Asymptotic properties of estimators for random fields induced by stationary germ–grain models

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Ursa Marianne Pantle
aus Langen in Hessen

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Amtierender Dekan:	Prof. Dr. F. Stehling
Erster Gutachter:	Prof. Dr. V. Schmidt
Zweiter Gutachter:	Jun.–Prof. Dr. Evgueni Spodarev
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To Marc–Christian

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Chapter 1

Introduction

1.1 Motivation

In various fields of practical application such as medicine, biology, geology or material sciences (see, e.g., [40, 43, 45, 53]) huge amounts of data sets are on–hand in form of images (like satellite photographs or microscopic images). In order to extract those structures in the images that are relevant for the analysis of the underlying material a segmentation process is applied. As a result one obtains a black–and–white or grayscale image highlighting the texture of interest. On the left hand side of Figure 1.1 one finds, for example, the microscopic image of a section from mammary gland tissue. The corresponding segmented image is given on the right hand side displaying the phases stroma, epithel and lumen given in black, white and gray, respectively. One task arising in this context is the characterization and classification of the data by means of quantitative image characteristics like the number of particles or their specific shape. In general, not only a single geometric feature but a combination of different structural properties identify affiliations between pattern.

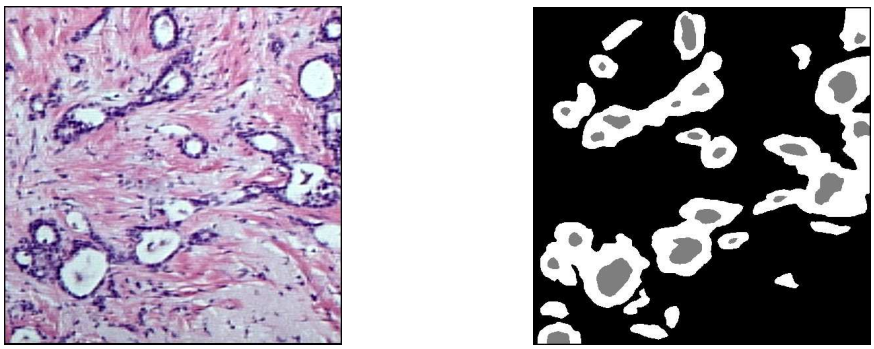


Figure 1.1: Sections of benignant mammary gland tissue; `left`: original microscopic image, `right`: segmented image (the image is taken from [41]).

The main objective of this thesis is to establish the theoretical background for confidence intervals and significance tests for these image characteristics allowing an

automated classification (and by this a fast prediagnostic) of the given data. For a general class of (image) characteristics, consistent estimators are derived and their asymptotic normality is proven. Under appropriate prerequisites, statistical significance tests can be performed on a finite observation window in order to categorize data sets using the information from a single representative image. The publications [41, 56, 57], of which I am a coauthor, are partially based on the results of this thesis.

In [41], sections of mammary gland tissue are analyzed with the aim to decide whether the samples originate from benignant (tumor–free) or malignant (cancerous) tissue; see Figure 1.2. Since in the presence of cancer the lumen deform and diminish, a classification criterion might be a combination of the mean area and the mean boundary length of the gray phase of the representing image per unit reference area. As one can see in the images of Figure 1.2, the single parts of the lumen can be

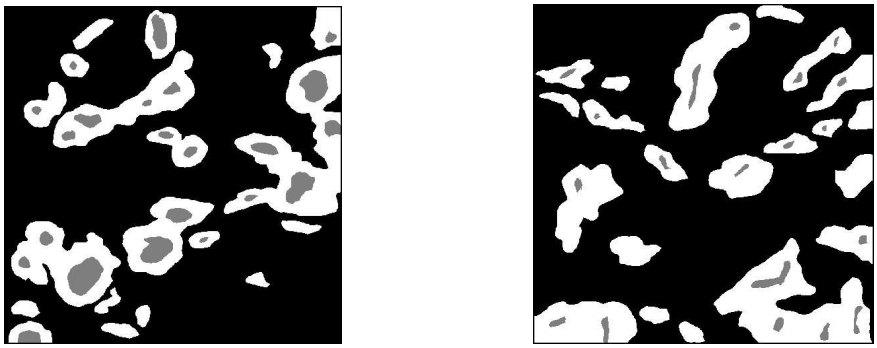


Figure 1.2: Sections of mammary gland tissue; `left`: benignant, `right`: malignant.

thought of as randomly scattered compact particles. A suitable mathematical model is given by so–called germ–grain models, which describe the location (germ) and structure (grain) of these particles; compare Figure 1.3. These stochastic models will form the basis for the subsequent considerations.

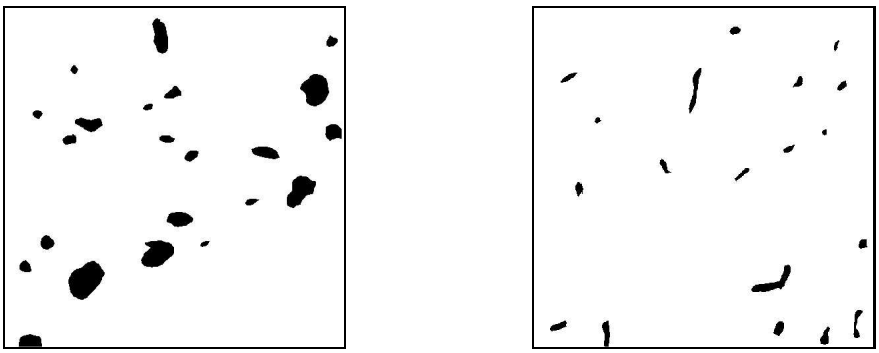


Figure 1.3: Extracting the gray phase (lumen) of the images in Figure 1.2; `left`: benignant, `right`: malignant tissue.

1.2 Overview

Germ–grain models make up a special class of random closed sets; see, e.g., [18, 39]. They can be used as a mathematical model for complex structures of various forms, which arise, for instance, in connection with biological cells, mineral grains, geological maps, or coverage processes in wireless communications. In this thesis, we will only consider stationary random sets Ξ with realizations in the extended convex ring $\mathcal{S}$. In particular, we assume that

$$\Xi \;=\; \bigcup_{l=1}^{\infty} (M_l + X_l) \tag{1.1}$$

for some point process $X = \{X_l, l \geq 1\}$ of germs, and a set $\{M_l, l \geq 1\}$ of independent and identically distributed copies of a random closed set M_0 (called "the grains"), where M_0 is a compact polyconvex set almost surely. The germs X_l, $l \geq 1$, can be thought of as the location of some affiliated object M_l; take, for example, a satellite antenna together with its coverage area. The family of random $\mathcal{S}$–sets is a simple but at the same time sufficiently flexible family of stochastic models. On the one hand it admits a rich variety of functionals such as the classical intrinsic volumes, and on the other hand most of the relevant assumptions and conditions are expressible in a tangible way.

As indicated before, we do not consider Ξ itself but a stationary random field $\{Y(x), x \in \mathbb{R}^d\}$ that is associated with Ξ via the relation

$$Y(x) = \varphi((\Xi - x) \cap K), \qquad x \in \mathbb{R}^d : K + x \subset W_n\,. \tag{1.2}$$

At this, W_n denotes a compact sampling window, and φ is a measurable (conditionally bounded and additive) functional describing some "local" geometric feature of Ξ, that is computed inside of a small (compact and convex) scanning window K. On a sequence of observation windows $\{W_n\}$, the mean value $\mu = \mathbb{E}\,Y(o)$ is estimated by the spatial averages

$$\widehat{\mu}_n \;=\; \int_{W_n} Y(x)\, G(W_n, x)\, dx\,, \qquad n \geq 1 \tag{1.3}$$

(see, e.g., [28]), where $G(W_n, x)$ is a weighting function that integrates to one over W_n and is equal to zero if $K + x$ is not entirely contained in W_n. Given that $\lim_{n\to\infty} W_n = \mathbb{R}^d$ in a "regular" way, conditions are derived that ensure mean–square consistency of $\{\widehat{\mu}_n, n \geq 1\}$ with respect to μ. The primary question, however, will be under which prerequisites $\{\sqrt{|W_n|}(\widehat{\mu}_n - \mu), n \geq 1\}$ is asymptotically normally distributed as n tends to infinity. From the general theory of stationary random fields (see, e.g., Section 1.7 of [28]) it is known that $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ is asymptotically normally distributed provided that Y fulfills certain mixing conditions and that $\mathbb{E}(|Y(x)|^{2+\delta}) < \infty$ holds for some $\delta > 0$. This means, roughly speaking, that the dependence structure of Y has to be decreasing sufficiently fast. In the case where $\{X_l, l \geq 1\}$ is a Poisson point process and M_0 is a convex and compact set almost

surely, we shall deduce a central limit theorem using a sequence of m_n–dependent random fields $U_n = \{U_n(z), z \in \mathbb{Z}^d\}$, $n \geq 1$. For such a sequence $\{U_n, n \geq 1\}$, a Lindeberg–type central limit theorem developed in [17] is applicable. It will be shown that the required conditions on Y, respectively on U_n, reduce to the tractable sufficient condition

$$\mathbb{E}\big(|M_0 \oplus \check{K}|^2\big) \; < \; \infty\,, \tag{1.4}$$

where $\oplus$ denotes the Minkowski addition and $\check{K}$ the reflection of test set K with respect to the origin $o \in \mathbb{R}^d$.
In the case of more general germ–grain models, conditions for the asymptotic normality of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ are derived from a central limit theorem for stationary random fields $\{U(z), z \in \mathbb{R}^d\}$ that satisfy certain β–mixing conditions (see [21]). In this case, we will prove that it is sufficient to assume that both $\mathbb{E}(|Y(o)|^{2+\delta})$ and $\mathbb{E}(||M_0||^{2d(\frac{\delta+1}{\delta})+\varepsilon})$ are finite for some $\delta, \varepsilon > 0$, where M_0 is a compact and polyconvex set almost surely and $||B|| = \sup\{|x| : x \in B\}$ for any set $B \subset \mathbb{R}^d$. In addition, the underlying point process X has to fulfill appropriate β–mixing conditions.

The asymptotic variance σ^2 of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ is given by

$$\sigma^2 \; = \; \theta \int_{\mathbb{R}^d} \mathrm{Cov}(Y(o), Y(x))\, dx \tag{1.5}$$

for a certain constant $\theta \in (0, \infty)$, where we assume that the covariance function of Y is absolutely integrable. Unfortunately, the variance σ^2 is in general unknown. Therefore, consistent estimators of σ^2 are constructed. First, we consider a nonparametric estimator of the form

$$\widehat{\sigma}_n^2 \; = \; \int_{\mathbb{R}^d} \widehat{\mathrm{Cov}}_{Y,n}(x)\, \gamma_n(x)\, dx\,, \qquad n \geq 1, \tag{1.6}$$

where $\widehat{\mathrm{Cov}}_{Y,n}(x)$ is a consistent estimator of $\mathrm{Cov}_Y(x) = \mathrm{Cov}(Y(o), Y(x))$ and $\gamma_n(x)$ is an appropriate weighting function. Secondly, an empirical variance estimator $\widetilde{\sigma}_n^2$ is introduced by

$$\widetilde{\sigma}_n^2 \; = \; \frac{|\tilde{W}_n|}{m-1} \sum_{k=1}^{m} \big(\widehat{\mu}_{kn} - \overline{\mu}_n\big)^2\,, \tag{1.7}$$

where $\overline{\mu}_n = \frac{1}{m}\sum_{k=1}^m \widehat{\mu}_{kn}$ for some integer $m = m(n)$ and estimates $\widehat{\mu}_{kn}$ of μ computed on subwindows $\tilde{W}_{kn}$, $k = 1, \ldots, m$ of W_n.
With regard to the mean–square consistency of the estimators $\widehat{\sigma}_n^2$ and $\widetilde{\sigma}_n^2$, certain integrability conditions are to be imposed on mixed moments of the random field Y. In the case of the Boolean model, we shall prove that the conditions needed for the consistency of $\widehat{\sigma}_n^2$ hold given that $\mathbb{E}(|M_0 \oplus \check{K}|^2) < \infty$ as required in (1.4) for the asymptotic normality of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$.

A multivariate version of the above–described topics is treated as well, that is, we consider several characteristics $\mu_1 = \mathbb{E}\, Y_1(o), \ldots, \mu_q = \mathbb{E}\, Y_q(o)$, $q \geq 2$ of Ξ simultaneously, where $Y_1, \ldots, Y_q$ are jointly stationary random fields of the form (1.2).

Note that the presented results hold for any measurable conditionally bounded and additive functional φ on the convex ring $\mathcal{R}$. A specific example is treated, for instance, in [3] and [38], where the 0–1 random field $\{\mathbb{1}_\Xi(x), x \in \mathbb{R}^d\}$ with $\mathbb{1}_\Xi(x) = \mathbb{1}((\Xi - x) \cap \{o\} \neq \emptyset)$ is discussed. The corresponding mean value is the volume fraction $p = \overline{V}_d(\Xi)$ of the germ–grain model Ξ. In [60] and [70], estimators of the vector of intrinsic volume densities $(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))^\top$ are introduced, which are based on statistics of the form (1.3) with uniform weights. For further estimators of geometric characteristics of Ξ such as the contact distribution function, the germ intensity, the surface area or the Euler characteristic confer, amongst others, [20], [48] and [75]. A general concept for a family of stationary random measures defined on the non–overlapping parts of Ξ is presented in [23].
Regarding the estimation of the asymptotic variance via the weighted covariance estimator defined in (1.6), confer Section 3 of [3] for 0–1–valued random fields and Section 3.5 of [60] for general random fields and uniform weights. Besides this, see Chapter 4 of [28] for properties of the estimator $\widehat{\mathrm{Cov}}_{Y,n}(x)$ of $\mathrm{Cov}(Y(o), Y(x))$ for fixed $x \in \mathbb{R}^d$. In contrast to [3], we do not restrict to uniformly bounded random fields in the present considerations. As a consequence, we need additional integrability conditions. Compared to the paper [60], we define weighting functional G more generally and use different arguments in the proof. The latter leads to more tractable integrability conditions, for which we derive a simple sufficient condition in the case of a Boolean model.

In practical applications, spatial data sets can often be transformed into binary or grayscale images extracting only the most relevant parts for analysis purposes. To categorize these data sets by statistical methods, we can make use of the asymptotic normality of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ and the consistent estimation of the asymptotic variance σ^2 to formulate asymptotic significance tests for the mean value μ. In the present work, we shall have a closer look on tests for the vector $\mathbf{v} = (\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))^\top$ of intrinsic volume densities. When examining, for instance, the images in Figure 1.2 displaying mammary gland tissue, a statistical test can yield a decision whether both images originate from the same class of cell structure or if the underlying models are significantly different in terms of characteristic $\mathbf{v}$ (or μ in general).

The classes of estimators of $\mathbf{v}$ employed here are introduced in [60] and [70]. The corresponding evaluation algorithms for the computer–based analysis can be found in [33] and [12], respectively. In the first step, we shall extend the algorithm derived in [33] so that it is also applicable for the estimation of the asymptotic covariance matrix. As a positive side–effect, we find that the new algorithm also improves the procedures described in [12]. In the second step, we discuss some empirical properties (like computation time, variation, or normal distribution) of the estimators and evaluation algorithms introduced in this thesis.

1.3 Organization of the thesis

An overview of some fundamental definitions and results from stochastic geometry, which are used throughout this manuscript, is given in Chapter 2. First, we introduce an adequate notion of measurability on the space of closed sets before turning to random closed sets and their main characteristics; see Section 2.1 and [39]. The second object of interest are point processes. The definition of these random counting measures is given on an arbitrary locally compact and second countable Hausdorff space in Section 2.2. As special cases, marked point processes and Poisson point processes are discussed. Random closed sets and point processes are related in various ways. On the one hand, the support of a point process is a locally finite random closed set, and, on the other hand, we will consider so–called particle processes, which are random collections of compact sets (i.e., point processes on the space of compact sets). The union $\Xi = \bigcup_{l=1}^{\infty} Z_l$ of the elements of a particle process $N = \{Z_l, l \geq 1\}$ is referred to as a germ–grain model; see Section 2.3. If N is a Poisson process, then Ξ is called a Boolean model. In the definition of Ξ, the particle process N can be replaced by a marked point process $\Psi = \{[X_l, M_l], l \geq 1\}$, where Ξ hence reads as

$$\Xi \;=\; \bigcup_{l=1}^{\infty} (M_l + X_l)\,.$$

In what follows, we are not interested in the germ–grain models themselves, but rather in certain characteristics expressed by means of random fields introduced in Section 2.4. The latter are of the form $Y = \{Y(x), x \in \mathbb{R}^d\}$, where

$$Y(x) \;=\; \varphi((\Xi - x) \cap K), \qquad x \in \mathbb{R}^d$$

for some measurable, conditionally bounded valuation $\varphi : \mathcal{R} \to \mathbb{R}$, a non–empty, convex and compact test set K and a stationary germ–grain model Ξ such that $\Xi \cap K$ belongs to the convex ring $\mathcal{R}$ almost surely. Important representants of functional φ are the intrinsic volumes $V_0(\Xi \cap K), \ldots, V_d(\Xi \cap K)$ considered in Paragraph 2.4.3. With regard to these, one objective will be to estimate the corresponding densities $\overline{V}_j(\Xi) = \lim_{n\to\infty} V_j(\Xi \cap (nK))/V_d(nK)$. The estimation is done indirectly by means of linear transformations resulting from the additive extension of the intrinsic volumes on the convex ring $\mathcal{R}$ or from the principal kinematic formula.

In Chapter 3, we focus on the case where Ξ is a Boolean model and estimate the mean value $\mu = \mathbb{E}\,Y(o)$ of a stationary random field Y defined as indicated above. The estimates are based on observations from Y in a sequence of compact Borel sets $\{W_n, n \geq 1\}$. To each measurement $Y(x)$, $x \in W_n$ a weight $G(W_n, x)$ is assigned, where $G(W_n, \cdot\,)$ integrates to one over W_n and vanishes for those x for which $Y(x)$ is not observable. The weighted spatial averages

$$\widehat{\mu}_n = \int_{W_n} Y(x)\, G(W_n, x)\, dx, \qquad n \geq 1$$

define a sequence of unbiased and mean–square consistent estimators of mean value μ. Unbiased estimation requires no further conditions; see Lemma 3.2. Mean–square

consistency, on the other hand, is proven in Lemma 3.3 with respect to the so–called van Hove property of the sequence $\{W_n\}$, on which weighting function G fulfills the condition

$$\lim_{n\to\infty} |W_n| \int_{\mathbb{R}^d} G(W_n, y+x)\, G(W_n, y)\, dy \;=\; \theta, \qquad x \in \mathbb{R}^d$$

for a certain constant $\theta \in (0,\infty)$. In addition, absolute integrability of the covariance function $\mathrm{Cov}_Y(x)$, $x \in \mathbb{R}^d$ is to be assumed, that is, we suppose that

$$\int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)|\, dx \;<\; \infty.$$

In Lemma 3.4, we show that $\mathrm{Cov}_Y(x)$, $x \in \mathbb{R}^d$ admits the upper bound

$$h(x) \;=\; 24\,\lambda\, c^2(K)\, e^{5\lambda \mathbb{E}|M_0 \oplus \check{K}|}\, \mathbb{E}\big|(M_0 \oplus \check{K}) \cap ((M_0 \oplus \check{K}) - x)\big|, \qquad x \in \mathbb{R}^d,$$

where $c(K) = \sup\{\varphi(K') : K' \in \mathcal{K}, K' \subseteq K\}$ and λ is the intensity of the germ process X. The function $h(x)$, $x \in \mathbb{R}^d$ is integrable provided that $\mathbb{E}\,(|M_0 \oplus \check{K}|^2) < \infty$.

The second main result of Chapter 3 is a central limit theorem for $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$, $n \in \mathbb{N}$. For this, the Boolean model $\Xi = \bigcup_{l\geq 1}(M_l + X_l)$ is approximated by a sequence of Boolean models $\Xi_n = \bigcup_{l\geq 1}(M_{ln} + X_l)$ with truncated grains $M_{ln} = M_l \cap A_n$, $l \geq 1$ for some d–dimensional cube $A_n = [-a_n, a_n]^d$ with $\lim_{n\to\infty} a_n = \infty$; see Paragraph 3.4.1. The associated random fields Y_n converge to the original random field Y in the (pointwise) L^2–sense, i.e., $\lim_{n\to\infty} \mathbb{E}\,((Y_n(x) - Y(x))^2) = 0$ for fixed $x \in \mathbb{R}^d$ as we shall deduce in Lemma 3.5. Asymptotic normality of $\sqrt{|W_n|}(\widetilde{\mu}_n - \mu)$, $n \geq 1$, where $\widetilde{\mu}_n = \int_{W_n} Y_n(x)\, G(W_n, x)\, dx$, is proven in Theorem 3.2 using a Lindeberg–type central limit theorem for m_n–dependent random fields presented in Theorem 2 of [17]. Given the absolute integrability of $\mathrm{Cov}_Y(x)$, $x \in \mathbb{R}^d$, the asymptotic distributions of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ and $\sqrt{|W_n|}(\widetilde{\mu}_n - \mu)$ coincide; see Theorem 3.3.

A multivariate version of the estimation problem is the final objective of Chapter 3. Here, we are given a vector–valued random field $\mathbf{Y} = \{(Y_1(x), \ldots, Y_q(x))^\top, x \in \mathbb{R}^d\}$ for some $q \geq 2$ and a set of jointly stationary random fields $Y_1, \ldots, Y_q$. Similar to the univariate case, we get by Theorem 3.4 that

$$\begin{pmatrix} \sqrt{|W_n|}\,(\widehat{\mu}_{1,n} - \mu_1) \\ \vdots \\ \sqrt{|W_n|}\,(\widehat{\mu}_{q,n} - \mu_q) \end{pmatrix} \xrightarrow{D} \mathcal{N}_q(o, \Sigma), \qquad n \to \infty,$$

where $\mathcal{N}_q(o,\Sigma)$ is a q–dimensional Gaussian random vector with zero mean vector and covariance matrix $\Sigma = (\sigma_{ij})_{i,j=1,\ldots,q}$, whose entries are $\sigma_{ij} = \theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx$ for certain constants $\theta_{ij} \in (0,\infty)$ and $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}(Y_i(o), Y_j(x))$.

Due to the specific independence structure of the Poisson point process, Theorem 3.2 can only be used given a "Poisson–like" germ process. In Chapter 4, we discuss conditions for asymptotic normality of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ for more general germ–grain models. The basic tool in this case is a stationary random field $U = \{U(z), z \in \mathbb{R}^d\}$

that fulfills certain β–mixing conditions; see [21, Theorem 6.1] and [23]. In the first step, we establish the link to m–dependent random fields in Lemma 4.1. Thereafter, we address the coherence between mixing properties of random field Y and the induced random field U in Section 4.2. Besides this, an upper bound for the absolute value of $\mathrm{Cov}_Y(x)$, $x \in \mathbb{R}^d$ is derived. By means of these auxiliary results, we prove asymptotic normality of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ in Theorem 4.2 under the condition that $\mathbb{E}(|Y(o)|^{2+\delta}) < \infty$ and $\mathbb{E}\big(||M_0||^{2d(\frac{\delta+1}{\delta})+\varepsilon}\big) < \infty$ for some $\delta > 0$, $\varepsilon > 0$, and that germ process X satisfies appropriate β–mixing conditions. Examples of point processes, that fulfill the imposed mixing conditions, and of valuations, for which the conditions of Theorem 4.2 can be relaxed, are discussed subsequently. A multivariate version of the central limit theorem closes the chapter.

Unfortunately, the asymptotic covariance matrix $\Sigma = (\sigma_{ij})_{i,j=1,\ldots,q}$ of the integrated covariance functions $\sigma_{ij} = \theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx$ is, in general, unknown or can not be provided by simple analytical formulae. For this reason, Chapter 5 is dedicated to the asymptotically unbiased and mean–square consistent estimation of Σ. Here, we pursue two approaches. The first class of estimators, defined in Sections 5.1 and 5.2, makes use of the fact that we basically have to estimate integrals of $\mathrm{Cov}_{ij}(x)$ over $\mathbb{R}^d$. For each pair $i, j = 1, \ldots, q$ and any $n \in \mathbb{N}$, we therefore consider a weighted integral of a consistent estimator $\widehat{\mathrm{Cov}}_{ij,n}(x)$ of $\mathrm{Cov}_{ij}(x)$ restricted to some averaging window $U_{ij,n}$. The resulting estimator of Σ is of the form

$$\widehat{\Sigma}_n = \left(\int_{\mathbb{R}^d} \widehat{\mathrm{Cov}}_{ij,n}(x)\, \gamma_{ij,n}(x)\, dx \right)_{i,j=1,\ldots,q}, \qquad n \geq 1,$$

where $\gamma_{ij,n}(x) = |W_n| \mathbb{1}(x \in U_{ij,n}) \int_{\mathbb{R}^d} G_i(W_n, y)\, G_j(W_n, y + x)\, dy$, $x \in \mathbb{R}^d$.
In Section 5.2, a modification $\widehat{\Sigma}'_n$ of the estimator $\widehat{\Sigma}_n$ is looked at, motivated by aspects of the efficient computation of $\widehat{\Sigma}_n$. A fast evaluation algorithm for $\widehat{\Sigma}_n$ can be designed using the well–known algorithm called Fast Fourier Transform (FFT). To enhance the efficiency of evaluation via FFT, a subdivision of the sampling window into smaller sub–windows is suitable in certain cases, as we shall also see in Section 7.3 along some examples.
Regarding the asymptotic properties of $\widehat{\Sigma}_n$ and $\widehat{\Sigma}'_n$, we first show in Lemma 5.1 that, for any fixed $x \in \mathbb{R}^d$, the estimator $\widehat{\mathrm{Cov}}_{ij,n}(x)$ is asymptotically unbiased for $\mathrm{Cov}_{ij}(x)$ given that the latter is absolutely integrable. Mean–square consistency of $\widehat{\mathrm{Cov}}_{ij,n}(x)$ is derived in Lemma 5.2 under the integrability conditions

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathrm{Cov}(Y_i(o)Y_j(x), Y_i(y)Y_j(x+y))\big|\, dy \;<\; \infty$$

and

$$\sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathbb{E}\big((Y_i(o) - \mu_i)(Y_i(y) - \mu_i)Y_j(x_1)Y_j(x_2)\big)\big|\, dy \;<\; \infty\,.$$

Both properties can be transferred to $\widehat{\Sigma}_n$ and $\widehat{\Sigma}'_n$, whereat the mean–square consistency of the integrated covariance estimators requires the additional assumption

$$\frac{1}{|U_{ij,n}|^2} \int_{U_{ij,n}} \int_{U_{ij,n}} \int_{\mathbb{R}^d} \big|\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big)\big|\, dy\, dx_1 dx_2 \;<\; \infty\,;$$

see Theorem 5.1. In Section 5.3 we shall prove that the above stated integrability conditions are satisfied in the case of a stationary vector–valued random field associated with a Boolean model Ξ having typical grain M_0 such that $\mathbb{E}\,(|M_0 \oplus \check{K}|^2) < \infty$ for all relevant compact and convex sets K.

The idea of estimation on sub–sampling windows $\tilde{W}_{kn}$, $k = 1, \ldots, m$ for some $m \in \mathbb{N}$ can be evolved into an empirical estimation procedure. This approach is presented in Section 5.4 and yields the second class of estimators $\widetilde{\Sigma}_n$, $n \geq 1$ of Σ. By subdivision of the observation window W_n into partially or non–overlapping windows $\tilde{W}_{kn}$, $k = 1, \ldots, m$, we obtain samples $\widehat{\mu}_{i,nk}$, $k = 1, \ldots, m$ for each $i = 1, \ldots, q$. Consequently, we can employ the empirical covariance $\widetilde{\Sigma}_n = (\widetilde{\sigma}_{ij,n})_{i,j=1,\ldots,q}$, $n \geq 1$ of these samples to estimate $\Sigma = (\sigma_{ij})_{i,j=1\ldots,q}$, where

$$\widetilde{\sigma}_{ij,n} \;=\; \frac{|\tilde{W}_n|}{m-1} \sum_{k=1}^{m} \left(\widehat{\mu}_{i,kn} - \overline{\mu}_{i,n}\right)\left(\widehat{\mu}_{j,kn} - \overline{\mu}_{j,n}\right)$$

for $\overline{\mu}_{i,n} = \frac{1}{m}\sum_{k=1}^{m} \widehat{\mu}_{i,kn}$, $i = 1, \ldots, q$. In Theorem 5.3 it is shown that $\widetilde{\Sigma}_n$ is consistent for Σ in the mean–square sense if

$$\int_{\mathbb{R}^{3d}} \mid s_{ij}^{(4)}(o, x_1, x_2, x_3) \mid \; d(x_1, x_2, x_3) \;<\; \infty\,,$$

where $s_{ij}^{(4)} : \mathbb{R}^{4d} \to \mathbb{R}$, $i, j = 1, \ldots, q$ denote certain fourth order cumulant functions of the random field $\mathbf{Y}$.

A practical implementation of estimators for the vector of intrinsic volume densities $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$ of a stationary germ–grain model Ξ in $\mathbb{R}^2$ and the associated asymptotic covariance matrices is the topic of Chapter 6. The estimation is based on the algorithms presented in Section 2.4 of [60] and Section 2 of [70]; see Section 6.1 for a short overview. The principal tool in the computer–based evaluation of the respective estimators (called $\widehat{\mathbf{v}}_n^{(1)}$ and $\widehat{\mathbf{v}}_n^{(2)}$) is a polygonal approximation $\xi_{\mathbb{F}}$ of the realization $\xi = \Xi(\omega) \cap W_n$ of germ–grain model Ξ within a given observation window W_n. Discretization and approximation issues are addressed in Section 6.2. Evaluation algorithms for the two classes of estimators are given in Section 3 of [33] and Section 2.4 of [12], respectively. In Paragraph 6.3.2, we introduce a different approach to evaluate the approximation formulae related to $\widehat{\mathbf{v}}_n^{(1)}$ with the perspective to be able to compute the corresponding weighted covariance estimators $\widehat{\Sigma}_{n,\mathbf{v}}^{(1)}$ and $\widehat{\Sigma}_{n,\mathbf{v}}^{\prime(1)}$. Chapter 6 closes with a brief introduction to the FFT algorithm.

Theoretical properties and evaluation algorithms can only provide a partial insight in the actual feasibility of an estimation procedure. In Chapter 7, we therefore give some numerical examples for the estimators discussed in Chapter 6 on the basis of Boolean models with uniformly bounded grains. Aspects of computation time, variation and asymptotic normality of the mean value estimators are presented in Section 7.2. Regarding the second issue, we check in which cases the appropriately transformed estimators $\widehat{\mathbf{v}}_n^{(1)}$ and $\widehat{\mathbf{v}}_n^{(2)}$ fit a multivariate normal distribution for increasing $n \in \mathbb{N}$, different grain distributions and germ intensities. A similar analysis is done for the

estimators of the asymptotic covariance matrix, where we have a closer look also on the positive semi–definiteness of the matrices; see Section 7.3. Finally, in Section 7.4 asymptotic tests for the vector $\mathbf{v}$ of intrinsic volume densities are performed. At this, we test if a given binary image matches a certain Boolean model with specific vector of intrinsic volume densities $\mathbf{v}^*$, and if two images can be considered as realizations of the same Boolean model.

Chapter 2

Fundamental definitions and results from stochastic geometry

2.1 Random closed sets

In stochastic geometry random closed sets serve as primary mathematical model to describe random structures. As the naming suggests, the term refers to random objects, whose realizations are closed subsets of an underlying topological space. It is surely possible to consider, for instance, random open sets as well. Random closed sets, however, have proven to be most convenient and general enough to work with. First related studies date back to the early thirties of the twenties century. The starting point of modern theory was laid by the work of Choquet [7] in 1953 concerning capacities of random sets. Subsequent studies on more general spaces have been published by Kendall [16] and Matheron [39]. In the following, we recall some definitions presented in [39]. At the beginning, most general spaces are treated. Later on, we will restrict our attention to the d–dimensional Euclidean space $\mathbb{R}^d$ and related spaces, as, for instance, the family of compact and convex sets in $\mathbb{R}^d$. Additional information on random closed sets are to be found in [49, 65, 71].

Before we get into the stochastic model, the underlying notion of measurability on certain topological spaces is introduced.

2.1.1 Topological and measure spaces

Throughout the following, let E be a topological space, which is locally compact, Hausdorff and second countable (short LCHS space); see Appendix A. Then E is metrizable with some metric $\varrho : E \times E \to \mathbb{R}_+$. The Borel σ–algebra on E is denoted by $\mathfrak{B}(E)$, and $\mathfrak{B}_0(E)$ refers to the class of bounded Borel sets.
On $E = \mathbb{R}^d$, we shortly write $\mathfrak{B}^d$ (respectively $\mathfrak{B}_0^d$) and choose the Euclidean metric $\varrho(x, y) = \sqrt{< x, y >}$, where $< x, y >= \sum_{i=1}^d x_i y_i$ is the scalar product of two points (or vectors) $x = (x_1, \ldots, x_d)^\top, y = (y_1, \ldots, y_d)^\top \in \mathbb{R}^d$. The norm of a vector $x \in \mathbb{R}^d$

is then given by $|x| = \sqrt{< x, x >}$, and $||B|| = \sup\{|x| : x \in B\}$ is the norm of a set $B \subset \mathbb{R}^d$. Moreover, write $|B|$ for the d–dimensional Lebesgue measure (or volume) of a set $B \in \mathfrak{B}^d$, ∂B for its boundary and $\mathrm{int}(B)$ for its interior. Besides this, let $\#B$ denote the cardinality of any countable set B. Finally, we define $B_r(x) = \{y \in \mathbb{R}^d : |x - y| \leq r\}$ to be a closed ball with center $x \in \mathbb{R}^d$ and radius $r > 0$ and represent the origin in $\mathbb{R}^d$ by $o = (0, \ldots, 0)^\top$.

In what follows, different classes of subsets of E are considered along with adequate topologies and σ–algebras. Denote by $\mathcal{F}(E), \mathcal{G}(E)$ and $\mathcal{C}(E)$ the families of all respectively closed, open, and compact subsets of E. To shorten notation, we shall simply write $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{C}$ and indicate the underlying space if necessary. The empty set belongs to each of these families. To indicate its exclusion, put $\mathcal{F}' = \mathcal{F} \setminus \{\emptyset\}$ and proceed accordingly for all other classes. The space $\mathcal{F}$ can be equipped with the so–called `hit-or-miss topology` $\mathcal{T}_\mathcal{F}$, which is also named `Fell topology` or `topology of closed convergence` (see [39, pp. 2–3]). To this end, define the sub–classes

$$\mathcal{F}_B \;=\; \{F \in \mathcal{F} : F \cap B \neq \emptyset\}, \qquad \mathcal{F}^B \;=\; \{F \in \mathcal{F} : F \cap B = \emptyset\}.$$

for any $B \subseteq E$. Through this, we obtain a basis for $\mathcal{T}_\mathcal{F}$ by the family

$$\mathcal{F}^C_{G_1,\ldots,G_n} \;=\; \mathcal{F}^C \cap \mathcal{F}_{G_1} \cap \ldots \cap \mathcal{F}_{G_n} \tag{2.1}$$

for $C \in \mathcal{C}$, $G_1, \ldots, G_n \in \mathcal{G}$ and $n \in \mathbb{N}_0$. The Borel σ–algebra induced by $\mathcal{T}_\mathcal{F}$ shall be denoted by $\mathfrak{B}(\mathcal{F})$. It is also known as `Effros σ-algebra` and admits the generating sets $\{F^C : C \in \mathcal{C}\}$ and $\{F_G : G \in \mathcal{G}\}$, respectively. The topologized space $\mathcal{F}$ is compact, Hausdorff and second countable, and $\mathcal{F}' = \mathcal{F} \setminus \{\emptyset\}$ is locally compact; see [39, Section 1.2].

The choice of $\mathcal{T}_\mathcal{F}$ and $\mathfrak{B}(\mathcal{F})$ is such that standard set operations are continuous or at least semi–continuous, and therefore measurable. Among these transformations are the union of two sets, i.e., the mapping $(F_1, F_2) \mapsto F_1 \cup F_2$, $F_1, F_2 \in \mathcal{F}$, which is continuous, and the intersection $(F_1, F_2) \mapsto F_1 \cap F_2$, which is upper semi–continuous. Here, a function $f : E \to \mathcal{F}$ is said to be `upper (lower) semi-continuous` if for any $C \in \mathcal{C}$ (for any $G \in \mathcal{G}$), the set $f^{-1}(\mathcal{F}^C) = \{x \in E : f(x) \in \mathcal{F}^C\}$ (the set $f^{-1}(\mathcal{F}_G) = \{x \in E : f(x) \in \mathcal{F}_G\}$) is open in $(E, \mathcal{T}_E)$. A function $f : E \to \mathcal{F}$ is called `continuous` if it is upper and lower semi–continuous.

Lemma 2.1 *On $E = \mathbb{R}^d$ or $E = \mathcal{F}'$, the shift operation $T_x : \mathcal{F}(E) \to \mathcal{F}(E)$ with $T_x(F) = F + x = \{y + x : y \in F\}$ for arbitary $x \in \mathbb{R}^d$ is a continuous mapping.*

Confer [65, pp. 18] for a *proof* on $E = \mathbb{R}^d$ and [65, pp. 38] on $E = \mathcal{F}'$. □

The family of compact sets $\mathcal{C}$ (and any subsequently considered subspace) is a LCHS space if $\mathcal{C}$ is endowed with the topology that is defined as in (2.1) for the subbasis $\{\mathcal{K}_G : G \in \mathcal{G}\} \cup \{\mathcal{K}^F : F \in \mathcal{F}\}$. Then $\mathcal{C}$ is metrizable by the Hausdorff metric

$$\varrho_H(C_1, C_2) \;=\; \max\{\sup_{x \in C_1} \inf_{y \in C_2} \varrho(x, y), \sup_{y \in C_2} \inf_{x \in C_1} \varrho(x, y)\}, \qquad C_1, C_2 \in \mathcal{C}$$

with $\varrho_H(C_1, \emptyset) = \infty, C_1 \in \mathcal{C}'$ and $\varrho_H(\emptyset, \emptyset) = 0$. According to Sections 1.3 and 2.1 of [65], the set $\mathcal{C}$ is a Borel set in $\mathcal{F}$. So it can be equipped with the induced σ–algebra $\mathfrak{B}(\mathcal{C}) = \{B \cap \mathcal{C} : B \in \mathfrak{B}(\mathcal{F})\}$, which coincides with the Borel σ–algebra under ϱ_H.

Some properties of sets, like convexity, will only be considered on the Euclidean space. On $E = \mathbb{R}^d$, we denote by $\mathcal{K}$ the class of all compact convex sets, which are also called `convex bodies`. The family of finite unions of convex bodies is the `convex ring` $\mathcal{R}$. Furthermore, the `extended convex ring` $\mathcal{S}$ is defined as the set

$$\mathcal{S} \;=\; \{F \in \mathcal{F} : F \cap K \in \mathcal{R} \text{ for all } K \in \mathcal{K}\}\,.$$

Just as $\mathcal{C}$, the sets $\mathcal{K}, \mathcal{R}$ and $\mathcal{S}$ are Borel sets in $\mathcal{F}$, so that the induced σ–algebras can be provided.

In addition to the set operations mentioned so far, further continuous functionals of subsets of $\mathbb{R}^d$ will be of interest later on. These are the reflection with respect to the origin $F \mapsto \check{F} = \{-x : x \in F\}$, positive homothetics $(a, F) \mapsto aF$ for $a \in \mathbb{R}_+$ with $aF = \{ax : x \in F\}$, and the `Minkowski addition` $(F_1, F_2) \mapsto F_1 \oplus F_2$, where

$$F_1 \oplus F_2 \;=\; \{x \in \mathbb{R}^d : x = x_1 + x_2\,,\; x_1 \in F_1,\; x_2 \in F_2\}\,, \qquad F_1, F_2 \subset \mathbb{R}^d.$$

If, in particular, $F_2 = B_r(o)$ for some $r > 0$, then $F_1 \oplus B_r(o)$ is called a `parallel set` of F_1 (see Figure 2.1). For translations $F \oplus \{x\}$ with $x \in \mathbb{R}^d$, we shortly write $F + x$. By definition it holds that $F_1 \oplus (F_2 \cup F_3) = (F_1 \oplus F_2) \cup (F_1 \oplus F_3)$. Besides this, notice that the Minkowski sum $F_1 \oplus F_2$ is equal to $\bigcup_{x \in F_2} (F_1 + x)$.

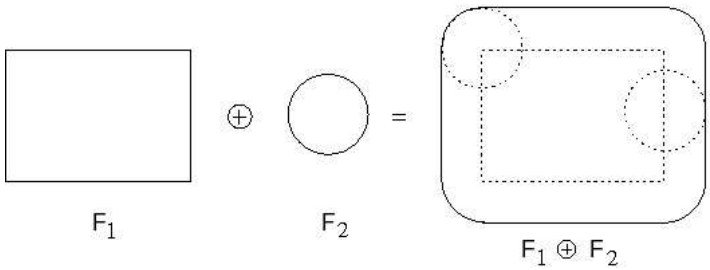


Figure 2.1: Minkowski addition of a rectangle F_1 and a ball F_2 ($\hat{=}$ parallel set of F_1).

In addition, we define the `Minkowski subtraction` (see Figure 2.2) as the mapping $(F_1, F_2) \mapsto F_1 \ominus F_2$ with

$$F_1 \ominus F_2 \;=\; \{x \in \mathbb{R}^d : \check{F}_2 + x \subseteq F_1\}\,, \qquad F_1, F_2 \subset \mathbb{R}^d.$$

Here, one has $F_1 \ominus (F_2 \cup F_3) = (F_1 \ominus F_2) \cap (F_1 \ominus F_3)$.

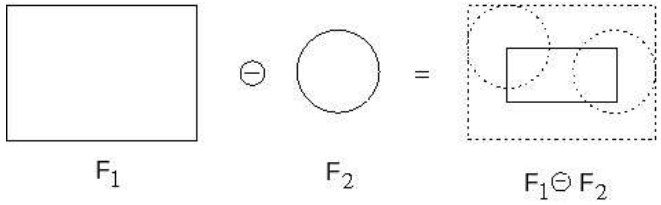


Figure 2.2: Minkowski subtraction of a rectangle F_1 and a ball F_2.

Lemma 2.2 *Let $E = \mathbb{R}^d$, then the following functionals are continuous*

(a) $(F, C) \mapsto F \oplus C$ *mapping from* $\mathcal{F} \times \mathcal{C}$ *into* $\mathcal{F}$,

(b) $F \mapsto \check{F}$ *mapping from* $\mathcal{F}$ *into* $\mathcal{F}$,

(c) $(a, F) \mapsto aF$ *mapping from* $\mathbb{R}_+ \times \mathcal{F}$ *into* $\mathcal{F}$.

Besides this, the Minkowski subtraction $(F, C) \mapsto F \ominus C$ is upper semi–continuous as a map from $\mathcal{F} \times \mathcal{C}'$ into $\mathcal{F}$.

Confer [39, pp. 19–20] for a *proof* of the above statements. □

It should be mentioned that $F_1 \oplus F_2$ and $F_1 \ominus F_2$ belong to $\mathcal{C}$ (to $\mathcal{K}$) for all compact sets (convex bodies) F_1, F_2. For further information on the set operations $\oplus$ and $\ominus$ see, for instance, Section 2.1.4 of [53].

2.1.2 Definition and main characteristics of random closed sets

Consider an arbitrary probability space $(\Omega, \mathcal{A}, P)$, and let E be some LCHS space.

Definition 2.1 *A* `random closed set` *Z in E is defined as a measurable mapping from $(\Omega, \mathcal{A}, P)$ into $(\mathcal{F}(E), \mathfrak{B}(\mathcal{F}(E)))$. The induced distribution shall be denoted by P_Z, that is to say, $P_Z(B) = P(Z \in B)$ for $B \in \mathfrak{B}(\mathcal{F}(E))$.*

A random closed set Z with values in $\mathcal{C}$ is called a `random compact set`, and we refer to it as a `random` $\mathcal{R}$`-set` (respectively a `random` $\mathcal{S}$`-set`) whenever $P(Z \in \mathcal{R}) = 1$ (respectively $P(Z \in \mathcal{S}) = 1$). Elementary examples of random closed sets in $\mathbb{R}^d$ are any deterministic closed sets, closed spheres with random radius or closed cubes with random side length and random orientation.

Due to the (semi–)continuity of the respective set operations, $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are random closed sets, whenever Z_1 and Z_2 are. In $\mathbb{R}^d$, we additionally have the random closed sets $\check{Z}_1$, $aZ_1, a \geq 0$, $Z_1 \oplus Z_2$ and $Z_1 \ominus Z_2$ provided that $Z_2 \in \mathcal{C}$ almost surely. Likewise, the shifted sets $Z_1 + x, x \in \mathbb{R}^d$ are random closed sets in $E = \mathcal{F}'$.

Definition 2.2 *A random closed set Z in $E = \mathbb{R}^d$ or $E = \mathcal{F}'$ is called* `stationary` *if $P_{Z+x} = P_Z$ holds for all $x \in \mathbb{R}^d$.*

To characterize the distribution of a random closed set uniquely, it is not necessary to know it entirely. As the definition of the underlying σ–algebra $\mathfrak{B}(\mathcal{F})$ suggests, it suffices to consider certain 'hitting probabilities'.

Definition 2.3 *For a random closed set Z in E, the* `capacity functional` *(or* `Choquet capacity`*) $T_Z : \mathcal{C} \to [0, 1]$ is defined by*

$$T_Z(C) \;=\; P_Z(\mathcal{F}_C) \;=\; P(Z \cap C \neq \emptyset)\,, \qquad C \in \mathcal{C}.$$

By the definition of T_Z and since $\{\mathcal{F}_C : C \in \mathcal{C}\}$ is a generating set for $\mathfrak{B}(\mathcal{F})$, two random sets Z_1 and Z_2 are identically distributed (write $Z_1 \stackrel{D}{=} Z_2$) if and only if $T_{Z_1}(C) = T_{Z_2}(C)$ for all $C \in \mathcal{C}$. Stationarity of a random closed set Z therefore

holds if and only if its capacity functional is invariant under translations, i.e., if $T_Z(C+x) = T_Z(C)$ for every $x \in \mathbb{R}^d$ and $C \in \mathcal{C}$. Furthermore, independence of Z_1 and Z_2 can be expressed by the property that $P(Z_1 \cap C_2 \neq \emptyset, Z_2 \cap C_2 \neq \emptyset) = T_{Z_1}(C_1)T_{Z_2}(C_2)$ for any $C_1, C_2 \in \mathcal{C}$.

A special role is assigned to random $\mathcal{R}$–sets by a decomposition into convex random compact sets. Convex bodies are quite tractable to work with, and several important characteristics of compact sets are merely defined under the assumption of convexity; see Section 2.4.3. With regard to this, the following functional will turn out useful.

Definition 2.4 *Introduce the auxiliary functional $N_{co} : \mathcal{R} \to \mathbb{N}_0$ by $N_{co}(\emptyset) = 0$ and*

$$N_{co}(K) = \min\{m \in \mathbb{N} : K = \bigcup_{\ell=1}^{m} K_\ell, K_\ell \in \mathcal{K}\}, \qquad K \in \mathcal{R}'. \tag{2.2}$$

That is to say, $N_{co}(K)$ is the smallest number of convex sets generating K.

By Lemma 4.4.1 in [65], the mapping $N_{co} : \mathcal{R} \to \mathbb{N}_0$ is Borel–measurable. Consequently, $N_{co}(Z \cap K)$ is an integer–valued random variable for all $K \in \mathcal{K}'$ and any random $\mathcal{R}$–set Z.

Further random variables, which are associated with random sets in $\mathbb{R}^d$ and which will be used later on, are the norm $||Z||$ of Z, its volume $|Z|$ and the indicator function

$$\mathbb{1}_Z(x) = \begin{cases} 0, & x \notin Z, \\ 1, & x \in Z \end{cases}$$

for some fixed $x \in \mathbb{R}^d$, where we also write $\mathbb{1}(x \in Z) = \mathbb{1}_Z(x)$.

In the sequel, it is often assumed that $\mathbb{E}\,|Z|^k < \infty$ for $k = 1$ or $k = 2$. For this, `Robbins' theorem` can be applied, which is essentially Fubini's theorem in a random set context.

Lemma 2.3 *Let Z be a random closed set in $\mathbb{R}^d$, and let μ be a locally finite measure on $\mathfrak{B}^d$. Then $\mu(Z)$ is a random variable and for $k = 1, 2, \ldots$ it holds that*

$$\mathbb{E}\,\mu^k(Z) \;=\; \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} P(\{x_1, \ldots, x_k\} \subseteq Z)\; \mu(dx_1) \ldots \mu(dx_k)\,.$$

We refer to Section 4.4 of [49] regarding a *proof* of Robbins' theorem. □

Finally, two essential characteristics of stationary random closed sets in $\mathbb{R}^d$ should be introduced. The so–called `volume fraction` $p = \mathbb{E}\,|Z \cap [0,1]^d|$, for which we have

$$p \;=\; T_Z(\{0\}) \;=\; P(o \in Z) \;=\; P(x \in Z), \qquad x \in \mathbb{R}^d, \tag{2.3}$$

and the covariance function $C(x) = \mathbb{E}|Z \cap (Z - x) \cap [0,1]^d|, x \in \mathbb{R}^d$ with

$$C(x) \;=\; 2p - T_Z(\{o, x\}) \;=\; P(o \in Z, x \in Z), \qquad x \in \mathbb{R}^d.$$

The volume fraction reflects the first–order structure of Z via $p = \mathbb{E}\,\mathbb{1}_Z(x)$, whereas $C(x)$ summarizes second–order properties, since $C(x) = \mathbb{E}\,(\mathbb{1}_Z(y)\mathbb{1}_Z(y+x))$ holds for any $x, y \in \mathbb{R}^d$.

2.2 Point processes

Many irregular patterns observable in satelite photographs, geological maps, microscopic images from biology or material sciences may be well represented by unions of random compact sets. The latter includes, in particular, random collections of singleton sets $\{x\}$ with $x \in \mathbb{R}^d$ or $x \in \mathcal{F}'$. A mathematical model for such collections is provided by point processes on E. We will restrict our attention to locally finite configurations of points, meaning that there exist no finite accumulation points. The following results are based on the books [8, 36, 49] and [65].

2.2.1 General definition and properties

Let E be an arbitrary LCHS space with Borel σ–algebra $\mathfrak{B}(E)$. A point process N on E can either be considered as a locally finite random closed set in E or as a random counting measure on $\mathfrak{B}(E)$.

Definition 2.5 *A* `counting measure` η *on* $\mathfrak{B}(E)$ *is defined as a measure on* $\mathfrak{B}(E)$ *taking values in* $\mathbb{N}_0 \cup \{\infty\}$. *It is called* `locally finite` *if* $\eta(B) < \infty$ *for any* $B \in \mathfrak{B}_0(E)$. *An* `atom` *of a counting measure* η *is an element* $x \in E$ *with* $\eta(\{x\}) > 0$. *The set of atoms of* η *is represented by its* `support` $\mathrm{supp}(\eta) = \{x \in E : \eta(\{x\}) > 0\}$. *If* $\eta(\{x\}) \in \{0, 1\}$ *for all* $x \in E$, *then the counting measure* η *is said to be* `simple`.

Let $\mathcal{N}_{lf} = \mathcal{N}_{lf}(E)$ be the family of all locally finite counting measures on $\mathfrak{B}(E)$, and denote by $\mathcal{N} = \mathcal{N}(E)$ the family of all $\eta \in \mathcal{N}_{lf}$ such that η is simple. On $\mathcal{N}_{lf}$, introduce the σ–algebra $\mathfrak{N}_{lf}$ generated by $\{\eta \in \mathcal{N}_{lf} : \eta(B) = k\}$ for $k \in \mathbb{N}_0$ and $B \in \mathfrak{B}_0(E)$. Likewise, set $\mathfrak{N} = \sigma(\{\eta \in \mathcal{N} : \eta(B) = k\}, k \in \mathbb{N}_0, B \in \mathfrak{B}_0(E))$ on $\mathcal{N}$.

Definition 2.6 *An* $(\mathcal{A}, \mathfrak{N}_{lf})$*–measurable mapping* $N : \Omega \to \mathcal{N}_{lf}$ *from an arbitrary probability space* $(\Omega, \mathcal{A}, P)$ *into* $(\mathcal{N}_{lf}, \mathfrak{N}_{lf})$ *is called a* `random counting measure` *or a* `point process` *on* E. *We say, a point process* N *is* `simple` *if* $P(\sup_x N(\{x\}) \leq 1) = 1$. *The distribution of* N *is denoted by* P_N *with* $P_N(A) = P(N \in A)$ *for* $A \in \mathfrak{N}_{lf}$.

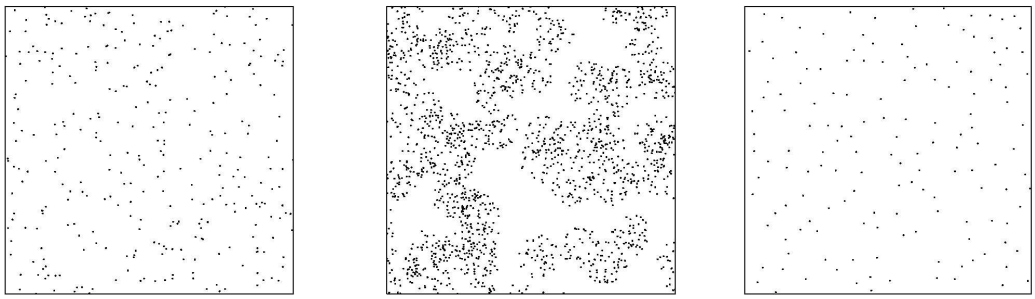


Figure 2.3: Stationary point processes on $\mathbb{R}^2$. `From Left to right`: Realization of a Poisson point process, a Matérn cluster process and a hard core process.

If a point process N on $\mathcal{F}'$ is concentrated on $\mathcal{C}'$, that is to say $N(\{C\}) = 0$ almost surely for all $C \notin \mathcal{C}'$, then it is referred to as a `particle process` on $\mathbb{R}^d$. We say N is a particle process on $\mathcal{K}$ (resp. on $\mathcal{R}$), if it is concentrated on $\mathcal{K}$ (resp. on $\mathcal{R}$).

A counting measure $\eta \in \mathcal{N}_{lf}$ can be expressed by the Dirac measures of its atoms (see [8, p. 198]), that is

$$\eta \;=\; \sum_{x \in \mathrm{supp}(\eta)} \eta(\{x\})\, \delta_x \,,$$

where, for any $A \in \mathfrak{B}(E)$, $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ else. For simple, random counting measures, we have the following.

Lemma 2.4 *Every simple point process N on E admits a representation*

$$N \;=\; \sum_{l=1}^{N(E)} \delta_{X_l} \,, \tag{2.4}$$

where $X_1, X_2, \ldots$ is a sequence of $(\mathcal{A}, \mathfrak{B}(E))$–measurable mappings $X_l : \Omega \to E$.

Confer [65, p. 69] regarding a *proof.* □

By Theorem 3.1.2 in [65], every simple counting measure $\eta \in \mathcal{N}$ is isomorphic to its support $\mathrm{supp}(\eta)$. In particular, N is a simple point process if and only if $\mathrm{supp}(N)$ is a locally finite random closed set. We will therefore use the symbol N for both the random counting measure and its (random) support. According to Lemma 2.4, there exists a measurable enumeration of the elements of $N = \mathrm{supp}(N)$, namely $N = \{X_1, X_2, \ldots\}$ or short $N = \{X_l\}$, for some random variables $X_l : \Omega \to E$, $l \in \mathbb{N}$. Thus, we can write

$$\sum_{x \in N} f(x) = \sum_{l=1}^{N(E)} f(X_l)$$

for any measurable function $f : E \to \mathbb{R}$.

By means of representation (2.4), we can now define stationarity of a point process.

Definition 2.7 *A point process N on $\mathbb{R}^d$ or $\mathcal{F}$ is called* `stationary` *if and only if its distribution is invariant under translations, in other words, $N \stackrel{D}{=} N + x$ for any $x \in \mathbb{R}^d$, where $N + x = \sum_{l=1}^{N(E)} \delta_{X_l + x}$ given $N = \sum_{l=1}^{N(E)} \delta_{X_l}$.*

Examples of stationary point processes on $\mathbb{R}^2$ are displayed in Figure 2.3; see [36] for corresponding definitions.

In addition to formula (2.4), there exist several other possibilities to characterize a point processe uniquely. One of them is the `void-probability functional` $z_N(B) = P(N(B) = 0), B \in \mathfrak{B}(E)$. Another characteristic is the `generating functional` G_N of N; see Chapter 7 of [8] or Chapter 3 of [36] for further details.

For $B \in \mathfrak{B}_0^d$, the generating function $g_{N(B)}(s) = \mathbb{E}(s^{N(B)})$ of the integer–valued random variable $N(B)$ will be an important tool later on. Here, all arguments $s \in \mathbb{R}$ may be considered for which $\mathbb{E}(s^{N(B)})$ is well defined. What is more, the expectation $\mathbb{E}\, N(B)$ of $N(B)$ plays a significant role as the `intensity measure` $\Lambda : \mathfrak{B}(E) \to [0, \infty]$ of point process N given by

$$\Lambda(B) = \mathbb{E}\, N(B), \qquad B \in \mathfrak{B}(E). \tag{2.5}$$

Note that $\Lambda(B)$ is the expected number of points inside of set B.

If the point process is stationary, then we have $\Lambda(B+x) = \Lambda(B)$ for every $x \in \mathbb{R}^d$ and each $B \in \mathfrak{B}(E)$. On $\mathfrak{B}^d$, in particular, it holds that

$$\Lambda(B) \;=\; \lambda|B|, \qquad B \in \mathfrak{B}^d,\; 0 \le \lambda \le \infty.$$

This is due to the fact that every translation invariant Radon measure on $\mathfrak{B}^d$ is a constant multiple of the Lebesgue measure. The constant λ is known as the `intensity` of the point process.

The following theorem underlines the importance of the intensity measure Λ. In literature, Theorem 2.1 is referred to as `Campbell's theorem`.

Theorem 2.1 *Let N be a simple point process on E, and let $f : E \to [0,\infty)$ be some non–negative measurable function. Then $\sum_{x\in N} f(x)$ is measurable, and its expectation can be calculated by means of*

$$\mathbb{E}\Big(\sum_{x\in N} f(x)\Big) \;=\; \int_E f(x)\,\Lambda(dx)\,. \tag{2.6}$$

Proof It is easily seen that (2.6) holds for simple functions $f(x) = \sum_{i=1}^k a_i \mathbb{1}_{B_i}(x)$ with $a_i \in \mathbb{R}_+, B_i \in \mathfrak{B}_0^d$, and thus for arbitrary non–negative, measurable functions using monotone convergence. □

All point processes considered in this manuscript are assumed to be simple, so the term is henceforth omitted. We point out, however, that several results hold for general point processes as well and refer to [8] and [36]. Despite the fact that a point process N is locally finite almost surely, it may happen that $\mathbb{E}\,N(B) = \infty$ for some $B \in \mathfrak{B}_0^d(E)$. Thus, we henceforth presume that Λ is finite on bounded Borel sets to avoid troubles. Moreover, Λ is supposed to be different from the zero measure (in symbols $\Lambda \not\equiv 0$). For the intensity of stationary point processes on $\mathbb{R}^d$, this implies that $0 < \lambda < \infty$.

2.2.2 Marked point processes

Particle processes are closely related to so–called marked point process on $\mathbb{R}^d$. In general, a marked point process can be thought of as a random collection of locations in $\mathbb{R}^d$ to which certain marks (specific characteristics like type or shape) are associated. In order to distinguish easily between the different types of point processes, we shall use the symbol $\Psi = \{[X_l, M_l]\}$ for marked point processes, $X = \{X_l\}$ for the process of locations (which will turn out to be a point process on $\mathbb{R}^d$) and $N = \{Z_l\}$ for particle processes.

Definition 2.8 *Let M be a LCHS space, and let $E = \mathbb{R}^d \times M$ be equipped with the product topology. A simple point process Ψ on E is called a* `marked point process` *on $\mathbb{R}^d$ with* `mark space` *M if its intensity measure fulfills the requirement*

$$\Lambda(C \times M) < \infty\,, \qquad \textit{for all } C \in \mathcal{C}. \tag{2.7}$$

According to Lemma 2.4, there exists a measurable enumeration $\{[X_l, M_l]\}$ of the elements of Ψ, where $X_l : \Omega \to \mathbb{R}^d$ and $M_l : \Omega \to M$ are measurable functions for each $l \in \mathbb{N}$ representing the locations and marks, respectively. Condition (2.7) assures that the marginal process of locations $X = \{X_l\}$ is a point process on $\mathbb{R}^d$.

Definition 2.9 *Let $\Psi = \{[X_l, M_l]\}$ be an arbitrary marked point process. If the sequences of locations $\{X_l\}$ and marks $\{M_l\}$ are independent and $\{M_l\}$ consists of independent and identically distributed variates, then we say that Ψ is an* `independently marked point process`. *Besides this, Ψ is called* `stationary` *if $\Psi + x \stackrel{D}{=} \Psi$ for all $x \in \mathbb{R}^d$, where $\Psi + x = \{[X_l + x, M_l]\}$ for any $x \in \mathbb{R}^d$.*

The intensity measure Λ of a stationary marked point process can be represented by the intensity $\lambda \in (0, \infty)$ of the marginal point process X and a probability measure $\mathbb{Q}$ on $\mathfrak{B}(M)$, where $\mathbb{Q}$ is known as the `mark distribution`.

Theorem 2.2 *Let Ψ be a stationary marked point process on $\mathbb{R}^d$ with mark space M and intensity measure $\Lambda \not\equiv 0$. Then there exists a constant $\lambda \in (0, \infty)$ and a probability measure $\mathbb{Q}$ on $\mathfrak{B}(M)$, both uniquely defined, such that*

$$\Lambda(B \times L) \;=\; \lambda |B|\, \mathbb{Q}(L)\,, \qquad B \times L \in \mathfrak{B}^d \otimes \mathfrak{B}(M). \tag{2.8}$$

A *proof* of Theorem 2.2 is given in [65, p. 190].

A stationary, independently marked point process can therefore be generated by constructing a stationary point process $X = \{X_l\}$ on $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$ and attaching to each point X_l an independent copy M_l of a random variable M_0 with distribution $\mathbb{Q}$ (short $M_0 \sim \mathbb{Q}$), where $\mathbb{Q}$ is the corresponding mark distribution.

In combination with Fubini's theorem, the decomposition (2.8) of intensity measure Λ enables us to reformulate Campbell's formula (2.6) for marked point processes.

Corollary 2.1 *Let Ψ be a stationary marked point process with intensity $\lambda \in (0, \infty)$ and mark distribution $\mathbb{Q}$ on $\mathfrak{B}(M)$. For any measurable $f : \mathbb{R}^d \times M \to \mathbb{R}_+$ and $M_0 \sim \mathbb{Q}$, it holds that*

$$\mathbb{E}\Big(\sum_{[x,m] \in \Psi} f(x, m) \Big) \;=\; \lambda\, \mathbb{E}\Big(\int_{\mathbb{R}^d} f(x, M_0)\, dx \Big)\,. \tag{2.9}$$

We close this section by presenting the relationship between stationary marked point processes with mark space $\mathcal{C}'$ and stationary particle processes. With regard to the following results, see Chapter 13 of [36] and Chapter 4 of [65].

Theorem 2.3 *Let Ψ be a stationary marked point process on $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$ and mark distribution $\mathbb{Q}$ on $\mathcal{C}'$. Furthermore, let M_0 be a random compact set with distribution $\mathbb{Q}$ such that $\mathbb{E}\,|M_0 \oplus B_r(o)| < \infty$ for some $r > 0$. Then the induced random counting measure $N_\Psi = \sum_{[x,m] \in \Psi} \delta_{(m+x)}$ on $\mathcal{C}'$ is a stationary particle process with locally finite intensity measure Λ, which can be written as*

$$\Lambda(B) = \lambda \int_{\mathbb{R}^d} \int_{\mathcal{C}'} \mathbb{1}_B(m + x)\, \mathbb{Q}(dm)\, dx, \qquad B \in \mathfrak{B}(\mathcal{C}'). \tag{2.10}$$

Proof It suffices to show that the number of elements $(M_l + X_l) \in N_\Psi, l \geq 1$ hitting an arbitrary compact set is finite. Let $r > 0$ be given such that $\mathbb{E}\,|M_0 \oplus B_r(o)| < \infty$. Since for any $C \in \mathcal{C}$ there exists a finite covering by translates of $B_r(o)$, one has $\mathbb{E}\,|M_0 \oplus C| < \infty$ for all $C \in \mathcal{C}$. From Corollary 2.1, we deduce that

$$\begin{aligned}
\mathbb{E}\Big(\#\{l : (M_l + X_l) \cap C \neq \emptyset\}\Big) &= \mathbb{E}\Big(\sum_{[x,m]\in\Psi} \mathbb{1}\big((m+x) \cap C \neq \emptyset\big)\Big) \\
&= \lambda\,\mathbb{E}\Big(\int_{\mathbb{R}^d} \mathbb{1}\big((M_0 + x) \cap C \neq \emptyset\big)\,dx\Big) \\
&= \lambda\,\mathbb{E}\,|M_0 \oplus \check{C}| < \infty\,.
\end{aligned}$$

Regarding the final equality observe that

$$\begin{aligned}
(A+x) \cap B \neq \emptyset \;&\Leftrightarrow\; \exists\, a \in A, b \in B \text{ such that } a + x = b \\
&\Leftrightarrow\; \exists\, a \in A, b \in B \text{ such that } x = (-a) + b\,,
\end{aligned}$$

and that $|\check{A} \oplus B| = |A \oplus \check{B}|$. Likewise, equation (2.10) is obtained by Campbell's formula (2.9) letting $f(x, m) = \mathbb{1}_B(m + x)$ for arbitrary $B \in \mathfrak{B}(\mathcal{C}')$. □

In Theorem 2.3 one can also do without stationarity of Ψ, but then other conditions (replacing $\mathbb{E}\,|M_0 \oplus B_r(o)| < \infty, r > 0$) are required to guarantee local finiteness of the induced counting measure N_Ψ.

For a converse of Theorem 2.3, define $c_o : \mathcal{C}' \to \mathbb{R}^d$ as the mapping which associates to each compact set $C \in \mathcal{C}'$ the midpoint of its uniquely determined circumsphere. Then $c_o : \mathcal{C}' \to \mathbb{R}^d$ is measurable, since continuous; see Lemma 4.2.1 in [65]. In addition, put $\mathcal{C}_o = \{C \in \mathcal{C}' : c_o(C) = o\}$.

Theorem 2.4 *Let $N = \{Z_l\}$ be a stationary particle process with locally finite intensity measure $\Lambda \not\equiv 0$. Then there exists a $\lambda \in (0, \infty)$ and a distribution $\mathbb{Q}_o$ on $\mathcal{C}_o$, uniquely determined by Λ and c_o, such that*

$$\Lambda(B) = \lambda \int_{\mathbb{R}^d} \int_{\mathcal{C}_o} \mathbb{1}_B(m + x)\,\mathbb{Q}_o(dm)\,dx\,, \qquad B \in \mathfrak{B}(\mathcal{C}_o).$$

Furthermore, $X_{c_0} = \sum_{l\geq 1} \delta_{c_o(Z_l)}$ defines a stationary point process on $\mathbb{R}^d$ with intensity λ, and $\Psi_{c_0} = \sum_{l\geq 1} \delta_{[c_o(Z_l), Z_l - c_o(Z_l)]}$ is a stationary marked point process with mark distribution $\mathbb{Q}_o$ and intensity λ.

Confer [65, pp. 136–139] for a detailed *proof*. □

The measure $\mathbb{Q}_o$ in Theorem 2.4 is called the `grain distribution` of N. It should be mentioned that other centering maps c_o' may be considered as well, which eventually yield different distributions $\mathbb{Q}_o'$. Yet, using the midpoint of the circumsphere has the advantage that the grain distribution of N coincides with the mark distribution of Ψ_{c_0} as introduced above.

2.2.3 Poisson point processes

As does the Wiener process for stochastic processes with continuous trajectories, the Poisson case plays a crucial role among point processes.

