion

In this paper, we investigated the effect of multiple, segregated cover funds equipped with different equity ratios on the short-term fluctuation risk as well as on the distribution of the terminal benefit of different retirement savings products. We found that smoothing mechanisms embedded in such cover funds can significantly reduce interim fluctuations while hardly impacting the long term risk-return profile. Hence, the existence of multiple, segregated cover funds that can be used as building blocks for more complex products enables insurers to offer different risk-return profiles of the terminal benefit in combination with a rather low pathwise volatility (when compared to investments without smoothing mechanisms). This appears highly relevant for the subjective attractiveness.

We analyzed various static and dynamic investment products based on cover funds with different stock ratios and compared them to purely market-based investment products. The results show that dynamic products based on the insurer's cover fund have a significantly higher return potential compared to dynamic market-based products with the same risk characteristics (in

particular for conservative product settings, which is of particular interest for rather risk averse investors).

Moreover, we examined the effect of collective cover funds on the objective utility and subjective attractiveness. To this end, we compared the products under Expected Utility Theory as well as Cumulative Prospect Theory and extensions of it (MCPT and PMCPT). Overall, we found that in most cases static products based on collective cover funds (mostly equipped with high stock ratios) outperform all other products with respect to objective utility and subjective attractiveness. Although, unfortunately, not the same product design is at the same time objectively optimal and subjectively preferable. Also, the main results are qualitatively independent of the smoothing period and also hold for other (reasonable) interest rate environments. Consequently, our results clearly demonstrate the advantages of segregated cover funds with different equity ratios.

The current challenges caused mainly by consumers' demand for low-risk products and the difficulty to offer high guarantees in the low-interest environment, could potentially be overcome with products without guarantee that still come with a rather high degree of safety. Therefore, our findings should be of high topical interest for insurance companies and legislators/regulators. Suitably designed products based on insurer's cover funds with return smoothing elements can serve the consumers' desire for safety (in particular, avoiding high short-term losses) without limiting the long-term return potential. Nevertheless, there are several practical obstacles that need to be overcome. For instance, in some countries, e.g., Germany, regulation for surplus distribution in participating life insurance products was designed to distribute returns that exceed a guaranteed rate of return. It is not always clear, if and how this can be applied to products without a guarantee. Even if it might be formally straightforward to set the annual guaranteed rate to -100% (and hence let any and all return play the role of "surplus"), some laws and regulation needs to be rewritten to make sense also in such settings. Also, in particular when cover funds with a rather high stock ratio are being offered, there need to be rules that prevent individual consumers to speculate against the pool of policyholders, e.g. by surrendering their contract after a market crash, when the smoothed policy value significantly exceeds the market value of the corresponding assets. Hence, appropriate regulation for fair market value adjustments in case of surrender is needed. Moreover, there is a range of practical issues (operationally as well as economically) that need to be considered when a new, segregated fund is being set up, in particular in the current low interest rate environment.

There are numerous suggestions for further research: Mainly the impact of different kinds of smoothing mechanisms on the risk-return characteristics of the resulting products should be further analyzed. We have seen that the considered – purely formula based – smoothing algorithm is very effective to reduce interim fluctuation but does hardly impact the distribution

of terminal wealth. While this may be desirable in some circumstances, one might also be interested in smoothing mechanisms that reduce the volatility of the probability distribution of terminal wealth (ideally without impacting expected return). It can be expected that path-dependent return smoothing (e.g., by building up certain "collective buffers" in good years which are used to increase returns in bad years or benefits at maturity that are particularly low) increases the advantages of smoothed products even more. Hence, a systematic analysis which type of smoothing mechanism is suitable to achieve which goal could stimulate the design of products that more effectively reduce those risks that are relevant to an individual consumer without harming the return potential too much. Since some smoothing mechanisms applied in practice have characteristics of "collective buffers" described above, such analyses might reveal that purely formula based smoothing rules systematically underestimate the effect of actually used smoothing mechanisms on the probability distribution of terminal wealth. This might impact the question how smoothing should be approximated, e.g., in models that calculate risk-return profiles and risk-return classes of retirement savings products.