Definition 2.10 *Let $\Lambda : \mathfrak{B}(E) \to [0,\infty]$ be a diffuse and locally finite measure, meaning that $\Lambda(\{x\}) = 0$ for all $x \in E$ and $\Lambda(B) < \infty$ for all $B \in \mathfrak{B}_0(E)$. A simple point process N on E with intensity measure Λ is called a* `Poisson (point) process` *if the random variable $N(B)$ obeys a (degenerate) Poisson distribution with parameter $\Lambda(B)$ for all $B \in \mathfrak{B}(E)$. In other words, it holds that $P(N(B) = k) = \frac{\Lambda(B)^k}{k!} e^{-\Lambda(B)}$ for any $B \in \mathfrak{B}(E)$ with $\Lambda(B) < \infty$ and $P(N(B) = \infty) = 1$ if $\Lambda(B) = \infty$.*

The following theorem contains some useful properties regarding the construction and the characterization of Poisson processes.

Theorem 2.5 *Let $N = \sum_{l\geq 1} \delta_{X_l}$ be a Poisson process with intensity measure Λ.*

(a) *For any pairwise disjoint Borel sets $B_1, B_2, \ldots$ in E satisfying $\Lambda(B_j) < \infty$, the random variables $N(B_1), N(B_2), \ldots$ are independent. More generally, the point processes $N_{|B_j} = \sum_{l\geq 1} \delta_{X_l}(B_j \cap \cdot)$, $j = 1, 2, \ldots$ are independent.*

(b) *Consider $B \in \mathfrak{B}(E)$ with $\Lambda(B) \in (0,\infty)$ and let ξ be a random element on E with distribution $P_\xi(A) = \Lambda(B \cap A)/\Lambda(B)$ for $A \in \mathfrak{B}(E)$. Then the conditional distribution of $N_{|B} = \sum_{l\geq 1} \delta_{X_l}(B \cap \cdot)$ given $N(B) = k$ is equal to the k–fold convolution of P_{δ_ξ}, that is $P\big(N_{|B} \in A \mid N(B) = k\big) = P\big(\sum_{i=1}^k \delta_{\xi_i} \in A\big)$ for $A \in \mathfrak{N}$, where $\xi_1, \ldots, \xi_k$ are independent copies of ξ.*

(c) *Fix $B \in \mathfrak{B}(E)$ with $\Lambda(B) < \infty$. The generating function $g_{N(B)}(s)$ of $N(B)$ is well defined on whole $\mathbb{R}$, and it holds that $g_{N(B)}(s) = e^{(s-1)\Lambda(B)}, s \in \mathbb{R}$.*

A *proof* of assertions (a) and (b) can be found in Section 3.2 of [65], whereas (c) follows directly from Definition 2.10. □

Part (b) of Theorem 2.5 enables us to define a construction method for stationary Poisson process N on $\mathbb{R}^d$ with intensity λ inside of $B \in \mathfrak{B}_0^d$ arbitrarily, but fixed. First, generate a Poisson random variable $N(B)$ with parameter $\Lambda(B) = \lambda|B|$. Secondly, if $N(B) = k$, generate k independent and uniformly distributed points $X_1, \ldots, X_k$ in B. Furthermore, if one considers stationary Poisson particle processes on $\mathbb{R}^d$ with grain distribution $\mathbb{Q}$, then it suffices to attach k independent and identically distributed copies of $M_0 \sim \mathbb{Q}$ to each of the generated points in order to obtain a realization, provided that $\{M_1, ..., M_k\}$ and $\{X_1, \ldots, X_k\}$ are also independent.

Corollary 2.2 *Let Ψ be a marked point process with mark space M. If Ψ is independently marked and the marginal point process X is a Poisson process on $\mathbb{R}^d$, then Ψ is a Poisson process on $\mathbb{R}^d \times M$. If, conversely, Ψ is a stationary Poisson process on $\mathbb{R}^d \times M$, then Ψ is independently marked.*

Regarding a *proof* see Theorems 3.4.7 and 3.4.8 in [65]. □

The subsequent corollary will be of essential use later on. It is a direct consequence of Theorem 2.4 and Corollary 2.2.

Corollary 2.3 *If $N = \sum_{l=1}^{\infty} \delta_{Z_l}$ is a stationary Poisson particle process on $\mathbb{R}^d$ and $c_o(C)$ represents the midpoint of circumsphere for any $C \in \mathcal{C}'$, then the random counting measure $\Psi = \sum_{l=1}^{\infty} \delta_{[c_o(Z_l), Z_l - c_o(Z_l)]}$ is an independently marked stationary Poisson process on $\mathbb{R}^d$ with mark space $\mathcal{C}_0$. On the other hand, it holds that $N = \sum_{l=1}^{\infty} \delta_{(M_l + X_l)}$ is a stationary Poisson particle process if $\Psi = \sum_{l=1}^{\infty} \delta_{[X_l, M_l]}$ is a stationary, independently marked Poisson process on $\mathbb{R}^d$ with mark space $\mathcal{C}'$ satisfying $\mathbb{E}|M_0 \oplus B_r(o)| < \infty$ for some $r > 0$.*

2.3 Germ–grain models

2.3.1 General definitions and results

We now discuss particular random closed sets, which arise as the set–theoretic union of particle processes. Given a marked point process $\Psi = \{[X_l, M_l]\}$ with $M_l \in \mathcal{C}'$ almost surely, consider the union set

$$\Xi = \bigcup_{l=1}^{\infty} (M_l + X_l) \,, \tag{2.11}$$

the so–called `germ-grain model`. Examples of germ–rain models on $\mathbb{R}^2$ with underlying Poisson point process $X = \{X_l\}$ are shown in Figure 2.4.

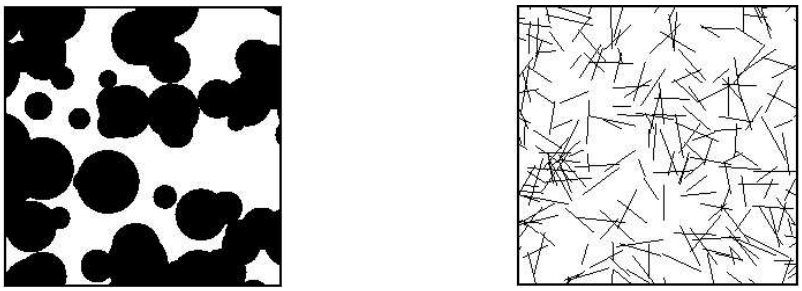


Figure 2.4: Germ–grain model of random discs (`left`) and of line segments (`right`).

Sufficient conditions for (2.11) to be well defined were established in [18], including aspects of ergodicity and mixing.

Theorem 2.6 *Let $\Psi = \{[X_l, M_l]\}$ be a stationary and independently marked point process on $\mathbb{R}^d$ with mark distribution $\mathbb{Q}$ on $\mathcal{C}'$. Then $\Xi = \bigcup_{l=1}^{\infty} (M_l + X_l)$ is a stationary random closed set that is different from $\mathbb{R}^d$ if $\mathbb{E}\,|M_0 \oplus B_r(o)| < \infty$ for some $M_0 \sim \mathbb{Q}$ and $r > 0$.*

Proof Measurability of the mapping $(x, m) \mapsto (m + x)$ for $(x, m) \in \mathbb{R}^d \times \mathcal{F}$ is shown in Lemma 1 of [18]. The remainder is given in Lemma 3 of the same paper. □

Recall from Theorem 2.3 that the integrability condition $\mathbb{E}\,|M_0 \oplus B_r(o)| < \infty$ ensures that only finitely many particles $M_l + X_l, l \geq 1$ hit a given compact set. With regard to this, we point out that it does not suffice to assume finiteness of $\mathbb{E}\,|M_0|$. If $\mathbb{E}\,|M_0| < \infty$ and $\mathbb{E}\,|M_0 \oplus B_r(o)| = \infty$ for all $r > 0$, Proposition 2 in [22] states that $\Xi = \bigcup_{l=1}^{\infty}(M_l + X_l)$ is closed in $\mathbb{R}^d$ with probability 0. As an example, consider M_0 to be a rectangle in $\mathbb{R}^2$ with random side lengths X and $1/X$, where $\mathbb{E}\,X = \infty$. Then we have $\mathbb{E}\,|M_0| = 1$ and $\mathbb{E}\,|M_0 \oplus B_r(o)| = \infty$ for any $r > 0$.

Given a particle process $N = \{Z_l\}$ on $\mathcal{F}'$, it readily follows from local finiteness of N that $\Xi = \bigcup_{l=1}^{\infty} Z_l$ is a random closed set. As to the converse, the following decomposition theorem was first presented in [75] (see also [65, pp. 147–149]).

Theorem 2.7 *For any random closed set Ξ in $\mathbb{R}^d$, there exists a particle process $N = \{Z_l\}$ such that $\Xi = \bigcup_{l=1}^{\infty} Z_l$. If Ξ belongs to the extended convex ring $\mathcal{S}$ almost surely, then the particles can be chosen such that $P(Z_l \in \mathcal{K}') = 1$ for each $l \geq 1$. Furthermore, if Ξ is stationary, then so is N.*

Analogously, we find a marked point process $\Psi = \{[X_l, M_l]\}$ that constitues Ξ via (2.11), putting $X_l = c_o(Z_l)$ and $M_l = Z_l - c_o(Z_l)$, $l \geq 1$ (compare Theorem 2.4).

Let us summarize the related nomenclature.

Definition 2.11 *If $\Psi = \{[X_l, M_l]\}$ is a stationary and independently marked point process on $\mathbb{R}^d$ with mark distribution $\mathbb{Q}$ on $\mathcal{C}'$ such that $\mathbb{E}\,|M_0 \oplus B_r(o)| < \infty$ for $r > 0$ and $M_0 \sim \mathbb{Q}$, then Ψ is called a* `germ-grain process` *and $\Xi = \bigcup_{l=1}^{\infty}(M_l + X_l)$ is said to be a* `germ-grain model`*. The underlying point process $X = \{X_l\}$ on $\mathbb{R}^d$ is known as* `germ process`*, where the points X_l are the* `germs`*. The marks M_l, $l \geq 1$ are referred to as* (`primary`) `grains` *and a random closed set $M_0 \sim \mathbb{Q}$ is called a* `typical grain` *of Ξ (or of Ψ).*

2.3.2 Boolean models

Boolean models are the best–known and most–studied germ–grain models. They are applied, whenever complex particle patterns are to be approximated by means of a Poisson process of random closed sets. The advantage of Boolean models lies in their flexibility together with rather simple formulae, which often depend on the grain distribution and the germ process intensity alone.

Definition 2.12 *A germ–grain model $\Xi = \bigcup_{l=1}^{\infty}(M_l + X_l)$ is called a* `Boolean model` *if the generating germ process $\{X_l\}$ is a stationary Poisson process.*

First thourough studies of Boolean models are due to Matheron and Serra, amongst others, see [39, 66] and references therein. Confer [71] for a historical overview of earlies studies with applications in geology, forestry and material sciences. Statistical methods for Boolean models are presented, for example, in [48], including test on the Boolean model assumption. A discussion of coverage and connectivity properties is given in [14], where, for instance, the number of grains covering a given set or the number of connected components per unit area are considered.

Lemma 2.5 *Let Ξ be a Boolean model with intensity $\lambda \in (0,\infty)$ and typical grain M_0. Then the capacity functional of Ξ reads as*

$$T_\Xi(C) = 1 - e^{-\lambda\,\mathbb{E}|M_0 \oplus \check{C}|}, \qquad C \in \mathcal{C}'.$$

Proof By Corollary 2.3 we know that the induced particle process $N_\Xi = \{(M_l + X_l)\}$ is a Poisson process with intensity measure Λ satisfying $\Lambda(\mathcal{F}_C) = \lambda\,\mathbb{E}\,|M_0 \oplus \check{C}|$ for any $C \in \mathcal{C}$ by Theorem 2.3. By this, it holds that

$$T_\Xi(C) = 1 - P(N_\Xi(\mathcal{F}_C) = 0) = 1 - e^{-\Lambda(\mathcal{F}_C)} = 1 - e^{-\lambda\,\mathbb{E}|M_0 \oplus \check{C}|}, \quad C \in \mathcal{C}'. \qquad \square$$

By Lemma 2.5, explicit formulae for the volume fraction $p = T_\Xi(\{o\})$ and the covariance function $C(x) = 2p - T_\Xi(\{o,x\})$ are obtained. If Ξ is a Boolean model, then we have

$$p \;=\; 1 - e^{-\lambda\,\mathbb{E}|M_0|} \tag{2.12}$$

and

$$C(x) \;=\; 2p - 1 + (1-p)^2 e^{\lambda\,\mathbb{E}|M_0 \cap (M_0 - x)|}, \qquad x \in \mathbb{R}^d. \tag{2.13}$$

2.4 Functionals associated with random closed sets

A principal task of spacial statistics is the estimation of characteristic features of random closed sets Ξ using observations within some window $W \subset \mathbb{R}^d$. We assume that there exist local information about Ξ, that is to say, one can make use of quantities $\varphi((\Xi - x) \cap K)$ for certain functionals $\varphi : \mathcal{F} \to \mathbb{R}$, test sets $K \in \mathcal{C}'$ and locations $x \in \mathbb{R}^d$ such that $K + x \subseteq W$. In what follows, the class of functionals considered is specified and important examples are introduced.

2.4.1 Basics on random fields

Let a random closed set Ξ, a functional $\varphi : \mathcal{F} \to \mathbb{R}$ and a set $K \in \mathcal{C}'$ be given. Then $\varphi((\Xi - x) \cap K)$ is a real–valued random variable for every fixed $x \in \mathbb{R}^d$ if φ is measurable. Considering the collection $\{\varphi((\Xi - x) \cap K), x \in \mathbb{R}^d\}$ of all these random variables, we get a `random field` on $\mathbb{R}^d$ induced by Ξ and K. In the upcoming chapters, we will need to impose certain conditions on functional φ and will also make use of random fields $\{U(z), z \in \mathbb{Z}^d\}$ on $\mathbb{Z}^d$. To this end, let $T \in \{\mathbb{R}^d, \mathbb{Z}^d\}$ and denote by S the class of functions $f : T \to \mathbb{R}^q$ for some $q \in \mathbb{N}$. On S, define $\mathcal{B}(S)$ to be the σ–algebra induced by the `evaluation maps` $\pi_x : S \to \mathbb{R}^q$, $x \in T$ with $\pi_x(f) = f(x)$ for any $f \in S$.

Definition 2.13 *Fix $S' \subseteq S$ arbitrarily. A measurable mapping $Y : \Omega \to S'$ from a probability space $(\Omega, \mathcal{A}, P)$ into $(S', S' \cap \mathcal{B}(S))$ is said to be a* `random field on` T `with paths in` S'. *If $q \geq 2$, we call Y a* `vector-valued random field`.

By measurability of the evaluation maps $\pi_x, x \in \mathbb{R}^d$, the random variables $Y(x) = \pi_x(Y) : \Omega \to \mathbb{R}$ are well defined for every $x \in T$, and we write $Y = \{Y(x), x \in T\}$. The realization of a random field Y at some point $x \in T$ will be denoted by $Y(\omega, x)$ rather than $Y(\omega)(x)$, $\omega \in \Omega$.

Lemma 2.6 *The mapping $Y : \Omega \to S', S' \subseteq S$ is $(\mathcal{A}, S' \cap \mathcal{B}(S))$–measurable if and only if $Y(x) : \Omega \to \mathbb{R}^q$ is $(\mathcal{A}, \mathcal{B}^q)$–measurable for every $x \in T$.*
Let Y and Y' be two arbitrary (vector–valued) random fields, then $Y \stackrel{D}{=} Y'$ if and only if $(Y(x_1), \ldots, Y(x_k))^\top \stackrel{D}{=} (Y'(x_1), \ldots, Y'(x_k))^\top$ for any $x_1, \ldots, x_k \in T, k \in \mathbb{N}$.

A *proof* of this Lemma is given in [30, Lemma 3.1, Proposition 3.2]. □

The second statement in Lemma 2.6 states that the distribution of a random field $Y = \{Y(x), x \in T\}$ is uniquely determined by its `finite dimensional distributions`, meaning by the distributions of the random vectors $(Y(x_1), \ldots, Y(x_k))^\top$ for arbitrary $x_1, \ldots, x_k \in T$ and $k \in \mathbb{N}$.

Definition 2.14 *A random field $\{Y(x), x \in T\}$ with $T \in \{\mathbb{R}^d, \mathbb{Z}^d\}$ is (strictly)* `stationary` *if its finite dimensional distributions are invariant with respect to translations, i.e., it holds that $(Y(x_1 + h), \ldots, Y(x_k + h))^\top \stackrel{D}{=} (Y(x_1), \ldots, Y(x_k))^\top$ for all $h \in T$ and any $x_1, \ldots, x_k \in T$, $k \in \mathbb{N}$.*

If Y is stationary, then one has, in particular, $Y(x) \stackrel{D}{=} Y(o)$ for each $x \in T$. Accordingly, the random variable $Y(o)$ associated with the origin is henceforth used as typical variate of Y. As a consequence, the `mean function` $\mu(x) = \mathbb{E}\, Y(x), x \in T$ of a stationary random field Y is constant given it exists. Thus, we shall write

$$\mu = \mathbb{E}\, Y(x)$$

for any $x \in T$. Moreover, if $\mathbb{E}\, Y^2(o) < \infty$, then the covariance of the random variables $Y(x)$ and $Y(y)$ is well defined for all $x, y \in T$ and depends on the difference $y - x$ only, since $(Y(x), Y(y)) \stackrel{D}{=} (Y(o), Y(y - x))$. The `covariance function` of a stationary random field Y is represented by

$$\mathrm{Cov}_Y(x) = \mathrm{Cov}(Y(o), Y(x)), \qquad x \in T\,.$$

Regarding further definitions and properties related to random fields we refer, among others, to [1, 28, 30].

2.4.2 Conditionally bounded valuations

Let us consider a random closed Ξ with values in the extended convex ring $\mathcal{S}$ almost surely. This implies that, for almost all $\omega \in \Omega$ and $K \in \mathcal{C}'$, there exist convex bodies $K_1, \ldots, K_{n_{co}}$ such that $\Xi(\omega) \cap K = \bigcup_{\ell=1}^{n_{co}} K_\ell$ with $n_{co} = N_{co}(\Xi(\omega) \cap K)$ as defined in (2.2). Furthermore, the convex ring $\mathcal{R}$ is closed under intersections and finite unions. This gives rise to a special, but sufficiently general class of functionals, for which only the values on $\mathcal{K}$ need to be defined in order to uniquely determine the functional on whole $\mathcal{R}$.

Definition 2.15 *Let $E \subseteq \mathfrak{B}^d$ be an arbitrary family of sets such that $\emptyset \in E$ and $K_1 \cap K_2 \in E$ for $K_1, K_2 \in E$. A functional $\varphi : E \to \mathbb{R}$ is called* `additive` *or a* `valuation` *if $\varphi(\emptyset) = 0$ and*

$$\varphi(K_1 \cup K_2) \;=\; \varphi(K_1) + \varphi(K_2) - \varphi(K_1 \cap K_2)$$

for any $K_1, K_2 \in E$ with $K_1 \cup K_2 \in E$. Moreover, φ is said to be `fully additive` *if*

$$\varphi(K_1 \cup \ldots \cup K_m) \;=\; \sum_{k=1}^{m} (-1)^{k-1} \Big[\sum_{1 \le \ell_1 < \ldots < \ell_k \le m} \varphi(K_{\ell_1} \cap \ldots \cap K_{\ell_k}) \Big] \qquad (2.14)$$

for all $K_1, \ldots, K_m \in E$ and any $m \in N$ such that $K_1 \cup \ldots \cup K_m \in E$.

Formula (2.14) is knows as `inclusion-exclusion principle`. Applying the additivity property inductively, one finds that any valuation on $\mathcal{R}$ is fully additive. Besides this, the value of $\varphi(K), K \in \mathcal{R}$ does not depend on the chosen decomposition of K into convex sets; see [61, p. 174].

Lemma 2.7 *If φ is a valuation on $\mathcal{K}$ which is fully additive or continuous, then there exists a unique additive extension of φ on $\mathcal{R}$.*

Provided that the inclusion–exclusion principle is fulfilled, the stated assertion is a special case of Theorem 3.4.11 in [62]. For continuous valuations the proof is due to Groemer (1978) and can be found, for example, in [64, pp. 52–54]. □

In the context of random sets, a note on measurability is mandatory (see Appendix II of [64] on this issue).

Lemma 2.8 *A valuation on $\mathcal{R}$ is Borel–measurable if its restriction to $\mathcal{K}$ is Borel–measurable.*

Next to additivity, we will make use of another helpful property of functionals on $\mathcal{R}$.

Definition 2.16 *A functional $\varphi : \mathcal{R} \to \mathbb{R}$ is said to be* `conditionally bounded` *if $c(K) = \sup\{|\varphi(K')| : K' \in \mathcal{K}\,,\, K' \subseteq K\}$ is finite for every $K \in \mathcal{K}'$.*

Throughout the following, we consider some measurable, conditional bounded valuation $\varphi : \mathcal{R} \to \mathbb{R}$, a test set $K \in \mathcal{K}'$ and a stationary random $\mathcal{S}$–set Ξ. Herefrom, we derive the (real–valued) random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ via

$$Y(x) = \begin{cases} \varphi((\Xi - x) \cap K) & \text{if } (\Xi - x) \cap K \in \mathcal{R}, \\ 0 & \text{else,} \end{cases} \qquad (2.15)$$

where we omit the case $(\Xi(\omega) - x) \cap K \notin \mathcal{R}$ in the sequel. Due to stationarity of Ξ, the random field Y given in (2.15) is stationary by definition.

A simple example for a random field of the form (2.15) is the indicator random field $\{\mathbb{1}_{\Xi \oplus \check{K}}(x), x \in \mathbb{R}^d\}$, where $\mathbb{1}_{\Xi \oplus \check{K}}(x) = \mathbb{1}((\Xi - x) \cap K \neq \emptyset)$ and $|\mathbb{1}_{\Xi \oplus \check{K}}(x)| \le 1$ for each $x \in \mathbb{R}^d$. What is more, we have $\mathbb{E}\,\mathbb{1}_{\Xi \oplus \check{K}}(x) = T_\Xi(K)$ for any $x \in \mathbb{R}^d$.

A sufficient condition for finiteness of the p–th moments of random field Y can be formulated in terms of the generating function $g_{N_{co}(\Xi\cap K)}$ of the random variable $N_{co}(\Xi \cap K)$ as introduced in (2.2).

Lemma 2.9 *Let Ξ be a random closed set with $P(\Xi \in \mathcal{S}) = 1$. If Y is defined by (2.15), then it holds that*

$$\mathbb{E}\,|Y^p(x)| \;\le\; c^p(K) \cdot g_{N_{co}(\Xi\cap K)}(2^p) \tag{2.16}$$

for any $p > 0$ and $x \in \mathbb{R}^d$.

Proof For almost all $\omega \in \Omega$ there exist $n_{co} = N_{co}\big((\Xi(\omega) - x) \cap K\big)$ convex sets $K_1(\omega, x), \ldots, K_{n_{co}}(\omega, x) \in \mathcal{K}'$ such that

$$\big(\Xi(\omega) - x\big) \cap K = \bigcup_{\ell=1}^{n_{co}} K_\ell(\omega, x)\,.$$

Applying formula (2.14), we get

$$\begin{aligned} |Y(\omega,x)| &= \big|\varphi((\Xi(\omega) - x) \cap K)\big| \\ &= \Big|\sum_{k=1}^{n_{co}} (-1)^{k-1} \sum_{\ell_1<\ldots<\ell_k} f\big(K_{\ell_1}(\omega,x) \cap \ldots \cap K_{\ell_k}(\omega,x)\big)\Big| \\ &\le \sum_{k=1}^{n_{co}} \binom{n_{co}}{k} c(K) \;\le\; 2^{n_{co}} c(K) \end{aligned}$$

taking into account that $K_k(\omega, x) \subseteq K$ for every $x \in \mathbb{R}^d$, so that the upper bound does not depend on $x \in \mathbb{R}^d$. Raising both sides to the p–th power and taking the expectation yields the desired inequality

$$\begin{aligned} \mathbb{E}\,|Y^p(x)| &= \mathbb{E}\,\big|\varphi((\Xi(\omega) - x) \cap K)\big|^p \\ &\le c^p(K) \cdot \mathbb{E}\,2^{p\cdot N_{co}(\Xi\cap K)} \;=\; c^p(K) \cdot g_{N_{co}(\Xi\cap K)}(2^p)\,. \end{aligned}$$

□

For germ–grain models $\Xi = \bigcup_{l=1}^{\infty}(M_l + X_l)$ with convex grains $M_l, l \ge 1$, Lemma 2.9 can be reformulated employing the generating function of $N_\Xi(K)$, where $N_\Xi = \{M_l + X_l\}$ is the induced particle process on $\mathcal{K}'$. This makes the upper bound in (2.16) more tractable in many cases.

Lemma 2.10 *Let Ξ be a germ–grain model with $P(M_0 \in \mathcal{K}') = 1$, and let Y be given as in (2.15). Then it holds that*

$$\mathbb{E}\,|Y^p(x)| \;\le\; c^p(K) \cdot g_{N_\Xi(\mathcal{F}_K)}(2^p)$$

for any $x \in \mathbb{R}^d$ and $p > 0$. If, in particular, Ξ is a Boolean model, then we get

$$\mathbb{E}\,|Y^p(x)| \;\le\; c^p(K) \cdot e^{\lambda(2^p-1)\mathbb{E}|M_0\oplus\check{K}|}\,,$$

so that $\mathbb{E}\,|Y^p(x)|$ is finite for all $p > 0$.

Proof The first assertion is due to the fact that $N_{co}(\Xi(\omega) \cap K) \leq N_{\Xi(\omega)}(\mathcal{F}_K)$ for all $\omega \in \Omega$. Moreover, Theorem 2.5 implies the representation in case of a Poisson germ process. □

The posed setting can be easily extended to the multivariate case. For each $i = 1, \ldots, q$ with $q \geq 2$, consider a triple (Ξ_i, φ_i, K_i) of a random closed set Ξ_i, a functional φ_i on $\mathcal{R}$ and a test set $K_i \in \mathcal{K}'$, all satisfying the same properties as before. In addition, we assume that the random sets $\Xi_1, \ldots, \Xi_q$ are `jointly stationary`, that is to say, $(\Xi_1 + h, \ldots, \Xi_q + h) \stackrel{D}{=} (\Xi_1, \ldots, \Xi_q)$ holds for all $h \in \mathbb{R}^d$. Then the random fields $Y_1 = \{Y_1(x), x \in \mathbb{R}^d\}, \ldots, Y_q = \{Y_q(x), x \in \mathbb{R}^d\}$ with $Y_i(x) = \varphi_i((\Xi_i - x) \cap K_i)$ are jointly stationary as well, and the vector–valued random field $\mathbf{Y} = \{\mathbf{Y}(x), x \in \mathbb{R}^d\}$ with $\mathbf{Y}(x) = (Y_1(x), \ldots, Y_q(x))^\top$ is stationary. If $\mathbb{E}\, Y_i^2(o) < \infty$ for $i = 1, \ldots, q$, put $\mu_i = \mathbb{E}\, Y_i(o)$ and define

$$\boldsymbol{\mu} \;=\; (\mu_1, \ldots, \mu_q)^\top. \tag{2.17}$$

For each pair $i, j \in \{1, \ldots, q\}$, consider the `(cross-)covariance functions`

$$\mathrm{Cov}(Y_i(x), Y_j(y)) \;=\; \mathbb{E}(Y_i(x) Y_j(y)) - \mu_i \mu_j\,, \qquad x, y \in \mathbb{R}^d,$$

where $\mathrm{Cov}(Y_i(x), Y_j(y)) = \mathrm{Cov}(Y_i(o), Y_j(y - x))$ holds by stationarity. With regard to this, we introduce the notation

$$\mathrm{Cov}_{ij}(x) \;=\; \mathrm{Cov}(Y_i(o), Y_j(x))\,, \qquad x \in \mathbb{R}^d.$$

Since covariances are symmetric, one has $\mathrm{Cov}_{ii}(x) = \mathrm{Cov}_{ii}(-x)$ for each $x \in \mathbb{R}^d$. However, we observe a change of indices whenever $i \neq j$, i.e., $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}_{ji}(-x)$ for any $x \in \mathbb{R}^d$.

2.4.3 Intrinsic volumes

In this paragraph, we discuss important examples of conditionally bounded valuations on the convex ring, the so–called `intrinsic volumes` $V_0, \ldots, V_d$. At this, V_d is simply the functional mapping any $K \in \mathcal{R}$ onto its volume $|K|$. Moreover, there exists a unique valuation χ on $\mathcal{R}$ satisfying $\chi(K) = \mathbb{1}(K \neq \emptyset)$ for any $K \in \mathcal{K}$, see [61, p. 175]. This functional is known as `Euler characteristic`, and we put $V_0(K) = \chi(K)$, $K \in \mathcal{R}$. On $\mathbb{R}^2$, the Euler characteristic of a set $K \in \mathcal{R}$ is equal to the number of connected components minus the number of "holes", meaning the bounded connected components of the complement. The definition (and existence) of the other intrinsic volumes is given by `Steiner's formula` for the volume of parallel bodies. Before stating the corresponding theorem, we define κ_j to be the volume of the unit ball in $\mathbb{R}^j$, $j = 0, \ldots, d$, that is $\kappa_j = \pi^{j/2}/\Gamma(1 + j/2)$ with Gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$, $z \in (0, \infty)$.

Theorem 2.8 *Choose $K \in \mathcal{K}$ and $r \geq 0$. The volume $V_d(K \oplus B_r(o))$ of the parallel set $K \oplus B_r(o)$ can be represented as a polynomial in r of degree d with non–negative coefficients $V_0(K), \ldots, V_d(K)$ depending only on K, in other words,*

$$V_d(K \oplus B_r(o)) \;=\; \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K)\,. \tag{2.18}$$

A *proof* of Steiner's formula (2.18) is given in [64, p. 37]. □

The functional $V_j : \mathcal{K} \to [0, \infty)$ with $j \in \{0, \ldots, d\}$ appearing in formula (2.18) is called the j-`th intrinsic volume`. In particular, $V_0(K)$ and $V_d(K)$ are, as mentioned above, the Euler characteristic and the volume of K, respectively. Additionally, there exist nice geometrical interpretation for V_{d-1} and V_1. Since $\kappa_1 = 2$ and $\kappa_1 V_{d-1}(K) = \lim_{r \downarrow 0} \frac{1}{r}[V_d(K \oplus B_r(o)) - V_d(K)]$ for $K \in \mathcal{K}$, the quantity $V_{d-1}(K)$ can be thought of as one half of the surface area for a d–dimensional convex set K. Next, let $b(K)$ denote the mean width of $K \in \mathcal{K}$, that is the mean distance between two parallel, supporting hyperplanes of K, where the mean is understood with respect to the orientation of the hyperplanes (see also Appendix B and [62, p. 42]). Then one has $V_1(K) = d\kappa_d / 2\kappa_{d-1} \cdot b(K)$.

To stress the importance of the intrinsic volumes in convex geometry, we present `Hadwiger's characterization theorem` which states that the only continuous or monotone, motion invariant valuations on $\mathcal{K}$ are the linear combinations of $V_0, \ldots, V_d$.

Theorem 2.9 *Let $\varphi : \mathcal{K} \to \mathbb{R}$ be a motion invariant valuation on $\mathcal{K}$. If φ is continuous, then there exists constants $c_0, \ldots, c_d \in \mathbb{R}$ such that*

$$\varphi(K) = \sum_{j=0}^{d} c_j V_j(K), \qquad K \in \mathcal{K}. \tag{2.19}$$

If φ is monotone with respect to set inclusions, then representation formula (2.19) *holds with non–negative constants $c_0, \ldots, c_d \in \mathbb{R}_+$.*

For the complete, original *proof* see [13, pp. 221–225]. A shortened and simplified verification for the first statement of Hadwiger's theorem is presented in [32]. □

Up to now, the maps V_j are merely defined on the set of convex bodies $\mathcal{K}$, but they can be additively extended to $\mathcal{R}$.

Lemma 2.11 *For $j = 0, \ldots, d$, the j–th intrinsic volume $V_j : \mathcal{K} \to [0, \infty)$ is additive, continuous with respect to the Hausdorff metric, invariant under translations and rotations, increasing under set inclusions and homogeneous of degree j, i.e., $V_j(\alpha K) = \alpha^j V_j(K)$ for all $\alpha \geq 0, K \in \mathcal{K}$.*

The properties of the intrinsic volumes are mainly derived from Steiner's formula. For a detailed proof, we refer to [64, Section 2.3 and 2.4] and [62, Section 4.2]. □

As a consequence of Lemmas 2.7 and 2.11, there exists a unique additive extension of V_j on $\mathcal{R}$ for each $j = 0, \ldots, d$, which we denote by V_j as well. The extensions of V_d and V_{d-1} keep their geometrical interpretation and remain positive ([65, p. 300]). All functionals $V_0, \ldots, V_d : \mathcal{R} \to \mathbb{R}$ are invariant under rigid motions (i.e., invariant under translations and rotations; short: `motion invariant`). Continuity, however, is lost, and V_j may take negative values for $j \leq d - 2$. Nevertheless, each map $K \mapsto V_j(K)$ is Borel–measurable on $\mathcal{R}$, since it is continuous and by this Borel–measurable on $\mathcal{K}$; compare Lemma 2.8. Moreover, $V_j : \mathcal{R} \to \mathbb{R}$ is conditionally bounded with $c(K) = V_j(K)$ for any $K \in \mathcal{K}$ due to its monotonicity on $\mathcal{K}$.

Unfortunately, Lemma 2.7 guarantees merely the existence of such extensions, a tangible description of V_j on $\mathcal{R}$, however, is not provided. The volume V_d exhibits of course no problem. An explicit construction of V_0 on $\mathcal{R}$ can be done by induction on the dimension $d \in \mathbb{N}$ and is presented, for example, in [13, Section 6.3.3]. As to the remaining functionals $V_1, \dots, V_{d-1}$, one can use the Euler characteristic V_0 on $\mathcal{R}$ in order to characterize their extensions. In what follows, we explain the employed method and refer to [61, Section 2] concerning details like existence and properties of auxiliary functionals.

In a first step, one introduces the `index` of a set $K \in \mathcal{R}$ at point $u \in \mathbb{R}^d$ with respect to $x \in \mathbb{R}^d, x \neq u$, by the equation

$$J(K,u,x) = \begin{cases} 1 - \lim\limits_{\delta \downarrow 0} \lim\limits_{\varepsilon \downarrow 0} V_0\big(K \cap B_{|x-u|-\varepsilon}(x) \cap B_\delta(u)\big)\,, & u \in \partial K \\ 0, & u \notin \partial K, \end{cases} \tag{2.20}$$

see Figure 2.5 for an example of J on $\mathbb{R}^2$. The mapping J is additive in the first component. Let $K \neq \emptyset$ be convex, then $J(K,u,x) = 1$ if u is the metric projection of x onto K (i.e., $|x-u| \leq |x-y|$ for all $y \in K$) which is uniquely determined.

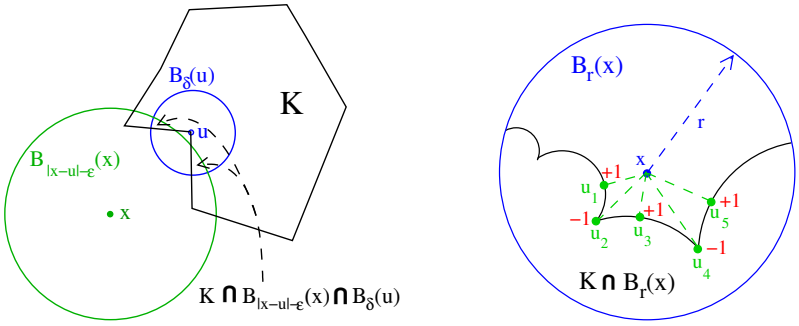


Figure 2.5: `Left:` Euler characteristic $V_0(K \cap B_{|x-u|-\varepsilon} \cap B_\delta(u)) = 2$ with $\varepsilon, \delta > 0$ and $J(K,x,u) = -1$. `Right:` Points of projection of x onto K with corresponding indices $J(K,x,u) \in \{-1,1\}$. Here one has $z_r(K,x) = 1$.

Next, put

$$z_r(K,x) = \sum_{u \in \partial K \cap B_r(x),\, u \neq x} J(K \cap B_r(x), u, x)\,, \qquad K \in \mathcal{R},\, x \in \mathbb{R}^d \tag{2.21}$$

for $r > 0$ and define the additive functional $\rho_r : \mathcal{R} \to \mathbb{R}$ by

$$\rho_r(K) = \int_{\mathbb{R}^d} z_r(K,x)\, dx\,, \qquad K \in \mathcal{R}.$$

It should be mentioned that only finitely many summands in the defining formula (2.21) of $z_r(K,x)$ are different from zero, since $K \cap B_r(x) = \bigcup_{\ell=1}^m K_\ell$ for some $K_\ell \in \mathcal{K}'$, $l = 1, \dots, m$ and $m \in \mathbb{N}$. By additivity of $J(\cdot, x, u)$ and uniqueness of the metric projections onto convex sets, at most 2^m summands are non–zero (see also

[61, pp. 220–221]). Finally, it can be shown that the following `Steiner's formula on` $\mathcal{R}$ holds

$$\rho_r(K) = \sum_{j=0}^{d-1} r^{d-j} \kappa_{d-j} V_j(K)\,, \qquad K \in \mathcal{R}. \tag{2.22}$$

By this, $V_1, \ldots, V_{d-1}$ are unambiguously characterized on the convex ring.

Notice that there exist also positive extensions $V_j^+ : \mathcal{R} \to [0, \infty)$ of the intrinsic volumes $V_0, \ldots, V_{d-1}$ on $\mathcal{R}$. A corresponding construction method was first presented in [39] and studied in more detail in [61]. These extensions, however, do not preserve additivity.

2.4.4 Densities of valuations

As we have seen in Lemma 2.9, the expected value $\mathbb{E}\,\varphi(\Xi \cap K)$ is well defined for any measurable, conditionally bounded valuation φ, random $\mathcal{S}$–set Ξ and convex body $K \in \mathcal{K}$, provided that $\mathbb{E}\,2^{N_{co}(\Xi \cap K)}$ is finite. Yet, in general, the quantity $\mathbb{E}\,\varphi(\Xi \cap K)$ will depend on the shape and location of K. In order to eliminate this dependence, we concentrate (for the moment) on translation invariant functionals and address the question, whether the limit

$$\overline{\varphi}(\Xi) \;=\; \lim_{n \to \infty} \frac{\mathbb{E}\,\varphi(\Xi \cap nK)}{V_d(nK)}$$

exists and is independent of K. In case of existence, $\overline{\varphi}(\Xi)$ is known as `φ-density` of Ξ. An answer to the posed question can be provided for translation invariant and conditionally bounded valuations on $\mathcal{R}$ as follows.

Theorem 2.10 *Let $\varphi : \mathcal{R} \to \mathbb{R}$ be a measurable, translation invariant and conditionally bounded valuation. Furthermore, let Ξ be a stationary random closed set taking values in $\mathcal{S}$ almost surely and satisfying*

$$\mathbb{E}\,2^{N_{co}(\Xi \cap B_r(o))} < \infty\,, \qquad r > 0. \tag{2.23}$$

Then the limit

$$\overline{\varphi}(\Xi) \;=\; \lim_{n \to \infty} \frac{\mathbb{E}\,\varphi(\Xi \cap nK)}{V_d(nK)}$$

exists for any $\mathcal{K} \in \mathcal{K}'$ with $V_d(K) > 0$ and is independent of K.

Proof Recall that the prerequisite that $\mathbb{E}\,2^{N_{co}(\Xi \cap B_r(o))} < \infty$ for some $r > 0$ guarantees finiteness of $\mathbb{E}\,2^{N_{co}(\Xi \cap K)}$ for any compact set K. By additivity and Lemma 2.9 the mapping $K \mapsto \varphi'(K) = \mathbb{E}\,\varphi(\Xi \cap K)$ is well defined for all $K \in \mathcal{R}$. At this, recall that due to compactness of $K \in \mathcal{K}'$, the condition $E\,2^{N_{co}(\Xi \cap B_r(o))} < \infty$ guarantees finiteness of $E\,2^{N_{co}(\Xi \cap B_r(o))} < \infty$. Moreover, $\varphi' : \mathcal{R} \to \mathbb{R}$ is a translation invariant and conditionally bounded valuation, taking into account that Ξ is stationary. Thus, we obtain the desired by Lemma 5.1.2 in [65]. □

For the remainder of this section, assume that Ξ satisfies condition (2.23). One finds that the intrinsic volumes $V_0, \ldots, V_d : \mathcal{R} \to \mathbb{R}$ fulfill the requirements of Theorem 2.10; see Lemma 2.11. Consequently, we can define the `intrinsic volume densities` (or `specific intrinsic volumes`) by

$$\overline{V}_j(\Xi) \;=\; \lim_{n\to\infty} \frac{\mathbb{E}\, V_j(\Xi \cap nK)}{V_d(nK)}\,, \qquad j = 0, \ldots, d, \tag{2.24}$$

where $K \in \mathcal{K}'$ with $V_d(K) > 0$. Closed form expressions for the intrinsic volume densities for Boolean models are given in Appendix B.

For $j = d$, the limit in (2.24) is redundant, since by stationarity of Ξ

$$\frac{\mathbb{E}\, V_d(\Xi \cap nK)}{V_d(nK)} \;=\; \frac{1}{V_d(nK)} \int_{nK} P(x \in \Xi)\, dx = \frac{V_d(nK)}{V_d(nK)} P(o \in \Xi)\,.$$

The final quantity is the volume fraction p (compare (2.3)) and is also known as `volume density`. In fact, $\mathbb{E}\, V_d(\Xi \cap K) = V_d(K)\overline{V}_d(\Xi)$ is valid for any stationary random closed set Ξ and all $K \in \mathfrak{B}_0^d$. Since V_{d-1} is non–negative on $\mathcal{R}$, the `surface density` $\overline{V}_{d-1}(\Xi)$ is well defined for Ξ even when neglecting condition (2.23). Furthermore, it holds that $\mathbb{E}\, V_{d-1}(\Xi \cap K) = V_{d-1}(K)\overline{V}_d(\Xi) + V_d(K)\overline{V}_{d-1}(\Xi)$. The latter equation is due to the `principal kinematic formula`, see [64, Chapter 3], which, in the present context, reads as

$$\mathbb{E}\, V_j(\Xi \cap K) \;=\; \sum_{k=j}^{d} \alpha_{djk} V_k(K) \overline{V}_{d+j-k}(\Xi) \tag{2.25}$$

for coefficients $\alpha_{djk} = k!\,\kappa_k\,(d+j-k)!\,\kappa_{d+j-k}/(j!\,\kappa_j\,d!\,\kappa_d)$ (compare [65, Section 5.3]).

Putting $j = 0$ and $K = B_{r_0}(o)$ for some $r_0 > 0$ in formula (2.25), we obtain

$$\mathbb{E}\, V_0(\Xi \cap B_{r_0}(o)) \;=\; \sum_{j=0}^{d} {r_0}^{d-j} \kappa_{d-j} \overline{V}_j(\Xi)\,. \tag{2.26}$$

Here, we used the facts that $\kappa_0 = 1$ and $V_j(B_{r_0}(o)) = d!\,\kappa_d\,{r_0}^{d-j}/(d-j)!$, which is easily deduced from Steiner's formula (2.18), considering $K = B_{r_0}(o)$ and taking the j-th derivative in $r = 0$. The quantity $\mathbb{E}\, V_0(\Xi \cap B_{r_0}(o))$ is refered to as `local Euler characteristic` of Ξ. With it, the intrinsic volume densities are uniqely characterized. What is more, $\mathbb{E}\, V_0(\Xi \cap B_{r_0}(o))$ is the mean of the stationary random field

$$\left\{ V_0(\Xi \cap B_{r_0}(x)), x \in \mathbb{R}^d \right\}, \qquad r_0 > 0,$$

that is of the form (2.15), since $V_0(\Xi \cap B_{r_0}(o)) = V_0((\Xi - x) \cap B_{r_0}(x))$ and $V_0(K) = \mathbb{1}(K \neq \emptyset)$ for any $K \in \mathcal{K}$. The principal kinematic formula represents an essential tool in integral geometry, but is rather complicated to obtain. We therefore point out that there exists an alternative proof of equation (2.26) based on Steiner's formula (2.22); confer Proposition 3.1 in [70].

Another random field describing the intrinsic volume densities $\overline{V}_0(\Xi), \ldots, \overline{V}_{d-1}(\Xi)$ by its mean is deduced from mapping z_r, which was introduced in (2.21) to construct

explicit extensions of the intrinsic volume $V_1, \ldots, V_{d-1}$ on $\mathcal{R}$. On $\mathcal{S}$, we analogously write

$$z_r(K,x) \;=\; \sum_{u \in \partial K \cap B_r(x),\, u \neq x} J(K \cap B_r(x), u, x)\,, \qquad K \in \mathcal{S}, x \in \mathbb{R}^d. \tag{2.27}$$

For any $r > 0$, define $\mathcal{Z}_r = \{\mathcal{Z}_r(x), x \in \mathbb{R}^d\}$ by

$$\mathcal{Z}_r(x) \;=\; z_r(\Xi, x) \;=\; \sum_{u \in \partial \Xi \cap B_r(x),\, u \neq x} J(\Xi \cap B_r(x), u, x)\,. \tag{2.28}$$

In reference to (2.15), put $z(K) = \sum_{u \in \partial K,\, u \neq o} J(K, u, o)$ for $K \in \mathcal{R}$. Thus we find $\mathcal{Z}_r(x) = z((\Xi - x) \cap B_r(o))$, since

$$\begin{aligned} \mathcal{Z}_r(x) \;&=\; \sum_{\tilde{u} \in \partial(\Xi - x) \cap B_r(o),\, \tilde{u} \neq o} J((\Xi - x) \cap B_r(o), \tilde{u}, o) \\ &=\; \sum_{\tilde{u} \in \partial((\Xi - x) \cap B_r(o)),\, \tilde{u} \neq o} J((\Xi - x) \cap B_r(o), \tilde{u}, o), \end{aligned}$$

noting that $\partial((\Xi - x) \cap B_r(o)) = (\partial(\Xi - x) \cap B_r(o)) \cup (\partial B_r(o) \cap \operatorname{int}(\Xi - x))$ and $J((\Xi - x) \cap B_r(o), \tilde{u}, o) = 0$ for $\tilde{u} \in \partial B_r(o) \cap \operatorname{int}(\Xi - x)$. The functional $z : \mathcal{R} \to \mathbb{R}$ is measurable, additive and conditionally bounded. The first two properties arise from corresponding properties of the Euler characteristic and finiteness of summation. Moreover, it holds that $z(K) = \mathbb{1}(o \notin K, K \neq \emptyset)$ for any $K \in \mathcal{K}$, so that z is bounded on $\mathcal{K}$. Yet, it should be mentioned that functional z is not translation invariant. Employing Steiner's formula (2.22) and the stationarity of the random field $\mathcal{Z}_r$, one finds a similar result as for the local Euler characteristic $\mathbb{E}\, V_0(\Xi \cap B_r(x))$, namely,

$$\mathbb{E}\, \mathcal{Z}_r(o) = \sum_{j=0}^{d-1} r^{d-j} \kappa_{d-j} \overline{V}_j(\Xi)\,, \qquad r > 0, \tag{2.29}$$

see Section 3.2 of [60] for the details. In addition, note that $\mathcal{Z}_r(x) = V_0(\Xi \cap B_r(x))$ for all $x \notin \Xi$ almost surely; see [62, p. 224].

The aim of the following chapters is to establish a general theory for the estimation of expected values like those on the left hand side of formulae (2.26) and (2.29), respectively. By means of this, unbiased and asymptotically normal estimators of the vector of intrinsic volume densities $\mathbf{v} = (\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))$ can be constructed.

Chapter 3

Mean value estimation for Boolean models

In this chapter, we consider a stationary Boolean model Ξ and a random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ associated with Ξ via $Y(x) = \varphi((\Xi - x) \cap K)$. Here, φ is a measurable, conditionally bounded valuation on $\mathcal{R}$ and $K \in \mathcal{K}'$ is a non–empty test set. In what follows, a class of unbiased estimators $\widehat{\mu}_n$, $n \in \mathbb{N}$ of the mean value $\mu = \mathbb{E}\, Y(o)$ is discussed. For any $n \geq 1$, the estimate $\widehat{\mu}_n$ is based on observations of Y within some compact Borel set W_n, where W_n expands in a "regular" way. Mean–square consistency of the sequence $\{\widehat{\mu}_n, n \in \mathbb{N}\}$ is proven under integrability conditions imposed on the covariance function Cov_Y of Y. It is shown, in particular, that $|\mathrm{Cov}_Y(x)|$, $x \in \mathbb{R}^d$ admits an integrable upper bound if $\mathbb{E}(|M_0 \oplus \check{K}|^2) < \infty$ holds for a typical grain M_0 of Ξ. To prove the asymptotic normality of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ as $W_n \uparrow$, a Lindeberg–type central limit theorem for m–dependent random fields on $\mathbb{Z}^d$ can be used. This results from the fact that the dependence structure of a Boolean model is determined by the independence property of the underlying Poisson point process $\{X_l, l \geq 1\}$ and the reach of the grains $\{M_l, l \geq 1\}$. A multivariate version of the central limit theorem rounds up the discussion.

We point out that the presented technique is applicable to prove asymptotic normality of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ for any measurable, conditionally bounded valuation φ. In literature, there exist several results in this direction for particular functionals φ. Confer, for instance, [3, 38] regarding estimation of the volume fraction $p = \overline{V}_d(\Xi)$. In [48, 50], estimators of the germ process intensity λ and of the surface density $\overline{V}_{d-1}(\Xi)$ are considered that admit asymptotic normality, where for $\overline{V}_{d-1}(\Xi)$ it is assumed that $M_0 \subset C_0$ almost surely for some $C_0 \in \mathcal{C}'$. Given that M_0 is rotation–invariant with parametrical distribution P_ϑ, $\vartheta \in \Theta$ for $\Theta \in \mathcal{C}'(\mathbb{R}^q)$, strongly consistent and asymptotically normally distributed estimators of the vector $(\lambda, \vartheta)^\top$ and thus of $(\lambda, \mathbb{E}\, V_0(M_0), \ldots, \mathbb{E}\, V_{d-1}(M_0))^\top$ are discussed in [20], amongst others. More general classes of random measures associated with Ξ can be found in [23]; see also Chapter 4. For further examples, confer [48, pp. 30–43] and references therein.

3.1 Preliminaries

As mentioned before, for any $n \in \mathbb{N}$, the estimator $\widehat{\mu}_n$ of $\mu = \mathbb{E}\,Y(o)$ is based on observations of Y within a compact sampling window $W_n \in \mathfrak{B}_0^d$. Whereas unbiased estimation can be performed on a single set W, asymptotic properties are understood with respect to a sequence of expanding observation windows $\{W_n, n \in \mathbb{N}\}$.

Definition 3.1 *A sequence $\{W_n, n \in \mathbb{N}\}$ of compact Borel sets is called a* `van Hove sequence` *if*

$$\lim_{n\to\infty} |W_n| \;=\; \infty \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(o)|}{|W_n|} \;=\; 0, \qquad r > 0. \tag{3.1}$$

A possible choice of such a sequence is given by $W_n = nW_0, n \geq 1$, for some convex body $W_0 \in \mathcal{K}'$ with $o \in \mathrm{int}(W_0)$ and $|W_0| > 0$. The first limit in (3.1) is easily obtained noting that $|W_n| = n^d |W_0|$. With regard to the second limit, we find that $\partial W_n \oplus B_r(o) \subset (W_n \oplus B_r(o)) \setminus (W_n \ominus B_{r+\varepsilon}(o))$ for any r, $\varepsilon > 0$. This leads to the estimate

$$\begin{aligned} 0 \;\leq\; \frac{|\partial W_n \oplus B_r(o)|}{|W_n|} \;&\leq\; \frac{|W_n \oplus B_r(o)|}{|W_n|} - \frac{|W_n \ominus B_{r+\varepsilon}(o)|}{|W_n|} \\ &=\; \frac{\left|W_0 \oplus B_{r/n}(o)\right|}{|W_0|} - \frac{\left|W_0 \ominus B_{(r+\varepsilon)/n}(o)\right|}{|W_0|}, \end{aligned}$$

Since $\lim_{n\to\infty}(W_0 \oplus B_{r/n}(o)) = \lim_{n\to\infty}(W_0 \ominus B_{(r+\varepsilon)/n}(o)) = W_0$ with respect to the Hausdorff metric ϱ_H and since the mapping $V_d : K \mapsto |K|$ is continuous on $(\mathcal{K}, \varrho_H)$ (see [64, Theorem 7.1.15]), one has

$$\lim_{n\to\infty} \frac{\left|W_0 \oplus B_{r/n}(o)\right|}{|W_0|} \;=\; \lim_{n\to\infty} \frac{\left|W_0 \ominus B_{(r+\varepsilon)/n}(o)\right|}{|W_0|} \;=\; 1\,.$$

This result can be formulated more generally.

Lemma 3.1 *Let $\{W_n, n \in \mathbb{N}\}$ be a sequence of compact Borel sets in $\mathbb{R}^d$ satisfying $\lim_{n\to\infty} |W_n| = \infty$. Then $\{W_n, n \in \mathbb{N}\}$ is a van Hove sequence if and only if*

$$\lim_{n\to\infty} \frac{|W_n \oplus B_r(o)|}{|W_n|} \;=\; \lim_{n\to\infty} \frac{|W_n \ominus B_r(o)|}{|W_n|} \;=\; 1\,, \qquad r > 0. \tag{3.2}$$

Furthermore, define $I_n^- = \{z \in \mathbb{Z}^d : [z, z+e) \subset W_n\}$ and $I_n^+ = \{z \in \mathbb{Z}^d : [z, z+e) \cap W_n \neq \emptyset\}$, where $[z, z+e) = [z_1, z_1+1) \times \ldots \times [z_d, z_d+1)$, $z \in \mathbb{Z}^d$. If $\{W_n, n \in \mathbb{N}\}$ is a van Hove sequence, then it holds that

$$\lim_{n\to\infty} \#I_n^- \;=\; \infty \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{\#I_n^-}{\#I_n^+} \;=\; 1\,. \tag{3.3}$$

Proof Firstly, assume that (3.1) is fulfilled. With $W_n \setminus (\partial W_n \oplus B_r(o)) \subset W_n \ominus B_r(o)$ it follows that

$$1 \;\geq\; \limsup_{n\to\infty} \frac{|W_n \ominus B_r(o)|}{|W_n|} \;\geq\; \liminf_{n\to\infty} \frac{|W_n \ominus B_r(o)|}{|W_n|} \geq 1 - \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(o)|}{|W_n|} \;=\; 1.$$

Besides this, we have $W_n \oplus B_r(o) = (W_n \ominus B_r(o)) \cup (\partial W_n \oplus B_r(o))$ so that

$$\begin{aligned} 1 \;&\leq\; \liminf_{n\to\infty} \frac{|W_n \oplus B_r(o)|}{|W_n|} \;\leq\; \limsup_{n\to\infty} \frac{|W_n \oplus B_r(o)|}{|W_n|} \\ &\leq\; \lim_{n\to\infty} \frac{|W_n \ominus B_r(o)|}{|W_n|} + \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(o)|}{|W_n|} \;=\; 1\,, \end{aligned}$$

which proves (3.2). The converse direction is readily given by the fact that $\partial W_n \oplus B_r(o) \subset (W_n \oplus B_r(o)) \setminus (W_n \ominus B_{r+\varepsilon}(o))$ for any $r,\, \varepsilon > 0$.

Suppose now that $\{W_n, n \in \mathbb{N}\}$ is a van Hove sequence. Put $V_n^- = \bigcup_{z \in I_n^-} [z, z+e)$ and $V_n^+ = \bigcup_{z \in I_n^+} [z, z+e)$ with $|V_n^-| = \#I_n^-$ and $|V_n^+| = \#I_n^+$. Then it holds that $W_n \ominus B_{\sqrt{d}}(o) = \{x \in \mathbb{R}^d : B_r(x) \subset W_n\} \subset V_n^-$, which implies

$$\#I_n^- \;=\; |V_n^-| \;\geq\; |W_n \ominus B_{\sqrt{d}}(o)| \;\to\; \infty\,, \qquad n \to \infty$$

due to (3.2). Observing that $V_n^+ \subset W_n \oplus B_{\sqrt{d}}(o) = \{x \in \mathbb{R}^d : B_r(x) \cap W_n \neq \emptyset\}$, we analogously get

$$1 \;\geq\; \frac{\#I_n^-}{\#I_n^+} \;\geq\; \frac{|W_n \ominus B_{\sqrt{d}}(o)|}{|W_n \oplus B_{\sqrt{d}}(o)|} \;\to\; 1\,, \qquad n \to \infty. \qquad \square$$

Considering more general rectangular parallelepipeds like I_n^- and I_n^+, it is possible to formulate an "extended version" of property (3.3) that is equivalent to (3.1); confer [28, p. 2]. Even if not explicitly mentioned, we will always assume in this manuscript that the sequence $\{W_n, n \in \mathbb{N}\}$ of observation windows is a van Hove sequence.

Let $\mathcal{C}^* = \{C \in \mathcal{C} : |C| > 0\}$ and pick $W \in \mathcal{C}^*$. To each observation $Y(x)$, $x \in W$ within W a weight $G(W,x)$ is assigned. More precisely, let $G : \mathcal{C}^* \times \mathbb{R}^d \to [0,\infty)$ be some nonnegative function, which is Borel–measurable in the second component and fulfills

$$G(W,x) \;=\; 0, \quad x \in \mathbb{R}^d \setminus (W \ominus \check{K}), \qquad \int_W G(W,x)\,dx \;=\; 1\,. \tag{3.4}$$

Recall that $Y(x) = \varphi((\Xi - x) \cap K)$. Therefore, the first condition in (3.4) ensures that $Y(x)$ is taken into account only if $\Xi \cap (K+x)$ is entirely contained in W. An elementary example of a weighting function G satisfying (3.4) on a given van Hove sequence $\{W_n, n \geq 1\}$ is

$$G(W_n,x) \;=\; \frac{\mathbb{1}(x \in W_n \ominus \check{K})}{|W_n \ominus \check{K}|}, \qquad x \in \mathbb{R}^d,\, n \in \mathbb{N}.$$

The denominator is non–zero, without loss of generality, due to (3.2) and since $\check{K} \subseteq B_r(o)$ for some $r > 0$ by compactness of K.

With regard to the asymptotic behavior of the estimator $\widehat{\mu}_n$ (defined below), further conditions have to be imposed on the weighting function $G : \mathcal{C}^* \times \mathbb{R}^d \to [0, \infty)$. In addition to (3.4), assume that there exist constants c_0, $\theta \in (0, \infty)$ such that

$$\sup_{y \in W_n} G(W_n, y) \;\leq\; \frac{c_0}{|W_n|} \tag{3.5}$$

for any $n \geq 1$ and

$$\lim_{n \to \infty} |W_n|\, \Gamma_n(x) \;=\; \theta \tag{3.6}$$

for each $x \in \mathbb{R}^d$, where

$$\Gamma_n(x) \;=\; \int_{\mathbb{R}^d} G(W_n, y)\, G(W_n, x+y)\, dy\,, \qquad x \in \mathbb{R}^d.$$

Notice that both conditions hold for the uniform weights considered before, where $c_0 = 2$ and $\theta = 1$. This follows from the fact that, for any fixed $x \in \mathbb{R}^d$, we have

$$\Gamma_n(x) \;=\; \int_{\mathbb{R}^d} \frac{\mathbb{1}(y \in W_n^K)}{|W_n^K|}\, \frac{\mathbb{1}((x+y) \in W_n^K)}{|W_n^K|}\; dy \;=\; \frac{|W_n^K \cap (W_n^K - x)|}{|W_n^K|^2}$$

with $W_n^K = W_n \ominus \check{K}$ and $|W_n^K \ominus B_{|x|}(o)|/|W_n^K| \leq |W_n^K \cap (W_n^K - x)|/|W_n^K| \leq 1$. The left hand side of the latter inequality tends to 1 for all $x \in \mathbb{R}^d$ according to (3.2).

3.2 Unbiased and consistent estimation of the mean value

By virtue of Lemma 2.10, the stationary random field Y has finite moments of any order, in particular, we have

$$\mathbb{E}|Y^p(x)| \;\leq\; c^p(K)\, e^{\lambda(2^p-1)\mathbb{E}|M_0 \oplus \check{K}|} \;<\; \infty, \qquad p > 0.$$

Then $\mu = \mathbb{E}\, Y(o)$ is well defined, and we introduce the estimator $\widehat{\mu}_n$ of μ by

$$\widehat{\mu}_n \;=\; \int_{W_n} Y(x)\, G(W_n, x)\, dx\,, \tag{3.7}$$

where $W_n \subset \mathbb{R}^d$ is a compact Borel set for each $n \geq 1$ and $G : \mathcal{C}^* \times \mathbb{R}^d \to [0, \infty)$ is measurable in the second component (compare, e.g., Section 1.7 of [28]).

Lemma 3.2 *If weighting function G satisfies condition* (3.4)*, then $\widehat{\mu}_n$ is an unbiased estimator of $\mu = \mathbb{E}\, Y(o)$ for any $n \geq 1$, and its variance is given by*

$$\operatorname{Var} \widehat{\mu}_n \;=\; \int_{\mathbb{R}^d} \operatorname{Cov}_Y(x)\, \Gamma_n(x)\, dx\,.$$

Proof Unbiasedness follows directly from Fubini's theorem and condition (3.4), whereby we obtain

$$\mathbb{E}\, \widehat{\mu}_n \;=\; \int\limits_{W_n} \mathbb{E}\big(Y(x)\big)\, G(W_n, x)\, dx = \mu \int\limits_{W_n} G(W_n, x)\, dx \;= \mu\,.$$

Furthermore, we have

$$\begin{aligned}
\mathrm{Var}\,\widehat{\mu}_n &= \mathbb{E}\Big(\int_{W_n}(Y(u)-\mu)\,G(W_n,u)\,du\int_{W_n}(Y(v)-\mu)\,G(W_n,v)\,dv\Big)\\
&= \int_{W_n}\int_{W_n}\mathbb{E}\left((Y(o)-\mu)(Y(v-u)-\mu)\right)\,G(W_n,u)G(W_n,v)\;dv\,du\\
&= \int_{W_n\oplus\check{W}_n}\mathrm{Cov}_Y(x)\int_{W_n\cap(W_n-x)}G(W_n,y)G(W_n,x+y)\,dy\;\,dx\\
&= \int_{\mathbb{R}^d}\mathrm{Cov}_Y(x)\,\Gamma_n(x)\,dx\,,
\end{aligned}$$

where the final equality is due to the fact that $W_n\cap(W_n-x)=\emptyset$ for any $x\notin W_n\oplus\check{W}_n$. □

By the subsequent lemma, we see that under some additional condition it holds that $\lim_{n\to\infty}\mathrm{Var}\,\widehat{\mu}_n=0$, which is the same as to say that the unbiased estimator $\widehat{\mu}_n$ is consistent for μ in mean–square sense.

Lemma 3.3 *Let $\{W_n, n\geq 1\}$ be a van Hove sequence, and suppose that $G:\mathcal{C}^*\times\mathbb{R}^d\to[0,\infty)$ fulfills conditions* (3.4)–(3.6). *Furthermore, assume that*

$$\int_{\mathbb{R}^d}|\mathrm{Cov}_Y(x)|\,dx\;<\;\infty \tag{3.8}$$

holds. Then the estimator $\widehat{\mu}_n$ defined in (3.7) *is mean–square consistent for μ, i.e., $\lim_{n\to\infty}\mathrm{Var}\,\widehat{\mu}_n=0$, with*

$$\lim_{n\to\infty}|W_n|\mathrm{Var}\;\widehat{\mu}_n\;=\;\theta\int_{\mathbb{R}^d}\mathrm{Cov}_Y(x)\,dx\,.$$

Proof Observe that

$$\Gamma_n(x)\;=\int\limits_{W_n\cap(W_n-x)}G(W_n,y)G(W_n,x+y)\,dy\;\leq\;c_0^2\,\frac{|W_n\cap(W_n-x)|}{|W_n|^2}$$

for $c_0<\infty$ due to condition (3.5). From this we deduce that

$$\big|\mathrm{Cov}_Y(x)|W_n|\Gamma_n(x)\big|\;\leq\;c_0^2\,|\mathrm{Cov}_Y(x)|\,,$$

where the right hand side is integrable by condition (3.8). Hence, the claim follows from Lebesgue's dominated convergence theorem, Lemma 3.2 and the pre–requisite that $\lim_{n\to\infty}|W_n|\Gamma_n(x)=\theta<\infty$. □

Notice that by the above results we get an upper bound for the mean–square error of $\widehat{\mu}_n$, that is

$$\mathbb{E}\,(\widehat{\mu}_n-\mu)^2\;\leq\;\frac{c_0^2}{|W_n|}\int_{\mathbb{R}^d}|\mathrm{Cov}_Y(x)|\,dx\,,\qquad n\geq 1.$$

3.3 Integrability of the covariance function

Integrability of the covariance function of Y is an essential assumption not only for consistency of $\widehat{\mu}_n$, but also for further results obtained later on. A special case arises if the grains are uniformly bounded, i.e., $P(M_0 \subseteq B_{r_0}(o)) = 1$ for some $r_0 > 0$. If so, then $Y(x)$ and $Y(y)$ are independent whenever $|x-y| > 2r_0 + ||K||$, and we have $\int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)|\,dx \le (2r_0 + ||K||)^d \kappa_d \,\mathrm{Var}\,(Y(o)) < \infty$. Another example is given by the random field $Y(x) = \mathbb{1}_{\Xi \oplus \check{K}}(x), x \in \mathbb{R}^d$. In this case, it holds that $\int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)|\,dx \le (1 - T_\Xi(K))\,\lambda\,\mathbb{E}(|M_0 \oplus \check{K}|^2)$, see [48, p. 62]. The following lemma reveals that the finiteness of $\mathbb{E}(|M_0 \oplus \check{K}|^2)$ is a sufficient condition for the integrability of $\mathrm{Cov}_Y(x)$, $x \in \mathbb{R}^d$ also in the general case.

Lemma 3.4 *Assume that* $\mathbb{E}\left(|M_0 \oplus \check{K}|^2\right) < \infty$. *Then* $\int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)|\,dx < \infty$ *holds.*

Proof To improve readability, we use the representation of Ξ as the set–theoretic union of the generating Poisson particle process $N_\Xi = \{Z_l\}$ with $Z_l = M_l + X_l$. For simplicity, write $N = N_\Xi$, and let Λ denote the intensity measure of N.

We first introduce some notation, which will be helpful in upcoming calculations. To start with, recall that the value of $Y(x) = \varphi((\Xi - x) \cap K)$ does depend only on grains Z_l that intersect $K + x$. With regard to this, define $\mathcal{K}_{K+x} = \mathcal{F}_{K+x} \cap \mathcal{K} = \{K' \in \mathcal{K} : K' \cap (K+x) \neq \emptyset\}$, $x \in \mathbb{R}^d$, where we put $\mathcal{K}_K = \mathcal{K}_{K+o}$. Note that $\mathcal{K}_{K+x} \in \mathfrak{B}(\mathcal{K})$ for any x, and $\Lambda(\mathcal{K}_{K+x}) = \lambda\,\mathbb{E}|M_0 \oplus \check{K}| < \infty$, since by Campbell's formula (2.6)

$$\begin{aligned}\Lambda(\mathcal{K}_{K+x}) &= \lambda\,\mathbb{E}\Big(\int_{\mathbb{R}^d} \mathbb{1}\big((M_0+y) \cap (K+x) \neq \emptyset\big)\,dy\Big) \\ &= \lambda\,\mathbb{E}\Big(\int_{\mathbb{R}^d} \mathbb{1}\big((x-y) \in M_0 \oplus \check{K}\big)\,dy\Big) \;=\; \lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|\,.\end{aligned}$$

In addition, fix $x \in \mathbb{R}^d$ and let $\mathcal{K}_x^* = \mathcal{K}_K \cap \mathcal{K}_{K+x}$. Then it holds that $\mathcal{K}_K \backslash \mathcal{K}_x^*$, $\mathcal{K}_{K+x} \backslash \mathcal{K}_x^*$ and $\mathcal{K}_x^*$ are pairwise disjoint Borel sets on $\mathcal{K}$ with finite measure $\Lambda(\,\cdot\,)$. By means of Theorem 2.5(a), the restricted processes $N_{|\mathcal{K}_K \backslash \mathcal{K}_x^*}$, $N_{|\mathcal{K}_{K+x} \backslash \mathcal{K}_x^*}$ and $N_{|\mathcal{K}_x^*}$ are mutually independent. Finally, put $Y_{|B}(x) = \varphi(\bigcup_{l:\, Z_l \in B}(Z_l - x) \cap K)$ for any $B \in \mathfrak{B}(\mathcal{K})$.