A The Hull-White Model and the Stock Market Model

The two-factor Hull-White model (cf., Brigo & Mercurio (2007)) is given by the following dynamics for $t \in [0, T]$ with $T \in \mathbb{N}$:

$$dx(t) = -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0, \tag{8}$$

$$dy(t) = -by(t)dt + \eta\Big(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\Big), \quad y(0) = 0, \tag{9}$$

$$r^*(t) = x(t) + y(t) + \zeta(t), \tag{10}$$

where $r^*(t)$ denotes the short rate at time t under the risk-neutral measure Q, $\{W_i(t)\}_{t\geq 0}$ are independent one-dimensional Brownian motions for $i = 1, 2$, and $\zeta(t)$ is a deterministic function given by

$$\zeta(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 + \frac{\eta^2}{2b^2}(1-e^{-bt})^2 + \rho\frac{\sigma\eta}{ab}(1-e^{-at})(1-e^{-bt}), \tag{11}$$

where $f^M(0,t)$ is the initial market forward rate curve and can be obtained by the Nelson-Siegel-Svensson curve (NSS). The interest rate model is exponentially affine. Hence, based on $r(t)$ we can determine the value of a zero coupon bond $P(t,T)$ at time t with maturity T.

Under the real-world measure P, the short rate r is given by,

$$r(t) = x(t) + d_x(1-e^{-at}) + y(t) + d_y(1-e^{-bt}) + \zeta(t), \tag{12}$$

where $\{x(t)\}_{t\geq 0}$ and $\{y(t)\}_{t\geq 0}$ are the processes under Q and d_x and d_y are constants where $d_x + d_y$ can be interpreted as the long-run risk premium of the short-rate, cf., Berninger & Pfeiffer (2020).

The dynamics of the stock market is modeled by a generalized Black-Scholes model, particularly the price of a stock at time t is given by

$$S^*(t) = s_0 \exp\left(\int_0^t r^*(s)ds - 0.5\sigma_S^2 t + \sigma_S W(t)\right) \quad \text{under } Q, \tag{13}$$

$$S(t) = s_0 \exp\left(\int_0^t r(s)ds + (\lambda_S - 0.5\sigma_S^2)t + \sigma_S W(t)\right) \quad \text{under } P, \tag{14}$$

where s_0 is the initial price of the stock (at time 0), λ_S is the risk premium, σ_S is the volatility of the stock market, and $\{W(t)\}_{t\geq 0}$ is a one-dimensional Brownian motion independent of the short rate processes. The parameters used in our numerical analyses are given in Tables 9-11.

HW-Model	a	b	σ	η	ρ	d_x	d_y
	0.3912	0.0785	0.0201	0.0135	-0.6450	-0.0033	0.0255
NSS Curve	β_1	β_2	β_3	β_4	τ_1	τ_2	
	0.27173	-0.37865	-2.5003	-1.43785	2.95077	0.21103	

Table 9: The base case parameters of the short rate model (in line with Graf *et al.* (2021)).

NSS Curve	β_1	β_2	β_3	β_4	τ_1	τ_2	
	2.14449	-2.37645	26.11241	-29.99782	1.8297	1.99969	
History	2012	2013	2014	History	2012	2013	2014
Stock	17.65%	21.61%	3.93%	Rolling Bond	9.44%	-3.34%	18.19%

Table 10: The NSS-parameters and the historical returns (stock and rolling bond) as at the end of 2014 (used for the sensitivity analysis, cf., Section 4.2.1).

Year	2010/1	2011/2	2012/3	2013/4	2014/5	2015/6	2016/7	2017/8	2018/9	2019/20
Stock	14.36%	-17.29%	19.2%	26.68%	8.83%	-13.89%	23.27%	1.28%	5.12%	-5.05%
R-Bond	1.10%	19.39%	2.61%	8.15%	6.83%	11.11%	-4.49%	2.33%	7.40%	1.81%

Table 11: The historical stock and rolling bond returns for a smoothing period of 10 years, cf., Section 4.2.4.

B Cumulative Prospect Theory

In Cumulative Prospect Theory (CPT), cf., Kahneman & Tversky (1979) and Tversky & Kahneman (1992), an investment A with (random) final outcomes E is valued with an S-shaped value function v and relative to a given reference point χ. The gains and losses are described by the random variable $X := E - \chi$. Then the CPT utility is defined as

$$CPT(X) = \int_{-\infty}^{0} v(x) d\left(w\left(F(x)\right)\right) + \int_{0}^{\infty} v(x) d\left(-w\left(1 - F(x)\right)\right), \tag{15}$$

where $F(x) = \mathbb{P}(X \leq x)$ and v is the investor's value-function which is defined as $v(x) := x^a \mathbb{1}\{x \geq 0\} - \lambda |x|^a \mathbb{1}\{x < 0\}$ where $\lambda > 0$ is the loss aversion parameter and $a \in \mathbb{R}_+$ controls the risk appetite. The probability distortion function is given by $w^s(p) = \exp(-\beta^s(-\ln(p))^{\alpha^s})$, where $\alpha > 0$ governs the curvature, $\beta > 0$ the elevation, and $s \in \{+, -\}$ indicates gains or losses since probabilities for losses (α^-, β^-) and gains (α^+, β^+) are distorted differently.

In Multi Cumulative Prospect Theory (MCPT), cf., Ruß & Schelling (2018), the annual gains and losses (X_t) of an investment A are taken into account, i.e., $X_t := A_t - \chi_t$, where $t \in \{1, \ldots, T\}$, T is the maturity of the investment, A_t is the account value at time t, and χ_t is the reference point at time t. The MCPT value of investment A is then defined by

$$MCPT(A) := \sum_{t=1}^{T} CPT(X_t) \tag{16}$$

with $CPT(X)$ as defined in (15) and we assume no subjective discounting.

In Partial Multi Cumulative Prospect Theory (PMCPT) the terminal value (X) of an investment A as well as potential interim changes (X_t) are considered and thus it combines features of CPT and MCPT. The combination is defined as

$$PMCPT(A) := \omega CPT(X) + (1 - \omega) MCPT(A), \tag{17}$$

where $\omega \in [0, 1]$ denotes the weight, $CPT(X)$ as defined in (15) and $MCPT(A)$ as defined in (16).

C Computation of the Certainty Equivalents

For all preference formulations it is possible to compute certainty equivalent returns which describe the fixed annual return that an investor would regard equally desirable as the considered contract. We denote the value of choice c under a preference formulation G by $G(c)$, where $G \in \{EUT, CPT, MCPT, PMCPT\}$. Further, we compute the corresponding fixed annual

returns r^G by the following formulas:

- Under EUT:

$$EUT(c)^{\frac{1}{\gamma}} = e^{T \cdot rEUT}.$$

- Under CPT:

$$CPT(c) = \left(e^{T \cdot r^{CPT}} - 1\right)^a \quad \text{if } CPT(c) \geq 0$$
$$CPT(c) = -\lambda \left|e^{T \cdot r^{CPT}} - 1\right|^a \quad \text{if } CPT(c) < 0.$$

- Under MCPT:

$$MCPT(c) = \sum_{t=1}^{T} \left(e^{t \cdot r^{MCPT}} - e^{(t-1) \cdot r^{MCPT}}\right)^a \quad \text{if } MCPT(c) \geq 0$$
$$MCPT(c) = -\lambda \sum_{t=1}^{T} \left|e^{t \cdot r^{MCPT}} - e^{(t-1) \cdot r^{MCPT}}\right|^a \quad \text{if } MCPT(c) < 0.$$

- Under PMCPT:

$$PMCPT(c) = \omega \left(e^{T \cdot r^{PMCPT}} - 1\right)^a + (1-\omega) \sum_{t=1}^{T} \left(e^{t \cdot r^{PMCPT}} - e^{(t-1) \cdot r^{PMCPT}}\right)^a$$
$$\text{if } PMCPT(c) \geq 0$$
$$PMCPT(c) = -\lambda \left[\omega \left|e^{T \cdot r^{PMCPT}} - 1\right|^a + (1-\omega) \sum_{t=1}^{T} \left|e^{t \cdot r^{PMCPT}} - e^{(t-1) \cdot r^{PMCPT}}\right|^a\right]$$
$$\text{if } PMCPT(c) < 0.$$

Note that for EUT and CPT we can also derive certainty equivalent values which describe the fixed terminal payoff that an investor would regard as equally desirable as the considered contract. For these preference formulations, the certainty equivalent returns are equal to the annualized returns which correspond to the certainty equivalent values. However, under MCPT and PMCPT the preferences are based on interim value changes (in our case annual), i.e., from the paths which result in the terminal payoff. Therefore, the consideration of a single certainty equivalent value is not possible and we rather rely on fixed annual returns.