Now, we consider the event $A = \{N(\mathcal{K}_x^*) > 0\}$ together with its complement A^c and write

$$\begin{aligned}\mathrm{Cov}_Y(x) &= \mathbb{E}\big((Y(o) - \mu)\,(Y(x) - \mu)\big) \;=\; \mathbb{E}\big(Y(o)\,(Y(x) - \mu)\big) \\ &= \mathbb{E}\big(Y(o)\,(Y(x) - \mu)\,\mathbb{1}_A\big) + \mathbb{E}\big(Y(o)\,(Y(x) - \mu)\mathbb{1}_{A^c}\big). \qquad (3.9)\end{aligned}$$

With regard to the first summand in (3.9), observe that

$$\begin{aligned}&|\mathbb{E}\big(Y(o)\,(Y(x) - \mu)\,\mathbb{1}_A\big)| \\ &\quad\le\; |\mathbb{E}\big(Y(o)\,Y(x)\,\mathbb{1}_A\big)| + |\mu|\,|\mathbb{E}\big(Y(o)\,\mathbb{1}_A\big)| \\ &\quad\le\; c^2(K)\,\mathbb{E}\big(2^{N(\mathcal{K}_K) + N(\mathcal{K}_{K+x})}\,\mathbb{1}_A\big) + c(K)\,|\mu|\,\mathbb{E}\big(2^{N(\mathcal{K}_K)}\,\mathbb{1}_A\big)\,.\end{aligned}$$

The second inequality is obtained in similar fashion to the proof of Lemma 2.9, i.e.,

$$\begin{aligned} |Y(o,\omega)\,Y(x,\omega)| &\leq c^2(K)\,2^{N_{co}(\Xi(\omega)\cap K)+N_{co}(\Xi(\omega)\cap(K+x))} \\ &\leq c^2(K)\,2^{N(\omega)(\mathcal{K}_K)+N(\omega)(\mathcal{K}_{K+x})} \end{aligned} \tag{3.10}$$

for almost all $\omega \in \Omega$. Due to the independence of $N(\mathcal{K}_K\backslash\mathcal{K}_x^*)$, $N(\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*)$ and $N(\mathcal{K}_x^*)$, we find

$$\begin{aligned} &|\mathbb{E}\big(Y(o)\,(Y(x)-\mu)\,\mathbb{1}_A\big)| \\ &\leq c^2(K)\,\mathbb{E}\Big(2^{N(\mathcal{K}_K\backslash\mathcal{K}_x^*)}\Big)\,\mathbb{E}\Big(2^{N(\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*)}\Big)\,\mathbb{E}\Big(2^{2N(\mathcal{K}_x^*)}\,\mathbb{1}_A\Big) \\ &\quad + c(K)\,|\mu|\,\mathbb{E}\Big(2^{N(\mathcal{K}_K\backslash\mathcal{K}_x^*)}\Big)\,\mathbb{E}\Big(2^{N(\mathcal{K}_x^*)}\,\mathbb{1}_A\Big)\,, \end{aligned} \tag{3.11}$$

By Theorem 2.5 (c), one has $\mathbb{E}(2^{N(\mathcal{K}_{K+y}\backslash\mathcal{K}_x^*)}) \leq e^{\lambda\,\mathbb{E}|M_0\oplus\check{K}|} < \infty$ for $y \in \{o,x\}$, taking into account that $\Lambda(B') \leq \Lambda(B)$ for all pairs $B, B' \in \mathfrak{B}(\mathcal{K})$ with $B' \subseteq B$.

As to the second summand in (3.9), employ the identities

$$\begin{aligned} Y(o)\,\mathbb{1}_{A^c} &= \varphi\Big(\Big(\bigcup_{l:\,Z_l\in\mathcal{K}_K\backslash\mathcal{K}_x^*} Z_l \;\cup \bigcup_{l:\,Z_l\in\mathcal{K}_x^*} Z_l\Big)\cap K\Big)\,\mathbb{1}\big(N(\mathcal{K}_x^*)=0\big) \\ &= \varphi\Big(\bigcup_{l:\,Z_l\in\mathcal{K}_K\backslash\mathcal{K}_x^*} Z_l \;\cap K\Big)\,\mathbb{1}\big(N(\mathcal{K}_x^*)=0\big) \;=\; Y_{|\mathcal{K}_K\backslash\mathcal{K}_x^*}(o)\,\mathbb{1}_{A^c} \end{aligned}$$

and analogously

$$Y(o)\,Y(x)\,\mathbb{1}_{A^c} \;=\; Y_{|\mathcal{K}_K\backslash\mathcal{K}_x^*}(o)\,Y_{|\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*}(x)\,\mathbb{1}_{A^c}\,.$$

As indicated above, the random variables $Y_{|\mathcal{K}_K\backslash\mathcal{K}_x^*}(o)$, $Y_{|\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*}(x)$ and $\mathbb{1}_{A^c}$ are mutually independent. Combining these results with the representation $\mu = \mathbb{E}(Y(x)\,\mathbb{1}_A) + \mathbb{E}(Y_{|\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*}(x)\,\mathbb{1}_{A^c})$, it follows that

$$\begin{aligned} &\mathbb{E}\big(Y(o)\,(Y(x)-\mu)\,\mathbb{1}_{A^c}\big) \\ &= \mathbb{E}\big(Y_{|\mathcal{K}_K\backslash\mathcal{K}_x^*}(o)\,\mathbb{1}_{A^c}\big)\,\big(\mathbb{E}\big(Y_{|\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*}(x)\big) - \mathbb{E}\,(Y(x)\,\mathbb{1}_A) - \mathbb{E}\,(Y_{|\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*}(x)\,\mathbb{1}_{A^c})\big) \\ &= \mathbb{E}\,(Y(o)\,\mathbb{1}_{A^c})\,\mathbb{E}\big(Y_{|\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*}(x)\,\mathbb{1}_A - Y(x)\,\mathbb{1}_A\big) \end{aligned}$$

By similar arguments as used for (3.11), we deduce

$$\begin{aligned} |\mathbb{E}\big(Y(o)\,(Y(x)-\mu)\,\mathbb{1}_{A^c}\big)| &\leq \mathbb{E}|Y(o)|\,\mathbb{E}\big(\big(2^{N(\mathcal{K}_{K+x}\backslash\mathcal{K}_x^*)}+2^{N(\mathcal{K}_{K+x})}\big)\,\mathbb{1}_A\big) \\ &\leq 2\,c(K)\,\mathbb{E}|Y(o)|\,e^{\lambda\,\mathbb{E}|M_0\oplus\check{K}|}\,\mathbb{E}\big((1+2^{N(\mathcal{K}_x^*)})\mathbb{1}_A\big). \end{aligned} \tag{3.12}$$

Finally, joining both estimates (3.11) and (3.12), one has

$$|\mathrm{Cov}_Y(x)| \;\leq\; \tau\big(\lambda\mathbb{E}(|M_0\oplus\check{K}|), c(K)\big)\,\mathbb{E}\big(4^{N(\mathcal{K}_x^*)}\mathbb{1}_A\big)$$

for some finite constant $\tau(\lambda\mathbb{E}(|M_0\oplus\check{K}|), c(K))$ – keeping in mind that $\mathbb{E}|Y(o)| \leq c(K)e^{\lambda\mathbb{E}(|M_o\oplus\check{K}|)}$. Thus, it suffices to show that $\mathbb{E}(4^{N(\mathcal{K}_x^*)}\,\mathbb{1}_A)$ is integrable with respect to $x \in \mathbb{R}^d$ in order to prove the claimed integrability of $\mathrm{Cov}_Y(x)$, $x \in \mathbb{R}^d$.

Since N is a Poisson process, we have

$$\begin{aligned}\mathbb{E}(4^{N(\mathcal{K}_x^*)}\,\mathbb{1}_A) &= \mathbb{E}\,4^{N(\mathcal{K}_x^*)} - \mathbb{E}\,\mathbb{1}\big(N(\mathcal{K}_x^*)=0\big)\\ &= e^{3\Lambda(\mathcal{K}_x^*)}\big(1-e^{-4\Lambda(\mathcal{K}_x^*)}\big) \;\le\; 4\,e^{3\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\Lambda(\mathcal{K}_x^*)\,,\end{aligned}$$

where we used the inequality $1-e^{-s}\le s$, $s\ge 0$ for the final bound. By Campbell's formula (2.9) and Fubini's theorem, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^d}\Lambda(\mathcal{K}_x^*)\,dx\\ &= \int_{\mathbb{R}^d}\Big[\mathbb{E}\sum_{l=1}^{\infty}\mathbb{1}\Big((M_l+X_l)\cap K\neq\emptyset,(M_l+X_l)\cap(K+x)\neq\emptyset\Big)\Big]\,dx\\ &= \int_{\mathbb{R}^d}\Big[\lambda\,\mathbb{E}\int_{\mathbb{R}^d}\mathbb{1}\Big((M_0+y)\cap K\neq\emptyset,(M_0+y)\cap(K+x)\neq\emptyset\Big)\,dy\Big]\,dx\\ &= \lambda\,\mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{1}\big(y\in\check{M}_0\oplus K\big)\,\mathbb{1}\big((y-x)\in\check{M}_0\oplus K\big)\,dy\,dx\\ &= \lambda\,\mathbb{E}\int_{\mathbb{R}^d}\mathbb{1}\big(y\in M_0\oplus\check{K}\big)\Big[\int_{\mathbb{R}^d}\mathbb{1}\big(x\in\big((M_0\oplus\check{K})-y\big)\big)\,dx\Big]\,dy\\ &= \lambda\,\mathbb{E}\big(|M_0\oplus\check{K}|^2\big)\,,\end{aligned}$$

which is finite by assumption. □

We point out that by the proof of Lemma 3.4 an integrable upper bound for $|\mathrm{Cov}_Y|$ is provided, for instance, by $h:\mathbb{R}^d\to\mathbb{R}_+$ with

$$h(x) \;=\; 24\,\lambda\,c^2(K)\,e^{5\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\mathbb{E}\big|(M_0\oplus\check{K})\cap((M_0\oplus\check{K})-x)\big|\,, \tag{3.13}$$

where $\int_{\mathbb{R}^d}h(x)\,dx \;=\; 24\,\lambda\,c^2(K)\,e^{5\lambda\mathbb{E}|M_0\oplus\check{K}|}\mathbb{E}(|M_0\oplus\check{K}|^2)$. Function h depends on φ, K and Ξ through $c(K)<\infty$ and first and second moments of (translations of) $(M_0\oplus\check{K})$. Since the dependence of h on M_0 is monotone with respect to set inclusion, we sometimes write $h(x)=h(x,M_0)$. Namely, for any M_0, $M_0'\in\mathcal{K}$ with $M_0'\subseteq M_0$ almost surely, it holds that $h(x,M_0')\le h(x,M_0)$ for all $x\in\mathbb{R}^d$.

3.4 A central limit theorem for the estimator of the mean value

The aim of this section is to provide conditions for asymptotic normality of the sequence $\sqrt{|W_n|}(\widehat{\mu}_n-\mu)$ as $n\to\infty$. An essential tool is a Lindeberg–type central limit theorem for m–dependent random fields with unbounded m, as given in Section 3 of [17]. To this end, the germ–grain model Ξ with grains $\{M_l\}$ is approximated by a sequence of germ–grain models Ξ_n with bounded grains $\{M_{ln}\}$, where we put $M_{ln}=M_l\cap A_n$ for a compact set $A_n\in\mathfrak{B}^d$ satisfying $A_n\uparrow\mathbb{R}^d$ as $n\to\infty$. On the basis of the truncated Boolean models Ξ_n, asymptotic normality of a corresponding version

of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ is shown. Thereafter, the convergence can be easily transferred to the original model. In what follows, we always assume that conditions (3.4) – (3.6) are fulfilled on the van Hove sequence $\{W_n, n \in \mathbb{N}\}$ and that $\mathbb{E}\big(|M_0 \oplus \check{K}|^2\big) < \infty$ holds. We close this section with a multivariate analog of the univariate central limit theorem.

3.4.1 Truncated Boolean models

The Boolean model Ξ and the associated random field Y shall be approximated by germ–grain models Ξ_n with truncated grains and related random fields Y_n, respectively. To be more precise, let $\{a_n, n \in \mathbb{N}\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = \infty$. Furthermore, put $A_n = [-a_n, a_n]^d$ for all $n \geq 1$. Next, introduce the auxiliary germ–grain model Ξ_n, $n \geq 1$ by

$$\Xi_n \;=\; \bigcup_{l=1}^{\infty} (M_{ln} + X_l)\,, \tag{3.14}$$

where $M_{ln} = M_l \cap A_n \in \mathcal{K}$ for any i,$n \in \mathbb{N}$. Each random set Ξ_n represents a stationary Boolean model, since the driving point process $\{X_l\}$ remains unchanged as do the assumptions on identical distribution and independence of the grains. Finally, define the associated random field $Y_n = \{Y_n(x),\, x \in \mathbb{R}^d\}$ by

$$Y_n(x) \;=\; \varphi\left((\Xi_n - x) \cap K\right), \qquad x \in \mathbb{R}^d.$$

This construction provides an approximation of Y by $\{Y_n, n \in \mathbb{N}\}$ in L^2–sense.

Lemma 3.5 *The random fields Y_n are of second order for any $n \geq 1$. Furthermore, the sequence $\{Y_n(x), n \in \mathbb{N}\}$ converges to $Y(x)$ in mean–square sense as $n \to \infty$, that is, $\lim_{n\to\infty} \mathbb{E}\,(Y(x) - Y_n(x))^2 = 0$ for all $x \in \mathbb{R}^d$.*

Proof Due to stationarity, we may assume that $x = o$ without loss of generality. Because Ξ_n is a Boolean model with typical grain M_{0n} for any $n \in \mathbb{N}$, we conclude by Lemma 2.10 that

$$\mathbb{E}\big(Y_n^2(o)\big) \;\leq\; c^2(K)\, e^{3\lambda\, \mathbb{E}|M_{0n} \oplus \check{K}|} \;\leq\; c^2(K)\, e^{3\lambda\, \mathbb{E}|M_0 \oplus \check{K}|},$$

where $M_{0n} = M_0 \cap A_n \subseteq M_0$ for each realization so that $\mathbb{E}|M_{0n} \oplus \check{K}| \leq \mathbb{E}|M_0 \oplus \check{K}|$ for all $n \geq 1$, and the uniform upper bound $c^2(K)\, e^{3\lambda\, \mathbb{E}|M_0 \oplus \check{K}|}$ is finite by assumption.

To validate the second assertion, define

$$N_{|A_n}(\Xi \cap K) \;=\; \#\{[X_l, M_l] \in \Psi_\Xi : (M_l + X_l) \cap K \neq \emptyset, K \not\subseteq (A_n + X_l)\}\,,$$

where Ψ_Ξ is the independently marked Poisson process associated with Ξ, see Corollary 2.3. Using this notation, we get

$$\begin{aligned} |Y(o,\omega) - Y_n(o,\omega)|^2 \;&=\; |Y(o,\omega) - Y_n(o,\omega)|^2\, \mathbb{1}\big(N_{|A_n}(\Xi(\omega) \cap K) > 0\big) \\ &\leq\; 2\left(|Y(o,\omega)|^2 + |Y_n(o,\omega)|^2\right) \mathbb{1}\big(N_{|A_n}(\Xi(\omega) \cap K) > 0\big) \end{aligned}$$

for any $\omega \in \Omega$. By similar arguments as employed, for instance, for (3.10), the latter inequality yields

$$\mathbb{E}\left(|Y(o) - Y_n(o)|^2\right) \;\leq\; 4\,c^2(K)\,\mathbb{E}\Big(4^{N(\Xi\cap K)}\,\mathbb{1}\big(N_{|A_n}(\Xi\cap K) > 0\big)\Big).$$

Since $\mathbb{E}\big(4^{N(\Xi\cap K)}\big) = e^{3\lambda\,\mathbb{E}|M_0\oplus\check{K}|} < \infty$, it suffices to show that

$$\lim_{n\to\infty} P(N_{|A_n}(\Xi\cap K) > 0) \;=\; 0\,.$$

Apply Campbell's formula as given in (2.9) to obtain

$$\begin{aligned} P(N_{|A_n}(\Xi\cap K) > 0) \;&\leq\; \mathbb{E}\big(N_{|A_n}(\Xi\cap K)\big) \\ &=\; \mathbb{E}\sum_{[X_l,M_l]\in\Psi_\Xi} \mathbb{1}\big((M_l+X_l)\cap K\neq\emptyset, K\not\subseteq (A_n+X_l)\big) \\ &=\; \lambda\,\mathbb{E}\int_{\mathbb{R}^d} \mathbb{1}\big((M_0+y)\cap K\neq\emptyset, K\not\subseteq (A_n+y)\big)\,dy \\ &=\; \lambda\,\mathbb{E}\int_{\mathbb{R}^d} \mathbb{1}\big(y\in(M_0\oplus\check{K})\big)\,\mathbb{1}\big(y\notin(A_n\ominus\check{K})\big)\,dy\,. \end{aligned}$$

Notice that $\big(A_n\ominus\check{K}\big)\uparrow\mathbb{R}^d$ uniformly in all directions with $A_n = [-a_n,a_n]^d$ and $\lim_{n\to\infty} a_n = \infty$. Consequently, for all $y\in\mathbb{R}^d$, there exists an $n_0(y)\in\mathbb{N}$ such that $\mathbb{1}(y\notin(A_n\ominus\check{K})) = 0$ for all $n\geq n_0(y)$. By this and since $\mathbb{E}\,|M_0\oplus\check{K}| < \infty$ by assumption, the final expression of the above equations converges to zero by Lebesgue's dominated convergence theorem. □

In accordance with previous notation and assumptions, let $\mu_n = \mathbb{E}\,Y_n(o)$ and set

$$\widetilde{\mu}_n \;=\; \int_{W_n} Y_n(x)G(W_n,x)\,dx\,, \qquad n\geq 1, \tag{3.15}$$

where $\mathbb{E}\,\widetilde{\mu}_n = \mu_n$ by stationarity of Ξ_n for every $n\geq 1$.

Lemma 3.6 *The estimator $\widetilde{\mu}_n$ is asymptotically unbiased for μ as $n\to\infty$.*

Proof The claim is a direct consequence of Lemmas 3.2 and 3.5 observing that $|\mathbb{E}\,\widetilde{\mu}_n - \mu| \leq |\mathbb{E}\,\widehat{\mu}_n - \mu| + |\mathbb{E}\,\widehat{\mu}_n - \widetilde{\mu}_n|$ and $|\mathbb{E}\,(\widehat{\mu}_n - \widetilde{\mu}_n)|^2 \leq \mathbb{E}((Y_n(o)-Y(o))^2)$ by Fubini's theorem and the Cauchy–Schwarz inequality. □

As one would expect from the construction of Y_n, $n\geq 1$, the covariance function $\mathrm{Cov}_{Y_n}(x) = \mathrm{Cov}(Y_n(o),Y_n(x))$, $x\in\mathbb{R}^d$ of random field Y_n is absolutely integrable under the same sufficient condition as derived for $\mathrm{Cov}_Y(x)$, $x\in\mathbb{R}^d$. Recall that we postulate that $\mathbb{E}(|M_0\oplus\check{K}|^2) < \infty$ holds.

Lemma 3.7 *For any $n\geq 1$ and $x\in\mathbb{R}^d$, one has*

$$|\,\mathrm{Cov}(Y_n(o),Y_n(x))\,| \;\leq\; h(x) \qquad \textit{and} \qquad |\,\mathrm{Cov}(Y(o),Y_n(x))\,| \;\leq\; h(x)\,,$$

with integrable function $h:\mathbb{R}^d\to\mathbb{R}_+$ defined in (3.13). This means, in particular, that $\mathrm{Cov}_{Y_n}(x)$, $x\in\mathbb{R}^d$ is absolutely integrable.

Proof For each $n \in \mathbb{N}$, the germ–grain model Ξ_n is a Boolean model with typical grain $M_{0n} = M_0 \cap A_n$. Since functional φ and test set K remain unchanged, one has $|\mathrm{Cov}(Y_n(o), Y_n(x))| \leq h(x, M_{0n})$ by Lemma 3.4 and formula (3.13). Now, we find that $M_{0n} = M_0 \cap A_n \subseteq M_0$ for each realization and deduce from monotonicity of h with respect to M_0 that

$$|\mathrm{Cov}(Y_n(o), Y_n(x))| \;\leq\; h(x, M_{0n}) \;\leq\; h(x, M_0) = h(x), \qquad x \in \mathbb{R}^d.$$

Next, let N_{Ξ_n} and N_{Ξ} be the particle processes associated with Ξ_n and Ξ, respectively. The random variable $N_{\Xi_n}(\mathcal{F}_K)$ counts the (random) number of particles $M_{ln}+X_l \in N_{\Xi_n}$ hitting the test set K. But, if a truncated grain $M_{ln} = M_l \cap A_n$ has a non–empty intersection with K, then so does the original random set M_l. Hence, it follows that $N_{\Xi_n}(\omega)(\mathcal{F}_K) \leq N_{\Xi}(\omega)(\mathcal{F}_K)$ for all $\omega \in \Omega$. On the lines of the proof of Lemma 3.4, one can now easily reason that $|\mathrm{Cov}(Y(o), Y_n(x))| \leq h(x)$ holds. □

In the next step, we show that $\sqrt{|W_n|}\,\widetilde{\mu}_n$ and $\sqrt{|W_n|}\,\widehat{\mu}_n$ have the same asymptotic variance. This implies, amongst others, mean–square consistency of $\widetilde{\mu}_n$ for μ.

Lemma 3.8 *Under the imposed conditions, it holds that*

$$\lim_{n\to\infty} |W_n| \mathrm{Var}\, \widetilde{\mu}_n \;=\; \theta \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx\,.$$

Proof For any fixed $n \geq 1$, Lemma 3.2 states that

$$|W_n| \mathrm{Var}\, \widetilde{\mu}_n \;=\; \int_{\mathbb{R}^d} \mathrm{Cov}_{Y_n}(x)\, |W_n| \Gamma_n(x)\, dx\,.$$

Furthermore, we have $|\mathrm{Cov}_{Y_n}(x)| \leq h_Y(x)$ by Lemma 3.7 and $\lim_{n\to\infty} |W_n|\Gamma_n(x) = \theta$ due to condition (3.6). Employing Lebesgue's dominated convergence theorem once again, it therefore suffices to verify that $\lim_{n\to\infty} \mathrm{Cov}_{Y_n(x)} = \mathrm{Cov}_Y(x)$ for all $x \in \mathbb{R}^d$. This result is obtained as follows

$$\begin{aligned}
&|\,\mathrm{Cov}_{Y_n}(x) - \mathrm{Cov}_Y(x)\,| \\
&\quad\leq\; \bigl|\mathbb{E}\bigl[Y_n(o)Y_n(x) - Y(o)Y(x)\bigr]\bigr| + \bigl|\mu_n^2 - \mu^2\bigr| \\
&\quad=\; \bigl|\mathbb{E}\bigl[(Y_n(o) - Y(o))Y_n(x) + (Y_n(x) - Y(x))Y(o)\bigr]\bigr| + \bigl|\mu_n^2 - \mu^2\bigr| \\
&\quad=\; \bigl|\mathbb{E}\bigl[(Y_n(o) - Y(o))\,(Y_n(x) + Y(-x))\bigr]\bigr| + \bigl|\mu_n^2 - \mu^2\bigr| \\
&\quad\leq\; \sqrt{\mathbb{E}\bigl[(Y_n(o) - Y(o))^2\bigr]\mathbb{E}\bigl[(Y_n(x) + Y(-x))^2\bigr]} + \bigl|\mu_n^2 - \mu^2\bigr|,
\end{aligned}$$

where the second equality is based on the fact that Ξ and Ξ_n are jointly stationary, since they are driven by the same stationary point process $X = \{X_l\}$. The last estimate is due to the Cauchy–Schwarz inequality. In the final line of the above calculations, the term $\mathbb{E}(Y_n(x) + Y(-x))^2$ is uniformly bounded in n and x by $4\,c^2(K)\,e^{3\lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|}$, if one applies Minkowski's inequality and inequality (3.7). Moreover, it holds that $\lim_{n\to\infty} \mathbb{E}\,(Y_n(o) - Y(o))^2 = 0$ (by Lemma 3.5) and $\lim_{n\to\infty} |\mu_n^2 - \mu^2| = 0$ (see Lemma 3.6), which completes the proof. □

3.4.2 On m–dependent random fields

In the subsequent paragraph m–dependent random fields will play a central role in order to prove the asymptotic normality of $\sqrt{|W_n|}(\widetilde{\mu}_n - \mu_n)$ as n tends to infinity.

Definition 3.2 *For $z = (z_1, \ldots, z_d)^\top \in \mathbb{Z}^d$, put $||z|| = \max\{|z_k| : k = 1, \ldots, d\}$. Furthermore, let $m > 0$ be an arbitrary integer. A family of random variables $\{U(z), z \in \mathbb{Z}^d\}$ is called an m-*`dependent random field`* if the random vectors $(U(z))_{z \in I}$ and $(U(z'))_{z' \in I'}$ are independent for all finite sets $I, I' \subset \mathbb{Z}^d$ of distance larger than m, that is to say, $\min\{||z - z'|| : z \in I, z' \in I'\} > m$.*

A simple example of an m–dependent random field in the context of the Boolean models Ξ_n with bounded grains $M_{ln} \subseteq A_n$ is $\{\mathbb{1}_{\Xi_n}(z), z \in \mathbb{Z}^d\}$, where $m \geq 2a_n$.

For sequences of m_n–dependent random fields, where m_n may be unboundedly increasing, the following central limit theorem holds (see [17, Theorem 2]).

Theorem 3.1 *Let $\{I_n, n \geq 1\}$ be a sequence of finite subsets of $\mathbb{Z}^d$ with $\#I_n \to \infty$ as $n \to \infty$. For each $n \geq 1$, let $\{U_n(z), z \in \mathbb{Z}^d\}$ be an m_n–dependent random field with $\mathbb{E}\, U_n(z) = 0$ for all $z \in \mathbb{Z}^d$. Put $S_n = \sum_{z \in I_n} U_n(z)$ and assume that $\mathbb{E}\, S_n^2 = \sigma_n^2 \to \sigma^2$ as $n \to \infty$ with $\sigma^2 < \infty$. Then S_n converges in distribution to a Gaussian random variable with mean zero and variance σ^2 if there exists a finite constant $c_1 > 0$ such that*

$$\sum_{z \in I_n} \mathbb{E}\, U_n^2(z) \;\leq\; c_1 \tag{3.16}$$

for any $n \geq 1$ and if for every $\varepsilon > 0$ it holds that

$$\lim_{n \to \infty} L_n(\varepsilon) \;=\; \lim_{n \to \infty} m_n^{2d} \sum_{z \in I_n} \mathbb{E}\big(U_n^2(z)\, \mathbb{1}\big(|U_n(z)| \geq \varepsilon \cdot m_n^{-2d}\big)\big) \;=\; 0\,. \tag{3.17}$$

Proof Provided that $\lim_{n \to \infty} \sigma_n^2 = 0$, the claim follows from Chebychev's inequality, since in this case S_n converges to zero in probability and thus also in distribution. Now, assume that $\lim_{n \to \infty} \sigma_n^2 = \sigma^2$ with $\sigma^2 > 0$ and hence $\sigma_n^2 > 0$ for n large enough. By Theorem 2 in [17] it holds that $(S_n/\sigma_n) \xrightarrow{D} \mathcal{N}(0,1)$ as $n \to \infty$, which gives the desired convergence by the continuous mapping theorem. □

3.4.3 Asymptotic normality under m–dependence

We will now see that the sequence $\sqrt{|W_n|}(\widetilde{\mu}_n - \mu_n)$, $n \geq 1$ can be represented by a sequence of m_n–dependent random fields $\{U_n(z), z \in \mathbb{Z}^d\}$ that satisfies the conditions of Theorem 3.1 if the truncation parameters a_n, $n \in \mathbb{N}$ are chosen appropriately.

Theorem 3.2 *Let Ξ_n be given as in (3.14) with grains $M_{ln} = M_l \cap [-a_n, a_n]^d$, and consider the estimator $\widetilde{\mu}_n$, $n \in \mathbb{N}$ introduced in (3.15). If $\{a_n, n \in \mathbb{N}\}$ is a sequence of positive integers such that $\lim_{n\to\infty} a_n = \infty$ and*

$$\lim_{n\to\infty} \frac{a_n^{4d(1+\delta)/\delta}}{|W_n|} = 0 \tag{3.18}$$

for some $\delta > 0$, then the random variables $\widetilde{S}_n = \sqrt{|W_n|}\big(\widetilde{\mu}_n - \mu_n\big)$ are asymptotically normally distributed with mean zero and variance $\sigma^2 = \theta \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx$, that is, $\widetilde{S}_n \xrightarrow{D} \mathcal{N}(0,1)$ as $n \to \infty$.

Proof The principal idea of this proof is to find a representation of $\widetilde{S}_n$ of the form

$$\widetilde{S}_n = S_n + R_n = \sum_{z\in I_n} U_n(z) + R_n\,,$$

where $\{U_n(z), z \in \mathbb{Z}^d\}$ is a random field on $\mathbb{Z}^d$ fulfilling the requirements of Theorem 3.1 with $I_n \subset \mathbb{Z}^d$ such that $\lim_{n\to\infty} \#I_n = \infty$, and R_n converges to zero in probability as $n \to \infty$. Then the assertion follows by Slutsky's theorem.

To begin with, recall the notation $[z, z+e) = [z_1, z_1+1) \times \ldots \times [z_d, z_d+1)$ for $z \in \mathbb{Z}^d$, set $I_n = I_n^- = \{z \in \mathbb{Z}^d : [z, z+e) \subseteq W_n\}$ and introduce $V_n = \bigcup_{z\in I_n} [z, z+e)$, $n \geq 1$. By Lemma 3.1, we have $\lim_{n\to\infty} \#I_n = \infty$. Now, consider the decomposition of $\widetilde{S}_n$ into S_n and R_n with

$$S_n = \sqrt{|W_n|} \int_{V_n} \big(Y_n(x) - \mu_n\big)\, G(W_n, x)\, dx$$

and

$$R_n = \sqrt{|W_n|} \int_{W_n\setminus V_n} \big(Y_n(x) - \mu_n\big)\, G(W_n, x)\, dx\,.$$

With regard to R_n, we show that $R_n \xrightarrow{L^2} 0$ and therefore $R_n \xrightarrow{P} 0$ by Chebychev's inequality. To see that $\lim_{n\to\infty} \mathbb{E}(R_n^2) = 0$ holds, employ condition (3.5) and the integrable upper bound h (as defined in (3.13); see Lemma 3.8) of $|\mathrm{Cov}_{Y_n}(x)|$, $x \in \mathbb{R}^d$ to get

$$\begin{aligned}
&\mathbb{E}\left(R_n^2\right) \\
&= \int_{W_n\setminus V_n} \int_{W_n\setminus V_n} |W_n|\, \mathbb{E}\left(\left(Y_n(x) - \mu_n\right)\left(Y_n(y) - \mu_n\right)\right) G(W_n, x)\, G(W_n, y)\, dx\, dy \\
&= \int_{\mathbb{R}^d} \left(\mathrm{Cov}_{Y_n}(x)\, |W_n| \left(\int_{(W_n\setminus V_n)\,\cap\,(W_n\setminus V_n - x)} G(W_n, y)\, G(W_n, x+y)\, dy \right) \right) dx \\
&\leq c_0^2 \left(\int_{\mathbb{R}^d} h(x)\, dx \right) \cdot \frac{|W_n \setminus V_n|}{|W_n|} \\
&\leq c_0^2 \left(\int_{\mathbb{R}^d} h(x)\, dx \right) \cdot \frac{\left|\partial W_n \oplus B_{\sqrt{d}}(o)\right|}{|W_n|}\,,
\end{aligned}$$

since $W_n \setminus V_n \subseteq W_n \setminus (W_n \ominus B_r(o)) \subset \partial W_n \oplus B_{\sqrt{d}}(o)$. The prefactor $c_0^2 \int_{\mathbb{R}^d} h(x)\,dx$ in the final line is a finite constant by assumption, whereas condition (3.1) ensures that $\lim_{n\to\infty} |\partial W_n \oplus B_{\sqrt{d}}(o)|/|W_n| = 0$. This verifies the claim regarding R_n.

Next, we observe that $S_n = \sum_{z\in I_n} U_n(z)$ for

$$U_n(z) = \begin{cases} \sqrt{|W_n|}\, \int_{[z,z+e)} \big(Y_n(x) - \mu_n\big)\, G(W_n, x)\, dx, & z \in I_n, \\ 0, & \text{else.} \end{cases}$$

Set $m_n = 2(a_n + ||K||) + 1$ for any $n \geq 1$. Since Ξ_n is generated by an independently marked Poisson point process with marks $M_{ln} \subseteq A_n$, the random fields $\{Y_n(x), x \in [z, z+e)\}$ and $\{Y_n(y), y \in [z', z'+e)\}$ are independent, whenever $||z - z'|| > m_n$. For this, pick an arbitrary pair $z, z' \in \mathbb{Z}^d$ satisfying $||z - z'|| > m_n$ and arbitrary points $x \in [z, z+e), y \in [z', z'+e)$. If we assumed that a particle $(M_{ln} + X_l)$ hits $K+x$ as well as $K+y$, then we have $X_l \in (A_n \oplus (K+x)) \cap (A_n \oplus (K+y))$ almost surely. The latter set, however, is equal to the empty set, since in the considered scenario $|x_k - y_k| > 2\,(a_n + ||K||)$ for at least one $k \in \{1, \dots d\}$. Consequently, $U_n(z)$ and $U_n(z')$ are independent if $||z - z'|| > m_n$, and $\{U_n(z), z \in \mathbb{Z}^d\}$ forms an m_n–dependent random field on $\mathbb{Z}^d$.

Let us now check the conditions of Theorem 3.1. Firstly, we have $\mathbb{E}\, U_n(z) = 0$, since $\mu_n = \mathbb{E}\, Y_n(x)$ for all $x \in \mathbb{R}^d$. Secondly, it holds that $\lim_{n\to\infty} \mathbb{E}\, S_n^2 = \sigma^2$ by Lemma 3.8 and since $S_n = \widetilde{S}_n - R_n$ with $\mathbb{E}\big(R_n^2\big) \to 0$ as shown above. In order to complete the proof, it remains to validate conditions (3.16) and (3.17).

With respect to condition (3.16), employ Fubini's theorem and (3.5) to get

$$\begin{aligned} \sum_{z\in I_n} \mathbb{E}\, U_n^2(z) &\leq \sum_{z\in I_n} \frac{c_0^2}{|W_n|} \int_{[z,z+e)} \int_{[z,z+e)} |\mathrm{Cov}_{Y_n}(x-y)|\, dx\, dy \\ &\leq \sum_{z\in I_n} \frac{c_0^2\, \mathrm{Var}(Y_n(o))}{|W_n|} \;=\; c_0^2\, \mathrm{Var}(Y_n(o))\, \frac{|V_n|}{|W_n|} \;\leq\; c_0^2\, h(o) \end{aligned}$$

for all sufficiently large n, where the uniform, final bound follows from the facts that $\mathrm{Var}(Y_n(o)) \leq h(o) < \infty$ and $|V_n|/|W_n| \leq 1$ for any $n \geq 1$. Thus, (3.16) holds.

Regarding (3.17), observe that $|U_n(z)| \leq c_0\, \widetilde{U}_n(z)$ by condition (3.5) putting

$$\widetilde{U}_n(z) \;=\; \int_{[z,z+e)} \frac{|Y_n(x) - \mu_n|}{\sqrt{|W_n|}}\, dx\,, \qquad z \in I_n.$$

The random variables $\widetilde{U}_n(z)$, $z \in I_n$ are identically distributed due to the stationarity

of Ξ_n. This yields the following estimates

$$\begin{aligned} L_n(\varepsilon) &= m_n^{2d} \sum_{z \in I_n} \mathbb{E}\left(U_n^2(z)\, \mathbb{1}\Big(m_n^{2d}|U_n(z)| \geq \varepsilon\Big)\right) \\ &\leq m_n^{2d}\, \#I_n\, c_0^2\, \mathbb{E}\left(\widetilde{U}_n^2(o)\, \mathbb{1}\Big(\widetilde{U}_n^{\delta}(o) \geq \varepsilon^{\delta} c_0^{-\delta}\, m_n^{-2d\delta}\Big)\right) \\ &\leq \varepsilon^{-\delta}\, c_0^{2+\delta}\, m_n^{2d(1+\delta)}\, \#I_n\, \mathbb{E}\big(\widetilde{U}_n^{2+\delta}(o)\big) \\ &\leq \frac{c_0^{2+\delta}}{\varepsilon^{\delta}} \cdot \left(\frac{m_n^{4d(1+\delta)/\delta}}{|W_n|}\right)^{\delta/2} \cdot \frac{\#I_n}{|W_n|} \cdot \Big(|W_n|^{(2+\delta)/2}\, \mathbb{E}\big(\widetilde{U}_n^{2+\delta}(o)\big)\Big) \end{aligned}$$

for every $\delta > 0$. If $\delta > 0$ is chosen in such a way that (3.18) holds, the second factor in the last line converges to zero as n tends to infinity, since $m_n = 2(a_n + ||K||)+1$. The remaining components are uniformly bounded in n, because we have $\#I_n/|W_n| = |V_n|\,/\,|W_n| \leq 1$, and by Minkowski's inequality it holds that

$$\begin{aligned} \sqrt{|W_n|}\left(\mathbb{E}\big(\widetilde{U}_n^{2+\delta}(o)\big)\right)^{\frac{1}{2+\delta}} &\leq \left(\mathbb{E}\left(\int_{[o,e)} |Y_n(x)|\, dx + |\mu_n|\right)^{2+\delta}\right)^{\frac{1}{2+\delta}} \\ &\leq \left(\mathbb{E}\int_{[o,e)} |Y_n(x)|^{2+\delta}\, dx\right)^{\frac{1}{2+\delta}} + |\mu_n| \\ &\leq c(K)\left(g_{N_{\Xi}(\mathcal{F}_K)}\big(2^{2+\delta}\big)\right)^{\frac{1}{2+\delta}} + c(K) g_{N_{\Xi}(\mathcal{F}_K)}(2) \end{aligned}$$

for any $n \geq 1$, where the last inequality is obtained by Lemma 2.10 giving a finite uniform upper bound (see also (3.7)). Hence, $\lim_{n\to\infty} L_n(\varepsilon) = 0$, i.e., condition (3.17) of Theorem 3.1 is fulfilled as well. □

A feasible choice of $\{a_n, n \in \mathbb{N}\}$ meeting the requirements of Theorem 3.2 is given by $a_n = \lfloor w_n^{\varepsilon} \rfloor$, where $0 < \varepsilon < \delta/(4(1+\delta))$ and w_n is the radius of the largest ball that can be inscribed in W_n. Note that $\lim_{n\to\infty} w_n = \infty$, since assuming that $w_n \leq w$ for some $w < \infty$ and all $n \geq 1$ implies $W_n \ominus B_{w+1}(o) = \emptyset$, which contradicts (3.2).

3.4.4 Asymptotic normality for the original model

Combining the results from Sections 3.4.1 and 3.4.3, the weak convergence, proven in Theorem 3.2, of the mean–value estimator associated with the truncated Boolean models implies an equivalent statement for the original model.

Theorem 3.3 *Let the conditions* (3.1) *and* (3.4) – (3.6) *be fulfilled, and suppose that* $\mathbb{E}(|M_0 \oplus \check{K}|^2) < \infty$ *holds. Then we have*

$$\sqrt{|W_n|}\,(\widehat{\mu}_n - \mu) \xrightarrow{D} \mathcal{N}(0, \sigma^2)\,, \qquad n \to \infty$$

where $\sigma^2 = \theta \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx$.

Proof For $n \geq 1$, put $R_n = \sqrt{|W_n|}\big((\widetilde{\mu}_n - \mu_n) - (\widehat{\mu}_n - \mu)\big)$ and observe that

$$\begin{aligned}
\mathbb{E}(R_n^2) &= |W_n|\,\mathbb{E}\Big(\int_{\mathbb{R}^d} \big((Y_n(x) - Y(x)) - (\mu_n - \mu)\big)\, G(W_n, x)\, dx\Big)^2 \\
&= |W_n| \int_{\mathbb{R}^d} \mathrm{Cov}(Y_n(o) - Y(o), Y_n(x) - Y(x))\, \Gamma_n(x)\, dx \\
&\leq \frac{c_0^2}{|W_n|} \int_{\mathbb{R}^d} |\mathrm{Cov}(Y_n(o) - Y(o), Y_n(x) - Y(x))|\, dx \\
&\leq \frac{4\, c_0^2}{|W_n|} \int_{\mathbb{R}^d} h(x)\, dx
\end{aligned}$$

due to Lemma 3.7, where we used (3.5) in the last but one line and $h(x)$ is given as in (3.13) so that the final bound converges to zero as n tends to ∞. An application of Slutsky's theorem to

$$\sqrt{|W_n|}(\widehat{\mu}_n - \mu) \;=\; \sqrt{|W_n|}(\widetilde{\mu}_n - \mu_n) + R_n$$

with $\sqrt{|W_n|}(\widetilde{\mu}_n - \mu_n) \xrightarrow{D} \mathcal{N}(0, \sigma^2)$ completes the proof. □

3.5 A multivariate central limit theorem

The results from the preceeding paragraphs will now be extended to the multivariate case. As in Section 2.4, let $q \geq 2$ and choose for each $i = 1, \ldots, q$ a triple (Ξ_i, φ_i, K_i), where Ξ_i is a stationary Boolean model, φ_i is a measurable, conditionally bounded valuation on $\mathcal{R}$ and K_i is a compact and convex test set. Associate with each tuple (Ξ_i, φ_i, K_i), $i \in \{1, \ldots, q\}$ a stationary random field $Y_i = \{Y_i(x), x \in \mathbb{R}^d\}$ via the equation $Y_i(x) = \varphi_i((\Xi_i - x) \cap K_i)$ for $x \in \mathbb{R}^d$. With respect to stationarity, we additionally assume that the vector–valued random field $\mathbf{Y} = \{(Y_1(x), \ldots, Y_q(x))^\top, x \in \mathbb{R}^d\}$ is stationary. This condition is met, for instance, if the Boolean models $\Xi_1, \ldots, \Xi_q$ are jointly stationary or, in particular, if the same Boolean model Ξ is chosen for every $i \in \{1, \ldots, q\}$.

3.5.1 Weighting functions

As before, let $\{W_n, n \in \mathbb{N}\}$ be a van Hove sequence, and define $G_i : \mathcal{C}^* \times \mathbb{R}^d \to [0, \infty)$, $i = 1, \ldots, q$ to be a set of weighting functions satisfying the conditions (3.4) and (3.5) on $\{W_n, n \in \mathbb{N}\}$, respectively. That is to say,

$$G_i(W_n, x) = 0, \quad x \in \mathbb{R}^d \setminus (W_n \ominus \check{K}_i) \quad \text{and} \quad \int_{W_n} G_i(W_n, x)\, dx = 1\,. \tag{3.19}$$

and

$$\sup_{y \in W_n} G_i(W_n, y) \leq \frac{c_0}{|W_n|} \tag{3.20}$$

for any $n \geq 1$ and each $i \in \{1, \ldots, q\}$. The upper bound in (3.20) can be chosen uniformly in i, since only finitely many triples (Ξ_i, φ_i, K_i) are considered.

Furthermore, for any pair $i, j \in \{1, \ldots, q\}$, introduce the auxiliary function $\Gamma_{ij,n} : \mathbb{R}^d \to [0, \infty)$ by

$$\Gamma_{ij,n}(x) \;=\; \int_{\mathbb{R}^d} G_i(W_n, y) G_j(W_n, x+y)\, dy\,, \qquad x \in \mathbb{R}^d \tag{3.21}$$

assuming that

$$\lim_{n\to\infty} |W_n| \Gamma_{ij,n}(x) \;=\; \theta_{ij} \tag{3.22}$$

for all $x \in \mathbb{R}^d$ and some constants $\theta_{ij} \in (0, \infty)$. Notice that $\Gamma_{ij,n}(x) = \Gamma_{ji,n}(-x)$ for any $x \in \mathbb{R}^d$, and therefore $\theta_{ij} = \theta_{ji}$.

3.5.2 An estimator of the mean vector

For any $n \geq 1$, the estimator

$$\widehat{\boldsymbol{\mu}}_n \;=\; \begin{pmatrix} \widehat{\mu}_{1,n} \\ \vdots \\ \widehat{\mu}_{q,n} \end{pmatrix} \qquad \text{with} \quad \widehat{\mu}_{i,n} \;=\; \int_{W_n} Y_i(x)\, G_i(W_n, x)\, dx\,, \;\; i = 1, \ldots, q \tag{3.23}$$

is unbiased for $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_q)^\top$ with $\mu_i = \mathbb{E}\, Y_i(o)$. Following arguments in the proof of Lemma 3.2, we see that the covariances $\mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n})$, $i, j = 1, \ldots, q$ are given by

$$\mathrm{Cov}\,(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n}) \;=\; \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, \Gamma_{ij,n}(x)\, dx\,,$$

where $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}(Y_i(o), Y_j(x))$, $x \in \mathbb{R}^d$.

Suppose now that $\mathbb{E}(|M_{0i} \oplus \check{K}_i|^2) < \infty$ for $i = 1, \ldots, q$, where M_{0i} is a typical grain of Ξ_i. Convergence of $\widehat{\boldsymbol{\mu}}_n$ to $\boldsymbol{\mu}$ in mean–square sense, that is $\lim_{n\to\infty} \mathbb{E}\,(|\widehat{\boldsymbol{\mu}}_n - \boldsymbol{\mu}|^2) = 0$, follows from the convergence of its components, i.e., $\lim_{n\to\infty} \mathbb{E}\,(|\widehat{\mu}_{i,n} - \mu_i|^2) = 0$. By virtue of Lemma 3.3, integrability of $\mathrm{Cov}_{ii}(x)$, $x \in \mathbb{R}^d$, $i = 1, \ldots, q$ is a sufficient condition for mean–square consistency of $\widehat{\boldsymbol{\mu}}_n$, which in turn is guaranteed by the condition $\mathbb{E}\,(|M_{i0} \oplus \check{K}_i|^2) < \infty$; see Lemma 3.4. As shown in Lemma 3.3, condition (3.22) implies that

$$\lim_{n\to\infty} |W_n| \int_{\mathbb{R}^d} \mathrm{Cov}_{ii}(x)\, \Gamma_{ii,n}(x)\, dx \;=\; \theta_{ii} \int_{\mathbb{R}^d} \mathrm{Cov}_{ii}(x)\, dx\,. \tag{3.24}$$

With respect to the cross–covariances $\mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n})$ for $i \neq j$, we analogously obtain

$$\lim_{n\to\infty} |W_n| \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, \Gamma_{ij,n}(x)\, dx \;=\; \theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx\,, \tag{3.25}$$

provided that

$$\int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx \;<\; \infty, \qquad i, j \in \{1, \ldots, q\}\,. \tag{3.26}$$

Simple sufficient conditions for (3.26) to hold in the general case, where $\Xi_1, \ldots, \Xi_q$ are arbitrarily chosen Boolean models, are unknown so far. Yet two special cases will be looked at. First, assume that the germ–grain models $\Xi_1, \ldots, \Xi_q$ are pairwise independent. Then the cross–covariances $\mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n})$ need not be considered for any $i \neq j$, since they are equal to zero. Secondly, suppose that $\Xi_i = \Xi$, $i = 1, \ldots, q$ for some Boolean model Ξ with typical grain M_0. In this case, an integrable upper bound for $|\mathrm{Cov}_{ij}(x)|$, $x \in \mathbb{R}^d$ is once more provided by the proof of Lemma 3.4. Here, one finds that

$$\begin{aligned} |\mathrm{Cov}_{ij}(x)| \;&\leq\; 24\,\lambda\, c^2(K)\, e^{3\lambda\,(\mathbb{E}|M_0 \oplus \check{K}_i| + \mathbb{E}|M_0 \oplus \check{K}_j|)} \\ &\qquad \times \mathbb{E}\big|(M_0 \oplus \check{K}_i) \cap ((M_0 \oplus \check{K}_j) - x)\big| \end{aligned} \tag{3.27}$$

for all $x \in \mathbb{R}^d$ and $c(K) = \max\{c(K_1), \ldots, c(K_q)\}$. Denote the right hand side of (3.27) by $h_{ij}(x)$. As is easily derived by Fubini's theorem, one has

$$\int_{\mathbb{R}^d} h_{ij}(x)\, dx \;=\; c(K_i, K_j, \varphi_i, \varphi_j, M_0)\, \mathbb{E}\big(|M_0 \oplus \check{K}_i|\, |M_0 \oplus \check{K}_j|\big)$$

with the finite constant $c(K_i, K_j, \varphi_i, \varphi_j, M_0) = 24\,\lambda\, c^2(K)\, e^{3\lambda\,(\mathbb{E}|M_0 \oplus \check{K}_i| + \mathbb{E}|M_0 \oplus \check{K}_j|)}$. By this and the Cauchy–Schwarz inequality, we see that $\mathbb{E}\,(|M_0 \oplus \check{K}_i|^2) < \infty$ for $i = 1, \ldots, q$ implies integrability of the cross–covariance function $\mathrm{Cov}_{ij}(x)$, $x \in \mathbb{R}^d$ for any pair $i, j \in \{1, \ldots, q\}$.

We are now in a position to formulate a multivariate analog of Theorem 3.3.

3.5.3 Cramér–Wold device

If $\Xi_1, \ldots, \Xi_q$ are mutually independent, stationary Boolean models, which satisfy the conditions of Theorem 3.3 individually, then it readily follows that $\sqrt{|W_n|}(\widehat{\boldsymbol{\mu}}_n - \boldsymbol{\mu})$ converges in distribution to a q–dimensional Gaussian random variable as $n \to \infty$. Let us therefore assume that $\Xi_i = \Xi$ for all $i \in \{1, \ldots, q\}$, where Ξ is some stationary Boolean model with typical grain M_0.

Theorem 3.4 *Let conditions* (3.1) *and* (3.19) – (3.22) *be fulfilled, and suppose that* $\mathbb{E}(|M_0 \oplus \check{K}_i|^2) < \infty$ *for* $i = 1, \ldots, q$. *Then it holds that*

$$\begin{pmatrix} \sqrt{|W_n|}\,(\widehat{\mu}_{1,n} - \mu_1) \\ \vdots \\ \sqrt{|W_n|}\,(\widehat{\mu}_{q,n} - \mu_q) \end{pmatrix} \;\xrightarrow{D}\; \mathcal{N}_q(o, \Sigma)\,, \qquad n \to \infty,$$

where $\mathcal{N}_q(o, \Sigma)$ *is a* q*–dimensional Gaussian random vector with mean vector zero and covariance matrix* $\Sigma = (\sigma_{ij})_{i,j=1,\ldots,q}$, *whose entries are* $\sigma_{ij} = \theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx$.

Proof By the well–known Cramér–Wold device the assertion is true if and only if

$$\sqrt{|W_n|} \sum_{i=1}^{q} t_i(\widehat{\mu}_{i,n} - \mu_i) \;\xrightarrow{D}\; \mathcal{N}\left(0, \mathbf{t}^\top \Sigma\, \mathbf{t}\right)$$

for all $\mathbf{t} = (t_1, \ldots, t_q)^\top \in \mathbb{R}^q \setminus \{o\}$. To this end, a similar proving technique can be applied as for Theorem 3.3. On that account, we only sketch the proof here covering differences. First, denote by $Y_{i,n}$, $i = 1, \ldots, q$ the random fields associated with the truncated model Ξ_n, and let $\widetilde{\mu}_{i,n}$ be the induced estimator of μ_i with $\mathbb{E}\,\widetilde{\mu}_{i,n} = \mu_{i,n}$. Then Lemma 3.5 holds without any necessity of change, so that $\lim_{n\to\infty} \mathbb{E}\,(Y_{i,n}(x) - Y_i(x))^2 = 0$ for $i = 1, \ldots, q$.

Regarding the proof of the multivariate analog of Lemma 3.8, we remark that

$$\begin{aligned}
&|\operatorname{Cov}(Y_{i,n}(o), Y_{j,n}(x)) - \operatorname{Cov}(Y_i(o), Y_j(x))| \\
&\quad\le \left|\mathbb{E}\big(Y_{i,n}(o)Y_{j,n}(x) - Y_i(o)Y_j(x)\big)\right| + \left|\mathbb{E}\,Y_{i,n}(o)\,\mathbb{E}\,Y_{j,n}(o) - \mathbb{E}\,Y_i(o)\,\mathbb{E}\,Y_j(o)\right| \\
&\quad\le \left|\mathbb{E}\big[(Y_{i,n}(o) - Y_i(o))Y_{j,n}(x)\big]\right| + \left|\mathbb{E}\big[(Y_{j,n}(o) - Y_j(o))Y_{i,n}(-x)\big]\right| \\
&\qquad + \left|\mathbb{E}(Y_{i,n}(o) - Y_i(o))\right| \left|\mathbb{E}\,Y_{j,n}(o)\right| + \left|\mathbb{E}(Y_{j,n}(o) - Y_j(o))\right| \left|\mathbb{E}\,Y_i(o)\right| \\
&\quad\le 2\sqrt{\mathbb{E}(Y_{i,n}(o) - Y_i(o))^2\,\mathbb{E}\,Y_{j,n}^2(o)} + 2\sqrt{\mathbb{E}(Y_{j,n}(o) - Y_j(o))^2\,\mathbb{E}\,Y_i^2(o)}
\end{aligned}$$

by the Cauchy–Schwarz inequality and by stationarity of Ξ_i and $\Xi_{i,n}$, $i = 1, \ldots, q$. The final inequality provides that $|\operatorname{Cov}(Y_{i,n}(o), Y_{j,n}(x)) - \operatorname{Cov}(Y_i(o), Y_j(x))|$ converges to zero as $n \to \infty$ due to Lemma 3.5, and since $\mathbb{E}\,Y_i^2(o)$ and $\mathbb{E}\,Y_{i,n}^2(o)$ admit a common finite upper bound. By the integrability of the functions h_{ij}, $i, j \in \{1, \ldots, q\}$ as introduced in (3.27), one finds

$$\lim_{n\to\infty} |W_n| \operatorname{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n}) \;=\; \lim_{n\to\infty} |W_n| \operatorname{Cov}(\widetilde{\mu}_{i,n}, \widetilde{\mu}_{j,n}) \;=\; \sigma_{ij}\,. \tag{3.28}$$

With respect to the proof of Theorem 3.2, note that $\{U_{\mathbf{t},n}(z), z \in \mathbb{Z}^d\}$ with

$$U_{\mathbf{t},n}(z) \;=\; \begin{cases} \sqrt{|W_n|} \sum\limits_{i=1}^{q} \Big[t_i \int_{[z,z+e)} \big(Y_{i,n}(x) - \mathbb{E}\,Y_{i,n}(x)\big)\, G_i(W_n, x)\, dx \Big], & z \in I_n, \\ 0, & \text{else} \end{cases}$$

for $I_n = \{z \in \mathbb{Z}^d : [z, z+e) \subseteq W_n\}$ is an m_n–dependent random field on $\mathbb{Z}^d$ for $m_n = 2(a_n + \max_{i=1,\ldots,q} ||K_i||) + 1$. At this, we have $\mathbb{E} \sum_{z\in I_n} U_{\mathbf{t},n}(z) = 0$ for all $n \ge 1$ and $\lim_{n\to\infty} \operatorname{Var} \sum_{z\in I_n} U_{\mathbf{t},n}(z) = \mathbf{t}^\top \Sigma\, \mathbf{t}$ due to equation (3.28). Moreover, we see as before that

$$\begin{aligned}
\sum_{z\in I_n} \mathbb{E}\,U_{\mathbf{t},n}^2(z) &\le \sum_{z\in I_n} \left[\frac{c_0^2}{|W_n|} \sum_{i,j=1}^{q} t_i t_j \sqrt{\operatorname{Var}\left(Y_{i,n}(o)\right) \operatorname{Var}\left(Y_{j,n}(o)\right)} \right] \\
&\le c_0^2 \sum_{i,j=1}^{q} t_i t_j \sqrt{h_{ii}(o)\, h_{jj}(o)}\,,
\end{aligned}$$

where the final bound provides the required finite constant c_1 in (3.16). It remains to verify that $L_n(\varepsilon)$ tends to zero as $n \to \infty$ for any $\varepsilon > 0$, where

$$L_n(\varepsilon) \;=\; m_n^2 \sum_{z\in I_n} \mathbb{E}\left(U_{\mathbf{t},n}^2(z)\, \mathbb{1}\left(m_n^{2d} |U_{\mathbf{t},n}(z)| \ge \varepsilon \right) \right).$$

For this purpose, define

$$\widetilde{U}_{\mathbf{t},n}(z) \;=\; \sum_{i=1}^{q} t_i \int_{[z,z+e)} \frac{|Y_{i,n}(x) - \mathbb{E}\, Y_{i,n}(x)|}{\sqrt{|W_n|}}\, dx\,, \qquad z \in I_n.$$

From Minkowski's inequality it follows that

$$\sqrt{|W_n|}\Big(\mathbb{E}\,\widetilde{U}_{\mathbf{t},n}^{2+\delta}(o)\Big)^{\frac{1}{2+\delta}} \;\le\; c(K) \sum_{i=1}^{q} |t_i| \left[\Big(g_{N_\Xi(\mathcal{F}_{K_i})}\big(2^{2+\delta}\big)\Big)^{\frac{1}{2+\delta}} + g_{N_\Xi(\mathcal{F}_{K_i})}(2)\right] \;<\; \infty\,.$$

As in the proof of Theorem 3.2, we then get

$$L_n(\varepsilon) \;\le\; \frac{c_0^{2+\delta}}{\varepsilon^\delta} \cdot \left(\frac{m_n^{4d(1+\delta)/\delta}}{|W_n|}\right)^{\delta/2} \cdot \frac{|V_n|}{|W_n|} \cdot \Big(|W_n|^{(2+\delta)/2}\, \mathbb{E}\big((\widetilde{U}_{\mathbf{t},n}(o))^{2+\delta}\big)\Big)\,,$$

which converges to zero as n tends to infinity by the assumptions on the sequence a_n, keeping in mind that $m_n = 2(a_n + \max_{i=1,\dots,q} ||K_i||) + 1$. Thus, we have shown that

$$\sqrt{|W_n|}\Big(\sum_{i=1}^{q} t_i(\widetilde{\mu}_{i,n} - \mu_{i,n})\Big) \;\xrightarrow{D}\; \mathcal{N}(0, \mathbf{t}^\top \Sigma\, \mathbf{t}), \qquad n \to \infty\,.$$

An application of Slutsky's theorem completes the proof, because

$$R_n \;=\; \sqrt{|W_n|}\Big(\sum_{i=1}^{q} t_i(\widehat{\mu}_{i,n} - \mu_i) - \sum_{i=1}^{q} t_i(\widetilde{\mu}_{i,n} - \mu_{i,n})\Big) \;\xrightarrow{L^2}\; 0\,.$$

The latter is established by Minkowski's inequality once again, which gives

$$\begin{aligned}\lim_{n\to\infty} \sqrt{\mathbb{E}\, R_n^2} \;&\le\; \lim_{n\to\infty} \sqrt{|W_n|} \sum_{i=1}^{q} |t_i| \sqrt{\mathbb{E}\big((\widehat{\mu}_{i,n} - \mu_i) - (\widetilde{\mu}_{i,n} - \mu_{i,n})\big)^2} \\ &\le\; 2\, c_0 \sum_{i=1}^{q} |t_i| \left(\int_{\mathbb{R}^d} h_i(x)\, dx\right)^{\frac{1}{2}} \cdot \lim_{n\to\infty} \frac{1}{\sqrt{|W_n|}} \;=\; 0\,.\end{aligned}$$

□

Rounding up the discussion of asymptotic normality in the case of Boolean models, it should be mentioned that the assumption

$$\mathbb{E}\big(|M_0 \oplus B_r(o)|^2\big) \;<\; \infty \qquad \text{for some } r > 0$$

is a universal sufficient condition for $\mathbb{E}(|M_0 \oplus \check{K}_i|^2) < \infty$, $i = 1, \dots, q$ by compactness of the test sets K_i. In contrast to the first moment condition (which quarantees that $\Xi_i = \bigcup_{l=1}^{\infty}(M_{li} + X_{li})$ is well defined, compare page 23), it may nevertheless be sufficient to suppose that $\mathbb{E}\,|M_0|^2 < \infty$; consider for instance $K = \{y\}$ for some $y \in \mathbb{R}^d$, where $|M_0 \oplus \{y\}| = |M_0|$.

Chapter 4

Mean value estimation for mixing random fields

The central limit theorem for $S_n^* = \sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ as considered in the previous chapter will now be generalized to the case where the defining germ–grain model Ξ is not necessarily a Boolean model. As before, let $\widehat{\mu}_n$ be the weighted average of a stationary random field Y associated with germ–grain model Ξ via some conditionally bounded valuation $\varphi : \mathcal{R} \to \mathbb{R}$, i.e., $Y(x) = \varphi((\Xi - x) \cap K)$, $x \in \mathbb{R}^d$, where $K \in \mathcal{K}'$ and $\mu = \mathbb{E}\, Y(o)$. Assume that Ξ is generated by some independently marked point process $\Psi = \{[X_l, M_l], l \geq 1\}$, where $X = \{X_l\}$ is an arbitrary stationary point process on $\mathbb{R}^d$ with intensity $\lambda > 0$. If the underlying point process X is such that, for any $B, B' \in \mathfrak{B}^d$, the restricted processes $X_{|B}$ and $X_{|B'}$ are independent, whenever B and B' are far enough apart (that is to say, $\varrho(B, B') = \inf\{||z - z'|| : z \in B, z' \in B'\}$ is large enough), then S_n^* can be approximated by sums of m_n–dependent random fields, and Theorem 3.1 is applicable. A similar idea is used in the more general case. The approximation is now provided by a single stationary random field $\{U(z), z \in \mathbb{Z}^d\}$ that fulfills certain mixing conditions. Mixing properties refer to the dependence structure of the random field $\{U(z), z \in \mathbb{Z}^d\}$. In our case, we mainly focus on so called β–mixing conditions. A variety of limit theorems for mixing random fields on $\mathbb{Z}^d$ is given, among others, in [5, 9, 27] or [52]. In addition, see Section 1.7 of [28] for central limit theorems based on α–mixing properties of random field $Y = \{Y(x), x \in \mathbb{R}^d\}$. A drawback of the latter theorems is that mixing conditions on Y are often difficult to verify or too restrictive. By contrast, the mixing properties of $\{U(z), z \in \mathbb{Z}^d\}$ can be related to tractable conditions on Ξ in several cases. In what follows, we make use of a technique presented in [23] for another general class of functionals induced by stationary germ–grain models. The basis for subsequent results is a central limit theorem for stationary random fields on $\mathbb{Z}^d$ with "quickly decreasing" dependence rate (see Theorem 4.1).

4.1 Mixing coefficients and asymptotic normality

Mixing properties of random fields are defined with respect to measures of dependence between related σ–algebras. In the present context, we merely mention two types of dependence measures. Regarding further details, confer [9] and references therein. Throughout the following, let $(\Omega, \mathcal{A}, P)$ be some probability space, and let $\mathcal{A}_1, \mathcal{A}_2$ represent two sub–σ–algebras of $\mathcal{A}$.

Definition 4.1 *The* α-`mixing coefficient` (*or* `Rosenblatt dependence rate`) *of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *is defined by*

$$\alpha(\mathcal{A}_1, \mathcal{A}_2) \;=\; \sup\{|P(A \cap B) - P(A)P(B)| \,:\, A \in \mathcal{A}_1, B \in \mathcal{A}_2\}\,.$$

The quantity

$$\beta(\mathcal{A}_1, \mathcal{A}_2) \;=\; \frac{1}{2} \sup \sum_k \sum_\ell |P(A_k \cap B_\ell) - P(A_k)P(B_\ell)|$$

is called the β-`mixing coefficient` (*or* `absolute regularity coefficient`) *of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, where the supremum is taken over all pairs of finite partitions* $\{A_k\}$ *and* $\{B_\ell\}$ *of* Ω *with* $A_k \in \mathcal{A}_1$ *for all* k *and* $B_\ell \in \mathcal{A}_2$ *for all* ℓ*.*

Note that one has $2\alpha(\mathcal{A}_1, \mathcal{A}_2) \leq \beta(\mathcal{A}_1, \mathcal{A}_2) < 1$ as shown by Proposition 1 in [9].

For a stationary random field $U = \{U(z), z \in \mathbb{Z}^d\}$, we denote the σ–algebra induced by the random variables $\{U(z), z \in B\}$ for $B \subseteq \mathbb{Z}^d$ by $\sigma(U(z), z \in B)$. For any $k \in \mathbb{N}$ and $u, v \in \mathbb{N} \cup \{\infty\}$, put

$$\begin{aligned} \beta_U(k; u, v) \;=\; & \sup\{\beta\left(\sigma(U(z), z \in B_1), \sigma(U(z'), z' \in B_2)\right) : \\ & B_1, B_2 \subset \mathbb{Z}^d, \#(B_1) \leq u, \#(B_2) \leq v, \varrho(B_1, B_2) \geq k\} \end{aligned} \tag{4.1}$$

with $\varrho(B_1, B_2) = \inf\{||x - y|| : x \in B_1, y \in B_2\}$. If $\lim_{k\to\infty} \beta_U(k; u, v) = 0$ for all integers $u, v \in \mathbb{N}$, then we say U is β-`mixing` (and define α–mixing analogously). Any m–dependent random field is obviously β–mixing, since $\beta_U(k; u, v) = 0$ for all $k \geq m$. Without restricting the cardinality of the sets B_1, B_2 in (4.1), that is to say, letting $u = v = \infty$, β–mixing is equivalent to m–dependence.

Lemma 4.1 *If* $\beta_U(k; \infty, \infty) \to 0$ *as* $k \to \infty$*, then the stationary random field* $U = \{U(z), z \in \mathbb{Z}^d\}$ *is* m*–dependent for some* $m \in \mathbb{N}$*.*

Confer Theorem 2 in [9, p. 17] for the *proof* of the stated relationship. □

For an m–dependent stationary random field $\{U(z), z \in \mathbb{Z}^d\}$, Theorem 3.1 gives asymptotic normality of $S_n = (\#I_n)^{-\frac{1}{2}} \sum_{z \in I_n} U(z)$, where $I_n = \{-n, \ldots, n\}^d$.

Lemma 4.2 *Let* $U = \{U(z), z \in \mathbb{Z}^d\}$ *be an* m*–dependent, stationary random field on* $\mathbb{Z}^d$ *with* $\mathbb{E}\,U(o) = 0$ *and* $\mathbb{E}\,U^2(o) < \infty$*. Then it holds that*

$$S_n \;=\; (\#I_n)^{-\frac{1}{2}} \sum_{z \in I_n} U(z) \;\xrightarrow{D}\; \mathcal{N}(0, \sigma^2)$$

and the asymptotic variance is given by $\sigma^2 = \sum_{z \in \mathbb{Z}^d : ||z|| \leq m} \mathbb{E}\,(U(o)U(z))$*.*

Proof Put $U_n(z) = (\#I_n)^{-1/2}\, U(z)$ for each $z \in \mathbb{Z}^d$ and $n \geq 1$. Then $\{U_n(z), z \in \mathbb{Z}^d\}$, $n \geq 1$ is a sequence of m–dependent random fields with mean zero and $S_n = \sum_{z \in I_n} U_n(z)$ for any $n \geq 1$ as in Theorem 3.1. By stationarity, we have $\sum_{z \in I_n} \mathbb{E}\, U_n^2(z) = \mathbb{E}\, U^2(o)$ for all $n \geq 1$ and $\lim_{n\to\infty} L_n(\varepsilon) = \lim_{n\to\infty} m^{2d}\, \mathbb{E}(U^2(o) \cdot \mathbb{1}(|U(o)| > \sqrt{\#I_n}\, \varepsilon\, m^{-2d})) = 0$. Next, observe that $\mathbb{E}(U_n(z)U_n(y)) = 0$ for all $z, y \in \mathbb{Z}^d$ with $||z - y|| > m$ by the definition of m–dependence and by the fact that $\mathbb{E}\, U_n(z) = 0$ for all $z \in \mathbb{Z}^d$. With $I_m = \{z \in \mathbb{Z}^d : ||z|| \leq m\}$ we get

$$
\begin{aligned}
&\big|\, \mathrm{Var}\,(S_n) - \sigma^2 \,\big| \\
&= \Big| \frac{1}{\#I_n} \sum_{z \in I_n} \sum_{y \in I_n} \mathbb{E}\,(U_n(z)U_n(y)) - \sum_{z' \in I_m} \mathbb{E}(U_n(o)U_n(z'))\Big| \\
&= \frac{1}{\#I_n} \Big| \sum_{z \in I_n \setminus \{o\}} \Big(\sum_{y \in I_n} \mathbb{E}\,(U_n(o)U_n(y-z)) - \sum_{z' \in I_m} \mathbb{E}\,(U_n(o)U_n(z'))\Big)\Big| \\
&= \frac{1}{\#I_n} \Big| \sum_{z \in I_n \setminus \{o\}} \Big(\sum_{z' \in I_m \,:\, z'+z \in I_n} \mathbb{E}\,(U_n(o)U_n(z')) - \sum_{z' \in I_m} \mathbb{E}\,(U_n(o)U_n(z'))\Big)\Big| \\
&\leq \frac{1}{\#I_n} \sum_{z \in I_n \setminus \{o\}} \sum_{z' \in I_m \,:\, z'+z \notin I_n} \big|\, \mathbb{E}\,(U(o)U(z'))\,\big| \\
&\leq \frac{1}{\#I_n} \sum_{z' \in I_m} \big(n^d - \prod_{j=1}^{d} (n - z_j')\big)\, \big|\, \mathbb{E}\,(U(o)U(z'))\,\big| \\
&\leq \frac{(\#I_m)\,(m+1)^d\, n^{d-1}}{(2n+1)^d}\, \mathbb{E}\big|U^2(o)\big|\,.
\end{aligned}
$$

The final bound tends to zero as $n \to \infty$. In the second inequality we used the fact that the number of $z \in I_n$ such that $z' \in I_n - z$ is equal to $\prod_{j=1}^{d}(n - z_j')$. For the third estimate, note that $|z_j'| \leq m$, $j = 1, \ldots, d$, $\#I_n = (2n+1)^d$ and

$$
(n^d - \prod_{j=1}^{d} (n - z_j')) \;\leq\; \sum_{k=1}^{d} \binom{d}{k} m^k n^{d-k} \;\leq\; (m+1)^d\, n^{d-1}\,.
$$

□

As we have seen in Chapter 3, m–dependence is a quite strong assumption. Even for Boolean models, boundedness of the grains had to be imposed in order to derive an m–dependent random field. Various central limit theorems for stationary random fields on $\mathbb{Z}^d$ require less restrictive assumptions with respect to the mixing properties, namely, considering $u < \infty$ and $v = \infty$ (i.e., only one of the sets in (4.1) may be unbounded). In return, higher moment conditions are needed. In the present context, we especially refer to [5, Theorem 1], [9, Section 1.5], [21, Theorem 6.1] and [52, Chapter 7] on this issue.