D Additional Numerical Results

Table 12 shows the shortfall probability, the expected shortfall and the mean for different dynamic strategies and protection levels.

		Shortfall probability in %				Exp. shortfall				Mean			
	PL	50%	70%	80%	90%	50%	70%	80%	90%	50%	70%	80%	90%
$m=1$	ICF^{10}	0.94	3.37	6.12	11.04	0.4606	0.6324	0.7158	0.7958	4.1351	3.4098	3.0474	2.6875
	ICF^{30}	2.43	5.94	8.68	12.06	0.4353	0.599	0.6796	0.7557	4.4258	3.821	3.5228	3.2306
	ICF^{50}	4.46	8.87	11.53	14.51	0.4028	0.5533	0.6263	0.6971	4.6716	4.1767	3.9389	3.7125
	RB	0.75	4.12	9.06	18.71	0.4587	0.6276	0.7074	0.7805	3.6734	2.765	2.3171	1.8944
	TB	0	0	0	0	-	-	-	-	3.3235	2.2734	1.7484	1.2234
$m=2$	ICF^{10}	6.16	11.05	14.31	18.69	0.4477	0.6216	0.7062	0.7878	5.7042	5.2553	4.9143	4.4812
	ICF^{30}	8.39	13.57	16.5	19.8	0.4279	0.5939	0.6763	0.7566	5.6321	5.1585	4.8511	4.5011
	ICF^{50}	10.62	16.33	19.11	21.81	0.3978	0.5513	0.627	0.6998	5.5546	5.0494	4.7653	4.4784
	RB	6.15	12.53	17.76	26.29	0.4479	0.6203	0.7035	0.7817	5.5208	4.5851	3.8995	3.0821
	TB	0	0	0	0	-	-	-	-	5.441	4.257	3.3268	2.0273
$m=3$	ICF^{10}	10.7	17.92	21.6	26.04	0.4391	0.6092	0.6935	0.7767	5.7765	5.4521	5.2025	4.8669
	ICF^{30}	11.92	18.82	22.09	25.46	0.4231	0.5872	0.6693	0.7507	5.7221	5.3459	5.0937	4.7962
	ICF^{50}	13.26	20.42	23.57	26.48	0.3943	0.5466	0.6229	0.698	5.6669	5.235	4.9727	4.6981
	RB	11.25	20.85	26.93	34.74	0.4423	0.6137	0.6984	0.779	5.6551	4.9366	4.3042	3.4653
	TB	0	0	0	0	-	-	-	-	5.5984	4.6636	3.7922	2.4251
$m=4$	ICF^{10}	13.08	21.76	26.16	30.69	0.4336	0.6009	0.6847	0.7675	5.7937	5.4948	5.2708	4.9761
	ICF^{30}	13.59	21.58	25.22	28.67	0.4201	0.5829	0.6647	0.746	5.7484	5.3991	5.1667	4.8935
	ICF^{50}	14.38	22.38	25.88	29.03	0.3923	0.5438	0.6204	0.6968	5.7032	5.3003	5.0519	4.7863
	RB	14.03	25.57	32.11	39.97	0.4388	0.6089	0.6927	0.7747	5.6785	5.0271	4.4356	3.6075
	TB	0	0	0	0	-	-	-	-	5.6124	4.7149	3.854	2.4983
$m=5$	ICF^{10}	14.37	24.05	28.75	33.42	0.4295	0.5951	0.6779	0.7601	5.8014	5.5099	5.2976	5.0187
	ICF^{30}	14.43	23.12	27.07	30.65	0.4176	0.5798	0.6613	0.7425	5.7612	5.4214	5.2007	4.9382
	ICF^{50}	14.97	23.54	27.27	30.58	0.3911	0.5426	0.6193	0.6957	5.7212	5.3304	5.0942	4.8328
	RB	15.7	28.35	35.37	43.31	0.4365	0.6059	0.6898	0.7721	5.6868	5.0545	4.4907	3.6674
	TB	0	0	0	0	-	-	-	-	5.6058	4.684	3.8117	2.4562

Table 12: Risk measures for different dynamic strategies and protection levels (PL).

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Curriculum Vitae

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