The following theorem is going to build the basis for the proof of asymptotic normality of $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$, $n \geq 1$. For this, let $U = \{U(z), z \in \mathbb{Z}^d\}$ be a stationary random field with mean zero and $\mathbb{E}(|U(o)|^{2+\delta}) < \infty$ for some $\delta > 0$. Furthermore, define $\sigma(U_{[k]}) = \sigma(U(z), ||z|| < k)$ and $\sigma(U_{[k]}^c) = \sigma(U(z), ||z|| \geq k)$ for $k \in \mathbb{N}$. Regarding

the mixing properties of U, we assume that there exist non–negative functions β_U^* and β_U^{**} such that

$$\beta\big(\sigma\big(U_{[p+1]}\big), \sigma\big(U_{[p+q]}^c\big)\big) \;\leq\; \begin{cases} \beta_U^*(q)\,, & p=0, q\geq 1, \\ p^{d-1}\beta_U^{**}(q)\,, & 1\leq q\leq p \end{cases} \tag{4.2}$$

for any $p, q \in \mathbb{N}$.

Theorem 4.1 *Let U, β_U^* and β_U^{**} be given as above. For any $n \geq 1$, set $S_n = (\#I_n)^{-1/2} \sum_{z\in I_n} U(z)$ with $I_n = \{-n, \ldots, n\}^d$. Under the condition that*

$$\sum_{q\geq 1} q^{d-1} \big(\beta_U^*(q)\big)^{\delta/(2+\delta)} \;<\; \infty \tag{4.3}$$

is satisfied, we have $\sum_{z\in\mathbb{Z}^d} |\mathbb{E}(U(o)\,U(z))| < \infty$ and $\operatorname{Var} S_n \to \sigma^2$ as $n\to\infty$, where $\sigma^2 = \sum_{z\in\mathbb{Z}^d} \mathbb{E}(U(o)\,U(z))$. If, in addition,

$$q^{2d-1}\beta_U^{**}(q) \;\to\; 0\,, \qquad q\to\infty, \tag{4.4}$$

then S_n converges in distribution as $n\to\infty$ to a Gaussian random variable with mean zero and variance σ^2.

Proof A detailed proof of Theorem 4.1 is given in [21, pp. 21–23], but we shortly sketch the idea here as well. In the first step, one shows that $\lim_{n\to\infty} \operatorname{Var}(S_n) = \sigma^2$ with $\sigma^2 < \infty$, provided that (4.3) holds together with $\mathbb{E}\,(|U(o)|^{2+\delta}) < \infty$. The absolute convergence of the series σ^2 can be deduced from (4.3) as follows. We have

$$\begin{aligned} \sum_{z\in\mathbb{Z}^d} |\mathbb{E}(U(o)\,U(z))| \;&\leq\; c_{\delta,U} \sum_{z\in\mathbb{Z}^d} \beta^{\delta/(2+\delta)}\big(\sigma(U(o)), \sigma(U(z))\big) \\ &=\; c_{\delta,U} \sum_{q=1}^{\infty} \sum_{z\in\mathbb{Z}^d\,:\,||z||=q} \beta^{\delta/(2+\delta)}\big(\sigma(U(o)), \sigma(U(z))\big) \\ &\leq\; c_{\delta,U} \sum_{q=1}^{\infty} 2^{2d-1} d\, q^{d-1}\, \beta^{\delta/(2+\delta)}\big(\sigma\big(U_{[1]}\big), \sigma\big(U_{[q]}^c\big)\big) \\ &\leq\; c_{\delta,U}\, 2^{2d-1} d \sum_{q=1}^{\infty} q^{d-1} \big(\beta^*(q)\big)^{\delta/(2+\delta)} \;<\; \infty\,, \end{aligned}$$

where $c_{\delta,U} = 8\,\big(\mathbb{E}\,(|U(o)|^{2+\delta})\big)^{\frac{2}{2+\delta}} < \infty$. By Theorem 3 in [9] it holds that $|\mathbb{E}(U(o)\,U(z))| \leq c_{\delta,U}\,\beta(\sigma(U(o)), \sigma(U(z)))^{\delta/(2+\delta)}$. Besides this, we have $\sigma(U(z)) \subset \sigma(U_{[q]}^c)$ for any $z\in\mathbb{Z}^d$ with $||z|| = q$, and $\#\{z\in\mathbb{Z}^d : ||z|| = q\} = (2q+1)^d - (2q-1)^d \leq 2^{2d-1} d\, q^{d-1}$ for all $q\geq 1$. The remainder is shown in [21] (see also [52, Chapter 3]).

Next, the random variables $U(z)$ are approximated by truncated versions $U_{a_n}(z) = U(z)\mathbb{1}(|U(z)| \leq a_n) - \mathbb{E}(U(z)\mathbb{1}(|U(z)| \leq a_n))$, where the sequence a_n is chosen such that $\lim_{n\to\infty} \mathbb{E}(S_n - S_{a_n})^2 = 0$ for $S_{a_n} = (\#I_n)^{-1/2} \sum_{z\in I_n} U_{a_n}(z)$. This has

the advantage that a multivariate analog of Bernstein's sectioning technique can be applied to S_{a_n}. Namely, S_{a_n} is split into two sums, say S_{a_n,J_n} and $S_{a_n,I_n\setminus J_n}$, where $\lim_{n\to\infty}\mathbb{E}\,(S^2_{a_n,I_n\setminus J_n}) = 0$ if (4.3) holds, and $S_{a_n,J_n} \xrightarrow{D} \mathcal{N}(0,\sigma^2)$ under condition (4.4). The sectioning is done in the following way. In each coordinate, blocks of $p(n)$ consecutive indices ("long' segments) are separated by blocks of $q(n)$ consecutive indices ("short" segments), where $p(n) \gg q(n)$. Then J_n contains all $z = (z_1,\ldots,z_d)^\top \in I_n$ such that, for $k = 1,\ldots,d$, coordinate z_k belongs to a "long" segment, and each $z \in I_n \setminus J_n$ has at least one coordinate that is contained in a "short" segment. The block sizes $p(n)$ and $q(n)$ are chosen in such a way that $S_{a_n,I_n\setminus J_n}$ is negligible if $q(n)$ is small in comparison to $p(n)$, and if $q(n)$ is large enough, then S_{a_n,J_n} is approximately a sum of independent blocks. An application of Berry–Esseen's inequality completes the proof. □

For uniformly bounded random fields, condition (4.3) can be weakened.

Lemma 4.3 *Let the setup of Theorem* 4.1 *be given, where we assume, in addition, that* $|U(o)|$ *is bounded almost surely. Then it holds that* $S_n \xrightarrow{D} \mathcal{N}(0,\sigma^2)$, *where* $\sigma^2 = \sum_{z\in\mathbb{Z}^d}\mathbb{E}(U(o)\,U(z))$ *is absolutely convergent, if condition* (4.4) *is fulfilled and if one has*

$$\sum_{q\geq 1} q^{d-1}\beta^*_U(q) \;<\; \infty\,.$$

Proof The proof can be given in an analogous way as for Theorem 4.1. At this, note that $|\mathbb{E}(U(o)\,U(z))| \leq 4\,c_U\,\beta(\sigma(U(o)),\sigma(U(z)))$ holds for any $z \in \mathbb{Z}^d$ and some constant $c_U \in (0,\infty)$ such that $P(|U(o)| \leq c_U) = 1$; see Lemma 3 in [9]. □

4.2 Coherence of mixing properties

Returning to the original problem, consider a stationary germ–grain model $\Xi = \bigcup_{l=1}^{\infty}(M_l + X_l)$ with typical grain $M_0 \in \mathcal{R}'$. Furthermore, suppose that the stationary random field Y defined in (2.15) has finite moments of order $2+\delta$ for some $\delta > 0$ and $\mu = \mathbb{E}\,Y(o)$. Let $W_n = [-n,n)^d$ with $n \in \mathbb{N}$ such that $n > ||K|| + 1$ and choose uniform weights $G(W_n,x) = \mathbb{1}(x \in W_n^K)/|W_n^K|$ for each $x \in \mathbb{R}^d$ and $n \geq 1$, where $W_n^K = [-n_K, n_K+1)^d$ with maximal $n_K \in \mathbb{N}$ satisfying $W_n^K \subseteq W_n \ominus \check{K}$. Note that $n_K \geq \lfloor n - ||K|| \rfloor - 1$, which yields $\lim_{n\to\infty} n_K = \infty$.

Now, we introduce the auxiliary random field $U = \{U(z), z \in \mathbb{Z}^d\}$ by

$$U(z) \;=\; \int_{[z,z+e)} \big(Y(x) - \mu\big)\,dx\,, \qquad z \in \mathbb{Z}^d. \tag{4.5}$$

Then one finds that U is stationary with $\mathbb{E}\,U(o) = 0$.

For the estimator $\widehat{\mu}_n = \int_{W_n} Y(x)G(W_n,x)dx$ of μ, it holds that

$$\sqrt{|W_n|}\big(\widehat{\mu}_n - \mu\big) \;=\; \frac{\sqrt{|W_n|}}{\sqrt{|W_n^K|}}\,(\#I_n)^{-1/2}\sum_{z\in I_n} U(z), \qquad I_n = \{-n_K,\ldots,n_K\}^d, \tag{4.6}$$

putting $I_n = \{-n_K, \ldots, n_K\}^d$ and observing that $\#I_n = |W_n^K|$. Moreover, we have $\lim_{n\to\infty} |W_n^K|/|W_n| = 1$, since $|W_n \ominus B_{||K||+1}(o)| \le |W_n^K|$ for all $n \in \mathbb{N}$ large enough. Consequently, one can neglect the pre–factor on the right hand side of (4.6) when analyzing the asymptotic properties of $S_n^* = \sqrt{|W_n|}\big(\widehat{\mu}_n - \mu\big)$, $n \ge 1$.

The sequence of observation windows W_n, $n \ge 1$ and weighting function G may be chosen more generally than stated above. Yet the present choice simplifies notation. In contrast to Chapter 3, however, a certain uniformity in G is mandatory here in order to be able to construct a random field U of the form (4.5) that is, in particular, stationary. Assume, for instance, that $G(W_n, x) = G_o([o,e), x-z)\,\mathbb{1}(x \in W_n^K)$ for $z \in \mathbb{Z}^d$ such that $x \in [z, z+e)$ and for some feasible weighting function G_0 satisfying (3.4) with respect to $W = [o,e)$. Then one can also use the random field U defined by $U(z) = \int_{[z,z+e)} (Y(x) - \mu)\, G_o([0,e), x-z)\, dx$, $z \in \mathbb{Z}^d$.

Equations (4.2)–(4.4) specify sufficient conditions for asymptotic normality of S_n^* in terms of β–mixing conditions on random field $\{U(z), z \in \mathbb{Z}^d\}$ defined in (4.5). The aim of the subsequent considerations is to relate (4.2)–(4.4) to conditions imposed on the grain distribution and on the mixing behavior of underlying germ process $X = \{X_l\}$. To this end, let $\sigma(X_{|B}), B \in \mathfrak{B}^d$ represent the σ–algebra generated by the events $\{X(B') = k\}$, $k \in \mathbb{N}$, $B' \in \mathfrak{B}_0^d$, $B' \subseteq B$. The concept employed below follows the ideas presented in Section 5 of [23] and Section 3 of [38].

Lemma 4.4 *For any $p \ge 0$ and $q \ge 1$, put*

$$F = \big[-p-1, p+1\big)^d, \qquad G = \mathbb{R}^d \setminus \big[-p-q, p+q\big)^d$$

and

$$\widetilde{F} = \big[-p-\frac{q+3}{4}, p+\frac{q+3}{4}\big)^d, \qquad \widetilde{G} = \mathbb{R}^d \setminus \big[-p-\frac{3q+1}{4}, p+\frac{3q+1}{4}\big)^d$$

so that $F \subset \widetilde{F}$, $G \subset \widetilde{G}$ and $\widetilde{F} \cap \widetilde{G} = \emptyset$. Then it holds that

$$\begin{aligned}&\beta\big(\sigma\big(U_{[p+1]}\big), \sigma\big(U_{[p+q]}^c\big)\big)\\ &\quad\le\ \beta\big(\sigma(X_{|\widetilde{F}}), \sigma(X_{|\widetilde{G}})\big) + 2P\big(\Xi_{|\widetilde{F}^c} \cap (F \oplus K) \ne \emptyset\big) + 2P\big(\Xi_{|\widetilde{G}^c} \cap (G \oplus K) \ne \emptyset\big)\,.\end{aligned}$$

If, in particular, X is a stationary Poisson process, then one has

$$\beta\big(\sigma\big(U_{[p+1]}\big), \sigma\big(U_{[p+q]}^c\big)\big) \ \le\ 2P\big(\Xi_{|\widetilde{F}^c} \cap (F \oplus K) \ne \emptyset\big) + 2P\big(\Xi_{|\widetilde{G}^c} \cap (G \oplus K) \ne \emptyset\big)\,.$$

Proof To start with, we introduce an auxiliary functional $\eta : \mathfrak{B}_0^d \to \mathbb{R}$ by

$$\eta(B) \ = \ \int_B (Y(x) - \mu)\, dx, \qquad B \in \mathfrak{B}_0^d$$

and consider the σ-algebras $\sigma_\eta(B') = \sigma\big(\{\eta(B),\ B \subseteq B', B \in \mathfrak{B}_0^d\}\big), B' \in \mathfrak{B}^d$. By augmentation of the involved σ–algebras, one has

$$\begin{aligned}\beta\big(\sigma\big(U_{[p+1]}\big), \sigma\big(U_{[p+q]}^c\big)\big) \ &= \ \beta\big(\sigma\big(U(z), ||z|| \le p\big), \sigma\big(U(z), ||z|| \ge p+q\big)\big)\\ &\le \ \beta\big(\sigma_\eta\big([-p-1, p+1)^d\big), \sigma_\eta\big(\mathbb{R}^d \setminus [-p-q, p+q)^d\big)\big)\end{aligned}$$

for any $p \geq 0$ and $q \geq 1$. Following the lines of the proof on Lemma 5.1 in [23], we find that

$$\begin{aligned} &\beta\big(\sigma_\eta(F), \sigma_\eta(G)\big) \qquad (4.7)\\ &\quad \leq \quad \beta\big(\sigma(X_{|\widetilde{F}}), \sigma(X_{|\widetilde{G}})\big) + 2P\big(\Xi_{|\widetilde{F}^c} \cap (F \oplus K) \neq \emptyset\big) + 2P\big(\Xi_{|\widetilde{G}^c} \cap (G \oplus K) \neq \emptyset\big)\,. \end{aligned}$$

We shortly sketch the argument here, because of a differing definition of functional η, and refer to [23, pp. 293–294] regarding the details. Assume for the moment that $G, \widetilde{G} \in \mathfrak{B}_0^d$ and consider the following two partitions of Ω respectively into the sets

$$A_k \quad = \quad \{\eta(F_1) \in I_{k_1}^{(1)}, \ldots, \eta(F_a) \in I_{k_a}^{(a)}\}\,, \qquad k = (k_1, \ldots, k_a), \qquad (4.8)$$

where $k_i = 1, \ldots, m_i$, $m_i \in \mathbb{N}$ for $i = 1, \ldots, a$, $a \in \mathbb{N}$, and into

$$B_\ell \quad = \quad \{\eta(G_1) \in J_{\ell_1}^{(1)}, \ldots, \eta(G_b) \in J_{\ell_b}^{(b)}\}\,, \qquad \ell = (\ell_1, \ldots, \ell_b), \qquad (4.9)$$

where $\ell_j = 1, \ldots, n_j$, $n_j \in \mathbb{N}$ for $j = 1, \ldots, b$, $b \in \mathbb{N}$. The sets $I_1^{(i)}, \ldots, I_{m_i}^{(i)}$ ($J_1^{(j)}, \ldots, J_{n_j}^{(j)}$) are disjoint intervals spanning $\mathbb{R}$ for $i = 1 \ldots, a$ ($j = 1 \ldots, b$), respectively, and $F_1, \ldots, F_a \in \mathfrak{B}_0^d$ ($G_1, \ldots, G_b \in \mathfrak{B}_0^d$) represent an arbitrary partition of F (of G). Then

$$\beta\big(\sigma_\eta(F), \sigma_\eta(G)\big) \;=\; \frac{1}{2} \sup \sum_k \sum_\ell \big|P(A_k \cap B_\ell) - P(A_k)P(B_\ell)\big|\,,$$

where the supremum is taken over all partitions $\{A_k\}$ and $\{B_\ell\}$ as described above.

Next, put $\eta_{|\widetilde{F}}(B) = \int_B \varphi((\Xi_{|\widetilde{F}} - x) \cap K)\, dx$, $B \in \mathfrak{B}_0^d$ and define $\eta_{|\widetilde{G}}(B)$ analogously. Replace η by $\eta_{|\widetilde{F}}$ in (4.8) and by $\eta_{|\widetilde{G}}$ in (4.9) and denote the resulting sets by $\widetilde{A}_k$ and $\widetilde{B}_\ell$, respectively. For any $F' \in \mathfrak{B}_0^d$ with $F' \subseteq F$, we have $\eta_{|\widetilde{F}}(F') = \eta(F')$ whenever $\Xi_{|\widetilde{F}} \cap (F \oplus K) = \emptyset$, and likewise $\eta_{|\widetilde{G}}(G') = \eta(G')$ for any $G' \in \mathfrak{B}_0^d$ with $G' \subseteq G$ assuming that $\Xi_{|\widetilde{G}} \cap (G \oplus K) = \emptyset$. From this one deduces

$$\begin{aligned} &\sum_k \sum_\ell \big|P(A_k \cap B_\ell) - P(A_k)P(B_\ell) - P(\widetilde{A}_k \cap \widetilde{B}_\ell) - P(\widetilde{A}_k)P(\widetilde{B}_\ell)\big| \\ &\qquad \leq \; 4\,P\big(\Xi_{|\widetilde{F}^c} \cap (F \oplus K) \neq \emptyset\big) + 4\,P\big(\Xi_{|\widetilde{G}^c} \cap (G \oplus K) \neq \emptyset\big)\,. \end{aligned} \qquad (4.10)$$

We point out that in contrast to the proof of Lemma 5.1 in [23] we need to include the Minkowski addition of K on the right hand side of (4.10). Moreover, one has

$$\sum_k \sum_\ell \big|P(\widetilde{A}_k \cap \widetilde{B}_\ell) - P(\widetilde{A}_k)P(\widetilde{B}_\ell)\big| \quad \leq \quad \beta\big(\sigma(X_{|\widetilde{F}}), \sigma(X_{|\widetilde{G}})\big) \qquad (4.11)$$

due to formula (6.4) in [21] and conditional independence of $\widetilde{A}_k$ and $\widetilde{B}_\ell$ given $X_{|\widetilde{F}}, X_{|\widetilde{G}}$ (recall that we assume independent marking). Combining (4.10), (4.11) with the triangle inequality, we obtain (4.7) for bounded sets $F, \widetilde{F}, G$ and $\widetilde{G} \in \mathfrak{B}_0^d$. An approximation of G and $\widetilde{G}$ by bounded Borel sets completes the proof for arbitrary germ processes X. If X is a Poisson process on $\mathbb{R}^d$, then we additionally observe that $\beta(\sigma(X_{|\widetilde{F}}), \sigma(X_{|\widetilde{G}})) = 0$ by independence of $X_{|\widetilde{F}}$ and $X_{|\widetilde{G}}$. $\square$

Lemma 4.5 *Assume that* $\mathbb{E}\big(||M_0 \oplus \check{K}||^d\big) < \infty$ *and let* $F \subset \widetilde{F}$ *and* $G \subset \widetilde{G}$ *be the sets introduced in Lemma* 4.4. *For* $p \geq 0$ *and* $q \geq 5$, *it holds that*

$$\begin{aligned} &2\,P\big(\Xi_{|\widetilde{F}^c} \cap (F \oplus K) \neq \emptyset\big) + 2\,P\big(\Xi_{|\widetilde{G}^c} \cap (G \oplus K) \neq \emptyset\big) \\ &\leq \lambda\, d\, 2^{3d} \left(\frac{p+1}{\min\big\{p+1, \frac{q-1}{4}\big\}}\right)^{d-1} \mathbb{E}\left(||M_0 \oplus \check{K}||^d\, \mathbb{1}\big(||M_0 \oplus \check{K}|| > \frac{q-1}{4}\big)\right). \end{aligned}$$

Proof To simplify notation, put $a = p+1$ and $\Delta = (q-1)/4$ so that $a, \Delta \geq 1$. An application of Campbell's formula (2.9) yields

$$\begin{aligned} P\big(\Xi_{|\widetilde{F}^c} \cap (F \oplus K) \neq \emptyset\big) &= \mathbb{E}\Big(\mathbb{1}\big(\bigcup_{l:\, X_l \in \widetilde{F}^c} (M_l + X_l) \cap (F \oplus K) \neq \emptyset\big)\Big) \\ &\leq \mathbb{E} \sum_{l:\, X_l \in \widetilde{F}^c} \mathbb{1}\big((M_l + X_l) \cap (F \oplus K) \neq \emptyset\big) \\ &= \lambda\, \mathbb{E} \int_{\mathbb{R}^d} \mathbb{1}\big(y \in \widetilde{F}^c, (M_0 + y) \cap (F \oplus K) \neq \emptyset\big)\, dy \\ &= \lambda\, \mathbb{E}\big|((M_0 \oplus \check{K}) \oplus (-a, a]^d) \setminus (-a - \Delta, a + \Delta]^d\big|. \end{aligned}$$

Let $F_{||M_0 \oplus \check{K}||}$ be the distribution function of random variable $||M_0 \oplus \check{K}||$. Observing that $(M_0 \oplus \check{K}) \oplus (-a, a]^d \subseteq (-a - ||M_0 \oplus \check{K}||, a + ||M_0 \oplus \check{K}||\,]^d$, we get

$$\begin{aligned} &P\big(\Xi_{|\widetilde{F}^c} \cap (F \oplus K) \neq \emptyset\big) \\ &\leq \lambda\, \mathbb{E}\Big(\big|\big(-a - ||M_0 \oplus \check{K}||, a + ||M_0 \oplus \check{K}||\,\big]^d \setminus (-a - \Delta, a + \Delta]^d\big|\Big) \\ &= \lambda \int_\Delta^\infty \big|(-a - v, a + v]^d \setminus (-a - \Delta, a + \Delta]^d\big|\, dF_{||M_0 \oplus \check{K}||}(v) \\ &= \lambda\, 2^d \int_\Delta^\infty \big((a+v)^d - (a+\Delta)^d\big)\, dF_{||M_0 \oplus \check{K}||}(v) \\ &\leq \lambda\, d\, 2^d \int_\Delta^\infty \Big(v\,(a+v)^{d-1} - \Delta\,(a+\Delta)^{d-1}\Big)\, dF_{||M_0 \oplus \check{K}||}(v) \\ &< \lambda\, d\, 2^d \Big(1 + \frac{a}{\Delta}\Big)^{d-1} \int_\Delta^\infty v^d\, dF_{||M_0 \oplus \check{K}||}(v), \end{aligned}$$

since $\Delta > 0$ and $1 + a/v \leq 1 + a/\Delta$ in the final line. The last but one inequality results from the binomial identity, keeping in mind that $v \geq \Delta$, $\Delta \geq 1$. That is, more precisely,

$$\begin{aligned} (a+v)^d - (a+\Delta)^d &= \sum_{k=0}^{d} \binom{d}{k} a^k \big(v^{d-k} - \Delta^{d-k}\big) \\ &= \sum_{k=0}^{d-1} \binom{d-1}{k} a^k \Big(v \cdot v^{d-1-k} - \Delta \cdot \Delta^{d-1-k}\Big)\, \frac{d}{(d-k)} \\ &\leq d\,v\,(a+v)^{d-1} - d\,\Delta\,(a+\Delta)^{d-1}, \end{aligned}$$

where $v^{d-k} - \Delta^{d-k} \geq 0$. Differentiating the cases $a \leq \Delta$ and $a > \Delta$ gives

$$P\big(\Xi_{|\widetilde{F}^c} \cap (F \oplus K) \neq \emptyset\big) \;\leq\; \lambda\, d\, 2^{2d-1} \left(\frac{a}{\min\{a,\Delta\}}\right)^{d-1} \int_{\Delta}^{\infty} v^d \, dF_{||M_0 \oplus \check{K}||}(v)\,.$$

Likewise, put $b = p + q$ and find that

$$\begin{aligned} & P\big(\Xi_{|\widetilde{G}^c} \cap (G \oplus K) \neq \emptyset\big) \\ & \quad \leq \;\; \lambda\, \mathbb{E}\Big(\big|(-b+\Delta, b-\Delta]^d \setminus (-b+||M_0 \oplus \check{K}||, b-||M_0 \oplus \check{K}||\,]^d\big|\Big) \\ & \quad \leq \;\; \lambda\, d\, 2^d \int_{\Delta}^{b} \Big(v\,(b-v)^{d-1} - \Delta\,(b-\Delta)^{d-1}\Big)\, dF_{||M_0 \oplus \check{K}||}(v) \\ & \quad < \;\; \lambda\, d\, 2^d \Big(\frac{b}{\Delta} - 1\Big)^{d-1} \int_{\Delta}^{\infty} v^d \, dF_{||M_0 \oplus \check{K}||}(v)\,. \end{aligned}$$

In the first inequality, we made use of the fact that $F_1^c \oplus F_2 = (F_1 \ominus F_2)^c$ for any $F_1, F_2 \in \mathcal{F}$ (see [53, p. 19]), where in our case $F_1^c = G$ and $F_2 = M_0 \oplus \check{K}$ (by realization). Since $b = a + 4\Delta$ and thus $b/\Delta - 1 \leq 4\,a/\min\{a,\Delta\}$, we finally conclude

$$P\big(\Xi_{|\widetilde{G}^c} \cap (G \oplus K) \neq \emptyset\big) \;\leq\; \lambda\, d\, 2^{3d-2} \left(\frac{a}{\min\{a,\Delta\}}\right)^{d-1} \int_{\Delta}^{\infty} v^d \, dF_{||M_0 \oplus \check{K}||}(v)\,. \qquad \square$$

To give a closed form expression of the asymptotic variance $\sigma^2 = \sum_{z \in \mathbb{Z}^d} \mathbb{E}\,(U(o)U(z))$ of $S_n = (\#I_n)^{-1/2} \sum_{z \in I_n} U(z)$, we will use absolute integrability of $\mathrm{Cov}_Y(x), x \in \mathbb{R}^d$; see the upcoming Theorem 4.2. A helpful estimate with respect to this issue is provided by the following lemma.

Lemma 4.6 *Set $a = ||K|| + 1$ and let $|x| > 8a$. If $\mathbb{E}\big(|Y(o)|^{2+\delta}\big) < \infty$ for some $\delta > 0$, then one has*

$$\begin{aligned} |\,\mathrm{Cov}_Y(x)\,| \;\; \leq \;\; & 16\,\big(\mathbb{E}\big(|Y(o)|^{2+\delta}\big)\big)^{\frac{2}{2+\delta}} \Big(\beta\big(\sigma(X_{|B_1}), \sigma(X_{|B_2})\big) \\ & + P(\Xi_{|B_1^c} \cap A_1 \neq \emptyset) + P(\Xi_{|B_2^c} \cap A_2 \neq \emptyset)\Big)^{\delta/(2+\delta)} \end{aligned}$$

for $A_1 = [-a,a)^d$, $A_2 = \mathbb{R}^d \setminus [-|x|+3a, |x|-3a)^d$ and $B_1 = [-\frac{1}{4}|x|, \frac{1}{4}|x|)^d$, $B_2 = \mathbb{R}^d \setminus [-\frac{3}{4}|x|+2a, \frac{3}{4}|x|-2a)^d$. If, in addition, $\mathbb{E}\,(||M_0||^d) < \infty$, then it holds

$$P(\Xi_{|B_1^c} \cap A_1 \neq \emptyset) + P(\Xi_{|B_2^c} \cap A_2 \neq \emptyset) \;\; \leq \;\; \lambda\, d\, 2^{3d}\, \mathbb{E}\Big(||M_0||^d\, \mathbb{1}\big(||M_0|| > \frac{1}{4}|x| + a\big)\Big)\,.$$

Proof For all $x \in \mathbb{R}^d$, one finds that

$$\begin{aligned} |\,\mathrm{Cov}_Y(x)\,| \;\; & \leq \;\; c_{\delta,Y}\, \alpha^{\delta/(2+\delta)}\big(\sigma(Y(o)), \sigma(Y(x))\big) \\ & \leq \;\; c_{\delta,Y}\, \alpha^{\delta/(2+\delta)}\big(\sigma(\Xi \cap K), \sigma(\Xi \cap (K+x))\big) \end{aligned}$$

with $c_{\delta,Y} = 8\left(\mathbb{E}\left(|Y(o)|^{2+\delta}\right)\right)^{\frac{2}{2+\delta}} < \infty$. In the first inequality, we employed once again Theorem 3 of [9]. Next, apply Lemma 3.1 of [38] for all $x \in B_{8a}^c(o)$ to get

$$\alpha\big(\sigma(\Xi \cap K), \sigma(\Xi \cap (K+x))\big) \;\leq\; 4\,\alpha\big(\sigma(X_{|B_1}), \sigma(X_{|B_2})\big) + 2\sum_{i=1}^{2} P(\Xi_{|B_i^c} \cap A_i \neq \emptyset)\,,$$

where $K \subset A_1 \subset B_1$, $(K+x) \subset A_2 \subset B_2$ and $B_1 \cap B_2 \neq \emptyset$. With regard to the first summand, we have $2\alpha(\sigma(X_{|B_1}), \sigma(X_{|B_2})) \leq \beta(\sigma(X_{|B_1}), \sigma(X_{|B_2}))$ as mentioned before. The first assertion of the lemma is therefore verified, since $\delta/(2+\delta) \leq 1$ and thus $2^{\delta/(2+\delta)} \leq 2$.

Now, assume that $\mathbb{E}\left(||M_0||^d\right) < \infty$ holds. Putting $b = |x| - 3a$ and $\Delta = |x|/4 - a$, one derives just as in the proof of Lemma 4.5 that

$$\begin{aligned} P\big(\Xi_{|B_1^c} \cap A_1 \neq \emptyset\big) &= P\big(\Xi_{|\mathbb{R}^d \setminus [-a-\Delta, a+\Delta)^d} \cap [-a,a)^d \neq \emptyset\big) \\ &\leq \lambda\, d\, 2^d \left(1 + \frac{a}{\Delta}\right)^{d-1} \mathbb{E}\big(||M_0||^d\, \mathbb{1}\big(||M_0|| > \Delta\big)\big)\,. \end{aligned}$$

Since $a/\Delta = 4a/(|x| - 4a) \leq 1$ for all $x \in \mathbb{R}^d$ with $|x| > 8a$, we conclude that

$$P\big(\Xi_{|B_1^c} \cap A_1 \neq \emptyset\big) \;\leq\; \lambda\, d\, 2^{2d-1}\, \mathbb{E}\big(||M_0||^d\, \mathbb{1}\big(||M_0|| > \Delta\big)\big)\,.$$

Analogously, it follows that

$$\begin{aligned} P\big(\Xi_{|B_2^c} \cap A_2 \neq \emptyset\big) &= P\big(\Xi_{|[-b+\Delta, b-\Delta)^d} \cap [-b,b)^d \neq \emptyset\big) \\ &< \lambda\, d\, 2^d \left(\frac{b}{\Delta} - 1\right)^{d-1} \mathbb{E}\big(||M_0||^d\, \mathbb{1}\big(||M_0|| > \Delta\big)\big) \\ &\leq \lambda\, d\, 2^{3d-1}\, \mathbb{E}\big(||M_0||^d\, \mathbb{1}\big(||M_0|| > \Delta\big)\big)\,, \end{aligned}$$

which completes the proof. □

4.3 Asymptotic normality under mixing conditions

Consider a germ–grain model $\Xi = \bigcup_{l=1}^{\infty} (M_l + X_l)$ with underlying stationary point process $X = \{X_l\}$ and typical grain $M_0 \in \mathcal{R}'$. Moreover, let W_n, W_n^K and G be given as in Section 4.2. Combining the results presented so far, a central limit theorem for $S_n^* = \sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ can be formulated in terms of the moment conditions on $Y(o)$ and $||M_0||$ and mixing conditions imposed on X. Several examples that meet the requirements stated below are discussed subsequently to the theorem.

Theorem 4.2 *Suppose that* $\mathbb{E}(|Y(o)|^{2+\delta}) < \infty$ *and* $\mathbb{E}\big(||M_0||^{2d(\frac{\delta+1}{\delta})+\varepsilon}\big) < \infty$ *for some* $\delta > 0$, $\varepsilon > 0$. *Furthermore, assume that there exists a non–increasing function* $\beta_X : [1,\infty) \to [0,\infty)$ *such that*

$$\beta\big(\sigma\big(X_{|[-\tilde{a},\tilde{a}]^d}\big), \sigma\big(X_{|\mathbb{R}^d \setminus [-\tilde{a}-\tilde{\Delta}, \tilde{a}+\tilde{\Delta}]^d}\big)\big) \;\leq\; \beta_X(\tilde{\Delta}) \left(\frac{\tilde{a}}{\min\{\tilde{a}, \tilde{\Delta}\}}\right)^{d-1} \qquad (4.12)$$

for all $\tilde{a},\ \tilde{\Delta} \geq 1$. *If it holds that*

$$\sum_{r=1}^{\infty} r^{d-1} \beta_X^{\delta/(2+\delta)}(r) \ < \ \infty, \tag{4.13}$$

then

$$\sqrt{|W_n|} \int_{W_n^K} \frac{(Y(x)-\mu)}{|W_n^K|}\, dx \ \xrightarrow{D} \ N(0,\sigma^2)\,, \qquad n \to \infty$$

with $\sigma^2 = \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx < \infty$.

Proof With regard to the asymptotic distribution of S_n^*, it suffices to verify the conditions of Theorem 4.1 for $S_n = (\#I_n)^{1/2} \sum_{z \in I_n} U(z)$, where $I_n = \{-n_K, \ldots, n_K\}$ and $\{U(z), z \in \mathbb{Z}^d\}$ defined in (4.5), due to representation formula (4.6).

First, note that $\mathbb{E}\,(|U(o)|^{2+\delta}) < \infty$ under the assumption that $\mathbb{E}(|Y(o)|^{2+\delta}) < \infty$ for some $\delta > 0$. This can derived as follows. Apply Jensen's inequality and Fubini's theorem to get

$$\mathbb{E}\big(|U(o)|^{2+\delta}\big) \ \leq \ \int_{[0,e)} \mathbb{E}\big(|Y(x)-\mu|^{2+\delta}\big)\, dx = \mathbb{E}\big(|Y(o)-\mu|^{2+\delta}\big)\,,$$

where $(\mathbb{E}\big(|Y(o)-\mu|^{2+\delta}\big))^{1/(2+\delta)} \leq (\mathbb{E}\big(|Y(o)|^{2+\delta}\big))^{1/(2+\delta)} + |\mu| < \infty$.

In the second step, we construct β_U^* and β_U^{**} (introduced in (4.2)) employing condition (4.13) and moments of $||M_0 \oplus \check{K}||$. By Lemma 4.5, we know that, for any $p \geq 0$ and $q \geq 1$,

$$\begin{aligned} &\beta\big(\sigma\big(U_{[p+1]}\big), \sigma\big(U^c_{[p+q]}\big)\big) \\ &\quad \leq \ \beta\big(\sigma(X_{|\tilde{F}}), \sigma(X_{|\tilde{G}})\big) + c_{d,\lambda} \left(\frac{a}{\min\{a,\Delta\}}\right)^{d-1} \int_{\Delta}^{\infty} v^d\, dF_{||M_0 \oplus \check{K}||}(v) \end{aligned}$$

putting $a = p+1$, $\Delta = (q-1)/4$ and $c_{d,\lambda} = \lambda\, d\, 2^{3d}$. Since the β–mixing coefficient is bounded by 1, we may assume that $q \geq 5$ without loss of generality. For $\tilde{a} = a + \Delta \geq 1$ and $\tilde{\Delta} = 2\Delta \geq 1$, condition (4.12) yields

$$\beta\big(\sigma\big(U_{[p+1]}\big), \sigma\big(U^c_{[p+q]}\big)\big) \ \leq \ \left(\frac{a}{\min\{a,\Delta\}}\right)^{d-1} \Big(\beta_X(2\Delta) + c_{d,\lambda} \int_{\Delta}^{\infty} v^d\, dF_{||M_0 \oplus \check{K}||}(v)\Big).$$

If $p = 0$ and $q \geq 5$, then $a = 1$ and $\Delta \geq 1$. Consequently, put

$$\beta_U^*(q) \ = \ \beta_X\Big(\frac{q-1}{2}\Big) + c_{d,\lambda} \int_{\frac{q-1}{4}}^{\infty} v^d\, dF_{||M_0 \oplus \check{K}||}(v)\,.$$

For any $1 < q \leq p$, one has $a = p+1 \geq (q-1)/4 = \Delta$. The second required function β_U^{**} is therefore given by

$$\beta_U^{**}(q) \ = \ \Big(\frac{4}{q-1}\Big)^{d-1} \beta_U^*(q)\,.$$

It remains to show that β^* and β^{**} meet conditions (4.3) and (4.4) of Theorem 4.1. As to (4.4), we find that

$$\begin{aligned}
\lim_{q\to\infty} q^{2d-1}\beta_U^{**}(q) &= \lim_{q\to\infty}\Big(q^d\beta_X\Big(\frac{q-1}{2}\Big) + c_{d,\lambda}\, q^d \int_{\frac{q-1}{4}}^{\infty} v^d\, dF_{||M_0\oplus\check{K}||}(v)\Big) \\
&\le 4^d \lim_{r\to\infty} r^d\beta_X(r) + c_{d,\lambda}\, 8^d \lim_{s\to\infty} s^d\, \mathbb{E}\big(||M_0\oplus\check{K}||^d\, \mathbb{1}(||M_0\oplus\check{K}|| > s)\big) \\
&\le 4^d \lim_{r\to\infty} \mathbb{E}\big(||M_0\oplus\check{K}||^{(2+\delta)/\delta}\, \mathbb{1}(||M_0\oplus\check{K}|| > r)\big) \lim_{r\to\infty} r^{(d-1)(2+\delta)/\delta}\beta_X(r) \\
&\quad + c_{d,\lambda}\, 8^d \lim_{s\to\infty} \mathbb{E}\big(||M_0\oplus\check{K}||^{2d}\, \mathbb{1}(||M_0\oplus\check{K}|| > s)\big) \\
&= 0\,.
\end{aligned}$$

In the first inequality, we substituted q by $r = \lfloor\frac{q-1}{2}\rfloor$ and $s = \lfloor\frac{q-1}{4}\rfloor$, respectively, employing the fact that β_X is non–increasing and that $q/r \le 4$ and $q/s \le 8$ for all $q \ge 5$. The final limits follow, on the one hand, from the dominated convergence theorem, since $\mathbb{E}(||M_0\oplus\check{K}||^\nu) \le \mathbb{E}((||M_0|| + ||K||)^\nu) < \infty$ for all $1 \le \nu \le 2d(\frac{1+\delta}{\delta})$ by Minkowski's inequality, and, on the other hand, from condition (4.13).

Regarding requirement (4.3), apply the same substitution to get

$$\begin{aligned}
\sum_{q\ge5} q^{d-1}\big(\beta_U^*(q)\big)^{\delta/(2+\delta)} &\le 4^d \sum_{r\ge1} r^{d-1}\big(\beta_X(r)\big)^{\delta/(2+\delta)} \\
&\quad + c_{d,\lambda}^{\delta/(2+\delta)}\, 8^d \sum_{s\ge1} s^{d-1}\bigg(\int_s^\infty v^d\, dF_{||M_0\oplus\check{K}||}(v)\bigg)^{\delta/(2+\delta)}
\end{aligned}$$

using $(a+b)^{\delta/(2+\delta)} \le 2\,a^{\delta/(2+\delta)} + 2\,b^{\delta/(2+\delta)}$ for any $a, b > 0$. The first summand on the right hand side of the above estimate is finite under the imposed mixing condition (4.13) on point process X. In order to ensure that the second series converges as well, put $\varepsilon' = \varepsilon\,\delta/(2+\delta)$, where $\varepsilon > 0$ is chosen such that $\mathbb{E}\big(||M_0||^{2d(\frac{\delta+1}{\delta})+\varepsilon}\big) < \infty$. Then all conditions Theorem 4.1 are satisfied, since it holds that

$$\begin{aligned}
&\sum_{s\ge1} s^{d-1}\bigg(\int_s^\infty v^d\, dF_{||M_0\oplus\check{K}||}(v)\bigg)^{\delta/(2+\delta)} \\
&\quad\le \sum_{s\ge1} s^{d-1}\mathbb{E}\Big(||M_0\oplus\check{K}||^d\, \mathbb{1}(||M_0\oplus\check{K}|| > s)\Big) \\
&\quad\le \Big[\mathbb{E}\Big(||M_0\oplus\check{K}||^{2d(\frac{\delta+1}{\delta})+\varepsilon}\Big)\Big]^{\delta/(2+\delta)} \sum_{s\ge1} s^{-(1+\varepsilon')} \;<\; \infty\,. \qquad (4.14)
\end{aligned}$$

In the last step, we analyze the variance of the asymptotic distribution. According to the first statement in Theorem 4.1 one has $\sigma^2 = \sum_{z\in\mathbb{Z}^d} \mathbb{E}(U(o)U(z))$, where the series converges absolutely. In fact, one has

$$\begin{aligned}
\sum_{z\in\mathbb{Z}^d} \mathbb{E}(U(o)\,U(z)) &= \sum_{z\in\mathbb{Z}^d} \int_{[z,z+e)} \int_{[o,e)} \mathbb{E}\big((Y(u)-\mu)(Y(v)-\mu)\big)\, du\, dv \\
&= \int_{\mathbb{R}^d} \int_{[o,e)} \mathrm{Cov}_Y(v-u)\, du\, dv\,.
\end{aligned}$$

Under the additional assumption that $\int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)|\,dx < \infty$, Fubini's theorem can be applied to deduce

$$\sigma^2 \;=\; \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\,dx\,.$$

Since $|\mathrm{Cov}_Y(x)| \le \mathrm{Var}\,(Y(o)) < \infty$ for all $x \in \mathbb{R}^d$, we only prove that

$$\int_{\{x\in\mathbb{R}^d\,:\,|x|>8(||K||+1)\}} |\mathrm{Cov}_Y(x)|\,dx < \infty\,.$$

Recalling the notation and results of Lemma 4.6, it holds that

$$\begin{aligned} |\,\mathrm{Cov}_Y(x)\,| \;&\le\; 4\,c_{\delta_Y}\,\beta^{\delta/(2+\delta)}\big(\sigma(X_{|B_1}),\sigma(X_{|B_2})\big) \\ &\quad + 4\,c_{\delta_Y}\,\Big(\lambda\,d\,2^{3d}\,\mathbb{E}\big(||M_0||^d\,\mathbb{1}\big(||M_0|| > \frac{1}{4}|x| + a\big)\big)\Big)^{\delta/(2+\delta)}\,, \end{aligned}$$

where $c_{\delta,Y} = 8\big(\mathbb{E}\big(|Y(o)|^{2+\delta}\big)\big)^2$, $a = ||K|| + 1, B_1 = [-\frac{1}{4}|x|, \frac{1}{4}|x|)^d$ and $B_2 = \mathbb{R}^d \setminus [-\frac{3}{4}|x| + 2a, \frac{3}{4}|x| - 2a)^d$. For the first summand, condition (4.12) gives

$$\beta\big(\sigma(X_{|B_1}),\sigma(X_{|B_2})\big) \;\le\; 2^{d-1}\,\beta_X\left(\frac{|x|}{2} - 2a\right),$$

where we also used the inequality $|x|/(|x|-4a) \le 2$ for all $x \in \mathbb{R}^d$ satisfying $|x| > 8a$. Moreover, it follows from (4.13) and from monotonicity of β_X that

$$\begin{aligned} \int_{\mathbb{R}^d} \beta_X^{\delta/(2+\delta)}\Big(\frac{|x|}{2} - 2a\Big)\,dx \;&\le\; \sum_{r=1}^{\infty} \beta_X^{\delta/(2+\delta)}(r)\,\big|B_{2(r+a+1)}(o) \setminus B_{2(r+a)}(o)\big| \\ &\le\; \kappa_d\,d\,2^d\,(a+2)^d \sum_{r=1}^{\infty} r^{d-1}\beta_X^{\delta/(2+\delta)}(r) \\ &<\; \infty \end{aligned}$$

substituting $r = \lfloor |x|/2 - 2a \rfloor$. Secondly, we find with $s = \lfloor |x|/8 - a \rfloor$ that

$$\begin{aligned} &\int_{\mathbb{R}^d} \Big(\mathbb{E}\big(||M_0||^d\,\mathbb{1}\big(||M_0|| > |x|/4 - a\big)\big)\Big)^{\delta/(2+\delta)}\,dx \\ &\le\; \sum_{s=1}^{\infty} \Big(\mathbb{E}\big(||M_0||^d\,\mathbb{1}\big(||M_0|| > s\big)\big)\Big)^{\delta/(2+\delta)}\,\big|B_{8(s+1)+a}(o) \setminus B_{8s+a}(o)\big| \\ &\le\; \kappa_d\,d\,2^{3(d-1)}\,(a+2)^d \sum_{s\ge 1} s^{d-1}\mathbb{E}\big(||M_0 \oplus \check{K}||^d \mathbb{1}(||M_0 \oplus \check{K}|| > s)\big)\,, \end{aligned}$$

which is finite given $\mathbb{E}\big(||M_0 \oplus \check{K}||^{2d(\frac{\delta+1}{\delta})+\varepsilon}\big) < \infty$ for some $\varepsilon > 0$ as shown above. □

The conditions of Theorem 4.2 can be reformulated and weakened in certain cases. As in Lemma 4.3, one can put $\delta = \infty$ if random field Y is bounded, this means, it sufficed to assume that $\mathbb{E}(||M_0 \oplus \check{K}||^{2d+\varepsilon}) < \infty$ for some $\varepsilon > 0$ and

$$\sum_{r=1}^{\infty} r^{d-1}\beta_X(r) \;<\; \infty\,.$$

Notice, however, that $\mathbb{E}(||M_0 \oplus \check{K}||^{2d}) < \infty$ already implies finiteness of the second moment of the volume $|M_0 \oplus \check{K}|$. Considering Boolean models with convex grains, $\mathbb{E}(|M_0 \oplus \check{K}|^2) < \infty$ is a sufficient condition for Theorem 3.3. Not only is this a weaker condition with respect to the grain distribution, but weighting function G can be chosen more generally as well.

The moment conditions on M_0 are certainly fulfilled for uniformly bounded grains. Furthermore, given $M_0 \in \mathcal{R}'$ almost surely with $P(N_{co}(M_0) \leq n_{co}) = 1$ for some finite number $n_{co} \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{E}\big(|Y(o)|^{2+\delta}\big) &\leq \mathbb{E}\big(2^{(2+\delta)N_{co}(\Xi \cap K)}\big) \\ &\leq \mathbb{E}\big(2^{n_{co}(2+\delta)N_{\Xi}(\mathcal{F}_K)}\big) \\ &\leq \int_0^\infty \mathbb{E}\big(2^{n_{co}(2+\delta)X(K \oplus B_v(o))}\big)\, dF_{||M_0||}(v)\,. \end{aligned}$$

At this, recall the notation from Chapter 2, where $N_{co}(\Xi \cap K)$ denotes the random number of convex components of $\Xi \cap K$, and $N_{\Xi}(\mathcal{F}_K)$ is the random number of grains $M_l, l \geq 1$ hitting test set K; compare Lemmas 2.9 and 2.10. Lemma 2.10 states, in particular, that $\mathbb{E}\big(|Y(o)|^{2+\delta}\big) < \infty$ for any $\delta > 0$ if Ξ is a Boolean model.

Next, let $M_0 \in \mathcal{K}'$ and consider the random field $Y = \mathcal{Z}_r$ introduced in (2.28) with $r > 0$. As shown in Section 4 of [60], $\mathbb{E}\, N_{co}^{(2+\delta)d}(\Xi \cap B_r(o)) < \infty$ implies that $\mathbb{E}(|Y(x)|^{2+\delta}) < \infty$ for any $\delta > 0$ provided that there exists almost surely no boundary point of Ξ, where more than d grains overlap (as in the case of a Boolean model).

Examples of point processes satisfying the mixing condition of Theorem 4.2 are given in [23, p. 297]. Among these are, for example, `Poisson cluster processes`. With regard to the latter, let $\Psi = \{[Z_l, N_l]\}$ be an independently marked Poisson point process with intensity $\lambda > 0$ and mark space $\mathcal{N}_f = \{\eta \in \mathcal{N}_{lf} : \eta(\mathbb{R}^d) < \infty\}$ (Corollary 7.1.IV in [8] provides the basis for the existences of Ψ), and let N_0 be a typical mark. If $\mathbb{E}\, N_0(\mathbb{R}^d) < \infty$, then $X = \sum_{l=1}^{\infty} N_l(\,\cdot - Z_l)$ is said to be a Poisson cluster process. The random variable $\varrho_{N_0} = \inf\{r > 0 : N_0(B_r(o)) = N_0(\mathbb{R}^d)\}$ is referred to as `cluster radius`. Given a Poisson cluster process X, one can choose

$$\beta_X(\tilde{\Delta}) \;=\; \tilde{c}_{d,\lambda}\, \mathbb{E}\big(\varrho_{N_0}^d\, \mathbb{1}\big(\varrho_{N_0} \geq \tilde{\Delta}/2\big)\big), \qquad \tilde{\Delta} > 0$$

for some constant $\tilde{c}_{d,\lambda} < \infty$. Condition (4.13) is fulfilled if $\mathbb{E}(\varrho_{N_0}^{2d(\frac{\delta+1}{\delta})+\varepsilon})$ is finite for some $\varepsilon > 0$, which can be derived by using similar arguments as for (4.14).

Analogously to Section 3.5, we deduce from Theorem 4.2 a multivariate central limit theorem with respect to the vector–valued random field $\mathbf{Y} = \{(Y_1(x), \ldots, Y_q(x))^\top, x \in \mathbb{R}^d\}$ for $q \geq 2$; confer the mentioned section for related notation. The associated germ–grain models Ξ_i, $i = 1, \ldots, q$ may either be mutually independent, satisfying the conditions of Theorem 4.2 individually, or they coincide. Assume for the following corollary that $\Xi_i = \Xi$, $i = 1, \ldots, q$ for some germ–grain model Ξ.

Corollary 4.1 *If Ξ fulfills the conditions of Theorem 4.2, then it holds that*

$$\begin{pmatrix} \sqrt{|W_n|}\,(\widehat{\mu}_{1,n} - \mu_1) \\ \vdots \\ \sqrt{|W_n|}\,(\widehat{\mu}_{q,n} - \mu_q) \end{pmatrix} \xrightarrow{D} \mathcal{N}_q(o, \Sigma)\,, \qquad n \to \infty\,,$$

where $\widehat{\mu}_{i,n} = \int_{W_n} (Y_i(x) - \mu_i)\, \mathbb{1}(x \in W_n^{K_i})/|W_n^{K_i}|\, dx, i = 1, \ldots, q$ and $\mathcal{N}_q(o, \Sigma)$ is a q–dimensional Gaussian random vector with zero mean vector and covariance matrix $\Sigma = (\sigma_{ij})_{i,j=1,\ldots,q}$, whose entries are $\sigma_{ij} = \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx$, $i, j = 1, \ldots, q$.

Proof The assertion follows directly from the Cramér–Wold device, taking into account that $F \oplus K_i \subseteq F \oplus B_r(o)$ for any $F \in \mathcal{F}$ and $r > 0$ such that $K_i \subseteq B_r(o)$ for each $i = 1, \ldots, q$. □

Chapter 5

Consistent estimation of the asymptotic covariance matrix

In the previous two chapters, we have seen that under appropriate conditions the random vector $\sqrt{|W_n|}(\widehat{\boldsymbol{\mu}}_n - \boldsymbol{\mu})$, where $\boldsymbol{\mu} = \mathbb{E}\mathbf{Y}(o)$ and $\widehat{\boldsymbol{\mu}} = (\widehat{\mu}_{1,n}, \ldots, \widehat{\mu}_{q,n})^\top$ defined in (3.23), is asymptotically normally distributed with zero mean vector and asymptotic covariance matrix

$$\Sigma = \begin{pmatrix} \theta_{11} \int_{\mathbb{R}^d} \mathrm{Cov}_{11}(x)\,dx & \ldots & \theta_{1q} \int_{\mathbb{R}^d} \mathrm{Cov}_{1q}(x)\,dx \\ \vdots & \ddots & \vdots \\ \theta_{q1} \int_{\mathbb{R}^d} \mathrm{Cov}_{q1}(x)\,dx & \ldots & \theta_{qq} \int_{\mathbb{R}^d} \mathrm{Cov}_{qq}(x)\,dx \end{pmatrix}, \tag{5.1}$$

where $\theta_{ij} = \theta_{ji}$ and $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}(Y_i(o), Y_j(x)) = \mathrm{Cov}_{ji}(-x)$ for $i, j = 1, \ldots, q$. In the case that Σ is known, this result can be used, for instance, to perform asymptotic tests for the mean vector $\boldsymbol{\mu}$. In general, however, Σ is unknown. The aim of the present chapter is therefore to determine asymptotically unbiased and consistent estimators of covariance matrix Σ. First, we will approach this estimation problem from the point of view that σ_{ij} is basically the integral of $\mathrm{Cov}_{ij}(x)$ over $\mathbb{R}^d$. Thus, we introduce an estimator $\widehat{\mathrm{Cov}}_{ij,n}(x)$, $n \geq 1$ of $\mathrm{Cov}_{ij}(x)$, $x \in \mathbb{R}^d$ and integrate over an unboundedly increasing sequence of averaging windows. Second, we make use of the fact that σ_{ij} is the asymptotic covariance of $\sqrt{|W_n|}(\widehat{\mu}_{i,n} - \mu_i)$ and $\sqrt{|W_n|}(\widehat{\mu}_{j,n} - \mu_j)$, and approximate σ_{ij} by an empirical covariance. Both procedures require certain integrability conditions in order the obtain mean–square consistent estimates. For conditionally bounded valuations of Boolean models, these conditions hold once again under the assumption that $\mathbb{E}(|M_0 \oplus \check{K}_i|^2) < \infty$, $i = 1, \ldots, q$ (compare Lemma 3.4).

In what follows, it is not always necessary to assume the setup of Chapters 3 or 4. Hence, if not mentioned otherwise, let $\mathbf{Y} = \{(Y_1(x), \ldots, Y_q(x))^\top,\ x \in \mathbb{R}^d\}$ for $q \geq 2$ be an arbitrary stationary, vector–valued random field on $\mathbb{R}^d$, and choose weights $G_1, \ldots, G_q$ as in Section 3.5.1. On observation window W_n, an unbiased estimator of the mean vector $\boldsymbol{\mu} = \mathbb{E}\mathbf{Y}(o)$ is then given by $\widehat{\boldsymbol{\mu}}_n = (\widehat{\mu}_{1,n}, \ldots, \widehat{\mu}_{q,n})^\top$ with $\widehat{\mu}_{i,n} = \int_{W_n} Y_i(x)\, G_i(W_n, x)\,dx$ for $i = 1, \ldots, q$ and $n \geq 1$. Throughout this

chapter, we assume that $\mathbb{E}\, Y_i^2(o) < \infty$ for $i = 1, \ldots, q$ and that

$$\int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx \;<\; \infty\,, \qquad i, j = 1, \ldots, q. \tag{5.2}$$

Then, we get $\lim_{n\to\infty} |W_n| \mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n}) = \sigma_{ij}$ with $\sigma_{ij} = \theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx$ (see Lemma 3.3, where only basic properties of Y_i and G_i, but no specific structure is used in the proof).

5.1 A weighted covariance estimator

The first estimator to be presented is a nonparametrical estimator $\widehat{\Sigma}_n$ of the form

$$\widehat{\Sigma}_n = \left(\int_{\mathbb{R}^d} \widehat{\mathrm{Cov}}_{ij,n}(x)\, \gamma_{ij,n}(x)\, dx \right)_{i,j=1,\ldots,q}, \qquad n \geq 1,$$

where $\widehat{\mathrm{Cov}}_{ij,n}(x)$ is a consistent estimator of $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}(Y_i(o), Y_j(x))$ weighted by $\gamma_{ij,n}(x) = |W_n| \Gamma_{ij,n}(x) \mathbb{1}(x \in U_{ij,n})$ with $\Gamma_{ij,n}(x)$ defined in (3.21) and $U_{ij,n} \in \mathfrak{B}_0^d$. The construction principle employed here is similar to the techniques used, for instance, in [3, Section 3] for 0–1–valued vector fields or [60, Section 3.5] for arbitrary stationary vector fields with uniform weights.

5.1.1 Definition and basic assumptions

Let $\{W_n\}$ be an increasing sequence of bounded Borel sets satisfying the van Hove condition (3.1), and let $Y_i = \{Y_i(x), x \in \mathbb{R}^d\}$ be observable on $W_{i,n} = W_n \ominus \check{K}_i$ for some $K_i \in \mathcal{K}', i = 1, \ldots, q$. For any pair $i \leq j$, choose an increasing sequence $\{U_{ij,n}\}$ of compact Borel sets such that $U_{ij,n} \subseteq W_{i,n} \oplus \check{W}_{j,n}$ and $o \in U_{ij,n}$. Furthermore, put $U_{ji,n} = \check{U}_{ij,n}$ for $j < i$ to preserve symmetry in the estimate of the covariance matrix. Next, denote by $\mathrm{supp}(\mathrm{Cov}_{ij})$ the support of $\mathrm{Cov}_{ij}(x)$ (i.e., the closure of the set $\{x \in \mathbb{R}^d : \mathrm{Cov}_{ij}(x) \neq 0\}$), and assume that $\lim_{n\to\infty} U_{ij,n} \;\supseteq\; \mathrm{supp}(\mathrm{Cov}_{ij})$. The above–stated conditions on $\{U_{ij,n}\}$ ensure that we obtain the correct domain of integration. In addition, we suppose that

$$\lim_{n\to\infty} |U_{ij,n}|^2 / |W_n| \;=\; 0 \tag{5.3}$$

and

$$\lim_{n\to\infty} \sup_{x \in U_{ij,n}} |\, \theta_{ij} - |W_n| \Gamma_{ij,n}(x)\,| \;=\; 0 \tag{5.4}$$

is valid for all $i, j = 1, \ldots, q$. Without loss of generality, assume that $|U_{ij,n}| > 0$ and $\Gamma_{ij,n}(x) > 0$ for all $x \in U_{ij,n}$, $i, j = 1, \ldots, q$ and any $n \in \mathbb{N}$.

As an example, let $\mathrm{supp}(\mathrm{Cov}_{ij}) = \mathbb{R}^d$ and $W_n = nW_0$, $n \geq 1$ for $W_0 \in \mathcal{K}'$ with $|W_0| > 0$ and $o \in \mathrm{int}(W_0)$. Define $w_n = \sup\{r > 0 : B_r(o) \subseteq nW_0\}$; that is in fact $w_n = nw_1$. Choose uniform weights $G_i(W_n, x) = \mathbb{1}(x \in B_{w_n - r_0}(o)) / |B_{w_n - r_0}(o)|$, where $r_0 > 0$ satisfies $K_i \subseteq B_{r_0}(o)$ for all $i = 1, \ldots, q$, and $n \in \mathbb{N}$ is large enough so

that $w_n > r_0$. Then conditions (5.3) and (5.4) are fulfilled with $\theta_{ij} \equiv 1$ if one puts $U_{ij,n} = B_{\sqrt{w_n}\,\varepsilon_n - r_0}(o)$ for some positive sequence $\{\varepsilon_n\}$ satisfying $\varepsilon_n \downarrow 0$ and $\lim_{n\to\infty} \sqrt{n}\,\varepsilon_n = \infty$; see [3, p. 5]. If, on the other hand, $\mathrm{supp}(\mathrm{Cov}_{ij})$ is a bounded Borel set itself, then one can define $U_{ij,n} = \mathrm{supp}(\mathrm{Cov}_{ij})$ provided that $n \geq 1$ is sufficiently large ensuring $U_{ij,n} \subseteq W_{i,n} \oplus \check{W}_{j,n}$.

Based on the stated pre–requisites, introduce the estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{ij,n})$, $n \geq 1$ by

$$\widehat{\sigma}_{ij,n} \;=\; \int_{U_{ij,n}} \widehat{\mathrm{Cov}}_{ij,n}(x)\, |W_n| \Gamma_{ij,n}(x)\, dx\,, \qquad i,j = 1\ldots, q, \tag{5.5}$$

where

$$\begin{aligned} &\widehat{\mathrm{Cov}}_{ij,n}(x) \\ &\;=\; \int_{W_{i,n}\cap(W_{j,n}-x)} Y_i(y) Y_j(x+y)\, \frac{G_i(W_n,y) G_j(W_n,x+y)}{\Gamma_{ij,n}(x)}\, dy - \widehat{\mu}_{i,n}\, \widehat{\mu}_{j,n}. \end{aligned} \tag{5.6}$$

Notice, in particular, that $\widehat{\sigma}_{ij,n} = \widehat{\sigma}_{ji,n}$ for any pair $i,j \in \{1,\ldots,q\}$.

5.1.2 Pointwise unbiasedness and consistency

By the subsequent lemmas, we find that $\widehat{\mathrm{Cov}}_{ij,n}(x)$, considered separately for any fixed $x \in \mathbb{R}^d$, is an asymptotically unbiased and consistent estimator of $\mathrm{Cov}_{ij}(x)$.

Lemma 5.1 *For any fixed $x \in \mathbb{R}^d$, it holds that $\lim_{n\to\infty} \mathbb{E}\,\widehat{\mathrm{Cov}}_{ij,n}(x) = \mathrm{Cov}_{ij}(x)$ for $i,j = 1,\ldots,q$.*

Proof Inserting $\pm\mu_i\mu_j$ into the defining equation (5.6) of $\widehat{\mathrm{Cov}}_{ij,n}(x)$ yields

$$\begin{aligned} \mathbb{E}\,\widehat{\mathrm{Cov}}_{ij,n}(x) \;&=\; \mathbb{E}\big(Y_i(o)Y_j(x) - \mu_i\mu_j\big) \int_{W_{i,n}\cap(W_{j,n}-x)} \frac{G_i(W_n,y) G_j(W_n,x+y)}{\Gamma_{ij,n}(x)}\, dy \\ &\quad - \mathbb{E}\big(\widehat{\mu}_{i,n}\,\widehat{\mu}_{j,n} - \mu_i\mu_j\big) \\ &\;=\; \mathrm{Cov}_{ij}(x) - \mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n}) \end{aligned}$$

for any $i,j = 1,\ldots,q$. Given (5.2), one has $\lim_{n\to\infty} |W_n|\,\mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n}) = \sigma_{ij} < \infty$ by formulae (3.24) and (3.25), so it readily follows that $\lim_{n\to\infty} \mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n}) = 0$ and thus $\lim_{n\to\infty} \mathbb{E}\,\widehat{\mathrm{Cov}}_{ij,n}(x) = \mathrm{Cov}_{ij}(x)$ for any fixed $x \in \mathbb{R}^d$. □

With respect to mean–square consistency, it is, in addition, necessary to assume that $\mathbb{E}\,Y_i^4(o)$ is finite for $i = 1,\ldots,q$. Besides this, the following integrability conditions are required

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathrm{Cov}(Y_i(o)Y_j(x), Y_i(y)Y_j(x+y))\big|\, dy \;<\; c_2 \tag{5.7}$$

and

$$\sup_{x_1,x_2\in\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathbb{E}\big((Y_i(o)-\mu_i)(Y_i(y)-\mu_i)Y_j(x_1)Y_j(x_2)\big)\big|\, dy \;<\; c_3 \tag{5.8}$$

for some constants $c_2, c_3 < \infty$ and any pair $i,j = 1,\ldots,q$.

Lemma 5.2 *Suppose that* $\mathbb{E}\, Y_i^4(o) < \infty$ *for* $i = 1, \ldots, q$ *and that conditions* (5.7) *and* (5.8) *are satisfied. Given some* $x \in \mathbb{R}^d$, *the estimator* $\widehat{\mathrm{Cov}}_{ij,n}(x)$ *is mean–square consistent for* $\mathrm{Cov}_{ij}(x)$.

Proof By the asymptotic unbiasedness of $\widehat{\mathrm{Cov}}_{ij,n}(x)$ for $\mathrm{Cov}_{ij}(x)$, it suffices to show that the variance of $\widehat{\mathrm{Cov}}_{ij,n}(x)$ tends to zero as $n \to \infty$. For this, put $\omega_{ij,n}(y, x+y) = G_i(W_n, y) G_j(W_n, x+y)$ for any $x, y \in \mathbb{R}^d$ and observe that

$$\begin{aligned}
&\mathbb{E}\Big(\widehat{\mathrm{Cov}}_{ij,n}(x) - \mathbb{E}\,\widehat{\mathrm{Cov}}_{ij,n}(x)\Big)^2 \\
&\quad = \mathbb{E}\bigg(\int_{W_n} \big(Y_i(y)Y_j(x+y) - \mathbb{E}(Y_i(y)Y_j(x+y))\big)\, \frac{\omega_{ij,n}(y, x+y)}{\Gamma_{ij,n}(x)}\, dy \\
&\qquad - \big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)\bigg)^2 \\
&\quad \le 2\,\mathbb{E}\bigg(\int_{W_n} \big(Y_i(y)Y_j(x+y) - \mathbb{E}(Y_i(y)Y_j(x+y))\big)\, \frac{\omega_{ij,n}(y, x+y)}{\Gamma_{ij,n}(x)}\, dy\bigg)^2 \qquad (5.9.\mathrm{a}) \\
&\qquad + 2\,\mathbb{E}\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)^2 \qquad (5.9.\mathrm{b})
\end{aligned}$$

using Hölder's inequality in the final estimate.

In the first step, we now prove that

$$\begin{aligned}
\mathbb{E}\bigg(\int_{W_n} &\big(Y_i(y)Y_j(x+y) - \mathbb{E}(Y_i(y)Y_j(x+y))\big) \\
&\times |W_n|\, \omega_{ij,n}(y, x+y)\, dy\bigg)^2 \;\le\; \frac{c_0^3\, c_2}{|W_n|}\,,
\end{aligned} \qquad (5.10)$$

where c_0 is the finite constant appearing in (3.20). Since $|W_n|\Gamma_{ij,n}(x) \to \theta_{ij} \in (0, \infty)$, the estimate (5.10) yields that the summand in (5.9.a) tends to zero as $n \to \infty$. To show (5.10), let condition (5.7) be fulfilled. Then it follows from (3.20) that

$$\begin{aligned}
&\mathbb{E}\Big(\int_{W_n} \big(Y_i(y)Y_j(x+y) - \mathbb{E}(Y_i(y)Y_j(x+y))\big)\, |W_n|\, \omega_{ij,n}(y, x+y)\, dy\Big)^2 \\
&\quad = \int_{W_n}\int_{W_n} \mathrm{Cov}\big(Y_i(y_1)Y_j(x+y_1), Y_i(y_2)Y_j(x+y_2)\big) \\
&\qquad\qquad \times |W_n|^2\, \omega_{ij,n}(y_1, x+y_1)\, \omega_{ij,n}(y_2, x+y_2)\, dy_1 dy_2 \\
&\quad = \int_{\mathbb{R}^d} \mathrm{Cov}\big(Y_i(o)Y_j(x), Y_i(y)Y_j(x+y)\big) \\
&\qquad\qquad \times |W_n|^2 \int_{W_n} \omega_{ij,n}(u, x+u)\, \omega_{ij,n}(y+u, x+y+u)\, du\, dy \\
&\quad \le \frac{c_0^3\, |W_n|^2}{|W_n|^3} \int_{\mathbb{R}^d} \big|\mathrm{Cov}\big(Y_i(o)Y_j(x), Y_i(y)Y_j(x+y)\big)\big|\, dy \\
&\quad \le \frac{c_0^3\, c_2}{|W_n|}\,.
\end{aligned}$$

As to the convergence of the summand in (5.9.b), we show

$$\limsup_{n\to\infty} |W_n|\,\mathbb{E}\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)^2 \;<\; \infty\,, \tag{5.11}$$

which results in $\lim_{n\to\infty} \mathbb{E}(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n}))^2 = 0$. Inserting $\pm\mu_i\mu_j$ and $\pm\mu_i\widehat{\mu}_{j,n}$ into the expectation gives

$$\begin{aligned}
&|W_n|\,\mathbb{E}\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)^2 \\
&\quad= \;|W_n|\,\mathbb{E}\Big(\Big[(\widehat{\mu}_{i,n}-\mu_i)\,\widehat{\mu}_{j,n} + \mu_i\,(\widehat{\mu}_{j,n}-\mu_j) - (\mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n}) - \mu_i\mu_j)\Big]^2\Big) \\
&\quad\le \;3\,|W_n|\Big(\mathbb{E}\,((\widehat{\mu}_{i,n}-\mu_i)^2\,\widehat{\mu}_{j,n}^2) + \mu_i^2\,\mathbb{E}\,(\widehat{\mu}_{j,n}-\mu_j)^2 + (\mathrm{Cov}\,(\widehat{\mu}_{i,n},\widehat{\mu}_{j,n}))^2\Big).
\end{aligned}$$

The inequality is once again obtained by Hölder's inequality. In the final bound, we have $\lim_{n\to\infty} |W_n|\mathbb{E}\,(\widehat{\mu}_{j,n}-\mu_j)^2 = \sigma_{jj} < \infty$ and

$$\lim_{n\to\infty} |W_n|\big(\mathrm{Cov}\,(\widehat{\mu}_{i,n},\widehat{\mu}_{j,n})\big)^2 \;=\; \lim_{n\to\infty} \frac{\big(|W_n|\mathrm{Cov}\,(\widehat{\mu}_{i,n},\widehat{\mu}_{j,n})\big)^2}{|W_n|} \;=\; \lim_{n\to\infty} \frac{\sigma_{ij}^2}{|W_n|} \;=\; 0\,. \tag{5.12}$$

Furthermore, from the definition of $\widehat{\mu}_{i,n}$, $i = 1, ..., q$ it follows that

$$\begin{aligned}
&|W_n|\,\mathbb{E}\,((\widehat{\mu}_{i,n}-\mu_i)^2\,\widehat{\mu}_{j,n}^2) \\
&\quad= \int_{W_n}\int_{W_n}\int_{W_n}\int_{W_n} \mathbb{E}\big((Y_i(y_1)-\mu_i)(Y_i(y_2)-\mu_i)\,Y_j(y_3)Y_j(y_4)\big) \\
&\qquad\qquad \times |W_n|\,\omega_{ii,n}(y_1,y_2)\,\omega_{jj,n}(y_3,y_4)\,dy_1\,dy_2\,dy_3\,dy_4 \\
&\quad= \int_{W_n}\int_{W_n-h}\int_{W_n-h}\int_{W_n-h} \mathbb{E}\big((Y_i(o)-\mu_i)(Y_i(y)-\mu_i)\,Y_j(x_1)Y_j(x_2)\big) \\
&\qquad\qquad \times |W_n|\,\omega_{ii,n}(h,y+h)\,\omega_{jj,n}(x_1+h,x_2+h)\,dy\,dx_2\,dx_1\,dh \\
&\quad\le c_0\int_{W_n}\int_{W_n-h}\int_{W_n-h}\int_{\mathbb{R}^d} \big|\mathbb{E}\big((Y_i(o)-\mu_i)(Y_i(y)-\mu_i)\,Y_j(x_1)Y_j(x_2)\big)\big|\,dy \\
&\qquad\qquad \times G_i(W_n,h)\,G_j(W_n,x_1+h)\,G_j(W_n,x_2+h)\,dx_2\,dx_1\,dh \\
&\quad\le c_0\sup_{x_1,x_2\in\mathbb{R}^d}\int_{\mathbb{R}^d} \big|\mathbb{E}\big((Y_i(o)-\mu_i)(Y_i(y)-\mu_i)\,Y_j(x_1)Y_j(x_2)\big)\big|\,dy \\
&\qquad\quad \times \int_{W_n}\int_{W_n-h}\int_{W_n} G_i(W_n,h)\,G_j(W_n,v_1+h)\,G_j(W_n,v_2+h)\,dv_2\,dv_1\,dh \\
&\quad= c_0\sup_{x_1,x_2\in\mathbb{R}^d}\int_{\mathbb{R}^d} \big|\mathbb{E}\big((Y_i(o)-\mu_i)(Y_i(y)-\mu_i)\,Y_j(x_1)Y_j(x_2)\big)\big|\,dy\,.
\end{aligned}$$

The first inequality is due to (3.19), that is $G_i(W_n,x) \le c_0/|W_n|$. The final line follows from (3.20), meaning that the integral of $G(W_n,x)$ over W_n is one. By virtue of condition (5.8), the proof is complete. □

For centered random fields and uniform weights, further results for $\widehat{\mathrm{Cov}}_{ij,n}(x)$, $x \in \mathbb{R}^d$, like uniform convergence on compact sets or asymptotic normality, are presented in [28, Chapter 4].

5.1.3 Unbiasedness and consistency of the integrated covariances

Asymptotic unbiasedness and mean–square consistency of $\widehat{\mathrm{Cov}}_{ij,n}(x)$ for fixed x will now be transferred to the estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{ij,n})_{i,j=1,\ldots,q}$, $n \geq 1$ with $\widehat{\sigma}_{ij,n} = \int_{U_{ij,n}} \widehat{\mathrm{Cov}}_{ij}(x)\, |W_n| \Gamma_{ij,n}(x)\, dx$.

Lemma 5.3 *The estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{ij,n})_{i,j=1,\ldots,q}$, $n \geq 1$ defined in (5.5) and (5.6) is asymptotically unbiased for covariance matrix Σ defined in (5.1) as $n \to \infty$.*

Proof Analogously to the proof of Lemma 5.1 we find that

$$\mathbb{E}\,\widehat{\sigma}_{ij,n} \;=\; \int_{U_{ij,n}} \mathrm{Cov}_{ij}(x)\, |W_n| \Gamma_{ij,n}(x)\, dx \tag{5.13.a}$$

$$- |W_n| \mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n}) \int_{U_{ij,n}} \Gamma_{ij,n}(x)\, dx\,. \tag{5.13.b}$$

Since $\lim_{n\to\infty} |W_n| \mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n}) = \sigma_{ij} < \infty$ and

$$0 \;\leq\; \limsup_{n\to\infty} \int_{U_{ij,n}} \Gamma_{ij,n}(x)\, dx \;\leq\; c_0 \cdot \lim_{n\to\infty} \frac{|U_{ij,n}|}{|W_n|} \;=\; 0\,,$$

the expression in (5.13.b) converges to zero as $n \to \infty$.

On the other hand, the summand in (5.13.a) can be split up as follows. Add and subtract the desired limit $\theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx$ to get

$$\int_{U_{ij,n}} \mathrm{Cov}_{ij}(x)\, |W_n| \Gamma_{ij,n}(x)\, dx \;=\; \theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx$$
$$- \int_{U_{ij,n}} \mathrm{Cov}_{ij}(x) \big(\theta_{ij} - |W_n| \Gamma_{ij,n}(x)\big)\, dx - \theta_{ij} \int_{\mathbb{R}^d \setminus U_{ij,n}} \mathrm{Cov}_{ij}(x)\, dx\,.$$

Now, we show that the surplus summands vanish as $n \to \infty$. Due to the absolute integrability of $\mathrm{Cov}_{ij}(x)$, $x \in \mathbb{R}^d$ (by assumption) and since $\theta_{ij} < \infty$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d \setminus U_{ij,n}} \mathrm{Cov}_{ij}(x)\, dx \;=\; 0\,.$$

Moreover, it follows from condition (5.4) that

$$\limsup_{n\to\infty} \Big| \int_{U_{ij,n}} \mathrm{Cov}_{ij}(x) (\theta_{ij} - |W_n| \Gamma_{ij,n}(x))\, dx \Big|$$
$$\leq \quad \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx \cdot \lim_{n\to\infty} \sup_{x \in U_{ij,n}} \big|\, \theta_{ij} - |W_n| \Gamma_{ij,n}(x)\, \big| \;=\; 0\,.$$

□

According to Lemma 5.3, asymptotic unbiasedness of $\widehat{\Sigma}_n$ holds under the same integrability conditions needed to define $\Sigma = (\sigma_{ij})_{i,j=1,\dots,q}$ with $\sigma_{ij} < \infty$. Like for $\widehat{\mathrm{Cov}}_{ij,n}(x)$, additional assumptions are required to ensure the mean–square consistency. The latter is understood with respect to the matrix norm $||A|| = (\sum_{i,j=1}^{q} a_{ij}^2)^{1/2}$ for a real–valued matrix $A = (a_{ij})_{i,j=1,\dots,q}$.

Theorem 5.1 *Let condition* (5.8) *be fulfilled and suppose that there exists a constant* $c_4 < \infty$ *such that*

$$\frac{1}{|U_{ij,n}|^2} \int_{U_{ij,n}} \int_{U_{ij,n}} \int_{\mathbb{R}^d} \left| \mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big) \right| \, dy \, dx_1 dx_2 \; < \; c_4 \quad (5.14)$$

for any $i, j = 1, \dots, q$ *and* $n \geq 1$. *Then it holds that* $\lim_{n\to\infty} \mathbb{E}\,||\widehat{\Sigma}_n - \Sigma||^2 = 0$.

Proof By Minkowski's inequality and since $\lim_{n\to\infty} \mathbb{E}\,\widehat{\sigma}_{ij,n} = \sigma_{ij}$ holds according to Lemma 5.3, it suffices to show that the convergence in mean–square sense holds for each component, i.e., that $\mathbb{E}\,(\widehat{\sigma}_{ij,n} - \mathbb{E}\,\widehat{\sigma}_{ij,n})^2$ tends to zero for all $i, j = 1, \dots, q$ as $n \to \infty$. To this end, write $\widehat{\sigma}_{ij,n} - \mathbb{E}\,\widehat{\sigma}_{ij,n} = R_{n1} + R_{n2}$, where

$$\begin{aligned} R_{n1} &= \int_{U_{ij,n}} \int_{W_{i,n} \cap (W_{j,n} - x)} \Big(Y_i(y)Y_j(x+y) - \mathbb{E}\,(Y_i(y)Y_j(x+y)) \Big) \\ &\qquad \times |W_n| \, G_i(W_n, y) G_j(W_n, x+y) \, dy \, dx \end{aligned}$$

and

$$R_{n2} = -\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big) \int_{U_{ij,n}} |W_n| \Gamma_{ij,n}(x) \, dx \, .$$

Both summands R_{n1} and R_{n2} converge to zero in mean–square sense as we will verify in the following.

Firstly, one observes like in the proof of Lemma 5.2 that

$$\begin{aligned} \mathbb{E}\big(R_{n1}^2\big) &= \int_{U_{ij,n}} \int_{U_{ij,n}} \int_{W_n} \int_{W_n} \mathrm{Cov}\big(Y_i(y_1)Y_j(y_1+x_1), Y_i(y_2)Y_j(x_2+y_2)\big) \\ &\qquad \times |W_n| \, \omega_{ij,n}(y_1, x_1+y_1) \omega_{ij,n}(y_2, x_2+y_2) \, dy_2 \, dy_1 \, dx_2 \, dx_1 \\ &\leq \frac{c_0^3}{|W_n|} \int_{U_{ij,n}} \int_{U_{ij,n}} \int_{\mathbb{R}^d} \left| \mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big) \right| \, dy \, dx_1 dx_2 \\ &\leq c_0^3 \, c_4 \, \frac{|U_{ij,n}|^2}{|W_n|} \, , \end{aligned}$$

where $\omega_{ij,n}(y, x+y) = G_i(W_n, y) G_j(y+x)$ and the constant $c_4 < \infty$ is provided by condition (5.14). Hence, given (5.3), i.e., $\lim_{n\to\infty} |U_{ij,n}|^2/|W_n| = 0$, the final bound of the above inequalities converges to zero.

Secondly, it readily follows from condition (3.19) that

$$\begin{aligned} \mathbb{E}\big(R_{n2}^2\big) &= \mathbb{E}\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)^2 \left(\int_{U_{ij,n}} |W_n| \Gamma_{ij,n}(x) \, dx \right)^2 \\ &\leq |W_n| \mathbb{E}\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)^2 \, c_0^2 \, \frac{|U_{ij,n}|^2}{|W_n|} \, . \end{aligned}$$

The latter bound tends to zero as $n \to \infty$, since in addition to condition (5.14) we have seen in (5.11) that $\limsup_{n\to\infty} |W_n| \, \mathbb{E}\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)^2 < \infty$ holds. □

As mentioned above, similar estimation methods for Σ are used in [3, Section 3] for 0–1–valued vector fields or [60, Section 3.5] for arbitrary stationary vector fields with uniform weights. Both variants assume that conditions (3.8) and (5.7) hold to prove mean-square consistency. In the case of 0–1–valued random fields no further assumptions are needed, since $Y(x) = \mathbb{1}(x \in \Xi)$, $x \in \mathbb{R}^d$ is uniformly bounded on $\mathbb{R}^d$. For the more general setting considered in [60], mean–square consistency is shown using different arguments in the proof, which require stronger integrability conditions. We point out that formula (56) in [60] is a sufficient condition for (5.7). In addition, one have to assume that

$$\begin{aligned}\int_{\mathbb{R}^{3d}} \Big| \mathbb{E}\big((Y_i(o)-\mu_i)(Y_j(x_1)-\mu_i)(Y_i(x_2)-\mu_i)(Y_j(x_3)-\mu_i)\big) \\ -\mathrm{Cov}_{ij}(x_1)\mathrm{Cov}_{ij}(x_2-x_3) - \mathrm{Cov}_{ii}(x_2)\mathrm{Cov}_{jj}(x_1-x_3) \\ -\mathrm{Cov}_{ij}(x_3)\mathrm{Cov}_{ji}(x_1-x_2)\Big| \, d(x_1,x_2,x_3) \;<\; c_5\end{aligned} \tag{5.15}$$

for some finite constant c_5 and $i, j = 1, \ldots, m$. Notice that the last three terms of (5.15) are missing in formula (57) in [60] by mistake.

For conditionally bounded valuations of Boolean models, we shall discuss integrability conditions (5.7), (5.8) and (5.14) in the Section 5.3. Some special examples of functionals Y_i, $i = 1, \ldots, q$ satisfying the requirements of Theorem 5.1 (resp. Corollary 5.1) are uniformly bounded random and "regionally independent" random fields.

Corollary 5.1 *If the random fields $\{Y_i(x), x \in \mathbb{R}^d\}$ are uniformly bounded for $i = 1, \ldots, q$, that is to say, if there exist finite constants κ_i such that $|Y_i(x)| \le \kappa_i$ for all $x \in \mathbb{R}^d$ and $i = 1, \ldots, q$, then $\widehat{\Sigma}_n$ converges in the mean–square sense to Σ as $n \to \infty$ given* (5.14).

Proof By the proof of Theorem 5.3, it suffices to show that condition (5.8) is redundant for uniformly bounded random fields. Put $\kappa = \max\{\kappa_i : i = 1, \ldots, q\}$. Then we have $|\widehat{\mu}_{i,n}| \le \kappa$ for all $i = 1, \ldots, q$. As in the proof of (5.11), we find that

$$\begin{aligned}&\limsup_{n\to\infty} |W_n| \, \mathbb{E}\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)^2 \\ &\quad= \limsup_{n\to\infty} |W_n| \, \mathbb{E}\big(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n} \pm \mu_i\widehat{\mu}_{j,n} \pm \mu_i\mu_j - \mathbb{E}\,(\widehat{\mu}_{i,n}\widehat{\mu}_{j,n})\big)^2 \\ &\quad\le 3 \limsup_{n\to\infty} |W_n| \, \Big(\mathbb{E}\big((\widehat{\mu}_{i,n}-\mu_i)^2\,\widehat{\mu}_{j,n}^2\big) + \mu_i^2\,\mathbb{E}\big((\widehat{\mu}_{j,n}-\mu_j)^2\big) + \big(\mathrm{Cov}\,(\widehat{\mu}_{i,n},\widehat{\mu}_{j,n})\big)^2\Big) \\ &\quad\le 3\Big(\kappa^2 \lim_{n\to\infty} |W_n| \, \mathrm{Var}\,\widehat{\mu}_{i,n} + \kappa^2 \lim_{n\to\infty} |W_n| \, \mathrm{Var}\,\widehat{\mu}_{j,n} + \lim_{n\to\infty} |W_n| \big(\mathrm{Cov}\,(\widehat{\mu}_{i,n},\widehat{\mu}_{j,n})\big)^2\Big) \\ &\quad= 3\,\kappa^2\,(\sigma_{ii}+\sigma_{jj}) \;<\; \infty\end{aligned}$$

for $i, j = 1, \ldots, q$. At this, recall that $\lim_{n\to\infty} \sqrt{|W_n|}\mathrm{Cov}(\widehat{\mu}_{i,n},\widehat{\mu}_{j,n}) = 0$; see (5.12). In total, we have shown that in the present case equation (5.11) holds without assuming (5.8) as required in the more general situation of Lemma 5.1. □

Corollary 5.2 *The estimator $\widehat{\Sigma}_n$ is asymptotically unbiased and mean–square consistent for Σ if there exists some $r_0 > 0$ such that $\{\mathbf{Y}(x), x \in B_1\}$ and $\{\mathbf{Y}(y), x \in B_2\}$ are independent for any $B_1, B_2 \in \mathfrak{B}_0^d$ with $\inf\{|x_1 - x_2| : x_1 \in B_1, x_2 \in B_2\} > r_0$.*

Proof Under the imposed condition, the random vectors $(Y_i(o), Y_j(x_1))^\top$ and $(Y_i(y), Y_j(x_2 + y))^\top$ are independent whenever $|y| > r_0$, $|y - x_1| > r_0$ and $|y + x_2| > r_0$, $|y - (x_1 - x_2)| > r_0$. For this, put $B_1 = \{o, x_1\}$ and $B_2 = \{y, x_2 + y\}$. As a result, we get $\mathrm{Cov}(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2 + y)) = 0$ for any $y \notin B_{r_o}(o, x_1, x_2) = B_{r_0}(o) \cup B_{r_0}(x_1) \cup B_{r_0}(-x_2) \cup B_{r_0}(x_1 - x_2)$.

Now, we show that condition (5.14) of Theorem 5.1 is satisfied. Using the Cauchy–Schwarz inequality and the stationarity of $\mathbf{Y}$, we find that

$$
\begin{aligned}
&\sup_{x_1,x_2\in\mathbb{R}^d} \int_{\mathbb{R}^d} \bigl|\mathrm{Cov}\bigl(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\bigr)\bigr|\, dy \\
&\quad = \sup_{x_1,x_2\in\mathbb{R}^d} \int_{B_{r_o}(o,x_1,x_2)} \bigl|\mathrm{Cov}\bigl(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\bigr)\bigr|\, dy \\
&\quad \le \sup_{x_1,x_2\in\mathbb{R}^d} \int_{B_{r_o}(o,x_1,x_2)} \Bigl(\bigl|\mathbb{E}\bigl(Y_i(o)Y_j(x_1)Y_i(y)Y_j(x_2+y)\bigr)\bigr| \\
&\qquad\qquad + \bigl|\mathbb{E}\bigl(Y_i(o)Y_j(x_1)\bigr)\,\mathbb{E}\bigl(Y_i(y)Y_j(x_2+y)\bigr)\bigr|\Bigr) dy \\
&\quad \le \sup_{x_1,x_2\in\mathbb{R}^d} |\, B_{r_o}(o,x_1,x_2)\,| \cdot \Bigl(\sqrt{\mathbb{E}(Y_i^4(o))\,\mathbb{E}(Y_j^4(o))} + \mathbb{E}(Y_i^2(o))\,\mathbb{E}(Y_j^2(o))\Bigr) \\
&\quad \le 4\, r_0^d\, \kappa_d \cdot 2\sqrt{\mathbb{E}\, Y_i^4(o)\,\mathbb{E}\, Y_j^4(o)}\,,
\end{aligned}
$$

where $\mathbb{E}\, Y_i^4(o) < \infty$ for any $i \in \{1, \ldots, q\}$ by assumption.

With regard to condition (5.8), note that $\mathbb{E}((Y_i(o) - \mu_i)(Y_i(y) - \mu_i)Y_j(x_1)Y_j(x_2)) = 0$ for any $y \notin B_{r_0}(o) \cup B_{r_0}(x_1) \cup B_{r_0}(x_2)$. At this, put $B_1 = \{o, x_1, x_2\}$ and $B_2 = \{y\}$. Furthermore, by repeated application of the Cauchy–Schwarz inequality, one gets

$$
\begin{aligned}
&\bigl|\,\mathbb{E}\bigl((Y_i(o) - \mu_i)(Y_j(y) - \mu_j)\, Y_i(x_1)\, Y_j(x_2)\bigr)\bigr| \\
&\qquad \le \sqrt[4]{\mathbb{E}(Y_i(o) - \mu_i)^4\,\mathbb{E}(Y_j(o) - \mu_j)^4\,\mathbb{E}\, Y_i^4(o)\,\mathbb{E}\, Y_j^4(o)}\,.
\end{aligned}
$$

Hence, it holds that

$$
\begin{aligned}
&\sup_{x_1,x_2\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\,\mathbb{E}\bigl((Y_i(o) - \mu_i)(Y_j(y) - \mu_j)\, Y_i(x_1)\, Y_j(x_2)\bigr)|\, dy \\
&\quad \le \sup_{x_1,x_2\in\mathbb{R}^d} |B_{r_0}(o) \cup B_{r_0}(x_1) \cup B_{r_0}(x_2)| \\
&\qquad\qquad \times \sqrt[4]{\mathbb{E}(Y_i(o) - \mu_i)^4\,\mathbb{E}(Y_j(o) - \mu_j)^4\,\mathbb{E}\, Y_i^4(o)\,\mathbb{E}\, Y_j^4(o)} \\
&\quad \le 3\, r_0^d\, \kappa_d \sqrt[4]{\mathbb{E}(Y_i(o) - \mu_i)^4\,\mathbb{E}(Y_j(o) - \mu_j)^4\,\mathbb{E}\, Y_i^4(o)\,\mathbb{E}\, Y_j^4(o)}\,.
\end{aligned}
$$

By the above considerations, we have verified that the conditions of Theorem 5.3 are fulfilled, and the assertion of the corollary follows. □

5.2 Modified weighted covariance estimator

Motivated by the task of finding an efficient implementation for the computation of $\widehat{\Sigma}_n$, we now introduce a modification $\widehat{\Sigma}'_n$ of the estimator $\widehat{\Sigma}_n$, $n \geq 1$.
The mean–squared distance between $\widehat{\Sigma}_n$ and Σ tends to zero with expanding window W_n, i.e., with increasing information on $\mathbf{Y}$. At the same time, the computational effort needed to evaluate $\widehat{\mathrm{Cov}}_{ij,n}(x)$ for each $x \in U_{ij,n}$ usually rises. One way to use as much information as possible and to cut down the running time simultaneously is to divide W_n into smaller subwindows and perform the estimation on each subwindow separately. Afterwards the estimates are combined by means of the arithmetic average.

5.2.1 Decomposition of the observation window

To be more precise, we need some additional definitions and notation. Let $\{\tilde{W}_n\}$ with $\tilde{W}_n \subset W_n$ be a van Hove sequence, and let $\{m_n\}$ an arbitrary non–decreasing sequence of integers. For each $n \in \mathbb{N}$ and $k = 1, \ldots, m_n$, choose a vector $h_{kn} \in \mathbb{R}^d$ and define $\tilde{W}_{kn} = \tilde{W}_n + h_{kn}$ under the constraint that $\bigcup_{k=1}^{m_n} \tilde{W}_{kn} \subseteq W_n$. In addition, suppose that G_i, $i = 1, \ldots, q$ satisfy (3.19) and (3.20) with respect to $\{\tilde{W}_n\}$ and constant $\tilde{c}_0 \in (0, \infty)$, and that

$$G_i(\tilde{W}_{kn}, x) \;=\; G_i(\tilde{W}_n, x - h_{kn}), \qquad x \in \mathbb{R}^d,\ i = 1, \ldots, q\,. \tag{5.16}$$

For $i, j = 1, \ldots, q$, put

$$\tilde{\Gamma}_{ij,n}(x) = \int_{\mathbb{R}^d} G_i(\tilde{W}_n, y) G_j(\tilde{W}_n, x + y)\, dy$$

and assume that

$$\lim_{n \to \infty} |\tilde{W}_n| \tilde{\Gamma}_{ij,n}(x) \;=\; \tilde{\theta}_{ij}$$

for some $\tilde{\theta}_{ij} \in (0, \infty)$. Without loss of generality, we may suppose that $\tilde{c}_0 = c_0$ and $\tilde{\theta}_{ij} = \theta_{ij}$ for all $i, j = 1, \ldots, q$ (elsewise multiply the estimate by $\theta_{ij}/\tilde{\theta}_{ij}$).

As before, let $\tilde{W}_{i,n} = \tilde{W}_n \ominus K_i$ and let $\tilde{W}_{i,kn}$ denote the translates $\tilde{W}_{i,n}$ by h_{kn}. The corresponding averaging sequences $\{\tilde{U}_{ij,n}\}$, $i, j = 1, \ldots, q$ (see page 72) are chosen in such a way that $\tilde{U}_{ji,n} = -\tilde{U}_{ij,n}$ for $j < i$, $o \in \tilde{U}_{ij,n}$, $\tilde{U}_{ij,n} \subseteq \tilde{W}_{i,n} \oplus (-\tilde{W}_{j,n})$, $|\tilde{U}_{ij,n}|^2/|\tilde{W}_n| \to 0$ and $\sup_{x \in \tilde{U}_{ij,n}} |\theta_{ij} - |\tilde{W}_n| \tilde{\Gamma}_{ij,n}(x)| \to 0$ as $n \to \infty$. Note that

$$\tilde{W}_{i,kn} \oplus (-\tilde{W}_{j,kn}) = \tilde{W}_{i,n} \oplus (-\tilde{W}_{j,n})$$

for all $i, j = 1, \ldots, q$, $k = 1, \ldots, m_n$ and $n \geq 1$.

For $k = 1, \ldots, m_n$ and $n \geq 1$, represent by $\widehat{\mu}_{i,kn}$ and $\widehat{\mathrm{Cov}}_{ij,kn}(x)$, $x \in \mathbb{R}^d$ the estimators of $\mu_i = \mathbb{E}\, Y_i(o)$ and $\mathrm{Cov}_{ij}(x)$, $x \in \mathbb{R}^d$ as defined in (3.23) and (5.6), respectively, but based on observations within $\tilde{W}_{kn}$ alone.

By virtue of these definitions, we introduce the modified estimator $\widehat{\Sigma}'_n = (\widehat{\sigma}'_{ij,n})_{i,j}$ of covariance matrix Σ by

$$\widehat{\sigma}'_{ij,n} \;=\; \frac{1}{m_n}\sum_{k=1}^{m_n}\widehat{\sigma}_{ij,kn} \quad \text{with} \quad \widehat{\sigma}_{ij,kn} = \int_{\tilde{U}_{ij,n}} \widehat{\mathrm{Cov}}_{ij,kn}(x)|\tilde{W}_n|\tilde{\Gamma}_{ij,kn}(x)\,dx\,, \qquad (5.17)$$

where $\tilde{\Gamma}_{ij,kn}(x) = \int_{\mathbb{R}^d} G_i(\tilde{W}_{kn},y)G_j(\tilde{W}_{kn},x+y)\,dy = \tilde{\Gamma}_{ij,n}(x)$ for all $x\in\mathbb{R}^d$.

Corollary 5.3 *The estimator $\widehat{\Sigma}'_n$ defined in* (5.17) *is an asymptotically unbiased estimator of Σ. If the conditions* (5.14) *and* (5.8) *of Theorem* 5.1 *are fulfilled with respect to $\tilde{U}_{ij,n}, i,j=1,\ldots,q$, then it holds that* $\lim_{n\to\infty}\mathbb{E}\,||\widehat{\Sigma}'_n-\Sigma||^2 = 0$.

Proof Notice that each estimator $\widehat{\sigma}_{ij,kn}$, $k\in\{1,\ldots,m_n\}$ has the same asymptotic properties as $\widehat{\sigma}_{ij,n}$ defined in (5.5). This follows from condition (5.16) and from the stationarity of $\mathbf{Y}$. as we shall see below.

In the first step, we deduce from (5.16) that

$$\begin{aligned}\widehat{\mathrm{Cov}}_{ij,kn}(x) \;&=\; \int_{\tilde{W}_{i,kn}\cap(\tilde{W}_{j,kn}-x)} \Big(Y_i(y)Y_j(x+y) - \widehat{\mu}_{i,kn}\,\widehat{\mu}_{j,kn}\Big)\\ &\qquad\qquad \times G_i(\tilde{W}_{kn},y)\,G_j(\tilde{W}_{kn},x+y)\,\tilde{\Gamma}^{-1}_{ij,kn}(x)\,dy\\ &=\; \int_{\tilde{W}_{i,n}\cap(\tilde{W}_{j,n}-x)} \Big(Y_i(y+h_{kn})Y_j(x+y+h_{kn}) - \widehat{\mu}_{i,kn}\,\widehat{\mu}_{j,kn}\Big)\\ &\qquad\qquad \times G_i(\tilde{W}_n,y)\,G_j(\tilde{W}_n,x+y)\,\tilde{\Gamma}^{-1}_{ij,n}(x)\,dy\end{aligned}$$

for any $x\in\tilde{U}_{ij,n}$, where furthermore

$$\widehat{\mu}_{i,kn} \;=\; \int_{\tilde{W}_{kn}} Y_i(x)\,G_i(\tilde{W}_{kn},x)\,dx \;=\; \int_{\tilde{W}_n} Y_i(x+h_{kn})\,G_i(\tilde{W}_n,x)\,dx \qquad (5.18)$$

Due to the stationarity of $\mathbf{Y}$, we deduce from the above representation that

$$\begin{aligned}\mathbb{E}\,\widehat{\sigma}_{ij,kn} \;&=\; \int_{\tilde{U}_{ij,n}}\int_{W_{i,n}\cap(W_{j,n}-x)} \mathbb{E}\big(Y_i(y+h_{kn})Y_j(x+y+h_{kn}) - \mu_i\mu_j\big)\\ &\qquad\qquad \times |\tilde{W}_n|\,G_i(\tilde{W}_n,y)\,G_j(\tilde{W}_n,x+y)\,dy\\ &\quad - |\tilde{W}_n|\int_{\tilde{U}_{ij,n}}\tilde{\Gamma}_{ij,n}(x)\int_{W_n}\int_{W_n} \mathbb{E}\big(Y_i(u+h_{kn})\,Y_j(v+h_{kn}) - \mu_i\mu_j\big)\\ &\qquad\qquad \times\; G_i(\tilde{W}_n,u)G_j(\tilde{W}_n,v)\,du\,dv\\ &=\; \int_{\tilde{U}_{ij,n}}\mathrm{Cov}_{ij}(x)|\tilde{W}_n|\tilde{\Gamma}_{ij,n}(x)\,dx \;-\; |\tilde{W}_n|\mathrm{Cov}(\widehat{\mu}_{i,0n},\widehat{\mu}_{j,0n})\int_{\tilde{U}_{ij,n}}\tilde{\Gamma}_{ij,n}(x)\,dx\,,\end{aligned}$$

where we put $\widehat{\mu}_{\nu,0n} = \int_{\tilde{W}_n} Y_\nu(x)\,G_\nu(\tilde{W}_n,x)\,dx$, $\nu=1,\ldots,q$. By this, we see that the

expectation of $\widehat{\sigma}_{ij,kn}$ is independent of index k. Consequently, the arithmetic average $\widehat{\sigma}'_{ij,n}$ is an asymptotically unbiased estimator of σ_{ij} under condition (5.2) as has been shown in the proof of Theorem 5.3.

Analogous equations hold for the second moments of $\widehat{\sigma}_{ij,kn}$, $i,j = 1,\dots,q$, $k = 1,\dots,m_n$, $n \geq 1$. In fact, one finds that

$$
\begin{aligned}
&\mathbb{E}(\widehat{\sigma}_{ij,kn} - \mathbb{E}\,\widehat{\sigma}_{ij,kn})^2 \\
&\leq\ 2\,\mathbb{E}\Big(\int_{\tilde{U}_{ij,n}} \int_{\tilde{W}_{i,n}\cap(\tilde{W}_{j,n}-x)} \Big[Y_i(y+h_{kn}) Y_j(y+h_{kn}+x) \\
&\qquad - \mathbb{E}\big(Y_i(y+h_{kn})Y_j(y+h_{kn}+x)\big)\Big]\, |\tilde{W}_n|\, G_i(\tilde{W}_n,y)\, G_j(\tilde{W}_n,x+y)\, dy\, dx \Big)^2 \\
&\quad + 2\,\mathbb{E}\Big(\int_{\tilde{W}_n} Y_i(x+h_{kn})\, G_i(\tilde{W}_n,x)\, dx \int_{\tilde{W}_n} Y_j(y+h_{kn})\, G_j(\tilde{W}_n,y)\, dy \\
&\qquad - \mathbb{E}\Big(\int_{\tilde{W}_n} Y_i(x+h_{kn})\, G_i(\tilde{W}_n,x)\, dx \int_{\tilde{W}_n} Y_j(y+h_{kn})\, G_j(\tilde{W}_n,y)\, dy \Big)\Big)^2 \\
&\quad \times \Big(\int_{\tilde{U}_{ij,n}} |\tilde{W}_n|\, \tilde{\Gamma}_{ij,n}(x)\, dx \Big)^2 \\
&\leq\ 2\, c_0^3\, \frac{|\tilde{U}_{ij,n}|}{|\tilde{W}_n|}\, c_4 + 2\, c_0^2\, \frac{|\tilde{U}_{ij,n}|}{|\tilde{W}_n|}\, |\tilde{W}_n|\, \mathbb{E}\big(\widehat{\mu}_{i,0n}\, \widehat{\mu}_{j,0n} - \mathbb{E}\big(\widehat{\mu}_{i,0n}\, \widehat{\mu}_{j,0n}\big)\big)^2
\end{aligned}
$$

by (3.5), (5.14) and (5.18) and using $\mathbb{E}(\widehat{\mu}_{i,kn}\,\widehat{\mu}_{j,kn})^\nu = \mathbb{E}(\widehat{\mu}_{i,0n}\,\widehat{\mu}_{j,0n})^\nu$ for $\nu = 1,2$ and all $k = 1,\dots,m_n$. Like in (5.11) we have

$$
\limsup_{n\to\infty} |\tilde{W}_n|\, \mathbb{E}\big(\widehat{\mu}_{i,0n}\, \widehat{\mu}_{j,0n} - \mathbb{E}\big(\widehat{\mu}_{i,0n}\, \widehat{\mu}_{j,0n}\big)\big)^2 \ =\ 0
$$

so that $\mathbb{E}(\widehat{\sigma}_{ij,kn} - \mathbb{E}\,\widehat{\sigma}_{ij,kn})^2 = 0$ as in the proof of Lemma 5.2. Combining this result with Hölder's inequality, we finally deduce that

$$
\lim_{n\to\infty} \mathbb{E}(\widehat{\sigma}'_{ij,n} - \sigma_{ij})^2 \ \leq\ \lim_{n\to\infty} \frac{1}{m_n} \sum_{k=1}^{m_n} \mathbb{E}(\widehat{\sigma}_{ij,kn} - \sigma_{ij})^2 \ =\ \lim_{n\to\infty} \mathbb{E}(\widehat{\sigma}_{ij,1n} - \sigma_{ij})^2 \ =\ 0\,.
$$

□

5.2.2 Some computational issues

To give an idea, why the subdivision of the sampling window is useful when considering the running time, we briefly discuss the algorithm used for the computation of $\widehat{\Sigma}_n$ and $\widehat{\Sigma}'_n$. The algorithm is called the `Fast Fourier Transform` (in short FFT); see [6] for details. It can be applied to obtain a fast approximation of the Fourier transform $F(y) = \int_{\mathbb{R}^d} f(x) e^{-2\pi \mathrm{i} <x,y>}\, dx$, $y \in \mathbb{R}^d$ of an integrable function $f : \mathbb{R}^d \to \mathbb{C}$ as well as the inverse Fourier transform $f(x) = \int_{\mathbb{R}^d} F(y) e^{2\pi \mathrm{i} <x,y>}\, dy$. As we shall see below, we are interested in evaluating the convolution of two functions f_1 and

f_2. Denote the convolution operator by " $*$ " and let F_1 and F_2 be the Fourier transforms of f_1 and f_2, respectively. Then it holds by the `(continuous) convolution theorem` (compare [6, p. 78]) that

$$(f_1 * f_2)(x) \;=\; \int_{\mathbb{R}^d} F_1(y) \cdot F_2(y)\, e^{2\pi \mathrm{i} <x,y>}\, dy\,, \qquad x \in \mathbb{R}^d.$$

To see how this comes into play in our computation problem, notice that for any $x \in \check{U}_{ij,n}$, we can write $\widehat{\mathrm{Cov}}_{ij,n}(-x)\Gamma_{ij,n}(-x)$ as

$$\widehat{\mathrm{Cov}}_{ij,n}(-x)\Gamma_{ij,n}(-x) \;=\; \widehat{\mathrm{C}}_{ij,n}(x) - \widehat{\mu}_{i,n}\widehat{\mu}_{j,n}\Gamma_{ji,n}(x)\,,$$

where $\widehat{\mathrm{C}}_{ij,n}(x) = \int_{W_{i,n}\cap(W_{j,n}+x)} Y_i(y)Y_j(y-x)G_i(W_n,y)G_j(W_n,y-x)\,dy$. It turns out that

$$\widehat{\mathrm{C}}_{ij,n}(x) \;=\; \int_{\mathbb{R}^d} f_{1i}(y) \cdot f_{2j}(x-y)\,dy \;=\; (f_{1i} * f_{2j})(x), \qquad x \in \check{U}_{ij,n}$$

for $f_{1i}(u) = Y_i(u)G_i(W_n,u)$ and $f_{2j}(u) = f_{1j}(-u)$, $u \in \mathbb{R}^d$. Besides this, we have

$$\Gamma_{ji,n}(x) \;=\; \int_{\mathbb{R}^d} \tilde{f}_{1i}(y) \cdot \tilde{f}_{2j}(x-y)\,dy \;=\; (\tilde{f}_{1i} * \tilde{f}_{2j})(x), \qquad x \in \check{U}_{ij,n}$$

with $\tilde{f}_{1i}(u) = G_i(W_n,u)$ and $\tilde{f}_{2j}(u) = \tilde{f}_{1j}(-u)$, $u \in \mathbb{R}^d$. Suppose now, for instance, that the integral in $\widehat{\mathrm{C}}_{ij,n}(x)$ is discretized into a sum in terms of n measurement points $y_1,\ldots,y_n \in W_{i,n}\cap(W_{j,n}+x)$. Then the computation of one Fourier transform using FFT has complexity $O(n^d \log_2(n))$; see also Section 6.4. If the measurement points and $\tilde{W}_n$ are chosen so that approximately n/m_n points lie inside of $\tilde{W}_{i,kn}\cap(\tilde{W}_{j,kn}+x)$ for each $k = 1,...,m_n$, then the same computation using the subdivision of the observation window requires only $O(n^d \log_2(n/m_n))$ operations. Confer Section 7.3 on numerical results.

5.3 Integrability of mixed moments for Boolean models

Let Ξ be a stationary Boolean model with typical grain $M_0 \in \mathcal{K}'$, and let $Y_i(x) = \varphi_i((\Xi - x)\cap K_i)$, $x \in \mathbb{R}^d$, $i = 1,\ldots,q$ be given as in Section 3.5. In the following, we show that the corresponding vector field $\mathbf{Y} = (Y_1,\ldots,Y_q)^\top$ satisfies the integrability conditions (5.7), (5.8) and (5.14) under second moment assumptions on the volume of the dilated typical grain $M_0 \oplus \check{K}$, $K \in \mathcal{K}$. To this end, we use the representation of Ξ as the union set of the Poisson particle process $N = \{Z_l\}$ with $Z_l = M_l + X_l$, $l \geq 1$ and intensity measure Λ on $\mathfrak{B}(\mathcal{K})$. By Lemma 3.4, we already know that $\mathrm{Cov}_{ij}(x)$, $x \in \mathbb{R}^d$ is absolutely integrable over $\mathbb{R}^d$ provided that $\mathbb{E}\,(|M_0 \oplus \check{K}_i|^2) < \infty$ holds for $i = 1,\ldots,q$. The subsequent theorem shows that conditions (5.7), (5.8) and (5.14) are satisfied under the very same assumption. At this, (5.19) implies (5.7) and (5.14), and (5.20) is identical to (5.8).

Theorem 5.2 *If* $\mathbb{E}\big(|M_0 \oplus \check{K}_i|^2\big) < \infty$ *for* $i = 1, \ldots, q$, *then there exist constants* $\tilde{c}_4 < \infty$ *and* $c_3 < \infty$ *such that*

$$\sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big)\big|\, dy \;<\; \tilde{c}_4 \tag{5.19}$$

and

$$\sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathbb{E}\big((Y_i(o)-\mu_i)(Y_i(y)-\mu_i)Y_j(x_1)Y_j(x_2)\big)\big|\, dy \;<\; c_3 \tag{5.20}$$

for all $i, j = 1, \ldots, q$.

In the proof of Theorem 5.2, we will make use of the following two technical results.

Lemma 5.4 *Let* $N = \{M_l + X_l\}$ *be the Poisson particle process representing the Boolean model* Ξ *with typical grain* $M_0 \in \mathcal{K}'$ *and intensity* $\lambda > 0$. *Suppose that* $Y = \{Y(x), x \in \mathbb{R}^d\}$ *is an arbitrary stationary random field of the form* (2.15) *with test set* $K \in \mathcal{K}$. *Define* $\mathcal{K}_{K+x} = \{K' \in \mathcal{K} : K' \cap (K+x) \neq \emptyset\}$ *for any* $x \in \mathbb{R}^d$ *and put*

$$\mathcal{K}^*(x_1, x_2, y) = (\mathcal{K}_K \cup \mathcal{K}_{K+x_1}) \cap (\mathcal{K}_{K+y} \cup \mathcal{K}_{K+x_2+y})$$

for $x_1, x_2, y \in \mathbb{R}^d$. *Then* $\Lambda(\mathcal{K}^*(x_1, x_2, y))$ *is finite for any* $x_1, x_2, y \in \mathbb{R}^d$, *and it holds that*

$$\begin{aligned} &\mathbb{E}\left(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+y})+N(\mathcal{K}_{K+x_2+y})}\, \mathbb{1}\big[N(\mathcal{K}^*(x_1,x_2,y)) > 0\big]\right) \\ &\quad\leq\; 16\, e^{36\lambda \mathbb{E}|M_0 \oplus \check{K}|}\, \Lambda(\mathcal{K}^*(x_1,x_2,y))\,. \end{aligned} \tag{5.21}$$

Furthermore, one has

$$\sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \Lambda(\mathcal{K}^*(x_1,x_2,y))\, dy \;\;\leq\;\; 4\,\lambda\, \mathbb{E}\big(|M_0 \oplus \check{K}|^2\big)\,.$$

Proof To start with, fix $x_1, x_2, y \in \mathbb{R}^d$ arbitrarily and write $\mathcal{K}^* = \mathcal{K}^*(x_1, x_2, y)$. Recall from the proof of Lemma 3.4 that $\mathcal{K}_{K+x} \in \mathfrak{B}(\mathcal{K})$ with

$$\Lambda(\mathcal{K}_{K+x}) = \lambda\, \mathbb{E}|M_0 \oplus \check{K}| < \infty\,.$$

For any $B, B' \in \mathfrak{B}(\mathcal{K})$, it holds that $\Lambda(B \cap B') \leq \min\{\Lambda(B), \Lambda(B')\}$ and $\Lambda(B \cup B') \leq \Lambda(B) + \Lambda(B')$. Thus, we conclude that

$$\begin{aligned} \Lambda(\mathcal{K}^*) \;&\leq\; \Lambda(\mathcal{K}_K) + \Lambda(\mathcal{K}_{K+x_1}) \\ &=\; \lambda\, \mathbb{E}|M_0 \oplus \check{K}| + \lambda\, \mathbb{E}|(M_0 \oplus \check{K}) - x_1| \;=\; 2\,\lambda\, \mathbb{E}|M_0 \oplus \check{K}| \;<\; \infty\,. \end{aligned} \tag{5.22}$$

As to the upper bound of the expectation in (5.21), note that by Theorem 2.5 (a), the random variables $N(\mathcal{K}_K \setminus \mathcal{K}^*) + N(\mathcal{K}_{K+x_1} \setminus \mathcal{K}^*)$, $N(\mathcal{K}_{K+y} \setminus \mathcal{K}^*) + N(\mathcal{K}_{K+x_2+y} \setminus \mathcal{K}^*)$ and $N(\mathcal{K}^*)$ are independent, so that one has

$$\begin{aligned} &\mathbb{E}\left(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+y})+N(\mathcal{K}_{K+x_2+y})}\, \mathbb{1}\big[N(\mathcal{K}^* > 0\big]\right) \\ &\quad=\; \mathbb{E}\, 2^{N(\mathcal{K}_K \setminus \mathcal{K}^*)+N(\mathcal{K}_{K+x_1} \setminus \mathcal{K}^*)}\; \mathbb{E}\, 2^{N(\mathcal{K}_{K+y} \setminus \mathcal{K}^*)+N(\mathcal{K}_{K+x_2+y} \setminus \mathcal{K}^*)} \\ &\qquad\times \mathbb{E}\left(2^{4N(\mathcal{K}^*)} \mathbb{1}\big[N(\mathcal{K}^*) > 0\big]\right). \end{aligned}$$

From the Cauchy–Schwarz inequality and Theorem 2.5 (c) it follows that

$$\begin{aligned}\mathbb{E}\, 2^{N(\mathcal{K}_{K+u}\setminus\mathcal{K}^*)+N(\mathcal{K}_{K+v}\setminus\mathcal{K}^*)} &\leq \sqrt{\mathbb{E}\, 4^{N(\mathcal{K}_{K+u})}\, \mathbb{E}\, 4^{N(\mathcal{K}_{K+v})}} \\ &\leq e^{3\lambda E|M_0\oplus\check{K}|} < \infty\end{aligned}$$

for any $u, v \in \mathbb{R}^d$. Besides this, we deduce from the Poisson distribution of $N(\mathcal{K}^*)$ that

$$\begin{aligned}\mathbb{E}\left(16^{N(\mathcal{K}^*)}\mathbb{1}\big[N(\mathcal{K}^*)>0\big]\right) &= \mathbb{E}\, 16^{N(\mathcal{K}^*)} - \mathbb{E}\,\mathbb{1}\big[N(\mathcal{K}^*)=0\big] \\ &= e^{15\Lambda(\mathcal{K}^*)}\big(1-e^{-16\Lambda(\mathcal{K}^*)}\big) \\ &\leq 16\, e^{30\lambda\,\mathbb{E}\,|M_0\oplus\check{K}|}\Lambda(\mathcal{K}^*)\,,\end{aligned}$$

using $1-e^{-s}\leq s$, $s\geq 0$ and (5.22) in the inequality. This completes the proof of the second assertion of the lemma.

It remains to verify the integrability of $\Lambda(\mathcal{K}^*) = \Lambda(\mathcal{K}^*(x_1,x_2,y))$ with respect to $y\in\mathbb{R}^d$ and that the integral can be bounded by a finite constant which is independent of $x_1, x_2 \in \mathbb{R}^d$. An application of Campbell's formula (2.9) yields

$$\begin{aligned}&\int_{\mathbb{R}^d}\Lambda(\mathcal{K}^*(x_1,x_2,y))\,dy \\ &= \lambda\,\mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{1}\big(z\in(\check{M}_0\oplus K)\cup(\check{M}_0\oplus(K+x_1))\big) \\ &\qquad\qquad \times\mathbb{1}\big(z-y\in(\check{M}_0\oplus K)\cup(\check{M}_0\oplus(K+x_2))\big)\,dy\,dz \\ &\leq \lambda\,\mathbb{E}\int_{\mathbb{R}^d}\big[\mathbb{1}(z\in\check{M}_0\oplus K)+\mathbb{1}(z\in\check{M}\oplus(K+x_1))\big] \\ &\qquad\qquad\times\int_{\mathbb{R}^d}\big[\mathbb{1}(y\in(M_0\oplus\check{K})+z)+\mathbb{1}(y\in(M_0\oplus\check{K})-x_2+z))\big]\,dy\,dz \\ &= 4\lambda\,\mathbb{E}\big(|M_0\oplus\check{K}|^2\big)<\infty\end{aligned}$$

for arbitrary $x_1, x_2 \in \mathbb{R}^d$. □

Note that the bounds derived in Lemma 5.4 do not provide the best estimates, but sufficiently simple ones. The same holds true for the upcoming Lemma 5.5.

Lemma 5.5 *Given the setup of Lemma* 5.4, *let*

$$\mathcal{K}^{**}(x_1,x_2,y) = \mathcal{K}_{K+y}\cap(\mathcal{K}_K\cup\mathcal{K}_{K+x_1}\cup\mathcal{K}_{K+x_2})$$

with arbitrary $x_1, x_2, y \in \mathbb{R}^d$. *Then it holds that* $\Lambda(\mathcal{K}^{**}(x_1,x_2,y))<\infty$ *for any* x_1, x_2 *and* $y\in\mathbb{R}^d$. *Moreover, one has*

$$\begin{aligned}&\mathbb{E}\left(2^{\,N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+x_2})+N(\mathcal{K}_{K+y})}\,\mathbb{1}\big[N(\mathcal{K}^{**}(x_1,x_2,y))>0\big]\right) \\ &\quad\leq 16\, e^{23\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\Lambda(\mathcal{K}^{**}(x_1,x_2,y))\,,\end{aligned} \tag{5.23}$$

where

$$\sup_{x_1,x_2\in\mathbb{R}^d}\int_{\mathbb{R}^d}\Lambda(\mathcal{K}^{**}(x_1,x_2,y))\,dy \;\leq\; 3\,\lambda\,\mathbb{E}\big(|M_0\oplus\check{K}|^2\big)\,.$$

Proof Put $\mathcal{K}^{**} = \mathcal{K}^{**}(x_1, x_2, y)$ and observe that $\mathcal{K}^{**} \subset \mathcal{K}_{K+y}$. This results in

$$\Lambda(\mathcal{K}^{**}) \;\le\; \Lambda(\mathcal{K}_{K+y}) \;=\; \mathbb{E}\,|M_0 \oplus \check{K}| \;<\; \infty\,.$$

Analogously to the proof of Lemma 5.4, we get by the independence of the involved random variables

$$\begin{aligned}
&\mathbb{E}\left(2^{\,N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+x_2})+N(\mathcal{K}_{K+y})}\,\mathbb{1}\big[N(\mathcal{K}^{**}(x_1,x_2,y))>0\big]\right)\\
&\quad= \mathbb{E}\,2^{\,N(\mathcal{K}_K\setminus\mathcal{K}^{**})+N(\mathcal{K}_{K+x_1}\setminus\mathcal{K}^{**})+N(\mathcal{K}_{K+x_2}\setminus\mathcal{K}^{**})}\;\mathbb{E}\,2^{N(\mathcal{K}_{K+y}\setminus\mathcal{K}^{**})}\\
&\qquad\times\mathbb{E}\left(2^{4N(\mathcal{K}^{**})}\,\mathbb{1}\big[N(\mathcal{K}^{**})>0\big]\right)\\
&\quad\le\; 16\,e^{7\lambda\mathbb{E}|M_0\oplus\check{K}|}\,e^{\lambda\mathbb{E}|M_0\oplus\check{K}|}\,e^{15\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\Lambda(\mathcal{K}^{**})\,,
\end{aligned}$$

where the inequality is due to Hölder's inequality and $1-e^{-s}\le s$, $s\ge 0$ as before. Finally, one finds that for any $x_1, x_2 \in \mathbb{R}^d$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Lambda\big(\mathcal{K}^{**}(x_1,x_2,y)\big)\,dy\\
&\quad=\;\lambda\,\mathbb{E}\int_{\mathbb{R}^d}\mathbb{1}\big(z\in(\check{M}_0\oplus K)\cup(\check{M}_0\oplus K+x_1)\cup(\check{M}_0\oplus K+x_2)\big)\\
&\qquad\qquad\times\int_{\mathbb{R}^d}\mathbb{1}\big(y\in M_0\oplus\check{K}+z\big)\,dy\,dz\\
&\quad\le\;3\lambda\,\mathbb{E}\big(|M_0\oplus\check{K}|^2\big)<\infty\,.
\end{aligned}$$

□

Proof of Theorem 5.2 In the first step we consider the case $i=j$ and neglect the indices to improve readability.

In what follows, we are going to prove assertion (5.19) of Theorem 5.2 using the definitions and results of Lemma 5.4. Conditioning on the event $A=\{N(\mathcal{K}^*)>0\}$, one can write

$$\begin{aligned}
&\mathrm{Cov}\big(Y(o)Y(x_1),Y(y)Y(x_2+y)\big)\\
&\quad=\;\mathbb{E}\big(\,Y(o)Y(x_1)\big[Y(y)Y(x_2+y)-\mathbb{E}\,(Y(y)Y(x_2+y))\big]\,\mathbb{1}_A\big) &\text{(5.24.a)}\\
&\qquad+\mathbb{E}\big(\,Y(o)Y(x_1)\big[Y(y)Y(x_2+y)-\mathbb{E}\,(Y(y)Y(x_2+y))\big]\,\mathbb{1}_{A^c}\big)\,. &\text{(5.24.b)}
\end{aligned}$$

The summand in (5.24.a) can be bounded above as follows

$$\begin{aligned}
&|\mathbb{E}\big(\,Y(o)Y(x_1)\big[Y(y)Y(x_2+y)-\mathbb{E}\,(Y(y)Y(x_2+y))\big]\,\mathbb{1}_A\big)|\\
&\quad\le\;\mathbb{E}\big(\,|Y(o)Y(x_1)Y(y)Y(x_2+y)|\,\mathbb{1}_A\big)+\mathbb{E}\big(\,|Y(o)Y(x_1)|\,\mathbb{1}_A\big)\,\mathbb{E}\big(|Y(o)Y(x_2)|\big)\\
&\quad\le\;c^4(K)\,\mathbb{E}\big(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+y})+N(\mathcal{K}_{K+x_2+y})}\,\mathbb{1}_A\big)\\
&\qquad+c^4(K)\,\mathbb{E}\big(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})}\,\mathbb{1}_A\big)\,\mathbb{E}\big(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_2})}\big)\,,
\end{aligned}\tag{5.25}$$

where $c(K)$ is the finite constant introduced in (2.16). Next to the triangle inequality, we used the stationarity of Y in the first inequality. The second estimate is derived in

similar fashion to the proof of Lemma 2.9 and inequality (3.10), that is, it pointwise holds that

$$|Y(v)| \;\leq\; c(K)\, 2^{N(\mathcal{K}_{K+v})}, \qquad v \in \mathbb{R}^d. \tag{5.26}$$

When investigating the summand in (5.24.b), recall that for every $x \in \mathbb{R}^d$, one has $Y(x)\mathbb{1}_{A^c} = Y_{|\mathcal{K}_{K+x}\setminus\mathcal{K}^*}(x)\mathbb{1}_{A^c}$ almost surely, where $Y_{|B}(x)$ was defined as $Y_{|B}(x) = \varphi(\bigcup_{l:\, Z_l\in B}(Z_l - x)\cap K)$ for any $B \in \mathfrak{B}(\mathcal{K})$; compare the considerations on page 41. In particular, it almost surely holds that

$$Y(o)Y(x_1)\cdot\mathbb{1}_{A^c} \;=\; Y_{|\mathcal{K}_K\setminus\mathcal{K}^*}(o)Y_{|\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\cdot\mathbb{1}_{A^c} \tag{5.27}$$

and

$$Y(y)Y(x_2+y)\cdot\mathbb{1}_{A^c} \;=\; Y_{|\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{|\mathcal{K}_{K+x_2+y}\setminus\mathcal{K}^*}(x_2+y)\cdot\mathbb{1}_{A^c}\,. \tag{5.28}$$

In formulae (5.27) and (5.28), the random variables $Y_{|\mathcal{K}_K\setminus\mathcal{K}^*}(o)\,Y_{|\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\,\mathbb{1}_{A^c}$ and $Y_{|\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)\,Y_{|\mathcal{K}_{K+x_2+y}\setminus\mathcal{K}^*}(x_2+y)$ are independent, since the involved particles are mutually independent. Hence, we can write the summand in (5.24.b) as

$$\begin{aligned}
&\mathbb{E}\big(\,Y(o)Y(x_1)\big[Y(y)Y(x_2+y) - \mathbb{E}\,(Y(y)Y(x_2+y))\big]\,\mathbb{1}_{A^c}\,\big)\\
&=\; \mathbb{E}\big(\,Y_{|\mathcal{K}_K\setminus\mathcal{K}^*}(o)Y_{|\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\,\mathbb{1}_{A^c}\big)\\
&\qquad\times\big[\mathbb{E}\big(Y_{|\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{|\mathcal{K}_{K+x_2+y}\setminus\mathcal{K}^*}(x_2+y)\big) - \mathbb{E}\big(Y(y)Y(x_2+y)\big)\big]\\
&=\; \mathbb{E}\big(\,Y(o)Y(x_1)\,\mathbb{1}_{A^c}\big)\\
&\qquad\times\mathbb{E}\big((Y_{|\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{|\mathcal{K}_{K+x_2+y}\setminus\mathcal{K}^*}(x_2+y) - Y(y)Y(x_2+y))(\mathbb{1}_A+\mathbb{1}_{A^c})\big)\\
&=\; \mathbb{E}\big(Y(o)Y(x_1)\,\mathbb{1}_{A^c}\big)\mathbb{E}\big(\big[Y_{|\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{|\mathcal{K}_{K+x_2+y}\setminus\mathcal{K}^*}(x_2+y) - Y(y)Y(x_2+y)\big]\,\mathbb{1}_A\,\big)\,.
\end{aligned}$$

From the final bound of the above equations and an adequate version of (5.26), we see that

$$\begin{aligned}
&\big|\mathbb{E}\big(\,Y(o)Y(x_1)\big[Y(y)Y(x_2+y) - \mathbb{E}\,(Y(y)Y(x_2+y))\big]\,\mathbb{1}_{A^c}\,\big)\big|\\
&\quad\leq\; \mathbb{E}\big(|Y(o)Y(x_1)|\big)\\
&\qquad\times\mathbb{E}\big(\big[|Y_{|\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{|\mathcal{K}_{K+x_2+y}\setminus\mathcal{K}^*}(x_2+y)| + |Y(y)Y(x_2+y)|\big]\,\mathbb{1}_A\big)\\
&\quad\leq\; c^4(K)\,\mathbb{E}\big(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})}\big)\\
&\qquad\times\mathbb{E}\big(\big[2^{N(\mathcal{K}_{K+y}\setminus\mathcal{K}^*)+N(\mathcal{K}_{K+x_2+y}\setminus\mathcal{K}^*)} + 2^{N(\mathcal{K}_{K+y})+N(\mathcal{K}_{K+x_2+y})}\big]\,\mathbb{1}_A\big)\,.
\end{aligned} \tag{5.29}$$

Now, we insert the upper bounds (5.25) and (5.29) for (5.24.a) and (5.24.b), respectively, and deduce from Lemma 5.4 that

$$\begin{aligned}
&\big|\mathrm{Cov}\big(Y(o)Y(x_1), Y(y)Y(x_2+y)\big)\big|\\
&\leq\; c^4(K)\,\mathbb{E}\big(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+y})+N(\mathcal{K}_{K+x_2+y})}\,\mathbb{1}_A\big)\\
&\quad + c^4(K)\,\mathbb{E}\big(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})}\,\mathbb{1}_A\big)\,\mathbb{E}\big(2^{N(\mathcal{K}_K+y)+N(\mathcal{K}_{K+x_2+y})}\big)\\
&\quad + c^4(K)\,\mathbb{E}\big(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})}\big)\\
&\qquad\qquad\times\mathbb{E}\big(\big(2^{N(\mathcal{K}_{K+y}\setminus\mathcal{K}^*)+N(\mathcal{K}_{K+x_2+y}\setminus\mathcal{K}^*)} + 2^{N(\mathcal{K}_{K+y})+N(\mathcal{K}_{K+x_2+y})}\big)\,\mathbb{1}_A\big)
\end{aligned}$$

$$\begin{aligned} &\leq 4\,c^4(K)\,e^{3\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\mathbb{E}\big(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+y})+N(\mathcal{K}_{K+x_2+y})}\,\mathbb{1}_A\big)\\ &\leq 64\,c^4(K)\,e^{39\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\Lambda(\mathcal{K}^*(x_1,x_2,y)) \end{aligned}$$

for all $x_1, x_2, y \in \mathbb{R}^d$. This finally leads to the estimate

$$\begin{aligned} &\sup_{x_1,x_2\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathrm{Cov}\big(Y(o)Y(x_1),Y(y)Y(x_2+y)\big)\big|\,dy\\ &\quad\leq 256\,c^4(K)\,\lambda\,e^{39\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\mathbb{E}\big(|M_0\oplus\check{K}|^2\big), \end{aligned}$$

where $\mathbb{E}\big(|M_0\oplus\check{K}|^2\big) < \infty$ by assumption. Defining constant $\tilde{c}_4$ as the right hand side of the above inequality yields (5.19).

The existence of a constant $c_3 < \infty$ such that (5.20) is satisfied can be proven following the same idea as used for (5.19). Here, let $\mathcal{K}^{**} = \mathcal{K}^{**}(x_1,x_2,y) = \mathcal{K}_{K+y} \cap (\mathcal{K}_K \cup \mathcal{K}_{K+x_1} \cup \mathcal{K}_{K+x_2})$ for $x_1, x_2, y \in \mathbb{R}^d$ and consider the event $A' = \{N(\mathcal{K}^{**}) > 0\}$ so that

$$\begin{aligned} &\big|\mathbb{E}\big((Y(o)-\mu)(Y(y)-\mu)Y(x_1)Y(x_2)\big)\big|\\ &\quad\leq \big|\mathbb{E}\big((Y(o)-\mu)Y(x_1)Y(x_2)\,(Y(y)-\mu)\mathbb{1}_{A'}\big)\big|\\ &\qquad+\mathbb{E}\big|(Y(o)-\mu)Y(x_1)Y(x_2)\big|\,\big|\mathbb{E}\big((Y_{|\mathcal{K}_{K+y}\setminus\mathcal{K}^{**}}(y)-Y(y))\mathbb{1}_{A'}\big)\big|. \end{aligned}$$

As before, we firstly see that

$$\begin{aligned} &\big|\mathbb{E}\big((Y(o)-\mu)(Y(y)-\mu)Y(x_1)Y(x_2)\,\mathbb{1}_{A'}\big)\big|\\ &\quad\leq c^4(K)\,\mathbb{E}\Big(\big[2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+y})}+2^{N(\mathcal{K}_K)}\,\mathbb{E}\,2^{N(\mathcal{K}_K)}+2^{N(\mathcal{K}_{K+y})}\,\mathbb{E}\,2^{N(\mathcal{K}_K)}\\ &\qquad\qquad+\big(\mathbb{E}\,2^{N(\mathcal{K}_K)}\big)^2\big]\,2^{N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+x_2})}\,\mathbb{1}_{A'}\Big)\\ &\quad\leq 4\,c^4(K)\,e^{2\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\mathbb{E}\left(2^{N(\mathcal{K}_K)+N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+x_2})+N(\mathcal{K}_{K+y})}\,\mathbb{1}_{A'}\right) \end{aligned}$$

and secondly

$$\begin{aligned} &\mathbb{E}\big|(Y(o)-\mu)Y(x_1)Y(x_2)\big|\,\big|\mathbb{E}\big((Y_{|\mathcal{K}_{K+y}\setminus\mathcal{K}^{**}}(y)-Y(y))\mathbb{1}_{A'}\big)\big|\\ &\quad\leq c^4(K)\,\mathbb{E}\big(\big[2^{N(\mathcal{K}_K)}+\mathbb{E}\,2^{N(\mathcal{K}_K)}\big]\,2^{N(\mathcal{K}_{K+x_1})+N(\mathcal{K}_{K+x_2})}\big)\\ &\qquad\times\mathbb{E}\,2^{N(\mathcal{K}_{K+y}\setminus\mathcal{K}^{**})}\,\mathbb{E}\big(\big[1+2^{N(\mathcal{K}^{**})}\big]\,\mathbb{1}_{A'}\big)\\ &\quad\leq 2\,c^4(K)\,\big(e^{8\lambda\mathbb{E}|M_0\oplus\check{K}|}+e^{5\lambda\mathbb{E}|M_0\oplus\check{K}|}\big)\,\mathbb{E}\left(2^{N(\mathcal{K}_{K+y})}\,\mathbb{1}_{A'}\right). \end{aligned}$$

As a consequence of Lemma 5.5 one therefore has

$$\begin{aligned} &\sup_{x_1,x_2\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathbb{E}\big((Y(o)-\mu)(Y(y)-\mu)Y(x_1)Y(x_2)\big)\big|\,dy\\ &\quad\leq\Big(4\,c^4(K)\,e^{2\lambda\mathbb{E}|M_0\oplus\check{K}|}+2\,c^4(K)\,e^{8\lambda\mathbb{E}|M_0\oplus\check{K}|}+2\,c^4(K)\,e^{5\lambda\mathbb{E}|M_0\oplus\check{K}|}\Big)\\ &\qquad\times 16\,\lambda\,e^{23\lambda\mathbb{E}|M_0\oplus\check{K}|}\int_{\mathbb{R}^d}\Lambda\big(\mathcal{K}^{**}(x_1,x_2,y)\big)\,dy\\ &\quad\leq 384\,c^4(K)\,\lambda\,e^{31\lambda\mathbb{E}|M_0\oplus\check{K}|}\,\mathbb{E}\big(|M_0\oplus\check{K}|^2\big) \;<\; \infty. \end{aligned}$$

This completes the proof for $i = j$. The case $i \neq j$ can be treated analogously, putting $c(K) = \max\{c(K_i) : i = 1, \ldots, q\}$ and replacing $\mathbb{E}\big(|M_0 \oplus \check{K}|^\nu\big)$ by $\max\{\mathbb{E}\big(|M_0 \oplus \check{K}_i|^\nu\big) : i = 1, \ldots, q\}$ for $\nu = 1, 2$. □

As a direct consequence of Theorems 5.1 and 5.2 we obtain the following result.

Corollary 5.4 *Suppose that Ξ is a Boolean model with convex and compact typical grain M_0 satisfying $\mathbb{E}\big(|M_0 \oplus \check{K}_i|^2\big) < \infty$ for $i = 1, \ldots, q$. Then the estimators $\widehat{\Sigma}_n$ and $\widehat{\Sigma}'_n$ defined in (5.5) and (5.17), respectively, are asymptotically unbiased and mean–square consistent estimators of covariance matrix Σ.*

5.4 An empirical covariance matrix

The estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{ij,n})_{i,j=1,\ldots,q}$ introduced in (5.5) makes use of the fact that σ_{ij} is basically the integral of $\mathrm{Cov}_{ij}(x)$ over $\mathbb{R}^d$ for each pair $i, j = 1, \ldots, q,$. Thus, integrating the estimate $\widehat{\mathrm{Cov}}_{ij}(x)$ over an unboundedly increasing averaging set $U_{ij,n}$ is a quite intuitive approach. Another possibility is to employ the equations $\sigma_{ij} = \lim_{n\to\infty} |W_n| \mathrm{Cov}(\widehat{\mu}_{i,n}, \widehat{\mu}_{j,n})$ for $i, j = 1, \ldots, q$. Provided that there exist samples $\sqrt{|W_n|}\,\widehat{\mu}_{i,kn}$ and $\sqrt{|W_n|}\,\widehat{\mu}_{j,kn}$ for $k = 1, \ldots, m_n$ and $n \geq 1$, another natural estimator of σ_{ij} is given by the empirical covariance. In order to obtain these samples, we pick up the idea of Section 5.2.

Let $\{\tilde{W}_n\}$ and $\{\tilde{U}_{ij,n}\}$, $i, j = 1, \ldots, q$ be given as in Section 5.2, and suppose that $\{m_n\}$ is an non–decreasing sequence of integers that tends to infinity. With respect to $\tilde{W}_{kn}$, $k = 1, \ldots, m_n$, we additionally assume that there exists some $r > 0$ such that

$$\tilde{W}_{kn} \cap \tilde{W}_{\ell n} \subset \partial \tilde{W}_{kn} \oplus B_r(o) \qquad \text{for } k, \ell \in \{1, \ldots, m_n\} \text{ with } k \neq \ell. \tag{5.30}$$

Given this setup, define the estimator $\widetilde{\Sigma}_n = (\widetilde{\sigma}_{ij,n})_{i,j=1,\ldots,q}$, $n \geq 1$ by the formula

$$\widetilde{\sigma}_{ij,n} = \frac{|\tilde{W}_n|}{m_n - 1} \sum_{k=1}^{m_n} \big(\widehat{\mu}_{i,kn} - \overline{\mu}_{i,n}\big)\big(\widehat{\mu}_{j,kn} - \overline{\mu}_{j,n}\big), \tag{5.31}$$

where $\overline{\mu}_{i,n} = \dfrac{1}{m_n} \sum_{k=1}^{m_n} \widehat{\mu}_{i,kn}$ for $i = 1, \ldots, q$.

5.4.1 Asymptotic unbiasedness

Lemma 5.6 *If $\int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx < \infty$ for $i, j = 1, \ldots, q$ (i.e., (5.2) holds), then the estimator $\widetilde{\Sigma}_n$, $n \geq 1$ is asymptotically unbiased for Σ.*

Proof Fix $i, j \in \{1, \ldots, q\}$ arbitrarily. By elementary transformation, one finds

$$\begin{aligned}
\widetilde{\sigma}_{ij,n} &= \frac{|\tilde{W}_n|}{m_n - 1} \sum_{k=1}^{m_n} \big(\widehat{\mu}_{i,kn} - \overline{\mu}_{i,n} \pm \mu_i\big)\big(\widehat{\mu}_{j,kn} - \overline{\mu}_{j,n} \pm \mu_j\big) \\
&= \frac{|\tilde{W}_n|}{m_n - 1} \sum_{k=1}^{m_n} \big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,kn} - \mu_j\big) \\
&\quad - \frac{|\tilde{W}_n|}{m_n(m_n - 1)} \sum_{k=1}^{m_n} \sum_{\ell=1}^{m_n} \big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,\ell n} - \mu_j\big) \\
&= \frac{|\tilde{W}_n|}{m_n} \sum_{k=1}^{m_n} \big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,kn} - \mu_j\big) \qquad (5.32.a) \\
&\quad - \frac{|\tilde{W}_n|}{m_n(m_n - 1)} \sum_{k=1}^{m_n} \sum_{\ell=1, \ell \neq k}^{m_n} \big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,\ell n} - \mu_j\big)\,. \qquad (5.32.b)
\end{aligned}$$

Regarding the expectation of the summand in (5.32.a), it holds that

$$\begin{aligned}
&\frac{|\tilde{W}_n|}{m_n} \sum_{k=1}^{m_n} \mathbb{E}\,\big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,kn} - \mu_j\big) \\
&= \frac{|\tilde{W}_n|}{m_n} \sum_{k=1}^{m_n} \int_{\tilde{W}_{kn}} \int_{\tilde{W}_{kn}} \mathrm{Cov}_{ij}(y - x)\, G_i(\tilde{W}_{kn}, x) G_j(\tilde{W}_{kn}, y)\, dx\, dy \\
&= \frac{|\tilde{W}_n|}{m_n} m_n \int_{\tilde{W}_n} \int_{\tilde{W}_n} \mathrm{Cov}_{ij}(y - x)\, G_i(\tilde{W}_n, x) G_j(\tilde{W}_n, y)\, dx\, dy \\
&= \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, |\tilde{W}_n| \tilde{\Gamma}_{ij,n}(x)\, dx\,.
\end{aligned}$$

As we have seen before, the final expression converges to $\sigma_{ij} < \infty$ as $n \to \infty$ by Lebesgue's dominated convergence theorem provided that (5.2) holds.

The second step is to show that the expectation of the summand in (5.32.b) tends to zero. Then the assertion of the lemma is proven. In fact, one has

$$\mathbb{E}\,\big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,\ell n} - \mu_j\big) \;\leq\; c_0^2 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, \frac{|\tilde{W}_{kn} \cap (\tilde{W}_{\ell n} - x)|}{|\tilde{W}_n|^2}\, dx$$

for any $k \neq \ell$, $k, l = 1, \ldots, m_n$, $n \geq 1$. Since $\tilde{W}_{kn} \cap \tilde{W}_\ell \subset \partial \tilde{W}_{kn} \oplus B_r(o)$ by condition (5.30), it follows that

$$\begin{aligned}
\sum_{\ell=1, \ell \neq k}^{m_n} \big|\tilde{W}_{kn} \cap \big(\tilde{W}_{\ell n} - x\big)\big| &\leq \sum_{\ell=1, \ell \neq k}^{m_n} \big|\tilde{W}_{kn} \cap \big(\tilde{W}_{\ell n} \oplus B_{|x|}(o)\big)\big| \\
&\leq (m_n - 1)\,\big|\tilde{W}_{kn} \cap \big(\partial \tilde{W}_{kn} \oplus B_{|x|+r}(o)\big)\big| \\
&= (m_n - 1)\,\big|\tilde{W}_n \cap \big(\partial \tilde{W}_n \oplus B_{|x|+r}(o)\big)\big|
\end{aligned}$$

for each $k = 1, \ldots, m_n$, $n \geq 1$. By virtue of conditions (3.1) and (5.2), we finally conclude that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\Big(\frac{|\tilde{W}_n|}{m_n(m_n-1)} \sum_{k=1}^{m_n} \sum_{\ell=1,\ell\neq k}^{m_n} \big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,\ell n} - \mu_j\big)\Big) \\
&\quad \leq \quad c_0^2 \lim_{n\to\infty} \frac{|\tilde{W}_n|}{m_n} \sum_{k=1}^{m_n} \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)| \, \frac{\big|\tilde{W}_{kn} \cap \big(\partial\tilde{W}_{kn} \oplus B_{|x|+r}(o)\big)\big|}{|\tilde{W}_n|^2} \, dx \\
&\quad = \quad c_0^2 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)| \lim_{n\to\infty} \frac{\big|\tilde{W}_n \cap \big(\partial\tilde{W}_n \oplus B_{|x|+r}(o)\big)\big|}{|\tilde{W}_n|} \, dx \;=\; 0\,. \qquad \square
\end{aligned}$$

According to the final line of the proof of Lemma 5.6, the bias of $\widetilde{\Sigma}_n$ tends to zero even without the imposed condition that $\lim_{n\to\infty} m_n = \infty$.

5.4.2 Mean–square consistency

Mean–square consistency of $\widetilde{\Sigma}_n$ is subject to integrability conditions imposed on certain fourth order cumulant functions $s_{ij}^{(4)} : \mathbb{R}^{4d} \to \mathbb{R}$, $i, j = 1, \ldots, q$ of random field $\mathbf{Y}$. These characteristics are defined by

$$\begin{aligned}
s_{ij}^{(4)}(x_1, x_2, x_3, x_4) \;=\; & \mathbb{E}\left((Y_i(x_1) - \mu_i)(Y_j(x_2) - \mu_j)(Y_i(x_3) - \mu_i)(Y_j(x_4) - \mu_j)\right) \\
& - \mathrm{Cov}_{ij}(x_2 - x_1)\, \mathrm{Cov}_{ij}(x_4 - x_3) \\
& - \mathrm{Cov}_{ii}(x_3 - x_1)\, \mathrm{Cov}_{jj}(x_4 - x_2) \\
& - \mathrm{Cov}_{ij}(x_4 - x_1)\, \mathrm{Cov}_{ij}(x_2 - x_3).
\end{aligned}$$

for any $x_1, x_2, x_3, x_4 \in \mathbb{R}^d$.

Theorem 5.3 *If there exists a finite constant c_5 such that*

$$\int_{\mathbb{R}^{3d}} \mid s_{ij}^{(4)}(o, x_1, x_2, x_3) \mid \, d(x_1, x_2, x_3) \;<\; c_5\,, \tag{5.33}$$

then $\widetilde{\Sigma}_n$ is a mean-square consistent estimator of Σ.

Proof By virtue of Lemma 5.6, it suffices to verify that $\lim_{n\to\infty} \mathrm{Var}\,(\widetilde{\sigma}_{ij,n}) = 0$. Write $\omega_{ij,kn}(x, y) = G_i(\tilde{W}_{kn}, x) G_j(\tilde{W}_{kn}, y)$ for all $i, j = 1, \ldots, q$, $x, y \in \mathbb{R}^d$. In addition, put $Y_i'(x) = Y_i(x) - \mu_i$ for $x \in \mathbb{R}^d$. By application of Minkowski's inequality, we once more consider the expressions (5.32.a) and (5.32.b) separately. For the variance of the summand in (5.32.a), one finds that

$$\begin{aligned}
&\mathbb{E}\Big(\frac{|\tilde{W}_n|}{m_n} \sum_{k=1}^{m_n} \Big[\big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,kn} - \mu_j\big) - \int_{\tilde{W}_{kn}^2} \mathrm{Cov}_{ij}(y - x)\, \omega_{ij,kn}(x, y)\, d(x, y) \Big] \Big)^2 \\
&\quad = \quad \frac{|\tilde{W}_n|^2}{m_n^2} \sum_{k,\ell=1}^{m_n} \int_{\tilde{W}_{kn}^2 \times \tilde{W}_{\ell n}^2} \Big[\mathbb{E}\big(Y_i'(y_1)\, Y_j'(y_2)\, Y_i'(y_3)\, Y_j'(y_4)\big) \\
&\qquad - \mathrm{Cov}_{ij}(y_2 - y_1) \mathrm{Cov}_{ij}(y_4 - y_3) \Big]\, \omega_{ij,kn}(y_1, y_2)\, \omega_{ij,\ell n}(y_3, y_4)\, d(y_1, y_2, y_3, y_4)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|\tilde{W}_n|^2}{m_n^2} \sum_{k,\ell=1}^{m_n} \int_{\tilde{W}_{kn}^2 \times \tilde{W}_{\ell n}^2} \Big[s^{(4)}(o, y_2 - y_1, y_3 - y_1, y_4 - y_1) \\
&\quad + \mathrm{Cov}_{ii}(y_3 - y_1)\mathrm{Cov}_{jj}(y_4 - y_2) + \mathrm{Cov}_{ij}(y_4 - y_1)\mathrm{Cov}_{ij}(y_2 - y_3) \Big] \\
&\quad \times \omega_{ij,kn}(y_1, y_2)\, \omega_{ij,\ell n}(y_3, y_4)\, d(y_1, y_2, y_3, y_4) \\
&\leq \frac{|\tilde{W}_n|^2}{m_n^2} \sum_{k,\ell=1}^{m_n} \int_{\mathbb{R}^{3d}} \big| s^{(4)}(o, x_1, x_2, x_3) \big| \, \frac{c_0^3}{|\tilde{W}_n|^3}\, d(x_1, x_2, x_3) \\
&\quad + \int_{\mathbb{R}^d} \big| \mathrm{Cov}_{ii}(x) \big| \, c_0^2 \, \frac{|\tilde{W}_{kn} \cap (\tilde{W}_{\ell n} - x)|}{|\tilde{W}_n|^2}\, dx \int_{\mathbb{R}^d} \big| \mathrm{Cov}_{jj}(y) \big| \, \frac{c_0}{|\tilde{W}_n|}\, dy \\
&\quad + \int_{\mathbb{R}^d} \big| \mathrm{Cov}_{ij}(x) \big| \, c_0^2 \, \frac{|\tilde{W}_{kn} \cap (\tilde{W}_{\ell n} - x)|}{|\tilde{W}_n|^2}\, dx \int_{\mathbb{R}^d} \big| \mathrm{Cov}_{ij}(y) \big| \, \frac{c_0}{|\tilde{W}_n|}\, dy \,,
\end{aligned}
$$

where the inequality is due to (3.19) and (3.20). Repeating the arguments of the proof of Lemma 5.6, it follows that

$$
\begin{aligned}
&\mathbb{E}\bigg(\frac{|\tilde{W}_n|}{m_n} \sum_{k=1}^{m_n} \Big[\big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,kn} - \mu_j\big) - \int_{\tilde{W}_{kn}^2} \mathrm{Cov}_{ij}(y - x)\, \omega_{ij,k}(x, y)\, d(x, y) \Big] \bigg)^2 \\
&\leq \frac{c_0^3}{|\tilde{W}_n|} \int_{\mathbb{R}^{3d}} |\, s_{ij}^{(4)}(o, x_1, x_2, x_3)|\, d(x_1, x_2, x_3) \\
&\quad + c_0^3 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ii}(x)| \bigg(\frac{1}{m_n} + \frac{|\tilde{W}_n \cap (\partial \tilde{W}_n \oplus B_{|x|+r}(o))|}{|\tilde{W}_n|} \bigg) dx \int_{\mathbb{R}^d} |\mathrm{Cov}_{jj}(x)|\, dx \\
&\quad + c_0^3 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)| \bigg(\frac{1}{m_n} + \frac{|\tilde{W}_n \cap (\partial \tilde{W}_n \oplus B_{|x|+r}(o))|}{|\tilde{W}_n|} \bigg) dx \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx \,.
\end{aligned}
$$

The final bound tends to zero as $n \to \infty$, since both $\lim_{n\to\infty} |\partial \tilde{W}_n \oplus B_{|x|+r}(o)|/|\tilde{W}_n| = 0$ and $\lim_{n\to\infty} m_n = \infty$.

It can be easily seen that the variance of the second summand, (5.32.b), admits similar estimates and, consequently, converges to zero as well. In fact, we have

$$
\begin{aligned}
&\mathbb{E}\bigg(\frac{|\tilde{W}_n|}{m_n(m_n - 1)} \sum_{k=1}^{m_n} \sum_{\ell=1, \ell \neq k}^{m_n} \Big[\big(\widehat{\mu}_{i,kn} - \mu_i\big)\big(\widehat{\mu}_{j,\ell n} - \mu_j\big) - \mathrm{Cov}\big(\widehat{\mu}_{i,kn}, \widehat{\mu}_{j,\ell n}\big) \Big] \bigg)^2 \\
&= \frac{|\tilde{W}_n|^2}{m_n^2(m_n - 1)^2} \sum_{\substack{k,\ell,k',\ell' = 1 \\ \ell \neq k, \ell' \neq k'}}^{m_n} \int_{\tilde{W}_{kn} \times \tilde{W}_{\ell n} \times \tilde{W}_{k'n} \times \tilde{W}_{\ell' n}} \Big[\mathbb{E}\big(Y_i'(y_1)\, Y_j'(y_2)\, Y_i'(y_3)\, Y_j'(y_4) \big) \\
&\quad - \mathrm{Cov}_{ij}(y_2 - y_1)\mathrm{Cov}_{ij}(y_4 - y_3) \Big] \, G_i(\tilde{W}_{kn}, y_1)\, G_j(\tilde{W}_{\ell n}, y_2)\, G_i(\tilde{W}_{k'n}, y_3) \\
&\qquad \times G_j(\tilde{W}_{\ell' n}, y_4)\, d(y_1, y_2, y_3, y_4)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{c_0^3}{|\tilde{W}_n|} \int_{\mathbb{R}^{3d}} |\, s_{ij}^{(4)}(o,x_1,x_2,x_3)|\, d(x_1,x_2,x_3) \\
& + c_0^3 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ii}(x)|\, \frac{|\tilde{W}_n \cap (\partial \tilde{W}_n \oplus B_{|x|+r}(o))|}{|\tilde{W}_n|}\, dx \int_{\mathbb{R}^d} |\mathrm{Cov}_{jj}(x)|\, dx \\
& + c_0^3 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, \frac{|\tilde{W}_n \cap (\partial \tilde{W}_n \oplus B_{|x|+r}(o))|}{|\tilde{W}_n|}\, dx \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx\,.
\end{aligned}
$$

Note that only indices $l \neq k$ and $l' \neq k'$ are included in the summation.

Condition (5.33) and the absolute integrability of the covariance functions combined with condition (3.1) now imply that

$$
\lim_{n\to\infty} \mathbb{E}\biggl(\frac{|\tilde{W}_n|}{m_n(m_n-1)} \sum_{k=1}^{m_n} \sum_{\ell=1,\ell\neq k}^{m_n} \Bigl[\bigl(\widehat{\mu}_{i,kn} - \mu_i\bigr)\bigl(\widehat{\mu}_{j,\ell n} - \mu_j\bigr) - \mathrm{Cov}\bigl(\widehat{\mu}_{i,kn}, \widehat{\mu}_{j,\ell n}\bigr) \Bigr] \biggr)^2 \;=\; 0\,.
$$

□

5.4.3 Discussion of integrability conditions

If Ξ is a Boolean model and $Y_i(x) = \mathbb{1}(x \in \Xi \oplus \check{K}_i)$, $i = 1,\ldots,q$, then it follows from Lemma 7 in [22] that $\mathbb{E}\,|M_0 \oplus \check{K}_i|^4 < \infty$ implies integrability of the fourth order cumulant functions over $\mathbb{R}^{3d}$ as required in (5.33). Unfortunately, even for conditionally bounded valuations of Boolean models, a general sufficient condition with respect to the grain distribution, which guarantees (5.33), is unknown so far. A special case is arises when the grains are uniformly bounded.

Corollary 5.5 *The estimator $\widetilde{\Sigma}_n$ is asymptotically unbiased and mean–square consistent for Σ if Ξ is a Boolean model with $M_0 \subseteq B_{r_0}(o)$ almost surely for some $r_0 > 0$.*

Proof As in Corollary 5.2, the claim follows from the independence of the random variables involved. Put $r_0' = 2(r_0 + ||K||)$. Considering the different positions of x_1, x_2 and x_3 with respect to the origin o and with respect to each other, one finds that c_5 in (5.33) can be defined as

$$
c_5 \;=\; (3\,r_0')^{3d}\, \kappa_d^3 \sqrt{\mathbb{E}\bigl(Y_i(o)-\mu_i\bigr)^4\, \mathbb{E}\bigl(Y_j(o)-\mu_j\bigr)^4}\,.
$$

Let for instance $x_1 \in B_{r_0'}(o)$ and $x_2, x_3 \notin B_{2r_0'}(o)$ then it holds that $\mathrm{Cov}_{ii}(x_2) = \mathrm{Cov}_{jj}(x_3 - x_1) = 0$ and $\mathbb{E}((Y_i(o)-\mu_i)(Y_j(x_1)-\mu_j)(Y_i(x_2)-\mu_i)(Y_j(x_3)-\mu_j)) = \mathrm{Cov}_{ij}(x_1)\,\mathrm{Cov}_{ij}(x_3 - x_2)$, so that $s^{(4)}(o,x_1,x_2,x_3) = 0$. □

Chapter 6

Estimating the intrinsic volume densities in the plane

Throughout this chapter, let Ξ be a stationary germ–grain model in $\mathbb{R}^2$ with non–empty, compact and polyconvex grains such that the conditions of Theorem 3.4 or Theorem 4.2 are fulfilled. This implies, in particular, that the vector of specific intrinsic volumes $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$ is well defined; see Section 2.4.3 regarding the definition. The following considerations are dedicated to the computer–based realization of the estimation of vector $\mathbf{v}$. In a first step, we present two estimation methods (compare [60, 70]), which meet the construction principle and requirements of Chapters 3 to 5. As indicated at the beginning of Chapter 3, there exist also other estimation methods for the specific intrinsic volumes; see, amongst others, [51] and Chapter 4 of [53]. A comparative simulation study of these methods is given in [12]. In the present context, however, we are only interested in estimators of the form $\widehat{\mu}_n = \int_{W_n} Y(x)\, G(W_n, x)\, dx$, $n \geq 1$ as introduced in Section 3.2. At this, let $W_n = [-n, n]^2$ for $n \geq 1$ and assign uniform weights to all observations $Y(x)$ with $x \in W_n \ominus B_r(o)$, where radius $r > 0$ is specified later on. Procedures for differently defined observation windows or weights are easily deduced from the methods described below. In the second part of this chapter, we explain how to reconstruct (or rather approximate) a deterministic polyconvex set ξ from its digitalization, that is, the intersection of ξ with a regular lattice $\mathbb{L}$. Here, ξ will represent a realization of Ξ inside of W_n for arbitrary but fixed $n \in \mathbb{N}$, i.e., $\xi = \Xi(\omega) \cap W_n$ for $\omega \in \Omega$. Reconstruction of ξ is necessary, since in practice often not ξ itself, but the discretized set $\xi \cap \mathbb{L}$ is available for estimation purposes. In the final section, computation algorithms for the estimators of $\mathbf{v}$ and the corresponding asymptotic covariance matrices are discussed. Two of the presented methods are given in [12] and [33], whereas the third method is newly derived for the estimation of the asymptotic covariances.

6.1 Estimation of intrinsic volume densities

The content of this section is closely related to the manuscripts [60] and [70]. See these on details.

6.1.1 Using Steiner's formula (SF)

To estimate the specific Euler characteristic $\overline{V}_0(\Xi)$ and $\overline{V}_1(\Xi)$ (where $2\overline{V}_1(\Xi)$ is also called the specific boundary length of Ξ), we can make use of equation as given in (2.29), which is based on Steiner's formula on $\mathcal{R}$ stated in (2.22). Recall that the random field $\mathcal{Z}_r = \{\mathcal{Z}_r(x), x \in \mathbb{R}^2\}$ introduced in (2.28) for arbitrary $r > 0$ is defined by

$$\mathcal{Z}_r(x) \;=\; z_r(\Xi, x) \;=\; \sum_{u \in \partial\Xi \cap B_r(x),\, u \neq x} J(\Xi \cap B_r(x), u, x)\,, \qquad x \in \mathbb{R}^2$$

with index function J as in (2.20). By formula (2.29), it holds that

$$\mathbb{E}\,\mathcal{Z}_r(o) = \pi\, r^2\, \overline{V}_0(\Xi) + 2\, r\, \overline{V}_1(\Xi)\,, \qquad r > 0. \tag{6.1}$$

The right hand side of equation (6.1) will be denoted by $\overline{\rho}_r(\Xi)$. According to the results obtained in Chapters 3 and 4, an unbiased and mean–square consistent estimator of $\overline{\rho}_r(\Xi)$ is given by

$$\widehat{\rho}_{r,n}(\Xi) = \int_{W_n} \mathcal{Z}_r(x)\, G_r(W_n, x)\, dx\,, \qquad n \geq 1,\ r > 0 \tag{6.2}$$

for some weight G_r that fulfills $G_r(W_n, x) = 0$ for all $x \notin W_n \ominus B_r(o)$. From this, an estimator $(\widehat{v}_{0,n}, \widehat{v}_{1,n})^\top$ of $(\overline{V}_0(\Xi), \overline{V}_1(\Xi))^\top$ can be derived as follows. Choose $q - 1 \gg 2$ radii $r_1, \ldots, r_{q-1}$ satisfying $0 < r_1 < r_2 < \ldots < r_{q-1}$ (w.l.o.g.), and put

$$\begin{pmatrix} \widehat{v}_{0,n} \\ \widehat{v}_{1,n} \end{pmatrix} \;=\; \left(A_{q-1}^\top A_{q-1}\right)^{-1} A_{q-1}^\top\, \widehat{\boldsymbol{\rho}}_n\,, \qquad n \geq 1, \tag{6.3}$$

where

$$A_{q-1} \;=\; \begin{pmatrix} \pi r_1^2 & 2r_1 \\ \vdots & \vdots \\ \pi r_{q-1}^2 & 2r_{q-1} \end{pmatrix} \qquad \text{and} \qquad \widehat{\boldsymbol{\rho}}_n = \begin{pmatrix} \widehat{\rho}_{r_1,n}(\Xi) \\ \vdots \\ \widehat{\rho}_{r_{q-1},n}(\Xi) \end{pmatrix}.$$

Then $(\widehat{v}_{0,n}, \widehat{v}_{1,n})^\top$ is the unique solution of the least squares minimization problem

$$|\,\widehat{\boldsymbol{\rho}}_n - A_{q-1}\,\mathbf{v}\,| \;\longrightarrow\; \min\,, \qquad \mathbf{v} \in \mathbb{R}^2.$$

To complete the estimation of $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$, we need to estimate the area fraction $p = \overline{V}_2(\Xi)$ (compare (2.3)). Since $p = \mathbb{E}\,\mathbb{1}(o \in \Xi)$, a natural estimator is given by

$$\widehat{p}_n(\Xi) = \int_{W_n} \mathbb{1}(x \in \Xi)\, G_q(W_n, x)\, dx\,, \qquad n \geq 1 \tag{6.4}$$

for some feasible weighting function G_q. Regarding studies on the volume fraction of jointly stationary random fields confer also [3] and [38].

As a combination of (6.3) and (6.4), one obtains an estimator $\widehat{\mathbf{v}}_n^{(1)}$ of vector $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$ by

$$\widehat{\mathbf{v}}_n^{(1)} \;=\; (A^\top A)^{-1} A^\top \widehat{\boldsymbol{\mu}}_n^{(1)}\,, \qquad n \geq 1 \tag{6.5}$$

for

$$A \;=\; \begin{pmatrix} \pi r_1^2 & 2r_1 & 0 \\ \vdots & \vdots & \vdots \\ \pi r_{q-1}^2 & 2r_{q-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \widehat{\boldsymbol{\mu}}_n^{(1)} = \begin{pmatrix} \widehat{\rho}_{r_1,n}(\Xi) \\ \vdots \\ \widehat{\rho}_{r_{q-1},n}(\Xi) \\ \widehat{p}_n(\Xi) \end{pmatrix}.$$

In the sequel, we will refer to this estimation approach as the `SF method` due to its relation to Steiner's formula (2.22). For the actual computation of $\widehat{\boldsymbol{\mu}}_n^{(1)}$, it turns out to be most convenient to put $G_q(W_n, x) = G_{r_i}(W_n, x) = \mathbb{1}(x \in W_n^-)/|W_n^-|$ for all $i = 1, \ldots, q-1$, where $W_n^- = W_n \ominus B_{r_{\max}}(o) = [-n + r_{\max}, n - r_{\max}]^2$ with $r_{\max} = \max\{r_i : i = 1, \ldots, q-1\}$.

Let $\boldsymbol{\mu}^{(1)} = (\overline{\rho}_{r_1}(\Xi), \ldots, \overline{\rho}_{r_{q-1}}(\Xi), p)^\top$. In the previous chapters, we have seen that, under appropriate assumptions, the vector $\sqrt{W_n}(\widehat{\boldsymbol{\mu}}_n^{(1)} - \boldsymbol{\mu}^{(1)})$ converges in distribution (as $n \to \infty$) to a q–dimensional Gaussian random variable with covariance matrix $\Sigma^{(1)} = (\sigma_{ij}^{(1)})_{i,j=1,\ldots,q}$ given by

$$\begin{aligned}
\sigma_{ij}^{(1)} \;&=\; \int_{\mathbb{R}^2} \mathrm{Cov}(\mathcal{Z}_{r_i}(o), \mathcal{Z}_{r_j}(x))\, dx\,, && i,j = 1, \ldots, q-1,\\
\sigma_{iq}^{(1)} \;&=\; \sigma_{qi}^{(1)} \;=\; \int_{\mathbb{R}^2} \mathrm{Cov}\,(\mathcal{Z}_{r_i}(o), \mathbb{1}(x \in \Xi))\, dx\,, && i = 1, \ldots, q-1,\\
\sigma_{qq}^{(1)} \;&=\; \int_{\mathbb{R}^2} \mathrm{Cov}\,(\mathbb{1}(o \in \Xi)\,, \mathbb{1}(x \in \Xi))\, dx\,.
\end{aligned}$$

Consequently, we have $\sqrt{W_n}(\widehat{\mathbf{v}}_n^{(1)} - \mathbf{v}) \xrightarrow{D} \mathcal{N}_3(o, \Sigma_{\mathbf{v}}^{(1)})$ with

$$\Sigma_{\mathbf{v}}^{(1)} \;=\; (A^\top A)^{-1} A^\top\, \Sigma^{(1)} A\, (A^\top A)^{-1}\,. \tag{6.6}$$

The weighted covariance estimator $\widehat{\Sigma}_n^{(1)} = (\widehat{\sigma}_{ij,n}^{(1)})_{i,j=1,\ldots,q}$, $n \geq 1$ of $\Sigma^{(1)}$ as introduced in (5.5)–(5.6) evolves into

$$\begin{aligned}
\widehat{\sigma}_{ij,n}^{(1)} \;=\; \frac{|W_n|}{|W_n^-|} \int_{U_{ij,n}} \Bigg(\int_{W_n^- \cap (W_n^- - x)} \frac{Y_i(y)\, Y_j(x+y)}{|W_n^- \cap (W_n^- - x)|}\, dy \;-\; \widehat{\mu}_{i,n}^{(1)}\, \widehat{\mu}_{j,n}^{(1)} \Bigg)\\
\times\; \frac{|W_n^- \cap (W_n^- - x)|}{|W_n^-|}\, dx\,,
\end{aligned} \tag{6.7}$$

where $Y_v(x) = \mathcal{Z}_{r_v}(x)$, $x \in \mathbb{R}^2$ for $v \in \{i, j\}$, $i, j = 1, \ldots, q-1$ and $Y_q(x) = \mathbb{1}(x \in \Xi)$, $x \in \mathbb{R}^2$. Accordingly, we write

$$\widehat{\Sigma}_{\mathbf{v},n}^{(1)} \;=\; (A^\top A)^{-1} A^\top\, \widehat{\Sigma}_n^{(1)}\, A\, (A^\top A)^{-1}\,, \qquad n \geq 1 \tag{6.8}$$

for the corresponding estimator of $\Sigma_{\mathbf{v}}^{(1)}$. The notation of the remaining estimator of covariance matrix $\Sigma_{\mathbf{v}}^{(1)}$ is chosen in similar fashion. The modified version of the weighted covariance estimator as defined in (5.17) induces the estimate $\widehat{\Sigma}_{\mathbf{v},n}^{\prime(1)}$, and $\widetilde{\Sigma}_{\mathbf{v},n}^{(1)}$ denotes the transformation of the empirical covariance matrix given in (5.31).

6.1.2 Using the principal kinematic formula (PKF)

The second estimation procedure for $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$ is based on the local Euler characteristic and formula (2.26), which is a special case of the principle kinematic formula. Consequently, we call this approach the `PKF method` and refer to [70, Section 3] for details. For some $r > 0$, consider the random field $\{V_0(\Xi \cap B_r(x)), x \in \mathbb{R}^2\}$ and observe that on $\mathbb{R}^2$ it holds that

$$\mathbb{E}\, V_0(\Xi \cap B_r(o)) \;=\; \pi\, r^2\, \overline{V}_0(\Xi) + 2\, r\, \overline{V}_1(\Xi) + \overline{V}_2(\Xi)\,.$$

In this case, we choose $q \gg 3$ radii $r_1, \ldots, r_q$ satisfying $0 < r_1 < r_2 < \ldots < r_q$, and introduce the estimator $\widehat{\mathbf{v}}_n^{(2)}$ by

$$\widehat{\mathbf{v}}_n^{(2)} \;=\; \big(\tilde{A}_q^\top \tilde{A}_q\big)^{-1} \tilde{A}_q^\top\, \widehat{\boldsymbol{\mu}}_n^{(2)}\,, \qquad n \geq 1, \tag{6.9}$$

where

$$\tilde{A}_q \;=\; \begin{pmatrix} \pi r_1^2 & 2r_1 & 1 \\ \vdots & \vdots & \\ \pi r_q^2 & 2r_q & 1 \end{pmatrix} \qquad \text{and} \qquad \widehat{\boldsymbol{\mu}}_n^{(2)} \;=\; \begin{pmatrix} \widehat{\mu}_{1,n}^{(2)}(\Xi) \\ \vdots \\ \widehat{\mu}_{q,n}^{(2)}(\Xi) \end{pmatrix}$$

with

$$\widehat{\mu}_{i,n}^{(2)}(\Xi) \;=\; \int_{W_n} V_0(\Xi \cap B_{r_i}(x))\, G_{r_i}(W_n, x)\, dx\,, \qquad i = 1, \ldots, q. \tag{6.10}$$

As before, put $G_{r_i}(W_n, x) = \mathbb{1}(x \in W_n \ominus B_{r_{\max}}(o))/|W_n \ominus B_{r_{\max}}(o)|$ for all $i = 1, \ldots, q$ and $r_{\max} = \max\{r_i : i = 1, \ldots, q\}$.

Analogously to the previous paragraph, we find that, under appropriate assumptions, $\sqrt{W_n}(\widehat{\boldsymbol{\mu}}_n^{(2)} - \boldsymbol{\mu}^{(2)}) \xrightarrow{D} \mathcal{N}_q(o, \Sigma^{(2)})$ as $n \to \infty$, where $\boldsymbol{\mu}^{(2)} = (\mu_1^{(2)}, \ldots, \mu_q^{(2)})^\top$ for $\mu_i^{(2)} = \mathbb{E}\, V_0(\Xi \cap B_{r_i}(o))$, $i = 1, \ldots, q$ and

$$\Sigma^{(2)} \;=\; \left(\int_{\mathbb{R}^2} \mathrm{Cov}\big(V_0(\Xi \cap B_{r_i}(o)), V_0(\Xi \cap B_{r_j}(x))\big)\, dx\right)_{i,j=1,\ldots,q}.$$

This yields that $\sqrt{W_n}(\widehat{\mathbf{v}}_n^{(2)} - \mathbf{v}) \xrightarrow{D} \mathcal{N}_3(o, \Sigma_{\mathbf{v}}^{(2)})$ as n tends to ∞ with

$$\Sigma_{\mathbf{v}}^{(2)} \;=\; (\tilde{A}_q^\top \tilde{A}_q)^{-1} \tilde{A}_q^\top\, \Sigma^{(2)}\, \tilde{A}_q\, (\tilde{A}_q^\top \tilde{A}_q)^{-1}\,. \tag{6.11}$$

The covariance matrix estimator $\widehat{\Sigma}_n^{(2)} = (\widehat{\sigma}_{ij,n}^{(2)})_{i,j=1,\ldots,q}$ of $\Sigma^{(2)}$ reads as in (6.7) with $Y_v(x) = V_0(\Xi \cap B_{r_v}(x))$, $x \in \mathbb{R}^2$ for $v \in \{i, j\}$. Like before, we define

$$\widehat{\Sigma}_{\mathbf{v},n}^{(2)} \;=\; (\tilde{A}_q^\top \tilde{A}_q)^{-1} \tilde{A}_q^\top\, \widehat{\Sigma}_n^{(2)}\, \tilde{A}_q\, (\tilde{A}_q^\top \tilde{A}_q)^{-1}, \qquad n \geq 1 \tag{6.12}$$

and use the notation $\widehat{\Sigma}_{\mathbf{v},n}^{\prime(2)}$ and $\widetilde{\Sigma}_{\mathbf{v},n}^{(2)}$ for the estimates derived respectively from the modified weighted covariance matrix and from the empirical covariance matrix.

6.2 Discretization and approximation

The computation of $\widehat{\boldsymbol{\mu}}_n^{(1)}$ and $\widehat{\boldsymbol{\mu}}_n^{(2)}$ (and likewise of all other estimators) is subject to two approximations. First, for fixed $n \in \mathbb{N}$ and $\omega \in \Omega$, the realization $\xi = \Xi(\omega) \cap W_n$ of the random closed set Ξ inside of the observation window W_n is approximated by means of a discretized image $\mathbb{B}(\xi)$ on a square lattice $\mathbb{L} = \Delta\mathbb{Z}^2$ for some $\Delta > 0$. Thereafter, the integrals in (6.2) and (6.10) are approximated by corresponding sums over points $x \in W_n^- \cap \mathbb{L}_\tau$, where $\mathbb{L}_\tau = \Delta_\tau \mathbb{Z}^2$ with $\Delta_\tau = \tau\Delta$ for some $\tau \in \mathbb{N}$. Both issues are shortly described below. Regarding the details and a general concept on the d–dimensional Euclidean space see [33] and references therein.

6.2.1 Discretized set

In practice, data is often on–hand in form of digitalized images rather than continuous data sets. Assume, for instance, that ξ is the microscopic image of a biological cell structure, in which it is recorded whether a certain point is covered by a cell or not. Then all information on ξ that is actually available for computer–based analysis is stored in a binary image $\mathbb{B}(\xi) = \{\mathbb{1}(x \in \xi), x \in W_n \cap \mathbb{L}\}$. In the context of binary images, a point $x \in W_n \cap \mathbb{L}$ is called a `foreground` (or black) `pixel` if $\mathbb{1}(x \in \xi) = 1$ and a `background` (or white) `pixel` if $\mathbb{1}(x \in \xi) = 0$.

In order to estimate the area fraction p, no further processing of the image $\mathbb{B}(\xi)$ is required. The estimate $\widehat{p}_n(\Xi(\omega))$ defined in (6.4) can be approximated as

$$\widehat{p}_n(\Xi(\omega)) \quad \approx \quad \frac{1}{\#(W_n^- \cap \mathbb{L}_\tau)} \sum_{x \in W_n^- \cap \mathbb{L}_\tau} \mathbb{1}(x \in \xi)\,.$$

With regard to $z_r(\Xi(\omega), \cdot)$ and $V_0(\Xi(\omega) \cap B_r(\cdot))$, however, one has to reconstruct the boundary structure of ξ as well.

6.2.2 Approximation system

The reconstruction of ξ from $\mathbb{B}(\xi)$ is performed by means of points, segments, triangles and squares having their vertices on $\mathbb{L}$. Feasible "construction bricks" are displayed in Figure 6.1. They constitute a so–called `approximation system` $\mathbb{F}$. As a result, one obtains a `polygonal approximation` $\xi_{\mathbb{F}}$ of the set ξ; compare Figure 6.2.

To be more precise, denote by $\mathcal{F}^0(P)$ the set of vertices of a polytope $P \subset \mathbb{R}^2$, by $\mathcal{F}^1(P)$ the set of edges and by $\mathcal{F}^2(P)$ the polytope itself (these sets are also called k–facets of P). Next, let $\mathbb{T}^2$ be the group of rigid motions that map $\mathbb{L}$ onto itself, in other words, all combinations of translations by integer multiples of $\ell_1 = (\Delta, 0)^\top$, $\ell_2 = (\Delta, \Delta)^\top$ or $\ell_3 = (0, \Delta)^\top$ and rotations of degree $k \cdot 90°$, $k = 1, 2, 3$. Then the approximation system $\mathbb{F}$ is defined by

$$\mathbb{F} = \mathcal{F}^0(G) \cup \mathcal{F}^1(G) \cup \mathcal{F}^2(G), \qquad \mathcal{F}^k(G) = \bigcup_{P \in \mathcal{G}} \mathcal{F}^k(P),\ k = 0, 1, 2,$$

where $G = \text{conv}\{o, \ell_1, \ell_2\}$ is the triangle with vertices o, ℓ_1 and ℓ_2 (or equivalently, the convex hull of o, ℓ_1 and ℓ_2), and $\mathcal{G} = \{T(G) : T \in \mathbb{T}^2\}$.

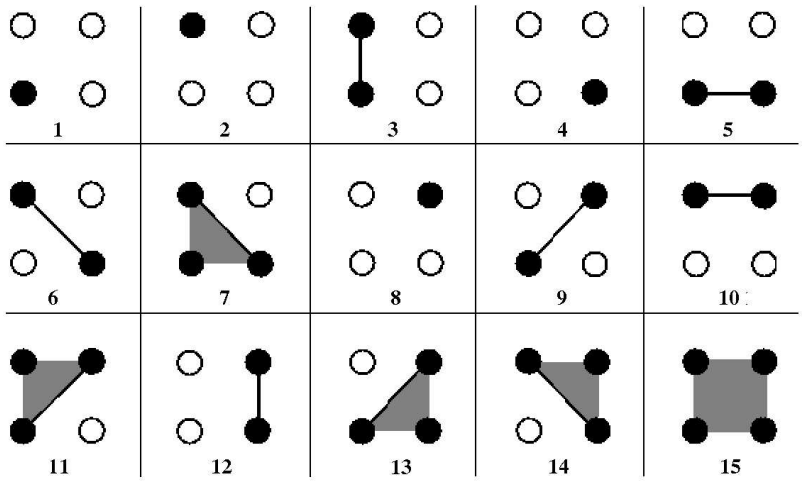


Figure 6.1: Building blocks for the polygonal approximation.

As a result, the polygonal approximation $\xi_{\mathbb{F}}$ of ξ is given by

$$\xi_{\mathbb{F}} = \bigcup_{P \in \xi \sqcap \mathbb{F}} P, \qquad \text{for } \xi \sqcap \mathbb{F} = \{\tilde{P} \in \mathbb{F} : \mathcal{F}^0(\tilde{P}) \subseteq \xi \cap \mathbb{L}\}. \tag{6.13}$$

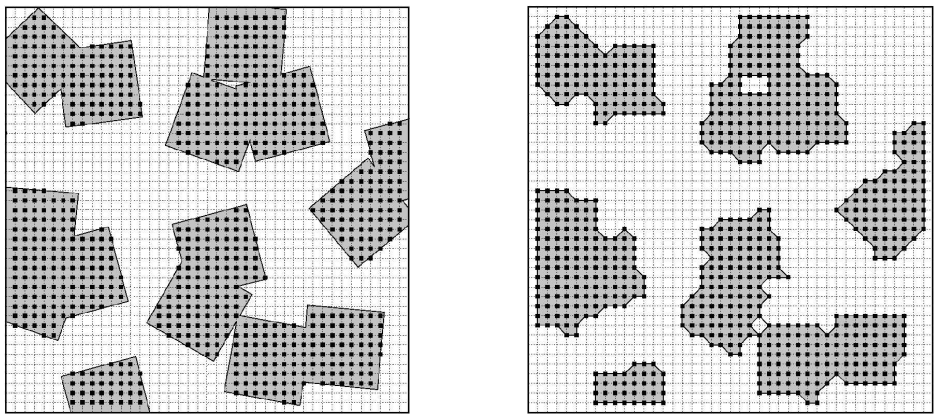


Figure 6.2: Polygonal approximation $\xi_{\mathbb{F}}$ of a set ξ, which originates from a Boolean model with randomly rotated square grains.

The presented approximation system $\mathbb{F}$ is based on the so–called `8-neighborhood relation`; see Figure 6.3 (right) and [69, p. 33].

Definition 6.1 *Two pixels $x = (x_1, x_2)^\top$ and $x' = (x'_1, x'_2)^\top$ are called neighbors in $\mathbb{L}$ with respect to the* `8-neighborhood relation` *if and only if $||x - x'|| = \max\{|x_1 - x'_1|, |x_2 - x'_2|\} \leq \Delta$. Given $||x - x'|| \leq \Delta$ but $|x_1 - x'_1| \neq |x_2 - x'_2|$, then x and x' are said to be neighbors in $\mathbb{L}$ with respect to the* `4-neighborhood relation`.

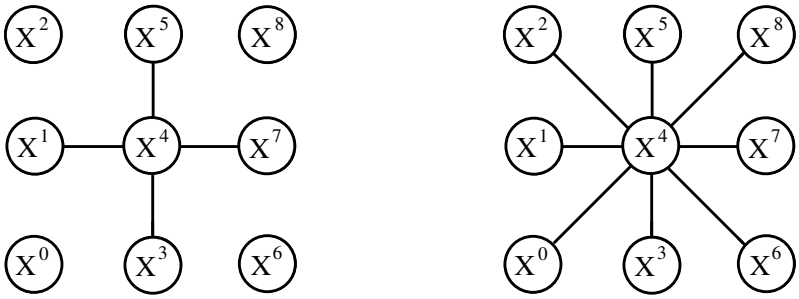


Figure 6.3: 4–neighborhood relation (`left`) and 8–neighborhood relation (`right`).

In compliance with Definition 6.1 and Figure 6.3, we define the 4-`neighborhood` of pixel $x^4 = x$ by the set $N^4(x) = \{x^1, x^3, x^5, x^7\}$, where $x^1 = x - \ell_1$, $x^3 = x - \ell_3$, $x^5 = x + \ell_3$ and $x^7 = x + \ell_1$. The 8-`neighborhood` of x is $N^8(x) = \{x^0, \ldots, x^8\} \backslash \{x^4\}$, where we additionally put $x^0 = x - \ell_2$, $x^2 = x - \ell_1 + \ell_3$, $x^6 = x + \ell_1 - \ell_3$ and $x^8 = x + \ell_2$. The chosen enumeration of the neighborhood pixels admits an efficient coding of the elements of $\mathbb{B}(\xi)$, which will be employed later on.

In a last step, it remains to specify, which of the pixels make up the boundary of $\xi_{\mathbb{F}}$.

Definition 6.2 *A pixel $x \in \xi \cap \mathbb{L}$ is said to be a* `boundary pixel` *of ξ with respect to the 8–neighborhood relation if x is a foreground pixel and if there exists at least one background pixel in the 4–neighborhood of x.*

In total, there exist 240 different 3×3–pixel configurations $\{x^0, \ldots, x^8\}$ that identify the center pixel as a boundary pixel with respect to the 8–neighborhood relation (compare [69, p. 153]). This number can be reduced to 45 configurations by coupling those which are equal up to rotations by $90°, 180°$ or $270°$ degree, or reflections parallel or diagonal to the axes (confer [33]). In Figure 6.4 of the following section, 11 basic boundary elements are displayed which build up $\partial\xi_{\mathbb{F}}$.

6.2.3 Approximation formulae

Now, we put $z_r(\Xi(\omega), x) \approx z_r(\xi_{\mathbb{F}}, x)$, where $z_r(\xi_{\mathbb{F}}, x) = \sum_{u \in \partial\xi_{\mathbb{F}}} J(\xi_{\mathbb{F}} \cap B_r(x), u, x)$, and $V_0(\Xi(\omega) \cap B_r(x)) \approx V_0(\xi_{\mathbb{F}} \cap B_r(x))$ for any $x \in W_n^- \cap \mathbb{L}_\tau$ and all $r > 0$. This yields the approximations

$$\widehat{\rho}_{r_i,n}(\Xi(\omega)) \;\approx\; \widehat{\rho}_{r_i,n}(\xi_{\mathbb{F}}) \;=\; \frac{1}{\#(W_n^- \cap \mathbb{L}_\tau)} \sum_{x \in W_n^- \cap \mathbb{L}_\tau} z_{r_i}(\xi_{\mathbb{F}}, x) \tag{6.14}$$

for $i = 1, \ldots, q-1$, $n \geq 1$ and

$$\widehat{\mu}^{(2)}_{i,n}(\Xi(\omega)) \;\approx\; \widehat{\mu}^{(2)}_{i,n}(\xi_{\mathbb{F}}) \;=\; \frac{1}{\#(W_n^- \cap \mathbb{L}_\tau)} \sum_{x \in W_n^- \cap \mathbb{L}_\tau} V_0(\xi_{\mathbb{F}} \cap B_{r_i}(x)) \tag{6.15}$$

for $i = 1, \ldots, q$, $n \geq 1$.

In similar fashion, we approximate the components $\widehat{\sigma}_{ij,n}^{(1)}$ of the estimator $\widehat{\Sigma}_n^{(1)}$ over some feasible averaging window $U_{ij,n}$ (see Section 5.1). As we will see in Section 7.3.3, it is sometimes advisable to choose a larger lattice spacing $\Delta_{\tau'} \geq \Delta_\tau$ for the estimation of the covariances $\sigma_{ij}^{(1)}$ (as well as for $\sigma_{ij}^{(2)}$). Therefore, fix some $\tau' \in \mathbb{N}$ with $\tau' \geq \tau$ and put

$$\begin{aligned} \widehat{\sigma}_{ij,n}^{(1)} \;\approx\; & \frac{\#(W_n \cap \mathbb{L}_{\tau'})}{\#(W_n^- \cap \mathbb{L}_{\tau'})} \sum_{x \in U_{ij,n} \cap \mathbb{L}_{\tau'}} \Bigg[\frac{\#(W_n^- \cap (W_n^- - x) \cap \mathbb{L}_{\tau'})}{\#(W_n^- \cap \mathbb{L}_{\tau'})} \\ & \times \Bigg(\sum_{y:\, y,\, y+x \in W_n^- \cap \mathbb{L}_{\tau'}} \frac{\mathrm{y}_i(y)\, \mathrm{y}_j(x+y)}{\#(W_n^- \cap (W_n^- - x) \cap \mathbb{L}_{\tau'})} \;-\; \widehat{\mu}_{i,n}^{(1)}(\xi_\mathbb{F})\, \widehat{\mu}_{j,n}^{(1)}(\xi_\mathbb{F}) \Bigg) \Bigg], \end{aligned} \tag{6.16}$$

where $\mathrm{y}_i(x) = z_{r_i}(\xi_\mathbb{F}, x)$ for $i = 1, \ldots, q-1$ and $\mathrm{y}_q(x) = \mathbb{1}(x \in \xi_\mathbb{F})$, together with $\widehat{\mu}_{i,n}^{(1)}(\xi_\mathbb{F}) = \widehat{\rho}_{r_i,n}(\xi_\mathbb{F})$, $i = 1, \ldots, q-1$ (where $\mathbb{L}_\tau$ is to be replaced by $\mathbb{L}_{\tau'}$ in (6.14)) and

$$\widehat{\mu}_{q,n}^{(1)}(\xi_\mathbb{F}) \;=\; \frac{1}{\#(W_n^- \cap \mathbb{L}_{\tau'})} \sum_{x \in W_n^- \cap \mathbb{L}_{\tau'}} \mathbb{1}(x \in \xi_\mathbb{F})\,.$$

An approximation of $\widehat{\sigma}_{ij,n}^{(2)}$ is obtained analogously. For $\upsilon \in \{i, j\}$ with $i, j = 1, \ldots, q$, insert $\mathrm{y}_\upsilon(x) = V_0(\xi_\mathbb{F} \cap B_{r_\upsilon}(x))$ and $\widehat{\mu}_{\upsilon,n}^{(2)}(\xi_\mathbb{F})$ as defined in (6.15) into equation (6.16) with $\mathbb{L}_\tau$ replacing $\mathbb{L}_{\tau'}$.

6.3 Evaluation algorithms

The reference to a specific radius $r_i > 0$ with $i \in \{1, \ldots, q\}$ will not be used in the algorithms described below. We therefore omit the indices and consider $\widehat{\rho}_{r,n}(\xi_\mathbb{F})$, $z_r(\xi_\mathbb{F}, \cdot)$ and $V_0(\xi_\mathbb{F} \cap B_r(\cdot))$ for arbitrary $r > 0$.

6.3.1 SF method I

An efficient algorithm for the computation of $\widehat{\rho}_{r,n}(\xi_\mathbb{F})$ is presented in Section 3 of [33]. Leaving out the details, the idea is essentially the following. Notice that $\partial\xi_\mathbb{F}$ is comprised by both pixels and connecting segments (vertical, horizontal or diagonal), in symbols $\partial\xi_\mathbb{F} = (\partial\xi_\mathbb{F} \cap \mathbb{L}) \cup (\partial\xi_\mathbb{F} \setminus (\partial\xi_\mathbb{F} \cap \mathbb{L}))$. Considering these sets separately, we can write

$$z_r(\xi_\mathbb{F}, x) \;=\; \sum_{u \in \partial\xi_\mathbb{F} \cap \mathbb{L}} J(\xi_\mathbb{F} \cap B_r(x), u, x) \;+\; \sum_{u \in \partial\xi_\mathbb{F} \setminus (\partial\xi_\mathbb{F} \cap \mathbb{L})} J(\xi_\mathbb{F} \cap B_r(x), u, x) \tag{6.17}$$

so that

$$\#(W_n^- \cap \mathbb{L}_\tau)\, \widehat{\rho}_{r,n}(\xi_\mathbb{F}) \;=\; \sum_{x \in W_n^- \cap \mathbb{L}_\tau} \sum_{u \in \partial\xi_\mathbb{F} \cap \mathbb{L}} J(\xi_\mathbb{F} \cap B_r(x), u, x) \tag{6.18.a}$$

$$+ \sum_{x \in W_n^- \cap \mathbb{L}_\tau} \sum_{u \in \partial\xi_\mathbb{F} \setminus (\partial\xi_\mathbb{F} \cap \mathbb{L})} J(\xi_\mathbb{F} \cap B_r(x), u, x) \tag{6.18.b}$$

Interchanging the order of summation in formula (6.18.a) leads to the representation

$$\sum_{x\in W_n^- \cap \mathbb{L}_\tau} \sum_{u\in\partial\xi_{\mathbb{F}}\cap\mathbb{L}} J(\xi_{\mathbb{F}} \cap B_r(x), u, x) \;=\; \sum_{u\in\partial\xi_{\mathbb{F}}\cap\mathbb{L}} S_r(u)$$

with

$$S_r(u) \;=\; \sum_{x\in\mathbb{L}_\tau,\, 0<|x-u|\le r} J(\xi_{\mathbb{F}} \cap B_r(x), u, x)\,.$$

Since $u \in \partial\xi_{\mathbb{F}} \cap \mathbb{L}$, one has to consider (up to rotations and reflections) only 45 configurations of $3\times 3-$pixels, whose center pixel (here u) is black. In Figure 6.4, one finds the afore–mentioned 11 elementary boundary elements together with the corresponding index value. This value is the same for all pixel x inside of the displayed index region, that is a sector of a circle (or line segment) of radius $r > 0$. For points of concavity, meaning $J(\xi_{\mathbb{F}} \cap B_r(x), u, x) = -1$, the sector is understood without the bounding radii.

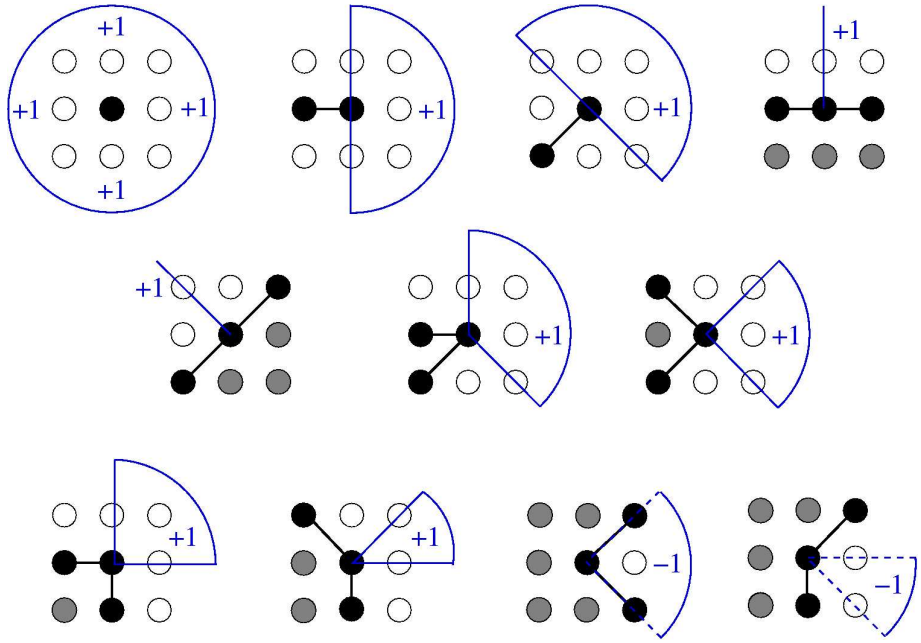


Figure 6.4: Elementary boundary elements with the corresponding index value and index sector. All boundary elements are made up of these primary structures.

If $u \in W_n \ominus B_{2r}(o)$, then the index sector is entirely contained in $W_n^- = W_n \ominus B_r(o)$. Hence, the value of $S_r(u)$ does not depend on the location of u, and one can apply the following fast evaluation method. Let $h_{S,k}$ denote the frequency of boundary elements of type $k \in 1,\ldots,45$ inside of $W_n \ominus B_{2r}(o)$, and represent the index sectors by IS_ν, $\nu = 1,\ldots,11$. Then we have

$$\sum_{u\in\partial\xi_{\mathbb{F}}\cap\mathbb{L}\cap(W_n\ominus B_{2r}(o))} S_r(u) \;=\; \sum_{k=1}^{45} h_{S,k} \sum_{\nu=1}^{11} \omega_{k\nu}\cdot J_\nu \cdot \#(IS_\nu) \qquad (6.19)$$

for $J_\nu \in \{-1,+1\}$ as displayed and certain weights $\omega_{k\nu} \in \{0,1,2,3,4\}$, $k = 1,\ldots,45$, $\nu = 1,\ldots,11$, which are specified in Section 4.2 of [33]. Notice that the cardinality $\#(IS_\nu)$ of IS_ν, $\nu \in \{1,\ldots,11\}$ can be calculated (and stored) beforehand for each relevant radius $r > 0$. For $u \in W_n \setminus (W_n \ominus B_{2r}(o))$, the cardinality of the index sector has to be calculated according to the position of u so that only pixels x inside of $W_n^- \cap \mathbb{L}_\tau$ are counted.

A pixel $x \in W_n^- \cap \mathbb{L}_\tau$ can also be projected onto non–lattice points $u \in \partial\xi_\mathbb{F} \setminus (\partial\xi_\mathbb{F} \cap \mathbb{L}_\tau)$. These points of projection lie on diagonal segments (notice that no pixel x is ever projected onto the interior of a vertical or a horizontal segment). So one defines

$$M_r(P) \;=\; \sum_{x \in D_r(P)} J(\xi_\mathbb{F} \cap B_r(x), u, x)\,, \qquad P \in \mathcal{F}^1(G), P \subset \partial\xi_\mathbb{F}\,,$$

where $D_r(P)$ is the set of all lattice points x, whose projection u belongs to the interior of a (diagonal) segment P and such that $0 < |x-u| \le r$. In fact, one has $M_r(P) = \#(D_r(P))$. Summand (6.18.b) can now be written as

$$\sum_{x \in W_n^- \cap \mathbb{L}_\tau} \; \sum_{u \in \partial\xi_\mathbb{F} \setminus (\partial\xi_\mathbb{F} \cap \mathbb{L})} J(\xi_\mathbb{F} \cap B_r(x), u, x) \;=\; \sum_{P \in \mathcal{F}^1(G),\, P \subset \partial\xi_\mathbb{F}} M_r(P)\,. \qquad (6.20)$$

For $P \subset W_n \ominus B_{2r}(o)$, we have $M_r(P) = \lfloor r/(\sqrt{2}\Delta_\tau) - 1/2 \rfloor + 1$ and conclude that

$$\sum_{P \in \mathcal{F}^1(G),\, P \subset \partial\xi_\mathbb{F} \cap (W_n \ominus B_{2r}(o))} M_r(P) \;=\; h_{M,1} \cdot \left(\left\lfloor \frac{r}{\sqrt{2}\Delta_\tau} - \frac{1}{2} \right\rfloor + 1 \right), \qquad (6.21)$$

where $h_{M,1}$ is one half of the total number of diagonal segments in the 3×3–pixel configurations contained in $W_n \ominus B_{2r}(o)$. Again, one has to compute $M(P)$ individually for each segment P close to the boundary, i.e., $P \subset W_n \setminus W_n \ominus B_{2r}(o)$.

On $W_n \ominus B_{2r}(o)$, the computation of $\widehat{\rho}_{r,n}(\xi_\mathbb{F})$ via formulae (6.19) and (6.20) has complexity $O(\tilde{m}_1)$, where $\tilde{m}_1 = \#((W_n \ominus B_{2r}(o)) \cap \mathbb{L})$. This is due to the fact that each pixel $u \in W_n \ominus B_{2r}(o) \cap \mathbb{L}$ has to be considered only once (together with its 8–neighborhood) in order to determine the frequencies $h_{S,k}$ for $k = 1,\ldots,45$ and $h_{M,1}$. If radius $r > 0$ is small in comparison to $n \in \mathbb{N}$ (and thus the set $W_n \setminus W_n \ominus B_{2r}(o)$ is negligible), then the above–described method is very efficient; see Section 7.2.1 on a comparison of running times. Note that in any case the complexity of the computations does not depend on Δ_τ. Thus, one can put $\Delta_\tau = \Delta$ without causing any deceleration.

6.3.2 SF method II

A drawback of the previously described method is that it can not be used to compute the approximation of $\widehat{\Sigma}_n^{(1)}$ as presented in (6.16), where one has to evaluate, amongst others, $c_r(x) = \sum_y z_r(\xi_\mathbb{F}, y)\, z_r(\xi_\mathbb{F}, y+x)$ for any $x \in W_n^- \cap \mathbb{L}_\tau$ and all y such that y, $y+x \in W_n^- \cap \mathbb{L}_\tau$. This means, in particular, that it is necessary to compute $z_r(\xi_\mathbb{F}, v)$ for any $v \in W_n^- \cap \mathbb{L}_\tau$. The essential idea of the evaluation algorithm SF I, however, is

to determine the arithmetic average $\widehat{\rho}_{r,n}(\xi_{\mathbb{F}}) = \#(W_n^- \cap \mathbb{L}_\tau)^{-1} \sum_{v \in W_n^- \cap \mathbb{L}_\tau} z_{r_i}(\xi_{\mathbb{F}}, v)$ without evaluating $z_r(\xi_{\mathbb{F}}, v)$ explicitely. Due to this reason, we present in the following a new algorithm for the direct computation of formula (6.14) by employing (6.17). This variant requires $O(m_1 + m_2 \frac{r^2}{\Delta})$ operations, where $m_1 = \#(W_n \cap \mathbb{L})$ and $m_2 = \#(W_n^- \cap \mathbb{L}_\tau)$. The running time of this algorithm is therefore heavily dependent on the magnitude of m_2 and thus on the choice of Δ_τ.

In the first step, the binary image $\mathbb{B}(\xi)$ is transferred into a gray–scale image $\mathbb{G}(\xi)$, which assigns to each pixel $u \in W_n \ominus B_\Delta(o)$ a unique code, its so–called `neighborhood configuration`. To this end, set $\overline{N^8}(u) = \{u^0, \ldots, u^8\} = N^8(u) \cup \{u\}$, where $N^8(u) = \{u^0, \ldots, \not{u}^4, \ldots, u^8\}$ is the 8–neighborhood of pixel $u = u^4$. Next, introduce the mapping

$$\{\mathbb{1}_\xi(\tilde{u}), \tilde{u} \in \overline{N^8}(u)\} \quad \mapsto \quad g_8(\xi, u) = \sum_{k=0}^{8} 2^k \, \mathbb{1}_\xi(u^k) \, .$$

Note that a code can only be assigned to those pixels, whose 8–neighborhood is completely contained in W_n, i.e., $u \in W_n \ominus B_\Delta(o)$. To overcome this edge–effect, one may assume that the boundary of W_n does not hit $\Xi(\omega)$, or equivalently expand $\mathbb{B}(\xi)$ by a strip of white pixels on each side, so that each pixel $u \in W_n$ can be encoded. The mentioned gray–scale image is now defined as $\mathbb{G}(\xi) = \{g_8(\xi, u), u \in W_n \cap \mathbb{L}\}$, where $g_8(\xi, u)$ takes values in $\{1, \ldots, 512\}$. This coding procedure is also the first step to be performed in the algorithm described in the previous paragraph (yielding the neighborhood histogram $\{h_{S,1}, \ldots, h_{S,45}, h_{M,1}\}$).

In the second step, we fix a pixel $x \in W_n^- \cap \mathbb{L}_\tau$ arbitrarily and consider all $u \in B_r(x) \cap \mathbb{L}$. Dependent on the position of u with respect to x, the index value $J(\xi_{\mathbb{F}} \cap B_r(x), u, x) \in \{-1, 0, 1\}$ is uniquely determined by the neighborhood configuration at u; compare Figure 6.4. With regard to this, define the index sectors $IS_0(x), \ldots, IS_{15}(x)$ of x as

$$IS_\nu(x) \, = \, \{\tilde{u} \in B_r(x) : \tilde{u}_1 = x_1 + s\cos(t), \tilde{u}_2 = x_2 + s\sin(t), s \in (0, r], t \in \mathrm{Arc}(\nu)\}$$

with $\mathrm{Arc}(\nu) = \left\{\frac{\nu\pi}{8}\right\}$ for $\nu \in \{0, 2, \ldots, 12, 14\}$ and $\mathrm{Arc}(\nu) = \left(\frac{(\nu-1)\pi}{8}, \frac{(\nu+1)\pi}{8}\right)$ else. In other words, we distinguish between the interior of the 45° degree sectors of $B_r(x)$ and the bounding radii; see Figure 6.5.

For each sector, one can determine (and store) the list of neighborhood configurations that contribute either index value -1 or $+1$ to the value of $z_r(\xi_{\mathbb{F}}, x)$. Accordingly, we write $J_\nu(g_8(\xi, u)) \in \{-1, 0, 1\}$ and define

$$S_r'(x) \quad = \quad \sum_{u \in B_r(x) \cap \mathbb{L}} \; \sum_{\nu=0}^{15} \mathbb{1}(u \in IS_\nu(x)) \cdot J_\nu(g_8(\xi, u)) \, . \tag{6.22}$$

Regarding the summand in (6.17.b), we consider once again the projections onto diagonal segments of $\partial\xi_{\mathbb{F}}$. The projection points $u \in \partial\xi_{\mathbb{F}} \setminus (\xi_{\mathbb{F}} \cap \mathbb{L})$ belong to the index sectors $IS_\nu(x)$ for $\nu \in \{2, 6, 10, 14\}$ and increase $z_r(\xi_{\mathbb{F}}, x)$ by 1. To write down

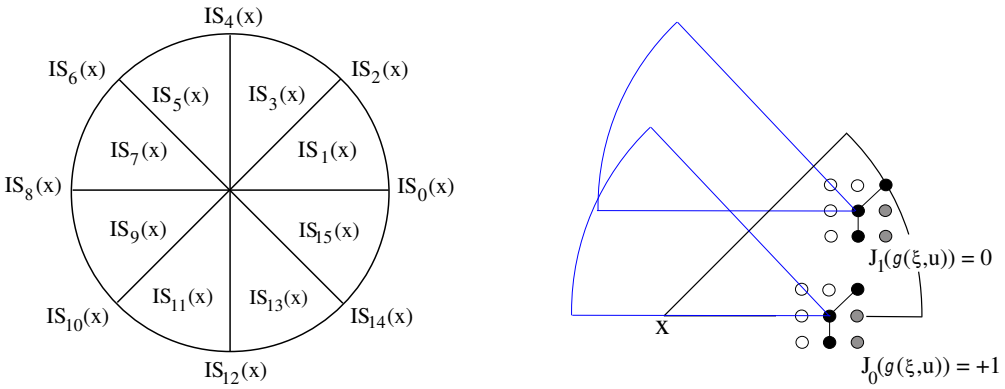


Figure 6.5: Index sectors $IS_0(x), \ldots, IS_{15}(x)$ on the `left`. Example for the value of $J_\nu(g_8(\xi,u))$, $\nu = 0,1$ on the `right` with the same neighborhood configuration.

a closed form expression for these points of projection, introduce the shifted sectors $\widetilde{IS}_0(x) = IS_2(x) + \left(\frac{\Delta}{2}, -\frac{\Delta}{2}\right)$, $\widetilde{IS}_1(x) = IS_6(x) - \left(\frac{\Delta}{2}, \frac{\Delta}{2}\right)$, $\widetilde{IS}_2(x) = IS_{10}(x) + \left(\frac{\Delta}{2}, -\frac{\Delta}{2}\right)$ and $\widetilde{IS}_3(x) = IS_{14}(x) - \left(\frac{\Delta}{2}, \frac{\Delta}{2}\right)$. Consequently, one can define

$$\widetilde{J}_\upsilon(\xi,u) \;=\; \begin{cases} 1\,, & \text{if } \upsilon = 0, u \in \widetilde{IS}_0 : \mathbb{1}_\xi(u) = \mathbb{1}_\xi(u - \ell_1 + \ell_3) = 1 - \mathbb{1}_\xi(u - \ell_1) = 1, \\ & \text{or } \upsilon = 1, u \in \widetilde{IS}_1 : \mathbb{1}_\xi(u) = \mathbb{1}_\xi(u + \ell_1 + \ell_3) = 1 - \mathbb{1}_\xi(u + \ell_1) = 1, \\ & \text{or } \upsilon = 2, u \in \widetilde{IS}_2 : \mathbb{1}_\xi(u) = \mathbb{1}_\xi(u - \ell_1 + \ell_3) = 1 - \mathbb{1}_\xi(u + \ell_3) = 1, \\ & \text{or } \upsilon = 3, u \in \widetilde{IS}_3 : \mathbb{1}_\xi(u) = \mathbb{1}_\xi(u + \ell_1 + \ell_3) = 1 - \mathbb{1}_\xi(u + \ell_3) = 1, \\ 0\,, & \text{else.} \end{cases}$$

Combining this with (6.22) results in

$$z_r(\xi,x) \;=\; S'_r(x) + M'_r(x)\,,$$

where

$$M'_r(x) \;=\; \sum_{\upsilon=0}^{3} \sum_{u \in IS_\upsilon} \widetilde{J}_\upsilon(\xi,u).$$

Finally, we rewrite formula (6.18) as follows

$$\widehat{\rho}_{r,n}(\xi_{\mathbb{F}}) \;=\; \frac{1}{\#(W_n^- \cap \mathbb{L}_\tau)} \sum_{x \in W_n^- \cap \mathbb{L}_\tau} S'_r(x) + \frac{1}{\#(W_n^- \cap \mathbb{L}_\tau)} \sum_{x \in W_n^- \cap \mathbb{L}_\tau} M'_r(x)\,.$$

The approximation error $||\overline{\rho}_r(\xi) - \widehat{\rho}_{r,n}(\xi_{\mathbb{F}})||$ of the algorithm presented above is the same as for the method described in Section 6.3.2, since these algorithms differ only in the way of evaluating the sums in (6.18.a) and in (6.18.b). An upper bound to the approximation error can be found in [33]; see formula (3.9) therein.

6.3.3 PKF evaluation method

In order to approximate the Euler characteristic $V_0(\xi \cap B_r(x))$ on the basis of the polygonal approximation $\xi_{\mathbb{F}}$, one can apply the inclusion–exclusion principle (2.14) to

the elementary polytopes ("construction bricks") that make up $\xi_{\mathbb{F}}$. For each $x \in \mathbb{L}_\tau$, let $\xi_{|B_r(x)} = \xi \cap B_r(x)$ and denote by $(\xi_{|B_r(x)})_{\mathbb{F}}$ the polygonal approximation of ξ by polytopes that are completely contained in $B_r(x)$, i.e.,

$$(\xi_{|B_r(x)})_{\mathbb{F}} \;=\; \bigcup_{P \in \xi \sqcap \mathbb{F}, P \subset B_r(x)} P$$

with $\xi \sqcap \mathbb{F}$ as given in (6.13). Correspondingly, put

$$\mathbb{B}(\xi_{|B_r(x)}) = \left\{ \mathbb{1}_{\xi_{|B_r(x)}}(u), u \in ([-\lfloor r \rfloor - 1, \lfloor r \rfloor + 1]^2 + x) \cap \mathbb{L} \right\}.$$

In the latter definition, white pixels are added around $B_r(x)$, on the one hand to simplify evaluations, and on the other hand to be able to assign codes to each 2×2–pixel configurations of which at least one pixel lies in $B_r(x)$. To this end, define $\overline{N_4}(u) = \{{}^0u, {}^1u, {}^2u, {}^3u\}$, where ${}^0u = u$, ${}^1u = u + \ell_3$, ${}^2u = u + \ell_1$ and ${}^3u = u + \ell_2$. Now, the binary image $\mathbb{B}(\xi_{|B_r(x)})$ is encoded by means of the mapping

$$\{\mathbb{1}_\xi(\tilde{u}), \tilde{u} \in \overline{N_4}(u)\} \;\mapsto\; g_4(\xi_{|B_r(x)}, u) = \sum_{k=0}^{3} 2^k \, \mathbb{1}_{\xi_{|B_r(x)}}({}^k u)\,.$$

The resulting codes are displayed in Figure 6.1, except for the 2×2–configurations that contains only background pixels and bears code 0. Next, we count the frequencies $h_1(x), \ldots, h_{15}(x)$ of the 15 elementary polytopes that are contained in $\xi_{\mathbb{F}} \cap B_r(x)$. Note that vertical and horizontal edges are counted twice, and vertices are counted four times. This problem is corrected by application of the inclusion–exclusion principle. Following the details in Section 3.4 of [55], we get an approximation of $V_0(\xi_{\mathbb{F}} \cap B_r(x))$ by the weighted sum

$$V_0((\xi_{|B_r(x)})_{\mathbb{F}}) \;=\; \sum_{i=1}^{15} w_i h_i(x)\,,$$

for $w = (1, 1, 0, 1, 0, -\frac{1}{2}, -\frac{1}{4}, 1, -\frac{1}{2}, 0, -\frac{1}{4}, 0, -\frac{1}{4}, -\frac{1}{4}, 0)$; compare formula (10) in [55]. This results in

$$\widehat{\mu}_n^{(2)}(\xi_{\mathbb{F}}) \;\approx\; \frac{1}{\#(W_n^- \cap \mathbb{L}_\tau)} \sum_{x \in W_n^- \cap \mathbb{L}_\tau} \sum_{i=1}^{15} w_i h_i(x)\,. \tag{6.23}$$

The computation of $\widehat{\mu}_n^{(2)}(\xi_{\mathbb{F}})$, as described above, has complexity $O(m_2 \frac{r^2}{\Delta_\tau})$ with $m_2 = \#(W_n^- \cap \mathbb{L}_\tau)$. Returning to the original setup of $q \geq 2$ distinct radii $r_1, \ldots, r_q$, one could therefore expect a shorter running time for the evaluation of estimation vector $\widehat{\boldsymbol{\mu}}_n^{(2)}$ than for $\widehat{\boldsymbol{\mu}}_n^{(1)}$ with method SF II and the same choice of radii (here $O(m_1 + m_2 \frac{r^2}{\Delta_\tau})$ operations are required). Yet, the problem is that formula (6.23) does heavily depend on the considered radius, meaning that a pixel might be coded differently for different radii $r_i \neq r_j$. An efficient way to cope with this phenomenon is presented in Section 2.4 of [12]. In addition, we point out that the computational time for $\widehat{\mu}_n^{(2)}$ can be improved even more by employing the relationship $z_r(\xi_{\mathbb{F}}, x) = V_0(\xi_{\mathbb{F}} \cap B_r(x))$ for

all $x \notin \xi_{\mathbb{F}}$ and $r > 0$. The first advantage of computing $(z_{r_1}(\xi_{\mathbb{F}}), \ldots, z_{r_q}(\xi_{\mathbb{F}}))^\top$ is that the image $\mathbb{B}(\xi)$ has to be encoded only once, i.e, we do not have to determine $\{g_4(\xi_{|B_r(x)}, u) : u \in B_{r_i}(x)\}$, $i = 1, \ldots, q$ for any $x \in W_n^- \setminus \xi_{\mathbb{F}}$. Secondly, all values $z_{r_i}(\xi_{\mathbb{F}}, x)$ can be calculated in one step. In addition to finding the correct index sector and index value at u, we only have to check if $||u - x|| \leq r_i$ for $i = q, \ldots, 1$ (recall that $r_q > \ldots > r_1$). A numerical comparison of the running times is given in Section 7.2.1.

It should be mentioned that there exist also other approaches to evaluate the Euler characteristic $V_0(\xi \cap B_r(x))$, $x \in W_n^-$. Ohser and Nagel ([54]), for instance, consider a procedure based on other formulae from integral geometry. A graph theoretic approach is presented in Chapter VI of [66]. All methods (including the one described above) have the computational advantage that they are easily obtained from the coded image. However, although $(\xi_{|B_r(x)})_{\mathbb{F}} \to \xi_{|B_r(x)}$ with respect to the Hausdorff metric as $\Delta \to 0$, one can not expect that $V_0((\xi_{|B_r(x)})_{\mathbb{F}}) \to V_0(\xi \cap B_r(x))$, since the Euler characteristic is not continuous on $\mathcal{R}$. Confer [31] and pages 216–220, 493–494 of [66] on the approximation error. Recently, another algorithm on the basis of the polygonal approximation $(\xi_{|B_r(x)})_{\mathbb{F}}$ was suggested by Kiderlen (see [31]). The principle idea is to impose additional conditions on the underlying polyconvex set ξ and to take into account only those connected components of $(\xi_{|B_r(x)})_{\mathbb{F}}$ and of its complement that have inner points. The subclass $\tilde{\mathcal{R}}$ of feasible polyconvex sets $\xi = \bigcup_{i=1}^m K_i$ with $K_i \in \mathcal{K}', m \in \mathbb{N}$ is such that $\bigcap_{i \in I} K_i$ is empty or has interior points for any $I \subseteq \{1, \ldots, m\}$ and such that $\#(\partial K_i \cap \partial K_j) < \infty$ for all $i \neq j \in \{1, \ldots, m\}$. The first condition excludes cases, where two grains merely touch. Under the above conditions, there exists a $\Delta(\xi) > 0$ such that $V_0((\xi_{|B_r(x)})_{\mathbb{F}}) = V_0(\xi_{|B_r(x)})$ for all $0 < \Delta \leq \Delta(\xi)$; see Theorem 1 in [31]. Speaking of Boolean models with 2–dimensional, convex grains, this property holds almost surely. This is due to the fact that for any 2–dimensional convex bodies $M, M' \in \mathcal{K}'$, the set $\{x : (M + x) \cap M' \neq \emptyset, \mathrm{int}((M + x) \cap M') = \emptyset\} = \partial(M \oplus \check{M}')$ has Lebesgue measure zero, and that $\partial M \cap (\partial M' + x)$ is finite for almost all $x \in \mathbb{R}^2$ (see [26, p. 102]). Kiderlen states that his new algorithm is not more than twice as time consuming than the approaches mentioned above.

6.4 Fast Fourier transform

In the previous section, we presented methods to compute $\{\mathrm{y_i}(x), x \in W_n^- \cap \mathbb{L}_\tau\}$ for $i = 1, \ldots, q$ with $\mathrm{y_i}(x) \in \{z_{r_i}(\xi_{\mathbb{F}}, \cdot), \mathbb{1}_{\xi_{\mathbb{F}}}, V_0(\xi_{\mathbb{F}} \cap B_{r_i}(\cdot))\}$ (including $\mathbb{1}_{\xi_{\mathbb{F}}}$ without the necessity of further considerations). We want to close this chapter by briefly discussing the procedure used to evaluate the approximation of $\widehat{\sigma}_{ij,n}^{(\upsilon)}, \upsilon \in \{1, 2\}$ as given in (6.16) efficiently. In particular, we are interested in

$$\mathrm{c}_{ij}(x) \quad = \sum_{y:\, y,\, y+x \in W_n^- \cap \mathbb{L}_{\tau'}} \mathrm{y}_i(y)\, \mathrm{y}_j(x+y), \qquad x \in U_{ij,n}. \tag{6.24}$$

Due to condition (5.3) and since $o \in W_n$ by assumption, we may suppose in the following that $U_{ij,n} \subseteq W_n^-$ (w.l.o.g.).

6.4.1 Discrete convolution theorem

As indicated in Chapter 5, we are going to apply the (discrete) convolution theorem to $\check{c}_{ij}(x) = c_{ij}(-x), x \in \check{U}_{ij,n}$. To this end, we first introduce some notation and represent $\check{c}_{ij}(x)$ by a discrete convolution. For $M = 2\lfloor (n-r)/\Delta_{\tau'} \rfloor + 1$ and $k_1, k_2 = 0, \dots M-1$, define

$$f_{1i}(k_1,k_2) \;=\; \mathrm{y}_i\Big((k_1\Delta_{\tau'}, k_2\Delta_{\tau'})^\top - \Big(\Big\lfloor \frac{n-r}{\Delta_{\tau'}}\Big\rfloor \Delta_{\tau'}, \Big\lfloor \frac{n-r}{\Delta_{\tau'}}\Big\rfloor \Delta_{\tau'}\Big)^\top\Big)$$

and

$$f_{2j}(k_1,k_2) \;=\; \mathrm{y}_j(-(k_1\Delta_{\tau'}, k_2\Delta_{\tau'})^\top).$$

Both functions are to be expanded periodically, where the periodicity is chosen as a power of 2 for reasons that will become clear shortly. To avoid overlapping effects, let M_* be the smallest power of 2 that is greater or equal to $2M$ (see [6, pp. 130, 140] and [55, p. 9] on aliasing effects). Put $f_{1i}(k_1,k_2) = f_{2j}(k_1,k_2) = 0$ whenever k_1 or $k_2 \in \{M, \dots, M_*-1\}$ and define

$$f_{\cdot\cdot}(k_1 + l_1 M_*, k_2 + l_2 M_*) \;=\; f_{\cdot\cdot}(k_1,k_2), \qquad l_1, l_2 \in \mathbb{Z}\,.$$

Writing $\check{c}_{ij}(k_1,k_2) = \check{c}_{ij}(x)$ for $x = ((k_1 - \lfloor \frac{(n-r)}{\Delta_{\tau'}} \rfloor)\Delta_{\tau'}, (k_2 - \lfloor \frac{(n-r)}{\Delta_{\tau'}} \rfloor)\Delta_{\tau'})^\top$ with $k_1, k_2 \in \{0, \dots, M_*-1\}$, it follows that

$$\check{c}_{ij}(k_1,k_2) \;=\; \sum_{k_1'=0}^{M_*-1} \sum_{k_2'=0}^{M_*-1} f_{1i}(k_1',k_2')\, f_{2j}(k_1-k_1', k_2-k_2')\,. \tag{6.25}$$

Denote by DF_{1i} and DF_{2i} the `discrete Fourier transforms` (short `DFT`) of f_{1i} and f_{2j}, respectively. This is to say, we have

$$\mathrm{DF}_{\cdot\cdot}(m_1,m_2) \;=\; \sum_{k_1=0}^{M_*-1} \sum_{k_2=0}^{M_*-1} f_{\cdot\cdot}(k_1,k_2)\, e^{-2\pi \mathrm{i}\,(k_1 m_1 + k_2 m_2)/M_*}. \tag{6.26}$$

for $m_1, m_2 = 0, \dots, M_*-1$. By virtue of the `discrete convolution theorem` (see [6, p. 145]), it follows that $\check{c}_{ij}$ is equal to the inverse DFT of $\mathrm{DF}_{1i} \cdot \mathrm{DF}_{2j}$, meaning

$$\check{c}_{ij}(k_1,k_2) \;=\; \frac{1}{M_*^2} \sum_{m_1=0}^{M_*-1} \sum_{m_2=0}^{M_*-1} \mathrm{DF}_{1i}(m_1,m_2) \cdot \mathrm{DF}_{2j}(m_1,m_2)\, e^{2\pi \mathrm{i}\,(k_1 m_1 + k_2 m_2)/M_*}.$$

Hence, we have to evaluate two DFT and one inverse DFT in order to derive $\check{c}_{ij}(k_1,k_2)$. A direct computation of a 2–dimensional DFT via formula (6.26), however, has complexity $O((M_*^2)^2)$. This means, in particular, that we do not gain anything compared to the convolution and shift operation in (6.25). Yet, a computation time proportional to $M_*^2 \log_2(M_*)$ is achieved by using a `fast Fourier transform` algorithm.

6.4.2 FFT algorithm

As is easily seen, the evaluation of a 2–dimensional DFT using formula (6.26) can be performed by means of two 1–dimensional DFTs when considering each dimension separately. Therefore, we focus only on 1–dimensional DFT and introduce

$$\mathrm{DF}(m) \;=\; \sum_{k=0}^{M_*-1} f(k)\,\exp\bigl(-\frac{2\pi\mathrm{i}\,km}{M_*}\bigr)\,, \qquad m=0,\dots M_*-1. \tag{6.27}$$

In the first step, one decomposes the sum on the right hand side of (6.27) into

$$\begin{aligned}\mathrm{DF}^{\mathrm{even}}(m) &= \sum_{k=0}^{M_*/2-1} f(2k)\,\exp\bigl(-\frac{2\pi\mathrm{i}\,km}{M_*/2}\bigr),\\ \mathrm{DF}^{\mathrm{odd}}(m) &= \sum_{k=0}^{M_*/2-1} f(2k+1)\,\exp\bigl(-\frac{2\pi\mathrm{i}\,km}{M_*/2}\bigr).\end{aligned}$$

for $m=0,\dots,M_*/2-1$. This yields

$$\begin{aligned}\mathrm{DF}(m) &= \mathrm{DF}^{\mathrm{even}}(m)+\mathrm{w}_{M_*}^m\mathrm{DF}^{\mathrm{odd}}(m),\\ \mathrm{DF}(m+M_*/2) &= \mathrm{DF}^{\mathrm{even}}(m)-\mathrm{w}_{M_*}^m\mathrm{DF}^{\mathrm{odd}}(m), \qquad m=0,\dots,M_*/2-1,\end{aligned}$$

where $\mathrm{w}_{M_*}=\exp(-2\pi\mathrm{i}/M_*)$. By definition, $\mathrm{DF}^{\mathrm{even}}$ and $\mathrm{DF}^{\mathrm{odd}}$ are again DFTs, this time with $M_*'=M_*/2$ summands. This explains, why we have chosen M_* to be a power of 2. The decomposition is recursively applied $\log_2(M_*)$ times. In the last step, $M_*/2$ DFTs consisting of 2 summands each are split up into two DFTs with only one summand, respectively. The evaluation of the final DFTs is trivial, since it is the identity map. In total, $(M_*/2)\log_2(M_*)$ complex multiplications and $M_*\log_2(M_*)$ complex additions have to be performed to evaluate the original DFT (6.27). Further details and different FFT algorithms are found, for instance, in Chapter 10 of [6] and Section 2.5 of [29].

Chapter 7

Numerical results for planar Boolean models

Based on the algorithms described in Sections 6.3.2 and 6.3.3, we now discuss some numerical examples for the estimation of the intrinsic volume densities of Boolean models in $\mathbb{R}^2$. On $W_n = [-500\,n, 500\,n]^2$ with $n \geq 1$, we analyze realizations of $m \in \mathbb{N}$ independent samples $\mathbf{v}_{n,k}^{(1)}$, $k = 1, \ldots, m$ and $\mathbf{v}_{n,k}^{(2)}$, $k = 1, \ldots, m$ of the estimators $\widehat{\mathbf{v}}_n^{(1)}$ and $\widehat{\mathbf{v}}_n^{(2)}$ introduced in (6.3) and (6.5), respectively. At this, we select different grain distributions and varying intensities of the underlying Poisson germ process. The first objective will be to check whether the samples fit to a multivariate normal distribution depending on the size of the observation window. The second target is to compute estimators of the corresponding asymptotic covariance matrices. The methods introduced in Sections 5.1, 5.2 and 5.4 are compared in terms of stability of the results and the computation time needed. Finally, asymptotic tests for the intrinsic volume densities $\overline{V}_0(\Xi), \overline{V}_1(\Xi)$ and $\overline{V}_2(\Xi)$ of some Boolean Models Ξ in $\mathbb{R}^2$ are performed. Based on the representing binary images, the tests yield a decision whether a given spatial data set is a realization of a certain Boolean model or if two data sets differ significantly.

Besides the choice of shape and size of the observation window W_n, $n \geq 1$, all presently addressed estimation methods for $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$ have several parameters to select in advance. These are, on the one hand, the scaling parameters Δ, Δ_τ and $\Delta_{\tau'}$ introduced in Section 6.2, and, on the other hand, the number $\tilde{q} \in \mathbb{N}$ and the size of the radii $r_i > 0$, $i = 1, \ldots, \tilde{q}$ (see Section 6.1), where we put $\tilde{q} = q - 1$ when considering $\widehat{\mathbf{v}}_n^{(1)}$ and $\tilde{q} = q$, otherwise. It should be pointed out that the following investigations are not designed to (and can not) provide a guideline of how to choose these parameters. This is due to the fact that the estimation outcome does heavily depend on the underlying germ–grain model. In order to adjust the parameters, an adequate learning sample has to be on–hand.

7.1 Description of the models

In this chapter we deal with three types of planar Boolean models with uniformly bounded grains, calling them Model A, Model B and Model C. At this, denote by $Sq(S,\alpha)$ a square, which is centered at the origin, rotated by an angle $\alpha \in [0, 2\pi)$ and has side length S. In Model A, we fix $\alpha = 0$ and choose a random or deterministic side length $S_1 > 0$ with $P(S_1 < s) = 1$ for some $s \in (0, \infty)$, that is, $M_0 = Sq(S_1, 0)$. For Model B, let α be uniformly distributed on $[0, 2\pi)$ (in short, $\alpha \sim U[0, 2\pi)$) and select side length S_2 in similar manner to S_1. Finally, let an almost surely bounded radius $R > 0$ be given, and put $M_0 = B_R(o)$ in Model C.

For these Boolean models, explicit formulae for the vector of intrinsic volume densities $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$ exist. The theoretical background is presented in Appendix B. According to Lemma B.1 and equation (B.4), we have

$$\text{Model A:} \qquad \mathbf{v}_{\mathrm{A}}^\top = \Big(\big(\lambda - \lambda^2 (\mathbb{E}S_1)^2\big)\, e^{-\lambda \mathbb{E}(S_1^2)}, 2\,\lambda\, \mathbb{E}S_1\, e^{-\lambda \mathbb{E}(S_1^2)}, 1 - e^{-\lambda \mathbb{E}(S_1^2)} \Big),$$

$$\text{Model B:} \qquad \mathbf{v}_{\mathrm{B}}^\top = \Big(\big(\lambda - 4\,\frac{\lambda^2}{\pi} (\mathbb{E}S_2)^2\big)\, e^{-\lambda \mathbb{E}(S_2^2)}, 2\,\lambda\, \mathbb{E}S_2\, e^{-\lambda \mathbb{E}(S_2^2)}, 1 - e^{-\lambda \mathbb{E}(S_2^2)} \Big),$$

$$\text{Model C:} \qquad \mathbf{v}_{\mathrm{C}}^\top = \Big(\big(\lambda - \pi\,\lambda^2 (\mathbb{E}R)^2\big)\, e^{-\lambda \pi \mathbb{E}(R^2)}, \pi \lambda \mathbb{E}R\, e^{-\lambda \pi \mathbb{E}(R^2)}, 1 - e^{-\lambda \pi \mathbb{E}(R^2)} \Big).$$

7.2 Estimating the mean value vector

Given a Boolean model with uniformly bounded grains, the estimators $\widehat{\boldsymbol{\mu}}_n^{(1)}$ and $\widehat{\boldsymbol{\mu}}_n^{(2)}$ defined in Section 6.1 satisfy the conditions of Theorem 3.3. Therefore, the quantities $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(1)} - \mathbf{v})$ and $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(2)} - \mathbf{v})$ are asymptotically normally distributed. The aim of the present section is to perform tests for multivariate normality subject to the size of the observation window, the grain distribution and the germ intensity. Before turning to the tests themselves, we compare the running time of the respective algorithms and the variability of the empirical values.

Since the presently considered examples are based on synthetic data, we can fix $\Delta = 1$. Scaling parameter $\Delta_\tau \in \mathbb{N}$ will be specified later on. In order to have a starting point on how to choose the radii $r_1, \ldots, r_{\tilde{q}}$, we follow the recommendations in [12]. For this, define $P_{\tilde{q},r_1} = \{(\tilde{q}, r_1, \ldots, r_{\tilde{q}}) : r_i = r_1 + 1.3\,(i-1), i = 2, \ldots, \tilde{q}\}$ for $\tilde{q} \in \mathbb{N}$ and $r_1 > 0$. As to the evaluation of $\mathbf{v}_n^{(1)}$, we employ the parameter constellation $P^{(1)} = P_{30,5.0}$. Here, we differ from [12], since the setting $P_{50,5.0}$ suggested therein turned out to be problematic for the estimation of the asymptotic covariance matrix; see Section 7.3.1. For the computation of $\mathbf{v}_n^{(2)}$, put $P^{(2)} = P_{16,4.2}$.

7.2.1 Computation time

Having fast evaluation methods is one target in the estimation process. As we have seen in Section 6.3, there exist two ways to compute $\widehat{\mathbf{v}}_n^{(1)}$. Method SF I has the

advantage that we can put $\Delta_\tau = 1$ without changing the running time, whereas the advantage of method SF II is that it can also be used for the computation of the estimators of $\Sigma_{\mathbf{v}}^{(1)}$ introduced in Sections 5.1 and 5.2. Table 7.1 displays the differences in running time of both methods on two different observation windows $W_1 = [-500, 500]^2$ and $W_2 = [-1000, 1000]^2$. Regarding method SF I, one should keep in mind that the percentage of pixels $u \in W_n \ominus B_{2r_{\max}}(o)$ (here $r_{\max} = r_{\check{q}}$), for which the efficient formulae (6.19) and (6.21) are applicable, increases with $n \in \mathbb{N}$. As a consequence, the computation time does not rise proportional to n, but much slower. In contrast, the increment in the running time of method SF II grows proportional to $\#(W_n^- \cap \mathbb{L}_\tau)$.

	SF I	SF II		
	$(\Delta_\tau = 1)$	$\Delta_\tau = 10$	$\Delta_\tau = 5$	$\Delta_\tau = 1$
$W_1 = [-500, 500]^2$	3 sec	0.5 sec	2 sec	52 sec
$W_2 = [-1000, 1000]^2$	9 sec	2 sec	9 sec	3 min 56 sec

Table 7.1: Average running time (of 100 runs) for the computation of $\widehat{\mathbf{v}}_n^{(1)}$, $n = 1, 2$ using methods SF I with $\Delta_\tau = 1$ and SF II with varying resolutions Δ_τ.

As to the evaluation of $\widehat{\mathbf{v}}_n^{(2)}$, we also compare two methods; calling them PKF I and PKF II. Both methods are based on the algorithm suggested in Section 2.4 of [12]. In PKF II, however, we make use of the relationship $V_0(\xi_{\mathbb{F}} \cap B_r(x)) = z_r(\xi_{\mathbb{F}}, x)$ for all $x \notin \xi_{\mathbb{F}}$ and $r > 0$. That is, we employ the algorithm SF II with parameter setting $P^{(2)}$ in order to compute $V_0(\xi_{\mathbb{F}} \cap B_{r_i}(x))$, $i = 1, \ldots, 16$ outside of $\xi_{\mathbb{F}}$. In doing so, we observe that the running time decreases approximately by factor two; compare Table 7.2. This phenomenom is easily explained. The Boolean model Ξ, for which the running times were measured, is chosen such that $\overline{V}_2(\Xi) = 0.5$, meaning that the computation of $V_0(\xi_{\mathbb{F}} \cap B_{r_i}(\cdot))$ is performed via SF II for 50 % of the pixels on average. Yet on W_1, it takes less than a second to evaluate $z_{r_1}(\xi_{\mathbb{F}}, x), \ldots, z_{r_{16}}(\xi_{\mathbb{F}}, x)$ with $\Delta_\tau \in \{5, 10\}$ and about 19 seconds if $\Delta_\tau = 1$. On W_2, the computation requires roughly 3 seconds for $\Delta_\tau = 5$ and about 65 seconds for $\Delta_\tau = 1$. The gain with respect to the running time increases with decreasing area fraction $p = \overline{V}_2(\Xi)$ (and vice versa).

PKF I	$\Delta_\tau = 10$	$\Delta_\tau = 5$	$\Delta_\tau = 1$
$W_1 = [-500, 500]^2$	14 sec	58 sec	24 min 10 sec
$W_2 = [-1000, 1000]^2$	1 min 0 sec	4 min 6 sec	1 h 43 min 14 sec

PKF II	$\Delta_\tau = 10$	$\Delta_\tau = 5$	$\Delta_\tau = 1$
$W_1 = [-500, 500]^2$	8 sec	33 sec	14 min 8 sec
$W_2 = [-1000, 1000]^2$	37 sec	2 min 24 sec	56 min 52 sec

Table 7.2: Average running time (of 100 runs) for the computation of $\widehat{\mathbf{v}}_n^{(2)}$, $n = 1, 2$ using methods PKF I and PKF II with $P^{(2)}$ and varying resolutions Δ_τ.

All measurements were performed on a dual–core AMD Opteron processor 875 (note that a Pentium 4 with 3 GHz requires approximately twice the indicated time). The underlying Boolean model is Model C with $R = 15$, i.e., $M_0 = B_{15}(o)$, and intensity $\lambda > 0$ corresponding to the area fraction $p = 0.5$, that is, $\lambda = -\log(0.5)/(\pi R^2)$.

7.2.2 Variation of the estimates

Studying the exactness of the estimation of the intrinsic volume densities $v_j = \overline{V}_j(\Xi)$, $j = 0, 1, 2$ is a multilayered task, since there are many influencing variables to take into account. In practical application, theoretical values to which the estimates could be compared to are rarely at hand. Besides this, the polyconvex set $\xi_{\mathbb{F}}$ can not be considered as a realization of the Boolean model Ξ inside of W_n, since the grains a deformed by the discretization. Hence, the formulae for the intrinsic volume densities presented in Section 7.1 are themselves only approximate reference values. Therefore, our focus lies mainly on the variability of the estimates rather than on the departure from the theoretical values. Nevertheless, these values are included in Tables 7.4 – 7.6 in order to give an idea of the magnitude of deviation. The displayed quantities are typical results for the estimates obtained for Models A and B with $S_1 = S_2 = 30$ and Model C with $R = 15$. Additionally, the coefficient of variation of the samples $\widehat{v}_{j,n,k}^{(\cdot)}$, $k = 1, \ldots, m$, $j = 0, 1, 2$ and the relative errors with respect to the theoretical values $v_j = \overline{V}_j(\Xi)$, $j = 0, 1, 2$ are given in percent. More precisely, we have

$$\delta\big(\widehat{v}_{j,n}^{(\cdot)}\big) = \frac{\sqrt{\dfrac{1}{m-1}\sum_{k=1}^{m}\big(\widehat{v}_{j,n,k}^{(\cdot)} - \overline{v}_{j,n}^{(\cdot)}\big)^2}}{\overline{v}_{j,n}^{(\cdot)}} \cdot 100\%\,, \qquad j = 0, 1, 2,$$

$$\delta\big(\widehat{v}_{j,n}^{(\cdot)}, v_j\big) = \frac{\sqrt{\dfrac{1}{m-1}\sum_{k=1}^{m}\big(\widehat{v}_{j,n,k}^{(\cdot)} - v_j\big)^2}}{v_j} \cdot 100\%\,, \qquad j = 0, 1, 2,$$

where $\overline{v}_{j,n}^{(\cdot)} = \frac{1}{m}\sum_{k=1}^{m}\widehat{v}_{j,n,k}^{(\cdot)}$, $j = 0, 1, 2$ and $n \geq 1$. Here and in the following, superscript $(\cdot)$ is a placeholder for (1), i.e., the SF algorithm presented in Section 6.1.1 or (2), i.e., the PKF method described in Section 6.1.2.

As mentioned before, we use the parameter constellation $P^{(1)} = P_{30,5.0}$ for the computation of $\widehat{\mathbf{v}}_n^{(1)}$ and $P^{(2)} = P_{16,4.2}$ for $\widehat{\mathbf{v}}_n^{(2)}$. Furthermore, we put $\Delta_\tau = 6$. The choice of Δ_τ originates from the tests for multivariate normality. Experiments showed, that a large lattic spacing Δ_τ is not advisable in terms of a high percentage of rejection of the hypothesis of 3–dimensional normal distribution. Values of Δ_τ that are smaller than $\Delta_\tau = 6$ have a rather negligible effect on the distributional assumption or the variation (compare SF I, where $\Delta_\tau = 1$, and SF II, where $\Delta_\tau = 6$, in Tables 7.4 – 7.6), but a considerable drawback with respect to the running time.

To summarize, firstly notice that the variability of the sample values decreases (as expected) with expanding observation window, i.e., with the available information

on $\mathbf{Y}$ (compare Table 7.3). Secondly, the variation in the estimates of the Euler density $\overline{V}_0(\Xi)$ is the highest (see Tables 7.4 – 7.6). It is a well–known problem that among the intrinsic volume densities, the Euler density is the most difficult one to estimate (see [66]). Not surprising, in addition, is that Model A exhibits the best results, since the approximation error is the smallest for this model. Recall that the typical grain in Model A has sides parallel to the coordinate axis so that the construction bricks of approximation system $\mathbb{F}$ (see Section 6.2.2) match the shape of M_0 quite well. A randomly rotated square or a ball, in contrast, is severely deformed when discretized. The results for the estimation of $\mathbf{v}$ using $\widehat{\mathbf{v}}_n^{(2)}$ can be improved with respect to the deviation from the theoretical vector when using more radii or larger $r_i > 0$, $i = 1, \ldots, q$ (see, e.g., Table 7.6). Yet there is typically no (noteworthy) decay in the fluctuations, whereas the running time for the evaluation of $\widehat{\mathbf{v}}_n^{(2)}$ becomes unnecessarily large.

PKF II	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\delta\big(\widehat{v}_{0,n}^{(2)}\big)$	-42.13 %	-20.21 %	-13.46 %	-10.03 %
$\delta\big(\widehat{v}_{1,n}^{(2)}\big)$	2.39 %	1.13 %	0.75 %	0.55 %
$\delta\big(\widehat{v}_{2,n}^{(2)}\big)$	2.97 %	1.46 %	0.96 %	0.71 %

Table 7.3: Decay of variation coefficients for increasing observation window. Estimation method PKF II on Model B with $M_0 = Sq(S_1, \alpha)$, $S_1 = 30$, $\alpha \sim U[0, 2\pi)$. Average values of 1000 samples.

Model A	SF I	SF II	PKF II, $P^{(2)}$	PKF II, $P^{(1)}$	$\mathbf{v}$
$\widehat{v}_{0,3}^{(\cdot)}$	1.1159e-4	1.0868e-4	1.0649e-4	1.0687e-4	
$\delta\big(\widehat{v}_{0,3}^{(\cdot)}\big)$	4.44 %	4.43 %	4.94 %	4.38 %	
$\delta\big(\widehat{v}_{0,3}^{(\cdot)}, v_0\big)$	6.97 %	9.06 %	10.84 %	10.35 %	1.1816e-4
$\widehat{v}_{1,3}^{(\cdot)}$	0.02307	0.02348	0.02220	0.02228	
$\delta\big(\widehat{v}_{1,3}^{(\cdot)}\big)$	0.73 %	0.71 %	0.81 %	0.71 %	
$\delta\big(\widehat{v}_{1,3}^{(\cdot)}, v_1\big)$	0.84 %	1.75 %	3.99 %	3.64 %	0.02310
$\widehat{v}_{2,3}^{(\cdot)}$	0.49953	0.50016	0.48339	0.48170	
$\delta\big(\widehat{v}_{2,3}^{(\cdot)}\big)$	0.88 %	0.93 %	0.98 %	0.96 %	
$\delta\big(\widehat{v}_{2,3}^{(\cdot)}, v_2\big)$	0.88 %	0.93 %	3.45 %	3.77 %	0.50000

Table 7.4: Comparison of variations in 1000 samples for Model A with $S_1 = 30$, i.e., $M_0 = Sq(30, 0)$, on $W_3 = [-1500, 1500]^2$. Note that SF I computes $\widehat{\mathbf{v}}_n^{(1)}$ with $\Delta_\tau = 1$ and $P^{(1)} = P_{30,5.0}$, and SF II with $\Delta_\tau = 6$. Results for $\widehat{\mathbf{v}}_n^{(2)}$ using PKF II are given w.r.t. parameter constellations $P^{(1)} = P_{30,5.0}$ and $P^{(2)} = P_{16,4.2}$.

Model B	SF I	SF II	PKF II, $P^{(2)}$	PKF II, $P^{(1)}$	$\mathbf{v}$
$\widehat{v}_{0,3}^{(\cdot)}$	-4.7059e-5	-5.0837e-5	-6.8986e-5	-5.3898e-5	
$\delta\big(\widehat{v}_{0,3}^{(\cdot)}\big)$	-19.42 %	-17.55 %	-13.46 %	-15.92 %	
$\delta\big(\widehat{v}_{0,3}^{(\cdot)}, v_0\big)$	205.42 %	213.31 %	253.34 %	219.98 %	4.5231e-5
$\widehat{v}_{1,3}^{(\cdot)}$	0.02169	0.02205	0.02453	0.02386	
$\delta\big(\widehat{v}_{1,3}^{(\cdot)}\big)$	0.74 %	0.71 %	0.75 %	0.75 %	
$\delta\big(\widehat{v}_{1,3}^{(\cdot)}, v_1\big)$	6.13 %	4.61 %	6.21 %	3.36 %	0.02310
$\widehat{v}_{2,3}^{(\cdot)}$	0.49923	0.49988	0.47237	0.48149	
$\delta\big(\widehat{v}_{2,3}^{(\cdot)}\big)$	0.88 %	0.72 %	0.96 %	0.99 %	
$\delta\big(\widehat{v}_{2,3}^{(\cdot)}, v_2\big)$	0.89 %	0.93 %	5.60 %	3.82 %	0.50000

Table 7.5: Comparison of variations in 1000 samples for Model B with $S_2 = 30$, i.e., $M_0 = Sq(30, \alpha)$ for $\alpha \sim U[0, 2\pi)$, on $W_3 = [-1500, 1500]^2$. Note that SF I computes $\widehat{\mathbf{v}}_n^{(1)}$ with $\Delta_\tau = 1$ and $P^{(1)}$, and SF II with $\Delta_\tau = 6$. Results for $\widehat{\mathbf{v}}_n^{(2)}$ using PKF II are given w.r.t. parameter constellations $P^{(1)}$ and $P^{(2)}$.

Model C	SF I	SF II	PKF II, $P^{(2)}$	PKF II, $P^{(1)}$	$\mathbf{v}$
$\widehat{v}_0^{(\cdot)}$	4.7742e-5	4.3853e-5	2.8104e-5	4.0895e-5	
$\delta\big(\widehat{v}_{0,n}^{(\cdot)}\big)$	17.75 %	18.88 %	29.61 %	20.00 %	
$\delta\big(\widehat{v}_{0,n}^{(\cdot)}, v_0\big)$	68.50 %	71.07 %	80.51 %	73.02 %	1.5045e-4
$\widehat{v}_1^{(\cdot)}$	0.02306	0.02347	0.02445	0.02385	
$\delta\big(\widehat{v}_{1,n}^{(\cdot)}\big)$	0.74 %	0.65 %	0.65 %	0.75 %	
$\delta\big(\widehat{v}_{1,n}^{(\cdot)}, v_1\big)$	0.77 %	1.71 %	5.89 %	3.32 %	0.02310
$\widehat{v}_2^{(\cdot)}$	0.50382	0.50451	0.48728	0.48562	
$\delta\big(\widehat{v}_{2,n}^{(\cdot)}\big)$	0.83 %	0.87 %	0.84 %	0.97 %	
$\delta\big(\widehat{v}_{2,n}^{(\cdot)}, v_2\big)$	1.13 %	1.26 %	4.61 %	3.03 %	0.50000

Table 7.6: Comparison of variations in 1000 samples for Model C with $R = 15$, i.e., $M_0 = B_{15}(o)$, on $W_3 = [-1500, 1500]^2$. Note that SF I computes $\widehat{\mathbf{v}}_n^{(1)}$ with $\Delta_\tau = 1$ and $P^{(1)}$, and $\Delta_\tau = 6$ in SF II. Results for $\widehat{\mathbf{v}}_n^{(2)}$ using PKF II are given w.r.t. parameter constellations $P^{(1)}$ and $P^{(2)}$.

7.2.3 Tests for asymptotic normality

By virtue of Theorem 3.3 and the continuous mapping theorem, it holds that the vector $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v})$ is asymptotically $\mathcal{N}_3(o, \Sigma_{\mathbf{v}}^{(\cdot)})$–distributed as W_n approaches $\mathbb{R}^2$ in the van Hove sense. For different grain distributions and germ–intensities, we investigate samples $\sqrt{|W_n|}(\widehat{\mathbf{v}}_{n,k}^{(\cdot)} - \mathbf{v})$, $k = 1, \ldots, m$ of size $m \in \mathbb{N}$ on $W_n = [-500\,n, 500\,n]^2$ for increasing $n \in \mathbb{N}$ and $m \in \{1000, 3000\}$. The samples are tested for multivariate normality by means of Mardia's multivariate skewness and kurtosis tests and the Henze–Zirkler test. Confer Appendix C for a description of these tests. For the Henze–Zirkler test, we use $\beta = 0.5$ in order to test against distributions with heavy tails and $\beta_3(m) = 2^{-1/2}\,(7\,m/4)^{1/7}$ as suggested in [25]. Each univariate subsample was additionally tested for one–dimensional normal distribution. The respective Kolmogorov–Smirnov tests, however, lead very rarely to a rejection (if at all, then for the component $v_0 = \overline{V}_0(\Xi)$ on small observation windows), so we omit the test results in the tables.

The overall behavior is similar for $\widehat{\mathbf{v}}_n^{(1)}$ and $\widehat{\mathbf{v}}_n^{(2)}$. We therefore refrain from listing the results for $\widehat{\mathbf{v}}_n^{(2)}$ in the tables and restrict to the data for $\widehat{\mathbf{v}}_n^{(1)}$. For each test, the tables display the P-value derived from the asymptotic distribution of the respective test statistic. Any P-value less than significance level $\alpha = 0.05$ is marked with an asterisk to allow a quick overview. Regarding the estimation parameters, we employ $P^{(1)}$ for $\widehat{\mathbf{v}}_n^{(1)}$ and $P^{(2)}$ for $\widehat{\mathbf{v}}_n^{(2)}$. As mentioned before, the choice $\Delta_\tau = 6$ turned out convenient with respect to the running time and the required minimal size of W_n so that the hypothesis of 3–dimensional normality of $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v})$ is not rejected. An ill–behaved example is given, for instance, when choosing $M_0 = Sq(30, \alpha)$ with $\alpha \sim U[0, 2\pi)$ and $\Delta_\tau = 10$. For this scenario, all tests for multivariate normal distribution of $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v})$ are still rejected on $W_5 = [-2500, 2500]^2$. Yet putting $\Delta_\tau = 6$ gives to good results (i.e., no rejection of any performed test) on $W_4 = [-2000, 2000]^2$ already.

The outcome of the simulation study for the investigated models can be summarized as follows (all tables are to be found in Appendix D). The smaller the area fraction $p = \overline{V}_2(\Xi)$ of a Boolean model Ξ is, the higher one has to choose $n \in \mathbb{N}$ so that the null-hypothesis is not rejected; see Table D.1 for a typical example. The dependence of the test decision on the grain distribution can be observed in Tables D.2 and D.3. In Table D.2, compare the results for Model A and Model B, where the only difference with respect to the grain distribution lies in the rotation. Table D.3 displays results for Model C with grain radius $R = 20, 50$ and 80. The samples contradict the hypothesis of multivariate normality for fixed n more likely with increasing grain diameter.

The presented results are based on the same parameter constellations for all models. When considering one specific model, then the parameters $\{(\tilde{q}, r_1, \ldots, r_{\tilde{q}})\}$ and Δ_τ should be adjusted by means of a learning sample. At this, two phenomena should be taken into account. Firstly, even if the null–hypothesis is not rejected for small sample sizes $m \in \mathbb{N}$ (here $m = 1000$), it might be rejected for larger samples (here $m = 3000$). Secondly, tests which do not yield a rejection on small windows eventu-

ally do so on larger ones. We therefore performed all tests for different sample sizes and also for the next level of n even if no rejection occured on the previous level.

7.3 Estimating the asymptotic covariance matrix

In Chapter 5, we have introduced the estimators $\widehat{\Sigma}_n$, $\widehat{\Sigma}'_n$ and $\widetilde{\Sigma}_n$, $n \geq 1$ of the asymptotic covariance matrix Σ of $\sqrt{|W_n|}(\widehat{\boldsymbol{\mu}}_n - \boldsymbol{\mu})$. Now, we discuss some empirical properties of the corresponding transforms $\widehat{\Sigma}^{(\cdot)}_{n,\mathbf{v}}$, $\widehat{\Sigma}'^{(\cdot)}_{n,\mathbf{v}}$ and $\widetilde{\Sigma}^{(\cdot)}_{n,\mathbf{v}}$, $n \geq 1$ (see, e.g., (6.8) and (6.12)), which estimate the asymptotic covariance matrix $\Sigma^{(\cdot)}_{\mathbf{v}}$ of $\sqrt{|W_n|}(\widehat{\mathbf{v}}^{(\cdot)}_n - \mathbf{v})$. At this, the following questions are approached. Firstly, we have once again to select an appropriate set of parameters. The constellation of radii $0 < r_1 \leq \ldots \leq r_{\tilde{q}}$, $\tilde{q} \in \mathbb{N}$ is predefined by the respective choice for $\widehat{\mathbf{v}}^{(1)}_n$ and $\widehat{\mathbf{v}}^{(2)}_n$. Additional parameters for the evaluation of the estimators $\widehat{\Sigma}_n$ and $\widehat{\Sigma}'_n$ are the lattice spacing $\Delta_{\tau'}$ (see (6.16)) and the averaging windows $U_{ij,n}$, $i,j = 1,\ldots,q$ (see Section 5.1.1). Simulation studies showed that $\Delta_{\tau'}$ highly influences the definiteness of the estimation matrices. Despite the fact that theoretically neither $\widehat{\Sigma}^{(\cdot)}_{n,\mathbf{v}}$ nor $\widehat{\Sigma}'^{(\cdot)}_{n,\mathbf{v}}$ need to be positive semi–definite for fixed $n \in \mathbb{N}$, their limit $\Sigma^{(\cdot)}_{\mathbf{v}}$ has to be positive semi–definite as a covariance matrix. Therefore, we select $\Delta_{\tau'}$ is such a way that all eigenvalues of $\widehat{\Sigma}^{(\cdot)}_{n,\mathbf{v}}$ and $\widehat{\Sigma}'^{(\cdot)}_{n,\mathbf{v}}$ are non–negative (which is equivalent to positive semi–definiteness). With regard to $\widehat{\Sigma}'_n$ and $\widetilde{\Sigma}_n$, we furthermore have to define subsampling windows $\tilde{W}_{kn} = \tilde{W}_n + h_{kn}$, $k = 1,\ldots,m_n$ with $h_{kn} \in \mathbb{R}^2$ and $m_n \in \mathbb{N}$ as introduced in Sections 5.2 and 5.4, respectively. In the second step, we compare the running time for the computation of the estimates of $\Sigma^{(\cdot)}_{\mathbf{v}}$. A look at the empirical variations is the topic of the final paragraph.

In what follows, we will only use algorithms SF II and PKF II (and not SF I or PKF I). Hence, we shortly write SF and PKF to indicate the underlying method.

7.3.1 Choosing the parameters

Regarding $\widehat{\Sigma}'_n$ and $\widetilde{\Sigma}_n$, the definition of the subwindows $\tilde{W}_{kn}, k = 1,\ldots,m_n$ is rather arbitrary. The only restrictions are given by $\bigcup_{k=1}^{m_n} \tilde{W}_{kn} \subseteq W_n$ for any $n \geq 1$ and by the condition that the intersection of two subwindows is negligible for large n when considering $\widetilde{\Sigma}_n$; compare condition (5.30). For the empirical results, we fix $n = 3$ and look at partitions of $W_3 = [-1500, 1500]^2$ into $m_3 \in \{2^2, 3^2, 4^2, \ldots, 8^2\}$ squares of side length $3000/\sqrt{m_3}$ that have no interior points in common, i.e., m_3 translates of $\tilde{W}_3 = [-1500/\sqrt{m_3}, 1500/\sqrt{m_3}]^2$ that overlap at the utmost on the boundary.

For the computation of $\widehat{\Sigma}_n$ and $\widehat{\Sigma}'_n$, $n \geq 1$, one has to select a sequence $\{U_{ij,n}\}$ of compact Borel sets such that, amongst others, $\mathrm{supp}(\mathrm{Cov}_{ij}) \subseteq \lim_{n\to\infty} U_{ij,n}$, where $\mathrm{supp}(\mathrm{Cov}_{ij})$ is the support of $\mathrm{Cov}_{ij}(x)$, $x \in \mathbb{R}^2$. As we have seen in Chapter 3, there exists a finite constant $\tilde{c}$ depending on $\mathbb{E}|M_0 \oplus B_{r_{\max}}(o)|$ and λ such that $|\mathrm{Cov}_{ij}(x)| \leq \tilde{c}\,\mathbb{E}\,|(M_0 \oplus B_{r_i}(x)) \cap (M_0 \oplus B_{r_j}(o))|$ for all $x \in \mathbb{R}^2$. Since the grains in Models A, B and C are assumed to be uniformly bounded, say $P(M_0 \subset B_{r_0}(o)) = 1$ for some

finite $r_0 > 0$, this results in $\mathrm{Cov}_{ij}(x) = 0$ for any $x \in \mathbb{R}^2$ with $|x| > 2r_0 + r_i + r_j$. In Models A and B, we put $U_{ij,n} = B_{\sqrt{2}s_\nu + r_i + r_j}(o)$ for $s_\nu > 0$ satisfying $P(S_\nu \leq s_\nu) = 1$, $\nu = 1, 2$, and $U_{ij,n} = B_{2r_0 + r_i + r_j}(o)$ in case of Model C, whenever $P(R \leq r_0) = 1$.

Next, we choose the value of lattice spacings Δ, Δ_τ and $\Delta_{\tau'}$, where $\Delta_{\tau'} \geq \Delta_\tau \geq \Delta > 0$. As before, put $\Delta = 1$, and let $\Delta_\tau = 6$ for the computation of $\widetilde{\Sigma}^{(\cdot)}_{n,\mathbf{v}}$. By definition, matrix $\widetilde{\Sigma}^{(\cdot)}_{n,\mathbf{v}}$ is always positive semi–definite and turned out to be even positive definite for all mentioned choices of m_3. The question of an adequate value of $\Delta_{\tau'}$ is more complicated. From the theoretical point of view, we only know that in the limit $\widehat{\Sigma}^{(\cdot)}_{n,\mathbf{v}}$ and $\widehat{\Sigma}'^{(\cdot)}_{n,\mathbf{v}}$ are at least positive semi–definite. In Table 7.7 we list some typical values for the percentage of estimates from $\widehat{\Sigma}^{(1)}_{3,\mathbf{v}}$ and $\widehat{\Sigma}^{(2)}_{3,\mathbf{v}}$, respectively, which have at least one negative eigenvalue. Herein lies also the answer to why we do not choose parameter setting $P_{50,5.0}$ for method SF as suggested in [12], but $P_{30,5.0}$. When using $P_{50,5.0}$ and $\Delta_{\tau'} = 6$, about 45 % of the matrices derived from $\widehat{\Sigma}^{(1)}_{3,\mathbf{v}}$ were not positive semi–definite. For $\Delta_{\tau'} = 24$, the percentage was still at 6 %. The situation improved by reducing the size of the radii $r_1, \ldots, r_{q-1}$. No improvement, however, was achieved when only the number but not the size of the radii was reduced. An explanation of this behavior might be that the correlation of random variables $Y_i(y)$ and $Y_j(x+y)$, which are part of the estimation of $\mathrm{Cov}_{ij}(x)$, $x \in U_{ij,n}$, with $y \in W_{in} \cap (W_{jn} - x)$ and $x \in U_{ij,n}$ is bounded above by a (finite) multiple of $\mathbb{E}|(M_0 \oplus B_{r_i}(x)) \cap (M_0 \oplus B_{r_j}(o))|$; see (3.13) and (5.6). The latter expression decreases with decreasing size of the radii r_i and r_j and increasing lattice spacing $\Delta_{\tau'}$ (recall that $x \in \mathbb{L}_{\tau'}$). Yet, no notable influence of the grain size is observed. Note that for the tests on asymptotic normality of $\sqrt{|W_n|}(\widehat{\mathbf{v}}^{(\cdot)}_n - \mathbf{v})$ the converse applies, i.e., there is only a marginal influence of the number and size of the radii, but a high influence of the grain size.

$n = 3$	$\Delta_{\tau'} = 6$	$\Delta_{\tau'} = 10$	$\Delta_{\tau'} = 14$	$\Delta_{\tau'} = 18$	$\Delta_{\tau'} = 22$
SF, $P_{30,5.0}$	12 %	8 %	2 %	0 %	0 %
SF, $P_{50,5.0}$	45 %	30 %	25 %	10 %	10 %

$n = 3$	$\Delta_{\tau'} = 6$	$\Delta_{\tau'} = 8$	$\Delta_{\tau'} = 10$	$\Delta_{\tau'} = 12$	$\Delta_{\tau'} = 14$
PKF, $P_{16,4.2}$	10 %	5 %	1 %	0 %	0 %

Table 7.7: Average percentage of estimates from $\widehat{\Sigma}^{(1)}_{3,\mathbf{v}}$ (SF) and $\widehat{\Sigma}^{(2)}_{3,\mathbf{v}}$ (PKF). For each constellation, 100 simulations were performed. The underlying Boolean model is Model C with $M_0 = B_{15}(o)$.

7.3.2 Comparison of running times

In view of Table 7.7, we use $P^{(1)} = P_{30,5.0}$ with $\Delta^{(1)}_{\tau'} = 18$ for the estimation of $\Sigma^{(1)}_{\mathbf{v}}$ on $W_3 = [-1500, 1500]^2$, and $P^{(2)} = P_{16,4.2}$ with $\Delta^{(2)}_{\tau'} = 12$ when considering $\Sigma^{(2)}_{\mathbf{v}}$. For this parameter setup, we give an overview over the running times for the computation of $\widehat{\Sigma}^{(\cdot)}_{3,\mathbf{v}}$ on W_3, and of $\widehat{\Sigma}'^{(\cdot)}_{3,\mathbf{v}}$ and $\widetilde{\Sigma}^{(\cdot)}_{3,\mathbf{v}}$ on different subdivisions of W_3

into $m_3 \in \{4, 9, 16, \ldots, 64\}$ disjoint subwindows $\tilde{W}_{k3}$, $k = 1, \ldots, m_3$, each of them a translate of $\tilde{W}_3 = [-1500/\sqrt{m_3}, 1500/\sqrt{m_3}]^2$.

Recall that the definition of $\widehat{\Sigma}'_n$ in Section 5.2 was motivated by the FFT algorithm that is employed for the computation of the weighted covariance estimator $\widehat{\Sigma}_n$; confer Sections 5.2.2 and 6.4. The complexity of the FFT depends on the number M of evaluation points. More precisely, it depends on the smallest power of 2 that is greater or equal to $2M$. This quantity is represented by M_*. The actual value of M_* is listed in Table 7.8 for the different partitions, where $m_3 = 1$ stands for the computation of the weighted covariance estimator on whole W_3, i.e., for $\widehat{\Sigma}^{(\cdot)}_{3,\mathbf{v}}$. At this, one should keep in mind that only evaluation points $x \in W_3 \ominus B_{r_{\tilde{q}}}(o)$ are taken into account. Furthermore, the covariance matrix $\Sigma^{(1)}$ has size 31×31, whereas $\Sigma^{(2)}$ has only 16×16 entries. This means that 496 and 136 covariances have to be estimated, respectively. For the computation of 10 Fourier transforms with $M_* = 512$ approximately 17 seconds are required, if $M_* = 256$ it takes about 1.5 seconds and less than 0.4 seconds if $M_* = 128$. Again all running times are measured on a dual–core AMD Opteron processor 875. The underlying Boolean model is Model C with $M_0 = B_{15}(o)$. We have not listed any results for partitions of W_3 finer than $m_3 = 16$, since some of the realizations of $\widehat{\Sigma}'^{(\cdot)}_{3,\mathbf{v}}$ with $m_3 \geq 16$ possess negative eigenvalues (for $m_3 = 16$ the percentage lies at 0.5 – 1 %).

	$m_3 = 1$	$m_3 = 4$	$m_3 = 9$	$m_3 = 16$
SF FFT	13 min 4 sec $M_* = 512$	3 min 43 sec $M_* = 256$	1 min 39 sec $M_* = 128$	3 min 2 sec $M_* = 128$
PKF FFT	4 min 9 sec $M_* = 512$	1 min 53 sec $M_* = 256$	3 min 16 sec $M_* = 256$	2 min 2 sec $M_* = 128$

Table 7.8: Average running time (of 200 runs) for the computation of $\widehat{\Sigma}^{(\cdot)}_{3,\mathbf{v}}$, represented by $m_3 = 1$, and $\widehat{\Sigma}'^{(\cdot)}_{3,\mathbf{v}}$ for partitions into $m_3 = 4, 9$ and 16 subwindows.

With regard to $\widetilde{\Sigma}^{(\cdot)}_{\mathbf{v},n}$, introduce $\Delta_{\tau''} \geq \Delta_\tau$ as the lattice spacing between the measurement points that are included in the computation of $\widetilde{\Sigma}^{(\cdot)}_{\mathbf{v},n}$. For an appropriate choice of $\Delta_{\tau''}$, positive semi–definiteness is no usefull criterion, since $\widetilde{\Sigma}^{(\cdot)}_{\mathbf{v},n}$ fulfills this property by definition. Yet, the running time for $\widetilde{\Sigma}^{(2)}_{3,\mathbf{v}}$ is much higher if $\Delta_{\tau''} = 6$ compared to $\Delta_{\tau''} = 12$; see Table 7.9. In Table D.10, however, we find that the variation of the empirical values increases drastically when choosing $\Delta_{\tau''} = 12$.

7.3.3 Variation of estimates

When comparing $\widehat{\Sigma}^{(\cdot)}_{3,\mathbf{v}}$ and $\widehat{\Sigma}'^{(\cdot)}_{3,\mathbf{v}}$ on different subdivisions of the observation window, one of the questions is, to what extend the variation of the empirical values is affected. In Tables D.4 and D.5, one finds some results for SF and PKF with respect to Model C for $R = 15$, where $\widehat{\Sigma}'^{(\cdot)}_{3,\mathbf{v}}$ is computed using $m_3 = 4, 9$ and 36. Note that 2 % of the realizations from $\widehat{\Sigma}'^{(1)}_{3,\mathbf{v}}$ (out of 200) have at least one negative eigenvalue if

$\Delta_{\tau''} = 6$	$m_3 = 9$	$m_3 = 16$	$m_3 = 36$	$m_3 = 49$	$m_3 = 64$
SF	1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec
PKF	3 min 52 sec	3 min 37 sec	3 min 46 sec	3 min 14 sec	2 min 58 sec

$\Delta_{\tau''} = 12$	$m_3 = 9$	$m_3 = 16$	$m_3 = 36$	$m_3 = 49$	$m_3 = 64$
PKF	49 sec	50 sec	36 sec	48 sec	21 sec

Table 7.9: Average running time (of 200 runs) for the computation of $\widetilde{\Sigma}^{(\cdot)}_{3,\mathbf{v}}$ for partitions into $m_n = 3^2, 4^2, \ldots, 8^2$ subwindows.

$m_3 = 36$ and 1 % of those from $\widehat{\Sigma}'^{(2)}_{3,\mathbf{v}}$. For Model A with $S_1 = 30$, these percentages are respectively about 1 % and 0.5 %, whereas for Model B with $S_2 = 30$, we have respectively 5 % and 3 %. The results on whole W_3 and on the first two subdivisions ($m_3 \in \{4, 9\}$) are comparable with respect to domain and variation of the estimated covariances. Besides the fact that we want $\widehat{\Sigma}'^{(\cdot)}_{3,\mathbf{v}}$ to be positive semi–definite, the increasing variation also disapproves finer partitions than $m_3 = 9$ in the present example. In Tables D.7 and D.8, we also give some results for Model A with $S_1 = 30$ and Model B with $S_2 = 30$ on the first two partitions. With regard to the running time of $\widehat{\Sigma}^{(\cdot)}_{3,\mathbf{v}}$, one could argue that larger values of $\Delta_{\tau'}$ should be used, in particular, since the estimates are still positive definite in all simulated examples; see Table 7.7. As indicated in Table D.6, this is not always advisable, because the variation may increase significantly; examine, for instance, the variation in the components $\widehat{\sigma}'^{(2)}_{v_0 v_2}$ and $\widehat{\sigma}'^{(2)}_{v_1 v_2}$.

As to the estimates from $\widetilde{\Sigma}^{(\cdot)}_{\mathbf{v},3}$, we observe a differing behavior in the methods SF and PKF. As expected, most of the estimates get better in terms of the variation with increasing number m_3 of subsamples, i.e., with increasing subdivision of the observation window; see Table D.9 for an example from SF. An exception occurs for the realizations of $\widehat{\sigma}'^{(2)}_{v_1 v_2}$, where the variation increases (in the considered examples); see Table D.10. For Model C with $R = 30$ and $m_3 = 25$ the variation lies at 65 %, for $m_3 = 36$ at 87 %, and for $m_3 = 64$, we have a variation of 187 %. An explanation might be the reduced number of measurement points which make up the estimates $\widehat{\mathbf{v}}^{(2)}_{3,k}$, $k = 1, \ldots, m_3$ on the subwindows. This hypothesis is underlined by the fact that for $\Delta_{\tau''} = 12$, we get variations up to 1258 %; see Table D.10. In addition, we observe a change of sign in the average value of $\widehat{\sigma}'^{(2)}_{v_1 v_2}$. The empirical covariance of 1000 independent samples from $\widehat{\mathbf{v}}^{(1)}_n$ on W_3, however, is positive (and approximately 3.3310). Finer resolutions Δ_τ could be examined as well, but the running time increases so fast that a simulation study can not be performed in a reasonable time horizon. A similar tendency as described for Model C was also observed for Model B with $S_2 = 30$. With regard to Model A with $S_1 = 30$, however, the results were comparable to those displayed in Table D.9. In the following, we will always put $\Delta_{\tau''} = 6$ and $m_3 = 25$ when treating $\widetilde{\Sigma}^{(2)}_{\mathbf{v},3}$, because these values gave the best results for Models B and C and sufficiently good results for Model A.

7.4 Asymptotic tests for the mean vector

Based on the empirical results presented in Sections 7.2 and 7.3, we will now consider diverse tests for the mean vector $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$ of some Boolean model Ξ. Examining the binary images of two Boolean models with circular grains in Figure 7.1, the human eye quickly characterizes these pictures as different. The picture on the left hand side shows a realization of Model C with $M_0 = B_{15}(o)$, whereas $M_0 = B_R(o)$ with $R \sim U(5, 21.5)$ in the picture on the right hand side. The decision on whether or not the realizations originate from the same germ–grain model shall now be answered by means of a statistical test.

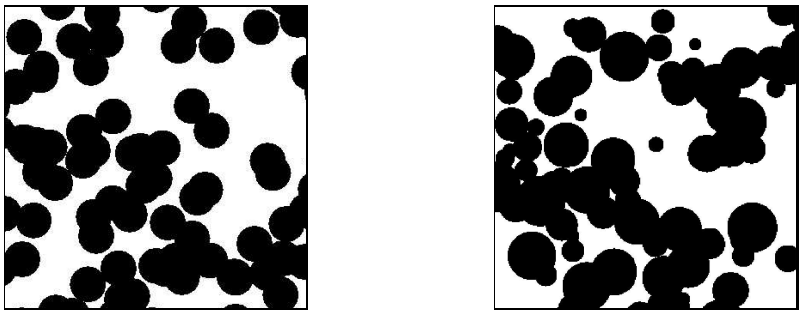


Figure 7.1: Realizations of Boolean Model C with $M_0 = B_{15}(o)$ (`left`) and $M_0 = B_R(o)$ for $R \sim U(5, 21.5)$ (`right`).

7.4.1 Theoretical background

By virtue of Theorem 3.3 and the continuous mapping theorem, we know that $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v})$ converges in distribution as $n \to \infty$ to a Gaussian random vector with mean vector zero and covariance matrix $\Sigma_{\mathbf{v}}^{(\cdot)}$ (specified in (6.6) and (6.11)) under appropriate assumptions on Ξ. That is, it holds that

$$\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v}) \quad \xrightarrow{D} \quad \mathcal{N}_3(o, \Sigma_{\mathbf{v}}^{(\cdot)}) \,. \tag{7.1}$$

Assuming that the asymptotic covariance matrix is positive definite, and therefore regular, (7.1) can be reformulated as

$$\sqrt{|W_n|}\left(\Sigma_{\mathbf{v}}^{(\cdot)}\right)^{-\frac{1}{2}}(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v}) \quad \xrightarrow{D} \quad \mathcal{N}_3(o, \mathrm{I}_3) \,,$$

where $\left(\Sigma_{\mathbf{v}}^{(\cdot)}\right)^{-\frac{1}{2}}\left(\Sigma_{\mathbf{v}}^{(\cdot)}\right)^{-\frac{1}{2}} = \left(\Sigma_{\mathbf{v}}^{(\cdot)}\right)^{-1}$ and I_3 is the 3×3 identity matrix. Since $\widehat{\Sigma}_{\mathbf{v},n}^{(\cdot)}$ is mean–square consistent, we apply Slutsky's theorem and deduce from (7.2) that

$$|W_n|(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v})^\top \left(\widehat{\Sigma}_{\mathbf{v},n}^{(\cdot)}\right)^{-1}(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v}) \quad \xrightarrow{D} \quad \mathcal{X}_3^2 \,, \tag{7.2}$$

where $\mathcal{X}_3^2$ represents a χ^2–distributed random variable with 3 degrees of freedom. Formula (7.2) provides the basis of an asymptotic test for the vector of intrinsic

volume densities $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))$. To this end, define

$$T_n \quad = \quad |W_n|(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v}^*)^\top \left(\widehat{\Sigma}_{\mathbf{v},n}^{(\cdot)}\right)^{-1} (\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v}^*), \qquad n \geq 1 \tag{7.3}$$

for some potential mean vector $\mathbf{v}^* \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$. Under the hypothesis $H_0 : \mathbf{v} = \mathbf{v}^*$, it holds that $T_n \xrightarrow{D} \mathcal{X}_3^2$. Consequently, the null–hypothesis is rejected at significance level α if

$$T_n \quad > \quad \chi^2_{3,1-\alpha}, \qquad \alpha \in (0,1) \tag{7.4}$$

for the $(1-\alpha)$–quantile $\chi^2_{3,1-\alpha}$ of the χ^2_3–distribution. Note that the estimator $\widehat{\Sigma}_{\mathbf{v},n}^{(\cdot)}$ can be replaced by $\widehat{\Sigma}_{\mathbf{v},n}^{\prime(\cdot)}$ or $\widetilde{\Sigma}_{\mathbf{v},n}^{(\cdot)}$ in formula (7.2) and thus in (7.3). Moreover, analogous statements can be given for the single components v_0, v_1 and v_2 (with one degree of freedom in (7.4)), and the pairs (v_0, v_1), (v_0, v_2) and (v_1, v_2) (with two degrees of freedom).

The asymptotic test described above can be employed to check if a given binary image represents a realization of a certain Boolean model, where in our case the test decision is based on the vector of intrinsic volume densities $\mathbf{v}^*$. The same idea is applicable for the comparison of two images. For this, let Ξ_1 and Ξ_2 be two independent Boolean models, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ represent the corresponding vectors of intrinsic volume densities. Next, denote by $\widehat{\mathbf{v}}_{n,1}^{(\cdot)}$ and $\widehat{\mathbf{v}}_{n,2}^{(\cdot)}$ the estimators of $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, and represent by $\widehat{\Sigma}_{\mathbf{v},n,1}^{(\cdot)}$ and $\widehat{\Sigma}_{\mathbf{v},n,2}^{(\cdot)}$ the estimators for the corresponding asymptotic covariance matrices. Under the hypothesis $H_0 : \mathbf{v}_1 = \mathbf{v}_2$ one has

$$|W_n|(\widehat{\mathbf{v}}_{n,1}^{(\cdot)} - \widehat{\mathbf{v}}_{n,2}^{(\cdot)})^\top \left(\widehat{\Sigma}_{\mathbf{v},n,1}^{(\cdot)} + \widehat{\Sigma}_{\mathbf{v},n,2}^{(\cdot)}\right)^{-1} (\widehat{\mathbf{v}}_{n,1}^{(\cdot)} - \widehat{\mathbf{v}}_{n,2}^{(\cdot)}) \quad \xrightarrow{D} \quad \chi^2_3\,. \tag{7.5}$$

The null–hypothesis is rejected if the left hand side in formula (7.5) is greater than the $(1-\alpha)$–quantile $\chi^2_{3,1-\alpha}$ of the χ^2_3–distribution. As before, the test can be reduced to the single components or any pair of intrinsic volume densities by adjusting the degrees of freedom appropriately and extracting the relevant components of the covariance matrices.

7.4.2 Numerical examples

The tests for the vector of intrinsic volume densities are performed on observation window $W_3 = [-1500, 1500]^2$, i.e., $n = 3$. The estimation parameters are chosen such that all samples from $\widehat{\mathbf{v}}_n^{(\cdot)}$ do not significantly depart from a multivariate normal distribution, and such that the estimators for the asymptotic covariance matrix are positive definite. More precisely, we use the following setup. For the estimation based on method SF, we select the constellation $P_{30,5.0}$ of radii and scaling parameters $\Delta_\tau = 6$, $\Delta_{\tau'} = 18$. With regard to $\widehat{\Sigma}_{\mathbf{v},n}^{\prime(1)}$, we let $m_3 = 9$ and put $m_3 = 64$ when considering $\widetilde{\Sigma}_{\mathbf{v},n}^{(1)}$. As to method PKF, employ $P_{16,4.2}$, $\Delta_\tau = 6$ and $\Delta_{\tau'} = 12$. For $\widehat{\Sigma}_{\mathbf{v},n}^{\prime(2)}$, we use $m_3 = 4$, and fix $m_3 = 25$ for the computation of $\widetilde{\Sigma}_{\mathbf{v},n}^{(2)}$. Due to the running time, the estimator $\widehat{\Sigma}_{\mathbf{v},n}^{(\cdot)}$ was only considered in some of the examples. The

results did not differ noteworthy from those of the modified version $\widehat{\Sigma}'^{(\cdot)}_{\mathbf{v},n}$ and will therefore not be mentioned in the sequel.

As a first example, we generate 1000 realizations of Boolean model C with $M_0 = B_{15}(o)$ and area fraction $p = 0.5$. Comparing two realizations at a time, we perform 500 tests for equality of the vector of intrinsic volume densities based on (7.5). The percentages of rejections on significance level $\alpha = 0.05$ are displayed in Table 7.10 for the different estimation methods.

Secondly, we compare 500 estimates of the vector of intrinsic volume densities on W_3 originating from a Boolean model Ξ_1 with typical grain $B_{15}(o)$ and germ intensity $\lambda_1 = -\log(0.5)/225\pi$ to 500 realizations from a Boolean model Ξ_2 with $B_R(o)$, where $R \sim U(5, 21.5)$, and $\lambda_2 = -\log(0.5)/\mathbb{E}(R^2)\pi$ (as illustrated in Figure 7.1). The first impression of Figure 7.1 might be misleading. The theoretical vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of Ξ_1 and Ξ_2 do not differ much. In fact, one has $\mathbf{v}_1 \approx (1.5045\text{e}{-4}, 0.02310, 0.5)^\top$ and $\mathbf{v}_2 \approx (1.7075\text{e}{-4}, 0.02316, 0.5)^\top$. As one can see in Table 7.11, only the tests for the specific Euler characteristic reveal the difference of the Boolean models satisfactorily. Testing for all three components simultaneously identifies nearly 100 % of the triples as significantly different.

	$H_0 : \mathbf{v}_1 = \mathbf{v}_2$
SF, $\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}$	3.4 % of the tests yield rejection
SF, $\widetilde{\Sigma}^{(1)}_{\mathbf{v},n}$	4.8 % of the tests yield rejection
PKF, $\widehat{\Sigma}'^{(2)}_{\mathbf{v},n}$	3.2 % of the tests yield rejection
PKF, $\widetilde{\Sigma}^{(2)}_{\mathbf{v},n}$	4.0 % of the tests yield rejection

Table 7.10: Test results for $H_0 : \mathbf{v}_1 = \mathbf{v}_2$, where $\mathbf{v}_\nu = (\overline{V}_0(\Xi_\nu), \overline{V}_1(\Xi_\nu), \overline{V}_2(\Xi_\nu))^\top$, $\nu = 1, 2$ for two independent Boolean models Ξ_1 and Ξ_2 each of them having intensity $\lambda = -\log(0.5)/(R^2\pi)$ and typical grain $M_0 = B_R(o)$, $R = 15$.

	$H_0 : v_{1,0} = v_{2,0}$	$H_0 : v_{1,1} = v_{2,1}$	$H_0 : v_{1,2} = v_{2,2}$	$H_0 : \mathbf{v}_1 = \mathbf{v}_2$
SF, $\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}$	99.6 %	5.6 %	8.2 %	99.8 %
SF, $\widetilde{\Sigma}^{(1)}_{\mathbf{v},n}$	99.2 %	6.4 %	7.4 %	99.4 %
PKF, $\widehat{\Sigma}'^{(2)}_{\mathbf{v},n}$	100.0 %	2.5 %	6.5 %	100.0 %
PKF, $\widetilde{\Sigma}^{(2)}_{\mathbf{v},n}$	98.4 %	3.4 %	10.0 %	100.0 %

Table 7.11: Test results for Boolean models Ξ_1 with typical grain $B_{15}(o)$ and germ intensity $\lambda_1 = -\log(0.5)/225\pi$, and Ξ_2 with typical grain $B_R(o)$, where $R \sim U(5, 21.5)$, and $\lambda_2 = -\log(0.5)/\mathbb{E}(R^2)\pi$.

One can construct different examples of Boolean models for which a single component of the vector of intrinsic volume densities suffices to differentiate distinct models;

see Table 7.11 for the specific Euler characteristic, Table 7.12 for (one half of) the specific boundary length and Table 7.13 for the specific area fraction. There exist also examples, where a combination of at least two of the intrinsic volume densities is needed to detect differences; see Table 7.14.

Testing for a potential vector $\mathbf{v}^*$ can be done in different ways. Firstly, the estimate $\widehat{\mathbf{v}}_n^{(\cdot)}$ can be compared to the theoretical vector of intrinsic volume densities, or it is tested against a reference value obtained from an independent learning sample. In Table 7.15, we give the results obtained for Boolean Model C with $M_0 = B_R(o)$, $R \sim U(15, 30)$. Since the estimated values depart too much from the theoretical ones, nearly all tests result in a rejection when $\mathbf{v}^*$ is defined to be the theoretical vector given in Section 7.1.

	$H_0 : v_{1,0} = v_{2,0}$	$H_0 : v_{1,1} = v_{2,1}$	$H_0 : v_{1,2} = v_{2,2}$	$H_0 : \mathbf{v}_1 = \mathbf{v}_2$
SF, $\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}$	7.4 %	99.8 %	5.0 %	100.0 %
PKF, $\widehat{\Sigma}'^{(2)}_{\mathbf{v},n}$	5.8 %	100.0 %	4.6 %	100.0 %

Table 7.12: Test results for Boolean models Ξ_1 with typical grain $B_{15}(o)$ and germ intensity $\lambda_1 = -\log(0.5)/225\pi$ so that $\mathbf{v}_1 = (1.5045\text{e}{-4}, 0.02310, 0.5)^\top$, and Ξ_2 with typical grain $B_{16}(o)$ and intensity $\lambda_2 = -\log(0.5)/256\pi$, where $\mathbf{v}_2 = (1.3223\text{e}{-4}, 0.02166, 0.5)^\top$.

	$H_0 : v_{1,0} = v_{2,0}$	$H_0 : v_{1,1} = v_{2,1}$	$H_0 : v_{1,2} = v_{2,2}$	$H_0 : \mathbf{v}_1 = \mathbf{v}_2$
SF, $\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}$	69.2 %	75.2 %	99.0 %	96.8 %

Table 7.13: Test results for Boolean models Ξ_1 with typical grain $B_{15}(o)$ and $\overline{V}_2(\Xi_1) = 0.48$, and Ξ_2 with typical grain $B_{15}(o)$ and $\overline{V}_2(\Xi_1) = 0.52$. Here, we have $\mathbf{v}_1 = (1.6648\text{e}{-4}, 0.02267, 0.48)^\top$ and $\mathbf{v}_2 = (1.3259\text{e}{-4}, 0.02348, 0.52)^\top$.

PKF, $\widehat{\Sigma}'^{(2)}_{\mathbf{v},n}$	$H_0 : v_{1,0} = v_{2,0}$	$H_0 : v_{1,1} = v_{2,1}$	$H_0 : v_{1,2} = v_{2,2}$
	44.2 %	13.6 %	93.4 %
	$H_0 : (v_{1,0}, v_{1,1}) = (v_{2,0}, v_{2,1})$		$H_0 : (v_{1,0}, v_{1,2}) = (v_{2,0}, v_{2,2})$
	69.8 %		88.2 %
	$H_0 : (v_{1,1}, v_{1,2}) = (v_{2,1}, v_{2,2})$		$H_0 : \mathbf{v}_1 = \mathbf{v}_2$
	99.0 %		98.6 %

Table 7.14: Test results for Boolean models Ξ_1 with typical grain $Sq(30, \alpha)$, where $\alpha \sim U[0, 2\pi)$, and germ intensity $\lambda_1 = -\log(0.5)/900\pi$, and Ξ_2 with typical grain $Sq(31, \alpha)$ and intensity $\lambda_2 = \lambda_1$. This results in $\mathbf{v}_1 = (4.5231\text{e}{-5}, 0.02310, 0.5)^\top$ and $\mathbf{v}_2 = (4.3155\text{e}{-5}, 0.02278, 0.523)^\top$.

The fact that the test result depend on the choice of estimation parameters is illustrated in Table 7.16. Here, we have generated 1000 realizations of Boolean model A with $M_0 = Sq(30, 0)$ and area fraction $p = 0.5$, once again comparing two images at a time. Yet, the estimation on which the results in Table 7.16 are based on, are computed, on the one hand, via $\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}$ using $P_{16,4.2}$ and, on the other hand, via $\widetilde{\Sigma}^{(1)}_{\mathbf{v},n}$ using $P_{30,5.0}$ but a partition into $m_3 = 9$ subwindows only. At this, it should be mentioned that the estimators from $\widehat{\mathbf{v}}^{(1)}_n$ show a high variability with up to 54 % when considering $P_{16,4.2}$. Moreover, the coefficients of variation for $\widetilde{\Sigma}^{(1)}_{\mathbf{v},n}$ with the above mentioned parameters lie between 74 % and 542 %.

	$H_0 : v_0 = v_0^*$	$H_0 : v_1 = v_1^*$	$H_0 : v_2 = v_2^*$	$H_0 : \mathbf{v} = \mathbf{v}^*$
PKF, $\widehat{\Sigma}'^{(2)}_{\mathbf{v},n}$	71.4 %	100.0 %	100.0 %	100.0 %

	$H_0 : v_0 = v_0^{**}$	$H_0 : v_1 = v_1^{**}$	$H_0 : v_2 = v_2^{**}$	$H_0 : \mathbf{v} = \mathbf{v}^{**}$
PKF, $\widehat{\Sigma}'^{(2)}_{\mathbf{v},n}$	3.0 %	2.6 %	5.0 %	4.6 %

Table 7.15: Testing Boolean model Ξ with typical grain $B_R(o)$, where $R \sim U(15, 30)$, and $\lambda = -\log(0.5)/525\pi$ against the theoretical vector $\mathbf{v}^* \approx (1.7602\text{e-}4, 0.01485, 0.5)^\top$ and the reference values $\mathbf{v}^{**} \approx (0.1799\text{e-}4, 0.01583, 0.48344)^\top$ derived from another set of 1000 independent samples.

SF	$H_0 : v_{1,0} = v_{2,0}$	$H_0 : v_{1,1} = v_{2,1}$	$H_0 : v_{1,2} = v_{2,2}$	$H_0 : \mathbf{v}_1 = \mathbf{v}_2$
$\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}, P_{30,5.0}$	4.8 %	1.8 %	4.4 %	3.0 %
$\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}, P_{16,4.2}$	29.0 %	18.0 %	30.5 %	47.0 %
$\widetilde{\Sigma}'^{(1)}_{\mathbf{v},n}, m_3 = 36$	3.4 %	3.8 %	4.2 %	4.2 %
$\widetilde{\Sigma}'^{(1)}_{\mathbf{v},n}, m_3 = 9$	15.2 %	17.6 %	16.0 %	31.2 %

Table 7.16: Test results for two independent Boolean models Ξ_1 and Ξ_2 each with intensity $\lambda = -\log(0.5)/900\pi$ and typical grain $M_0 = Sq(30, 0)$.

7.5 Summary

At the beginning of this chapter we have compared the evaluation methods SF I (see [33]) and SF II (introduced in Section 6.3.2) for the estimator $\widehat{\mathbf{v}}^{(1)}_n$. With respect to the computation time, we have seen that method SF I is to be favored over SF II when choosing $\Delta_\tau = 1$ in both cases. Putting $\Delta_\tau = 6$ for SF II leads to acceptable running times, which are moreover comparable to those of SF I on observations windows up to

$W_3 = [-1500, 1500]^2$. An examination of Tables 7.4–7.6 shows that the variation of the estimates derived from SF II is of the same order as the one related to SF I. The deviation from the theoretical values is yet slightly higher. The latter observation is negligible in practical application, since, in general, the theoretical values (in terms of $\xi_{\mathbb{F}}$) are not available. One advantage of the new evaluation algorithm SF II is that it can also be used for the computation of the weighted covariance estimators $\widehat{\Sigma}^{(1)}_{\mathbf{v},n}$ and $\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}$. Both methods are surely applicable for the estimation of the asymptotic covariance matrix Σ by means of $\widetilde{\Sigma}^{(1)}_{\mathbf{v},n}$. A second advantage of method SF II is that it can be employed to compute the Euler characteristic $V_0(\xi_{\mathbb{F}} \cap B_{r_i}(x))$ for all $x \notin \xi_{\mathbb{F}}$ and for any $i \in \{1, \ldots, q\}$. This alternative evaluation procedure is represented by method PKF II. Compared to PKF I, where we strictly follow the algorithm described Section 2.4 of [12], one observes substantial shorter running times without changing the estimated values. The actual gain does, however, depend on the area fraction; as mentioned before, it is roughly proportional to $p = \overline{V}_2(\Xi)$ (subject to the size of the observation window). With regard to $\widehat{\mathbf{v}}^{(2)}_n$, we furthermore notice that the variations of the estimates are comparable to those of $\widehat{\mathbf{v}}^{(1)}_n$. Yet, the computation time is still much higher.

As to the tests for multivariate normality of $\sqrt{|W_n|}(\widehat{\mathbf{v}}^{(\cdot)}_n - \mathbf{v})$ for fixed $n \in \mathbb{N}$, one should keep in mind the dependency on the germ intensity (and by this on the volume fraction) and the grain size. Before running any test for the mean value $\mathbf{v}$, one has to make sure that the samples fit to a multivariate normal distribution, unless the distribution for fixed n is known or can be approximated otherwise.

The final tool for the asymptotic tests described in Section 7.4.2 is the (consistent) estimation of the respective asymptotic covariance matrix $\Sigma^{(1)}_{\mathbf{v}}$ and $\Sigma^{(2)}_{\mathbf{v}}$. Despite the fact that we do not have any closed–form expression for the asymptotic covariances $\sigma^{(1)}_{ij}$, $i = 1, \ldots, q$, $j = 1 \ldots, q-1$ or $\sigma^{(2)}_{ij}$, $i, j = 1, \ldots, q$, we know that the matrices are at least positive semi–definite. Since also the estimator $\widetilde{\Sigma}^{(\cdot)}_{\mathbf{v},n}$ is positive semi–definite for any $n \in \mathbb{N}$, we have chosen scaling parameter $\Delta_{\tau'}$ such that $\widehat{\Sigma}^{(\cdot)}_{\mathbf{v},n}$ and $\widehat{\Sigma}'^{(\cdot)}_{\mathbf{v},n}$ fulfill this property as well. Speaking of algorithm SF, we note that the estimator $\widetilde{\Sigma}^{(1)}_{\mathbf{v},n}$ with $m_n = 64$ is preferable over the weighted covariance estimators due to its distinctly shorter running time. The variation of the empirical values is of the same order for both estimation approaches when choosing sufficiently many subwindows. Only the variation in component $\sigma^{(1)}_{v_2 v_2}$ is much higher. This, however, has so far not shown any effect on the test results in the examined examples. On smaller observation windows, the weighted covariance estimators have the advantage that no (or only a marginal) subdivision of the window is necessary for their computation. When considering a set of large radii on a relatively small observation window, $\widehat{\Sigma}^{(1)}_{\mathbf{v},n}$ and $\widehat{\Sigma}'^{(1)}_{\mathbf{v},n}$ should be used, since due to the minus sampling the actual sampling windows $\check{W}_{nk} \ominus K_i$, $k = 1, \ldots, m_n$, $i = 1, \ldots, q$ get too small. With regard to the estimation of $\Sigma^{(2)}_{\mathbf{v}}$, we find that the estimator $\widehat{\Sigma}'^{(2)}_{\mathbf{v},n}$ is the most favorable in terms of running time and variability. The usage of $\widetilde{\Sigma}'^{(1)}_{\mathbf{v},n}$ is somewhat critical with regard to the variation. In addition, there is no gain in running time in comparison to the weighted covariance estimators as in the case of method SF.

As indicated in the respective sections, the parameters for the estimation of the mean vector and of the asymptotic covariance matrix have to be chosen in an appropriate way. Without an acceptable order of variation and basic properties like an (approximate) normal distribution or positive (semi–)definiteness the tests will probably not provide satisfactory results. The parameter adjustment requires most often an extensive analysis.

Appendix A

Locally compact, Hausdorff, second countable spaces

The definitions and properties of random closed sets and related functionals in Chapter 2 are based on the assumption of an underlying locally compact, Hausdorff and second countable space (LCHS space). In what follows, we give a short summary of relevant definitions and facts on these spaces and refer to [58] for details. With regard to the properties, the corresponding theorems in [58] are given in brackets.

A.1 Definitions

Topological space
Let E be an arbitrary set. A system $\mathcal{T}_E$ of subsets of E is called a `topology` on E if $\mathcal{T}_E$ is closed under arbitrary unions and finite intersections. The elements of $\mathcal{T}_E$ are said to be `open`, their complements are `closed`. The pair $(E, \mathcal{T}_E)$ is referred to as `topological space`, where we simply write $E = (E, \mathcal{T}_E)$. Let $E' \subset E$ be some subset of E and consider the `induced topology` $\mathcal{T}_{E'} = \{G \cap E' : G \in \mathcal{T}_E\}$. Then $(E', \mathcal{T}_{E'})$ is called a `(topological) subspace` of $(E, \mathcal{T}_E)$.

Basis and subbasis
A system $\mathcal{E} \subset \mathcal{T}_E$ is a `basis` for the topology $\mathcal{T}_E$ if each set in $\mathcal{T}_E$ can be represented as a union of elements of $\mathcal{E}$. A `subbasis` for the topology $\mathcal{T}_E$ is a collection $\mathcal{E}' \subset \mathcal{T}_E$ such that the set of all finite intersections of elements of $\mathcal{E}'$ is a basis for $\mathcal{T}_E$.

Second countable
A topological space is called `second countable` if its topology has a countable basis.

Neighborhood
Let $U \subseteq E$, then we say U is a `neighborhood` of $x, x \in E$ if there exists an open set $G \subseteq E$ such that $x \in G \subseteq U$ holds.

Hausdorff space
A topological space E is `Hausdorff` if any two distinct points $x, y \in E$ have disjoint neighborhoods.

Compact
A set $C \subseteq E$ is said to be `compact` if every open cover of C admits a finite subcover, i.e., given $C \subseteq \bigcup_{i\in I} G_i$ with $G_i \in \mathcal{T}_E, i \in I$ then there is a finite subset $\{i_1, \ldots, i_m\} \subseteq I$ with $m \in \mathbb{N}$ satisfying $C \subseteq \bigcup_{k=1}^{m} G_{i_k}$.

Locally compact
A topological space E is called `compact` if E is itself a compact set. If each point $x \in E$ has a compact neighborhood, then we say E is `locally compact`.

Metric space
Let E be some set. A mapping $d : E \times E \to [0, \infty)$ is a `metric on` E provided that

- $d(x, y) = 0 \Leftrightarrow x = y$,
- $d(x, y) = d(y, x)$ for all $x, y \in E$,
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in E$.

The pair (E, d) is referred to as `metric space`.

Metrizable
A topological space $(E, \mathcal{T}_E)$ is said to be `metrizable` if there exists a metric $d : E \times E \to [0, \infty)$ generating $\mathcal{T}_E$, i.e.,

$$\mathcal{T}_E = \{G \subseteq E : \forall\, x \in G\ \exists\, r > 0 \text{ such that } \{y \in E : d(x, y) < r\} \subseteq G\}\,.$$

Borel σ–algebra
Let E be some topological space, and denote by $\mathfrak{B}(E)$ the minimal system of subsets of E that contains $\mathcal{T}_E$ and is closed under countable unions (or intersections) and under the operation of taking complements (in other words, $\mathfrak{B}(E)$ is the `σ-algebra` generated by the open sets in E). Then $\mathfrak{B}(E)$ is called the `Borel σ-algebra` and its elements are the `Borel sets`.

Closure, Interior, Boundary
Fix $B \subset E$ arbitrarily. The `closure` $\mathrm{clos}(B)$ of B is defined as the smallest closed set containing B, the `interior` $\mathrm{int}(B)$ of B is the largest open subset of B and the `boundary` ∂B is given by $\partial B = \mathrm{clos}(B) \setminus \mathrm{int}(B)$.

A.2 Properties

Every metric space is a Hausdorff space; see [58, Thm. 1.25].

Every subspace of a Hausdorff space is Hausdorff; see [58, Thm. 6.10].

Every singleton $\{x\}, x \in E$ is closed in $(E, \mathcal{T}_E)$ if E is Hausdorff; see [58, Thm. 6.3].

Every compact Hausdorff space is locally compact; by definition.

Every subspace of a second countable space is second countable; by definition.

Every LCHS space is metrizable; First Metrication Theorem of Urysohn, see [58, Thm. 10.16].

Appendix B

Intrinsic volume densities

In Section 2.4.4, we have introduced the intrinsic volume densities $\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)$ of a stationary random closed set Ξ in $\mathbb{R}^d$ with $P(\Xi \in \mathcal{S}) = 1$. Here, we show how to calculate $\overline{V}_0(\Xi), \overline{V}_1(\Xi)$ and $\overline{V}_2(\Xi)$ for a stationary Boolean model in $\mathbb{R}^2$. To this end, we need some auxiliary variables.

Definition B.1 *Let $K, K' \in \mathcal{K}$, then the quantity $V_2(K, K')$ satisfying the equation $2V_2(K, K') = V_2(K \oplus K') - V_2(K) - V_2(K')$ is called the* `mixed area` *of K and K'.*

Since $K \oplus K = 2K$ and $V_2(2K) = 4\,V_2(K)$, one finds that $V_2(K, K) = V_2(K)$ for any convex set K.

Definition B.2 *The* `support function` *of a compact set $C \subset \mathbb{R}^2$ is defined as the mapping $s_C(\vartheta) : [0, 2\pi] \to \mathbb{R}$ with*

$$s_C(\vartheta) = \max\{x_1 \cos\vartheta + x_2 \sin\vartheta : (x_1, x_2)^\top \in C\}\,.$$

If $M_0 \in \mathcal{K}$ almost surely, then the convex set $\mathbb{E}_A M_0$ with support function

$$s_{\mathbb{E}_A M_0}(\vartheta) = \mathbb{E}(s_{M_0}(\vartheta))\,, \qquad \vartheta \in [0, 2\pi]$$

is referred to as the `Aumann expectation` *of M_0.*

Confer [72, pp. 109–110] on a more general definition of the Aumann expectation of random compact sets. In Section 2.4.3, we defined the mean width $b(C)$ of a set $C \in \mathcal{C}$ as the mean distance between any pair of parallel, supporting hyperplanes of K. Note that we have

$$b(C) = \frac{1}{\pi} \int_0^\pi \Big(s_C(\vartheta) + s_C(\vartheta + \pi)\Big) d\vartheta\,,$$

see [72, pp. 109–110].

One further property of random closed sets will be used in the following lemma.

Definition B.3 *A random closed set M_0 is said to be* `isotropic` *if its distribution is invariant with respect to rotations around the origin, meaning that $M_0 \stackrel{D}{=} g_\vartheta(M_0) = \{(x_1 \cos\vartheta - x_2 \sin\vartheta, x_1 \sin\vartheta + x_2 \cos\vartheta)^\top : (x_1, x_2)^\top \in M_0\}$ for any $\vartheta \in (0, 2\pi)$.*

Lemma B.1 *Let Ξ be a stationary planar Boolean model with intensity $\lambda > 0$ and typical grain $M_0 \sim \mathbb{Q}$, $M_0 \in \mathcal{K}$. Then it holds that*

$$\overline{V}_0(\Xi) = \left(\lambda - \lambda^2 V_2(\mathbb{E}_A M_0, -\mathbb{E}_A M_0)\right) e^{-\lambda \mathbb{E} V_2(M_0)}, \tag{B.1}$$

$$\overline{V}_1(\Xi) = \lambda \mathbb{E} V_1(M_0)\, e^{-\lambda \mathbb{E} V_2(M_0)}, \tag{B.2}$$

$$\overline{V}_2(\Xi) = 1 - e^{-\lambda \mathbb{E} V_2(M_0)}. \tag{B.3}$$

If $M_0 \stackrel{D}{=} -M_0$, then we have

$$\overline{V}_0(\Xi) = \left(\lambda - \lambda^2 V_2(\mathbb{E}_A M_0)\right) e^{-\lambda \mathbb{E} V_2(M_0)}. \tag{B.1$'$}$$

If M_0 is isotropic, then one can write

$$\overline{V}_0(\Xi) = \left(\lambda - \frac{\lambda^2}{\pi} (\mathbb{E} V_1(M_0))^2\right) e^{-\lambda \mathbb{E} V_2(M_0)}. \tag{B.1$''$}$$

Proof Formulas (B.1$''$), (B.2) and (B.3) are derived in Corollary 5.4.4 of [65]. With regard to equation (B.1) see Lemma 4.1 of [73] and Theorem 5.5.3 of [65].

Given $M_0 \stackrel{D}{=} -M_0$, we get $\mathbb{E}_A M_0 = -\mathbb{E}_A M_0$ by the definition of the Aumann expectation. Following the remarks after Definition B.1, we get $V_2(\mathbb{E}_A M_0, -\mathbb{E}_A M_0) = V_2(\mathbb{E}_A M_0)$, since $\mathbb{E}_A M_0$ is convex. Formula (B.1$'$) therefore follows from (B.1).

That (B.1$''$) is a direct consequence of (B.1) can easily be seen as well. For any $\vartheta, \vartheta' \in [0, 2\pi]$ with $\theta'' = \vartheta - \vartheta' \in (0, 2\pi)$, we have

$$\begin{aligned}
\mathbb{E} s_{M_0}(\vartheta) &= \int_{\mathcal{K}} \max\{x_1 \cos\vartheta + x_2 \sin\vartheta : (x_1, x_2)^\top \in M\}\, \mathbb{Q}(dM) \\
&= \int_{\mathcal{K}} \max\{(x_1 \cos\vartheta'' + x_2 \sin\vartheta'') \sin\vartheta' + (-x_1 \sin\vartheta'' + x_2 \cos\vartheta'') \sin\vartheta' : \\
&\qquad (x_1, x_2)^\top \in M\}\, \mathbb{Q}(dM) \\
&= \int_{\mathcal{K}} \max\{x_1 \cos\vartheta' + x_2 \sin\vartheta' : (x_1, x_2)^\top \in g_{2\pi-\theta''}(M)\}\, \mathbb{Q}(dM) \\
&= \mathbb{E} s_{M_0}(\vartheta'),
\end{aligned}$$

since $M_0 \stackrel{D}{=} g_{2\pi-\theta''}(M_0)$. Consequently, we obtain $\mathbb{E}_A M_0 = B_r(o)$ with radius $r = \mathbb{E} s_{M_0}(0) = \frac{1}{2}\mathbb{E}\, b(M_0) = \frac{1}{\pi}\mathbb{E} V_1(M_0)$. □

If M_0 is a uniformly rotated square $Sq(S_1, \alpha), \alpha \sim U(0, 2\pi)$ or a ball $B_R(o)$ (Models B and C of Chapter 7), then we can directly apply Lemma B.1 (with (B.1$''$)) to determine the vector $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))$. With regard to Model A, where $M_0 = Sq(S_1, 0)$ is a square with sides parallel to the coordinate axes it remains to compute the volume of the Aumann expectation of M_0. For this, we show that $\mathbb{E}_A(Sq(S_1, 0)) = Sq(\mathbb{E} S_1, 0)$ and thus

$$\overline{V}_0(\Xi) = \left(\lambda - \lambda^2 (\mathbb{E} S_1)^2\right) e^{\lambda \mathbb{E} V_2(M_0)}. \tag{B.4}$$

Consider the support function $s_{M_0}(\vartheta)$ of $M_0 = Sq(S_1, 0)$. For $k \in \{0, 1, 2, 3\}$, one finds that $s_{M_0}(\vartheta) = S_1 \cos((2k+1)\pi/4 - \vartheta)/\sqrt{2}$ if $\vartheta \in [k\pi/2, (2k+1)\pi/4]$ and $s_{M_0}(\vartheta) = S_1(\omega) \sin(\vartheta - (2k+1)\pi/4)/\sqrt{2}$ given $\vartheta \in [(2k+1)\pi/4, (k+1)\pi/2]$. Taking the expectation on both sides, yields the claimed relationship.

Appendix C

Tests of multivariate normality

There exist at least 50 test procedures of multivariate normality (short MVN) as well as several surveys investigating their properties, see, e.g., [11], [24] or [59]. A nice overview is given in [46] including an extensive bibliography. It is generally agreed, that there does not exist one method, which is entirely reliable if used alone. Hence, a combination of three well–established test procedures is used in Section 7.2.3 to approach the question of multivariate normality. Since under the MVN hypothesis the marginals are univariate normal distributions, a Kolmogorov–Smirnov test is applied to each univariate samples. A description of the Kolmogorov–Smirnov test can be found in almost every basic statistic literature discussing goodness–of–fit tests and is therefore omitted here, confer e.g. [34]. In addition, tests for MVN employing multivariate measures of skewness and kurtosis, as presented in [37], are used. These tests can provide evidence for multivariate outliner. The third test procedure is based on the comparison of empirical characteristic functions and was introduced in [25]. This class of test procedures is often called Henze–Zirkler tests, but are also known as BHEP tests due to the underlying papers [2] and [10].

In what follows, Mardia's MVN tests and the Henze–Zirkler test are shortly described. Confer the corresponding papers on the theoretical background. Let $\mathbf{X}_1, \ldots, \mathbf{X}_m, m \geq 1$ be an i.i.d random sample of q–dimensional random vectors with sample mean vector $\overline{\mathbf{X}}_m = \frac{1}{m}\sum_{i=1}^{m} \mathbf{X}_i$ and sample covariance matrix

$$\Sigma_m \;=\; \frac{1}{m}\sum_{i=1}^{m}(\mathbf{X}_i - \overline{\mathbf{X}}_m)(\mathbf{X}_i - \overline{\mathbf{X}}_m)^\top.$$

We assume that Σ_m is regular and denote its inverse by Σ_m^{-1}. As stated in Proposition 2.1 in [24], any affine invariant test statistic for MVN is a function of the *Mahalanobis distances* and *angles*

$$D_{m,ij} \;=\; (\mathbf{X}_i - \overline{\mathbf{X}}_m)^\top \Sigma_m^{-1}(\mathbf{X}_j - \overline{\mathbf{X}}_m), \qquad i,j = 1, ..., m.$$

The test statistic in Mardia's multivariate skewness test is given by

$$b_{1,m} \;=\; \frac{1}{m^2}\sum_{i,j=1}^{m} D_{m,ij}^3\,,$$

and the multivariate kurtosis is estimated by

$$b_{2,m} \;=\; \frac{1}{m}\sum_{i=1}^{m} D_{m,ii}^2 \,.$$

Skewness is a measure of symmetry (spherical, elliptical, central or angular) of a distribution, and kurtosis of peakedness and tailweight. Critical values of these statistics for small sample sizes are listed for instance in the Appendix of [59]. For a sample size $m \gg 200$ the distributions of the test statistics can be approximated as follows. Under the MVN hypothesis, $\frac{m}{6} b_{1,m}$ is asymptotically $\chi^2_{q(q+1)(q+2)/6}$ distributed. The null–hypothesis is therefore rejected on a significance level α if $\frac{m}{6} b_{1,m}$ is greater that the $(1-\alpha)$–quantile of the $\chi^2_{q(q+1)(q+2)/6}$–distribution. Furthermore, the random variable

$$K_{q,m} \;=\; \sqrt{m}\,\frac{b_{2,m} - q(q+2)}{\sqrt{8q(q+2)}}$$

converges in distribution to a standard normal variate provided that the MVN assumption holds. Therefore, both large and small values of $K_{q,m}$ lead to rejection of the null–hypothesis.

The Henze–Zirkler test is based on the test statistic

$$\begin{aligned} T_{m,\beta} \;&=\; \frac{1}{m}\sum_{i,j=1}^{m} \exp\Big(-\frac{\beta^2}{2}\,(D_{m,ii} - 2D_{m,ij} + D_{m,jj})\Big) \\ &\quad -2\big(1+\beta^2\big)^{-\frac{q}{2}} \sum_{i=1}^{m} \exp\Big(-\frac{\beta^2}{2(1+\beta^2)}\,D_{m,ii}\Big) + m\big(1+2\beta^2\big)^{-\frac{q}{2}} \end{aligned}$$

for some parameter $\beta > 0$ and $m \in \mathbb{N}$, provided that Σ_m^{-1} is regular, and one puts $T_{m,\beta} = 4m$ elsewise. Note, that $T_{m,\beta}$ also can be written as

$$T_{m,\beta} \;=\; m \int_{\mathbb{R}^q} \Big|\psi_m(\mathbf{t}) - \exp\big(-\frac{1}{2}|\mathbf{t}|^2\big)\Big|^2 \varphi_\beta(\mathbf{t})\, d\mathbf{t}\,,$$

where $\psi_m(\mathbf{t}) = \frac{1}{m}\sum_{j=1}^{m} \exp(i\,\mathbf{t}^\top \mathbf{Y}_j)$ is the empirical characteristic function of the scaled residuals $\mathbf{Y}_j = \Sigma_m^{-1/2}(\mathbf{X}_j - \overline{\mathbf{X}}_m)$ and $\varphi_\beta(\mathbf{t})$ is the density of the q–variate normal distribution $\mathcal{N}_q(o, \beta^2 I_q)$. The value of the integral in the above formula ranges from 0 to 4, which explains the definition of $T_{m,\beta}$ for non–regular Σ_m. Besides this, $T_{m,\beta}$ can be represented as the L^2–distance between the density of a $\mathcal{N}_q(o, (1+(2\beta^2)^{-1})I_q)$ distribution and a kernel density estimator for $\mathbf{Y}_1, \ldots, \mathbf{Y}_m$ with standard normal kernel and bandwidth $(2\beta^2)^{-1}$. Large values of the smoothing parameter β increase the variance of the kernel density estimator, whereas small values of β reduce the random variation, but entail a systematic estimation error. Besides this, small values of β yield powerful tests against alternative distributions with heavy tails. When estimating the q–dimensional standard normal density, the (approximate) mean integrated square error is minimized by

$$\beta \;=\; \beta_q(m) \;=\; \frac{1}{\sqrt{2}}\left(\frac{2q+1}{4}\,m\right)^{\frac{1}{q+4}}, \qquad q,\; m \in \mathbb{N}\,,$$

see [68, pp. 86–87]. Empirical studies show that the Henze–Zirkler tests behaves similar to Mardia's skewness test if $\beta \to 0$ and similar to Mardia's kurtosis test if $\beta \to \infty$. Theoretical results on the power of the Henze–Zirkler test in terms of parameter β are still open research. Some critical values for $T_{m,\beta}$ with $m \in \{20, 30, 50\}$ are given in Section 3 of [25] for $\beta \in \{0.5, 1, 3\}$ and $\beta_q(m)$. For large sample sizes, i.e., $m \geq 50$, the distribution of the Henze–Zirkler test statistic can be approximated by a lognormal distribution with mean

$$\mu_\beta \;=\; 1 - v(\beta)^{-\frac{q}{2}} \left(1 + \frac{q\,\beta^2}{v(\beta)} + \frac{q(q+2)\beta^4}{2v(\beta)^2}\right)$$

and variance

$$\begin{aligned}\sigma_\beta^2 \;=\;& 2(1+4\beta^2)^{-\frac{q}{2}} + 2v(\beta)^{-q}\left(1 + \frac{2q\,\beta^4}{v(\beta)^2} + \frac{2q(q+2)\beta^8}{4v(\beta)^4}\right) \\ & -w(\beta)^{-\frac{q}{2}}\left(1 + \frac{3q\,\beta^4}{2w(\beta)} + \frac{q(q+2)\beta^8}{2w(\beta)^2}\right),\end{aligned}$$

where $v(\beta) = 1 + 2\beta^2$ and $w(\beta) = (1+\beta^2)(1+3\beta^2)$. The respective lognormal distributions were used to obtain approximate P–values for the tests displayed in Tables (D.1)–(D.3).

Appendix D

Numerical results

The subsequently appended tables show some numerical results for the estimation of the vector of intrinsic volume densities $\mathbf{v} = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))^\top$ of a planar Boolean model Ξ as discussed in Chapter 7.

Tables D.1–D.3 correspond to Section 7.2.3. Herein, one finds the approximate P–values corresponding to the tests on multivariate normality of $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(\cdot)} - \mathbf{v})$, $n \in \{1, \ldots, 5\}$ as described in Appendix C. Any P–value less than significance level $\alpha = 0.05$ is marked with an asterisk to allow a quick overview in which cases the null–hypothesis is rejected. Parameter n defines the size of the observation window $W_n = [-500n, 500n]^2$, and m represents the sample size.

Tables D.4–D.10, display some results for the estimation of the asymptotic covariance matrix $\Sigma_{\mathbf{v}}^{(\cdot)}$ of $\sqrt{|W_n|}(\widehat{v}_n^{(\cdot)} - \mathbf{v})$ with $n = 3$ as discussed in Section 7.3.3. The displayed percentages (`CoV`) represent the coefficient of variation of the respective components of the matrices, i.e., the quotient of the mean and the standard deviation in percent. Observation window W_3 is partitioned into $m_3 \in \{2^2, 3^3, \ldots, 8^2\}$ disjoint subwindows, each of them being a translated copy of $\check{W}_3 = [-1500/\sqrt{m_3}, 1500/\sqrt{m_3}]^2$. The sample size is equal to 200 if not indicated otherwise.

D.1 Tests for multivariate normality

	n	m	$p = 0.25$	$p = 0.5$	$p = 0.75$
Mardia's skewness test	1	1000	0.0000 *	0.0017 *	0.0009 *
Mardia's kurtosis test			0.5436	0.8066	0.5770
Henze–Zirkler, $\beta = 0.5$			0.0000 *	0.0002 *	0.0387 *
Henze–Zirkler, $\beta \approx 2.0548$			0.0166 *	0.0003 *	0.2519
Mardia's skewness test	2	1000	0.0031 *	0.0180 *	0.8917
Mardia's kurtosis test			0.6583	0.5364	0.4275
Henze–Zirkler, $\beta = 0.5$			0.0088 *	0.0421 *	0.6736
Henze–Zirkler, $\beta \approx 2.0548$			0.0502	0.1894	0.5131
Mardia's skewness test	3	1000	0.0025 *	0.9167	0.7907
Mardia's kurtosis test			0.7232	0.4474	0.9151
Henze–Zirkler, $\beta = 0.5$			0.0127 *	0.9094	0.7800
Henze–Zirkler, $\beta \approx 2.0548$			0.0377 *	0.8354	0.7697
Mardia's skewness test	3	3000	0.0146 *	0.3361	0.7718
Mardia's kurtosis test			0.4516	0.0412 *	0.4126
Henze–Zirkler, $\beta = 0.5$			0.0208 *	0.1824	0.5158
Henze–Zirkler, $\beta \approx 2.4040$			0.0098 *	0.0353 *	0.1888
Mardia's skewness test	4	3000	0.0105 *	0.2498	0.6820
Mardia's kurtosis test			0.3270	0.8925	0.3180
Henze–Zirkler, $\beta = 0.5$			0.0115 *	0.3568	0.4657
Henze–Zirkler, $\beta \approx 2.4040$			0.5139	0.1108	0.3068

Table D.1: Testing for 3–dimensional normal distribution of $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(1)} - \mathbf{v})^\top, n = 1, \ldots, 4$ for Model C with $M_0 = B_R(o)$, where $R \sim U(15, 30)$, and area fraction $p \in \{0.25, 0.5, 0.75\}$. The P–values for Mardia's skewness and kurtosis tests and the Henze–Zirkler test are derived from the asymptotic distributions of the respective test statistics. None of the tests were rejected on W_4 with $m = 1000$. For $p = 0.25$, the null–hypothesis was not rejected on W_5 with $m = 3000$.

	n	m	Model A	Model B	Model C
Mardia's skewness test	1	1000	0.3345	0.1869	0.5319
Mardia's kurtosis test			0.3871	0.0275 $*$	0.1563
Henze–Zirkler, $\beta = 0.5$			0.1148	0.0145 $*$	0.2455
Henze–Zirkler, $\beta \approx 2.0548$			0.6030	0.0491 $*$	0.4823
Mardia's skewness test	1	3000	0.0139 $*$	0.0028 $*$	0.0198 $*$
Mardia's kurtosis test			0.4860	0.0064 $*$	0.4196
Henze–Zirkler, $\beta = 0.5$			0.0011 $*$	0.0011 $*$	0.0294 $*$
Henze–Zirkler, $\beta \approx 2.4040$			0.0806	0.1750	0.4908
Mardia's skewness test	2	3000	0.6260	0.0687	0.1227
Mardia's kurtosis test			0.7181	0.1845	0.6436
Henze–Zirkler, $\beta = 0.5$			0.5796	0.0333 $*$	0.0336 $*$
Henze–Zirkler, $\beta \approx 2.4040$			0.4276	0.0353 $*$	0.3303
Mardia's skewness test	3	3000	0.3634	0.3816	0.6943
Mardia's kurtosis test			0.6089	0.6145	0.9394
Henze–Zirkler, $\beta = 0.5$			0.5099	0.1761	0.7008
Henze–Zirkler, $\beta \approx 2.4040$			0.5792	0.2480	0.4595
Mardia's skewness test	4	3000	0.6685	0.7539	0.8361
Mardia's kurtosis test			0.9297	0.5658	0.5007
Henze–Zirkler, $\beta = 0.5$			0.5382	0.3661	0.4960
Henze–Zirkler, $\beta \approx 2.4040$			0.5444	0.6933	0.2180

Table D.2: Testing for 3–dimensional normal distribution of $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(1)} - \mathbf{v})^\top, n = 1, \ldots, 4$ in case of Model A with $S_1 = 30$, Model B with $S_2 = 30$ and Model C with $R = 15$. The germ intensity of the respective models is chosen such that $p = 0.5$.

	n	m	$R = 20$	$R = 50$	$R = 80$
Mardia's skewness test	2	1000	0.1604	0.1216	0.0009 $*$
Mardia's kurtosis test			0.0757	0.0047 $*$	0.1795
Henze–Zirkler, $\beta = 0.5$			0.1333	0.1285	0.0000 $*$
Henze–Zirkler, $\beta \approx 2.0548$			0.3179	0.1179	0.0089 $*$
Mardia's skewness test	2	3000	0.0859	0.0062 $*$	0.0000 $*$
Mardia's kurtosis test			0.1448	0.0077 $*$	0.0000 $*$
Henze–Zirkler, $\beta = 0.5$			0.0487 $*$	0.0004 $*$	0.0000 $*$
Henze–Zirkler, $\beta \approx 2.4040$			0.2368	0.0628	0.0000 $*$
Mardia's skewness test	3	3000	0.3593	0.0388 $*$	0.0000 $*$
Mardia's kurtosis test			0.8374	0.660	0.0738
Henze–Zirkler, $\beta = 0.5$			0.7661	0.0105 $*$	0.0000 $*$
Henze–Zirkler, $\beta \approx 2.4040$			0.2916	0.4324	0.1502
Mardia's skewness test	4	3000	0.6087	0.0160 $*$	0.0002 $*$
Mardia's kurtosis test			0.8226	0.5363	0.3201
Henze–Zirkler, $\beta = 0.5$			0.8073	0.0530	0.0033 $*$
Henze–Zirkler, $\beta \approx 2.4040$			0.3097	0.2105	0.2023
Mardia's skewness test	5	3000	0.8621	0.1385	0.0003 $*$
Mardia's kurtosis test			0.4131	0.4269	0.2575
Henze–Zirkler, $\beta = 0.5$			0.4090	0.2205	0.0018 $*$
Henze–Zirkler, $\beta \approx 2.4040$			0.5357	0.3736	0.5379

Table D.3: Testing for 3–dimensional normal distribution of $\sqrt{|W_n|}(\widehat{\mathbf{v}}_n^{(1)} - \mathbf{v})^\top$ with $n = 2, \ldots, 5$ for Model C with $M_0 = B_R(o)$, where $R \in \{20, 50, 80\}$, and $p = 0.5$. On W_1, the Kolmogorov–Smirnov test for 1–dimensional normality of the component $\sqrt{|W_n|}(\widehat{v}_{n0}^{(1)} - v_0)$, $n = 1$ was rejected. None of the tests for $R = 80$ was rejected on W_5 with $m = 1000$ and on W_6 with $m \in \{1000, 3000\}$.

D.2 Estimation of the asymptotic covariance matrix

$\widehat{\Sigma}^{(1)}_{\mathbf{v},3}$	$j=0$	$j=1$	$j=2$
$i=0$	6.08582e-4	-3.4742e-3	-0.15734
CoV	13.02 %	-41.21 %	-10.99 %
$i=1$		0.25269	2.85763
CoV		19.73 %	15.14 %
$i=2$			178.8199
CoV			1.79 %

$\widehat{\Sigma}'^{(1)}_{\mathbf{v},3}$, $m_3=4$	$j=0$	$j=1$	$j=2$
$i=0$	6.1213e-4	-3.4582e-3	-0.16183
CoV	13.69 %	-46.98 %	-16.18 %
$i=1$		0.26935	2.85963
CoV		20.30 %	17.26 %
$i=2$			186.8540
CoV			1.78 %

$\widehat{\Sigma}'^{(1)}_{\mathbf{v},3}$, $m_3=9$	$j=0$	$j=1$	$j=2$
$i=0$	6.2199e-4	-3.3109e-3	-0.16839
CoV	15.01 %	-52.55 %	-13.06 %
$i=1$		0.28791	2.88117
CoV		21.36 %	18.92 %
$i=2$			198.1557
CoV			1.98 %

$\widehat{\Sigma}'^{(1)}_{\mathbf{v},3}$, $m_3=36$	$j=0$	$j=1$	$j=2$
$i=0$	5.52402e-4	-1.7209e-3	-0.18275
CoV	20.38 %	-128.50 %	-16.26 %
$i=1$		0.34180	2.73127
CoV		23.68 %	28.19 %
$i=2$			236.2194
CoV			2.25 %

Table D.4: Estimation results for $\Sigma^{(1)}_{\mathbf{v}}$ on W_3 using method HSS II with $P_{30,5.0}$ and $\Delta_{\tau'}=18$. The underlying Boolean model is Model C with $M_0=B_{15}(o)$ and $p=0.5$.

$\widehat{\Sigma}^{(2)}_{\mathbf{v},3}$	$j=0$	$j=1$	$j=2$
$i=0$	7.0076e-4	-7.1259e-3	-0.12581
CoV	11.19 %	-25.56 %	-16.89 %
$i=1$		0.37318	2.07876
CoV		16.31 %	36.50 %
$i=2$			163.2947
CoV			5.47 %

$\widehat{\Sigma}'^{(2)}_{\mathbf{v},3}$, $m_3=4$	$j=0$	$j=1$	$j=2$
$i=0$	7.1257e-4	-7.3000e-3	-0.12844
CoV	12.19 %	-27.47 %	-18.39 %
$i=1$		0.38836	1.99666
CoV		17.33 %	38.72 %
$i=2$			168.8983
CoV			5.70 %

$\widehat{\Sigma}'^{(2)}_{\mathbf{v},3}$, $m_3=9$	$j=0$	$j=1$	$j=2$
$i=0$	7.0536e-4	-7.2983e-3	-0.13170
CoV	12.99 %	-30.29 %	-20.03 %
$i=1$		0.39401	1.97677
CoV		19.12 %	42.11 %
$i=2$			171.1035
CoV			5.48 %

$\widehat{\Sigma}'^{(2)}_{\mathbf{v},3}$, $m_3=36$	$j=0$	$j=1$	$j=2$
$i=0$	6.8660e-4	-7.0165e-3	-0.14396
CoV	15.76 %	-38.20 %	-20.46 %
$i=1$		0.42687	1.69368
CoV		22.07 %	57.72 %
$i=2$			188.4459
CoV			5.99 %

Table D.5: Estimation results for $\Sigma^{(2)}_{\mathbf{v}}$ on W_3 using method PKF II with $P_{16,4.2}$ and $\Delta_{\tau'}=12$. The underlying Boolean model is Model C with $M_0=B_{15}(o)$ and $p=0.5$.

$\widehat{\Sigma}'^{(2)}_{\mathbf{v},3}$, $m_3 = 4$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	8.6008e-4	-0.01363	-0.05789
CoV	11.25 %	-18.20 %	-54.12 %
$i = 1$		0.64630	-0.81445
CoV		15.29 %	125.87 %
$i = 2$			198.8630
CoV			6.62 %

Table D.6: Estimation results for $\Sigma^{(2)}_{\mathbf{v}}$ on W_3 using method PKF II with $P_{16,4.2}$ and $\Delta_{\tau'} = 14$. The underlying Boolean model is Model C with $M_0 = B_{15}(o)$ and $p = 0.5$. Sample size = 1000.

$\widehat{\Sigma}'^{(1)}_{\mathbf{v},3}$, $m_3 = 4$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	2.1808e-4	-1.6783e-3	-0.08964
CoV	20.78 %	-63.61 %	-19.39 %
$i = 1$		0.33012	2.76647
CoV		13.43 %	18.17 %
$i = 2$			243.6825
CoV			3.36 %

$\widehat{\Sigma}'^{(1)}_{\mathbf{v},3}$, $m_3 = 9$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	2.1824e-4	-1.7477e-3	-0.09676
CoV	24.74 %	-61.28 %	-21.95 %
$i = 1$		0.34829	2.82671
CoV		12.34 %	19.73 %
$i = 2$			256.3089
CoV			3.41 %

Table D.7: Estimation results for $\Sigma^{(1)}_{\mathbf{v}}$ on W_3 using method HSS II with $P_{30,5.0}$ and $\Delta_{\tau'} = 18$. The underlying Boolean model is Model A with $M_0 = Sq(30, 0)$ and $p = 0.5$. Sample size = 1000.

$\widehat{\Sigma}'^{(1)}_{\mathbf{v},3}$, $m_3 = 4$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	7.2301e-4	-5.1863e-3	-0.23009
CoV	12.59 %	-33.17 %	-11.44 %
$i = 1$		0.30795	4.00297
CoV		21.44 %	16.25 %
$i = 2$			226.7397
CoV			3.16 %

$\widehat{\Sigma}'^{(1)}_{\mathbf{v},3}$, $m_3 = 9$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	7.2572e-4	-4.8422e-3	-0.23976
CoV	14.64 %	-40.79 %	-13.92 %
$i = 1$		0.31383	4.17154
CoV		26.60 %	18.99 %
$i = 2$			235.9122
CoV			3.84 %

Table D.8: Estimation results for $\Sigma^{(1)}_{\mathbf{v}}$ on W_3 using method HSS II with $P_{30,5.0}$ and $\Delta_{\tau'} = 18$. The underlying Boolean model is Model B with $M_0 = Sq(30, \alpha)$, $\alpha \sim U(0, 2\pi)$, and $p = 0.5$. Sample size = 1000.

$\widetilde{\Sigma}^{(1)}_{\mathbf{v},3}$, $m_3 = 4$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	6.0012e-4	-2.4867e-3	-0.16847
CoV	81.31 %	-256.21 %	-121.93 %
$i = 1$		0.20623	3.95441
CoV		81.85 %	104.43 %
$i = 2$			158.0438
CoV			85.85 %

$\widetilde{\Sigma}^{(1)}_{\mathbf{v},3}$, $m_3 = 16$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	6.6727e-4	-3.1071e-3	-0.19681
CoV	36.93 %	-107.66 %	-52.72 %
$i = 1$		0.24309	4.35508
CoV		37.56 %	47.18 %
$i = 2$			180.3999
CoV			36.87 %

$\widetilde{\Sigma}^{(1)}_{\mathbf{v},3}$, $m_3 = 36$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	7.3305e-4	-3.3046e-3	-0.22064
CoV	24.63 %	-74.24 %	-34.66 %
$i = 1$		0.28701	4.63297
CoV		24.71 %	33.09 %
$i = 2$			203.5369
CoV			24.07 %

$\widetilde{\Sigma}^{(1)}_{\mathbf{v},3}$, $m_3 = 64$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	1.8708e-4	-3.6681e-3	-0.23731
CoV	19.54 %	-56.71 %	-26.24 %
$i = 1$		0.33811	4.84975
CoV		18.25 %	25.91 %
$i = 2$			223.6724
CoV			17.99 %

Table D.9: Estimation results for $\Sigma^{(1)}_{\mathbf{v}}$ on a partition of W_3 into m_3 subwindows using estimator $\widetilde{\Sigma}^{(1)}_{\mathbf{v},n}$ with $P_{30,5.0}$ and $\Delta_\tau = 6$. The underlying Boolean model is Model C with $M_0 = B_{15}(o)$.

$\widetilde{\Sigma}^{(2)}_{\mathbf{v},3}$, $m_3 = 64$, $\Delta_\tau = 6$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	7.8558e-4	-4.2715e-3	-0.23948
CoV	17.39 %	-63.90 %	-27.82 %
$i = 1$		0.55571	1.20315
CoV		16.82 %	75.23 %
$i = 2$			303.5185
CoV			18.35 %

$\widetilde{\Sigma}^{(2)}_{\mathbf{v},3}$, $m_3 = 64$, $\Delta_\tau = 12$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	7.7135e-4	-5.2552e-3	-0.21730
CoV	18.47 %	-54.96 %	-30.17 %
$i = 1$		0.58368	-0.13098
CoV		18.00 %	-258.25 %
$i = 2$			308.3025
CoV			17.81 %

Table D.10: Estimation results for $\Sigma^{(2)}_{\mathbf{v}}$ on a partition of W_3 into $m_3 = 64$ subwindows using estimator $\widetilde{\Sigma}^{(2)}_{\mathbf{v},n}$ with $P_{16,4.2}$ and $\Delta_\tau = 6$ or $\Delta_{\tau''} = 12$. The underlying Boolean model is Model C with $M_0 = B_{15}(o)$.

List of notation

$\mathbb{N}$	set of positive integers: $\mathbb{N} = \{1, 2, \dots\}$
$\mathbb{N}_0$	set of non–negative integers: $\mathbb{N} = \{0, 1, 2, \dots\}$
$\mathbb{Z}^d$	$z = (z_1, ..., z_d)^\top \in \mathbb{Z}^d$ iff $z_i \in \{\dots, -1, 0, 1, \dots\}$ for $i = 1, ..., d, d \in \mathbb{N}$
$\mathbb{R}^d$	d–dimensional Euclidean space, $d \in \mathbb{N}$
$\mathbb{R}_+$	nonnegative real line: $\mathbb{R}_+ = [0, \infty)$
$\mathfrak{B}^d$	Borel σ-algebra on $\mathbb{R}^d$
$\mathfrak{B}_0^d$	σ-algebra of all bounded Borel-sets
$\mathcal{F}$	family of all closed Borel sets
$\mathcal{C}$	family of all compact Borel sets
$\mathcal{K}$	family of all compact and convex Borel sets (convex bodies)
$\mathcal{R}$	convex ring, i.e., familiy of all finite unions of convex bodies
$\mathcal{S}$	extended convex ring: $B \in \mathcal{S}$ iff $B \cap K \in \mathcal{R}$ for all $K \in \mathcal{K}$
$\mathfrak{B}(\mathcal{F})$	σ-algebra on $\mathcal{F}$, generated by $\{F \in \mathcal{F} : F \cap C \neq \emptyset\}$, $C \in \mathcal{C}$
$\mathfrak{B}(\mathcal{K})$	induced Borel σ–algebra on $\mathcal{K}$
$< , >$	scalar product, $< x, y >= \sum_{i=1}^d x_i y_i$
$\lvert \cdot \rvert$	$\lvert x \rvert = \sqrt{< x, x >}$ length of a vector, $\lvert B \rvert$ Lebesgue-measure (volume) of a set
$\lvert\lvert \cdot \rvert\rvert$	norm of a set: $\lvert\lvert B \rvert\rvert = \sup\{\lvert x \rvert : x \in B\}$
$\#$	cardinality number, number of elements of a set
$\cap$	intersection of two sets
$\cup$	union of two sets
$\bigcup_{i=1}^m$	union of $m \in \mathbb{N}$ or countably many ($m = \infty$) sets
$\setminus$	set difference, $A \setminus B = A \cap B^c$
∂A	boundary of a set A
$\mathrm{int}(A)$	interior of a set A
B^c	complement of a set or event B
$\check{B}$	reflection, $\check{B} = -B$

$\oplus$	operator of the Minkowski addition (p. 13)
$\ominus$	operator of the Minkowski subtraction (p. 13)
$B + x$	translation, $B + x = B \oplus \{x\} = \{b + x : b \in B\}$, $x \in \mathbb{R}^d$
o	origin in $\mathbb{R}^d$, $o = (0, \ldots, 0)^\top$
$B_r(x)$	closed ball in $\mathbb{R}^d$ with radius $r > 0$ and center $x \in \mathbb{R}^d$
$\mathbb{1}(A), \mathbb{1}_A$	indicator function of an event or a set
$\lfloor \cdot \rfloor$	floor function; $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$
Z, Ξ, M	random closed sets (p. 14)
X	simple point process on $\mathbb{R}^d$, germ process $X = \{X_l\}$ (p. 16)
$X_{\lvert B}$	restriction of X to B, $B \in \mathfrak{B}^d$, $X_{\lvert B} = \{X_l \in X : X_l \in B\}$
M_0	typical grain of a germ–grain process (p. 14)
$\{M_l\}$	set of i.i.d copies of M_0, set of grains
Ξ	germ–grain model, $\Xi = \bigcup_{l=1}^{\infty}(M_l + X_l)$ (p. 22)
$\Xi_{\lvert B}$	restriction of Ξ to $B, B \in \mathfrak{B}^d$, $\Xi_{\lvert B} = \bigcup_{l: X_l \in B}(M_l + X_l)$
N, N_Ξ	particle process on $\mathcal{K}'$, $N = \{Z_l\}$ (p. 22)
	in particular: $Z_l = (M_l + X_l)$
Ψ, Ψ_Ξ	marked point process, in particular: $\Psi = \{[M_l, X_l]\}$ (p. 19)
$V_j(\Xi \cap K)$	intrinsic volumes of $\Xi \cap K$ for $\Xi \in \mathcal{S}$ a.s., $K \in \mathcal{K}'$ (p. 28)
$\overline{V}_j(\Xi)$	intrinsic volume densities of RACS $\Xi \in \mathcal{S}$ a.s. (p. 32)
φ	measurable, conditionaly bounded valuation on $\mathcal{R}$ (p. 26)
$c(K)$	$c(K) = \sup\{\lvert\varphi(K')\rvert : K' \in \mathcal{K}', K' \subseteq K\}, K \in \mathcal{K}'$.
Y	random field on $\mathbb{R}^d$, $Y = \{Y(x), x \in \mathbb{R}^d\}$ (p. 25), in particular:
	$Y(x) = \varphi((\Xi - x) \cap K)$ with $\varphi : \mathcal{R} \to \mathbb{R}$ and $K \in \mathcal{K}'$
$Y_{\lvert B}$	defined by $Y_{\lvert B}(x) = \varphi((\Xi_{\lvert B} - x) \cap K)$, $x \in \mathbb{R}^d$
U	random field on $\mathbb{Z}^d$, $U = \{U(z), z \in \mathbb{Z}^d\}$
$(\Omega, \mathcal{A}, P)$	arbitrary probability space
$\mathbb{E}$	expectation
Var	variance
Cov	covariance
$\stackrel{D}{=}$	equal in distribution (law)
$\stackrel{D}{\longrightarrow}$	convergence in distribution
$\stackrel{L^2}{\longrightarrow}$	convergence in mean–square sense
$\mathcal{N}_q(\boldsymbol{\mu}, \Sigma)$	q–dimensional normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ; write $\mathcal{N}(\mu, \sigma^2)$ if $q = 1$

Zusammenfassung

In zahlreichen Anwendungsgebieten der räumlichen Statistik wie der Medizin, der Biologie oder der Geologie liegen große Datenmengen in Form von Bildern vor. Eine sich hieraus ergebende Aufgabenstellung ist die automatisierte Klassifizierung dieser Bilddaten anhand von bestimmten Charakteristiken. Denkt man zum Beispiel an das mikroskopische Bild eines Zellgewebes, so kann eine Kombination von verschiedenen geometrischen Eigenschaften (wie die spezifischen inneren Volumina) der Zellstruktur einen Hinweis auf signifikante und gegebenenfalls krankhafte Veränderungen geben (siehe [41]). Das zu Grunde liegende Material (zum Beispiel das Zellgewebe) lässt sich häufig durch ein Keim–Korn–Modell $\Xi = \bigcup_{l\geq 1}(M_l + X_l)$ mathematisch beschreiben. Die angesprochenen Charakteristiken werden in der vorliegenden Arbeit durch stationäre Zufallsfelder $\{Y(x), x \in \mathbb{R}^d\}$ ausgedrückt, die sich über eine Vorschrift der Art

$$Y(x) = \varphi((\Xi - x) \cap K)\,, \qquad x \in \mathbb{R}^d : K + x \subset W_n$$

von Ξ ableiten. Der Wert von Y an der Stelle x im Beobachtungsfenster W_n, $n \in \mathbb{N}$ repräsentiert dabei ein (zufälliges) "lokales" Merkmal von Ξ, welches über eine messbare, bedingt beschränkte und additive Funktion φ in einer kompakten, konvexen Nachbarschaft K von x bestimmt wird. Das primäre Ziel dieser Arbeit ist es, hinreichende theoretische Grundlagen für asymptotische Signifikanztests für den Erwartungswert $\mu = \mathbb{E}\,Y(x)$ bereit zu stellen.
Ein erwartungstreuer Schätzer für μ ist auf dem Beobachtungsfenster W_n, $n \in \mathbb{N}$ gegeben durch

$$\widehat{\mu}_n \;=\; \int_{W_n} Y(x)\, G(W_n, x)\, dx\,,$$

vgl. [28]. Hierbei bezeichnet $G(W_n, x)$ eine auf W_n normierte Gewichtsfunktion, die allen Messpunkten $x \in \mathbb{R}^d$ den Wert Null zuweist, für die $K + x$ nicht vollständig in W_n enthalten ist. Für eine Folge von "sich unbegrenzt ausdehnenden" Fenstern $\{W_n, n \in \mathbb{N}\}$ zeigen wir, dass $\{\widehat{\mu}_n,\ n \in \mathbb{N}\}$ konsistent im quadratischen Mittel bezüglich μ ist, falls die Kovarianzfunktion $\mathrm{Cov}_Y(x)$, $x \in \mathbb{R}^d$ des Zufallsfeldes Y absolut integrierbar ist. Ist Ξ ein Boolesches Modell mit konvexen und kompakten Körnern, dann ist durch $\mathbb{E}(|M_0 \oplus \check{K}|^2) < \infty$ eine hinreichende Bedingung für die Integrierbarkeit von $\mathrm{Cov}_Y(x)$, $x \in \mathbb{R}^d$ gegeben. Für allgemeine Keim–Korn–Modelle setzen wir voraus, dass $\mathbb{E}(|Y(x)|^{2+\delta}) < \infty$ für ein $\delta > 0$ erfüllt ist und dass der Durchmesser des Primärkorns M_0 endliche Momente der Ordnung $2d(\frac{\delta+1}{\delta}) + \varepsilon$ mit $\varepsilon > 0$ besitzt. Darüber hinaus wird angenommen, dass der zu Grunde liegende Punktprozess $\{X_l, l \geq 1\}$ gewissen Mischungsbedingungen genügt. Unter den jeweiligen

Voraussetzungen lassen sich zentrale Grenzwertsätze für $\{\sqrt{|W_n|}(\widehat{\mu}_n - \mu), n \in \mathbb{N}\}$ angeben. Im Fall des Booleschen Modells liefert ein Grenzwertsatz für Folgen m_n–abhängiger Zufallsfelder auf $\mathbb{Z}^d$ (siehe [17]) die Grundlage für die asymptotische Normalverteiltheit von $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$, $n \in \mathbb{N}$. Dahingegen lässt sich für beliebige Keim–Korn–Modelle ein zentraler Grenzwertsatz für β-mischende Zufallsfelder verwenden (siehe [21]).
Die asymptotische Varianz σ^2 von $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ ist gegeben durch

$$\sigma^2 \;=\; \theta \int_{\mathbb{R}^d} \mathrm{Cov}(Y(o), Y(x))\, dx\,.$$

für eine gewisse Konstante $\theta \in (0, \infty)$. Meist ist jedoch der Wert von σ^2 nicht bekannt. Daher wenden wir uns auch der Frage der konsistenten Schätzung der asymptotische Varianz σ^2 zu. In einem ersten Schätzverfahren betrachten wir einen nichtparametrischen Schätzer der Form

$$\widehat{\sigma}_n^2 \;=\; \int_{\mathbb{R}^d} \widehat{\mathrm{Cov}}_{Y,n}(x)\, \gamma_n(x)\, dx\,, \qquad n \in \mathbb{N},$$

wobei $\gamma_n(x)$ eine geeignete Gewichtsfunktion und $\widehat{\mathrm{Cov}}_{Y,n}(x)$ einen konsistenten Schätzer für $\mathrm{Cov}_Y(x) = \mathrm{Cov}(Y(o), Y(x))$ bezeichnet. Eine effiziente Auswertung von $\widehat{\sigma}_n^2$, $n \in \mathbb{N}$ kann mit Hilfe der Fast–Fourier–Transformation erreicht werden. Hierdurch motiviert definieren wir des Weiteren eine Variante des Schätzers $\widehat{\sigma}_n^2$, bei der die Berechnung auf mehreren Teilfenstern von W_n erfolgt und die einzelnen Schätzwerte anschließend über das arithmetische Mittel kombiniert werden. Die Idee der Unterteilung des Beobachtungsfensters greifen wir in einem dritten Schätzansatz erneut auf und betrachten die empirische Varianz

$$\widetilde{\sigma}_n^2 \;=\; \frac{|\tilde{W}_n|}{m-1} \sum_{k=1}^{m} \left(\widehat{\mu}_{kn} - \overline{\mu}_n\right)^2 \qquad \text{mit} \qquad \overline{\mu}_n = \frac{1}{m}\sum_{k=1}^{m} \widehat{\mu}_{kn}, \quad n \in \mathbb{N}$$

von Schätzern $\widehat{\mu}_{kn}$ für μ auf Teilfenstern $\tilde{W}_{kn}$, $k = 1, \ldots, m$ von W_n. Alle drei genannten Methoden liefern konsistente Schätzer für σ^2 unter gewissen Integrierbarkeitsbedingungen, die an gemischte Momente des Zufallsfeldes Y gestellt werden. Die Konsistenz von $\widehat{\sigma}_n^2$ und der oben erwähnten Variante kann im Fall des Booleschen Modells auf die Momentenbedingung $\mathbb{E}(|M_0 \oplus \check{K}|^2) < \infty$ zurückgeführt werden.
Die beschriebenen Schätzmethoden und Eigenschaften gelten nicht nur für eindimensionale, sondern auch für vektorielle Zufallsfelder. Die Ergebnisse lassen sich daher auf die simultane Schätzung der spezifischen inneren Volumina $\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)$ übertragen. In den letzten beiden Abschnitten der vorliegenden Arbeit befassen wir uns genauer mit dem Vektor der spezifischen inneren Volumina Boolescher Modelle im $\mathbb{R}^2$. Für die in [60] und [70] vorgeschlagenen Schätzverfahren geben wir einen neuen Algorithmus zur Berechnung der Werte der zu Grunde liegenden Zufallsfelder an. Dieser Algorithmus ermöglicht zum einen die Auswertung des Schätzansatzes aus [60] auch für die Varianzschätzung (im Gegensatz zu der effizienten Prozedur aus [33]), zum anderen wird die Evaluierung der Schätzwerte nach [70] beschleunigt (verglichen mit der Methode aus [12]). Numerische Beispiele zur (asymptotischen) multivariaten Normalverteiltheit, zur Laufzeit und zur Variabilität der Erwartungswert- und Varianzschätzer schließen die Arbeit ab.

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Erklärungen

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe.

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